Inverse of a Matrix

1. What is the inverse of a matrix
   a) If B is an inverse of A, then A is an inverse of B, A and B are inverses to each other
   b) A is called invertible if there is a matrix B, such that AB = I and BA = I
   c) Non-square matrix cannot be invertible
   d) The inverse of a matrix is unique
   e) A and B are invertible n*n matrices, AB is invertible: $(AB)^{-1} = B^{-1}A^{-1}$
   f) A is invertible, A^T is invertible: $(A^T)^{-1} = (A^{-1})^T$
2. Invertible
   a) Let A be an n x n matrix. A is invertible if and only if:
      - The columns of A span R^n
      - For every b in R^n, the system Ax=b is consistent
      - The rank of A is n
      - The columns of A are linear independent
      - The only solution to Ax=0 is the zero vector
      - The nullity of A is zero
      - The reduced row echelon form of A is I_n
      - A is a product of elementary matrices
      - There exists an n x n matrix B such that BA = I_n
      - There exists an n x n matrix C such that AC = I_n
   b) One-to-one
      i) If co-domain is smaller than domain, f cannot be one-to-one
      ii) If a matrix is dwarf-fat, it cannot be one-to-one
      iii) f(x) = b has at most one solution
      iv) if a matrix is one-to-one, its columns are independent
   c) Onto
      i) If co-domain is larger than domain, f cannot be onto
      ii) If a matrix is high-thin, it cannot be onto
      iii) f(x) = b always have solution
      iv) if a matrix is onto, rank A = num of rows
   d) one-to-one and onto
      i) the domain and co-domain must have the same size
      ii) the corresponding matrix is square
      iii) one-to-one ⇔ onto
3. Inverse of elementary matrices
   a) Every elementary row operation can be performed by matrix multiplication
   b) Find elementary matrix: apply the desired ERO on Identity matrix
   c) Inverse of elementary matrix: reverse ERO on identity matrix
4. Inverse of general invertible matrices
   a) Let A be n*n matrix, A is invertible if and only if RREF of A is I_n
   b) $E_k E_{k-1} \dots E_2 E_1 A = R = I_n$ ➔ $A^{-1} = E_k E_{k-1} \dots E_2 E_1$
   c) Let A be n*n matrix, transform [A I_n] into its RREF [R B], R is the RREF of A, B is an n*n matrix (not RREF), if R = I_n, then A is invertible and B = A^{-1}