第四讲　　数字特征

综述

1. 求数字特征
2. 应用

一、概念

1. 数学期望(EX)与方差(DX)

(1) 期望定义.

①$X \sim p_i \Rightarrow EX = \sum_i x_i p_i$.

②$X \sim f(x) \Rightarrow EX = \int_{-\infty}^{+\infty} x f(x)\mathrm{d}x$.

③$X \sim p_i, Y = g(X) \Rightarrow EY = \sum_i g(x_i) p_i$.

④$X \sim f(x), Y = g(X) \Rightarrow EY = \int_{-\infty}^{+\infty} g(x) f(x)\mathrm{d}x$.

⑤$(X,Y) \sim p_{ij}, Z = g(X,Y) \Rightarrow EZ = \sum\sum g(x_i, y_j) p_{ij}$.

⑥$(X,Y) \sim f(x,y), Z = g(X,Y) \Rightarrow EZ = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y) f(x,y)\mathrm{d}x\mathrm{d}y$.

(2) 方差定义.

$$DX \overset{\triangle}{=} E[(X-EX)^2].$$

① 定义法$\begin{cases} X \sim p_i \Rightarrow DX = E[(X-EX)^2] = \sum_i (x_i - EX)^2 p_i, \\ X \sim f(x) \Rightarrow DX = E[(X-EX)^2] = \int_{-\infty}^{+\infty} (x-EX)^2 f(x)\mathrm{d}x. \end{cases}$

② 公式法. $DX = E[(X-EX)^2] = E[X^2 - 2\cdot X \cdot EX + (EX)^2]$

$= E(X^2) - 2\cdot EX \cdot EX + (EX)^2$,

$DX = E(X^2) - (EX)^2$.

(3) 性质.

①$Ea = a, E(EX) = EX$.

(无条件打开)

②$E(aX+bY) = aEX + bEY, E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i EX_i$(无条件).

$E(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1EX_1 + a_2EX_2 + \cdots + a_nEX_n$.

③ 若 X,Y 相互独立,则 $E(XY) = EXEY$.

④$Da = 0, D(EX) = 0, D(DX) = 0$.

⑤ 若 X,Y 相互独立,则 $D(X \pm Y) = DX + DY$.

⑥$D(aX+b) = a^2DX, E(aX+b) = aEX + b$.

⑦ 一般,$D(X \pm Y) = DX + DY \pm 2\mathrm{Cov}(X,Y)$

$$D\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} DX_i + 2\sum_{1 \leqslant i < j \leqslant n} \mathrm{Cov}(x_i, x_j),$$ C_n^2 个

$D(X_1 + X_2 + \cdots + X_n) = DX_1 + DX_2 + \cdots + DX_n + \sum_{1 \leqslant i < j \leqslant n} 2\mathrm{Cov}(x_i, x_j)$.

【注】记住如下 EX, DX.

①$0-1$ 分布,$EX = p, DX = p - p^2 = (1-p)p, X \sim \begin{pmatrix} 1 & 0 \\ p & 1-p \end{pmatrix}$.

②$X \sim B(n,p), EX = np, DX = np(1-p)$.

③$X \sim P(\lambda), EX = \lambda, DX = \lambda$.

④$X \sim Ge(p), EX = \frac{1}{p}, DX = \frac{1-p}{p^2}$.

⑤$X \sim U[a,b], EX = \frac{a+b}{2}, DX = \frac{(b-a)^2}{12}$.

⑥$X \sim E_X(\lambda), EX = \frac{1}{\lambda}, DX = \frac{1}{\lambda^2}$.

⑦$X \sim N(\mu, \sigma^2), EX = \mu, DX = \sigma^2$.

⑧$X \sim \chi^2(n), EX = n, DX = 2n$.

2. 协方差 $\mathrm{Cov}(X,Y)$ 与相关系数 ρ_{XY}

(1)$\mathrm{Cov}(X,Y) = E[(X-EX)(Y-EY)]$,

$\mathrm{Cov}(X,X) = E[(X-EX)(X-EX)]$

$= E[(X-EX)^2] = DX$.

① 定义法.

$$\begin{cases} (X,Y) \sim p_{ij} \Rightarrow \mathrm{Cov}(X,Y) = \sum_i \sum_j (x_i - EX)(y_i - EY) p_{ij}, \\ (X,Y) \sim f(x,y) \Rightarrow \mathrm{Cov}(X,Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x-EX)(y-EY) f(x,y)\mathrm{d}x\mathrm{d}y. \end{cases}$$

② 公式法.

$\mathrm{Cov}(X,Y) = E(XY - X\cdot EY - EX \cdot Y + EX \cdot EY)$

$= E(XY) - EX \cdot EY$

②$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{DX}\sqrt{DY}} \begin{cases} \rho = 0 \Leftrightarrow X,Y \text{ 不相关}, \\ \rho \neq 0 \Leftrightarrow X,Y \text{ 相关}. \end{cases}$

(量纲为 1,无单位)

【例】设 $X \sim U\left(-\frac{1}{2}, \frac{1}{2}\right), Y = \cos X$,求 ρ_{XY}.

【分析】

$$\rho_{XY} = \frac{EXY - EX\,EY}{\sqrt{DX}\sqrt{DY}}$$

$$EX = \frac{a+b}{2} = \frac{-\frac{1}{2}+\frac{1}{2}}{2} = 0$$

$$X \sim f(x) = \begin{cases} 1, & -\frac{1}{2} < x < \frac{1}{2} \\ 0, & \text{其他} \end{cases}$$

$$EXY = E(\underbrace{X\cos X}_{g(X)}) = \int_{-\infty}^{+\infty} x\cos x\, f(x)$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} x\cos x\,\mathrm{d}x = 0$$

$\rho = 0 \Rightarrow X,Y$ 不相关

【例 5】将一枚硬币重复掷 n 次,以 X 和 Y 分别表示正面向上、反面向上的次数,则 $\rho_{XY} =$ ________.

【分析】

$$X + Y = n$$

$$\boxed{Y = -X + n}$$

$$Y = aX + b,\ a < 0 \Rightarrow \rho_{XY} = -1$$

(3) 性质.

①$\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$.

②$\mathrm{Cov}(aX, bY) = ab\mathrm{Cov}(Y,X)$.

③$\mathrm{Cov}(X_1 + X_2, Y) = \mathrm{Cov}(X_1, Y) + \mathrm{Cov}(X_2, Y)$.

④$|\rho_{XY}| \leqslant 1$.

⑤$\rho_{XY} = 1 \Leftrightarrow P\{Y = aX + b\} = 1(a > 0)$.

$\rho_{XY} = -1 \Leftrightarrow P\{Y = aX + b\} = 1(a < 0)$.

考试时:$Y = aX + b, a > 0 \Rightarrow \rho_{XY} = 1$.

$Y = aX + b, a < 0 \Rightarrow \rho_{XY} = -1$.

【小结】五个充要条件:

$\rho_{XY} = 0 \Leftrightarrow \mathrm{Cov}(X,Y) = 0 \Leftrightarrow E(XY) = EX \cdot EY$

$\Leftrightarrow D(X+Y) = DX + DY \Leftrightarrow D(X-Y) = DX + DY$.

X,Y 独立 $\Rightarrow \rho_{XY} = 0$.

若$(X,Y) \sim N \Rightarrow$

X,Y 独立 $\Leftrightarrow X,Y$ 不相关($\rho_{XY} = 0$).

二、综合题解析

① 求,② 用.

【例 1】试验成功的概率为$\frac{3}{4}$,失败的概率为$\frac{1}{4}$,独立重复试验直到成功两次为止,求试验次数 X 的 EX.

【分析】

$$P\{X=k\} = C_{k-1}^{1}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^{k-2} \quad \underbrace{\times\cdots\times\checkmark\times\cdots\times}_{k-1\text{次}} \quad \underset{k\text{th}}{\checkmark}$$

$$EX = \sum_{k=2}^{\infty} k \cdot C_{k-1}^{1}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^{k-2} \quad (k = 2, 3, \cdots)$$ 无穷多个

$$= \frac{9}{16}\sum_{k=2}^{\infty} k(k-1)\left(\frac{1}{4}\right)^{k-2}$$

$$= \frac{9}{16}\cdot\sum_{k=2}^{\infty} k(k-1)x^{k-2}\Big|_{x=\frac{1}{4}}$$

$$= \frac{9}{16}\cdot\sum_{k=2}^{\infty} (x^k)''\Big|_{x=\frac{1}{4}}$$

$$= \frac{9}{16}\cdot\left(\frac{x^2}{1-x}\right)''\Big|_{x=\frac{1}{4}}$$

$$= \frac{9}{16}\cdot\frac{2}{(1-x)^3}\Big|_{x=\frac{1}{4}} = \frac{8}{3}\checkmark$$