Part 1. Preliminaries

supplementary materials

(further-information)

Topic-1. trace 'tr(A)' and '$A^H = \bar{A}^T$'

Notations

Real $m\times n$ matrix: $\mathrm{R}^{m\times n} = \mathrm{R}^{m,n}$ (实矩阵)； Complex $m\times n$-matrix: $\mathrm{C}^{m\times n} = \mathrm{C}^{m,n}$ (复矩阵)

$\mathrm{R}^{m\times n} = \mathrm{R}^{m,n} = \{A = A_{m,n} = (a_{ij}) \mid a_{ij} \in \mathrm{R}(\text{real numbers}),\ 1\le \mathrm{i} \le \mathrm{m},\ 1 \le \mathrm{j} \le n\}$.

$\mathrm{C}^{m\times n} = \mathrm{C}^{m,n} = \{A = A_{m,n} = (a_{ij}) \mid a_{ij} \in \mathrm{C}(\text{complex numbers}),\ 1\le \mathrm{i} \le \mathrm{m},\ 1 \le \mathrm{j} \le n\}$.

$$\mathrm{R}^{m,n} \subset \mathrm{C}^{m,n}\ \ (m\times n \text{ matrixes}).$$

Square-matrix (if "$\mathrm{m} = n$ "): $\mathrm{C}^{\mathrm{n},\mathrm{n}} = \mathrm{C}^{\mathrm{n},\mathrm{n}} = \{A = A_{\mathrm{n,n}} = (a_{ij}) \mid a_{ij} \in \mathrm{C},\ 1 \le \mathrm{i},\ \mathrm{j} \le \mathrm{n}\}$

$$\mathrm{R}^{n,n} \subset \mathrm{C}^{n,n} (n\times n \text{ square matrixes})$$

Real real vector—space: $\mathrm{R}^n = \mathrm{R}^{\mathrm{n}\times 1} = \left\{ \mathrm{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} :\ x_1,\cdots,x_n \in \mathrm{R} \right\}$ (column--vectors),

Complex **vector-- space**: $\mathrm{C}^n = \mathrm{C}^{\mathrm{n}\times 1} = \left\{ \mathrm{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} :\ x_1,\cdots,x_n \in \mathrm{C} \right\}$ (column--vectors!)

$$\mathrm{R}^{\mathrm{n}} \subset \mathrm{C}^{\mathrm{n}}$$

Row--vector--space: $\mathrm{R}_{\mathrm{n}} = \mathrm{R}^{1\times n} = \{\mathrm{X} = (x_1,\cdots,x_n) \mid x_1,\cdots,x_n \in \mathrm{R}\}$ (row—vectors),

Row--vector--space: $\mathrm{C}_{\mathrm{n}} = \mathrm{C}^{1\times n} = \{\ \mathrm{X} = (x_1,\cdots,x_n) \mid x_1,\cdots,x_n \in \mathrm{C}\}$ (row—vectors), We can write a column vector as $X=(x_1, \cdots x_n)^{\mathrm{T}}=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$,

here "T" means "transpose".

eg(例子). a column vector : $\alpha=(1, \mathrm{i})^{\mathrm{T}}=\begin{pmatrix} 1 \\ i \end{pmatrix} \in \mathrm{C}^2$.

Let

$$A=A_{m\times n}=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n} \in \mathrm{C}^{m?}$$

We can write $A=(\alpha_1, \alpha_2, \cdots \alpha_n)$ (according to columns in A)

$$Here, \alpha_1=\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \alpha_2=\begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix}, \cdots, \alpha_n=\begin{pmatrix} a_{1,n} \\ \vdots \\ a_{m,n} \end{pmatrix} \in \mathrm{C}^{\mathrm{m}}, and$$

$$A=(\alpha_1, \alpha_2, \cdots \alpha_n) \in \mathrm{C}^{m?}$$

Recall. "Conjugate" of $w=a+ib$, for $a,b\in \mathrm{R}$ (are real), $(i=\sqrt{-1}, i^2=-1)$ is as follows.

$$\overline{w}=\overline{a+bi}=a-bi.$$

Or, complex number "$w=a+ib$" has its conjugat: $\overline{w}=\overline{a+bi}=a-bi$

And $(a+bi)\overline{(a+bi)}=(a+bi)(a-bi)=a^2+b^2\geq 0$. That is we get following remark.

Rk. $w=a+bi \Rightarrow w\cdot\overline{w}=|w|^2=a^2+b^2$.

The Conjugate of $A=(a_{i,j})$ is "$\overline{A}=(\overline{a_{i,j}})$" (共轭) as follows.

Let $A=(a_{i,j})=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n} \in \mathrm{C}^{m,n}$, then $\overline{A}=(\overline{a_{i,j}})=\begin{pmatrix} \overline{a_{11}} & \cdots & \overline{a_{1n}} \\ \vdots & \ddots & \vdots \\ \overline{a_{m1}} & \cdots & \overline{a_{mn}} \end{pmatrix}_{n\times m} \in \mathrm{C}^{m,n}$

eg(例). $A=\begin{pmatrix} 1 & i \\ i & 2 \end{pmatrix}$, $\Rightarrow \overline{A}=\begin{pmatrix} \overline{1} & \overline{i} \\ \overline{i} & \overline{2} \end{pmatrix}=\begin{pmatrix} 1 & -i \\ -i & 2 \end{pmatrix}$.

eg. For real matrix: $A=\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \in \mathrm{R}^{2,2}$, $\Rightarrow \overline{A}=\begin{pmatrix} \overline{1} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}=A.$

Rk (Remark)(备注). For any real matrix: $A=(a_{i,j})\in \mathrm{R}^{m,n}$, then

$$\overline{\overline{A}} = (\overline{\overline{a_{i,j}}}) = (a_{i,j}) = A.$$

Rk : $\overline{(AB)} = (\overline{A})(\overline{B})$ for any $A = A_{m\times n} \in \mathrm{C}^{m,n}$, $B = B_{n\times p} \in \mathrm{C}^{n,p}$

Conjugate-transpose: " $A^H = \overline{A}^T$ " or " $A^* = A^H = \overline{A}^T$ " (共轭转置记号).

$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$ has the conjugate-transpose: $A^H = \overline{A}^T = \begin{pmatrix} \overline{a_{11}} & \cdots & \overline{a_{m1}} \\ \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \cdots & \overline{a_{mn}} \end{pmatrix}_{n\times m} \in \mathrm{C}^{n\times m}$.

Rk (Remark). A^H is also called the "Hermite—transpose" or "H—transpose" of A.

Rk . $A = (a_{i,j}) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n} \in \mathrm{C}^{m,n}$, then $A^H = \begin{pmatrix} \overline{a_{11}} & \cdots & \overline{a_{m1}} \\ \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \cdots & \overline{a_{mn}} \end{pmatrix}_{n\times m} \in \mathrm{C}^{n\times m}$.

ie. $A = (a_{i,j}) \in \mathrm{C}^{m,n} \Rightarrow A^H = (\overline{a_{i,j}})^T \in \mathrm{C}^{n,m}$

eg(例) $A = \begin{pmatrix} 1 & i \\ 1 & i \\ 1 & i \end{pmatrix} \in \mathrm{C}^{3\times 2}$, $\Rightarrow A^H = \overline{A}^T = \begin{pmatrix} 1 & 1 & 1 \\ -i & -i & -i \end{pmatrix} \in \mathrm{C}^{2\times 3}$

Rk : Remark. If $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathrm{C}^n$, then $X^H = \begin{pmatrix} \overline{x_1}, & \overline{x_2}, & \cdots & \overline{x_n} \end{pmatrix}$ is a row-vector.

eg. $\mathrm{X} = \begin{pmatrix} 1 \\ i \end{pmatrix} \in \mathrm{C}^2$, then $\mathrm{X}^H = \overline{\mathrm{X}}^T = (\overline{1}, \ \overline{i}) = (1, \ -i)$ is a row-vector.

Rk . If $A \in \mathrm{R}^{m,n}$ is real, then $A^H = A^T$ (recall $\overline{A} = A$ for any real matrix A).

For a real vector $X \in \mathrm{R}^n$ (实向量), then $X^H = X^T$.

Rk . If $a \in \mathrm{C}$ is cplx-number(复数), then $(a)^H = (\overline{a})^T = \overline{a}$, ie. $\overline{a} = (a)^H$

Recall that " $a \in \mathrm{C}$ is real-number iff(if and only if) $\overline{a} = a$ ".

ie. " a is real-number $\Leftrightarrow$ (iff) $\overline{a} = a$ ".

Rk . a is a real-number $\Leftrightarrow$ (iff) $\overline{a} = a$ $\Leftrightarrow$ (iff) $(a)^H = \overline{a} = a$ "

Some properties (laws or rules)

① $(A^H)^H = A$ and $(A+B)^H = A^H + B^H$;

②$(kA)^H = \bar{k}(A^H)$, $k \in \mathbb{C}$ is cplx-number(复数),

③$(AB)^H = B^H A^H$, and $(ABC)^H = C^H B^H A^H$

Recall that $(AB)^T = B^T A^T$, and $(ABC)^T = C^T B^T A^T$ （穿脱公式）

Pf(proof) $\because \overline{ABC} = \bar{A}\,\bar{B}\,\bar{C} \Rightarrow (ABC)^H = (\overline{ABC})^T = (\bar{A}\,\bar{B}\,\bar{C})^T$

$$= (\bar{C})^T (\bar{B})^T (\bar{A})^T = C^H B^H A^H$$

$$\Rightarrow (ABC)^H = C^H B^H A^H$$

Recall that $(AB)^{-1} = B^{-1}A^{-1}$, $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$, if A, B, C are invertible.

Hermitian (Hermite-matrix)

Def. If $A^H = A \in \mathbb{C}^{n,n}$, $\boldsymbol{A}$ is called an Hermite-matrix,

we can say $\boldsymbol{A}$ is Hermitian .

eg(例): $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ is Hermite. (验证) check. $\because A^H = \begin{pmatrix} \bar{1} & \bar{i} \\ \overline{-i} & \bar{1} \end{pmatrix}^T = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = A$.

$B = \begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix}$ $(B^H = B)$ is Hermite.

Rk(注): If $A = A^H$ is Hermite, then all (对角元) $a_{11}, a_{22}, \cdots\cdots, a_{nn}$ are real numbers

Check $\because A = \begin{pmatrix} a_{11} & & & * \\ & a_{22} & & \\ & & \ddots & \\ * & & & a_{nn} \end{pmatrix} = A^H = \begin{pmatrix} \overline{a_{11}} & & & * \\ & \overline{a_{22}} & & \\ & & \ddots & \\ * & & & \overline{a_{nn}} \end{pmatrix}$

$\Rightarrow a_{11} = \overline{a_{11}}, \cdots, a_{nn} = \overline{a_{nn}}$ (they are real numbers) （实数）

Rk(注): If $A = (a_{i,j})$ is Hermite $(A = A^H)$, then every $a_{i,j} = \overline{a_{j,i}}$ for $1 \le i, j \le n$.

The checking is easy.

Rk: If $A^T = A \in \mathbb{R}^{n\times n}$ (real symmetry), then $A^H = A$ is Hermit.

The checking is easy $(A^H = A^T = A)$.

eg(例) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$ (real symmetry) $\Rightarrow A^H = A$ (A is Hermit).

Skew-Hermit(斜 Hermite). A is skew-Hermite, if $A^H = -A$,

Eg. $B=\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$ **is skew-Hermit**, $\because B^H=\begin{pmatrix} \bar{i} & \overline{-1} \\ \bar{1} & \bar{i} \end{pmatrix}=\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}=-\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}=-B$.

Rk(1). If B **is skew-Hermit(**$B^H=-B$**)**then iB and $\frac{B}{i}$ are both **Hermit**.

Rk(2). If A **is Hermit(**$A^H=A$**)**then iA and $\frac{A}{i}$ are both **skew-Hermit.**

$\because \quad (iB)^H=\overline{(i)}B^H=-i(-B)=iB$, recall $(kA)^H=\bar{k}A^H$

Rk(3) A **is Hermit** $\Leftrightarrow$ (iff) iA **is skew-Hermit.**

Thm(结论): Any $A=A_{m\times n}\Rightarrow\Rightarrow A^H A$ and AA^H are both **Hermit**

Pf: $\because (A^H A)^H = A^H (A^H)^H = A^H A$

Eg. $A=\begin{pmatrix} 1 \\ i \end{pmatrix}_{2\times 1}$ $A^H=(1 \ , \ \bar{i})=(1 \ , \ -i)_{1\times 2}$ then,

$AA^H=\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$ and $A^H A=(1^2+|i|^2)=(2)=2$ (1×1) are both **Hermit.**

$A=A_{n\times n}=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}_{n?} \in C^{n?}$, **trace**: $\mathrm{tr}(A)=a_{11}+a_{22}+\cdots\cdots a_{nn}$

Eigenvalues (Eigen-roots): $\lambda(A)=\{\lambda_1,\lambda_2,\cdots,\lambda_n\}$

Rk : Remark(备注) $\mathrm{tr}(A)=a_{11}+a_{22}+\cdots\cdots a_{nn}=\lambda_1+\lambda_2+\cdots\cdots\lambda_n=\mathrm{tr}(A)$.

Put. $A=A_{m\times n}=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}\in C^{m?}$; $A^H=\begin{pmatrix} \overline{a_{11}} & \cdots & \overline{a_{m1}} \\ \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \cdots & \overline{a_{mn}} \end{pmatrix}\in C^{n\times m}$

Trace-formula. $\mathrm{tr}(A^H A)=\mathrm{tr}(AA^H)=\sum |a_{i,j}|^2$, ie.

$$\mathrm{tr}(A^H A)=\mathrm{tr}(AA^H)=(|a_{11}|^2+|a_{12}|^2+\cdots\cdots|a_{1n}|^2)+\cdots\cdots(|a_{m1}|^2+|a_{m2}|^2+\cdots\cdots|a_{mn}|^2)$$

Eg(check). $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}_{3\times 2}$, $A^H=\begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} & \overline{a_{31}} \\ \overline{a_{12}} & \overline{a_{22}} & \overline{a_{32}} \end{pmatrix}_{2\times 3}$

$$AA^H = \begin{pmatrix} |a_{11}|^2+|a_{12}|^2 & * & * \\ * & |a_{21}|^2+|a_{22}|^2 & * \\ * & * & |a_{31}|^2+|a_{32}|^2 \end{pmatrix}_{3\times 3}$$

$$A^H A = \begin{pmatrix} |a_{11}|^2+|a_{21}|^2+|a_{31}|^2 & * \\ * & |a_{12}|^2+|a_{22}|^2+|a_{32}|^2 \end{pmatrix}_{2\times 2}$$

$$\therefore tr(AA^H) = tr(A^H A) = |a_{11}|^2+|a_{21}|^2+|a_{31}|^2+|a_{12}|^2+|a_{22}|^2+|a_{32}|^2 = \sum |a_{ij}|^2 .$$

Rk . For a column-vector $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}$, $X^H = (\overline{x_1}, \cdots\cdots, \overline{x_n})_{1\times n}$, then

$$\mathrm{tr}(X^H X) = \mathrm{tr}(XX^H) = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 = \sum |x_j|^2$$

Eg: $X = (1,\ i,\ i)^T = \begin{pmatrix} 1 \\ \mathrm{i} \\ \mathrm{i} \end{pmatrix} \in \mathrm{C}^3$, $X^H X = \mathrm{tr}(XX^H) = 1^2 + |i|^2 + |i|^2 = 3$

Trace-commutative(interchanged) formula for $\mathrm{tr}(AB^T)$

Rk . Put. $A = A_{m\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n} \in \mathrm{C}^{m,\,n}$, $B = B_{m\times n} = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}_{m\times n} \in \mathrm{C}^{m,\,n}$

$$B^T = \begin{pmatrix} b_{11} & \cdots & b_{m1} \\ \vdots & \ddots & \vdots \\ b_{1n} & \cdots & b_{mn} \end{pmatrix}_{m\times n} \in \mathrm{C}^{n,\,m} \quad \Rightarrow \quad AB^T \in \mathrm{C}^{m,\,m}, B^T A \in \mathrm{C}^{n,\,n} \text{ (They are squared)}$$

Tr-commutative formula(1). $\mathrm{tr}(AB^T) = \mathrm{tr}(B^T A) = \sum a_{i,j} b_{i,j}$

Eg(check) . $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}_{3\times 2}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}_{3\times 2}$, $B^T = \begin{pmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \end{pmatrix}_{2\times 3} \in \mathrm{C}^{2\times 3}$

$$\Rightarrow \quad AB^T = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{12} & * & * \\ * & a_{21}b_{21}+a_{22}b_{22} & * \\ * & * & a_{31}b_{31}+a_{32}b_{32} \end{pmatrix}_{3\times 3}$$

also, $B^H A = \begin{pmatrix} a_{11}b_{11}+a_{21}b_{21}+a_{31}b_{31} & * \\ * & a_{12}b_{12}+a_{22}b_{22}+a_{32}b_{32} \end{pmatrix}_{2\times 2}$

thus $\mathrm{tr}(AB^T) = \mathrm{tr}(B^T A) = a_{11}b_{11} + a_{21}b_{21} + a_{31}b_{31} + a_{12}b_{12} + a_{22}b_{22} + a_{32}b_{32}$,

That is to say (we can write) $\mathrm{tr}(AB^T) = \mathrm{tr}(B^T A) = \sum a_{i,j} b_{i,j}$

Rk. **Replacing** $B = (b_{i,j})$ by $(\overline{B}) = (\overline{b_{i,j}})$ in $tr(AB^T) = tr(B^T A) = \sum a_{i,j} b_{i,j}$

we get $tr(A(\overline{B})^T) = tr((\overline{B})^T A) = \sum a_{i,j} \overline{b_{i,j}}$, and note $(\overline{B})^T = B^H$.

we get again $tr(AB^H) = tr(B^H A) = \sum a_{ij} \overline{b_{ij}}$.

Tr-commutative formula(2): $\mathrm{tr}(AB^H) = \mathrm{tr}(B^H A) = \sum a_{ij} \overline{b_{ij}}$

Eg . $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}_{3\times 2}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}_{3\times 2}$, $B^H = \begin{pmatrix} \overline{b_{11}} & \overline{b_{21}} & \overline{b_{31}} \\ \overline{b_{12}} & \overline{b_{22}} & \overline{b_{32}} \end{pmatrix}_{2\times 3} \in \mathrm{C}^{2\times 3}$

$$AB^H = \begin{pmatrix} a_{11}\overline{b_{11}} + a_{12}\overline{b_{12}} & * & * \\ * & a_{21}\overline{b_{21}} + a_{22}\overline{b_{22}} & * \\ * & * & a_{31}\overline{b_{31}} + a_{32}\overline{b_{32}} \end{pmatrix}_{3\times 3}$$

$$B^H A = \begin{pmatrix} a_{11}\overline{b_{11}} + a_{21}\overline{b_{21}} + a_{31}\overline{b_{31}} & * \\ * & a_{12}\overline{b_{12}} + a_{22}\overline{b_{22}} + a_{32}\overline{b_{32}} \end{pmatrix}_{2\times 2}$$

$$\Rightarrow tr(AB^H) = tr(B^H A) = a_{11}\overline{b_{11}} + a_{21}\overline{b_{21}} + a_{31}\overline{b_{31}} + a_{12}\overline{b_{12}} + a_{22}\overline{b_{22}} + a_{32}\overline{b_{32}} = \sum a_{ij} \overline{b_{ij}}$$

Let A, B $\in \mathrm{C}^{m\times n}$

$$A = A_{m\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n} \qquad B = B_{m\times n} = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}_{m\times n} \in \mathrm{C}^{m\times n}$$

Here $B^H = (\overline{B})^T = \begin{pmatrix} \overline{b_{11}} & \cdots & \overline{b_{m1}} \\ \vdots & \ddots & \vdots \\ \overline{b_{1n}} & \cdots & \overline{b_{mn}} \end{pmatrix} \in \mathrm{C}^{n,\,m}$,and $AB^H \in \mathrm{C}^{m,\,m}$, $B^H A \in \mathrm{C}^{n,\,n}$

Rk. **Puting** $A = B$ $(a_{ij} = b_{ij})$ in $\mathrm{tr}(AB^H) = \mathrm{tr}(B^H A) = \sum a_{ij} \overline{b_{ij}}$

we get again $\mathrm{tr}(A^H A) = \mathrm{tr}(AA^H) = \sum a_{i,j} \overline{a_{i,j}} = \sum |a_{i,j}|^2$.

Recall. "Conjugate" of $w = a + ib$, for $a, b \in \mathrm{R}$ (are real), $(i = \sqrt{-1}, i^2 = -1)$ is as follows.

$$\overline{w} = \overline{a + bi} = a - bi.$$

Or, complex number "$w = a + ib$" has its conjugat: $\overline{w} = \overline{a + bi} = a - bi$

And $(a + bi)\overline{(a + bi)} = (a + bi)(a - bi) = a^2 + b^2 \geq 0$. That is

Rk. (we have) $w=a+bi \Rightarrow w\cdot\overline{w}=|w|^2=a^2+b^2$

Put $X=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$, $Y=\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}\in \mathrm{C}^n$, $Y^H=(\overline{y_1},\cdots\cdots\overline{y_n})$, then,

$$XY^H=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}(\overline{y_1},\cdots\cdots\overline{y_n})=\begin{pmatrix}x_1\overline{y_1} & x_2\overline{y_2} & \cdots & x_1\overline{y_n}\\ x_2\overline{y_1} & x_2\overline{y_2} & \cdots & x_2\overline{y_n}\\ \vdots & \vdots & \ddots & \\ x_n\overline{y_1} & x_n\overline{y_2} & \cdots & x_n\overline{y_n}\end{pmatrix}_{n\times n}$$

and $$Y^HX=(\overline{y_1},\cdots,\overline{y_n})\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=(x_1\overline{y_1}+x_2\overline{y_2}+\cdots+x_n\overline{y_n})_{1\times 1}$$

$$\Rightarrow \mathrm{tr}(XY^H)=\mathrm{tr}(Y^HX)=Y^HX=x_1\overline{y_1}+\cdots+x_n\overline{y_n}$$

Rk. $\mathrm{tr}(XY^H)=\mathrm{tr}(Y^HX)=Y^HX=x_1\overline{y_1}+\cdots+x_n\overline{y_n}$. for $X,Y\in\mathrm{C}^n$.

Inner product for C^n

Def. **The standard-inner-product**(标内积) in C^n is defined as follows

$$(X,Y)=x_1\overline{y_1}+y_2\overline{y_2}+\cdots+x_n\overline{y_n}, \quad \text{where } X=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}, Y=\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}\in\mathrm{C}^n$$

Or, we can write following definition.

Def. The Inner product in C^n is defined as (X,Y):

$$(X,Y)=Y^HX=\mathrm{tr}(XY^H)=x_1\overline{y_1}+\cdots+x_n\overline{y_n}, \quad \text{for } X,Y\in\mathrm{C}^n$$

that is $(X,Y)=x_1\overline{y_1}+\cdots+x_n\overline{y_n}$, for $X,Y\in\mathrm{C}^n$.

C^n with (X,Y) is called Unitary-space (U-space) .

Rk. $(X,X)=\mathrm{tr}(XX^H)=X^HX=x_1\overline{x_1}+\cdots+x_n\overline{x_n}=|x_1|^2+\cdots+|x_n|^2=|X|^2$

Here, $|X|=\sqrt{(X,X)}=\sqrt{|x_1|^2+|x_2|^2+\cdots+|x_n|^2}\ge 0$ is called the norm (length) of X .

Rk. $|X|^2=(X,X)=\mathrm{tr}(XX^H)=\mathrm{tr}(X^HX)=X^HX=|X|^2$.

i.e. $X^HX=(X,X)=|X|^2$, here $|X|^2=\sum|x_j|^2=|x_1|^2+\cdots+|x_n|^2$

Rk. $Y^H X = (X,Y)$, $X^H Y = (Y,X) = \overline{(X,Y)}$. .

Eg: $X = \begin{pmatrix} 1 \\ i \\ i \end{pmatrix} \in C^3$, $|X|^2 = X^H X = 1^2 + |i|^2 + |i|^2 = 3$; $|X| = \sqrt{3}$

Rk. $|kX| = |k||X|$, $|\frac{X}{k}| = \frac{|X|}{|k|}, (k \neq 0)$; and $|X \pm Y| \leq |X| + |Y|$.

Rk. If $X \neq 0$, $\frac{X}{|X|}$ is unit-vector ($\because$ $|\frac{X}{|X|}| = \frac{|X|}{|X|} = 1$).

Eg: For $X = \begin{pmatrix} 1 \\ i \\ i \end{pmatrix}$, $\frac{X}{|X|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ i \\ i \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ i/\sqrt{3} \\ i/\sqrt{3} \end{pmatrix}$ is a unit-vector.

Rk. If $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in R^n$ （real-vectors 实向量）, , then

$$(x,y) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n. \quad \text{here} \quad R^n \subset C^n$$

Some properties for inner product.

(P_1). $(X,X) \geq 0$, and $(X,X) > 0$ if $X \neq 0$;

(P_2): $(Y,X) = \overline{(X,Y)}$; (P_3): $(kX,Y) = k(X,Y)$, $(X,kY) = \bar{k}(X,Y)$, for $k \in C$

(P_4): $(X+Y,W) = (X,W) + (Y,W)$, $(W, X+Y) = (W,X) + (W,Y)$.

Rk. $|(X,Y)|^2 \leq (X,X)(Y,Y)$, i.e. $|(X,Y)| \leq |X| \cdot |Y|$

Inner product for $C^{m,n}$.

Def. The inner product in $C^{m,n}$ is defined by (A,B):

$$(A,B) = tr(AB^H) = tr(B^H A) = \sum a_{ij} \overline{b_{ij}},$$

where $A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{m \times n} \in C^{m,n}$

Some properties.

(P_1): $(A,A) = tr(AA^H) = \sum |a_{ij}|^2 \geq 0$, and $(A,A) > 0$ if $A \neq 0$.

(P_2): $(B,A) = \overline{(A,B)}$; (P_3): $(kA,B) = k(A,B)$, $(A,kB) = \bar{k}(A,B)$, $k \in C$

(P_4): $(A+B,D)=(A,D)+(B,D), \quad (D,A+B)=(D,A)+(D,B)$.

Rk. $|(A,B)|^2 \le (A,A)(B,B)$, i.e. $|(A,B)| \le \|A\| \cdot \|B\|$

Def. $\|A\| = \sqrt{(A,A)} = \sqrt{tr(AA^H)} = \sqrt{\sum |a_{i,j}|^2}$ **is called the norm of** A.

Rk. $\|A\|^2 = tr(AA^H) = tr(A^H A) = \sum |a_{ij}|^2$

Eg. $A = \begin{pmatrix} 1 & i \\ 1 & i \\ 1 & i \end{pmatrix}$, $\quad \|A\| = \sqrt{1^2+1^2+1^2+|i|^2+|i|^2+|i|^2} = \sqrt{6}$.

$$\text{Put. } A = A_{n\times p} = \begin{pmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,p} \end{pmatrix} \in \mathrm{C}^{n,p}, \quad A^H = \begin{pmatrix} \overline{a_{11}} & \cdots & \overline{a_{n1}} \\ \vdots & \ddots & \vdots \\ \overline{a_{1,p}} & \cdots & \overline{a_{n,p}} \end{pmatrix} \in \mathrm{C}^{p,n}$$

We can write $A = (\alpha_1, \ \alpha_2, \ \cdots \ , \alpha_p)$ (according to **columns in A**)

$$Here, \alpha_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \alpha_2 = \begin{pmatrix} a_{12} \\ \vdots \\ a_{n,2} \end{pmatrix}, \cdots, \alpha_p = \begin{pmatrix} a_{1,n} \\ \vdots \\ a_{n,p} \end{pmatrix} \in \mathrm{C}^n, and$$

$$A = (\alpha_1, \quad \alpha_2, \quad \cdots \quad \alpha_p) \in \mathrm{C}^{n,p}$$

Rk. $$A^H A = \left(\overline{(\alpha_i,\alpha_j)}\right) = \begin{pmatrix} \overline{(\alpha_1,\alpha_1)} & \overline{(\alpha_1,\alpha_2)} & \cdots & \overline{(\alpha_1,\alpha_p)} \\ \overline{(\alpha_2,\alpha_1)} & \overline{(\alpha_2,\alpha_2)} & \cdots & \overline{(\alpha_2,\alpha_p)} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{(\alpha_p,\alpha_1)} & \overline{(\alpha_p,\alpha_2)} & \cdots & \overline{(\alpha_p,\alpha_p)} \end{pmatrix} \in \mathrm{C}^{p,p}, \text{ if } A = A_{n,p} \in \mathrm{C}^{n,p}.$$

$$\text{That is } \quad A^H A = \begin{pmatrix} |\alpha_1|^2 & \overline{(\alpha_1,\alpha_2)} & \cdots & \overline{(\alpha_1,\alpha_p)} \\ \overline{(\alpha_2,\alpha_1)} & |\alpha_2|^2 & \cdots & \overline{(\alpha_2,\alpha_p)} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{(\alpha_p,\alpha_1)} & \overline{(\alpha_p,\alpha_2)} & \cdots & |\alpha_p|^2 \end{pmatrix}, \text{ for } (X,X) = |X|^2$$

Pf. (proof). $A = (\alpha_1, \quad \alpha_2, \quad \cdots \quad \alpha_p) \in \mathrm{C}^{n,p} \Rightarrow A^H = \begin{pmatrix} \alpha_1^H \\ \alpha_2^H \\ \vdots \\ \alpha_p^H \end{pmatrix} \in \mathrm{C}^{p,n}$.

Recall $\quad X^H Y = (Y,X) = \overline{(X,Y)}, \quad Y^H X = (X,Y)$. $\quad$ we get

$$A^{\mathrm{H}}A=\begin{pmatrix}\alpha_1^{\mathrm{H}}\\ \alpha_2^{\mathrm{H}}\\ \vdots\\ \alpha_p^{\mathrm{H}}\end{pmatrix}\begin{pmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_p\end{pmatrix}=\begin{pmatrix}\overline{(\alpha_1,\alpha_1)} & \overline{(\alpha_1,\alpha_2)} & \cdots & \overline{(\alpha_1,\alpha_p)}\\ \overline{(\alpha_2,\alpha_1)} & \overline{(\alpha_2,\alpha_2)} & \cdots & \overline{(\alpha_2,\alpha_p)}\\ \vdots & \vdots & \ddots & \vdots\\ \overline{(\alpha_p,\alpha_1)} & \overline{(\alpha_p,\alpha_2)} & \cdots & \overline{(\alpha_p,\alpha_p)}\end{pmatrix}$$

Rk. If (n = p) $A=(\alpha_1, \quad \alpha_2, \quad \cdots \quad ,\alpha_n)\in \mathrm{C}^{n,n}$ is square, then

$$A^H A=\left(\overline{(\alpha_i,\alpha_j)}\right)=\begin{pmatrix}\overline{(\alpha_1,\alpha_1)} & \overline{(\alpha_1,\alpha_2)} & \cdots & \overline{(\alpha_1,\alpha_n)}\\ \overline{(\alpha_2,\alpha_1)} & \overline{(\alpha_2,\alpha_2)} & \cdots & \overline{(\alpha_2,\alpha_n)}\\ \vdots & \vdots & \ddots & \vdots\\ \overline{(\alpha_n,\alpha_1)} & \overline{(\alpha_n,\alpha_2)} & \cdots & \overline{(\alpha_n,\alpha_n)}\end{pmatrix}\in \mathrm{C}^{n,n}$$

i.e. $$A^H A=\begin{pmatrix}|\alpha_1|^2 & \overline{(\alpha_1,\alpha_2)} & \cdots & \overline{(\alpha_1,\alpha_n)}\\ \overline{(\alpha_2,\alpha_1)} & |\alpha_2|^2 & \cdots & \overline{(\alpha_2,\alpha_n)}\\ \vdots & \vdots & \ddots & \vdots\\ \overline{(\alpha_n,\alpha_1)} & \overline{(\alpha_n,\alpha_2)} & \cdots & |\alpha_n|^2\end{pmatrix},$$ for $(X,X)=|X|^2$.

Rk.(row-vector formula) Put $X=(x_1,\cdots,x_n), Y=(y_1,\cdots,y_n)\in \mathrm{C}^{1\times n}$ are row-vectors. the inner-formula is $(X,Y)=XY^H=x_1\overline{y_1}+\cdots+x_n\overline{y_n}$

Put. $A=A_{m\times n}=\begin{pmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{m,1} & \cdots & a_{m,n}\end{pmatrix}\in \mathrm{C}^{m,n}$, $A^H=\begin{pmatrix}\overline{a_{11}} & \cdots & \overline{a_{m1}}\\ \vdots & \ddots & \vdots\\ \overline{a_{1,n}} & \cdots & \overline{a_{m,n}}\end{pmatrix}\in \mathrm{C}^{n,m}$

write $A=A_{m\times n}=\begin{pmatrix}A_1\\ \vdots\\ A_m\end{pmatrix}$(row − block), here $A_1=(a_{11},\cdots,a_{1n}),\cdots$

here $A^H=(A_1^H,\cdots,A_m^H)\in \mathrm{C}^{n,m}$

then $$AA^H==\begin{pmatrix}(A_1,A_1) & (A_1,A_2) & \cdots & (A_1,A_m)\\ (A_2,A_1) & (A_2,A_2) & \cdots & (A_2,A_m)\\ \vdots & \vdots & \ddots & \vdots\\ (A_m,A_1) & (A_m,A_2) & \cdots & (A_m,A_{mn})\end{pmatrix}\in \mathrm{C}^{m,m},$$ if $A\in \mathrm{C}^{m,n}$

Ortho.-vectors. **Put** $X=(x_1, \quad \cdots \quad ,x_n)^T, Y=(y_1, \quad \cdots \quad ,y_n)^T\in \mathrm{C}^n$.

$X\perp Y$(orthogonal) $\Leftrightarrow$ $(X,Y)=x_1\overline{y_1}+y_2\overline{y_2}+\cdots+x_n\overline{y_n}=0$, where $X,Y\in \mathrm{C}^n$

Rk. $X\perp Y \Leftrightarrow (Y,X)=\overline{(X,Y)}=y_1\overline{x_1}+y_2\overline{x_2}+\cdots+y_n\overline{x_n}=0.$

Rk. $X \perp Y \Leftrightarrow (Y,X)=0 \Leftrightarrow (X,Y)=0$.

Recall $X^H Y=(Y,X)=\overline{(X,Y)}$, $Y^H X=(X,Y)$.

Rk. $X \perp Y \Leftrightarrow X^H Y=0, \Leftrightarrow Y^H X=0$

Rk. $X \perp Y \Rightarrow aX \perp bY$, for $(aX, bY)=a\overline{b}(X,Y)=0$.

Eg. (check) : $X=\begin{pmatrix}1\\i\end{pmatrix} \perp Y=\begin{pmatrix}i\\1\end{pmatrix}$, $\because (X,Y)=1\cdot\overline{i}+i\cdot\overline{1}=-i+i=0 \ \therefore X \perp Y$

Eg. $X=\begin{pmatrix}1\\i\\i\end{pmatrix} \perp Y=\begin{pmatrix}2i\\1\\1\end{pmatrix}$. $\because (X,Y)=1\cdot\overline{2i}+i\cdot\overline{1}+i\cdot\overline{1}=-2i+i+i=0, \ \therefore X \perp Y$

Ortho-formula (1) $X \perp Y \Rightarrow |X \pm Y|^2=|X|^2+|Y|^2$

Pf. $|X+Y|^2=(X+Y,X+Y)=(X,X)+(X,Y)+(Y,X)+(Y,Y)$

$=(X,X)+0+0+(Y,Y)=|X|^2+|Y|^2$

Ortho-formula (2) $X \perp Y \Rightarrow |aX+bY|^2=|aX|^2+|bY|^2$, (for $X \perp Y \Rightarrow aX \perp bY$)

Ortho-formula (3) $X \perp Y \perp W \Rightarrow |aX+bY+cW|^2=|aX|^2+|bY|^2+|cW|^2$

Here "$X \perp Y \perp W$" means X, Y, W are mutualy othogonal(any two vectors are otho.).

Rk."$X_1 \perp X_2 \perp \cdots \perp X_p$" means they are mutualy othogonal(any two vectors are otho.)

Rk. $X_1 \perp X_2 \perp \cdots \perp X_p \Rightarrow |c_1X_1+c_2X_2+\cdots+c_pX_p|^2=|c_1X_1|^2+|c_2X_2|^2+\cdots+|c_pX_p|^2$

Def. If "$X_1 \perp X_2 \perp \cdots \perp X_p$", and any one $X_j \neq 0$ (nonzero!)

we say "$X_1, X_2, \cdots, X_p$"is an "**ortho-group**".

Eg. $X=\begin{pmatrix}1\\i\\i\end{pmatrix}, Y=\begin{pmatrix}2i\\1\\1\end{pmatrix}, W=\begin{pmatrix}0\\1\\-1\end{pmatrix}$ $(\because X \perp Y \perp W)$ is one "**ortho-group**"

Def.(pre-u) If $\alpha_1=\begin{pmatrix}*\\\vdots\\*\end{pmatrix}, \alpha_2=\begin{pmatrix}*\\\vdots\\*\end{pmatrix}, \cdots \alpha_n=\begin{pmatrix}*\\\vdots\\*\end{pmatrix} \in C^n$ is one **ortho-group,**

i.e. "$\alpha_1 \perp \alpha_2 \perp \cdots \perp \alpha_n$"and any $\alpha_j \neq 0$ (nonzero), we say the $n \times n$ matrix

$A=(\alpha_1, \ \alpha_2, \ \cdots \ \alpha_n)_{n\times n}$ is **pre-unitary(pre-u.) (**预备 U 阵)

Eg. Put $X_1=\begin{pmatrix}1\\i\\i\end{pmatrix}, X_2=\begin{pmatrix}2i\\1\\1\end{pmatrix}, X_3=\begin{pmatrix}0\\1\\-1\end{pmatrix}$ $(X_1 \perp X_2 \perp X_2)$, then

$$A=(X_1,X_2,X_3)=\begin{pmatrix}1 & 2i & 0\\ i & 1 & 1\\ i & 1 & -1\end{pmatrix} \text{ is pre-U.}$$

Eg. Put $\alpha_1=\begin{pmatrix}1\\i\end{pmatrix}$, $\alpha_2=\begin{pmatrix}i\\1\end{pmatrix}(\alpha_1 \perp \alpha_2)$, then $A=(\alpha_1,\alpha_2)=\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}$ is pre-U

Eg. $A=(\alpha_1,\alpha_2)=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$ is pre-U ; $\mathrm{B}=(\beta_1,\beta_2)=\begin{pmatrix}3 & -4\\ 4 & 3\end{pmatrix}$ is pre-U.

Def.(pre-c-u) If $\alpha_1=\begin{pmatrix}*\\ \vdots\\ *\end{pmatrix}, \alpha_2=\begin{pmatrix}*\\ \vdots\\ *\end{pmatrix}, \cdots \alpha_p=\begin{pmatrix}*\\ \vdots\\ *\end{pmatrix} \in \mathrm{C}^n$ $(p \le n)$ is one **ortho-group,**

i.e. "$\alpha_1 \perp \alpha_2 \perp \cdots \perp \alpha_p$ "and any $\alpha_j \neq 0$ (nonzero), we say the $n \times p$ matrix

$A=\left(\alpha_1, \ \alpha_2, \ \cdots \ \alpha_p\right)_{n\times p}$ is **pre-column-unitary(pre-c-u) (半预 U)**

Rk. When $p=n$ we get $A=\left(\alpha_1, \ \alpha_2, \ \cdots \ \alpha_n\right)_{n\times n}$ is pre-u(also pre-c-u).

Here,"$\alpha_1 \perp \alpha_2 \perp \cdots \perp \alpha_n$ "

Thm.1 $A=(\alpha_1,\alpha_2,\cdots,\alpha_p)_{n\times p}$ is pre--c--u $\Leftrightarrow$ $A^H A=\begin{pmatrix}|\alpha_1|^2 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & |\alpha_p|^2\end{pmatrix}$

i.e. $A=A_{n,p}$ is pre-u $\Leftrightarrow$ $A^H A=\begin{pmatrix}\mathrm{d}_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \mathrm{d}_p\end{pmatrix}$ is diagonal (对角形)

Pf. Recall $A^H A=\begin{pmatrix}\overline{(\alpha_1,\alpha_1)} & \overline{(\alpha_1,\alpha_2)} & \cdots & \overline{(\alpha_1,\alpha_p)}\\ \overline{(\alpha_2,\alpha_1)} & \overline{(\alpha_2,\alpha_2)} & \cdots & \overline{(\alpha_2,\alpha_p)}\\ \vdots & \vdots & \ddots & \vdots\\ \overline{(\alpha_p,\alpha_1)} & \overline{(\alpha_p,\alpha_2)} & \cdots & \overline{(\alpha_p,\alpha_p)}\end{pmatrix}$, $\because (\alpha_1,\alpha_2)=\cdots=(\alpha_1,\alpha_p)=0$.

thus, $A^H A=\begin{pmatrix}|\alpha_1|^2 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & |\alpha_p|^2\end{pmatrix}$ (is OK)

When $p=n$ we get similar result for a pre-u: $A=\begin{pmatrix}\alpha_1, & \alpha_2, & \cdots & \alpha_n\end{pmatrix}_{n\times n}$.

Thm.2 $A=(\alpha_1,\alpha_2,\cdots,\alpha_n)_{\mathrm{n\times n}}$ is pre--u $\Leftrightarrow$ $A^H A=\begin{pmatrix}|\alpha_1|^2 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & |\alpha_n|^2\end{pmatrix}_{\mathrm{n,n}}$

i.e. $A=A_{n,n}$ is pre-u $\Leftrightarrow$ $A^H A=\begin{pmatrix}\mathrm{d}_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \mathrm{d}_n\end{pmatrix}_{\mathrm{n,n}}$ is diagonal (对角形)

Eg. $A=(X_1,X_2)=\begin{pmatrix}1 & 2i\\ i & 1\\ i & 1\end{pmatrix}_{3\times 2}$ is pre-c-U. $\because$ $X_1=\begin{pmatrix}1\\ i\\ i\end{pmatrix}, X_2=\begin{pmatrix}2i\\ 1\\ 1\end{pmatrix}$ $(X_1\perp X_2)$

Eg. $A=(\alpha_1,\alpha_2)=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}_{2\times 2}$ is pre-c-u (also pre-u).

"U-matrix".

Def. (1) If $A=\begin{pmatrix}\alpha_1, & \alpha_2, & \cdots & \alpha_n\end{pmatrix}_{n\times n}$ is **pre-u, and** $|\alpha_1|=|\alpha_2|=\cdots=|\alpha_n|=1$

, we say $A=A_{n\times n}$ is **unitary, or** $A=A_{n\times n}$ is a **U-matrix.**

Rk. $A=\begin{pmatrix}\alpha_1, & \cdots & \cdots & \alpha_n\end{pmatrix}_{n\times n}$ is U $\Leftrightarrow$ $\alpha_1\perp\alpha_2\perp\cdots\perp\alpha_n$, $|\alpha_1|=\cdots=|\alpha_n|=1.$

Thm. $A=(\alpha_1,\alpha_2,\cdots,\alpha_n)_{\mathrm{n\times n}}$ is U. $\Leftrightarrow$ $A^H A=\begin{pmatrix}1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 1\end{pmatrix}_{\mathrm{n,n}}=I$.

Rk. $A=A_{n,n}$ is U $\Leftrightarrow$ $A^H A=I_{\mathrm{n}}$ $\Leftrightarrow$ $A^{-1}=A^H$ $\Leftrightarrow$ $AA^H=I_{\mathrm{n}}$ $(\because AA^{-1}=I)$

Pf. $\because A=(\alpha_1,\alpha_2,\cdots,\alpha_n)_{\mathrm{n\times n}}$ is U $\Leftrightarrow$ $A^H A=\begin{pmatrix}|\alpha_1|^2 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & |\alpha_n|^2\end{pmatrix}=\begin{pmatrix}1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 1\end{pmatrix}_{\mathrm{n,n}}=I$

Rk. The following are equivalent ! Put $A=A_{n,n}=(\alpha_1,\alpha_2,\cdots,\alpha_n)\in\mathrm{C}^{\mathrm{n,n}}$.

(1) $A=A_{n,n}=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ is U i.e. $\alpha_1\perp\alpha_2\perp\cdots\perp\alpha_n$, $|\alpha_1|=\cdots=|\alpha_n|=1$.

(2) $A=A_{n,n}$ is U $\Leftrightarrow$ $A^H A=I_{\mathrm{n}}$ **;** (3) $A=A_{n,n}$ is U $\Leftrightarrow$ $A^{-1}=A^H$ **.**

(4) $A=A_{n,n}$ is U $\Leftrightarrow$ $AA^H=I_{\mathrm{n}}$

Rk. If $A=A_{n,n}$, $\mathrm{B}=\mathrm{B}_{n,n}$ are both U, then AB is U.

Pf. $\because A=A_{n,n}, \mathrm{B}=\mathrm{B}_{n,n}$ are U $\Rightarrow (AB)^H = B^H A^H = B^{-1}A^{-1} = (AB)^{-1} \therefore AB$ is U .

Or, $(AB)^H (AB) = B^H (A^H A) B = B^H B = I \ \therefore AB$ is U

Properties for U-matrix. If $A=A_{n,n}$ is U , and $X,Y \in C^n$, then

(1) $|Ax|^2 = |x|^2$, $\because |AX|^2 = (AX)^H (AX) = X^H A^H AX = X^H IX = X^H X = |X|^2$

(2) $x \perp y \Rightarrow Ax \perp Ay.$

$\because x \perp y \Rightarrow (x,y) = y^H x = 0, \therefore (Ax,Ay) = (Ay)^H Ax = y^H A^H Ax = y^H x = 0 \Rightarrow Ax \perp Ay$

(3) $(Ax,Ay) = (x,y).$

$\because (Ax,Ay) = (Ay)^H Ax = y^H A^H Ax = y^H x = (x,y)$

Rk. $A=(\alpha_1, \quad \cdots \quad \cdots \quad \alpha_n)$ is pre-U $\Rightarrow \widetilde{A} = \left(\frac{\alpha_1}{|\alpha_1|}, \quad \cdots \quad \cdots \quad \frac{\alpha_n}{|\alpha_n|}\right)$ is U

Eg. $A=(X_1,X_2,X_3)=\begin{pmatrix} 1 & 2i & 0 \\ i & 1 & 1 \\ i & 1 & -1 \end{pmatrix}$ is pre-U, and $|X_1|=\sqrt{3}, |X_2|=\sqrt{6}, |X_3|=\sqrt{2}$, then

$$\widetilde{A} = \left(\frac{X_1}{|X_1|}, \frac{X_2}{|X_2|}, \frac{X_3}{|X_3|}\right) = \left(\frac{X_1}{\sqrt{3}}, \frac{X_2}{\sqrt{6}}, \frac{X_3}{\sqrt{2}}\right) = \begin{pmatrix} 1/\sqrt{3} & 2i/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{3} \end{pmatrix} \text{ is U.}$$

Eg. $A=(\alpha_1,\alpha_2)=\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}_{2\times 2}$ is pre-u, $|\alpha_1|=|\alpha_2|=\sqrt{2}$, then

$$\widetilde{A} = \left(\frac{\alpha_1}{|\alpha_1|}, \frac{\alpha_2}{|\alpha_2|}\right) = \left(\frac{\alpha_1}{\sqrt{2}}, \frac{\alpha_2}{\sqrt{2}}\right) = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \text{is U.}$$

Eg. $A=(\alpha_1,\alpha_2)=\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ is pre-u, $|\alpha_1|=|\alpha_2|=\sqrt{2}$, then $A=(\alpha_1,\alpha_2)=\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$

$$\widetilde{A} = \left(\frac{\alpha_1}{\sqrt{2}}, \frac{\alpha_2}{\sqrt{2}}\right) = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \text{ is U.}$$

Eg. $\mathrm{B}=(\beta_1,\beta_2)=\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$ is pre-U, then $\widetilde{B} = \left(\frac{\beta_1}{\sqrt{5}}, \frac{\beta_2}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$ is U.

Def.(c-u-matrix):

$A=(\alpha_1,\alpha_2,\cdots,\alpha_p)_{n\times p}$ is c--u (column-unitary) if $A=A_{n\times p}$ is pre-c-u, and $|\alpha_1|=\cdots=|\alpha_p|=1$.

Rk. $A=(\alpha_1,\alpha_2,\cdots,\alpha_p)_{n\times p}$ is c--u $\Leftrightarrow$ $\alpha_1\perp\alpha_2\perp\cdots\perp\alpha_P$, $|\alpha_1|=\cdots=|\alpha_P|=1$.

Thm. $A=(\alpha_1,\alpha_2,\cdots,\alpha_p)_{n\times p}$ is C- U. $\Leftrightarrow$ $A^HA=\begin{pmatrix}1&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&1\end{pmatrix}_{p,p}=I_p$, i.e. $A^HA=I$

Pf. $\because A=(\alpha_1,\cdots,\alpha_n)_{n\times n}$ is C-U $\Leftrightarrow$ $A^HA=\begin{pmatrix}|\alpha_1|^2&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&|\alpha_p|^2\end{pmatrix}=\begin{pmatrix}1&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&1\end{pmatrix}_{p,p}=I_p$

Rk. When p<n, $A=(\alpha_1,\cdots,\alpha_p)_{n\times p}$ is C- U. $\Leftrightarrow$ $A^HA=I_p$, but $AA^H\neq I$!

QR- formula: Let $A=\left(\alpha_1,\alpha_2,\cdots,\alpha_p\right)_{n\times p}$,(p $\le$ n), $\operatorname{rank}\left(A\right)=p$.

Then $A=QR$,

here $Q=Q_{n\times p}$ is C--U (i.e. $Q^HQ=I_p$) ;

$$R=R_{p\times p}=\begin{pmatrix}r_1&&*\\&r_2&&\\&&\ddots&\\O&&&r_p\end{pmatrix}(\text{is upper}-\Delta),$$

and $R=Q^HA.$

Esp. for (p = n) $A=A_{n\times n}$, we have ,

$A=QR$； $Q=Q_{n\times n}$ 为 U 阵'

（here $Q=Q_{n\times n}$ is U （$Q^HQ=I_n$, i.e. $Q^{-1}=Q^H$)）

Pf. recall “Schmidt-formula”.

Schmidt-otho-process. in C^n. Put

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2,\ \alpha_1)}{|\alpha_1|^2}\alpha_1$$

$$\vdots$$

$$\beta_p = \alpha_p - \frac{(\alpha_p, \alpha_1)}{|\alpha_1|^2}\alpha_1 - \frac{(\alpha_p, \beta_2)}{|\beta_2|^2}\beta_2 - \cdots - \frac{(\alpha_p, \beta_{p-1})}{|\beta_{p-1}|^2}\beta_{p-1}$$

Then $\beta_1 \perp \beta_2 \perp \cdots \perp \beta_p$

We can write

$$\begin{cases} \alpha_1 = \beta_1 \\ \alpha_2 = (*)\beta_1 + \beta_2 \\ \vdots \\ \alpha_s = (*)\beta_1 + (*)\beta_2 + \cdots + \beta_p \end{cases} \Rightarrow (\alpha_1, \alpha_2, \cdots, \alpha_p) = (\beta_1, \beta_2, \cdots, \beta_p)\begin{pmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Set $\varepsilon_1 = \frac{\beta_1}{|\beta_1|}, \varepsilon_2 = \frac{\beta_2}{|\beta_2|}, \cdots, \varepsilon_p = \frac{\beta_p}{|\beta_p|}$

i.e. $\beta_1 = |\beta_1|\varepsilon_1,\ \beta_2 = |\beta_2|\varepsilon_2,\ \cdots,\ \beta_p = |\beta_p|\varepsilon_p$

$$\Rightarrow (\beta_1, \beta_2, \cdots, \beta_p) = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_p)\begin{pmatrix} |\beta_1| & & & O \\ & |\beta_2| & & \\ & & \ddots & \\ O & & & |\beta_p| \end{pmatrix}_{p\times p}$$

$$\Rightarrow A = (\alpha_1, \alpha_2, \cdots, \alpha_p)(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_p)\begin{pmatrix} |\beta_1| & & & * \\ & |\beta_2| & & \\ & & \ddots & \\ O & & & |\beta_p| \end{pmatrix}\begin{pmatrix} 1 & & (*) \\ & 1 & \\ & & \ddots & \\ O & & & 1 \end{pmatrix}$$

Set $R = \begin{pmatrix} |\beta_1| & & & * \\ & |\beta_2| & & \\ & & \ddots & \\ O & & & |\beta_p| \end{pmatrix}\begin{pmatrix} 1 & & (*) \\ & 1 & \\ & & \ddots & \\ O & & & 1 \end{pmatrix} = \begin{pmatrix} |\beta_1| & & & \widetilde{(*)} \\ & |\beta_2| & & \\ & & \ddots & \\ O & & & |\beta_p| \end{pmatrix}$ $(upper - \Delta)$

$\Rightarrow \quad Q = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_p)_{n\times p}$ is C-U.

$\Rightarrow A = QR = Q_{n\times p}\ R_{p\times p}$., Note $Q^H Q = I$, and $A = QR \;\Rightarrow\; R = Q^H A$.

Eg. $A = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 4 & 2 \\ 0 & 0 & 1 \end{pmatrix}$, and $A = QR$, here , $Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$. Find $R = ?$

Ans. $R = Q^H A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & -2 & 0 \\ 1 & 4 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & 3\sqrt{2} & \sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}$,

and we get $A = QR$ as follows:

$$A = QR = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & 3\sqrt{2} & \sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

Eg. $A = \begin{pmatrix} 1 & 2i \\ i & 1 \\ i & 0 \end{pmatrix} = (\alpha_1, \alpha_2)$, find $A = QR$.

Ans. Set $\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ i \\ i \end{pmatrix}$,

$$\beta_2 = \alpha_2 - \frac{(\alpha_2, \alpha_1)}{|\alpha_1|^2}\alpha_1 = \alpha_2 - \frac{(\alpha_1^H \alpha_2)}{|\alpha_1|^2}\alpha_1 = \frac{1}{3}\begin{pmatrix} 5i \\ 4 \\ 1 \end{pmatrix}$$

Put $\varepsilon_1 = \frac{\beta_1}{|\beta_1|} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ i \\ i \end{pmatrix}$, $\varepsilon_2 = \frac{\beta_2}{|\beta_2|} = \frac{1}{\sqrt{42}}\begin{pmatrix} 5i \\ 4 \\ 1 \end{pmatrix}$

$$\Rightarrow \quad Q=(\varepsilon_1,\varepsilon_2)=\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{5i}{\sqrt{42}} \\ \frac{i}{\sqrt{3}} & \frac{4}{\sqrt{42}} \\ \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{42}} \end{pmatrix} \text{ (is C-U)}, \quad Q^H Q = I$$

$$\text{Let } R=Q^H A=\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-i}{\sqrt{3}} & \frac{-i}{\sqrt{3}} \\ \frac{-5i}{\sqrt{42}} & \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{42}} \end{pmatrix} A=\begin{pmatrix} \sqrt{3} & \frac{i}{\sqrt{3}} \\ 0 & \frac{\sqrt{14}}{\sqrt{3}} \end{pmatrix}$$

$$\Rightarrow \quad A=QR=\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{5i}{\sqrt{42}} \\ \frac{i}{\sqrt{3}} & \frac{4}{\sqrt{42}} \\ \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{42}} \end{pmatrix}\begin{pmatrix} \sqrt{3} & \frac{i}{\sqrt{3}} \\ 0 & \frac{\sqrt{14}}{\sqrt{3}} \end{pmatrix}.$$

Eg. $A=\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}=(\alpha_1,\alpha_2)$, find $A=QR.$

Here, $\alpha_1=(1,\ i)^T$, $\alpha_2=(i,\ 1)^T$,

Put $\beta_1=\alpha_1=(1,\ \ i)^T$, $|\beta_1|^2=2$, $|\beta_1|=\sqrt{2}$,

$$\beta_2=\alpha_2-\frac{(\alpha_2,\alpha_1)}{|\alpha_1|^2}\alpha_1=\alpha_2-0\bullet\alpha_1=\alpha_2=(i,\ 1)^T$$

$\Rightarrow \beta_1\perp\beta_2$ (i.e $\alpha_1\perp\alpha_2$)

Set $\varepsilon_1=\frac{\beta_1}{|\beta_1|}=\frac{1}{\sqrt{2}}\beta_1$, $\quad \varepsilon_2=\frac{\beta_2}{|\beta_2|}=\frac{1}{\sqrt{2}}\beta_2$,

$$Q=\left(\varepsilon_1,\varepsilon_2\right)=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} \text{ is U.}$$

Put $R=Q^H A=\begin{pmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}\end{pmatrix}$, we get $A=QR=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}\end{pmatrix}$

Eg. $A=\left(\alpha_1,\alpha_2,\alpha_3\right)=\begin{pmatrix}1 & -1 & 4\\ 1 & 4 & -2\\ 1 & 4 & 2\\ 1 & -1 & 0\end{pmatrix}_{4\times 3}$, find $A=QR$.

Ans. $\alpha_1=\left(1,1,1,1\right)^T$, $\alpha_2=\left(-1,4,4,-1\right)^T$, $\alpha_3=\left(4,-2,2,0\right)^T$

Let $\beta_1=\alpha_1=\left(1,\ 1,1,1\right)^T$, $\left|\beta_1\right|^2=4$, $\left|\beta_1\right|=2$.

$$\beta_2=\alpha_2-\frac{\left(\alpha_2,\alpha_1\right)}{\left|\alpha_1\right|^2}\alpha_1=\left(-\frac{5}{2},\frac{5}{2},\frac{5}{2},-\frac{5}{2}\right)^T=\frac{5}{2}\left(-1,1,1,-1\right)^T,\quad \left|\beta_2\right|=5$$

$$\beta_3=\alpha_3-\frac{\left(\alpha_3,\alpha_1\right)}{\left|\alpha_1\right|^2}\alpha_1-\frac{\left(\alpha_3,\ \beta_2\right)}{\left|\beta_2\right|^2}\beta_2=\left(2,-2,2,-2\right)^T,\quad \left|\beta_3\right|=4.$$

Put $\varepsilon_1=\frac{\beta_1}{\left|\beta_1\right|}=\frac{1}{2}\left(1,\ 1,1,1\right)^T$, $\varepsilon_2=\frac{\beta_2}{\left|\beta_2\right|}=\frac{1}{2}\left(-1,1,1,-1\right)^T$, $\varepsilon_3=\frac{\beta_3}{\left|\beta_3\right|}=\frac{1}{2}\left(1,-1,1,-1\right)^T$

Put $Q=\left(\varepsilon_1,\varepsilon_2,\varepsilon_3\right)=\begin{pmatrix}1/2 & -1/2 & 1/2\\ 1/2 & 1/2 & -1/2\\ 1/2 & 1/2 & 1/2\\ 1/2 & -1/2 & -1/2\end{pmatrix}=1/2\begin{pmatrix}1 & -1 & 1\\ 1 & 1 & -1\\ 1 & 1 & 1\\ 1 & -1 & -1\end{pmatrix}_{4\times 3}$ (it is C-U)

Set $R=Q^H A=\frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ -1 & 1 & 1 & -1\\ 1 & -1 & 1 & -1\end{pmatrix}A=\begin{pmatrix}2 & 3 & 2\\ 0 & 5 & -2\\ 0 & 0 & 4\end{pmatrix}$

$$\Rightarrow A=QR=1/2\begin{pmatrix}1 & -1 & 1\\ 1 & 1 & -1\\ 1 & 1 & 1\\ 1 & -1 & -1\end{pmatrix}\begin{pmatrix}2 & 3 & 2\\ 0 & 5 & -2\\ 0 & 0 & 4\end{pmatrix}.$$

Eg. $A=(\alpha_1,\alpha_2,\alpha_3)=\begin{pmatrix}0&1&1\\1&1&0\\1&0&1\end{pmatrix}$, find $A=QR.$

Ans. $\alpha_1=(0,1,1)^T,\alpha_2=(1,1,0)^T,\alpha_3=(1,0,1)^T$. By Schmidt ,

$\because\ \beta_1=\alpha_1=(0,1,1)^T,\ |\beta_1|=\sqrt{2},\ \varepsilon_1==\dfrac{\beta_1}{|\beta_1|}=\dfrac{1}{\sqrt{2}}(0,1,1)^T$

$\beta_2=\alpha_2-(\alpha_2,\varepsilon_1)\varepsilon_1=\left(1,\dfrac{1}{2},-\dfrac{1}{2}\right)^T,|\beta_2|=\dfrac{\sqrt{6}}{2},\varepsilon_2=\dfrac{\beta_2}{|\beta_2|}=\dfrac{1}{\sqrt{6}}(2,1-1)^T$

$\beta_3=\alpha_3-(\alpha_3,\varepsilon_1)\varepsilon_1-(\alpha_3,\varepsilon_2)\varepsilon_2=\left(1,\dfrac{1}{2},-\dfrac{1}{2}\right)^T,\ |\beta_3|=\dfrac{2}{\sqrt{3}},\ \varepsilon_3=\dfrac{\beta_3}{|\beta_3|}=\dfrac{1}{\sqrt{3}}(1,-1,1)^T$

Put $Q=(\varepsilon_1,\varepsilon_2,\varepsilon_3)=\begin{pmatrix}0&\dfrac{2}{\sqrt{6}}&\dfrac{1}{\sqrt{3}}\\\dfrac{1}{\sqrt{2}}&\dfrac{1}{\sqrt{6}}&-\dfrac{1}{\sqrt{3}}\\\dfrac{1}{\sqrt{2}}&-\dfrac{1}{\sqrt{6}}&\dfrac{1}{\sqrt{3}}\end{pmatrix}$ (is U), $R=Q^HA=\begin{pmatrix}\sqrt{2}&\dfrac{1}{\sqrt{2}}&\dfrac{1}{\sqrt{2}}\\0&\dfrac{\sqrt{6}}{2}&\dfrac{1}{\sqrt{6}}\\0&0&\dfrac{2}{\sqrt{3}}\end{pmatrix}$

$\Rightarrow\quad A=QR.$

Ex. find $A=QR.$ （1） $A=\begin{pmatrix}1&i\\1&-i\end{pmatrix}$ （2） $A=\begin{pmatrix}1&i\\1&1\\1&-1\\i&0\end{pmatrix}$

(3) $A=\begin{pmatrix}0&3&1\\0&4&-2\\1&1&2\end{pmatrix}$, (4) $A=\begin{pmatrix}1&1\\1&-1\\1&1\\1&0\end{pmatrix}_{4\times2}$

许米（**Schmidt--otho-process**）

已知α_1， α_2， α_3 $\in C^n$ 线性无关，

令$\beta_1=\alpha_1$，

$$\beta_2=\alpha_2-\frac{(\alpha_2,\alpha_1)}{|\alpha_1|^2}\alpha_1$$

$$\beta_3=\alpha_3-\frac{(\alpha_3,\alpha_1)}{|\alpha_1|^2}\alpha_1-\frac{(\alpha_3,\beta_2)}{|\beta_2|^2}\beta_2$$ 则 β_1,β_2,β_3 互正交。

Check：$(\beta_2,\ \beta_1)=\left(\alpha_2-\frac{(\alpha_2,\alpha_1)}{|\alpha_1|^2}\alpha_1,\ \alpha_1\right)=(\alpha_2,\ \alpha_1)-\frac{(\alpha_2,\alpha_1)}{|\alpha_1|^2}(\alpha_1,\ \alpha_1)$

$=(\alpha_2,\ \alpha_1)-(\alpha_2,\ \alpha_1)=0$ $\quad\therefore \beta_2\perp\beta_1$

and, $(\beta_3,\ \beta_1)=(\beta_3,\ \alpha_1)$ （代入 β_3）

$$=(\alpha_3,\ \alpha_1)-\frac{(\alpha_3,\alpha_1)}{|\alpha_1|^2}(\alpha_1,\ \alpha_1)-\frac{(\alpha_3,\beta_2)}{|\beta_2|^2}(\beta_2,\ \beta_1)$$

$=(\alpha_3,\ \alpha_1)-(\alpha_3,\ \alpha_1)-0=0$ $\quad\therefore \beta_3\perp\beta_1$

$$\beta_2=\alpha_2-\frac{(\alpha_2,\beta_1)}{|\beta_1|^2}\beta_1 \qquad (\because \beta_1=\alpha_1)$$

$$\beta_3=\alpha_3-\frac{(\alpha_3,\beta_1)}{|\beta_1|^2}\beta_1-\frac{(\alpha_3,\beta_2)}{|\beta_2|^2}\beta_2$$

$$(\beta_3,\ \beta_2)=(\alpha_3,\ \beta_2)-\frac{(\alpha_3,\beta_1)}{|\beta_1|^2}(\beta_1,\ \beta_2)-\frac{(\alpha_3,\beta_2)}{|\beta_2|^2}(\beta_3,\ \beta_2)$$

$=(\alpha_3,\ \beta_2)-0-(\alpha_3,\ \beta_2)=0$ $\quad\therefore \beta_3\perp\beta_2$

正交化的目的是造 U 阵，用“许米”正交方法可作一个列 U 阵

由 $\alpha_1,\alpha_2,\alpha_3$ 产生 β_1,β_2,β_3 （互正交）

令 $Q=\left(\frac{\beta_1}{|\beta_1|},\frac{\beta_2}{|\beta_2|},\frac{\beta_3}{|\beta_3|}\right)$ 是列 U 阵

可知 $Q^HQ=I$

A=QR（i.e.　A= UR）

Case1：Let $A=(\alpha_1,\cdots,\alpha_t)_{n\times t}$ 列无关（列满秩）

（或 $r(A)=t$），则有 QR 分解 A=QR

其中$Q=(\varepsilon_1,\cdots,\varepsilon_t)_{n\times t}$是列 U 阵，$R=\begin{pmatrix} b_1 & & * \\ & \ddots & \\ 0 & & b_t \end{pmatrix}$是上三角，且$b_1>0,\cdots,b_t>0$

注：Q 可用“许米公式”构造

由（许米）产生$\beta_1,\cdots,\beta_t$ （互正交）

令$\varepsilon_1=\dfrac{\beta_1}{|\beta_1|},\cdots,\varepsilon_t=\dfrac{\beta_t}{|\beta_t|}$

令$Q=(\varepsilon_1,\cdots,\varepsilon_t)$ 则 Q 是列 U 阵，$Q^HQ=I$

Case2：设方阵$A=(\alpha_1,\cdots,\alpha_n)_{n\times n}$ 列无关（$|A|\neq 0$，可逆）

则有 A=QR，其中$Q=Q_{n\times n}$是 U 阵（正交阵），

$R=\begin{pmatrix} b_1 & & * \\ & \ddots & \\ 0 & & b_n \end{pmatrix}$是上三角，且$b_1>0,\cdots,b_n>0$

Case3：任一方阵$A=A_{n\times n}$ 都有分解

A=QR， Q 是 U 阵，$R=\begin{pmatrix} b_1 & & * \\ & \ddots & \\ 0 & & b_n \end{pmatrix}$是上三角

注：在公式 A=QR 中（$Q^HQ=I$），可解出$R=Q^HA$

因为$A=QR\Rightarrow Q^HA=Q^H(QR)=(Q^HQ)R=R\Rightarrow R=Q^HA$

U 阵等价条件：设$A=\left(\alpha_1,\quad \alpha_2,\quad \cdots\quad \alpha_n\right)_{n\times n}$为方阵，则以下互相等价

1. $A=A_{n\times n}$为u阵（$A^HA=I_n$），即 A 的列$\alpha_1,\quad \alpha_2,\quad \cdots\quad \alpha_n$互正交，长度都为 1.

2. $A^{-1}=A^H$ 或 $A^H=A^{-1}$

3. $A^HA=I$ 且 $AA^H=I$

4. $AA^H=I$

5. A 的各行向量互正交，且长度为 1

半 U 阵（列u阵）性质：

1. 若 A 为列 u 阵，则 $|Ax|^2=|x|^2$ （保长性）

证：用长度平方公式 $|x|^2=x^Hx, x\in C^n$

$|Ax|^2=(Ax)^H(Ax)=x^HA^HAx=x^HIx=x^Hx=|x|^2$

2. 若 A 为半 U 阵（列 U），$x\perp y$ 则 $Ax \perp Ay$ （保持正交性）

证：$\because x\perp y \therefore (x,y)=y^Hx=0$

$\because (Ax,Ay)=(Ay)^HAx=y^HA^HAx=y^Hx=0\Rightarrow Ax\perp Ay$

Rk.

$$D=\begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_n \end{pmatrix}_{n\times n} \quad D^H=\begin{pmatrix} \overline{b_1} & & & \\ & \overline{b_2} & & \\ & & \ddots & \\ & & & \overline{b_n} \end{pmatrix}_{n\times n}$$

$$\Rightarrow \quad DD^H=D^HD=\begin{pmatrix} |b_1|^2 & & 0 \\ & \ddots & \\ 0 & & |b_n|^2 \end{pmatrix}; \qquad A=\begin{pmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}(\text{upper}-\Delta),$$

$$B=\begin{pmatrix} b_1 & & & * \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_n \end{pmatrix}_{n\times n}$$ (Upper-Δ). Here, $\lambda(A)=\{\lambda_1,\lambda_2,\cdots\cdots\lambda_n\}$, $\lambda(B)=\{b_1,b_2,\cdots\cdots b_n\}$

$$:AB=\begin{pmatrix} \lambda_1b_1 & & & * \\ & \lambda_2b_2 & & \\ & & \ddots & \\ 0 & & & \lambda_nb_n \end{pmatrix}$$

Esp. $$A^k=\begin{pmatrix} \lambda_1^{\ k} & & & * \\ & \lambda_2^{\ k} & & \\ & & \ddots & \\ 0 & & & \lambda_{n1}^{\ k} \end{pmatrix}$$ （k=1，2，…）

Let

$$A=A_{m\times n}=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n} \in \mathrm{C}^{m?}$$

We can write $A=\begin{pmatrix}\alpha_1, & \alpha_2, & \cdots & \alpha_n\end{pmatrix}$ (according to columns in "A")

$$Here, \alpha_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \alpha_2 = \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix}, \cdots, \alpha_n = \begin{pmatrix} a_{1,n} \\ \vdots \\ a_{m,n} \end{pmatrix} \in \mathrm{C}^{\mathrm{m}}, and$$

$$A = \begin{pmatrix}\alpha_1, & \alpha_2, & \cdots & \alpha_n\end{pmatrix} \in \mathrm{C}^{\mathrm{m,\ n}}$$