Part 1. Preliminaries

supplementary materials

trace 'tr(A)' and '$A^H = \bar{A}^T$'

Notations

Real $m\times n$ matrix: $\mathrm{R}^{m\times n}=\mathrm{R}^{m,n}$(实矩阵)；Complex $m\times n$-matrix: $\mathrm{C}^{m\times n}=\mathrm{C}^{m,n}$(复矩阵)

$\mathrm{R}^{m\times n}=\mathrm{R}^{m,n}=\{A=A_{m,n}=(a_{ij})\,|\,a_{ij}\in\mathrm{R}(\text{real numbers}),\ 1\le \mathrm{i}\le \mathrm{m},\ 1\le \mathrm{j}\le n\}$.

$\mathrm{C}^{m\times n}=\mathrm{C}^{m,n}=\{A=A_{m,n}=(a_{ij})\,|\,a_{ij}\in\mathrm{C}(\text{complex numbers}),\ 1\le \mathrm{i}\le \mathrm{m},\ 1\le \mathrm{j}\le n\}$.

$$\mathrm{R}^{m,n}\subset\mathrm{C}^{m,n}\ \ (m\times n \text{ matrixes}).$$

Square-matrixs (方阵 $\mathrm{m}=n$): $\mathrm{C}^{\mathrm{n,n}}=\mathrm{C}^{\mathrm{n,n}}=\{A=A_{\mathrm{n,n}}=(a_{ij})\,|\,a_{ij}\in\mathrm{C},\ 1\le\mathrm{i},\ \mathrm{j}\le\mathrm{n}\}$

$$\mathrm{R}^{n,n}\subset\mathrm{C}^{n,n}(n\times n \text{ square matrixes})$$

Real real vector—space: $\mathrm{R}^n=\mathrm{R}^{\mathrm{n}\times 1}=\left\{\mathrm{X}=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}:\ x_1,\cdots,x_n\in\mathrm{R}\right\}$ (column--vectors!),

Complex **vector-- space**: $\mathrm{C}^n=\mathrm{C}^{\mathrm{n}\times 1}=\left\{\mathrm{X}=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}:\ x_1,\cdots,x_n\in\mathrm{C}\right\}$ (column--vectors!)

$$\mathrm{R}^{\mathrm{n}}\subset\mathrm{C}^{\mathrm{n}}$$

Row--vector--space: $\mathrm{R}_{\mathrm{n}}=\mathrm{R}^{1\times n}=\{\mathrm{X}=(x_1,\cdots,x_n)\,|\ x_1,\cdots,x_n\in\mathrm{R}\}$ (row—vectors),

Row--vector--space: $\mathrm{C}_{\mathrm{n}}=\mathrm{C}^{1\times n}=\{\ \mathrm{X}=(x_1,\cdots,x_n)\,|\ x_1,\cdots,x_n\in\mathrm{C}\}$ (row—vectors),

We can write a column vector as $X=\begin{pmatrix}x_1, & \cdots & x_n\end{pmatrix}^{\mathrm{T}}=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$,

here "T" means "transpose".

eg(例子). a column vector : $\alpha = (1,\ i)^T = \begin{pmatrix} 1 \\ i \end{pmatrix} \in \mathrm{C}^2$.

Let

$$A = A_{m\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n} \in \mathrm{C}^{m?n}$$

We can write $A = (\alpha_1,\ \alpha_2,\ \cdots\ \alpha_n)$ (according to columns in A)

$$Here, \alpha_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \alpha_2 = \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix}, \cdots, \alpha_n = \begin{pmatrix} a_{1,n} \\ \vdots \\ a_{m,n} \end{pmatrix} \in \mathrm{C}^m, and$$

$$A = (\alpha_1,\ \alpha_2,\ \cdots\ \alpha_n) \in \mathrm{C}^{m?n}$$

Recall. "Conjugate" of $w = a + ib$, for $a, b \in \mathrm{R}$ (are real), ($i = \sqrt{-1}, i^2 = -1$) is as follows.

共轭: $\overline{w} = \overline{a+bi} = a - bi$.

Or, complex number "$w = a + ib$" has its conjugat: $\overline{w} = \overline{a+bi} = a - bi$

And $(a+bi)\overline{(a+bi)} = (a+bi)(a-bi) = a^2 + b^2 \geq 0$. That is we get following remark.

Rk. $w = a + bi \Rightarrow w \cdot \overline{w} = |w|^2 = a^2 + b^2$.

The Conjugate of $A = (a_{i,j})$ is "$\overline{A} = (\overline{a_{i,j}})$" (共轭) as follows.

Let $A = (a_{i,j}) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n} \in \mathrm{C}^{m,n}$, then $\overline{A} = (\overline{a_{i,j}}) = \begin{pmatrix} \overline{a_{11}} & \cdots & \overline{a_{1n}} \\ \vdots & \ddots & \vdots \\ \overline{a_{m1}} & \cdots & \overline{a_{mn}} \end{pmatrix}_{n\times m} \in \mathrm{C}^{m,n}$

eg(例). $A = \begin{pmatrix} 1 & i \\ i & 2 \end{pmatrix}$, $\Rightarrow \overline{A} = \begin{pmatrix} \overline{1} & \overline{i} \\ \overline{i} & \overline{2} \end{pmatrix} = \begin{pmatrix} 1 & -i \\ -i & 2 \end{pmatrix}$.

eg. For real matrix: $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \in \mathrm{R}^{2,2}$, $\Rightarrow \overline{A} = \begin{pmatrix} \overline{1} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = A$.

Rk (Remark)(备注). For any real matrix: $A = (a_{i,j}) \in \mathrm{R}^{m,n}$, then

$$\overline{A} = (\overline{a_{i,j}}) = (a_{i,j}) = A.$$

Rk : $\overline{(AB)} = (\overline{A})(\overline{B})$ for any $A = A_{m\times n} \in \mathrm{C}^{m,n}$, $B = B_{n\times p} \in \mathrm{C}^{n,p}$

Conjugate-transpose(共轭转置)： " $A^H=\overline{A}^T$ " or " $A^*=A^H=\overline{A}^T$ " .

$$A=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$$ has the conjugate-transpose: $A^H=\overline{A}^T=\begin{pmatrix} \overline{a_{11}} & \cdots & \overline{a_{m1}} \\ \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \cdots & \overline{a_{mn}} \end{pmatrix}_{n\times m} \in C^{n\times m}$.

Rk (Remark). A^H is also called the "Hermite—transpose" or "H—transpose" of A.

Rk . $A=(a_{i,j})=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n} \in C^{m,n}$, then $A^H=\begin{pmatrix} \overline{a_{11}} & \cdots & \overline{a_{m1}} \\ \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \cdots & \overline{a_{mn}} \end{pmatrix}_{n\times m} \in C^{n\times m}$.

ie. $A=(a_{i,j})\in C^{m,n} \Rightarrow A^H=(\overline{a_{i,j}})^T\in C^{n,m}$

eg(例) $A=\begin{pmatrix} 1 & i \\ 1 & i \\ 1 & i \end{pmatrix}\in C^{3\times 2}$, $\Rightarrow A^H=\overline{A}^T=\begin{pmatrix} 1 & 1 & 1 \\ -i & -i & -i \end{pmatrix}\in C^{2\times 3}$

Rk : Remark. If $X=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\in C^n$, then $X^H=\begin{pmatrix} \overline{x_1}, & \overline{x_2}, & \cdots & \overline{x_n} \end{pmatrix}$ is a row-vector.

eg. $X=\begin{pmatrix} 1 \\ i \end{pmatrix}\in C^2$, then $X^H=\overline{X}^T=(\bar{1},\ \bar{i})=(1,\ -i)$ is a row-vector.

Rk . If $A\in R^{m,n}$ is real, then $A^H=A^T$ (recall $\overline{A}=A$ for any real matrix A).

For a real vector $X\in R^n$(实向量), then $X^H=X^T$.

Rk . If $a\in C$ is cplx-number(复数), then $(a)^H=(\overline{a})^T=\overline{a}$, ie. $\overline{a}=(a)^H$

Recall that " $a\in C$ is real-number iff(if and only if) $\overline{a}=a$ " .

ie. " a is real-number $\Leftrightarrow$ **(iff)** $\overline{a}=a$ " .

Rk . a is a real-number $\Leftrightarrow$ (iff) $\overline{a}=a$ $\Leftrightarrow$ (iff) $(a)^H=\overline{a}=a$ "

Some properties (laws or rules)

① $(A^H)^H=A$ and $(A+B)^H=A^H+B^H$;

② $(kA)^H=\overline{k}(A^H)$, $k\in C$ is cplx-number(复数),

③ $(AB)^H=B^HA^H$, and $(ABC)^H=C^HB^HA^H$

Recall that $(AB)^T = B^T A^T$, and $(ABC)^T = C^T B^T A^T$ （穿脱公式）

Pf(proof) $\because \overline{ABC} = \overline{A}\ \overline{B}\ \overline{C} \Rightarrow (ABC)^H = (\overline{ABC})^T = (\overline{A}\ \overline{B}\ \overline{C})^T$

$$= (\overline{C})^T (\overline{B})^T (\overline{A})^T = C^H B^H A^H$$

$$\Rightarrow (ABC)^H = C^H B^H A^H$$

Recall that $(AB)^{-1} = B^{-1}A^{-1}$, $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$, if A, B, C are invertible.

Hermitian (Hermite-matrix)

Def. If $A^H = A \in C^{n,n}$, $\boldsymbol{A}$ is called an Hermite-matrix,

we can say $\boldsymbol{A}$ is Hermitian .

eg(例)： $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ is Hermite. (验证) check. $\because A^H = \begin{pmatrix} \overline{1} & \overline{i} \\ \overline{-i} & \overline{1} \end{pmatrix}^T = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = A$.

$B = \begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix}$ $(B^H = B)$ is Hermite.

Rk(注): If $A = A^H$ is **Hermite, then all (对角元)** $a_{11}, a_{22}, \cdots\cdots, a_{nn}$ **are real numbers**

Check $\because A = \begin{pmatrix} a_{11} & & & * \\ & a_{22} & & \\ & & \ddots & \\ * & & & a_{nn} \end{pmatrix} = A^H = \begin{pmatrix} \overline{a_{11}} & & & * \\ & \overline{a_{22}} & & \\ & & \ddots & \\ * & & & \overline{a_{nn}} \end{pmatrix}$

$\Rightarrow a_{11} = \overline{a_{11}}, \cdots, a_{nn} = \overline{a_{nn}}$ (they are real numbers) （实数）

Rk(注): If $A = (a_{i,j})$ is **Hermite** $(A = A^H)$**, then every** $a_{i,j} = \overline{a_{j,i}}$ **for** $1 \le i, j \le n$.

The checking is easy.

Rk: If $A^T = A \in R^{n\times n}$ (real symmetry), then $A^H = A$ is Hermit.

The checking is easy $(A^H = A^T = A)$.

eg(例) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \in R^{2\times 2}$ (real symmetry) $\Rightarrow A^H = A$ (A is Hermit).

Skew-Hermit(斜 Hermite). *A* is skew-Hermite, if $A^H = -A$,

Eg. $B = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$ **is skew-Hermit**, $\because B^H = \begin{pmatrix} \overline{i} & \overline{-1} \\ \overline{1} & \overline{i} \end{pmatrix} = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} = -\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} = -B$.

Rk(1).If B **is skew-Hermit(** $B^H = -B$ **)**then iB and $\frac{B}{i}$ are both **Hermit**.

Rk(2). If A **is Hermit(** $A^H = A$ **)** then iA and $\frac{A}{i}$ are both **skew-Hermit.**

$\because \quad (iB)^H = \overline{(i)}B^H = -i(-B) = iB, \quad$ recall $(kA)^H = \bar{k}A^H$

Rk(3) A **is Hermit** $\Leftrightarrow$ (iff) iA is **skew-Hermit.**

Thm(结论): Any $A = A_{m\times n} \Longrightarrow\Longrightarrow A^H A$ and AA^H are both **Hermit**

Pf: $\because (A^H A)^H = A^H (A^H)^H = A^H A$

Eg. $A = \begin{pmatrix} 1 \\ i \end{pmatrix}_{2\times 1} \quad A^H = (1 \;,\; \bar{i}) = (1 \;,\; -i)_{1\times 2}$ then,

$AA^H = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$ and $A^H A = (1^2 + |i|^2) = (2) = 2 \; (1\times 1)$ are both **Hermit.**

$$A = A_{n\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}_{n?n} \in C^{n,n},$$

trace(迹): $\mathrm{tr}(A) = a_{11} + a_{22} + \cdots\cdots a_{nn}$

Eigenvalues (Eigen-roots)特征根集合: $\lambda(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$

Rk : Remark(备注) $\mathrm{tr}(A) = a_{11} + a_{22} + \cdots\cdots a_{nn} = \lambda_1 + \lambda_2 + \cdots\cdots \lambda_n = \mathrm{tr}(A)$.

Put. $A = A_{m\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \in C^{m?n}$; $A^H = \begin{pmatrix} \overline{a_{11}} & \cdots & \overline{a_{m1}} \\ \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \cdots & \overline{a_{mn}} \end{pmatrix} \in C^{n\times m}$

Trace-formula. $\mathrm{tr}(A^H A) = \mathrm{tr}(AA^H) = \sum |a_{i,j}|^2$, ie.

$$\mathrm{tr}(A^H A) = \mathrm{tr}(AA^H) = (|a_{11}|^2 + |a_{12}|^2 + \cdots\cdots |a_{1n}|^2) + \cdots\cdots (|a_{m1}|^2 + |a_{m2}|^2 + \cdots\cdots |a_{mn}|^2)$$

Eg(check). $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}_{3\times 2}$, $A^H = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} & \overline{a_{31}} \\ \overline{a_{12}} & \overline{a_{22}} & \overline{a_{32}} \end{pmatrix}_{2\times 3}$

$$AA^H = \begin{pmatrix} |a_{11}|^2 + |a_{12}|^2 & * & * \\ * & |a_{21}|^2 + |a_{22}|^2 & * \\ * & * & |a_{31}|^2 + |a_{32}|^2 \end{pmatrix}_{3\times 3}$$

$$A^H A = \begin{pmatrix} |a_{11}|^2 + |a_{21}|^2 + |a_{31}|^2 & * \\ * & |a_{12}|^2 + |a_{22}|^2 + |a_{32}|^2 \end{pmatrix}_{2\times 2}$$

$$\therefore tr(AA^H) = tr(A^H A) = |a_{11}|^2 + |a_{21}|^2 + |a_{31}|^2 + |a_{12}|^2 + |a_{22}|^2 + |a_{32}|^2 = \sum |a_{ij}|^2 .$$

Rk . For a column-vector $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}$, $X^H = (\overline{x_1}, \cdots\cdots, \overline{x_n})_{1\times n}$, then

$$\mathrm{tr}(X^H X) = \mathrm{tr}(XX^H) = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 = \sum |x_j|^2$$

Eg: $X = (1,\ i,\ i)^T = \begin{pmatrix} 1 \\ \mathrm{i} \\ \mathrm{i} \end{pmatrix} \in \mathrm{C}^3$, $X^H X = \mathrm{tr}(XX^H) = 1^2 + |i|^2 + |i|^2 = 3$

Trace-commutative(interchanged) formula for $\mathrm{tr}(AB^T)$

Rk . Put. $A = A_{m\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n} \in \mathrm{C}^{m,n}$, $B = B_{m\times n} = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}_{m\times n} \in \mathrm{C}^{m,n}$

$$B^T = \begin{pmatrix} b_{11} & \cdots & b_{m1} \\ \vdots & \ddots & \vdots \\ b_{1n} & \cdots & b_{mn} \end{pmatrix}_{m\times n} \in \mathrm{C}^{n,m} \quad \Rightarrow \quad AB^T \in \mathrm{C}^{m,m}, B^T A \in \mathrm{C}^{n,n} \text{ (They are squared)}$$

Tr-commutative formula(1). $\mathrm{tr}(AB^T) = \mathrm{tr}(B^T A) = \sum a_{i,j} b_{i,j}$

Eg(check) . $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}_{3\times 2}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}_{3\times 2}$, $B^T = \begin{pmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \end{pmatrix}_{2\times 3} \in \mathrm{C}^{2\times 3}$

$$\Rightarrow \ AB^T = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & * & * \\ * & a_{21}b_{21} + a_{22}b_{22} & * \\ * & * & a_{31}b_{31} + a_{32}b_{32} \end{pmatrix}_{3\times 3}$$

also, $B^H A = \begin{pmatrix} a_{11}b_{11} + a_{21}b_{21} + a_{31}b_{31} & * \\ * & a_{12}b_{12} + a_{22}b_{22} + a_{32}b_{32} \end{pmatrix}_{2\times 2}$

thus $\mathrm{tr}(AB^T) = \mathrm{tr}(B^T A) = a_{11}b_{11} + a_{21}b_{21} + a_{31}b_{31} + a_{12}b_{12} + a_{22}b_{22} + a_{32}b_{32}$,

That is to say (we can write) $\mathrm{tr}(AB^T) = \mathrm{tr}(B^T A) = \sum a_{i,j} b_{i,j}$

Rk. Replacing $B = (b_{i,j})$ by $(\overline{B}) = (\overline{b_{i,j}})$ in $tr(AB^T) = tr(B^T A) = \sum a_{i,j} b_{i,j}$

we get $tr(A(\overline{B})^T) = tr((\overline{B})^T A) = \sum a_{i,j} \overline{b_{i,j}}$, and note $(\overline{B})^T = B^H$.

we get again $tr(AB^H)=tr(B^HA)=\sum a_{ij}\overline{b_{ij}}$.

Tr-commutative formula(2): $\mathrm{tr}(AB^H)=\mathrm{tr}(B^HA)=\sum a_{ij}\overline{b_{ij}}$

Eg . $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}_{3\times 2}$, $B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}_{3\times 2}$, $B^H=\begin{pmatrix} \overline{b_{11}} & \overline{b_{21}} & \overline{b_{31}} \\ \overline{b_{12}} & \overline{b_{22}} & \overline{b_{32}} \end{pmatrix}_{2\times 3} \in \mathrm{C}^{2\times 3}$

$$AB^H=\begin{pmatrix} a_{11}\overline{b_{11}}+a_{12}\overline{b_{12}} & * & * \\ * & a_{21}\overline{b_{21}}+a_{22}\overline{b_{22}} & * \\ * & * & a_{31}\overline{b_{31}}+a_{32}\overline{b_{32}} \end{pmatrix}_{3\times 3}$$

$$B^HA=\begin{pmatrix} a_{11}\overline{b_{11}}+a_{21}\overline{b_{21}}+a_{31}\overline{b_{31}} & * \\ * & a_{12}\overline{b_{12}}+a_{22}\overline{b_{22}}+a_{32}\overline{b_{32}} \end{pmatrix}_{2\times 2}$$

$$\Rightarrow tr(AB^H)=tr(B^HA)=a_{11}\overline{b_{11}}+a_{21}\overline{b_{21}}+a_{31}\overline{b_{31}}+a_{12}\overline{b_{12}}+a_{22}\overline{b_{22}}+a_{32}\overline{b_{32}}=\sum a_{ij}\overline{b_{ij}}$$

Let A, B $\in \mathrm{C}^{m\times n}$

$$A=A_{m\times n}=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n} \qquad B=B_{m\times n}=\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}_{m\times n} \in \mathrm{C}^{m\times n}$$

Here $B^H=(\overline{B})^T=\begin{pmatrix} \overline{b_{11}} & \cdots & \overline{b_{m1}} \\ \vdots & \ddots & \vdots \\ \overline{b_{1n}} & \cdots & \overline{b_{mn}} \end{pmatrix} \in \mathrm{C}^{n,\,m}$,and $AB^H\in \mathrm{C}^{m,\,m}$, $B^HA\in \mathrm{C}^{n,\,n}$

Rk. Puting $A=B\ (a_{ij}=b_{ij})$ in $\mathrm{tr}(AB^H)=\mathrm{tr}(B^HA)=\sum a_{ij}\overline{b_{ij}}$

we get again $\mathrm{tr}(A^HA)=\mathrm{tr}(AA^H)=\sum a_{i,j}\overline{a_{i,j}}=\sum |a_{i,j}|^2$.

Recall. "Conjugate" of $w=a+ib$, for $a,b\in \mathrm{R}$ (are real), $(i=\sqrt{-1}, i^2=-1)$ is as follows.

共轭：$\overline{w}=\overline{a+bi}=a-bi$.

Or, complex number "$w=a+ib$" has its conjugat: $\overline{w}=\overline{a+bi}=a-bi$

And $(a+bi)\overline{(a+bi)}=(a+bi)(a-bi)=a^2+b^2\geq 0$. That is

Rk. (we have) $w=a+bi \Rightarrow w\cdot\overline{w}=|w|^2=a^2+b^2$

Put $X=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $Y=\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathrm{C}^n$, $Y^H=(\overline{y_1},\cdots\cdots\overline{y_n})$, then,

$$XY^H = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} (\overline{y_1}, \cdots\cdots \overline{y_n}) = \begin{pmatrix} x_1\overline{y_1} & x_2\overline{y_2} & \cdots & x_1\overline{y_n} \\ x_2\overline{y_1} & x_2\overline{y_2} & \cdots & x_2\overline{y_n} \\ \vdots & \vdots & \ddots & \\ x_n\overline{y_1} & x_n\overline{y_2} & \cdots & x_n\overline{y_n} \end{pmatrix}_{n\times n}$$

and $Y^H X = (\overline{y_1}, \cdots, \overline{y_n}) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_n\overline{y_n})_{1\times 1}$

$$\Rightarrow \mathrm{tr}(XY^H) = \mathrm{tr}(Y^H X) = Y^H X = x_1\overline{y_1} + \cdots + x_n\overline{y_n}$$

Rk. $\mathrm{tr}(XY^H) = \mathrm{tr}(Y^H X) = Y^H X = x_1\overline{y_1} + \cdots + x_n\overline{y_n}$. for $X, Y \in \mathrm{C}^n$.

Inner **product** for C^n

Def. **Standard-inner-product**(标准内积) in C^n is defined as follows

$$(X,Y) = x_1\overline{y_1} + y_2\overline{y_2} + \cdots + x_n\overline{y_n}, \quad \text{where } X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathrm{C}^n$$

Or, we can write following definition.

Def. The Inner product in C^n is defined as (X,Y):

$$(X,Y) = Y^H X = \mathrm{tr}(XY^H) = x_1\overline{y_1} + \cdots + x_n\overline{y_n}, \quad \text{for } X, Y \in \mathrm{C}^n$$

that is $(X,Y) = x_1\overline{y_1} + \cdots + x_n\overline{y_n}$, for $X, Y \in \mathrm{C}^n$.

C^n with (X,Y) is called Unitary-space (U-space) .

Rk. $(X,X) = \mathrm{tr}(XX^H) = X^H X = x_1\overline{x_1} + \cdots + x_n\overline{x_n} = |x_1|^2 + \cdots + |x_n|^2 = |X|^2$

Here,模长: $|X| = \sqrt{(X,X)} = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2} \geq 0$ is the norm (length) of X .

Rk. $|X|^2 = (X,X) = \mathrm{tr}(XX^H) = \mathrm{tr}(X^H X) = X^H X = |X|^2$.

i.e. $X^H X = (X,X) = |X|^2$, here $|X|^2 = \sum |x_j|^2 = |x_1|^2 + \cdots + |x_n|^2$

Rk. $Y^H X = (X,Y)$, $X^H Y = (Y,X) = \overline{(X,Y)}$. .

Eg: $X = \begin{pmatrix} 1 \\ \mathrm{i} \\ \mathrm{i} \end{pmatrix} \in \mathrm{C}^3$, $|X|^2 = X^H X = 1^2 + |i|^2 + |i|^2 = 3$; $|X| = \sqrt{3}$

Rk. 模长性质：$|kX|=|k||X|$, $|\frac{X}{k}|=\frac{|X|}{|k|}, (k\neq 0)$; and $|X\pm Y|\leq|X|+|Y|$.

Rk.单位化公式： If $X\neq 0$, $\frac{X}{|X|}$ is unit-vector ($\because$ $|\frac{X}{|X|}|=\frac{|X|}{|X|}=1$).

Eg: For $X=\begin{pmatrix}1\\ \mathrm{i}\\ \mathrm{i}\end{pmatrix}$, $\frac{X}{|X|}=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\ i\\ i\end{pmatrix}=\begin{pmatrix}1/\sqrt{3}\\ i/\sqrt{3}\\ i/\sqrt{3}\end{pmatrix}$ is a unit-vector.

Rk. If $x=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}, y=\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}\in \mathrm{R}^n$ （real-vectors 实向量），, then

$$(x,y)=x_1y_1+x_2y_2+\cdots+x_ny_n.\quad \text{here}\quad \mathrm{R}^n\subset \mathrm{C}^n$$

Some properties for inner product 内积性质：

(P_1). $(X,X)\geq 0$, and $(X,X)>0$ if $X\neq 0$;

(P_2): $(Y,X)=\overline{(X,Y)}$; (P_3): $(kX,Y)=k(X,Y)$, $(X,kY)=\bar{k}(X,Y)$, for $k\in \mathrm{C}$

(P_4): $(X+Y,W)=(X,W)+(Y,W)$, $(W,X+Y)=(W,X)+(W,Y)$.

Rk. $|(X,Y)|^2\leq (X,X)(Y,Y)$, i.e. $|(X,Y)|\leq |X|\cdot|Y|$

补充：Inner product for $\mathrm{C}^{m,n}$（复矩阵空间的内积）.

Def. The inner product in $\mathrm{C}^{m,n}$ is defined by (A,B):

$$(A,B)=tr(AB^H)=tr(B^HA)=\sum a_{ij}\overline{b_{ij}},$$

$$\text{where}\quad A=(a_{ij})_{m\times n},\ B=(b_{ij})_{m\times n}\in \mathrm{C}^{m,n}$$

Some properties.

(P_1): $(A,A)=tr(AA^H)=\sum|a_{ij}|^2\geq 0$, and $(A,A)>0$ if $A\neq 0$.

(P_2): $(B,A)=\overline{(A,B)}$; (P_3): $(kA,B)=k(A,B)$, $(A,kB)=\bar{k}(A,B)$, $k\in \mathrm{C}$

(P_4): $(A+B,D)=(A,D)+(B,D)$, $(D,A+B)=(D,A)+(D,B)$.

Rk. $|(A,B)|^2\leq (A,A)(B,B)$, i.e. $|(A,B)|\leq\|A\|\cdot\|B\|$

Def. $\|A\|=\sqrt{(A,A)}=\sqrt{tr(AA^H)}=\sqrt{\sum|a_{i,j}|^2}$ **is called the norm of** A.

Rk. $\|A\|^2=tr(AA^H)=tr(A^HA)=\sum|a_{ij}|^2$ **(矩阵模长公式)**

Eg(例). $A=\begin{pmatrix}1 & i\\ 1 & i\\ 1 & i\end{pmatrix}$, $\|A\|=\sqrt{1^2+1^2+1^2+|i|^2+|i|^2+|i|^2}=\sqrt{6}$.

$$\text{Put. } A=A_{n\times p}=\begin{pmatrix}a_{11} & \cdots & a_{1p}\\ \vdots & \ddots & \vdots\\ a_{n,1} & \cdots & a_{n,p}\end{pmatrix}\in \mathrm{C}^{n,p},\quad A^H=\begin{pmatrix}\overline{a_{11}} & \cdots & \overline{a_{n1}}\\ \vdots & \ddots & \vdots\\ \overline{a_{1,p}} & \cdots & \overline{a_{n,p}}\end{pmatrix}\in \mathrm{C}^{p,n}$$

We can write $A=(\alpha_1,\ \alpha_2,\ \cdots\ ,\alpha_p)$ (according to **columns in A**)

$$Here,\ \alpha_1=\begin{pmatrix}a_{11}\\ \vdots\\ a_{n1}\end{pmatrix},\alpha_2=\begin{pmatrix}a_{12}\\ \vdots\\ a_{n,2}\end{pmatrix},\cdots,\alpha_p=\begin{pmatrix}a_{1,n}\\ \vdots\\ a_{n,p}\end{pmatrix}\in \mathrm{C}^n,\ and$$

$$A=(\alpha_1,\ \ \alpha_2,\ \ \cdots\ \ \alpha_p)\in \mathrm{C}^{n,p}$$

Rk. $A^HA=\left(\overline{(\alpha_i,\alpha_j)}\right)=\begin{pmatrix}\overline{(\alpha_1,\alpha_1)} & \overline{(\alpha_1,\alpha_2)} & \cdots & \overline{(\alpha_1,\alpha_p)}\\ \overline{(\alpha_2,\alpha_1)} & \overline{(\alpha_2,\alpha_2)} & \cdots & \overline{(\alpha_2,\alpha_p)}\\ \vdots & \vdots & \ddots & \vdots\\ \overline{(\alpha_p,\alpha_1)} & \overline{(\alpha_p,\alpha_2)} & \cdots & \overline{(\alpha_p,\alpha_p)}\end{pmatrix}\in \mathrm{C}^{p,p}$, if $A=A_{n,p}\in \mathrm{C}^{n,p}$.

That is $A^HA=\begin{pmatrix}|\alpha_1|^2 & \overline{(\alpha_1,\alpha_2)} & \cdots & \overline{(\alpha_1,\alpha_p)}\\ \overline{(\alpha_2,\alpha_1)} & |\alpha_2|^2 & \cdots & \overline{(\alpha_2,\alpha_p)}\\ \vdots & \vdots & \ddots & \vdots\\ \overline{(\alpha_p,\alpha_1)} & \overline{(\alpha_p,\alpha_2)} & \cdots & |\alpha_p|^2\end{pmatrix}$, for $(X,X)=|X|^2$

Pf. (proof). $A=(\alpha_1,\ \ \alpha_2,\ \ \cdots\ \ \alpha_p)\in \mathrm{C}^{n,p} \Rightarrow A^H=\begin{pmatrix}\alpha_1{}^H\\ \alpha_2{}^H\\ \vdots\\ \alpha_p{}^H\end{pmatrix}\in \mathrm{C}^{p,n}$.

Recall $X^HY=(Y,X)=\overline{(X,Y)}$, $Y^HX=(X,Y)$. we get

$$A^{\mathrm{H}}A=\begin{pmatrix}\alpha_1{}^{\mathrm{H}}\\ \alpha_2{}^{\mathrm{H}}\\ \vdots\\ \alpha_p{}^{\mathrm{H}}\end{pmatrix}\begin{pmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_p\end{pmatrix}=\begin{pmatrix}\overline{(\alpha_1,\alpha_1)} & \overline{(\alpha_1,\alpha_2)} & \cdots & \overline{(\alpha_1,\alpha_p)}\\ \overline{(\alpha_2,\alpha_1)} & \overline{(\alpha_2,\alpha_2)} & \cdots & \overline{(\alpha_2,\alpha_p)}\\ \vdots & \vdots & \ddots & \vdots\\ \overline{(\alpha_p,\alpha_1)} & \overline{(\alpha_p,\alpha_2)} & \cdots & \overline{(\alpha_p,\alpha_p)}\end{pmatrix}$$

Rk. If (n = p) $A=(\alpha_1,\quad \alpha_2,\quad \cdots\quad ,\alpha_n)\in \mathrm{C}^{n,n}$ is square, then

$$A^{H}A=\left(\overline{(\alpha_i,\alpha_j)}\right)=\begin{pmatrix}\overline{(\alpha_1,\alpha_1)} & \overline{(\alpha_1,\alpha_2)} & \cdots & \overline{(\alpha_1,\alpha_n)}\\ \overline{(\alpha_2,\alpha_1)} & \overline{(\alpha_2,\alpha_2)} & \cdots & \overline{(\alpha_2,\alpha_n)}\\ \vdots & \vdots & \ddots & \vdots\\ \overline{(\alpha_n,\alpha_1)} & \overline{(\alpha_n,\alpha_2)} & \cdots & \overline{(\alpha_n,\alpha_n)}\end{pmatrix}\in \mathrm{C}^{n,n}$$

i.e. $$A^{H}A=\begin{pmatrix}|\alpha_1|^2 & \overline{(\alpha_1,\alpha_2)} & \cdots & \overline{(\alpha_1,\alpha_n)}\\ \overline{(\alpha_2,\alpha_1)} & |\alpha_2|^2 & \cdots & \overline{(\alpha_2,\alpha_n)}\\ \vdots & \vdots & \ddots & \vdots\\ \overline{(\alpha_n,\alpha_1)} & \overline{(\alpha_n,\alpha_2)} & \cdots & |\alpha_n|^2\end{pmatrix},\ \text{for } (X,X)=|X|^2.$$

Rk.(row-vector formula) Put $X=(x_1,\cdots,x_n), Y=(y_1,\cdots,y_n)\in \mathrm{C}^{1\times n}$ are row-vectors.
the inner-formula is $(X,Y)=XY^{H}=x_1\overline{y_1}+\cdots+x_n\overline{y_n}$

Put. $$A=A_{m\times n}=\begin{pmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{m,1} & \cdots & a_{m,n}\end{pmatrix}\in \mathrm{C}^{m,n},\quad A^{H}=\begin{pmatrix}\overline{a_{11}} & \cdots & \overline{a_{m1}}\\ \vdots & \ddots & \vdots\\ \overline{a_{1,n}} & \cdots & \overline{a_{m,n}}\end{pmatrix}\in \mathrm{C}^{n,m}$$

write $$A=A_{m\times n}=\begin{pmatrix}A_1\\ \vdots\\ A_m\end{pmatrix}(\text{row}-\text{block}),\ \text{here } A_1=(a_{11},\cdots,a_{1n}),\cdots$$

$$\text{here}\qquad A^{H}=(A_1{}^{H},\cdots,A_m{}^{H})\in \mathrm{C}^{n,m}$$

then $$AA^{H}==\begin{pmatrix}(A_1,A_1) & (A_1,A_2) & \cdots & (A_1,A_m)\\ (A_2,A_1) & (A_2,A_2) & \cdots & (A_2,A_m)\\ \vdots & \vdots & \ddots & \vdots\\ (A_m,A_1) & (A_m,A_2) & \cdots & (A_m,A_{mn})\end{pmatrix}\in \mathrm{C}^{m,m},\ \text{if } A\in \mathrm{C}^{m,n}$$

Ortho.-vectors. **Put** $X=(x_1,\quad \cdots\quad ,x_n)^T, Y=(y_1,\quad \cdots\quad ,y_n)^T\in \mathrm{C}^{n}$.

$X\perp Y$(orthogonal) $\Leftrightarrow$ $(X,Y)=x_1\overline{y_1}+y_2\overline{y_2}+\cdots+x_n\overline{y_n}=0$, where $X,Y\in \mathrm{C}^{n}$

Rk. $X\perp Y \Leftrightarrow (Y,X)=\overline{(X,Y)}=y_1\overline{x_1}+y_2\overline{x_2}+\cdots+y_n\overline{x_n}=0.$

Rk. $X \perp Y \Leftrightarrow (Y,X)=0 \Leftrightarrow (X,Y)=0$.

Recall $X^H Y=(Y,X)=\overline{(X,Y)}, \quad Y^H X=(X,Y)$.

Rk. $X \perp Y \Leftrightarrow X^H Y=0, \Leftrightarrow Y^H X=0$

Rk. $X \perp Y \Rightarrow aX \perp bY$, for $(aX,bY)=a\bar{b}(X,Y)=0$.

Eg. (check) : $X=\begin{pmatrix}1\\ i\end{pmatrix} \perp Y=\begin{pmatrix}i\\ 1\end{pmatrix}$, $\because (X,Y)=1\cdot\bar{i}+i\cdot\bar{1}=-i+i=0 \quad \therefore X \perp Y$

Eg. $X=\begin{pmatrix}1\\ i\\ i\end{pmatrix} \perp Y=\begin{pmatrix}2i\\ 1\\ 1\end{pmatrix}. \because (X,Y)=1\cdot\overline{2i}+i\cdot\bar{1}+i\cdot\bar{1}=-2i+i+i=0, \quad \therefore X \perp Y$

Ortho-formula (1) $X \perp Y \Rightarrow |X \pm Y|^2=|X|^2+|Y|^2$

Pf. $|X+Y|^2=(X+Y,X+Y)=(X,X)+(X,Y)+(Y,X)+(Y,Y)$
$=(X,X)+0+0+(Y,Y)=|X|^2+|Y|^2$

Ortho-formula (2) $X \perp Y \Rightarrow |aX+bY|^2=|aX|^2+|bY|^2$, (for $X \perp Y \Rightarrow aX \perp bY$)

Ortho-formula (3) $X \perp Y \perp W \Rightarrow |aX+bY+cW|^2=|aX|^2+|bY|^2+|cW|^2$

Here "$X \perp Y \perp W$" means X, Y, W are mutualy othogonal(any two vectors are otho.).

Rk. "$X_1 \perp X_2 \perp \cdots \perp X_p$" means they are mutualy othogonal(any two vectors are otho.)

Rk. $X_1 \perp X_2 \perp \cdots \perp X_p \Rightarrow |c_1X_1+c_2X_2+\cdots+c_pX_p|^2=|c_1X_1|^2+|c_2X_2|^2+\cdots+|c_pX_p|^2$

Def. If "$X_1 \perp X_2 \perp \cdots \perp X_p$", and any one $X_j \neq 0$ (nonzero!)

we say "$X_1, X_2, \cdots, X_p$" is an "**ortho-group**".

Eg. $X=\begin{pmatrix}1\\ i\\ i\end{pmatrix}, Y=\begin{pmatrix}2i\\ 1\\ 1\end{pmatrix}, W=\begin{pmatrix}0\\ 1\\ -1\end{pmatrix}$ $(\because X \perp Y \perp W)$ is one "**ortho-group**"

Def.(pre-u 预U) If $\alpha_1=\begin{pmatrix}*\\ \vdots\\ *\end{pmatrix}, \alpha_2=\begin{pmatrix}*\\ \vdots\\ *\end{pmatrix}, \cdots \alpha_n=\begin{pmatrix}*\\ \vdots\\ *\end{pmatrix} \in C^n$ is an **ortho-group,**

i.e. "$\alpha_1 \perp \alpha_2 \perp \cdots \perp \alpha_n$" and any $\alpha_j \neq 0$ (nonzero), we say the $n \times n$ matrix

$A=(\alpha_1, \ \alpha_2, \ \cdots \ \alpha_n)_{n\times n}$ is **pre-unitary(pre-u.) (预U阵)**

Eg. Put $X_1=\begin{pmatrix}1\\i\\i\end{pmatrix}, X_2=\begin{pmatrix}2i\\1\\1\end{pmatrix}, X_3=\begin{pmatrix}0\\1\\-1\end{pmatrix}$ $(X_1\perp X_2\perp X_2)$, then

$$A=(X_1,X_2,X_3)=\begin{pmatrix}1&2i&0\\i&1&1\\i&1&-1\end{pmatrix} \text{ is pre-U.}$$

Eg. Put $\alpha_1=\begin{pmatrix}1\\i\end{pmatrix}$, $\alpha_2=\begin{pmatrix}i\\1\end{pmatrix}$$(\alpha_1\perp\alpha_2)$, then $A=(\alpha_1,\alpha_2)=\begin{pmatrix}1&i\\i&1\end{pmatrix}$ is pre-U

Eg. $A=(\alpha_1,\alpha_2)=\begin{pmatrix}1&-1\\1&1\end{pmatrix}$ is pre-U ; $\mathrm{B}=(\beta_1,\beta_2)=\begin{pmatrix}3&-4\\4&3\end{pmatrix}$ is pre-U.

Def.(pre-c-u 预半优) If $\alpha_1=\begin{pmatrix}*\\\vdots\\*\end{pmatrix},\alpha_2=\begin{pmatrix}*\\\vdots\\*\end{pmatrix},\cdots\alpha_p=\begin{pmatrix}*\\\vdots\\*\end{pmatrix}\in\mathrm{C}^n\ (p\le n)$ **is an**

ortho-group:"$\alpha_1\perp\alpha_2\perp\cdots\perp\alpha_p$ "and any $\alpha_j\neq 0$ (nonzero), we say the $n\times p$ matrix

$A=\begin{pmatrix}\alpha_1, & \alpha_2, & \cdots & \alpha_p\end{pmatrix}_{n\times p}$ is **pre-column-unitary(pre-c-u) (预半 U)**

Rk. When $p=n$ we get $A=\begin{pmatrix}\alpha_1, & \alpha_2, & \cdots & \alpha_n\end{pmatrix}_{n\times n}$ is pre-u(also pre-c-u).

Here,"$\alpha_1\perp\alpha_2\perp\cdots\perp\alpha_n$ "

Thm.1 $A=(\alpha_1,\alpha_2,\cdots,\alpha_p)_{n\times p}$ is pre--c--u $\Leftrightarrow$ $A^HA=\begin{pmatrix}|\alpha_1|^2&\cdots&0\\\vdots&\ddots&\vdots\\0&\cdots&|\alpha_p|^2\end{pmatrix}$

i.e. $A=A_{n,p}$ is pre-u $\Leftrightarrow$ $A^HA=\begin{pmatrix}\mathrm{d}_1&\cdots&0\\\vdots&\ddots&\vdots\\0&\cdots&\mathrm{d}_p\end{pmatrix}$ is diagonal (对角形)

Pf. Recall $A^HA=\begin{pmatrix}\overline{(\alpha_1,\alpha_1)}&\overline{(\alpha_1,\alpha_2)}&\cdots&\overline{(\alpha_1,\alpha_p)}\\\overline{(\alpha_2,\alpha_1)}&\overline{(\alpha_2,\alpha_2)}&\cdots&\overline{(\alpha_2,\alpha_p)}\\\vdots&\vdots&\ddots&\vdots\\\overline{(\alpha_p,\alpha_1)}&\overline{(\alpha_p,\alpha_2)}&\cdots&\overline{(\alpha_p,\alpha_p)}\end{pmatrix}$, $\because(\alpha_1,\alpha_2)=\cdots=(\alpha_1,\alpha_p)=0$.

thus, $A^HA=\begin{pmatrix}|\alpha_1|^2&\cdots&0\\\vdots&\ddots&\vdots\\0&\cdots&|\alpha_p|^2\end{pmatrix}$ (OK)

When $p=n$ we get similar result for a pre-u: $A=\begin{pmatrix}\alpha_1, & \alpha_2, & \cdots & \alpha_n\end{pmatrix}_{n\times n}$.

Thm.2 $A=(\alpha_1,\alpha_2,\cdots,\alpha_n)_{\mathrm{n}\times\mathrm{n}}$ is pre--u $\Leftrightarrow$ $A^HA=\begin{pmatrix}|\alpha_1|^2 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & |\alpha_n|^2\end{pmatrix}_{\mathrm{n},\mathrm{n}}$

i.e. $A=A_{n,n}$ is pre-u $\Leftrightarrow$ $A^HA=\begin{pmatrix}\mathrm{d}_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \mathrm{d}_n\end{pmatrix}_{\mathrm{n},\mathrm{n}}$ is diagonal (对角形)

Eg. $A=(X_1,X_2)=\begin{pmatrix}1 & 2i\\ i & 1\\ i & 1\end{pmatrix}_{3\times2}$ is pre-c-U. $\because$ $X_1=\begin{pmatrix}1\\ i\\ i\end{pmatrix}, X_2=\begin{pmatrix}2i\\ 1\\ 1\end{pmatrix}$ $(X_1\perp X_2)$

Eg. $A=(\alpha_1,\alpha_2)=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}_{2\times2}$ is pre-c-u (also pre-u).

"U-matrix".

Def. (1) If $A=\begin{pmatrix}\alpha_1, & \alpha_2, & \cdots & \alpha_n\end{pmatrix}_{n\times n}$ is **pre-u, and** $|\alpha_1|=|\alpha_2|=\cdots=|\alpha_n|=1$

, we say $A=A_{n\times n}$ **is unitary, or** $A=A_{n\times n}$ **is a U-matrix.**

Rk. $A=\begin{pmatrix}\alpha_1, & \cdots & \cdots & \alpha_n\end{pmatrix}_{n\times n}$ is U $\Leftrightarrow$ $\alpha_1\perp\alpha_2\perp\cdots\perp\alpha_n$,$|\alpha_1|=\cdots=|\alpha_n|=1$.

Thm. $A=(\alpha_1,\alpha_2,\cdots,\alpha_n)_{\mathrm{n}\times\mathrm{n}}$ is U. $\Leftrightarrow$ $A^HA=\begin{pmatrix}1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 1\end{pmatrix}_{\mathrm{n},\mathrm{n}}=I$.

Rk. $A=A_{n,n}$ is U $\Leftrightarrow$ $A^HA=I_{\mathrm{n}}$ $\Leftrightarrow$ $A^{-1}=A^H$ $\Leftrightarrow AA^H=I_{\mathrm{n}}$ $(\because AA^{-1}=I)$

Pf. $\because A=(\alpha_1,\alpha_2,\cdots,\alpha_n)_{\mathrm{n}\times\mathrm{n}}$ is U $\Leftrightarrow$ $A^HA=\begin{pmatrix}|\alpha_1|^2 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & |\alpha_n|^2\end{pmatrix}=\begin{pmatrix}1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 1\end{pmatrix}_{\mathrm{n},\mathrm{n}}=I$

Rk. The following are equivalent ! Put $A=A_{n,n}=(\alpha_1,\alpha_2,\cdots,\alpha_n)\in\mathrm{C}^{\mathrm{n,n}}$.

(1) $A=A_{n,n}=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ is U i.e. $\alpha_1\perp\alpha_2\perp\cdots\perp\alpha_n$, $|\alpha_1|=\cdots=|\alpha_n|=1$.

(2) $A=A_{n,n}$ is U $\Leftrightarrow$ $A^HA=I_{\mathrm{n}}$ **;** (3) $A=A_{n,n}$ is U $\Leftrightarrow$ $A^{-1}=A^H$**.**

(4) $A=A_{n,n}$ is U $\Leftrightarrow$ $AA^H=I_{\mathrm{n}}$

Rk. If $A=A_{n,n}$, $\mathrm{B}=\mathrm{B}_{n,n}$ are both U , then AB is U .

Pf. $\because A=A_{n,n}, \mathrm{B}=\mathrm{B}_{n,n}$ are U $\Rightarrow (AB)^H = B^H A^H = B^{-1}A^{-1} = (AB)^{-1} \therefore AB$ is U .

Or, $(AB)^H(AB) = B^H(A^H A)B = B^H B = I \ \therefore AB$ is U

Properties for **U-matrix.** If $A=A_{n,n}$ is U , and $X,Y\in C^n$, then

(1) $|Ax|^2 = |x|^2$, $\because |AX|^2 = (AX)^H(AX) = X^H A^H AX = X^H IX = X^H X = |X|^2$

(2) $x \perp y \Rightarrow Ax \perp Ay.$

$\because x \perp y \Rightarrow (x,y) = y^H x = 0, \therefore (Ax,Ay) = (Ay)^H Ax = y^H A^H Ax = y^H x = 0 \Rightarrow Ax \perp Ay$

(3) $(Ax,Ay) = (x,y).$

$\because (Ax,Ay) = (Ay)^H Ax = y^H A^H Ax = y^H x = (x,y)$

Rk. $A=\left(\alpha_1, \cdots \cdots \alpha_n\right)$ is pre-U $\Rightarrow A=\left(\dfrac{\alpha_1}{|\alpha_1|}, \cdots \cdots \dfrac{\alpha_n}{|\alpha_n|}\right)$ is U

Eg. $A=(X_1,X_2,X_3)=\begin{pmatrix}1 & 2i & 0\\ i & 1 & 1\\ i & 1 & -1\end{pmatrix}$ is pre-U, and $|X_1|=\sqrt{3}, |X_2|=\sqrt{6}, |X_3|=\sqrt{2}$, then

$$A=(\frac{X_1}{|X_1|},\frac{X_2}{|X_2|},\frac{X_3}{|X_3|})=(\frac{X_1}{\sqrt{3}},\frac{X_2}{\sqrt{6}},\frac{X_3}{\sqrt{2}})=\begin{pmatrix}1/\sqrt{3} & 2i/\sqrt{6} & 0\\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2}\\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{3}\end{pmatrix} \text{ is U.}$$

Eg. $A=(\alpha_1,\alpha_2)=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}_{2\times 2}$ is pre-u, $|\alpha_1|=|\alpha_2|=\sqrt{2}$, then

$$A=(\frac{\alpha_1}{|\alpha_1|},\frac{\alpha_2}{|\alpha_2|})=(\frac{\alpha_1}{\sqrt{2}},\frac{\alpha_2}{\sqrt{2}})=\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} \text{is U.}$$

Eg. $A=(\alpha_1,\alpha_2)=\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}$ is pre-u, $|\alpha_1|=|\alpha_2|=\sqrt{2}$, then $A=(\alpha_1,\alpha_2)=\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}$

$$A=(\frac{\alpha_1}{\sqrt{2}},\frac{\alpha_2}{\sqrt{2}})=\begin{pmatrix}1/\sqrt{2} & i/\sqrt{2}\\ i/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} \text{ is U.}$$

Eg. $B=(\beta_1,\beta_2)=\begin{pmatrix}3 & -4\\ 4 & 3\end{pmatrix}$ is pre-U, then $B=(\dfrac{\beta_1}{\sqrt{5}},\dfrac{\beta_2}{\sqrt{5}})=\dfrac{1}{\sqrt{5}}\begin{pmatrix}3 & -4\\ 4 & 3\end{pmatrix}$ is U.

Def.(c-u-matrix 列优或半优阵)

$A=(\alpha_1,\alpha_2,\cdots,\alpha_p)_{n\times p}$ is c--u (column-unitary) if $A=A_{n\times p}$ is pre-c-u, and

$$|\alpha_1|=\cdots=|\alpha_p|=1.$$

Rk. $A=(\alpha_1,\alpha_2,\cdots,\alpha_p)_{n\times p}$ is c--u $\Leftrightarrow$ $\alpha_1\perp\alpha_2\perp\cdots\perp\alpha_P$, $|\alpha_1|=\cdots=|\alpha_P|=1.$

Thm. $A=(\alpha_1,\alpha_2,\cdots,\alpha_p)_{n\times p}$ is C- U. $\Leftrightarrow$ $A^H A=\begin{pmatrix}1&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&1\end{pmatrix}_{p,p}=I_p$, i.e. $A^H A=I$

Pf. $\because A=(\alpha_1,\cdots,\alpha_n)_{n\times n}$ is C-U $\Leftrightarrow$ $A^H A=\begin{pmatrix}|\alpha_1|^2&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&|\alpha_p|^2\end{pmatrix}=\begin{pmatrix}1&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&1\end{pmatrix}_{p,p}=I_p$

Rk. When p<n, $A=(\alpha_1,\cdots,\alpha_p)_{n\times p}$ is C- U. $\Leftrightarrow$ $A^H A=I_p$, but $AA^H\neq I$

U 阵等价条件：设 $A=\left(\alpha_1,\quad \alpha_2,\quad \cdots\quad \alpha_n\right)_{n\times n}$ 为方阵，则下列条件互等价

1. $A=A_{n\times n}$ 为 u 阵（$A^H A=I_n$），即 A 的列 $\alpha_1,\quad \alpha_2,\quad \cdots\quad \alpha_n$ 互正交，长度都为 1.
2. $A^{-1}=A^H$ 或 $A^H=A^{-1}$
3. $A^H A=I$ 且 $AA^H=I$，　　4. $AA^H=I$
5. A 的各行向量互正交，且模长为 1

半 U 阵(列 U 阵)性质：

1. 若 A 为列优阵，则 $|Ax|^2=|x|^2$（保模长）

证：用长度平方公式 $|x|^2=x^H x, x\in C^n$

$$|Ax|^2=(Ax)^H(Ax)=x^H A^H Ax=x^H Ix=x^H x=|x|^2$$

2. 若 A 为半 U 阵（列 U），$x\perp y$ 则 $Ax\perp Ay$（保正交性）

证 $\because x\perp y \therefore (x,y)=y^H x=0 \because (Ax,Ay)=(Ay)^H Ax=y^H A^H Ax=y^H x=0\Rightarrow Ax\perp Ay$

Rk. 设对角阵： $D=\begin{pmatrix}b_1&&&\\&b_2&&\\&&\ddots&\\&&&b_n\end{pmatrix}_{n\times n}$ 与 $D^H=\begin{pmatrix}\overline{b_1}&&&\\&\overline{b_2}&&\\&&\ddots&\\&&&\overline{b_n}\end{pmatrix}_{n\times n}$

则有 $DD^H=D^H D=\begin{pmatrix}|b_1|^2&&0\\&\ddots&\\0&&|b_n|^2\end{pmatrix}$.　再令

$$A=\begin{pmatrix}\lambda_1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & \lambda_n\end{pmatrix}(\text{上三角}),\quad B=\begin{pmatrix}b_1 & & & * \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_n\end{pmatrix}_{n\times n}$$ （上三角）

全体特根为：$\lambda(A)=\{\lambda_1,\lambda_2,\cdots\cdots\lambda_n\}$， $\lambda(B)=\{b_1,b_2,\cdots\cdots b_n\}$

则 $$AB=\begin{pmatrix}\lambda_1 b_1 & & & * \\ & \lambda_2 b_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n b_n\end{pmatrix}$$

特别 $$A^k=\begin{pmatrix}\lambda_1{}^k & & & * \\ & \lambda_2{}^k & & \\ & & \ddots & \\ 0 & & & \lambda_{n1}{}^k\end{pmatrix}$$ （k=1，2，…）

Let(令)任一矩阵：

$$A=A_{m\times n}=\begin{pmatrix}a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn}\end{pmatrix}_{m\times n}\in \mathrm{C}^{m,n}$$

We write(按列分块)： $A=(\alpha_1,\quad \alpha_2,\quad \cdots\quad \alpha_n)$（according to columns of A）

$$Here,\alpha_1=\begin{pmatrix}a_{11}\\ \vdots \\ a_{m1}\end{pmatrix},\alpha_2=\begin{pmatrix}a_{12}\\ \vdots \\ a_{m2}\end{pmatrix},\cdots,\alpha_n=\begin{pmatrix}a_{1,n}\\ \vdots \\ a_{m,n}\end{pmatrix}\in \mathrm{C}^{m}, and$$

$$A=(\alpha_1,\quad \alpha_2,\quad \cdots\quad \alpha_n)\in \mathrm{C}^{m,n}$$