ECON 5020

Microeconomic Theory

Lecture Notes I.2

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These notes are summaries of the materials discussed in classes. They should be used as a complement to, rather than a substitute for, attending classes. Reproduction of the relevant materials in these notes as the answer to an exam question will not necessarily get you full marks for the question

2. Utility Maximization

Price vector: $\mathbf{p} = (p_1, p_2, ..., p_k)$, where p_i is the price of good i

Consumer's income: m

Suppose $\mathbf{p} \gg \mathbf{0}$ and $m > 0$.

The budget constraint: $p_1x_1 + p_2x_2 + ... + p_kx_k \leq m$ or $\mathbf{p} \cdot \mathbf{x} \leq m$

The budget set: The set of affordable bundles

$$B = \{\boldsymbol{x} \in \boldsymbol{X} | \boldsymbol{p} \cdot \boldsymbol{x} \leq m\}$$

Assume (A1) - (A3), (A4") and (A5), i.e., preferences are complete, transitive, continuous, locally non-satiated and strictly convex.

The utility maximization problem:

$$\max_{\mathbf{x}} u(\mathbf{x}) \text{ s.t. } \mathbf{p}\cdot\mathbf{x} \le m \text{ and } \mathbf{x} \in X$$

or: $$\max_{\{x_1,\dots,x_k\}} u(x_1,\dots,x_k) \text{ s.t. } p_1x_1 + \dots + p_kx_k \le m \text{ and } \mathbf{x} \in X$$

The solution $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_k^*)$ is a function of ($\mathbf{p}$, m).

$x_i^* = x_i(\mathbf{p}, m)$: Marshallian demand function of good i, $i \in \{1, 2, \dots, k\}$.

Proposition 2.1: (A5) implies that $\mathbf{x}^*$ is unique.

Proof: Suppose not. Let $\mathbf{x}'$ and $\mathbf{x}''$ be the two solutions to $\max_{\boldsymbol{x}} u(\boldsymbol{x})$ s.t. $\boldsymbol{p} \cdot \boldsymbol{x} \leq m$ and $\boldsymbol{x} \in X$

Then $u(\mathbf{x}') = u(\mathbf{x}'')$, i.e. $\mathbf{x}' \sim \mathbf{x}''$.

Let $\mathbf{y} = t\mathbf{x}' + (1-t)\mathbf{x}''$ for $0 < t < 1$.

By (A5), $\mathbf{y} \succ \mathbf{x}'$ and $\mathbf{y} \succ \mathbf{x}''$.

Furthermore,

$$\begin{aligned} \mathbf{p} \cdot \mathbf{y} = \mathbf{p} \cdot [t\mathbf{x}' + (1-t)\mathbf{x}''] &= t\mathbf{p} \cdot \mathbf{x}' + (1-t)\mathbf{p} \cdot \mathbf{x}'' \\ &\leq tm + (1-t)m = m. \end{aligned}$$

$\therefore$ $\mathbf{y} = t\mathbf{x}' + (1-t)\mathbf{x}''$ satisfies the budget constraint.

This contradicts the assumption that $\mathbf{x}'$ and $\mathbf{x}''$ are solutions to the utility-maximization problem. ■

Proposition 2.2: $\mathbf{x}^*$ is homogenous of degree zero in ($\mathbf{p}$, m).

Proof: Let $\mathbf{x}^*(t\mathbf{p}, tm)$, where $t > 0$, be the solution to

$$\max_{\mathbf{x}} u(\mathbf{x}) \text{ s.t. } (t\mathbf{p})\cdot\mathbf{x} \le tm \text{ and } \mathbf{x} \in X. \qquad (1)$$

$$(t\mathbf{p})\cdot\mathbf{x} \le tm \Leftrightarrow tp_1x_1 + \ldots + tp_kx_k \le tm \Leftrightarrow \mathbf{p}\cdot\mathbf{x} \le m.$$

$\therefore$ Problem (1) is equivalent to

$$\max_{\mathbf{x}} u(\mathbf{x}) \text{ s.t. } \mathbf{p}\cdot\mathbf{x} \le m \text{ and } \mathbf{x} \in X. \qquad (2)$$

$\because$ $\mathbf{x}^*(\mathbf{p}, m)$ is the solution to problem (2).

$\therefore$ $\mathbf{x}^*(t\mathbf{p}, tm) = \mathbf{x}^*(p, m)$. ■

Proposition 2.3: (A4") implies that $\mathbf{p} \cdot \mathbf{x}^{*} = m$.

Proof: Suppose not. Then $\mathbf{p} \cdot \mathbf{x}^{*} < m$.

Local non-satiation implies that $\exists \mathbf{x}' \in X$ such that

$$\mathbf{x}' \succ \mathbf{x}^{*} \text{ and } \mathbf{p} \cdot \mathbf{x}' \leq m.$$

This contradicts the assumption that $\mathbf{x}^{*}$ is a solution to the utility-maximization problem. ■

Indirect utility functions

Substitute $\mathbf{x}^* = \mathbf{x}(\mathbf{p}, m)$ into $u(\mathbf{x})$, we obtain

Indirect utility function: $v(\mathbf{p}, m) = u(\mathbf{x}^*)$

An alternative way to define the indirect utility function:

$$v(\mathbf{p}, m) = \max_{\mathbf{x} \in B} u(\mathbf{x}),$$

where B is the budget set.

The indirect utility function represents the maximum utility level that the consumer can achieve at prices $\mathbf{p}$ and income m.

Properties of the indirect utility function:

(1) $v(\mathbf{p}, m)$ is non-increasing in $\mathbf{p}$; that is,
if $\mathbf{p}' \geq \mathbf{p}$, then $v(\mathbf{p}', m) \leq v(\mathbf{p}, m)$.

Proof: Let $B = \{\mathbf{x} \mid \mathbf{p} \cdot \mathbf{x} \leq m\}$ and $B' = \{\mathbf{x} \mid \mathbf{p}' \cdot \mathbf{x} \leq m\}$.

Since $\mathbf{p}' \geq \mathbf{p}$, $B' \subset B$.

$$\Rightarrow \max_{\mathbf{x} \in B} u(\mathbf{x}) \geq \max_{\mathbf{x} \in B'} u(\mathbf{x}) \Rightarrow v(\mathbf{p}', m) \leq v(\mathbf{p}, m) \qquad \blacksquare$$

(2) $v(\mathbf{p},m)$ is strictly increasing in m; that is, if $m' > m$, then $v(\mathbf{p},m') > v(\mathbf{p},m)$. $\mathbf{p}\cdot\mathbf{x}^* \leq m < m'. \Rightarrow \mathbf{x}^* \in B'$

Proof: Let $B = \{\mathbf{x} \mid \mathbf{p}\cdot\mathbf{x} \leq m\}$ and $B' = \{\mathbf{x} \mid \mathbf{p}\cdot\mathbf{x} \leq m'\}$.

Suppose $\mathbf{x}^* = \arg\max_{\mathbf{x}\in B} u(\mathbf{x})$.

Then $\mathbf{p}\cdot\mathbf{x}^* \leq m < m' \Rightarrow \boldsymbol{x}^* \in \langle\, B' \,\rangle$.

By local nonsatiation, $\exists \mathbf{x}' \in B'$ such that $\mathbf{x}' \succ \mathbf{x}^*$

$\Longrightarrow v(\mathbf{p},m') > v(\mathbf{p},m)$ ■

(3) $v(\mathbf{p},m)$ is homogenous of degree zero in $(\mathbf{p}, m)$.

Proof: Use Proposition 2.2 and note that $v(\mathbf{p},m) = u(x(\mathbf{p},m))$.

■

(4) $v(\mathbf{p}, m)$ is quasi-convex in ($\mathbf{p}$, m).

Proof: A function $f(\mathbf{x})$ is quasi-convex iff the set $\{\mathbf{x} \mid f(\mathbf{x}) \leq \alpha\}$ is a convex set for all α. Thus, we want to prove that the set $\{(\mathbf{p}, m) \mid v(\mathbf{p}, m) \leq \alpha\}$ is a convex set for all α.

Suppose $(\mathbf{p}, m)$ and $(\mathbf{p}', m')$ are such that $v(\mathbf{p}, m) \leq \alpha$ and $v(\mathbf{p}', m') \leq \alpha$.

Let $\mathbf{p}'' = t\mathbf{p} + (1-t)\mathbf{p}'$ and $m'' = tm + (1-t)m'$ (for $0 < t < 1$)

We want to show that $v(\mathbf{p}'', m'') \leq \alpha$.

Define $B = \{\mathbf{x} \mid \mathbf{p} \cdot \mathbf{x} \leq m\}$, $B' = \{\mathbf{x} \mid \mathbf{p}' \cdot \mathbf{x} \leq m'\}$ and $B'' = \{\mathbf{x} \mid \mathbf{p}'' \cdot \mathbf{x} \leq m''\}$

Claim: $B'' \subset (B \cup B')$.

Why? Suppose not. Then there exists some $\mathbf{x} \in B''$ with $\mathbf{p}'' \cdot \mathbf{x} \leq m''$ such that $\mathbf{x} \notin B$ and $\mathbf{x} \notin B'$.

$\Rightarrow \mathbf{p} \cdot \mathbf{x} > m$ and $\mathbf{p}' \cdot \mathbf{x} > m'$. Then,

$(t\mathbf{p}) \cdot \mathbf{x} + (1-t)\mathbf{p}' \cdot \mathbf{x} > tm + (1-t)m' = m''$

$\Rightarrow \mathbf{p}'' \cdot \mathbf{x} > m'' \quad \Rightarrow$ A contradiction.

Since $B'' \subset (B \cup B')$, then $\max_{\mathbf{x} \in B''} u(\mathbf{x}) \leq \max_{\mathbf{x} \in (B \cup B')} u(\mathbf{x})$

Since $\max_{\mathbf{x} \in B} u(\mathbf{x}) \leq \alpha$ and $\max_{\mathbf{x} \in B'} u(\mathbf{x}) \leq \alpha$, then

$\max_{\mathbf{x} \in B''} u(\mathbf{x}) \leq \alpha$.

$\therefore \{(\mathbf{p}, m) \mid v(\mathbf{p}, m) \leq \alpha\}$ is a convex set.

(5) $v(\mathbf{p}, m)$ is continuous in ($\mathbf{p}$, m) at all $\mathbf{p} \gg \mathbf{0}$, $m > 0$.

This property can be proven using the Theorem of Maximum.

Differentiable Utility Function: Assume that $u(\mathbf{x})$ is C^2 (twice continuously differentiable)

Based on Proposition 2.3, we can rewrite the utility maximization problem as

$$\max_{\boldsymbol{x}} u(\boldsymbol{x}) \quad \text{s.t.}\ \boldsymbol{p} \cdot \boldsymbol{x} = m$$

Use the Lagrange method to solve this problem:

$$L = u(\mathbf{x}) + \lambda(m - \mathbf{p} \cdot \mathbf{x})$$

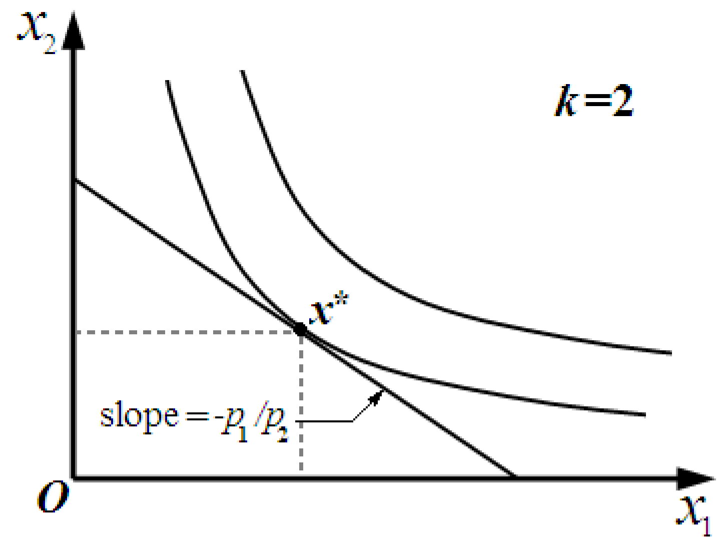
The first-order conditions (FOCs) for an *interior* solution:

$$\frac{\partial L}{\partial x_i} = \frac{\partial u}{\partial x_i} - \lambda p_i = 0, \quad i = 1, 2, \ldots, k.$$

$$\frac{\partial L}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 \ldots - p_k x_k = 0.$$

$$FOC: \begin{cases} \dfrac{\partial L}{\partial x_1} = \dfrac{\partial u}{\partial x_1} - \lambda p_1 = 0 \\ \dfrac{\partial L}{\partial x_2} = \dfrac{\partial u}{\partial x_2} - \lambda p_2 = 0 \\ \vdots \\ \dfrac{\partial L}{\partial x_k} = \dfrac{\partial u}{\partial x_k} - \lambda p_k = 0 \\ \dfrac{\partial L}{\partial \lambda} = m - \mathbf{p} \cdot \mathbf{x} = 0 \end{cases}$$

$$\Rightarrow \frac{\partial u / \partial x_i}{\partial u / \partial x_j} = \frac{p_i}{p_j}$$



The solution to the FOCs solves the consumer's utility-maximization problem if $u(\mathbf{x})$ is quasi-concave on X (for $\mathbf{x} >> 0$, more generally Kuhn-Tucker conditions).

Recall: (A5) $\Longrightarrow$ $u(\mathbf{x})$ is quasi-concave on X.

Example: (see Jehle & Reny for details)

$$u = (x_1^{\rho} + x_2^{\rho})^{1/\rho}, \text{ where } 0 \neq \rho < 1$$

(The CES utility function)

The Marshallian demand functions:

$$x_1^* = \frac{p_1^{1/(\rho-1)} m}{p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}}, \; x_2^* = \frac{p_2^{1/(\rho-1)} m}{p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}}$$

The indirect utility function:

$$v(p, m) = \frac{m}{\left[p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}\right]^{(\rho-1)/\rho}}$$

The elasticity of substitution:

$$\sigma = \frac{p_2/p_1}{x_1^*/x_2^*} \cdot \frac{\partial(x_1^*/x_2^*)}{\partial(p_2/p_1)} = \frac{1}{1-\rho}$$