练　习　十　九

1．求下列像函数$F(s)$的拉氏逆变换。

（1）$\dfrac{s}{(s-a)(s-b)}$

解：原式$=\dfrac{a}{a-b}\cdot\dfrac{1}{s-a}-\dfrac{b}{a-b}\cdot\dfrac{1}{s-b}$

$\therefore \mathscr{L}^{-1}[F(s)]=\dfrac{a}{a-b}e^{at}-\dfrac{b}{a-b}e^{bt}$

或$f(t)=\dfrac{s}{(s-a)+(s-b)}e^{st}\Big|_{s=a}+\dfrac{s}{(s-a)+(s-b)}e^{st}\Big|_{s=b}$

$=\dfrac{a}{a-b}e^{at}+\dfrac{b}{b-a}e^{bt}$

（2）$\dfrac{1}{s^4+5s^2+4}$

解：原式=$\dfrac{1}{3}(\dfrac{1}{s^2+1}-\dfrac{1}{s^2+4})$

$\therefore \mathscr{L}^{-1}[F(s)]=\dfrac{1}{3}\sin t-\dfrac{1}{6}\sin 2t$

（3）$\ln\dfrac{s^2-1}{s^2}$

解：$f(t)=-\dfrac{1}{t}\mathscr{L}^{-1}[F'(s)]$

$\therefore F(s)=\ln(s^2-1)-\ln s^2=\ln(s+1)+\ln(s-1)2\ln s$

$\therefore F'(s)=\dfrac{1}{s+1}+\dfrac{1}{s-1}-\dfrac{2}{s}$

故$f(t)=-\dfrac{1}{t}(e^{-t}+e^{t}-2)=\dfrac{1}{t}(2-2\cdot\dfrac{e^{t}+2^{-t}}{2})=\dfrac{2}{t}(1-cht)$

（4）$\dfrac{1+e^{-2s}}{s^2}$

解：$f(t)=\mathscr{L}^{-1}[F(s)]=\mathscr{L}^{-1}[\frac{1}{s^2}]+\mathscr{L}[e^{-2s}\cdot\frac{1}{s^2}]$

$$= t+\mathscr{L}^{-1}[e^{-2s}\cdot\frac{1}{s}]$$

$\mathscr{L}^{-1}[e^{-as}F(s)]=f(t-a)u(t-a)\quad(a>0)$

$\therefore f(t)=t+(t-2)u(t-2)$

$$=\begin{cases}t, & 0\le t<2\\ 2(t-1), & t\ge 2\end{cases}$$

2．利用卷积定理证明：$\mathscr{L}[\int_0^t f(t)dt]=\mathscr{L}[f(t)*u(t)]=\frac{F(s)}{s}$

证：$\mathscr{L}[\int_0^t f(t)dt]=\mathscr{L}[\int_0^t f(t)\cdot dt]=\mathscr{L}[f(t)*u(t)]=t+(t-2)u(t-2)$

$$=\int_0^{+\infty}[f(t)*u(t)]e^{-st}dt=\int_0^{+\infty}e^{st}[\int_0^t f(\tau)u(t-\tau)]dt$$

$$=\begin{cases}t, & 0\le t<2\\ 2(t-1), & t\ge 2\end{cases}$$

$$=\int_0^{\infty}f(\tau)e^{-st}dt=\int_0^{\infty}u(t)e^{-st}du=\frac{F(s)}{s}$$

3．用拉氏变换求下列微分方程。

（1） $y''-2y'+y=e^t$， $y(0)=y'(0)=0$

解：取拉氏变换有：

$$s^2\mathcal{L}[y(t)]-sy(0)-y'(0)-2(s\,\mathcal{L}[y(t)]-y(0)]+\mathcal{L}[y(t)]=\frac{1}{s-1}$$

$$\therefore (s^2-2s+1)\,\mathcal{L}[y(t)]=\frac{1}{s-1}$$

$$\therefore\mathcal{L}[y(t)]=\frac{1}{(s-1)^3}$$

取拉氏逆变换有：$y(t)=\frac{1}{2}t^2e^t$

（2） $y^{(4)}+y'''=\cos t$， $y(0)=y'''(0)=0$， $y''(0)=C$（常数）

解：$s^4y(s)-sy''(0)+s^3y(s)-y''(0)$

$$=\frac{s}{s^2+1}$$

$$\therefore y(s)=\frac{1}{s^2(s+1)(s^2+1)}+\frac{c}{s^3}$$

$$=-\frac{1}{s}+\frac{1}{s^2}+\frac{1}{2(s+1)}+\frac{s-1}{2(s^2+1)}+\frac{c}{s^3}$$

$$\therefore y(t)=-1+t+\frac{1}{2}e^{-t}+\frac{1}{2}\cos t-\frac{1}{2}\sin t+\frac{c}{2}t^2$$

（3）$\begin{cases} y''-x''+x'-y=e^t-2, & x(0)=x'(0)=0 \\ 2y''-x''-2y'+x=-t, & y(0)=y'(0)=0 \end{cases}$

解：设$\mathscr{L}[y(t)]=Y(s)$，$\mathscr{L}[x(t)]=X(s)$

则：$\begin{cases} s^2Y(s)-s^2X(s)+sX(s)-Y(s)=\dfrac{1}{s-1}-\dfrac{2}{s} \\ 2s^2Y(s)-s^2X(s)-2sY(s)+X(s)=-\dfrac{1}{s^2} \end{cases}$

$$\begin{cases} (s+1)Y(s)-sX(s)=\dfrac{-s+2}{s(s-1)^2} \\ 2sY(s)-(s+1)X(s)=-\dfrac{1}{s^2(s-1)} \end{cases}$$

可得：$\begin{cases} X(s)=\dfrac{2s-1}{s^2(s-1)^2} \\ Y(s)=\dfrac{1}{s(s-1)^2} \end{cases}$ $\quad\therefore\begin{cases} x(t)=-t+te^t \\ y(t)=1-e^t+te^t \end{cases}$

$$\therefore y(t)=1+te^t-e^t$$

（$x(t)=\lim\limits_{s\to 0}\dfrac{d}{ds}[\dfrac{2s-1}{(s-1)^2}e^{st}]+\lim\limits_{s\to 1}\dfrac{d}{ds}[\dfrac{2s-1}{s^2}e^{st}]=-t+te^t$）