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HW #3

3. $n = 12$

$$M_1 = \frac{\Sigma_i X_i}{n} = \frac{\Sigma_i X_i}{12}$$

a. 1st moment of Poisson:

$$E(X) = \lambda_s$$
$$\lambda_s = \frac{\Sigma_i X_i}{12}$$

b. $\Sigma_i X_i = 8+0+2+5+3+7+0+1+9+10+12+6 = 63$

$$\lambda_s = \frac{63}{12} = \underline{5.25}$$

c. 1 yard = 3 ft. → 1 yd.² = 9 ft.²

Average number of flaws per sq. ft.:

$$5.25 \cdot 9 = \underline{47.25}$$

4. a. $E(X) = \lambda_s$

$$\lambda_s = \frac{\Sigma_i X_i}{9}$$

b. $\Sigma_i X_i = 25+30+10+20+24+23+20+15+4 = 171$

$$\lambda_s = \frac{171}{9} = 19$$

c. $\frac{1}{4}\lambda_s = \frac{19}{4} = 4.75$ requests/hr.

5. a. $\hat{p} = \frac{\overline{X}}{10}$

$$\begin{aligned} E(\hat{p}) &= p \\ &= E\left(\frac{\overline{X}}{10}\right) \\ &= \frac{1}{10} E(\overline{X}) \\ &= \frac{1}{10} E\left(\frac{\Sigma X_i}{5}\right) \\ &= \frac{1}{10} E\left(\frac{X_1 + X_2 + \dots + X_5}{5}\right) \\ &= \frac{1}{50}\left[E(X_1) + E(X_2) + \dots + E(X_5)\right] \\ &= \frac{1}{50}(10p + 10p + \dots + 10p) \\ &= \frac{1}{50} \cdot 50p \\ &= p \end{aligned}$$

b. $\overline{X} = \frac{\Sigma_i X_i}{m}$

$$= \frac{3+4+4+5+6}{5} = \frac{22}{5} = 4.4$$

$$p = \frac{4.4}{10} = \underline{0.44}$$

6. a. $E(X) = np = \overline{X}$

$$p = \frac{\overline{X}}{8}$$

b. $\overline{X} = \frac{\Sigma_i X_i}{m}$

$$= \frac{\left(\begin{array}{l}1+0+0+0+0+0+1+1+0+1+2+\\1+1+0+1+0+2+2+3+0\end{array}\right)}{20}$$

$$= \frac{16}{20} = 0.8$$

$$p = \frac{\overline{X}}{n} = \frac{0.8}{8} = 0.1$$

c. $n = 64$

$$P(X \leq 1) = \binom{64}{0} p^0 (1-p)^{64} + \binom{64}{1} p^1 (1-p)^{63}$$

$$= (1-0.8)^{64} + 64(0.8)(1-0.8)^{63}$$

$$\approx 0$$

7. a. X: Uniformly distributed over interval $(0, b)$

Density function of X:

$$f(X) = \frac{1}{b}$$

$$\text{Mean} = \mu = E(X) = \frac{b}{2}$$

$$\text{Variance} = \sigma^2 = \frac{b^2}{12}$$

a. $\overline{X} = \frac{\Sigma X_i}{n}$

$= \frac{10+8+7+9+11+10+12+9+8+13}{10}$

$= \frac{97}{10} = 9.7$

b. $S^2 = \frac{\Sigma (X_i - \overline{X})^2}{n-1}$

$= \frac{(10-9.7)^2 + (8-9.7)^2 + (7-9.7)^2 + (9-9.7)^2 + (11-9.7)^2 + (10-9.7)^2 + (12-9.7)^2 + (9-9.7)^2 + (8-9.7)^2 + (13-9.7)^2}{9}$

$= \underline{3.567}$

c. $\overline{X} = E(X)$

$9.7 = \frac{b}{2}$

$b = \underline{19.4}$

d. $\sigma = \sqrt{\frac{b^2}{12}} = \frac{b}{\sqrt{12}} = \frac{19.4}{3.46} = \underline{5.607}$

10. a. $n=4$

$p=0.5$

$\mu = E(X) = np = (4)(0.5) = 2$

$\text{Var } X = npq = (4)(0.5)(1-0.5) = 1$

b. 3, 0, 2, 2, 1, 3, 1, 0, 3, 4

c. $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

$= \frac{3+0+2+2+1+3+1+0+3+4}{10} = \underline{\underline{1.4}}$

$\overline{X} < E(X)$ //

d. $\overline{X} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$

$= \frac{1}{9}\left[(3-1.4)^2 + (0-1.4)^2 + \dots + (3-1.4)^2 + (4-1.4)^2\right]$

$= \frac{2.56 + 1.96 + \dots + 2.56 + 6.76}{9}$

$= 78281.8$

$s^2 > \sigma^2$ //

11. $n=4$

$p=\frac{1}{2}$

$\mu = E(X) = (4)(0.5) = 2$

$\overline{X} = 2$

$\text{Var } X = \sigma^2 = npq = (4)(0.5)(1-0.5) = 1$

a. $\{3, 4, 1\}$

$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{3}(3+4+1) = 2.67$ //

b. $\{0,1,2\}$, $\{0,3,3\}$, $\{3,3,0\}$, $\{3,1,0\}$, $\{3,2,2\}$, $\{3,3,0\}$, $\{2,2,2\}$, $\{2,3,2\}$, $\{0,1,1\}$ $\longrightarrow$ 9 more times

2.67, 1, 2, 2, 1.33, 2.33, 2, 2, 2.33, 0.67

c. $E(X) = \frac{1}{n}\sum_{i=1}^{n} X_i$

$= \frac{2.67+1+2+1.33+2.33+2+2+2.33+0.67}{10}$

$= 1.834$

$E(X) < 2$, but close to 2

d. $\text{Var } X = \frac{\sigma^2}{n} = \frac{1}{3} = 0.33$ //

$$s^2 = \frac{\Sigma(\bar{x} - E(\bar{x}))^2}{n-1}$$

$$= \frac{1}{10-1}\left[(2.67-1.834)^2 + (1-1.834)^2 + \dots + (0.67-1.834)^2\right]$$

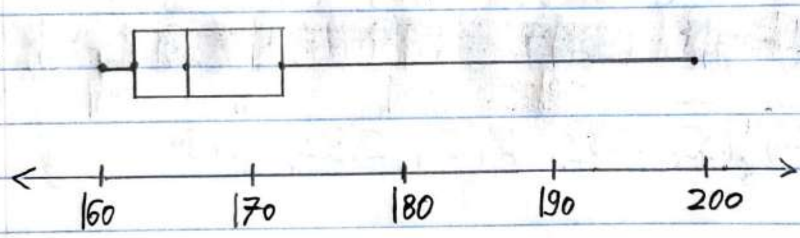
$$= \underline{0.324} \rightarrow \text{Very close to } 0.33$$ //

13\. a.

16	051320278317293 43
17	026398
18	50
19	6

Skewed right //

b.



$\tilde{x} = 167$ $\quad f_1 = 148$ $\quad F_1 = 133$

$q_1 = 163$ $\quad f_3 = 188$ $\quad F_3 = 203$

$q_3 = 173$

$$IQR = q_3 - q_1$$
$$= 173 - 163$$
$$= 10$$

196 is flagged.

Distribution : NOT bell-shaped

"Outlier" : probably NOT unusual

c. $\bar{X} = \dfrac{162 + 2 \cdot 166 + 172.75}{4}$

$= \underline{\underline{166.94}}$

15. $n = 20$, $p = \text{unknown}$

$E(X) = np = 20p$

$\overline{X} = 20\hat{p}$

$$\hat{p} = \frac{\overline{X}}{20} = \frac{1}{20} \sum_{i=1}^{5} \frac{X_i}{5}$$

$$= \frac{1}{100}(13 + 12 + 15 + 10 + 17) = \underline{\underline{0.67}}$$

17. $M_1 = \frac{\sum X_i}{n}$

$E(X) = \lambda s$

$$\hat{\lambda s} = \frac{\sum X_i}{n} = \overline{X}$$

$$\hat{\lambda} = \frac{\hat{\lambda s}}{s} = \frac{\overline{X}}{s}$$

16. Methods of moments
to derive an estimator of p :

$$M_1 = \frac{\sum_i X_i}{m} = \overline{X}$$

$E(X) = np$

$n\hat{p} = \overline{X}$

$$\hat{p} = \frac{\overline{X}}{n}$$

18. $\hat{\lambda} = \frac{\sum X_i}{n} = \overline{X} \longrightarrow n = 10$

$$\overline{X} = \frac{\sum X_i}{n}$$

$$= \frac{2+3+5+0+1+8+3+2+2+5}{10} = \underline{\underline{3.1}}$$

21. $M_k = \sum_{i=1}^{n} \frac{X_i^k}{n}$

$$M_1 = \sum_{i=1}^{25} \frac{X_i}{25}$$

$$= \frac{160 + 165 + 170 + \ldots + 179 + 163 + 178}{25} = \underline{\underline{169}}$$

$$M_2 = \sum_{i=1}^{25} \frac{X_i^2}{25}$$

$$= \frac{25600 + 27225 + \ldots + 26569 + 31684}{25}$$

$= 28652.8$

$E(X) = M_1 = 169$

$\sigma^2 = E(X^2) - (E(X))^2$

$= M_2 - M_1^2$

$= 28652.8 - 28651$

$= \underline{\underline{1.8}}$

23. $P(X=K) = p(1-p)^{K-1}$, $K = 1, 2, 3, \ldots$

$$p \downarrow \text{P(success)}$$

$$E(X) = \frac{1}{p}$$

Method of moments estimator p:
- involves equating 1st sample moment with corresponding population moment
- solve for p

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_n$$

$$p = \frac{1}{\overline{X}}$$

Method of moments estimator:

$$\hat{p} = \frac{1}{\overline{X}}$$

25. $\hat{p} = \frac{1}{\overline{X}} \rightarrow$ sample mean

$$\text{Var } X = \frac{1-p}{p^2}$$

Solving for p:

$$p = \frac{1}{1 + \frac{Var\ X}{\overline{X}^2}}$$

$$\hat{p} = \frac{1}{\overline{X}}$$

$$\hat{\sigma}^2 = \frac{1-\hat{p}}{\hat{p}^2} = \frac{1-\frac{1}{\overline{X}}}{\frac{1}{\overline{X}^2}}$$

$$= \overline{X}^2\left(1 - \frac{1}{\overline{X}}\right) = \overline{X}^2 - \overline{X}$$

$$\overline{X} = \frac{\begin{array}{c}24+28+33+34+45+32+33+39+26+33+35+ \\ 46+36+33+37+26+28+45+36+24+42+ \\ 30+33+33+48\end{array}}{25-1}$$

$$= \underline{34.36}$$

28. $P(X_1 = x_1, \ldots, X_{10} = x_{10})$

$$= P(X_1 = x_1) \cdot \ldots \cdot P(X_{10} = x_{10})$$

$$= e^{-\lambda} \cdot \frac{\lambda^{x_1}}{x_1!} \cdot \ldots \cdot e^{-\lambda} \cdot \frac{\lambda^{x_{10}}}{10!}$$

$$= e^{-10\lambda} \cdot \frac{\lambda^{x_1 + \ldots + x_{10}}}{x_1! \ldots x_{10}!} = L(\lambda)$$

$$g(\lambda) = \ln[L(\lambda)]$$
$$= -10\lambda + \ln(\lambda^{x_1+\ldots+x_{10}}) - \ln(x_1! \ldots x_{10}!)$$
$$= -10\lambda + (x_1 + \ldots + x_{10})\ln(\lambda) - \ln(x_1! \ldots x_{10}!)$$

$$g'(\lambda) = -10 + (x_1 + \ldots + x_{10})\frac{1}{\lambda} + 0 = 0$$

$$(x_1 + \ldots + x_{10})\frac{1}{\lambda} = 10$$

$$\frac{1}{\lambda} = \frac{10}{x_1 + \ldots + x_{10}}$$

$$\lambda = \frac{x_1 + \ldots + x_{10}}{10}$$
$$= \frac{2+5+1+3+2+3+0+8+2+5}{10}$$
$$= \underline{3.1}$$

30. $P(X_1 = x_1, \ldots, X_m = x_m)$

$$= P(X_1 = x_1) \cdot \ldots \cdot P(X_m = x_m)$$

$$= e^{-\lambda}\frac{\lambda^{x_1}}{x_1!} \cdot \ldots \cdot e^{-\lambda}\frac{\lambda^{x_m}}{m!}$$

$$= e^{-10\lambda}\frac{\lambda^{x_1+\ldots+x_m}}{x_m!}$$

$$\ln L(p) = (x_1 + x_2 + \ldots + x_m)(\ln p) + [mn - (x_1 + x_2 + \ldots + x_m)] \ln(1-p)$$

$$\frac{d}{dp}\left[\ln L(p)\right] = \frac{X_1+X_2+\ldots+X_m}{p} - \frac{mn-(X_1+X_2+\ldots+X_m)}{1-p} = 0$$

$$p = \frac{X_1+X_2+\ldots+X_m}{mn}$$

$\#16: \overline{X} = n\hat{p}$

$$\hat{p} = \frac{\overline{X}}{n} = \frac{X_1+X_2+\ldots+X_m}{mn}$$

31. $L(\beta) = f(W_1)\cdot f(W_2)\cdot \ldots \cdot f(W_n)$

$$= \frac{1}{\beta}e^{-W_1/\beta}\cdot\frac{1}{\beta}e^{-W_2/\beta}\cdot \ldots \cdot\frac{1}{\beta}e^{-W_n/\beta}$$

$$= \frac{1}{\beta^n}e^{-(W_1+W_2+\ldots+W_n)/\beta}$$

$$\ln L(\beta) = -n\ln(\beta) - \frac{W_1+W_2+\ldots+W_n}{\beta}$$

$$\frac{d}{d\beta}\ln L(\beta) = -\frac{n}{\beta} + \frac{W_1+W_2+\ldots+W_n}{\beta^2} = 0$$

$$\beta = \frac{W_1+W_2+\ldots+W_n}{n}$$

34. $L(\beta) = f(X_1)\, f(X_2)\ldots f(X_n)$

$$= \frac{1}{\beta}e^{-X_1/\beta}\cdot\frac{1}{\beta}e^{-X_2/\beta}\cdot \ldots \cdot\frac{1}{\beta}e^{-X_n/\beta}$$

$$= \frac{1}{\beta^n}e^{-(X_1+X_2+\ldots+X_n)/\beta}$$

$$\ln L(\beta) = -n\ln(\beta) - \frac{X_1+X_2+\ldots+X_n}{\beta}$$

$$\frac{d}{d\beta}\ln\left[L(\beta)\right] = -\frac{n}{\beta} + \frac{X_1+X_2+\ldots+X_n}{\beta^2} = 0$$

$$\frac{n}{\beta} = \frac{X_1+X_2+\ldots+X_n}{\beta^2} \quad \cdot\beta^2$$

$$\beta n = X_1+X_2+\ldots+X_n$$

$$\beta = \frac{X_1+X_2+\ldots+X_n}{n}$$

$$= \frac{1.7+2.1+\ldots+7.0+1.6}{20} = \underline{\underline{2.86}}$$

35. $n=5$, Binomial $\rightarrow$ Find p

$L(p) = f(X_1)\; f(X_2)\;\ldots\; f(X_n)$

$$= p^{X_1}(1-p)^{5-X_1}\cdot p^{X_2}(1-p)^{5-X_2}\ldots p^{X_n}(1-p)^{5-X_n}$$

$$= p^1(1-p)^{5-1}\cdot p^0(1-p)^{5-0}\ldots p^1(1-p)^{5-1}\cdot p^0(1-p)^{5-0}$$

$$L(p) = p^5 (1-p)^{45}$$

$$\ln L(p) = \ln[p^5 (1-p)^{45}]$$

$$= \ln(p^5) + \ln[(1-p)^{45}]$$

$$= 5 \ln p + 45 \ln(1-p)$$

$$\frac{d}{dp} \ln L(p) = \frac{5}{p} - \frac{45}{1-p} = 0$$

$$5(1-p) - 45p = 0$$
$$5 - 5p - 45p = 0$$
$$5 - 50p = 0$$
$$-50p = -5$$
$$p = \underline{0.1}$$

36. $X \geq 75000$

$n = 4$, Binomial $\longrightarrow p = ?$

$$L(p) = f(x_1)\, f(x_2) \ldots f(x_n)$$

$$= p^{x_1}(1-p)^{n-x_1} \cdot p^{x_2}(1-p)^{n-x_2} \ldots p^{x_n}(1-p)^{n-x_n}$$

$$= p^1(1-p)^{4-1} \cdot p^0(1-p)^{4-0} \ldots p^1(1-p)^{4-1}) \cdot p^0(1-p)^{4-0}$$

$$= p^8 (1-p)^{60}$$

$$\ln L(p) = 8(\ln p) + 60 \ln(1-p)$$

$$\frac{d}{dp} \ln L(p) = \frac{8}{p} - \frac{60}{1-p} = 0$$

$$8(1-p) - 60p = 0$$
$$8 - 8p - 60p = 0$$
$$8 - 68p = 0$$
$$-68p = -8$$
$$p = \frac{2}{17} = \underline{0.1176}$$

I would have some doubts
if early failure rate > 0.1

Reason: $p = 0.1176 > 0.1$

New material may not be suitable

46. X: Time required to do computation using an algorithm written in programming language A

Y: Time required to do same calculation in prog lang. B

X: → normally distributed
→ $N = 10$ sec
→ $\sigma = 3$ sec

Y: → normally distributed
→ $N = 9$ sec
→ $\sigma = 4$ sec

a. Distribution of random var X–Y:

X–Y : Normal

$N_{X-Y} = N_X - N_Y = 10 - 9 = 1$

$$Var(X-Y) = (Var\ X)^2 + (Var\ Y)^2$$
$$= 3^2 + 4^2$$
$$= 25$$

$\sigma_{X-Y} = \sqrt{25} = 5$

$X-Y$: → Normally distributed
→ $N = 1$
→ $\sigma = 5$ //

b. $P(X-Y<0)$
$$= P\left(\frac{X-Y-1}{5} < \frac{0-1}{5}\right)$$
$$= P(Z < -0.2)$$
$$= \underline{0.4207}$$

47. a. $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{20}\sum_{i=1}^{20} X_i = \underline{0.6436}$

Unbiased point estimate for μ: $\hat{\mu} = \overline{X} = \underline{0.6436}$

b. Formula for confidence interval:
$\overline{X} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$

where $\alpha = 0.05$
$Z_{\alpha/2} = 1.96$ for 95% C.I.
$\sigma = 0.01$
$n = 20$

$$0.6436 \pm 1.96 \cdot \frac{0.01}{\sqrt{20}} = 0.6436 \pm 0.0055$$

95% C.I.: $(0.6381, 0.6491)$

c. 90% C.I. shorter than 95% C.I.
$$0.6436 \pm 1.645 \cdot \frac{0.01}{\sqrt{20}}$$

90% C.I.: $(0.6390, 0.6482)$ → Shorter //

d. 99% C.I:
$$0.6436 \pm 2.576 \cdot \frac{0.01}{\sqrt{20}}$$
↓
$(0.6366, 0.6506)$ → Longer than 95% C.I. //

48. a.

0	879973
1	68294641681646433284363651415 65
2	010

- Feasible to be normally distributed
- can also be skewed right

b. $\overline{X} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{16+20+\dots+16+15}{40} = 13.85$

c. $C.I. = \overline{X} \pm Z \cdot \frac{\sigma}{\sqrt{n}}$ $\qquad Z = 1.96$ when 95%
$$= 13.85 \pm 1.96 \cdot \frac{4}{\sqrt{40}}$$
$$= \underline{(12.61, 15.09)}$$

$N = 17$ days $\rightarrow$ surprising, outside interval

Nevertheless, estimation only based on sample; uncertainty exists.

52. a. $E(X) = \int x \, f(x) \, dx$

$= \int_0^\theta x \cdot \frac{1}{\theta} \, dx$

$= \left[\frac{1}{2} x^2\right]_0^\theta \cdot \frac{1}{\theta}$

$= \frac{1}{2} \theta^2 \cdot \frac{1}{\theta}$

$= \frac{1}{2} \theta$

b. $\overline{X} = E(X) = \frac{1}{2} \theta$

$\theta = 2\overline{X} \rightarrow$ Unbiased for θ

Reason: $E(\theta) = E(2\overline{X}) = 2E(\overline{X}) = \frac{2\theta}{2} = \theta$

c. $\overline{X} = E(X) = \frac{\theta}{2}$

$\theta = 2\overline{X} = 2(0.83) = \underline{1.66}$

53. a. Normal Distribution
- Mean N
- Standard deviation $\frac{2}{\sqrt{16}} = 0.5$
- Variance : $\sigma^2 = 0.25$

b. $\overline{X} = \dfrac{42.65 + 45.15 + \ldots + 43.28 + 40.7}{16}$

$= \underline{42.88}$

c. 95% C.I. for N:

$\overline{X} \pm \left(\frac{t\alpha}{2}, n-1\right) \cdot \frac{S}{\sqrt{n}}$

$S = \sqrt{\dfrac{\Sigma(X_i - \overline{X})^2}{n-1}}$

$= \frac{1}{\sqrt{16-1}} \sqrt{(42.65 - 42.88)^2 + \ldots + (40.70 - 42.88)^2}$

$= \underline{2.072}$

$\left(\frac{t\alpha}{2}, n-1\right) = (0.025 t, 15) = 2.131$

95% C.I. for N:

$42.88 \pm 2.131 \cdot \frac{2.072}{\sqrt{16}} = 42.88 \pm 1.104$

$\downarrow$

$(41.776, 43.984)$

Not surprised: 42.88 lies within 95% C.I.

54. a. $\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$

$= \dfrac{3.83 + 3.54 + \ldots + 4.09 + 3.93}{10}$

$= \underline{3.698}$

b. $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2$

$= \frac{(3.83-3.698)^2 + \dots + (3.93-3.698)^2}{10-1}$

$= \underline{0.0487}$

c. $s = \sqrt{s^2} = \sqrt{0.102} \approx 0.319 \rightarrow$ unbiased estimate for σ //

d. $L(\mu, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2\right)$

$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (X_i - \mu)^2 = 0 \quad \cdot (-2\sigma^4)$

$\sigma^2 \cdot n - \sum_{i=1}^{n} (X_i - \mu)^2 = 0$

$\sigma^2 \cdot n = \sum_{i=1}^{n} (X_i - \mu)^2$

$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} (X_i - \mu)^2}{n}$

$\hat{\sigma}^2 = \frac{(3.83-3.698)^2 + \dots + (3.93-3.698)^2}{10}$

$= \underline{0.044}$

only slightly different $\rightarrow$ agree to (b) //

55. a. Binomial with p as probability success

b. Central Limit Theorem:

$\overline{X} = \frac{X_1 + X_2 + \dots + X_{100}}{100} \rightarrow$ Sample mean

$E(\overline{X}) = E(X_i) = p \longrightarrow$ Mean

$\mathrm{Var}\,\overline{X} = \frac{\mathrm{Var}\,X_i}{100} = \frac{p(1-p)}{100} \longrightarrow$ Variance //

c. Point estimate $\rightarrow$ Sample proportion:

$\hat{p} = \frac{\text{num of packages in sample}}{\text{sample size}} = \frac{5}{100} = \underline{0.05}$

59. $f(x) = \frac{1}{\theta^2} x e^{-x/\theta} \qquad x > 0, \; \theta > \theta$

a. Gamma Dist. with $\alpha = 2$ and $\beta = \theta$

b. $E(X) = \int x f(x)\, dx$

$= \int_0^\infty x \cdot \frac{1}{\theta^2} x e^{-x/\theta}\, dx$

$= \frac{1}{\theta^2} \int_0^\infty x^2 e^{-x/\theta}\, dx$

$= \frac{1}{\theta^2} \cdot 2\theta^3$

$= \underline{2\theta}$

c. $\overline{X} = E(X) = 2\theta$

$\theta = \frac{\overline{X}}{2}$

d. $L(\theta) = f(X_1)\, f(X_2) \ldots f(X_n)$

$= \frac{1}{\theta^2} X_1 e^{-X_1/\theta} \cdot \frac{1}{\theta^2} X_2 e^{-X_2/\theta} \cdot \ldots \cdot \frac{1}{\theta^2} X_n e^{-X_n/\theta}$

$= \theta^{-2n} X_1 X_2 \ldots X_n \, e^{-(X_1+X_2+\ldots+X_n)/\theta}$

$\ln L(\theta) = \ln(\theta^{-2n}) + \ln(X_1 X_2 \ldots X_n) - \frac{X_1+X_2+\ldots+X_n}{\theta}$

$= -2n \ln\theta + \ln(X_1 X_2 \ldots X_n) - \theta^{-1}(X_1+X_2+\ldots+X_n)$

$\frac{d}{d\theta} \ln L(\theta) = -\frac{2n}{\theta} + \theta^{-2}(X_1+X_2+\ldots+X_n)$

$= -\frac{2n}{\theta} + \frac{X_1+X_2+\ldots+X_n}{\theta^2} = 0 \qquad \cdot\theta^2$

$-2n\cdot\theta + X_1+X_2+\ldots+X_n = 0$

$2n\cdot\theta = X_1+X_2+\ldots+X_n$

$\theta = \frac{X_1+X_2+\ldots+X_n}{2n}$ //

60. a. $\frac{X-2}{5} \rightarrow$ Standard normal distribution

b. $\left(\frac{X-2}{5}\right)^2 \rightarrow$ Square of normal distribution $\rightarrow \chi_1^2$
Chi-squared distribution

c. Summing independent chi-squared distributions:
also summing their respective degrees of freedom.

In this case:

$\boxed{\text{10 independent variables; Each with 1 degree of freedom}}$ $\quad \sum_{i=1}^{10} \left(\frac{X_i-2}{5}\right)^2$

$\downarrow$

produce 10 degrees of freedom

63. a. $\hat{\mu} = \overline{X}$

$= \frac{1}{n} \sum_{i=1}^{n} X_i$

$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2 = \frac{1}{\lambda^2}$

$\hat{\lambda} = \frac{1}{\overline{X}} = \frac{1}{1274.14} = 7.85 \cdot 10^{-4}$

Method of moments estimator:
reciprocal of sample mean //

b. $L(\beta) = f(X_1)\, f(X_2) \ldots f(X_n)$

$= \beta e^{-\beta X_1} \cdot \beta e^{-\beta X_2} \cdot \ldots \cdot \beta e^{-\beta X_n}$

$= \beta^n e^{-\beta(X_1+X_2+\ldots+X_n)}$

$\ln L(\beta) = n\ln(\beta) - \beta(X_1+X_2+\ldots+X_n)$

$\frac{d}{d\beta} \ln L(\beta) = \frac{n}{\beta} - (X_1+X_2+\ldots+X_n) = 0$

$$\frac{n}{\beta} = X_1 + X_2 + \dots + X_n$$

$$\hat{\beta} = \frac{n}{\Sigma X_i} = \frac{50}{63707} = 7.85 \cdot 10^{-4}$$

c. Both are apparently same.

d. $P(X \geq 1000) = e^{-\beta \cdot 1000}$
$= e^{-7.85 \cdot 10^{-4} \cdot 1000}$
$= \underline{0.456}$

64. a. $\bar{X} \pm Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \rightarrow$ 95% C.I: $\alpha = 1 - 0.95 = 0.05$

$n = 30$
Dist. of X : Discrete & uniform

$$\mu_X = E(X) = \frac{1+2+3+4+5+6}{6} = 3.5$$

$$\sigma_X = \sqrt{\text{Var } X}$$
$$= \sqrt{\frac{1}{6}\left[(1-3.5)^2 + (2-3.5)^2 + \dots + (6-3.5)^2\right]}$$
$$= 1.7078$$

$$\bar{X} \pm t_{\left(\frac{\alpha}{2}, n-1\right)} \cdot \frac{\sigma}{\sqrt{n}}$$
$$= 2.83 \pm t_{(0.025, 29)} \cdot \frac{1.7078}{\sqrt{30}}$$
$$= 2.83 \pm (2.045)(0.311) = (2.21, 3.45)$$

3.5 is within (2.21, 3.45):
Cannot reject hypothesis that true mean is 3.5 at 5% significance level. //

b. Lower chance of trapping μ
Reason: 90% C.I. $\rightarrow$ narrower than 95% C.I. //

c. Higher chance of trapping true mean
Reason: 99% C.I. $\rightarrow$ Wider than 95% C.I. //

d. 99% C.I.:
$\alpha = 1 - 0.99 = 0.01$

$$\bar{X} \pm t_{(\alpha/2, n-1)} \cdot \frac{\sigma}{\sqrt{n}}$$
$$= 2.83 \pm t_{(0.005, 29)} \cdot 0.311$$
$$= 2.83 \pm (2.756)(0.311) = (1.03, 4.63)$$

3.5 is within (1.03, 4.63):
Cannot reject hypothesis that true mean is 3.5 at sigficance level.

8.1:

1. a.

1.2	68
1.3	07
1.4	89333876
1.5	1321611537
1.6	0484015
1.7	4

b. $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$

$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$

$= \frac{1}{30} \sum_{i=1}^{30} x_i$

$= \frac{1}{30} (1.48 + 1.30 + \ldots + 1.65 + 1.79)$

$= \frac{1}{30} (45.34)$

$= 1.5113$

$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$

$= \frac{1}{30-1} \left[(1.48 - 1.5113)^2 + (1.26 - 1.5113)^2 + \ldots + (1.79 - 1.5113)^2\right]$

$= \frac{0.3745}{29}$

$= \underline{\underline{0.0129}}$

c. 95% C.I. $\rightarrow \alpha = 1 - 0.95 = 0.05$

$L_1 = \frac{(n-1)s^2}{\chi^2_{\alpha/2}} = \frac{29 \cdot 0.0129}{45.7223} = 0.0818$

$L_2 = \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}} = \frac{29 \cdot 0.0129}{16.0471} = 0.0233$

$(0.0818, 0.0233) \rightarrow$ 95% C.I. on σ^2 //

d. $\sqrt{L_1} = \sqrt{0.00818} = 0.09044$

$\sqrt{L_2} = \sqrt{0.0233} = 0.152643$

95% C.I. on σ : $(0.09044, 0.152643)$ //

e. Surprised : 0.2 lies outside of 95% C.I. on σ //

2. a. $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$

$= \frac{1}{30} (10.9 + 1.7 + \ldots + 2.9 + 16.2)$

$= \frac{258.6}{30}$

$= 8.62$

$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$

$= \frac{1}{30-1} \left[(10.9 - 8.62)^2 + (1.7 - 8.62)^2 + \ldots + (16.2 - 8.62)^2\right]$

$= 20.428$

90% C.I. $\rightarrow \alpha = 1 - 0.9 = 0.1$

b. $L_1 = \frac{(n-1)s^2}{\chi^2_{\alpha/2}} = \frac{29 \cdot 20.428}{42.557} = 13.9204$

$L_2 = \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}} = \frac{29 \cdot 20.428}{17.7084} = 33.4537$

90% C.I. on σ^2 : $(13.9204, 33.4537)$ //

90% C.I. on σ : $(\sqrt{L_1}, \sqrt{L_2})$
$= (3.731, 5.789)$

3. a.

21	9
22	011
23	478
24	08825796
25	15243
26	755
27	28

Key: $21|9 = 21.9$

Values : multiplied by 10

Normality → Reasonable :
Mean = median = mode

Distribution : symmetric, NOT skewed

b. $\overline{X} = \frac{\Sigma x_i}{n} = \frac{21.9 + 23.4 + \ldots + 23.7 + 25.4}{27}$

$= 24.59$

$s^2 = \frac{\Sigma (X_i - \overline{X})^2}{n-1}$

$= \frac{1}{27-1}\left[(21.9-24.59)^2 + (23.4-24.59)^2 + \ldots + (25.4-24.59)^2\right]$

$= \underline{2.455}$

c. 99% C.I. → $\alpha = 1 - 0.99 = 0.01$

$L_1 = \frac{(n-1)s^2}{\chi^2_{0.01/2}} = \frac{26 \cdot 2.455}{48.2899} = 1.3218$

$L_2 = \frac{(n-1)s^2}{\chi^2_{1-0.01/2}} = \frac{26 \cdot 2.455}{11.1602} = 5.719$

99% C.I. for σ^2 : $(1.3218, 5.719)$

d. 99% C.I. for σ :
$(\sqrt{L_1}, \sqrt{L_2})$
$= (\sqrt{1.3218}, \sqrt{5.719})$
$= \underline{(1.15, 2.39)}$

5. a. Values : multiplied by 1000

0 | 17611233833451323445 26436

- Does NOT show much about dist.
→ Cannot conclude normality if reasonable or NOT.

b. $\overline{X} = \frac{\Sigma X_i}{n} = \frac{0.001 + 0.002 + \ldots + 0.005 + 0.006}{25}$

$= \underline{0.0036}$

$s^2 = \frac{\Sigma (X_i - \overline{X})^2}{n-1}$

$= \frac{(0.001-0.0036)^2 + (0.002-0.0036)^2 + \ldots + (0.006-0.0036)^2}{25-1}$

$S^2 = 3.75 \cdot 10^{-6}$

c. $\alpha = 0.1$

$L = \frac{(n-1)S^2}{\chi^2_{\alpha/2}} = \frac{(24)(3.75 \cdot 10^{-6})}{42.557} = 2.11 \cdot 10^{-6}$

$\sqrt{L} = \sqrt{2.11 \cdot 10^{-6}}$

90% one-sided C.I. :
$\sigma^2 = 2.11 \cdot 10^{-6}$
$\sigma = 1.454 \cdot 10^{-3}$

d. Robot : acceptable
if σ: NOT exceed 0.005

7. $n = 15$
$s = 7.5$

$r = 1 - 0.95 = 0.05$

$\chi^2_{0.025} = 0.5\left(z_{0.025} + \sqrt{2(15)-1}\right)^2$
$= 0.5\left(-1.96 + \sqrt{29}\right)^2$
$= 5.866$

$\chi^2_{0.975} = 0.5\left(z_{0.975} + \sqrt{29}\right)^2$
$= 0.5\left(1.96 + \sqrt{29}\right)^2$
$= 26.976$

$L_1 = \frac{(n-1)S^2}{\chi^2_{\alpha/2}} = \frac{(14)(7.5)^2}{5.866} = 134.248$

$L_2 = \frac{(14)(7.5)^2}{26.976} = 29.193$

95% C.I. on σ :
$\left(\sqrt{29.193}, \sqrt{134.248}\right)$
$= \underline{\underline{(5.403, 11.587)}}$

8.2:

9. Answers come directly from various tables:

a. $t_{0.05}(\gamma = 8) = 1.8595$

d. $t_{0.025}(\gamma = 12) = 2.1788$

f. $t_{0.05}(\gamma = 150) = 1.6575$

h. $P(-t \leq T_{25} \leq t) = 0.95$

$P(t \leq T_{25}) - P(-t \leq T_{25}) = 0.95$

$t = 2.0595$

j. $P(T_{20} \geq t) = 0.1 \longrightarrow t = 1.3253$

k. $P(T_{30} \leq -t) = 0.1 \longrightarrow t = 1.3368$

11. a. $\overline{X} = \frac{1.281 + 1.288 + \ldots + 1.291 + 1.286}{20} = 1.2905$

$S^2 = \frac{\sum(X_i - \overline{X})^2}{n-1} = \frac{(1.281 - 1.2905)^2 + \ldots + (1.286 - 1.2905)^2}{19}$

$s^2 = 6.9026 \cdot 10^{-4}$

$s = 0.02628$

b. $\alpha = 1 - 0.95 = 0.05$

$\bar{x} \pm t_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$

$= 1.2905 \pm 2.093 \cdot \frac{0.0066}{\sqrt{20}}$

$= (1.2877,\ 1.2933)$

$\downarrow$

95% C.I. on true mean

c. 1.29 : Within 95% C.I. on μ

$\rightarrow$ NOT suspicious

13. a. $\bar{X} = \frac{2.0 + 0.1 + \ldots + 2.5 + 3.7}{30} = \underline{2.35}$

$s = \frac{\Sigma(x_i - \bar{x})^2}{n-1}$

$= \frac{(2.0-2.35)^2 + (0.1-2.35)^2 + \ldots + (3.7-2.35)^2}{29}$

$= \underline{0.892}$

b. $\alpha = 1 - 0.99 = 0.01$; $n - 1 = 30 - 1 = 29$

$\bar{x} \pm t_{\alpha/2,\, n-1} \cdot \frac{s}{\sqrt{n}}$

$= 2.35 \pm (2.76) \cdot \frac{0.892}{\sqrt{30}} = (2.041,\ 2.722)$

c. Yes.

99% confident $\rightarrow$ New mean time is at most 2.8 sec.

15. $\alpha = 0.05$ $\quad \bar{X} = 11$

$n = 150$ $\quad s = 3$

$\bar{x} \pm t_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$

$= 11 \pm (2.093) \cdot \frac{3}{\sqrt{150}} = (10.487,\ 11.513)$

8.3:

21. a. Null hypothesis:

The proportion of household waste that is metal is still 8% or more in the United States.

Alternate hypothesis:

The proportion of household waste that is metal is less than 8% in the United States due to the increase in recycling efforts.

b. If Type I error $\rightarrow$ Committed:

Null hypothesis $\rightarrow$ rejected

despite its truthfulness

Concluded that:

Proportion of household waste that is metal:

decreased

Reason: Increase in recycling efforts when in reality it has NOT

Effect: Costly changes in policy and behavior based on inaccurate information

c. If Type II error → Committed:
Null hypothesis → accepted
despite being false

Concluded that:
Proportion of household waste (metal):
NOT decreased

Reason:
Increase in recycling efforts
when in reality it has

Effect:
Missed opportunity to take action and make further improvements in waste management and recycling.

d. If H_0 has been rejected at 0.05 level of significance:
We have found evidence: $\mu \neq$ hypothesized value of 8%

Yet: NOT indicate Alternative Hypothesis is true
NOT provide info about practical significance of the difference.

23. a. Type I error:
- Reject a credible model
- Meaning: Model → NOT valid
when it actually is

- In this case:
Model builder's risk:
Wasting resources in trying to fix something that is NOT broken.

6. Type II error:

Accepting a non-credible model

Stating that model : Valid when NOT actually

Meaning:

Risky toward base decisions on an inaccurate model

→ Leading to undesirable outcomes //