CHAPTER

4

CONTINUOUS DISTRIBUTIONS

In Chap. 3 we learned to distinguish a discrete random variable from one that is not discrete. In this chapter we consider a large class of nondiscrete random variables. In particular, we consider random variables that are called *continuous*. We first study the general properties of variables of the continuous type and then present some important families of continuous random variables.

4.1 CONTINUOUS DENSITIES

In Chap. 3 we considered the random variable T, the time of the peak demand for electricity at a particular power plant. We agreed that this random variable is not discrete since, "a priori"—before the fact—we cannot limit the set of possible values for T to some finite or countably infinite collection of times. Time is measured continuously, and T can conceivably assume any value in the time interval $[0, 24)$, where 0 denotes 12 midnight one day and 24 denotes 12 midnight the next day. Furthermore, if we ask *before* the day begins, What is the probability that the peak demand will occur exactly 12.013 278 650 931 271? the answer is 0. It is virtually impossible for the peak load to occur at this split second in time, not the slightest bit earlier or later. These two properties, possible values occurring as intervals and the a priori probability of assuming any specific value being 0, are the characteristics that identify a random variable as being continuous. This leads us to our next definition.

Definition 4.1.1 (Continuous random variable). A random variable is continuous if it can assume any value in some interval or intervals of real numbers and the probability that it assumes any specific value is 0.

Note that the statement that the probability that a continuous random variable assumes any specific value is 0 is essential to the definition. Discrete variables have no such restriction. For this reason, we calculate probabilities in the continuous case differently than we do in the discrete case. In the discrete case we defined a function f, called the density, which enabled us to compute probabilities associated with the random variable X. This function is given by

$$f(x) = P[X = x] \qquad x \text{ real}$$

This definition cannot be used in the continuous case because $P[X = x]$ is always 0. However, we do need a function that will enable us to compute probabilities associated with a continuous random variable. Such a function is also called a density.

Definition 4.1.2 (Continuous density). Let X be a continuous random variable. A function f such that

1. $f(x) \geq 0 \qquad$ for x real
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$
3. $P[a \leq X \leq b] = \int_{a}^{b} f(x)\,dx \qquad$ for a and b real

is called a density for X.

Although this definition may look arbitrary at first glance, it is not. Note that, as in the discrete case, f is defined over the entire real line and is nonnegative. Recall from elementary calculus that integration is the natural extension of summation in the sense that the integral is the limit of a sequence of Riemann sums. In the discrete case we require that $\sum_{\text{all } x} f(x) = 1$. The natural extension of this requirement to the continuous case is that the density integrate to 1. Therefore the necessary and sufficient conditions for a function to be a density for a continuous random variable are as follows:

Necessary and Sufficient Conditions for a Function to be a Continuous Density

1. $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

In the discrete case we find the probability that X assumes a value in some set A by summing $f(x)$ over all values of x in A. That is,

$$P[X \in A] = \sum_{x \in A} f(x).$$

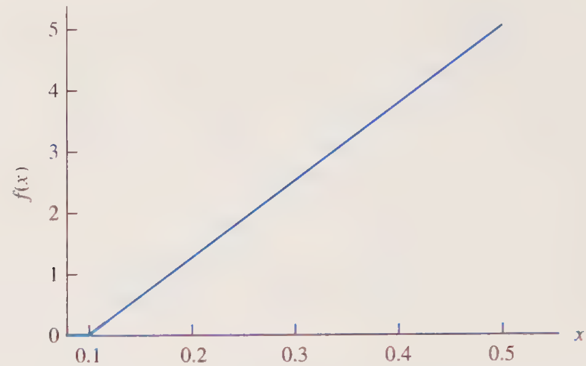


FIGURE 4.1
Graph of
$$f(x) = \begin{cases} 12.5x - 1.25 & .1 \le x \le .5 \\ 0 & \text{elsewhere} \end{cases}$$

In the continuous case we shall be interested in finding the probability that X assumes values in some interval $[a, b]$. Replacing A by $[a, b]$ and substituting integration for summation in the previous expression suggest property 3 of Definition 4.1.2. That is,

$$P[a \le X \le b] = \int_a^b f(x)dx$$

It is evident that the term "density" in the continuous case is just an extension of the ideas presented in the discrete case, with summation being replaced by integration. This is an important notion, as it will allow us to define the concept of expected value in the continuous case quite naturally.

Example 4.1.1. The lead concentration in gasoline currently ranges from .1 to .5 grams per liter. What is the probability that the lead concentration in a randomly selected liter of gasoline will lie between .2 and .3 grams inclusive? To answer this question, we need a density, f, for the random variable X, the number of grams of lead per liter of gasoline. Consider the function

$$f(x) = \begin{cases} 12.5x - 1.25 & .1 \le x \le .5 \\ 0 & \text{elsewhere} \end{cases}$$

The graph of f is shown in Fig. 4.1. The function is nonnegative. Furthermore,

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= \int_{.1}^{.5} (12.5x - 1.25)dx \\ &= \left[\frac{12.5x^2}{2} - 1.25x\right]_{.1}^{.5} \\ &= \left[\frac{12.5(.5)^2}{2} - 1.25(.5)\right] - \left[\frac{12.5(.1)^2}{2} - 1.25(.1)\right] \\ &= .9375 - (-.0625) = 1 \end{aligned}$$

Thus f satisfies properties 1 and 2 of Definition 4.1.2. Property 3 allows us to use f to find the desired probability. In particular,

$$\begin{aligned}
P[.2 \leq X \leq .3] &= \int_{.2}^{.3} f(x)dx \\
&= \int_{.2}^{.3} (12.5x - 1.25)dx \\
&= \left[\frac{12.5x^2}{2} - 1.25x\right]_{.2}^{.3} \\
&= \left[\frac{12.5(.3)^2}{2} - 1.25(.3)\right] - \left[\frac{12.5(.2)^2}{2} - 1.25(.2)\right] \\
&= .1875
\end{aligned}$$

There are several important points to be made concerning the density in the continuous case. First, we shall follow the convention of defining f only over intervals for which $f(x)$ may be nonzero. For values of x not explicitly mentioned, $f(x)$ is assumed to be 0. In Example 4.1.1 we could have written f as

$$f(x) = 12.5x - 1.25 \qquad .1 \leq x \leq .5$$

with the understanding that $f(x) = 0$ elsewhere. Second, since the integral of a nonnegative function can be thought of as an area, properties 2 and 3 of Definition 4.1.2 can be expressed in terms of areas. In particular, property 2 requires that *the total area under the graph of f be 1.* Property 3 implies that the probability that the variable assumes a value between two points a and b is *the area under the graph of f between $x = a$ and $x = b$.* These ideas as they apply to Example 4.1.1 are demonstrated in Figs. 4.2(a) and (b), respectively. Third, since $P[X = a] = P[X = b] = 0$ in the continuous case,

$$P[a \leq X \leq b] = P[a \leq X < b] = P[a < X \leq b] = P[a < X < b].$$

In Example 4.1.1 the probability that the lead concentration in a liter of gasoline lies between .2 and .3 gram inclusive, $P[.2 \leq X \leq .3]$, is the same as $P[.2 < X < .3]$, the probability that it lies strictly between .2 and .3 gram. See Fig. 4.2(c). Fourth, properties 1 and 2 of Definition 4.1.2 are necessary and sufficient conditions for a function to be a density for a continuous random variable X. However, the density chosen for X cannot be just any function satisfying these conditions. It should be a function that assigns reasonable probabilities to events via property 3 of Definition 4.1.2. Whether or not the function f given in Example 4.1.1 satisfies this criteria is debatable. It was chosen for illustrative purposes only. Finding an appropriate density is not always easy. Some methods for helping in the selection of a density are discussed in Chap. 6.

Cumulative Distribution

The idea of a cumulative distribution function in the continuous case is useful. It is defined exactly as in the discrete case although found by using integration rather than summation.

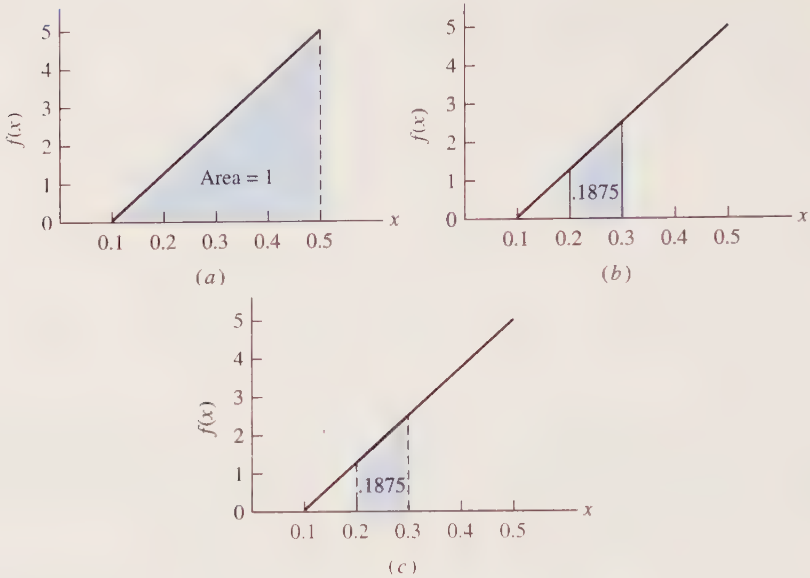


FIGURE 4.2
(a) $\int_{-\infty}^{\infty} f(x)dx = 1$ implies that the total area under the graph of f is 1; (b) $P[.2 \le X \le .3] = \int_{.2}^{.3}(12.5x - 1.25)dx = .1875$ implies that the area under the graph of f between $x = .2$ and $x = .3$ is .1875; (c) $P[.2 < X < .3] = P[.2 \le X \le .3] = .1875$.

Definition 4.1.3 (Cumulative distribution—continuous). Let X be continuous with density f. The cumulative distribution function for X, denoted by F, is defined by

$$F(x) = P[X \le x] \qquad x \text{ real}$$

To find $F(x)$ for a specific real number x, we integrate the density over all real numbers that are less than or equal to x.

Computing F Continuous Case

$$P[X \le x] = F(x) = \int_{-\infty}^{x} f(t)dt \qquad x \text{ real}$$

Graphically, this probability corresponds to the area under the graph of the density to the left of and including the point x.

Example 4.1.2. The density for the random variable X, the lead content in a liter of gasoline, is

$$f(x) = 12.5x - 1.25 \qquad .1 \le x \le .5$$

The cumulative distribution function for X is

$$P[X \leq x] = F(x) = \int_{-\infty}^{x} f(t)\,dt$$

For $x < .1$ this integral has value 0 since for these values of x, $f(t)$ is itself 0. For $.1 \leq x \leq .5$,

$$\begin{aligned} F(x) = \int_{-\infty}^{x} f(t)\,dt &= \int_{.1}^{x} (12.5t - 1.25)\,dt \\ &= \left[\frac{12.5t^2}{2} - 1.25t\right]_{.1}^{x} \\ &= 6.25x^2 - 1.25x + .0625 \end{aligned}$$

For $x > .5$ the integral has value 1 since for these values of x we have integrated the density over its entire set of possible values. Summarizing, F is given by

$$F(x) = \begin{cases} 0 & x < .1 \\ 6.25x^2 - 1.25x + .0625 & .1 \leq x \leq .5 \\ 1 & x > .5 \end{cases}$$

What is the probability that the lead concentration in a randomly selected liter of gasoline will lie between .2 and .3 gram per liter? To answer this question, we rewrite it in terms of the cumulative distribution

$$\begin{aligned} P[.2 \leq X \leq .3] &= P[X \leq .3] - P[X < .2] \\ &= P[X \leq .3] - P[X \leq .2] \qquad (X \text{ is continuous}) \\ &= F(.3) - F(.2) \end{aligned}$$

By substitution,

$$F(.3) = 6.25(.3)^2 - 1.25(.3) + .0625 = .2500$$
$$F(.2) = 6.25(.2)^2 - 1.25(.2) + .0625 = .0625$$

Thus

$$\begin{aligned} P[.2 \leq X \leq .3] &= F(.3) - F(.2) \\ &= .2500 - .0625 = .1875 \end{aligned}$$

Note that this agrees with the result obtained in Example 4.1.1 using direct integration. Note also that $F(.3)$ gives the area to the left of .3 shown in Fig. 4.3(*a*); $F(.2)$ gives the area to the left of .2 shown in Fig. 4.3(*b*). When we form the difference $F(.3) - F(.2)$, we naturally obtain the area between .2 and .3 given in Fig. 4.3(*c*).

Recall that in the discrete case, the cumulative distribution, F, was obtained from the density by addition; if F was available, f could be obtained by subtraction, the operation that reverses addition. The same sort of thing happens in the continuous case. We obtain the cumulative distribution from the density by integrating f; if F is available, we can retrieve f by reversing the integration operation via differentiation. That is, in the continuous case,

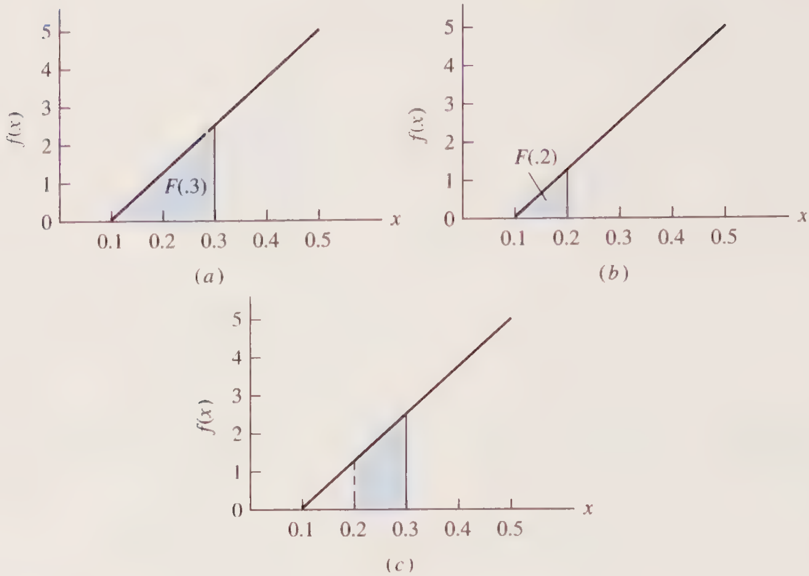


FIGURE 4.3
(a) $F(.3) = P[X \leq .3]$; (b) $F(.2) = P[X \leq .2]$; (c) $F(.3) - F(.2) = P[.2 \leq X \leq .3]$.

Obtaining f from F in the Continuous Case

$$f(x) = F'(x)$$

Example 4.1.3. In Example 4.1.2, we derived the cumulative distribution

$$F(x) = 6.25x^2 - 1.25x + .0625 \qquad .1 \leq x \leq .5$$

Note that

$$F'(x) = 12.5x - 1.25 \qquad .1 \leq x \leq .5$$

This is, as expected, the expression for the density for X that was given in Example 4.1.2.

Uniform Distribution

Perhaps the simplest continuous distribution with which to work is the *uniform* distribution. This distribution parallels the discrete uniform distribution presented in Exercise 34 of Chap. 3 in that, in a sense, events occur with equal or uniform probability. Since it is easy and instructive to develop the properties of this family of random variables directly from the definition, we leave the derivations to you. Important properties and applications are given in Exercises 5, 6, 10, 11, 18, and 19.

4.2 EXPECTATION AND DISTRIBUTION PARAMETERS

In this section we define the term *expected value for continuous random variables.* We also discuss how to use the definition to find the moment generating function, the mean, and the variance of a variable of the continuous type. As you will see, the definition parallels that given in the discrete case, with the summation operation being replaced by integration.

Definition 4.2.1 (Expected value). Let X be a continuous random variable with density f. Let $H(X)$ be a random variable. The expected value of $H(X)$, denoted by $E[H(X)]$, is given by

$$E[H(X)] = \int_{-\infty}^{\infty} H(x)f(x)\,dx$$

provided

$$\int_{-\infty}^{\infty} |H(x)|\,f(x)\,dx$$

is finite.

As in the discrete case, the mean or expected value of X is a special case of the above definition.

Expected Value of X

$$E[X] = \int_{-\infty}^{\infty} xf(x)\,dx = \mu$$

We illustrate the use of this definition by finding the mean and variance of the random variable X of Example 4.1.1. Recall that, by Theorem 3.3.2, the variance for X can be found via the computational shortcut

$$\sigma^2 = \operatorname{Var}(X) = E[X^2] - (E[X])^2$$

Example 4.2.1. The density for X, the lead concentration in gasoline in grams per liter, is given by

$$f(x) = 12.5x - 1.25 \qquad .1 \le x \le .5$$

The mean or expected value of X is

$$\begin{aligned}
\mu = E[X] &= \int_{-\infty}^{\infty} xf(x)\,dx \\
&= \int_{.1}^{.5} x(12.5x - 1.25)\,dx \\
&= \left[\frac{12.5x^3}{3} - \frac{1.25x^2}{2}\right]_{.1}^{.5} \\
&= \left[\frac{(12.5)(.5)^3}{3} - \frac{1.25(.5)^2}{2}\right] - \left[\frac{12.5(.1)^3}{3} - \frac{1.25(.1)^2}{2}\right] \\
&\doteq .3667 \text{ g/liter}
\end{aligned}$$

Since integration is over an interval of finite length

$$\int_{-\infty}^{\infty} |x|\, f(x)\, dx$$

exists. We can conclude that, on the average, a liter of gasoline contains approximately .3667 g of lead. How much variability is there from liter to liter? To answer this question, we find $E[X^2]$ and apply Theorem 3.3.2 to find the variance of X:

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x)\, dx \\ &= \int_{.1}^{.5} x^2 (12.5x - 1.25)\, dx \\ &= \left[\frac{12.5x^4}{4} - \frac{1.25x^3}{3} \right]_{.1}^{.5} \doteq .1433 \end{aligned}$$

By Theorem 3.3.2,

$$\operatorname{Var} X = E[X^2] - (E[X])^2 \doteq .1433 - (.3667)^2 \doteq .00883$$

The standard deviation of X is

$$\sigma = \sqrt{\operatorname{Var} X} = \sqrt{.00883} = .09396 \text{ g/liter}$$

As in the discrete case, the moment generating function for a continuous random variable X is defined as $E[e^{tX}]$ provided this expectation exists for t in some open interval about 0. Its use is illustrated in the following example.

Example 4.2.2. The spontaneous flipping of a bit stored in a computer memory is called a "soft fail." Let X denote the time in millions of hours before the first soft fail is observed. Suppose that the density for X is given by

$$f(x) = e^{-x} \qquad x > 0$$

The mean and variance for X can be found directly using the method of Example 4.2.1. However, to find $E[X]$ and $E[X^2]$, integration by parts is required. This method of integration, although not difficult, is time-consuming. Let us find the moment generating function for X and use it to compute the mean and variance. By definition,

$$m_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

In this case,

$$\begin{aligned} m_X(t) &= \int_0^{\infty} e^{tx} e^{-x}\, dx \\ &= \int_0^{\infty} e^{(t-1)x}\, dx \\ &= \frac{1}{t-1} e^{(t-1)x} \Big|_0^{\infty} \end{aligned}$$

Assume that $|t| < 1$. This guarantees that the exponent $(t - 1)x < 0$, allowing us to evaluate the above integral. In particular,

$$m_X(t) = \frac{1}{1-t} \qquad |t| < 1$$

Since $e^{tx} > 0$, $|e^{tx}| = e^{tx}$. Thus the above argument has shown that

$$\int_{-\infty}^{\infty} |e^{tx}| f(x)\,dx$$

exists, as required in Definition 4.2.1. To use $m_X(t)$ to find $E[X]$ and $E[X^2]$, we apply Theorem 3.4.2. Note that

$$\begin{aligned}
\frac{dm_X(t)}{dt} &= \frac{d(1-t)^{-1}}{dt} = (1-t)^{-2} \\
\frac{d^2 m_X(t)}{dt^2} &= 2(1-t)^{-3} \\
E[X] &= \left.\frac{dm_X(t)}{dt}\right|_{t=0} = 1 \\
E[X^2] &= \left.\frac{d^2 m_X(t)}{dt^2}\right|_{t=0} = 2 \\
\operatorname{Var} X &= E[X^2] - (E[X])^2 = 2 - 1^2 = 1
\end{aligned}$$

The average or mean time that one must wait to observe the first soft fail is 1 million hours. The variance in waiting time is 1, and the standard deviation is 1 million hours.

To find the distribution parameters μ, σ^2, and σ, we can use either Definition 4.2.1 or the moment generating function technique. In practice, use whichever method is easier.

It should be pointed out that there is a nice geometric interpretation of the mean in the case of a continuous random variable. Imagine cutting out of a piece of thin rigid metal the region bounded by the graph of f and the x axis, and attempting to balance this region on a knife-edge held parallel to the vertical axis. The point at which the region would balance, if such a point exists, is the mean of X. Thus, μ_X is a "location" parameter in that it indicates the position of the center of the density along the x axis. The variance can also be interpreted pictorially. In the continuous case variance is a "shape" parameter in the sense that a random variable with small variance will have a compact density; one with a large variance will have a density that is rather spread out or flat.

4.3 GAMMA, EXPONENTIAL, AND CHI-SQUARED DISTRIBUTIONS

In this section we consider the gamma distribution. This distribution is especially important in that it allows us to define two families of random variables, the exponential and chi-squared, that are used extensively in applied statistics. The theoretical basis for the gamma distribution is the gamma function, a mathematical function defined in terms of an integral.

Definition 4.3.1 (Gamma function). The function Γ defined by

$$\Gamma(\alpha) = \int_0^\infty z^{\alpha-1}e^{-z}dz \qquad \alpha > 0$$

is called the gamma function.

Theorem 4.3.1 presents two numerical properties of the gamma function that are useful in evaluating the function for various values of α. Its proof is outlined in Exercise 26.

Theorem 4.3.1 (Properties of the gamma function)

1. $\Gamma(1) = 1$.
2. For $\alpha > 1$, $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$.

The use of Theorem 4.3.1 is illustrated in the next example.

Example 4.3.1

(a) Evaluate $\int_0^\infty z^3e^{-z}\,dz$. To evaluate this integral using the methods of elementary calculus requires repeated applications of integration by parts. To evaluate the integral quickly, rewrite it as

$$\int_0^\infty z^3e^{-z}dz = \int_0^\infty z^{4-1}e^{-z}dz$$

The integral on the right is $\Gamma(4)$. By applying Theorem 4.3.1 repeatedly, it can be seen that

$$\begin{aligned}\int_0^\infty z^3e^{-z}dz &= \Gamma(4) = 3\cdot\Gamma(3)\\ &= 3\cdot 2\cdot\Gamma(2)\\ &= 3\cdot 2\cdot 1\cdot\Gamma(1)\\ &= 3\cdot 2\cdot 1 = 6\end{aligned}$$

(b) Evaluate $\int_0^\infty (1/54)x^2e^{-x/3}\,dx$. To evaluate this integral, we make a change of variable, a technique that is used extensively in deriving the properties of the gamma distribution. In particular, let $z = x/3$ or $3z = x$. Then $3\,dz = dx$ and the problem becomes

$$\begin{aligned}\int_0^\infty (1/54)x^2e^{-x/3}dx &= \int_0^\infty 1/54(3z)^2e^{-z}3dz\\ &= 27/54\int_0^\infty z^2e^{-z}dz\end{aligned}$$

However,

$$\int_0^\infty z^2 e^{-z} dz = \int_0^\infty z^{3-1} e^{-z} dz = \Gamma(3)$$
$$= 2 \cdot \Gamma(2)$$
$$= 2 \cdot 1 \cdot \Gamma(1)$$
$$= 2 \cdot 1 = 2$$

Thus

$$\int_0^\infty (1/54) x^2 e^{-x/3} dx = 27/54 \cdot 2 = 1$$

Note that since the nonnegative function

$$f(x) = (1/54)x^2 e^{-x/3}$$

has been shown to integrate to 1, it can be thought of as being a density for a continuous random variable X.

It is now possible to define the gamma distribution.

Gamma Distribution

Definition 4.3.2 (Gamma distribution). A random variable X with density

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \qquad x > 0, \quad \alpha > 0, \quad \beta > 0$$

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$$ USING $\lambda = \frac{1}{\beta}$

is said to have a gamma distribution with parameters α and β.

Although the mean and variance of a gamma random variable can be found easily from the definitions of these parameters (see Exercise 31), we shall use the moment generating function technique. As you will see later, it is very helpful to know the form of the moment generating function for a random variable.

Theorem 4.3.2. Let X be a gamma random variable with parameters α and β. Then

1. The moment generating function for X is given by
$$m_X(t) = (1 - \beta t)^{-\alpha} \qquad t < 1/\beta$$
2. $E[X] = \alpha\beta$
3. $\text{Var}\, X = \alpha\beta^2$

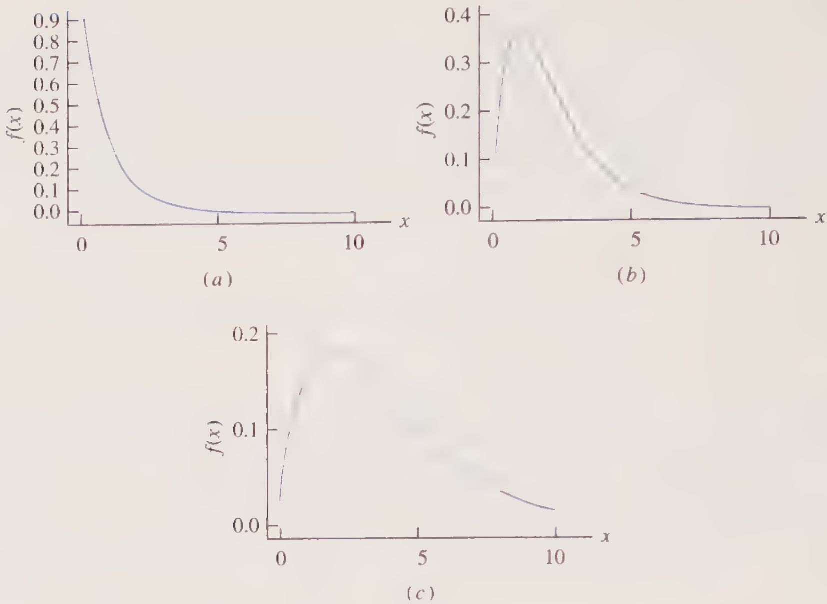


FIGURE 4.4
(*a*) $\alpha = 1, \beta = 1, \mu_X = 1, \sigma_X^2 = 1$; (*b*) $\alpha = 2, \beta = 1, \mu_X = 2, \sigma_X^2 = 2$; (*c*) $\alpha = 2, \beta = 2, \mu_X = 4$, $\sigma_X^2 = 8$.

The proof of this theorem is found in Appendix C.

Figure 4.4 shows the graphs of some gamma densities for a few values of α and β. Note that α and β both play a role in determining the mean and the variance of the random variable. Note also that the curves are not symmetric and are located entirely to the right of the vertical axis. It can be shown that for $\alpha > 1$, the maximum value of the density occurs at the point $x = (\alpha - 1)\beta$. (See Exercise 32.)

Exponential Distribution

As mentioned earlier, the gamma distribution gives rise to a family of random variables known as the *exponential* family. These variables are each gamma random variables with $\alpha = 1$. The density for an exponential random variable therefore assumes the form

Exponential density

$$f(x) = \frac{1}{\beta} e^{-x/\beta} \qquad x > 0$$

$$\beta > 0$$

The graph of a typical exponential density is shown in Fig. 4.4(a). This distribution arises often in practice in conjunction with the study of Poisson processes, which were discussed in Sec. 3.8. Recall that in a Poisson process discrete events are being observed over a continuous time interval. If we let W denote the time of the occurrence of the first event, then W is a continuous random variable. Theorem 4.3.3 shows that W has an exponential distribution.

Theorem 4.3.3. Consider a Poisson process with parameter λ. Let W denote the time of the occurrence of the first event. W has an exponential distribution with $\beta = 1/\lambda$.

Proof. The distribution function F for W is given by

$$F(w) = P[W \leq w] = 1 - P[W > w]$$

The first occurrence of the event will take place after time w only if no occurrences of the event are recorded in the time interval $[0, w]$. Let X denote the number of occurrences of the event in this time interval. X is a Poisson random variable with parameter λw. Thus

$$P[W > w] = P[X = 0] = \frac{e^{-\lambda w}(\lambda w)^0}{0!} = e^{-\lambda w}$$

By substitution we obtain

$$F(w) = 1 - P[W > w] = 1 - e^{-\lambda w}$$

Since in the continuous case the derivative of the cumulative distribution function is the density

$$F'(w) = f(w) = \lambda e^{-\lambda w}$$

This is the density for an exponential random variable with $\beta = 1/\lambda$.

The next example illustrates the use of this theorem.

Example 4.3.2. Some strains of paramecia produce and secrete "killer" particles that will cause the death of a sensitive individual if contact is made. All paramecia unable to produce killer particles are sensitive. The mean number of killer particles emitted by a killer paramecium is 1 every 5 hours. In observing such a paramecium, what is the probability that we must wait at most 4 hours before the first particle is emitted? Considering the measurement unit to be one hour, we are observing a Poisson process with $\lambda = 1/5$. By Theorem 4.3.3, W, the time at which the first killer particle is emitted, has an exponential distribution with $\beta = 1/\lambda = 5$. The density for W is

$$f(w) = (1/5)e^{-w/5} \qquad w > 0$$

The desired probability is given by

$$\begin{aligned} P[W \leq 4] &= \int_0^4 (1/5)e^{-w/5}dw \\ &= -e^{-w/5}\Big|_0^4 \\ &= 1 - e^{-4/5} \doteq .5507 \end{aligned}$$

Since an exponential random variable is also a gamma random variable, the average time that we must wait until the first killer particle is emitted is

$$E[W] = \alpha\beta = 1 \cdot 5 = 5 \text{ hours}$$

Chi-Squared Distribution

The gamma distribution gives rise to another important family of random variables, namely, the *chi-squared* family. This distribution is used extensively in applied statistics. Among other things, it provides the basis for making inferences about the variance of a population based on a sample. At this time we consider only the theoretical properties of the chi-squared distribution. You will see many examples of its use in later chapters.

> **Definition 4.3.3 (Chi-squared distribution).** Let X be a gamma random variable with $\beta = 2$ and $\alpha = \gamma/2$ for γ a positive integer. X is said to have a chi-squared distribution with γ degrees of freedom. We denote this variable by X_γ^2.

Note that a chi-squared random variable is completely specified by stating its degrees of freedom. By applying Theorem 4.3.2, we see that the mean of a chi-squared random variable is γ, its degrees of freedom; its variance is 2γ, twice its degrees of freedom. Figure 4.4(c) gives the graph of the density of a chi-squared random variable with 4 degrees of freedom.

Since the chi-squared distribution arises so often in practice, extensive tables of its cumulative distribution function have been derived. One such table is Table IV of App. A. In the table, degrees of freedom appear as row headings, probabilities appear as column headings, and points associated with those probabilities are listed in the body of the table. Notationally, we shall use χ_r^2 to denote that point associated with a chi-squared random variable such that

$$P[X_\gamma^2 \geq \chi_r^2] = r$$

That is, χ_r^2 is the point such that the area to its *right* is r. Technically speaking, we should write $\chi_{r,\gamma}^2$ since the value of the point does depend on both the probability desired and the number of degrees of freedom associated with the random variable. However, in applications the value of γ will be obvious. Therefore to simplify notation, we use only a single subscript. The use of this notation is illustrated in the following example.

Example 4.3.3. Consider a chi-squared random variable with 10 degrees of freedom. Find the value of $\chi_{.05}^2$. This point is shown in Fig. 4.5. By definition the area to the right of this point is .05; the area to its left is .95. The column probabilities in Table IV give the area to the *left* of the point listed. Thus to find $\chi_{.05}^2$, we look in row 10 and column .95 and see that $\chi_{.05}^2 = 18.3$.

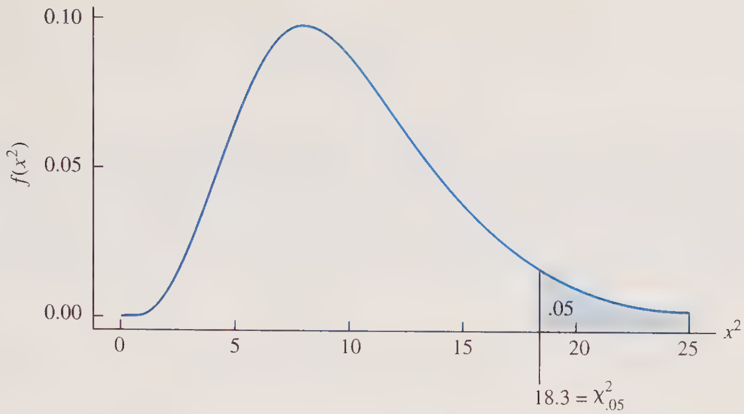


FIGURE 4.5
$P[X^2_{10} \geq \chi^2_{.05}] = .05$ and $P[X^2_{10} < \chi^2_{.05}] = .95$.

4.4 NORMAL DISTRIBUTION

The normal distribution is a distribution that underlies many of the statistical methods used in data analysis. It was first described in 1733 by De Moivre as being the limiting form of the binomial density as the number of trials becomes infinite. This discovery did not get much attention, and the distribution was "discovered" again by both Laplace and Gauss a half-century later. Both men dealt with problems of astronomy, and each derived the normal distribution as a distribution that seemingly described the behavior of errors in astronomical measurements. The distribution is often referred to as the "gaussian" distribution.

Definition 4.4.1 (Normal distribution). A random variable X with density

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(1/2)[(x-\mu)/\sigma]^2} \qquad -\infty < x < \infty$$

$$-\infty < \mu < \infty$$

$$\sigma > 0$$

is said to have a normal distribution with parameters μ and σ.

One implication of this definition is that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(1/2)[(x-\mu)/\sigma]^2} dx = 1$$

To verify this requires a transformation to polar coordinates. This technique is beyond the mathematical level assumed here. A detailed proof can be found in [49]. Note that Definition 4.4.1 states only that μ is a real number and that σ is positive. As you might suspect from the notation used, the parameters that appear in the equation for the density for a normal random variable are, in fact, its mean and its standard deviation. This can be verified once we know the moment generating function for X. Our next theorem gives us the form for this important function.

Theorem 4.4.1. Let X be normally distributed with parameters μ and σ. The moment generating function for X is given by

$$m_X(t) = e^{\mu t + \sigma^2 t^2/2}$$

For the proof of this theorem, see Appendix C.

It is now easy to show that the parameters that appear in the definition of the normal density are actually the mean and the standard deviation of the variable.

Theorem 4.4.2. Let X be a normal random variable with parameters μ and σ. Then μ is the mean of X and σ is its standard deviation.

Proof. The moment generating function for X is

$$m_X(t) = e^{\mu t + \sigma^2 t^2/2}$$

and

$$\frac{dm_X(t)}{dt} = e^{\mu t + \sigma^2 t^2/2}(\mu + \sigma^2 t)$$

By Theorem 3.4.2 the mean of X is given by

$$E[X] = \left.\frac{dm_X(t)}{dt}\right|_{t=0} = e^{\mu \cdot 0 + \sigma^2 0^2/2}(\mu + \sigma^2 \cdot 0) = \mu$$

as claimed. The proof of the remainder of the theorem is left as an exercise.

The graph of the density of a normal random variable is a symmetric, bell-shaped curve centered at its mean. The points of inflection occur at $\mu \pm \sigma$.

Example 4.4.1. One of the major contributors to air pollution is hydrocarbons emitted from the exhaust system of automobiles. Let X denote the number of grams of hydrocarbons emitted by an automobile per mile. Assume that X is normally distributed with a mean of 1 gram and a standard deviation of .25 gram. The density for X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}(.25)} e^{-(1/2)[(x-1)/.25]^2}$$

The graph of this density is a symmetric, bell-shaped curve centered at $\mu = 1$ with inflection points at $\mu \pm \sigma$, or $1 \pm .25$. A sketch of the density is given in Fig. 4.6.

One point must be made. Theoretically speaking, a normal random variable must be able to assume any value whatsoever. This is clearly unrealistic here. It is

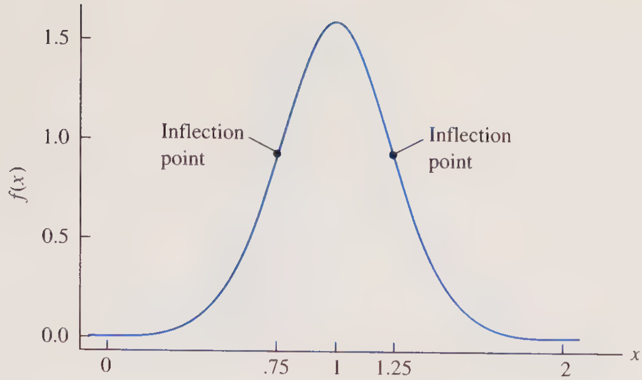


FIGURE 4.6
Graph of the density for a normal random variable with mean 1 and standard deviation .25.

impossible for an automobile to emit a negative amount of hydrocarbons. When we say that X is normally distributed, we mean that over the range of physically reasonable values of X, the given normal curve yields acceptable probabilities. With this understanding, we can at least approximate, for example, the probability that a randomly selected automobile will emit between .9 and 1.54 grams of hydrocarbons by finding the area under the graph of f between these two points.

Standard Normal Distribution

There are infinitely many normal random variables each of which is uniquely characterized by the two parameters μ and σ. To calculate probabilities associated with a specific normal curve requires that one integrate the normal density over a particular interval. However, the normal density is not integrable in closed form. To find areas under the normal curve requires the use of numerical integration techniques. A simple algebraic transformation is employed to overcome this problem. By means of this transformation, called the *standardization procedure,* any question about any normal random variable can be transformed to an equivalent question concerning a normal random variable with mean 0 and standard deviation 1. This particular normal random variable is denoted by Z and is called the *standard normal* variable.

Theorem 4.4.3 (Standardization theorem). Let X be normal with mean μ and standard deviation σ. The variable $(X - \mu)/\sigma$ is standard normal.

You have already verified that the transformation yields a random variable with mean 0 and standard deviation 1 (see Chap. 3, Exercise 21). To prove that the transformed variable is normal requires the use of moment generating function techniques to be introduced in Chap. 7.

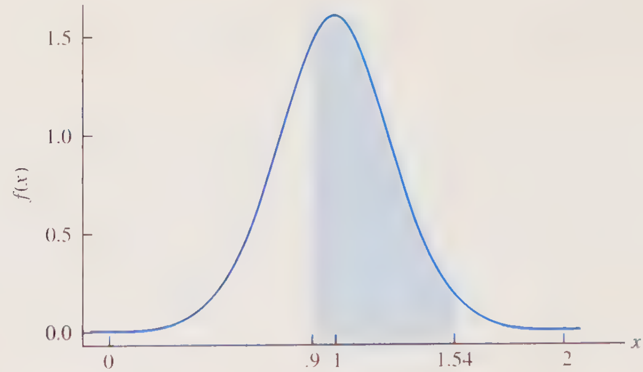


FIGURE 4.7
Shaded area = $P[.9 \le X \le 1.54]$.

The cumulative distribution function for the standard normal random variable is given in Table V of App. A. The use of the standardization theorem and this table is illustrated in the following example.

Example 4.4.2. Let X denote the number of grams of hydrocarbons emitted by an automobile per mile. Assuming that X is normal with $\mu = 1$ gram and $\sigma = .25$ gram, find the probability that a randomly selected automobile will emit between .9 and 1.54 grams of hydrocarbons per mile. The desired probability is shown in Fig. 4.7. To find $P[.9 \le X \le 1.54]$, we first standardize by subtracting the mean of 1 and dividing by the standard deviation of .25 across the inequality. That is,

$$P[.9 \le X \le 1.54] = P[(.9 - 1)/.25 \le (X - 1)/.25 \le (1.54 - 1)/.25]$$

The random variable $(X - 1)/.25$ is now Z. Therefore the problem is to find $P[-.4 \le Z \le 2.16]$ from Table V. We first express the desired probability in terms of the cumulative distribution as follows:

$$\begin{aligned} P[-.4 \le Z \le 2.16] &= P[Z \le 2.16] - P[Z < -.4] \\ &= P[Z \le 2.16] - P[Z \le -.4] \qquad (Z \text{ is continuous}) \\ &= F(2.16) - F(-.4) \end{aligned}$$

$F(2.16)$ is found by locating the first two digits (2.1) in the column headed z; since the third digit is 6, the desired probability of .9846 is found in the row labeled 2.1 and the column labeled .06. Similarly, $F(-.4)$ or .3446 is found in the row labeled -0.4 and the column labeled .00. We now see that the probability that a randomly selected automobile will emit between .9 and 1.54 grams of hydrocarbons per mile is

$$\begin{aligned} P[.9 \le X \le 1.54] &= P[-.4 \le Z \le 2.16] \\ &= F(2.16) - F(-.4) \\ &= .9846 - .3446 = .64 \end{aligned}$$

Interpreting this probability as a percentage, we can say that 64% of the automobiles in operation emit between .9 and 1.54 grams of hydrocarbons per mile driven.

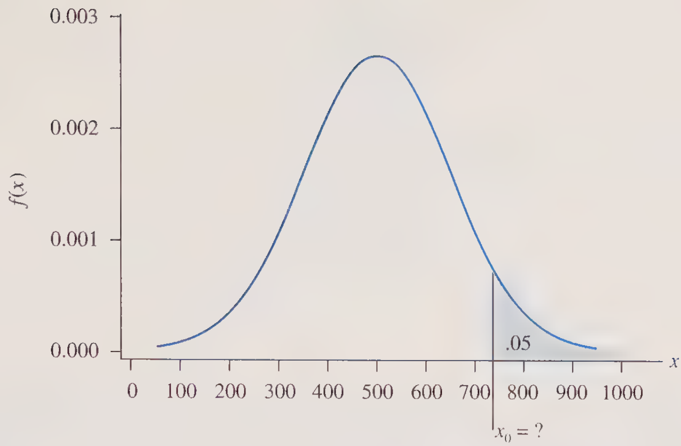


FIGURE 4.8
$P[X \geq x_0] = .05$.

We shall have occasion to read Table V in reverse. That is, given a particular probability r we shall need to find the point with r of the area to its right. This point is denoted by z_r. Thus, notationally, z_r denotes that point associated with a standard normal random variable such that

$$P[Z \geq z_r] = r$$

To see how this need arises, consider Example 4.4.3.

Example 4.4.3. Let X denote the amount of radiation that can be absorbed by an individual before death ensues. Assume that X is normal with a mean of 500 roentgens and a standard deviation of 150 roentgens. Above what dosage level will only 5% of those exposed survive? Here we are asked to find the point x_0 shown in Fig. 4.8. In terms of probabilities, we want to find the point x_0 such that

$$P[X \geq x_0] = .05$$

Standardizing gives

$$\begin{aligned} P[X \geq x_0] &= P\left[\frac{X - 500}{150} \geq \frac{x_0 - 500}{150}\right] \\ &= P\left[Z \geq \frac{x_0 - 500}{150}\right] = .05 \end{aligned}$$

Thus $(x_0 - 500)/150$ is the point on the standard normal curve with 5% of the area under the curve to its right and 95% to its left. That is, $(x_0 - 500)/150$ is the point $z_{.05}$. From Table V the numerical value of this point is approximately 1.645 (we have interpolated). Equating these, we get

$$\frac{x_0 - 500}{150} = 1.645$$

Solving this equation for x_0 gives the desired dosage level:

$$x_0 = 150(1.645) + 500 = 746.75 \text{ roentgens}$$

4.5 NORMAL PROBABILITY RULE AND CHEBYSHEV'S INEQUALITY

It is sometimes useful to have a quick way of determining which values of a random variable are common and which are considered to be rare. In the case of a normally distributed random variable, a rule of thumb, called the *normal probability rule*, can be developed easily. This rule is given in Theorem 4.5.1.

Theorem 4.5.1 (Normal probability rule). Let X be normally distributed with parameters μ and σ. Then

$$P[-\sigma < X - \mu < \sigma] \doteq .68$$
$$P[-2\sigma < X - \mu < 2\sigma] \doteq .95$$
$$P[-3\sigma < X - \mu < 3\sigma] \doteq .997$$

Proof. Note that division by σ yields

$$P[-\sigma < X - \mu < \sigma] = P\left[-1 < \frac{X-\mu}{\sigma} < 1\right]$$

By Theorem 4.4.3, $(X - \mu)/\sigma$ follows the standard normal distribution. From Table V of App. A,

$$P[-1 < Z < 1] = .8413 - .1587 = .6826$$

This probability can be rounded to .68. The other results given in the theorem are proved similarly.

The normal probability rule can be expressed in terms of percentages. In particular, it implies that in repeated sampling from a normal distribution approximately 68% of the observed values of X should lie within 1 standard deviation of its mean; 95% should lie within two standard deviations, and 99.7% within 3 standard deviations of the mean. Thus an observed value that falls farther than 3 standard deviations from μ is indeed rare, since such values occur with probability .003. This rule will be used later to obtain a quick estimate of the standard deviation of a normally distributed random variable.

Figure 4.9 illustrates the normal probability rule as it applies to the standard normal distribution. Recall that for this distribution $\sigma = 1$, $2\sigma = 2$, and $3\sigma = 3$.

Chebyshev's Inequality

A second rule of thumb that can be used to gauge the rarity of observed values of a random variable is *Chebyshev's inequality*. This inequality was derived by the

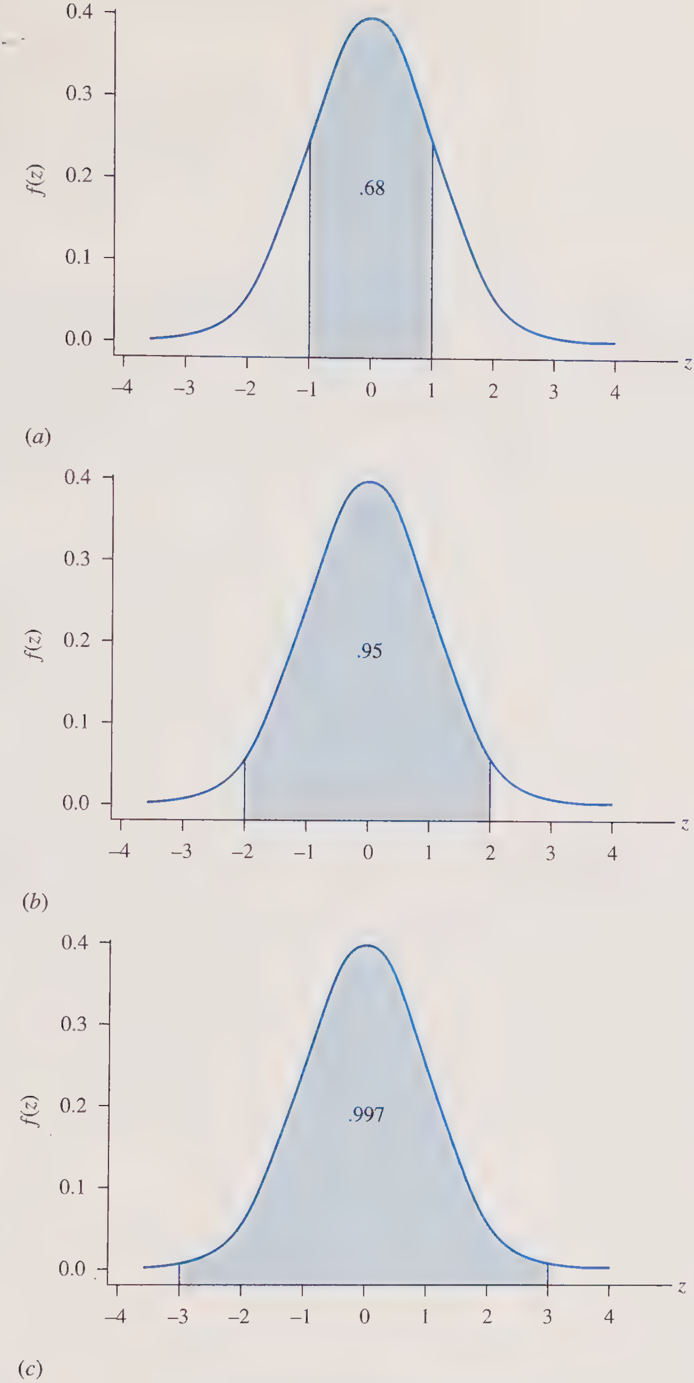


FIGURE 4.9
(*a*) The probability that a normally distributed random variable will lie within one standard deviation of its mean is approximately .68 or 68%.
(*b*) The probability that a normally distributed random variable will lie within two standard deviations of its mean is approximately .95 or 95%.
(*c*) The probability that a normally distributed random variable will lie within three standard deviations of its mean is approximately .997 or 99.7%.

Russian probabilist P. L. Chebyshev (Tchebysheff, 1821–1894). The inequality differs from the normal probability rule in that it does *not* require that the random variable involved be normally distributed. Although we shall prove the theorem in the continuous setting, continuity is *not* required. The inequality holds for any random variable.

Theorem 4.5.2 (Chebyshev's inequality). Let X be a random variable with mean μ and standard deviation σ. Then for any positive number k,

$$P[|X-\mu| < k\sigma] \geq 1 - \frac{1}{k^2}$$

See Appendix C for the proof of this theorem.

Some examples will clarify the difference between Theorems 4.5.1 and 4.5.2.

Example 4.5.1. The viscosity of a fluid can be measured roughly by dropping a small ball into a calibrated tube containing the fluid and observing X, the time that it takes for the ball to drop a measured distance. Assume that this random variable is normally distributed with a mean of 20 s and a standard deviation of .5 s. By the normal probability rule, approximately 95% of the observed values of X will lie within 1 s (2 standard deviations) of the mean. That is, X will fall between 19 and 21 s with probability .95. Since Chebyshev's inequality applies to any random variable, it is appropriate here. This inequality guarantees that X will fall between 19 and 21 s (within $k = 2$ standard deviations of its mean) with probability *at least* $1 - 1/k^2 = .75$. Note that when the random variable in question is normally distributed, the normal probability rule yields a stronger statement than does Chebyshev's inequality.

Example 4.5.2. The safety record of an industrial plant is measured in terms of M, the total staffing-hours worked without a serious accident. Past experience indicates that M has a mean of 2 million with a standard deviation of .1 million. A serious accident has just occurred. Would it be unusual for the next serious accident to occur within the next 1.6 million staffing-hours? To answer this question, we must assess $P[M \leq 1.6]$. Since we have no reason to assume that M is normally distributed, the normal probability rule is inappropriate here. However, we know from Chebyshev's inequality with $k = 4$ that

$$P[1.6 < M < 2.4] \geq 1 - (1/16) = .9375$$

This implies that

$$P[M \leq 1.6] + P[M \geq 2.4] \leq .0625$$

Since it is possible for M to exceed 2.4, we can safely say that

$$P[M \leq 1.6] < .0625$$

No stronger statement can be made without some knowledge of the shape of the density of M. However, if it is known that the density is symmetric, then we can go one step further and state that

$$P[M \leq 1.6] \leq .0625/2 = .03125$$

4.6 NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION

The binomial tables given in this text or in any other text are necessarily limited in scope due to the fact that n can vary from 1 to infinity and p can assume any value between 0 and 1. It is impossible to table every combination of n and p. Due to the advances in computer and calculator technology, it is now possible to find exact binomial probabilities for any combination of n and p. Prior to this time, the normal curve was used to give good approximations of binomial probabilities. The technique introduced in this section is still useful in situations in which the needed technology tools are not readily available. To see how such approximations were suggested, we consider four binomial random variables each with probability of success .4 but with differing values for n. The densities for these variables, obtained from Table I of App. A, together with a sketch for each, are given in Fig. 4.10(a) to (d).

The point to note from these diagrams is made in Fig. 4.10(d). Namely, it is not hard to imagine a smooth bell curve that closely fits the block diagram shown. This suggests that binomial probabilities represented by one or more blocks in the diagram can be approximated reasonably well by a carefully selected area under an appropriately chosen normal curve. Which of the infinitely many normal curves is appropriate? Common sense indicates that the normal variable selected should have the same mean and variance as the binomial variable that it approximates. Theorem 4.6.1 summarizes these ideas.

Theorem 4.6.1 (Normal approximation to the binomial distribution). Let X be binomial with parameters n and p. For large n, X is approximately normal with mean np and variance $np(1-p)$.

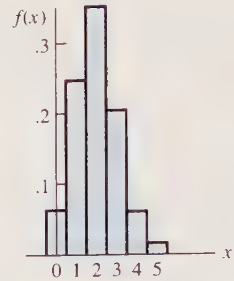
The proof of this theorem is based on the Central Limit Theorem, which will be considered in Chap. 7. Admittedly, Theorem 4.6.1 is a bit vague in the sense that the word "large" is not well defined. In the strictest mathematical sense, "large" means as n approaches infinity. For most practical purposes the approximation is acceptable for values of n and p such that either $p \leq .5$ and $np > 5$ or $p > .5$ and $n(1-p) > 5$.

Example 4.6.1. A study is performed to investigate the connection between maternal smoking during pregnancy and birth defects in children. Of the mothers studied, 40% smoke and 60% do not. When the babies were born, 20 were found to have some sort of birth defect. Let X denote the number of children whose mother smoked while pregnant. If there is no relationship between maternal smoking and birth defects, then X is binomial with $n = 20$ and $p = .4$. What is the probability that 12 or more of the affected children had mothers who smoked?

To answer this question, we need to find $P[X \geq 12]$ under the assumption that X is binomial with $n = 20$ and $p = .4$. This probability, .0565, can be found from Table I of App. A. Note that since $p = .4 \leq .5$ and $np = 20(.4) = 8 > 5$, the normal approximation should give a result quite close to .0565. We shall approximate probabilities associated with X using a normal random variable Y with mean $np = 20(.4) = 8$ and standard deviation $\sqrt{np(1-p)} = \sqrt{20(.4)(.6)} = \sqrt{4.8}$.

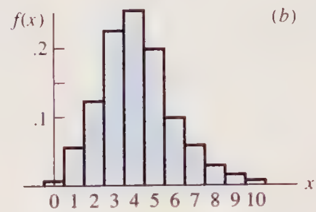
$n = 5$
$p = .4$

x	$f(x)$
0	.0778
1	.2592
2	.3456
3	.2304
4	.0768
5	.0102



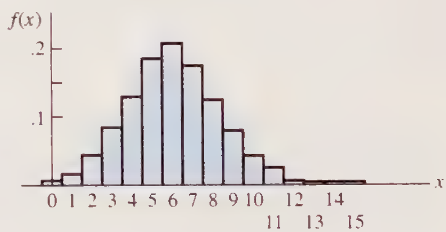
$n = 10$
$p = .4$

x	$f(x)$
0	.0060
1	.0404
2	.1209
3	.2150
4	.2508
5	.2007
6	.1114
7	.0425
8	.0106
9	.0016
10	.0001



$n = 15$
$p = .4$

x	$f(x)$	x	$f(x)$
0	.0005	8	.1181
1	.0047	9	.0612
2	.0219	10	.0245
3	.0634	11	.0074
4	.1268	12	.0016
5	.1859	13	.0003
6	.2066	14	~0
7	.1771	15	~0



$n = 20$
$p = .4$

x	$f(x)$	x	$f(x)$
0	~0	11	.0710
1	.0005	12	.0355
2	.0031	13	.0145
3	.0124	14	.0049
4	.0350	15	.0013
5	.0746	16	.0003
6	.1244	17	~0
7	.1659	18	~0
8	.1797	19	~0
9	.1597	20	~0
10	.1172		

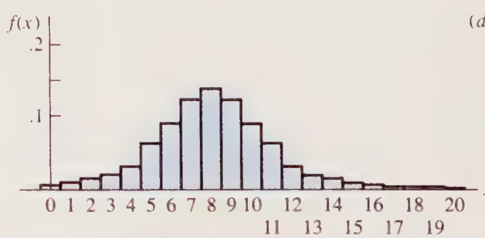


FIGURE 4.10
Density for X binomial: (*a*) $n = 5, p = .4$; (*b*) $n = 10, p = .4$; (*c*) $n = 15, p = .4$; (*d*) $n = 20, p = .4$.

The exact probability of .0565 is given by the sum of the areas of the blocks centered at 12, 13, 14, 15, 16, 17, 18, 19, and 20, as shown in Fig. 4.11. The approximate probability is given by the area under the normal curve shown above 11.5. That is,

$$P[X \geq 12] \doteq P[Y \geq 11.5]$$

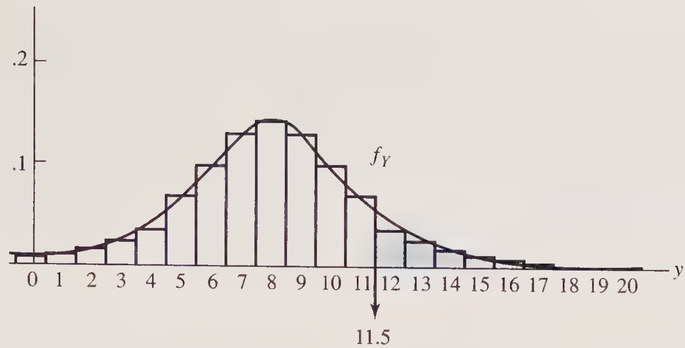


FIGURE 4.11
$P[X \geq 12]$ = area of shaded blocks $\doteq$ area under curve beyond 11.5.

The number .5 is called the *half-unit correction* for continuity. It is subtracted from 12 in the approximation because otherwise half the area of the block centered at 12 will be inadvertently ignored, leading to an unnecessary error in the calculation. From this point on the calculation is routine:

$$
\begin{aligned}
P[X \geq 12] &\doteq P[Y \geq 11.5] \\
&= P\left[\frac{Y-8}{\sqrt{4.8}} \geq \frac{11.5-8}{\sqrt{4.8}}\right] \\
&= P[Z \geq 1.59] \\
&= 1 - .9441 = .0559
\end{aligned}
$$

Note that even with n as small as 20, the approximated value of .0559 compares quite favorably with the exact value of .0565. In practice, of course, one would not approximate a probability that could be found directly from a binomial table. This was done here only for comparative purposes.

4.7 WEIBULL DISTRIBUTION AND RELIABILITY

In 1951 W. Weibull introduced a distribution that has been found to be useful in a variety of physical applications. It arises quite naturally in the study of reliability as we shall show. The most general form for the Weibull density is given by

$$
f(x) = \alpha\beta(x-\gamma)^{\beta-1}e^{-\alpha(x-\gamma)^{\beta}} \qquad \begin{aligned} x &> \gamma \\ \alpha &> 0 \\ \beta &> 0 \end{aligned}
$$

The implication of this definition of the density is that there is some minimum or "threshold" value γ below which the random variable X cannot fall. In most physical applications this value is 0. For this reason, we shall define the Weibull density with this fact in mind. Be careful when reading scientific literature to note the form of the Weibull density being used.

Definition 4.7.1 (Weibull distribution). A random variable X is said to have a Weibull distribution with parameters α and β if its density is given by

$$f(x) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}} \qquad \begin{array}{l} x > 0 \\ \alpha > 0 \\ \beta > 0 \end{array}$$

It is easy to verify that the function given in Definition 4.7.1 is a density. (See Exercise 61.) We shall find the mean of this distribution directly rather than by means of the moment generating function.

Theorem 4.7.1. Let X be a Weibull random variable with parameters α and β. The mean and variance of X are given by

$$\mu = \alpha^{-1/\beta}\,\Gamma(1 + 1/\beta)$$

and

$$\sigma^2 = \alpha^{-2/\beta}\,\Gamma(1 + 2/\beta) - \mu^2$$

Proof. By Definition 4.2.1,

$$\begin{aligned} E[X] &= \int_0^\infty x\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}dx \\ &= \int_0^\infty \alpha\beta x^{\beta}e^{-\alpha x^{\beta}}dx \end{aligned}$$

Let $z = \alpha x^{\beta}$. This implies that

$$x = (z/\alpha)^{1/\beta} \qquad \text{and} \qquad dx = (1/\alpha\beta)(z/\alpha)^{1/\beta-1}\,dz$$

By substitution, it is seen that

$$\begin{aligned} E[X] &= \int_0^\infty \alpha\beta(z/\alpha)e^{-z}(1/\alpha\beta)(z/\alpha)^{1/\beta-1}dz \\ &= \int_0^\infty (z/\alpha)^{1/\beta}e^{-z}dz \\ &= \alpha^{-1/\beta}\int_0^\infty z^{1/\beta}e^{-z}dz \end{aligned}$$

The integral on the right is, by definition, $\Gamma(1 + 1/\beta)$. (See Definition 4.3.1.) Thus we have shown that the mean of the Weibull distribution is

$$\mu = E[X] = \alpha^{-1/\beta}\Gamma(1 + 1/\beta)$$

as claimed. The remainder of the proof is outlined as an exercise. (See Exercises 62 and 63.)

The graph of the Weibull density varies depending on the values of α and β. The general shape resembles that of the gamma density with the curve becoming more symmetric as the value of β increases.

Example 4.7.1. Let X be a Weibull random variable with $\beta = 1$. The density for X is

$$f(x) = \alpha e^{-\alpha x} \qquad x > 0, \quad \alpha > 0$$

Note that this is the density for an *exponential* random variable. That is, the exponential distribution is a special case of the Weibull distribution with $\beta = 1$. By Theorem 4.7.1

$$\begin{aligned} \mu &= \alpha^{-1/\beta}\Gamma(1 + 1/\beta) = (1/\alpha)\Gamma(2) = 1/\alpha \cdot 1! = 1/\alpha \\ \sigma^2 &= \alpha^{-2/\beta}\Gamma(1 + 2/\beta) - \mu^2 \\ &= 1/\alpha^2\Gamma(3) - (1/\alpha)^2 \\ &= 2/\alpha^2 - 1/\alpha^2 = 1/\alpha^2 \end{aligned}$$

Note that these results are consistent with those obtained by viewing this random variable as being exponential. (See Exercise 33.)

Reliability

As we have said, the Weibull distribution frequently arises in the study of reliability. Reliability studies are concerned with assessing whether or not a system functions adequately under the conditions for which it was designed. Interest centers on describing the behavior of the random variable X, the time to failure of a system that cannot be repaired once it fails to operate. Three functions come into play when assessing reliability. These are the failure density f, the reliability function R, and ρ, the failure or hazard rate of the distribution. To understand how these functions are defined, consider some system being put into operation at time $t = 0$. We observe the system until it eventually fails. Let X denote the time of the failure. This random variable is continuous and a priori can assume any value in the interval $(0, \infty)$. The density f, for X, is called the *failure density* for the component. The *reliability function, R,* is defined to be the probability that the component will not fail before time t. Thus

$$\begin{aligned} R(t) &= 1 - P[\text{component will fail before time } t] \\ &= 1 - \int_0^t f(x)\,dx \\ &= 1 - F(t) \end{aligned}$$

where F is the cumulative distribution function for X. To define ρ, the hazard rate function, consider a time interval $[t, t + \Delta t]$ of length Δt. We define the force of mortality or hazard rate function over this interval by

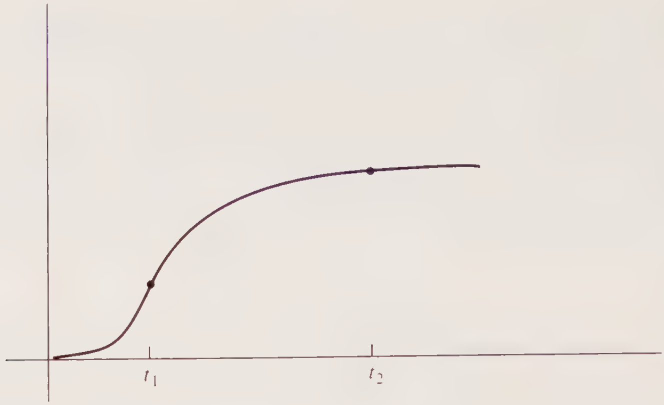


FIGURE 4.12
At time t_1, the graph of F is steep. Failures are likely to occur in the time interval near t_1. At time t_2, the graph of F is rather flat. Failures are not highly likely in the time interval near t_2.

$$\rho(t) = \lim_{\Delta t \to 0} P(t \le X \le t + \Delta t | t \le X) \frac{1}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\text{probability of failure in } [t, t + \Delta t]}{\text{probability of failure in } [t, \infty]} \cdot \frac{1}{\Delta t}$$

Notice that

$$\lim_{\Delta t \to 0} \frac{\text{probability of failure in } [t, t + \Delta t]}{\Delta t} = \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t}$$

This, by definition, is the derivative of the cumulative distribution function for X. Since the derivative of a function in general can be interpreted as giving the "instantaneous rate of change" of the function, this portion of the definition of $\rho(t)$ gives the instantaneous rate of change of F at time t. Since a cumulative distribution function cannot decrease, the derivative of F will always be nonnegative. Its magnitude tells us how fast failures are occurring at any given time. A large value of $F'(t)$ implies a steep curve at t, which in turn implies that failures are coming rapidly in an interval near t; a small value of $F'(t)$ implies that failures are occurring at a slower pace. (See Fig. 4.12.) Thus we can say that $\rho(t)$ gives us a picture of the instantaneous rate of failure at times t given that the system was operable prior to this time.

Theorem 4.7.2 relates the three functions f, R, and ρ.

Theorem 4.7.2. Let X be a random variable with failure density f, reliability function R, and hazard rate function ρ. Then

$$\rho(t) = \frac{f(t)}{R(t)}$$

Proof. By definition,

$$\begin{aligned}
\rho(t) &= \lim_{\Delta t \to 0} \frac{\text{probability of failure in } [t, t + \Delta t]}{\text{probability of failure in } [t, \infty]} \cdot \frac{1}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\int_t^{t+\Delta t} f(x)\,dx}{\int_t^{\infty} f(x)\,dx} \cdot \frac{1}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{1 - F(t)} \cdot \frac{1}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t} \cdot \frac{1}{R(t)} \\
&= \frac{F'(t)}{R(t)} = \frac{f(t)}{R(t)}
\end{aligned}$$

The job of the scientist is to find the form of these functions for the problem at hand. In practice, one often begins by assuming a particular form for the hazard rate function based on empirical evidence. To do so, one must have some practical way to interpret ρ. A rough interpretation is as follows:

Interpretation of the Hazard Rate

1. If ρ is increasing over an interval, then as time goes by a failure is more likely to occur. This normally happens for systems that begin to fail primarily due to wear.
2. If ρ is decreasing over an interval, then as time goes by a failure is less likely to occur than it was earlier in the time interval. This happens in situations in which defective systems tend to fail early. As time goes by, the hazard rate for a well-made system decreases.
3. A steady hazard rate is expected over the useful life span of a component. A failure tends to occur during this period due mainly to random factors.

Since one often has an idea of the form only of ρ, the natural question to ask is: "Can we derive the failure density and the reliability function from knowledge of ρ?" Theorem 4.7.3 shows how this can be done.

Theorem 4.7.3. Let X be a random variable with failure density f, reliability function R, and hazard rate ρ. Then

$$R(t) = \exp\left[-\int_0^t \rho(x)\,dx\right]$$

and $f(t) = \rho(t)R(t)$.

Proof. Note that since $R(x) = 1 - F(x)$, $R'(x) = -F'(x)$. Therefore

$$\rho(x) = \frac{f(x)}{R(x)} = \frac{F'(x)}{R(x)} = \frac{-R'(x)}{R(x)}$$

We integrate each side of this equation to obtain

$$\int_0^t \rho(x)\,dx = -\int_0^t \frac{R'(x)}{R(x)}\,dx = -[\ln R(t) - \ln R(0)]$$

Note that $R(0) = 1$ since the component will not fail before time $t = 0$, the moment that it is put into operation. Since $\ln R(0) = \ln 1 = 0$, we see that

$$-\int_0^t \rho(x)\,dx = \ln R(t)$$

or that

$$\exp\left[-\int_0^t \rho(x)\,dx\right] = e^{\ln R(t)} = R(t)$$

as claimed.

Example 4.7.2 illustrates the use of Theorem 4.7.3 and shows how the Weibull distribution arises in reliability studies.

Example 4.7.2. One hazard rate function in widespread use is the function

$$\rho(t) = \alpha\beta t^{\beta-1} \qquad \begin{array}{l} t > 0 \\ \alpha > 0 \\ \beta > 0 \end{array}$$

This function has the property that if $\beta = 1$, the hazard rate is constant, indicating that the occurrence of a failure is due primarily to random factors; if $\beta > 1$, the hazard rate is increasing, indicating that a failure is due primarily to a system wearing out over time; if $\beta < 1$, the hazard rate is decreasing, indicating that an early failure is likely due to a malfunctioning system. (See Exercise 64.) The reliability function is given by

$$\begin{aligned} R(t) &= \exp\left[-\int_0^t \alpha\beta x^{\beta-1}\,dx\right] \\ &= \exp\left[-\alpha x^{\beta}\Big|_0^t\right] = e^{-\alpha[t^{\beta}-0^{\beta}]} = e^{-\alpha t^{\beta}} \end{aligned}$$

The failure density is given by

$$f(t) = \rho(t)R(t) = \alpha\beta t^{\beta-1}e^{-\alpha t^{\beta}}$$

This is the density for a Weibull random variable with parameters α and β.

This section can be summarized as follows:

Properties of Reliability Studies

1. The random variable of interest is X, the time of failure of a system or a component of a system.

2. The failure density, f, is the probability density function for X.
3. The reliability function, R, gives the probability that the system or component will not fail before time t.
4. The function F is the cumulative distribution function for X.
5. The hazard rate function, ρ, gives a picture of the instantaneous rate of failures at time t given that the system or component was operable prior to time t.
6. These functions are connected to one another through the following relationships:

$$\rho(t) = f(t)/R(t)$$

$$R(t) = \exp\left[-\int_0^t \rho(x)\,dx\right]$$

7. The Weibull distribution is often appropriate as a failure density in applied engineering problems.

Reliability of Series and Parallel Systems

Components in multiple component systems can be installed within the system in various ways. Many systems are arranged in a "series" configuration, some are in "parallel," and others are combinations of the two designs. These terms are defined as follows:

Definition 4.7.2 (Series system). A system whose components are arranged in such a way that the system fails whenever any of its components fail is called a series system.

Definition 4.7.3 (Parallel system). A system whose components are arranged in such a way that the system fails only if all of its components fail is called a parallel system.

Recall that the reliability function for a component is the probability that it will not fail before time t. Consider a system consisting of k components connected in series. Let $R_i(t)$ denote the reliability of component i and assume that the components are independent in the sense that the reliability of one is unaffected by the reliability of the others. The reliability of the entire system is the probability that the system will not fail before time t. The system will not fail if and only if no component fails before time t. Thus the reliability of the system, $R_s(t)$, is given by

$$R_s(t) = \prod_{i=1}^{k} R_i(t)$$

The next two examples illustrate the use of this equation.

Example 4.7.3. Consider a system with five components connected in series. If each component has reliability .95 at time t, then the system reliability at that time is $R_s(t) = (.95)^5 = .774$.

Example 4.7.4. Suppose that we are designing a system of five independent components and we want the system reliability at time t to be at least .95. If the reliability of each component at t is to be the same, what is the minimum reliability required per component? Here we want $x^5 \geq .95$, where x is the reliability of each component. The solution is $x = (.95)^{1/5} = .9898$.

A more practical design for most equipment is the parallel system. Consider k independent components arranged in parallel. When the first fails, the second is used; when the second fails, the third comes on line. This continues until the last component fails, at which time the system fails. The system reliability at time t in this case is the probability that at least one of the k components does not fail before time t. This probability is given by

$$R_s(t) = 1 - P[\text{all components fail}]$$
$$= 1 - \prod_{i=1}^{k} [1 - R_i(t)]$$

It should be noted that in both series and parallel systems the reliability of individual components can differ. Example 4.7.5 illustrates a system that makes use of both types of configurations.

Example 4.7.5. Consider a system consisting of eight independent components connected as shown in Fig. 4.13. Note that the system consists of five assemblies in series, where assembly I consists of component 1; assembly II consists of components 2 and 3 in parallel; assembly III consists of components 4, 5, and 6 in parallel; assemblies IV and V consist of components 7 and 8, respectively. To calculate the system reliability, we first calculate the reliability of the two parallel assemblies. The reliability of assembly II is

$$[1 - (1 - .95)^2] = .9975$$

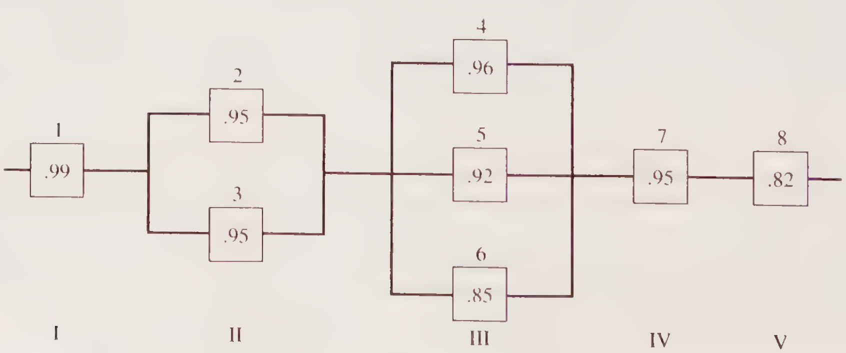


FIGURE 4.13
System with five assemblies, with assemblies II and III in parallel.

and that of assembly III is

$$[1 - (1 - .96)(1 - .92)(1 - .85)] = .99952$$

The system reliability is the product of the reliabilities of the five assemblies and is given by

$$R_s(t) = (.99)(.9975)(.99952)(.95)(.82) = .7689$$

It is evident that a system with many independent components connected in series may have a very low system reliability, even if each component alone is highly reliable. For example, a system of 20 components, each with a reliability of .95 connected in series, has a system reliability of $(.95)^{20} = .358$. One way to increase system reliability is to replace single components with several similar components arranged in parallel. Of course, the cost of providing this sort of redundancy is usually high.

4.8 TRANSFORMATION OF VARIABLES

Consider a continuous random variable X with density f_X. Suppose that interest centers on some random variable Y, where Y is a function of X. Can we determine the density for Y based on knowledge of the distribution of X? The next theorem allows us to answer this question whenever Y is a strictly monotonic function of X.

Theorem 4.8.1. Let X be a continuous random variable with density f_X. Let $Y = g(X)$, where g is strictly monotonic and differentiable. The density for Y is denoted by f_Y and is given by

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|$$

Proof. Assume that $Y = g(X)$ is a strictly decreasing function of X. By definition the cumulative distribution function for Y is

$$F_Y(y) = P[Y \le y] = P[g(X) \le y]$$

Since g is strictly decreasing, g^{-1} exists and is also decreasing. Therefore

$$\begin{aligned} P[g(X) \le y] &= P[g^{-1}(g(X)) \ge g^{-1}(y)] \\ &= P[X \ge g^{-1}(y)] \\ &= 1 - P[X \le g^{-1}(y)] \end{aligned}$$

By definition $P[X \le g^{-1}(y)] = F_X(g^{-1}(y))$, and thus substitution yields

$$F_Y(y) = 1 - F_X(g^{-1}(y))$$

Since the derivative of the cumulative distribution function yields the density,

$$f_Y(y) = (-1)f_X(g^{-1}(y))\,\frac{dg^{-1}(y)}{dy}$$

Note that since g^{-1} is decreasing, $dg^{-1}(y)/dy < 0$ and

$$-\frac{dg^{-1}(y)}{dy} = \left|\frac{dg^{-1}(y)}{dy}\right|$$

By substitution $f_Y(y)$ can be written as

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|$$

as claimed. The proof in the case in which g is increasing is similar and is left as an exercise. (See Exercise 69.)

An example will illustrate the idea.

Example 4.8.1. Let X be a random variable with density

$$f_X(x) = 2x \qquad 0 < x < 1$$

and let $g(X) = Y = 3X + 6$. Since $g(x) = 3x + 6$ is strictly increasing and differentiable, Theorem 4.8.1 is applicable. To obtain the expression for g^{-1}, we solve the equation $y = 3x + 6$ for x and see that

$$x = g^{-1}(y) = \frac{y-6}{3}$$

and

$$\frac{dg^{-1}(y)}{dy} = \frac{1}{3}$$

An application of Theorem 4.8.1 yields

$$f_Y(y) = f_X(g^{-1}(y))\frac{dg^{-1}(y)}{dy}$$

or

$$f_Y(y) = 2\frac{(y-6)}{3}\cdot\frac{1}{3} = \frac{2}{9}(y-6) \qquad 6 < y < 9$$

It should be pointed out that the results given in Theorem 4.8.1 can be applied to piecewise monotonic functions as well as to those that are strictly monotonic. In this case several different equations might be required to define the density for Y. The idea is illustrated in the next example.

Example 4.8.2. Let X be uniformly distributed over $(0, 4)$ and let $g(X) = Y = (X-3)^2$. The graph of this function is shown in Fig. 4.14. Note that since g is strictly decreasing on $(0, 3)$ and strictly increasing on $(3, 4)$, it is piecewise monotonic. It can be defined in terms of the two one-to-one functions g_1 and g_2 given by

$$g(x) = \begin{cases} g_1(x) = (x-3)^2 & 0 < x \leq 3 \\ g_2(x) = (x-3)^2 & 3 < x < 4 \end{cases}$$

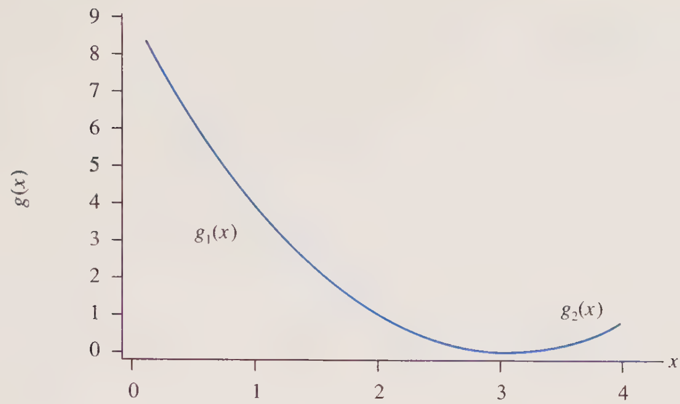


FIGURE 4.14
Graph of $g(x) = (x - 3)^2$, $0 < x < 4$ partitioned into two monotonic functions $g_1(x) = (x - 3)^2$, $0 < x \leq 3$ and $g_2(x) = (x - 3)^2$, $3 < x < 4$.

Each of the functions g_1 and g_2 is invertible, and their inverses are given by

$$g_1^{-1}(y) = 3 - \sqrt{y} \qquad 0 \leq y < 9$$
$$g_2^{-1}(y) = 3 + \sqrt{y} \qquad 0 < y < 1$$

Functions h_1 and h_2 used to determine the density for Y are found by applying Theorem 4.8.1 to each of the above. With this done, we then add those functions having common domains to obtain the final expression for f_Y. In this case the density is formed from the functions

$$h_1(y) = f_X(g_1^{-1}(y))\left|\frac{dg_1^{-1}(y)}{dy}\right| = \frac{1}{4}\left| -\frac{1}{2\sqrt{y}}\right| = \frac{1}{8\sqrt{y}}$$

$$h_2(y) = f_X(g_2^{-1}(y))\left|\frac{dg_2^{-1}(y)}{dy}\right| = \frac{1}{4}\left|\frac{1}{2\sqrt{y}}\right| = \frac{1}{8\sqrt{y}}$$

To obtain the density for Y, we note that the interval [0, 1] is common to the domain of both h_1 and h_2. Thus

$$f_Y(y) = h_1(y) + h_2(y) = \frac{2}{8\sqrt{y}} \qquad 0 \leq y < 1$$

The interval [1, 9] is contained only in the domain of h_2. Hence

$$f_Y(y) = h_2(y) = \frac{1}{8\sqrt{y}} \qquad 1 \leq y < 9$$

You can verify for yourself that f_Y is a valid density.

4.9 SIMULATING A CONTINUOUS DISTRIBUTION

In Sec. 3.9 we showed how to simulate a discrete distribution using a random digit table. The table also can be used to simulate a continuous distribution. The idea is as follows:

1. We find the cumulative distribution function F for the random variable and its inverse.
2. We select a random two- (or three-) digit number from Table III of App. A and interpret this number as a probability, that is, as a number between 0 and 1.
3. We evaluate F^{-1} at this randomly selected point to obtain a randomly generated value for the random variable X.

This procedure is illustrated in Example 4.9.1.

Example 4.9.1. Consider the random variable X, the time to failure of a computer chip. Assume that X has a Weibull distribution with parameters $\alpha = .02$ and $\beta = 1$. The density for X is

$$f(x) = .02e^{-.02x} \qquad x > 0$$

and its cumulative distribution is

$$y = F(x) = 1 - e^{-.02x}$$

The inverse of F is found by solving this equation for x as follows:

$$\begin{aligned} y &= 1 - e^{-.02x} \\ e^{-.02x} &= 1 - y \\ -.02x &= \ln(1 - y) \\ x &= \frac{-\ln(1 - y)}{.02} \end{aligned}$$

To simulate an observation on X, we select a random two-digit number from Table III of App. A. Suppose the number selected is 77, which is interpreted as the probability $y = .77$. For this value of y our simulated observation on X is

$$x = \frac{-\ln(1 - .77)}{.02} = 73.48 \text{ years}$$

This procedure can be repeated to generate as many random values for X as desired. Figure 4.15 illustrates this procedure graphically.

CHAPTER SUMMARY

In this chapter we considered the general properties underlying random variables of the continuous type. These are random variables that assume their values in intervals of real numbers rather than at isolated points. The density function was introduced

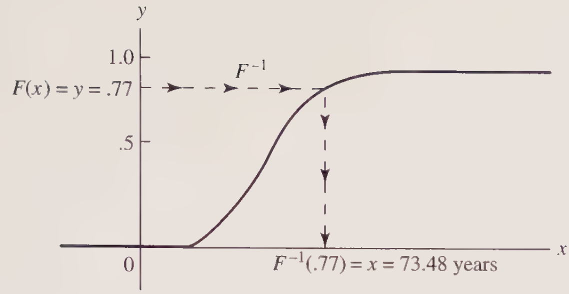


FIGURE 4.15
$F(x) = y = .77$ if and only if $F^{-1}(.77) = x = 73.48$ years.

as a means of computing probabilities. These densities are defined in such a way that probabilities correspond to areas. The ideas of expected value and moment generating function were defined by replacing the summation operation, used in the discrete case, with integration. A number of continuous distributions were studied. See Table 4.1. The gamma distribution was presented. We noted that the exponential distribution and the chi-squared distribution are special cases of the gamma distribution. We studied the normal distribution and showed how to use this distribution to approximate binomial and Poisson probabilities. The Weibull distribution was introduced, and its use in reliability studies was examined. The log-normal, uniform, and Cauchy distributions were introduced as exercises. We saw how to simulate continuous distributions. We introduced and defined important terms that you should know. These are:

Continuous random variable	Continuous density
Continuous distribution function	Gamma function
Half-unit correction	Failure density
Reliability function	Hazard rate function
Standard normal	

In the last two chapters we have presented some commonly encountered discrete and continuous distributions and have looked at some of the relationships that exist among them. The chart given in Fig. 4.16 summarizes the results that have been obtained. It is an adaptation of the more complete chart developed by Lawrence Leemis in "Relationships Among Common Univariate Distributions," *The American Statistician,* May 1986, vol. 40, no. 2. (Used with permission of the author.) In the chart two types of relationships, namely, special cases and approximations, are depicted. Special cases are indicated by a solid arrow, and approximations are shown by a dashed arrow. In each case the name of the distribution along with its associated parameters are given.

TABLE 4.1
Continuous distributions

Name	Density $= f_x(x)$		Moment generating function		Mean	Variance
Gamma	$\frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}$	$\alpha > 0$ $\beta > 0$ $x > 0$	$(1-\beta t)^{-\alpha}$		$\alpha\beta$	$\alpha\beta^2$
Exponential	$\frac{1}{\beta} e^{-x/\beta}$ $\to \lambda e$	$x > 0$ $\beta > 0$	$(1-\beta t)^{-1}$		β	β^2
Chi-squared	$\frac{1}{\Gamma(\gamma/2)2^{\gamma/2}} x^{\gamma/2-1} e^{-x/2}$	$x > 0$ γ a positive integer	$(1-2t)^{-\gamma/2}$		γ	2γ
Uniform	$\frac{1}{b-a}$	$a < x < b$	$\frac{e^{tb}-e^{ta}}{t(b-a)}$ 1	$t \neq 0$ $t = 0$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Cauchy	$\frac{1}{\pi} \frac{a}{a^2+(x-b)^2}$	$-\infty < x < \infty$ $-\infty < b < \infty$ $a > 0$	Does not exist		Does not exist	Does not exist
Normal	$\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-1/2\left(\frac{x-\mu}{\sigma}\right)^2\right]$	$-\infty < x < \infty$ $\sigma > 0$ $-\infty < \mu < \infty$	$e^{\mu t+\sigma^2 t^2/2}$		μ	σ^2
Weibull	$\alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}$	$x > 0$ $\alpha > 0$ $\beta > 0$			$\alpha^{-1/\beta}\Gamma\left(1+\frac{1}{\beta}\right)$	$\alpha^{-2/\beta}\Gamma\left(1+\frac{2}{\beta}\right)-\mu^2$

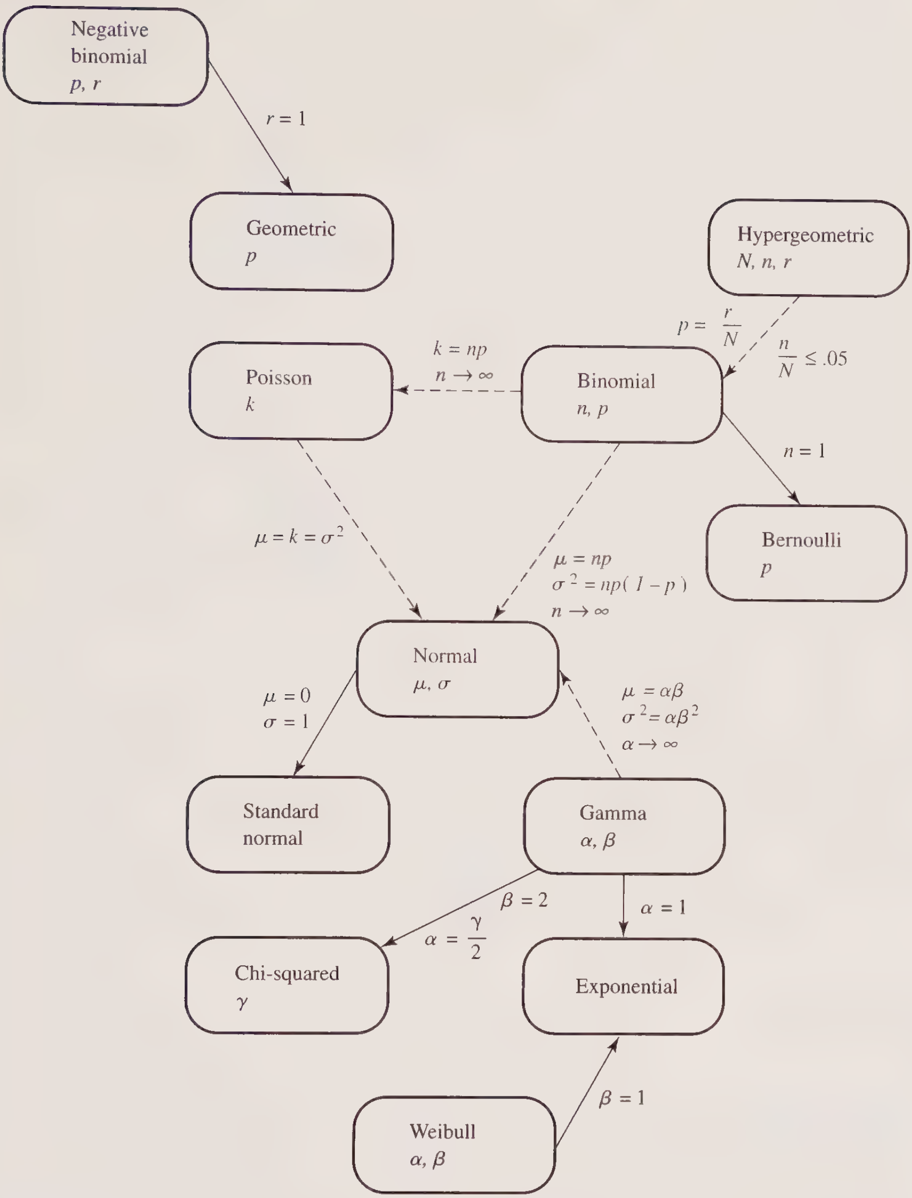


FIGURE 4.16
Some interrelationships among common distributions.

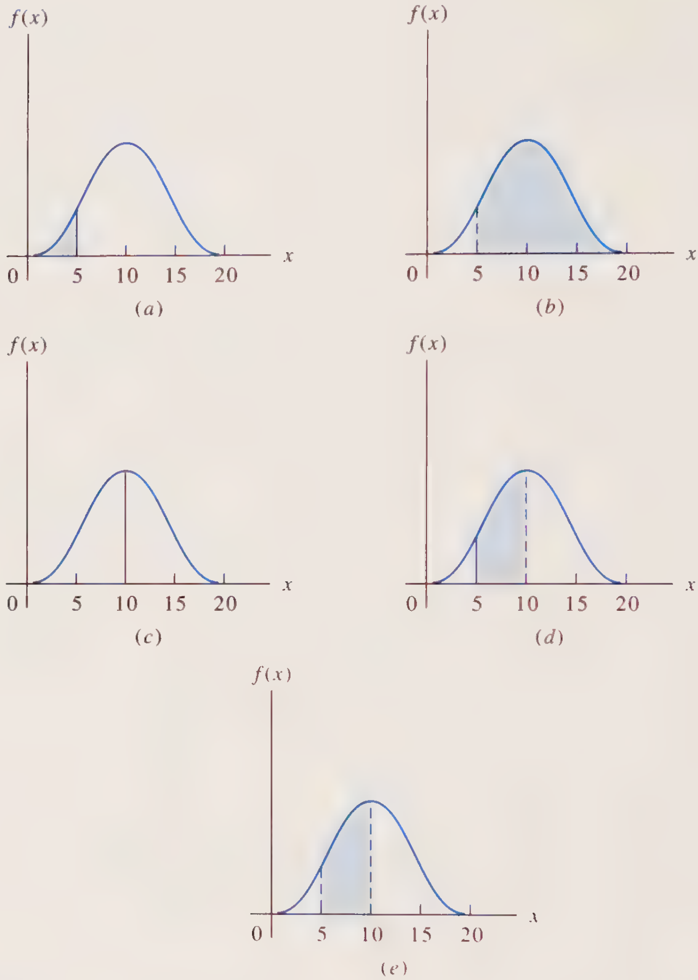


FIGURE 14.17

EXERCISES

Section 4.1

1. Consider the function

$$f(x) = kx \qquad 2 \leq x \leq 4$$

(a) Find the value of k that makes this a density for a continuous random variable.

(b) Find $P[2.5 \leq X \leq 3]$.

(c) Find $P[X = 2.5]$.

(d) Find $P[2.5 < X \leq 3]$.

2. Consider the areas shown in Fig. 4.17. In each case, state what probability is being depicted. What is the relationship between the areas depicted in Figs. 4.17(*a*) and (*b*)? Between those in Figs. 4.17(*d*) and (*e*)?

3. Let X denote the length in minutes of a long-distance telephone conversation. Assume that the density for X is given by

$$f(x) = (1/10)e^{-x/10} \qquad x > 0$$

(*a*) Verify that f is a density for a continuous random variable.
(*b*) Assuming that f adequately describes the behavior of the random variable X, find the probability that a randomly selected call will last at most 7 minutes; at least 7 minutes; exactly 7 minutes.
(*c*) Would it be unusual for a call to last between 1 and 2 minutes? Explain, based on the probability of this occurring.
(*d*) Sketch the graph of f and indicate in the sketch the area corresponding to each of the probabilities found in part (*b*).

4. Some plastics in scrapped cars can be stripped out and broken down to recover the chemical components. The greatest success has been in processing the flexible polyurethane cushioning found in these cars. Let X denote the amount of this material, in pounds, found per car. Assume that the density for X is given by

$$f(x) = \frac{1}{\ln 2}\frac{1}{x} \qquad 25 \le x \le 50$$

(*a*) Verify that f is a density for a continuous random variable.
(*b*) Use f to find the probability that a randomly selected auto will contain between 30 and 40 pounds of polyurethane cushioning.
(*c*) Sketch the graph of f, and indicate in the sketch the area corresponding to the probability found in part (*b*).

5. *(Continuous uniform distribution.)* A random variable X is said to be uniformly distributed over an interval (a, b) if its density is given by

$$f(x) = \frac{1}{b-a} \qquad a < x < b$$

(*a*) Show that this is a density for a continuous random variable.
(*b*) Sketch the graph of the uniform density.
(*c*) Shade the area in the graph of part (*b*) that represents $P[X \le (a+b)/2]$.
(*d*) Find the probability pictured in part (*c*).
(*e*) Let (c, d) and (e, f) be subintervals of (a, b) of equal length. What is the relationship between $P[c \le X \le d]$ and $P[e \le X \le f]$? Generalize the idea suggested by this example, thus justifying the name "uniform" distribution.

6. If a pair of coils were placed around a homing pigeon and a magnetic field was applied that reverses the earth's field, it is thought that the bird would become disoriented. Under these circumstances it is just as likely to fly in one direction as in any other. Let θ denote the direction in radians of the bird's initial flight. See Fig. 4.18. θ is uniformly distributed over the interval $[0, 2\pi]$.
(*a*) Find the density for θ.

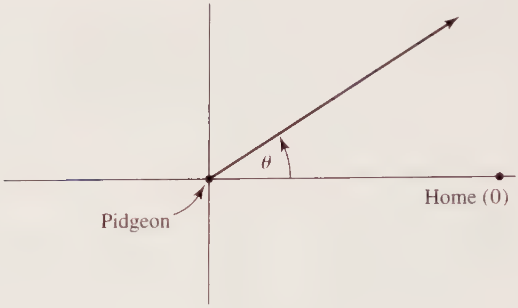


FIGURE 4.18
θ = direction of the initial flight of a homing pigeon measured in radians.

(*b*) Sketch the graph of the density. The uniform distribution is sometimes called the "rectangular" distribution. Do you see why?
(*c*) Shade the area corresponding to the probability that a bird will orient within $\pi/4$ radians of home, and find this area using plane geometry.
(*d*) Find the probability that a bird will orient within $\pi/4$ radians of home by integrating the density over the appropriate region(s), and compare your answer to that obtained in part (*c*).
(*e*) If 10 birds are released independently and at least seven orient within $\pi/4$ radians of home, would you suspect that perhaps the coils are not disorienting the birds to the extent expected? Explain, based on the probability of this occurring.

7. Use Definition 4.1.2 to show that for a continuous random variable X, $P[X = a] = 0$ for every real number a. *Hint:* Write $P[X = a]$ as $P[a \leq X \leq a]$.

8. Express each of the probabilities depicted in Fig. 4.16 in terms of the cumulative distribution function F.

9. Consider the random variable of Exercise 1.
(*a*) Find the cumulative distribution function F.
(*b*) Use F to find $P[2.5 \leq X \leq 3]$, and compare your answer to that obtained previously.
(*c*) Find $F'(x)$, and verify that your result is the density given in Exercise 1.

10. *(Uniform distribution.)* Find the general expression for the cumulative distribution function for a random variable X that is uniformly distributed over the interval (a, b). See Exercise 5.

11. *(Uniform distribution.)* Consider the random variable of Exercise 6.
(*a*) Use Exercise 10 to find the cumulative distribution function F.
(*b*) Find $F'(x)$, and verify that your result is, as expected, the uniform density over the interval $[0, 2\pi]$.

12. Find the cumulative distribution function for the random variable of Exercise 3. Use F to find $P[1 \leq X \leq 2]$, and compare your answer to that obtained previously.

13. Find the cumulative distribution function for the random variable of Exercise 4. Use F to find $P[30 \leq X \leq 40]$, and compare your answer to that obtained previously.

14. In parts (a) and (b) proposed cumulative distributions are given. In each case, find the "density" that would be associated with each, and decide whether it really does define a valid continuous density. If it does not, explain what property fails.

(a) Consider the function F defined by

$$F(x) = \begin{cases} 0 & x < -1 \\ x + 1 & -1 \leq x \leq 0 \\ 1 & x > 0 \end{cases}$$

(b) Consider the function defined by

$$F(x) = \begin{cases} 0 & x \leq 0 \\ x^2 & 0 < x \leq 1/2 \\ (1/2)x & 1/2 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

Section 4.2

15. Consider the random variable X with density

$$f(x) = (1/6)x \qquad 2 \leq x \leq 4$$

(a) Find $E[X]$.
(b) Find $E[X^2]$.
(c) Find σ^2 and σ.

16. Let X denote the amount in pounds of polyurethane cushioning found in a car. (See Exercise 4.) The density for X is given by

$$f(x) = \frac{1}{\ln 2}\frac{1}{x} \qquad 25 \leq x \leq 50$$

Find the mean, variance, and standard deviation for X.

17. Let X denote the length in minutes of a long-distance telephone conversation. The density for X is given by

$$f(x) = (1/10)e^{-x/10} \qquad x > 0$$

(a) Find the moment generating function, $m_X(t)$.
(b) Use $m_X(t)$ to find the average length of such a call.
(c) Find the variance and standard deviation for X.

18. *(Uniform distribution.)* The density for a random variable X distributed uniformly over (a, b) is

$$f(x) = \frac{1}{b-a} \qquad a < x < b$$

Use Definition 4.2.1 to show that

$$E[X] = \frac{a+b}{2} \qquad \text{and} \qquad \operatorname{Var} X = \frac{(b-a)^2}{12}$$

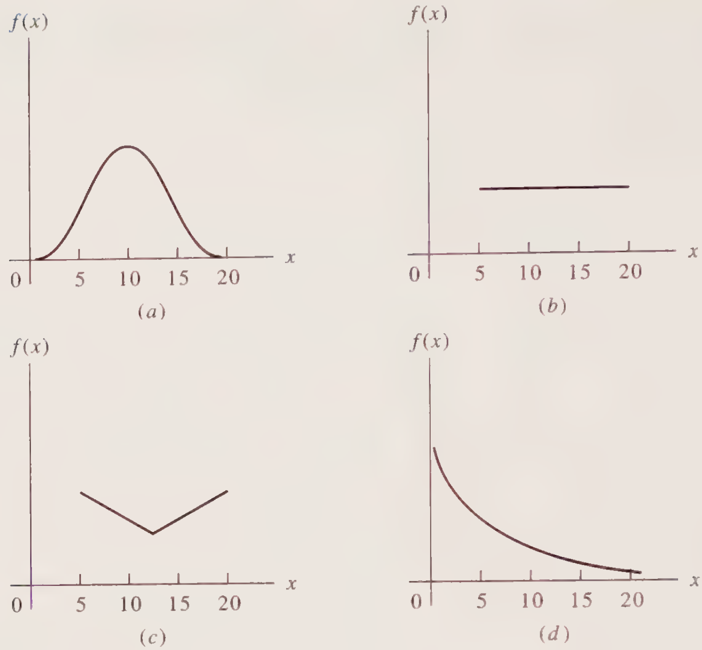


FIGURE 4.19

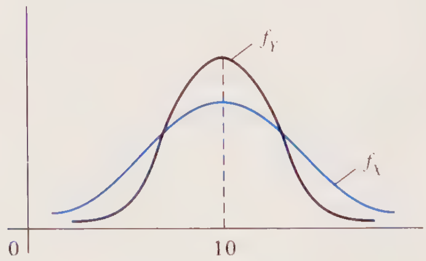


FIGURE 4.20

19. *(Uniform distribution.)* Let θ denote the direction in radians of the flight of a bird whose sense of direction has been disoriented as described in Exercise 6. Assume that θ is uniformly distributed over the interval $[0, 2\pi]$. Use the results of Exercise 18 to find the mean, variance, and standard deviation of θ.

20. Figure 4.19 gives the graphs of the densities of four continuous random variables whose means do exist. In each case, approximate the value of μ_X from the graph.

21. Consider the two densities given in Fig. 4.20. What is μ_X? What is μ_Y? Which random variable has the larger variance?

22. *(Cauchy distribution.)* A random variable X with density

$$f(x) = \frac{1}{\pi}\frac{a}{a^2 + (x - b)^2} \qquad \begin{array}{l} -\infty < x < \infty \\ -\infty < b < \infty \\ a > 0 \end{array}$$

is said to have a Cauchy distribution with parameters a and b. This distribution is interesting in that it provides an example of a continuous random variable whose mean does not exist. Let $a = 1$ and $b = 0$ to obtain a special case of the Cauchy distribution with density

$$f(x) = \frac{1}{\pi}\frac{1}{1 + x^2} \qquad -\infty < x < \infty$$

Show that $\int_{-\infty}^{\infty}|x| f(x)\, dx$ does not exist, thus showing that $E[X]$ does not exist. *Hint:* Write

$$\int_{-\infty}^{\infty}|x|\frac{1}{\pi}\frac{1}{1 + x^2}\, dx = \frac{1}{\pi}\int_{-\infty}^{0}\frac{-x}{1 + x^2}\, dx + \frac{1}{\pi}\int_{0}^{\infty}\frac{x}{1 + x^2}\, dx$$

and recall that $\int(du/u) = \ln|u|$.

23. Let X denote the amount of time in hours that a battery on a solar calculator will operate adequately between exposures to light sufficient to recharge the battery. Assume that the density for X is given by

$$f(x) = (50/6)x^{-3} \qquad 2 < x < 10$$

(*a*) Verify that this is a valid continuous density.
(*b*) Find the expression for the cumulative distribution function for X, and use it to find the probability that a randomly selected solar battery will last at most 4 hours before needing to be recharged.
(*c*) Find the average time that a battery will last before needing to be recharged.
(*d*) Find $E[X^2]$, and use this to find the variance of X.

24. Assume that the increase in demand for electric power in millions of kilowatt hours over the next 2 years in a particular area is a random variable whose density is given by

$$f(x) = (1/64)x^3 \qquad 0 < x < 4$$

(*a*) Verify that this is a valid density.
(*b*) Find the expression for the cumulative distribution for X, and use it to find the probability that the demand will be at most 2 million kilowatt hours.
(*c*) If the area only has the capacity to generate an additional 3 million kilowatt hours, what is the probability that demand will exceed supply?
(*d*) Find the average increase in demand.

Section 4.3

25. Evaluate each of these integrals:
(*a*) $\int_0^\infty z^2 e^{-z}\, dz$
(*b*) $\int_0^\infty z^7 e^{-z}\, dz$
(*c*) $\int_0^\infty x^3 e^{-x/2}\, dx$
(*d*) $\int_0^\infty (1/16) x e^{-x/4}\, dx$

26. Prove Theorem 4.3.1. *Hint:* To prove part 1, evaluate $\Gamma(1)$ directly from the definition of the gamma function. To prove part 2, use integration by parts with

$$u = z^{\alpha-1} \qquad dv = \int e^{-z}dz$$

$$du = (\alpha-1)z^{\alpha-2}dz \qquad v = -e^{-z}$$

Use L'Hospital's rule repeatedly to show that

$$-z^{\alpha-1}e^{-z}\Big|_0^\infty = 0$$

27. (*a*) Use Theorem 4.3.1 to evaluate $\Gamma(2)$, $\Gamma(3)$, $\Gamma(4)$, $\Gamma(5)$, and $\Gamma(6)$.
(*b*) Can you generalize the pattern suggested in part (*a*)?
(*c*) Does the result of part (*b*) hold even if $n = 1$?
(*d*) Evaluate $\Gamma(15)$ using the result of part (*b*).

28. Show that for $\alpha > 0$ and $\beta > 0$,

$$\int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1}e^{-x/\beta}\, dx = 1$$

thereby showing that the function given in Definition 4.3.2 is a density for a continuous random variable. *Hint:* Change the variable by letting $z = x/\beta$.

29. Let X be a gamma random variable with $\alpha = 3$ and $\beta = 4$.
(*a*) What is the expression for the density for X?
(*b*) What is the moment generating function for X?
(*c*) Find μ, σ^2, and σ.

30. Let X be a gamma random variable with parameters α and β. Use the moment generating function to find $E[X]$ and $E[X^2]$. Use these expectations to show that $\text{Var } X = \alpha\beta^2$.

31. Let X be a gamma random variable with parameters α and β.
(*a*) Use Definition 4.2.1, the definition of expected value, to find $E[X]$ and $E[X^2]$ directly. *Hint:* $z^\alpha = z^{(\alpha+1)-1}$ and $z^{\alpha+1} = z^{(\alpha+2)-1}$.
(*b*) Use the results of (*a*) to verify that $\text{Var } X = \alpha\beta^2$.

32. Show that the graph of the density for a gamma random variable with parameters α and β assumes its maximum value at $x = \beta(\alpha - 1)$ for $\alpha > 1$. Sketch a rough graph of the density for a gamma random variable with $\alpha = 3$ and $\beta = 4$. *Hint:* Find the first derivative of the density, set this derivative equal to 0, and solve for x.

33. Let X be an exponential random variable with parameter β. Find general expressions for the moment generating function, mean, and variance for X.

34. A particular nuclear plant releases a detectable amount of radioactive gases twice a month on the average. Find the probability that at least 3 months will elapse before the release of the first detectable emission. What is the average time that one must wait to observe the first emission?

35. The average number of lightning strikes on transformers during the severe thunderstorm season in a given area is two per week. Assume that a Poisson process is in operation, and find the probability that during the next storm season one must wait at most 1 week in order to see the first transformer strike.

36. Rock noise in an underground mine occurs at an average rate of three per hour. (See Exercise 65, Chap. 3.) Find the probability that no rock noise will be recorded for at least 30 minutes.

37. California is hit every year by approximately 500 earthquakes that are large enough to be felt. However, those of destructive magnitude occur, on the average, once a year. Find the probability that at least 3 months elapse before the first earthquake of destructive magnitude occurs. (See Exercise 64, Chap. 3.)

38. Consider a chi-squared random variable with 15 degrees of freedom.
(*a*) What is the mean of X^2_{15}? What is its variance?
(*b*) What is the expression for the density for X^2_{15}?
(*c*) What is the expression for the moment generating function for X^2_{15}?
(*d*) Use Table IV of App. A to find each of the following:

$P[X^2_{15} \le 5.23]$ $\quad P[6.26 \le X^2_{15} \le 27.5]$ $\quad \chi^2_{.05}$

$P[X^2_{15} \ge 22.3]$ $\quad \chi^2_{.01}$ $\quad \chi^2_{.95}$

Section 4.4

39. Use Table V of App. A to find each of the following:
(*a*) $P[Z \le 1.57]$. (*b*) $P[Z < 1.57]$.
(*c*) $P[Z = 1.57]$. (*d*) $P[Z > 1.57]$.
(*e*) $P[-1.25 \le Z \le 1.75]$. (*f*) $z_{.10}$.
(*g*) $z_{.90}$.
(*h*) The point z such that $P[-z \le Z \le z] = .95$.
(*i*) The point z such that $P[-z \le Z \le z] = .90$.

40. The bulk density of soil is defined as the mass of dry solids per unit bulk volume. A high bulk density implies a compact soil with few pores. Bulk density is an important factor in influencing root development, seedling emergence, and aeration. Let X denote the bulk density of Pima clay loam. Studies show that X is normally distributed with $\mu = 1.5$ and $\sigma = .2$ g/cm^3.
(*a*) What is the density for X? Sketch a graph of the density function. Indicate on this graph the probability that X lies between 1.1 and 1.9. Find this probability.
(*b*) Find the probability that a randomly selected sample of Pima clay loam will have bulk density less than .9 g/cm^3.
(*c*) Would you be surprised if a randomly selected sample of this type of soil has a bulk density in excess of 2.0 g/cm^3? Explain, based on the probability of this occurring.
(*d*) What point has the property that only 10% of the soil samples have bulk density this high or higher?
(*e*) What is the moment generating function for X?

41. Most galaxies take the form of a flattened disc, with the major part of the light coming from this very thin fundamental plane. The degree of flattening differs from galaxy to galaxy. In the Milky Way Galaxy most gases are concentrated near the center of the fundamental plane. Let X denote the perpendicular distance from this center to a gaseous mass. X is normally distributed with mean

0 and standard deviation 100 parsecs. (A parsec is equal to approximately 19.2 trillion miles.)

(a) Sketch a graph of the density for X. Indicate on this graph the probability that a gaseous mass is located within 200 parsecs of the center of the fundamental plane. Find this probability.

(b) Approximately what percentage of the gaseous masses are located more than 250 parsecs from the center of the plane?

(c) What distance has the property that 20% of the gaseous masses are at least this far from the fundamental plane?

(d) What is the moment generating function for X?

42. Among diabetics, the fasting blood glucose level X may be assumed to be approximately normally distributed with mean 106 milligrams per 100 milliliters and standard deviation 8 milligrams per 100 milliliters.

(a) Sketch a graph of the density for X. Indicate on this graph the probability that a randomly selected diabetic will have a blood glucose level between 90 and 122 mg/100 ml. Find this probability.

(b) Find $P[X \leq 120 \text{ mg/100 ml}]$.

(c) Find the point that has the property that 25% of all diabetics have a fasting glucose level of this value or lower.

(d) If a randomly selected diabetic is found to have fasting blood glucose level in excess of 130, do you think there is cause for concern? Explain, based on the probability of this occurring naturally.

43. Let X denote the time in hours needed to locate and correct a problem in the software that governs the timing of traffic lights in the downtown area of a large city. Assume that X is normally distributed with mean 10 hours and variance 9.

(a) Find the probability that the next problem will require at most 15 hours to find and correct.

(b) The fastest 5% of repairs take at most how many hours to complete?

44. Assume that during seasons of normal rainfall the water level in feet at a particular lake follows a normal distribution with mean of 1876 feet and standard deviation of 6 inches.

(a) During such a season, would it be unusual to observe a water level of at most 1875 feet? Explain based on the probability of this occurring.

(b) Suppose that the water will crest the spillway if the level exceeds 1878 feet. What is the probability that this will occur during a season of normal rainfall?

45. *(Log-normal distribution.)* The log-normal distribution is the distribution of a random variable whose natural logarithm follows a normal distribution. Thus if X is a normal random variable, then $Y = e^X$ follows a log-normal distribution. Complete the argument below, thus deriving the density for a log-normal random variable.

Let X be normal with mean μ and variance σ^2. Let G denote the cumulative distribution function for $Y = e^X$, and let F denote the cumulative distribution function for X.

(*a*) Show that $G(y) = F(\ln y)$.
(*b*) Show that $G'(y) = F'(\ln y)/y$.
(*c*) Show that the density for Y is given by

$$g(y) = \frac{1}{\sqrt{2\pi}\sigma y} \exp\left[-\frac{1}{2}\frac{(\ln y - \mu)^2}{\sigma^2}\right] \qquad \begin{array}{c} -\infty < \mu < \infty \\ \sigma > 0 \\ y > 0 \end{array}$$

Note that μ and σ are the mean and standard deviation of the underlying normal distribution; they are not the mean and standard deviation of Y itself.

46. Let Y denote the diameter in millimeters of Styrofoam pellets used in packing. Assume that Y has a log-normal distribution with parameters $\mu = .8$ and $\sigma = .1$.
(*a*) Find the probability that a randomly selected pellet has a diameter that exceeds 2.7 millimeters.
(*b*) Between what two values will Y fall with probability approximately .95?

Section 4.5

47. Verify the normal probability rule.

48. The number of Btu's of petroleum and petroleum products used per person in the United States in 1975 was normally distributed with mean 153 million Btu's and standard deviation 25 million Btu's. Approximately what percentage of the population used between 128 and 178 million Btu's during that year? Approximately what percentage of the population used in excess of 228 million Btu's?

49. Reconsider Exercises 40(*a*), 41(*a*), and 42(*a*) in light of the normal probability rule.

50. For a normal random variable, $P[|X - \mu| < 3\sigma] \doteq .997$. What value is assigned to this probability via Chebyshev's inequality? Are the results consistent? Which rule gives a stronger statement in the case of a normal variable?

51. Animals have an excellent spatial memory. In an experiment to confirm this statement, an eight-armed maze such as that shown in Fig. 4.21 is used. At the beginning of a test, one pellet of food is placed at the end of each arm. A hungry animal is placed at the center of the maze and is allowed to choose freely from among the arms. The optimal strategy is to run to the end of each arm exactly once. This requires that the animal remember where it has been. Let X denote the number of correct arms (arms still containing food) selected among its first eight choices. Studies indicate that $\mu = 7.9$.
(*a*) Is X normally distributed?
(*b*) State and interpret Chebyshev's inequality in the context of this problem for $k = .5$, 1, 2, and 3. At what point does the inequality begin to give us some practical information?

Section 4.6

52. Let X be binomial with $n = 20$ and $p = .3$. Use the normal approximation to approximate each of the following. Compare your results with the values obtained from Table I of App. A.

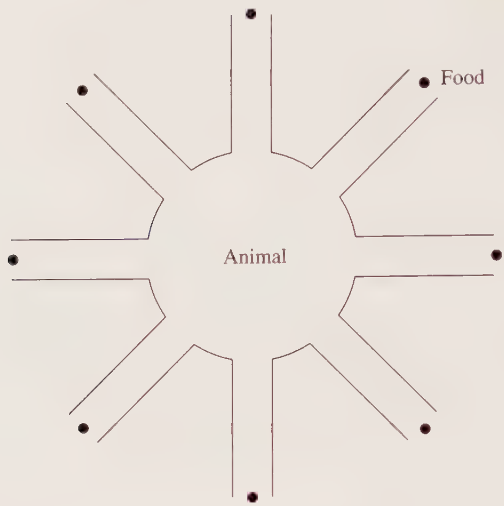


FIGURE 4.21
An eight-armed maze.

(*a*) $P[X \leq 3]$.
(*b*) $P[3 \leq X \leq 6]$.
(*c*) $P[X \geq 4]$.
(*d*) $P[X = 4]$.

53. Although errors are likely when taking measurements from photographic images, these errors are often very small. For sharp images with negligible distortion, errors in measuring distances are often no larger than .0004 inch. Assume that the probability of a serious measurement error is .05. A series of 150 independent measurements are made. Let X denote the number of serious errors made.
(*a*) In finding the probability of making at least one serious error, is the normal approximation appropriate? If so, approximate the probability using this method.
(*b*) Approximate the probability that at most three serious errors will be made.

54. A chemical reaction is run in which the usual yield is 70%. A new process has been devised that should improve the yield. Proponents of the new process claim that it produces better yields than the old process more than 90% of the time. The new process is tested 60 times. Let X denote the number of trials in which the yield exceeds 70%.
(*a*) If the probability of an increased yield is .9, is the normal approximation appropriate?
(*b*) If $p = .9$, what is $E[X]$?
(*c*) If $p > .9$ as claimed, then, on the average, more than 54 of every 60 trials will result in an increased yield. Let us agree to accept the claim if X is at

least 59. What is the probability that we will accept the claim if p is really only .9?

(*d*) What is the probability that we shall not accept the claim ($X \leq 58$) if it is true, and p is really .95?

55. Opponents of a nuclear power project claim that the majority of those living near a proposed site are opposed to the project. To justify this statement, a random sample of 75 residents is selected and their opinions are sought. Let X denote the number opposed to the project.

(*a*) If the probability that an individual is opposed to the project is .5, is the normal approximation appropriate?

(*b*) If $p = .5$, what is $E[X]$?

(*c*) If $p > .5$ as claimed, then, on the average, more than 37.5 of every 75 individuals are opposed to the project. Let us agree to accept the claim if X is at least 46. What is the probability that we shall accept the claim if p is really only .5?

(*d*) What is the probability that we shall not accept the claim ($X \leq 45$) even though it is true and p is really .7?

56. (*Normal approximation to the Poisson distribution.*) Let X be Poisson with parameter λs. Then for large values of λs, X is approximately normal with mean λs and variance λs. (The proof of this theorem is also based on the Central Limit Theorem and will be considered in Chap. 7.) Let X be a Poisson random variable with parameter $\lambda s = 15$. Find $P[X \leq 12]$ from Table II of App. A. Approximate this probability using a normal curve. Be sure to employ the half-unit correction factor.

57. The average number of jets either arriving at or departing from O'Hare Airport is one every 40 seconds. What is the approximate probability that at least 75 such flights will occur during a randomly selected hour? What is the probability that fewer than 100 such flights will take place in an hour?

Section 4.7

58. The length of time in hours that a rechargeable calculator battery will hold a charge is a random variable. Assume that this variable has a Weibull distribution with $\alpha = .01$ and $\beta = 2$.

(*a*) What is the density for X?

(*b*) What are the mean and variance for X? *Hint:* It can be shown that $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ for any $\alpha > 1$. Furthermore, $\Gamma(1/2) = \sqrt{\pi}$.

(*c*) What is the reliability function for this random variable?

(*d*) What is the reliability of such a battery at $t = 3$ hours? At $t = 12$ hours? At $t = 20$ hours?

(*e*) What is the hazard rate function for these batteries?

(*f*) What is the failure rate at $t = 3$ hours? At $t = 12$ hours? At $t = 20$ hours?

(*g*) Is the hazard rate function an increasing or a decreasing function? Does this seem to be reasonable from a practical point of view? Explain.

59. Computer chips do not "wear out" in the ordinary sense. Assuming that defective chips have been removed from the market by factory inspection, it is reasonable to assume that these chips exhibit a constant hazard rate. Let the hazard rate be given by $\rho(t) = .02$. (Time is in years.)

(*a*) In a practical sense, what are the main causes of failure of these chips?
(*b*) What is the reliability function for chips of this type?
(*c*) What is the reliability of a chip 20 years after it has been put into use?
(*d*) What is the failure density for these chips?
(*e*) What type of random variable is X, the time to failure of a chip?
(*f*) What is the mean and variance for X?
(*g*) What is the probability that a chip will be operable for at least 30 years?

60. The random variable X, the time to failure (in thousands of miles driven) of the signal lights on an automobile has a Weibull distribution with $\alpha = .04$ and $\beta = 2$.
(*a*) Find the density, mean, and variance for X.
(*b*) Find the reliability function for X.
(*c*) What is the reliability of these lights at 5000 miles? At 10,000 miles?
(*d*) What is the hazard rate function?
(*e*) What is the hazard rate at 5000 miles? At 10,000 miles?
(*f*) What is the probability that the lights will fail during the first 3000 miles driven?

61. Show that for $\alpha > 0$ and $\beta > 0$,

$$\int_0^\infty \alpha\beta x^{\beta-1} e^{-\alpha x^\beta}\, dx = 1$$

thereby showing that the nonnegative function given in Definition 4.7.1 is a density for a continuous random variable. *Hint:* Let $z = \alpha x^\beta$.

62. Let X be a Weibull random variable with parameters α and β. Show that $E[X^2] = \alpha^{-2/\beta}\Gamma(1 + 2/\beta)$. *Hint:* In evaluating

$$\int_0^\infty x^2 \alpha\beta x^{\beta-1} e^{-\alpha x^\beta}\, dx$$

let $z = \alpha x^\beta$. Evaluate the integral in a manner similar to that used in the proof of Theorem 4.7.1.

63. Use the result of Exercise 62 to find Var X for a Weibull random variable with parameters α and β, thus completing the proof of Theorem 4.7.1.

64. Consider the hazard rate function

$$\rho(t) = \alpha\beta t^{\beta-1} \qquad \begin{aligned} t &> 0 \\ \alpha &> 0 \\ \beta &> 0 \end{aligned}$$

(*a*) Show that $\rho(t)$ is constant if $\beta = 1$.
(*b*) Find $\rho'(t)$. Argue that $\rho'(t) > 0$ if $\beta > 1$, thus producing an increasing hazard rate. Argue that $\rho'(t) < 0$ if $\beta < 1$, thus producing a decreasing hazard rate.

65. A system has eight components connected as shown in Fig. 4.22.

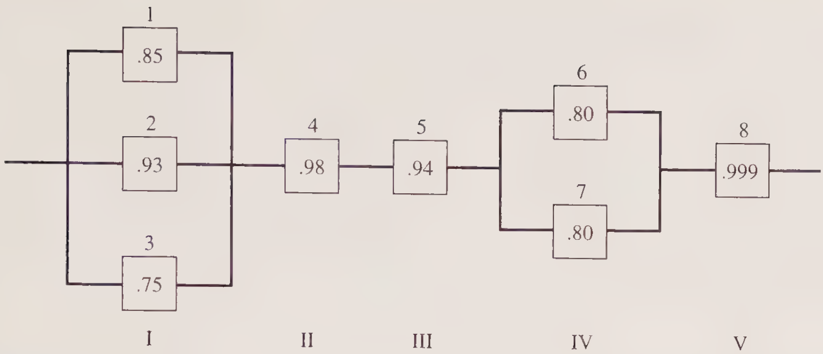


FIGURE 4.22

(*a*) Find the reliability of each of the parallel assemblies.
(*b*) Find the system reliability.
(*c*) Suppose that assembly II is replaced by two identical components in parallel, each with reliability .98. What is the reliability of the new assembly?
(*d*) What is the new system reliability after making the change suggested in part (*c*)?
(*e*) Make changes analogous to that of part (*c*) in each of the remaining single component assemblies. Compute the new system reliability.

66. A system consists of two independent components connected in series. The life span of the first component follows a Weibull distribution with $\alpha = .006$ and $\beta = .5$; the second has a life span that follows the exponential distribution with $\beta = .00004$.
(*a*) Find the reliability of the system at 2500 hours.
(*b*) Find the probability that the system will fail before 2000 hours.
(*c*) If the two components are connected in parallel, what is the system reliability at 2500 hours?

67. Suppose that a missile can have several independent and identical computers, each with reliability .9 connected in parallel so that the system will continue to function as long as at least one computer is operating. If it is desired to have a system reliability of at least .999, how many computers should be connected in parallel?

68. Three independent and identical components, each with a reliability of .9, are to be used in an assembly.
(*a*) The assembly will function if at least one of the components is operable. Find the system reliability.
(*b*) The assembly will function if at least two of the components are operable. Find the reliability of the system.
(*c*) The assembly will function only if all three of the components are operable. Find the reliability of the system.

Section 4.8

69. Prove Theorem 4.8.1 in the case in which g is strictly increasing.

70. Let X be a random variable with density

$$f_X(x) = (1/4)x \qquad 0 \leq x \leq \sqrt{8}$$

and let $Y = X + 3$.

(*a*) Find $E[X]$, and then use the rules for expectation to find $E[Y]$.

(*b*) Find the density for Y.

(c) Use the density for Y to find $E[Y]$, and compare your answer to that found in part (*a*).

71. Let X be a random variable with density

$$f_X(x) = (1/4)xe^{-x/2} \qquad x \geq 0$$

and let $Y = (-1/2)X + 2$. Find the density for Y.

72. Let X be a random variable with density

$$f_X(x) = e^{-x} \qquad x > 0$$

and let $Y = e^X$. Find the density for Y.

73. Let C denote the temperature in degrees Celsius to which a computer will be subjected in the field. Assume that C is uniformly distributed over the interval $(15, 21)$. Let F denote the field temperature in degrees Fahrenheit so that $F = (9/5)C + 32$. Find the density for F.

74. Let X denote the velocity of a random gas molecule. According to the Maxwell-Boltzmann law, the density for X is given by

$$f_X(x) = cx^2e^{-\beta x^2} \qquad x > 0$$

Here c is a constant that depends on the gas involved, and β is a constant whose value depends on the mass of the molecule and its absolute temperature. The kinetic energy of the molecule, Y, is given by $Y = (1/2)mX^2$ where $m > 0$. Find the density for Y.

75. Let X be a continuous random variable with density f_X, and let $Y = X^2$.

(*a*) Show that for $y \geq 0$,

$$F_Y(y) = P\left[-\sqrt{y} \leq X \leq \sqrt{y}\right]$$

(*b*) Show that for $y \geq 0$,

$$F_Y(y) = F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right)$$

(*c*) Use the technique given in the proof of Theorem 4.8.1 to show that

$$f_Y(y) = 1/\left(2\sqrt{y}\right)\left[f_X\left(\sqrt{y}\right) + f_X\left(-\sqrt{y}\right)\right]$$

(*d*) Use the technique illustrated in Example 4.8.2 to show that

$$f_Y(y) = 1/\left(2\sqrt{y}\right)\left[f_X\left(\sqrt{y}\right) + f_X\left(-\sqrt{y}\right)\right]$$

76. Let Z be a standard normal random variable and let $Y = Z^2$.

(*a*) Show that $\Gamma(1/2) = \int_0^\infty x^{-1/2}e^{-x}\,dx$.

(*b*) Show that $\Gamma(1/2) = \sqrt{\pi}$. *Hint:* Use the results of part (*a*) with $x = t^2/2$ and make use of the fact that the standard normal density integrates to 1 when integrated over the set of real numbers.
(*c*) Use the results of Exercise 75 to find f_Y.
(*d*) Argue that Y follows a chi-squared distribution with 1 degree of freedom.

77. Let X be normally distributed with mean μ and variance σ^2. Let $Y = e^X$. Show that Y follows the log-normal distribution. (See Exercise 45.)

78. Let Z be a standard normal random variable and let $Y = 2Z^2 - 1$. Find the density for Y.

Section 4.9

79. Use Table III of App. A to generate nine more observations on the random variable X, the time to failure of a computer chip. (See Example 4.9.1.) Based on these data, approximate the average time to failure by finding the arithmetic average of the values of X simulated in the experiment. Does this value agree well with the theoretical mean value of 50 years?

80. Simulate 20 observations on the random variable X, the time to failure of the signal lights on an automobile. (See Exercise 60.) Approximate the average time to failure for these lights based on the simulated data. Does this value agree well with the theoretical mean value for X?

81. A satellite has malfunctioned and is expected to reenter the earth's atmosphere sometime during a 4-hour period. Let X denote the time of reentry. Assume that X is uniformly distributed over the interval $[0, 4]$. Simulate 20 observations on X. (See Exercise 18.)

REVIEW EXERCISES

82. Let X be a continuous random variable with density

$$f(x) = cx^2 \qquad -3 \le x \le 3$$

(*a*) Assuming that $f(x) = 0$ elsewhere, find the value of c that makes this a density.
(*b*) Find $E[X]$ and $E[X^2]$ from the definitions of these terms.
(*c*) Find Var X and σ.
(*d*) Find $P[X \le 2]$; $P[-1 \le X \le 2]$; $P[X > 1]$ by direct integration.
(*e*) Find the closed-form expression for the cumulative distribution function F.
(*f*) Use F to find each of the probabilities of part (*d*), and compare your answers to those obtained earlier.

83. Find $\int_0^\infty z^{10}e^{-z}\,dz$.

84. A computer firm introduces a new home computer. Past experience shows that the random variable X, the time of peak demand measured in months after its introduction, follows a gamma distribution with variance 36.
(*a*) If the expected value of X is 18 months, find α and β.
(*b*) Find $P[X \le 7.01]$; $P[X \ge 26]$; $P[13.7 \le X \le 31.5]$.

85. Let X denote the lag time in a printing queue at a particular computer center. That is, X denotes the difference between the time that a program is placed in the queue and the time at which printing begins. Assume that X is normally distributed with mean 15 minutes and variance 25.

(*a*) Find the expression for the density for X.

(*b*) Find the probability that a program will reach the printer within 3 minutes of arriving in the queue.

(*c*) Would it be unusual for a program to stay in the queue between 10 and 20 minutes? Explain, based on the approximate probability of this occurring. You do not have to use the Z table to answer this question!

(*d*) Would you be surprised if it took longer than 30 minutes for the program to reach the printer? Explain, based on the probability of this occurring.

86. A computer center maintains a telephone consulting service to troubleshoot for its users. The service is available from 9 a.m. to 5 p.m. each working day. Past experience shows that the random variable X, the number of calls received per day, follows a Poisson distribution with $\lambda = 50$. For a given day, find the probability that the first call of the day will be received by 9:15 a.m.; after 3 p.m.; between 9:30 a.m. and 10 a.m.

87. Let $H(X) = X^2 + 3X + 2$. Find $E[H(X)]$ if

(*a*) X is normally distributed with mean 3 and variance 4.

(*b*) X has a gamma distribution with $\alpha = 2$ and $\beta = 4$.

(*c*) X has a chi-squared distribution with 10 degrees of freedom.

(*d*) X has an exponential distribution with $\beta = 5$.

(*e*) X has a Weibull distribution with $\alpha = 2$ and $\beta = 1$.

88. Let X denote the time required to upgrade a computer system in hours. Assume that the density for X is given by

$$f(x) = k \exp(-2x) \qquad 0 < x < \infty$$

(*a*) Find the numerical value of k that makes this a valid density.

(*b*) Find the probability that it will take at most 1 hour to upgrade a given system.

(*c*) Find the average time required to upgrade a system.

(*d*) Find the standard deviation in the time required for the upgrade.

89. Let X denote the time to failure in years of a telephone modem used to access a mainframe computer from a remote terminal. Assume that the hazard rate function for X is given by

$$\rho(t) = \alpha\beta t^{\beta-1}$$

where $\alpha = 2$ and $\beta = 1/5$.

(*a*) Find the failure density for X.

(*b*) Find the expected value of X.

(*c*) Find the reliability function for X.

(*d*) Find the probability that the modem will last for at least 2 years.

(*e*) What is the hazard rate at $t = 1$ year?

(*f*) Describe roughly the theoretical pattern in the causes of failure in these modems.

90. Past evidence shows that when a customer complains of an out-of-order phone there is an 8% chance that the problem is with the inside wiring. During a 1-month period, 100 complaints are lodged. Assume that there have been no wide-scale problems that could be expected to affect many phones at once, and that, for this reason, these failures are considered to be independent. Find the expected number of failures due to a problem with the inside wiring. Find the probability that at least 10 failures are due to a problem with the inside wiring. Would it be unusual if at most 5 were due to problems with the inside wiring? Explain, based on the probability of this occurring.

91. The cumulative distribution function for a continuous random variable X is defined by

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x^3 + x^2}{2} & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

Find the density for X.

92. The density for a continuous random variable is given by

$$f(x) = xe^{-x} \qquad 0 < x < \infty$$

(*a*) Show that $\int_0^\infty xe^{-x}\,dx = 1$. *Hint:* Use the gamma function.
(*b*) Find $E[X]$, $E[X^2]$, and Var X.
(*c*) Show that $m_X(t) = 1/(1-t)^2$, where $t < 1$.
(*d*) Use $m_X(t)$ to find $E[X]$.

93. An electronic counter records the number of vehicles exiting the interstate at a particular point. Assume that the average number of vehicles leaving in a 5-minute period is 10. Approximate the probability that between 100 and 120 vehicles inclusive will exit at this point in a 1-hour period.

94. Consider the following moment generating functions. In each case, identify the distribution involved completely. Be sure to specify the numerical value of all parameters that identify the distribution. For example, if X is normal, give the numerical value of μ and σ^2; if gamma, state α and β.
(*a*) $e^{3t+16t^2/2}$
(*b*) $(1-3t)^{-7}$
(*c*) $(1-2t)^{-12}$
(*d*) $\dfrac{e^{3t}-e^t}{2t}$
(*e*) $e^{t^2/2}$
(*f*) $(1-7t)^{-1}$
(*g*) $e^{3t+t^2/2}$

95. For each random variable in Exercise 94, state the numerical value of the average for X and its variance.