CHAPTER 1

INTRODUCTION TO PROBABILITY AND COUNTING

What is "statistics," and why is its study important to engineers and scientists? To answer this question, let us describe an aspect of the work of a scientist known as "model building."

Basically, the job of a scientist is to describe what he or she sees, to try to explain what is observed, and to use this knowledge to predict events in the world in which we live. The explanation often takes the form of a physical model. A *model* is a theoretical explanation of the phenomenon under study and, at the outset, is usually expressed verbally. To use the model for predictive purposes, this verbal description must be translated into one or more mathematical equations. These equations can be used to determine the value of a specific variable in the model based on the knowledge of the values assumed by other model variables. For example, the Perfect Gas Law states that the pressure and volume of a gas may both vary simultaneously when the temperature of the gas is changed. This verbal model can be translated into a mathematical equation by writing

$$\text{Perfect Gas Law: } PV = RT$$

where P is the pressure of the gas, V is its volume, T is its temperature, and R is a constant, called the *gas constant*. The numerical value of the gas constant depends on the physical units chosen for the other terms in the model. Once we know the values assumed by two of the three variables P, V, or T, we can calculate the value of the third via this mathematical model. For example, under a pressure of 760 mm mercury and a temperature of 273 kelvins, a mole of any gas is thought to have a volume of 22.4 liters. The gas constant in this case has a value of approximately 62.36. Based on the Perfect Gas Law, a gas with a volume of 5 liters at a temperature of 100 kelvins has pressure P given by

$$PV = RT = 62.36T$$

or

$$P(5) = 62.36(100)$$
$$P = 1247.2 \text{ mm mercury}$$

That is, our model leads us to expect the pressure to be 1247.2 mm mercury. A model such as the Perfect Gas Law is said to be "deterministic." It is deterministic in the sense that it allows us to determine an exact value for the variable of interest under specified experimental conditions. The Perfect Gas Law does describe *some* real gases at moderate temperatures and pressures. Unfortunately, many real gases cannot be described by this or any other deterministic model, especially at extreme temperatures and pressures! Under these circumstances we must find another way to predict the behavior of the gas with some degree of certainty. This can be done with the aid of statistical methods.

What do we mean by statistical methods? These are methods by which decisions are made based on the analysis of data gathered in carefully designed experiments. Since experiments cannot be designed to account for every conceivable contingency, there is always some uncertainty in experimental science. Statistical methods are designed to *allow us to assess the degree of uncertainty present in our results*. These methods can be classed roughly into three categories: descriptive statistics, inferential statistics, and model building. By descriptive statistics we mean those techniques, both analytic and graphical, that allow us to describe or picture a data set. Inferential statistics concerns methods by which conclusions can be drawn about a large group of objects, based on observing only a portion of the objects in the larger group.

This idea leads to the following definition:

Definition: The overall group of objects about which conclusions are to be drawn is called the *population*. A subset or portion of the population that is actually obtained and that is used to draw conclusions about the population is called a *sample*.

Model building entails the development of prediction equations from experimental data. These equations are called statistical models; they are models that allow us to predict the behavior of a complex system and to assess our probability of error. These categories are not mutually exclusive. That is, methods developed to solve problems in one area often find application in another. We shall be concerned with all three areas in this text.

A statistician or user of statistics is always working in two worlds. The ideal world is at the population level and is theoretical in nature. It is the world that we would like to see. The world of reality is the sample world. This is the level at which we really operate. We hope that the characteristics of our sample reflect well the characteristics of the population. That is, we treat our sample as a microcosm that mirrors the population. This idea is illustrated in Fig. 1.1.

The mathematics on which statistical methods rest is called probability theory. For this reason, we begin the study of statistics by considering the basic concepts of probability.

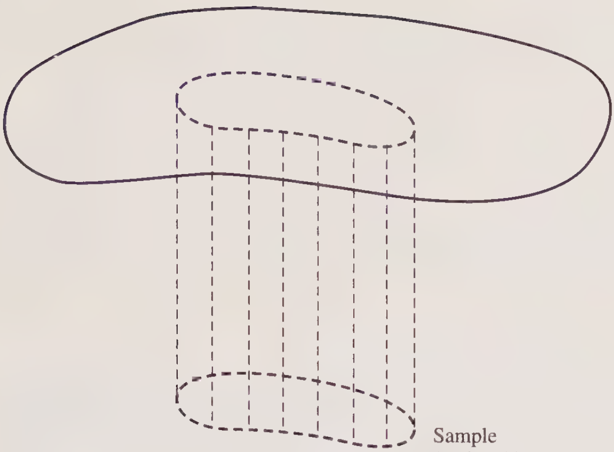

FIGURE 1.1
The sample is viewed as a miniature population. We hope that the behavior of the variable under study over the sample gives an accurate picture of its behavior in the population.

1.1 INTERPRETING PROBABILITIES

When asked, "Do you know anything about probability?" most people are quick to answer, "no!" Usually that is not the case at all. The ability to interpret probabilities is assumed in our culture. One hears the phrases "the probability of rain today is 95%" or "there is a 0% chance of rain today." It is assumed that the general public can interpret these values correctly. The interpretation of probabilities is summarized as follows:

Interpretation of Probabilities

1. Probabilities are numbers between 0 and 1, inclusive, that reflect the chances of a physical event occurring.
2. Probabilities near 1 indicate that the event is extremely likely to occur. They mean not that the event will occur, only that the event is considered to be a common occurrence.
3. Probabilities near zero indicate that the event is not very likely to occur. They do not mean that the event will fail to occur, only that the event is considered to be rare.
4. Probabilities near 1/2 indicate that the event is just as likely to occur as not.
5. Since numbers between 0 and 1 can be expressed as percentages between 0 and 100, probabilities are often expressed as percentages. This is particularly common in writings of a nontechnical nature.

These properties are guidelines for interpreting probabilities once they are available, but they do not indicate how to assign probabilities to events. Three methods are widely used: the *personal* approach, the *relative frequency* approach, and the *classical approach*. These methods are illustrated in the following examples.

Example 1.1.1. An oil spill has occurred. An environmental scientist asks, "What is the probability that this spill can be contained before it causes widespread damage to nearby beaches?" Many factors come into play, among them the type of spill, the amount of oil spilled, the wind and water conditions during the clean-up operation, and the nearness of the beaches. These factors make this spill unique. The scientist is called upon to make a value judgment, that is, to assign a probability to the event based on informed *personal opinion*.

The main advantage of the personal approach is that it is always applicable. Anyone can have a personal opinion about anything. Its main disadvantage is, of course, that its accuracy depends on the accuracy of the information available and the ability of the scientist to assess that information correctly.

Example 1.1.2. An electrical engineer is studying the peak demand at a power plant. It is observed that on 80 of the 100 days randomly selected for study from past records, the peak demand occurred between 6 and 7 p.m. It is natural to assume that the probability of this occurring on another day is at least *approximately*

$$\frac{80}{100} = .80$$

This figure is not simply a personal opinion. It is a figure based on repeated experimentation and observation. It is a *relative frequency*.

The relative frequency approach can be used whenever the experiment can be repeated many times and the results observed. In such cases, the probability of the occurrence of event A, denoted by $P[A]$, is approximated as follows:

Relative Frequency Approximation

$$P[A] \doteq \frac{f}{n} = \frac{\text{number of times event } A \text{ occurred}}{\text{number of times experiment was run}}$$

The disadvantage in this approach is that the experiment cannot be a one-shot situation; it must be repeatable. Remember that any probability obtained this way is an approximation. It is a value based on n trials. Further testing might result in a different approximate value. However, as the number of trials increases, the changes in the approximate values obtained tend to become slight. Thus for a large number of trials, the approximate probability obtained by using the relative frequency approach is usually quite accurate.

Example 1.1.3. What is the probability that a child born to a couple heterozygous for eye color (each with genes for both brown and blue eyes) will be brown-eyed? To answer this question, we note that since the child receives one gene from each parent, the possibilities for the child are (brown, blue), (blue, brown), (blue, blue) and (brown, brown), where the first member of each pair represents the gene received from the father. Since each parent is just as likely to contribute a gene for brown eyes as for blue eyes, all four possibilities are equally likely. Since the gene for brown eyes is dominant, three of the four possibilities lead to a brown-eyed child. Hence the probability that the child will be brown-eyed is $3/4 = .75$.

The above probability is not a personal opinion, nor is it based on repeated experimentation. In fact, we found this probability by the *classical* method. This method can be used *only* when it is reasonable to assume that the possible outcomes of the experiment are equally likely. In this case, the probability of the occurrence of event A is given by the following classical formula:

Classical Formula

$$P[A] = \frac{n(A)}{n(S)} = \frac{\text{number of ways } A \text{ can occur}}{\text{number of ways the experiment can proceed}}$$

One advantage to this method is that it does not require experimentation. Furthermore, if the outcomes are truly equally likely, then the probability assigned to event A is not an approximation. It is an accurate description of the frequency with which event A will occur.

1.2 SAMPLE SPACES AND EVENTS

To determine what is "probable" in an experiment, we first must determine what is "possible." That is, the first step in analyzing most experiments is to make a list of possibilities for the experiment. Such a list is called a *sample space*. We define this term as follows:

Definition 1.2.1 (Sample space and sample point). A sample space for an experiment is a set S with the property that each physical outcome of the experiment corresponds to exactly one element of S. An element of S is called a sample point.

When the number of possibilities is small, an appropriate sample space usually can be found without difficulty. For instance, we have seen that when a couple heterozygous for eye color parents a child, the possible genotypes for the child are given by

$$S = \{(\text{brown, blue}), (\text{blue, brown}), (\text{blue, blue}), (\text{brown, brown})\}$$

As the number of possibilities becomes larger, it is helpful to have a system for developing a sample space. One such system is the *tree diagram*. The next example illustrates the idea.

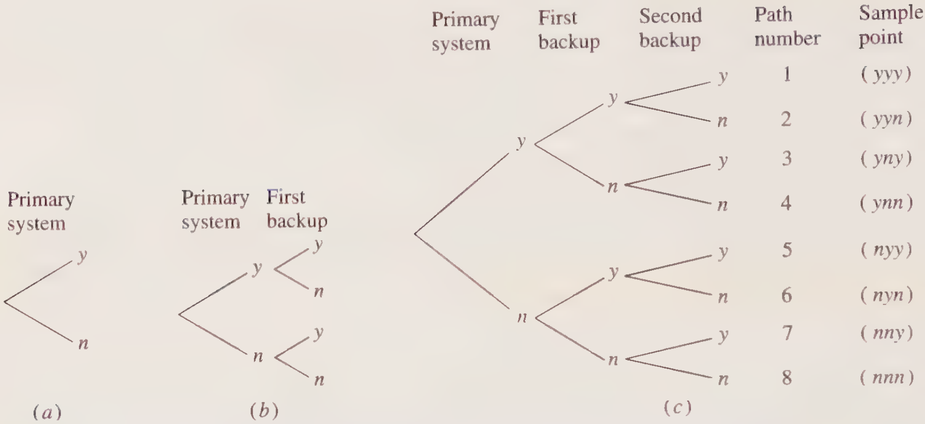

FIGURE 1.2
Constructing a tree diagram.

Example 1.2.1. During a space shot the primary computer system is backed up by two secondary systems. They operate independently of one another in that the failure of one has no effect on any of the others. We are interested in the readiness of these three systems at launch time. What is an appropriate sample space for this experiment?

Since we are primarily concerned with whether each system is operable at launch, we need only find a sample space that gives that information. To generate the sample space, we use a *tree*. The primary system is either operable (yes) or not operable (no) at the time of launch. This is indicated in the tree diagram of Fig. 1.2(a), where yes $= y$ and no $= n$. Likewise the first backup system either is or is not operable. This is shown in Fig. 1.2(b). Finally, the second backup system either is or is not operable. The tree is completed as shown in Fig. 1.2(c). A sample space S for the experiment can be read from the tree by following each of the eight distinct paths through the tree. Thus

$$S = \{yyy, yyn, yny, ynn, nyy, nyn, nny, nnn\}$$

Once a suitable sample space has been found, elementary set theory can be used to describe physical occurrences associated with the experiment. This is done by considering what are called *events* in the mathematical sense.

Definition 1.2.2 (Event). Any subset A of a sample space is called an event. The empty set $\varnothing$ is called the *impossible* event; the subset S is called the *certain* event.

Example 1.2.2. Consider a space shot in which a primary computer system is backed up by two secondary systems. The sample space for this experiment is

$$S = \{yyy, yyn, yny, ynn, nyy, nyn, nny, nnn\}$$

where, for example, *yny* denotes the fact that the primary system and second backup are operable at launch, whereas the first backup is inoperable (see Example 1.2.1). Let

A: primary system is operable

B: first backup is operable

C: second backup is operable

The mathematical event corresponding to each of these physical events is found by listing the sample points that represent the occurrence of the event. Thus we write

$$A = \{yyy, yyn, yny, ynn\}$$
$$B = \{yyy, yyn, nyy, nyn\}$$
$$C = \{yyy, yny, nyy, nny\}$$

Other events can be described using these events as building blocks. For example, the event that "the primary system *or* the first backup is operable" is given by the set $A \cup B$, the union of set A with set B. Recall from elementary mathematics that the *union of A with B consists of all sample points that are in set A or set B or are in both.* Thus

$$A \cup B = \begin{array}{c}\text{primary } \textit{or} \text{ first}\\ \text{backup is operable}\end{array} = \{yyy, yyn, yny, ynn, nyy, nyn\}$$

Note that the word "or" will denote set union. The event that "the primary system *and* the first backup is operable" is given by the set $A \cap B$, the intersection of set A with set B. *The intersection of two sets consists of all sample points that are in both sets.* That is, it is the set of points that they have in common. Here

$$A \cap B = \text{primary } \textit{and} \text{ first backup operable} = \{yyy, yyn\}$$

Note that the word "and" will denote the set intersection. The event that "the primary system or the first backup is operable but the second backup is inoperable" is given by $(A \cap B) \cap C'$, where C' denotes the complement of set C. *The complement of a set consists of the sample points in the sample space that are not in the given set.* Thus

$$(A \cup B) \cap C' = \begin{array}{c}\text{primary } \textit{or} \text{ first backup operable}\\ \textit{but} \text{ second backup inoperable}\end{array} = \{yyn, ynn, nyn\}$$

Note that the word "but" is also translated as a set intersection; the word "not" translates as a set complement.

Let us pause briefly to consider a basic difference between the sample space

$$S_1 = \{(\text{brown, blue}), (\text{blue, brown}), (\text{blue, blue}), (\text{brown, brown})\}$$

of Example 1.1.3 and

$$S_2 = \{yyy, yyn, yny, ynn, nyy, nyn, nny, nnn\}$$

of Example 1.2.1. Since each parent is just as likely to contribute a gene for brown eyes as for blue eyes, the sample points of S_1 are equally likely. This allows us to use the classical method to find the probability that a child born to a couple heterozygous for eye color will be brown-eyed. If we denote this event by A, then we can conclude that

$$\begin{aligned} P[A] &= P[\{(\text{brown, blue}), (\text{blue, brown}), (\text{brown, brown})\}] \\ &= \frac{n(A)}{n(S)} = \frac{3}{4} \end{aligned}$$

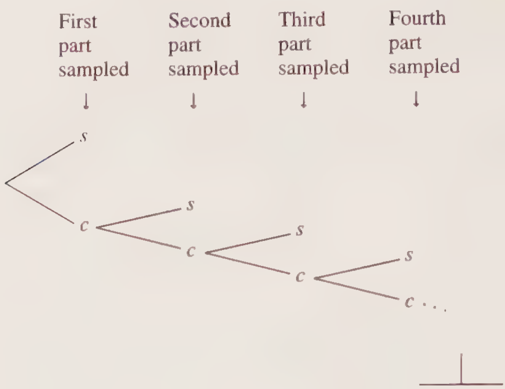

FIGURE 1.3
Sampling a production line for defective parts.

However, it is not correct to assume that the sample points of S_2 are equally likely. This would be true if and only if each of the three computer systems is just as likely to fail as to be operable at launch time. Our technology is much better than that! The primary question to be answered is "What is the probability that at least one system will be operable at the time of the launch?" That is, what is

$$P[\{yyy, yyn, yny, ynn, nyy, nyn, nny\}]?$$

As will be shown later, this question can be answered. However, since the sample points are not equally likely, it cannot be answered using the classical method.

Not all trees are symmetric as is that pictured in Fig. 1.2. In some settings, paths end at different stages of the game. Example 1.2.3 illustrates an experiment of this sort.

Example 1.2.3. Consider a production process that is known to produce defective parts at the rate of one per hundred. The process is monitored by testing randomly selected parts during the production process. Suppose that as soon as a defective part is found, the process will be stopped and all machine settings will be checked. We are interested in studying the number of parts that are tested in order to obtain the first defective part. In the tree of Fig. 1.3, c represents that the sampling continues and s represents that production is stopped. Notice that as soon as a defective item is found, the process ends and the path also ends. For this reason, some paths are much shorter than others. Notice also that theoretically this tree continues indefinitely. The sample space generated by the tree is

$$S = \{s, cs, ccs, cccs, ccccs, \ldots\}$$

Since defective parts occur with probability .01, it should be evident that the paths of this tree are not equally likely.

Mutually Exclusive Events

Occasionally interest centers on two or more events that cannot occur at the same time. That is, the occurrence of one event precludes the occurrence of the other. Such events are said to be *mutually exclusive*.

Example 1.2.4. Consider the sample space

$$S = \{yyy, yyn, yny, ynn, nyy, nyn, nny, nnn\}$$

of Example 1.2.1. The events

$$A_1\text{: primary system operable} = \{yyy, yyn, yny, ynn\}$$
$$A_2\text{: primary system inoperable} = \{nyy, nyn, nny, nnn\}$$

are mutually exclusive. It is impossible for the primary system to be both operable and inoperable at the same time. Mathematically, A_1 and A_2 have no sample points in common. That is, $A_1 \cap A_2 = \varnothing$.

Example 1.2.4 suggests the mathematical definition of the term "mutually exclusive events."

Definition 1.2.3 (Mutually exclusive events). Two events A_1 and A_2 are mutually exclusive if and only if $A_1 \cap A_2 = \varnothing$. Events $A_1, A_2, A_3, \ldots$ are mutually exclusive if and only if $A_i \cup A_j = \varnothing$ for $i \neq j$.

1.3 PERMUTATIONS AND COMBINATIONS

As indicated in Sec. 1.1, there are several ways to determine the probability of an event. When the physical description of the experiment leads us to believe that the possible outcomes are equally likely, then we can compute the probability of the occurrence of an event using the classical method. In this case the probability of an event A is given by

$$P[A] = \frac{n(A)}{n(S)}$$

Thus to compute a probability using the classical approach, you must be able to count two things: $n(A)$, the number of ways in which event A can occur, and $n(S)$, the number of ways in which the experiment can proceed. As the experiment becomes more complex, lists and trees become cumbersome. Alternative methods for counting must be found.

Two types of counting problems are common. The first involves *permutations* and the second, *combinations*. These terms are defined as follows:

Definition 1.3.1 (Permutation). A permutation is an arrangement of objects in a definite order.

Definition 1.3.2 (Combination). A combination is a selection of objects without regard to order.

Note that the characteristic that distinguishes a permutation from a combination is *order*. If the order in which some action is taken is important, then the problem is a permutation problem and can be solved using a technique called the multiplication principle. If order is irrelevant, then it is a combination problem and can be solved using a formula that we shall develop.

Example 1.3.1

1. Twenty different amino acids are commonly found in peptides and proteins. A pentapeptide consisting of the five amino acids

   alanine-valine-glycine-cysteine-tryptophan

   has different properties and is, in fact, a different compound from the pentapeptide

   alanine-glycine-valine-cysteine-tryptophan

   which contains the same amino acids. Peptides are permutations of amino acid units because the sequence, or order, of the amino acids in the chain is important.
2. A foundry ships engine blocks in lots of size 20. Before a lot is accepted, three blocks are selected at random and tested for hardness. Only three are tested because the testing requires that the blocks be cut in half, and is therefore destructive. The three blocks selected constitute a combination of engine blocks. We are interested only in which three are selected; we are not interested in the order in which they are chosen.

Counting Permutations

Once a problem has been identified as being one in which order is important, the next question to be answered is, "How many permutations or arrangements of the given objects are possible?" This question usually can be answered by means of the *multiplication principle*.

Multiplication principle. Consider an experiment taking place in k stages. Let n_i denote the number of ways in which stage i can occur for $i = 1, 2, 3, \ldots, k$. Altogether the experiment can occur in $\Pi_{i=1}^{k} n_i = n_1 \cdot n_2 \cdot n_3 \cdots n_k$ ways.

The next example illustrates the use of this principle.

Example 1.3.2. In how many ways can the five amino acids, alanine, valine, glycine, cysteine, tryptophan, be arranged to form a pentapeptide? This is a five-stage experiment, since there are five amino acids that must fall into place in the chain. This is indicated by drawing five slots and mentally noting what they represent.

1st acid in chain	2d acid in chain	3d acid in chain	4th acid in chain	5th acid in chain

In how many ways can the first stage of the experiment occur? Answer: Five. There are five acids available, any one of which could fall into the first position. Indicate this by placing a 5 in the first slot.

5				
1st acid in chain	2d acid in chain	3d acid in chain	4th acid in chain	5th acid in chain

Once the first stage is complete, in how many ways can stage 2 be performed? Answer: Four. Since each pentapeptide is to contain the five amino acids mentioned, repetition of the acid first in the chain is not permitted. The second member of the chain must be one of the four acids remaining. Indicate this by placing a 4 in the second slot.

5	4			
1st acid in chain	2d acid in chain	3d acid in chain	4th acid in chain	5th acid in chain

Similar reasoning leads us to conclude that stage 3 can take place in 3 ways, stage 4 in 2 ways, and stage 5 in 1 way. By the multiplication principle there are

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

5	4	3	2	1
1st acid in chain	2d acid in chain	3d acid in chain	4th acid in chain	5th acid in chain

pentapeptides that can be formed from these five amino acids.

There are several guidelines to keep in mind when using the multiplication principle:

Guidelines for Using the Multiplication Principle

1. Watch out for repetition versus nonrepetition. Sometimes objects can be repeated; sometimes they cannot. Whether or not repetition is allowed is determined by the physical context of the problem.
2. Watch out for subtraction. Consider event A. Occasionally it will be difficult, if not impossible, to find $n(A)$ directly. However, $S = A \cup A'$. Since A and A' have no points in common, $n(S) = n(A) + n(A')$. This implies that $n(A) = n(S) - n(A')$.

3. If there is a stage in the experiment with a special restriction, then you should think about the restriction first.

These points are illustrated in the next example.

Example 1.3.3. The DNA-RNA code is a molecular code in which the sequence of molecules provides significant genetic information. Each segment of RNA is composed of "words." Each word specifies a particular amino acid and is composed of a chain of three ribonucleotides. Each of the ribonucleotides in the chain is either adenine (A), uracil (U), guanine (G), or cytosine (C).

1. How many words can be formed? Here repetition is allowed. By the multiplication principle there are $4 \cdot 4 \cdot 4 = 64$ possible RNA words.
2. How many of these words involve some repetition? To answer this question, we use subtraction. There are 64 words possible. By the multiplication principle, $4 \cdot 3 \cdot 2 = 24$ of these have no repeated nucleotides. The remaining $64 - 24 = 40$ words must involve some repetition.
3. How many of the 64 words end with the nucleotides uracil or cytosine and have no repetition? Since there is a restriction on the last position of the chain, we consider it first by placing a 2 in the third position.

$$\underset{\text{1st}}{\underline{\qquad}} \quad \underset{\text{2d}}{\underline{\qquad}} \quad \underset{\text{3d}}{\underline{\ \ 2\ \ }}$$

Once this restriction has been taken care of, we note that repetition is not allowed. This means that the nucleotide in position 3 cannot be used again. The first position can be filled with any of the three remaining nucleotides, and the second by either of the two that will be left at that point. By the multiplication principle the number of words that end with uracil or cytosine and have no repetition is

$$\underset{\text{1st}}{\underline{\ \ 3\ \ }} \cdot \underset{\text{2d}}{\underline{\ \ 2\ \ }} \cdot \underset{\text{3d}}{\underline{\ \ 2\ \ }} = 12$$

The use of the multiplication principle often results in a product of the form $n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$, where n is a positive integer. For example, we found that the number of pentapeptides that can be formed from the five different amino acids is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. This product can be denoted by using what is called *factorial* notation.

Definition 1.3.3 (Factorial notation). Let n be a positive integer. The product $n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$ is called n factorial and is denoted by $n!$. Zero factorial, denoted by $0!$, is defined to be 1.

When we use this notation, the number of pentapeptides that can be formed from five different amino acids is $5!$. Even though the need for zero factorial is not obvious yet, its purpose will become apparent soon.

One formula for counting permutations can be derived easily from the multiplication principle. Suppose that we have n distinct objects but we are going to use only r of these objects in each arrangement. How many permutations are possible in this case? Let us denote this number by ${}_nP_r$. Note that the subscript on the left denotes the number of distinct objects available, the P denotes the fact that we are counting permutations, and the subscript on the right denotes the number of objects used per arrangement. Since each permutation is to be an arrangement of r different objects, we need r slots:

$$\underset{\substack{\text{1st}\\ \text{object}}}{\underline{\qquad}} \quad \underset{\substack{\text{2d}\\ \text{object}}}{\underline{\qquad}} \quad \underset{\substack{\text{3d}\\ \text{object}}}{\underline{\qquad}} \quad \cdots \quad \underset{\substack{r\text{th}\\ \text{object}}}{\underline{\qquad}}$$

Since n distinct objects are available, we have n choices for the first slot. Repetition is not allowed, so the number of permutations is given by

$$\underset{\substack{\text{1st}\\ \text{object}}}{\underline{\quad n \quad}} \cdot \underset{\substack{\text{2d}\\ \text{object}}}{\underline{(n-1)}} \cdot \underset{\substack{\text{3d}\\ \text{object}}}{\underline{(n-2)}} \quad \cdots \quad \underset{\substack{r\text{th}\\ \text{object}}}{\underline{\quad (?) \quad}}$$

To find the last number in the product, note that the number subtracted from n in each factor is one less than the slot number. Thus the rth factor will be $n - (r - 1) = n - r + 1$. We now have that

$${}_nP_r = n(n-1)(n-2)\cdots(n-r+1)$$

Note that

$$\begin{aligned}\frac{n!}{(n-r)!} &= \frac{n(n-1)(n-2)\cdots(n-r+1)\cancel{(n-r)}\cancel{(n-r-1)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{(n-r)}\cancel{(n-r-1)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}\\ &= {}_nP_r\end{aligned}$$

Substituting, we have shown that the formula for finding the number of permutations of n distinct objects taken r at a time is as stated in the next theorem.

Theorem 1.3.1. The number of permutations of n distinct objects used r at a time, denoted by ${}_nP_r$, is

$${}_nP_r = \frac{n!}{(n-r)!}$$

Example 1.3.4

1. $${}_9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5!}{5!} = 3024$$

2. $${}_7P_7 = \frac{7!}{(7-7)!} = \frac{7!}{0!} = \frac{7!}{1} = 5040$$

Note that in order to apply Theorem 1.3.1, the objects to be arranged must be distinct, no repetition is allowed, and there can be no restrictions on any position in the arrangement. This formula will not solve all your permutation problems! The multiplication principle should be the first thing that comes to mind once you realize that a problem involves order, either natural or imposed.

Counting Combinations

Thus far we have considered counting problems in which order is important. We now turn our attention to situations in which order is irrelevant. That is, we now consider problems involving combinations rather than permutations. One very useful formula for finding the number of combinations of n distinct objects selected r at a time can be derived. Note that arranging r objects taken from n that are available is a two-stage process. The r objects must first be selected; denote the number of ways to select these objects by ${}_nC_r$. The r objects selected must then be arranged in order; this can be done in $r!$ ways. By the multiplication principle the number of arrangements of r objects taken from n is

$${}_nP_r = {}_nC_r \cdot r!$$

Solving this equation for ${}_nC_r$ and applying Theorem 1.3.1, we see that

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

This result is summarized in the next theorem and illustrated in Example 1.3.5.

Theorem 1.3.2. The number of combinations of n distinct objects selected r at a time, denoted by ${}_nC_r$, or $\binom{n}{r}$, is given by

$${}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example 1.3.5

1. $${}_5C_3 = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{5 \cdot 4 \cdot 3!}{3!2 \cdot 1} = 10$$

2. $$\binom{5}{0} = {}_5C_0 = \frac{5!}{0!(5-0)!} = \frac{5!}{0!5!} = 1$$

It is usually difficult at first to distinguish combinations from permutations. Look for the key words "select" and "arrange." The former signals that the problem involves combinations; the latter, that a permutation is sought.

Example 1.3.6. A foundry ships a lot of 20 engine blocks of which five contain internal flaws. The purchaser will select three blocks at random and test them for hardness. The lot will be accepted if no flaws are found. What is the *probability* that this lot will be accepted? To answer this question, we must count two things: the number of ways to select 3 engine blocks from 20, and the number of ways to select 3 engine blocks from 20 and obtain no flawed engines. The former quantity is given by

$$_{20}C_3 = \frac{20!}{3!17!} = \frac{20 \cdot 19 \cdot 18 \cdot 17!}{3 \cdot 2 \cdot 1 \cdot 17!} = 1140$$

In order to obtain no flawed engines, all 3 of the sampled engines must be selected from among the 15 unflawed engines in the lot. This can be done in

$$_{15}C_3 = \frac{15!}{12!3!} = \frac{15 \cdot 14 \cdot 13 \cdot 12!}{3 \cdot 2 \cdot 1 \cdot 12!} = 455$$

ways. Since the engines selected for testing are chosen at random, each of the 1140 possible samples is equally likely. Using the classical approach to probability, we have

$$P[\text{lot is accepted}] = \frac{455}{1140}$$

Permutations of Indistinguishable Objects

Thus far we have been concerned only with permutation problems that may or may not involve repetition. Now we consider situations in which repetition is inevitable. The question to be answered is, "How many distinct arrangements of n objects are possible if some of the objects are identical and therefore cannot be distinguished one from the other?" An example will show that we already have available the tools to answer this question.

Example 1.3.7. Consider a computer with 16 ports and assume that at any given time each port is either in use (u), not in use but operable (n), or inoperable (i). How many configurations are possible in which 10 ports are in use, 4 are not in use but are operable, and 2 are inoperable? A typical sequence of this sort is

$$u\ u\ i\ u\ i\ n\ n\ u\ n\ u\ u\ n\ u\ u\ u\ u$$

To determine the number of ways that these letters can be permuted to form other arrangements, consider the 16 ports. If we could control port usage, then we are faced with a three-step process. These steps are:

Step 1: Select 10 ports for usage. This can be done in $_{16}C_{10} = 8008$ ways.

Step 2: Select 4 of the remaining 6 ports to represent ports that are not in use but which are operable. This can be done in $_6C_4 = 15$ ways.

Step 3: Select the remaining 2 ports to represent inoperable ports. This can be done in $_2C_2 = 1$ way.

The multiplication principle guarantees that the entire three-step process can be done in

$$
\begin{aligned}
{}_{16}C_{10} \cdot {}_6C_4 \cdot {}_2C_2 &= (8008)(15)(1) \\
&= 120{,}120 \text{ ways}
\end{aligned}
$$

Let us use the solution to the above problem to suggest a general formula that can be used to solve other problems involving permutations of indistinguishable objects. The expression ${}_{16}C_{10} \cdot {}_6C_4 \cdot {}_2C_2$ can be written as

$$
\begin{aligned}
{}_{16}C_{10} \cdot {}_6C_4 \cdot {}_2C_2 &= \frac{16!}{10!6!}\,\frac{6!}{4!2!}\,\frac{2!}{2!0!} \\
&= \frac{16!}{10!4!2!}
\end{aligned}
$$

Notice that the terms of this product are predictable from the original problem. The 16! appears in the numerator because there is a total of 16 ports. The three terms 10!, 4!, and 2! arise due to the fact that there are three types of letters, 10 u's, 4 n's, and 2 i's, being permuted. This suggests that to solve a permutation problem of this sort, we need to determine n, the total number of objects being permuted, and $n_1, n_2, \ldots n_k$, the number of each of the k types of objects involved. The general formula for the total number of distinguishable arrangements of these objects is then given by

Permutations of Indistinguishable Objects

$$
\frac{n!}{n_1!n_2! \cdots n_k!} \qquad n = n_1 + n_2 + \cdots + n_k
$$

A general argument similar to that given above will show that this formula does hold for any values of $n, n_1, n_2, \ldots n_k$.

Example 1.3.8. A traffic engineer is setting the timing on a series of 10 stoplights on the main street of a small town. At any given time a light can be either red, yellow, or green. How many color patterns are possible for the series of lights at startup? If the lights come on at random at startup, what is the probability that the initial setting will consist of 3 red, 5 yellow, and 2 green lights?

This is a 10-step process with 3 choices for each step. By the multiplication principle, the number of ways to form color patterns is

$$
3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^{10} = 59049
$$

The formula for permutations of indistinguishable objects yields

$$
\frac{10!}{3!\,5!\,2!} = 2520
$$

ways to obtain a 3 red, 5 yellow, 2 green color split. Thus, the probability of obtaining this split at startup is given by

$$\frac{2520}{59049} = .0427$$

CHAPTER SUMMARY

In this chapter we discussed how to interpret probabilities. We also presented three methods for assigning probabilities to events. These are called the personal, relative frequency, and classical approaches. We also introduced and defined important terms that you should know. These are

Sample space	Mutually exclusive events
Sample point	Permutation
Event	Combination
Impossible event	$n!$
Certain event	$0!$

In solving permutation problems, we used the multiplication principle. This principle was used to derive a formula for ${}_nP_r$, the number of permutations of n distinct objects arranged r at a time. We also derived a formula for finding ${}_nC_r$, the number of combinations of n distinct objects selected r at a time.

EXERCISES

Section 1.1

1. One environmental hazard recently identified is overexposure to airborne asbestos. In a sample of 10 public buildings over 20 years old, three were found to be insulated with materials that produced an excess number of airborne asbestos bodies. What is the approximate probability that another building of this type will have this problem? What method are you using to assign this probability?
2. A sample of 75 bridges in a given state is selected, and the bridges chosen are inspected for structural weaknesses. If 30 of the bridges sampled are found to have serious problems, what is the estimated probability that the next bridge sampled in the state will have serious structural problems? What method for assigning probabilities are you using to obtain this estimate?
3. Hemophilia is a sex-linked hereditary blood defect of males characterized by delayed clotting of the blood which makes it difficult to control bleeding, even in the case of a minor injury. When a woman is a carrier of classical hemophilia, there is a 50% chance that a male child will inherit the disease. If a carrier gives birth to two sons, what is the probability that both boys will have the disease? What approach to probability are you using to answer this question?

4. A foundry produces brake pads for use in Ford motor cars. A particular lot of 50 such pads contains 2 that have burrs (or rough spots) that were missed in the grinding process. If one part is selected at random from the lot to be installed in your car, what is the probability that it will have a burr? Is this a relative frequency approximation or a classical probability?

Section 1.2

5. Fission occurs when the nucleus of an atom captures a subatomic particle called a neutron and splits into two lighter nuclei. This causes energy to be released. At the same time other neutrons are emitted, two or three on the average. If at least one of these is captured by another fissionable nucleus, then a chain reaction is possible.

(*a*) Consider a reaction in which three neutrons are emitted initially. Let c denote that a given neutron is captured by another nucleus; let n denote that the neutron is not captured by another nucleus. Construct a tree denoting the possible behavior for these three neutrons.

(*b*) List the sample points generated by the tree.

(*c*) List the sample points that constitute each of these events:
A_1: a chain reaction is possible
A_2: all three neutrons are captured
A_3: a chain reaction is not possible

(*d*) Are A_1 and A_2 mutually exclusive?
Are A_1 and A_3 mutually exclusive?
Are A_2 and A_3 mutually exclusive?
Are A_1, A_2, and A_3 mutually exclusive?

(*e*) The probability that a neutron will be captured depends on its neutron energy and is not the same for each neutron. Under these circumstances, is it correct to say that the probability that all three neutrons will be captured is 1/8 because this can occur in only one way and there are eight paths through the tree of part (*a*)? Explain your answer.

6. In ballistics studies conducted during World War II it was found that, in ground-to-ground firing, artillery shells tended to fall in an elliptical pattern such as that of Fig. 1.4. The probability that a shell would fall in the inner ellipse is .50; the probability that it would fall in the outer ellipse is .95. ("Statistics and Probability Applied to Problems of Antiaircraft Fire in World War II," E. S. Pearson, *Statistics: A Guide to the Unknown*, Holden-Day, San Francisco, 1972, pp. 407–415.)

(*a*) A firing is considered to be a success (s) if the shell falls within the inner ellipse; otherwise it is failure (f). Construct a tree to represent the firing of three shells in succession.

(*b*) List the sample points generated by the tree.

(*c*) Let A_1 denote the event that the first firing is successful, A_2 the event that the second firing is successful, and A_3 the event that the third firing is successful. List the sample points that make up each of these three events. Are the events mutually exclusive? Explain from both a practical and a mathematical point of view.

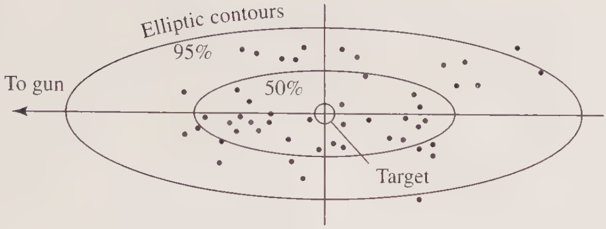

FIGURE 1.4
50% of the shells fall in the inner ellipse.

(*d*) Describe the event A_1' verbally, and then list the sample points that make up this event.
(*e*) Describe the event $A_1 \cap A_2' \cap A_3'$ verbally, and list the sample points that make up this event.
(*f*) Explain why classical probability can be used to find the probability of the event described in part (*e*), and find this probability.

7. A home computer is tied to a mainframe computer via a telephone modem. The home computer will dial repeatedly until contact is made. Once contact has been established, the dialing process will, of course, end. Let c denote the fact that contact is made on a particular attempt and n denote that contact is not made.
(*a*) Construct a tree diagram to represent the dialing process.
(*b*) Are the paths through the tree equally likely?
(*c*) List the sample points generated by the tree. Can this list ever be completed?
(*d*) List the sample points that constitute event A: contact is made in at most four attempts.
(*e*) Give an example of two events that are not impossible but that are mutually exclusive.

8. A missile battery can fire five missiles in rapid succession. As soon as the target is hit, firing will cease. Let h denote a hit and m a miss.
(*a*) Draw the tree to represent the possible firing of these missiles at a single incoming target.
(*b*) Is there any difference in the tree drawn here and that illustrated in Example 1.2.3? Explain.
(*c*) List the sample points generated by the tree.
(*d*) List the sample points that constitute the events
A_1: exactly two shots are fired
A_2: at most two shots are fired
Are these events mutually exclusive? Explain.

Section 1.3

9. Evaluate each of these expressions:
(*a*) $9!$ (*b*) $6!$
(*c*) ${}_7P_3$ (*d*) ${}_6P_2$
(*e*) ${}_5P_5$ (*f*) ${}_6P_6$

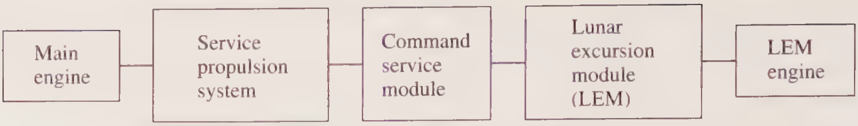

FIGURE 1.5
A simplified diagram of the Apollo system.

10. In investigating the Ideal Gas Law, experiments are to be run at four different pressures and three different temperatures.
  (*a*) How many experimental conditions are to be studied?
  (*b*) If each experimental condition is replicated (repeated) 5 times, how many experiments will be conducted on a given gas?
  (*c*) How many experiments must be conducted to obtain five replications on each experimental condition for each of six different gases?

11. In setting up a computer system for the home firm to use in quality control, an engineer has four choices for the main unit: IBM, VAX, Honeywell, or HP. There are six brands of CRTs that can be purchased and three types of graphics printers.
  (*a*) If all equipment is compatible, in how many ways can the system be designed?
  (*b*) If the engineer wants to be able to use a statistical software package that is only available on IBM and VAX equipment, in how many ways can the system be designed?

12. In Exercise 6 we considered the experiment of firing three artillery shells in succession. Each firing was classed as being either a success or a failure. Use the multiplication principle to verify that the number of paths through the tree representing this experiment is 8.

13. The Apollo mission to land humans on the moon made use of a system whose basic structure is shown in Fig. 1.5. For the system to operate successfully, all five components shown must function properly. Let us identify each component as being either operable (O) or inoperable (i). Thus the sequence $OOOOi$ denotes a state in which all components except the LEM engine are operable. ("Striving for Reliability," Gerald Lieberman, *Statistics: A Guide to the Unknown,* Holden-Day, San Francisco, 1972, pp. 400–406.)
  (*a*) How many states are possible?
  (*b*) How many states are possible in which the LEM engine is inoperable?
  (*c*) The mission is deemed at least partially successful if the first three components are operable. How many states represent at least a partially successful mission?
  (*d*) The mission is a total success if and only if all five components are operable. How many states represent a totally successful mission?

14. The basic storage unit of a digital computer is a "bit." A bit is a storage position that can be designated as either on (1) or off (0) at any given time. In converting picture images to a form that can be transmitted electronically, a picture element, called a *pixel* is used. Each pixel is quantized into gray levels and coded using a binary code. For example, a pixel with four gray levels can be coded using two bits by designating the gray levels by 00, 01, 10, and 11.

(*a*) How many gray levels can be quantized using a four-bit code?

(*b*) How many bits are necessary to code a pixel quantized to 32 gray levels?

15. Tests will be run on five different coatings used to protect fiber optics cables from extreme cold. These tests will be conducted in random order.

(*a*) In how many orders can the tests be run?

(*b*) If two of the coatings are made by the same manufacturer, what is the probability that tests on these coatings will be run back to back?

16. The effectiveness of irradiated polymers in the removal of benzene from water is being investigated. Three polymers are to be studied. Each is to be tested at four different temperatures and three different radiation levels.

(*a*) How many different experimental conditions are under study?

(*b*) If each experimental condition is to be replicated (repeated) five times, how many experiments must be conducted?

17. Evaluate each of these expressions:

(*a*) ${}_9C_4$ (*b*) ${}_8C_3$

(*c*) $\binom{8}{5}$ (*d*) $\binom{8}{0}$

18. A contractor has 8 suppliers from which to purchase electrical supplies. He will select 3 of these at random and ask each supplier to submit a project bid. In how many ways can the selection of bidders be made? If your firm is one of the 8 suppliers, what is the probability that you will get the opportunity to bid on the project?

19. A chemical engineer has 7 different treatments that she wishes to compare for effectiveness in producing a sand cast to be used in casting molten iron. She wants to compare each treatment to each of the others. How many pairwise comparisons will she have to make? That is, in how many ways can these treatments be selected two at a time?

20. To get the opportunity to enter the McNeill River Brown (Grizzly) Bear Sanctuary in Alaska, one must enter a lottery. For a given year there are 2000 individuals entered, and of these a set of 120 names will be randomly selected. Assume that you and a friend are both entered into the lottery.

(*a*) In how many ways can a set of 120 names be randomly selected from among the 2000 entered in the drawing?

(*b*) In how many ways can the drawing be done in such a way that you and your friend are both selected?

(*c*) What is the probability that you and your friend will both be chosen?

21. A firm employs 10 programmers, 8 systems analysts, 4 computer engineers, and 3 statisticians. A "team" is to be chosen to handle a new long-term project. The team will consist of 3 programmers, 2 systems analysts, 2 computer engineers, and 1 statistician.

(*a*) In how many ways can the team be chosen?

(*b*) If the customer insists that one particular engineer with whom he or she has worked before be assigned to the project, in how many ways can the team be chosen?

22. A company receives a shipment of 20 hard drives. Before accepting the shipment, 5 of them will be randomly selected and tested. If all 5 meet specifications, then the shipment will be accepted; otherwise all 20 will be returned to the manufacturer. If, in fact, 3 of the 20 drives are defective, what is the probability that the shipment will not be accepted?

23. A control chart is used to monitor the average thread count produced by a machine making spandex cloth. Samples are taken periodically, and each sample is classified into one of 5 categories. These are: in control but above average, in control and average, in control but below average, out of control and high, and out of control and low. In taking a series of 20 samples, in how many ways can we obtain a series in which there are exactly

(*a*) 5 samples in control but above average, 5 samples in control but below average, 5 samples in control and average, 3 samples out of control and high, and the rest out of control and low?

(*b*) 18 samples in control and 2 out of control?

24. The oil embargo of 1973 spurred a study of the possibility of using automatic meter readings to reduce costs to power companies. One procedure studied entailed the use of 128-bit messages. Occasionally transmission errors occur resulting in a digit reversal of one or more bits. How many messages can be sent that contain exactly two transmission errors? *Hint:* Think of a message as being a permutation of 128 objects, each of which is either correct (c) or not correct (n).

25. In studying a chemical reaction, 12 experiments will be conducted. Four different temperatures will be used 3 times, each with the temperatures run in random order. In how many orders can the series of experiments be conducted?

26. A garage door opener has six toggle switches, each with three settings: up, center, and down.

(*a*) In how many ways can these switches be set?

(*b*) If a thief knows the type of opener involved but does not know the setting, what is the probability that he or she can guess the setting on the first attempt?

(*c*) How many settings are possible in which two switches are up, two are down, and two are in the center?

27. Consider Example 1.2.1.

(*a*) Without looking at the tree diagram, how many paths through the tree will represent the fact that exactly two of the three computers are ready at the time of the launch? Verify your answer by listing these paths.

(*b*) If 10 computers were used instead of 3, the tree given in Fig. 1.1 could be expanded to answer questions posed concerning the number of computers that are ready at launch time. How many paths would such a tree entail? How many of these paths would represent the fact that exactly 7 of the 10 computers are ready at launch time?

REVIEW EXERCISES

28. Find n if $\binom{n}{2} = 21$; if $\binom{n}{2} = 105$.

29. The configuration of a particular computer terminal consists of a baud-rate setting, a duplex setting, and a parity setting. There are 11 possible baud-rate settings, two parity settings (even or odd), and two duplex settings (half or full).
  (*a*) How many configurations are possible for this terminal?
  (*b*) In how many of these configurations is the parity even and the duplex full?
  (*c*) A line surge occurs that causes these settings to change at random. What is the probability that the resulting configuration will have even parity and be full duplex?

30. A firm offers a choice of 10 free software packages to buyers of their new home computer. There are 25 packages from which to choose. In how many ways can the selection be made? Five of the packages are computer games. How many selections are possible if exactly three computer games are selected?

31. A project manager has 10 chemical engineers on her staff. Four are women and six are men. These engineers are equally qualified. In a random selection of three workers, what is the probability that no women will be selected? Would you consider it unusual for no women to be selected under these circumstances? Explain.

32. A computer system uses passwords that consist of five letters followed by a single digit.
  (*a*) How many passwords are possible?
  (*b*) How many passwords consist of three A's and two B's, and end in an even digit?
  (*c*) If you forget your password but remember that it has the characteristics described in part (*b*), what is the probability that you will guess the password correctly on the first attempt?

33. A mainframe computer has 16 ports. At any given time each port is either in use or not in use. How many possibilities are there for overall port usage of this computer? How many of these entail the use of at least 1 port?

34. A flashlight operates on two batteries. Eight batteries are available, but three are dead. In a random selection of batteries, what is the probability that exactly one dead battery will be selected?

35. An electrical control panel has three toggle switches labeled I, II, and III, each of which can be either on (O) or off (F).
  (*a*) Construct a tree to represent the possible configurations for these three switches.
  (*b*) List the elements of the sample space generated by the tree.
  (*c*) List the sample points that constitute the events
    A: at least one switch is on
    B: switch I is on
    C: no switch is on
    D: four switches are on
  (*d*) Are events A and B mutually exclusive? Are events A and C mutually exclusive? Are events A and D mutually exclusive?

(*e*) What is the name given to an event such as D?
(*f*) If at any given time each switch is just as likely to be on as off, what is the probability that no switch is on?

36. Two items are randomly selected one at a time from an assembly line and classed as to whether they are of superior quality (+), average quality (0), or inferior quality (−).
(*a*) Construct a tree for this two-stage experiment.
(*b*) List the elements of the sample space generated by the tree.
(*c*) List the sample points that constitute the events
A: the first item selected is of inferior quality
B: the quality of each of the items is the same
C: the quality of the first item exceeds that of the second
(*d*) Are the events A and B mutually exclusive? Are the events A and C mutually exclusive?
(*e*) Give a brief verbal description of these events:
$A' \cap B \quad A' \cap B'$
$A \cap B' \quad A \cap C' \cap B$
(*f*) It is known that 90% of the items produced are of average quality, 1% are of superior quality, and the rest are of inferior quality. It is argued that since the classification experiment can proceed in nine ways with only one of these resulting in two items of average quality, the probability of obtaining two such items is 1/9. Criticize this argument.

37. An experiment consists of selecting a digit from among the digits 0 to 9 in such a way that each digit has the same chance of being selected as any other. We name the digit selected A. These lines of code are then executed:

```
IF A < 2 THEN B = 12; ELSE B = 17;
IF B = 12 THEN C = A - 1; ELSE C = 0;
```

(*a*) Construct a tree to illustrate the ways in which values can be assigned to the variables A, B, and C.
(*b*) Find the sample space generated by the tree.
(*c*) Are the 10 possible outcomes for this experiment equally likely?
(*d*) Find the probability that A is an even number.
(*e*) Find the probability that C is negative.
(*f*) Find the probability that $C = 0$.
(*g*) Find the probability that $C \leq 1$.

38. Consider Exercise 16. If experimental runs are to be done in random order, how many different sequences are possible? (Set up only!) In experiments of this sort, runs are not usually done randomly. Rather, they are carefully designed so that the researcher has control of the order of experimentation. Can you think of some practical reasons for why this is necessary?