CHAPTER 2

# SOME PROBABILITY LAWS

In Chap. 1 we considered how to interpret probabilities. In this chapter we consider some laws that govern their behavior. The laws that we shall present are those that will have a direct application to problem solving. These laws will be stated and illustrated numerically. Their derivations are not hard, and most of them are left as exercises.

## 2.1 AXIOMS OF PROBABILITY

You have probably seen the development of a mathematical system in your study of high school geometry. In developing any mathematical system, one begins by stating a few basic definitions and axioms that underlie the system. The definitions are the technical terms of the system; axioms are statements that are assumed to be true and therefore require no proof. Usually one starts with as few axioms as possible and then uses these axioms and the technical definitions to develop whatever theorems follow logically. Some technical terms such as sample space, sample point, event, and mutually exclusive events have already been introduced. One can develop a useful system of theorems pertaining to probability with the aid of these definitions and three axioms, called the axioms of probability.

**Axioms of probability.**

1. Let $S$ denote a sample space for an experiment:

$$P[S] = 1$$

2. $P[A] \geq 0$ for every event $A$.
3. Let $A_1, A_2, A_3, \ldots$ be a finite or an infinite collection of mutually exclusive events. Then $P[A_1 \cup A_2 \cup A_3 \cdots] = P[A_1] + P[A_2] + P[A_3] + \cdots$.

Axiom 1 states a fact that most people regard as obvious; namely, the probability assigned to the certain event $S$ is 1. Axiom 2 ensures that probabilities can never be negative. Axiom 3 guarantees that when one deals with mutually exclusive events, the probability that at least one of the events will occur can be found by adding the individual probabilities. An important consequence of this axiom is that it gives us the ability to find the probability of an event when the sample points in the same space for the experiment are not equally likely. Example 2.1.1 illustrates this point.

**Example 2.1.1.** The distribution of blood types in the United States is roughly 41% type A, 9% type B, 4% type AB, and 46% type O. An individual is brought into an emergency room and is to be blood-typed. What is the probability that the type will be A, B, or AB?

The sample space for this experiment is

$$S = \{\text{A, B, AB, O}\}$$

The sample points are not equally likely, so the classical approach to probability is not applicable. That is, we cannot say that since there are four blood types and three of them are A, B, or AB the probability of obtaining one of these types is ¾. Let $A_1$, $A_2$, and $A_3$ denote the events that the patient has type A, B, and AB blood, respectively. The events $A_1$, $A_2$, and $A_3$ are mutually exclusive because one cannot have two different blood types at the same time. We are looking for $P[A_1 \cup A_2 \cup A_3]$. By axiom 3,

$$\begin{aligned} P[A_1 \cup A_2 \cup A_3] &= P[A_1] + P[A_2] + P[A_3] \\ &= .41 + .09 + .04 \\ &= .54 \end{aligned}$$

An immediate consequence of these axioms is the fact that the probability assigned to the impossible event is 0, as you should suspect. The derivation of this result is outlined in Exercise 12.

**Theorem 2.1.1.** $P[\varnothing] = 0$.

Another consequence of the axioms is that the probability that an event will *not* occur is equal to 1 minus the probability that it will occur. For example, if the probability of a successful space shuttle mission is .99, then the probability that it will not be successful is $1 - .99 = .01$. This idea is stated in Theorem 2.1.2. Its derivation is outlined in Exercise 12.

**Theorem 2.1.2.** $P[A'] = 1 - P[A]$.

## The General Addition Rule

We have seen how to handle questions concerning the probability of one or another event occurring if those events are mutually exclusive. We now develop a more general rule that will allow us to find the probability that at least one of two events

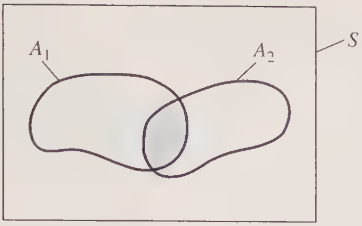


**FIGURE 2.1**
$A_1 \cap A_2 \neq \varnothing$.

will occur when the events are not necessarily mutually exclusive. This rule is suggested by considering the Venn diagram of Fig. 2.1. Assume that the shaded region in the diagram, $A_1 \cap A_2$, is not empty so that $A_1$ and $A_2$ are not mutually exclusive. If we claim that

$$P[A_1 \cup A_2] = P[A_1] + P[A_2]$$

we have committed an obvious error. Since $A_1 \cap A_2$ is contained in $A_1$ and $A_1 \cap A_2$ is contained in $A_2$, $P[A_1 \cap A_2]$ has been included twice in our calculation. To correct this error, we subtract $P[A_1 \cap A_2]$ from the right-hand side of the equation to obtain the general addition rule:

**General addition rule**

$$P[A_1 \cup A_2] = P[A_1] + P[A_2] - P[A_1 \cap A_2]$$

This rule can be derived from the axioms of probability and the theorems that we have already developed. Its proof is outlined in Exercise 12. The key word that signals its use is the word "or."

**Example 2.1.2.** Components of a propulsion system can be arranged in series. However, this arrangement has a serious drawback; if one component fails, the system fails. This is obviously a risky arrangement for space travel! Consider a system in which the main engine has a backup. These engines are designed to operate independently in that the success or failure of one has no effect on the other. The engine component is operable if one *or* the other of these two engines is operable. Such a system is said to have the engine component in parallel. Assume that each engine is 90% reliable. That is, each functions correctly with probability .9. As we shall show later, it is then reasonable to assume that both engines operate correctly with probability .81. Find the probability that the engine component is operable. Let $A_1$: the main engine is operable, and $A_2$: the backup engine is operable. We are given that $P[A_1] = P[A_2] = .9$ and that $P[A_1 \cap A_2] = .81$. We want to find $P[A_1 \cup A_2]$. By the addition rule

$$\begin{aligned} P[A_1 \cup A_2] &= P[A_1] + P[A_2] - P[A_1 \cap A_2] \\ &= .9 + .9 - .81 = .99 \end{aligned}$$

The addition rule links the operations of union and intersection. If $P[A_1 \cap A_2]$ is known, the addition rule can be used to find $P[A_1 \cup A_2]$. Similarly, if $P[A_1 \cup A_2]$

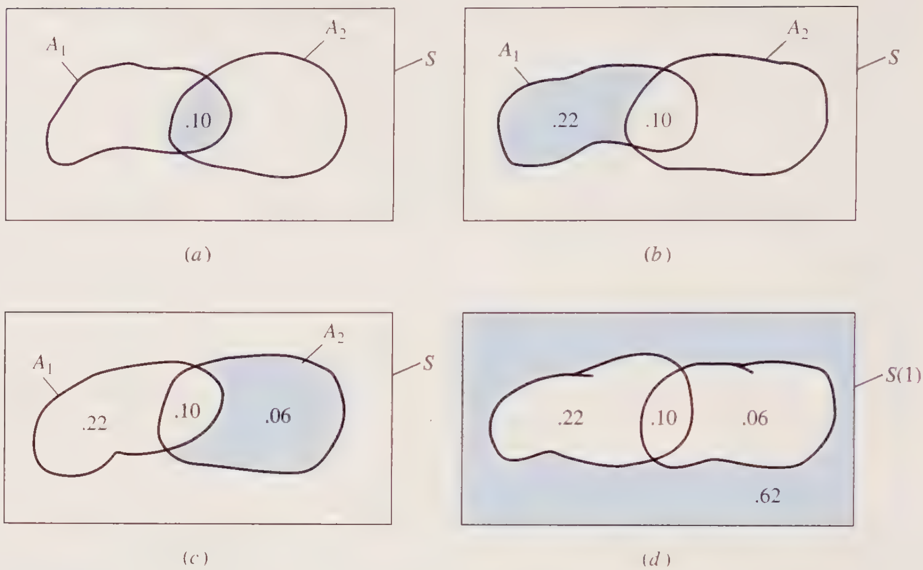


**FIGURE 2.2**
(*a*) $P[A_1 \cap A_2] = .10$; (*b*) $P[A_1 \cap A_2'] = .22$; (*c*) $P[A_1' \cap A_2] = .06$; (*d*) $P[A_1' \cap A_2'] = .62$.

is known, we can use the rule to find $P[A_1 \cap A_2]$. Venn diagrams are helpful when using this rule.

**Example 2.1.3.** A chemist analyzes seawater samples for two heavy metals: lead and mercury. Past experience indicates that 38% of the samples taken from near the mouth of a river on which numerous industrial plants are located contain toxic levels of lead or mercury: 32% contain toxic levels of lead and 16% contain toxic levels of mercury. What is the probability that a randomly selected sample will contain toxic levels of lead only? Let $A_1$ denote the event that the sample contains toxic levels of lead, and let $A_2$ denote that the sample contains toxic levels of mercury. We are given that $P[A_1] = .32$, $P[A_2] = .16$, and $P[A_1 \cup A_2] = .38$. By the addition rule

$$P[A_1 \cup A_2] = P[A_1] + P[A_2] - P[A_1 \cap A_2]$$

or

$$.38 = .32 + .16 - P[A_1 \cap A_2]$$

Solving this equation, we obtain $P[A_1 \cap A_2] = .10$. This is indicated in Fig. 2.2(*a*). Since $P[A_1] = .32$ and $A_1 \cap A_2$ is contained in $A_1$, the probability associated with the shaded region in Fig. 2.2(*b*) is .22. Similarly, since $A_1 \cap A_2$ is contained in $A_2$, a probability of .06 is associated with the shaded region of Fig. 2.2(*c*). Finally, since $P[S] = 1$, the probability assigned to the shaded area in Fig. 2.2(*d*) is .62. We are asked to find the probability that the sample will contain only lead. That is, we want to find $P[A_1 \cap A_2']$. This probability, .22, can be read from Fig. 2.2(*b*).

Notice that if the percentages reported in problems such as these are based on population data, then the probabilities calculated by use of the general addition rule are exact. However, if the percentages reported are based on samples drawn from a

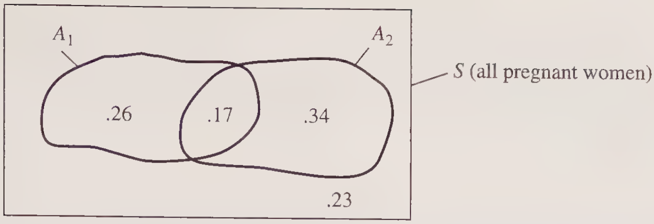


**FIGURE 2.3**
Partition of $S$.

larger population, then the probabilities computed are relative frequency probabilities. They are *approximations* to the true probability of the occurrence of the event in question. Since most percentages reported in the literature are based on samples, most of them are properly viewed as being relative frequency probabilities. We use the word "probability" with the understanding that the probabilities given and computed by using the theorems in this chapter are, in most cases, only approximations.

## 2.2 CONDITIONAL PROBABILITY

In this section we introduce the notion of conditional probability. The name itself is indicative of what is to be done. We wish to determine the probability that some event $A_2$ will occur, "conditional on" the assumption that some other event $A_1$ has occurred. The key words to look for in identifying a conditional question are "if" and "given that." We use the notation $P[A_2|A_1]$ to denote the conditional probability of event $A_2$ occurring given that event $A_1$ has occurred. A simple example will suggest the way to define this probability.

**Example 2.2.1.** In trying to determine the sex of a child a pregnancy test called "starch gel electrophoresis" is used. This test may reveal the presence of a protein zone called the pregnancy zone. This zone is present in 43% of all pregnant women. Furthermore, it is known that 51% of all children born are male. Seventeen percent of all children born are male and the pregnancy zone is present. The Venn diagram for these data is shown in Fig. 2.3. Let $A_1$ denote the event that the pregnancy zone is present, and $A_2$ that the child is male. We know that, for a randomly selected pregnant woman, $P[A_1] = .43$, $P[A_2] = .51$, $P[A_1 \cap A_2] = .17$. If asked, "What is the probability that the child is male?" the answer is .51. Suppose we are *given* the information that the pregnancy zone is present and asked, "What is the probability that the child is male?" We now have information that was not available originally. What effect, if any, does this new information have on our belief that the child is male? That is, what is $P[A_2|A_1]$? Once we know that the pregnancy zone is present, our sample space no longer includes all pregnant women; it consists only of the 43% with this characteristic. Of these, $.17/.43 \doteq .395$ have male children. Logic implies that

$$P[\text{male}|\text{zone present}] = P[A_2|A_1] = .395$$

Receipt of the information that the pregnancy zone is present reduces from .51 to .395 the probability that the child is male.

To formalize the reasoning used in the previous example, note that $P[A_2 | A_1]$ is found by forming a ratio whose denominator is $P[A_1]$, the probability that the *given* event will occur. The numerator is $P[A_1 \cap A_2]$, the probability that *both* the given event and the event in question will occur. That is, we define the conditional probability as follows:

**Definition 2.2.1 (Conditional probability).** Let $A_1$ and $A_2$ be events such that $P[A_1] \neq 0$. The conditional probability of $A_2$ given $A_1$, denoted by $P[A_2|A_1]$, is defined by

$$P[A_2|A_1] = \frac{P[A_1 \cap A_2]}{P[A_1]}$$

Sometimes receipt of the information that event $A_1$ has occurred has no effect on the probability assigned to event $A_2$. That is,

$$P[A_2|A_1] = P[A_2]$$

When this happens, $A_1$ and $A_2$ have a special relationship to one another. The nature of this relationship will be explored in the next section. In the meantime don't be surprised if you find that a particular conditional probability does not differ from the original probability assigned to the event!

## 2.3 INDEPENDENCE AND THE MULTIPLICATION RULE

We have used the word "independent" informally in several previous examples. Webster's dictionary defines independent objects as objects acting "irrespective of each other." Thus two events are independent if one may occur irrespective of the other. That is, the occurrence or nonoccurrence of one does not alter the likelihood of occurrence or nonoccurrence of the other. In some cases it is reasonable to assume that two events are independent from the physical description of the events themselves. For example, suppose that a couple heterozygous for eye color has two children. Since the eye color of a child is affected only by the genetic makeup of the parents and not by the eye color of the other child, it is reasonable to assume that the events $A_1$: the first child has brown eyes, and $A_2$: the second child has brown eyes, are independent. However, in most instances the issue is not clear-cut. In these cases we need a mathematical definition of the term to determine without a doubt whether two events are, in fact, independent.

To see how to characterize independence, let us consider a simple experiment that consists of rolling a single fair die once and then tossing a fair coin once. Let the first member of each ordered pair denote the number appearing on the die and the second, the face showing on the coin ($H$ = heads, $T$ = tails). A sample space for this experiment is

$$S = \{(1, H), (1, T), (2, H), (2, T), (3, H), (3, T), (4, H), (4, T), (5, H), (5, T), (6, H), (6, T)\}$$

Since the die and the coin are considered to be fair, these 12 outcomes are equally likely. Consider these events:

$A$: the die shows one or two

$B$: the coin shows heads

$A \cap B$: the die shows one or two and the coin shows heads

Since knowing the result of the die roll gives us no additional information on how the coin will land, it is reasonable to assume that the events $A$ and $B$ are independent. Using classical probability, we easily see that

$$\begin{aligned} P[A] &= P[\{(1, H), (1, T), (2, H), (2, T)\}] = 4/12 = 1/3 \\ P[B] &= P[\{(1, H), (2, H), (3, H), (4, H), (5, H), (6, H)\}] \\ &= 6/12 = 1/2 \\ P[A \cap B] &= P[\{(1, H), (2, H)\}] = 2/12 = 1/6 \end{aligned}$$

More importantly, it is easy to see that for these physically independent events

$$P[A \cap B] = P[A] \cdot P[B]$$

Consider now an experiment that consists of drawing two coins in succession from a box containing a nickel ($N$), a dime ($D$), and a quarter ($Q$). The first coin is not replaced before the second is drawn. A sample space for this experiment is

$$S = \{(N, D), (N, Q), (D, N), (D, Q), (Q, N), (Q, D)\}$$

These outcomes are equally likely. Consider these events:

$A$: the first coin is a dime

$B$: the second coin is a dime

Since we do not replace the first coin before the second draw, it is evident that if event $A$ occurs, event $B$ cannot occur. That is, knowledge that event $A$ has occurred does give us information on whether or not event $B$ will occur! These events are not independent. Using classical probability, we easily see that

$$\begin{aligned} P[A] &= P[\{(D, N), (D, Q)\}] = 2/6 \\ P[B] &= P[\{(N, D), (Q, D)\}] = 2/6 \\ P[A \cap B] &= P[\varnothing] = 0 \end{aligned}$$

More importantly, it is easy to see that for these events that are not independent

$$P[A \cap B] \neq P[A]P[B]$$

Thus we have noticed that when $A$ and $B$ are clearly independent, $P[A \cap B] = P[A]P[B]$; when they are clearly dependent, $P[A \cap B] \neq P[A]P[B]$. This is not coincidental. It is natural to use this mathematical characterization as our technical definition of the term "independent events."

**Definition 2.3.1 (Independent events).** Events $A_1$ and $A_2$ are independent if and only if

$$P[A_1 \cap A_2] = P[A_1]P[A_2]$$

This definition is useful in two ways. If exact probabilities are available, then it serves as a test for independence. However, since most probabilities encountered in scientific studies are approximations, it is most useful as a way to find the probability that two events will occur when the events are clearly independent. Example 2.3.1 illustrates its use as a test for independence.

**Example 2.3.1.** Consider the experiment of drawing a card from a well-shuffled deck of 52 cards. Let

$A_1$: a spade is drawn

$A_2$: an honor (10, J, Q, K, A) is drawn

Classical probability is used to see that $P[A_1] = 13/52$ and $P[A_2] = 20/52$. The probability that a spade and an honor, $P[A_1 \cap A_2]$, is drawn is 5/52. Notice that these probabilities are exact. They are not approximations based on observations of card draws. Are the events $A_1$ and $A_2$ independent? To decide, note that

$$P[A_1]P[A_2] = (13/52)(20/52) = 5/52$$

and

$$P[A_1 \cap A_2] = 5/52$$

Since $P[A_1 \cap A_2] = P[A_1]P[A_2]$, we can conclude that these events are independent.

In Chap. 15 a test for independence will be developed that can be used when working with real data rather than with classical probabilities. Its derivation is based on the definition of independent events just discussed.

Example 2.3.2 illustrates the use of Definition 2.3.1 in finding the probability that two events will occur simultaneously when the events are clearly independent.

**Example 2.3.2.** In Example 1.1.3, we found that the probability that a couple heterozygous for eye color will parent a brown-eyed child is 3/4 for each child. Genetic studies indicate that the eye color of one child is independent of that of the other. Thus if the couple has two children, then the probability that both will be brown-eyed is

$$P\left[\begin{matrix}\text{first} \\ \text{brown}\end{matrix} \text{ and } \begin{matrix}\text{second} \\ \text{brown}\end{matrix}\right] = P\left[\begin{matrix}\text{first} \\ \text{brown}\end{matrix}\right] P\left[\begin{matrix}\text{second} \\ \text{brown}\end{matrix}\right]$$
$$= \frac{3}{4} \cdot \frac{3}{4}$$
$$= \frac{9}{16}$$

Definition 2.3.1 defines independence for *any* events $A_1$ and $A_2$. If at least one of the events $A_1$ or $A_2$ occurs with *nonzero* probability, then an appealing characterization of independence can be obtained. To see how this is done, assume that $P[A_1] \neq 0$. By Definition 2.3.1, $A_1$ and $A_2$ are independent if and only if

$$P[A_1 \cap A_2] = P[A_1]P[A_2]$$

Dividing by $P[A_1]$, we can conclude that $A_1$ and $A_2$ are independent if and only if

$$\frac{P[A_1 \cap A_2]}{P[A_1]} = P[A_2|A_1] = P[A_2]$$

A similar argument holds if $P[A_2] \neq 0$. We have thus derived the result given in Theorem 2.3.1.

**Theorem 2.3.1.** Let $A_1$ and $A_2$ be events such that at least one of $P[A_1]$ or $P[A_2]$ is nonzero. $A_1$ and $A_2$ are independent if and only if

$$P[A_2|A_1] = P[A_2] \qquad \text{if } P[A_1] \neq 0 \qquad \text{and}$$
$$P[A_1|A_2] = P[A_1] \qquad \text{if } P[A_2] \neq 0$$

Since most events of real interest do occur with nonzero probability, Theorem 2.3.1 is used as a test for independence. To understand the logic behind the theorem, let us reconsider the data of Example 2.3.1.

**Example 2.3.3.** Consider the events $A_1$, a spade is drawn, and $A_2$, an honor is drawn. We know that $P[A_1] = 13/52$, $P[A_2] = 20/52$, and $P[A_1 \cap A_2] = 5/52$. Suppose we are asked, "What is the probability that a randomly selected card is an honor?" Our answer is 20/52. Suppose we are now told that the card is a spade and are asked, "What is the probability that the card is an honor?" That is, "What is $P[A_2|A_1]$?" If $A_1$ and $A_2$ are independent, the new information is irrelevant and our answer should not change. That is, $P[A_2|A_1] = P[A_2]$. Otherwise our answer should change, and $P[A_2|A_1] \neq P[A_2]$. In this setting, is $P[A_2|A_1] = P[A_2]$? To answer this question, note that

$$P[A_2|A_1] = \frac{P[A_1 \cap A_2]}{P[A_1]} = \frac{5/52}{13/52} = 5/13$$

and

$$P[A_2] = 20/52 = 5/13$$

Since these probabilities are the same, we conclude via Theorem 2.3.1 that $A_1$ and $A_2$ are independent.

Occasionally we must deal with more than two events. Again, the question arises, "When are these events considered independent?" Definition 2.3.2 answers this question by extending our previous definition to include more than two events.

**Definition 2.3.2.** Let $C = \{A_i: i = 1, 2, \ldots, n\}$ be a finite collection of events. These events are independent if and only if, given any subcollection $A_{(1)}, A_{(2)}, \ldots, A_{(m)}$ of elements of $C$,

$$P[A_{(1)} \cap A_{(2)} \cap \cdots \cap A_{(m)}] = P[A_{(1)}]P[A_{(2)}] \cdots P[A_{(m)}]$$

Although this definition can be used to test a collection of events for independence, its main purpose is to provide a way to find the probability that a series of events that are assumed to be independent will occur. To illustrate, we reconsider a problem encountered in Chap. 1 (Example 1.2.1).

**Example 2.3.4.** During a space shot, the primary computer system is backed up by two secondary systems. They operate independently of one another, and each is 90% reliable. What is the probability that all three systems will be operable at the time of the launch? Let

$A_1$: the main system is operable

$A_2$: the first backup is operable

$A_3$: the second backup is operable

We are given that $P[A_1] = P[A_2] = P[A_3] = .9$. We want $P[A_1 \cap A_2 \cap A_3]$. Since these events are assumed to be independent,

$$\begin{aligned} P[A_1 \cap A_2 \cap A_3] &= P[A_1]P[A_2]P[A_3] \\ &= (.9)(.9)(.9) \\ &= .729 \end{aligned}$$

Definition 2.3.2 must be used with care. In particular, one must be certain that it is reasonable to assume that events are independent before it is applied to compute the probability that a series of events will occur. The danger of erroneously assumed independence is illustrated in Example 2.3.5.

**Example 2.3.5.** An Atomic Energy Commission Study, WASH 1400, reported the probability of a nuclear accident such as that which occurred at Three Mile Island in March 1978 to be one in 10 million. Yet the accident did occur. According to Mark Stephens, "The methodology of WASH 1400 made use of event trees—sequences of actions that would be necessary for accidents to take place. These event trees did not assume any interrelation between events—that they might be caused by the same error in judgment or as part of the same mistaken action. The statisticians who assigned probabilities in the writing of WASH 1400 said, for example, that there was a one-in-a-thousand risk of one of the auxiliary feed-water control valves—the twelves—being closed. And if there is a one-in-a-thousand chance of one valve being closed, the chances of both valves being closed is one-thousandth of that, or a million to one. But both of the twelves were closed by the same man on March 26—and one had never been closed without the other." The events $A_1$: the first valve is closed, and $A_2$: the second valve is closed were not independent. However, they were treated as such when calculating the probability of an accident. This, among other things, led to an underestimate of the accident potential (from *Three Mile Island* by Mark Stephens, Random House, 1980).

## The Multiplication Rule

There is one further point to be made before we conclude this section. We can find $P[A_1 \cap A_2]$ if the events are assumed to be independent. Furthermore, if the proper information is given, the general addition rule can be used to find this probability.

Is there any other way to find the probability of the simultaneous occurrence of two events if the events are not independent? The answer is yes, and the method is easy to derive. We know that

$$P[A_2|A_1] = \frac{P[A_1 \cap A_2]}{P[A_1]} \qquad P[A_1] \neq 0$$

regardless of whether the events are independent. Multiplying each side of this equation by $P[A_1]$, we obtain the following formula, called the *multiplication rule:*

**Multiplication rule**

$$P[A_1 \cap A_2] = P[A_2|A_1]P[A_1]$$

The use of this rule is illustrated in Example 2.3.6.

**Example 2.3.6.** Recent research indicates that approximately 49% of all infections involve anaerobic bacteria. Furthermore, 70% of all anaerobic infections are polymicrobic; that is, they involve more than one anaerobe. What is the probability that a given infection involves anaerobic bacteria *and* is polymicrobic? Let $A_1$ denote the event that the infection is anaerobic, and $A_2$ that it is polymicrobic. We are given that $P[A_1] = .49$ and that $P[A_2|A_1] = .70$. We want to find $P[A_1 \cap A_2]$. By the multiplication rule,

$$\begin{aligned} P[A_1 \cap A_2] &= P[A_2|A_1]P[A_1] \\ &= (.70)(.49) \\ &= .343 \end{aligned}$$

## 2.4 BAYES' THEOREM

The topic of this section is the theorem formulated by the Reverend Thomas Bayes (1761). It deals with conditional probability. Bayes' theorem is used to find $P[A|B]$ when the available information is not immediately compatible with that required to apply the definition of conditional probability directly.

Example 2.4.1 is a typical problem calling for the use of Bayes' theorem. You will find applying Bayes' rule quite natural without having seen a formal statement of the theorem!

**Example 2.4.1.** Assume that 40% of all interstate highway accidents involve excessive speed on the part of at least one of the drivers (event $E$) and that 30% involve alcohol use by at least one driver (event $A$). If alcohol is involved there is a 60% chance that excessive speed is also involved; otherwise, this probability is only 10%. An accident involves speeding. What is the probability that alcohol is involved? We are given these probabilities:

$$\begin{array}{lll} P[E] = .40 & P[A] = .30 & P[E|A] = .60 \\ P[E'] = .60 & P[A'] = .70 & P[E|A'] = .10 \end{array}$$

We are being asked to find $P[A|E]$. Since this is a conditional question, it is natural to turn to the definition of conditional probability for a solution. In this case,

$$P[A|E] = \frac{P[E \cap A]}{P[E]}$$

Unfortunately, neither of the probabilities needed for the solution is immediately available. However, each can be obtained easily. By the multiplication rule,

$$P[E \cap A] = P[E|A]P[A]$$

Note that if excessive speed was involved, alcohol use either was or was not also involved. Hence event $E$ can be subdivided into two mutually exclusive events as follows:

$$E = (E \cap A) \cup (E \cap A')$$

Thus $$P[E] = P[E \cap A] + P[E \cap A']$$

An expression has already been found for the first probability on the right; the multiplication rule can be applied to the second probability to see that

$$P[E \cap A'] = P[E|A']P[A']$$

Substitution now yields

$$P[A|E] = \frac{P[E \cap A]}{P[E]}$$
$$= \frac{P[E|A]P[A]}{P[E|A]P[A] + P[E|A']P[A']}$$

Note the pattern in this solution. In the numerator the conditional expression is the reverse of that in the original question; in the denominator, the conditional expressions run through all of the alternatives to the event in question, in this case $A$ and $A'$. The numerical solution can now be obtained by substitution as follows:

$$P[A|E] = \frac{P[E|A]P[A]}{P[E|A]P[A] + P[E|A']P[A']}$$
$$= \frac{(.60)(.30)}{(.60)(.30) + (.10)(.70)}$$
$$= .72$$

If excessive speed was involved in an accident, there is a 72% chance that alcohol was also involved.

In the previous example, there were two mutually exclusive events, $A$ and $A'$, whose union is $S$. Bayes' theorem can also be applied when $S$ is subdivided into more than two mutually exclusive events. We state the theorem in this more general setting.

**Theorem 2.4.1 (Bayes' theorem).** Let $A_1, A_2, A_3, \ldots, A_n$ be a collection of mutually exclusive events whose union is $S$. Let $B$ be an event such that $P[B] \neq 0$. Then for any of the events $A_j$, $j = 1, 2, 3, \ldots, n$,

$$P[A_j|B] = \frac{P[B|A_j]P[A_j]}{\sum_{i=1}^{n} P[B|A_i]P[A_i]}$$

To see that Bayes' theorem could have been used directly to answer the question posed in Example 2.4.1, note that events $A$ and $A'$ are mutually exclusive events whose union is $S$ and that event $E$ occurs with nonzero probability. Hence we can make the following identifications:

$$A_1 = A \qquad A_2 = A' \qquad B = E$$

By applying Bayes' theorem directly we obtain

$$P[A_1|B] = \frac{P[B|A_1]P[A_1]}{P[B|A_1]P[A_1] + P[B|A_2]P[A_2]}$$

or

$$P[A|E] = \frac{P[E|A]P[A]}{P[E|A]P[A] + P[E|A']P[A']}$$

A quick comparison will show that this is the same as the solution derived in Example 2.4.1 using the multiplication rule.

The next example illustrates the use of Bayes' theorem in a setting in which the sample space is subdivided into four mutually exclusive events rather than two.

**Example 2.4.2.** The blood type distribution in the United States is type A, 41%; type B, 9%; type AB, 4%; and type O, 46%. It is estimated that during World War II, 4% of inductees with type O blood were typed as having type A; 88% of those with type A were correctly typed; 4% with type B blood were typed as A; and 10% with type AB were typed as A. A soldier was wounded and brought to surgery. He was typed as having type A blood. What is the probability that this is his true blood type? Let

$A_1$: he has type A blood

$A_2$: he has type B blood

$A_3$: he has type AB blood

$A_4$: he has type O blood

$B$: he is typed as type A

Note that the events $A_1, A_2, A_3, A_4$ are mutually exclusive, and their union is $S$ because each individual can have only one blood type and all possible blood types have been listed. We are being asked to find $P[A_1B]$. We are given that

$$\begin{aligned} P[A_1] &= .41 & P[B|A_1] &= .88 \\ P[A_2] &= .09 & P[B|A_2] &= .04 \\ P[A_3] &= .04 & P[B|A_3] &= .10 \\ P[A_4] &= .46 & P[B|A_4] &= .04 \end{aligned}$$

Substitution into the expression given by Bayes' theorem yields

$$\begin{aligned} P[A_1B] &= \frac{(.88)(.41)}{(.88)(.41) + (.04)(.09) + (.10)(.04) + (.04)(.46)} \\ &\doteq .93 \end{aligned}$$

If a person was typed as having type A blood, there was approximately a 93% chance that his true type was in fact type A.

## CHAPTER SUMMARY

In this chapter we presented some of the laws that govern the behavior of probabilities. We began with the axioms, and from those we were able to derive the remaining laws. In particular, we derived the addition rule, which deals with the probability of the union of two events; the multiplication rule, which deals with the probability of the intersection of two events; and Bayes' theorem, which deals with conditional probability. We introduced and defined important terms that you should know. These are:

| | |
|---|---|
| Conditional probability | Independent events |

Care must be taken when using the concept of independence. In an applied problem, be sure that it is reasonable to assume that events $A$ and $B$ are independent before finding the probability of their joint occurrence via the definition $P[A \cap B] = P[A]P[B]$.

## EXERCISES

### Section 2.1

1. The probability that a wildcat well will produce oil is 1/13. What is the probability that it will not be productive?
2. The theft of precious metals from companies in the United States was and is a serious problem. The estimated probability that such a theft will involve a particular metal is given below: (Based on data reported in "Materials Theft," *Materials Engineering*, February 1982, pp. 27–31.)

| | | |
|---|---|---|
| tin: 1/35 | platinum: 1/35 | nickel: 1/35 |
| steel: 11/35 | gold: 5/35 | zinc: 1/35 |
| copper: 8/35 | aluminum: 2/35 | silver: 4/35 |
| titanium: 1/35 | | |

(Note that these events are assumed to be mutually exclusive.)

(*a*) What is the probability that a theft of precious metal will involve gold, silver, or platinum?

(*b*) What is the probability that a theft will not involve steel?

3. Assuming the blood type distribution to be A: 41%, B: 9%, AB: 4%, O: 46%, what is the probability that the blood of a randomly selected individual will contain the A antigen? That it will contain the B antigen? That it will contain neither the A nor the B antigen?
4. Assume that the engine component of a spacecraft consists of two engines in parallel. If the main engine is 95% reliable, the backup is 80% reliable, and the engine component as a whole is 99% reliable, what is the probability that both engines will be operable? Use a Venn diagram to find the probability that the main engine will fail but the backup will be operable. Find the probability that the backup engine will fail but the main engine will be operable. What is the probability that the engine component will fail?
5. When an individual is exposed to radiation, death may ensue. Factors affecting the outcome are the size of the dose, the length and intensity of the exposure,

and the biological makeup of the individual. The term $LD_{50}$ is used to denote the dose that is usually lethal for 50% of the individuals exposed to it. Assume that in a nuclear accident 30% of the workers are exposed to the $LD_{50}$ and die; 40% of the workers die; and 68% are exposed to the $LD_{50}$ or die. What is the probability that a randomly selected worker is exposed to the $LD_{50}$? Use a Venn diagram to find the probability that a randomly selected worker is exposed to the $LD_{50}$ but does not die. Find the probability that a randomly selected worker is not exposed to the $LD_{50}$ but dies.

6. When a computer goes down, there is a 75% chance that it is due to an overload and a 15% chance that it is due to a software problem. There is an 85% chance that it is due to an overload or a software problem. What is the probability that both of these problems are at fault? What is the probability that there is a software problem but no overload?
7. Due to the recent energy crisis in California, rolling blackouts were necessary and more might be necessary in the future. Assume that there is a 60% chance that the temperature will exceed 85° F on any given day in July in a particular area. Assume that there is a 30% chance that a rolling blackout will be needed in that area. There is a 20% chance that both events will occur. Find the probability that the temperature will exceed 85° F on a given July day but that no rolling blackout will be needed on that day.
8. Experience shows that 25% of all complaints about home telephone lines involve static on the line. Fifty percent involve line deterioration. Thirty-five percent involve only line deterioration. What is the probability that a randomly selected complaint will involve both problems? Will involve neither problem?
9. Assume that in a particular military exercise involving two units, Red and Blue, there is a 60% chance that the Red unit will successfully meet its objectives and a 70% chance that the Blue unit will do so. There is an 18% chance that only the Red unit will be successful. What is the probability that both units will meet their objectives? What is the probability that one or the other but not both of the units will be successful?
10. It has been found that 80% of all accidents at foundries involve human error and 40% involve equipment malfunction. Thirty-five percent involve both problems. An accident at a foundry is investigated. What is the probability that human error alone was involved?
11. Assume that 1% of all tires of a particular brand are defective due to a problem with a supplier of an important chemical component of the tire. Assume that .5% of this brand of tire will eventually fail due to sidewall blowouts. Also, 1.4% of this brand of tire experience at least one of these problems. What is the probability that in a future accident involving these tires, a blowout will occur but there will be no problem found with the chemical composition of the tire?
12. (*a*) Derive Theorem 2.1.1.
    *Hint*: Note that $S = S \cup \varnothing$ and that $S$ and $\varnothing$ are mutually exclusive. Apply axioms 3 and 1.

    (*b*) Derive Theorem 2.1.2.
    *Hint*: Note that $S = A \cup A'$ and that $A$ and $A'$ are mutually exclusive. Apply axioms 3 and 1.

(*c*) Let $A$ be a subset of $B$. Show that $P[A] \leq P[B]$.
*Hint:* $B = A \cup (A' \cap B)$. Apply axioms 3 and 2.

(*d*) Show that the probability of any event $A$ is at most 1.
*Hint:* $A \subseteq S$. Apply Exercise 12C and axiom 1.

(*e*) Let $A_1$ and $A_2$ be mutually exclusive. By axiom 3, $P[A_1 \cup A_2] = P[A_1] + P[A_2]$. Show that the general addition rule yields the same result.

**Section 2.2**

**13.** Use the data of Exercise 5 to answer these questions.

(*a*) What is the probability that a randomly selected worker will die given that he is exposed to the lethal dose of radiation?

(*b*) What is the probability that a randomly selected worker will not die given that he is exposed to the lethal dose of radiation?

(*c*) What theorem allows you to determine the answer to (*b*) from knowledge of the answer to (*a*)?

(*d*) What is the probability that a randomly selected worker will die given that he is not exposed to the lethal dose?

(*e*) Is $P[\text{die}] = P[\text{die}|\text{exposed to lethal dose}]$? Did you expect these to be the same? Explain.

**14.** Use the data of Exercise 4 to answer these questions.

(*a*) What is the probability that in an engine system such as that described the backup engine will function given that the main engine fails?

(*b*) Is $P[\text{backup functions}] = P[\text{backup functions}|\text{main fails}]$? Did you expect these to be the same? Explain.

**15.** In a study of waters near power plants and other industrial plants that release wastewater into the water system it was found that 5% showed signs of chemical and thermal pollution, 40% showed signs of chemical pollution, and 35% showed evidence of thermal pollution. Assume that the results of the study accurately reflect the general situation. What is the probability that a stream that shows some thermal pollution will also show signs of chemical pollution? What is the probability that a stream showing chemical pollution will not show signs of thermal pollution?

**16.** A random digit generator on an electronic calculator is activated twice to simulate a random two-digit number. Theoretically, each digit from 0 to 9 is just as likely to appear on a given trial as any other digit.

(*a*) How many random two-digit numbers are possible?

(*b*) How many of these numbers begin with the digit 2?

(*c*) How many of these numbers end with the digit 9?

(*d*) How many of these numbers begin with the digit 2 and end with the digit 9?

(*e*) What is the probability that a randomly formed number ends with 9 given that it begins with a 2. Did you anticipate this result?

**17.** In studying the causes of power failures, these data have been gathered.

5% are due to transformer damage
80% are due to line damage
1% involve both problems

Based on these percentages, approximate the probability that a given power failure involves
(*a*) line damage given that there is transformer damage
(*b*) transformer damage given that there is line damage
(*c*) transformer damage but not line damage
(*d*) transformer damage given that there is no line damage
(*e*) transformer damage or line damage

**Section 2.3**

**18.** Let $A_1$ and $A_2$ be events such that $P[A_1] = .5$, $P[A_2] = .7$. What must $P[A_1 \cap A_2]$ equal for $A_1$ and $A_2$ to be independent?

**19.** Let $A_1$ and $A_2$ be events such that $P[A_1] = .6$, $P[A_2] = .4$, and $P[A_1 \cup A_2] = .8$. Are $A_1$and $A_2$ independent?

**20.** Consider your answer to Exercise 14(*b*). Are the events $A_1$: the backup engine functions, and $A_2$: the main engine fails independent?

**21.** Studies in population genetics indicate that 39% of the available genes for determining the Rh blood factor are negative. Rh negative blood occurs if and only if the individual has two negative genes. One gene is inherited independently from each parent. What is the probability that a randomly selected individual will have Rh negative blood?

**22.** An individual's blood group (A, B, AB, O) is independent of the Rh classification. Find the probability that a randomly selected individual will have AB negative blood. *Hint:* See Example 2.1.1 and Exercise 21.

**23.** The use of plant appearance in prospecting for ore deposits is called geobotanical prospecting. One indicator of copper is a small mint with a mauve-colored flower. Suppose that, for a given region, there is a 30% chance that the soil has a high copper content and a 23% chance that the mint will be present there. If the copper content is high, there is a 70% chance that the mint will be present.
(*a*) Find the probability that the copper content will be high and the mint will be present.
(*b*) Find the probability that the copper content will be high given that the mint is present.

**24.** The most common water pollutants are organic. Since most organic materials are broken down by bacteria that require oxygen, an excess of organic matter may result in a depletion of available oxygen. In turn this can be harmful to other organisms living in the water. The demand for oxygen by the bacteria is called the biological oxygen demand (BOD). A study of streams located near an industrial complex revealed that 35% have a high BOD, 10% show high acidity, and 40% of streams with high acidity have a high BOD. Find the probability that a randomly selected stream will exhibit both characteristics.

**25.** A study of major flash floods that occurred over the last 15 years indicates that the probability that a flash flood warning will be issued is .5 and that the probability of dam failure during the flood is .33. The probability of dam failure given that a warning is issued is .17. Find the probability that a flash flood warning will be issued and a dam failure will occur. (Based on data reported in *McGraw-Hill Yearbook of Science and Technology,* 1980, pp. 185–186.)

**26.** The ability to observe and recall details is important in science. Unfortunately, the power of suggestion can distort memory. A study of recall is conducted as follows: Subjects are shown a film in which a car is moving along a country road. There is no barn in the film. The subjects are then asked a series of questions concerning the film. Half the subjects are asked, "How fast was the car moving when it passed the barn?" The other half is not asked the question. Later each subject is asked, "Is there a barn in the film?" Of those asked the first question concerning the barn, 17% answer "yes"; only 3% of the others answer "yes." What is the probability that a randomly selected participant in this study claims to have seen the nonexistent barn? Is claiming to see the barn independent of being asked the first question about the barn? *Hint:*

$$P[\text{yes}] = P[\text{yes and asked about barn}] + P[\text{yes and not asked about barn}]$$

(Based on a study reported in *McGraw-Hill Yearbook of Science and Technology,* 1981, pp. 249–251.)

**27.** The probability that a unit of blood was donated by a paid donor is .67. If the donor was paid, the probability of contracting serum hepatitis from the unit is .0144. If the donor was not paid, this probability is .0012. A patient receives a unit of blood. What is the probability of the patient's contracting serum hepatitis from this source?

**28.** Show that the impossible event is independent of every other event.

**29.** Consider the percentages given in Exercise 7. Find the probability of a rolling blackout occurring on a day on which the temperature exceeds 85° F. If the probabilities given are assumed to be exact, is the event that a rolling blackout occurs independent of the event that the temperature exceed 85° F ? Explain based on the probability that you just computed.

**30.** Assume that there is a 50% chance of hard drive damage if a power line to which a computer is connected is hit during an electrical storm. There is a 5% chance that an electrical storm will occur on any given summer day in a given area. If there is a .1% chance that the line will be hit during a storm, what is the probability that the line will be hit and there will be hard drive damage during the next electrical storm in this area?

**31.** A foundry is producing cast iron parts to be used in the automatic transmissions of trucks. There are two crucial dimensions to the part, $A$ and $B$. Assume that if the part meets specifications on dimension $A$ then there is a 98% chance that it will also meet specifications on dimension $B$. There is a 95% chance that it will meet specifications on dimension $A$ and a 97% chance that it will meet specifications on dimension $B$. A part is randomly selected and inspected. What is the probability that it will meet specifications on both dimensions?

**32.** Let $A_1$ and $A_2$ be mutually exclusive events such that $P[A_1]P[A_2] > 0$. Show that these events are not independent.

**33.** Let $A_1$ and $A_2$ be independent events such that $P[A_1]P[A_2] > 0$. Show that these events are not mutually exclusive.

**Section 2.4**

**34.** Use the data of Example 2.4.2 to find the probability that an inductee who was typed as having type A blood actually had type B blood.

**35.** A test has been developed to detect a particular type of arthritis in individuals over 50 years old. From a national survey it is known that approximately 10% of the individuals in this age group suffer from this form of arthritis. The proposed test was given to individuals with confirmed arthritic disease, and a correct test result was obtained in 85% of the cases. When the test was administered to individuals of the same age group who were known to be free of the disease, 4% were reported to have the disease. What is the probability that an individual has this disease given that the test indicates its presence?

**36.** It is reported that 50% of all computer chips produced are defective. Inspection ensures that only 5% of the chips legally marketed are defective. Unfortunately, some chips are stolen before inspection. If 1% of all chips on the market are stolen, find the probability that a given chip is stolen given that it is defective.

**37.** As society becomes dependent on computers, data must be communicated via public communication networks such as satellites, microwave systems, and telephones. When a message is received, it must be authenticated. This is done by using a secret enciphering key. Even though the key is secret, there is always the possibility that it will fall into the wrong hands, thus allowing an unauthentic message to appear to be authentic. Assume that 95% of all messages received are authentic. Furthermore, assume that only .1% of all unauthentic messages are sent using the correct key and that all authentic messages are sent using the correct key. Find the probability that a message is authentic given that the correct key is used.

## REVIEW EXERCISES

**38.** A survey of engineering firms reveals that 80% have their own mainframe computer ($M$), 10% anticipate purchasing a mainframe computer in the near future ($B$), and 5% have a mainframe computer and anticipate buying another in the near future. Find the probability that a randomly selected firm:

(*a*) has a mainframe computer or anticipates purchasing one in the near future

(*b*) does not have a mainframe computer and does not anticipate purchasing one in the near future

(*c*) anticipates purchasing a mainframe computer given that it does not currently have one

(*d*) has a mainframe computer given that it anticipates purchasing one in the near future

**39.** In a simulation program, three random two-digit numbers will be generated independently of one another. These numbers assume the values 00, 01, 02, . . . , 99 with equal probability.

(*a*) What is the probability that a given number will be less than 50?

(*b*) What is the probability that each of the three numbers generated will be less than 50?

**40.** A power network involves three substations *A*, *B*, and *C*. Overloads at any of these substations might result in a blackout of the entire network. Past history has shown that if substation *A* alone experiences an overload, then there is a 1% chance of a network blackout. For stations *B* and *C* alone these percentages are 2% and 3%, respectively. Overloads at two or more substations simultaneously result in a blackout 5% of the time. During a heat wave there is a 60% chance that substation *A* alone will experience an overload. For stations *B* and *C* these percentages are 20 and 15%, respectively. There is a 5% chance of an overload at two or more substations simultaneously. During a particular heat wave a blackout due to an overload occurred. Find the probability that the overload occurred at substation *A* alone; substation *B* alone; substation *C* alone; two or more substations simultaneously.

**41.** A computer center has three printers, *A*, *B*, and *C*, which print at different speeds. Programs are routed to the first available printer. The probability that a program is routed to printers *A*, *B*, and *C* are .6, .3, and .1, respectively. Occasionally a printer will jam and destroy a printout. The probability that printers *A*, *B*, and *C* will jam are .01, .05, and .04, respectively. Your program is destroyed when a printer jams. What is the probability that printer *A* is involved? Printer *B* is involved? Printer *C* is involved?

**42.** A chemical engineer is in charge of a particular process at an oil refinery. Past experience indicates that 10% of all shutdowns are due to equipment failure *alone*, 5% are due to a combination of equipment failure and operator error, and 40% involve operator error. A shutdown occurs. Find the probability that

(*a*) equipment failure or operator error is involved

(*b*) operator error alone is involved

(*c*) neither operator error nor equipment failure is involved

(*d*) operator error is involved given that equipment failure occurs

(*e*) operator error is involved given that equipment failure does not occur

**43.** Assume that the probability that the air brakes on large trucks will fail on a particularly long downgrade is .001. Assume also that the emergency brakes on such trucks can stop a truck on this downgrade with probability .8. These braking systems operate independently of one another. Find the probability that

(*a*) the air brakes fail but the emergency brakes can stop the truck

(*b*) the air brakes fail and the emergency brakes cannot stop the truck

(*c*) the emergency brakes cannot stop the truck given that the air brakes fail

**44.** Consider the problem of Example 1.2.3. Assume that sampling is independent and that at each stage the probability of obtaining a defective part when the process is working correctly is .01. If the process is working correctly, what is the probability that the first defective part will be obtained on the fourth sample? On or before the fourth sample?

CHAPTER
# 3

# DISCRETE DISTRIBUTIONS

In the sciences one often deals with "variables." Webster's dictionary defines a variable as a "quantity that may assume any one of a set of values." In statistics we deal with *random variables*—variables whose observed value is determined by chance. Many of the examples presented in previous chapters involved random variables even though the term was not used at the time. Random variables usually fall into one of two categories; they are either discrete or continuous. We begin by learning to recognize discrete random variables. The remainder of the chapter is devoted to the study of random variables of this type.

## 3.1 RANDOM VARIABLES

We begin by considering three examples, each of which involves a random variable. Random variables will be denoted by uppercase letters and their observed numerical values by lowercase letters.

**Example 3.1.1.** Consider the random variable $X$, the number of brown-eyed children born to a couple heterozygous for eye color. If the couple is assumed to have two children, a priori, before the fact, the variable $X$ can assume any one of the values 0, 1, or 2. The variable is random in that brown eyes depend on the chance inheritance of a dominant gene at conception. If for a particular couple there are two brown-eyed children, we write $x = 2$.

**Example 3.1.2.** The basic premise underlying the field of immunology is that an animal is immunized by injection of a suitable antigen. In one study malignant plasmacytoma cells are exposed to lymphocytes carrying a specific antigen. It is hoped that these cells will fuse, because the fused cells retain the ability to grow continuously and also to retain the antibody characteristics of the antigen fused. In this way the animal is quickly immunized. Cells are exposed to the lymphocytes one at a time in the presence