Computational methods for Medical Physics

Lecture 5: Fourier transform and backprojection

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Contents

- Last lecture:
  - Convolution
  - Pencil beam algorithm for dose calculation

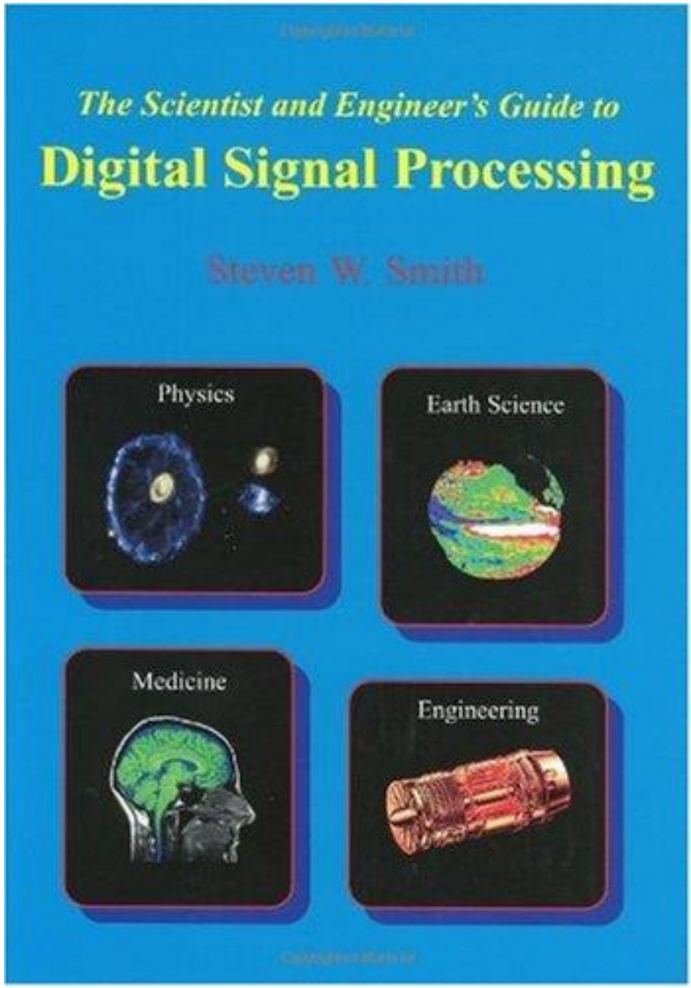
- This Lecture:
  - Fourier transform
  - Introduction to Computed tomography and (Filtered) Backprojection

Lecture material

- Material for the tomography part was taken from:

The Scientist and Engineer's Guide to Digital Signal Processing, copyright ©1997-1998 by Steven W. Smith

http://www.dspguide.com/



The Fourier Transform

- A mathematical operation which maps a function f defined in the domain of space or time (x), to a function F in the domain of wavelength or frequency (ξ)

$$F(\xi) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi i x \xi} dx$$

$$f(x) = \int_{-\infty}^{+\infty} F(\xi) e^{2\pi i x \xi} dx$$

- Where $e^{-2\pi i \xi}$ is the Euler's formula:

$$e^{-2\pi i \xi} = \cos(2\pi\xi) - i \sin(2\pi\xi)$$

The Fourier Transform

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- It is a complex formula:
  - Imaginary part: describes the phase shift of each sinusoid with frequency (ξ)
  - Magnitude: describes the amplitude (strength of the contribution) of a given frequency (ξ)

The Fourier Transform

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$$f(x) = \int_{-\infty}^{+\infty} F(\xi) e^{2\pi i x \xi} dx$$

- In two dimensions:

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-2\pi i x u} e^{-2\pi i y v} dx dy$$

The Fourier Transform

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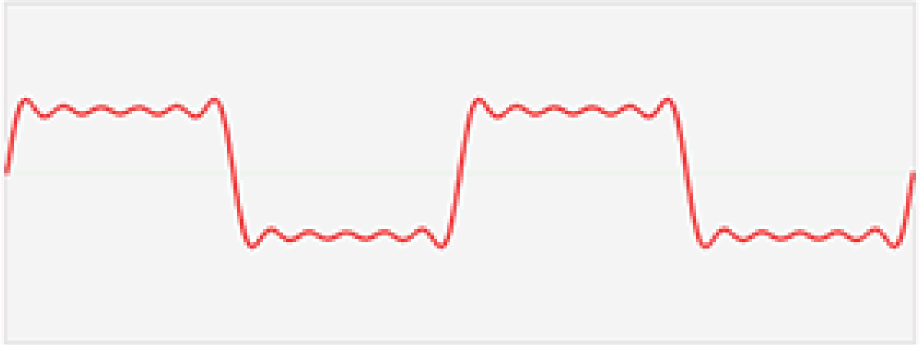


https://en.wikipedia.org/wiki/Fourier_transform

The Fourier Transform

- Some of properties of the Fourier Transform generally notated as: $F\{f\} = F(\xi)$

Linearity:

$$F\{a \cdot f + b \cdot g\} = a \cdot F\{f\} + b \cdot F\{g\}$$

Shift property:

$$F\{f(x - x_0)\} = e^{-2\pi i \xi x_0} F\{f(x)\}$$

....

Derivative property:

$$F\left\{\frac{d^n f(x)}{dx^n}\right\} = (2\pi i \xi)^n F\{f(x)\}$$

The Fourier Transform

- Why is it so useful?
  - On differential equations:

$$\frac{d^2 f(x)}{dx^2} - f(x) = -g(x)$$

$$F\left\{\frac{d^2 f(x)}{dx^2}\right\} - F = -G$$

$$F\left\{\frac{d^n f(x)}{dx^n}\right\} = (2\pi i\xi)^n F$$

$$F = \frac{G}{1 + 4\pi^2\xi^2}$$

  - Through Fourier transform, differential equations are transformed into simple algebraic equations

The Fourier Transform

- Why is it so useful?
  - The convolution theorem:

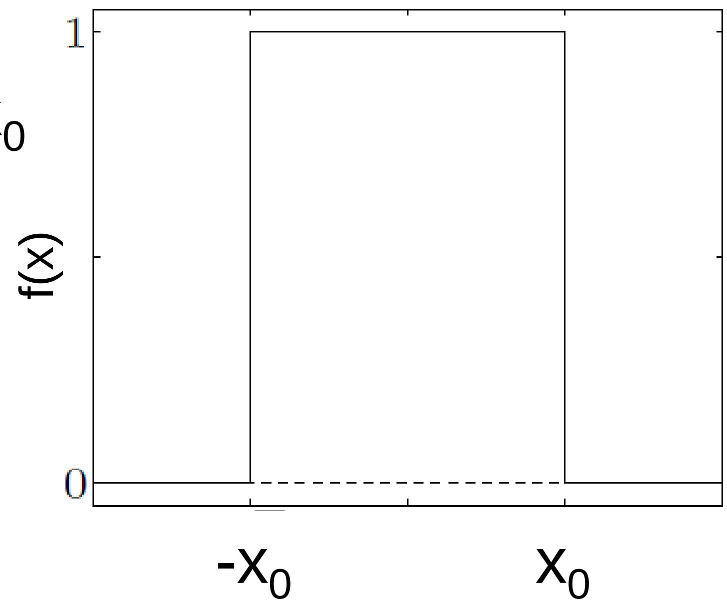
$$F\{f * g\} = F\{f\} \cdot F\{g\}$$

  - Through Fourier transform, convolution operations are transformed into multiplications
    - Filtering in signal/image processing can take place in the frequency domain and then the signal can be converted back to space domain with inverse Fourier transform

The Fourier Transform

http://web.stanford.edu/class/ee102/lectures/fourtran

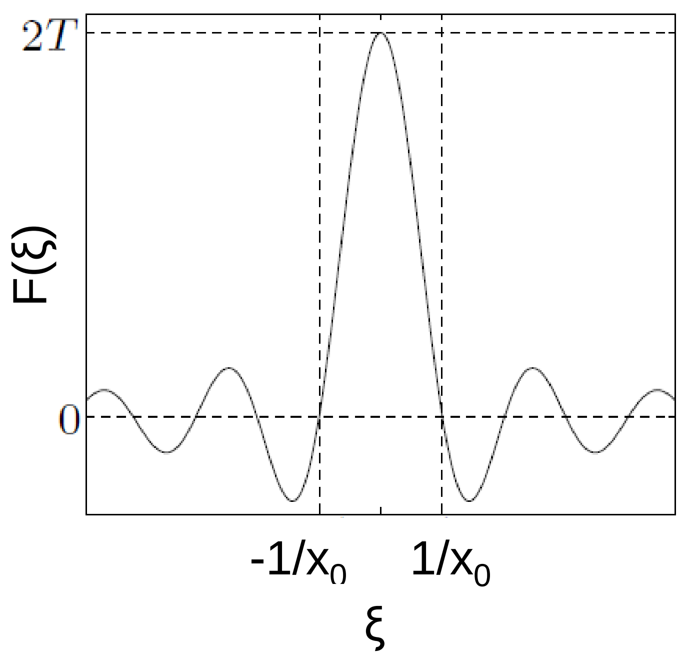
- Simple example:
  - Rectangular pulse: $f(x) = \begin{cases} 1, & -x_0 \le x \le x_0 \\ 0, & |x| > x_0 \end{cases}$



$$F(\xi) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi i x \xi} dx$$

$$F(\xi) = \int_{-x_0}^{x_0} e^{-2\pi i x \xi} dx = \frac{-1}{2\pi i}\left(e^{-2\pi i x_0 \xi} + e^{2\pi i x_0 \xi}\right)$$

$$= \frac{\sin(2\pi x_0 \xi)}{\pi \xi}$$



Discrete Fourier Transform

- In signal processing, the signal are very often in discrete form
- A discrete version of the Fourier Transform is the used:
  - Transforms a finite sequence of equally spaced values x_n into an equally spaced complex-valued samples of the same length in the frequency domain

$$F(\xi) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi i x \xi} dx \quad \Longrightarrow \quad F[k] = \sum_{n=0}^{N-1} f[n] e^{-2\pi i k n / N}$$

Where:
N is the length of x_n (0->N-1)
k is an integer that usually ranges in [0,N-1]. It signifies the frequency component
F(x) is the set of complex numbers indicating the amplitude and frequency

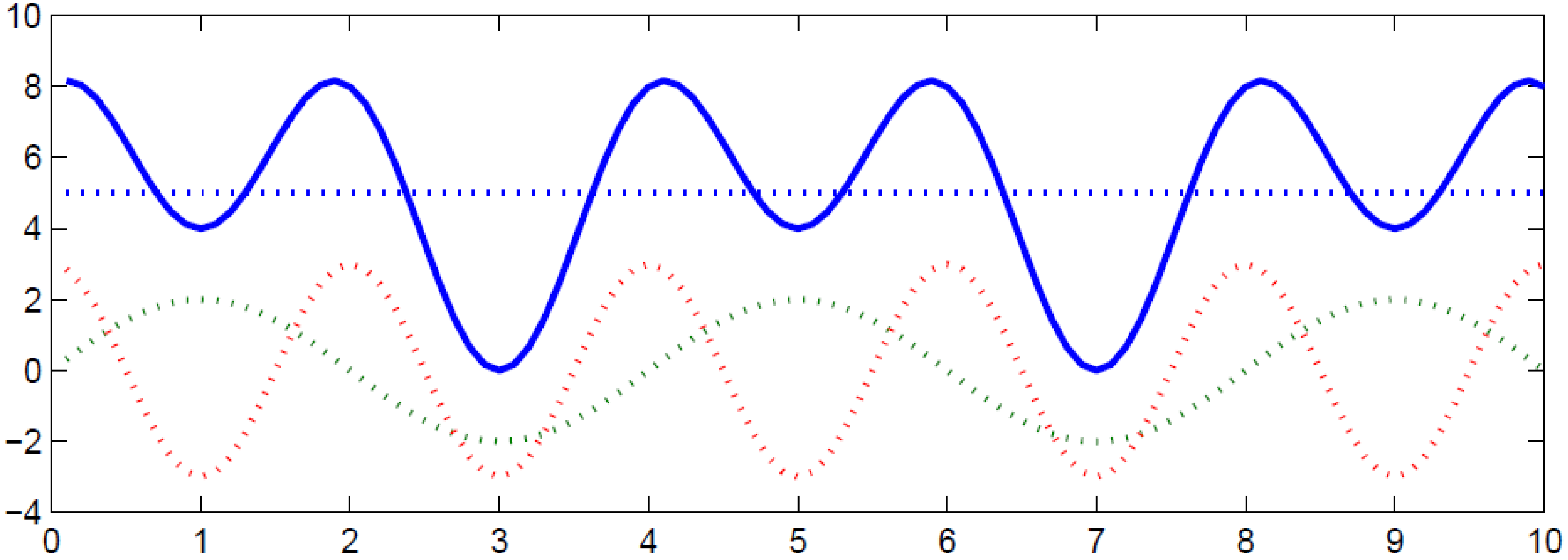
Often: $$F[k] = \sum_{n=0}^{N-1} f[n] \left(\cos\left(-2\pi k \frac{n}{N} \right) + i \sin\left(-2\pi k \frac{n}{N} \right) \right)$$

Discrete Fourier Transform

http://www.robots.ox.ac.uk/~sjrob/Teaching/SP/l7.pdf

- Simple example:
  - A continuous signal in the form of:

$$f(t) = 5 + 2\cos(2\pi t - \pi/2) + 3\cos(4\pi t)$$



Discrete Fourier Transform

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- Simple example:
  - A continuous signal in the form of:

$$f(t) = 5 + 2\cos(2\pi t - \pi/2) + 3\cos(4\pi t)$$

  - Then we create an equidistant discrete set of data by sampling f 4 times per second (4Hz), from t = 0 to t = 3/4

$$f[n] = 5 + 2\cos(k\pi/2 - \pi/2) + 3\cos(\pi k)$$

  - By setting n=k=[0,1,2,3] and t=k/4:

$$f[0] = 8, \quad f[1] = 4, \quad f[2] = 8, \quad f[3] = 0$$

Discrete Fourier Transform

http://www.robots.ox.ac.uk/~sjrob/Teaching/SP/l7.pdf

- Simple example:

$$F[k] = \sum_{n=0}^{N-1} f[n] e^{-2\pi ikn/N} \Rightarrow F[k] = \sum_{n=0}^{3} f[n] e^{-\pi ikn/2}$$

- Using Euler's formula:

$$F[k] = \sum_{n=0}^{N-1} f[n] e^{-2\pi ikn/N} \Rightarrow F[k] = \sum_{n=0}^{3} f[n]\left(\cos(nk\pi/2) - i\sin(nk\pi/2)\right)$$

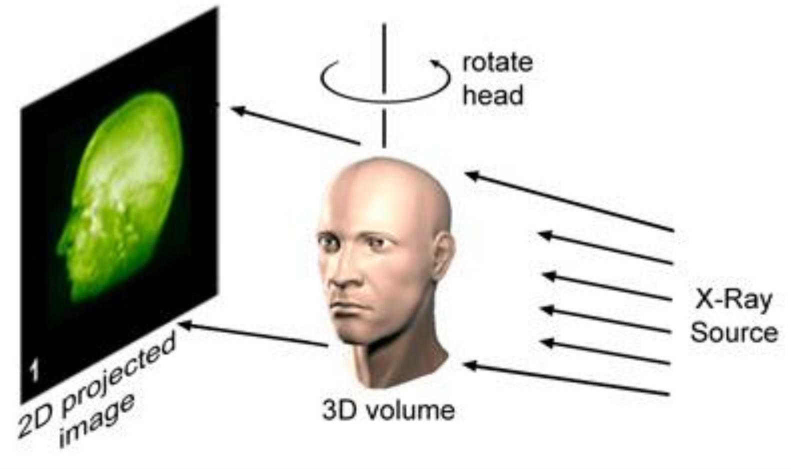
and

$$f[0] = 8, \quad f[1] = 4, \quad f[2] = 8, \quad f[3] = 0$$

$$F[0] = 20, \quad F[1] = -4i, \quad F[2] = 12, \quad F[3] = 4i$$

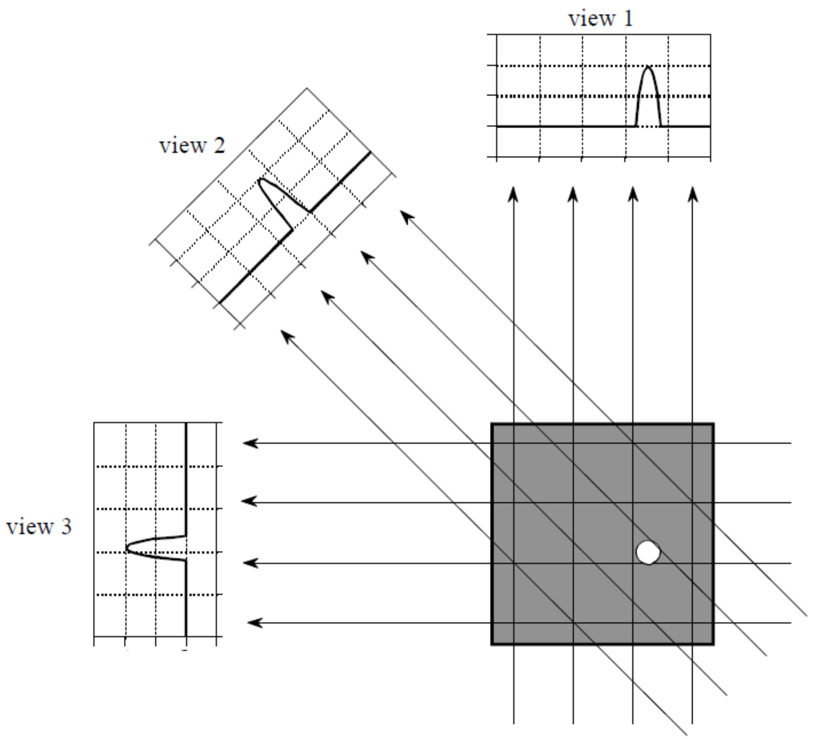
Computed Tomography

- Basic problem: the image obtained from a 3D object is 2D (overlap of structures and characteristics)



- The problem was solved in the 70s
  with the development of Computed Tomography (CT)
- First commercial CT scanner invented by Hounsfield, at EMI
- The first patient brain-scan was done on 1 October 1971

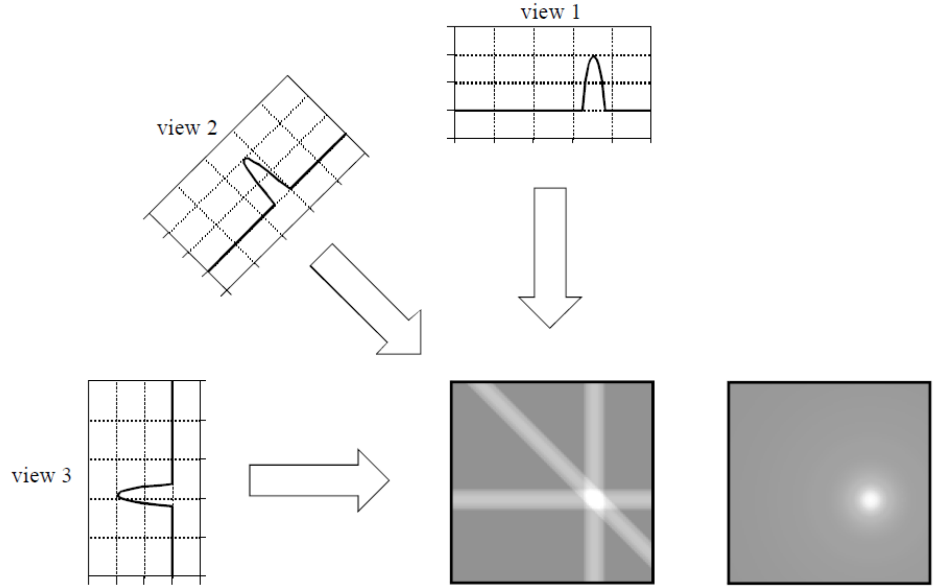
- There have been 4 main categories of methods for CT image reconstruction:
  - Solving of linear equations simultaneously
    - For each measurement in each view the results is the sum of the effect of the particular group of pixels traversed
    - For N pixels, N equations can be written from N measurements
    - In practice, an overdetermined system is needed in order to reduce noise and artifacts
    - Solving a large number of linear equations is computationally a very demanding task



- There have been 4 main categories of methods for CT image reconstruction:
    - Iterative reconstructions
      (more in the next lectures by Dr. Gianoli)
        - An example is the
          Algebraic Reconstruction Technique (ART)
        - All pixels of the image are set to arbitrary values
        - The image is changed in an iterative procedure in order to match the measurements
        - The measurement is compared to the sum of the assumed image effect along a ray pointing to that sample/measurement
        - ART was used in the first EMI commercial scanner

Computed Tomography

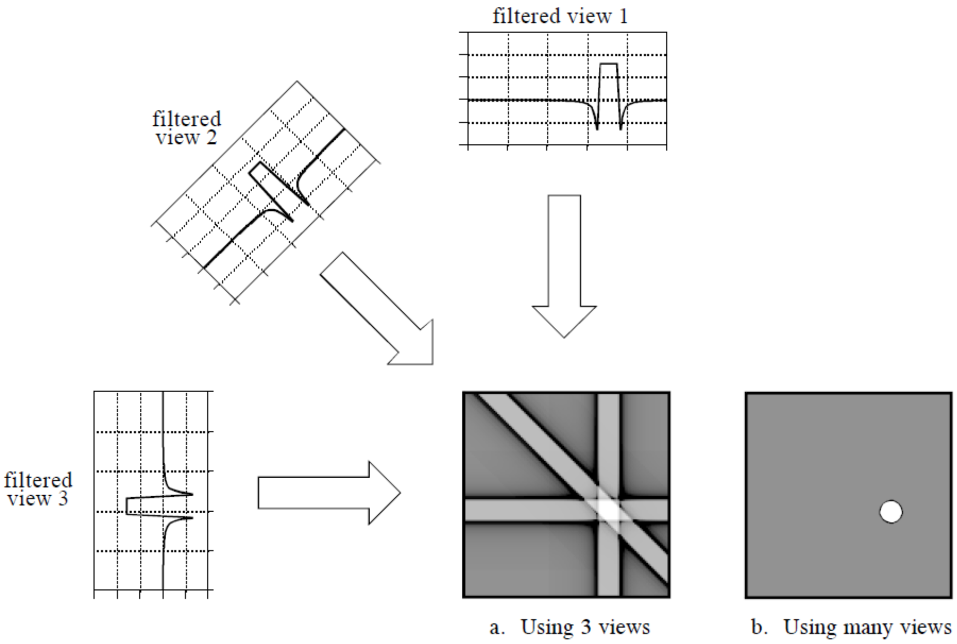
- There have been 4 main categories of methods for CT image reconstruction:
  - (Simple) Backprojection
    - Essentially, each measurement is backprojected by setting all pixel values along the ray for which the measurement was obtained, to the same value
    - The final image is the sum of all backprojected views
    - Simple but not adequate.
      The final image obtained with simple backprojection is blurred



a. Using 3 views
b. Using many views

Computed Tomography

- There have been 4 main categories of methods for CT image reconstruction:
  - Filtered Backprojection
    - A correction (filter) is applied in order to compensate for the blurring
    - Each view is convolved with a filter kernel to create a set of filtered views
    - The filtered views are backprojected
    - The important new element is examining the problem in the frequency domain (far simpler than dealing with it in the space domain)
    - Cornerstone of CT technology (Fourier slice theorem)

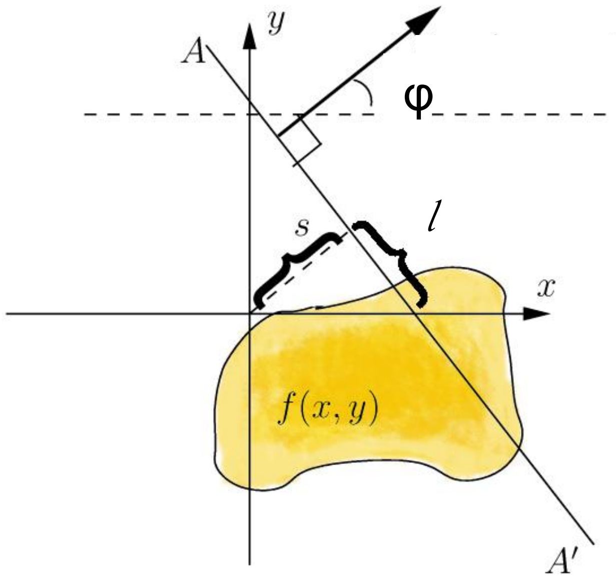


Filtered Backprojection

- The first step in using filtered backprojection is the Radon transform (Johann Radon in 1917)

- For an unknown density f(x,y), the Radon transform g(φ,s) is the integral of the density f along a line L

$$g(\phi, s) = \int_L f(x, y)\, dl$$



- As all the points on a line with distance s from the origin and angle φ, obey the following equation:

$$x \cos\phi + y \sin\phi = s$$

$$g(\phi, s) = \iint f(x, y)\, \delta(x \cos\varphi + y \sin\varphi - s)\, dx\, dy$$

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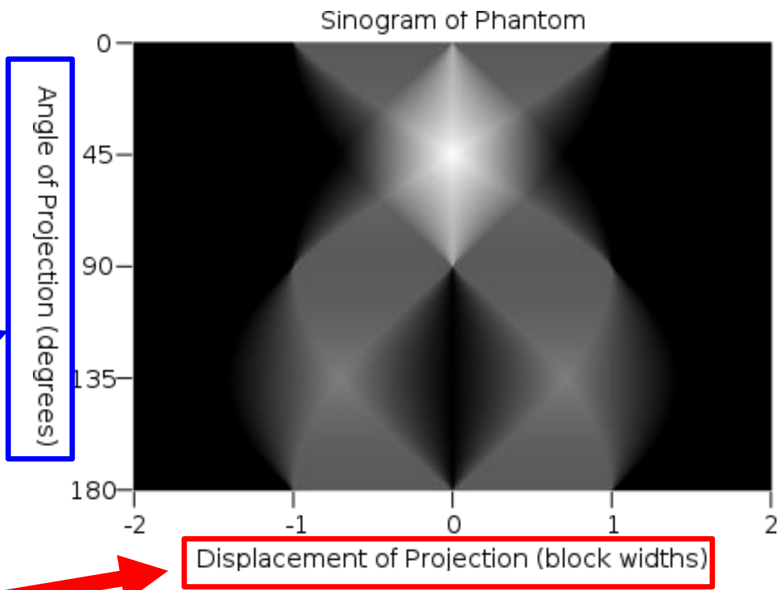

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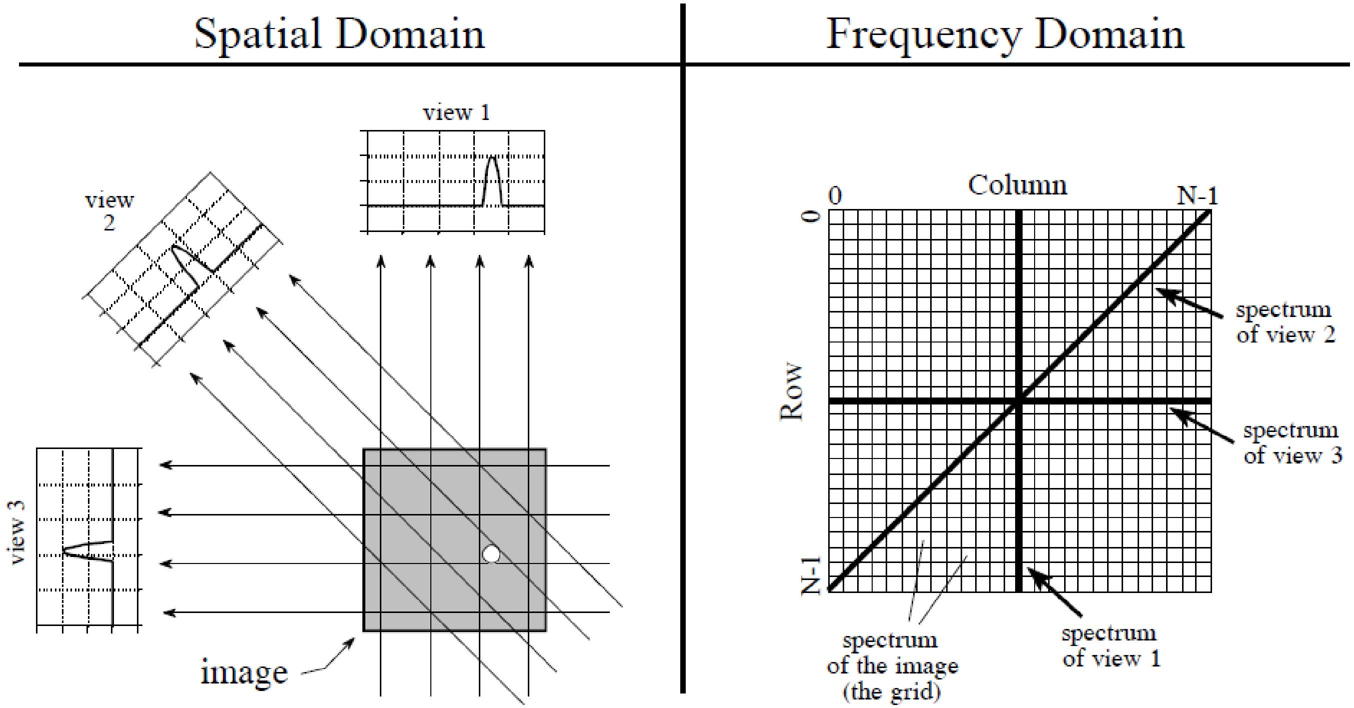
Filtered Backprojection

- Fourier slice theorem or projection-slice theorem:

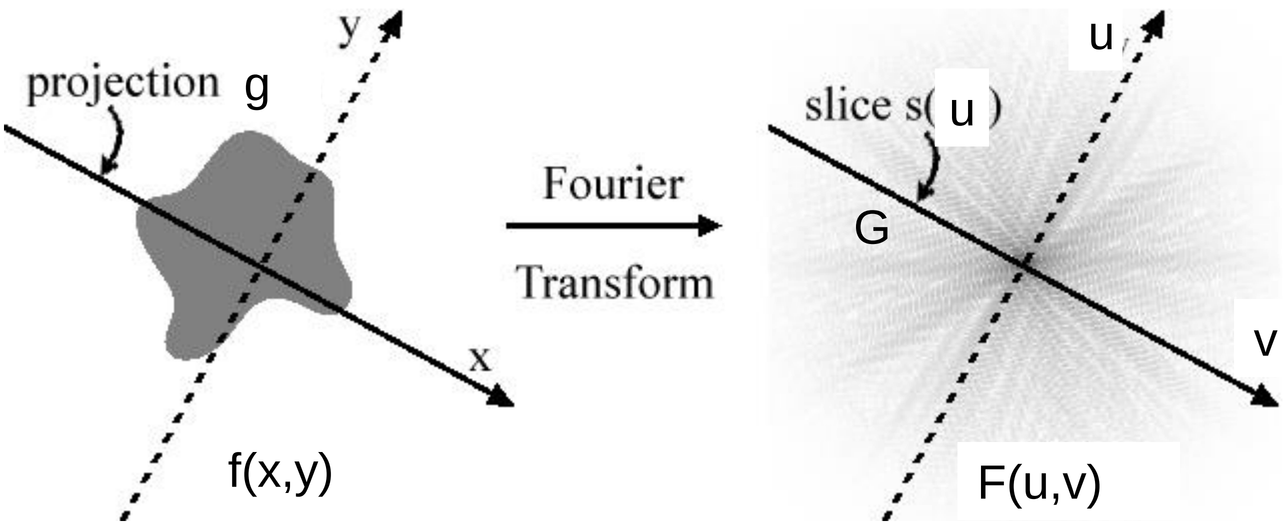


- The 1D projection function g of a 2D function f (Radon transform) when represented in the frequency domain (Fourier transform), is equal to a “slice” through the origin of the 2D Fourier transform of the initial 2D f function, when the “slice” and the projection function correspond to the same angle

Filtered Backprojection

- Fourier transform of the projections:

$$G(\phi,\omega) = \int e^{-i\omega s} g(\phi,s)\,ds$$



Filtered Backprojection

- Fourier transform of the projections:

$$G(\phi, \omega) = \int e^{-i\omega s} g(\phi, s) ds$$

$$G(\phi, \omega) = \iiint e^{-i\omega s} f(x, y) \delta(x \cos \varphi + y \sin \varphi - s) dxdyds$$

$$G(\phi, \omega) = \iint e^{-i\omega(x \sin \phi + y \cos \phi)} f(x, y) dxdy$$

- Which is similar to the 2D Fourier transform definition (for each φ):

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-ixu} e^{-iyv} dxdy$$

- With $u = \omega \cos \phi$ and $v = \omega \sin \phi$

Filtered Backprojection

- The inverse Fourier transform in 2D is:

$$f\left(x,y\right)=\int\int F(u,v)e^{ixu}e^{iyv}\,dudv$$

- In our case:

$$f\left(x,y\right)=\int\int G(\phi,\omega)e^{i\omega x\sin\phi}e^{iy\cos\phi}\left|\omega\right|d\omega d\phi$$

- We have made a change from cartesian to polar coordinates (u,v ->ω) within the integral

- The |ω| is the magnitude of the Jacobian (matrix with partial derivatives of the new coordinates with respect to the old coordinates)

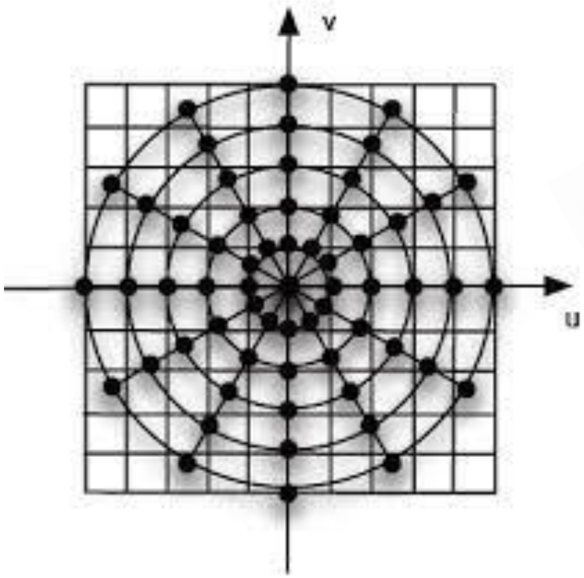
Filtered Backprojection

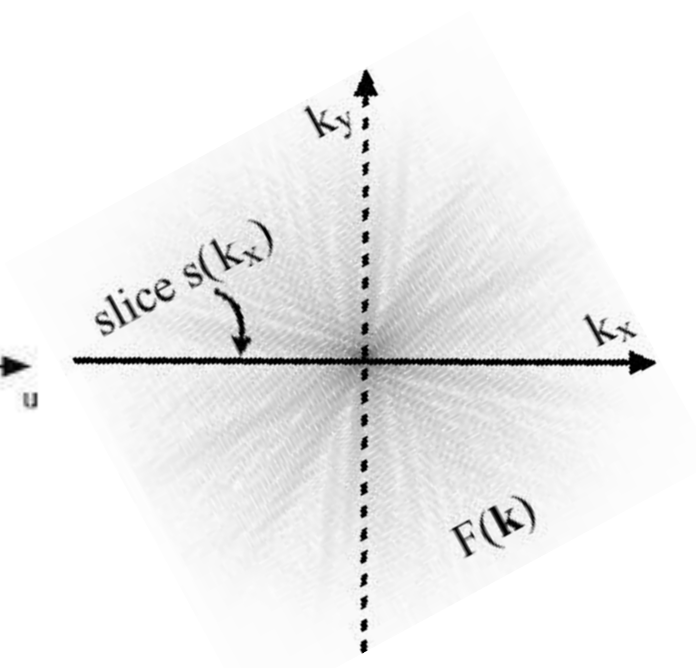
$$f(x,y) = \int\int G(\phi,\omega) e^{i\omega x \sin\phi} e^{iy\cos\phi} |\omega| \, d\omega \, d\phi$$

- The |ω| represents the filter applied to the projections, before backprojecting (inverse transform)

- What happens in essence is that our estimation of the initial 2D images from the 1D projections is not "equidistant"
  (more low frequency than high frequency information)

- The filter compensates for that





Next

- Next 3 lectures, by Dr. Chiara Gianoli

- Content:
  - Starting from imaging (iterative methods)
    as a motivation to optimization (applicable in treatment planning too)