Computational methods for medical physics

Lecture 6:
Tomographic image reconstruction: transmission imaging

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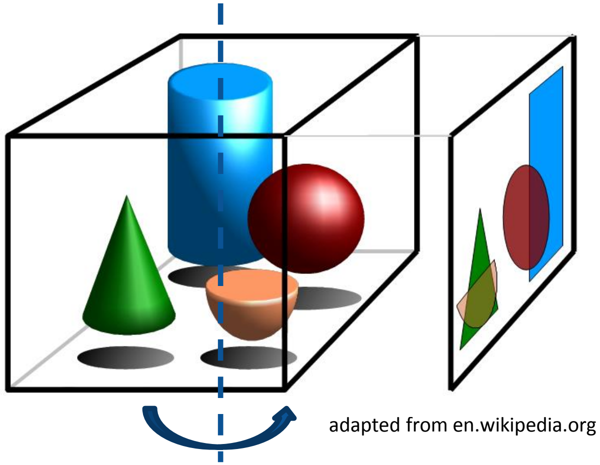
Tomographic imaging

- A tomographic image is a volumetric representation of the variable describing the object of interest

- The variable describes physical properties of the object of interest in terms of any variation of the energy source

- Depending on the energy source, two types of tomographic imaging are identified: transmission tomography (i.e. Computed Tomography, CT) and emission tomography (i.e. Positron Emission Tomography, PET)

- Tomographic image reconstruction is an inverse problem, aiming at the finding the cause of the observation (instead of observing the consequence)



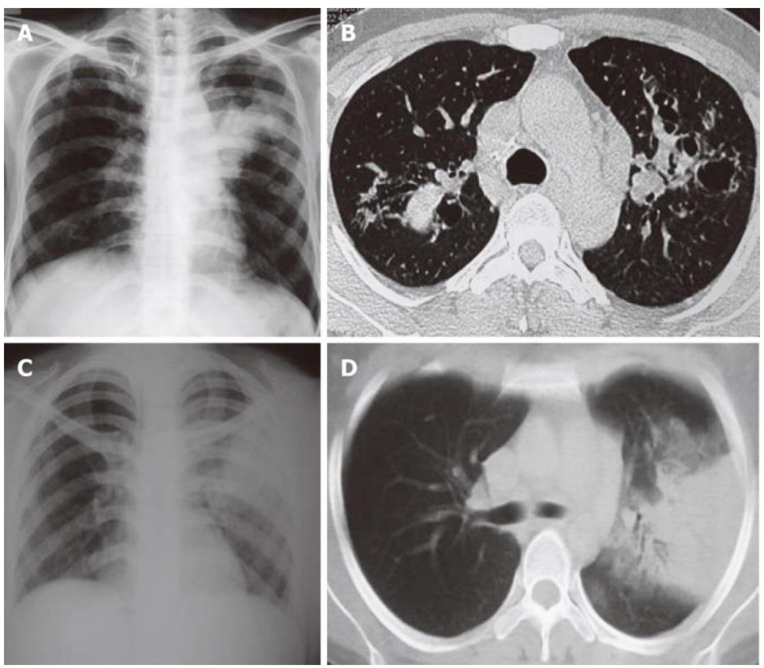
adapted from en.wikipedia.org

- Observation: the projections, at different projection angles with respect to the rotational axis of the imaging system

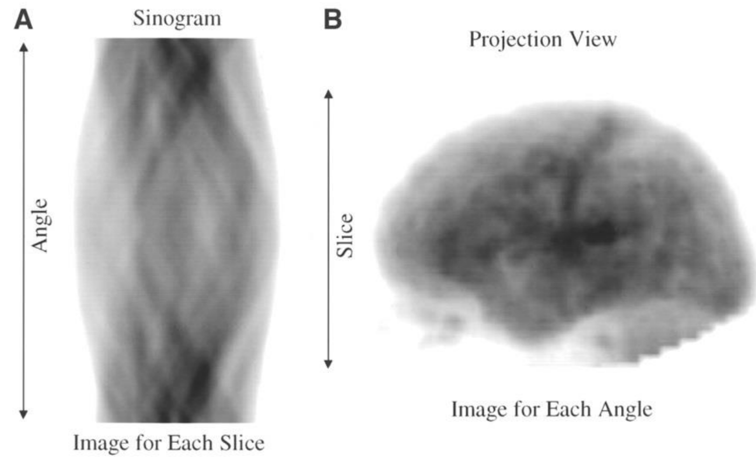
- Cause: physical properties of the object of interest

Tomographic imaging

- The rotational axis of the imaging system is the axis of the cylindrical scanner
  - In CT imaging, the projection is typically visualized as 2D radiography
  - In PET imaging, the projection is typically visualized as 2D sinogram or 2D view



Agarwal et al. *World J Radiol*, 2012;4(4):141.



Adapted from Fahey, *J Nucl Med Technol*, 2002;30(2):39

CT imaging

- The physical properties of the object of interest relate to attenuation of X-ray (or stopping power relative to water of ions), described through attenuation coefficients μ (or water equivalent path length), assumed to be constant for different energies

  - The projection expresses the intensity reduction due to photon attenuation in the object of interest

  - The attenuation is described by <u>Lambert Beer's law:</u>

$$I(x+\Delta x) = I(x) - \mu(x) I(x) \Delta x \qquad \frac{dI}{dx} = -\mu(x) I(x) \qquad I(x) = I(x=0) e^{-\mu x}$$

  - A (linear) tomographic image reconstruction of the attenuation coefficients μ is enable by modelling the projection according to Lambert Beer's law as:

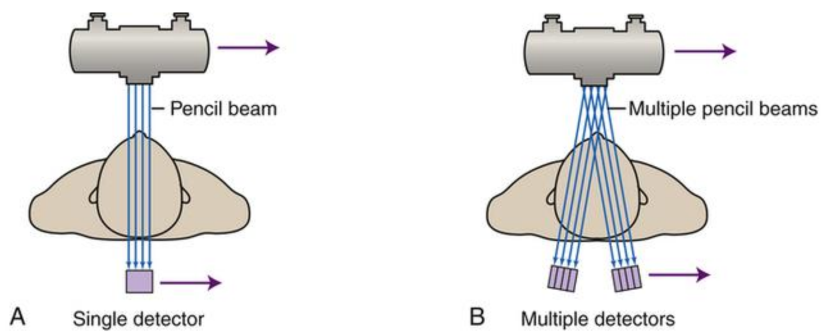
$$-\ln \frac{I(x=X)}{I(x=0)} = \int_0^X \mu(x) dx$$

The imaging system geometry in transmission imaging

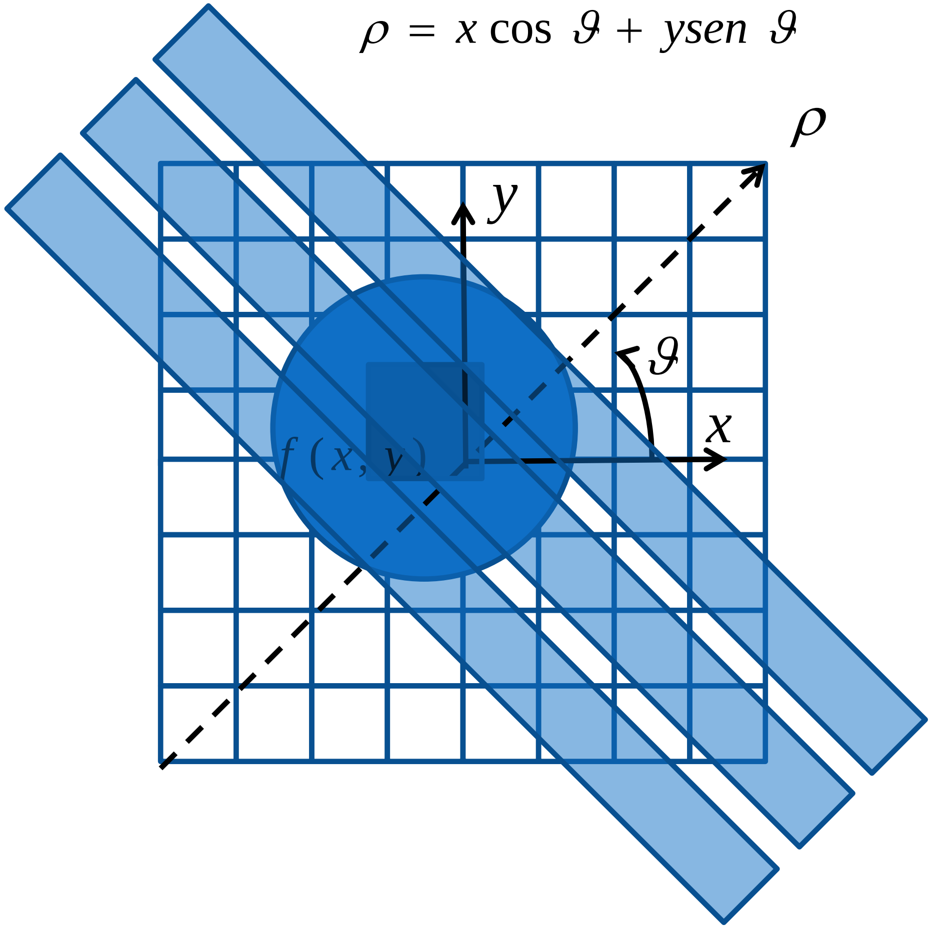
<u>Parallel geometry of the projection line</u>

The radiography is characterized by:

- Infinite center of projection, source to detector
- Fixed projection angle for the projection lines



$$\rho = x\cos\vartheta + y sen\ \vartheta$$



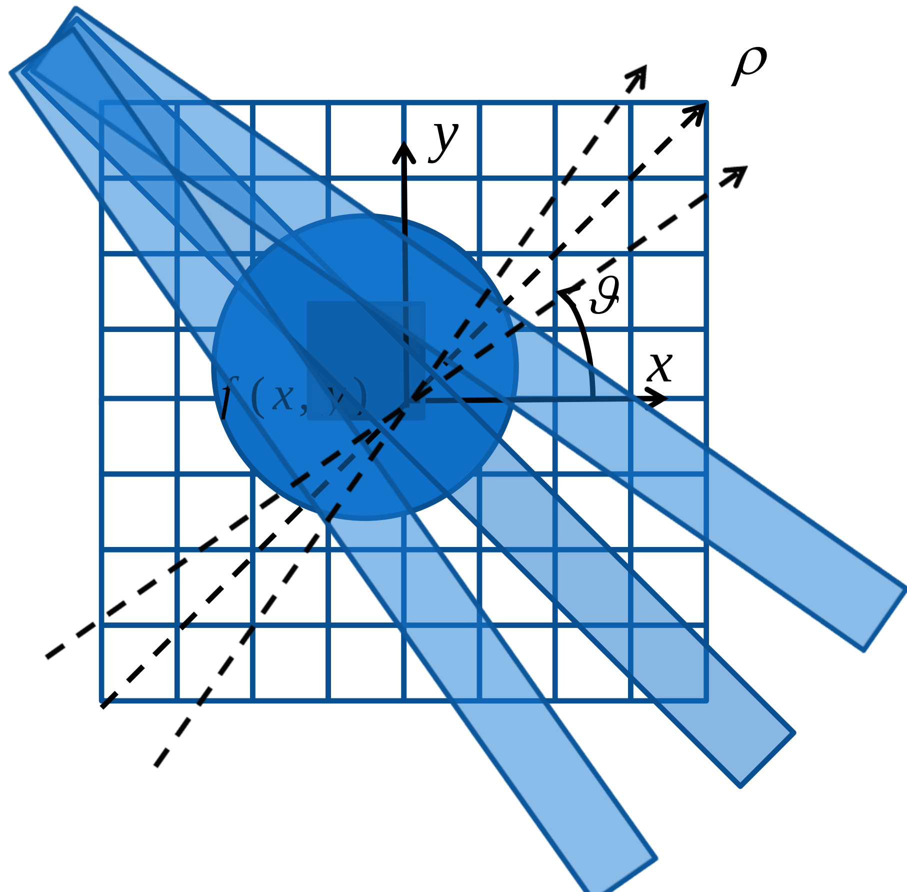
The imaging system geometry in transmission imaging

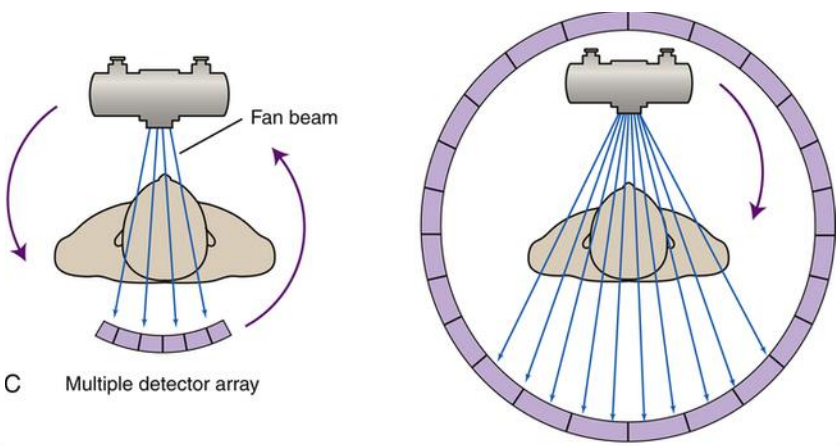
Fan geometry of the projection line

The radiography is characterized by:

- Finite center of projection, source to detector
- Variable projection angle for the projection lines, depending on the fan aperture

$$\rho = x\cos\vartheta + y\,sen\,\vartheta$$





C Multiple detector array

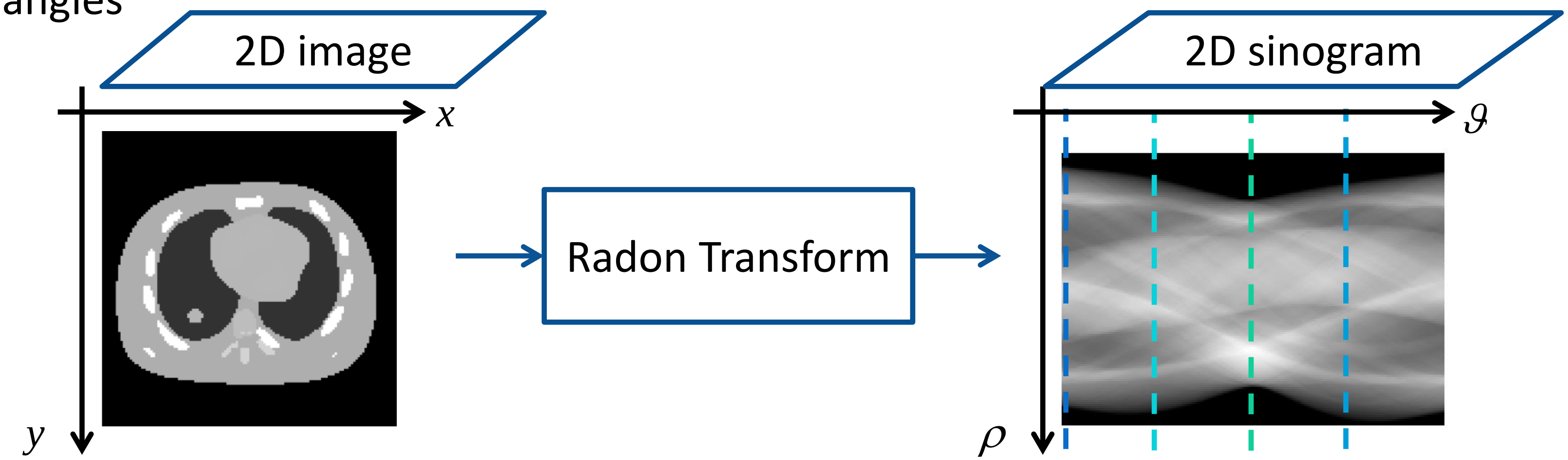
D Stationary ring of detectors

Tomographic image reconstruction: the Radon Transform

- Tomographic image reconstruction (analytical or numerical) is based on the Radon Transform (the 2D sinogram)

- Tomographic image acquisition can be modelled as a Radon Transform of the variable describing the physical properties of the object of interest

- The Radon Transform converts a 2D image from 2D spatial domain to 2D sinogram domain, by integrating the image along the projection lines, as a function of the projection angles



- The Radon Transform requires the description of the imaging system geometry in terms of geometry of the projection lines

Tomographic image reconstruction: the Radon Transform

Continuous form of the Radon Transform

$$g(\rho,\vartheta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,\delta(x\cos\vartheta + y\,sen\,\vartheta - \rho)\,dxdy = R(f)$$

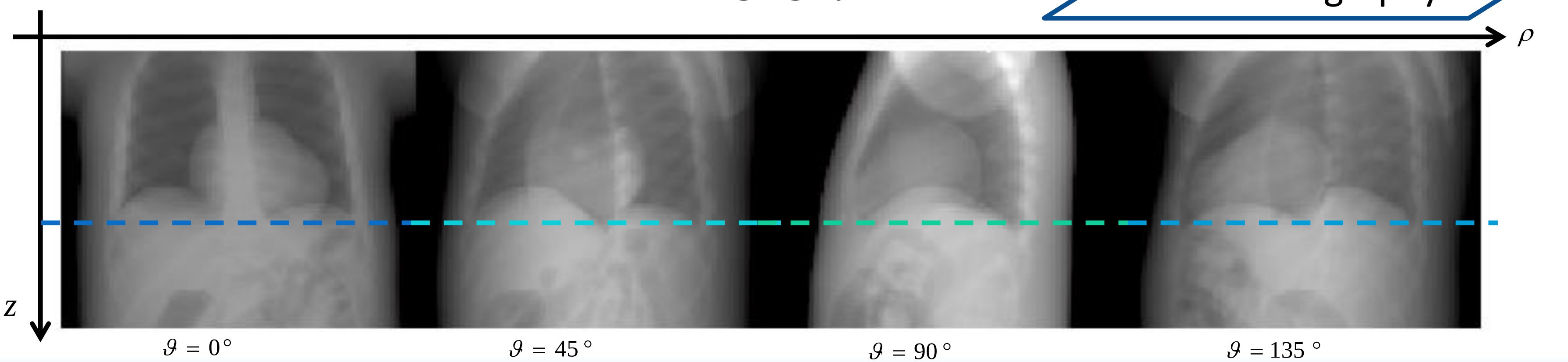
Discrete form of the Radon Transform (the 2D sinogram)

$$g(\rho,\vartheta) = \sum_x \sum_y f(x,y)\,\delta(x\cos\vartheta + y\,sen\,\vartheta - \rho)$$

- 1D radiography $\quad g_\vartheta(\rho) = \sum_x \sum_y f(x,y)\,\delta(x\cos\vartheta + y\,sen\,\vartheta - \rho)$
- 2D radiography $\quad g_\vartheta(\rho,z) = \sum_x \sum_y f(x,y,z)\,\delta(x\cos\vartheta + y\,sen\,\vartheta - \rho, z)$

where *z* is the rotational axis of the imaging system

2D radiography



Analytical image reconstruction

<u>Fourier Slice Theorem (or Central Section Theorem)</u>

- The Fourier Slice Theorem puts in correspondence the <u>Radon Transform</u> with the <u>Fourier Transform (FT)</u> of the 2D image

- The 2D FT of the image evaluated along the projection line in frequency domain, coincides with the 1D FT of the Radon Transform, for the same projection line in spatial domain:

$$\hat{f}_{\rho}(w_x, w_y) = \int_{-\infty}^{+\infty} R(f)\, e^{-2\pi i(\rho w_{\rho})}\, d\rho = \hat{R}(w_{\rho})$$

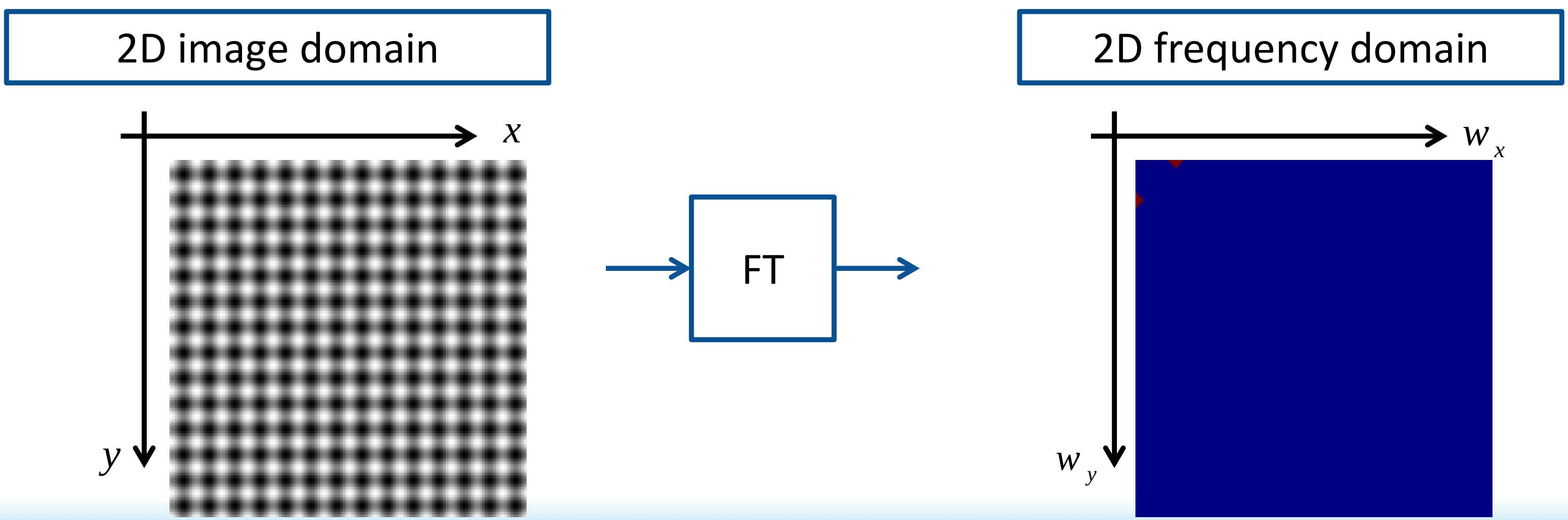
- The analytical image reconstruction is based on the discrete form of Fourier Slice Theorem, according to 2 different interpretations (method #1 and method #2 in the following)

Analytical image reconstruction: the Fourier Transform

- The 2D Fourier Transform (FT) converts an image from 2D spatial domain to 2D frequency domain, by decomposing the image into sine and cosine components (or basis functions)

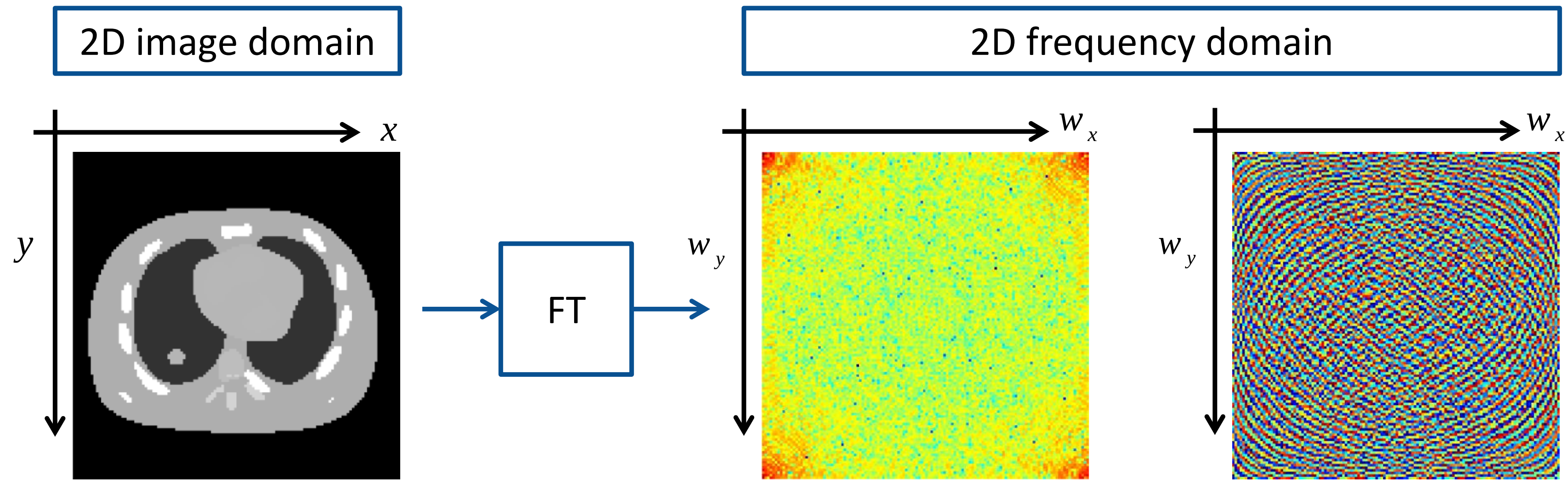
$$\hat{f}(w_x, w_y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{-2\pi i(xw_x + yw_y)}dxdy$$

  - Two sinusoidal components in spatial domain corresponds to two delta components in frequency domain



Analytical image reconstruction: the Fourier Transform

- The 2D FT of an image can be represented as real and imaginary parts
  - The real part represents the amplitude of the sinusoidal components
  - The imaginary part represents the phase of the sinusoidal components



Analytical image reconstruction

The analytical image reconstruction (method #1) is based on the Fourier Slice Theorem, following these equivalences:

$$f(x,y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\hat{f}\boxed{(w_x,w_y)}e^{2\pi i(xw_x+yw_y)}dw_x dw_y$$

$$f(x,y)=\int_{0}^{\pi}\int_{-\infty}^{+\infty}\hat{f}\boxed{(w_\rho\cos\vartheta,w_\rho\sin\vartheta)}e^{2\pi i w_\rho(x\cos\vartheta+y\sin\vartheta)}\left|w_\rho\right|dw_\rho d\vartheta$$

$$f(x,y)=\int_{0}^{\pi}\int_{-\infty}^{+\infty}\hat{f}(w_\rho\cos\vartheta,w_\rho\sin\vartheta)e^{2\pi i w_\rho \rho}\left|w_\rho\right|dw_\rho d\vartheta$$

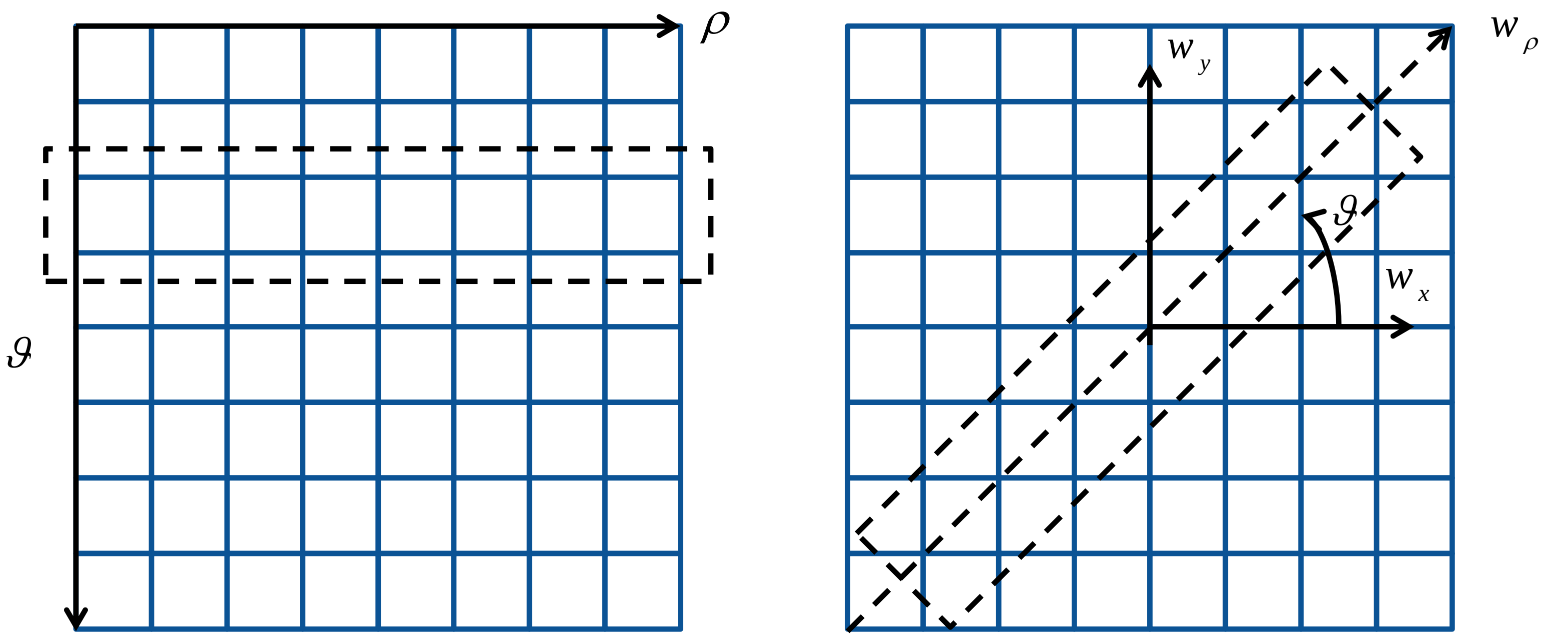
$$f(x,y)=\int_{0}^{\pi}\int_{-\infty}^{+\infty}\hat{R}(w_\rho)e^{2\pi i\rho w_\rho}\left|w_\rho\right|dw_\rho d\vartheta$$

- The image results as the inverse 2D FT of the 1D FT of the Radon Transform filtered by an high pass filter (Ramp filter) along each projection line in frequency domain

- The Ramp filter derives from variable substitution (from Cartesian coordinates to polar coordinates), in blue circle

Analytical image reconstruction

The analytical image reconstruction is based on the discrete form of the Fourier Slice Theorem, following these steps:



1D FT → **Ramp filter** as multiplication in frequency domain → **Interpolation** from polar to Cartesian coordinates → **Inverse 2D FT**

Analytical image reconstruction

The analytical image reconstruction (method #2) is based on the Fourier Slice Theorem, continuing the previous equivalences as:

$$f(x,y) = \int_0^{\pi} \int_{-\infty}^{+\infty} \hat{R}(w_\rho) e^{2\pi i \rho w_\rho} |w_\rho| dw_\rho d\vartheta$$

$$f(x,y) = \int_0^{\pi} \boxed{\int_{-\infty}^{+\infty} \hat{R}(w_\rho) e^{2\pi i \rho w_\rho} dw_\rho} |w_\rho| d\vartheta$$

$$f(x,y) = \int_0^{\pi} R(w_\rho) |w_\rho| d\vartheta$$

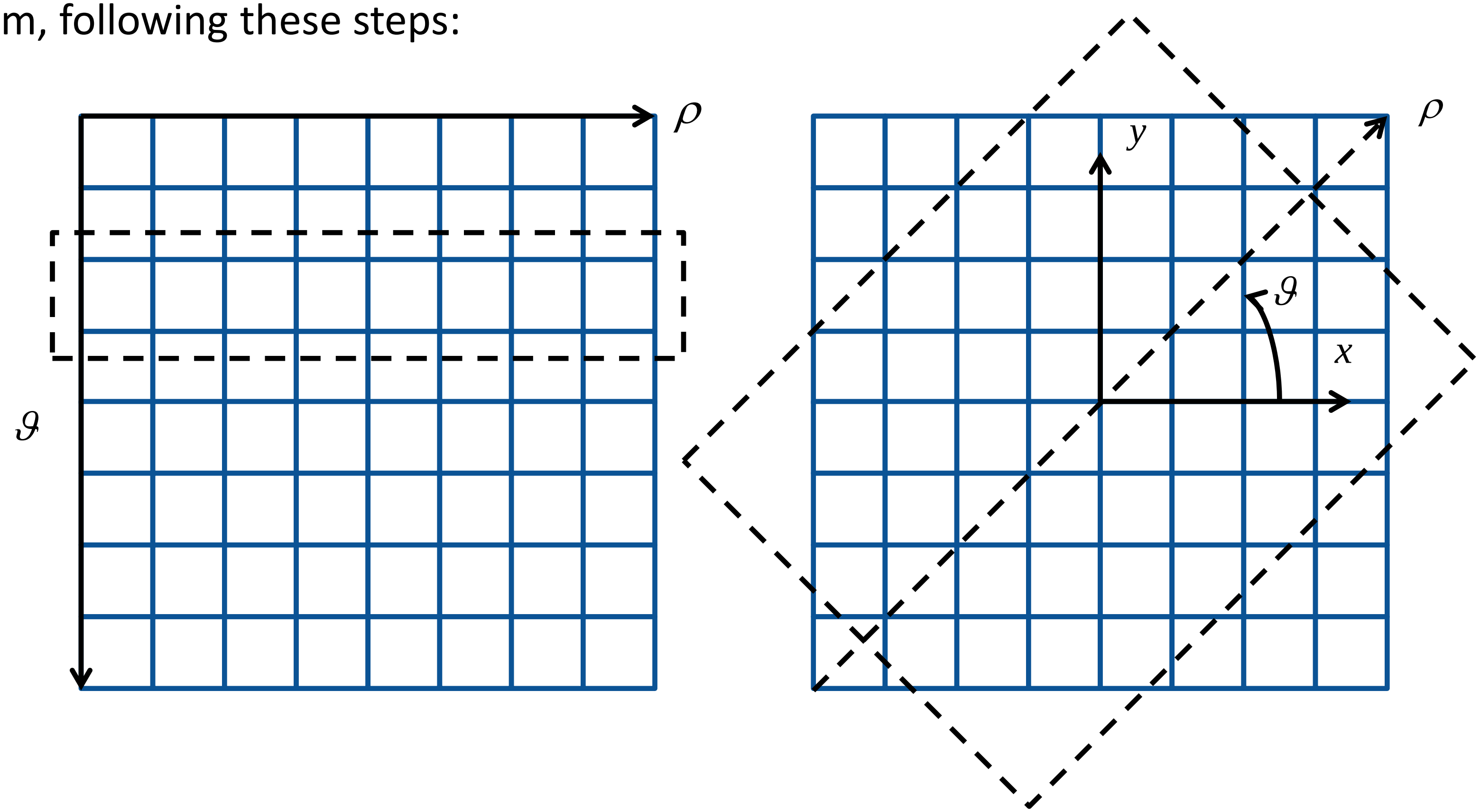
$$f(x,y) = \int_0^{\pi} g(\rho,\vartheta) * k_{ramp}(\rho) d\vartheta$$

- The image results as the back projection of the Radon Transform, filtered along each projection line in spatial domain by an high pass filter (Ramp filter), thus leading to the so called Filtered Back Projection (FBP)

- However, implementation of FBP typically relies on the method #1 (Matlab "*iradon.m*")

Analytical image reconstruction

The analytical image reconstruction is based on the discrete form of the Fourier Slice Theorem, following these steps:



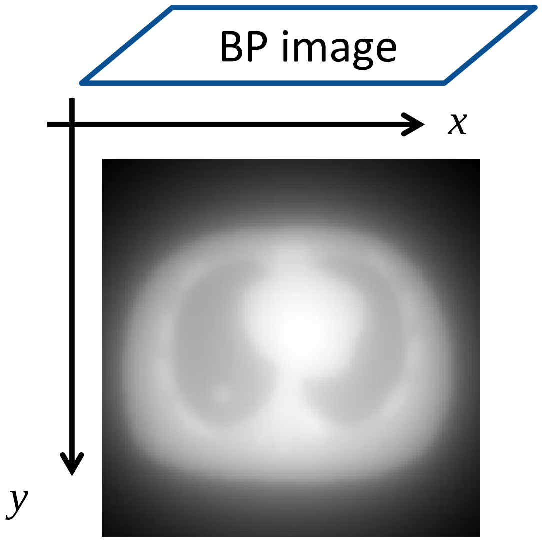
Ramp filter
as convolution in spatial domain

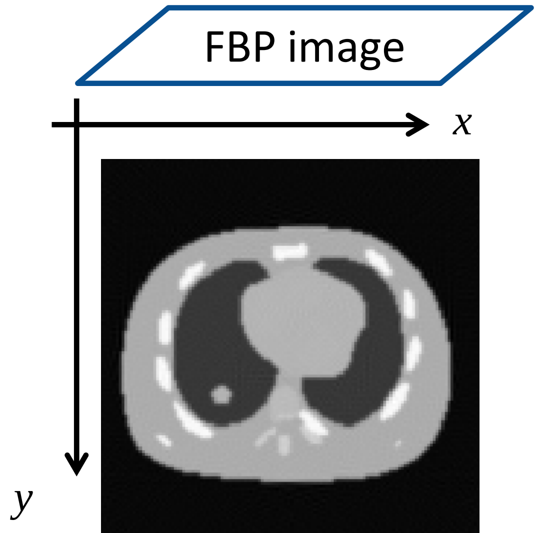
→

Back projection

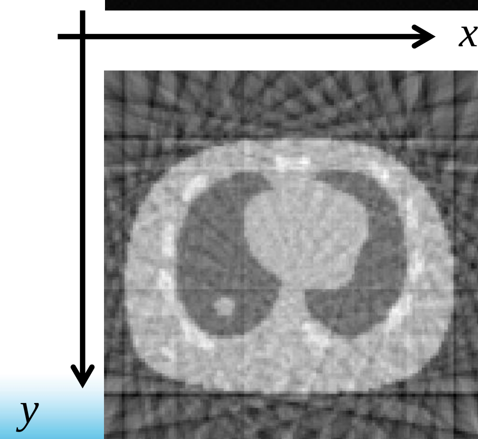
Analytical image reconstruction

- Image reconstructed according to Back Projection (BP) and Filtered Back Projection (FBP), assuming 180° rotation at 1°





- FBP assuming 180° rotation at 10°



Analytical image reconstruction

- The discrete form of the Fourier Slice Theorem relies on the Nyquist theorem of sampling
  - The Nyquist theorem establishes a sufficient condition on the sampling frequency *fs* for capturing (sampling) all the information of the continuous image up to the frequency *f*
  - The *fs* that guarantees the sufficient condition is: *fs=2f*
  - In other words, as the faster variation of the image in frequency domain requires at least 2 samples to be caught, the smaller variation in spatial domain is caught by at least 2 samples (two pixels!)
- The Nyquist theorem of sampling is therefore satisfied for: $$\Delta\vartheta = \arctan\left(\frac{1}{N/2}\right)$$

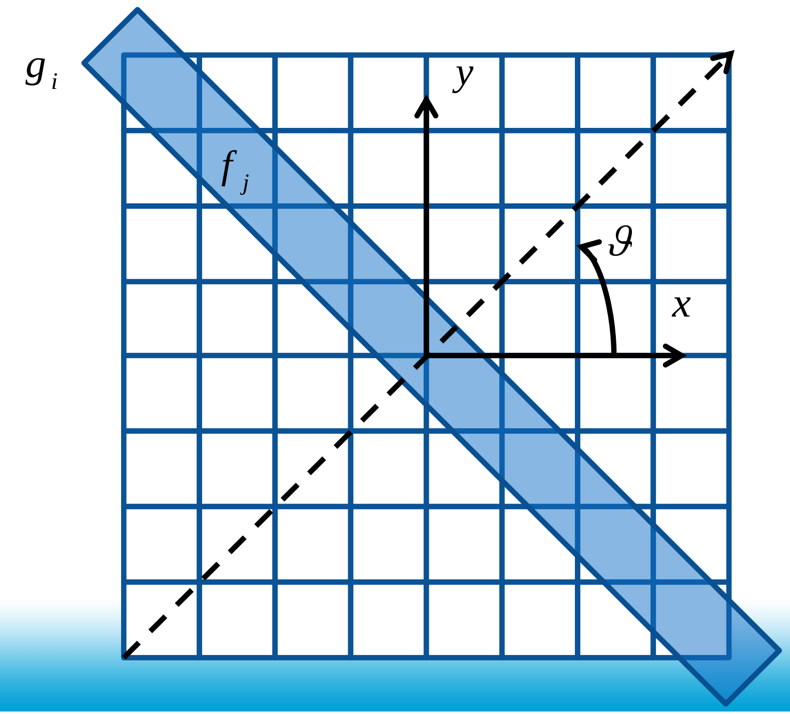
where *N* is the number of pixels along each dimension of the image (*NxN* total pixels)

- An analytical image reconstruction that violates this sufficient condition generates "streaks artifacts" (or star-artifacts) in the 2D image

Numerical image reconstruction

- Numerical image reconstruction does not rely on the Nyquist theorem of sampling
  - The 2D image and the 2D sinogram have not necessarily to be continuous, thus enabling the reconstruction in presence of dosimetric and/or geometrical constraints

- Numerical image reconstruction can be described as the solution of a linear system of equations
  - M equations, one for each projection
  - N unknowns, one for each pixel



$$f_{\min} = \arg\min_{f_j} F\left(\bar{g}_i, g_i\right)$$

$$g_i = \sum_{j=1}^{M} a_{ij} f_j \qquad \bar{g}_i = g_i + noise$$

  - Numerical image reconstruction is not as fast as the analytical one

Numerical image reconstruction

- Numerical image reconstruction requires the description of the imaging system geometry in terms of geometry of the projection lines (it is based on the Radon Transform, the 2D sinogram)

$$g_i = \sum_{j=1}^{M} a_{ij} f_j \qquad \bar{g}_i = g_i + noise$$

- The imaging system geometry is described in the system matrix of the numerical reconstruction $A=\{a_{ij}\}$, whose size is NxM

A

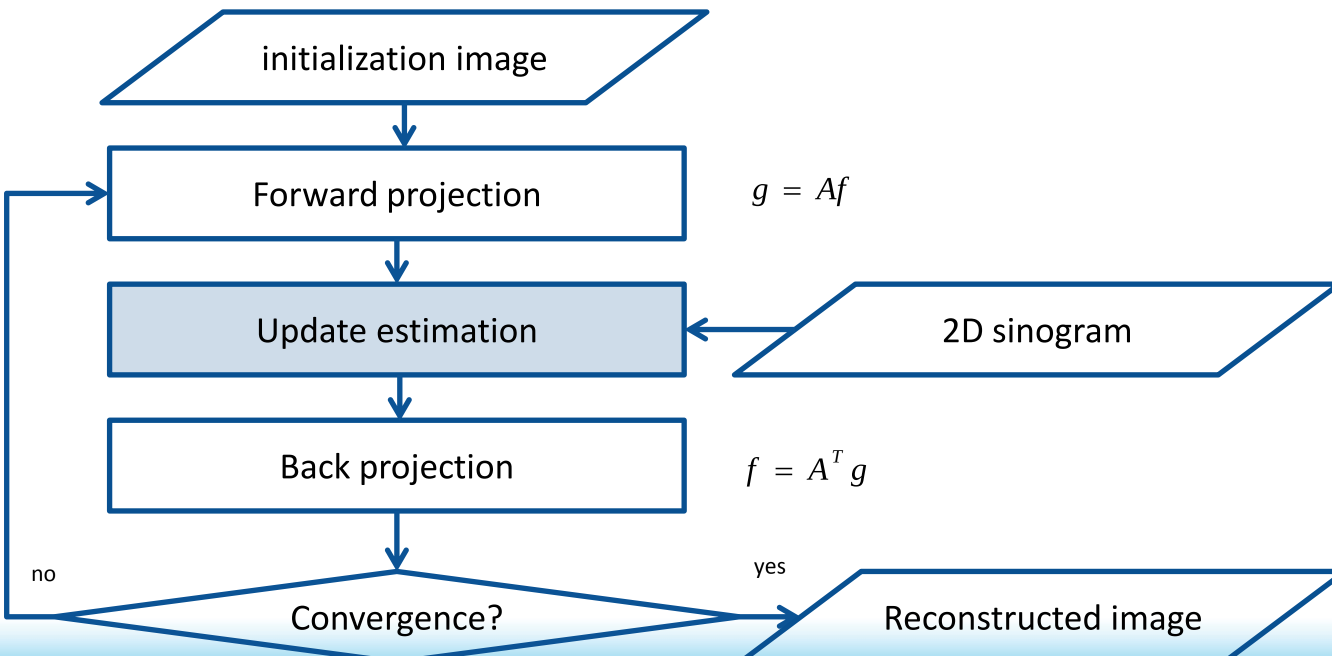
(g_1,f_1)	(g_2,f_1)						
(g_1,f_N)	(g_2,f_N)						(g_M,f_N)

Numerical image reconstruction

- Several computational methods aim at solving linear systems of equations
  - Least square optimization (for overdetermined system of equations)

$$f_{\min} = \arg\min \left\| \bar{g}_i - \sum_{i=1}^{M} a_{ij} f_j \right\|_2^2 \qquad f_{\min} = (A^T A)^{-1} A^T \bar{g}_i$$

  - Numerical (iterative) optimization

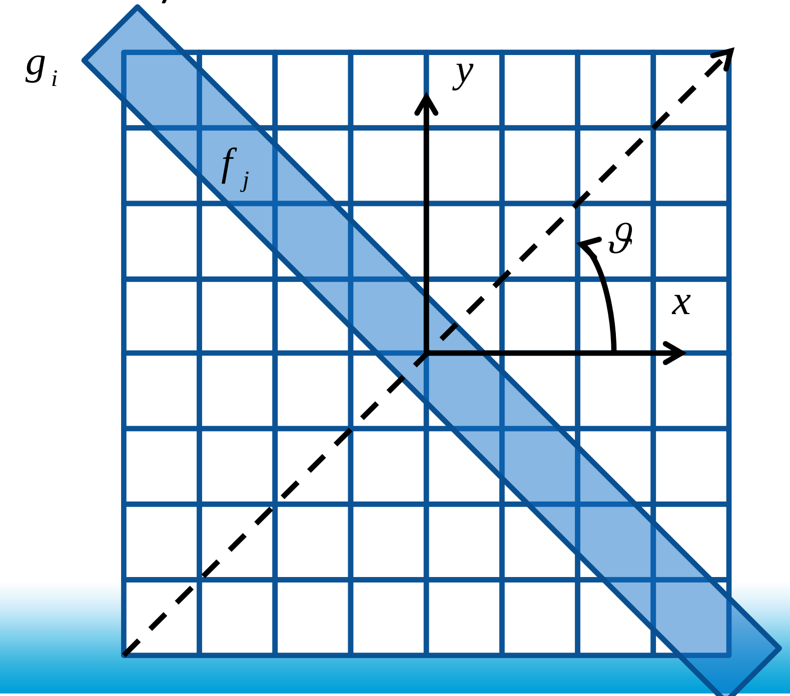


Numerical image reconstruction

ART: Algebraic Reconstruction Technique

- Each projection line is interpreted as an hyperplane in a N-dimensional space, where N is the degrees of freedom (the number of the unknowns *f* of the 2D image)
- The reconstruction consists in solving a system of M equations, where M is the number of boundaries (the number of parameters *p* of the 2D sinogram), relying on the constants a_{ij} describing the imaging system geometry (the system matrix)
- If existing, the intersection of the M hyperplanes represent the solution of the system of equations

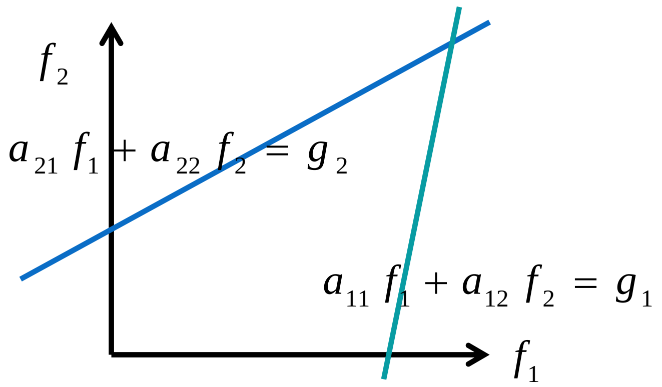
$$\begin{cases} a_{11} f_1 + a_{12} f_2 + \dots a_{1N} f_N = g_1 \\ \dots \\ a_{M1} f_1 + a_{M2} f_2 + \dots a_{MN} f_N = g_M \end{cases}$$

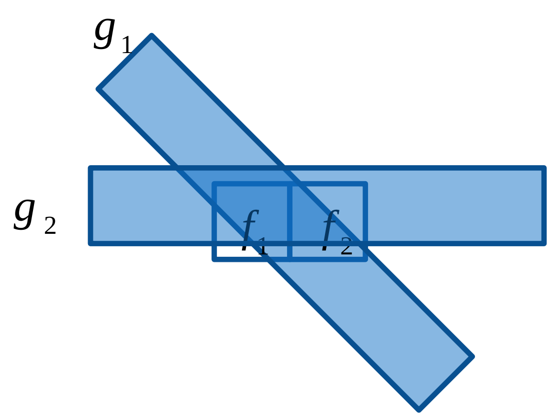


Numerical image reconstruction

Simplified imaging system geometry: 2 unknowns and 2 parameters

- The lines (N-1 is the dimension of the hyperplane) represent the boundaries (i.e. the projections)
- The intersection point represent the solution (i.e. the image)



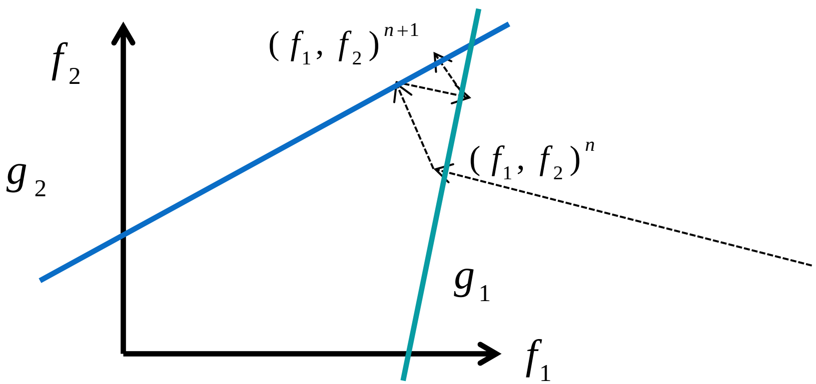


Numerical image reconstruction

ART: Algebraic Reconstruction Technique

- Additive update of the image, after the projection line has been considered (projection line per projection line)

- One iteration of ART is completed when all the projection lines have been considered



$$f^{n+1}_j = f^n_j + \frac{\boxed{\left(p_i - \sum_{j=1}^{N} a_{ij} f^n_j\right)}}{\sum_{j=1}^{N} a_{ij}^2} a_{ij}$$

  - The blue circle indicates the projection error

  - The update moves perpendicularly within boundaries

Numerical image reconstruction

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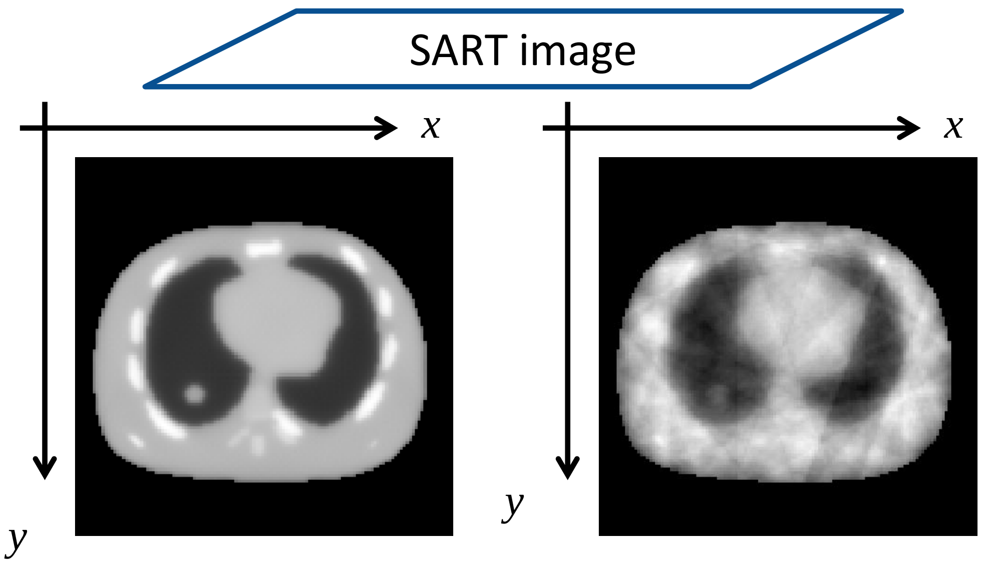
<u>SART: Simultaneous Algebraic Reconstruction Technique</u>

- Additive update of the image, contemporaneous for all the projection lines

- Under ideal conditions, one iteration of SART coincides with M updates of the ART

$$f^{n+1}{}_j = f^{n}{}_j + \frac{\displaystyle\sum_{i=1}^{M} a_{ij} \left(\frac{p_i - \displaystyle\sum_{j=1}^{N} a_{ij} f^{n}{}_j}{\displaystyle\sum_{j=1}^{N} a_{ij}} \right)}{\displaystyle\sum_{i=1}^{M} a_{ij}}$$



- SART image (40 iterations) and ART image (40 iterations, 7200 updates), 180° rotation at 1°

- SART image, 180° rotation at 10°

Outlook

- Tomographic image reconstruction, either analytical or numerical, makes use of the discrete form of the Radon Transform

- The Fourier Slice Theorem, provided with the Nyquist theorem of sampling, enables the implementation of analytical reconstruction algorithms

- Numerical reconstruction algorithms are simply enabled by the modeling of the imaging system geometry in a system matrix

- The choice of analytical or numerical reconstruction algorithms depends on the specific application in terms of geometry of the projection lines, angular coverage and angular sampling (i.e. geometrical constraints), noise level on the projections (i.e. dosimetric constraints)

  - If the continuity hypothesis of the 2D image and the 2D sinogram is matched and the noise level is low, FBP could be the first option

  - Otherwise, more flexible (but more expensive under a computational point of view) numerical algorithms should be preferred