Intro to Random processes - Markov Chain

January 15, 2021

Random process

- A random process is a mathematical model of a probabilistic experiment that evolves in time and generates a sequence of numerical values.
- Each numerical value in the sequence is modeled by a random variabl
- A collection of random variables

Example

- the sequence of daily prices of a stock;
- the sequence of scores in a football game;
- the sequence of failure times of a machine;
- the sequence of hourly traffic loads at a node of a communication network;
- the sequence of radar measurements of the position of an airplane

Discrete random process

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n \longrightarrow \ldots$$

- Sequence of random variable $X_0, X_1, \ldots, X_n$
- X_n: state of random process at time n
- All possible values of state: **State space**
- State space is countable $\rightarrow$ Discrete random process

Transition probability

Probability that the random process moves from state i at time n to state j at time $n+1$

$$\boxed{P(X_{n+1} = j | X_n = i) = P_{ij}}$$

1 step transition

Transition probability

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1 step transition

Markov property

$$X_0 \to X_1 \to \cdots \to X_{n-1} = i \to X_n = j \to \dots$$

Memoryless property *Given current state, the past does not matter*

$$P(X_n = j | X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i)$$
$$= P(X_n = j | X_{n-1} = i)$$

independent of n

Markov chain

- A Markov chain is a random process with Markov property
- Model specification
  - identify all possible states
  - identify the possible transition
  - identify the transition probability

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Transition probability

$$P_{ij} = P(X_{n+1} = j | X_n = i)$$

independent of n

Transition matrix

$$\begin{array}{cc} & \text{To} \\ & \begin{array}{ccc} 1 & 2 & \cdots \end{array} \\ \text{From} \begin{array}{c} 1 \\ \vdots \\ i \\ \vdots \end{array} & \begin{bmatrix} P_{11} & P_{12} & \dots \\ \vdots & \vdots & \vdots \\ P_{i1} & P_{i2} & \dots \\ \vdots & \vdots & \vdots \end{bmatrix} \end{array}$$

Index in **row**: current state (**from**)
Index in **column**: next/future state (**to**)

Transition probability

$$P_{ij} = P(X_{n+1} = j | X_n = i)$$

independent of n

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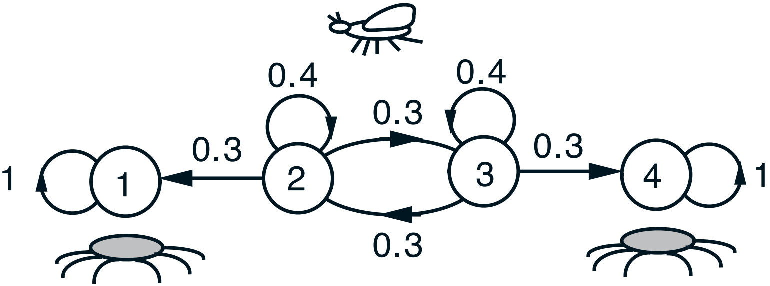
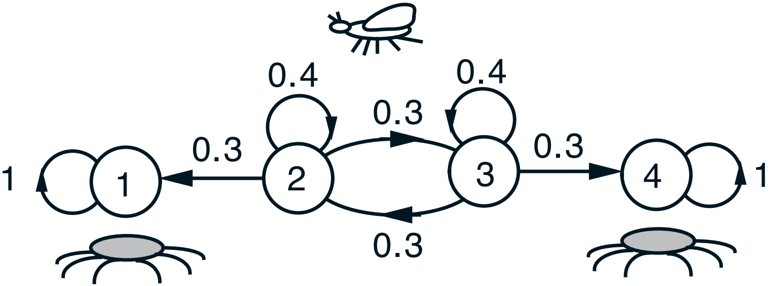
Example

A fly moves along a straight line in unit increments. At each time period, it moves one unit to the left with probability 0.3, one unit to the right with probability 0.3, and stays in place with probability 0.4, independently of the past history of movements. A spider is lurking at positions 1 and 4: if the fly lands there, it is captured by the spider, and the process terminates. Construct the Markov chain model, assuming that the fly starts in one of the positions, assuming that 2, 3



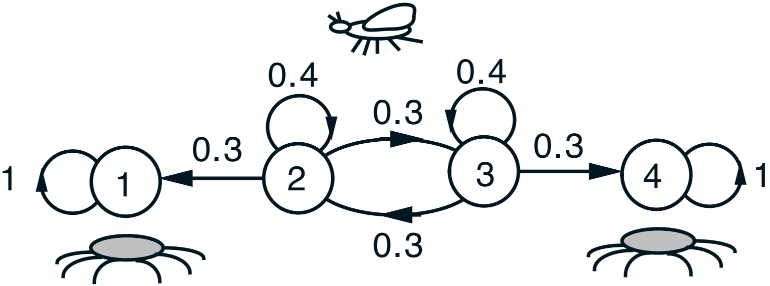
Sample episodes starting from 2:

- $2 \xrightarrow{.3} 1 \xrightarrow{1} 1 \xrightarrow{1} 1$
- $2 \xrightarrow{.3} 3 \xrightarrow{.3} 4 \xrightarrow{1} 4$
- $2 \xrightarrow{.3} 3 \xrightarrow{.4} 3 \xrightarrow{.3} 2 \xrightarrow{.4} 2 \xrightarrow{.3} 3 \xrightarrow{.3} 4$



Sample episodes starting from 2:

- $2 \xrightarrow{.3} 1 \xrightarrow{1} 1 \xrightarrow{1} 1$
- $2 \xrightarrow{.3} 3 \xrightarrow{.3} 4 \xrightarrow{1} 4$
- $2 \xrightarrow{.3} 3 \xrightarrow{.4} 3 \xrightarrow{.3} 2 \xrightarrow{.4} 2 \xrightarrow{.3} 3 \xrightarrow{.3} 4$



Sample episodes starting from 2:

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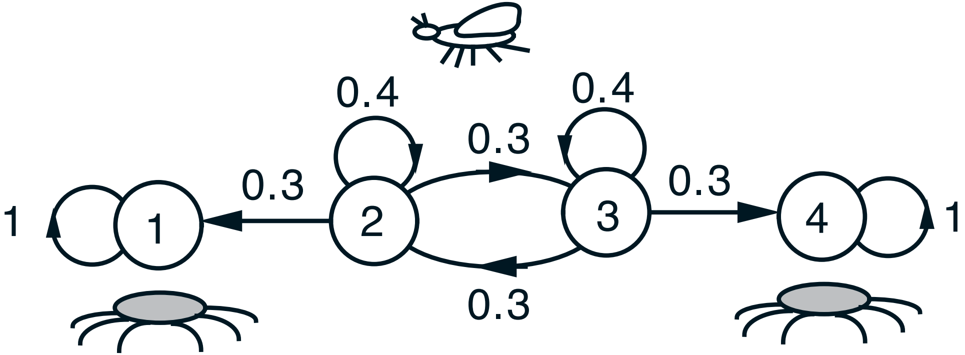
- $2 \xrightarrow{.3} 3 \xrightarrow{.4} 3 \xrightarrow{.3} 2 \xrightarrow{.4} 2 \xrightarrow{.3} 3 \xrightarrow{.3} 4$



1 and 4 are **absorbing states** that once entered, cannot left

Solution

- All possible states: 1, 2, 3, 4
- Transition probability
  - $p_{11} = 1$, $p_{44} = 1$
  - $$p_{ij} = \begin{cases} 0.3 & \text{if } j = i+1 \\ 0.4 & \text{if } j = i \\ 0.3 & \text{if } j = i-1 \end{cases} \quad \text{for } i = 2, 3, \ldots, m-1$$

Solution

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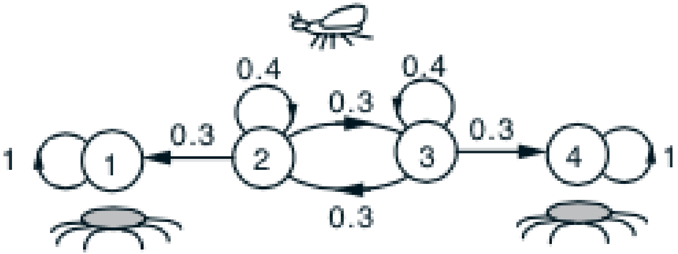


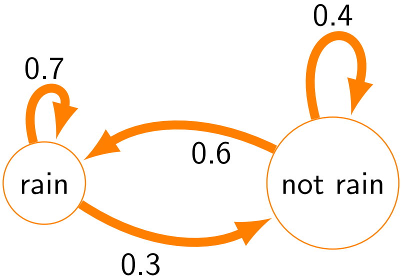
	1	2	3	4
1	1.0	0	0	0
2	0.3	0.4	0.3	0
3	0	0.3	0.4	0.3
4	0	0	0	1.0

p_{ij}

Example: Weather forecast

Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions. Suppose also that if it rains today, then it will rain tomorrow with probability .7; and if it does not rain today, then it will rain tomorrow with probability .4. Find a Markov chain that modeling the system.

Solution



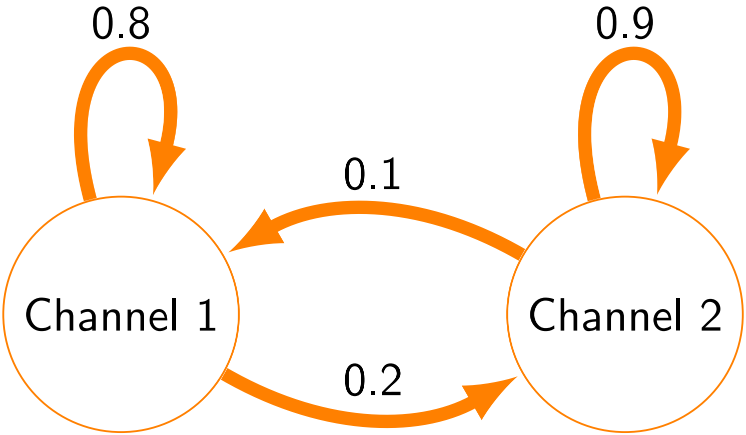
- State 1 (rain)
- State 2 (no rain)

- Transition matrix

$$
\begin{array}{cc}
 & \text{Next} \\
 & \begin{array}{cc} \text{Rain} & \text{No rain} \end{array} \\
\text{Current} \begin{array}{c} \text{Rain} \\ \text{No rain} \end{array} & \begin{bmatrix} .7 & .3 \\ .6 & .4 \end{bmatrix}
\end{array}
$$

Example: Market share

Suppose that two competing television channels. Assume that over each one-year period channel 1 captures 10% of channel 2's share, and channel 2 captures 20% of channel 1's share. Find a Markov chain that modeling the system.


0.8
0.9
0.1
Channel 1
Channel 2
0.2

Solution

- Consider one customer:
- State 1: he watches channel 1
- State 2: he watches channel 2

Transition matrix

$$
\begin{array}{c c c}
 & \multicolumn{2}{c}{\text{Next}} \\
 & \text{Chanel 1} & \text{Channel 2} \\
\text{Current} \quad \begin{array}{c}\text{Chanel 1} \\ \text{Channel 2}\end{array} & \multicolumn{2}{c}{\begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix}}
\end{array}
$$

n-steps transition

Given process initial state i, want to know probability that it will be in state j after n steps

$$r_{ij}(n) = P(X_n = j | X_0 = i)$$

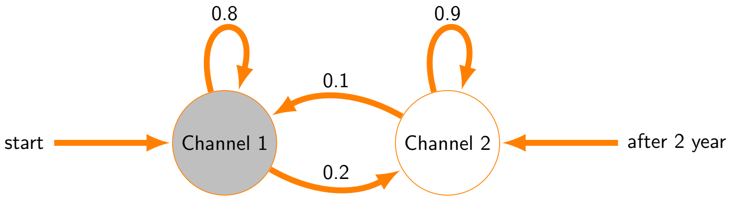
Remark

$$rij(1) = p_{ij}$$

Example- Market share

Find the probability that a customer watches Chanel 2 after 2 years given that he/she watches Chanel 1 at the beginning

$$r_{12}(2) = P(X_2 = 1 \mid X_0 = 1) = ?$$



Solution

There are 2 realizations

$$X_0 = 1 \to X_1 = 1 \to X_2 = 2$$

and

$$X_0 = 1 \to X_1 = 2 \to X_2 = 2$$

So

$$r_{12}(2) = P(X_2 = 2|X_0 = 1)$$
$$= P(X_2 = 2, X_1 = 1|X_0 = 1) + P(X_2 = 2, X_1 = 2|X_0 = 1)$$

$$
\begin{aligned}
&P(X_2 = 2, X_1 = 1 | X_0 = 1) \\
&= \underbrace{P(X_1 = 1 | X_0 = 1) P(X_2 = 2 | X_0 = 1, X_1 = 1)}_{\text{multiple rule}} \\
&= P(X_1 = 1 | X_0 = 1) \underbrace{P(X_2 = 2 | X_1 = 1)}_{\text{memoryless property}} \\
&= r_{11}(1) p_{12} = p_{11} p_{12}
\end{aligned}
$$

Similar

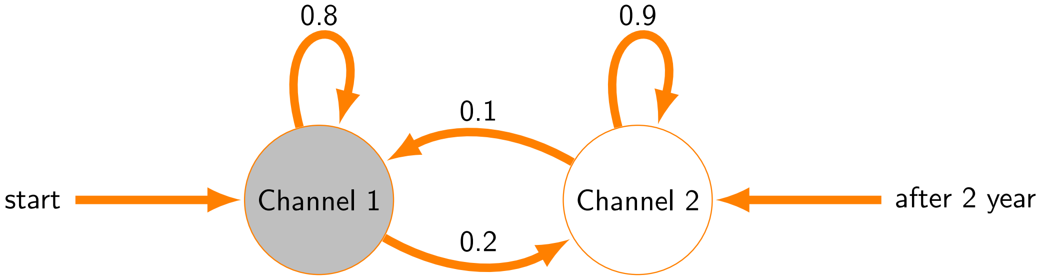
$$P(X_2 = 2, X_2 = 1 | X_0 = 1) = r_{12}(1)p_{22}$$

Hence

$$\begin{aligned} r_{12}(2) &= r_{11}p_{12} + r_{12}p_{22} \\ &= p_{11}p_{12} + p_{12}p_{22} \\ &= (.8)(.2) + (.2)(.9) \\ &= .34 \end{aligned}$$

After 2 years, channels 2 captures 34% of channel 1's share.

Practice

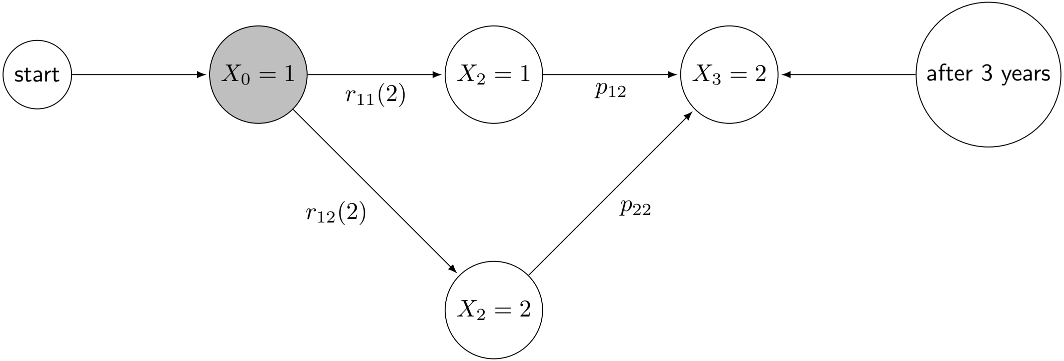


$$r_{11}(2) = P(X_2 = 1 \mid X_0 = 1) = ?$$

Question

$$r_{12}(3) = ?$$

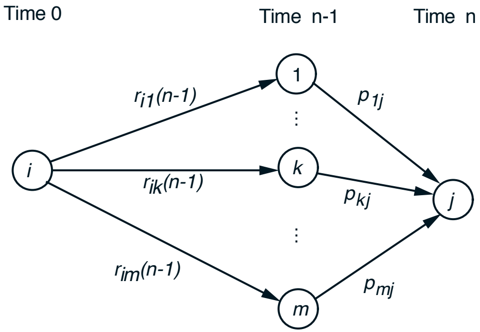
Answer



Thanks to memoryless property

$$r_{12}(3) = r_{11}(2)p_{12} + r_{12}(2)p_{22}$$

Key recursion



$$\boxed{r_{ij}(n) = \sum_k r_{ik}(n-1)p_{kj}}$$

starting with

$$r_{ij}(1) = p_{ij}$$

Proof

Thanks to total probability rule

$$r_{ij}(n) = P(X_n = j | X_0 = i) = \sum_k P(X_n = j, X_{n-1} = k | X_0 = i)$$

By multiple law

$$
\begin{aligned}
&P(X_n = j, X_{n-1} = k | X_0 = i) \\
&= \underbrace{P(X_n = j | X_{n-1} = 1, X_0 = i)}_{\text{memoryless property}} P(X_{n-1} = 1 | X_0 = i) \\
&= P(X_n = j | X_{n-1} = k) P(X_{n-1} = k | X_0 = i) \\
&= r_{kj}(1) r_{ik}(n-1) = p_{kj} r_{ik}(n-1)
\end{aligned}
$$

Hence

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) p_{kj}$$

General Chapman - Kolmogorov equation

Instead of considering all possible state of X_{n-1}, one can work with all possible state of X_q for some $q < n$ to get

$$\boxed{r_{ij}(n) = \sum_k r_{ik}(n-q) r_{kj}(q)}$$

Transitioning from i to j in n steps is equivalent to transitioning from i to some state k in $n-q$ steps and then moving from that state to j in the remaining q steps.

Matrix representation

Let

$$P^{(n)} = \begin{pmatrix} r_{11}(n) & \dots & r_{1m}(n) \\ \vdots & \vdots & \vdots \\ r_{m1}(n) & \dots & r_{mm}(n) \end{pmatrix}$$

with $P^{(1)} = P$ then

$$P^{(2)} = P^{(1)}P^{(1)} = P^2$$

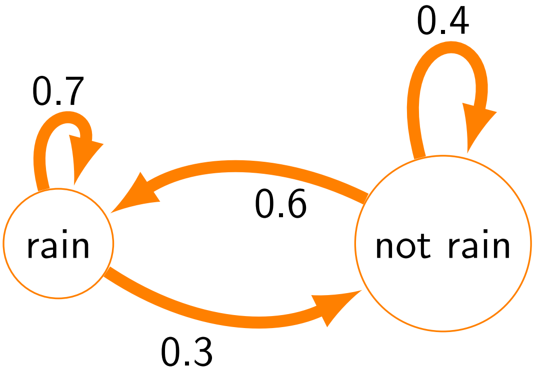
$$P^{(3)} = P^{(2)}P^{(1)} = P^3$$

n-step transition matrix

$$P^{(n)} = P^n = \underbrace{P.P \dots P}_{n \text{ times}}$$

- $P_{ij}^n = P(X_n = j | X_0 = i)$
- Row i of P^n: conditional distribution of X_n given $X_0 = i$

Example - Weather forecast



If it rains today, calculate the probability that it will rain 4 days from now.

Solution

- Transition matrix

$$P = \begin{pmatrix} .7 & .3 \\ .4 & .6 \end{pmatrix}$$

- Want to find $r_{11}(4)$
- Need to calculate P^4

After 4 days

$$\text{Current}\quad \begin{array}{r} \\ \text{Rain} \\ \text{No rain} \end{array} \begin{array}{c} \begin{array}{cc} \text{Rain} & \text{No rain} \end{array} \\ \begin{bmatrix} .5749 & .4251 \\ .5668 & .4332 \end{bmatrix} \end{array} = P^4$$

So $r_{11}(4) = P^4_{11} = 0.5749$

Unconditional distribution

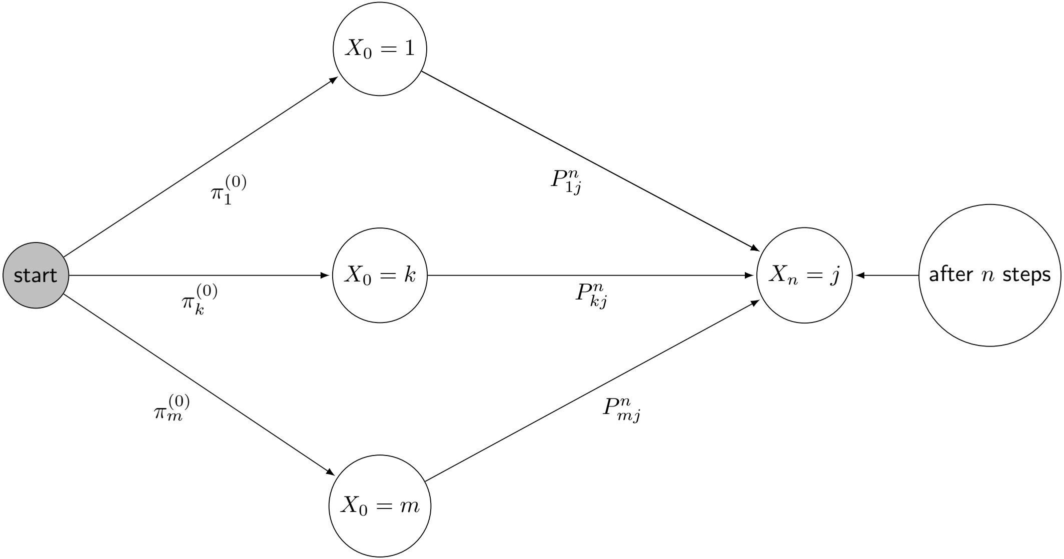
- Distribution of random initial state X_0

$$\pi^{(0)}(i) = P(X_0 = i)$$

- Distribution of X_n

$$\pi^{(n)}(i) = P(X_n = i)$$

Information about state X_n of Markov chain after n steps when you don't know the starting point of the process at initial time 0



start
$X_0 = 1$
$X_0 = k$
$X_0 = m$
$X_n = j$
after n steps
$\pi_1^{(0)}$
$\pi_k^{(0)}$
$\pi_m^{(0)}$
P_{1j}^n
P_{kj}^n
P_{mj}^n

Unconditional distribution of X_n

$$\pi^{(n)} = \pi^{(0)} P^n$$

where

$$\pi^{(0)} = \begin{pmatrix} \pi_1^{(0)} & \pi_2^{(0)} & \dots & \pi_m^{(0)} \end{pmatrix}$$

and

$$\pi^{(n)} = \begin{pmatrix} \pi_1^{(n)} & \pi_2^{(n)} & \dots & \pi_m^{(n)} \end{pmatrix}$$

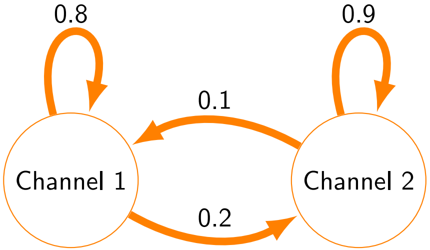
Proof

Thanks to Total rule probability

$$
\begin{aligned}
P(X_n = j) &= \sum_i P(X_n = j | X_0 = i) P(X_0 = i) \\
&= \sum_i P_{ij}^n P(X_0 = i) \\
&= \sum_i P(X_0 = i) P_{ij}^n \\
&= \sum_i \pi_i^{(0)} P_{ij}^n
\end{aligned}
$$

Example

Suppose that each have 50% of the viewer market at some initial point in time



After 1 year, the proportion viewer market ?

Solution

- Distribution of viewer market at initial time

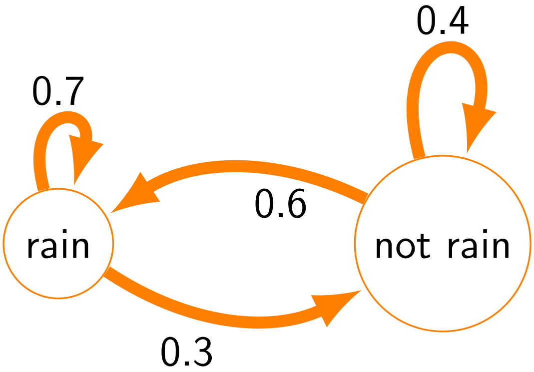
$$\pi^{(0)} = \begin{pmatrix} .5 & .5 \end{pmatrix}$$

- Distribution of viewer market after 1 year

$$\pi^{(1)} = \pi^{(0)} P = \begin{pmatrix} .5 & .5 \end{pmatrix} \begin{pmatrix} .8 & .2 \\ .1 & .9 \end{pmatrix} = \begin{pmatrix} .45 & .55 \end{pmatrix}$$

- After 1 year, 45% of viewer watches Channel 1, 55% of viewer watches Channel 2

Example - Weather forecast



Suppose probability rain today is .4, what is the probability that it will rain 4 days from now

Solution

- State: 1 = rain, 2 = not rain
- Initial probability for weather today

$$\pi^{(0)} = \begin{pmatrix} .4 & .6 \end{pmatrix}$$

- Transition matrix

$$P = \begin{pmatrix} .7 & .3 \\ .4 & .6 \end{pmatrix}$$

- Distribution for weather 4 days from now

$$\pi^{(4)} = \pi^{(0)} P^4 = \begin{pmatrix} .57 & .43 \end{pmatrix}$$

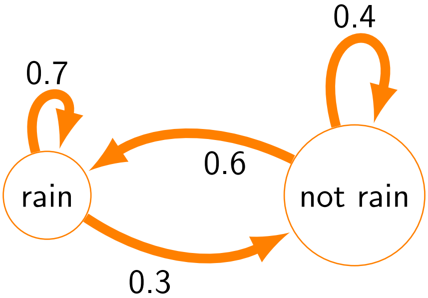
- Probability that it will rain 4 days from now

$$P(X_4 = 1) = \pi^{(4)}(1) = 0.5700$$

Long term behavior of Markov chain

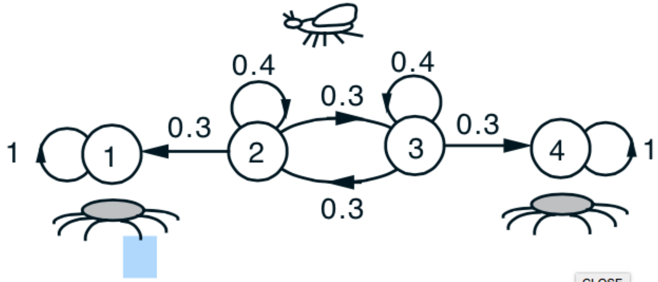
- Does $r_{ij}(n)$ converge to something?
- Does the limit depend on initial state?

Applications: Google Page's rank problem . . .



$$r_{ij}(1) = P = \begin{pmatrix} .7 & .3 \\ .6 & .4 \end{pmatrix}, \qquad r_{ij}(\infty) = \begin{pmatrix} .57 & .43 \\ .57 & .43 \end{pmatrix}$$

In long term, it will rain with probability .57 whatever the weather today is



CLOSE

	1	2	3	4
1	1.0	0	0	0
2	0.3	0.4	0.3	0
3	0	0.3	0.4	0.3
4	0	0	0	1.0

$r_{ij}(1)$

. . . .

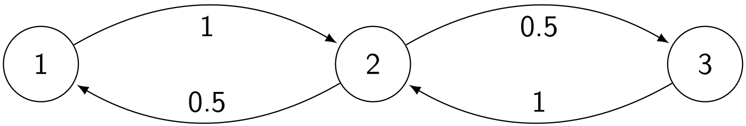
1.0	0	0	0
2/3	0	0	1/3
1/3	0	0	2/3
0	0	0	1.0

$r_{ij}(\infty)$

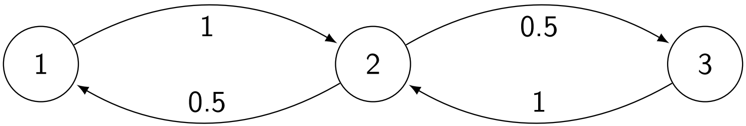
After a lot of transition, the fly is at position 4 with probability

- 1/3 if it starts at position 2
- 2/3 if it starts at state 3
- 0 if it starts at other state

Probability that the fly is at position *j* after long time depends on initial state



- n odd then $r_{22}(n) = 0$
- n even then $r_{22}(n) = 1$
- $r_{ij}(n)$ diverges



- n odd then $r_{22}(n) = 0$
- n even then $r_{22}(n) = 1$
- $r_{ij}(n)$ diverges

Question

Do $r_{ij}(n)$ converge to π_j independent of the initial state i?

1. Under which condition?
2. How to find π_j if it exists?

Answer for question 2

- Start from key recursion $r_{ij}(n) = \sum_k r_{ik}(n-1)p_{kj}$
- let $n \to \infty$

$$\pi_j = \sum_k \pi_k p_{kj} \text{ for all } j$$

- Addition equation $\sum_j \pi_j = 1$
- (π_j) is called the **stationary distribution** of the Markov chain

Interpretation

After some steps, the distribution of X_n is approximately $\{\pi_j\}$ and will not change much

$$P(X_n = j) \approx \pi_j \text{ for } n \text{ large enough}$$

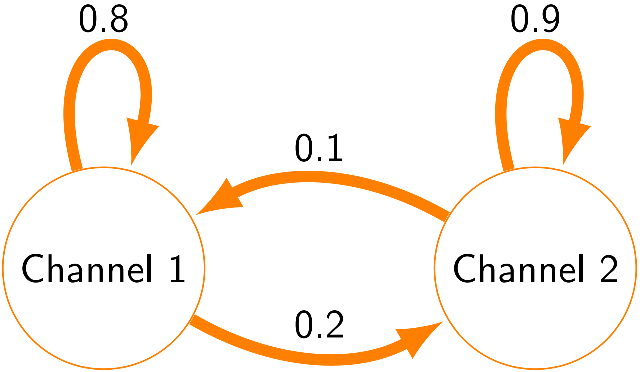
π_j: steady - state probability

Find stationary distribution

Solve

$$\begin{cases} \pi P = \pi \\ \sum \pi_i = 1 \end{cases}$$

Example



Initial market share of each channel is 50%. What will be the market share after a long time?

Solution

- Transition matrix $P = \begin{pmatrix} .8 & .2 \\ .1 & .9 \end{pmatrix}$
- Stationary distribution $\pi = \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix}$ satisfies

$$\begin{cases} \pi P = \pi \\ \pi_1 + \pi_2 = 1 \end{cases} \quad \text{or} \quad \begin{cases} .8\pi_1 + .1\pi_2 = \pi_1 \\ .2\pi_1 + .9\pi_2 = \pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases}$$

- Result $\pi_1 = 1/3$, $\pi_2 = 2/3$

After a long time, the market is stable. Each year, there is about

- 33% of customers watch channel 1
- 67% of customers watch channel 2

Find stationary distribution of the Markov chain with transition probability

$$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix}$$

Answer for question 1

If the Markov chain has the following properties

- recurrent states are all in a single class
- single recurrent class is not periodic

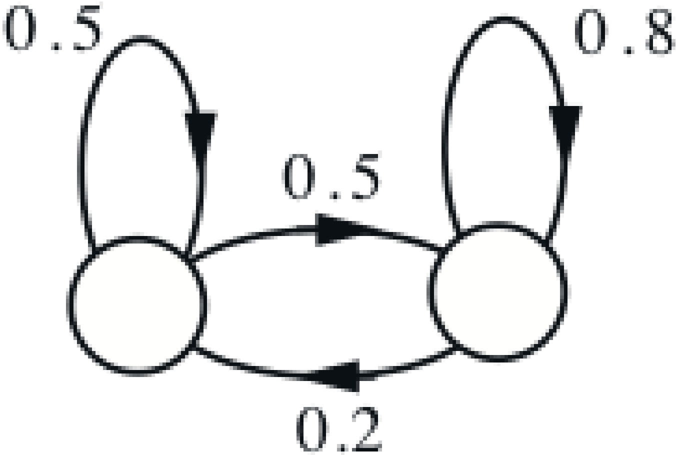
then the limit of $r_{ij}(n)$ exists and independent of initial state

Classification of states

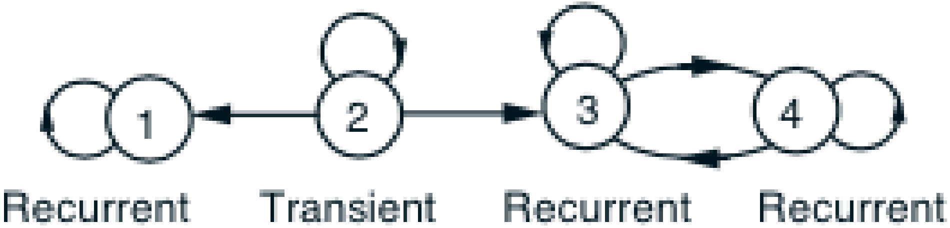
Types of state

- State j is accessible from state i if $P_{ij}^n > 0$ for some $n \geq 0$
- Two states that are accessible from each other are said to *communicate*
- If i communicates with j and j communicates with k then i communicates with k.
- Markov chain is *irreducible* if all states communicate with each other.

Example



- State i is **recurrent** if: starting from i, and from wherever you can go, there is a way of returning to i
- If not recurrent, called **transient**

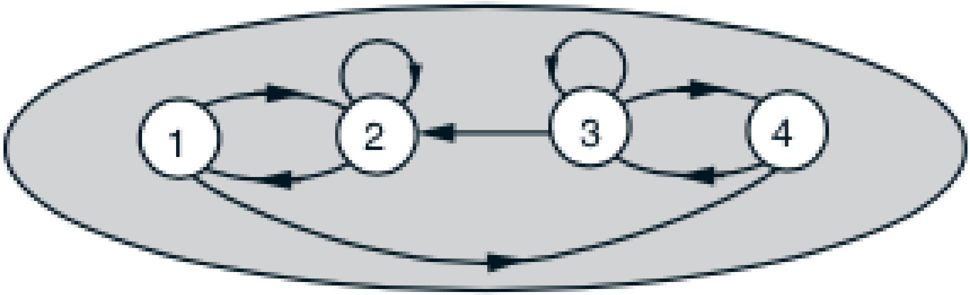


- If a recurrent state is visited once, it will be visited infinitely numbers of time
- a transient state will only be visited a finite number of times.

Reccurent Class

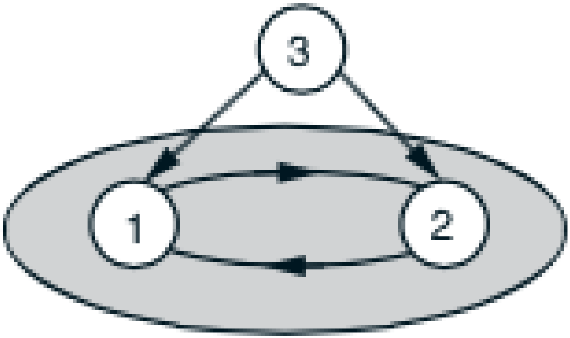
collection of recurrent states that “communicate” to each other and to no other state

Example



Single class of recurrent states

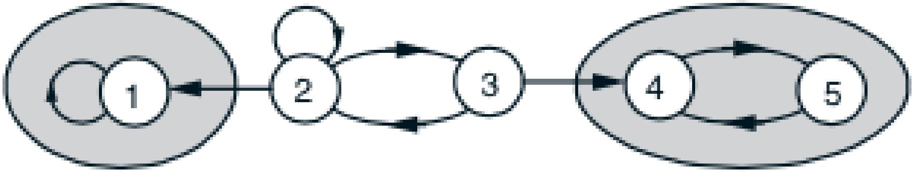
Example



Single class of recurrent states (1 and 2) and one transient state (3)

Example

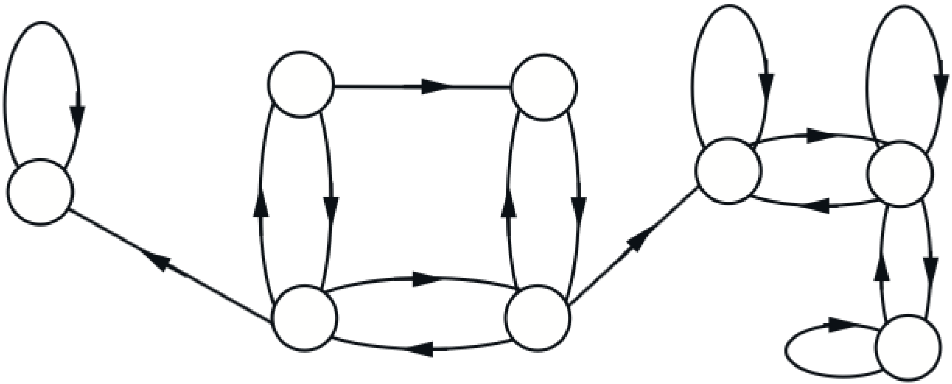
and one transient state ([illegible])



Two classes of recurrent states
(class of state1 and class of states 4 and 5)
and two transient states (2 and 3)

Practice

Determine classes of recurrent states of the Markov chain



Markov chain decomposition

- Transient states
- Recurrent classes

- once the state enters (or starts in) a class of recurrent states, it stays within that class; since all states in the class are accessible from each other, all states in the class will be visited an infinite number of times;
- if the initial state is transient, then the state trajectory contains an initial portion consisting of transient states and a final portion consisting of recurrent states from the same class

Analyze long - term behavior

- The Markov chain stays forever at a recurrent class that it visits first
- Need to analyze chains that consist of a single recurrent class

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Periodicity

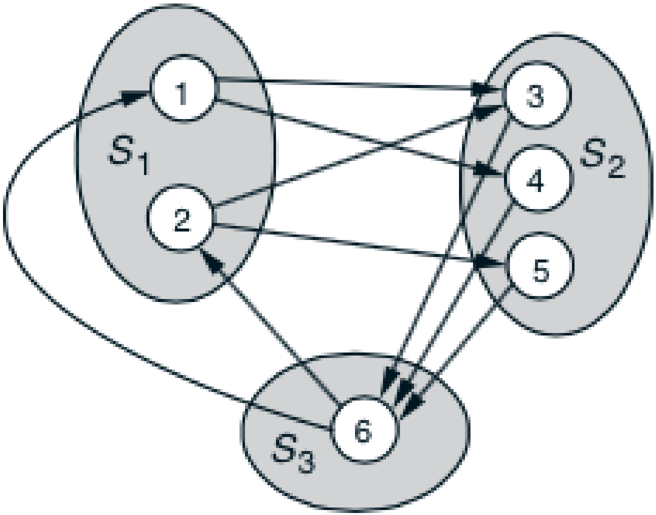
Consider a reccurrent class $\mathcal{R}$

1. $\mathcal{R}$ is said to be **periodic** if its states can be grouped in $d > 1$ disjoint subsets $S_1, \ldots, S_d$ so that all transitions from one subset lead to the next subset

$$\text{If } i \in S_k \text{ and } p_{ij} > 0 \text{ then } \begin{cases} j \in S_{k+1} & \text{if } k \leq d - 1 \\ j \in S_1 & \text{if } k = d \end{cases}$$

2. $\mathcal{R}$ is aperiodic if not periodic, i.e there exist a state s and a number n such that $r_{is}(n) > 0$ for all $i \in \mathcal{R}$

Structure of a periodic reccurrent class



- a periodic recurrent class, a positive time n, and a state j in the class, there must exist some state i such that $r_{ij}(n) = 0$ because he subset to which j belongs can be reached at time n from the states in only one of the subsets.
- thus a way to verify aperiodicity of a given recurrent class $\mathcal{R}$, is to check whether there is a special time $n \geq 1$ and a special state $s \in \mathcal{R}$ that can be reached at time n from all initial states in R, i.e., $r_{is}(n) > 0$ for all $i \in \mathcal{R}$

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Theorem

Let $\{X_n\}$ be a Markov chain with a single reccurent class and aperiodic. The steady-state probability π_j associated with the state j satisfies the following properites

1. $$\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j$$

2. π_j are the unique nonnegative solution of the **balance equation**

$$\pi_j = \sum_{i=1}^{\infty} \pi_i p_{ij}, \ \sum_{j=1}^{\infty} \pi_j = 1$$

$\{\pi_j\}$ is called the **stationary distribution**