Difficulty Factors Analysis towards Effective Instruction for Proof Writing

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Abstract: A literature survey on teaching / learning proof writing has provided us with a number of issues that should be considered to design a tutoring system for proof writing. We claim that to make students the good proof writers, one definitely needs to teach them search control (i.e., search strategy and the metacognitive skills). However, it is not the end of the story. We further claim that to become good proof writers the students must also understand the underlying geometric concepts, the language, and the proof structure. These latter skills are apparently what most of the students (>70%) have longed to learn, but they are not automatically acquired from a traditional instruction.

1. Introduction

The primary purpose of this article is to understand the bottom line in designing an effective tutor for proof writing in geometry.

We first show how difficult proof writing is for most of the students in the middle school, where the data show rather disappointed facts. To see what makes proof writing so hard, we go through the literature survey to reveal the difficulties students might have when learning proof writing. We then learn from the literature what kind of teaching strategy would be most effective for proof writing.

There are hundreds and thousands of studies on teaching and learning in proof writing. In this report, we focus only on the studies that have empirical evidence (i.e., the data). We also focus only on proof writing in geometry with some exceptions that have rather general results.

2. Students' Achievement in Proof Writing

Not so many studies have reported students' competences for geometry proof writing. The table below shows some of them in the chronological order:

Citation	Findings
Tubridy (1992)	Teaching postulates in the IF-THEN form with "three-part format." One semester long sessions. 89% of Ss in experimental group (N=26) achieved mastery level on CDASSG proof test (Usiskin, 1982), where the *masterly* is defined as having at least 3 proof problems (out of 4) scored 3 or 4 (on a scale of 0 to 4). Whereas 73% of Ss in control group (N=26) achieved mastery level.
Koedinger (1991)	Teaching "abstract planning" with diagrammatic schema. Four 2-hour sessions. N=30, high school students. Ss had 35.5% accuracy on proof writing at pre-test even though they had 67% accuracy on judgment of geometric statements. Accuracy of proof writing improved up to 61% on post-test in the experimental group. It turned out that his tutor is about equally effective as GPT, which teaches proof skills with proof tree.
Chaiyasang (1989)	Less than 15% Ss, even in VLH 4, achieved "Good" in proof writing. N=3047, 6th to 9th graders in Thailand. Students in and under VLH 3 couldn't write proofs at all.
Senk (1983)	30% achieved about 75% masterly level, 40% had some proof skills, and 30% gained no competence on proof writing at all. N=1520 from 74 classes in 11 schools. No more than half the students were capable of a non-trivial proof at the end of school year.
Lovell (1971)	45% of Ss failed to write proofs even though they could provide proper arguments. N=2956 in 1st, 3rd, 5th, and 6th grads.

3. Difficulty Factors Analysis in the Literature

The study on students' difficulties in proof writing had started as early as Smith's work (1940a; 1940b). Since then, many studies have been conducted around this issue. "Proof writing" can be implemented in different subjects in mathematics: set theory, number theory, calculus, analysis, geometry, and so forth. The literature survey in this section mainly focuses on geometry. Hence, reasoning skills required out side of the geometry proofs are not concerned in this document. Driscoll (1982), for example, discuss 7 different types of mathematical proof, whereas the geometry proof is most likely viewed as the "direct proof" in his terminology, which is characterized as a chain of postulates.

Different researches focus on different aspects of proof writing. The major issues are: (1) factual domain knowledge (i.e., definitions, axioms, theorems, etc.), (2) structure of proof, (3) heuristics, and (4) search strategy. The first two issues are different from the last two issues in a sense that that latter concerns with so called meta-knowledge that is the knowledge about how to deal with the first two issues. Thus, we split this section into two parts: the prerequisite knowledge and the search skills. Students' misconception in proof writing follows afterward.

3.1 Prerequisite Knowledge

Lack of prerequisite knowledge (or background knowledge, if you will) keeps students away from being successful in proof writing. There are several aspects that belong to the prerequisite knowledge. Each of them is summarized below.

3.1.1 Van Hiele Level

Probably the most dominant predictor on students' proof-writing performance is the van Hiele level (van Hiele, 1986). Studying a relationship between the van Hiele level (VHL) and proof-writing ability has been one of the central issues in geometry education. All of those studies showed strong correlation between VHL and proof writing.

One of the largest and the most extensive study on VHL is conducted by Senk (1982; 1985; 1989) as a part of the Chicago project (Usiskin, 1982) [Must read]. She found a strong correlation between the van Hiele level and the performance on proof writing. According her observation, "students entering the course with low van Hiele level could hardly become good proof writers." Over 70% of students begin high school geometry at level 1 or below, and only those students who enter level 2 or higher become competent with proof by the end of the course.

3.1.2 Basic facts

Apparently, to become a good proof writer, the student must understand the elements of proofs, namely, the geometric objects, properties, definitions, axioms, theorems, and so forth. Since these are factual knowledge, they should be rather easily learned. The question, then, is how could those elements be organized as the basic facts to be manipulated for proof writing.

Moore (1994) observed students and found that the major source of difficulty is in *concept understanding*. He then proposed *concept-understanding schema* as a model of proof writing. The "concept" here corresponds to the facts that constitute a proof; namely, definitions, axioms, and theorems etc. The basic idea is that unless a student understands geometric concepts, he or she would never be able to apply those concepts to build up a proof. Hart (1994) also conducted an experiment, in college level group theory, that support importance of conceptual schema.[1]

3.1.3 Language

Laborde (1990) argued the students' difficulty to speak in *mathematical language*. He found that the peculiarity of the mathematical language itself makes students hard to communicate (to understand what the teachers say as well as to express what the students think). Laborde then suggested letting students have *image* of the sentences to increase their comprehension.

Even if proof writing were described as a procedure with if-then operators, it would still be not univocally and completely well formalized, hence students would not be able to apply the procedure (Landa, 1975). The issue of ambiguity would be striking when one wish to teach "heuristics," which are usually described with highly general terms like the ones in (Polya, 1957).

3.1.4 Structure of proof and form of deduction

It might be the case that students simply don't pay attention to the IF-THEN structure that is essential in geometric postulates. Or, they can't restate the postulates into IF-THEN format. This observation has been supported at least three independent experiments that explicitly taught students the IF-THEN structure, which in turn enhanced students' proof-writing performances (Dreyfus & Hadas, 1987; Greeno, 1983; Tubridy, 1992).

In the materials used in a classroom experiment, Tubridy (1992) explicitly described geometric postulates with IF-THEN statements that is organized as the "three-part format," which consists of (1) configuration of the postulate, (2) conclusion of the postulate, and (3) verbal explanation of the postulate. Students were repeatedly exposed to the fact that (2) and (3) correspond to "if" and "then" parts.

Greeno (1983) discussed an important aspect of structure of proof and developed an instruction that teaches students proof checking. The result showed that teaching proof checking encourages proof writing.

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[1] Hart extended the concept understanding model by involving processes, outcomes, metacognition, and errors. This extended model could explain expert-novice difference and have some implication for instruction.

Schoenfeld (1986) emphasized the complicated structure of proofs. Students must have empirical knowledge and a deep understanding of the properties and elements that are to be manipulated in a geometric proof in addition to knowledge of the nature of deduction and proof.

Frerking (1995) conducted an intensive examination (N=58) on the relationships among various aspects around proof-writing including van Hiele level, proof-writing achievement, conjecturing, and justification. Various tests were used to measure those variables. He observed that *abilities to conjecture and justify conjectures* in a geometry class are directly related to proof-writing ability.

3.1.5 Formal and abstract reasoning

The National Assessment of Educational Progress reported on its 1992 evaluation that "most students in 4th (N=8738), 8th (N=9432), and 12th (N=8499) grades seem to be operating at the *holistic level* (i.e., VHL 1 or 2)," which means the students need concrete examples to infer geometric properties (Strutchens & Blume, 1997). The report also observed that many students have very low competence on both synthetic (i.e., to reason about a figure with given properties) and analytic (i.e., to reason about a property within a given figure) reasoning skills.

Renner and Stafford (1976) observed that 73% of the 10th graders (N=94) are not capable of formal-operational thinking (hence rely on concrete operational thinking). Lovell (1971) conducted a large experiment (2956 students in 1st, 3rd, 5th, and 6th grades) on various concepts on proof, namely, generalization, symbols, assumptions, converses, reductio ad absurdum, and deductions. On reductio ad absurdum, he found that 14% in 1st grade, 38% in 3rd , 60% in 5th & 6th non-math, 70% in 6th math students could made a complete proof.[2]

A survey with a small number of prospective math teachers (N=41) revealed that an amount of topics to learn and level of abstraction are two major reasons for students being biased against math (Algarabel & Dasi, 1996). This observation agrees that many students fail on theorem proving that is essentially so densely and highly abstracted. Hence Algarabel and Dasi's study also supports students' weakness on abstract reasoning skills.

3.2 Search Skills

The above observations all agree on one point. Namely, they all assume that once the students grasped the prerequisite knowledge, they should be fine with proof writing. We doubt this observation due to the fact that theorem proving has such a huge problem space hence need an effective search.

[2] It might be interesting how often do we use reductio ad absurdum in geometry thermo proving. It could be a good heuristic when proving hard theorems.

Schoenfeld (1985) also refuted above optimistic view by arguing that "it is a control of reasoning that counts." He observed, "even if the students were capable of accessing their knowledge, they could still chase wild goose." His experiment proved that teaching heuristics actually improved students problem-solving skills.

3.3 Misconception

There is little research on students' misconceptions in proof writing. Chaiyasang (1989) listed some of the observed bugs from a large experiment with 3074 sixth to ninth graders from 12 schools in eastern Thailand. The list includes: continuing to prove when the proof was finished; taking what was to be proved as the given; supplying incorrect or manufactures reasons; incorrectly assuming properties from a figure; deducing implicitly; and accepting facts as proven if they were previously demonstrating by inductive reasoning.

Schoenfeld (1988) discussed students' beliefs that hinder their problem-solving activities and how those misbeliefs could sneak into students mind.

4. Teaching Strategies in the Literature

There have been many attempts made to provide effective instructions for proof writing. Unfortunately, most of them have failed or at least as good as the traditional classroom lectures with traditional materials. This section first shows the successful instructional strategies followed by the not-so-great results.

(NOTE: Italicized references are cited in somewhere else without citation, most likely in (Tubridy, 1992), and I haven't read them yet.)

Successful results:

1. **The "thee-part format"** (Tubridy, 1992). Teaching geometric postulates in IF-THEN statement as well as the "three-part format," which consists of (1) configuration of the postulate, (2) conclusion of the postulate, and (3) verbal explanation of the postulate, had no effect on proof writing. An instructional strategy (not clearly explained) to help students develop insights and mental abilities to enhance proof writing was developed. One semester long sessions with 10th graders. N=26 for each experimental and control groups. Had a significant effect for low- and middle-level Ss. Difference by 5.6 (22%, p<.007) on a scale of 0-24 found for lowest level Ss (Geometry score 13). Difference by 2.9 (12%, p<.0008) for middle level Ss (CDASSG geometry score 16). No difference in effects found for good Ss (DCASSG geometry score 19). 89% of experimental group achieved masterly level,

which means at least three proofs (out of 4) get 3 or 4 (out of 4). 73% of control group achieved masterly.[3]

2. **The six principles for geometry teaching** (Dreyfus & Hadas, 1987). (1) A theorem has no exceptions: (2) Even "obvious" statements have to be proved: (3) A proof must be general: (4) The assumptions of a theorem must be clearly identified and distinguished from the conclusions: (5) The converse of a correct statement is not necessarily correct: (6) Complex figures consist of basic components whose identification may be indispensable in a proof. 2-hour par week for full year. 22 experimental vs. 10 control classes. Significant effect on geometry test at midyear and post-test.
3. **Teaching proof checking** (Greeno, 1983). Experimental group (N=30) had four 1-hour training sessions and did better on proof writing. IF-THEN aspect of postulates was highlighted in the proof checking procedure. Cf. (Ireland, 1974), which did almost the same thing but observed no effect.
4. **Using flow-diagram format** (Summa, 1982). Essentially same as a proof tree but no justification (postulate application) involved. Experiment lasted for 4-days. 10th graders (N=120). The experimentalgroup was significantly higher ($p<.01$) in means on the proof writing test. They were also more efficient in time ($p<.05$). Cf. *Harbeck* (*1972*) and *Martin* (*1970*), which reported neutral results on a use of flow diagram.
5. **Teaching "prescriptive plan"** (Landa, 1976) [Must read]. Did enhance proof writing on considerable difficult problems.

Most of the studies showed neutral results, i.e., "no treatment has a favorable record of a significant positive effect with respect to any outcome."

1. Using Mira manipulative (Cook-Bax, 1996). An effect of Mira manipulative for 9-10 graders in two-column proof writing (N=21 in experimental and 22 in control group). 3 weeks sessions.
2. Teaching proof plan by proof tree (Uccellini, 1994). Apprenticeship teaching method for 7 days sessions. N=98 eighth and ninth graders. 43% of the Ss always used proof tree for all posttest problems. 30% of Ss did so for some posttest problems. No Ss in control group used proof tree in posttest at all. *Q: Were the Ss taught well enough how to plan? Is it the apprenticeship teaching strategy that didn't work to teach proof planning, or it is just a matter of quality in implementation of the strategy?*
3. Using an expository writing component (Bell, 1988). Designed to enhance the critical thinking skills (no further explanation available in the abstract). 9 weeks. Size of the experiment is unknown.
4. van Hiele Phase-based instruction (Bobango, 1988). Providing instructions depending on student's

[3] I don't actually get all these results. Need to re-examine Tubridy's thesis.

VHL did improve S's VHL, but no significant difference in proof writing. N=72 from 2 regular and 2 honors geometry classes. 20 days treatment with Geometric Supposer.

5. van Hiele Phase-based instruction (Han, 1987). Initially providing informal instruction and delaying proof until the latter half of the course. 487 high school students in 2 different schools. Full one year treatment. Traditional (control) group did perform better than experimental group. *Q: Two schools were assigned to experimental and control groups as a whole. Difference in the school resulted in the difference in Ss performance?*
6. Using a formal study of algorithms (*Ernie, 1979*)
7. Using an enrichment material from arts and science (*Adams, 1977*)
8. Using a formal unit on logic (*Mueller, 1975; Platt, 1969*)
9. **Teaching a nature of proof** (Ireland, 1974) [Must read]. (1) Emphasis on proof concept, (2) employment of several short deductive systems, (3) careful treatment of conditional statement, (4) consideration of both valid and invalid reasoning, (5) practice in evaluating proof, (6) omission of formal treatment of length and angle measure, (7) explicit representation of the inference patters in proof, (8) definitions for only those terms in proof. 10th graders: 46 in experiment vs. 72 in control group. 12 weeks investigation. Did encourage S's competence of deductive processes, but no significant difference in proof writing. Cf. (Greeno, 1983) that reached a significant effect with the same kind of treatment.

There are many comparative studies showing no difference in effectiveness of the teaching method. It is not clear, however, how each of the method is effective for proof writing (i.e., some treatments could be effective for proof writing, but failed to show the difference):

1. Abstract planning vs. proof-tree (execution space) planning (Koedinger, 1991)
2. Two column vs. paragraph proof (Bell, 1988) (*van Akin, 1972*)
3. Transformational vs. traditional (*Durapau, 1979; Usiskin, 1969*)
4. Frequent vs. infrequent oral use of logic (*Elrod, 1979*)
5. Formal vs. informal (Wood, 1976). Formal consisted of traditional axiomatic proof with two-column format, whereas informal instruction has little emphasis on abstraction, no two-column proof, but greater emphasis on practical applications. 10th graders: 102 Ss in experimental and 117 Ss in control. Experiment ran for full school year.
6. Synthetic vs. analytic vs. combination (*Carroll, 1974*)
7. Cloze technique (filling-up blanks) vs. non-Cloze discussion (*King, 1974*)
8. Flow chart format vs. two-column proof for full year (*Harbeck, 1972; Martin, 1970*).

5. Discussion

Our educational objective is to make students competent in proof writing. We hope they to be able to write proofs for rather difficult problems. As the difficulty factor analysis reported in this document implies, to achieve this educational objective students must understand basic facts, language, formal deduction, proof structure, search skills, and metacognitive skills. These are *the target cognitive skills* that students must learn.

Each of the level throughout VHL 1 to 5 seems to have a relating to all of these skills. For example, each level has new facts to learn and there might be the language problem in each level, and so forth. Hence it is hard to assume that students at a certain level are stably competent in any of the target skills. The fact that the phase-based instruction along with Van Hiele level doesn't necessarily work well (Bobango, 1988; Han, 1987) agrees with this observation. Thus, we must take all the target skills into account when dealing with a student who has a trouble in writing proofs.

How could the target skills be organized into geometry curriculum for middle school students? Namely, in what order could all the target skills be taught? Should we start with teaching the proof structure? Should it be the formal deduction that must be taught first? Or, could it be simply jump into search skills? Our main concern is to build an ITS for students to be competent to write proofs hence we would let students learn those skills by problem solving. We assume that students would have learned proof writing in a traditional classroom lecture. In other words, we would not develop any curriculum for students to learn proof writing. As most of the tutoring systems do, our ITS is error-remedy oriented.[4]

Regarding the ingredients of effective instruction for proof writing, an interesting aspect we can read off the successful treatment, but what all the failed trials are commonly lucking, is explicit emphasis on the proof structure (Dreyfus & Hadas, 1987; Greeno, 1983; Tubridy, 1992). Especially, notion of the IF-THEN format for geometric postulate should not be overlooked.[5]

For advanced students, teaching search control and metacognitive skills must be taught. Schoenfeld's study (1985) shows two points: (1) it is possible to explicitly teach heuristics to the students, (2) the more knowledge students learn, the more likely they lose control of the search. Since the problems we would deal with (i.e., theorem proving with construction) have huge search space and low success ratio (i.e., the ratio of correct vs.

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[4] After reading lots and lots of "how to teach" articles written by mathematics educators, I am in a mood to emphasize this.

[5] There is one exception of this observation, namely, the study conducted by Ireland (Ireland, 1974). We must read his thesis to learn more about the limitation of our hypothesis.

garden paths in the search space), the issue of control must be critical.

Clearly, our attitude mentioned above is to improve performance on proof writing for the students with the wide variety of achievement levels. Unfortunately, we can't help all the students with various competences; let's assume that we only deal with the students who are capable of formal and abstract reasoning. The rational of this restriction is that learning those abilities seems to be beyond the scope of geometry proof writing.

In sum, our ITS would encourage students' proof-writing abilities by supporting following cognitive skills:

(1) **Basic facts**: Definitions, axioms, and postulates used in the proofs.
(2) **Language**: The way things are expressed in the mathematical language.
(3) **Formal deduction**: Premises and conclusions. Syllogism (application of postulates).
(4) **Proof structure**: This issue is two fold: (1) a proof is a sequence of postulate applications, and (2) a postulate application is described in the IF-THEN format. Since the postulates are commonly described in expository format (i.e., not in the IF-THEN format), students must learn to translate those non IF-THEN formats into IF-THEN format. Teaching all postulates in IF-THEN format vs. descriptive statements would be worth investigation.
(5) **Search skills**: Forward and backward chaining, and back up.
(6) **Metacognitive skills**: Not sure for now how could we help students to learn those skills.

We have learned, from the previous study on GRAMY, that geometry theorem proving is highly non-deterministic and the ratio of the correct and garden paths is quite low. So, the slight generalization of our work is as follows: Teaching proof writing is to teach problem solving in highly non-deterministic domain with a twist on knowledge taught. The twist is that a problem solver must translate domain postulates that are expository taught into the conditional statements in IF-THEN format so that they would fix into the proof structure. Hence, what we need to emphasize in teaching is (1) domain factual knowledge, (2) the twisting technique, and (3) the search control (i.e., heuristics and metacognition). This "twisting" part would explain why such a knowledgeable student fails to write proof.

Here comes a small complain: Teaching search control would be really interesting research topic, however, quite unfortunately, our clients (i.e., the students) seem to be wishing to get scaffolding a step behind the search.

Finally, the questions is that how could those skills be effectively taught through problem solving? – how should our ITS look like? There has been no study conducted to answer to this question. However, it would be interesting to hypothesize that dividing those skills into two parts facilitates learning processes, namely, the structure of proof and search control. The former consists of basic facts, language, formal deduction and proof structure, whereas the latter consists of search skills and metcognitive skills. This is because it might be possible to organize all concepts and skills involved in the former category into a single story on the structure of proof. It might also be plausible to assume that once a student understands the structure of proof, then he / she would likely to understand the search control.

Avoiding bottom-out on hint would be interesting research topic. To do so, we need a hierarchically organized domain knowledge that apparently should be written expository, so that the tutor can provide Socratic dialog? Socratic dialog would be suitable for instructions that need to elicit students' knowledge in a complex domain like proof writing where students' competence can be hide by a various reasons.

Teaching proof as explanation apparently doesn't necessarily enhance proof-writing ability. This is probably because they don't emphasize the structure of proof. While there is a strong trend on teaching proof as explanation instead of a formal deduction, our educational objective is to become capable of writing formal proofs after all. It would be great if teaching proof as explanation is an effective vehicle for students to learn proof writing, but we haven't been claimed that it is the case.

What about teaching construction? Honestly, for now, we are not quite sure why teach theorem proving with construction, besides its' really challenging problems for students hence should be fun!

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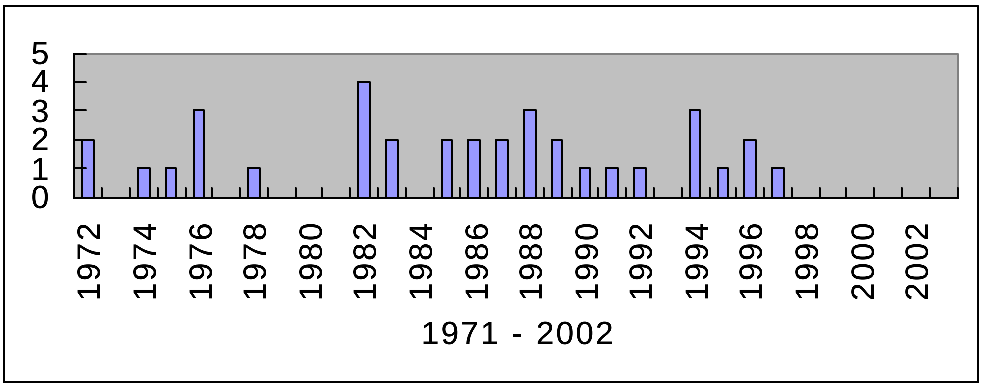
The “trend” of study on proof writing in education



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