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Australian Government
Department of Defence
Defence Science and Technology Group

Estimating Transonic Drag

(Optimisation and Parametrisation)

Land Simulation, Experimentation, and Wargaming

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2017

DST GROUP Science and Technology for Safeguarding Australia

Estimating Transonic Drag

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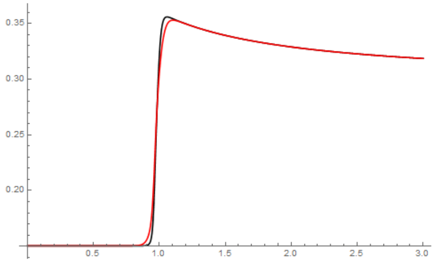
February 16, 2017

The problem



Range = 8659.45m, Range = 8598.72m
Difference = 61m (Relative error of 0.8 %)
Target ≈ 5m
High precision needed!

The basics

The trajectory is governed by gravity and drag forces.

$$m\frac{d\boldsymbol{v}}{dt} = F_g + F_d$$

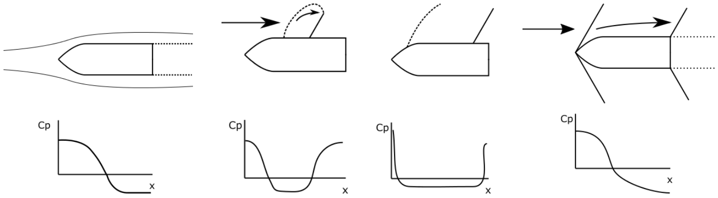
$$m\frac{d\boldsymbol{v}}{dt} = m\boldsymbol{g} - \frac{1}{2}A\rho_a C_d v\boldsymbol{v}$$

Issues:

- $C_d = C_d(M)$ is complicated,
- Mach number: $M = v/c(z)$
- The speed of sound changes with height via the density of air.

The physics

$M \ll c,\qquad M = 1 - \varepsilon,\qquad M = 1,\qquad M \gg 1.$



$C_d \propto \text{const},\qquad C_d \propto M^{-1}$

Fitting against firing tables.

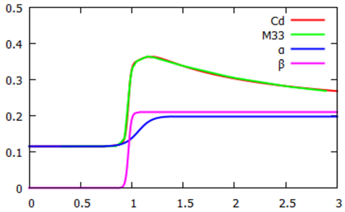
C_d for each projectile can be estimated by

- Point by point optimisation,
- Parameterisation:

$$C_d = \alpha + \frac{\beta}{M}$$
$$\alpha = \alpha_0 + \frac{\alpha_1}{1+\exp\left(-w_\alpha^{-1}(\mu-\alpha_\mu)\right)}$$
$$\beta = \frac{\beta_1}{1+\exp\left(-w_\beta^{-1}(\mu-\beta_\mu)\right)}$$

7 Free parameters

R	Angle	Fall	topH	topR	Time	Vf
100	-	0.0667	0.03	51	0.15	623
300	0.2333	0.3000	-	162	0.52	-
500	-	0.8000	1.3	-	1.04	334
600	-	1.1500	2.3	344	1.35	304
800	1.1000	2.1000	5.5	468	2.05	266

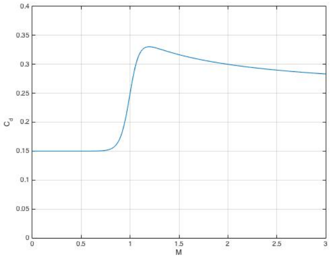


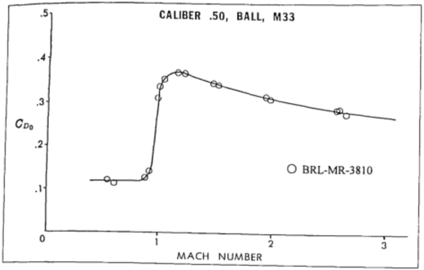
Our contributions

Functional forms for C_d

- Hyperbolic tan form with 3 free parameters.

$$C_d = \begin{cases} a + b \tanh\big(c(M-1)\big), & M \leq 1 \\ a + \frac{b}{M} \tanh\big(c(M-1)\big), & M > 1 \end{cases}$$

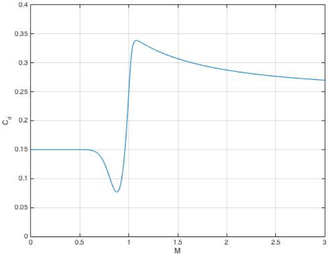


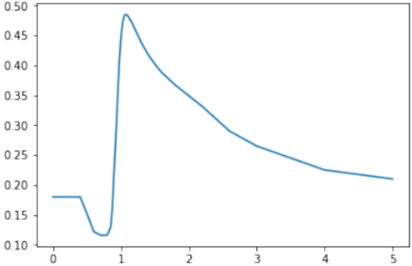


Functional forms for C_d

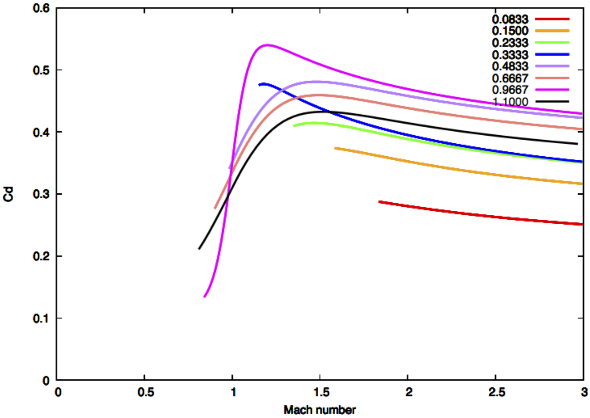
- Hyperbolic tan form with 5 free parameters.

$$C_d = \begin{cases} \Big(a + b\tanh\big(c(M-1)\big)\Big)\times \\ \Big(1 + d(M-1)\exp\big(-e(M-1)^2\big)\Big), & M \le 1 \\ a + \frac{b\tanh\left(c\left(1+\frac{ad}{bc}\right)(M-1)\right)}{M\left(1+\frac{M-1}{ad+bc}\right)}, & M > 1 \end{cases}$$





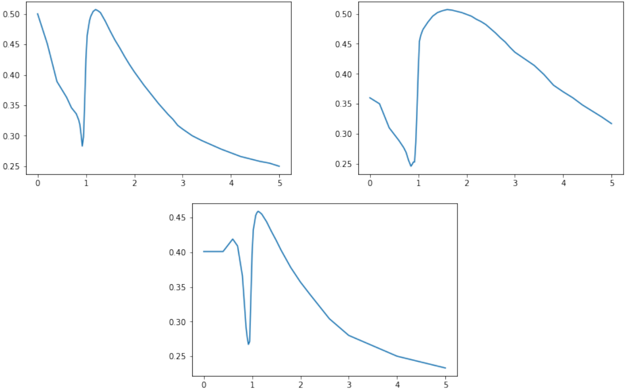
Optimising with the 3-parameter tanh



Angle	r(R)	r(θ)	r(Vf)	r(ToF)	t(Hmax)	r(TopR)	a	b	c
0.0833	1.38	1	1.00	1.38	1.73	1.38	0.191	0.18	2.9
0.15	1.119	1	1.00	1.1312	1.23	1.11	0.243	0.22	3
0.2333	1.03	1	1.00	1.04	1.07	1.02	0.274	0.23	3.1
0.3333	0.991	1	0.99	0.99	0.98	0.98	0.265	0.26	11
0.4833	0.960	1	1.00	0.98	1.01	0.96	0.351	0.215	3
0.6667	0.984	1	1.00	0.99	1.00	0.98	0.335	0.208	2.9
0.9667	0.991	1	1.00	1.04	1.12	0.98	0.35	0.238	9.5
1.1	1.00	1	1.00	0.99	0.98	0.98	0.31	0.21	2.7

Point-by-point optimisation

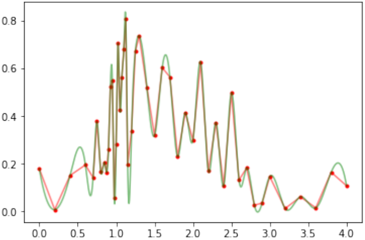
Functional forms cannot be used for all C_d profiles:

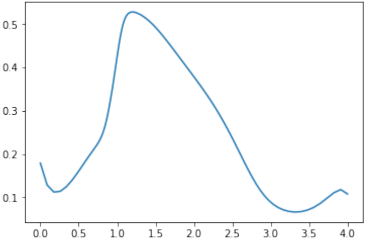


Experimental radar data shows many strange drag functions.

Point-by-point optimisation

Using Oleg's C++ Optimisation scheme can yield messy results, but these can be smoothed my Bezier curves





Mach number formulation

$$m\frac{d\boldsymbol{v}}{dt} = m\boldsymbol{g} - \frac{1}{2}A\rho_a C_d v\boldsymbol{v}$$

Becomes:

$$\frac{dM}{dt} = -\frac{g\dot{z}}{c(z)^2 M} - \frac{1}{2m}C_d(M)A\rho_a(z)M^2 c(z) - M\frac{c'(z)}{c(z)}\dot{z}.$$

$$\frac{d\dot{z}}{dt} = -g - \frac{1}{2m}C_d(M)\dot{z}A\rho_a(z)Mc(z)$$

- Advantage: Explicit formulation with three state variables means no recalculation of C_d each step.
- C_d is only dependent on state variable.
- Optimise with possible objective function:

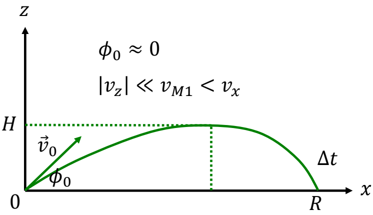
$$\mathcal{J}(C_d) = \lambda\int_0^T \left[C_d^{(k)}(M)\right]^2 dt + z(T)^2$$

Determining C_d from Horizontal shot data

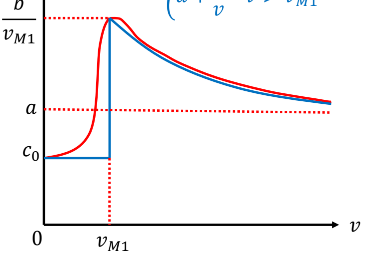
$$\dot{\vec{v}} = -\vec{g} - c(|\vec{v}|)|\vec{v}|\vec{v}$$

$$\phi_0 \approx 0$$

$$|v_z| \ll v_{M1} < v_x$$



$$c(v) \approx \begin{cases} c_0 & v < v_{M1} \\ a + \dfrac{b}{v} & v > v_{M1} \end{cases}$$



$$\left.\begin{array}{l} \dot{v}_x \approx -c(v_x)v_x^2 \qquad \Longrightarrow t = -\displaystyle\int_{v_0 \sin\phi_0}^{v_x} \frac{dv}{c(v)v^2} \\ \dot{v}_z = -g - c(v_x)v_x v_z \Longrightarrow \text{linear ODE.} \end{array}\right\} \Longrightarrow (x,z)(t; v_0, \phi_0; a, b)$$

$H, R, \Delta t \Longrightarrow$ equations for a, b

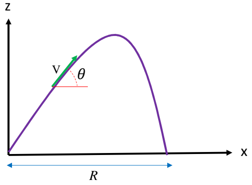
- Run simulations with given values for a, b, c_0
- Compute a, b with this algorithm and compare

Determining C_d using $|v|$ and θ

SR1

- Components of velocities in a projectile motion in presence of air resistance

$$V_x' = -\frac{1}{2}\frac{C_d A \rho_a}{m} V V_x,$$
$$V_z' = -g - \frac{1}{2}\frac{C_d A \rho_a}{m} V V_z,$$



- By Changing the variables to Θ (The velocity angle) and V(the absolute velocity) equations can be simplified in the form of

$$\frac{V}{\cos(\theta)}\frac{d\theta}{dt} = g$$

$$\int_0^T V\cos(\theta)dt = R$$

Measured Data

- ✓ V(0)
- ✓ V(T)
- ✓ R
- ✓ T
- ✓ Θ(0)

- Numerical techniques are required to compute V and Θ.

Drag coefficient ⇒ $$\boxed{C_d = \frac{-2m\frac{d}{dt}(V\cos(\theta))}{A\rho_a V^2 \cos(\theta)}}$$

2D phase plane

- 2 dimensional equations of motion for $\boldsymbol{v} = (u, v)$.
- Assuming constant air temperature/pressure.

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{g} - C_d \frac{A\rho_a}{2m} \boldsymbol{v}|\boldsymbol{v}|$$

- Scaled using $u = cu^*$, $v = cv^*$, and $t = \frac{2m}{A\rho_a c} t^*$ to give:

$$\frac{du}{dt} = -C_d u \sqrt{u^2 + v^2}$$

$$\frac{dv}{dt} = -J - C_d v \sqrt{u^2 + v^2} \quad \text{where} \quad J = \frac{2mg}{A\rho_a c^2}$$

2D phase plane

- Consider a drag coefficient rising sharply from constants a to b across Mach 1 ($\sqrt{u^2 + v^2} = 1$).
- Solving for regions where Mach 1 is a stable velocity gives: $J\sqrt{1 - u^2} = C_d$.
- This allows us to classify projectiles into three classes by their J number.

J value	Terminal velocity	Stable region
$J < a$	subsonic	none
$a < J < b$	sonic	$0 < u < \sqrt{1 - \frac{a^2}{J^2}}$
$J > b$	supersonic	$\sqrt{1 - \frac{b^2}{J^2}} < u < \sqrt{1 - \frac{a^2}{J^2}}$

2D phase plane

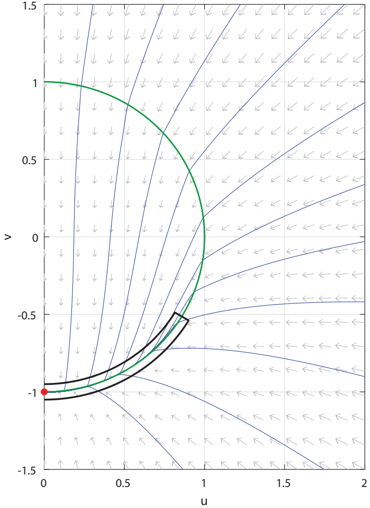
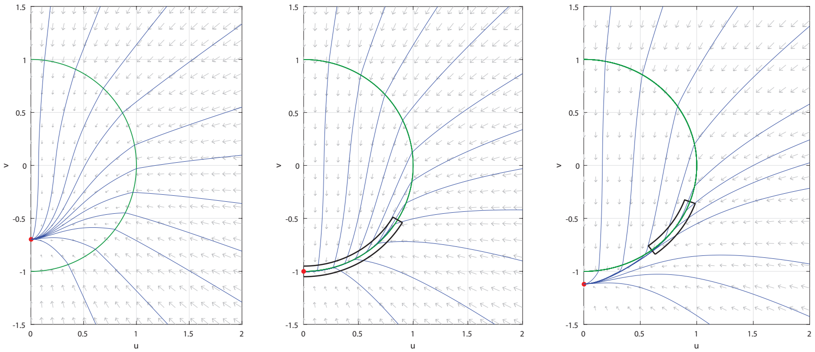


Figure: u, v phase plane with J number in sonic range

2D phase plane



Low angle trajectories

- We now consider a projectile shot at a low angle such that we may assume $v << u$ and $|\boldsymbol{v}| \approx u$.
- Under these assumptions the u equation becomes uncoupled.
- With $u > 0$ we have:

$$\frac{du}{dt} = -C_d(u)u^2$$

- Integrate to give:

$$\int_{u_f}^{u_i} \frac{du}{C_d(u)u^2} = T$$

Low angle trajectories

- Consider multiple trials with different launch angles giving data T, u_f and u_i. Label them by index x to give.

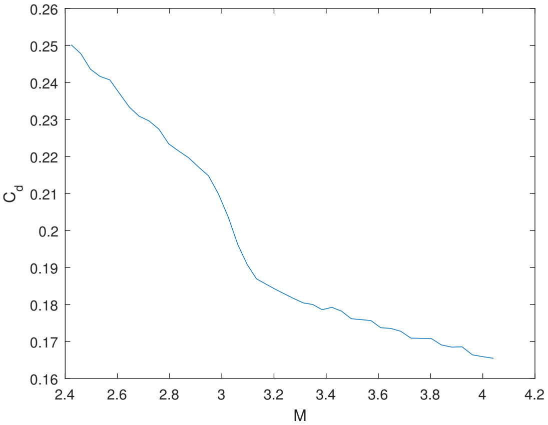
$$\int_{u_f(x)}^{u_i(x)} \frac{du}{C_d(u)u^2} = T(x)$$

- Numerically differentiating with respect to x gives:

$$C_d(u_f) = -\frac{1}{u_f^2}\frac{du_f}{dx}\frac{1}{dt/dx}$$

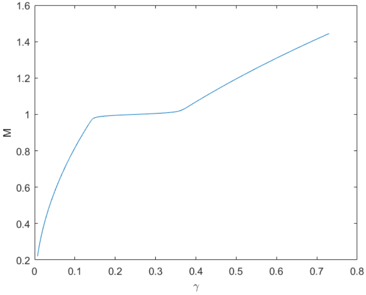
- This can be used to estimate C_d over a range of u

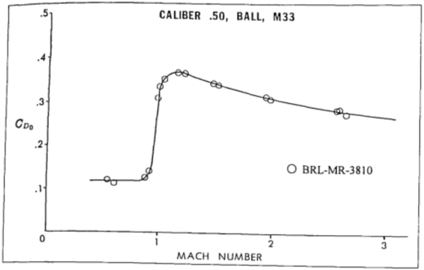
Low angle trajectories



Transonic C_d from terminal velocity

It is very difficult to get transonic C_d data from wind tunnel experiments.





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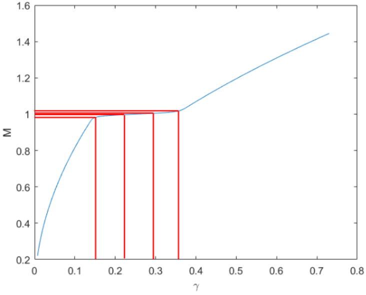

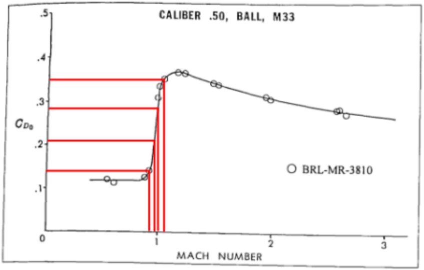

- The terminal velocity near Mach 1 is very insensitive to γ (dimensionless mass).
- We can use this to gain very fine control over the velocity in "dropping" experiments.