Solution4

HUDM 4125

3.3-1

(b)
$$P(-0.97 \le Z \le 1.27) = P(Z \le 1.27) - P(Z \le -0.97) = 0.8980 - 0.1660 = 0.7320$$

(c)
$$P(Z > -1.56) = 1 - P(Z \le -1.56) = 1 - 0.0594 = 0.9406$$

(d)
$$P(|Z| < 3) = P(-3 < Z < 3) = P(Z \le 3) - P(Z \le -3) = 0.9987 - 0.0013 = 0.9974$$

3.3-2

(c)
$$P(-2.31 \le Z \le -0.65) = P(Z \le -0.65) - P(Z \le -2.31) = 0.2578 - 0.0104 = 0.2474$$

(d)
$$P(|Z| > 1.93) = P(Z > 1.93) + P(Z < -1.93) = (1 - 0.9732) + 0.0268 = 0.0536$$

3.3-4

(a)
$$z_{0.10} \approx 1.28$$

(b)
$$-z_{0.0485} \approx -1.66$$

3.3-6

(a) The Mean of X

$$\mu = 166$$

(b) The Variance of X

$$\sigma^2 = 400$$

(c) $P(170 < X < 200)$

$$P(170 < X < 200) = P\left(0.2 < Z < 1.7\right) = 0.9554 - 0.5793 = 0.3761$$

(d) $P(148 \le X \le 172)$

$$P(148 \le X \le 172) = P\left(-0.9 \le Z \le 0.3\right) = 0.6179 - 0.1841 = 0.4338$$

```r
# Calculate P(Z <= 0.3)
p1 <- pnorm(0.3)

# Calculate P(Z <= -0.9)
p2 <- pnorm(-0.9)

# Calculate P(-0.9 <= Z <= 0.3)
probability <- p1 - p2
probability
```

```
## [1] 0.4338513
```

3.3-11

(a) Finding $P(X > 22.07)$

$$P(X > 22.07) = P\left(Z > \frac{22.07 - 21.37}{0.4}\right) = P(Z > 1.75)$$
$$= 1 - P(Z \le 1.75) = 1 - 0.9599 = 0.0401$$

(b)

$$P(X < 20.857) = P\left(Z < \frac{20.857 - 21.37}{0.4}\right) = P(Z < -1.2825) \approx 0.10$$

Using the binomial distribution $Y \sim \text{Binomial}(n = 15, p = 0.10)$:

$$P(Y \le 2) = P(Y = 0) + P(Y = 1) + P(Y = 2)$$

where

$$P(Y = 0) \approx 0.2059, \quad P(Y = 1) \approx 0.3434, \quad P(Y = 2) \approx 0.2751$$
$$P(Y \le 2) \approx 0.2059 + 0.3434 + 0.2751 = 0.8244$$

3.3-15

(a)

$$P(X < 12) = 0.01$$

Standardize the inequality:

$$P\left(Z < \frac{12 - 12.1}{\sigma}\right) = 0.01$$
$$\frac{-0.1}{\sigma} = -2.33$$

$$\sigma = \frac{0.1}{2.33} \approx 0.0429$$

So, $\sigma \approx 0.0429$.

(b)

$$P(X < 12) = 0.01$$

Standardize the inequality:

$$P\left(Z < \frac{12 - \mu}{0.05}\right) = 0.01$$
$$\frac{12 - \mu}{0.05} = -2.33$$

$$\mu = 12 + (2.33 \times 0.05) \approx 12.1165$$

So, $\mu \approx 12.1165$.

4.1-2

(a) Sample Space of X and Y

The sample space S is:

$$S = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$$

(b) Joint PMF of X and Y

Since each outcome is equally likely, the joint PMF is:

$$P(X = x, Y = y) = \frac{1}{16} \quad \text{for all } x, y \in \{1, 2, 3, 4\}$$

(c) Marginal PMF of X

$$P(X = x) = \frac{1}{4} \quad \text{for } x = 1, 2, 3, 4$$

(d) Marginal PMF of Y

$$P(Y = y) = \frac{1}{4} \quad \text{for } y = 1, 2, 3, 4$$

(e) Independence of X and Y

Since $P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$ for all x and y, X and Y are independent.

4.1-3

Given the joint PMF:

$$f(x, y) = \frac{x+y}{32}, \quad x = 1, 2, \quad y = 1, 2, 3, 4$$

The marginal PMF of X is:

$$f_X(x) = \sum_{y=1}^{4} f(x, y)$$

For $x = 1$:

$$f_X(1) = \frac{1+1}{32} + \frac{1+2}{32} + \frac{1+3}{32} + \frac{1+4}{32} = \frac{14}{32} = \frac{7}{16}$$

For $x = 2$:

$$f_X(2) = \frac{2+1}{32} + \frac{2+2}{32} + \frac{2+3}{32} + \frac{2+4}{32} = \frac{18}{32} = \frac{9}{16}$$

So,

$$f_X(x) = \begin{cases} \frac{7}{16} & \text{if } x = 1 \\ \frac{9}{16} & \text{if } x = 2 \end{cases}$$

(b) The marginal PMF of Y is:

$$f_Y(y) = \sum_{x=1}^{2} f(x, y)$$

For $y = 1$:

$$f_Y(1) = \frac{1+1}{32} + \frac{2+1}{32} = \frac{5}{32}$$

For $y = 2$:

$$f_Y(2) = \frac{1+2}{32} + \frac{2+2}{32} = \frac{7}{32}$$

For $y = 3$:

$$f_Y(3) = \frac{1+3}{32} + \frac{2+3}{32} = \frac{9}{32}$$

For $y = 4$:

$$f_Y(4) = \frac{1+4}{32} + \frac{2+4}{32} = \frac{11}{32}$$

So,

$$f_Y(y) = \begin{cases} \frac{5}{32} & \text{if } y = 1 \\ \frac{7}{32} & \text{if } y = 2 \\ \frac{9}{32} & \text{if } y = 3 \\ \frac{11}{32} & \text{if } y = 4 \end{cases}$$

(c)

$$P(X > Y) = f(2, 1) + f(2, 2) + f(2, 3) = \frac{3}{32} + \frac{4}{32} + \frac{5}{32} = \frac{12}{32} = \frac{3}{8}$$

(d) $P(Y = 2X)$

$$P(Y = 2X) = f(1,2) + f(2,4) = \frac{3}{32} + \frac{6}{32} = \frac{9}{32}$$

(e) $P(X + Y = 3)$

$$P(X + Y = 3) = f(1,2) + f(2,1) = \frac{3}{32} + \frac{3}{32} = \frac{6}{32} = \frac{3}{16}$$

(f) $P(X \le 3 - Y)$

$$P(X \le 3 - Y) = f(1,1) + f(2,1) + f(1,2) = \frac{2}{32} + \frac{3}{32} + \frac{3}{32} = \frac{8}{32} = \frac{1}{4}$$

(g) Since $f(x,y) \neq f_X(x) \cdot f_Y(y)$ for all x and y, X and Y are dependent.

(h)Means and Variances of X and Y

$$E(X) = \sum_{x=1}^{2} x \cdot f_X(x) = 1 \cdot \frac{7}{16} + 2 \cdot \frac{9}{16} = \frac{25}{16} = 1.5625$$

$$E(Y) = \sum_{y=1}^{4} y \cdot f_Y(y) = 1 \cdot \frac{5}{32} + 2 \cdot \frac{7}{32} + 3 \cdot \frac{9}{32} + 4 \cdot \frac{11}{32} = 2.8125$$

$$E(X^2) = \sum_{x=1}^{2} x^2 \cdot f_X(x) = 1^2 \cdot \frac{7}{16} + 2^2 \cdot \frac{9}{16} = \frac{7 + 36}{16} = \frac{43}{16} = 2.6875$$

$$\begin{aligned} \mathrm{Var}(X) &= E(X^2) - (E(X))^2 = 2.6875 - (1.5625)^2 \\ &= 2.6875 - 2.4414 = 0.2461 \end{aligned}$$

$$E(Y^2) = \sum_{y=1}^{4} y^2 \cdot f_Y(y) = \frac{145}{16} = 9.0625$$

$$\begin{aligned} \mathrm{Var}(Y) &= E(Y^2) - (E(Y))^2 = 9.0625 - (2.8125)^2 \\ &= \frac{295}{256} = 1.1523 \end{aligned}$$

- **Mean of X:** $E(X) = 1.5625$
- **Variance of X:** $\mathrm{Var}(X) = 0.2461$
- **Mean of Y:** $E(Y) = 2.1875$
- **Variance of Y:** $\mathrm{Var}(Y) = 4.2773$

4.1-5

(a) Let $x = 0, 1, 2, 3, 4$ and $y = 0, \ldots, x$.

The marginal pmf of X, $f_X(x)$, is given by:

$$f_X(x) = \frac{x+1}{15}, \quad x = 0, 1, 2, 3, 4;$$

The marginal pmf of Y, $f_Y(y)$, is given by:

$$f_Y(y) = \frac{5-y}{15}, \quad y = 0, 1, 2, 3, 4.$$

(d) Let $x = 0, 1, 2, 3, 4$ and $y = 4 - x, \ldots, 4$.

The marginal pmf of X, $f_X(x)$, is given by:

$$f_X(x) = \frac{x+1}{15}, \quad x = 0, 1, 2, 3, 4;$$

The marginal pmf of Y, $f_Y(y)$, is given by:

$$f_Y(y) = \frac{y+1}{15}, \quad y = 0, 1, 2, 3, 4.$$

4.1-6

$$P(X_1 = 7, X_2 = 8, X_3 = 6, X_4 = 4) = \frac{n!}{x_1!\, x_2!\, x_3!\, x_4!} p_1^{x_1} p_2^{x_2} p_3^{x_3} p_4^{x_4}$$

where: - $n = 25$, - $x_1 = 7$, $x_2 = 8$, $x_3 = 6$, $x_4 = 4$, - $p_1 = 0.30$, $p_2 = 0.40$, $p_3 = 0.20$, $p_4 = 0.10$.
Substitute these values:

$$P(X_1 = 7, X_2 = 8, X_3 = 6, X_4 = 4) = \frac{25!}{7!\, 8!\, 6!\, 4!} \times (0.30)^7 \times (0.40)^8 \times (0.20)^6 \times (0.10)^4 \approx 0.00405$$

4.2-1

$$f(x, y) = \frac{x+y}{32}, \quad x = 1, 2, \quad y = 1, 2, 3, 4$$

Marginal PMF of X:

$$f_X(x) = \sum_{y=1}^{4} f(x, y)$$

Marginal PMF of Y:

$$f_Y(y) = \sum_{x=1}^{2} f(x, y)$$

The mean (expected value) of X is:

$$\mu_X = E(X) = \sum_{x=1}^{2} x \cdot f_X(x) = 1.5625$$

The mean of Y is:

$$\mu_Y = E(Y) = \sum_{y=1}^{4} y \cdot f_Y(y) = 2.8125$$

The variance of X is calculated as:

$$\sigma_X^2 = E(X^2) - (E(X))^2 = 0.2461$$

The variance of Y is:

$$\sigma_Y^2 = E(Y^2) - (E(Y))^2 = 1.1523$$

The covariance between X and Y is:

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = -0.0195$$

The correlation coefficient ρ is:

$$\rho = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} = -0.0367$$

Since the joint PMF $f(x,y) \neq f_X(x) \cdot f_Y(y)$ for all x and y, X and Y are **dependent**.

4.2-6

$$f(x,y) = \frac{1}{6}, \quad 0 \leq x + y \leq 2, \quad x, y \text{ are nonnegative integers}$$

The support of (X,Y) is:

$$(0,0), (1,0), (0,1), (2,0), (1,1), (0,2)$$

Marginal PMF of X:

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{3} & \text{if } x = 1 \\ \frac{1}{6} & \text{if } x = 2 \end{cases}$$

Marginal PMF of Y:

$$f_Y(y) = \begin{cases} \frac{1}{2} & \text{if } y = 0 \\ \frac{1}{3} & \text{if } y = 1 \\ \frac{1}{6} & \text{if } y = 2 \end{cases}$$

- **Mean of X:** $\mu_X = 0.667$
- **Mean of Y:** $\mu_Y = 0.667$
- **Variance of X:** $\sigma_X^2 = 0.556$
- **Variance of Y:** $\sigma_Y^2 = 0.556$
- **Covariance of X and Y:** $\text{Cov}(X,Y) = -0.278$

The correlation coefficient ρ is:

$$\rho = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} = -0.500$$

The best-fitting line for predicting Y based on X is:

$$Y = 1.0 - 0.5X$$

4.2-7

(a) The correlation coefficient ρ is:

$$\rho = 0.5$$

(b) The correlation coefficient ρ is:

$$\rho = -0.5$$

4.2-10

Given the joint PMF table for X (moisture content) and Y (impurity content):

$Y \backslash X$	1	2	3	4
2	0.10	0.20	0.30	0.05
1	0.05	0.05	0.15	0.10

(a)
- **Marginal PMF of X:**

$$f_X(x) = \begin{cases} 0.15 & \text{if } x = 1 \\ 0.25 & \text{if } x = 2 \\ 0.45 & \text{if } x = 3 \\ 0.15 & \text{if } x = 4 \end{cases}$$

- **Marginal PMF of Y:**

$$f_Y(y) = \begin{cases} 0.35 & \text{if } y = 1 \\ 0.65 & \text{if } y = 2 \end{cases}$$

- **Mean of X (μ_X):** 2.6
- **Mean of Y (μ_Y):** 1.65
- **Variance of X (σ_X^2):** 0.84
- **Variance of Y (σ_Y^2):** 0.2275

(b) Covariance and Correlation Coefficient

- **Covariance of X and Y (Cov(X, Y)):** -0.09
- **Correlation coefficient** ρ: -0.206

(c)

$$E(C) = 34.7$$

4.3-2

(x, y)	$f(x, y)$
$(1, 1)$	$\frac{3}{8}$
$(2, 1)$	$\frac{1}{8}$
$(1, 2)$	$\frac{1}{8}$
$(2, 2)$	$\frac{3}{8}$

- **Marginal PMF of X:**

$$f_X(x) = \begin{cases} 0.5 & \text{if } x = 1 \\ 0.5 & \text{if } x = 2 \end{cases}$$

- **Marginal PMF of Y:**

$$f_Y(y) = \begin{cases} 0.5 & \text{if } y = 1 \\ 0.5 & \text{if } y = 2 \end{cases}$$

- **Conditional PMF of $X|Y$:**

$$f_{X|Y}(x|y) = \begin{cases} 0.75 & \text{if } x = 1, y = 1 \\ 0.25 & \text{if } x = 2, y = 1 \\ 0.25 & \text{if } x = 1, y = 2 \\ 0.75 & \text{if } x = 2, y = 2 \end{cases}$$

- **Conditional PMF of $Y|X$:**

$$f_{Y|X}(y|x) = \begin{cases} 0.75 & \text{if } y = 1, x = 1 \\ 0.25 & \text{if } y = 2, x = 1 \\ 0.25 & \text{if } y = 1, x = 2 \\ 0.75 & \text{if } y = 2, x = 2 \end{cases}$$

- **Mean and Variance of $X|Y = 1$:**

$$E(X|Y = 1) = 1.25, \quad \text{Var}(X|Y = 1) = 0.1875$$

- **Mean and Variance of $X|Y = 2$:**

$$E(X|Y = 2) = 1.75, \quad \text{Var}(X|Y = 2) = 0.1875$$

- **Mean and Variance of $Y|X = 1$:**

$$E(Y|X = 1) = 1.25, \quad \text{Var}(Y|X = 1) = 0.1875$$

- **Mean and Variance of $Y|X = 2$:**

$$E(Y|X = 2) = 1.75, \quad \text{Var}(Y|X = 2) = 0.1875$$

4.3-6

Given the joint probability table for X and Y:

$Y \backslash X$	100	500	1000
1000	0.05	0.10	0.15
500	0.10	0.20	0.05
100	0.20	0.10	0.05

(a) Probabilities

$P(X = 500) = 0.4$ $P(Y = 500) = 0.35$ $P(Y = 500|X = 500) = 0.5$ $P(Y = 100|X = 500) = 0.25$

(b) Means and Variances

- **Mean of X (μ_X):** 485
- **Mean of Y (μ_Y):** 510
- **Variance of X (σ_X^2):** 118275
- **Variance of Y (σ_Y^2):** 130900

(c) Conditional Means

$E(X|Y = 100) = 342.86$
$E(Y|X = 500) = 525$

(d) Covariance

- **Covariance of X and Y (Cov(X, Y)):** 49650

(e) Correlation Coefficient

- **Correlation coefficient ρ:** 0.399

4.3-9

(a) Marginal PMF $f_X(x)$

$$f_X(x) = 0.125 \quad \text{for each } x = 0, 1, 2, \ldots, 7$$

(b) Conditional PMF $h(y|x) = P(Y = y|X = x)$

- For $X = 0$: $h(y|0) = \begin{cases} 0.333 & \text{if } y = 0, 1, 2 \end{cases}$
- For $X = 1$: $h(y|1) = \begin{cases} 0.333 & \text{if } y = 1, 2, 3 \end{cases}$
- For $X = 2$: $h(y|2) = \begin{cases} 0.333 & \text{if } y = 2, 3, 4 \end{cases}$
- And similarly for each $X = x$ up to $X = 7$.

(c) Conditional Mean $E(Y|X = x)$

$$E(Y|X = x) = x + 1$$

(d) Conditional Variance $\sigma^2_{Y|X}$

$$\sigma^2_{Y|X} = 0.667 \quad \text{for all } X = x$$

(e) Marginal PMF $f_Y(y)$

$$f_Y(y) = \begin{cases} 0.0417 & \text{if } y = 0 \text{ or } 9 \\ 0.0833 & \text{if } y = 1 \text{ or } 8 \\ 0.125 & \text{if } y = 2, 3, 4, 5, 6, 7 \end{cases}$$