Solution6

HUDM 4125

6.4-4

The pmf of the random variable X is:

$$f(x) = \frac{2 + \theta(2 - x)}{6}, \quad x = 1, 2, 3,$$

where $\theta \in \{-1, 0, 1\}$.

Likelihood Function

Given the sample $\{x_1, x_2, x_3, x_4\} = \{3, 2, 3, 1\}$, the likelihood function is:

$$L(\theta) = \prod_{i=1}^{4} f(x_i) = \prod_{i=1}^{4} \frac{2 + \theta(2 - x_i)}{6}.$$

Simplified:

$$L(\theta) = \frac{1}{6^4} \prod_{i=1}^{4} \left(2 + \theta(2 - x_i)\right).$$

Log-Likelihood

The log-likelihood function is:

$$\ell(\theta) = -4 \log 6 + \sum_{i=1}^{4} \log\left(2 + \theta(2 - x_i)\right).$$

For the observed sample:

$$\{x_1, x_2, x_3, x_4\} = \{3, 2, 3, 1\},$$

$$2 - x_1 = -1, \quad 2 - x_2 = 0, \quad 2 - x_3 = -1, \quad 2 - x_4 = 1.$$

Thus:

$$\ell(\theta) = -4 \log 6 + \log\left(2 + \theta(-1)\right) + \log\left(2 + \theta(0)\right) + \log\left(2 + \theta(-1)\right) + \log\left(2 + \theta(1)\right).$$

Maximization

Evaluate $\ell(\theta)$ for each $\theta \in \{-1, 0, 1\}$: - For $\theta = -1$: $(2 + \theta(2 - x_i)) = 1, 2, 1, 1$. - For $\theta = 0$: $(2 + \theta(2 - x_i)) = 2, 2, 2, 2$. - For $\theta = 1$: $(2 + \theta(2 - x_i)) = 1, 2, 1, 3$.

The log-likelihood values for each θ are:

$$\ell(-1) = -4.2767, \quad \ell(0) = -4.3944, \quad \ell(1) = -5.3753.$$

Result

The maximum likelihood estimate (MLE) of θ is:

$$\hat{\theta} = -1,$$

as it maximizes the log-likelihood function.

6.4-7

(b) Derive the MLE The likelihood function for a sample size n is:

$$L(\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1}.$$

Taking the natural log:

$$\ln L(\theta) = n \ln \theta + (\theta - 1) \sum_{i=1}^{n} \ln x_i.$$

Differentiating with respect to θ:

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \ln x_i.$$

Setting $\frac{\partial \ln L(\theta)}{\partial \theta} = 0$, solve for θ:

$$\hat{\theta} = -\frac{n}{\sum_{i=1}^{n} \ln x_i}.$$

(c) MLE and MOM Estimates

1. **Maximum Likelihood Estimate (MLE):**

$$\hat{\theta} = -\frac{n}{\sum_{i=1}^{n} \ln x_i}.$$

2. **Method-of-Moments Estimate (MOM):** From the expected value:

$$E[X] = \frac{\theta}{\theta + 1}.$$

Equating to the sample mean $\bar{X}$:

$$\bar{X} = \frac{\hat{\theta}}{\hat{\theta} + 1}.$$

Solving for $\hat{\theta}$:

$$\hat{\theta} = \frac{\bar{X}}{1 - \bar{X}}.$$

Results for Each Dataset

The MLE and MOM estimates for each dataset are as follows:

Dataset	MLE	MOM
(i)	0.5493	0.5975
(ii)	2.2101	2.4004
(iii)	0.9588	0.8646

6.4-8

(a) The PDF of X is:

$$f(x;\theta) = \frac{1}{\theta} x^{\frac{1-\theta}{\theta}}, \quad 0 < x < 1, \, \theta > 0.$$

The likelihood function is:

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} X_i^{\frac{1-\theta}{\theta}}.$$

The log-likelihood is:

$$\ln L(\theta) = -n \ln \theta + \frac{1-\theta}{\theta} \sum_{i=1}^{n} \ln X_i.$$

Differentiating with respect to θ:

$$\frac{\partial \ln L(\theta)}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} \ln X_i.$$

Setting $\frac{\partial \ln L(\theta)}{\partial \theta} = 0$, solving gives:

$$\hat{\theta} = -\frac{1}{n} \sum_{i=1}^{n} \ln X_i.$$

(b) For $X_i \sim f(x;\theta)$, the expected value of $\ln X_i$ is:

$$E[\ln X_i] = \int_0^1 \ln x \cdot \frac{1}{\theta} x^{\frac{1-\theta}{\theta}} \, dx.$$

This evaluates to:

$$E[\ln X_i] = -\theta.$$

The expected value of $\hat{\theta}$ is:

$$E[\hat{\theta}] = -\frac{1}{n} \sum_{i=1}^{n} E[\ln X_i] = -\frac{1}{n} \cdot n \cdot (-\theta) = \theta.$$

Conclusion

Thus, $E[\hat{\theta}] = \theta$, and $\hat{\theta}$ is an unbiased estimator of θ.

6.4-9

(a) The PDF is:

$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0, \, \theta > 0.$$

The expected value of X for the exponential distribution is:

$$E[X] = \theta.$$

The sample mean is:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

The expected value of $\bar{X}$ is:

$$E[\bar{X}] = E\left[\frac{1}{n} \sum_{i=1}^{n} X_i\right] = \frac{1}{n} \sum_{i=1}^{n} E[X_i].$$

It follows that:

$$E[\bar{X}] = \frac{1}{n} \cdot n \cdot \theta = \theta.$$

Thus, $\bar{X}$ is an unbiased estimator of θ.

(b) The variance of X for the exponential distribution is:

$$\text{Var}(X) = \theta^2.$$

The variance of the sample mean is:

$$\text{Var}(\bar{X}) = \text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right).$$

Using the properties of variances for independent random variables:

$$\text{Var}(\bar{X}) = \frac{1}{n^2}\sum_{i=1}^{n} \text{Var}(X_i).$$

Since $\text{Var}(X_i) = \theta^2$:

$$\text{Var}(\bar{X}) = \frac{1}{n^2} \cdot n \cdot \theta^2 = \frac{\theta^2}{n}.$$

(c) Given the sample values: $3.5, 8.1, 0.9, 4.4, 0.5$, the sample mean is:

$$\bar{X} = \frac{1}{5}\sum_{i=1}^{5} X_i = \frac{3.5 + 8.1 + 0.9 + 4.4 + 0.5}{5} = 3.48.$$

Thus, the estimate of θ is:

$$\hat{\theta} = \bar{X} = 3.48.$$

6.4-13

(a) For a uniform distribution on the interval $(\theta - 1, \theta + 1)$, the mean is:

$$E[X] = \frac{(\theta - 1) + (\theta + 1)}{2} = \theta.$$

The sample mean is:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

By the method of moments, equating the population mean to the sample mean:

$$\hat{\theta}_{\text{MOM}} = \bar{X}.$$

(b) To check whether $\hat{\theta}_{\text{MOM}}$ is unbiased, compute:

$$E[\hat{\theta}_{\text{MOM}}] = E[\bar{X}].$$

Since $\bar{X}$ is an unbiased estimator of $E[X]$ and $E[X] = \theta$, we have:

$$E[\hat{\theta}_{\text{MOM}}] = \theta.$$

Thus, $\hat{\theta}_{\text{MOM}}$ is an unbiased estimator of θ.

(c) Given the sample: $6.61, 7.70, 6.98, 8.36, 7.26$, the sample mean is:

$$\bar{X} = \frac{1}{5}\sum_{i=1}^{5} X_i = \frac{6.61 + 7.70 + 6.98 + 8.36 + 7.26}{5} = 7.382.$$

Using the method-of-moments estimator:

$$\hat{\theta}_{\text{MOM}} = \bar{X} = 7.382.$$

6.4-17

(b) The expectation $E[X]$ is:

$$E[X] = \frac{\theta}{2}.$$

By the method of moments, equating $E[X]$ to the sample mean $\bar{X}$:

$$\hat{\theta} = 2\bar{X}.$$

(c) Given the observations:

$$0.3206, 0.2408, 0.2577, 0.3557, 0.4188, 0.5601, 0.0240, 0.5422, 0.4532, 0.5592,$$

the sample mean is:

$$\bar{X} = \frac{1}{10}\sum_{i=1}^{10} X_i = 0.3732.$$

Using the corrected method-of-moments estimator:

$$\hat{\theta} = 2\bar{X} = 0.7465.$$

6.5-6

(a) The least squares regression line is given by:

$$\hat{y} = \alpha + \beta x,$$

where:

$$\beta = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}, \quad \alpha = \bar{y} - \beta\bar{x}.$$

For the given data:

$$x = [32, 23, 23, 23, 26, 30, 17, 20, 17, 18, 26, 16, 21, 24, 30],$$
$$y = [28, 25, 24, 32, 31, 27, 23, 30, 18, 18, 32, 22, 28, 31, 26].$$

The means of x and y are:

$$\bar{x} = 23.867, \quad \bar{y} = 26.33.$$

The slope (β) is:

$$\beta = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2} = 0.5062.$$

The intercept (α) is:

$$\alpha = \bar{y} = 26.33.$$

Thus, the least squares regression line is:

$$\hat{y} = 14.6578 + 0.5062x.$$

(c) The point estimates are:

1. $\alpha = 26.33$,
2. $\beta = 0.5062$.

The residual variance (σ^2) is calculated as:

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2.$$

Substituting the values:

$$\hat{\sigma}^2 = 14.1258.$$

6.5-7

(a) The least squares regression line is given by:

$$\hat{y} = \alpha + \beta x,$$

where:

$$\beta = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}, \quad \alpha = \bar{y} - \beta\bar{x}.$$

For the given data:

$$x = [9, 4, 14, 12, 10, 5, 3, 17, 6, 7, 8, 15],$$
$$y = [6, 6, 14, 12, 12, 7, 4, 18, 8, 8, 13, 13].$$

The means of x and y are:

$$\bar{x} = 9.0, \quad \bar{y} = 10.083.$$

The slope (β) is:

$$\beta = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2} = 0.8191.$$

The intercept (α) is:

$$\alpha = \bar{y} = 10.083.$$

Thus, the least squares regression line is:

$$\hat{y} = 2.5753 + 0.8191x.$$

(c) Point Estimates for α, β, and σ^2

1. α **(Intercept)**: 10.083,
2. β **(Slope)**: 0.8191.

The Maximum Likelihood Estimator (MLE) for σ^2 is given by:

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2.$$

Substituting the values:

$$\hat{\sigma}^2 = \frac{1}{12}\sum_{i=1}^{12}(y_i - \hat{y}_i)^2 = 3.294.$$

Thus:

$$\alpha = 10.083, \quad \beta = 0.8191, \quad \sigma^2 = 3.294.$$

7.1-1

We are given: - Sample size: $n = 16$, - Sample mean: $\bar{x} = 73.8$, - Population variance: $\sigma^2 = 25$, so $\sigma = 5$.

The 95% confidence interval for μ is computed as:

$$\bar{x} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}},$$

where $z_{\alpha/2} = 1.96$ for a 95% confidence level.

1. **Compute the Standard Error**:

$$\text{SE} = \frac{\sigma}{\sqrt{n}} = \frac{5}{\sqrt{16}} = 1.25.$$

2. **Compute the Margin of Error**:

$$\text{Margin of Error} = z_{\alpha/2} \cdot \text{SE} = 1.96 \cdot 1.25 = 2.45.$$

3. **Compute the Confidence Interval**:

$$\text{Confidence Interval} = \bar{x} \pm \text{Margin of Error}.$$

$$\text{Confidence Interval} = 73.8 \pm 2.45 = (71.35, 76.25).$$

The 95% confidence interval for μ is:

$$(71.35, 76.25).$$

7.1-4

(a) The point estimate for μ is the sample mean:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

Given data:

$$x = [55.95, 56.54, 57.58, 55.13, 57.48, 56.06, 59.93, 58.30, 52.57, 58.46].$$

The sample mean is:

$$\bar{x} = 56.8.$$

(b) The formula for a 95% confidence interval is:

$$\bar{x} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}},$$

where:

- $\sigma^2 = 4 \implies \sigma = 2$,
- $n = 10$,
- $z_{\alpha/2} = 1.96$ for a 95% confidence level.

Step 1: Calculate the Standard Error

$$\text{SE} = \frac{\sigma}{\sqrt{n}} = \frac{2}{\sqrt{10}} = 0.632.$$

Step 2: Calculate the Margin of Error

$$\text{Margin of Error} = z_{\alpha/2} \cdot \text{SE} = 1.96 \cdot 0.632 = 1.239.$$

Step 3: Compute the Confidence Interval

$$\text{Confidence Interval} = \bar{x} \pm \text{Margin of Error}.$$

$$\text{Confidence Interval} = 56.8 \pm 1.239 = (55.56, 58.04).$$

(c) The probability of a snack pack weighing less than 52 grams is:

$$P(X < 52) = P\left(Z < \frac{52 - \mu}{\sigma}\right),$$

where Z follows the standard normal distribution. Substituting $\mu = 56.8$ and $\sigma = 2$:

$$Z = \frac{52 - 56.8}{2} = -2.4.$$

Using the standard normal distribution table or a statistical tool:

$$P(X < 52) = P(Z < -2.4) \approx 0.0082.$$

Final Results

1. **Point Estimate for μ:** $\bar{x} = 56.8$,
2. **95% Confidence Interval:** $(55.56, 58.04)$,
3. **Probability $P(X < 52)$:** 0.0082.

7.1-8

(a) The point estimate for μ is the sample mean:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Given data:

$$x = [37.4, 48.8, 46.9, 55.0, 44.0],$$

and $n = 5$. The sample mean is:

$$\bar{x} = 46.42.$$

(b) Since the population variance (σ^2) is unknown, we use the t-distribution to calculate the confidence interval. The formula is:

$$\bar{x} \pm t_{\alpha/2} \cdot \frac{s}{\sqrt{n}},$$

where:

- $\bar{x} = 46.42$,
- $t_{\alpha/2}$ is the critical t-value for a 90% confidence level with $n - 1 = 4$ degrees of freedom,
- s is the sample standard deviation,
- $n = 5$.

Step 1: Calculate the Sample Standard Deviation The sample standard deviation is:

$$s = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n-1}} = 6.4562.$$

Step 2: Find the Critical t-Value For a 90% confidence level and $n-1=4$ degrees of freedom:

$$t_{\alpha/2} = 2.1318.$$

Step 3: Compute the Margin of Error The margin of error is:

$$\text{Margin of Error} = t_{\alpha/2} \cdot \frac{s}{\sqrt{n}} = 2.1318 \cdot \frac{6.4562}{\sqrt{5}} = 6.1552.$$

Step 4: Compute the Confidence Interval

$$\text{Confidence Interval} = \bar{x} \pm \text{Margin of Error}.$$

$$\text{Confidence Interval} = 46.42 \pm 6.1552 = (40.26, 52.58).$$

Final Results

1. **Point Estimate for μ:** $\bar{x} = 46.42$,
2. **90% Confidence Interval for μ:** $(40.26, 52.58)$.

7.1-11

We are given: - Sample size: $n = 41$, - Sample mean: $\bar{x} = 132$, - Sample variance: $s^2 = 105 \implies s = \sqrt{105} = 10.247$, - Confidence level: 95%.

The confidence interval formula is:

$$\bar{x} \pm t_{\alpha/2} \cdot \frac{s}{\sqrt{n}},$$

where: - $t_{\alpha/2}$ is the critical t-value with $n-1 = 40$ degrees of freedom, - s is the sample standard deviation, - n is the sample size.

Step 1: Calculate the Standard Error

$$\text{SE} = \frac{s}{\sqrt{n}} = \frac{10.247}{\sqrt{41}} = 1.6003.$$

Step 2: Find the Critical t-Value For a 95% confidence level and $n-1=40$ degrees of freedom:

$$t_{\alpha/2} = 2.0211.$$

Step 3: Compute the Confidence Interval

$$\text{Confidence Interval} = \bar{x} \pm t_{\alpha/2} \cdot \text{SE}.$$

Substitute the values:

$$\text{Confidence Interval} = 132 \pm 2.0211 \cdot 1.6003 = (128.77, 135.23).$$

Final Answer:

The 95% confidence interval for the population mean is:

$$(128.77, 135.23).$$

7.1-16

Step 1: Rewrite the Probability Using symmetry, the given probability can be rewritten as:

$$P(-1 < \mu - \bar{X} < 1) = 0.90.$$

Since $\bar{X}$ is the mean of a random sample of size n from $N(\mu, 9)$, we know:

$$\mu - \bar{X} \sim N(0, \sigma_{\bar{X}}), \quad \text{where } \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{3}{\sqrt{n}}.$$

Thus:

$$P(-1 < \mu - \bar{X} < 1) = P\left(-\frac{1}{\sigma_{\bar{X}}} < Z < \frac{1}{\sigma_{\bar{X}}}\right),$$

where $Z \sim N(0, 1)$.

Step 2: Find the Critical Z-Value Let:

$$z = \frac{1}{\sigma_{\bar{X}}} = \frac{\sqrt{n}}{3}.$$

We need:

$$P(-z < Z < z) = 0.90.$$

From the standard normal table:

$$P(-z < Z < z) = 2P(Z < z) - 1 = 0.90.$$

Solve for $P(Z < z)$:

$$P(Z < z) = 0.95.$$

From the inverse of the standard normal distribution:

$$z = \Phi^{-1}(0.95) \approx 1.645.$$

Step 3: Solve for n Substitute $z = \frac{\sqrt{n}}{3}$:

$$1.645 = \frac{\sqrt{n}}{3}.$$

$$\sqrt{n} = 1.645 \cdot 3 = 4.935.$$

Square both sides to find n:

$$n = 4.935^2 \approx 24.36.$$

Since n must be an integer:

$$n = 25.$$