Solution 3

HUDM 4125

2.7-1

(a)

$$P(3 \le X \le 5) = P(X = 3) + P(X = 4) + P(X = 5)$$

$$P(X = 3) = \frac{4^3 e^{-4}}{3!} = \frac{64e^{-4}}{6}, \quad P(X = 4) = \frac{4^4 e^{-4}}{4!} = \frac{256e^{-4}}{24}, \quad P(X = 5) = \frac{4^5 e^{-4}}{5!} = \frac{1024e^{-4}}{120}$$

Thus,

$$P(3 \le X \le 5) = e^{-4}\left(\frac{64}{6} + \frac{256}{24} + \frac{1024}{120}\right) = 0.547$$

(b)

$$P(X \ge 3) = 1 - P(X \le 2)$$

$$P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$P(X = 0) = e^{-4}, \quad P(X = 1) = 4e^{-4}, \quad P(X = 2) = 8e^{-4}$$

$$P(X \le 2) = e^{-4} \times 13$$

Thus,

$$P(X \ge 3) = 1 - 13e^{-4} = 0.7619$$

(c) $P(X \le 3)$

$$P(X \le 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$P(X \le 3) = e^{-4}\left(1 + 4 + 8 + \frac{64}{6}\right)$$

Thus,

$$P(X \le 3) = e^{-4}\left(13 + \frac{64}{6}\right) = 0.4335$$

2.7-4

The probability mass function for a Poisson distribution is:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$3P(X = 1) = P(X = 2)$$

Substitute the Poisson PMF for $P(X = 1)$ and $P(X = 2)$:

$$3 \times \frac{\lambda^1 e^{-\lambda}}{1!} = \frac{\lambda^2 e^{-\lambda}}{2!}$$

$$3\lambda e^{-\lambda} = \frac{\lambda^2 e^{-\lambda}}{2}$$

$$3\lambda = \frac{\lambda^2}{2}$$

$$6\lambda = \lambda^2$$

$$\lambda^2 - 6\lambda = 0$$

$$\lambda(\lambda - 6) = 0$$

Thus, $\lambda = 6$

$$P(X = 4) = \frac{6^4 e^{-6}}{4!} = \frac{1296 e^{-6}}{24}$$

$$P(X = 4) \approx 0.1339$$

2.7-5

$$\lambda = \frac{225}{150} = 1.5$$

$$P(X \leq 1) = P(X = 0) + P(X = 1)$$

$$P(X = 0) = \frac{1.5^0 e^{-1.5}}{0!} = e^{-1.5}$$

$$P(X = 1) = \frac{1.5^1 e^{-1.5}}{1!} = 1.5e^{-1.5}$$

$$P(X \leq 1) = e^{-1.5} + 1.5e^{-1.5} = 0.5578$$

2.7-13

$$\frac{\lambda^2 e^{-\lambda}}{2!} = 4 \times \frac{\lambda^3 e^{-\lambda}}{3!}$$

$$\frac{\lambda^2}{2} = 4 \times \frac{\lambda^3}{6}$$

$$3\lambda^2 = 4\lambda^3$$

$$\lambda = \frac{3}{4}$$

$$E(X^2) = \lambda + \lambda^2$$

$$E(X^2) = \frac{3}{4} + \left(\frac{3}{4}\right)^2 = 1.3125$$

3.1-2

(a)

$$\int_0^1 c(x - x^2)\, dx = 1$$

$$c = 6$$

(b)

$$P(0.3 < X < 0.6) = \int_{0.3}^{0.6} 6(x - x^2)\, dx \approx 0.4320$$

3.1-5

(a)

$$W = a + (b - a)Y$$

The CDF of W is:

$$F_W(w) = P(W \leq w) = P\left(Y \leq \frac{w-a}{b-a}\right)$$

Since $Y \sim U(0,1)$, its CDF is:

$$F_Y(y) = y$$

Thus:

$$F_W(w) = \begin{cases} 0, & w < a \\ \frac{w-a}{b-a}, & a \leq w \leq b \\ 1, & w > b \end{cases}$$

(b) The random variable W is uniformly distributed between a and b:

$$W \sim U(a, b)$$

3.1-9

(i)

$$P(0 \leq X \leq 1/2) = \int_0^{1/2} 2(1-x)\,dx$$

$$P(0 \leq X \leq 1/2) = 0.75$$

(ii)

$$P(1/4 \leq X \leq 3/4) = \int_{1/4}^{3/4} 2(1-x)\,dx$$

$$P(1/4 \leq X \leq 3/4) = 0.5$$

(iii)

$$P(X = 3/4) = 0$$

(iv)

$$P(X \geq 3/4) = \int_{3/4}^{1} 2(1-x)\,dx$$

$$P(X \geq 3/4) = 0.0625$$

3.2-1

(a) The pdf of X is:

$$f(x) = \frac{1}{3}e^{-\frac{1}{3}x}, \quad x \geq 0$$

For an exponential distribution, the mean is $\frac{1}{\lambda}$. Therefore:

$$\text{Mean} = 3$$

The variance of an exponential distribution is $\frac{1}{\lambda^2}$. Therefore:

$$\text{Variance} = 9$$

(b) This is also the MGF for an exponential distribution with rate $\lambda = 3$.

The pdf of X is:

$$f(x) = 3e^{-3x}, \quad x \geq 0$$

$$\text{Mean} = 0.3333$$

$$\text{Variance} = 0.1111$$

3.2-13

(a)
$$P(14.85 < X < 32.01) = F(32.01) - F(14.85)$$

Using the chi-square cumulative distribution function (CDF):

$$P(14.85 < X < 32.01) \approx F_{23}(32.01) - F_{23}(14.85) = 0.9 - 0.1 = 0.8$$

Thus:

$$P(14.85 < X < 32.01) = 0.8$$

(b) We use the inverse CDF (quantile function):

$$a = \chi^2_{0.025}(23) = 11.69$$

We use the quantile function:

$$b = \chi^2_{0.975}(23) = 38.08$$

Thus:

$$P(a < X < b) = P(11.69 < X < 38.08) = 0.95$$

(c) For a chi-square distribution with k degrees of freedom:

- The mean is k, and
- The variance is $2k$.

Thus, for $X \sim \chi^2(23)$:

$$\text{Mean} = 23$$

$$\text{Variance} = 46$$

(d)

$$\chi^2_{0.05}(23) = 35.17$$
$$\chi^2_{0.95}(23) = 13.09$$

3.2-16

$$P(T > 26.30) = 1 - F(26.30)$$

Using R, we calculate:

```
# Gamma distribution parameters
k <- 8
lambda <- 0.5

# Calculate the probability
prob <- 1 - pgamma(26.30, shape = k, rate = lambda)
round(prob, 4)
```

```
## [1] 0.05
```

3.2-17

```
# Chi-square distribution parameter
df <- 4

# Find P(X > 7.779)
p_chisq <- 1 - pchisq(7.779, df = df)

# Binomial distribution parameters
n <- 15
p <- p_chisq

# Find P(Y <= 3)
prob <- pbinom(3, size = n, prob = p)
round(prob, 4)
```

```
## [1] 0.9444
```