Solution5

HUDM 4125

4.4-1

Let $f(x, y) = \frac{3}{16}xy^2$ for $0 \leq x \leq 2$ and $0 \leq y \leq 2$ be the joint pdf of X and Y.

(a) The marginal pdf of X, $f_X(x)$, is given by:

$$f_X(x) = \int_0^2 f(x, y)\, dy = \int_0^2 \frac{3}{16}xy^2\, dy.$$

Evaluating this integral:

$$f_X(x) = \frac{3}{16}x \int_0^2 y^2\, dy = \frac{3}{16}x \cdot \left[\frac{y^3}{3}\right]_0^2 = \frac{3}{16}x \cdot \frac{8}{3} = \frac{x}{2}.$$

Thus, the marginal pdf of X is:

$$f_X(x) = \frac{x}{2}, \quad 0 \leq x \leq 2.$$

Similarly, the marginal pdf of Y, $f_Y(y)$, is given by:

$$f_Y(y) = \int_0^2 f(x, y)\, dx = \int_0^2 \frac{3}{16}xy^2\, dx.$$

Evaluating this integral:

$$f_Y(y) = \frac{3}{16}y^2 \int_0^2 x\, dx = \frac{3}{16}y^2 \cdot \left[\frac{x^2}{2}\right]_0^2 = \frac{3}{16}y^2 \cdot 2 = \frac{3}{8}y^2.$$

Thus, the marginal pdf of Y is:

$$f_Y(y) = \frac{3}{8}y^2, \quad 0 \leq y \leq 2.$$

(b) To check if X and Y are independent, we verify if $f(x, y) = f_X(x)f_Y(y)$.

$$f_X(x)f_Y(y) = \frac{x}{2} \cdot \frac{3}{8}y^2 = \frac{3}{16}xy^2 = f(x, y).$$

Since $f(x, y) = f_X(x)f_Y(y)$, X and Y are independent.

(c) Mean and Variance of X and Y

Mean of X:

$$E(X) = \int_0^2 x f_X(x)\, dx = \int_0^2 x \cdot \frac{x}{2}\, dx = \int_0^2 \frac{x^2}{2}\, dx = \frac{1}{2} \cdot \frac{x^3}{3}\Big|_0^2 = \frac{1}{2} \cdot \frac{8}{3} = \frac{4}{3}.$$

Variance of X:

$$E(X^2) = \int_0^2 x^2 f_X(x)\,dx = \int_0^2 x^2 \cdot \frac{x}{2}\,dx = \int_0^2 \frac{x^3}{2}\,dx = \frac{1}{2} \cdot \frac{x^4}{4}\Big|_0^2 = \frac{1}{2} \cdot 4 = 2.$$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \frac{2}{9}.$$

Mean of Y:

$$E(Y) = \int_0^2 y f_Y(y)\,dy = \int_0^2 y \cdot \frac{3}{8}y^2\,dy = \int_0^2 \frac{3}{8}y^3\,dy = \frac{3}{8} \cdot \frac{y^4}{4}\Big|_0^2 = \frac{3}{8} \cdot \frac{16}{4} = \frac{3}{2}.$$

Variance of Y:

$$E(Y^2) = \int_0^2 y^2 f_Y(y)\,dy = \int_0^2 y^2 \cdot \frac{3}{8}y^2\,dy = \int_0^2 \frac{3}{8}y^4\,dy = \frac{3}{8} \cdot \frac{y^5}{5}\Big|_0^2 = \frac{3}{8} \cdot \frac{32}{5} = \frac{12}{5}.$$

$$\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2 = \frac{12}{5} - \left(\frac{3}{2}\right)^2 = \frac{12}{5} - \frac{9}{4} = \frac{3}{20}.$$

(d) To find $P(X \le Y)$, we need to compute:

$$P(X \le Y) = \int_0^2 \int_0^y f(x,y)\,dx\,dy = \int_0^2 \int_0^y \frac{3}{16}xy^2\,dx\,dy.$$

Evaluating the inner integral with respect to x:

$$= \int_0^2 \frac{3}{16}y^2 \cdot \frac{x^2}{2}\Big|_0^y\,dy = \int_0^2 \frac{3}{16}y^2 \cdot \frac{y^2}{2}\,dy = \int_0^2 \frac{3}{32}y^4\,dy.$$

Evaluating the outer integral with respect to y:

$$= \frac{3}{32} \cdot \frac{y^5}{5}\Big|_0^2 = \frac{3}{32} \cdot \frac{32}{5} = \frac{3}{5}.$$

Thus,

$$P(X \le Y) = \frac{3}{5}.$$

4.4-3

Given:

$$f(x,y) = 2e^{-x-y}, \quad 0 \le x \le y < \infty$$

This is the joint pdf of X and Y.

(a): Finding the Marginal PDFs $f_X(x)$ and $f_Y(y)$

Marginal PDF of X

The marginal pdf $f_X(x)$ is obtained by integrating out y from $f(x, y)$:

$$f_X(x) = \int_x^\infty f(x, y)\, dy = \int_x^\infty 2e^{-x-y}\, dy.$$

Since e^{-x} is constant with respect to y, we can factor it out:

$$f_X(x) = 2e^{-x} \int_x^\infty e^{-y}\, dy.$$

Evaluating the integral:

$$f_X(x) = 2e^{-x} \left[-e^{-y}\right]_x^\infty = 2e^{-x} \cdot e^{-x} = 2e^{-2x}.$$

Thus, the marginal pdf of X is:

$$f_X(x) = 2e^{-2x}, \quad x > 0.$$

Marginal PDF of Y

The marginal pdf $f_Y(y)$ is obtained by integrating out x from $f(x, y)$:

$$f_Y(y) = \int_0^y f(x, y)\, dx = \int_0^y 2e^{-x-y}\, dx.$$

Since e^{-y} is constant with respect to x, we can factor it out:

$$f_Y(y) = 2e^{-y} \int_0^y e^{-x}\, dx.$$

Evaluating the integral:

$$f_Y(y) = 2e^{-y} \left[-e^{-x}\right]_0^y = 2e^{-y} \left(1 - e^{-y}\right).$$

Thus, the marginal pdf of Y is:

$$f_Y(y) = 2e^{-y}(1 - e^{-y}), \quad y > 0.$$

(b): Checking for Independence of X and Y

For X and Y to be independent, $f(x, y)$ must equal $f_X(x)f_Y(y)$.

1. From part (a), we have:
$$f(x, y) = 2e^{-x-y}.$$
2. The product $f_X(x)f_Y(y)$ is:
$$f_X(x)f_Y(y) = \left(2e^{-2x}\right) \cdot \left(2e^{-y}(1 - e^{-y})\right) = 4e^{-2x}e^{-y}(1 - e^{-y}).$$

Since $f(x, y) \neq f_X(x)f_Y(y)$, X and Y are **not independent**.

4.4-4

(a) The probability $P(0 \le X \le 1/2)$ can be found by integrating the joint pdf over the region where X ranges from 0 to 1/2 and Y follows the constraints $x^2 \le y \le 1$.

$$P(0 \le X \le 1/2) = \int_0^{1/2} \int_{x^2}^1 \frac{3}{2}\, dy\, dx$$

1. **Inner Integral with respect to y:**

$$\int_{x^2}^{1} \frac{3}{2}\,dy = \frac{3}{2}\,[y]_{x^2}^{1} = \frac{3}{2}(1-x^2)$$

2. **Outer Integral with respect to x:**

$$P(0 \le X \le 1/2) = \int_0^{1/2} \frac{3}{2}(1-x^2)\,dx = \frac{3}{2}\int_0^{1/2}(1-x^2)\,dx$$

$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_0^{1/2} = \frac{3}{2}\left(\frac{1}{2} - \frac{(1/2)^3}{3}\right)$$

$$= \frac{3}{2}\left(\frac{1}{2} - \frac{1}{24}\right) = \frac{3}{2}\cdot\frac{11}{24} = \frac{33}{48} = \frac{11}{16}$$

Thus,

$$P(0 \le X \le 1/2) = \frac{11}{16}$$

(b)

$$f_Y(y) = \int_0^{\sqrt{y}} f(x,y)\,dx = \int_0^{\sqrt{y}} \frac{3}{2}\,dx = \frac{3}{2}\sqrt{y}, \quad 0 \le y \le 1$$

$$P\left(\frac{1}{2} \le Y \le 1\right) = \int_{1/2}^{1} f_Y(y)\,dy = \int_{1/2}^{1} \frac{3}{2}\sqrt{y}\,dy$$

$$= \frac{3}{2}\int_{1/2}^{1} y^{1/2}\,dy = \frac{3}{2}\cdot\frac{2}{3}y^{3/2}\Big|_{1/2}^{1}$$

$$= y^{3/2}\Big|_{1/2}^{1} = 1 - \left(\frac{1}{2}\right)^{3/2} = 1 - \frac{\sqrt{2}}{4}$$

Therefore,

$$P\left(\frac{1}{2} \le Y \le 1\right) = 1 - \frac{\sqrt{2}}{4}$$

(c)

$$P\left(X \ge \frac{1}{2}, Y \ge \frac{1}{2}\right) = \int_{1/2}^{1}\int_{\sqrt{y}}^{1} \frac{3}{2}\,dx\,dy$$

$$= \int_{1/2}^{1}\left[\frac{3}{2}x\right]_{\sqrt{y}}^{1} dy = \int_{1/2}^{1} \frac{3}{2}(1-\sqrt{y})\,dy$$

$$= \frac{3}{2}\int_{1/2}^{1}(1-\sqrt{y})\,dy = \frac{3}{2}\left(\int_{1/2}^{1} 1\,dy - \int_{1/2}^{1}\sqrt{y}\,dy\right)$$

$$= \frac{3}{2}\left([y]_{1/2}^{1} - \left[\frac{2}{3}y^{3/2}\right]_{1/2}^{1}\right)$$

$$= \frac{3}{2}\left(\left(1-\frac{1}{2}\right) - \frac{2}{3}\left(1-\frac{\sqrt{2}}{4}\right)\right)$$

$$= \frac{5-2\sqrt{2}}{8}$$

(d) X **and** Y **are not independent**.

4.4-15

$$E(Y|X=x) = \frac{(x-0.1)+(x+0.1)}{2} = x$$

Thus, the expected value of Y given $X = x$ is simply x.

$$E(Y) = E(E(Y|X)) = E(X)$$

$$E(X) = \int_{0.2}^{\infty} x \cdot 2e^{-2(x-0.2)}\,dx$$

$$E(X) = \int_{0}^{\infty} (u+0.2) \cdot 2e^{-2u}\,du$$

Separate the integral:

$$= 2\int_{0}^{\infty} ue^{-2u}\,du + 2 \cdot 0.2 \int_{0}^{\infty} e^{-2u}\,du$$

- **First Integral:**

$$\int_{0}^{\infty} ue^{-2u}\,du = \frac{1}{4}$$

  - **Second Integral:**

$$\int_{0}^{\infty} e^{-2u}\,du = \frac{1}{2}$$

  So,

$$E(X) = 2 \cdot \frac{1}{4} + 2 \cdot 0.2 \cdot \frac{1}{2} = \frac{1}{2} + 0.2 = 0.7$$

Therefore, the expected value of Y is:

$$E(Y) = 0.7$$

4.4-17

(b)

$$f_X(x) = \int_{10-x}^{14-x} \frac{1}{40}\, dy$$

$$f_X(x) = \frac{1}{40} \int_{10-x}^{14-x} 1\, dy = \frac{1}{40} \cdot ((14-x) - (10-x)) = \frac{1}{40} \cdot 4 = \frac{1}{10}$$

So, the marginal pdf of X is:

$$f_X(x) = \frac{1}{10} \quad \text{for } 0 \leq x \leq 10$$

(c)

$$h(y|x) = \frac{f(x,y)}{f_X(x)}$$

$$h(y|x) = \frac{\frac{1}{40}}{\frac{1}{10}} = \frac{1}{4}$$

$$h(y|x) = \frac{1}{4} \quad \text{for } 10 - x \leq y \leq 14 - x, 0 \leq x \leq 10$$

(d)

$$E(Y|X = x) = \int_{10-x}^{14-x} y \cdot h(y|x)\, dy$$

$$E(Y|X = x) = \int_{10-x}^{14-x} y \cdot \frac{1}{4}\, dy = \frac{1}{4} \int_{10-x}^{14-x} y\, dy$$

$$E(Y|X = x) = \frac{1}{4} \cdot \left[\frac{y^2}{2}\right]_{10-x}^{14-x} = \frac{1}{4} \cdot \left(\frac{(14-x)^2}{2} - \frac{(10-x)^2}{2}\right)$$

$$E(Y|X = x) = \frac{1}{8}\left((196 - 28x + x^2) - (100 - 20x + x^2)\right)$$

$$= \frac{1}{8}(96 - 8x) = 12 - x$$

$$E(Y|X = x) = 12 - x$$

4.4-21

1. **Joint Probability Definition**:
   We want to find $P(X+Y \geq 1)$, which can be expressed as:

$$P(X+Y \geq 1) = \int_0^1 P(Y \geq 1-X \mid X=x) f_X(x)\, dx$$

   Since $X \sim U(0,1)$, we have $f_X(x) = 1$ for $x \in (0,1)$.

2. **Conditional Probability Calculation**:
   Given $X = x$, $Y \sim U(0,x)$, so Y has a uniform distribution from 0 to x.

   Then:

$$P(Y \geq 1-x \mid X=x) = \frac{\text{Length of interval where } Y \geq 1-x}{\text{Total length of interval for } Y}$$

   For $Y \sim U(0,x)$:

   - The interval length where $Y \geq 1-x$ is $x-(1-x) = 2x-1$ (if $2x-1 \geq 0$, i.e., $x \geq \frac{1}{2}$).
   - The total interval length for Y is x.

   Thus:

$$P(Y \geq 1-x \mid X=x) = \begin{cases} 0, & \text{if } x < \frac{1}{2} \\ \frac{2x-1}{x}, & \text{if } x \geq \frac{1}{2} \end{cases}$$

$$P(X+Y \geq 1) = \int_{\frac{1}{2}}^1 \frac{2x-1}{x}\, dx$$

$$P(X+Y \geq 1) = \int_{\frac{1}{2}}^1 \left(2 - \frac{1}{x}\right) dx$$

$$= [2x - \ln x]_{\frac{1}{2}}^1$$

$$= (2(1) - \ln(1)) - \left(2 \cdot \frac{1}{2} - \ln\left(\frac{1}{2}\right)\right)$$

$$= (2-0) - (1+\ln 2) = 1 - \ln 2$$

So, the answer is:

$$P(X+Y \geq 1) = 1 - \ln 2$$

4.5-1

(a) $P(-5 < X < 5)$

Standardizing X:

$$P(-5 < X < 5) = P\left(-0.4 < Z < 1.6\right)$$

- $\Phi(1.6) \approx 0.9452$
- $\Phi(-0.4) \approx 0.3446$

$$P(-0.4 < Z < 1.6) = 0.9452 - 0.3446 = 0.6006$$

Answer (a): 0.6006

(b) $P(-5 < X < 5 \mid Y = 13)$

Given $Y = 13$, we have $X \mid Y = 13 \sim N(0, 16)$, so:

$$P(-5 < X < 5 \mid Y = 13) = P(-1.25 < Z < 1.25)$$

- $\Phi(1.25) \approx 0.8944$
- $\Phi(-1.25) \approx 0.1056$

$$P(-1.25 < Z < 1.25) = 0.8944 - 0.1056 = 0.7888$$

Answer (b): 0.7888

(c) $P(7 < Y < 16)$

Standardizing Y:

$$P(7 < Y < 16) = P(-1 < Z < 2)$$

- $\Phi(2) \approx 0.9772$ - $\Phi(-1) \approx 0.1587$

$$P(-1 < Z < 2) = 0.9772 - 0.1587 = 0.8185$$

Answer (c): 0.8185

(d)

$$\mu_{Y|X=2} = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X) = 10 + \frac{3}{5} \cdot \frac{3}{5}(2 + 3) = 11.8$$

$$\text{Var}(Y|X = 2) = \sigma_Y^2(1 - \rho^2) = 9\left(1 - \left(\frac{3}{5}\right)^2\right) = 5.76$$

$$P(7 < Y < 16|X = 2) = P\left(\frac{7 - 11.8}{\sqrt{5.76}} < Z < \frac{16 - 11.8}{\sqrt{5.76}}\right)$$

$$P(7 < Y < 16|X = 2) = P(-2 < Z < 1.75) = \Phi(1.75) - \Phi(-2)$$

Using the standard normal table:

$$\Phi(1.75) = 0.9599, \quad \Phi(-2) = 0.0228$$

$$P(-2 < Z < 1.75) = 0.9599 - 0.0228 = 0.9371$$

Answer (d): 0.9371

Summary of Answers:

- **(a)** 0.6006
- **(b)** 0.7888
- **(c)** 0.8185
- **(d)** 0.9371

4.5-3

(a) $P(108 < Y < 126)$

Since Y is normally distributed with mean $\mu_Y = 110$ and standard deviation $\sigma_Y = 10$, we standardize Y to calculate the probability.

1. **Standardize 108:**

$$Z_1 = \frac{108 - \mu_Y}{\sigma_Y} = \frac{108 - 110}{10} = -0.2$$

2. **Standardize 126:**

$$Z_2 = \frac{126 - \mu_Y}{\sigma_Y} = \frac{126 - 110}{10} = 1.6$$

Now, we need $P(-0.2 < Z < 1.6)$ where Z is a standard normal variable.

Using the standard normal table: - $\Phi(1.6) \approx 0.9452$ - $\Phi(-0.2) \approx 0.4207$

So,

$$P(-0.2 < Z < 1.6) = 0.9452 - 0.4207 = 0.5245$$

Answer for (a): 0.5245

(b) $P(108 < Y < 126 \mid X = 3.2)$

Given $X = 3.2$, the conditional distribution of Y is normally distributed with:

1. **Conditional Mean of Y:**

$$E(Y \mid X = 3.2) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(X - \mu_X)$$

Substituting the values:

$$E(Y \mid X = 3.2) = 110 + 0.6 \cdot \frac{10}{0.4} \cdot (3.2 - 2.8)$$
$$= 110 + 0.6 \cdot 25 \cdot 0.4 = 110 + 6 = 116$$

2. **Conditional Variance of Y:**

$$\text{Var}(Y \mid X = 3.2) = \sigma_Y^2(1 - \rho^2) = 100 \cdot (1 - 0.36) = 100 \cdot 0.64 = 64$$

Thus, the conditional standard deviation of Y is $\sigma_{Y|X} = \sqrt{64} = 8$.

Now, we want $P(108 < Y < 126 \mid X = 3.2)$.

3. **Standardize 108 and 126 under the conditional distribution of Y:**

- For 108:

$$Z_1 = \frac{108 - 116}{8} = \frac{-8}{8} = -1$$

- For 126:

$$Z_2 = \frac{126 - 116}{8} = \frac{10}{8} = 1.25$$

Now, we need $P(-1 < Z < 1.25)$.

Using the standard normal table: - $\Phi(1.25) \approx 0.8944$ - $\Phi(-1) \approx 0.1587$

So,

$$P(-1 < Z < 1.25) = 0.8944 - 0.1587 = 0.7357$$

Answer for (b): 0.7357

Summary of Answers

- **(a)** 0.5245
- **(b)** 0.7357

4.5-11

(a) $P(1.3 \leq Y \leq 5.8)$

Standardizing Y:

$$\begin{aligned} P(1.3 \leq Y \leq 5.8) &= P\left(\frac{1.3 - (-0.2)}{3.0} \leq Z \leq \frac{5.8 - (-0.2)}{3.0}\right) \\ &= P(0.5 \leq Z \leq 2.0) \end{aligned}$$

Using the standard normal distribution tables or a calculator:

$$P(0.5 \leq Z \leq 2.0) = 0.4772 - 0.1915 = 0.2857$$

Answer for (a): 0.2857

(b) Conditional Mean $\mu_{Y|X}$

The conditional mean $\mu_{Y|X}$ is calculated as follows:

$$\mu_{Y|X} = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X)$$

Substituting values:

$$\begin{aligned} \mu_{Y|X} &= -0.2 + (-0.32) \cdot \frac{3.0}{4.8} \cdot (x - 24.5) \\ &= 0.2 \cdot x + 4.7 \end{aligned}$$

(c) Conditional Variance $\sigma^2_{Y|X}$

The conditional variance $\sigma^2_{Y|X}$ is calculated as:

$$\sigma^2_{Y|X} = \sigma^2_Y(1 - \rho^2)$$

Substituting values:

$$\sigma^2_{Y|X} = 9.0 \cdot (1 - (-0.32)^2) = 9.0 \cdot (1 - 0.1024) = 9.0 \cdot 0.8976 = 8.0784$$

Thus, the conditional standard deviation $\sigma_{Y|X}$ is:

$$\sigma_{Y|X} = \sqrt{8.0784} \approx 2.8423$$

Answer for (c): $\sigma^2_{Y|X} = 8.0784$ and $\sigma_{Y|X} \approx 2.8423$

(d) $P(1.3 \leq Y \leq 5.8|X = 18)$

First, calculate $\mu_{Y|X=18}$:

$$\mu_{Y|X=18} = -0.2 - 0.2 \cdot (18 - 24.5) = -0.2 - 0.2 \cdot (-6.5) = -0.2 + 1.3 = 1.1$$

Using $\sigma_{Y|X} \approx 2.8423$ from part (c), standardize the bounds:

$$P(1.3 \leq Y \leq 5.8|X = 18) = P\left(\frac{1.3 - 1.1}{2.8423} \leq Z \leq \frac{5.8 - 1.1}{2.8423}\right) = 0.4228$$

Answer for (d): 0.4228

Summary of Answers

- **(a)** 0.2857
- **(b)** $\mu_{Y|X} = 0.2 \cdot x + 4.7$
- **(c)** $\sigma^2_{Y|X} = 8.0784$, $\sigma_{Y|X} \approx 2.8423$
- **(d)** 0.4228

5.1-3

1. **Determine the range of Y:**
   Since X ranges from 0 to 1, $Y = X^2$ will range from 0 to 1 as well.
2. **Find the cumulative distribution function (CDF) of Y:**
   To find $F_Y(y) = P(Y \leq y)$, we express this in terms of X:

   $$P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y})$$

   For $0 \leq y \leq 1$, the CDF of Y is:

   $$F_Y(y) = P(X \leq \sqrt{y}) = \int_0^{\sqrt{y}} 4x^3\,dx$$

Calculating the integral:

$$F_Y(y) = \int_0^{\sqrt{y}} 4x^3\,dx = \left[x^4\right]_0^{\sqrt{y}} = (\sqrt{y})^4 = y^2$$

Thus, $F_Y(y) = y^2$ for $0 \le y \le 1$.

3. **Find the pdf of Y:**

   To find the pdf $f_Y(y)$, we differentiate the CDF $F_Y(y)$ with respect to y:

   $$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}y^2 = 2y$$

   Therefore, the pdf of Y is:

   $$f_Y(y) = 2y, \quad 0 < y < 1$$

Final Answer

The pdf of $Y = X^2$ is:

$$f_Y(y) = 2y, \quad 0 < y < 1$$

5.1-12

1. **Determine the range of Y:**

   Since X is uniformly distributed from -1 to 3, $Y = X^2$ will range from 0 (when $X = 0$) to 9 (when $X = 3$).

   Therefore, $0 \le Y \le 9$.

2. **Find the cumulative distribution function (CDF) of Y:**

   The CDF $F_Y(y) = P(Y \le y)$ can be found by expressing it in terms of X:

   $$P(Y \le y) = P(X^2 \le y)$$

   This inequality $X^2 \le y$ implies $-\sqrt{y} \le X \le \sqrt{y}$.

   Since X is uniformly distributed over $(-1, 3)$, we need to consider different cases for y:

   - For $0 \le y < 1$:

     $$F_Y(y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{4}\,dx = \frac{\sqrt{y} - (-\sqrt{y})}{4} = \frac{\sqrt{y}}{2}$$

   - For $1 \le y \le 9$:

     $$F_Y(y) = P(-1 \le X \le \sqrt{y}) = \int_{-1}^{\sqrt{y}} \frac{1}{4}\,dx = \frac{\sqrt{y} + 1}{4}$$

3. **Differentiate to find the pdf $f_Y(y)$:**

   To get the pdf $f_Y(y)$, differentiate $F_Y(y)$ with respect to y:

   - For $0 \le y < 1$:

     $$f_Y(y) = \frac{d}{dy}\left(\frac{\sqrt{y}}{2}\right) = \frac{1}{4\sqrt{y}}$$

   - For $1 \le y \le 9$:

     $$f_Y(y) = \frac{d}{dy}\left(\frac{\sqrt{y} + 1}{4}\right) = \frac{1}{8\sqrt{y}}$$

Final Answer

The pdf of $Y = X^2$ is:

$$f_Y(y) = \begin{cases} \frac{1}{4\sqrt{y}}, & 0 \le y < 1 \\ \frac{1}{8\sqrt{y}}, & 1 \le y \le 9 \end{cases}$$

5.1-18

(a)

$$G(y) = P(Y \le y) = P(X^2 \le y) = \begin{cases} P(-\sqrt{y} \le X \le \sqrt{y}), & 0 \le y < 1 \\ P(-1 < X \le \sqrt{y}), & 1 \le y < 4 \end{cases}$$

When $0 \le y < 1$,

$$G(y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{2}{9}(x+1)\,dx = \frac{2}{9}\left(\frac{x^2}{2} + x\right)\Big|_{-\sqrt{y}}^{\sqrt{y}} = \frac{4}{9}\sqrt{y}$$

$$g(y) = G'(y) = \frac{2}{9\sqrt{y}}, \quad 0 \le y < 1$$

When $1 \le y < 4$,

$$G(y) = P(-1 < X \le \sqrt{y}) = \int_{-1}^{\sqrt{y}} \frac{2}{9}(x+1)\,dx = \frac{2}{9}\left(\frac{y}{2} + \sqrt{y} + \frac{1}{2}\right)$$

$$g(y) = G'(y) = \frac{1}{9} + \frac{1}{9\sqrt{y}}, \quad 1 \le y < 4$$

Therefore,

$$g(y) = \begin{cases} \frac{2}{9\sqrt{y}}, & 0 \le y < 1 \\ \frac{1}{9} + \frac{1}{9\sqrt{y}}, & 1 \le y < 4 \end{cases}$$

(b)

$$G(y) = P(Y \le y) = P(X^2 \le y) = \begin{cases} P(-\sqrt{y} \le X \le \sqrt{y}), & 0 \le y < 1 \\ P(-\sqrt{y} \le X < 1), & 1 \le y < 4 \end{cases}$$

When $0 \le y < 1$,

$$G(y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{2}{9}(x+2)\,dx = \frac{8}{9}\sqrt{y}$$

$$g(y) = G'(y) = \frac{4}{9\sqrt{y}}, \quad 0 \le y < 1$$

When $1 \le y < 4$,

$$G(y) = P(-\sqrt{y} \le X < 1) = \int_{-\sqrt{y}}^{1} \frac{2}{9}(x+2)\,dx = \frac{2}{9}\left(-\frac{y}{2} + 2\sqrt{y} + \frac{3}{2}\right)$$

$$g(y) = G'(y) = -\frac{1}{9} + \frac{2}{9\sqrt{y}}, \quad 1 \le y < 4$$

Therefore,

$$g(y) = \begin{cases} \frac{4}{9\sqrt{y}}, & 0 \le y < 1 \\ -\frac{1}{9} + \frac{2}{9\sqrt{y}}, & 1 \le y < 4 \end{cases}$$

5.2-1

1. **Joint PDF of Y_1 and Y_2**

   Since $Y_1 = X_1$ and $Y_2 = X_1 + X_2$, we have:

   $$X_1 = Y_1 \quad \text{and} \quad X_2 = Y_2 - Y_1$$

   The Jacobian determinant J for the transformation is:

   $$J = \begin{vmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

   The joint PDF $f_{Y_1,Y_2}(y_1, y_2)$ is:

   $$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1}(y_1) f_{X_2}(y_2 - y_1)$$

   where $f_{X_i}(x) = \frac{1}{2}e^{-x/2}$ for $x > 0$. Thus,

   $$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2}e^{-y_1/2} \cdot \frac{1}{2}e^{-(y_2-y_1)/2} = \frac{1}{4}e^{-y_2/2}, \quad 0 < y_1 < y_2 < \infty$$

2. **Marginal PDFs of Y_1 and Y_2**

   - For Y_1:

     $$f_{Y_1}(y_1) = \int_{y_1}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\, dy_2 = \int_{y_1}^{\infty} \frac{1}{4}e^{-y_2/2}\, dy_2$$
     $$= \frac{1}{4} \cdot 2e^{-y_1/2} = \frac{1}{2}e^{-y_1/2}, 0 < y1$$

   - For Y_2:

     $$f_{Y_2}(y_2) = \int_{0}^{y_2} f_{Y_1,Y_2}(y_1, y_2)\, dy_1 = \int_{0}^{y_2} \frac{1}{4}e^{-y_2/2}\, dy_1$$
     $$= \frac{1}{4}e^{-y_2/2} \cdot y_2, 0 < y2$$

3. **Independence of Y_1 and Y_2**

   Since $f_{Y_1,Y_2}(y_1, y_2) \ne f_{Y_1}(y_1) f_{Y_2}(y_2)$, Y_1 and Y_2 are not independent.

5.2-5

(a) 14.80;

(b) $1/7.01 = 0.1427$;

(c) 0.95.

5.3-1

Solution

(a) Since X_1 and X_2 are independent Poisson random variables:

$$P(X_1 = 3, X_2 = 5) = P(X_1 = 3) \cdot P(X_2 = 5)$$

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$P(X_1 = 3, X_2 = 5) = \left(\frac{2^3 e^{-2}}{3!}\right) \cdot \left(\frac{3^5 e^{-3}}{5!}\right) = 0.0182$$

(b) The event $X_1 + X_2 = 1$ can occur under two disjoint conditions:

- $X_1 = 1$ and $X_2 = 0$
- $X_1 = 0$ and $X_2 = 1$

$$P(X_1 + X_2 = 1) = P(X_1 = 1, X_2 = 0) + P(X_1 = 0, X_2 = 1)$$

Since X_1 and X_2 are independent:

$$P(X_1 + X_2 = 1) = P(X_1 = 1) \cdot P(X_2 = 0) + P(X_1 = 0) \cdot P(X_2 = 1)$$

$$P(X_1 = 1) = \frac{2^1 e^{-2}}{1!}, \quad P(X_1 = 0) = \frac{2^0 e^{-2}}{0!}$$
$$P(X_2 = 1) = \frac{3^1 e^{-3}}{1!}, \quad P(X_2 = 0) = \frac{3^0 e^{-3}}{0!}$$

$$P(X_1 + X_2 = 1) = \left(\frac{2^1 e^{-2}}{1!} \cdot \frac{3^0 e^{-3}}{0!}\right) + \left(\frac{2^0 e^{-2}}{0!} \cdot \frac{3^1 e^{-3}}{1!}\right) = 0.0337$$

5.3-6

Mean The expected value of X, $\mathbb{E}[X]$, is:

$$\mathbb{E}[X] = \int_0^1 x \cdot f(x)\, dx = \int_0^1 x \cdot 6x(1-x)\, dx$$

Compute the integral:

$$\mathbb{E}[X] = \int_0^1 6x^2 - 6x^3\, dx = \left[2x^3 - \frac{3x^4}{2}\right]_0^1 = 2 - \frac{3}{2} = \frac{1}{2}$$

Thus:

$$\mathbb{E}[X_1] = \mathbb{E}[X_2] = \frac{1}{2}$$

And:

$$\mathbb{E}[Y] = \mathbb{E}[X_1] + \mathbb{E}[X_2] = \frac{1}{2} + \frac{1}{2} = 1$$

Variance The variance of X, $\text{Var}(X)$, is:

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

Compute $\mathbb{E}[X^2]$:

$$\mathbb{E}[X^2] = \int_0^1 x^2 \cdot f(x)\,dx = \int_0^1 x^2 \cdot 6x(1-x)\,dx = \int_0^1 6x^3 - 6x^4\,dx$$

$$\mathbb{E}[X^2] = \left[\frac{6x^4}{4} - \frac{6x^5}{5}\right]_0^1 = \frac{3}{2} - \frac{6}{5} = \frac{15}{10} - \frac{12}{10} = \frac{3}{10}$$

Thus:

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{3}{10} - \frac{1}{4} = \frac{12}{40} - \frac{10}{40} = \frac{1}{20}$$

Since X_1 and X_2 are independent:

$$\text{Var}(Y) = \text{Var}(X_1) + \text{Var}(X_2) = \frac{1}{20} + \frac{1}{20} = \frac{1}{10}$$

5.5-1

(a)

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

For $\bar{X} = 77$:

$$Z_1 = \frac{77 - 77}{1.25} = 0$$

For $\bar{X} = 79.5$:

$$Z_2 = \frac{79.5 - 77}{1.25} = 2$$

Therefore:

$$P(77 < \bar{X} < 79.5) = P(0 < Z < 2)$$

Using the standard normal table:

$$P(0 < Z < 2) = \Phi(2) - \Phi(0) = 0.9772 - 0.5 = 0.4772$$

(b) For $\bar{X} = 74.2$:

$$Z_1 = \frac{74.2 - 77}{1.25} = -2.24$$

For $\bar{X} = 78.4$:

$$Z_2 = \frac{78.4 - 77}{1.25} = 1.12$$

Therefore:

$$P(74.2 < \bar{X} < 78.4) = P(-2.24 < Z < 1.12)$$

Using the standard normal table:

$$P(-2.24 < Z < 1.12) = \Phi(1.12) - \Phi(-2.24)$$

$$\Phi(1.12) = 0.8686, \quad \Phi(-2.24) = 1 - \Phi(2.24) = 1 - 0.9875 = 0.0125$$

Thus:

$$P(-2.24 < Z < 1.12) = 0.8686 - 0.0125 = 0.8561$$

5.5-3

(a) Given $X \sim N(46.58, 40.96)$, the sample mean $\bar{X}$ of $n = 16$ observations has:

$$\mathbb{E}[\bar{X}] = \mu = 46.58, \quad \mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n} = \frac{40.96}{16} = 2.56$$

The standard deviation of $\bar{X}$:

$$\mathrm{SD}(\bar{X}) = \sqrt{2.56} = 1.6$$

(b)

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

For $\bar{X} = 44.42$:

$$Z_1 = \frac{44.42 - 46.58}{1.6} = \frac{-2.16}{1.6} = -1.35$$

For $\bar{X} = 48.98$:

$$Z_2 = \frac{48.98 - 46.58}{1.6} = \frac{2.4}{1.6} = 1.5$$

Thus:

$$P(44.42 \leq \bar{X} \leq 48.98) = P(-1.35 \leq Z \leq 1.5)$$

Using the standard normal table:

$$\Phi(1.5) = 0.9332, \quad \Phi(-1.35) = 1 - \Phi(1.35) = 1 - 0.9115 = 0.0885$$

Therefore:

$$P(-1.35 \leq Z \leq 1.5) = 0.9332 - 0.0885 = 0.8447$$

5.5-6

(a)

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}(\bar{Y})$$

Substitute the variances:

$$\frac{4}{100} = \frac{9}{n}$$

Solve for n:

$$n = \frac{9 \cdot 100}{4} = 225$$

(b) For $n = 225$, find $P(\bar{Y} - \bar{X} > 0.2)$.

Since $\bar{Y}$ and $\bar{X}$ are independent:

$$\bar{Y} - \bar{X} \sim N(0, \mathrm{Var}(\bar{Y}) + \mathrm{Var}(\bar{X}))$$

Substitute the variances:

$$\mathrm{Var}(\bar{Y} - \bar{X}) = \frac{9}{225} + \frac{4}{100} = 0.04 + 0.04 = 0.08$$

Standard deviation:

$$\mathrm{SD}(\bar{Y} - \bar{X}) = \sqrt{0.08} = 0.2828$$

Convert to the standard normal distribution:

$$Z = \frac{\bar{Y} - \bar{X} - 0.2}{\mathrm{SD}(\bar{Y} - \bar{X})} = \frac{0.2}{0.2828} \approx 0.707$$

Using the standard normal table:

$$P(\bar{Y} - \bar{X} > 0.2) = P(Z > 0.707)$$

$$P(Z > 0.707) = 1 - \Phi(0.707) = 1 - 0.7602 = 0.2398$$

5.5-7

Step 1: Distribution of Y Since the sum of independent normal random variables is also normal:

$$Y \sim N\left(\mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3], \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \mathrm{Var}(X_3)\right)$$

Substitute the values:

$$\mathbb{E}[Y] = 1.18 + 1.18 + 1.18 = 3.54$$

$$\mathrm{Var}(Y) = 0.07^2 + 0.07^2 + 0.07^2 = 3 \times 0.0049 = 0.0147$$

$$Y \sim N(3.54, 0.0147)$$

Step 2: Find $P(Y > W)$ Let $Z = Y - W$. Since Y and W are independent:

$$Z \sim N\left(\mathbb{E}[Y] - \mathbb{E}[W], \mathrm{Var}(Y) + \mathrm{Var}(W)\right)$$

Substitute the values:

$$\mathbb{E}[Z] = \mathbb{E}[Y] - \mathbb{E}[W] = 3.54 - 3.22 = 0.32$$

$$\mathrm{Var}(Z) = \mathrm{Var}(Y) + \mathrm{Var}(W) = 0.0147 + 0.09^2 = 0.0147 + 0.0081 = 0.0228$$

$$Z \sim N(0.32, 0.0228)$$

The standard deviation of Z:

$$\mathrm{SD}(Z) = \sqrt{0.0228} \approx 0.151$$

Step 3: Compute $P(Y > W) = P(Z > 0)$ Convert Z to the standard normal distribution:

$$Z = \frac{Z - \mathbb{E}[Z]}{\mathrm{SD}(Z)} = \frac{0 - 0.32}{0.151} \approx -2.12$$

Using the standard normal table:

$$P(Z > 0) = 1 - P(Z \leq -2.12)$$

$$P(Z \leq -2.12) \approx 0.017$$

$$P(Z > 0) = 1 - 0.017 = 0.983$$

5.5-15

(a) 2.567;

(b) -1.740;

(c) 0.90.