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RANDOM GIBBS u-STATE FOR PARTIALLY HYPERBOLIC ON FIBERS SYSTEM

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Abstract. In this paper, we prove the existence of random Gibbs u-state for partially hyperbolic system driven by an external force. Several examples such as fiber partially hyperbolic maps on tori, random small perturbations of partially hyperbolic systems, and random composition of partially hyperbolic automorphisms on $\mathbb{T}^3$ with a fixed central direction are under consideration.

1. **Introduction.** In this paper, we consider partially hyperbolic systems (in the broad sense) driven by an external force. Let M be a connected compact smooth Riemannian manifold without boundary. Let $\theta : \Omega \to \Omega$ be a homeomorphism on a compact metric space (Ω, d_Ω). A dynamical system driven by an external force θ is a continuous mapping

$$\varphi : \mathbb{Z} \times \Omega \times M \to M, \ (n, \omega, x) \mapsto \varphi(n, \omega)(x)$$

satisfying the cocycle property, i.e.,

$$\begin{aligned} \varphi(0, \omega) &= id_M \text{ for all } \omega \in \Omega, \\ \varphi(n + m, \omega) &= \varphi(n, \theta^m \omega) \circ \varphi(m, \omega) \text{ for all } n, m \in \mathbb{Z}, \ \omega \in \Omega. \end{aligned}$$

When $(\Omega, \mathcal{B}(\Omega))$ is equipped with a $\theta-$invariant probability measure $\mathbb{P}$, φ is a random dynamical system [2]. We use f_ω to denote the time-one map $\varphi(1, \omega)$ of the cocycle. Putting f_ω and θ together forms a skew product transformation $\phi : M \times \Omega \to M \times \Omega$ given by

$$\phi(x, \omega) = (f_\omega x, \theta\omega) \text{ for all } (x, \omega) \in M \times \Omega.$$

Assume f_ω to be a diffeomorphism on M. The system ϕ is called partially hyperbolic on fibers if for every $(x, \omega) \in M \times \Omega$, there is a splitting of the tangent bundle of $M_\omega = M \times \{\omega\}$ at x into central-stable and strong unstable subbundles

$$T_x M_\omega = E^{cs}_{(x,\omega)} \oplus E^{uu}_{(x,\omega)}$$

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such that the splitting is continuous on $(x,\omega) \in M \times \Omega$ and is invariant in the following sense

$$D_x f_\omega E^\tau_{(x,\omega)} = E^\tau_{(f_\omega x, \theta\omega)}, \text{ for } \tau = cs, uu,$$

and there are constants λ, λ_0 with $\lambda < \lambda_0$ and $\lambda_0 > 0$, and $C_0 > 1$ such that

$$\begin{cases} |D_x f^n_\omega \xi| \geq C_0^{-1} e^{\lambda_0 n}|\xi|, & \forall \xi \in E^{uu}_{(x,\omega)}, \\ |D_x f^n_\omega \eta| \leq C_0 e^{\lambda n}|\eta|, & \forall \eta \in E^{cs}_{(x,\omega)}. \end{cases} \tag{1}$$

A typical example of such systems is the random perturbations of partially hyperbolic systems [23]. More examples will be given in Section 2.3.

A probability measure μ on $M \times \Omega$ with marginal $\mathbb{P}$ on Ω is called a random Gibbs $u-$state if for any measurable partition subordinate to the strong unstable manifolds, the conditional probability measure of μ on this partition is absolutely continuous with respect to the Riemmanian volume measure on the strong unstable manifold. Note that when the strong unstable manifolds coincide the unstable manifolds, the random Gibbs $u-$state is the random SRB measure [3]. In this article, we prove the existence of random Gibbs u-state for C^2 partially hyperbolic on fibers systems.

The study of Gibbs u-states/SRB measures for dynamical systems without an external driving force has a long and rich history, dates back to Sinai [28] for Anosov diffeomorphisms, Ruelle and Bowen [27, 8] for Axiom A diffeomorphisms and flows. There is an extensive literature on this subject, see [25, 18, 26, 4, 5, 32, 7, 1, 29, 12, 9, 23, 20, 21] and reference therein. For systems driven by an external forcing, there are a few works on this subject. Kifer and Gundlach proved the existence of random SRB measures for random hyperbolic systems. Recently, Wang, Wu and Zhu in [30] showed the existence of Gibbs $u-$state for a random dynamical system that has a uniformly dominated splitting, is uniformly expanding on the unstable subbundle, and has no positive Lyapunov exponent on central-stable subbundle. The system we study here may have small positive Lyapunov exponents on the central-stable subbundle, and the approach is totally different.

We organize this paper as follows. In section 2, we introduce the setting, the main theorem and give several examples of partially hyperbolic on fibers systems. In section 3, some lemmas for the proof of main result are given, such as the strong unstable manifolds theorem and a distortion lemma. In section 4, we prove the main theorem, i.e., the existence of random Gibbs u-state.

2. **Setting, main result and examples.**

In this section, we first introduce the basic concepts and notations. Although some of them have already been given in the previous section, we restate them for the sake of convenience. Then we state our main result. Examples are given in this section.

2.1. **Settings.**

Let M be a connected closed smooth Riemannian $n-$dimensional manifold, and d_M be the induced Riemannian metric on M. Let (Ω, d_Ω) be a compact metric space, and $\theta : \Omega \to \Omega$ be a homeomorphism. Let $\mathbb{P}$ be a complete ergodic Borel probability measure on Ω. Let $\mathcal{H}$ =diff$^2(M)$ be the space of C^2 diffeomorphisms on M with C^2 topology (see, e.g., [15]), and $f : \Omega \to \mathcal{H}$ be a continuous map.

The random diffeomorphism f generates the following random dynamical system

$$\varphi(n,\omega) := f_\omega^n = \begin{cases} f_{\theta^{n-1}\omega} \circ \cdots \circ f_\omega, & \text{if } n > 0 \\ id_M, & \text{if } n = 0 \\ (f_{\theta^n\omega})^{-1} \circ \cdots \circ (f_{\theta^{-1}\omega})^{-1}, & \text{if } n < 0, \end{cases}$$

where we rewrite $f(\omega)$ as f_ω. The skew product $\phi : M \times \Omega \to M \times \Omega$ induced by f and θ is defined by

$$\phi(x,\omega) = (f(\omega)x, \theta\omega) = (f_\omega x, \theta\omega), \text{ for all } \omega \in \Omega,\ x \in M.$$

The system ϕ is said to be partially hyperbolic in the broad sense on fibers (partially hyperbolic on fibers for short) if for every $(x,\omega) \in M \times \Omega$, there is a splitting of the tangent bundle of $M_\omega = M \times \{\omega\}$

$$T_x M_\omega = E^{uu}_{(x,\omega)} \oplus E^{cs}_{(x,\omega)},$$

such that the splitting depends continuously on $(x,\omega) \in M \times \Omega$ with $\dim E^{uu}_{(x,\omega)}$, $\dim E^{cs}_{(x,\omega)} > 0$, and for all $(x,\omega) \in M \times \Omega$

$$D_x f_\omega E^{uu}_{(x,\omega)} = E^{uu}_{(f_\omega x, \theta\omega)},\ D_x f_\omega E^{cs}_{(x,\omega)} = E^{cs}_{(f_\omega x,\theta\omega)},$$

and there exist constants $\lambda_0 > 0$, $\lambda < \lambda_0$ and $C_0 > 1$ such that for any $n \in \mathbb{N}$,

$$\begin{cases} |D_x f_\omega^n \xi| \geq C_0^{-1} e^{\lambda_0 n} |\xi|, & \forall \xi \in E^{uu}_{(x,\omega)} \\ |D_x f_\omega^n \eta| \leq C_0 e^{\lambda n} |\eta|, & \forall \eta \in E^{cs}_{(x,\omega)}. \end{cases} \tag{2}$$

Remark 2.1. For dynamical systems without an external driving force, the terminology 'partial hyperbolicity in the broad sense' is also named by 'pseudo hyperbolicity' [16] or $(e^\lambda, e^{\lambda_0})$-splitting [31], where e^λ and e^{λ_0} are the constants in (2).

2.2. **Main result.** Let $Pr(M)$ denote the space of probability measures on space $(M, \mathcal{B}(M))$ equipped with the narrow topology. A random probability measure is a measurable mapping $\mu : \Omega \to Pr(M)$ with respect to the Borel σ-algebra of the narrow topology on $Pr(M)$ (see [13, Remark 3.20]). We denote it by $\omega \mapsto \mu_\omega$ or $(\mu_\omega)_{\omega\in\Omega}$ and use $Pr_\Omega(M)$ to denote the space of all random probability measures. Since M is compact, $Pr_\Omega(M)$ is compact in the narrow topology.

Let $Pr_{\mathbb{P}}(M \times \Omega)$ be the space of probability measures on the product space $(M \times \Omega, \mathcal{B}(M \times \Omega))$ with marginal $\mathbb{P}$ on Ω. Notice that for every $\mu \in Pr_{\mathbb{P}}(M \times \Omega)$, there exists a random probability measure $\omega \mapsto \mu_\omega$ such that

$$\int_{M\times\Omega} h(x,\omega) d\mu(x,\omega) = \int_\Omega \int_M h(x,\omega) d\mu_\omega(x) d\mathbb{P}(\omega)$$

for every bounded measurable $h : M \times \Omega \to \mathbb{R}$ (see [13, Proposition 3.6]). Moreover, this disintegration $\omega \mapsto \mu_\omega$ is $\mathbb{P}$-a.e. unique. In addition, each random probability measure induces a unique probability measure on $(M \times \Omega, \mathcal{B}(M \times \Omega))$ through the above equation. For the sake of convenience, we will identify $\mu \in Pr_{\mathbb{P}}(M \times \Omega)$ with $\omega \mapsto \mu_\omega$.

For $\mu \in Pr_{\mathbb{P}}(M \times \Omega)$, we define $\phi^*\mu$ by $(\phi^*\mu)_\omega := (f_{\theta^{-1}\omega})_* \mu_{\theta^{-1}\omega}$, i.e.,

$$(\phi^*\mu)_\omega(B) = \mu_{\theta^{-1}\omega}((f_{\theta^{-1}\omega})^{-1}(B))$$

for each $B \in \mathcal{B}(M)$ and $\omega \in \Omega$. A random probability measure $\omega \mapsto \mu_\omega$ is called ϕ-invariant if $(\phi^*\mu)_\omega = \mu_\omega$ for P-a.e. $\omega \in \Omega$.

From the classical unstable manifolds theorem (see [19]), it follows that ϕ has the strong unstable manifold $\{W^{uu}(x,\omega) : (x,\omega) \in M\times\Omega\}$. Notice that $\{W^{uu}(x,\omega)\}$ forms a partition of $M\times\Omega$. But, in general, such a partition is not measurable.

Given a ϕ-invariant random probability measure μ, a measurable partition η of $M\times\Omega$ is called u-subordinate if for μ-a.e. $(x,\omega)\in M\times\Omega$, $\eta(x,\omega)\subset\{\omega\}\times M$, $\eta_\omega(x)=\{y\in M : (y,\omega)\subset\eta(x,\omega)\}\subset W^{uu}(x,\omega)$ and it contains an open neighborhood of x contained in $W^{uu}(x,\omega)$, this neighborhood being taken in the submanifold topology of $W^{uu}(x,\omega)$.

Definition 2.2. An invariant random probability measure μ is called a random Gibbs u-state if for every measurable u-subordinate partition η,

$$\mu^\eta_{(x,\omega)} \ll \lambda^{uu}_{(x,\omega)}, \quad \text{for } \mu\text{-a.e. } (x,\omega)\in M\times\Omega,$$

where $\{\mu^\eta_{(x,\omega)}\}_{(x,\omega)\in M\times\Omega}$ is a canonical system of conditional measures of μ associated with η, $\mu^\eta_{(x,\omega)}$ is regarded as a measure on $\eta_\omega(x)$ by identifying $\eta(x,\omega)=\eta_\omega(x)\times\{\omega\}$ with $\eta_\omega(x)$, and $\lambda^{uu}_{(x,\omega)}$ is the Riemannian volume measure on $W^{uu}(x,\omega)$ induced by its inherited Riemannian structure as a submanifold of M.

The following is our main result.

Theorem 2.3. *If ϕ is partially hyperbolic on fibers, then there exists at least one ϕ-invariant random Gibbs u-state.*

2.3. **Examples of partially hyperbolic on fibers systems.**

Example 2.1 (Fiber partially hyperbolic maps on Tori)**.** Let (Ω,θ) be any homeomorphism on a compact metric space, and let $\mathbb{P}$ be any complete ergodic θ-invariant measure on Ω. Define $\phi:\mathbb{T}^k\times\Omega\to\mathbb{T}^k\times\Omega$ for some $k\geq 4$ by

$$\phi\left(\vec{x},\omega\right)=\left(A\vec{x}+\vec{h}(\omega),\theta\omega\right),$$

where $\vec{h}$ is a continuous map from Ω to $\mathbb{T}^k$, and $A\in GL(k,\mathbb{Z})$ have characteristic polynomial $\chi_A(t)$ that is irreducible over $\mathbb{Q}$ and which has some but not all of its eigenvalues on the unit circle.

Note that in this example $df_\omega=A$, and according to [22, Example 3.4], there is an A-invariant splitting $\mathbb{R}^k=\mathcal{C}\oplus\mathcal{N}\oplus\mathcal{E}$, where A contracts on $\mathcal{C}$, expands on $\mathcal{E}$, and is an isometry on $\mathcal{N}$.

The following example is adapted from [23].

Example 2.2 (Random Small Perturbations of Partially Hyperbolic Systems)**.** Let M be a smooth compact Riemannian manifold without boundary, and let $\mathrm{Diff}^2(M)$ be the space of C^2 diffeomorphisms from M to M equipped with the C^2 topology [15]. Note that the C^2 topology on $\mathrm{Diff}^2(M)$ is metrizable, and we denote the metric generating the C^2 topology by d_{C^2}. Assume $h\in\mathrm{Diff}^2(M)$ is partially hyperbolic in the following sense that there is a continuous splitting

$$TM=E^{uu}\oplus E^{cs}$$

with $\dim E^{uu}>0$ and a number $\lambda_0>0$ such that for any $x\in M$

$$\begin{cases}\limsup_{n\to\infty}\frac{1}{n}\log|D_xh^n\xi|\geq\lambda_0, & \forall\xi\in E^{uu}_x,\ \xi\neq 0\\ \liminf_{n\to\infty}\frac{1}{n}\log|D_xh^n\eta|\leq 0, & \forall\eta\in E^{cs}_x,\ \eta\neq 0.\end{cases}$$

Note that h may not be structurally stable. Let $\mathcal{U}_\epsilon(h)$ be the $\epsilon-$neighborhood of h in the $\mathrm{Diff}^2(M)$ with respect to the C^2 topology. Let $K_\epsilon(h)\subset\mathcal{U}_\epsilon(h)$ be any compact set. For $\omega \in \Omega_\epsilon := K_\epsilon(h)^{\mathbb{Z}}$, we denote $(\cdots, g_{-1}(\omega), g_0(\omega), g_1(\omega), \cdots)$ to be the sequence of maps corresponding to ω and define the metric on Ω_ϵ by

$$d_{\Omega_\epsilon}(\omega, \omega') = \sum_{i \in \mathbb{Z}} \frac{d_{C^2}(g_i(\omega), g_i(\omega'))}{2^{|i|}}.$$

The metric d_{Ω_ϵ} generates the product topology on Ω_ϵ. As a consequence, Ω_ϵ is a compact metric space. Let $\theta : \Omega_\epsilon \to \Omega_\epsilon$ be the left shift operator, then θ is a homeomorphism. Define $f : \Omega_\epsilon \to \mathrm{Diff}^2(M)$ by $f(\omega) = g_0(\omega)$, then f is a continuous map. Denote

$$f_\omega^n := \begin{cases} g_{\theta^{n-1}\omega} \circ g_{\theta^{n-2}\omega} \circ \cdots \circ g_\omega, & \text{if } n > 0 \\ id, & \text{if } n = 0 \\ (g_{\theta^n \omega})^{-1} \circ \cdots \circ (g_{\theta^{-1}\omega})^{-1}, & \text{if } n < 0. \end{cases}$$

The following proposition is an adapted version of [23, Proposition 2.2], and it shows that the Example 2.2 is partially hyperbolic (in the broad sense) on fibers.

Proposition 2.4. *Given sufficiently small positive number $\delta \ll \lambda_0$, we can find $\epsilon_\delta > 0$ and a constant A_δ such that the following hold: for every $(x, \omega) \in M \times \Omega_{\epsilon_\delta}$, there is a splitting*

$$T_x M = E^{uu}_{(x,\omega)} \oplus E^{cs}_{(x,\omega)}$$

which depends continuously on (x, ω) and has the following properties:

1. *$D_x f_\omega E^\tau_{(x,\omega)} = E^\tau_{(f_\omega x, \theta\omega)}$ for $\tau = cs, uu$;*
2. *for all $n \geq 0$*

$$|D_x f_\omega^n \xi| \geq A_\delta^{-2} e^{(\lambda_0 - 3\delta)n} |\xi|, \ \forall \xi \in E^{uu}_{(x,\omega)},$$
$$|D_x f_\omega^n \eta| \leq A_\delta^2 e^{3\delta n} |\eta|, \ \forall \eta \in E^{cs}_{(x,\omega)}.$$

Example 2.3. All Anosov on fibers systems given in [17, Section 8] are partially hyperbolic on fibers systems, including fiber Anosov maps on 2-d tori and random composition of 2×2 area-preserving positive matrices. Such systems are shown to have complicated dynamical behavior.

Fiber Anosov maps on 2-d tori: Let (Ω, θ) be any homeomorphism on a compact metric space, and let $\mathbb{P}$ be an ergodic measure on Ω with respect to θ. Define $\phi : \mathbb{T}^2 \times \Omega \to \mathbb{T}^2 \times \Omega$ by

$$\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}, \omega\right) = \left(A \begin{pmatrix} x \\ y \end{pmatrix} + h(\omega), \theta\omega\right) = \left(\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h_1(\omega) \\ h_2(\omega) \end{pmatrix}, \theta\omega\right),$$

where $h(\cdot)$ is a continuous map from Ω to $\mathbb{T}^2$. In this example, $E^{uu}(x, \omega) \equiv \{t(1 + \sqrt{5})/2, 1)|\ t \in \mathbb{R}\}$, and $E^{cs}(x, \omega) \equiv \{t((1 - \sqrt{5})/2, 1)|\ t \in \mathbb{R}\}$, since $Df_\omega \equiv A$.

Random composition of 2×2 area-preserving positive matrices: let

$$\left\{ B_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \right\}_{1 \leq i \leq p}$$

be 2×2 matrices with $a_i, b_i, c_i, d_i \in \mathbb{Z}^+$, and $|a_i d_i - c_i b_i| = 1$ for any $i \in \{1, ..., p\}$. Let $\Omega = \mathcal{S}_p := \{1, ..., p\}^{\mathbb{Z}}$ with the left shift operator θ be the symbolic dynamical system with p symbols. For any $\omega = (..., \omega_{-1}, \omega_0, \omega_1, ...) \in \Omega$, we define $g(\omega) = B_{\omega_0}$. Then the skew product $\tilde{\phi} : \mathbb{T}^2 \times \Omega \to \mathbb{T}^2 \times \Omega$ defined by

$$\tilde{\phi}(x, \omega) = (g(\omega)x, \theta\omega)$$

is a Anosov on fibers system with continuous co-invariant splitting $\mathbb{R}^2 = E^u_\omega \oplus E^s_\omega$ for $\omega \in \mathcal{S}_p$, on which

$$|Dg_\omega v| \geq \kappa |v| \text{ for } v \in E^u_\omega;$$
$$|Dg_\omega w| \leq \kappa^{-1} |w| \text{ for } w \in E^s_\omega$$

for $\kappa := \min_{1\leq i\leq p} \min\{\sqrt{a_i^2 + c_i^2}, \sqrt{b_i^2 + d_i^2}\} \geq \sqrt{2}$ by Proposition 8.2 in [17].

Example 2.4 (Random composition of partially hyperbolic automorphisms on $\mathbb{T}^3$ with a fixed central direction)**.** Let

$$\left\{ A_i = \begin{pmatrix} a_i & b_i & 0 \\ c_i & d_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}_{1\leq i\leq p}$$

be 3×3 matrices with $a_i, b_i, c_i, d_i \in \mathbb{Z}^+$, and $|a_i d_i - c_i d_i| = 1$ for any $i \in \{1, ..., p\}$. Let $\Omega = \{1, ..., p\}^{\mathbb{Z}}$ with the left shift operator θ be the symbolic dynamical system with p symbols.

For any $\omega = (..., \omega_{-1}, \omega_0, \omega_1, ...) \in \Omega$, we define $f(\omega) = A_{\omega_0}$. Then the skew product $\phi : \mathbb{T}^3 \times \Omega \to \mathbb{T}^3 \times \Omega$ defined by

$$\phi(x, \omega) = (f(\omega)x, \theta\omega)$$

is partially hyperbolic on fibers.

In fact, consider the following system, random composition of 2×2 area-preserving positive matrices. Let

$$\left\{ B_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \right\}_{1\leq i\leq p}$$

be the top left 2×2 submatrix of $\{A_i\}_{i=1,...,p}$. For any $\omega = (..., \omega_{-1}, \omega_0, \omega_1, ...) \in \Omega$, we define $g(\omega) = B_{\omega_0}$. By the Example 2.3, the skew product $\tilde{\phi} : \mathbb{T}^2 \times \Omega \to \mathbb{T}^2 \times \Omega$ defined by

$$\tilde{\phi}(x, \omega) = (g(\omega)x, \theta\omega)$$

is a Anosov on fibers system with continuous co-invariant splitting $R^2 = E^u_\omega \oplus E^s_\omega$.

Now define $E^{uu}(\omega) := \{(v, 0)|\ v \in E^u_\omega\}$, $E^{cs}(\omega) := \{(w, 0) + t(0, 0, 1)|\ w \in E^s_\omega,\ t \in \mathbb{R}\}$. Then

$$|Df_\omega \eta| \geq \kappa |\eta| \text{ for } \eta \in E^{uu}(\omega);$$
$$|Df_\omega \gamma| \leq |\gamma| \text{ for } \gamma \in E^{cs}(\omega).$$

Hence ϕ is partially hyperbolic on fibers.

3. **Preliminary Lemmas.**

3.1. **Closed random sets and random probability measures.** In this section, we supply some necessary knowledge that is used in the proof of our main theorem. All of these are taken from [13].

Definition 3.1. A set valued map $C : \Omega \to 2^M$ is said to be a closed random set if

1. for each $\omega \in \Omega$, $C(\omega)$ is closed;
2. for each $x \in M$, the mapping $\omega \mapsto d(x, C(\omega))$ is measurable.

A set valued map $\omega \mapsto U(\omega)$ is said to be a open random set if its complement $\omega \mapsto U^c(\omega)$ is a closed random set.

The following lemma comes from corollary 2.10 in [13].

Lemma 3.2. *If $\omega \mapsto C(\omega)$ is a closed random set, then its interior $intC$ is an open random set.*

Lemma 3.3 (The Selection Theorem)**.** *A set valued map $C : \Omega \to 2^M$ is a closed random set if and only if there exists a sequence $\{c_n\}_{n\in\mathbb{N}}$ of measurable maps $c_n : \Omega \to M$, such that $C(\omega) = closure\{c_n(\omega) :\ n \in \mathbb{N}\}$ for all $\omega \in \Omega$.*

A function $c : M \times \Omega \to \mathbb{R}$ is called a random continuous function if $c(x, \omega)$ is measurable in ω and continuous in x, and $\sup_{x\in M} |c(x, \cdot)| \in L^1(\Omega, \mathbb{R})$. The narrow topology on $Pr_\Omega(M)$ is generated by the functions $\mu \mapsto \mu(c)$ for random continuous function c. Note that Ω is compact metric space, then $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$ is countably generated (mod $\mathbb{P}$). Let $\{G_m : m \in \mathbb{N}\}$ be a countable algebra generating $\mathcal{B}(\Omega)$ (mod $\mathbb{P}$) with $\Omega = G_0$. Then by [13, Theorem 4.16], the following metric on $Pr_\Omega(M)$ generates the narrow topology

$$\begin{aligned}\hat{d}(\mu, \nu) = \sum_{m\in\mathbb{N}} \frac{1}{2^m} \sup \Bigg\{ \int_{G_m} (\mu_\omega(g) - \nu_\omega(g)) d\mathbb{P}(\omega) :& \\ g \in BL(M), 0 \le g \le 1, [g]_L \le 1 \Bigg\}&\end{aligned} \tag{3}$$

where $BL(M)$ denotes the space of bounded Lipschitz functions, and $[\cdot]_L$ denotes the Lipschitz constant.

The following is part of the Portmanteau theorem for random probability measures.

Lemma 3.4 (The Portmanteau Theorem)**.** *If $\mu_n \in Pr_\Omega(M)$, then the following statements are equivalent:*

1. *$\mu_n \to \mu$ in the narrow topology ;*
2. *$\limsup_{n\to\infty} \mu_n(C) \le \mu(C)$ for all closed random sets C,*
3. *$\liminf_{n\to\infty} \mu_n(U) \ge \mu(U)$ for all open random sets U,*

where

$$\mu(C) = \int_\Omega \mu_\omega(C(\omega)) d\mathbb{P}(\omega),\ \ \mu(U) = \int_\Omega \mu_\omega(U(\omega)) d\mathbb{P}(\omega).$$

The method of [13, Lemma 3.19] can be modified to prove the following lemma.

Lemma 3.5. *Suppose that K is a closed random set. Let $c : graph(K) \to \mathbb{R}$ be a measurable map, and assume that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $x \mapsto c(x, \omega)$ is continuous on $K(\omega)$. Suppose further that*

$$\omega \mapsto \sup_{x\in K(\omega)} |c(x, \omega)|$$

is integrable with respect to $\mathbb{P}$. Then the map $\gamma \mapsto \int c d\gamma$ is continuous on the set of random probability measures supported on K.

Proof of Lemma 3.5. Consider the following function $\tilde{c} : M \times \Omega \to \mathbb{R}$,

$$\tilde{c}(x, \omega) = \begin{cases} c(x, \omega), & \text{if } x \in K(\omega) \\ \sup_{x\in K(\omega)} \left\{ \frac{c(z,\omega) - \inf_{y\in K(\omega)} c(y,\omega)}{(1+d(z,x)^2)^{1/d(x,K(\omega))}} \right\} + \inf_{y\in K(\omega)} c(y, \omega), & \text{otherwise.} \end{cases}$$

Then by [14, Section 2.6 Problem 7], $\tilde{c}$ is $\mathbb{P}$-a.s. continuous on M. Moreover, $\sup_{x\in M} |\tilde{c}(x, \omega)| \le \sup_{x\in K(\omega)} |c(x, \omega)|$. For any random probability measures γ supported on K, since $\tilde{c}$ coincides with c on K, we have $\int \tilde{c} d\gamma = \int c d\gamma$. Therefore, by the definition of narrow topology, $\gamma \mapsto \int c d\gamma = \int \tilde{c} d\gamma$ is continuous on the set of random probability measures supported on K. $\square$

3.2. **Strong unstable manifolds theorem.** In this section, we state a strong unstable manifolds theorem under the settings in section 2.1 and a distortion lemma.

Lemma 3.6 (Strong Unstable Manifolds Theorem)**.** *For partially hyperbolic on fibers system ϕ, the local strong unstable set is a $C^{1,1}$ embedded submanifold given by*

$$W_\delta^{uu}(x,\omega) = exp_x(Graph(h_{(x,\omega)}^{uu})) \tag{4}$$

satisfying that

1. *$h_{(x,\omega)}^{uu} : E_\delta^{uu}(x,\omega) \to E^{cs}(x,\omega)$ is a $C^{1,1}$-map with $h_{(x,\omega)}^{uu}(0) = 0$, $Dh_{(x,\omega)}^{uu}(0) = 0$, Lip $h_{(x,\omega)}^{uu}(\cdot) \leq \frac{1}{3}$, Lip $D.h_{(x,\omega)}^{uu} \leq L$, where $E_\delta^{uu}(x,\omega) = \{\eta \in E^{uu}(x,\omega) : |\eta| < \delta\}$ and $L > 1$ is a constant;*
2. *$W_\delta^{uu}(\phi(x,\omega)) \subset f_\omega(W_\delta^{uu}(x,\omega))$, and $W^{uu}(x,\omega) = \bigcup_{n\geq 1} f_{\theta^{-n}\omega}^n W_\delta^{uu}(\phi^{-n}(x,\omega))$, where the global strong unstable manifolds $W^{uu}(x,\omega)$ is given by*
$$W^{uu}(x,\omega) = \{y \in M : \limsup_{n\to+\infty} \frac{1}{n} \log d_M(f_\omega^{-n}x, f_\omega^{-n}y) \leq -\lambda_0\};$$
3. *$d^u(f_\omega^{-n}y, f_\omega^{-n}z) \leq \gamma_0 e^{-n(\lambda_0-\epsilon_0)} d^u(y,z)$ for any $y,z \in W_\delta^{uu}(x,\omega)$ where d^u denotes the distance along strong unstable manifolds, $\gamma_0 > 0$ and $0 < \epsilon_0 \ll \lambda_0$ are constants;*
4. *For any $\rho < \frac{1}{4}\delta$, if $W_\rho^{uu}(x,\omega) := exp_x(Gragh(h_{(x,\omega)}^{uu}|_{E_\rho^{uu}(x,\omega)}))$ intersects $W_\rho^{uu}(x',\omega)$, then*
$$W_\rho^{uu}(x,\omega) \subset W_\delta^{uu}(x',\omega);$$
5. *$W_\delta^{uu}(x,\omega)$ depends continuously on $(x,\omega) \in M \times \Omega$ (in C^1 topology).*

The proof of this lemma can be found in [24] except of the continuity in $(x,\omega) \in M \times \Omega$. Such continuity follows from the continuity of $E^{uu}(x,\omega)$ in $(x,\omega) \in M \times \Omega$ and the continuity of $\omega \mapsto f_\omega \in$Diff$^2(M)$.

Lemma 3.7. *Denote $J^u(y,\omega) := |det(D_y f_\omega|_{E^{uu}(y,\omega)})|$ for any $(y,\omega) \in M \times \Omega$. There exists a constant $C > 0$ such that for any $n \in \mathbb{N}$, $(y,\omega) \in M \times \Omega$, and $z \in W_\delta^{uu}(y,\omega)$, the following holds*

$$\frac{1}{C} \leq \prod_{k=0}^{n} \frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))} \leq C. \tag{5}$$

Furthermore, the function

$$D(z,y,\omega) := \lim_{n\to\infty} \prod_{k=0}^{n} \frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))} \tag{6}$$

is well defined if $z \in W_\delta^{uu}(y,\omega)$, and the sequence converges uniformly in $z \in W_\delta^{uu}(y,\omega)$.

Proof of Lemma 3.7. By the compactness of M, there exists a constant $\delta_0 > 0$ such that for any $y \in M$, the exponential map $\exp_y : B_{\delta_0}(0) \subset T_yM \to B_{\delta_0}(y) \subset M$ is a diffeomorphism and this gives a normal coordinate chart around y (see e.g. [10, Theorem 3.7]). In this paper, we may let $\delta < \delta_0$ such that $W_\delta^{uu}(y,\omega)$ lies in the normal coordinate chart given by exp_y.

We first prove the uniform Lipschitz variation of $J^u(x,\omega)$ along the local strong unstable manifolds, i.e., there exists a constant $K > 0$ such that for any $(x,\omega) \in M_\omega$, $y,z \in W_\delta^{uu}(x,\omega)$, one has

$$|J^u(y,\omega) - J^u(z,\omega)| \leq Kd^u(y,z). \tag{7}$$

Denote $\pi^{uu}_{(x,\omega)}, \pi^{cs}_{(x,\omega)}$ to be the projection from T_xM_ω to $E^{uu}_{(x,\omega)}$ and $E^{cs}_{(x,\omega)}$ respectively with respect to the splitting $T_xM_\omega = E^{uu}_{(x,\omega)} \oplus E^{cs}_{(x,\omega)}$. Notice that $E^{uu}_{(x,\omega)}$ and $E^{cs}_{(x,\omega)}$ are continuously depending on (x,ω), so $\|\pi^{uu}_{(x,\omega)}\|$ and $\|\pi^{cs}_{(x,\omega)}\|$ are uniformly bounded. Notice that $f:\Omega \to \mathrm{Diff}^2(M)$ is continuous, $\|D_xf_\omega\|$ and $\|D^2_xf_\omega\|$ are uniformly bounded. Let $R\geq 1$ be a constant such that

$$\max\left\{\sup_{(x,\omega)\in M\times\Omega}|D_xf_\omega|, \sup_{(x,\omega)\in M\times\Omega}|D^2_xf_\omega|, \sup_{(x,\omega)\in M\times\Omega} LipDh^{uu}_{(x,\omega)}, \sup_{(x,\omega)}\{\|\pi^{cs}_{(x,\omega)}\|, \|\pi^{uu}_{(x,\omega)}\|\}\right\} \leq R.$$

Notice that if $y,z\in W^{uu}_\delta(x,\omega)$, and $d^u(y,z)<\frac{\delta}{2R^3}$, then

$$\begin{aligned}
|\pi^{uu}_{(y,\omega)}(exp_y^{-1}z - exp_y^{-1}y)| &\leq Rd^u(y,z) < \frac{\delta}{2R^2};\\
|\pi^{uu}_{(f_\omega(y),\theta\omega)}(exp^{-1}_{f_\omega y}f_\omega(z) - exp^{-1}_{f_\omega y}f_\omega(y))| &\leq R|exp^{-1}_{f_\omega y}f_\omega(z) - exp^{-1}_{f_\omega y}f_\omega(y)|\\
&\leq R^2|exp_y^{-1}z - \exp_y^{-1}y| \leq \frac{\delta}{2R}.
\end{aligned}$$

Therefore, $z\in W^{uu}_\delta(y,\omega)$ and $f_\omega(z)\in W^{uu}_\delta(f_\omega(y),\theta\omega)$. So it is sufficient to prove that there exists a constant $K>0$ independent of x and ω such that for any $y\in W^{uu}_{\delta/(2R^3)}(x,\omega)$,

$$|J^u(x,\omega) - J^u(y,\omega)| \leq Kd^u(x,y).$$

With the help of normal coordinate chart, we identify $B_\delta(x)$ and $B_\delta(f_\omega(x))$ with Euclidean spaces. In particular, $W^{uu}_\delta(x,\omega)$ and $W^{uu}_\delta(f_\omega x,\theta\omega)$ are curves in Euclidean space. By the strong unstable manifolds theorem, there exist $\xi_y\in E^{uu}_{(x,\omega)}(\frac{\delta}{2R^2})$ and $\xi_{f_\omega(y)}\in E^{uu}_{(f_\omega(x),\theta\omega)}(\delta)$ such that

$$y = x+\xi_y + h^{uu}_{(x,\omega)}(\xi_y), \tag{8}$$

$$f_\omega(y) = f_\omega(x) + \xi_{f_\omega(y)} + h^{uu}_{(f_\omega(x),\theta\omega)}(\xi_{f_\omega(y)}), \tag{9}$$

and $E^{uu}_{(y,\omega)} = graph\left((Dh^{uu}_{(x,\omega)})_{\xi_y}\right)$, $E^{uu}_{(f_\omega(y),\theta\omega)} = graph\left((Dh^{uu}_{(f_\omega(x),\theta\omega)})_{\xi_{f_\omega(y)}}\right)$. By (8) and (9), we have

$$|x-y| = |\xi_y + h^{uu}_{(x,\omega)}(\xi_y)| \geq |\xi_y| - \frac{1}{3}|\xi_y| = \frac{2}{3}|\xi_y|,$$

and

$$(1-\tfrac{1}{3})|\xi_{f_\omega(y)}| \leq |\xi_{f_\omega(y)} + h^{uu}_{(f_\omega(x),\theta\omega)}(\xi_{f_\omega(y)})| = |f_\omega(y) - f_\omega(x)| \leq R|y-x| \leq R(1+\tfrac{1}{3})|\xi_y|,$$

so $|\xi_{f_\omega(y)}|\leq 2R|\xi_y|$.

Now, we define the following linear maps $L_{(x,\omega)}, L_{(y,\omega)}: E^{uu}_{(x,\omega)} \to E^{uu}_{(f_\omega(x),\theta\omega)}$ by

$$\begin{aligned}
L_{(x,\omega)} &= D_xf_\omega|_{E^{uu}_{(x,\omega)}},\\
\tilde{L}_{(y,\omega)} &= \pi^{uu}_{(f_\omega(x),\theta\omega)}D_yf_\omega|_{E^{uu}_{(y,\omega)}}(I + (Dh^{uu}_{(x,\omega)})_{\xi_y}).
\end{aligned}$$

Then $\|L_{(x,\omega)}\|, \|\tilde{L}_{(y,\omega)}\| \leq \frac{4}{3}R^2$. Now, we have

$$\begin{aligned}
&\sup_{v\in E^{uu}_{(x,\omega)},\ \|v\|=1} |D_xf_\omega v - \pi^{uu}_{(f_\omega(x),\theta\omega)}D_yf_\omega(I+(Dh^{uu}_{(x,\omega)})_{\xi_y})v|\\
&\leq R\left(\|D_xf_\omega - D_yf_\omega\| + \|D_yf_\omega(Dh^{uu}_{(x,\omega)})_{\xi_y}\|\right)\\
&\leq R^2|y-x| + R^3|\xi_y|\\
&\leq (R^2+\frac{3}{2}R^3)d^u(x,y).
\end{aligned}$$

Hence, $\|L_{(x,\omega)} - \tilde{L}_{(y,\omega)}\| \leq C_0(R^2 + \frac{3}{2}R^3)d^u(x,y)$, where C_0 only depends on the normal coordinate chart. Then by the property of the determinant,

$$|det(L_{(x,\omega)}) - det(\tilde{L}_{(y,\omega)})| \leq C_1 d^u(x,y), \tag{10}$$

where C_1 is a polynomial of R and $\dim E^{uu}(x,\omega)$.

Notice that

$$\|(\pi^{uu}_{(f_\omega(x),\theta\omega)}|_{E^{uu}_{(f_\omega(y),\theta\omega)}} - I)(I + (Dh^{uu}_{(f_\omega(x),\theta\omega)})_{\xi_{f_\omega(y)}})\| \leq \|(Dh^{uu}_{(f_\omega(x),\theta\omega)})_{\xi_{f_\omega(y)}}\|.$$

Therefore,

$$\begin{aligned}
\|\pi^{uu}_{(f_\omega(x),\theta\omega)}|_{E^{uu}_{(f_\omega(y),\theta\omega)}} - I\| &\leq \frac{\|(Dh^{uu}_{(f_\omega(x),\theta\omega)})_{\xi_{f_\omega(y)}}\|}{1 - \|(Dh^{uu}_{(f_\omega(x),\theta\omega)})_{\xi_{f_\omega(y)}}\|} \\
&\leq \frac{R|\xi_{f_\omega(y)}|}{1 - R|\xi_{f_\omega(y)}|} \leq \frac{2R^2|\xi_y|}{1 - 2R^2|\xi_y|} \leq \frac{2R^2|\xi_y|}{1 - 2R^2\frac{\delta}{2R^2}} \leq 4R^2|\xi_y| \\
&\leq 6R^2 d^u(x,y).
\end{aligned}$$

So we have

$$|det(\pi^{uu}_{(f_\omega(x),\theta\omega)}|_{E^{uu}_{(f_\omega(y),\theta\omega)}}) - 1| \leq C_2 d^u(x,y) \tag{11}$$

where C_2 is a polynomial of R and $\dim E^{uu}_{(x,\omega)}$. Also

$$\|I + (Dh^{uu}_{(x,\omega)})_{\xi_y} - I\| \leq R|\xi_y| \leq R d^u(x,y)$$

implies that there exists a constant C_3 such that

$$|det(I + (Dh^{uu}_{(x,\omega)})_{\xi_y}) - 1| \leq C_3 d^u(x,y). \tag{12}$$

Combining (10), (11), and (12),

$$|J^u(x,\omega) - J^u(y,\omega)| \leq K d^u(x,y),$$

where K is a constant.

Next, we prove (5). By (7), for any $y, z \in W^{uu}_\delta(x,\omega)$, $k \in \mathbb{N}$,

$$\begin{aligned}
|J^u(\phi^{-k}(y,\omega)) - J^u(\phi^{-k}(z,\omega))| &\leq K d^u(f^{-k}_\omega(y), f^{-k}_\omega(z)) \leq K\gamma_0 e^{-k(\lambda_0-\epsilon_0)} d^u(y,z) \\
&\leq K\gamma_0 e^{-k(\lambda_0-\epsilon_0)}\delta.
\end{aligned}$$

Notice that $J^u(x,\omega) \geq C_0^{-1}e^{\lambda_0} > 0$ for all $(x,\omega) \in M \times \Omega$, hence

$$\left|\frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))} - 1\right| \leq KC_0\gamma_0\delta e^{-\lambda_0}e^{-k(\lambda_0-\epsilon_0)}.$$

Thus, for any $l < n$, $l, n \in \mathbb{N}$,

$$\begin{aligned}
&\prod_{k=l}^{n-1} \frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))} \\
\leq &\prod_{k=l}^{n-1}(1 + KC_0\gamma_0\delta e^{-\lambda_0}e^{-k(\lambda_0-\epsilon_0)}) = e^{\sum_{k=l}^{n-1}\log(1+KC_0\gamma_0\delta e^{-\lambda_0}e^{-k(\lambda_0-\epsilon_0)})} \\
\leq &e^{\sum_{k=l}^{n-1} KC_0\gamma_0\delta e^{-\lambda_0}e^{-k(\lambda_0-\epsilon_0)}} \leq e^{\sum_{k=l}^{\infty} KC_0\gamma_0\delta e^{-\lambda_0}e^{-k(\lambda_0-\epsilon_0)}} \\
\leq &e^{\sum_{k=0}^{\infty} KC_0\gamma_0\delta e^{-\lambda_0}e^{-k(\lambda_0-\epsilon_0)}} := C.
\end{aligned}$$

Swap y and z to get

$$\frac{1}{C} \leq \prod_{k=0}^{n-1}\frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))} \leq C \text{ for any } n \in \mathbb{N}.$$

Finally, we prove the uniform convergence on local strong unstable leaf. Pick any $z \in W^{uu}_\delta(y,\omega)$, any $n > m > 0$,

$$\begin{aligned}
&\left|\prod_{k=0}^{n}\frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))} - \prod_{k=0}^{m}\frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))}\right| \\
&= \left|\prod_{k=0}^{m}\frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))}\right| \cdot \left|\prod_{k=m+1}^{n}\frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))} - 1\right| \\
&\leq C \cdot \left|\prod_{k=m+1}^{n}\frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))} - \prod_{k=m+1}^{n-1}\frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))} + \prod_{k=m+1}^{n-1}\frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))}\right. \\
&\qquad \left. - \cdots - \frac{J^u(\phi^{-(m+1)}(y,\omega))}{J^u(\phi^{-(m+1)}(z,\omega))} + \frac{J^u(\phi^{-(m+1)}(y,\omega))}{J^u(\phi^{-(m+1)}(z,\omega))} - 1\right| \\
&\leq C \cdot \left(C \cdot \left|\frac{J^u(\phi^{-(n)}(y,\omega))}{J^u(\phi^{-(n)}(z,\omega))} - 1\right| + \cdots + C\left|\frac{J^u(\phi^{-(m+1)}(y,\omega))}{J^u(\phi^{-(m+1)}(z,\omega))} - 1\right|\right) \\
&\leq C^2 K C_0 \gamma_0 \delta e^{-\lambda_0}(e^{-n(\lambda_0-\epsilon_0)} + \cdots e^{-(m+1)(\lambda_0-\epsilon_0)}) \\
&= C' e^{-m(\lambda_0-\epsilon_0)}
\end{aligned}$$

for some constant C'. Hence, the convergence $\lim_{n\to\infty}\prod_{k=0}^{n}\frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))}$ is uniform for all $z \in W^{uu}_\delta(y,\omega)$. The proof of Lemma 3.7 is complete. ☐

4. **Proof of main result.** In this section, we prove the Theorem 2.3 in four steps.

Step 1. *We start with a reference random probability measure λ_{x_0} whose disintegration coincides with the normalized intrinsic Riemmanian measure on a local strong unstable manifold. Then we consider the sequence of Krylov-Bogolyubov type measure $\frac{1}{n}\sum_{k=0}^{n-1}(\phi^*)^k\lambda_{x_0}$.*

For a fixed $x_0 \in M$, define $\mathcal{L}_{x_0} : \Omega \to 2^M$ by

$$\mathcal{L}_{x_0}(\omega) := W^{uu}_\delta(x_0,\omega).$$

Let $(\lambda_{x_0})_\omega \in Pr(M)$ be the normalized intrinsic Riemannian volume measure on $W^{uu}_\delta(x_0,\omega)$. Because $W^{uu}_\delta(x_0,\omega)$ varies continuously on $\omega \in \Omega$ in C^1 topology, $(\lambda_{x_0})_\omega$ induced by the restricted Riemannian metric on continuous submanifold $W^{uu}_\delta(x_0,\omega)$ depends continuously on ω. Then the disintegration $\{(\lambda_{x_0})_\omega\}_{\omega\in\Omega}$ defines a random probability measure, named λ_{x_0}.

Consider the sequence of random probability measures $\{\frac{1}{n}\sum_{k=0}^{n-1}(\phi^*)^k\lambda_{x_0}\}_{n=1}^\infty$. Since $Pr_\Omega(M)$ is compact with respect to the narrow topology, there exists a convergent subsequence of it

$$\frac{1}{n_i}\sum_{k=0}^{n_i-1}(\phi^*)^k\lambda_{x_0} \to \mu \in Pr_\Omega(M) \text{ as } i \to \infty \tag{13}$$

with respect to the narrow topology. Moreover, μ is clearly $\phi-$invariant.

Step 2. *We cover $M \times \Omega$ by finitely many graphs of closed random sets, then we convert the absolute continuity problem with respect to the $u-$subordinate partition to a absolute continuity problem on these graphs.*

Let $\{x_i\}_{i=1}^{\infty} \subset M$ be a countable dense subset. For each $\omega \in \Omega$, $\epsilon > 0$ and $\rho < \frac{1}{4}\delta$, we define

$$V_{x_i,\epsilon,\rho}(\omega) = \bigcup_{y \in \Xi_{x_i,\epsilon}(\omega)} W_\rho^{uu}(y,\omega) \setminus \partial\Big(\bigcup_{y \in \Xi_{x_i,\epsilon}(\omega)} W_\rho^{uu}(y,\omega)\Big),$$

where $\Xi_{x_i,\epsilon}(\omega) := exp_{x_i}(E_\epsilon^{cs}(x_i,\omega))$ and ∂ denotes the boundary. Here, by property (4) in Lemma 3.6, we pick ϵ sufficiently small such that $W_\rho^{uu}(y,\omega) \cap W_\rho^{uu}(z,\omega) = \emptyset$ for any $y, z \in \Xi_{x_i,\epsilon}(\omega), y \neq z$. This family of sets $\{V_{x_i,\epsilon,\rho}(\omega)\}_{i=1}^{\infty}$ forms an open cover of M on the fiber $\{\omega\}$. We choose a finite cover of it, named $\{V_{x_i,\epsilon,\rho}(\omega)\}_{i \in J(\omega)}$, where $J(\omega)$ is a finite index set. Denote $m(\omega) = \max\{i : i \in J(\omega)\}$. Then, $\{V_{x_i,\epsilon,\rho}(\omega)\}_{i=1}^{m(\omega)}$ is also a finite open cover. By the continuity of $W_\delta^{uu}(x,\omega)$ and $E^{cs}(x,\omega)$ on ω, there exists a $\gamma(\omega) > 0$ sufficiently small such that whenever $d(\omega,\omega') < \gamma(\omega)$,

$$\{V_{x_i,\epsilon,\rho}(\omega')\}_{i=1}^{m(\omega)} \text{ is an open cover of } M \text{ on the fiber } \omega'. \tag{14}$$

Let $B_{\gamma(\omega)}(\omega)$ be the ball in Ω centered at ω with radius $\gamma(\omega)$, then $\{B_{\gamma(\omega)}(\omega)\}_{\omega\in\Omega}$ is an open cover of Ω. By the compactness of Ω, there is a finite subcover, named $\{B_{\gamma(\omega_i)}(\omega_i)\}_{i=1}^n$. Define $F_1 = B_{\gamma(\omega_1)}(\omega_1)$, $F_i = B_{\gamma(\omega_i)}(\omega_i) \setminus \cup_{j=1}^{i-1} B_{\gamma(\omega_j)}(\omega_j)$ for $i = 2, ..., n$. Then we obtain

$$\text{a measurable partition of } \Omega,\ \{F_i\}_{i=1}^n,\ \text{with } F_i \subset B_{\gamma(\omega_i)}(\omega_i) \text{ for } i = 1, ..., n. \tag{15}$$

Without loss of generality, we assume $F_i \neq \emptyset$ for all $i \in \{1, ..., n\}$. Set $m_i = m(\omega_i)$. Then (14) and (15) imply

$$\{V_{x_j,\epsilon,\rho}(\omega)\}_{j=1}^{m_i} \text{ is an open cover of } M \text{ on the fiber } \omega,\ \text{whenever } \omega \in F_i. \tag{16}$$

Now we define $g_{j_1,j_2,...,j_n} \in L^\infty(\Omega, M)$ for $j_i \in \{1, ..., m_i\}$ by

$$g_{j_1,...,j_n}(\omega) = \begin{cases} x_{j_1}, & \text{if } \omega \in F_1 \\ \vdots, & \\ x_{j_n}, & \text{if } \omega \in F_n. \end{cases}$$

Now for each $g \in \{g_{j_1,...,j_n}\}$, we define $V_{g,\epsilon,\rho} : \Omega \to 2^M$ by

$$V_{g,\epsilon,\rho}(\omega) := V_{g(\omega),\epsilon,\rho}(\omega).$$

For any $\omega \in \Omega$, there exists $i \in \{1, ..., n\}$ such that $\omega \in F_i$,

$$\begin{aligned} M = \cup_{j=1}^{m_i} V_{x_j,\epsilon,\rho}(\omega) &= \bigcup_{k=1}^{m_i} V_{g_{j_1,...,j_{i-1},k,j_{i+1},...,j_n},\epsilon,\rho}(\omega) \\ &\subset \bigcup_{g \in \{g_{j_1,...,j_n}\}} V_{g,\epsilon,\rho}(\omega). \end{aligned} \tag{17}$$

Claim 4.1. $\overline{V_{g,\epsilon,\rho}} : \Omega \mapsto 2^M$ *defined by* $\omega \mapsto \overline{V_{g,\epsilon,\rho}}(\omega)$ *is a closed random set, where* $\overline{V_{g,\epsilon,\rho}}(\omega)$ *is the closure of* $V_{g,\epsilon,\rho}(\omega)$ *in* M *(see the definition of closed random set in Def. 3.1).*

Proof of Claim 4.1. For $g = g_{j_1,...,j_n}$, we first prove that $\omega \mapsto \overline{\Xi_{g(\omega),\epsilon}}(\omega)$ is a closed random set, where $\overline{\Xi_{g(\omega),\epsilon}}(\omega)$ is the closure of $\Xi_{g(\omega),\epsilon}(\omega)$ in M. For each $\omega \in \Omega$,

$\overline{\Xi_{g(\omega),\epsilon}}(\omega)$ is closed. Pick any $z \in M$, $r > 0$

$$\begin{aligned}(18)\quad \{\omega:\ d(z, \overline{\Xi_{g(\omega),\epsilon}}(\omega)) < r\} &= \cup_{i=1}^{n}\{\omega \in F_i:\ d(z, \overline{\Xi_{g(\omega),\epsilon}}(\omega)) < r\} \\ &= \cup_{i=1}^{n}\{\omega \in F_i:\ d(z, \overline{\Xi_{x_{j_i},\epsilon}}(\omega)) < r\} \\ &= \cup_{i=1}^{n}\{\omega \in F_i:\ d(z, \overline{exp_{x_{j_i}}(E_\epsilon^{cs}(x_{j_i},\omega))}) < r\}.\end{aligned}$$

Since $E_\epsilon^{cs}(x_{j_i},\omega)$ is continuous in ω, $d(z, \overline{exp_{x_{j_i}}(E_\epsilon^{cs}(x_{j_i},\omega))})$ is continuous in $\omega \in F_i$. As a consequence, the set $\{\omega :\ d(z, \overline{\Xi_{g(\omega),\epsilon}}(\omega)) < r\}$ in (18) is measurable. Hence $\omega \mapsto \overline{\Xi_{g(\omega),\epsilon}}(\omega)$ is a closed random set. By Lemma 3.3, there exist countably many measurable functions $c_n : \Omega \mapsto M$ for $n \in \mathbb{N}$ such that

$$\overline{\Xi_{g(\omega),\epsilon}}(\omega) = closure\{c_n(\omega):\ n \in \mathbb{N}\}.$$

Thus for any $z \in M$,

$$\{\omega:\ d(z, \overline{V_{g,\epsilon,\rho}}(\omega)) < r\} = \cup_{n\in N}\{\omega:\ d(z, \overline{W_\rho^{uu}(c_n(\omega),\omega)}) < r\},$$

which is measurable. Hence $\omega \mapsto \overline{V_{g,\epsilon,\rho}}(\omega)$ is a closed random set by checking definition. The proof of claim 4.1 is complete. $\square$

As a consequence, $V_{g,\epsilon,\rho}(\omega) = int(\overline{V_{g,\epsilon,\rho}}(\omega))$ is an open random set by Lemma 3.2. Now we have finitely many closed random sets $\{\overline{V_{g,\epsilon,\rho}}:\ g \in \{g_{j_1,\dots,j_n}\}\}$ and by (17),

$$M \times \Omega \quad = \bigcup_{g\in\{g_{j_1,\dots,j_n}\}} graph(\overline{V_{g,\epsilon,\rho}}), \tag{19}$$

where

$$graph(\overline{V_{g,\epsilon,\rho}}) = \{(x,\omega) \in M \times \Omega:\ x \in \overline{V_{g,\epsilon,\rho}}(\omega)\}.$$

Claim 4.2. *By shrinking ϵ and ρ if necessary, we can achieve (19) and*

$$\mu(\partial\overline{V_{g,\epsilon,\rho}}) = \textstyle\int_\Omega \mu_\omega(\partial\overline{V_{g,\epsilon,\rho}}(\omega))d\mathbb{P}(\omega) = 0 \text{ for all } g \in \{g_{j_1,\dots,j_n}\}$$

at the same time.

Proof of Claim 4.2. By the construction of $V_{x_i,\epsilon,\rho}(\omega)$, (14) and the finiteness of $\{F_i\}$, there exist two intervals $[\epsilon_1,\epsilon_2]$ and $[\rho_1,\rho_2]$, with $\epsilon_1 < \epsilon_2$ and $\rho_1 < \rho_2$, such that for any $\epsilon \in [\epsilon_1,\epsilon_2]$, $\rho \in [\rho_1,\rho_2]$, (19) still holds by using such ϵ and ρ in the construction.

We notice that the boundary $\omega \mapsto \partial(\overline{V_{g,\epsilon,\rho}}(\omega))$ consists of

$$\cup_{z\in\partial(\overline{\Xi_{g(\omega),\epsilon}}(\omega))} W_\rho^{uu}(z,\omega),\ \text{ and } \partial(\overline{V_{g,\epsilon,\rho}}(\omega)) \setminus \cup_{z\in\partial(\overline{\Xi_{g(\omega),\epsilon}}(\omega))} W_\rho^{uu}(z,\omega).$$

Note that for $\epsilon \neq \epsilon'$, we have

$$\cup_{z\in\partial(\overline{\Xi_{g(\omega),\epsilon}}(\omega))} W_{\rho_2}^{uu}(z,\omega) \cap \cup_{z\in\partial(\overline{\Xi_{g(\omega),\epsilon'}}(\omega))} W_{\rho_2}^{uu}(z,\omega) = \emptyset.$$

For $k \in \mathbb{N}$ and $g \in \{g_{j_1,\dots,g_n}\}$, we define

$$M_{g,k} = \left\{\epsilon \in [\epsilon_1,\epsilon_2]:\ \int_\Omega \mu_\omega(\cup_{z\in\partial(\overline{\Xi_{g(\omega),\epsilon}}(\omega))} W_{\rho_2}^{uu}(z,\omega))d\mathbb{P}(\omega) > \frac{1}{k}\right\}.$$

Then $M_{g,k}$ must be a finite set. Therefore, $\cup_{k\in\mathbb{N}} M_{g,k}$ is at most countable. We pick and fix

$$\epsilon \in [\epsilon_1,\epsilon_2] \setminus \left(\cup_{g\in\{g_{j_1,\dots,j_n}\}} \cup_{k\in\mathbb{N}} M_{g,k}\right). \tag{20}$$

Next we pick ρ by using the following fact: for different $\rho \neq \rho'$,

$$\left(\partial(\overline{V_{g,\epsilon,\rho}}(\omega)) \setminus \cup_{z\in\partial(\overline{\Xi_{g(\omega),\epsilon}}(\omega))} W_\rho^{uu}(z,\omega)\right) \cap \left(\partial(\overline{V_{g,\epsilon,\rho'}}(\omega)) \setminus \cup_{z\in\partial(\overline{\Xi_{g(\omega),\epsilon}}(\omega))} W_{\rho'}^{uu}(z,\omega)\right) = \emptyset.$$

For ϵ in (20), $k \in \mathbb{N}$ and $g \in \{g_{j_1,\ldots,g_n}\}$, we define

$$M'_{g,k} = \left\{ \rho \in [\rho_1, \rho_2] : \int_\Omega \mu_\omega(\partial(\overline{V_{g,\epsilon,\rho}}(\omega)) \setminus \cup_{z \in \partial(\overline{\Xi_{g(\omega),\epsilon}}(\omega))} W^{uu}_\rho(z,\omega)) d\mathbb{P}(\omega) > \frac{1}{k} \right\}.$$

Then $M'_{g,k}$ must be finite set. Therefore, $\cup_{k\in\mathbb{N}} M'_{g,k}$ is at most countable. We pick

$$\rho \in [\rho_1, \rho_2] \setminus \left(\cup_{g \in \{g_{j_1,\ldots,j_n}\}} \cup_{k\in\mathbb{N}} M'_{g,k} \right). \tag{21}$$

Therefore, ϵ and ρ chosen in (20) and (21) are desired numbers. The proof of claim 4.2 is complete. □

For any $g \in \{g_{j_1,\ldots,j_n}\}$, and $\omega \in \Omega$, we divide $\overline{V_{g,\epsilon,\rho}}(\omega)$ into pieces $\{\overline{W^{uu}_\rho(y,\omega)}\}$ for $y \in \overline{\Xi_{g(\omega),\epsilon}}(\omega)$, which produce a measurable partition of $\overline{V_{g,\epsilon,\rho}}(\omega)$. Recall that μ is given in (13). Let $\mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)}$ be the restriction of μ_ω on $\overline{V_{g,\epsilon,\rho}}(\omega)$. Let $(\mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)})_y$ be the conditional probability measure of $\mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)}$ on $\overline{W^{uu}_\rho(y,\omega)}$ with respect to the measurable partition $\{\overline{W^{uu}_\rho(y,\omega)}\}_{y\in\overline{\Xi_{g(\omega),\epsilon}}(\omega)}$. The space of partition is identified with the quotient space $\overline{\Xi_{g(\omega),\epsilon}}(\omega)$. Let the $\sigma-$algebra $\mathcal{B}(\overline{\Xi_{g(\omega),\epsilon}}(\omega))$ be the $\sigma-$algebra induced by the quotient, i.e., $B \in \mathcal{B}(\overline{\Xi_{g(\omega),\epsilon}}(\omega))$ if and only if $\cup_{y\in B}\overline{W^{uu}_\rho(y,\omega)}$ is measurable subset in $\overline{V_{g,\epsilon,\rho}}(\omega)$. Let $\widetilde{\mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)}}$ be the induced measure on quotient space $\overline{\Xi_{g(\omega),\epsilon}}(\omega)$, i.e.,

$$\widetilde{\mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)}}(B) = \mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)}(\cup_{y\in B}\overline{W^{uu}_\rho(y,\omega)}) \text{ for any } B \in \mathcal{B}(\Xi_{g(\omega),\epsilon}(\omega)). \tag{22}$$

Then by Rokhlin's disintegration theorem, for any Borel measurable subset $E \subset \overline{V_{g,\epsilon,\rho}}(\omega)$ we have

$$\mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)}(E) = \int_{\overline{\Xi_{g(\omega),\epsilon}}(\omega)} (\mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)})_y(E) d\widetilde{\mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)}}(y). \tag{23}$$

Claim 4.3. *If for each $g \in \{g_{j_1,\ldots,j_n}\}$, there exist a subset $\Omega_g \subset \Omega$ of $\mathbb{P}$-full measure, and a $\widetilde{\mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)}}$-full measure subset $N_g(\omega) \subset \overline{\Xi_{g(\omega),\epsilon}}(\omega)$ for $\omega \in \Omega_g$ such that the following holds for $\omega \in \Omega_g$ and $y \in N_g(\omega)$*

$$(\mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)})_y \ll \lambda^{uu}_{(y,\omega)} \text{ on every piece } W^{uu}_\rho(y,\omega), \; y \in \overline{\Xi_{g(\omega),\epsilon}}(\omega) \tag{24}$$

where $\lambda^{uu}_{(y,\omega)}$ is the intrinsic Riemannian volume measure on $W^{uu}_\rho(y,\omega)$, then μ is a random Gibbs $u-$state.

Proof of Claim 4.3. Let $\mathcal{P}$ be any u-subordinate partition of $M \times \Omega$. For any $P(x,\omega) \in \mathcal{P}$, we denote $P(x,\omega) = P_\omega(x) \times \{\omega\}$. Denote $\mathcal{F}(\mathcal{P})$ to be the $\sigma-$algebra on $\mathcal{P}$ such that $F \in \mathcal{F}(\mathcal{P})$ if $\cup_{P\in F} P \subset M \times \Omega$ is measurable. Let $\tilde{\mu}$ be the measure induced by μ on $(\mathcal{P}, \mathcal{F}(\mathcal{P}))$, i.e., $\tilde{\mu}(F) = \mu(\cup_{P\in F} P)$. For any $F \in \mathcal{F}(\mathcal{P})$, since $\mathcal{P}$ is $u-$subordinate, $F = \cup_{\omega\in\Omega} F(\omega)$, where $F(\omega) = \{P_\omega : P \in F, \; P \subset M \times \{\omega\}\}$. We denote $\mathcal{P}(\omega) := \{P_\omega : P \in \mathcal{P}\}$, which is a measurable partition of M on the fiber ω. Define $\mathcal{F}(\mathcal{P}(\omega))$ and $\tilde{\mu_\omega}$ on $(\mathcal{P}(\omega), \mathcal{F}(\mathcal{P}(\omega)))$ similarly. Then by Rokhlin's disintegration theorem, we have

$$\mu(c) = \int_{\mathcal{P}} \int_P c(z,\omega) d\mu_P d\tilde{\mu}(P), \tag{25}$$

and

$$\mu(c) = \int_\Omega \int_{\mathcal{P}(\omega)} \int_{P_\omega} c(z,\omega) d(\mu_\omega)_{P_\omega} d\tilde{\mu_\omega}(P_\omega) d\mathbb{P}(\omega) \tag{26}$$

for measurable function $c : M \times \Omega \to \mathbb{R}$, where μ_P is the conditional measure of μ on $P \in \mathcal{P}$ with respect to measurable partition $\mathcal{P}$ and $(\mu_\omega)_{P_\omega}$ is the conditional measure of μ_ω on $P_\omega \in \mathcal{P}(\omega)$ with respect to measurable partition $\mathcal{P}(\omega)$. Note that $P(x,\omega) = P_\omega(x) \times \{\omega\}$ for any $P(x,\omega) \in \mathcal{P}$, therefore we can regard $\mu_{P(x,\omega)}$ as a measure on $P_\omega(x)$, and we denote it by $\mu_{P_\omega(x)}$.

Now for any $F \in \mathcal{F}(\mathcal{P})$, we have

$$\begin{aligned}\int_{\mathcal{P}} 1_F(P) d\tilde{\mu}(P) &= \tilde{\mu}(F) = \mu(\cup_{P\in F} P)\\ &= \int_\Omega \mu_\omega(\cup_{P_\omega \in F(\omega)} P_\omega) d\mathbb{P}(\omega) = \int_\Omega \tilde{\mu_\omega}(F(\omega)) d\mathbb{P}(\omega) \qquad (27)\\ &= \int_\Omega \int_{\mathcal{P}(\omega)} 1_{F(\omega)}(P_\omega) d\tilde{\mu_\omega}(P_\omega) d\mathbb{P}(\omega).\end{aligned}$$

In (25), $P \mapsto \int_P c(z,\omega) d\mu_P$ is $\mathcal{F}(\mathcal{P})-$measurable. Hence by (27) and identifying μ_P and μ_{P_ω} for $P \subset M \times \{\omega\}$, we have

$$\int_{\mathcal{P}} \int_P c(z,\omega) d\mu_P d\tilde{\mu}(P) = \int_\Omega \int_{\mathcal{P}(\omega)} \int_{P_\omega} c(z,\omega) d\mu_{P_\omega} d\tilde{\mu_\omega}(P_\omega) d\mathbb{P}(\omega). \qquad (28)$$

Then by the uniqueness of disintegration, (26) and (28), neglecting a $\tilde{\mu}-$null set, $\mu_{P_\omega} = (\mu_\omega)_{P_\omega}$, i.e., for μ-a.e. $(z,\omega) \in M \times \Omega$, $\mu^{\mathcal{P}}_{(z,\omega)} = (\mu_\omega)^{\mathcal{P}(\omega)}_z$.

By the above and the assumption of claim 4.3, then the collection of $(z,\omega) \in M \times \Omega$ satisfying $\mu^{\mathcal{P}}_{(z,\omega)} = (\mu_\omega)^{\mathcal{P}(\omega)}_z$ and $\omega \in \cap_{g \in \{g_{j_1},\dots,j_n\}} \Omega_g$ has μ-full measure. Pick any such (z,ω) and $P(z,\omega) \in \mathcal{P}$. Without loss of generality, we assume $\mu_\omega(P_\omega(z)) > 0$. Recall that $P_\omega(z) \subset W^{uu}(z,\omega)$ and contains an open neighborhood of z contained in $W^{uu}(z,\omega)$. By (17), we have $P_\omega(z) = \cup_{g \in \{g_{j_1},\dots,j_n\}} P_\omega(z) \cap \overline{V_{g,\epsilon,\rho}}(\omega)$. We note that any unstable leaves in $\overline{V_{g,\epsilon,\rho}}(\omega)$ disjoint. Then $P_\omega(z) \cap \overline{V_{g,\epsilon,\rho}}(\omega)$ consists of disjoint pieces of strong unstable leaves. In this way, we divide $P_\omega(z)$ into strong unstable leaves. It is sufficient to prove $\mu^{\mathcal{P}}_{(z,\omega)}$ restricted on each strong unstable leaf is absolutely continuous with respect to the Riemannian volume measure on the leaf. Denote $W^{uu}_\delta(y,\omega)$ to be one of such leaf.

Note that if $(\mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)})_y(B) = 0$ for some Borel $B \subset W^{uu}_\delta(y,\omega) \cap P_\omega(z)$, then by (23), we have

$$\begin{aligned}\mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)}(B) &= \int_{\overline{\Xi_{g(\omega),\epsilon}}(\omega)} (\mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)})_{y'}(B) d\widetilde{\mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)}}(y')\\ &= (\mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)})_y(B) \cdot \mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)}(W^{uu}_\rho(y,\omega))\\ &= 0.\end{aligned}$$

As a consequence, $\mu_\omega(B) = 0$. Again using Rokhlin's disintegration theorem, we have

$$\begin{aligned}0 = \mu_\omega(B) &= \int_{\mathcal{P}(\omega)} (\mu_\omega)_{P_\omega}(B) d\tilde{\mu_\omega}(P_\omega)\\ &= (\mu_\omega)^{\mathcal{P}(\omega)}_z(B) \mu_\omega(P_\omega(z))\end{aligned}$$

So $(\mu_\omega)^{\mathcal{P}(\omega)}_z(B) = 0$. Therefore,

$$\mu^{\mathcal{P}}_{(z,\omega)}|_{W^{uu}_\delta(y,\omega)} = (\mu_\omega)^{\mathcal{P}(\omega)}_z|_{W^{uu}_\delta(y,\omega)} \ll (\mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)})_y.$$

If $y \in N_g(\omega)$, then by assumption (24), $\mu^{\mathcal{P}}_{(z,\omega)}|_{W^{uu}_\delta(y,\omega)} \ll \lambda^{uu}_{(y,\omega)}$. If $y \notin N_g(\omega)$, then $\mu_\omega(W^{uu}_\rho(y,\omega)) = 0$, and therefore, $\mu^{\mathcal{P}}_{(z,\omega)}|_{W^{uu}_\delta(y,\omega)} \ll \lambda^{uu}_{(y,\omega)}$ holds naturally. The proof of claim 4.3 is complete. $\square$

Remark 4.1. When the external forcing disappears, that is the deterministic partially hyperbolic system, one often defines Gibbs $u-$state by the condition of Claim 4.3 (see Definition 11.7 in [6] and Section 5.2 in [11]).

In the following steps, we fix a $V_{g,\epsilon,\rho}$ and prove that the conditions of claim 4.3 hold.

Step 3. *We construct the density functions of μ with respect to the Riemannian volume measure on each local strong unstable manifold in each $V_{g,\epsilon,\rho}$.*

For each $k \geq 0$ and all $\omega \in \Omega$, let

$$L_k(\omega) := \big\{ z \in \mathcal{L}_{x_0}(\omega) = W^{uu}_\delta(x_0,\omega) : \ f^k_\omega(z) \in W^{uu}_\rho(y,\theta^k\omega) \text{ for some } y \in \Xi_{g(\theta^k\omega),\epsilon}(\theta^k\omega) \text{ but } W^{uu}_\rho(y,\theta^k\omega) \not\subset f^k_\omega W^{uu}_\delta(x_0,\omega) \big\}.$$

We must have that for any $z \in L_k(\omega)$, $d^u(z, \partial\mathcal{L}_{x_0}(\omega)) \leq \delta\gamma_0 e^{-k(\lambda_0-\epsilon_0)}$. Otherwise, since for any $z' \in W^{uu}_\rho(y,\theta^k\omega)$, $d^u(f^k_\omega(z), z') < 2\rho$, then by Lemma 3.6,

$$d^u(z, f^{-k}_\omega z') \leq \gamma_0 e^{-k(\lambda_0-\epsilon_0)} \cdot 2\rho \leq \gamma_0 e^{-k(\lambda_0-\epsilon_0)}\delta < d^u(z, \partial\mathcal{L}_{x_0}(\omega)),$$

which implies that $z' \in f^n_\omega W^{uu}_\delta(x_0,\omega)$, and contradicts to the assumption that $W^{uu}_\rho(y,\theta^k\omega) \not\subset f^k_\omega W^{uu}_\delta(x_0,\omega)$.

By Lemma 3.6 (1), there exists a constant C independent of $\omega \in \Omega$ and $x \in M$ such that

$$(\lambda_x)_\omega(L_k(\omega)) \leq C(\delta\gamma_0 e^{-k(\lambda_0-\epsilon_0)})^{dim(E^{uu}(x,\omega))} \to 0 \text{ as } k \to \infty$$

uniformly for all $\omega \in \Omega$. Therefore,

$$(\lambda_{x_0})_\omega|_{\mathcal{L}_{x_0}(\omega)\setminus L_k(\omega)}(g) \to (\lambda_{x_0})_\omega|_{\mathcal{L}_{x_0}(\omega)}(g) = (\lambda_{x_0})_\omega(g) \text{ as } k \to \infty$$

uniformly for all $\omega \in \Omega$ for any $g \in BL(M)$ with $0 \leq g \leq 1$ and $[g]_L \leq 1$, where $(\lambda_{x_0})_\omega|_{\mathcal{L}_{x_0}(\omega)\setminus L_k(\omega)}$ is the restriction of $(\lambda_{x_0})_\omega$ on $\mathcal{L}_{x_0}(\omega) \setminus L_k(\omega)$. Thus by noticing (3), we have

$$\lim_{i\to\infty} \frac{1}{n_i} \sum_{k=0}^{n_i-1} (\phi^*)^k(\lambda_{x_0}|_{\mathcal{L}_{x_0}\setminus L_k}) = \lim_{i\to\infty} \frac{1}{n_i} \sum_{k=0}^{n_i-1} (\phi^*)^k(\lambda_{x_0}) \overset{(13)}{=} \mu \tag{29}$$

in narrow topology on $Pr_\Omega(M)$, where $\lambda_{x_0}|_{\mathcal{L}_{x_0}\setminus L_n}$ is the random measure with disintegration $\omega \mapsto (\lambda_{x_0})_\omega|_{\mathcal{L}_{x_0}(\omega)\setminus L_n(\omega)}$.

For each random set $\overline{V_{g,\epsilon,\rho}}$, by claim 4.2, we have $\int_\Omega \mu_\omega(\partial\overline{V_{g,\epsilon,\rho}}(\omega))d\mathbb{P}(\omega) = 0$, we claim that

$$\lim_{i\to\infty} \left(\frac{1}{n_i} \sum_{k=0}^{n_i-1} (\phi^*)^k(\lambda_{x_0}|_{\mathcal{L}_{x_0}\setminus L_k}) \right)(\overline{V_{g,\epsilon,\rho}}) = \mu(\overline{V_{g,\epsilon,\rho}}). \tag{30}$$

In fact, by the Portmanteau theorem (Lemma 3.4) and (29), and the fact that $\omega \mapsto \overline{V_{g,\epsilon,\rho}}(\omega)$ is closed random set and $\omega \mapsto V_{g,\epsilon,\rho}(\omega)$ is a open random set, we obtain

$$\mu(\overline{V_{g,\epsilon,\rho}}) = \int_\Omega \mu_\omega(\overline{V_{g,\epsilon,\rho}}(\omega))d\mathbb{P}(\omega) = \int_\Omega \mu_\omega(V_{g,\epsilon,\rho}(\omega))d\mathbb{P}(\omega)$$

$$\leq \liminf_{i\to\infty} \frac{1}{n_i} \sum_{k=0}^{n_i-1} (\phi^*)^k (\lambda_x|_{\mathcal{L}_{x_0}\setminus L_k})(V_{g,\epsilon,\rho})$$
$$\leq \liminf_{i\to\infty} \frac{1}{n_i} \sum_{k=0}^{n_i-1} (\phi^*)^k (\lambda_x|_{\mathcal{L}_{x_0}\setminus L_k})(\overline{V_{g,\epsilon,\rho}}),$$

where the claim 4.2 makes the second equality hold, and

$$\mu(\overline{V_{g,\epsilon,\rho}}) = \int_\Omega \mu_\omega(\overline{V_{g,\epsilon,\rho}(\omega)})d\mathbb{P}(\omega) \geq \limsup_{i\to\infty} \frac{1}{n_i} \sum_{k=0}^{n_i-1} (\phi^*)^k (\lambda_x|_{\mathcal{L}_{x_0}\setminus L_k})(\overline{V_{g,\epsilon,\rho}})$$

Hence the equality (30) holds.

Recall that $\Xi_{g(\omega),\epsilon}(\omega) = exp_{g(\omega)}(E^{cs}_\epsilon(g(\omega),\omega))$, and the random sets $\omega \mapsto \overline{\Xi_{g,\epsilon}}(\omega)$ is a closed random set. By [13, Proposition 2.4], both $graph(\overline{V_{g,\epsilon,\rho}})$ and $graph(\overline{\Xi_{g,\epsilon}})$ are measurable subsets of $M \times \Omega$. For each $\omega \in \Omega$, we define projection map along the strong unstable leaf $\pi_\omega : \overline{V_{g,\epsilon,\rho}}(\omega) \to \overline{\Xi_{g(\omega),\epsilon}}(\omega)$ by

$$\pi_\omega(z) \in \overline{\Xi_{g(\omega),\epsilon}}(\omega) \text{ be the point such that } z \in \overline{W^{uu}_\rho(\pi_\omega(z),\omega)}. \tag{31}$$

Note that $\pi_\omega(z) = \pi_\omega(z') = y$ for any $z, z' \in \overline{W^{uu}_\rho(y,\omega)}$, $y \in \overline{\Xi_{g(\omega),\epsilon}}(\omega)$. Now for any $n \in \mathbb{N}$, $\omega \in \Omega$ and $z \in \overline{V_{g,\epsilon,\rho}}(\omega)$, we define

$$\begin{aligned} h_n(z,\omega) &= \frac{\prod_{k=0}^n \frac{1}{J^u(\phi^{-k}(z,\omega))}}{\int_{\overline{W^{uu}_\rho(\pi_\omega(z),\omega)}} \prod_{k=0}^n \frac{1}{J^u(\phi^{-k}(z',\omega))} d\lambda^{uu}_{(\pi_\omega(z),\omega)}(z')} \\ &= \frac{\prod_{k=0}^n \frac{J^u(\phi^{-k}(\pi_\omega(z),\omega))}{J^u(\phi^{-k}(z,\omega))}}{\int_{\overline{W^{uu}_\rho(\pi_\omega(z),\omega)}} \prod_{k=0}^n \frac{J^u(\phi^{-k}(\pi_\omega(z),\omega))}{J^u(\phi^{-k}(z',\omega))} d\lambda^{uu}_{(\pi_\omega(z),\omega)}(z')}, \end{aligned} \tag{32}$$

where $J^u(y,\omega) := |det(D_y f_\omega|_{E^{uu}(y,\omega)})|$ for any $(y,\omega) \in M \times \Omega$.

Claim 4.4. *The map $h_n : graph(\overline{V_{g,\epsilon,\rho}}) \to \mathbb{R}$ is measurable with respect to the $\sigma-$algebra $graph(\overline{V_{g,\epsilon,\rho}}) \cap \mathcal{B}(M) \otimes \mathcal{B}(\Omega)$.*

Proof of Claim 4.4. We define $\overline{V_{g,\epsilon,\rho}} \times \overline{\Xi_{g,\epsilon}} : \Omega \to 2^{M\times M}$ by $(\overline{V_{g,\epsilon,\rho}} \times \overline{\Xi_{g,\epsilon}})(\omega) = \overline{V_{g,\epsilon,\rho}}(\omega) \times \overline{\Xi_{g(\omega),\epsilon}}(\omega)$. We define the metric on $M\times M$ by $d_{M\times M}((x_1,x_2),(y_1,y_2)) = \max\{d_M(x_1,y_1), d_M(x_2,y_2)\}$. For any $(x_1,x_2) \in M \times M$, $r > 0$,

$$\begin{aligned} &\{\omega : \ d_{M\times M}((x_1,x_2), \overline{V_{g,\epsilon,\rho}}(\omega) \times \overline{\Xi_{g(\omega),\epsilon}}(\omega)) < r\} \\ = &\{\omega : \ d_M(x_2, \overline{\Xi_{g(\omega),\epsilon}}(\omega)) < r\} \cap \{\omega : \ d_M(x_1, \overline{V_{g,\epsilon,\rho}}(\omega)) < r\}, \end{aligned}$$

which is measurable since $\overline{V_{g,\epsilon,\rho}}$ and $\overline{\Xi_{g,\epsilon}}$ are closed random sets. Then $\overline{V_{g,\epsilon,\rho}} \times \overline{\Xi_{g,\epsilon}}$ is a closed random set by definition. We define two functions $S_1, S_2 : graph(\overline{V_{g,\epsilon,\rho}} \times \overline{\Xi_{g,\epsilon}}) \to \mathbb{R}$ by

$$S_1(z,y,\omega) = \prod_{k=0}^n \frac{1}{J^u(\phi^{-k}(z,\omega))},$$
$$S_2(z,y,\omega) = \int_{\overline{W^{uu}_\rho(y,\omega)}} \prod_{k=0}^n \frac{1}{J^u(\phi^{-k}(z',\omega))} d\lambda^{uu}_{(y,\omega)}(z'),$$

for $(z,y) \in \overline{V_{g,\epsilon,\rho}}(\omega) \times \overline{\Xi_{g(\omega),\epsilon}}(\omega)$. Note that $S_1(z,y,\omega)$ is independent of y and $S_2(z,y,\omega)$ is independent of z. They are measurable with respect to the $\sigma-$algebra

$graph(\overline{V_{g,\epsilon,\rho}} \times \overline{\Xi_{g,\epsilon}}) \cap \mathcal{B}(M \times M) \otimes \mathcal{B}(\Omega)$. So $S_1/S_2 : graph(\overline{V_{g,\epsilon,\rho}} \times \overline{\Xi_{g,\epsilon}}) \to \mathbb{R}$ is measurable.

Note that $h_n(z,\omega) = (S_1/S_2)(z, \pi_\omega(z), \omega)$. To prove the measurability of h_n, it is sufficient to prove the following map is measurable

$$\pi : graph(\overline{V_{g,\epsilon,\rho}}) \to graph(\overline{V_{g,\epsilon,\rho}} \times \overline{\Xi_{g,\epsilon}}) \text{ by } (z,\omega) \mapsto (z, \pi_\omega(z), \omega)$$

with respect to the $\sigma-$algebras $graph(\overline{V_{g,\epsilon,\rho}}) \cap \mathcal{B}(M) \otimes \mathcal{B}(\Omega)$ and $graph(\overline{V_{g,\epsilon,\rho}} \times \overline{\Xi_{g,\epsilon}}) \cap \mathcal{B}(M \times M) \otimes \mathcal{B}(\Omega)$.

For any closed set $B \subset M$, $\omega \mapsto B \cap \overline{\Xi_{g(\omega),\epsilon}}(\omega)$ is a closed random set since both $\omega \mapsto B$ and $\omega \mapsto \overline{\Xi_{g(\omega),\epsilon}}(\omega)$ are closed random sets. Then by Lemma 3.3, there are countably many measurable functions $\{c_n : \Omega \to M\}_{n\in\mathbb{N}}$ such that for all $\omega \in \Omega$,

$$B \cap \overline{\Xi_{g(\omega),\epsilon}}(\omega) = closure\{c_n(\omega) : \ n \in \mathbb{N}\}.$$

Then the following random set

$$\omega \mapsto \bigcup_{y \in B \cap \overline{\Xi_{g(\omega),\epsilon}}(\omega)} \overline{W^{uu}_\rho(y,\omega)}$$

is a closed random set since for each $\omega \in \Omega$, $\bigcup_{y \in B \cap \overline{\Xi_{g(\omega),\epsilon}}(\omega)} \overline{W^{uu}_\rho(y,\omega)}$ is closed and for any $x \in M$ fixed, $r > 0$,

$$\left\{ \omega : \ d\left(x, \bigcup_{y \in B \cap \overline{\Xi_{g(\omega),\epsilon}}(\omega)} \overline{W^{uu}_\rho(y,\omega)} \right) < r \right\} = \bigcup_{n \in N} \{\omega : \ d_M(x, W^{uu}_\rho(c_n(\omega),\omega)) < r\}$$

is measurable. Therefore, $graph(\bigcup_{y \in B \cap \overline{\Xi_{g(\omega),\epsilon}}(\omega)} \overline{W^{uu}_\rho(y,\omega)})$ is measurable. Now for any $B_1, B_2 \subset M$ closed, and $F \subset \mathcal{B}(\Omega)$,

$$\begin{aligned}
&\pi^{-1}(B_1 \times B_2 \times F \cap graph(\overline{V_{g,\epsilon,\rho}} \times \overline{\Xi_{g,\epsilon}})) \\
&= \{(z,\omega) \in graph(\overline{V_{g,\epsilon,\rho}}) | \ (z, \pi_\omega(z), \omega) \in B_1 \times B_2 \times F\} \\
&= \{(z,\omega) \in graph(\overline{V_{g,\epsilon,\rho}}) | \ (z,\omega) \in B_1 \times F\} \\
&\quad \cap \{(z,\omega) \in graph(\overline{V_{g,\epsilon,\rho}}) | \ (\pi_\omega(z),\omega) \in B_2 \times F\} \\
&= (graph(\overline{V_{g,\epsilon,\rho}}) \cap B_1 \times F) \cap (graph(\bigcup_{y \in B_2 \cap \overline{\Xi_{g(\omega),\epsilon}}(\omega)} \overline{W^{uu}_\rho(y,\omega)}) \cap M \times F)
\end{aligned}$$

is measurable. Note that $\{B_1 \times B_2 \times F : \ \ B_1, B_2 \subset M \text{ closed}, F \subset \mathcal{B}(\Omega)\}$ is a π-system generating $\mathcal{B}(M \times M) \otimes \mathcal{B}(\Omega)$. Therefore, the mapping π is measurable. The proof of Claim 4.4 is complete. $\square$

For $y \in \Xi_{g(\omega),\epsilon}(\omega)$ satisfying $W^{uu}_\rho(y,\omega) \subset f^n_{\theta^{-n}\omega}(\mathcal{L}_{x_0}(\theta^{-n}\omega) \setminus L_n(\theta^{-n}\omega))$, let $m^n_{(y,\omega)}$ be the conditional prob. measure of $(f^n_{\theta^{-n}\omega})^*((\lambda_{x_0})_{\theta^{-n}\omega}|_{\mathcal{L}_{x_0}(\theta^{-n}\omega)\setminus L_n(\theta^{-n}\omega)})$ on $W^{uu}_\rho(y,\omega)$. Then by definition, we have

$$h_n(\cdot,\omega)|_{W^{uu}_\rho(y,\omega)} = \frac{dm^n_{(y,\omega)}}{d\lambda^{uu}_{(y,\omega)}}. \tag{33}$$

By (32) and (6), h_n converges to a measurable function $h : graph(\overline{V_{g,\epsilon,\rho}}) \to (0, +\infty)$, which is defined by

$$h(z,\omega) := \frac{\prod_{k=0}^\infty \frac{J^u(\phi^{-k}(\pi_\omega(z),\omega))}{J^u(\phi^{-k}(z,\omega))}}{\int_{W^{uu}_\rho(\pi_\omega(z),\omega)} \prod_{k=0}^\infty \frac{J^u(\phi^{-k}(\pi_\omega(z'),\omega))}{J^u(\phi^{-k}(z',\omega))} d\lambda^{uu}_{(\pi_\omega(z),\omega)}(z')} \tag{34}$$

for any $(z,\omega)\in graph(\overline{V_{g,\epsilon,\rho}})$. Moreover, for each $\omega\in\Omega$, note that $J^u(\cdot,\omega)$ is a continuous function and $\phi^{-1}(\cdot,\omega)$ is continuous function, so $h_n(\cdot,\omega)$ is a continuous function on each local strong unstable leaf. Since $h_n(\cdot,\omega)\to h(\cdot,\omega)$ uniformly on each local strong unstable leaf as $n\to\infty$ and π_ω is constant on each local strong unstable leaf, $h(\cdot,\omega)$ is continuous on each local strong unstable leaf.

Recall that $(\mu_\omega|_{\overline{V_{g,\epsilon,\rho}(\omega)}})_y$ is the conditional probability measure of $\mu_\omega|_{\overline{V_{g,\epsilon,\rho}(\omega)}}$ on $\overline{W^{uu}_\rho(y,\omega)}$ for $y\in\overline{\Xi_{g(\omega),\epsilon}}(\omega)$ with respect to the measurable partition $\{\overline{W^{uu}_\rho(y,\omega)}\}$ for $y\in\overline{\Xi_{g(\omega),\epsilon}}(\omega)$, and $\widetilde{\mu_\omega|_{\overline{V_{g,\epsilon,\rho}(\omega)}}}$ is the induced measure on the quotient space $\overline{\Xi_{g(\omega),\epsilon}}(\omega)$ satisfying (22). Now we define a random measure ν on $graph(\overline{V_{g,\epsilon,\rho}})$ by

$$\nu(A)=\int_\Omega\int_{\overline{\Xi_{g(\omega),\epsilon}}(\omega)}\int_{\overline{W^{uu}_\rho(y,\omega)}\cap A(\omega)}h(z,\omega)d\lambda^{uu}_{(y,\omega)}(z)d(\widetilde{\mu_\omega|_{\overline{V_{g,\epsilon,\rho}(\omega)}}})(y)d\mathbb{P}(\omega)\tag{35}$$

for any measurable set $A\subset graph(\overline{V_{g,\epsilon,\rho}})$, where $A(\omega):=\{x\in M:\ (x,\omega)\in A\}$ is the section of A.

Step 4. *We prove* $\mu|_{\overline{V_{g,\epsilon,\rho}}}=\nu$.

Let $c: graph(\overline{V_{g,\epsilon,\rho}})\to\mathbb{R}$ be any random continuous function on $graph(\overline{V_{g,\epsilon,\rho}})$, i.e., (i) c is measurable, (ii) $c(x,\omega)$ is continuous on $x\in V_{g,\epsilon,\rho}(\omega)$, and (iii) $\int_\Omega\sup_{x\in V_{g,\epsilon,\rho}(\omega)}|c(x,\omega)|d\mathbb{P}(\omega)<\infty$. By (30) and Lemma 3.5, we have

$$\mu|_{\overline{V_{g,\epsilon,\rho}}}(c)=\lim_{i\to\infty}\left(\frac{1}{n_i}\sum_{k=0}^{n_i-1}(\phi^*)^k(\lambda_{x_0}|_{\mathcal{L}_{x_0}\setminus L_k})\right)\Big|_{\overline{V_{g,\epsilon,\rho}}}(c).\tag{36}$$

For any $z\in\overline{V_{g,\epsilon,\rho}}(\omega)$, we define

$$\bar c(z,\omega)=\int_{\overline{W^{uu}_\rho(\pi_\omega(z),\omega)}}c(z',\omega)h(z',\omega)d\lambda^{uu}_{(\pi_\omega(z),\omega)}(z').\tag{37}$$

Then $\bar c(z,\omega)$ is a random continuous function defined on $graph(\overline{V_{g,\epsilon,\rho}})$, and for each fixed $\omega\in\Omega$, $y\in\overline{\Xi_{g(\omega),\epsilon}}(\omega)$, $\bar c(\cdot,\omega)$ is constant on each $\overline{W^{uu}_\rho(y,\omega)}$ since $\pi_\omega(z)=\pi_\omega(z')=y$ for $z,z'\in\overline{W^{uu}_\rho(y,\omega)}$.

We denote $\Lambda_k(\omega):=\overline{\Xi_{g(\omega),\epsilon}}(\omega)\cap f^k_{\theta^{-k}\omega}(\mathcal{L}_{x_0}(\theta^{-k}\omega)\setminus L_k(\theta^{-k}\omega))$. Then for any random continuous function $c: graph(\overline{V_{g,\epsilon,\rho}})\to\mathbb{R}$, by (36), we have

$$\begin{aligned}\mu|_{\overline{V_{g,\epsilon,\rho}}}(c)&=\lim_{i\to\infty}\left(\frac{1}{n_i}\sum_{k=0}^{n_i-1}(\phi^*)^k(\lambda_{x_0}|_{\mathcal{L}_{x_0}\setminus L_k})\right)\Big|_{\overline{V_{g,\epsilon,\rho}}}(c)\\&=\lim_{i\to\infty}\int_\Omega\int_{\overline{V_{g,\epsilon,\rho}(\omega)}}c(z,\omega)d(\frac{1}{n_i}\sum_{k=0}^{n_i-1}(\phi^*)^k(\lambda_{x_0}|_{\mathcal{L}_{x_0}\setminus L_k})|_{\overline{V_{g,\epsilon,\rho}}})_\omega d\mathbb{P}(\omega)\\&=\lim_{i\to\infty}\frac{1}{n_i}\sum_{k=0}^{n_i-1}\int_\Omega\int_{\overline{V_{g,\epsilon,\rho}(\omega)}}c(z,\omega)d((\phi^*)^k(\lambda_{x_0}|_{\mathcal{L}_{x_0}\setminus L_k})|_{\overline{V_{g,\epsilon,\rho}}})_\omega d\mathbb{P}(\omega).\end{aligned}$$

By (33), the above

$$=\lim_{i\to\infty}\frac{1}{n_i}\sum_{k=0}^{n_i-1}\int_\Omega\sum_{y\in\Lambda_k(\omega)}\left[((\phi^*)^k(\lambda_{x_0}|_{\mathcal{L}_{x_0}\setminus L_k}))_\omega(\overline{W^{uu}_\rho(y,\omega)})\int_{\overline{W^{uu}_\rho(y,\omega)}}c(z,\omega)h_k(z,\omega)d\lambda^{uu}_{(y,\omega)}(z)\right]d\mathbb{P}.$$

By Lemma 3.7, the above

$$=\lim_{i\to\infty}\frac{1}{n_i}\sum_{k=0}^{n_i-1}\int_\Omega\sum_{y\in\Lambda_k(\omega)}\left[((\phi^*)^k(\lambda_{x_0}|_{\mathcal{L}_{x_0}\setminus L_k}))_\omega(\overline{W^{uu}_\rho(y,\omega)})\int_{\overline{W^{uu}_\rho(y,\omega)}}c(z,\omega)h(z,\omega)d\lambda^{uu}_{(y,\omega)}(z)\right]d\mathbb{P}.$$

By the definition of $\bar{c}$ (37), the above

$$= \lim_{i\to\infty} \frac{1}{n_i}\sum_{k=0}^{n_i-1}\int_\Omega \sum_{y\in\Lambda_k(\omega)} \left[((\phi^*)^k(\lambda_{x_0}|_{\mathcal{L}_{x_0}\setminus L_k}))_\omega(\overline{W^{uu}_\rho(y,\omega)}) \int_{\overline{W^{uu}_\rho(y,\omega)}} \bar{c}(z,\omega)h(z,\omega)d\lambda^{uu}_{(y,\omega)}(z)\right] d\mathbb{P}.$$

Now the same computation can be repeated for $\bar{c}$, and so

$$\mu|_{\overline{V_{g,\epsilon,\rho}}}(c) = \mu|_{\overline{V_{g,\epsilon,\rho}}}(\bar{c}) = \int_\Omega \int_{\overline{V_{g,\epsilon,\rho}(\omega)}} \bar{c}(z,\omega)d(\mu|_{\overline{V_{g,\epsilon,\rho}}})_\omega(z)d\mathbb{P}(\omega).$$

Note that $\bar{c}(z,\omega)$ is constant on each local strong unstable leaf, therefore,

$$\begin{aligned}\mu|_{\overline{V_{g,\epsilon,\rho}}}(\bar{c}) &= \int_\Omega \int_{\overline{\Xi_{g(\omega),\epsilon}(\omega)}} \int_{\overline{W^{uu}_\rho(y,\omega)}} c(z,\omega)h(z,\omega)d\lambda^{uu}_{(y,\omega)}(z)d(\widetilde{\mu|_{\overline{V_{g,\epsilon,\rho}}}})_\omega(y)d\mathbb{P}(\omega)\\ &\overset{(35)}{=} \nu(c).\end{aligned}$$

Now we have that $\mu|_{\overline{V_{g,\epsilon,\rho}}}(c) = \nu(c)$ for all random continuous function c on $graph(\overline{V_{g,\epsilon,\rho}})$, so $\mu|_{\overline{V_{g,\epsilon,\rho}}} = \nu$. We notice that $\mu|_{\overline{V_{g,\epsilon,\rho}}}$ and ν are in $Pr_{\mathbb{P}}(M\times\Omega)$. So, their disintegration $\omega \mapsto (\mu|_{\overline{V_{g,\epsilon,\rho}}})_\omega = \mu_\omega|_{\overline{V_{g,\epsilon}(\omega)}}$ and $\omega\mapsto\nu_\omega$ coincide for P-a.e. $\omega\in\Omega$ by the uniqueness of disintegration. By (23), on the one hand, we have

$$d\mu_\omega|_{\overline{V_{g,\epsilon}(\omega)}} = d(\mu_\omega|_{\overline{V_{g,\epsilon,\rho}(\omega)}})_y d\widetilde{\mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)}}, \tag{38}$$

and this disintegration corresponds to measurable partition $\{\overline{W^{uu}_\rho(y,\omega)}\}_{y\in\overline{\Xi_{g(\omega),\epsilon}}(\omega)}$. On the other hand, by (35),

$$d\nu_\omega = h(z,\omega)d\lambda^{uu}_{(y,\omega)}d\widetilde{\mu_\omega|_{\overline{V_{g,\epsilon,\rho}}(\omega)}}. \tag{39}$$

Therefore, for $\widetilde{\mu_\omega|_{\overline{V_{g(\omega),\epsilon,\rho}}(\omega)}}$-a.e. $y\in\overline{\Xi_{g(\omega),\epsilon}}(\omega)$, $(\mu_\omega|_{\overline{V_{g,\epsilon,\rho}(\omega)}})_y \ll \lambda^{uu}_{(y,\omega)}$. Thus the conditions of claim 4.3 hold. This proves that μ is a random Gibbs $u-$state. The proof of Theorem 2.3 is complete.

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