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A NEW PROOF FOR THE DANIEL-STONE THEOREM FOR RANDOM PROBABILITY MEASURES

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ABSTRACT. In this paper, we give a new proof for the Daniel-Stone theorem for random probability measures without using the result of the classical Daniel-Stone theorem.

In this paper, we present a new proof for the Daniel-Stone theorem for random probability measures which is a fundamental theorem in random dynamical systems. Throughout this paper, let $(\Omega, \mathcal{F}, P)$ be a probability space, let $(X, \mathcal{B}, d)$ be a Polish space, and let $Pr(Y)$ be the set of probability measures on a measurable space Y. We first review some basic concepts taken from [1] and [2].

Definition 1. A **random probability measure** is a measurable map $\mu : \Omega \to Pr(X)$ by $\omega \mapsto \mu_\omega$ with respect to the Borel $\sigma-$algebra of the narrow topology on $Pr(X)$, where the narrow topology on $Pr(X)$ is the smallest topology that makes $\rho \mapsto \rho(g) := \int_X g d\rho$ continuous for $\rho \in Pr(X)$ and for every bounded and continuous function g on X. We denote by $Pr_\Omega(X)$ the collection of all random probability measures.

Definition 2. A **random Lipschitz function** is a function $f : X \times \Omega \to \mathbb{R}$ satisfying the following:

- for all $x \in X$ the $x-$section $\omega \mapsto f(x, \omega)$ is measurable,
- for all $\omega \in \Omega$, the $\omega-$section $x \mapsto f(x, \omega)$ is continuous and bounded,
- $\omega \to \|f(\cdot, \omega)\|_{BL} \leq C$ for some $C \in \mathbb{R}$ $P-a.s.$ where

$$\|f(\cdot, \omega)\|_{BL} = \max\{\sup_{x \in X} |f(x, \omega)|, \sup_{x \neq y} \frac{|f(x, \omega) - f(y, \omega)|}{d(x, y)}\}.$$

We denote by $BL_\Omega(X)$ the space of all random Lipschitz functions. We will prove the following theorem.

Theorem 1 (Daniel-Stone theorem for random probability measures)**.** *Assume that X is a Polish space and $(\Omega, \mathcal{F}, P)$ is a probability space. Suppose that $L : BL_\Omega(X) \to \mathbb{R}$ satisfies the following conditions:*

- *L is linear,*
- *if $f \geq 0$ for $f \in BL_\Omega(X)$, then $L(f) \geq 0$ (i.e., L is nonnegative),*

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- *there exists* $\rho \in Pr(X)$ *such that* $L(g) = \rho(g) = \int_X g d\rho$ *for every non-random* $g \in BL(X)$, *where* $BL(X) = \{g \in BL_\Omega(X) : \ g(\cdot, \omega) \equiv g(\cdot)\ P-a.s.\}$,
- $L(h) = E(h) = \int_\Omega h dP$ *for every* $h : \Omega \to \mathbb{R}$ *essentially bounded and measurable.*

Then there exists a random measure $\mu \in Pr_\Omega(X)$ *such that* $L(f) = \mu(f)$ *for all* $f \in BL_\Omega(X)$ *where*

$$\mu(f) = \int_\Omega \int_X f(x, \omega) d\mu_\omega(x) dP(\omega).$$

The proof of this theorem is based on an observation in [2] that there is an isomorphism between $Pr_\Omega(X)$ and $Pr_P(X \times \Omega)$ in the sense of disintegration, i.e., for any $\mu \in Pr_\Omega(X)$ there exists a $\nu \in Pr_P(X \times \Omega)$ such that for $A \in \mathcal{B} \otimes \mathcal{F}$

$$\int_{X\times\Omega} \mathbb{1}_A(x, \omega) d\nu = \int_\Omega \int_X \mathbb{1}_A(x, \omega) d\mu_\omega(x) dP(\omega), \tag{1}$$

where

$$Pr_P(X \times \Omega) = \{\nu \in Pr(X \times \Omega) : \ \pi_\Omega \nu = P\},$$

and vice versa. Applying the monotone class theorem for functions [4], one can see that

$$\int_{X\times\Omega} f(x, \omega) d\nu = \int_\Omega \int_X f(x, \omega) d\mu_\omega(x) dP(\omega)$$

is true for every bounded and measurable function f, in particular, for random Lipschitz functions. Hence, to prove the Daniel-Stone theorem for random probability measures is equivalent to prove the existence of $\nu \in Pr_P(X \times \Omega)$ such that $L(f) = \nu(f)$ for all $f \in BL_\Omega(X)$.

In [2], Crauel gave a proof by showing that L satisfies the conditions of the classical Daniel-Stone theorem [3] for the case $Y = X \times \Omega$ and $I = BL_\Omega(X)$. The following is the classical Daniel-Stone theorem.

Theorem 2 (Daniel-Stone)**.** *Given vector lattice* I *of real functions on a set* Y, *containing constants, denote by* $\sigma(I)$ *the* $\sigma-$*algebra generated by* I *on* Y, *and let* L *be a pre-integral on* I *which means* $L : I \to \mathbb{R}$ *such that*

- L *is linear:* $L(cf + g) = cL(f) + L(g)$ *for all* $c \in \mathbb{R}$ *and* $f, g \in I$,
- L *is nonnegative, in the sense that whenever* $f \in I$ *and* $f \geq 0$, *then* $L(f) \geq 0$,
- $L(f_n) \downarrow 0$ *whenever* $f_n \in I$ *and* $f_n(x) \downarrow 0$ *for all* x.

Then there is a unique measure μ *on* $(Y, \sigma(I))$ *such that* $L(f) = \int f d\mu$ *for all* $f \in I$.

In the proof of the classical Daniel-Stone theorem, one first constructs a semialgebra $\mathcal{S} := \{[f, g) : [f, g) = \{(y, t) \in Y \times \mathbb{R}, f(y) \leq t < g(y)\}; f, g \in I\}$. Define a premeasure ν on $\mathcal{S}$ by $\nu([f, g)) = L(g-f)$. Then one uses the A. C. Zaanen theorem [3] to extend ν to a countable additive measure ν on the $\sigma-$algebra $\tau = \sigma(\mathcal{S})$. Let $M = \{f^{-1}((1, \infty))$ for $f \in I\}$. One uses the approximation $[0, g_n) \uparrow f^{-1}((1, \infty)) \times [0, 1)$, where $g_n := (n(f - f \wedge 1)) \wedge 1$, to prove $\{A \times [0, c); \ A \in \sigma(M), c \in \mathbb{R}\} \subseteq \tau$. Define $\mu(A) = \nu(A \times [0, 1))$ for $A \in \sigma(M)$. One shows that μ is the measure corresponding to L using simple function approximation and the monotone convergence theorem. Finally, one proves the uniqueness by showing two measures agree on M which is a $\pi-$system and then applying Dynkin's $\pi - \lambda$ theorem.

We give a new proof for the Daniel-Stone theorem for random probability measures by using the idea of the Riesz representation theorem. We first construct a pre-measure μ on a semialgebra of $X \times \Omega$ which generates the $\sigma-$algebra $\mathcal{B} \otimes \mathcal{F}$; then extend μ to a measure on $\mathcal{B} \otimes \mathcal{F}$ by the first extension theorem and the Carathéodory extension theorem [5], and finally we prove the extended μ is what we want.

We first review a result, Proposition 4.12 in [2], which shows that for any sequence $\{f_n\}_{n\in\mathbb{N}} \in BL_\Omega(X)$ with $f_n(x,w) \downarrow 0$ for all (x,w), $L(f_n) \downarrow 0$. This actually shows the monotone convergence theorem for operator L, i.e., for any $f_n, f \in BL_\Omega(X)$ with f_n increasingly (or decreasingly) convergent to f, $L(f_n)$ increasingly (or decreasingly) converges to $L(f)$. In fact, if f_n increasingly (or decreasingly) converges to f, let $g_n = f - f_n$ (or $g_n = f_n - f$); then $g_n \in BL_\Omega(X)$ and $g_n \downarrow 0$, therefore $L(g_n) \downarrow 0$, i.e., $L(f_n) \uparrow L(f)$ (or $L(f_n) \downarrow L(f)$) by the linearity of L.

Moreover, we still need the following lemma to prove the theorem.

Lemma 1. *Let f_n, f, and g be in $BL_\Omega(X)$, $|f_n(x,\omega)| \leq g(x,\omega)$, and $f_n(x,\omega) \to f(x,\omega)$ for all $(x,\omega) \in X \times \Omega$. Then $L(f_n) \to L(f)$.*

Proof. Let $l_n := \inf\{f_m : \ m \geq n\}$ in $BL_\Omega(X)$ and $u_n := \sup\{f_m : \ m \geq n\}$ in $BL_\Omega(X)$. Then $l_n \leq f_n \leq u_n$. Since $l_n \uparrow f$ and $L(l_1) \geq -L(g) > -\infty$, we have monotone convergence $L(l_n) \uparrow L(f)$. Likewise, we have monotone convergence $L(u_n) \downarrow L(f)$. Since $L(l_n) \leq L(f_n) \leq L(u_n)$, we get $L(f_n) \to L(f)$. □

Proof of Theorem 1. We divide this proof into three steps.

Step 1. We construct a pre-measure μ on $X \times \Omega$ with the semialgebra $\mathcal{S} = \{\mathcal{B} \times \mathcal{F}\}$, i.e., a function $\mu : \mathcal{S} \to [0,1]$ such that $\mu(X \times \Omega) = 1$ and μ has $\sigma-$additivity.

For $K \subset X$ closed, let $k_n(x) = 1 - \min\{nd(x,K), 1\}$. Then $k_n(x) \in BL(X)$ and $k_n(x) \downarrow 1_K(x)$. Notice that, for every $F \in \mathcal{F}$, $k_n(x)1_F(\omega) \in BL_\Omega(X)$ and $0 \leq k_n(x)1_F(\omega) \leq 1$, so $0 \leq L(k_n(x)1_F(\omega)) \leq 1$. Moreover, L is nonnegative. If $f \geq g$, then $L(f) \geq L(g)$. So $\{L(k_n(x)1_F(\omega))\}$ is a decreasing sequence. Therefore, $\lim L(k_n(x)1_F(\omega))$ exists. We define

$$\mu(K \times F) := \lim_{n\to\infty} L(k_n(x)1_F(\omega)). \tag{2}$$

For $U \subset X$ open, choose $k_n(x) \downarrow 1_{U^c}$ with $k_n \in BL(X)$ as above; then $g_n(x) := 1 - k_n(x) \uparrow 1_U(x)$ and $g_n \in BL(X)$. We define

$$\begin{aligned}\mu(U \times F) &:= P(F) - \mu(U^c \times F)\\ &= P(F) - \lim_{n\to\infty} L\left(k_n(x)1_F(\omega)\right)\\ &= \lim_{n\to\infty} L\left(g_n(x)1_F(\omega)\right).\end{aligned}$$

Now for any $B \in \mathcal{B}$, $F \in \mathcal{F}$, we prove the following equality:

$$\sup\{\mu(K \times F),\ K \subset B,\ K \text{ closed}\} = \inf\{\mu(U \times F),\ B \subset U,\ U \text{ open}\}. \tag{3}$$

First, for any $K \subset B \subset U$ with K closed and U open, we have

$$\begin{aligned}\mu(U \times F) &= P(F) - \mu(U^c \times F)\\ &\geq P(F) - \mu(\overline{K^c} \times F)\\ &\geq \mu(K \times F),\end{aligned}$$

so $\sup\{\mu(K \times F)\} \leq \inf\{\mu(U \times F)\}$. On the other hand, for any $\epsilon > 0$, choose K, U with $K \subset B \subset U$ such that $\rho(U - B) \leq \frac{\epsilon}{2}$ and $\rho(B - K) \leq \frac{\epsilon}{2}$ (existence of U, K

is guaranteed by the regularity of ρ); then $\rho(U \cap K^c) \le \epsilon$. So $\rho(U^c \cup K) \ge 1-\epsilon$. Constructing $k_n^1(x) \downarrow 1_{U^c}(x)$ and $k_n^2 \downarrow 1_K(x)$ with $k_n^i \in BL(X)$ as above, we get

$$\begin{aligned}\rho(U^c \cup K) &= \lim_{n\to\infty} \rho(k_n^1(x) + k_n^2(x)) \\ &= \lim_{n\to\infty} L(k_n^1(x) + k_n^2(x)),\end{aligned}$$

where the monotone convergence theorem is used to get the first equality. Therefore,

$$\begin{aligned}P(F) - \mu(U^c \times F) - \mu(K \times F) &= \mu((U^c \cup K)^c \times F) \\ &\le \mu((U^c \cup K)^c \times \Omega) \\ &= 1 - \mu((U^c \cup K) \times \Omega) \\ &= 1 - \lim_{n\to\infty} L((k_n^1(x) + k_n^2(x))1_\Omega) \\ &= 1 - \lim_{n\to\infty} L(k_n^1(x) + k_n^2(x)) \\ &\le 1 - \rho(U^c \cup K) \\ &\le \epsilon.\end{aligned}$$

Consequently,

$$\begin{aligned}\inf\{\mu(U \times F)\} &= \inf\{P(F) - \mu(U^c \times F)\} \\ &\le \mu(K \times F) + \epsilon \le \sup\{\mu(K \times F)\} + \epsilon.\end{aligned}$$

Let $\epsilon \to 0$, $\inf\{\mu(U \times F)\} \le \sup\{\mu(K \times F)\}$. Hence, (3) holds. For all $B \times F \in \mathcal{S}$, define μ on $\mathcal{S}$ by

$$\begin{aligned}\mu(B \times F) &:= \sup\{\mu(K \times F),\ K \subset B,\ K \text{ closed}\} \\ &= \inf\{\mu(U \times F),\ B \subset U,\ U \text{ open}\}.\end{aligned} \tag{4}$$

Step 2. We prove μ is a pre-measure on $\mathcal{S}$, i.e., $\mu(X \times \Omega) = 1$ and $\mu(E) = \sum_{i=1}^\infty \mu(E_i)$ for any $E, E_i \in \mathcal{S}$ with E_i disjoint such that $E = \bigcup_{i=1}^\infty E_i$.

By definition of μ (see (2) and (4)). Choosing $k_n(x) = 1 - \min\{nd(x, X), 1\} = 1$, we obtain

$$\mu(X \times \Omega) = \lim_{n\to\infty} L(k_n \cdot \mathbb{1}_\Omega) = L(\mathbb{1}_\Omega) = E(\mathbb{1}_\Omega) = 1.$$

Next, check the $\sigma-$additivity of μ. First, we prove the case $E_i = U_i \times F_i$ where E_i are disjoint and U_i are open. There exists a sequence of functions $\{g_j^i\}_{j=1}^\infty$ in $BL_\Omega(X)$ such that $g_j^i \uparrow 1_{U_i \times F_i}$ as $j \to \infty$. Therefore, $\sum_{i=1}^\infty g_j^i \uparrow 1_E$ with $\sum_{i=1}^\infty g_j^i \in BL_\Omega(X)$ since the supports of g_j^i are disjoint for each i different. By (2), we have

$$\lim_{j\to\infty} L(g_j^i) = \mu(E_i).$$

We choose j_1 such that

$$|L(g_{j_1}^i) - \mu(E_i)| < \frac{1}{2^i},$$

and choose j_2 such that

$$|L(g_{j_2}^i) - \mu(E_i)| < \frac{1}{2^{i+1}},$$

and so on; then we get a subsequence $\{g_{j_{l(i)}}^i\}$ such that

$$|L(g_{j_l}^i) - \mu(E_i)| < \frac{1}{2^{i+l-1}}.$$

Thus, $\sum_{i=1}^{\infty} g^i_{j_{l(i)}}(x,w) \uparrow \mathbb{1}_E$ as l tends to infinity. Consequently,

$$\mu(E) = \lim_{l\to\infty} L(\sum_{i=1}^{\infty} g^i_{j_{l(i)}}(x,w)). \tag{5}$$

Using the monotone convergence theorem, we obtain

$$\begin{aligned}\lim_{l\to\infty} L(\sum_{i=1}^{\infty} g^i_{j_{l(i)}}(x,w)) &= \lim_{l\to\infty}\lim_{n\to\infty} L(\sum_{i=1}^{n} g^i_{j_{l(i)}}(x,w))\\ &= \lim_{l\to\infty}\lim_{n\to\infty} \sum_{i=1}^{n} L(g^i_{j_{l(i)}}(x,w))\\ &= \lim_{l\to\infty} \sum_{i=1}^{\infty} L(g^i_{j_{l(i)}}(x,w)).\end{aligned}$$

Next, we will prove that the following equality holds:

$$\lim_{l\to\infty} \sum_{i=1}^{\infty} L(g^i_{j_{l(i)}}(x,w)) = \sum_{i=1}^{\infty} \lim_{l\to\infty} L(g^i_{j_{l(i)}}(x,w)), \tag{6}$$

i.e., for any $\epsilon > 0$, there exists an $N > 0$ such that for any $l > N$,

$$|\sum_{i=1}^{\infty} L(g^i_{j_{l(i)}}(x,w)) - \sum_{i=1}^{\infty} \lim_{l\to\infty} L(g^i_{j_{l(i)}}(x,w))| < \epsilon.$$

Choose a sufficiently large N such that $\frac{1}{2^N} < \epsilon$; then for any $l > N+1$, we have

$$\begin{aligned}&|\sum_{i=1}^{\infty} L(g^i_{j_{l(i)}}(x,w)) - \sum_{i=1}^{\infty} \lim_{l\to\infty} L(g^i_{j_{l(i)}}(x,w))|\\ &\leq \sum_{i=1}^{\infty} |L(g^i_{j_{l(i)}}(x,w)) - \lim_{l\to\infty} L(g^i_{j_{l(i)}}(x,w))|\\ &= \sum_{i=1}^{\infty} |L(k^i_{j_{l(i)}}(x,w)) - \mu(E_i)|\\ &\leq \sum_{i=1}^{\infty} \frac{1}{2^{i+l-1}}\\ &= \frac{1}{2^{l-1}}\\ &< \epsilon.\end{aligned}$$

Hence (6) holds and (5) and (6) imply

$$\begin{aligned}\mu(E) &= \lim_{l\to\infty} \sum_{i=1}^{\infty} L(g^i_{j_{l(i)}}(x,w))\\ &= \sum_{i=1}^{\infty} \lim_{l\to\infty} L(g^i_{j_{l(i)}}(x,w))\\ &= \sum_{i=1}^{\infty} \mu(E_i).\end{aligned} \tag{7}$$

Therefore, $\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$ for the case $E_i = U_i \times F_i$ where E_i are disjoint and U_i are open.

Second, for the case $E_i = K_i \times F_i$ where E_i are disjoint and K_i are closed we have

$$\begin{aligned}
\mu(E) &= \mu\left(\bigcup_{i=1}^{\infty} K_i \times F_i\right) \\
&= \mu\left(\bigcup_{i=1}^{\infty}(X \times F_i - K_i^c \times F_i)\right) \\
&= \mu\left(\bigcup_{i=1}^{\infty} X \times F_i - \bigcup_{i=1}^{\infty} K_i^c \times F_i\right) \\
&= \mu\left(\bigcup_{i=1}^{\infty} X \times F_i\right) - \mu\left(\bigcup_{i=1}^{\infty} K_i^c \times F_i\right) \\
&= \sum_{i=1}^{\infty} P(F_i) - \sum_{i=1}^{\infty} \mu(K_i^c \times F_i) \\
&= \sum_{i=1}^{\infty} \mu(K_i \times F_i) \\
&= \sum_{i=1}^{\infty} \mu(E_i),
\end{aligned}$$

where the fourth equality holds by the linearity of L.

Now, for $E = B \times F \in \mathcal{S}$, $E_i = B_i \times F_i \in \mathcal{S}$ where E_i are disjoint and $E = \bigcup_{i=1}^{\infty} E_i$. We first prove that the following equality holds:

$$\begin{aligned}
&\sum_{i=1}^{\infty} \sup\{\mu(K_i \times F_i),\ K_i \subset B_i \text{ with } K_i \text{ closed}\} \\
&= \sup\{\sum_{i=1}^{\infty} \mu(K_i \times F_i),\ K_i \subset B_i \text{ with } K_i \text{ closed}\}.
\end{aligned} \tag{8}$$

For each K_i satisfing the above condition, we have

$$\sum_{i=1}^{\infty} \sup\{\mu(K_i \times F_i)\} \geq \sum_{i=1}^{\infty} \mu(K_i \times F_i).$$

On the other hand, for any $\epsilon > 0$, choose K_i such that

$$\mu(K_i \times F_i) \geq \sup\{\mu(K_i \times F_i)\} - \frac{\epsilon}{2^i}.$$

Then,

$$\sum_{i=1}^{\infty} \mu(K_i \times F_i) \geq \sum_{i=1}^{\infty} \left[\sup\{\mu(K_i \times F_i)\} - \frac{\epsilon}{2^i}\right] = \sum_{i=1}^{\infty} \sup\{\mu(K_i \times F_i)\} - \epsilon.$$

Thus,

$$\sum_{i=1}^{\infty} \sup\{\mu(K_i \times F_i)\} - \epsilon \leq \sup \sum_{i=1}^{\infty} \mu(K_i \times F_i).$$

Letting $\epsilon \to 0$ above, we get (8). By (4), (7), and (8), we have

$$\begin{aligned}\sum_{i=1}^{\infty} \mu(E_i) &= \sum_{i=1}^{\infty} \sup\{\mu(K_i \times F_i),\ K_i \subset B_i \text{ with } K_i \text{ closed}\} \\ &= \sup\{\sum_{i=1}^{\infty} \mu(K_i \times F_i),\ K_i \subset B_i \text{ with } K_i \text{ closed}\} \\ &= \sup\{\mu(\bigcup_{i=1}^{\infty} K_i \times F_i),\ K_i \subset B_i \text{ with } K_i \text{ closed}\} \\ &= \mu(E).\end{aligned}$$

Therefore, $\mu : \mathcal{S} \to [0,1]$ is $\sigma-$additive on $\mathcal{S}$ and satisfies $\mu(X \times \Omega) = 1$. Applying the first extension theorem and the Carathéodory extension theorem [5], we get a unique probability measure on $\sigma(S) = \mathcal{B} \otimes \mathcal{F}$ that extends μ, which we denote by μ again.

Step 3. We finish this proof by proving $\mu \in Pr_P(X \times \Omega)$ and $L(f) = \mu(f)$ for all $f \in BL_\Omega(X)$.

For any $F \in \mathcal{F}$,

$$\pi_X \mu(F) = \mu(X \times F) = \lim L(1 \cdot 1_F(\omega)) = E(1_F(\omega)) = P(F),$$

so $\mu \in Pr_P(X \times \Omega)$.

Now for any $f \in BL_\Omega(X)$, f is jointly measurable. For any $n \in N$, define

$$C^{n,l} := \{(x,\omega), \frac{l}{2^n} < f(x,\omega) \le \frac{l+1}{2^n}\},\ -\infty < l < \infty.$$

Note that $C^{n,l} \in \mathcal{B} \otimes \mathcal{F} = \sigma(\mathcal{S}) = \sigma(\mathcal{A}(\mathcal{S}))$, where $\mathcal{A}(\mathcal{S})$ is the algebra generated by $\mathcal{S}$. Since $\sup_{x \in X} |f(x,\omega)| \le C$ P-a.s., only finitely many $C^{n,l}$ are nonempty sets. Let

$$f_n(x,\omega) = \sum_{l \in \mathbb{N}, \text{finite}} \frac{l+1}{2^n} 1_{C^{n,l}}(x,\omega).$$

Then $f_n(x,\omega) \downarrow f(x,\omega)$.

For $C^{n,l} \in \mathcal{B} \otimes \mathcal{F}$, according to the proof of the first extension theorem and the Carathéodory extension theorem, we have

$$\mu(C^{n,l}) = \inf\{\sum_{i=1}^{\infty} \mu(E_i^{n,l});\ C^{n,l} \subset \bigcup_{i=1}^{\infty} E_i^{n,l},\ E_i^{n,l} \in \mathcal{A}(\mathcal{S}) \text{ are disjoint}\} \tag{9}$$

and

$$E_i^{n,l} = \bigcup_{j \text{ finite}} S_j^{n,l,i},\ S_j^{n,l,i} \in \mathcal{S} \text{ are disjoint.} \tag{10}$$

In view of definition (4), $\mu(B \times F) = \inf\{\mu(U \times F),\ B \subset U,\ U \text{ open}\}$. Choosing $g_k^{n,l,i,j}(x,\omega) \in BL_\Omega(X)$ such that $g_k^{n,l,i,j}(x,\omega) \to 1_{S_j^{n,l,i}}$ as $k \to \infty$ and $\mu(S_j^{n,l,i}) = \lim_{k\to\infty} L(g_k^{n,l,i,j})$, we obtain

$$g_k^{n,l,i} := \sum_{j \text{ finite}} g_k^{n,l,i,j} \in BL_\Omega(X) \to 1_{E_i^{n,l}} \text{ as } k \to \infty.$$

By the monotone convergence theorem and Fubini's theorem, we get

$$\begin{aligned}\sum_{i=1}^{\infty}\mu(E_i^{n,l}) &= \sum_{i=1}^{\infty}\lim_{k\to\infty}L(g_k^{n,l,i})\\ &= \lim_{k\to\infty}\sum_{i=1}^{\infty}L(g_k^{n,l,i})\\ &= \lim_{k\to\infty}L(\sum_{i=1}^{\infty}g_k^{n,l,i})\\ &:= \lim_{k\to\infty}L(g_k^{n,l}),\end{aligned}$$

where $g_k^{n,l} := \sum_{i=1}^{\infty} g_k^{n,l,i}$ which is still in $BL_\Omega(X)$ since $E_i^{n,l}$ are disjoint. Hence (9) becomes $\mu(C^{n,l}) = \inf\{\lim_{k\to\infty} L(g_k^{n,l})\}$. Applying diagonalization, we have a sequence of functions in $BL_\Omega(X)$, denoted by $g_k^{n,l}$ again, such that

$$\mu(C^{n,l}) = \lim_{k\to\infty} L(g_k^{n,l}). \tag{11}$$

By (11), we have

$$\begin{aligned}\mu(f_n) &= \sum_{l\in\mathbb{N},\text{finite}}\frac{l}{2^n}\mu(C_n^l)\\ &= \sum_{l\in\mathbb{N},\text{finite}}\frac{l}{2^n}\lim_{k\to\infty}L(g_k^{n,l})\\ &= \lim_{k\to\infty}L(\sum_{l\in\mathbb{N},\text{finite}}\frac{l}{2^n}g_k^{n,l})\\ &:= \lim_{k\to\infty}L(g_k^n),\end{aligned}$$

where

$$g_k^n = \sum_{l\in\mathbb{N},\text{finite}}\frac{l}{2^n}g_k^{n,l} \in BL_\Omega(X) \to f_n.$$

Now, we have $g_k^n \to f_n$ and $f_n(x) \to f$. Applying diagonalization again, we have a sequence of functions in $BL_\Omega(X)$, denoted by g_n, such that $g_n \to f$. Moreover, we can see that in the construction of g_n, g_n is bounded by the bound of f. Finally, using the monotone convergence theorem and Lemma 1, we conclude that

$$\mu(f) = \lim_{n\to\infty}\mu(f_n) = \lim_{n\to\infty}L(g_n) = L(f). \tag{12}$$

Thus, μ is the desired measure.

□

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