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Random semi-horseshoe for partially hyperbolic systems driven by an external force

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Abstract

In this paper, we consider the symbolic representation of systems driven by an invertible measurable mapping on a measurable space. We prove that when such a system possesses an equicontinuous uniformly expanding subbundle, then it has a subsystem of its iterations randomly semi-conjugating to the full shift on two symbols. Examples such as random perturbation of partially hyperbolic systems, random composition of partially hyperbolic automorphisms on 3-d tori with a fixed central direction, and fiber partially hyperbolic maps on 3-d tori without stable subbundles are under consideration.

1. Introduction

1.1. Background and motivation

To describe the chaotic behaviors of dynamical systems from a geometrical perspective, people relate the considered systems with the symbolic dynamical systems via conjugation or semi-conjugation. There is extensive literature on this subject since the work of Hadamard and Morse [37] for certain geodesic flows. Smale introduced the Smale horseshoe in [40], and proved its existence in the presence of the transverse homoclinicity. Sinai [39] and Bowen [7] coded the Anosov and Axiom A diffeomorphisms to the subshift of finite type by using Markov partitions. It is well known that Smale horseshoes relate to the entropy, including measure-theoretic entropy introduced by Kolmogorov [26] in 1958 and topological entropy introduced by Adler *et al* [1] in 1965. Particularly, if a system has a Smale horseshoe, then its topological entropy must be positive. It is natural to ask the vice versa, whether the positivity of topological entropy implies the existence of Smale horseshoe? Since Smale horseshoe is topologically conjugate to a full left shift on finite symbols, a dynamical system containing a Smale horseshoe should have uncountably many pairwise disjoint minimal subsets, see [18], lemma 2.3. However, a real analytic minimal system with positive topological entropy was constructed by Herman [16]. Therefore, the positive topological entropy is not sufficient to imply the existence of Smale horseshoe. In [23], Katok proved that for a $C^{1+\alpha}$ $(\alpha > 0)$ diffeomorphism on a connected compact smooth manifold with a hyperbolic ergodic measure, the positive entropy implies the existence of Smale horseshoes. Thereafter, Katok's result was extended to C^2 differentiable nonuniformly hyperbolic maps and semiflows in a separable Hilbert space [28, 29] and in a separable Banach space [27, 35].

When one considers a more general situation, systems driven by an external force come into the picture. Let $\theta: \Omega \to \Omega$ be an invertible measurable mapping on a measurable space $(\Omega, \mathscr{F})$, which describes the external force. A dynamical system on a connected smooth compact Riemannian manifold M driven by an external force $(\Omega, \mathscr{F}, \theta)$ is a measurable mapping

$$\varphi: \mathbb{Z} \times \Omega \times M \to M, \ (n, \omega, x) \mapsto \varphi(n, \omega)(x)$$

satisfying the cocycle property, i.e.

$$\varphi(0, \omega) = id_M \text{ for all } \omega \in \Omega,$$
$$\varphi(n + m, \omega) = \varphi(n, \theta^m\omega)\circ\varphi(m, \omega) \text{ for all } n, m \in \mathbb{Z}, \omega \in \Omega.$$

Denote f_ω to be the time-one map $\varphi(1, \omega)$, which is called the generator of the dynamical systems driven by an external force φ in the following sense

$$\varphi(n, \omega) = f_\omega^n := \begin{cases} f_{\theta^{n-1}\omega} \circ \cdots \circ f_\omega, & \text{if } n > 0; \\ id_M, & \text{if } n = 0; \\ f^{-1}_{\theta^n\omega} \circ \cdots \circ f^{-1}_{\theta^{-1}\omega}, & \text{if } n < 0. \end{cases} \tag{1.1}$$

See analogous definition of generator of a dynamical system in Chap. 2 of [2].

On the one hand, for a given dynamical system driven by an external force φ, the mapping

$$(\omega, x) \mapsto (\theta^n\omega, f_\omega^n x) := \phi^n(\omega, x), \forall n \in \mathbb{Z} \tag{1.2}$$

is measurable on $\mathscr{F} \otimes \mathcal{B}(M)$. Therefore, the iteration of mapping ϕ on $\Omega \times M$ forms a skew product system $(\Omega \times M, \phi)$. On the other hand, given the skew product system $(\Omega \times M, \phi)$ over θ, there is a corresponding system φ driven by an external force θ. For convenience, we consider the skew product system $(\Omega \times M, \phi)$ over θ and the dynamical systems φ driven by an external force θ as two synonymous concepts throughout this paper.

When $(\Omega, \mathscr{F})$ is equipped with a θ-invariant probability measure $\mathbb{P}$, φ is called a random dynamical system (abbr. RDS)[2]. In the scope of RDS, Bogenschütz and Gundlach [4] introduced random k-shift and random subshifts of finite type. These definitions are different from the classical ones as the number of symbols is a random variable. Gundlach and Kifer in [15] proved that there exists a random semi-conjugation between random subshifts of finite type and random hyperbolic invariant set.

If the driving force θ: $\Omega \to \Omega$ is aperiodic, then the skew product $(\Omega \times M, \phi)$ over θ has no periodic orbit. Therefore, the skew product $(\Omega \times M, \phi)$ has no Smale horseshoe due to the fact that the Smale horseshoe has abundance of periodic orbits. To overcome this obstacle, Huang, Lian, Lu in [19] considered the random periodic orbits in skew product systems that are Anosov and topological mixing on fibers. They proved the denseness of random periodic orbits and the existence of a strong version of Smale horseshoe for such systems.

All results for the existence of horseshoes or horseshoe-like structures we mentioned above (both in deterministic systems and systems driven by an external force) heavily rely on the hyperbolic structure of the system. In this paper, we endeavor to investigate the following question:

Are there any horseshoe-like structures for more kinds of systems driven by an external force without the presence of hyperbolic structure?

One step towards this question is [20], in which Huang and Lu considered an injective infinite-dimensional continuous random dynamical system φ on a Polish space X over an ergodic Polish system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$, and a φ-invariant compact random set $\omega \to K(\omega)$. If (K, φ) has positive topological entropy, then it has a weak horseshoe of two symbols. In [20], a random dynamical system (K, ϕ) is said to have a weak horseshoe of two symbols if one can find two subsets $U_0, U_1 \subset X$ such that the followings hold:

- U_0, U_1 are nonempty, bounded, closed and $d(U_0, U_1) > 0$, where d is the metric on X;
- There exist a constant $b > 0$ and $M_{b,\omega} \in \mathbb{N}$ defined for $\mathbb{P}$-a.e. $\omega \in \Omega$ such that for any natural number $m \geqslant M_{b,\omega}$, there is a subset $J_m \subset \{0, 1, \ldots, m\}$ satisfying: (a) $\#|J_m| \geqslant bm$ (positive density), where $\#|\cdot|$ is the cardinality of a subset of $\mathbb{Z}$; (b) for any $s \in \{0, 1\}^{J_m}$, there exists an $x_s \in K(\omega)$ with $\varphi(j, \omega)(x_s) \in U_{s(j)} \cap K(\theta^j\omega)$ for any $j \in J_m$.

The weak horseshoe is an extension of Smale's horseshoe and relates to a series of works in the study of topological dynamical systems, see [14] for symbolic dynamics, [21] for topological dynamical systems, and [25] for C^* dynamics.

Recently in [18], the authors considered analogous phenomena in partially hyperbolic systems. By only assuming the existence of a uniformly expanding direction for C^1 diffeomorphisms attractor, they proved that there is a topological semi-conjugation between a subsystem of the considered system and the full shift on two symbols. Such a subsystem is called semi-horseshoe (also named by horseshoe factors in [11] and topological horseshoes in [24]), and it is proved to be equivalent to the periodic weak horseshoe for deterministic dynamical systems.

In this paper, inspired by [15, 18, 20], we prove the following statement: If a system driven by an external force φ has an equicontinuous uniformly expanding subbundle, then it has a random semi-horseshoe, which means the system has a subsystem randomly semi-conjugating to the full shift of two symbols.

1.2. Setting and Results

Throughout this paper, we let (M,g) be a connected smooth compact m-dimensional Riemannian manifold with Riemannian metric g, and θ: $\Omega \mapsto \Omega$ be an invertible measurable mapping on a measurable space $(\Omega, \mathscr{F})$. The norm of a tangent vector $v \in T_xM$ for $x \in M$ induced by the given Riemannian metric is denoted by

$\|v\| = \|v\|_x = \sqrt{g_x\langle v, v\rangle}$. Denote d_M to be the metric on M induced by the Riemannian metric and $\mathcal{B}(M)$ to be the Borel σ-algebra on M. Denote $\mathrm{Diff}^1(M)$ to be the space of C^1 diffeomorphisms on M, which is equipped with the C^1-topology [17]. We assume that f_ω, the generator of the φ in (1.1), belongs to $\mathrm{Diff}^1(M)$ and $f\colon \Omega \to \mathrm{Diff}^1(M)$ by $\omega \mapsto f_\omega$ is a measurable mapping with respect to the σ-algebra generated by the C^1-topology on $\mathrm{Diff}^1(M)$.

Definition 1.1. In this paper, we consider the following type of systems: (**EE**) The system $(\Omega \times M, \phi)$ is called an (**EE**)-system if it satisfies the following two conditions:

(a) it has an **invariant uniformly expanding subbundle**: for every $(\omega, x) \in \Omega \times M$, there is an $m'-$dimensional subspace $E^u(\omega, x) \subset T_xM$ on the fiber $\{\omega\} \times M$ such that $E^u(\omega, x)$ measurably depends on $(\omega, x) \in \Omega \times M$, and satisfies

$$D_x f_\omega(E^u(\omega, x)) = E^u(\theta\omega, f_\omega x), \forall(\omega, x) \in \Omega \times M,$$

and

$$\|D_x f_\omega^n \xi\| \geqslant C^{-1}e^{\lambda n}\|\xi\|, \forall \xi \in E^u(\omega, x), n \in \mathbb{Z}^+, \tag{1.3}$$

where $m' \in (0, m) \cap \mathbb{Z}$, $\lambda > 0$ and $C > 1$ are constants.

(b) $E^u(\omega, x)$ is **equicontinuous** with respect to $(\omega, x) \in \Omega \times M$ in the following sense: for any given $x \in M$ and $\epsilon > 0$, there exists a $\delta > 0$ such that if $d_M(x, y) < \delta$, then

$$\mathrm{dist}(E^u(\omega, x), E^u(\omega, y)) < \epsilon \, for all \omega \in \Omega,$$

where $\mathrm{dist}(\cdot,\cdot)$ is the distance defined on the unstable subbundle (refer to section 3.3 for the complete definition).

Remark 1.1. The assumption of equicontinuity on the invariant uniformly expanding subbundle was inspired by [32], in which they proved that the invariant expanding subbundle for random perturbations of Anosov system is equi-Hölder continuous. Similarly, one can prove that the invariant expanding subbundle for random perturbations of partially hyperbolic system is equi-Hölder continuous (see Example 2.3). More examples of the (**EE**)-systems can be found in section 2, including fiber partially hyperbolic maps on 3-dimensional tori without stable subbundles, and random composition of partially hyperbolic automorphisms on 3-dimensional torus with a fixed central direction. Further discussion on the necessity of equicontinuity assumption is placed in section 1.3.

A set valued map $C\colon \Omega \to 2^M$ is said to be a compact random set if: for each $\omega \in \Omega$, $C(\omega)$ is a nonempty compact subset of M; and for each $x \in M$, the map $\omega \mapsto d_M(x, C(\omega))$ is measurable. If $\omega \mapsto C(\omega)$ is a compact random set, then by proposition 2.4 in [12] (see proposition A.1), $graph(C) = \{(\omega, x) \in \Omega \times M\colon x \in C(\omega)\}$ is a measurable subset of $\Omega \times M$ and it is equipped with the restricted σ-algebra $graph(C) \cap (\mathscr{F} \otimes \mathcal{B}(M))$. For some number $N \in \mathbb{N}$, a compact random set $C\colon \Omega \to 2^M$ is called ϕ^N-invariant if $f_\omega^N C(\omega) = C(\theta^N\omega)$ for all $\omega \in \Omega$, i.e. $\phi^N(graph(C)) = graph(C)$.

Let $\{0, 1\}^{\mathbb{Z}}$ with the left shift σ be the symbolic dynamical systems on two symbols. We take the product topology on $\{0, 1\}^{\mathbb{Z}}$ so that $\{0, 1\}^{\mathbb{Z}}$ is a compact space with a basis given by the cylinders

$$C^{n_1,\ldots,n_k}_{j_1,\ldots,j_k} = \{s \in \{0, 1\}^{\mathbb{Z}}\colon s(n_i) = j_i \, for \, i = 1,\ldots,k\},$$

where $n_1 < n_2 < \cdots < n_k$ are integers and $j_1,\ldots,j_k \in \{0, 1\}$. Denote $\mathcal{B}(\{0, 1\}^{\mathbb{Z}})$ to be the Borel σ-algebra on $\{0, 1\}^{\mathbb{Z}}$ generated by the product topology.

Definition 1.2 Random semi-horseshoe. The system $(\Omega \times M, \phi)$ is said to have a **random semi-horseshoe** if there exists a number $N \in \mathbb{N}$ and a ϕ^N-invariant compact random set $Y\colon \Omega \to 2^M$ such that $\phi^N\colon graph(Y) \to graph(Y)$ is random semi-conjugate to the left shift σ on $\{0, 1\}^{\mathbb{Z}}$, i.e. the following two properties are satisfied:

(SH1) There exists a family of surjective and continuous maps $\{\pi_\omega\colon Y(\omega) \to \{0, 1\}^{\mathbb{Z}}\}_{\omega\in\Omega}$ such that

$$\pi_{\theta^N\omega} \circ f_\omega^N = \sigma \circ \pi_\omega \text{ for all } \omega \in \Omega;$$

(SH2) the family of maps is measurable in the following sense: the mapping

$$\pi\colon (graph(Y),\ graph(Y)\cap(\mathscr{F}\otimes\mathcal{B}(M))) \to (\Omega\times\{0,1\}^{\mathbb{Z}},\ \mathscr{F}\otimes\mathcal{B}(\{0,1\}^{\mathbb{Z}}))$$
$$(\omega, x) \mapsto (\omega, \pi_\omega x)$$

is measurable.

Remark 1.2. When the driving force $(\Omega, \mathcal{F}, \theta)$ has a periodic point $\tilde{\omega}$ with period N, the subsystem $(\{\tilde{\omega}\}\times M, \phi^N)$ coincides with a deterministic dynamical system $(M, \tilde{f})$. Therefore, the problem of looking for horseshoe-like structure on $(\{\tilde{\omega}\}\times M, \phi^N)$ degenerates to the case Huang, *et al* considered in [18]. The random semi-conjugation defined in definition 1.2 is similar to the random semi-conjugation defined in [15] (see Example 2.5 in [15] for the definition of random conjugation and section 3 in [15] for the definition of random semi-conjugation). To be specific, when $(\Omega, \mathscr{F})$ is equipped with a θ-invariant probability measure $\mathbb{P}$ and (SH1) in definition 1.2 holds for $\mathbb{P}$-a.s. $\omega\in\Omega$ instead of all $\omega\in\Omega$, these two definitions of random semi-conjugation coincide.

The main result of this paper is that the chaotic behavior, random semi-horseshoe, can be observed in a class of systems driven by an external force.

Theorem 1. *Any (**EE**)-system $(\Omega\times M, \phi)$ has a random semi-horseshoe.*As a direct application of the above theorem, we have the following corollary.

Corollary 1.1. *For any system $(\Omega\times M, \phi)$ generated by f_ω over an external force $(\Omega, \mathscr{F}, \theta)$ such that $\theta\colon\Omega\to\Omega$ is a homeomorphism on a compact metric space (Ω, d_Ω) and $f\colon\Omega\to\mathrm{Diff}^1(M)$ is continuous, if there is an invariant uniformly expanding subbundle E^u (as in definition 1.1) continuously depending on $(\omega, x)\in\Omega\times M$, then $(\Omega\times M, \phi)$ has a random semi-horseshoe.*

Remark 1.3. The equicontinuity of E^u directly follows from the continuity of E^u and the compactness of $\Omega\times M$. The system described in corollary 1.1 has been studied in [19, 34], and such systems exhibit complicated dynamical behaviour both from the topological perspective and statistical perspective.

1.3. Technical summary

There are mainly three steps to proceed the proof of theorem 1. Firstly, we generalize the idea of weak horseshoe of two symbols in [20] to random weak horseshoe. The system $(\Omega\times M, \phi)$ has a random weak horseshoe if there are interpolating sets $J\subset\mathbb{Z}$ with positive density and two separated compact random sets $\omega\mapsto K_i(\omega)$, $i=0,1$, such that for any $s\in\{0,1\}^J$ and $\omega\in\Omega$, there is a point $x_{s,\omega}\in M$ satisfying that $f_\omega^j(x_s)\in K_{s(j)}(\theta^j\omega)$ for all $j\in J$. See section 4 for the precise definition. The main differences between the random weak horseshoe and full (or weak) horseshoe of two symbols in [20] (we mentioned in section 1.1) are:

(i) K_0, K_1 are compact random sets rather than constant subsets of the phase space;

(ii) the interpolating set is two-sided rather than one-sided;

(iii) the horseshoe phenomenon can be found on all $\omega\in\Omega$ rather than almost everywhere with respect to a given θ-invariant probability measure.

In particular, if $J\supset N\mathbb{Z}$ for some $N\in\mathbb{N}$, then the system $(\Omega\times M, \phi)$ is said to have a periodic random weak horseshoe. In section 5, we will show that if the system $(\Omega\times M, \phi)$ has a periodic random weak horseshoe, then it has a random semi-horseshoe. Therefore, it is sufficient to prove the existence of periodic random weak horseshoe.

To this end, we introduce Property 1 in section 4.1. Generally, such property says that there are a positive constant τ corresponding to a given $\delta>0$, and two τ-separated compact random sets $V_{0,\delta}$ and $V_{1,\delta}$, such that for any $(\omega, x)\in\Omega\times M$, there are curves starting from x and reaching $V_{0,\delta}(\omega)$ and $V_{1,\delta}(\omega)$ along the uniformly expanding direction within the given short distance δ respectively. With the benefit of an adapted version of Brin's result in [8] (see lemma 3.1) and the equicontinuity of the uniformly expanding subbundle, we can prove that any (**EE**)-system $(\Omega\times M, \phi)$ has Property 1.

With the help of Property 1, we can prove that for any (**EE**)-system, given $s\in\{0,1\}^{\mathbb{Z}}$ and $\omega\in\Omega$, there is a point $x_{s,\omega}$ whose orbit $\{f_\omega^{kN}(x_{s,\omega})\}_{k\in\mathbb{Z}}$ on the interpolating set $J=N\mathbb{Z}$ is shadowing $\{V_{s(k),\delta}(\theta^{Nk}\omega)\}_{k\in\mathbb{Z}}$ on each fiber within given precision by allowing N sufficiently large. The desired two separated compact random sets K_0 and K_1 in the definition of periodic random weak horseshoe are just the closed neighborhood of $V_{0,\delta}$ and $V_{1,\delta}$ respectively. In this way, the existence of periodic random weak horseshoe is proved.

Besides, one may ask whether the existence of random semi-horseshoe is equivalent to the existence of periodic random weak horseshoe. In section 5, we also show that if the σ-algebra $\mathscr{F}$ on Ω is universally complete, then the existence of random semi-horseshoe implies the existence of periodic random weak horseshoe.

Remark 1.4.

(i) If we remove the equicontinuity assumption of $E^u(\omega, x)$ in definition 1.1, then it has trouble in constructing the two compact random sets in constructing the periodic random weak horseshoe. To be specific, the measurability of the subsets $V_{0,\delta}$ and $V_{1,\delta}$, which we mentioned above, cannot be reached. By the discussion in remark 1.3, the equicontinuity assumption can be removed if further assume that the randomness is driven by a homeomorphism on a compact metric space and $E^u(\omega, x)$ is continuously depending on $(\omega, x) \in \Omega \times M$.

(ii) If there is a **dominated splitting**, and the generator $f: \Omega \mapsto \mathrm{Diff}^2(M)$ satisfies that $\sup_{\omega\in\Omega} \|f_\omega\|_{C^2} < \infty$, our method can be adapted to prove the existence of random semi-horse. By **dominated splitting**, we mean that for every $(\omega, x) \in \Omega \times M$, there is an invariant measurable splitting of the tangent space $T_x M = E^u(\omega, x) \oplus E^c(\omega, x)$, and satisfies

(1) for all $(\omega, x) \in \Omega \times M$, $D_x f_\omega E^\tau(\omega, x) = E^\tau(\theta\omega, f_\omega x)$, where $\tau = u, c$;

(2) there are constants $1 < \mu < \lambda < +\infty$, such that

$$\begin{aligned} &\|D_x f_\omega^n \xi\| \geqslant C^{-1}(\omega) e^{\lambda n}\|\xi\|, \ \forall\, \xi \in E^u(\omega, x),\ n \in \mathbb{Z}^+, \\ &\|D_x f_\omega^n \xi\| \leqslant C(\omega) e^{\mu n}\|\xi\|, \ \forall\, \xi \in E^c(\omega, x),\ n \in \mathbb{Z}^+. \end{aligned} \tag{1.4}$$

Here, C in (1.4) is a tempered random variable, i.e. a random variable $C: \Omega \to \mathbb{R}^+$ satisfying $\lim_{n\to\pm\infty} \frac{\log C(\theta^n\omega)}{n} = 0$ for P-a.s. $\omega \in \Omega$, where P is a θ-invariant probability measure on Ω. Clearly, we can define a random norm $\|\cdot\|'_{(\omega,x)}$ for all $(\omega, x) \in \Omega \times M$, which can absorb the random parameter C, and there is a tempered random variable $B: \Omega \to (1, +\infty)$ such that $\frac{1}{B(\omega)}\|\xi\|'_{(\omega,x)} \leqslant \|\xi\| \leqslant B(\omega)\|\xi\|'_{(\omega,x)}$ for each $\omega \in \Omega$, $\xi \in T_x M$. The readers can follow the framework in section III.4 of [33] to get the equi-Hölder continuity of the subbundles with respect to the new norm by noticing that $\sup_{\omega\in\Omega} \|f_\omega\|_{C^2} < \infty$.

(iii) The method and result in [15] cannot be applied to the systems we considered in this paper since that the random Markov partition may not exist as we only assume a uniformly expanding direction on fibers rather than hyperbolic.

1.4. Structure of this paper

Firstly, we introduce several examples of the (**EE**)-systems in section 2. In order to prove the theorem 1, we recall some notions and prove several preliminary lemmas in section 3. We define the random weak horseshoe and prove the existence of periodic random weak horseshoe for the (**EE**)-systems in section 4. The relationship between the periodic random weak horseshoe and the random semi-horseshoe will be investigated in section 5. In section 6, we will prove the main result, which is merely a corollary of propositions in section 4 and section 5.

2. Examples

In this section, we give several examples of the (**EE**)-systems.

Example 2.1 Fiber partially hyperbolic maps on Tori. Let θ be an invertible measurable mapping on a measurable space $(\Omega, \mathscr{F})$. Define $\phi: \Omega \times \mathbb{T}^k \to \Omega \times \mathbb{T}^k$ for some $k \geqslant 4$ by

$$\phi(\omega, x) = (\theta\omega, Ax + h(\omega)),$$

where h is a measurable mapping from Ω to $\mathbb{T}^k$, and $A \in GL(k, \mathbb{Z})$ have characteristic polynomial $\chi_A(t)$ that is irreducible over $\mathbb{Q}$ and which has some but not all of its roots on the unit circle. Note that in this example $D_x f_\omega = A$ for all $x \in \mathbb{T}^k$. According to Example 3.4 in [30], there is an A-invariant splitting $\mathbb{R}^k = \mathcal{C} \oplus \mathcal{N} \oplus \mathcal{E}$, where A contracts on $\mathcal{C}$, expands on $\mathcal{E}$, and is an isometry on $\mathcal{N}$. Note that $E^u(\omega, x) = \mathcal{E}$ is a constant subbundle, hence equicontinuous.

Example 2.2 Fiber partially hyperbolic maps on 3-d tori without stable subbundles. Let (Ω, θ) be an invertible measurable mapping on a measurable space $(\Omega, \mathscr{F})$. Define $\phi: \Omega \times \mathbb{T}^3 \to \Omega \times \mathbb{T}^3$ by

$$\phi(\omega, x) = (\theta\omega, f(x) + h(\omega)),$$

where h is a measurable mapping from Ω to $\mathbb{T}^k$, and $f\colon \mathbb{T}^3 \to \mathbb{T}^3$ is any transitive diffeomorphism without stable subbundle chosen from section 6.2 in [5]. In [5] section 6.2, the authors showed that f admits a Df-invariant dominated splitting $T\mathbb{T}^3 = E^{uu} \oplus E^c$ into a 1-dimensional strong unstable subbundle E^{uu} and a 2-dimensional subbundle E^c. Moreover, E^c is not uniformly hyperbolic and does not admit any invariant uniformly contracting subbundle. In this example, $f_\omega = f + h(\omega)$, and $Df_\omega = Df$. Therefore, $E^u(\omega, x) \equiv E^{uu}(x)$ is the desired uniformly expanding subbundle.

The following example comes from [31].

Example 2.3 Random perturbations of partially hyperbolic systems. Let M be a connected smooth compact Riemannian manifold, and Diff $^2(M)$ be the space of C^2 diffeomorphisms from M to M equipped with the C^2 topology [17]. Note that the C^2 topology on Diff $^2(M)$ is metrizable, we denote the metric generating the C^2 topology by d^2_C. Assume $h \in$ Diff $^2(M)$ is partially hyperbolic in the following sense: there is a continuous splitting

$$T_x M = E^u(x) \oplus E^{cs}(x)$$

with dim $E^u > 0$ and a number $\lambda_0 > 0$ such that for any $x \in M$

$$\limsup_{n\to\infty} \frac{1}{n} \log \|D_x h^n \xi\| \geqslant \lambda_0, \quad \forall\, \xi \in E^u_x,\ \xi \neq 0;$$
$$\liminf_{n\to\infty} \frac{1}{n} \log \|D_x h^n \eta\| \leqslant 0, \quad \forall\, \eta \in E^{cs}_x,\ \eta \neq 0.$$

Note that h may not be structurally stable. Given $\epsilon > 0$, let $\mathcal{U}_\epsilon(h)$ be the ϵ-neighborhood of h in the Diff $^2(M)$ with respect to the C^2 topology. We denote $\omega = (\cdots, g_{-1}(\omega), g_0(\omega), g_1(\omega), \cdots) \in \Omega_\epsilon := U_\epsilon(h)^{\mathbb{Z}}$ and define the metric on Ω_ϵ by

$$d_{\Omega_\epsilon}(\omega, \omega') = \sum_{i\in\mathbb{Z}} \frac{d_{C^2}(g_i(\omega), g_i(\omega'))}{2^{|i|}}.$$

The metric d_{Ω_ϵ} generates the product topology on Ω_ϵ. Let $\theta\colon \Omega_\epsilon \to \Omega_\epsilon$ be the left shift operator, then θ is a homeomorphism. Let $f\colon \Omega_\epsilon \to$ Diff $^2(M)$ by $f(\omega) = f_\omega = g_0(\omega)$, then f is a continuous map. Denote

$$f^n_\omega := \begin{cases} f_{\theta^{n-1}\omega} \circ f_{\theta^{n-2}\omega} \circ \cdots \circ f_\omega, & \text{if } n > 0; \\ id, & \text{if } n = 0; \\ (f_{\theta^n\omega})^{-1} \circ \cdots \circ (f_{\theta^{-1}\omega})^{-1}, & \text{if } n < 0. \end{cases}$$

The following proposition shows that the Example 2.3 belongs to the (**EE**)-systems.

Proposition 2.1. *Given sufficiently small $\delta > 0$, we can find $\epsilon_\delta > 0$ and a positive constant A_δ such that the following holds: for every $(\omega, x) \in \Omega_{\epsilon_\delta} \times M$, there is a splitting*

$$T_x M = E^u(\omega, x) \oplus E^{cs}(\omega, x)$$

which depends on (ω, x) continuously and satisfies

(1) $D_x f_\omega(E^\tau(\omega, x)) = E^\tau(\theta\omega, f_\omega x)$ for $\tau = cs, u$;

(2) for all $n \geqslant 0$

$$\|D_x f^n_\omega \xi\| \geqslant A_\delta^{-2} e^{(\lambda_0 - 3\delta)n} \|\xi\|,\ \forall\, \xi \in E^u(\omega, x),$$
$$\|D_x f^n_\omega \eta\| \leqslant A_\delta^2 e^{3\delta n} \|\eta\|,\ \forall\, \eta \in E^{cs}(\omega, x).$$

(3) the splitting $E^{cs}(\omega, x) \oplus E^u(\omega, x)$ is equi-Hölder continuous in the following sense: there exist constants $L > 0$ and $\alpha > 0$ depending on δ and $U_\epsilon(h)$ such that

$$\mathrm{dist}(E^\tau(\omega, x), E^\tau(\omega, y)) \leqslant L d_M(x, y)^\alpha,\ \tau = cs, u$$

for any $\omega \in \Omega$ and $x, y \in M$.

Proof of proposition 2.1. (1) and (2) are the results of proposition 2.2 in [31]. To prove (3), we introduce the following new norm $\|\cdot\|^*_{(\omega,x)}$ and a new inner product $\langle\cdot,\cdot\rangle^*_{(\omega,x)}$ on T_{xM} on the fiber $\{\omega\} \times M$:

$$\|\xi\|^*_{(\omega,x)} = \left(\sum_{n=0}^{\infty}\left(\frac{\|D_x f_\omega^{-n}\xi\|}{e^{(\lambda_0-4\delta)(-n)}}\right)^2\right)^{1/2}, \forall\, \xi \in E^u(\omega, x),$$

$$\|\eta\|^*_{(\omega,x)} = \left(\sum_{n=0}^{\infty}\left(\frac{\|D_x f_\omega^{n}\eta\|}{e^{(4\delta)n}}\right)^2\right)^{1/2}, \forall\, \eta \in E^{cs}(\omega, x),$$

$$\|\zeta\|^*_{(\omega,x)} = \sqrt{(\|\xi\|^*_{(\omega,x)})^2 + (\|\eta\|^*_{(\omega,x)})^2}, \forall\, \zeta = \xi + \eta \in E^u(\omega, x) \oplus E^{cs}(\omega, x);$$

$$<\zeta_1, \zeta_2>^*_{(\omega,x)} := \frac{1}{4}((\|\zeta_1 + \zeta_2\|^*_{(\omega,x)})^2 - (\|\zeta_1 - \zeta_2\|^*_{(\omega,x)})^2), \forall\, \zeta_1, \zeta_2 \in E^u(\omega, x) \oplus E^{cs}(\omega, x).$$

There exists a constant $A'_\delta > 0$ such that

$$\|\zeta\| \leqslant \|\zeta\|^*_{(\omega,x)} \leqslant A'_\delta\|\zeta\| \text{for all} \zeta \in T_x M_\omega, \text{ any} \omega \in \Omega,$$

i.e. $\|\cdot\|$ is equivalent to $\|\cdot\|^*_{(\omega,x)}$ uniformly for all $(\omega, x) \in \Omega \times M$. With respect to this new inner product $\langle\cdot,\cdot\rangle^*_{(\omega,x)}$, $E^u(\omega, x)$ and $E^{cs}(\omega, x)$ are orthogonal and

$$\|D_x f_\omega^n \xi\|^*_{(\omega,x)} \geqslant e^{(\lambda_0-4\delta)n}\|\xi\|^*_{(\omega,x)}, \forall\, \xi \in E^u(\omega, x),$$
$$\|D_x f_\omega^n \eta\|_{(\omega,x)} \leqslant e^{4\delta n}\|\eta\|^*_{(\omega,x)}, \forall\, \eta \in E^{cs}(\omega, x).$$

The left proof of (3) is the same as that of proposition VII.2.4 in [33]. □

In particular, random perturbations of Anosov systems belong to the (**EE**)-systems(see also proposition 1.5 in [32]).

Next two examples belong to systems described in corollary 1.1, so they belong to the (**EE**)-systems by remark 1.3.

Example 2.4. All Anosov on fibers systems given in [19] section 8 belong to the (**EE**)-systems, including fiber Anosov maps on 2-d tori and random composition of 2×2 area-preserving positive matrices.

Fiber Anosov maps on 2-d tori: Let θ be a homeomorphism on a compact metric space Ω and let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Define ϕ: $\Omega \times \mathbb{T}^2 \to \Omega \times \mathbb{T}^2$ by

$$\phi\left(\omega, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(\theta\omega, A\begin{pmatrix} x \\ y \end{pmatrix} + h(\omega)\right) = \left(\theta\omega, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h_1(\omega) \\ h_2(\omega) \end{pmatrix}\right),$$

where h is a continuous map from Ω to $\mathbb{T}^2$. In this example, $E^u(\omega, x) \equiv \{t((1 + \sqrt{5})/2, 1)|t \in \mathbb{R}\}$ and $\|Df_\omega\xi\| \geqslant (\frac{3+\sqrt{5}}{2})\|\xi\|$ for $\xi \in E^u(\omega, x)$.

Random composition of 2×2 area-preserving positive matrices: let

$$\left\{B_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\right\}_{1\leqslant i\leqslant p}$$

be 2×2 matrices with $a_i, b_i, c_i, d_i \in \mathbb{Z}^+$, and $|a_i d_i - c_i b_i| = 1$ for any $i \in \{1,\dots,p\}$. Let $\Omega = \mathcal{S}_p := \{1,\dots,p\}^{\mathbb{Z}}$ with the left shift operator θ be the symbolic dynamical system with p symbols. For any $\omega = (\dots, \omega_{-1}, \omega_0, \omega_1,\dots) \in \Omega$, we define $g(\omega) = B_{\omega_0}$. Then the skew product $\tilde{\phi}$: $\Omega \times \mathbb{T}^2 \to \Omega \times \mathbb{T}^2$ defined by

$$\tilde{\phi}(\omega, x) = (\theta\omega, g(\omega)x)$$

is an Anosov on fibers system with continuous co-invariant splitting $\mathbb{R}^2 = E^u_\omega \oplus E^s_\omega$ for $\omega \in \mathcal{S}_p$, on which

$$\|Dg_\omega v\| \geqslant \kappa\|v\| \text{for} v \in E^u_\omega;$$
$$\|Dg_\omega \eta\| \leqslant \kappa^{-1}\|\eta\| \text{for} \eta \in E^s_\omega$$

for $\kappa := \min_{1\leqslant i\leqslant p} \min\{\sqrt{a_i^2 + c_i^2}, \sqrt{b_i^2 + d_i^2}\} \geqslant \sqrt{2}$ by proposition 8.2 in [19].

Example 2.5 Random composition of partially hyperbolic automorphisms on $\mathbb{T}^3$ with a fixed central direction. Let

$$\left\{A_i = \begin{pmatrix} a_i & b_i & 0 \\ c_i & d_i & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}_{1\leqslant i\leqslant p}$$

be 3×3 matrices with $a_i, b_i, c_i, d_i \in \mathbb{Z}^+$, and $|a_i d_i - c_i d_i| = 1$ for any $i \in \{1,\dots,p\}$. Let $\Omega = \mathcal{S}_p := \{1,\dots,p\}^{\mathbb{Z}}$ with the left shift operator θ be the symbolic dynamical system with p symbols.

For any $\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) \in \Omega$, we define $f(\omega) = A_{\omega_0}$. Then the skew product $\phi: \Omega \times \mathbb{T}^3 \to \Omega \times \mathbb{T}^3$ defined by

$$\phi(\omega, x) = (\theta\omega, f(\omega)x)$$

belongs to the (**EE**)-systems. In fact, we can consider the following system, random composition of 2×2 area-preserving positive matrices. Let

$$\left\{ B_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \right\}_{1 \leqslant i \leqslant p}$$

be top left 2×2 submatrix of $\{A_i\}_{i=1,\ldots,p}$. For any $\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) \in \Omega$, we define $g(\omega) = B_{\omega_0}$. By the Example 2.4, the skew product $\tilde{\phi}: \Omega \times \mathbb{T}^2 \to \Omega \times \mathbb{T}^2$ defined by

$$\tilde{\phi}(\omega, x) = (\theta\omega, g(\omega)x)$$

is an Anosov on fibers system with continuous co-invariant splitting $\mathbb{R}^2 = \tilde{E}^u_\omega \oplus \tilde{E}^s_\omega$. Now define $E^u(\omega, x) := \{(v, 0) | \ v \in \tilde{E}^u_\omega\}$, then

$$\|Df_\omega \eta\| \geqslant \kappa \|\eta\| \text{for} \eta \in E^u(\omega, x).$$

3. Preliminary lemmas

In this section, we recall some notions and introduce several preliminary lemmas to pave the way for the further proof.

3.1. E-consistent curves

We first recall some notations from differential geometry. If $\gamma: [0, 1] \to M$ is a piecewise C^1 curve, then the length of γ is given by $L(\gamma) = \int_0^1 \|\gamma'(t)\| dt$. For two piecewise C^1 curves $\gamma_1, \gamma_2: [0, 1] \to M$ with $\gamma_2(0) = \gamma_1(1)$, the gluing of γ_1 and γ_2 is defined by

$$\gamma_2 \cdot \gamma_1(t) = \begin{cases} \gamma_1(2t), & \text{if } 0 \leqslant t \leqslant \dfrac{1}{2}; \\ \gamma_2(2t - 1), & \text{if } \dfrac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Clearly, the gluing $\gamma_2 \cdot \gamma_1$ is a piecewise C^1 curve and $L(\gamma_2 \cdot \gamma_1) = L(\gamma_1) + L(\gamma_2)$. Inductively, for piecewise C^1 curves $\gamma_i: [0, 1] \to M$ with $\gamma_i(0) = \gamma_{i-1}(1)$ for $i = 2, \ldots, n$, the gluing $\gamma_n \cdot \gamma_{n-1} \cdot \cdots \cdot \gamma_1$ is a well-defined piecewise C^1 curve and $L(\gamma_n \cdot \gamma_{n-1} \cdot \cdots \gamma_1) = \sum_{i=1}^n L(\gamma_i)$.

Let U be a nonempty open subset of M, and E be a subbundle of TU. A curve $\gamma: [0, 1] \to M$ is called **E-consistent** if it is continuous piecewise C^1 and $\gamma'(t) \in E(\gamma(t))$ for all $t \in [0, 1]$ except finitely many values. In this paper, for each $\omega \in \Omega$, the strong unstable subbundle $E^u(\omega, x)$ is a continuous subbundle of TM, and a curve γ is called E^u_ω-consistent if it is continuous piecewise C^1 and $\gamma'(t) \in E^u(\omega, \gamma(t))$ for all $t \in [0, 1]$ except finitely many values.

For non-trivial subspaces A, B in a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and induced norm $|\cdot|$, set

$$\Theta(A, B) = \inf\{\arccos(\tilde{\delta}(v, w)): v \in A, |v| = 1; w \in B, |w| = 1\}, \tag{3.1}$$

where $\tilde{\delta}(v, w) = \frac{|\langle v, w \rangle|}{|v| \cdot |w|} \in [0, 1]$ and $\arccos: [0, 1] \to [0, \frac{\pi}{2}]$. For $\varsigma \in (0, \frac{\pi}{2}]$, we say that A is ς-transverse to B if $\Theta(A, B) \geqslant \varsigma$.

Remark 3.1. In the past literature, one defines that A is ς-transverse to B for some $\varsigma \in [0, \sqrt{2}]$ if

$$\tilde{\Theta}(A, B) = \inf\{\|v - w\|: v \in A, |v| = 1; w \in B, |w| = 1\} \geqslant \varsigma.$$

(See e.g. Chap. 6.2 in [10] or section 6.2 in [3]). By using Law of Cosines, one has the following relation

$$\tilde{\Theta}(A, B) = 2\sin\left(\frac{\Theta(A, B)}{2}\right).$$

Therefore, $\tilde{\Theta}(A, B)$ is one-to-one corresponding to $\Theta(A, B)$, and preserving the order. Thus, these two definitions are equivalent. We use $\Theta(A, B)$ instead of $\tilde{\Theta}(A, B)$ for convenience in the later proof.

The following lemma is an adapted version of lemma 2 in [8].

Lemma 3.1. *Let $\mathbb{R}^m$ be an m-dimensional Euclidean space with a given inner product $\langle \cdot, \cdot \rangle$ and induced norm $|\cdot|$. Let $m' \in (0, m)$ be an integer. Assume $E = \{(u, E(u)): u \in B_{\mathbb{R}^m}(0, s)\} = \{(u, E(u)): \ |u - 0| < s\}$ is a continuous m'-dimensional subbundle of $TB_{\mathbb{R}^m}(0, s) = \{(u, \mathbb{R}^m): u \in B_{\mathbb{R}^m}(0, s)\}$ for some $s > 0$. If F is an*

$(m-m')$-dimensional nontrivial subspace of $\mathbb{R}^m$ which is ς-transverse to $\mathrm{E}(\mathrm{u})$ for all $u \in B_{\mathbb{R}^m}(0, s)$ with respect to the given inner product and some constant $\varsigma \in (0, \frac{\pi}{2})$, then for all $x, y \in B_{\mathbb{R}^m}(0, \frac{s}{2(C_0(\varsigma)+1)})$ with $C_0 = C_0(\varsigma) := 1 + \frac{1}{\sin\varsigma}$, there exists an E-consistent curve γ: $[0, 1] \to B_{\mathbb{R}^m}(0, s)$ satisfying

$$\gamma(0) = x,\ \gamma(1) \in \bar{B}_{F_y}(y, C_0|x-y|),\ \text{and} L(\gamma) \leqslant C_0|x-y|,$$

where $F_y = F + y$, and $\bar{B}_{F_y}(y, C_0|x-y|) = \{z \in F_y: |y-z| \leqslant C_0|x-y|\}$.

Proof of lemma 3.1. Pick any $x, y \in B_{\mathbb{R}^m}(0, \frac{s}{2(C_0(\varsigma)+1)})$ and fix it. We only consider the case $x \not\in F_y$. Otherwise, the curve $\gamma(t) \equiv x$ for $t \in [0, 1]$ satisfies the requirement. There is a point $z \in F_y$ such that $(x-z) \perp F_y$. In particular, we have $|x-z| \leqslant |x-y|$ and $|z-y| \leqslant |x-y|$. Let

$$Y = \bigcup\nolimits_{0\leqslant t\leqslant 1}\{F_y + (1-t)(x-z)\}$$

be the parallel translate of F_y along the segment connecting z and x. Then Y is an $(m-m'+1)$-dimensional flat surface. We refer to F_y and $(x-z)$ the horizontal and vertical component of Y respectively.

For any $u \in Y \cap B_{\mathbb{R}^m}(0, s)$, we define

$$V(u) = E(u) \cap T_u Y = E(u) \cap (F \oplus \{\lambda(x-z): \lambda \in \mathbb{R}\}).$$

$V(u)$ is a well-defined continuous line field since $E(u)$ is a continuous subbundle on $Y \cap B_{\mathbb{R}^m}(0, s)$ and ς-transverse to F. Therefore, for each $u \in Y \cap B_{\mathbb{R}^m}(0, s)$, there exists a unique vector $v(u) \in V(u) \subset E(u)$ whose vertical component is $(z-x)$, i.e.

$$v(u) = w(u) + (z-x) \text{for some} w(u) \in F. \tag{3.2}$$

By the ς-transversality of $V(u)$ and F, we have that

$$|v(u)| = \sqrt{|z-x|^2 + |w(u)|^2} \leqslant \sqrt{|z-x|^2 + \frac{1}{\tan^2\varsigma}|z-x|^2} \leqslant \frac{1}{\sin\varsigma}|z-x|. \tag{3.3}$$

Now, $v(u)$ is a continuous bounded vector field on $Y \cap B_{R^m}(0, s)$.

Denote D to be the collection of all continuous curves γ: $[0, 1] \to Y \cap \bar{B}_{\mathbb{R}^m}\left(0, C_0|x-y| + \frac{s}{2(C_0+1)}\right)$ such that $\gamma(0) = x$ and $\gamma(1) \in \bar{B}_{F_y}(y, C_0|x-y|)$. Then D is a non-empty closed convex subset of Banach space $X = \{\gamma: \gamma \in C([0, 1], \mathbb{R}^m)\}$ with the sup norm $\|\gamma\|_X = \max_{t\in[0,1]}|\gamma(t)|$. Now, we consider the following operator T on D by

$$(T\gamma)(t) = x + \int_0^t v(\gamma(\tau))d\tau, \forall t \in [0, 1].$$

Note that for any $\gamma \in D$, $\gamma(t) \in Y \cap \bar{B}_{\mathbb{R}^m}\left(0, C_0|x-y| + \frac{s}{2(C_0+1)}\right) \subset Y \cap B_{\mathbb{R}^m}(0, s)$, we have $|v(\gamma(s))| \leqslant \frac{1}{\sin\varsigma}|z-x|$ by (3.3). Therefore, the curve $T\gamma$ is continuous. Moreover, we have

$$\begin{aligned}|(T\gamma)(t) - 0| &\leqslant |(T\gamma)(t) - x| + |x - 0| \leqslant \int_0^t |v(\gamma(\tau))|d\tau + \frac{s}{2(C_0+1)}\\ &\leqslant \frac{1}{\sin\varsigma}|z-x| + \frac{s}{2(C_0+1)}\\ &< C_0|x-y| + \frac{s}{2(C_0+1)},\end{aligned}$$

and

$$\begin{aligned}(T\gamma)(t) &= x + \int_0^t v(\gamma(\tau))d\tau \overset{(3.2)}{=} = x + \int_0^t w(\gamma(\tau))d\tau + t(z-x)\\ &= z + \int_0^t w(\gamma(\tau))d\tau + (1-t)(x-z)\\ &\in F_y + F + (1-t)(x-z)\\ &\subset F_y + (1-t)(x-z) \subset Y, \forall t \in [0, 1].\end{aligned}$$

Therefore, $T\gamma$ is a continuous mapping from $[0, 1]$ to $Y \cap \bar{B}_{\mathbb{R}^m}\left(0, C_0|x-y| + \frac{s}{2(C_0+1)}\right)$ with $(T\gamma)(0) = x$. We also notice that

$$(T\gamma)(1) = z + \int_0^t w(\gamma(\tau))d\tau \in F_y + F \subset F_y, \tag{3.4}$$

and

$$|(T\gamma)(1) - y| \leqslant |x - y| + \int_0^1 |v(\gamma)(\tau)| d\tau \leqslant |x - y| + \frac{1}{\sin\varsigma}|x - y| = C_0|x - y|. \tag{3.5}$$

Therefore, $(T\gamma)(1) \in \bar{B}_{F_y}(y, C_0|x - y|)$. Thus, $TD \subset D$.

Note that $T\gamma$ is Lipschitz continuous function with Lipschitz constant $\frac{1}{\sin\varsigma}|z - x|$ for all $\gamma \in D$ since

$$|T\gamma(t_1) - T\gamma(t_2)| \leqslant \int_{t_1}^{t_2} |v(\gamma(\tau))| d\tau \leqslant \frac{1}{\sin\varsigma}|z - x||t_1 - t_2|, \forall\, 0 \leqslant t_1 < t_2 \leqslant 1.$$

Moreover, $\|T\gamma\|_X \leqslant C_0|x - y| + \frac{s}{2(C_0 + 1)}$ for all $\gamma \in D$. Thus, $T(D)$ is a sequentially compact subset of $C([0, 1], \mathbb{R}^m)$ by Ascoli's theorem (see e.g., theorem 45.4 in [38]).

For any $\gamma_1, \gamma_2 \in D$, we have

$$\|T\gamma_1 - T\gamma_2\|_X \leqslant \int_0^1 |v(\gamma_1(\tau)) - v(\gamma_2(\tau))| d\tau. \tag{3.6}$$

Note that $v(u)$ is continuous vector field on $Y \cap B_{\mathbb{R}^m}(0, s)$. Therefore, $v(u)$ is uniformly continuous on $u \in Y \cap \bar{B}_{\mathbb{R}^m}\left(0, C_0|x - y| + \frac{s}{2(C_0 + 1)}\right)$. Hence, the uniform continuity of v and (3.6) imply that T: $D \to D$ is a continuous mapping. The Schauder's fixed point theorem (see e.g., theorem 7.5 in [36]) implies that T has a fixed point $\gamma^* \in D$, which is

$$\gamma^*(t) = x + \int_0^t v(\gamma^*(\tau)) d\tau, \forall\, t \in [0, 1].$$

The fixed point γ^* is E-consistent since

$$(\gamma^*)'(t) = v(\gamma^*(t)) \in V(\gamma^*(t)) \subset E(\gamma^*(t))\ \forall\, t \in [0, 1].$$

The length of γ^*

$$L(\gamma^*) = \int_0^1 |v(\gamma^*(\tau))| d\tau \leqslant \frac{1}{\sin\varsigma}|x - z| \leqslant C_0|x - y|.$$

Therefore, γ^* is the desired curve in lemma 3.1. □

Lemma 3.2. *Let $\mathbb{R}^m$ be an m-dimensional Euclidean space with a given inner product $\langle\cdot,\cdot\rangle$ and an induced norm $|\cdot|$, and $m' \in (0, m)$ be an integer. Assume $E = \{(u, E(u)): u \in B_{\mathbb{R}^m}(0, s)\}$ is a continuous m'-dimensional subbundle of $TB_{\mathbb{R}^m}(0, s) = \{(u, \mathbb{R}^m): u \in B_{\mathbb{R}^m}(0, s)\}$ for some $s > 0$. If F is an $(m - m')$-dimensional subspace of $\mathbb{R}^m$ which is ς-transverse to E(u) for all $u \in B_{\mathbb{R}^m}(0, s)$ and for some constant $\varsigma \in (0, \frac{\pi}{2})$, then the followings hold:*

(1) Let

$$V = \bar{B}_F\left(0, \frac{s}{4(C_0(\varsigma) + 1)}\right) := \left\{z \in F: |z - 0| \leqslant \frac{s}{4(C_0(\varsigma) + 1)}\right\}, \tag{3.7}$$

where $C_0(\varsigma)$ is defined in lemma 3.1. For any $\delta > 0$, there exists an open neighborhood $U(\delta)$ of 0 in $B_{\mathbb{R}^m}(0, s)$, which is defined by

$$U(\delta) = B_{\mathbb{R}^m}\left(0, \min\left\{\frac{\delta}{C_0(\varsigma)}, \frac{s}{8(C_0(\varsigma) + 1)^2}\right\}\right), \tag{3.8}$$

such that for any $u \in U(\delta)$, one can find an E-consistent curve γ: $[0, 1] \to B_{\mathbb{R}^m}(0, s)$ with

$$\gamma(0) = u,\ \gamma(1) \in V,\ \text{and} L(\gamma) \leqslant \delta.$$

(2) For any $\epsilon > 0$, let

$$r(\epsilon, \varsigma) := \sin\varsigma \cdot \frac{1}{2}\min\left\{\frac{s}{16(C_0(\varsigma) + 1)}, \frac{\epsilon}{2C_0(\varsigma)}\right\}. \tag{3.9}$$

For any $u \in B_{\mathbb{R}^m}(0, s)$, there exists an E-consistent curve β: $[0, 1] \to B_{\mathbb{R}^m}(0, s)$ satisfying

$$\beta(0) = u,\ d(\beta(1), V) \geqslant r(\epsilon, \varsigma),\ \text{and} L(\beta) \leqslant \epsilon.$$

Proof of lemma 3.2. We first prove (1). Given any $\delta > 0$ and fix it. For any $u \in U(\delta)$ defined in (3.8), then

$$|u - 0| < \frac{s}{8(C_0(\varsigma) + 1)^2} < \frac{s}{2(C_0(\varsigma) + 1)}.$$

By lemma 3.1, there exists an E-consistent curve γ: $[0, 1] \to B_{\mathbb{R}^m}(0, s)$ such that $\gamma(0) = u$, and

$$\gamma(1) \in \bar{B}_F(0, C_0|u - 0|) \subset \bar{B}_F\left(0, C_0\frac{s}{8(C_0 + 1)^2}\right) \subset \bar{B}_F\left(0, \frac{s}{4(C_0 + 1)}\right) = V,$$

and $L(\gamma) \leqslant C_0|u - 0| < C_0 \cdot \frac{\delta}{C_0} = \delta$. Then (1) is proved.

Now we prove (2). Given $\epsilon > 0$ and fix it, we choose a vector $v(\epsilon) \in E(0)$ such that

$$|v(\epsilon)| = \frac{1}{2}\min\left\{\frac{s}{16(C_0 + 1)}, \frac{\epsilon}{2C_0}\right\}.$$

Since $E(0)$ is ς-transverse to F, then

$$d(v(\epsilon), F) \geqslant \sin\varsigma \cdot |v(\epsilon)| = r(\epsilon, \varsigma),$$

and

$$d\left(V, \bar{B}_{F_{v(\epsilon)}}\left(v(\epsilon), 2C_0|v(\epsilon)| + \frac{s}{4(C_0 + 1)}\right)\right) \geqslant d(F, F_{v(\epsilon)}) = d(v(\epsilon), F) \geqslant r(\epsilon, \varsigma). \tag{3.10}$$

For any $u \in B_{\mathbb{R}^m}(0, s)$, to get (2) of this Lemma, we only need to consider the case $d(u, V) < |v(\epsilon)|$. Otherwise, we take $\beta(t) \equiv u$. We can pick a $w \in V$ with $|u - w| < |v(\epsilon)|$. Then $|0 - w| \leqslant \frac{s}{4(C_0 + 1)}$ by the definition of V, and

$$|u - 0| \leqslant |u - w| + |w - 0| \leqslant |v(\epsilon)| + \frac{s}{4(C_0 + 1)} < \frac{s}{2(C_0 + 1)}.$$

Let $v = v(\epsilon) + w$, then

$$|v - 0| \leqslant |v(\epsilon)| + |w - 0| < \frac{s}{2(C_0 + 1)}.$$

Now $u, v \in B_{\mathbb{R}^m}(0, \frac{s}{2(C_0 + 1)})$, and

$$|u - v| = |u - (v(\epsilon) + w)| \leqslant |u - w| + |w - (w + v(\epsilon))| \leqslant 2|v(\epsilon)|.$$

We apply lemma 3.1 to u, v, then there exists an E-consistent curve β: $[0, 1] \to B_{\mathbb{R}^m}(0, s)$ such that $\beta(0) = u$,

$$\beta(1) \in \bar{B}_{F_v}(v, C_0|u - v|) \subset \bar{B}_{F_{v(\epsilon)}}(v(\epsilon), |w - 0| + C_0|u - v|)$$
$$\subset \bar{B}_{F_{v(\epsilon)}}\left(v(\epsilon), 2C_0|v(\epsilon)| + \frac{s}{4(C_0 + 1)}\right),$$

and $L(\gamma) \leqslant C_0|u - v| \leqslant 2C_0|v(\epsilon)| < \epsilon$. Since $\beta(1) \in \bar{B}_{F_{v(\epsilon)}}\left(v(\epsilon), 2C_0|v(\epsilon)| + \frac{s}{4(C_0 + 1)}\right)$, by (3.10), we have

$$d(\beta(1), V) \geqslant d\left(V, \bar{B}_{F_{v(\epsilon)}}\left(v(\epsilon), 2C_0|v(\epsilon)| + \frac{s}{4(C_0 + 1)}\right)\right) \geqslant r(\epsilon, \varsigma).$$

The proof is complete. □

Recall that a curve γ is called E^u_ω-consistent if it is continuous piecewise C^1 and $\gamma'(t) \in E^u(\omega, \gamma(t))$ for all $t \in [0, 1]$ except finitely many values.

Lemma 3.3. *For a system $(\Omega \times M, \phi)$ with a uniformly expanding subbundle on fibers as in definition 1.1, and any $\omega \in \Omega$, if γ_ω: $[0, 1] \to M$ is an E^u_ω-consistent curve, then for any $n \in \mathbb{N}$, $\beta_{\theta^{-n}\omega}(\cdot) := (f_\omega^{-n} \circ \gamma_\omega)(\cdot)$: $[0, 1] \to M$ is an $E^u_{\theta^{-n}\omega}$-consistent curve. Moreover,*

$$L(\beta_{\theta^{-n}\omega}) \leqslant C \cdot e^{-\lambda n}L(\gamma_\omega),$$

where C, λ come from definition 1.1.

Proof of lemma 3.3. Since $f_\omega \in \mathrm{Diff}^1(M)$ for all $\omega \in \Omega$, $\beta_{\theta^{-n}\omega}(t)$ is a continuous piecewise C^1 curve as $\gamma_\omega(t)$. By the definition of E^u_ω-consistent curve, $\gamma'_\omega(t) \in E^u(\omega, \gamma_\omega(t))$ for all $t \in [0, 1]$ except finite points, named $t_1,\ldots,t_q$. Now for $t \in [0, 1]\setminus\{t_1,\ldots,t_q\}$,

$$\beta'_{\theta^{-n}\omega}(t) = D_{\gamma_\omega(t)}f_\omega^{-n}(\gamma'_\omega(t)) \in E^u(\theta^{-n}\omega, f_\omega^{-n}\gamma_\omega(t))$$

by the invariance of E^u. Hence $\beta_{\theta^{-n}\omega}(\cdot)$ is an $E^u_{\theta^{-n}\omega}$-consistent curve. Moreover, by (1.3), we have

$$\begin{aligned}L(\beta_{\theta^{-n}\omega}) &= \int_0^1 \|\beta'_{\theta^{-n}\omega}(t)\|dt = \int_0^1 \|D_{\gamma_\omega(t)} f_\omega^{-n}(\gamma'_\omega(t))\|dt\\ &\leqslant Ce^{-\lambda n}\int_0^1 \|\gamma'_\omega(t)\|dt = Ce^{-\lambda n}L(\gamma_\omega),\end{aligned}$$

which completes the proof. □

3.2. Aperture and transversality

For nontrivial closed subspaces A, B in a Hilbert space H with given inner product $\langle\cdot,\cdot\rangle$ and induced norm $|\cdot|$, define the aperture (or gap) between A and B by

$$\Gamma(A, B) := \max\left\{\sup_{v\in A,|v|=1}\inf_{w\in B}|v-w|,\ \sup_{w\in B,|w|=1}\inf_{v\in A}|v-w|\right\}. \tag{3.11}$$

Then $\Gamma(A, B)\in[0, 1]$. The aperture Γ satisfies the triangle inequality since $\Gamma(A, B) = \|P_A - P_B\|$, where P_A and P_B are the orthogonal projections on A and B respectively (see more details in Chap. Four section 2 in [22]).

For any invertible bounded linear operator $\mathcal{T}\colon H\to H$ and nontrivial closed subspace A of H, we have

$$\begin{aligned}\Gamma(\mathcal{T}A, A) &= \max\left\{\sup_{v\in A,|\mathcal{T}v|=1}\inf_{w\in A}|\mathcal{T}v-w|,\ \sup_{w\in A,|w|=1}\inf_{v\in A}|\mathcal{T}v-w|\right\}\\ &\leqslant \max\left\{\sup_{v\in A,|\mathcal{T}v|=1}|\mathcal{T}v-v|,\ \sup_{w\in A,|w|=1}|\mathcal{T}w-w|\right\}\\ &= \max\left\{\sup_{v\in A,|\mathcal{T}v|=1}|\mathcal{T}v-\mathcal{T}^{-1}(\mathcal{T}v)|,\ \sup_{w\in A,|w|=1}|\mathcal{T}w-w|\right\}\\ &\leqslant \max\left\{\sup_{|\mathcal{T}v|=1}|\mathcal{T}v-\mathcal{T}^{-1}(\mathcal{T}v)|,\ \sup_{|w|=1}|\mathcal{T}w-w|\right\}\\ &\leqslant \max\{\|I-\mathcal{T}^{-1}\|, \|I-\mathcal{T}\|\},\end{aligned} \tag{3.12}$$

where the operator norm of $\mathcal{T}\colon H\to H$ is defined by $\|\mathcal{T}\| := \max_{|v|=1}|\mathcal{T}v|$.

Let H_1 and H_2 be two Hilbert spaces with inner product $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ respectively. For any nontrivial closed subspaces $A, B\subset H_1$ with the same dimension, and any invertible bounded linear operator $\mathcal{T}\colon H_1\to H_2$, one has

$$\Gamma_2(\mathcal{T}A, \mathcal{T}B)\leqslant \|\mathcal{T}\|\cdot\|\mathcal{T}^{-1}\|\cdot\Gamma_1(A, B), \tag{3.13}$$

where Γ_i is the aperture between subspaces in H_i with respect to the corresponding inner product. In fact, this is a direct corollary of the following two inequalities:

$$\begin{aligned}\sup_{v\in A,|\mathcal{T}v|_2=1}\inf_{w\in B}|\mathcal{T}v-\mathcal{T}w|_2 &\leqslant \|\mathcal{T}\|\sup_{v\in A,|\mathcal{T}v|_2=1}\inf_{w\in B}|v-w|_1\\ &= \|\mathcal{T}\|\sup_{v\in A,|\mathcal{T}v|_2=1}|v|_1\inf_{w\in B}\left|\frac{v}{|v|_1}-\frac{w}{|v|_1}\right|_1\\ &\leqslant \|\mathcal{T}\|\cdot\|\mathcal{T}^{-1}\|\sup_{v\in A,|\mathcal{T}v|_2=1}\inf_{w\in B}\left|\frac{v}{|v|_1}-\frac{w}{|v|_1}\right|\\ &\leqslant \|\mathcal{T}\|\cdot\|\mathcal{T}^{-1}\|\cdot\Gamma_1(A, B),\end{aligned} \tag{3.14}$$

and similarly,

$$\sup_{w\in B,|\mathcal{T}w|_2=1}\inf_{v\in A}|\mathcal{T}v-\mathcal{T}w|_2\leqslant \|\mathcal{T}\|\cdot\|\mathcal{T}^{-1}\|\cdot\Gamma_1(A, B). \tag{3.15}$$

Lemma 3.4. *Let* H *be a Hilbert space with inner product* $\langle\cdot,\cdot\rangle$ *and induced norm* $|\cdot|$*. For nontrivial proper closed subspaces* $A, B\subsetneq H$ *with* $\dim A = \dim B$*, if* $\Gamma(A, B) < \sin\theta$*, where* $\theta\in(0, \frac{\pi}{2}]$*, then* B *is* $(\frac{\pi}{2}-\theta)$*-transverse to* $A^\perp$ *with respect to the given inner product.*

Proof of lemma 3.4. It is sufficient to prove that for any $v\in B$ and $w\in A^\perp$ with $|v| = 1$, $|w| = 1$, $\arccos(\tilde{\delta}(v, w))\geqslant\frac{\pi}{2}-\theta$. There exist unique $v_1\in A$ and $v_2\in A^\perp$ such that $v = v_1 + v_2$. Then we have

$$|v_2| = |v - v_1| = \inf_{\eta \in A} |v - \eta| \leqslant \Gamma(B, A) < \sin \theta = \cos(\frac{\pi}{2} - \theta).$$

Now

$$\begin{aligned}\tilde{\delta}(v, w) &= \frac{|\langle v, w \rangle|}{|v| \cdot |w|} = |\langle v, w \rangle| \\ &= |\langle v_2, w \rangle| \leqslant |v_2||w| \\ &< \cos(\frac{\pi}{2} - \theta).\end{aligned}$$

Therefore, $\arccos(\tilde{\delta}(v, w)) \geqslant \frac{\pi}{2} - \theta$. The proof is complete. □

3.3. Distance between subbundles

In this subsection, we define the distance between subbundles by using apertures. This definition can be found in [9] and [33].

By the compactness of M, there exists a constant $\rho^* > 0$ such that for any $y \in M$, there exists a neighborhood $B_M(y, \rho^*) \subset M$ of y, the exponential map $\exp_y : B_{T_y M}(0, \rho^*) \to B_M(y, \rho^*)$ is a diffeomorphism and this gives a normal coordinate chart around y (see theorem 3.3.7 in [13]). Moreover, if $d(x, y) < \rho^*$, there is a unique geodesic connecting x and y. We pick

$$\rho \in \left(0, \frac{\rho^*}{2}\right). \tag{3.16}$$

Throughout this paper, we fix this ρ.

For any $x, y \in M$, if $d_M(x, y) < \rho$, then there exists a linear isometry from T_{xM} to $T_y M$ given by the parallel transport with respect to the Levi-Civita connection along the unique geodesic connecting x and y, named $P(x, y)$ (see proposition 2.2.6 in [13]). Then for any $x, y \in M$, $E(x)$, $E(y)$ are two nontrivial subspaces of T_{xM} and T_{yM} respectively, we can define

$$\operatorname{dist}(E(x), E(y)) := \begin{cases} \Gamma_x(E(x), P(y, x)E(y)), & \text{if } d_M(x, y) < \rho; \\ 1, & \text{otherwise}, \end{cases} \tag{3.17}$$

where Γ_x is the aperture between subspaces in T_{xM} with respect to the given Riemannian metric $g_x\langle \cdot, \cdot \rangle$. When $d(x, y) < \rho$, since $P(x, y)$: $T_x M \to T_y M$ is a linear isometry preserving the given inner product in T_{xM} and T_{yM} (see definition 3.1 and theorem 3.6 in [13]), it is clear that

$$\Gamma_x(E(x), P(y, x)E(y)) = \Gamma_y(E(y), P(x, y)E(x)).$$

Therefore, $\operatorname{dist}(E(x), E(y)) = \operatorname{dist}(E(y), E(x))$.

4. Periodic random weak horseshoes

In this section, we define the periodic random weak horseshoe and prove its existence in the (**EE**)-systems.

4.1. Definition of periodic random weak horseshoes

For a real random variable η: $\Omega \to \mathbb{R}^+$, two compact random sets C_0, C_1: $\Omega \to 2^M$ are said to be η-seperated if

$$d_M(C_0(\omega), C_1(\omega)) \geqslant \eta(\omega) \text{for all} \omega \in \Omega.$$

Remark 4.1. In [20], Huang and Lu defined weak horseshoe of two symbols and full horseshoe of two symbols. In their paper, they consider an injective infinite-dimensional continuous random dynamical system φ on a Polish space X over an ergodic Polish system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$, and a compact random set $\omega \to K(\omega)$ is φ-invariant in the sense of $\varphi(n, \omega)(K(\omega)) = K(\theta^n \omega)$ for $n \in \mathbb{N}$. If (K, φ) has positive topological entropy, then it has a weak horseshoe of two symbols, that means the system has the following properties:

- Let d be the metric on X, there are two nonempty, bounded and closed subsets $U_0, U_1 \subset X$ such that $d(U_0, U_1) > 0$;
- there exist a constant $b > 0$ and $M_{b,\omega} \in \mathbb{N}$ defined for $\mathbb{P}$-a.e. $\omega \in \Omega$ such that for any natural number $m \geqslant M_{b,\omega}$, there is a subset $J_m \subset \{0, 1, \ldots, m\}$ satisfying: (a) $\#|J_m| \geqslant bm$ (positive density), where $\#|\cdot|$ is the cardinality of a subset of $\mathbb{Z}$; (b) for any $s \in \{0, 1\}^{J_m}$, there exists an $x_s \in K(\omega)$ with $\varphi(j, \omega)(x_s) \in U_{s(j)} \cap K(\theta^j \omega)$ for any $j \in J_m$.

While in the definition of full horseshoe of two symbols, the interpolating set J_m is replaced with $J(\omega) \subset \mathbb{N}$ for $\mathbb{P}$-a.s., where $J(\omega)$ has positive density greater than b in the sense of

$$\lim_{m\to+\infty} \frac{\#|J(\omega)\cap\{0,\ldots,m-1\}|}{m} \geqslant b.$$

The set of time they considered were one-sided since the infinite-dimensional dynamical systems generated by parabolic PDEs are not invertible.

We mimic the definition of full horseshoe of two symbols to define the random weak horseshoe for systems driven by an external force.

Definition 4.1 Random weak horseshoe. The system $(\Omega \times M, \phi)$ is said to have a random weak horseshoe with an interpolating set $J \subset \mathbb{Z}$ if there exist two compact random sets $\omega \mapsto K_0(\omega)$ and $\omega \mapsto K_1(\omega)$, which are η-seperated for some real random variable η: $\Omega \to \mathbb{R}^+$, such that given any $s \in \{0, 1\}^J$ and $\omega \in \Omega$, there is a point $x_{s,\omega} \in M$ on the fiber $\{\omega\} \times M$ satisfying

$$f_\omega^j(x_{s,\omega}) \in K_{s(j)}(\theta^j\omega) \text{for all} j \in J.$$

If J has positive density, i.e. $\lim_{n\to\infty} \frac{\#|[-n,n]\cap J|}{2n+1}$ exists and is positive, where $\#|\cdot|$ is the cardinality of a subset of $\mathbb{Z}$, then $(\Omega \times M, \phi)$ is said to have a random weak horseshoe. In particular, if $J \supset N\mathbb{Z}$ for some number $N \in \mathbb{N}$, then $(\Omega \times M, \phi)$ is said to have a **periodic random weak horseshoe.** Equivalently, there exists an $N \in \mathbb{N}$ such that for any $s \in \{0, 1\}^{\mathbb{Z}}$ and $\omega \in \Omega$, there is a point $x_{s,\omega} \in M$ satisfying

$$f_\omega^{jN}(x_{s,\omega}) \in K_{s(j)}(\theta^{jN}\omega) \text{for all} j \in \mathbb{Z}.$$

Remark 4.2. The differences between the definition of random weak horseshoe in this paper and the definition of full horseshoe of two symbols in [20] are that we allow K_0, K_1 to be compact random sets; the interpolating set is two-sided; and the horseshoe phenomenon can be found for all $\omega \in \Omega$.

To prove the existence of periodic random weak horseshoes, we need the following proposition.

Proposition 4.1. *If $(\Omega \times M, \phi)$ belongs to the (**EE**)-systems, then $(\Omega \times M, \phi)$ has the following property:*

Property A 1. for any given $\delta > 0$, there exists a positive constant $\tau(\delta)$ and two $\tau(\delta)$-separated compact random sets $\omega \mapsto V_{0,\delta}(\omega)$ and $\omega \mapsto V_{1,\delta}(\omega)$ satisfying the following statements:

(1) for any $(\omega, x) \in \Omega \times M$, there is an E_ω^u-consistent curve $\gamma_{\omega,\delta}$: $[0, 1] \to M$ such that $\gamma_{\omega,\delta}(0) = x$, $\gamma_{\omega,\delta}(1) \in V_{0,\delta}(\omega)$ and $L(\gamma_{\omega,\delta}) \leqslant \delta$,

(2) for any $(\omega, z) \in \Omega \times M$, there is an E_ω^u-consistent curve $\beta_{\omega,\delta}$: $[0, 1] \to M$ such that $\beta_{\omega,\delta}(0) = z$, $\beta_{\omega,\delta}(1) \in V_{1,\delta}(\omega)$ and $L(\beta_{\omega,\delta}) \leqslant \delta$.

With the help of Property 1, we can prove the following proposition.

Proposition 4.2. *Any (**EE**)-system $(\Omega \times M, \phi)$ has a periodic random weak horseshoe.* In the rest of this section, we prove the above two propositions.

4.2. Proof of proposition 4.1

In this part, we prove proposition 4.1, i.e. (**EE**)-systems have Property 1.

Since $(\Omega \times M, \phi)$ belongs to the (**EE**)-systems, $E^u(\omega, x)$ is equicontinuous with respect to (ω, x), which is defined as in ((**EE**)**b**), **i.e. for any given** $x \in M$ **and** $\epsilon > 0$, **there exists a** $\delta(x, \epsilon) > 0$ **such that if** $d_M(x, y) < \delta$, **then** $\text{dist}(E^u(\omega, x), E^u(\omega, y)) < \epsilon$ **for all** $\omega \in \Omega$. **By the compactness of** M **and the Lebesgue's number lemma, we conclude that for each** $\epsilon > 0$, **there exists a** $\delta = \delta(\epsilon) > 0$ **such that if** $d_M(x, y) < \delta$, **then**

$$\text{dist}(E^u(\omega, x), E^u(\omega, y)) < \epsilon \text{ for all} \omega \in \Omega.$$

In order to translate lemma 3.2 to the case on the Riemannian manifold, we need the following lemma. Before we introduce the next lemma, we recall some notations and knowledge from differential geometry. The following can be found in Ch. 3.2 in [13]. For any $y \in M$, the local coordinate chart $(N_y, \exp_y)$ is given by the exponential map

$$\exp_y: B_{T_yM}(0, \rho) \to N_y \subset M, \text{ and } \exp_y^{-1}: N_y \to B_{T_yM}(0, \rho),$$

where ρ is picked in (3.16). Note that $\exp_y$ is a smooth diffeomorphism and so is $\exp_y^{-1}$ on its corresponding domain. Thus

$$D_v \exp_y: T_v T_y M \cong T_y M \to T_{\exp_y(v)} M, \text{ for any} v \in B_{T_y M}(0, \rho),$$

and

$$D_z \exp_y^{-1}: T_z M \to T_{\exp_y^{-1}(z)} T_y M \cong T_y M, \text{ for any} z \in N_y$$

are well-defined and smoothly differentiable. Besides, we have $D_0 \exp_y = I$: $T_0 T_y M \cong T_y M \to T_y M$ and $D_y \exp_y^{-1} = I$: $T_y M \to T_0 T_y M \cong T_y M$ (see proposition 3.2.9 in [13]). Since $\exp_y \circ \exp_y^{-1} = I$ and $\exp_y^{-1} \circ \exp_y = I$, by the chain rule, one has

$$I = D_z(\exp_y \circ \exp_y^{-1}) = (D_{\exp_y^{-1}(z)} \exp_y)(D_z \exp_y^{-1}), \forall z \in N_y; \tag{4.1}$$

$$I = D_v(\exp_y^{-1} \circ \exp_y) = (D_{\exp(v)} \exp_y^{-1})(D_v \exp_y), \forall v \in B_{T_y M}(0, \rho). \tag{4.2}$$

Lemma 4.1. *Let $(\Omega \times M, \phi)$ belong to the (**EE**)-systems, then for any $y \in M$, there exists a constant $\rho_y \in (0, \rho)$ such that for any $x \in B_M(y, \rho_y)$, one has*

$$\text{for all} \omega \in \Omega,\ E^u(\omega, y)^\perp \text{ is } \frac{\pi}{4} - \text{transverse to } (D_x \exp_y^{-1}) E^u(\omega, x) \text{ with respect to}$$
$$\text{the inner product on } T_y M \text{ induced by the Riemannian metric},$$

where $E^u(\omega, y)^\perp$ is the orthogonal complement of $E^u(\omega, y)$ in T_{yM}.

Proof of lemma 4.1. With respect to the normal coordinate chart $(N_y, \exp_y)$, given $x \in N_y$, we have

$$\begin{aligned}
&\Gamma_y((D_y \exp_y^{-1}) E^u(\omega, y), (D_x \exp_y^{-1}) P(y, x) E^u(\omega, y))\\
&\overset{(4.1)}{=} \Gamma_y((D_y \exp_y^{-1}) E^u(\omega, y), (D_x \exp_y^{-1}) P(y, x)(D_0 \exp_y)(D_y \exp_y^{-1}) E^u(\omega, y))\\
&\overset{(3.12)}{\leqslant} \max\{\|I - (D_x \exp_y^{-1}) P(y, x)(D_0 \exp_y)\|, \|I - (D_0 \exp_y)^{-1} P(y, x)^{-1} (D_x \exp_y^{-1})^{-1}\|\}\\
&= \max\{\|I - (D_x \exp_y^{-1}) P(y, x)(D_0 \exp_y)\|, \|I - (D_y \exp_y^{-1}) P(x, y)(D_{\exp_y^{-1}(x)} \exp_y)\|\},
\end{aligned} \tag{4.3}$$

where Γ_y is the aperture between subspaces of T_{yM} with respect to the inner product induced by the Riemannian metric. Since that $P(y, y) = I$: $T_y M \to T_y M$ and $P(x, x) = I$: $T_x M \to T_x M$, we have

$$\begin{aligned}
I &\overset{(4.2)}{=} (D_y \exp_y^{-1}) P(y, y)(D_0 \exp_y);\\
I &\overset{(4.2)}{=} = (D_x \exp_y^{-1}) P(x, x)(D_{\exp_y^{-1}(x)} \exp_y).
\end{aligned} \tag{4.4}$$

Notice that the following four mappings $(D_y \exp_y^{-1}) P(y, y)(D_0 \exp_y)$, $(D_x \exp_y^{-1}) P(y, x)(D_0 \exp_y)$, $(D_x \exp_y^{-1}) P(x, x)(D_{\exp_y^{-1}(x)} \exp_y)$ and $(D_y \exp_y^{-1}) P(x, y)(D_{\exp_y^{-1}(x)} \exp_y)$ are the local representation of $P(y, y)$, $P(y, x)$, $P(x, x)$ and $P(x, y)$ respectively with respect to the coordinate chart $(N_y, \exp_y)$. We note that the parallel vector field is the solution of a first order linear differential system with respect to the normal coordinate chart (see detail in VII.3 of [6] or proposition 2.6 in [13]). Therefore, there exists a positive constant $C_{1,y}$ only depending on the local coordinate chart such that

$$\begin{aligned}
&\|(D_y \exp_y^{-1}) P(y, y)(D_0 \exp_y) - (D_x \exp_y^{-1}) P(y, x)(D_0 \exp_y)\| \leqslant C_{1,y} d_M(y, x);\\
&\|(D_x \exp_y^{-1}) P(x, x)(D_{\exp_y^{-1}(x)} \exp_y) - (D_y \exp_y^{-1}) P(x, y)(D_{\exp_y^{-1}(x)} \exp_y)\| \leqslant C_{1,y} d_M(y, x).
\end{aligned} \tag{4.5}$$

Now (4.3), (4.4), and (4.5) imply that

$$\Gamma_y((D_y \exp_y^{-1}) E^u(\omega, y), (D_x \exp_y^{-1}) P(y, x) E^u(\omega, y)) \leqslant C_{1,y} d_M(y, x). \tag{4.6}$$

Since $\exp_y$ is a smooth diffeomorphism and $D_y \exp_y^{-1} = I$, for any $x \in N_y$, there is a $C_{2,y}$, which only depends on this chart, such that

$$\begin{aligned}
&\|(D_x \exp_y^{-1}) - (D_y \exp_y^{-1})\| = \|(D_x \exp_y^{-1}) - I\| \leqslant C_{2,y} d_M(x, y),\\
&\|(D_{\exp_y^{-1}(x)} \exp_y) - (D_0 \exp_y)\| = \|(D_{\exp_y^{-1}(x)} \exp_y) - I\| \leqslant C_{2,y} d_M(x, y).
\end{aligned}$$

Hence, if $x \in N_y \cap B_M(y, \frac{1}{C_{2,y}})$, where $B_M(y, \frac{1}{C_{2,y}})$ is the open ball around y with radius $\frac{1}{C_{2,y}}$ with respect to d_M, one has

$$\|D_x \exp_y^{-1}\| \leqslant 1 + C_{2,y} d_M(x, y), \tag{4.7}$$

and

$$\|D_{\exp_y^{-1}(x)}\exp_y\| \leqslant 1 + C_{2,y}d_M(x, y). \tag{4.8}$$

Therefore,

$$\begin{aligned}
&\Gamma_y((D_x \exp_y^{-1})P(y, x)E^u(\omega, y), (D_x \exp_y^{-1})E^u(\omega, x)) \\
&\overset{(3.13)}{\leqslant} \|D_x \exp_y^{-1}\| \cdot \|(D_x \exp_y^{-1})^{-1}\| \cdot \Gamma_x(P(y, x)E^u(\omega, y), E^u(\omega, x)) \\
&= \|D_x \exp_y^{-1}\| \cdot \|D_{\exp_y^{-1}(x)}\exp_y\| \cdot \Gamma_x(P(y, x)E^u(\omega, y), E^u(\omega, x)) \\
&\overset{(4.7),(4.8)}{\leqslant} (1 + C_{2,y}d_M(y, x))^2 \cdot \Gamma_x(P(y, x)E^u(\omega, y), E^u(\omega, x)) \\
&= (1 + C_{2,y}d_M(y, x))^2 \cdot \mathrm{dist}(E^u(\omega, y), E^u(\omega, x)).
\end{aligned} \tag{4.9}$$

By using the triangle inequality, (4.6) and (4.9), we have

$$\begin{aligned}
&\Gamma_y((D_y \exp_y^{-1})E^u(\omega, y), (D_x \exp_y^{-1})E^u(\omega, x)) \\
&\leqslant \Gamma_y((D_y \exp_y^{-1})E^u(\omega, y), (D_x \exp_y^{-1})P(y, x)E^u(\omega, y)) \\
&+ \Gamma_y(D_x \exp_y^{-1}P(y, x)E^u(\omega, y), D_x \exp_y^{-1}E^u(\omega, x)) \\
&\leqslant C_{1,y}d_M(y, x) + (1 + C_{2,y}d_M(y, x))^2\mathrm{dist}(E^u(\omega, y), E^u(\omega, x)).
\end{aligned} \tag{4.10}$$

We choose $\rho_y \in (0, \rho/2)$ to be sufficient small so that if $d_M(y, x) < \rho_y$, then

$$x \in N_y \cap B_M\left(y, \frac{1}{C_{2,y}}\right),$$

and

$$C_{1,y}d_M(y, x) + (1 + C_{2,y}d_M(y, x))^2 \cdot \mathrm{dist}(E^u(\omega, y), E^u(\omega, x)) < \sin\frac{\pi}{4}.$$

We note that ρ_y does not depend on ω since E^u is uniformly equicontinuous. Therefore, by (4.10), for any $x \in B_M(y, \rho_y)$, we have

$$\Gamma_y((D_y \exp_y^{-1})E^u(\omega, y), (D_x \exp_y^{-1})E^u(\omega, x)) < \sin\frac{\pi}{4}.$$

Then by lemma 3.4, we conclude that for any $x \in B_M(y, \rho_y)$,

$$(D_x \exp_y^{-1})E^u(\omega, x) \text{is} \tfrac{\pi}{4} - \text{transverse to}((D_y \exp_y^{-1})E^u(\omega, y))^\perp = E^u(\omega, y)^\perp$$

with respect to the inner product on T_{yM} induced by the Riemannian metric g. The proof of lemma 4.1 is complete. □

Recall that $\rho \in (0, \frac{\rho^*}{2})$ is chosen in (3.16), where ρ^* is the radius of the normal coordinate chart. Then the following constant exists:

$$C_y := \max\left\{\sup_{v\in \bar{B}_{T_yM}(0,\rho)} \|D_v \exp_y\|, \sup_{x\in \overline{N_y}} \|D_x \exp_y^{-1}\|\right\} < \infty, \tag{4.11}$$

which only depends on point y and its normal coordinate chart.

For any $y \in M$, let $\rho_y \in (0, \rho)$ be the number given in lemma 4.1. Recall that $C_0(\frac{\pi}{4})$ is defined in lemma 3.1. Now for any $\delta > 0$, we define

$$U_\delta(y) = \exp_y\left(B_{T_yM}\left(0, \min\left\{\frac{\delta/C_y}{C_0(\pi/4)}, \frac{\rho_y}{8(C_0(\pi/4) + 1)^2}\right\}\right)\right). \tag{4.12}$$

$\{U_\delta(y)\}_{y\in M}$ is an open cover of M. By the compactness of M, there exists a finite subcover, named $\{U_\delta(y_i)\}_{i=1}^n$. From now on, we fix such $\{y_1,\ldots,y_n\} \subset M$. For $i \in \{1,\ldots,n\}, \omega \in \Omega$, we define

$$V_{(\omega,y_i)} = \exp_{y_i}\left(\left\{v \in E^u(\omega, y_i)^\perp: \|v\| \leqslant \frac{\rho_{y_i}}{4(C_0(\pi/4) + 1)}\right\}\right). \tag{4.13}$$

On each open subset $B_M(y_i, \rho_{y_i})$ for $i \in \{1,\ldots,n\}$, by the choice of ρ_{y_i} and lemma 4.1, we have the following observation:

$$\text{for all } \omega \in \Omega, \ (D_x \exp_{y_i}^{-1})E^u(\omega, x) \text{is } \frac{\pi}{4} - \text{ transverse to } E^u(\omega, y_i)^{\perp} \text{ for all } x \in B_M(y_i, \rho_{y_i})$$

with respect to the inner product on $T_{y_i}M$ induced by the Riemannian metric g. By using the above observation, we can translate lemma 3.2 on each open subset $B_M(y_i, \rho_{y_i})$.

Lemma 4.2. *(Translation of lemma 3.2)For all $\omega \in \Omega$, $i \in \{1,\dots,n\}$, $E^u_\omega = \{(x, E^u(\omega, x)): x \in M\}$ is a continuous m'-dimensional subbundle, and $(D_x \exp_{y_i}^{-1})E^u(\omega, x)$ is $\frac{\pi}{4}$-transverse to $(D_{y_i} \exp_{y_i}^{-1})E^u(\omega, y_i)^{\perp}$ for all $x \in B_M(y_i, \rho_{y_i})$ with respect to the inner product on $T_{y_i}M$ induced by the Riemannian metric g. For $U_\delta(y_i)$ and $V_{(\omega,y_i)}$ defined by (4.12) and (4.13) respectively, we have the following statements:*

(a)For all $x \in U_\delta(y_i)$, we can find an E^u_ω-consistent curve $\gamma_\omega = \gamma_\omega(x, y_i)$: $[0, 1] \to B_M(y_i, \rho_{y_i})$ with $\gamma_\omega(0) = x$, $\gamma_\omega(1) \in V_{(\omega,y_i)}$, and $L(\gamma_\omega) \leqslant \delta$;

(b)For all $\epsilon > 0$, let

$$r(\epsilon, y_i, \frac{\pi}{4}) = \sin\frac{\pi}{4} \cdot \frac{1}{2} \cdot \min\left\{\frac{\rho_{y_i}}{16(C_0(\pi/4) + 1)}, \frac{\epsilon/C_{y_i}}{2C_0(\pi/4)}\right\}, \tag{4.14}$$

where C_{y_i} is defined as (4.11). Then for any $x \in B_M(y_i, \rho_{y_i})$, there exists an E^u_ω-consistent curve β_ω: $[0, 1] \to B_M(y_i, \rho_{y_i})$ satisfying $\beta_\omega(0) = x$, $d_M(\beta_\omega(1), V_{(\omega,y_i)}) \geqslant C_{y_i}^{-1} r(\epsilon, y_i, \frac{\pi}{4})$, and $L(\beta_\omega) \leqslant \epsilon$.

Proof of lemma 4.2. Given $i \in \{1,\dots,n\}$, it is well known that $T_{y_i}M$ is m-dimensional vector space. Pick any orthonormal basis on $T_{y_i}M$ and fix it, then $T_{y_i}M$ is naturally identified with $\mathbb{R}^m$, on which the inner product is inherited from the given Riemannian metric. We note that lemma 3.2 does not depend on the choice of basis. Now for each $\omega \in \Omega$, the following subbundle on $T_{y_i}M$

$$\tilde{E}_\omega(\exp_{y_i}^{-1}(x)) := (D_x \exp_{y_i}^{-1})E^u(\omega, x) \tag{4.15}$$

is $\frac{\pi}{4}$-transverse to $F_i := D_{y_i} \exp_{y_i}^{-1} E^u(\omega, y_i)^{\perp}$ for all $(\exp_{y_i}^{-1}(x)) \in B_{T_{y_i}M}(0, \rho_{y_i})$ with respect to the inner product on $T_{y_i}M$ induced by the Riemannian metric. According to lemma 3.2, for any $x \in U_\delta(y_i)$, we can find an $\tilde{E}_\omega$-consistent curve $\tilde{\gamma}_\omega$: $[0, 1] \to B_{T_{y_i}M}(0, \rho_{y_i})$ with

$$\tilde{\gamma}_\omega(0) = \exp_{y_i}^{-1}(x), \ \tilde{\gamma}_\omega(1) \in \exp_{y_i}^{-1} V_{(\omega,y_i)}, \ \text{and} L(\tilde{\gamma}_\omega) \leqslant \delta/C_{y_i}.$$

Let $\gamma_\omega = \exp_{y_i}(\tilde{\gamma}_\omega)$. Since $\tilde{\gamma}_\omega$ is $\tilde{E}_\omega-$ consistent, we have

$$\gamma'_\omega(t) = (D_{\tilde{\gamma}_\omega(t)}\exp_{y_i})\tilde{\gamma}'_\omega(t) \in (D_{\tilde{\gamma}_\omega(t)}\exp_{y_i})\tilde{E}_\omega(\tilde{\gamma}_\omega(t)).$$

By (4.15), we obtain

$$\begin{aligned}\gamma'_\omega(t) &\in (D_{\tilde{\gamma}_\omega(t)}\exp_{y_i})(D_{\exp_{y_i}(\tilde{\gamma}_\omega(t))}\exp_{y_i}^{-1})E^u(\omega, \exp_{y_i}(\tilde{\gamma}_\omega(t)))\\ &\overset{(4.1)}{=} E^u(\omega, \exp_{y_i}(\tilde{\gamma}_\omega(t))) = E^u(\omega, \gamma_\omega(t))\end{aligned}$$

except finite many values. Moreover, $\gamma_\omega(0) = x$, $\gamma_\omega(1) \in V_{(\omega,y_i)}$, and

$$L(\gamma_\omega) \leqslant \sup_{v \in B_{T_{y_i}M}(0,\rho)} \|D_v \exp_{y_i}\| \cdot L(\tilde{\gamma}_\omega) \leqslant \delta.$$

Part (b) can be proved similarly. The proof of lemma 4.2 is complete. □

Remark 4.3. Notice that the unstable subbundle $E^u(\omega, \cdot)$ is defined on all points of the manifold, we can change (b) in the above lemma to

(b') For all $\epsilon > 0$, let

$$r\left(\epsilon, y_i, \frac{\pi}{4}\right) = \sin\frac{\pi}{4} \cdot \frac{1}{2} \cdot \min\left\{\frac{\rho_{y_i}}{16(C_0(\pi/4) + 1)}, \frac{\epsilon/C_{y_i}}{2C_0(\pi/4)}\right\}.$$

Then for any $x \in M$, there exists an E^u_ω-consistent curve β_ω: $[0, 1] \to M$ satisfying $\beta_\omega(0) = x$, $d_M(\beta_\omega(1), V_{(\omega,y_i)}) \geqslant C_{y_i}^{-1} r(\epsilon, y_i, \frac{\pi}{4})$, and $L(\beta_\omega) \leqslant \epsilon$.

In fact, for $x \in M \backslash B_M(y_i, \rho_{y_i})$, we let $\beta_\omega(t) \equiv x$, for all $t \in [0, 1]$.

For each $i \in \{1,\ldots,n\}$, $\omega \mapsto V_{(\omega,y_i)}$ defined in (4.13) is a compact random set, since that for each ω, $V_{(\omega,y_i)}$ is compact and for any fixed $z \in M$,

$$\omega \mapsto d_M(z,\ V_{(\omega,y_i)}) = d_M\left(z,\ \exp_{y_i}\left(\left\{v \in E^u(\omega,\ y_i)^{\perp}\colon \|v\| \leqslant \frac{\rho_{y_i}}{4(C_0(\pi/4)+1)}\right\}\right)\right)$$

is measurable by the measurability of $\omega \mapsto E^u(\omega,\ y_i)^{\perp}$.

Now let us construct the compact random set $\omega \mapsto V_{0,\delta}(\omega)$ in the statement of Property 1. For all $\omega \in \Omega$, we define

$$\begin{aligned} V_{0,\delta}(\omega) &:= \bigcup_{i=1}^{n} V_{(\omega,y_i)} \\ &= \bigcup_{i=1}^{n} \exp_{y_i}\left(\left\{v \in E^u(\omega,\ y_i)^{\perp}\colon \|v\| \leqslant \frac{\rho_{y_i}}{4(C_0(\pi/4)+1)}\right\}\right). \end{aligned}$$

This set depends on δ since the choice of $\{y_i\}_{i=1}^{n}$ depends on δ. The random set $V_{0,\delta}(\omega)$ is a compact random set by using the fact that finite union of compact random sets is a compact random set.

Now, we are ready to check whether the **(EE)**-systems have the Property 1. For any $(\omega, x) \in \Omega \times M$, then $x \in U_\delta(y_i)$ for some $i \in \{1,\ldots,n\}$ since $\{U_\delta(y_i)\}_{i=1}^{n}$ is an open cover of M. By lemma 4.2, there exists an E_ω^u-consistent curve γ_ω: $[0, 1] \to M$ with $\gamma_\omega(0) = x$, $\gamma_\omega(1) \in V_{(\omega,y_i)} \subset V_{0,\delta}(\omega)$, and $L(\gamma_\omega) \leqslant \delta$. The proof of the first statement of Property 1 is complete.

Next, we prove the second statement of Property 1. Recall that $r(\epsilon,\ y_i,\ \frac{\pi}{4})$ is defined in (4.14) for any $\epsilon > 0$ and $i \in \{1,\ldots,n\}$. We define

$$\epsilon_1 = \frac{\delta}{2},\ \epsilon_{i+1} = \frac{1}{2}\min\{\epsilon_i,\ C_{y_i}^{-1} r(\epsilon_i,\ y_i,\ \frac{\pi}{4})\}\ \text{for}\ i = 1,\ldots,n-1.$$

By construction, we have

$$0 < \epsilon_i \leqslant \frac{\delta}{2^i}\ \text{for}\ i = 1,\ 2,\ldots,n; \tag{4.16}$$

$$\epsilon_{i'} \leqslant \frac{C_{y_i}^{-1} r(\epsilon_i,\ y_i,\ \frac{\pi}{4})}{2^{i'-i}}\ \text{for}\ 1 \leqslant i < i' \leqslant n. \tag{4.17}$$

Define

$$\tau(\delta) = \frac{1}{2^n}\min_{1\leqslant i\leqslant n}\{C_{y_i}^{-1} r(\epsilon_i,\ y_i,\ \frac{\pi}{4})\}, \tag{4.18}$$

and let

$$V_{1,\delta}(\omega) := B_M(V_{0,\delta}(\omega),\ \tau(\delta))^c = M\backslash B_M(V_{0,\delta}(\omega),\ \tau(\delta)). \tag{4.19}$$

For any $(\omega, z) \in \Omega \times M$, by lemma 4.2 and remark 4.3, we can find an E_ω^u-consistent curve $\beta_{1,\omega}$: $[0, 1] \to M$ such that

$$\beta_{1,\omega}(0) = z,\ d_M(\beta_{1,\omega}(1),\ V_{(\omega,y_1)}) \geqslant C_{y_1}^{-1} r(\epsilon_1,\ y_1,\ \pi/4),\ \text{and}\ L(\beta_{1,\omega}) \leqslant \epsilon_1.$$

Inductively, for $i = 2,\ldots,n$, we can find an E_ω^u-consistent curve $\beta_{i,\omega}$: $[0, 1] \to M$ such that

$$\beta_{i,\omega}(0) = \beta_{i-1,\omega}(1),\ d_M(\beta_{i,\omega}(1),\ V_{(\omega,y_i)}) \geqslant C_{y_i}^{-1} r(\epsilon_i,\ y_i,\ \pi/4),\ \text{and}\ L(\beta_{i,\omega}) \leqslant \epsilon_i.$$

For any $i \in \{1,\ldots,n\}$, by the above construction, (4.17) and (4.18), we conclude

$$\begin{aligned} & d_M(\beta_{n,\omega}(1),\ V_{(\omega,y_i)}) \geqslant d_M(\beta_{i,\omega}(1),\ V_{(\omega,y_i)}) - d_M(\beta_{n,\omega}(1),\ \beta_{i,\omega}(1)) \\ &\geqslant C_{y_i}^{-1} r(\epsilon_i,\ y_i,\ \pi/4) - \sum_{p=0}^{n-i-1} d_M(\beta_{i+p,\omega}(1),\ \beta_{i+p+1,\omega}(1)) \\ &= C_{y_i}^{-1} r(\epsilon_i,\ y_i,\ \pi/4) - \sum_{p=0}^{n-i-1} d_M(\beta_{i+p+1,\omega}(0),\ \beta_{i+p+1,\omega}(1)) \\ &\geqslant C_{y_i}^{-1} r(\epsilon_i,\ y_i,\ \pi/4) - \sum_{p=0}^{n-i-1} L(\beta_{i+p+1,\omega}) \\ &\geqslant C_{y_i}^{-1} r(\epsilon_i,\ y_i,\ \pi/4) - \sum_{p=0}^{n-i-1} \epsilon_{i+p+1,\omega} \end{aligned}$$

$$\overset{(4.17)}{\geqslant} C_{y_i}^{-1} r(\epsilon_i, y_i, \pi/4) - \sum_{p=0}^{n-i-1} \frac{C_{y_i}^{-1} r(\epsilon_i, y_i, \pi/4)}{2^{p+1}}$$
$$= \frac{1}{2^{n-i}} C_{y_i}^{-1} r(\epsilon_i, y_i, \pi/4) \overset{(4.17)}{\geqslant} \tau(\delta).$$

This implies that

$$d_M(\beta_{n,\omega}(1),\ V_{0,\delta}(\omega)) = \min_{1 \leqslant i \leqslant n} d_M(\beta_{n,\omega}(1),\ V_{(\omega, y_i)}) \geqslant \tau(\delta).$$

We define the gluing

$$\beta_{\omega,\delta} \coloneqq \beta_{n,\omega} \cdot \cdots \beta_{2,\omega} \cdot \beta_{1,\omega},$$

which is obviously an E_ω^u-consistent curve. Moreover, $\beta_{\omega,\delta}(0) = \beta_{1,\omega}(0) = z$, $d_M(\beta_{\omega,\delta}(1), V_{0,\delta}(\omega)) = d_M(\beta_{n,\omega}(1), V_{0,\delta}(\omega)) \geqslant \tau(\delta)$, i.e. $\beta_{\omega,\delta}(1) \in V_{1,\delta}(\omega)$, and

$$L(\beta_{\omega,\delta}) \leqslant \sum_{i=1}^{k(\omega_j)} L(\beta_{i,\omega}) \leqslant \sum_{i=1}^{k(\omega_j)} \epsilon_i \leqslant \sum_{i=1}^{k(\omega_j)} \frac{\delta}{2^i} \leqslant \delta.$$

It is left to check whether $V_{1,\delta}$: $\omega \mapsto V_{1,\delta}(\omega)$ is a compact random set.

Claim 4.1. $V_{1,\delta}$: $\omega \mapsto V_{1,\delta}(\omega)$ is a compact random set.

Proof of Claim 4.1. First, we prove that $\omega \mapsto \bar{B}_M(V_{0,\delta}(\omega), \tau(\delta)) \coloneqq \{x \in M\colon\ d(x, V_{0,\delta}(\omega)) \leqslant \tau(\delta)\}$ is a compact random set. It is compact for all $\omega \in \Omega$. For any nonempty open set $O \subset M$,

$$\{\omega\colon \bar{B}_M(V_{0,\delta}(\omega), \tau(\delta)) \cap O \neq \varnothing\} = \{\omega\colon B_M(V_{0,\delta}(\omega), \tau(\delta)) \cap O \neq \varnothing\}$$
$$= \{\omega\colon V_{0,\delta}(\omega) \cap B_M(O, \tau(\delta)) \neq \varnothing\},$$

which is measurable since $\omega \mapsto V_{0,\delta}(\omega)$ is a compact random set and proposition 2.4 in [12] (see proposition A.1). Therefore, $\omega \mapsto \bar{B}_M(V_{0,\delta}(\omega), \tau(\delta))$ is a compact random set again by proposition A.1. Note that

$$B_M(V_{0,\delta}(\omega), \tau(\delta)) = \{x \in M\colon\ d(x, V_{0,\delta}(\omega)) < \tau(\delta)\} = \mathrm{interior}(\bar{B}_M(V_{0,\delta}(\omega), \tau(\delta))),$$

so it is an open random set by corollary 2.10 in [12] (see proposition A.3). As a consequence,

$$\omega \mapsto V_{1,\delta}(\omega) = B_M(V_{0,\delta}(\omega), \tau(\delta))^c = M \backslash B_M(V_{0,\delta}(\omega), \tau(\delta))$$

cis compact random set by the definition of open random set (see the definition of open random sets in definition A.1), which completes the proof of Claim 4.1. □

Thus Property 1 holds for any (**EE**)-system $(\Omega \times M, \phi)$. The proof of proposition 4.1 is complete.

4.3. Proof of Proposition 4.2

In this part, we prove any (**EE**)-system $(\Omega \times M, \phi)$ has a periodic random weak horseshoe with the help of Property 1.

Pick any $\delta > 0$ and fix it. Let the constant $\tau(\delta) > 0$ and $\tau(\delta)$-separated compact random sets $\omega \mapsto V_{0,\delta}(\omega)$ and $\omega \mapsto V_{1,\delta}(\omega)$ be defined as in the statement of Property 1. Let $\eta \in (0,\ \min\{\delta, \frac{\tau(\delta)}{3}\})$ be any constant. For all $\omega \in \Omega$, we define

$$K_0(\omega) = \bar{B}_M(V_{0,\delta}(\omega), \eta) \coloneqq \{x \in M\colon d_M(x, V_{0,\delta}(\omega)) \leqslant \eta\},$$
$$K_1(\omega) = \bar{B}_M(V_{1,\delta}(\omega), \eta) \coloneqq \{x \in M\colon d_M(x, V_{1,\delta}(\omega)) \leqslant \eta\}.$$

For each $\omega \in \Omega$, $K_0(\omega)$ and $K_1(\omega)$ are compact. For any nonempty open set $O \subset M$,

$$\begin{aligned}\{\omega\colon K_i(\omega) \cap O \neq \varnothing\} &= \{\omega\colon \bar{B}_M(V_{i,\delta}(\omega), \eta) \cap O \neq \varnothing\}\\ &= \{\omega\colon B_M(V_{i,\delta}(\omega), \eta) \cap O \neq \varnothing\}\\ &= \{\omega\colon V_{i,\delta}(\omega) \cap B_M(O, \eta) \neq \varnothing\}\, \mathrm{for}\, i \;=\; 0,\ 1.\end{aligned} \tag{4.20}$$

Since $\omega \mapsto V_{i,\delta}(\omega)$ is a compact random set for i=0,1 and $B_M(O, \eta)$ is an open set, $\{\omega\colon\ V_{i,\delta}(\omega) \cap B_M(O, \eta) \neq \varnothing\}$ is measurable for i=0,1 by proposition 2.4 in [12] (see proposition A.1). By applying proposition 2.4 in [12] (see proposition A.1) and the measurability of (4.20), both $\omega \mapsto K_0(\omega)$ and $\omega \mapsto K_1(\omega)$ are compact random sets. Moreover, they are η-seperated by the choice of η.

We pick $N \in \mathbb{N}$ large enough so that

$$\frac{C\delta}{e^{\lambda N} - 1} < \eta, \tag{4.21}$$

where C and λ come from definition 1.1. From now on, we fix this N.

With the help of Property 1 and choosing such N, we have the following claim.

Claim 4.2. For any $\omega \in \Omega$, sequence $0 = a_0 < a_1 < a_2 < \cdots < a_q$ of $\mathbb{N}$ with $a_{i+1} - a_i \geqslant N$ for $i = 0, 1, \ldots, q-1$, and $s \in \{0, 1\}^{\{a_0, \ldots, a_q\}}$, there exists a point $z_{s,\omega} \in M$ on the fiber $\{\omega\} \times M$ such that

$$f_\omega^{a_i}(z_{s,\omega}) \in K_{s(a_i)}(\theta^{a_i}\omega) \text{ for all } i = 0, 1, \ldots, q.$$

Proof of Claim 4.2. We first pick a point $z_{0,\omega} \in V_{s(a_0),\delta}(\omega)$. For $(\theta^{a_1}\omega, f_\omega^{a_1}(z_{0,\omega})) \in \Omega \times M$, by Property 1, we can find an $E^u_{\theta^{a_1}\omega}$-consistent curve $\gamma_{\theta^{a_1}\omega}\colon [0, 1] \to M$ such that

$$\gamma_{\theta^{a_1}\omega}(0) = f_\omega^{a_1}(z_{0,\omega}),\ \gamma_{\theta^{a_1}\omega}(1) \in V_{s(a_1),\delta}(\theta^{a_1}\omega), \text{ and } L(\gamma_{\theta^{a_1}\omega}) \leqslant \delta.$$

For $(\theta^{a_2}\omega, f_{\theta^{a_1}\omega}^{a_2 - a_1}(\gamma_{\theta^{a_1}\omega}(1))) \in \Omega \times M$, using Property 1 again, we can find an $E^u_{\theta^{a_2}\omega}$-consistent curve $\gamma_{\theta^{a_2}\omega}\colon [0, 1] \to M$ such that

$$\gamma_{\theta^{a_2}\omega}(0) = f_{\theta^{a_1}\omega}^{a_2 - a_1}(\gamma_{\theta^{a_1}\omega}(1)),\ \gamma_{\theta^{a_2}\omega}(1) \in V_{s(a_2),\delta}(\theta^{a_2}\omega), \text{ and } L(\gamma_{\theta^{a_2}\omega}) \leqslant \delta.$$

Inductively, for $i \in \{2, \ldots, q-1\}$, $(\theta^{a_{i+1}}\omega, f_{\theta^{a_i}\omega}^{a_{i+1} - a_i}(\gamma_{\theta^{a_i}\omega}(1))) \in \Omega \times M$, there is an $E^u_{\theta^{a_{i+1}}\omega}$-consistent curve $\gamma_{\theta^{a_{i+1}}\omega}\colon [0, 1] \to M$ such that

$$\gamma_{\theta^{a_{i+1}}\omega}(0) = f_{\theta^{a_i}\omega}^{a_{i+1} - a_i}(\gamma_{\theta^{a_i}\omega}(1)),\ \gamma_{\theta^{a_{i+1}}\omega}(1) \in V_{s(a_{i+1}),\delta}(\theta^{a_{i+1}}\omega), \text{ and } L(\gamma_{\theta^{a_{i+1}}\omega}) \leqslant \delta. \tag{4.22}$$

We define $z_{s,\omega} := f_{\theta^{a_q}\omega}^{-a_q}(\gamma_{\theta^{a_q}\omega}(1))$.

Next, we will prove $f_\omega^{a_i}(z_{s,\omega}) \in K_{s(a_i)}(\theta^{a_i}\omega)$ for all $i = 1, \ldots, q$, that is

$$d_M(f_\omega^{a_i}(z_{s,\omega}), V_{s(a_i),\delta}(\theta^{a_i}\omega)) \leqslant \eta \text{ for all } i = 1, \ldots, q.$$

For $i = q$,

$$d_M(f_\omega^{a_q}(z_{s,\omega}), V_{s(a_q),\delta}(\theta^{a_q}\omega)) = d_M(\gamma_{\theta^{a_q}\omega}(1), V_{s(a_q),\delta}(\theta^{a_q}\omega)) = 0 \leqslant \eta.$$

For any $i \in \{1, \ldots, q-1\}$, notice that $\gamma_{\theta^{a_i}\omega}(1) \in V_{s(a_i),\delta}(\theta^{a_i}\omega)$, we have

$$d_M(f_\omega^{a_i}(z_{s,\omega}), V_{s(a_i),\delta}(\theta^{a_i}\omega)) \leqslant d_M(f_\omega^{a_i}(z_{s,\omega}), \gamma_{\theta^{a_i}\omega}(1)) = d_M(f_{\theta^{a_q}\omega}^{-(a_q - a_i)}(\gamma_{\theta^{a_q}\omega}(1)), \gamma_{\theta^{a_i}\omega}(1)). \tag{4.23}$$

By the induction step (4.22), for $j \in \{0, \ldots, q-i-1\}$, we have

$$f_{\theta^{a_{i+j}}\omega}^{-(a_{i+j} - a_i)}(\gamma_{\theta^{a_{i+j}}\omega}(1)) = f_{\theta^{a_{i+j+1}}\omega}^{-(a_{i+j+1} - a_i)}(f_{\theta^{a_{i+j}}\omega}^{a_{i+j+1} - a_{i+j}}(\gamma_{\theta^{a_{i+j}}\omega}(1))) = f_{\theta^{a_{i+j+1}}\omega}^{-(a_{i+j+1} - a_i)}\gamma_{\theta^{a_{i+j+1}}\omega}(0). \tag{4.24}$$

Using the triangle inequality, (4.23) and (4.24), we have

$$\begin{aligned}
&d_M(f_\omega^{a_i}(z_{s,\omega}), V_{s(a_i),\delta}(\theta^{a_i}\omega))\\
&\leqslant d_M(f_{\theta^{a_q}\omega}^{-(a_q - a_i)}(\gamma_{\theta^{a_q}\omega}(1)), f_{\theta^{a_q}\omega}^{-(a_q - a_i)}(\gamma_{\theta^{a_q}\omega}(0))) + d_M(f_{\theta^{a_q}\omega}^{-(a_q - a_i)}(\gamma_{\theta^{a_q}\omega}(0)), \gamma_{\theta^{a_i}\omega}(1))\\
&\overset{(4.24)}{=} d_M(f_{\theta^{a_q}\omega}^{-(a_q - a_i)}(\gamma_{\theta^{a_q}\omega}(1)), f_{\theta^{a_q}\omega}^{-(a_q - a_i)}(\gamma_{\theta^{a_q}\omega}(0))) + d_M(f_{\theta^{a_{q-1}}\omega}^{-(a_{q-1} - a_i)}(\gamma_{\theta^{a_{q-1}}\omega}(1)), \gamma_{\theta^{a_i}\omega}(1))\\
&\overset{(4.24)}{=} d_M(f_{\theta^{a_q}\omega}^{-(a_q - a_i)}(\gamma_{\theta^{a_q}\omega}(1)), f_{\theta^{a_q}\omega}^{-(a_q - a_i)}(\gamma_{\theta^{a_q}\omega}(0)))\\
&+ d_M(f_{\theta^{a_{q-1}}\omega}^{-(a_{q-1} - a_i)}(\gamma_{\theta^{a_{q-1}}\omega}(1)), f_{\theta^{i+1}\omega}^{-(a_{i+1} - a_i)}(\gamma_{\theta^{i+1}\omega}(0)))\\
&\leqslant \cdots\\
&\leqslant \sum_{j=0}^{q-i-1} d_M(f_{\theta^{a_{i+j+1}}\omega}^{-(a_{i+j+1} - a_i)}(\gamma_{\theta^{a_{i+j+1}}\omega}(1)), f_{\theta^{a_{i+j+1}}\omega}^{-(a_{i+j+1} - a_i)}(\gamma_{\theta^{a_{i+j+1}}\omega}(0)))\\
&\leqslant \sum_{j=0}^{q-i-1} L(f_{\theta^{a_{i+j+1}}\omega}^{-(a_{i+j+1} - a_i)} \circ \gamma_{\theta^{a_{i+j+1}}\omega}).
\end{aligned}$$

Moreover, by lemma 3.3 and the assumption that $a_{i+1} - a_i \geqslant N$ for $i = 0, 1, \ldots, q-1$, we have

$$\begin{aligned}
d_M(f_\omega^{a_i}(z_{s,\omega}), V_{s(a_i),\delta}(\theta^{a_i}\omega)) &\leqslant \sum_{j=0}^{q-i-1} C \cdot e^{-\lambda(a_{i+j+1} - a_i)} L(\gamma_{\theta^{a_{i+j+1}}\omega})\\
&\leqslant \sum_{j=0}^{q-i-1} C \cdot e^{-\lambda(j+1)N}\delta\\
&\leqslant \frac{C\delta}{e^{\lambda N} - 1} < \eta.
\end{aligned}$$

The proof of Claim 4.2 is complete. □

For the N defined in (4.21) and any given $s \in \{0, 1\}^{\mathbb{Z}}$, in order to find a point $(\omega, x_{s,\omega}) \in \Omega \times M$ such that $f_{\omega}^{jN}(x_{s,\omega}) \in K_{s(j)}(\theta^{jN}\omega)$, we consider

$$s_N \in \{0, 1\}^{N\mathbb{Z}} \text{such that} s_N(jN) = s(j).$$

For each $q \in \mathbb{N}$, we define

$$s_N^q \in \{0, 1\}^{\{0,N,\cdots,2qN\}} \text{by} s_N^q(iN) = s_N((i-q)N) \text{for} i = 0,\ldots,2q.$$

By Claim 4.2, we choose the starting fiber to be $\{\theta^{-qN}\omega\}$, then there exists a point $z_{s_N^q,\theta^{-qN}\omega} \in M$ such that

$$f_{\theta^{-qN}\omega}^{iN}(z_{s_N^q,\theta^{-qN}\omega}) \in K_{s_N^q(iN)}(\theta^{iN}\theta^{-qN}\omega) = K_{s_N((i-q)N)}(\theta^{(i-q)N}\omega) \text{for all} i = 0,\ldots,2q,$$

i.e.

$$z_{s_N^q,\theta^{-qN}\omega} \in \bigcap_{i=0}^{2q} f_{\theta^{(i-q)N}\omega}^{-iN}(K_{s_N((i-q)N)}(\theta^{(i-q)N}\omega)).$$

So

$$f_{\theta^{-qN}\omega}^{qN}(z_{s_N^q,\theta^{-qN}\omega}) \in \bigcap_{i=0}^{2q} f_{\theta^{(i-q)N}\omega}^{(q-i)N}(K_{s_N((i-q)N)}(\theta^{(i-q)N}\omega)) = \bigcap_{j=-q}^{q} f_{\theta^{jN}\omega}^{-jN}(K_{s_N(jN)}(\theta^{-jN}\omega))$$
$$= \bigcap_{j=-q}^{q} f_{\theta^{jN}\omega}^{-jN}(K_{s(j)}(\theta^{-jN}\omega)).$$

Therefore, for any $q \in \mathbb{N}$, $\bigcap_{j=-q}^{q} f_{\theta^{jN}\omega}^{-jN}(K_{s(j)}(\theta^{-jN}\omega)) \neq \emptyset$. Note that $\bigcap_{j=-q}^{q} f_{\theta^{jN}\omega}^{-jN}(K_{s(j)}(\theta^{-jN}\omega))$ is compact for each $q \in \mathbb{N}$. By Cantor's intersection theorem, we have

$$\bigcap_{j\in\mathbb{Z}} f_{\theta^{jN}\omega}^{-jN}(K_{s(j)}(\theta^{jN}\omega)) \neq \emptyset.$$

Pick any point $x_{s,\omega} \in \bigcap_{j\in\mathbb{Z}} f_{\theta^{jN}\omega}^{-jN} K_{s(j)}(\theta^{jN}\omega)$, then $f_{\omega}^{jN}(x_{s,\omega}) \in K_{s(j)}(\theta^{jN}\omega)$ for all $j \in \mathbb{Z}$. The proof of proposition 4.2 is complete.

5. Random semi-horseshoe and periodic random weak horseshoe

In this section, we explore the relationship between the random semi-horseshoe (see definition 1.2) and the periodic random weak horseshoe (see definition 4.1). Throughout this section, we consider the skew product $(\Omega \times M, \phi)$ only satisfying the settings in section 1.1.

Proposition 5.1. *If the skew product $(\Omega \times M, \phi)$ has a periodic random weak horseshoe, then it has a random semi-horseshoe. The vice versa holds if $\mathscr{F}$, the σ-algebra on Ω in the settings (see section 1.1), is universally complete.*

Here, we recall the definition of universally complete σ-algebra (see, e.g., definition 2.3 in [12]). Let $(\Omega, \mathscr{F})$ be a measurable space. The universal completion of $\mathscr{F}$ is $\bigcap_Q \mathscr{F}_Q$, where $\mathscr{F}_Q$ denotes the completion of $\mathscr{F}$ with respect to a positive finite measure Q on $(\Omega, \mathscr{F})$, and the intersection is taken over all positive finite measure Q. Any σ-algebra $\mathscr{F}$ containing its universal completion of $\mathscr{F}$ is called universally complete. In particular, the σ-algebra of a complete probability space $(\Omega, \mathscr{F}, P)$ is universally complete, since any σ-algebra which is complete with respect to some finite measure contains its universal completion.

We prove proposition 5.1 in the next two subsections.

5.1. Periodic random weak horseshoe implies random semi-horseshoe

We first assume that the skew product $(\Omega \times M, \phi)$ has a periodic random weak horseshoe.

As in definition 4.1, we denote $\eta\colon \Omega \to \mathbb{R}^+$ to be the positive random variable, $\omega \mapsto K_0(\omega)$ and $\omega \mapsto K_1(\omega)$ to be the two η-seperated compact random sets, and N to be the number in the definition. For any $s \in \{0, 1\}^{\mathbb{Z}}$, $\omega \in \Omega$, we define

$$Y_s(\omega) := \bigcap_{i\in\mathbb{Z}} f_{\theta^{iN}\omega}^{-iN}(K_{s(i)}(\theta^{iN}\omega)).$$

By the definition of periodic random weak horseshoe,

$$Y_s(\omega) \neq \emptyset \quad \text{for all} \quad s \in \{0, 1\}^{\mathbb{Z}} \text{and} \quad \omega \in \Omega. \tag{5.1}$$

For any $M \in \mathbb{N}$ and $\omega \in \Omega$, we define

$$X_M(\omega) = \bigcup_{t\in\{0,1\}^{\mathbb{Z}\bigcap[-M,M]}} \left(\bigcap_{i=-M}^{M} f_{\theta^{iN}\omega}^{-iN}(K_{t(i)}(\theta^{iN}\omega))\right).$$

Claim 5.1. For all $\omega \in \Omega$, we have

$$\bigcap_{M=1}^{\infty} X_M(\omega) = \bigcup_{s\in\{0,1\}^{\mathbb{Z}}} Y_s(\omega). \tag{5.2}$$

Proof of Claim 5.1. We first prove the direction '⊂'. For any $x \in \bigcap_{M=1}^{\infty} X_M(\omega)$ and fixed, the following holds:

$$\forall M \in \mathbb{N},\ \exists t_M \in \{0,\ 1\}^{\mathbb{Z}\cap[-M,M]},\ \text{such that} x \in \bigcap_{i=-M}^{M} f_{\theta^{iN}\omega}^{-iN}(K_{t_M(i)}(\theta^{iN}\omega)). \tag{5.3}$$

For each t_M in (5.3), we pick

$$s_M \in \{0,\ 1\}^{\mathbb{Z}} \text{such that} s_M(i) = t_M(i) \text{for all} i = -M,\ldots,0,\ldots,M-1,\ M. \tag{5.4}$$

By the compactness of $\{0,\ 1\}^{\mathbb{Z}}$, there exists a converging subsequence of s_M, named $\{s_{M_l}\}_{l=1}^{\infty} \to s^* \in \{0,\ 1\}^{\mathbb{Z}}$. Next, we prove $x \in Y_{s^*}(\omega)$. For all $j \in \mathbb{Z}$, since $s_{M_l} \to s^*$, there exists l_j such that

$$M_{l_j} > |j|,\ \text{and} s_{M_{l_j}}(i) = s^*(i) \text{for all} i \in \{-|j|,\ -|j|+1,\ldots,|j|-1,\ |j|\}. \tag{5.5}$$

Now by (5.3), (5.4), and (5.5), we have

$$\begin{aligned} x &\in \bigcap_{i=-M_{l_j}}^{M_{l_j}} f_{\theta^{iN}\omega}^{-iN}(K_{t_{M_{l_j}}(i)}(\theta^{iN}\omega)) = \bigcap_{i=-M_{l_j}}^{M_{l_j}} f_{\theta^{iN}\omega}^{-iN}(K_{s_{M_{l_j}}(i)}(\theta^{iN}\omega)) \\ &\subset f_{\theta^{jN}\omega}^{-jN}(K_{s^*(j)}(\theta^{jN}\omega)). \end{aligned}$$

Note that j is arbitrary, hence we conclude that

$$x \in \bigcap_{j\in\mathbb{Z}} f_{\theta^{jN}\omega}^{-jN}(K_{s^*(j)}(\theta^{jN}\omega)) = Y_{s^*}(\omega).$$

Secondly, we prove the direction '⊃'. For any $y \in \bigcup_{s\in\{0,1\}^{\mathbb{Z}}} Y_s(\omega)$, there exists $s \in \{0,\ 1\}^{\mathbb{Z}}$ such that $y \in Y_s(\omega)$. For arbitrary $M \in \mathbb{N}$, let $t_M \in \{0,\ 1\}^{\mathbb{Z}\cap[-M,M]}$ be the restriction of s on $\mathbb{Z}\cap[-M,\ M]$. Then

$$y \in Y_s(\omega) \subset \bigcap_{i=-M}^{M} f_{\theta^{iN}\omega}^{-iN}(K_{t_M(i)}(\theta^{iN}\omega)) \subset X_M(\omega).$$

Since M is arbitrary, $y \in \bigcap_{M=1}^{\infty} X_M(\omega)$. The proof of Claim 5.1 is complete. □

Due to Claim 5.1, we can denote

$$Y(\omega) := \bigcap_{M=1}^{\infty} X_M(\omega) = \bigcup_{s\in\{0,1\}^{\mathbb{Z}}} Y_s(\omega).$$

Claim 5.2. The set valued map $Y: \Omega \to 2^M$ defined by $\omega \mapsto Y(\omega)$ is a compact random set.

Proof of Claim 5.2. Firstly, note that for each $\omega \in \Omega$, $X_M(\omega)$ is compact since $f_{\theta^{iN}\omega}^{-iN}$ is a diffeomorphism and $K_{t(i)}(\theta^{iN}\omega)$ is compact. Hence $Y(\omega) = \bigcap_{M=1}^{\infty} X_M(\omega)$ is compact.

Secondly, we use proposition 2.4 in [12] (see proposition A.1) to prove that $\omega \mapsto Y(\omega)$ is a compact random set. By theorem 2.6 in [12] (Proposition A.2), there exist two countable sequences of measurable maps $\{c_n^0: \Omega \to M\}_{n=1}^{\infty}$ and $\{c_n^1: \Omega \to M\}_{n=1}^{\infty}$ such that for each $\omega \in \Omega$,

$$K_0(\omega) = closure\{c_n^0(\omega): n \in \mathbb{N}\} \text{ and} K_1(\omega) = closure\{c_n^1(\omega): n \in \mathbb{N}\}.$$

For each fixed $M \in \mathbb{N}$, fixed $t \in \{0,\ 1\}^{\mathbb{Z}\cap[-M,M]}$, and fixed $i \in \{-M,\ -M+1,\ldots,M\}$, $\omega \mapsto K_{t(i)}(\omega)$ is a compact random set, since $K_{t(i)}$ is either K_0 or K_1. For any $n \in \mathbb{N}$, we define

$$c_n^{M,t,i}(\omega) := f_{\theta^{iN}\omega}^{-iN}(c_n^{t(i)}(\theta^{iN}\omega)) \text{for all} \omega \in \Omega.$$

It is clear that $\{c_n^{M,t,i}: \Omega \mapsto M\}_{n\in\mathbb{N}}$ are measurable maps, and

$$\omega \mapsto f_{\theta^{iN}\omega}^{-iN}(K_{t(i)}(\theta^{iN}\omega)) = closure\{c_n^{M,t,i}(\omega): n \in \mathbb{N}\}.$$

Hence by the selection theorem (Proposition A.2) again,

$$\omega \mapsto f_{\theta^{iN}\omega}^{-iN}(K_{t(i)}(\theta^{iN}\omega)) \text{is a compact random set.} \tag{5.6}$$

For any open set $O \subset M$, by proposition 2.4 in [12] (see proposition A.1),

$$\{\omega: f_{\theta^{iN}\omega}^{-iN}(K_{t(i)}(\theta^{iN}\omega)) \cap O \neq \varnothing\} \text{is measurable.} \tag{5.7}$$

Therefore,

$$\{\omega: Y(\omega)\cap O \neq \varnothing\} = \bigcap_{M=1}^{\infty} \bigcup_{t\in\{0,1\}^{\mathbb{Z}\cap[-M,M]}} \left(\bigcap_{i=-M}^{M} \{\omega: f_{\theta^{iN}\omega}^{-iN}(K_{t(i)}(\theta^{iN}\omega))\cap O \neq \varnothing\}\right)$$

is measurable. Hence, by proposition 2.4 in [12] (see proposition A.1), $\omega \mapsto Y(\omega)$ is a compact random set. □

We define $\pi_\omega: Y(\omega) \to \{0,\ 1\}^{\mathbb{Z}}$ by $\pi_\omega(x) = s$ if $x \in Y_s(\omega)$. For $s \neq s' \in \{0,\ 1\}^{\mathbb{Z}}$, we have $Y_s(\omega) \cap Y_{s'}(\omega) = \varnothing$. Otherwise, there exists a point $x \in Y_s(\omega)\cap Y_{s'}(\omega)$, and suppose $s(i) \neq s'(i)$ for some $i \in \mathbb{Z}$, then it contradicts to the fact that $f_\omega^{iN}(x)$ cannot be in $K_{s(i)}(\theta^{iN}\omega)$ and $K_{s'(i)}(\theta^{iN}\omega)$ at the same time since $K_{s(i)}(\theta^{iN}\omega)$ and $K_{s'(i)}(\theta^{iN}\omega)$ are $\eta(\theta^{iN}\omega)$-separated. Therefore, π_ω is well-defined.

Recall that σ: $\{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ is the left shift, i.e. $(\sigma(s))(i) = s(i+1)$ for $i \in \mathbb{Z}$.

Claim 5.3. For all $\omega \in \Omega$, f_ω^N: $Y(\omega) \to Y(\theta^N\omega)$ is a bijection and π_ω is a continuous surjective map satisfying

$$\pi_{\theta^N\omega} \circ f_\omega^N = \sigma \circ \pi_\omega \text{ for all } \omega \in \Omega. \tag{5.8}$$

Proof of Claim 5.3. For any $x \in Y(\omega)$, there exists an $s \in \{0, 1\}^{\mathbb{Z}}$ such that $x \in Y_s(\omega)$, i.e.

$$x \in \bigcap_{i\in\mathbb{Z}} f_{\theta^{iN}\omega}^{-iN}(K_{s(i)}(\theta^{iN}\omega)).$$

Then we have

$$\begin{aligned} f_\omega^N(x) &\in \bigcap_{i\in\mathbb{Z}} f_\omega^N \circ f_{\theta^{iN}\omega}^{-iN}(K_{s(i)}(\theta^{iN}\omega)) = \bigcap_{i\in\mathbb{Z}} f_{\theta^{(i-1)N}\theta^N\omega}^{-(i-1)N}(K_{(\sigma(s))(i-1)}(\theta^{(i-1)N}\theta^N\omega)) \\ &= \bigcap_{j\in\mathbb{Z}} f_{\theta^{jN}\theta^N\omega}^{-jN}(K_{(\sigma(s))(j)}(\theta^{jN}\theta^N\omega)) \\ &= Y_{\sigma(s)}(\theta^N\omega). \end{aligned}$$

The above implies that $\pi_{\theta^N\omega}(f_\omega^N(x)) = \sigma(s) = \sigma(\pi_\omega(x))$. By reversing the time, similar proof can be applied to show that $f_{\theta^N\omega}^{-N}(y) \in Y_{\sigma^{-1}(s)}(\omega)$ for any $s \in \{0, 1\}^{\mathbb{Z}}$, $y \in Y_s(\theta^N\omega)$. Note that both f_ω^N and $f_{\theta^N\omega}^{-N}$ are injective. We conclude that f_ω^N: $Y(\omega) \to Y(\theta^N\omega)$ is bijective.

Next, we prove π_ω is surjective and continuous. The surjectivity of π_ω follows by that $Y_s(\omega) \neq \emptyset$ for all $s \in \{0, 1\}^{\mathbb{Z}}$ and $\omega \in \Omega$ by (5.1). Let $\{x_j\}_{j=1}^\infty$ be a sequence in $Y(\omega)$ converging to $x \in Y(\omega)$. There exists a unique $s^j \in \{0, 1\}^{\mathbb{Z}}$ for $j \in \mathbb{N}$ and $s \in \{0, 1\}^{\mathbb{Z}}$ such that $x_j \in Y_{s^j}(\omega)$ and $x \in Y_s(\omega)$ respectively. For each $i \in \mathbb{Z}$, we have

$$f_\omega^{iN}(x_j) \in K_{s^j(i)}(\theta^{iN}\omega). \tag{5.9}$$

Note that $f_\omega^{iN}(x) \in K_{s(i)}(\theta^{iN}\omega)$, then we fix i and let $j \to \infty$ in (5.9). By the continuity of f_ω^{iN}, we conclude that $s^j(i) = s(i)$ for large enough j. Thus $s^j \to s$. Hence π_ω is continuous.

The proof of Claim 5.3 is complete. □

Now we define π: $graph(Y) \to \Omega \times \{0, 1\}^{\mathbb{Z}}$ by

$$\pi(\omega, x) = (\omega, \pi_\omega(x)).$$

It is left to prove that π is a measurable map with respect to the σ-algberas $graph(Y) \cap (\mathscr{F} \otimes \mathcal{B}(M))$ and $\mathscr{F} \otimes \mathcal{B}(\{0, 1\}^{\mathbb{Z}})$. For any cylinder set

$$C_{j_1,\ldots,j_k}^{n_1,\ldots,n_k} = \{s \in \{0, 1\}^{\mathbb{Z}}: s(n_i) = j_i \text{ for } i = 1,\ldots,k\},$$

where $n_1 < n_2 < \cdots < n_k$ are integers and $j_1,\ldots,j_k \in \{0,1\}$, and $A \in \mathscr{F}$, we have

$$\begin{aligned} &\pi^{-1}(A \times C_{j_1,\ldots,j_k}^{n_1,\ldots,n_k}) \\ &= \{(\omega, x) \in graph(Y) |\ \pi_\omega(x)(n_i) = j_i \text{ for } i = 1,\ldots,k \text{ and } \omega \in A\} \\ &= \left\{(\omega, x) \in graph(Y) |\ x \in \bigcap_{i=1}^k f_{\theta^{n_i N}\omega}^{-n_i N}(K_{j_i}(\theta^{n_i N}\omega)) := Y_{j_1,\ldots,j_k}^{n_1,\ldots,n_k}(\omega) \text{ and } \omega \in A\right\} \\ &= graph(Y_{j_1,\ldots,j_k}^{n_1,\ldots,n_k}) \cap graph(Y) \cap (A \times M), \end{aligned} \tag{5.10}$$

where $Y_{j_1,\ldots,j_k}^{n_1,\ldots,n_k}$: $\Omega \to 2^M$ is defined by $Y_{j_1,\ldots,j_k}^{n_1,\ldots,n_k}(\omega) = \bigcap_{i=1}^k f_{\theta^{n_i N}}^{-n_i N}(K_{j_i}(\theta^{n_i N}\omega))$. By the same reason as (5.6), $\omega \mapsto f_{\theta^{n_i N}}^{-n_i N}(K_{j_i}(\theta^{n_i N}\omega))$ is a compact random set for each $i \in \{1,..,k\}$. Notice the fact that the finite intersection of compact random sets is still a compact random set, therefore $Y_{j_1,\ldots,j_k}^{n_1,\ldots,n_k}$ is a compact random set. So $graph(Y_{j_1,\ldots,j_k}^{n_1,\ldots,n_k})$ is a measurable subset of $\Omega \times M$ by proposition 2.4 in [12] (see proposition A.1). By (5.10), $\pi^{-1}(A \times C_{j_1,\ldots,j_k}^{n_1,\ldots,n_k})$ is measurable with respect to the σ-algebra $graph(Y) \cap (\mathscr{F} \otimes \mathcal{B}(M))$. Hence, π is measurable.

5.2. Random semi-horseshoe implies periodic random weak horseshoe when $\mathscr{F}$ is universally complete

In this section, we consider the case that the σ-algebra $\mathscr{F}$ on the space Ω in settings of this paper is replaced by a universal complete σ-algebra, named $\bar{\mathscr{F}}$. In particular, the σ-algebra $\mathscr{F}$ in the definition of random semi-horseshoe is replaced by $\bar{\mathscr{F}}$.

Suppose now we have a random semi-horseshoe. Let Y, N, $\{\pi_\omega\}_{\omega\in\Omega}$ and π be given in the definition of random semi-horseshoe (Definition 1.2). For any $\omega \in \Omega$, we define

$$K_i(\omega) := \pi_\omega^{-1}(\{s \in \{0, 1\}^{\mathbb{Z}}: s(0) = i\}) \text{ for } i = 0, 1. \tag{5.11}$$

By the measurability of π,

$$\pi^{-1}(\{(\omega, s) \in \Omega \times \{0, 1\}^{\mathbb{Z}}: s(0) = i\}) = graph(K_i)$$

is measurable in $graph(Y) \cap \bar{\mathscr{F}} \otimes \mathcal{B}(M)$. For each i=0,1 and $\omega \in \Omega$, $K_i(\omega)$ is compact by the continuity of π_ω. Note that $\bar{\mathscr{F}}$ is the universal completion of $\mathscr{F}$ on Ω, then proposition 2.4 in [12] (see proposition A.1) implies that $\omega \mapsto K_i(\omega)$ is a compact random set for $i = 0, 1$. By the Selection theorem, theorem 2.6 in [12] (Proposition A.2), there exist two countable sequences of measurable maps $\{c_n^0: \Omega \to M\}_{n=1}^{\infty}$ and $\{c_n^1: \Omega \to M\}_{n=1}^{\infty}$ such that for each $\omega \in \Omega$,

$$K_0(\omega) = closure\{c_n^0(\omega): n \in \mathbb{N}\} \text{ and} K_1(\omega) = closure\{c_n^1(\omega): n \in \mathbb{N}\}.$$

Let

$$\eta(\omega) = \inf_{n,k \in \mathbb{N}} \{d_M(c_n^0(\omega), c_k^1(\omega))\}.$$

Due to the measurability and countability of $\{c_n^i\}_{n \in \mathbb{N}}$ for i=0,1, $\eta: \Omega \to \mathbb{R}$ is measurable. By the disjointness of the cylinder set $C_0^0 := \{s \in \{0, 1\}^{\mathbb{Z}}: s(0) = 0\}$ and $C_1^0 := \{s \in \{0, 1\}^{\mathbb{Z}}: s(0) = 1\}$, and the continuity of π_ω, $K_0(\omega) = \pi_\omega^{-1}(C_0^0)$ and $K_1(\omega) = \pi_\omega^{-1}(C_1^0)$ are two disjoint compact sets. Therefore, $\eta(\omega) = d_M(K_0(\omega), K_1(\omega)) > 0$ for all $\omega \in \Omega$.

Now for any $s \in \{0, 1\}^{\mathbb{Z}}$, $\omega \in \Omega$, pick a point $x_{s,\omega} \in \pi_\omega^{-1}(s)$. For any $j \in \mathbb{Z}$, by (1) in definition 1.2,

$$\pi_{\theta^{jN}\omega}(f_\omega^{jN}(x_{s,\omega})) = \sigma^j(\pi_\omega(x_{s,\omega})) = \sigma^j(s).$$

Hence, for any $j \in \mathbb{Z}$, we have

$$\begin{aligned} f_\omega^{jN}(x_{s,\omega}) &\in \pi_{\theta^{jN}\omega}^{-1}(\{s' \in \{0, 1\}^{\mathbb{Z}}: s'(0) = (\sigma^j(s))(0)\}) \\ &= \pi_{\theta^{jN}\omega}^{-1}(\{s' \in \{0, 1\}^{\mathbb{Z}}: s'(0) = s(j)\}) \\ &= K_{s(j)}(\theta^{jN}\omega), \end{aligned}$$

which shows the existence of periodic random weak horseshoe.

6. Proof of theorem 1

Now, we are ready to prove that the (**EE**)-systems have random semi-horseshoe. Proposition 4.2 says that any (**EE**)-system has a periodic random weak horseshoe. By proposition 5.1, theorem 1 is proved.

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Data availability statement

No new data were created or analysed in this study.

Appendix

In this appendix, we supply some necessary knowledge that is used in the proof of our main theorem. All of these are borrowed from [12].

Let $(\Omega, \mathscr{F})$ be a measurable space, and (X,d) be a Polish space.

Definition Appendix A.1. (Definition 2.1 in [12]) A set valued map $C: \Omega \to 2^X$ is said to be a closed (compact respectively) random set if

(1) for each $\omega \in \Omega$, $C(\omega)$ is closed (compact respectively);

(2) for each $x \in M$, the map $\omega \mapsto d(x, C(\omega))$ is measurable.

A set valued map $\omega \mapsto U(\omega)$ is said to be an open random set if its complement $\omega \mapsto U^c(\omega)$ is a closed random set.

Remark Appendix A.1. In this paper, since M is compact, a random set C is a closed random set if and only if it is a compact random set.

Proposition Appendix A.1. *(Proposition 2.4 in [12]) For a set valued map $C: \Omega \to 2^X$, taking values in the closed subsets of a Polish space X, consider the following conditions:*

(i) $\omega \mapsto d(x, C(\omega))$ *is measurable for every $x \in X$ (i.e. C is a closed random set);*

(ii) *for all (nonempty) open set $U \subset X$, the set $\{\omega \mapsto C(\omega) \cap U \neq \varnothing\}$ is measurable;*

(iii) *for every $\delta > 0$, graph (C^δ) is a measurable subset of $\Omega \times X$, where graph (C^δ) is the graph of the δ-neighborhood of $\omega \mapsto C^\delta(\omega)$ of C;*

(iv) graph(C) *is a measurable subset of $\Omega \times X$.*

Then (i), (ii) and (iii) are equivalent, and either of them implies (iv). Furthermore, (iv) implies (i) if $\mathscr{F}$, the σ-algebra on Ω, is universally complete.

Proposition Appendix A.2 The Selection Theorem. *(Theorem 2.6 in [12]) A set valued map $C: \Omega \to 2^X$ is a closed random set if and only if there exists a sequence $\{c_n\}_{n\in\mathbb{N}}$ of measurable maps $c_n: \Omega \to X$, such that $C(\omega) =$ closure $\{c_n(\omega): n \in \mathbb{N}\}$ for all $\omega \in \Omega$.*

Proposition Appendix A.3. *(Corollary 2.10 in [12]) If $\omega \mapsto C(\omega)$ is a closed random set, then* interior (C) *is an open random set.*

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