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Exponential decay of random correlations for random Anosov systems mixing on fibers

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Abstract

In this paper, we study the statistical property of Anosov systems on surface driven by an external force. By utilizing the Birkhoff cone method, we show that if the systems on surface satisfying the Anosov and topological mixing on fibers property, then the quenched random correlation for Hölder observables with respect to the unique random SRB measures decays exponentially.

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Keywords: Random dynamical system; Exponential decay of quenched random correlation; Random SRB measure

Contents

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1. Introduction

In this paper, we investigate the decay rate of quenched random correlation for Anosov system on surface driven by an external force with respect to the random SRB measure. Let M be a surface, by which we mean a connected closed smooth 2-dimensional Riemannian manifold. Let $\theta : \Omega \to \Omega$ be a homeomorphism on a compact metric space (Ω, d_Ω), which will describe the external force. A dynamical system driven by (Ω, θ) is a mapping

$$F : \mathbb{Z} \times \Omega \times M \to M, \ (n, \omega, x) \mapsto F(n, \omega, x)$$

satisfying for each $n \in \mathbb{Z}$, $(\omega, x) \mapsto F(n, \omega, x)$ is continuous and the mappings $F(n, \omega) := F(n, \omega, \cdot) : M \to M$ form a cocycle over θ, i.e.,

$$F(0, \omega) = id_M \text{ for all } \omega \in \Omega,$$

$$F(n+m, \omega) = F(n, \theta^m \omega) \circ F(m, \omega) \text{ for all } n, m \in \mathbb{Z}, \ \omega \in \Omega.$$

When $(\Omega, \mathcal{B}(\Omega))$ is equipped with a θ-invariant probability measure $\mathbb{P}$, F is a continuous random dynamical system (abbr. RDS) [4]. We denote f_ω by the time-one map $F(1, \omega)$ of the RDS, and assume f_ω to be a diffeomorphism for all $\omega \in \Omega$. Putting f_ω and θ together forms a skew product transformation $\phi : M \times \Omega \to M \times \Omega$ by $\phi(x, \omega) = (f_\omega(x), \theta\omega)$. The skew product system ϕ is called Anosov on fibers if for every $(x, \omega) \in M \times \Omega$, there is a splitting of the tangent bundle of M at x

$$T_x M = E^s(x, \omega) \oplus E^u(x, \omega)$$

which depends continuously on $(x,\omega)\in M\times\Omega$ with $\dim E^s(x,\omega)=\dim E^u(x,\omega)=1$, and the splitting is invariant in the sense that

$$D_x f_\omega E^u(x,\omega)=E^u(f_\omega(x),\theta\omega),\quad D_x f_\omega E^s(x,\omega)=E^s(f_\omega(x),\theta\omega),$$

and

$$\begin{cases}|D_x f_\omega \xi|\ge e^{\lambda_0}|\xi|, & \forall \xi\in E^u(x,\omega),\\ |D_x f_\omega \eta|\le e^{-\lambda_0}|\eta|, & \forall \eta\in E^s(x,\omega),\end{cases}$$

where $\lambda_0>0$ is a constant. We assume that ϕ is topological mixing on fibers, that is, for any nonempty open sets $U,\ V\subset M$, there exists $N>0$ such that for any $n\ge N$ and $\omega\in\Omega$, $\phi^n(U\times\{\omega\})\cap(V\times\{\theta^n\omega\})\neq\emptyset$.

A random probability measure $\omega\mapsto\mu_\omega$ on M is a measurable map $\mu:\Omega\to Pr(M)$, where $Pr(M)$ is the space of Borel probability measures on M equipped with the Borel σ-algebra generated by the weak* topology. A random probability measure $\omega\mapsto\mu_\omega$ is said to be ϕ-invariant if $(f_\omega)_*\mu_\omega=\mu_{\theta\omega}$ for $\mathbb{P}$-a.s. $\omega\in\Omega$. Given a pair of regular observables φ and ψ on M, we say the (quenched) past random correlation function of φ and ψ with respect to the system ϕ and an invariant random probability measure $(\mu_\omega)_{\omega\in\Omega}$ decays exponentially if

$$\left|\int_M \psi(F(n,\theta^{-n}\omega,x))\varphi(x)d\mu_{\theta^{-n}\omega}-\int_M\psi(x)d\mu_\omega\int_M\varphi(x)d\mu_{\theta^{-n}\omega}\right|\to 0$$

exponentially as $n\to\infty$. We say that the (quenched) future random correlation function of φ and ψ decays exponentially if

$$\left|\int_M \psi(F(n,\omega,x))\varphi(x)d\mu_\omega-\int_M\psi(x)d\mu_{\theta^n\omega}\int_M\varphi(x)d\mu_\omega\right|\to 0$$

exponentially as $n\to\infty$. If ϕ is Anosov and topological mixing on fibers, then ϕ is a random topological transitive hyperbolic systems [22], so there exists a unique random SRB measure and the unique random SRB measure is given by $\mu_\omega:=\lim_{n\to\infty}(f^n_{\theta^{-n}\omega})_*m$, where m is the normalized Riemannian volume measure [20]. In this paper, we prove that such system ϕ and the unique random SRB measure have exponential decay of both past and future random correlations for Hölder observables ψ and φ. We emphasis that our result holds for all $\omega\in\Omega$, while most result of decay of quenched random correlation only holds for $\mathbb{P}$-a.s. $\omega\in\Omega$.

Recently, Anosov on fibers systems with the topological mixing on fibers property have been studied in [22] from the topological complexity perspective, in which the authors proved dynamical complexity such as the density of random periodic points, strong random horseshoe, the density of measure equi-distributed on a random periodic orbit, and a simplified random Livšic theorem. Examples such as fiber Anosov maps on 2-dimension torus driven by irrational rotation on the torus and random composition of 2×2 area-preserving positive matrices are under consideration. Moreover, the Anosov on fibers systems contain a class of partially hyperbolic systems. In fact, let Ω be a compact differentiable manifold, and let $\theta:\Omega\to\Omega$ be a diffeomorphism such that the expansion of $D\theta$ is weaker than e^{λ_0} and contraction of $D\theta$ is weaker than $e^{-\lambda_0}$.

Furthermore, we assume $f_\omega(x)$ and $f_\omega^{-1}(x)$ are C^1 in ω. Then the system ϕ is a partially hyperbolic system, and the dimension of central direction coincides with $\dim \Omega$ (see Proposition A.1 in Appendix).

For deterministic dynamical systems, there are a large number of results considering the exponential decay of correlations, for instance [2,9,12–14,18,19,26,30,32,35,40,42,43]. For RDS, the exponential decay of (quenched) random correlations was obtained for random Lasota–Yorke maps on intervals [10], for random perturbations of expanding maps [8], for i.i.d. unimodal maps [7] and for a class of non-uniformly expanding random dynamical systems [38]. In [25], the topological one-sided random shift of finite type with the fiber Gibbs measure was proved to have certain nonuniform ω-wise decay of correlations, and similar results held for random expanding in average transformations. Note that the system in this paper belongs to random hyperbolic system and it is randomly conjugated to a two-sided random shift [20]. Recently, the authors in [1] obtain the exponential decay of quenched future random correlations for random perturbation of topological mixing uniformly hyperbolic system and a family of equivariant physical measure. We note that our results are independent, since our main example can not be obtained by perturbation, and it is unknown whether the family of equivariant physical measure coincides the random SRB measure. Other decay rates of random correlations such as stretched exponential and polynomial decay were also considered for certain random dynamical systems [1,5,27,36].

In this paper, we prove the exponential decay of (quenched) random correlations for Anosov and topological mixing on fibers system by directly studying the fiber transfer operator L_ω, which is defined by

$$L_\omega\varphi : M \to \mathbb{R},\ (L_\omega\varphi)(x) := \frac{\varphi((f_\omega)^{-1}x)}{|\det D_{(f_\omega)^{-1}(x)} f_\omega|} \tag{1.1}$$

for any measurable observables $\varphi : M \to \mathbb{R}$. We construct a Birkhoff cone on each fiber and introduce the projective metric on each fiber Birkhoff cone. The construction of the Birkhoff cone on each fiber is inspired by the construction in [30] and [39]. The most technical analysis in this paper lies in estimating the diameter of the image of $L_\omega^N = L_{\theta^{N-1}\omega} \circ \cdots \circ L_\omega$ on fiber Birkhoff cone, where N comes from the mixing on fibers property. We prove that the image of L_ω^N acting on the ω-fiber Birkhoff cone has finite diameter with respect to the projective metric on $\theta^N\omega$-fiber Birkhoff cone. Moreover, this diameter is uniformly finite for all $\omega \in \Omega$. As a consequence of Birkhoff's inequality, L_ω^N is a contraction and the contraction rate is independent of $\omega \in \Omega$. The contraction implies weak*-limit of $(L_{\theta^{-n}\omega}^n 1)dm$ exists and it gives the unique random SRB measure, where m is the normalized Riemannian volume measure. The exponential decay of random correlations can be obtained from the contraction and the usual techniques in deterministic systems.

The Birkhoff cone approach has been used extensively to study the transfer operator and exponential decay of correlations. Liverani in [30] used it to prove the exponential decay of correlations for smooth uniformly hyperbolic area-preserving cases. Later, it was generalized to general Axiom A attractors in [6,39], and some partially hyperbolic systems [3,11]. For RDS, the Birkhoff cone approach was used in [8] and [38] for exponential decay of (quenched) random correlations.

This paper is organized as follows. In Section 2, we state the settings and formulate the main result. In Section 3, we introduce several preliminary lemmas and propositions to pave the way for the proof of the main result. Section 4 addresses the proof of the main result based on the

Birkhoff cone. We recall the definitions of convex cone, projective metric and Birkhoff's inequality in the Appendix.

2. Setting and main result

In this section, we begin with the setting of Anosov and mixing on fibers systems. After introducing several necessary notations, we formulate the main result.

2.1. Anosov and mixing on fibers systems

Let M be a connected closed smooth Riemannian manifold with $\dim M = 2$, and d_M be the induced Riemannian metric on M, (Ω, d_Ω) be a compact metric space, and $\theta : \Omega \to \Omega$ be a homeomorphism. Let $\mathbb{P}$ be an ergodic Borel probability measure on Ω. $M \times \Omega$ is a compact metric space with distance $d((x_1, \omega_1), (x_2, \omega_2)) = d_M(x_1, x_2) + d_\Omega(\omega_1, \omega_2)$ for any $x_1, x_2 \in M$ and $\omega_1, \omega_2 \in \Omega$. Let $\mathrm{Diff}^2(M)$ be the space of C^2 diffeomorphisms on M with C^2-topology (see, e.g., [21]), and $f : \Omega \to \mathrm{Diff}^2(M)$ be a continuous map. The skew product $\phi : M \times \Omega \to M \times \Omega$ induced by f and θ is defined by:

$$\phi(x, \omega) = (f(\omega)x, \theta\omega) = (f_\omega x, \theta\omega), \ \forall \omega \in \Omega, \ x \in M,$$

where we rewrite $f(\omega)$ as f_ω. Then inductively:

$$\phi^n(x, \omega) = (f_\omega^n x, \theta^n \omega) := \begin{cases} (f_{\theta^{n-1}\omega} \circ \cdots \circ f_\omega x, \theta^n \omega), & \text{if } n > 0 \\ (x, \omega), & \text{if } n = 0 \\ ((f_{\theta^n \omega})^{-1} \circ \cdots \circ (f_{\theta^{-1}\omega})^{-1} x, \theta^n \omega), & \text{if } n < 0. \end{cases} \tag{2.1}$$

The system ϕ is called Anosov on fibers if the following is true: for every $(x, \omega) \in M \times \Omega$, there is a splitting of the tangent bundle of M at x

$$T_x M = E^s(x, \omega) \oplus E^u(x, \omega)$$

which depends continuously on $(x, \omega) \in M \times \Omega$ with $\dim E^s(x, \omega) = \dim E^u(x, \omega) = 1$ and satisfies that

$$D_x f_\omega E^u(x, \omega) = E^u(f_\omega(x), \theta\omega), \ \ D_x f_\omega E^s(x, \omega) = E^s(f_\omega(x), \theta\omega)$$

and

$$\begin{cases} |D_x f_\omega \xi| \geq e^{\lambda_0} |\xi|, & \forall \xi \in E^u(x, \omega), \\ |D_x f_\omega \eta| \leq e^{-\lambda_0} |\eta|, & \forall \eta \in E^s(x, \omega), \end{cases}$$

where $\lambda_0 > 0$ is a constant. We say that $\phi : M \times \Omega \to M \times \Omega$ is topological mixing on fibers if for any nonempty open sets $U, V \subset M$, there exists $N > 0$ such that for any $n \geq N$ and $\omega \in \Omega$

$$\phi^n(U \times \{\omega\}) \cap V \times \{\theta^n \omega\} \neq \emptyset.$$

Examples of Anosov and topological mixing on fibers systems are given in Appendix A.1.

2.2. Random probability measures

The following notations are borrowed from [15]. Denote $Pr(M)$ to be the space of probability measures on $(M, \mathcal{B}(M))$ equipped with weak* topology.

A map $\mu : \mathcal{B}(M) \times \Omega \to [0,1]$ by $(B, \omega) \mapsto \mu_\omega(B)$ is said to be a random probability measure on M if it satisfies: for every $B \in \mathcal{B}(M)$, $\omega \mapsto \mu_\omega(B)$ is measurable; and for $\mathbb{P}$-almost every $\omega \in \Omega$, $B \to \mu_\omega(B)$ is a Borel probability measure. By Remark 3.2 in [15], the second condition can be relaxed to: for every D from a $\cap$-stable family $\mathfrak{F}$ of Borel sets of M which generates $\mathcal{B}(M)$ (i.e. $\sigma(\mathfrak{F}) = \mathcal{B}(M)$), $\omega \mapsto \mu_\omega(D)$ is measurable. A typical example of $\mathfrak{F}$ is the family of all closed sets in M. We denote a random probability measure by $\omega \mapsto \mu_\omega$ or $(\mu_\omega)_{\omega\in\Omega}$.

A random probability measure $\omega \mapsto \mu_\omega$ is said to be ϕ-invariant if $(f_\omega)_*\mu_\omega = \mu_{\theta\omega}$ for $\mathbb{P}$-a.s. $\omega \in \Omega$.

2.3. Main result

Let $C^0(M)$ be the collection of all continuous functions $\varphi : M \to \mathbb{R}$. For $\alpha \in (0,1)$, and $\varphi \in C^0(M)$, denote

$$\|\varphi\|_{C^0(M)} := \sup_{x\in M} |\varphi(x)| \text{ and } |\varphi|_\alpha := \sup_{x,y\in M, x\neq y} \frac{|\varphi(x) - \varphi(y)|}{d(x,y)^\alpha}.$$

We denote $C^{0,\alpha}(M) := \{\varphi \in C^0(M) : \ |\varphi|_\alpha < \infty\}$ to be the space of α-Hölder continuous functions on M. For $\varphi \in C^{0,\alpha}(M)$, we let

$$\|\varphi\|_{C^{0,\alpha}(M)} := \|\varphi\|_{C^0(M)} + |\varphi|_\alpha.$$

Theorem 1. *Assume that ϕ satisfies Anosov on fibers and topological mixing on fibers. Then*

(1) there exists an invariant random probability measure $\omega \mapsto \mu_\omega$ given by $\mu_\omega = \lim_{n\to\infty} (f^n_{\theta^{-n}\omega})_* m$ for all $\omega \in \Omega$, where m is the normalized Riemannian volume measure,
(2) there exists a constant ν_0 that only depends on the system ϕ such that, for Hölder exponents $\kappa, \nu \in (0,1)$ with

$$0 < \kappa + \nu < \nu_0$$

and $\psi \in C^{0,\kappa}(M)$, $\varphi \in C^{0,\nu}(M)$, the (quenched) past and future random correlation between φ and ψ exponential decay with respect to the random probability measure $(\mu_\omega)_{\omega\in\Omega}$ defined in (1), i.e. for any $n \in \mathbb{N}$, $\omega \in \Omega$,

$$\left| \int_M \psi(f^n_{\theta^{-n}\omega} x)\varphi(x) d\mu_{\theta^{-n}\omega} - \int_M \psi(x) d\mu_\omega \int_M \varphi(x) d\mu_{\theta^{-n}\omega} \right| \le K \|\psi\|_{C^{0,\kappa}(M)} \cdot \|\varphi\|_{C^{0,\nu}(M)} \cdot \Lambda^n;$$

$$\left| \int_M \psi(f^n_\omega x)\varphi(x) d\mu_\omega - \int_M \psi(x) d\mu_{\theta^n\omega} \int_M \varphi(x) d\mu_\omega \right| \le K \|\psi\|_{C^{0,\kappa}(M)} \cdot \|\varphi\|_{C^{0,\nu}(M)} \cdot \Lambda^n,$$

where constants $K > 1$ and $\Lambda \in (0,1)$ only depend on κ, ν and system ϕ.

We note that topological mixing on fibers property implies random topological transitivity by [22, Lemma A.1]. Then according to [20, Theorem 4.3], the measure μ_ω we constructed above is the unique random SRB measure (we state this lemma and this theorem in the Appendix).

3. Preliminary lemmas and propositions

In this section, we introduce several technical lemmas and propositions that will be used in the proof of the main result. Lemmas in Subsections 3.1 and 3.2 can be viewed as the combination of Lusin's theorem and corresponding results in general RDS [29] by noticing that our system f_ω, $E^s(x,\omega)$ and $E^u(x,\omega)$ are continuous depending on ω. We state and prove two distortion lemmas in Subsection 3.3. We formulate and prove the absolute continuity and Hölder continuity of the stable and unstable foliations on each fiber in Subsection 3.4 and 3.5 respectively. We discuss properties of holonomy maps between local stable leaves in Subsection 3.6. In Subsection 3.7, we prove a version of Fubini's theorem on any rectangle on each fiber.

3.1. Hölder continuity of stable and unstable subbundle on each fiber

In this subsection, we will formulate the Hölder continuity of $E^s(x,\omega)$ and $E^u(x,\omega)$ on $x \in M$ for any $\omega \in \Omega$, which is borrowed from [29].

For nontrivial closed subspaces A, B in a Hilbert space H with given inner product and induced norm $|\cdot|$, define the aperture between A and B by

$$\Gamma(A,B) := \max\left\{ \sup_{v\in A, |v|=1} \inf_{w\in B} |v-w|, \sup_{w\in B, |w|=1} \inf_{v\in A} |v-w| \right\}.$$

Then $\Gamma(A,B) \in [0,1]$. One also has $\Gamma(A,B) = \|P_A - P_B\|$, where P_A and P_B are the orthogonal projections on A and B respectively (see more details in [23, Chap. Four sec. 2]).

By the compactness of M, there exists a $\rho_0 > 0$ such that for any $x, y \in M$ with $d(x,y) < \rho_0$, there exists a unique geodesic connecting x and y. For any $x, y \in M$, if $d(x,y) < \rho_0$, then there exists an isometry from T_xM to T_yM given by the parallel transport on the unique geodesic connecting x and y, named $P(x,y)$. Then for any $x, y \in M$, $E(x) \subset T_xM$, $E(y) \subset T_yM$ subspaces, we can define

$$d(E(x), E(y)) := \begin{cases} \Gamma_x(E(x), P(y,x)E(y)), & \text{if } d(x,y) < \rho_0 \\ 1, & \text{otherwise,} \end{cases} \tag{3.1}$$

where Γ_x is the aperture between subspaces in T_xM defined by the given Riemannian metric. The following lemma is an adapted version of Theorem 4.1 in [29] by noticing that $f_\omega \in \mathrm{Diff}^2(M)$ is continuously depending on $\omega \in \Omega$.

Lemma 3.1. *There are constants $C_1 > 0$ and $\nu_1 \in (0,1)$ such that for each $\omega \in \Omega$, $E^s(x,\omega)$ and $E^u(x,\omega)$ are (C_1, ν_1)-Hölder continuous on x, i.e.,*

$$d(E^\tau(x,\omega), E^\tau(y,\omega)) \le C_1 d(x,y)^{\nu_1}, \quad \tau = s, u. \tag{3.2}$$

3.2. Stable and unstable invariant manifolds

We define the local stable and unstable manifolds as the following:

$$W_\epsilon^s(x,\omega) = \{y \in M \mid d(\phi^n(y,\omega), \phi^n(x,\omega)) \le \epsilon \text{ for all } n \ge 0\},$$
$$W_\epsilon^u(x,\omega) = \{y \in M \mid d(\phi^n(y,\omega), \phi^n(x,\omega)) \le \epsilon \text{ for all } n \le 0\}.$$

The following lemma can be found in [22, Lemma 3.1], and it is a special version of [20, Theorem 3.1].

For $\tau = s, u$, denote $P(E^\tau(x,\omega))$ to be the projection from T_xM to $E^\tau(x,\omega)$ with respect to the splitting $T_xM = E^s(x,\omega) \oplus E^u(x,\omega)$. Since $E^s(x,\omega),\ E^u(x,\omega)$ are uniformly continuous on x and ω, there exists a number $\mathcal{P} > 1$ such that

$$\sup\{\|P(E^s(x,\omega))\|, \|P(E^u(x,\omega))\| :\ (x,\omega) \in M \times \Omega\} < \mathcal{P}. \tag{3.3}$$

Lemma 3.2 *(Stable and unstable invariant manifolds). For any $\lambda \in (0,\lambda_0)$, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0,\epsilon_0]$, the followings hold:*

(1) $W_\epsilon^\tau(x,\omega)$ are C^2 embedded discs for all $(x,\omega) \in M \times \Omega$ with $T_xW^\tau(x,\omega) = E^\tau(x,\omega)$ for $\tau = u, s$. Moreover, there exist a constant $L > 1$ and C^2 maps

$$h^u_{(x,\omega)} : E^u(x,\omega)(\mathcal{P}\epsilon) \to E^s(x,\omega),\ h^s_{(x,\omega)} : E^s(x,\omega)(\mathcal{P}\epsilon) \to E^u(x,\omega)$$

such that $W_\epsilon^\tau(x,\omega) \subset Exp_x(graph(h^\tau_{(x,\omega)}))$ and $\|Dh^\tau_{(x,\omega)}\| < \frac{1}{3},\ Lip(Dh^\tau_{(x,\omega)}) < L$ for $\tau = u, s$, where $E^\tau(x,\omega)(\mathcal{P}\epsilon)$ is the $\mathcal{P}\epsilon$-ball of $E^\tau(x,\omega)$ centered at origin for $\tau = u, s$.

(2) $d_M(f^n_\omega x, f^n_\omega y) \le e^{-n\lambda} d_M(x,y)$ for $y \in W^s_\epsilon(x,\omega)$ and $n \ge 0$, and $d_M(f^{-n}_\omega x, f^{-n}_\omega y) \le e^{-n\lambda} d_M(x,y)$ for $y \in W^u_\epsilon(x,\omega)$ and $n \ge 0$.

(3) $W^s_\epsilon(x,\omega), W^u_\epsilon(x,\omega)$ vary continuously on (x,ω) in C^1 topology.

The following lemma is about the local product structure on each fiber.

Lemma 3.3. *[22, Lemma 3.2] For any $\epsilon \in (0,\epsilon_0)$, there is a $\delta \in (0,\epsilon)$ such that for any $x, y \in M$ with $d_M(x,y) < \delta$, $W^s_\epsilon(x,\omega) \cap W^u_\epsilon(y,\omega)$ consists of a single point, which is denoted by $[x,y]_\omega$. Furthermore*

$$[\cdot,\cdot]_\cdot : \{(x,y,\omega) \in M \times M \times \Omega \mid d_M(x,y) < \delta\} \to M$$

is continuous.

3.3. Two distortion lemmas

In this subsection, we state two distortion lemmas. The following Lemma is clear, and it is used for proving the absolutely continuity of stable and unstable foliations on each fiber.

Lemma 3.4. *For any $x, y \in M,\ E(x,\omega) \in T_xM,\ E(y,\omega) \in T_yM$ with $\dim E(x,\omega) = E(y,\omega)$, we have*

$$||\det(D_x f_\omega|_{E(x,\omega)})| - |\det(D_y f_\omega|_{E(y,\omega)})|| \le C_2 d(x,y) + C_2 d(E(x,\omega), E(y,\omega)), \tag{3.4}$$

$$||\det(D_x f_\omega^{-1}|_{E(x,\omega)})| - |\det(D_y f_\omega^{-1}|_{E(y,\omega)})|| \le C_2 d(x,y) + C_2 d(E(x,\omega), E(y,\omega)), \tag{3.5}$$

where $C_2 > 0$ is a constant depending on $\sup_{\omega\in\Omega} \|f_\omega\|_{C^2}$ and $\sup_{\omega\in\Omega} \|f_\omega^{-1}\|_{C^2}$.

Lemma 3.5 is used for the construction of the fiber convex cone of observables in Section 4, and its proof is parallel to the deterministic case, see [28, Lemma 3.2].

Lemma 3.5. *$J^s(x,\omega) = |\det(D_x f_\omega|_{E^s(x,\omega)})|$ has a uniform Lipschitz variation on the local stable manifolds, i.e., there is a constant $K_1 > 0$ independent of ω such that for any $z \in M$, and $x, y \in W^s_\epsilon(z,\omega)$,*

$$|J^s(x,\omega) - J^s(y,\omega)| \le K_1 d(x,y),$$

and

$$|\log J^s(x,\omega) - \log J^s(y,\omega)| \le K_1 d(x,y).$$

Proof of Lemma 3.5. Since $M \times \Omega$ is a compact space and $f : \Omega \to \mathrm{Diff}^2(M)$ is continuous, $|D_x f_\omega|$ and $|D_x^2 f_\omega|$ are uniformly bounded. Let $K \ge 1$ be a constant such that

$$\max\left\{ \sup_{(x,\omega)\in M\times\Omega} \|D_x f_\omega\|, \sup_{(x,\omega)\in M\times\Omega} \|D_x^2 f_\omega\|, Lip Dh^s_{(x,\omega)} \right\} \le K.$$

Notice that if $y, z \in W^s_\epsilon(x,\omega)$, and $d(y,z) < \frac{\epsilon}{2\mathcal{P}K^2}$, then

$$\|P(E^s(x,\omega))(Exp_x^{-1}(z) - Exp_x^{-1}(y))\| \le \mathcal{P} d(y,z) < \frac{\epsilon}{2K^2};$$

$$\|P(E^s(f_\omega y, \theta\omega))(Exp^{-1}_{f_\omega(x)}(f_\omega(z)) - Exp^{-1}_{f_\omega(x)}(f_\omega(y)))\| \le \mathcal{P} K d(y,z) \le \frac{\epsilon}{2K}.$$

Therefore, $z \in W^s_\epsilon(y,\omega)$ and $f_\omega(z) \in W^s_\epsilon(f_\omega(y), \theta\omega)$. So it is sufficient to prove that there exists a constant $K_1 > 0$ independent of x and ω such that for any $y \in W^s_{\frac{\epsilon}{2\mathcal{P}K^2}}(x,\omega)$,

$$|J^s(x,\omega) - J^s(y,\omega)| \le K_1 d(x,y). \tag{3.6}$$

With the help of the normal coordinate chart, and notice that $d(x,y) < \epsilon < \rho_0$ and $d(f_\omega x, f_\omega y) \le \epsilon < \rho_0$, we may pretend that x, y together with $W^s_{\frac{\epsilon}{2\mathcal{P}K}}(x,\omega)$ lie in a same Euclidean space, and $f_\omega x$, $f_\omega y$ together with $W^s_\epsilon(f_\omega x, \theta\omega)$ lie in a same Euclidean space. By the invariant stable manifolds theorem, there exists $\xi_y \in E^s(x,\omega)(\frac{\epsilon}{2K^2})$ and $\xi_{f_\omega(y)} \in E^u(f_\omega(x), \theta\omega)(\epsilon)$ such that

$$y = x + \xi_y + h^s_{(x,\omega)}(\xi_y); \tag{3.7}$$

$$f_\omega(y) = f_\omega(x) + \xi_{f_\omega(y)} + h^s_{(f_\omega(x),\theta\omega)}(\xi_{f_\omega(y)}), \tag{3.8}$$

and $E^s(y,\omega) = graph((Dh^s_{(x,\omega)})_{\xi_y})$, $E^s(f_\omega(y),\theta\omega) = graph((Dh^s_{(f_\omega(x),\theta\omega)})_{\xi_{f_\omega(y)}})$. From (3.7) and (3.8), we have

$$d(x,y) = \|x-y\| = \|\xi_y + h^s_{(x,\omega)}(\xi_y)\| \geq \|\xi_y\| - \frac{1}{3}\|\xi_y\| = \frac{2}{3}\|\xi_y\| \tag{3.9}$$

and

$$\begin{aligned}(1-\frac{1}{3})\|\xi_{f_\omega(y)}\| &\leq \|\xi_{f_\omega(y)} + h^s_{(f_\omega(x),\theta\omega)}(\xi_{f_\omega(y)})\| = \|f_\omega(y) - f_\omega(x)\| \\ &\leq K\|y-x\| \leq K(1+\frac{1}{3})\|\xi_y\|,\end{aligned}$$

so $\|\xi_{f_\omega(y)}\| \leq 2K\|\xi_y\|$.

Now, we define the following linear maps $L_{(x,\omega)}, L_{(y,\omega)} : E^s(x,\omega) \to E^s(f_\omega(x),\theta\omega)$ by

$$\begin{aligned}&L_{(x,\omega)} = D_x f_\omega|_{E^s(x,\omega)};\\ &L_{(y,\omega)} = P(E^s(f_\omega x,\theta\omega))D_y f_\omega|_{E^s(y,\omega)}(I + (Dh^s_{(x,\omega)})_{\xi_y}).\end{aligned}$$

We have $\|L_{(x,\omega)}\|, \|L_{(y,\omega)}\| \leq \frac{4}{3}\mathcal{P}K$. Hence, we have

$$\begin{aligned}&\sup_{v\in E^s(x,\omega),\|v\|=1} \|P(E^s(f_\omega x,\theta\omega))D_x f_\omega v - P(E^s(f_\omega x,\theta\omega))D_y f_\omega(I + (Dh^s_{(x,\omega)})_{\xi_y})v\|\\ &\leq \mathcal{P}(\|D_x f_\omega - D_y f_\omega\| + \|D_y f_\omega (Dh^s_{(x,\omega)})_{\xi_y}\|)\\ &\leq \mathcal{P}K\|y-x\| + \mathcal{P}K^2\|\xi_y\|\\ &= \mathcal{P}Kd(x,y) + \mathcal{P}K^2\|\xi_y\|\\ &\overset{(3.9)}{\leq} (\mathcal{P}K + \frac{3}{2}\mathcal{P}K^2)d(x,y).\end{aligned}$$

So $\|L_{(x,\omega)} - L_{(y,\omega)}\| \leq C(\mathcal{P}K + \frac{3}{2}\mathcal{P}K^2)d(x,y)$, where the constant C only depends on the normal coordinate chart. Then by properties of the determinant, we have that

$$|det(L_{(x,\omega)}) - det(L_{(y,\omega)})| \leq R_1 d(x,y), \tag{3.10}$$

where R_1 is a polynomial of $K, \mathcal{P}$ and $\dim E^s(x,\omega)$.

Notice that for $\xi_y \in E^s(x,\omega)(\frac{\epsilon}{2K^2})$

$$\begin{aligned}&\|P(E^s(f_\omega(x),\theta\omega))|_{E^s(f_\omega(y),\theta\omega)} - I\| \leq \frac{\|(Dh^s_{(f_\omega(x),\theta\omega)})_{\xi_{f_\omega(y)}}\|}{1 - \|(Dh^s_{(f_\omega(x),\theta\omega)})_{\xi_{f_\omega(y)}}\|}\\ &\leq \frac{K\|\xi_{f_\omega(y)}\|}{1-K\|\xi_{f_\omega(y)}\|} \leq \frac{2K^2\|\xi_y\|}{1-2K^2\|\xi_y\|} \leq \frac{2K^2\|\xi_y\|}{1-2K^2\frac{\epsilon}{2K^2}}\\ &\leq 4K^2\|\xi_y\| \leq 6K^2 d(x,y).\end{aligned}$$

So we have

$$|det(P(E^s(f_\omega(x),\theta\omega))|_{E^s(f_\omega(y),\theta\omega)})-1| \le R_2 d(x,y), \tag{3.11}$$

where R_2 is a polynomial of K and $\dim E^s(x,\omega)$. Also

$$\|I+(Dh^s_{(x,\omega)})_{\xi_y}-I\| \le K\|\xi_y\| \le \frac{3}{2}Kd(x,y)$$

implies that there exists a constant R_3 such that

$$|det(I+(Dh^s_{(x,\omega)})_{\xi_y})-1| \le R_3 d(x,y). \tag{3.12}$$

Combining (3.10), (3.11), and (3.12), we have

$$|J^s(x,\omega)-J^s(y,\omega)| \le K_0 d(x,y),$$

where K_0 only depends on K, $\mathcal{P}$ and $\dim E^s$. Notice that $\inf_{(x,\omega)\in M\times\Omega}|J^s(x,\omega)|>0$, as a consequence, there exists a $K_1>K_0$ such that

$$|\log J^s(x,\omega)-\log J^s(y,\omega)| \le K_1 d(x,y).$$

The proof of Lemma 3.5 is complete. □

3.4. Absolute continuity of the stable and unstable foliations on each fiber

The absolute continuity of $\{W^\tau_\epsilon(x,\omega)\}$ for fixed ω and $\tau=s,u$ was stated in [29] for general random dynamical systems without proof. In this subsection, we give a proof in our settings. Besides, we give a specific formula for the Jacobian of holonomy maps.

For any $\omega\in\Omega$, a smooth submanifold $U(\omega)\subset M$ is said to be transverse to the local stable manifolds if for any $x\in U(\omega)$, $T_xU(\omega)\oplus E^s(x,\omega)=T_xM$. The transversal angel can be measured by the following quantity. For subspaces $A,B\subset\mathbb{R}^N$ with a given norm $\|\cdot\|$, set

$$\Theta(A,B)=\min\left\{\min_{v\in A,\|v\|=1;w\in B}\|v-w\|;\ \min_{w\in B,\|w\|=1;v\in A}\|v-w\|\right\}. \tag{3.13}$$

For $\theta\in(0,1]$, we say that a subspace $A\subset\mathbb{R}^N$ is θ-transverse to a subspace $B\subset\mathbb{R}^N$ if $\Theta(A,B)\ge\theta$. Given smooth submanifolds $U(\omega)$ and $V(\omega)$ transversal to the local stable manifolds, we say that $\psi_\omega:U(\omega)\to V(\omega)$ is a fiber holonomy map induced by the local stable manifolds if ψ_ω is injective and continuous, and

$$\psi_\omega(x)\in W^s_\epsilon(x,\omega)\cap V(\omega) \text{ for every } x\in U(\omega).$$

We say that $\{W^s_\epsilon(x,\omega)\}$ is absolutely continuous on each fiber if every fiber holonomy map ψ_ω induced by the local stable manifolds is absolutely continuous with respect to $m_{U(\omega)}$ and $m_{V(\omega)}$, where $m_{V(\omega)}$ and $m_{U(\omega)}$ are the intrinsic Riemannian volume measure on manifolds $V(\omega)$ and $U(\omega)$ respectively. The absolute continuity of $\{W^u_\epsilon(x,\omega)\}$ can be defined similarly. Our proof of absolute continuity follows the idea given in [41, Theorem 6.2.6].

Proposition 3.1. *Suppose ϕ is C^2 Anosov on fibers, then $\{W^s_\epsilon(x,\omega)\}$ and $\{W^u_\epsilon(x,\omega)\}$ are absolutely continuous on each fiber respectively.*

Proof of Proposition 3.1. We only prove that $\{W^s_\epsilon(x,\omega)\}$ is absolutely continuous on each fiber since the case for $\{W^u_\epsilon(x,\omega)\}$ is similar. For any $\omega \in \Omega$ fixed, let $\psi_\omega : U(\omega) \to V(\omega)$ be the fiber holonomy map between two smooth pre-compact submanifolds that are transverse to the local stable manifolds. By the regularity of Riemannian volume measure, to prove the absolute continuity of ψ_ω, it is sufficient to prove that there exists a number $C(\omega) > 0$ such that

$$m_{V(\omega)}(\psi_\omega(A)) \le C(\omega) m_{U(\omega)}(A) \text{ for any compact subset } A \subset U(\omega). \tag{3.14}$$

Before starting the proof, we need some preparations.

Lemma 3.6. *There exists $C_3,\ C_4(\omega) \in (0,1)$ such that for any $n \in \mathbb{N}$, one has*

(1) for any $(x,\omega) \in M \times \Omega$, $v \in (E^s(x,\omega))^\perp$,

$$\|D_x f^n_\omega v\| \ge C_3 e^{\lambda n}\|v\|; \tag{3.15}$$

(2) for any $x \in U(\omega)$, $v \in T_x U(\omega)$ and $y \in V(\omega)$, $\beta \in T_y V(\omega)$,

$$\|D_x f^n_\omega v\| \ge C_4(\omega) e^{\lambda n}\|v\|, \text{ and } \|D_y f^n_\omega \beta\| \ge C_4(\omega) e^{\lambda n}\|\beta\|; \tag{3.16}$$

Proof of Lemma 3.6. First, we prove (1). We pick $N_0 \in \mathbb{N}$ such that for any $n \ge N_0$,

$$e^{\lambda n} - e^{-\lambda n}\mathcal{P} \ge \frac{1}{2} e^{\lambda n},$$

where $\mathcal{P}$ is the constant in (3.3). Now for any $(x,\omega) \in M \times \Omega$, $v \in (E^s(x,\omega))^\perp$, then v has decomposition $v = v_1 + v_2$ for $v_1 \in E^u(x,\omega)$ and $v_2 \in E^s(x,\omega)$. Note that

$$\langle v, v\rangle = \langle v, v_1 + v_2\rangle = \langle v, v_1\rangle \le \|v\| \cdot \|v_1\|,$$

which implies $\|v_1\| \ge \|v\|$. If $0 \le n < N_0$, we have

$$\|D_x f^n_\omega v\| \ge (\inf_{(x,\omega)\in M\times\Omega} m(D_x f_\omega))^{N_0}\|v\| \ge \frac{(\inf_{(x,\omega)\in M\times\Omega} m(D_x f_\omega))^{N_0}}{e^{\lambda N_0}} e^{\lambda n}\|v\|,$$

where $m(D_x f_\omega) = \|(D_x f_\omega)^{-1}\|^{-1} < 1$ denotes the co-norm of $D_x f_\omega$. If $n \ge N_0$, then

$$\begin{aligned}\|D_x f^n_\omega v\| &\ge \|D_x f^n_\omega v_1\| - \|D_x f^n_\omega v_2\| \ge e^{\lambda n}\|v_1\| - e^{-\lambda n}\|v_2\| \\ &\ge e^{\lambda n}\|v\| - e^{-\lambda n}\mathcal{P}\|v\| \ge \frac{1}{2} e^{\lambda n}\|v\|.\end{aligned}$$

These two cases imply that (3.15) holds for

$$C_3 := \min\left\{\frac{1}{2}, \frac{(\inf_{(x,\omega)\in M\times\Omega} m(D_x f_\omega))^{N_0}}{e^{\lambda N_0}}\right\}.$$

Second, we prove (2). We only prove the first inequality of (3.16), as the second one is similar. Note that $U(\omega)$ and $V(\omega)$ are transverse to the local stable manifolds and pre-compact, the following quantity is positive

$$\gamma_1(\omega) := \min\{\inf\{\Theta(T_x U(\omega), E^s(x,\omega))|\ x \in U(\omega)\}, \inf\{\Theta(T_y V(\omega), E^s(y,\omega))|\ y \in V(\omega)\}\}. \tag{3.17}$$

Now for any $x \in U(\omega)$ and any $v \in T_x U(\omega)$, then $v = v_1 + v_2$ for $v_1 \in (E^s(x,\omega))^\perp$ and $v_2 \in E^s(x,\omega)$. Then one has

$$\|v_1\| = \|v - v_2\| = \min_{\eta\in E^s(x,\omega)} \|v-\eta\| = \min_{\eta\in E^s(x,\omega)} \left\|\frac{v}{\|v\|} - \frac{\eta}{\|v\|}\right\| \cdot \|v\| \geq \gamma_1(\omega)\|v\|. \tag{3.18}$$

We pick $N_1(\omega)$ sufficiently large such that for any $n \geq N_1(\omega)$

$$(C_3 e^{\lambda n} - e^{-\lambda n})\gamma_1(\omega) - e^{-\lambda n} \geq \frac{1}{2} e^{\lambda n} C_3 \gamma_1(\omega). \tag{3.19}$$

If $0 < n < N_1(\omega)$, then

$$\|D_x f_\omega^n v\| \geq (\inf_{(x,\omega)\in M\times\Omega} m(D_x f_\omega))^{N_1(\omega)} \|v\| \geq \frac{(\inf_{(x,\omega)\in M\times\Omega} m(D_x f_\omega))^{N_1(\omega)}}{e^{\lambda N_1(\omega)}} e^{\lambda n}\|v\|.$$

If $n \geq N_1(\omega)$, we have

$$\begin{aligned}
\|D_x f_\omega^{-n} v\| &= \|D_x f_\omega^n (v_1 + v_2)\| \geq \|D_x f_\omega^n v_1\| - \|D_x f_\omega^n v_2\| \\
&\overset{(3.15)}{\geq} C_3 e^{\lambda n}\|v_1\| - e^{-\lambda n}\|v_2\| \geq C_3 e^{\lambda n}\|v_1\| - e^{-\lambda n}(\|v\| + \|v_1\|) \\
&= (C_3 e^{\lambda n} - e^{-\lambda n})\|v_1\| - e^{-\lambda n}\|v\| \overset{(3.18)}{\geq} (C_3 e^{\lambda n} - e^{-\lambda n})\gamma_1(\omega)\|v\| - e^{-\lambda n}\|v\| \\
&\overset{(3.19)}{\geq} \frac{1}{2}\gamma_1(\omega) C_3 e^{\lambda n}\|v\|.
\end{aligned}$$

These two cases imply (3.16) holds for

$$C_4(\omega) = \min\left\{\frac{1}{2}\gamma_1(\omega) C_3, \frac{(\inf_{(x,\omega)\in M\times\Omega} m(D_x f_\omega))^{N_1(\omega)}}{e^{\lambda N_1(\omega)}}\right\} \in (0,1).$$

The proof of Lemma 3.6 is complete. □

Lemma 3.7. *Recall that the distance between subbundle is defined in (3.1). There exists $C_6(\omega) > 1$ such that for any $n \in \mathbb{N}$,*

$$d(D_x f_\omega^n T_x U(\omega), D_x f_\omega^n E^u(x,\omega)) \leq C_6(\omega) e^{-\lambda n} d(T_x U(\omega), E^u(x,\omega)) \text{ for } x \in U(\omega); \tag{3.20}$$

$$d(D_y f_\omega^n T_y V(\omega), D_y f_\omega^n E^u(y,\omega)) \leq C_6(\omega) e^{-\lambda n} d(T_y V(\omega), E^u(y,\omega)) \text{ for } y \in V(\omega). \tag{3.21}$$

Proof of Lemma 3.7. The proof of inequalities (3.20) and (3.21) are similar, therefore we only prove (3.20). Recall that $\gamma_1(\omega)$ is defined in (3.17). For any $v \in T_x U(\omega)$ with $\|v\| = 1$, suppose v has decomposition $v = v_1 + v_2$ for some $v_1 \in E^u(x,\omega)$ $v_2 \in E^u(x,\omega)^{\perp}$, and decomposition $v = \beta_1 + \beta_2$ for some $\beta_1 \in E^u(x,\omega)$ and $\beta_2 \in E^s(x,\omega)$, then by trigonometry, we have

$$\|\beta_2\| \le \gamma_1(\omega)^{-1}\|v_2\| = \gamma_1(\omega)^{-1} \inf_{\eta \in E^u(x,\omega)} \|v - \eta\|. \tag{3.22}$$

Let $\theta_0 := \inf_{(x,\omega)\in M\times\Omega} \Theta(E^u(x,\omega), E^s(x,\omega)) > 0$. For any $v \in E^u(x,\omega)$ with $\|v\| = 1$, suppose v has decomposition $v = v_1 + v_2$ for some $v_1 \in T_x U(\omega)$, $v_2 \in (T_x U(\omega))^{\perp}$, and decomposition $v = \beta_1 + \beta_2$ for some $\beta_1 \in T_x U(\omega)$ and $\beta_2 \in E^s(x,\omega)$, then by trigonometry, we have

$$\|\beta_2\| \le \theta_0^{-1}\|v_2\| = \theta_0^{-1} \inf_{\eta \in T_x U(\omega)} \|v - \eta\|. \tag{3.23}$$

For any $n \in \mathbb{N}$, pick any $v \in D_x f_\omega^n T_x U(\omega)$ with $\|v\| = 1$. Denote $v = D_x f_\omega^n v'$ for $v' \in T_x U(\omega)$. Then there is decomposition $v' = v_1' + v_2'$ for $v_1' \in E^u(x,\omega)$ and $v_2' \in E^s(x,\omega)$. Then we have

$$\begin{aligned}
\inf_{\eta \in D_x f_\omega^n E^u(x,\omega)} \|v - \eta\| &\le \|v - D_x f_\omega^n v_1'\| = \|D_x f_\omega^n v_2'\| \le e^{-\lambda n}\|v_2'\| \\
&= e^{-\lambda n}\|v' - v_1'\| = e^{-\lambda n}\left\| \frac{v'}{\|v'\|} - \frac{v_1'}{\|v'\|} \right\| \cdot \|v'\| \\
&\overset{(3.16)}{\le} e^{-\lambda n} \cdot (C_4(\omega))^{-1} e^{-\lambda n}\|v\| \left\| \frac{v'}{\|v'\|} - \frac{v_1'}{\|v'\|} \right\| \\
&\overset{(3.22)}{\le} e^{-\lambda n} \cdot (C_4(\omega))^{-1} \cdot e^{-\lambda n} \cdot 1 \cdot \gamma_1(\omega)^{-1} \cdot \inf_{\beta \in E^u(x,\omega)} \left\| \frac{v'}{\|v'\|} - \beta \right\| \\
&\le e^{-\lambda n} \cdot (C_4(\omega))^{-1} \cdot \gamma_1(\omega)^{-1} \cdot \sup_{\zeta \in T_x U(\omega), \|\zeta\|=1} \inf_{\beta \in E^u(x,\omega)} \|\zeta - \beta\|.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\sup_{v \in D_x f_\omega^n T_x U(\omega), \|v\|=1} \inf_{\eta \in D_x f_\omega^n E^u(x,\omega)} \|v - \eta\| \\
\le & e^{-\lambda n} \cdot (C_4(\omega))^{-1} \cdot \gamma_1(\omega)^{-1} \cdot \sup_{\zeta \in T_x U(\omega), \|\zeta\|=1} \inf_{\beta \in E^u(x,\omega)} \|\zeta - \beta\|.
\end{aligned} \tag{3.24}$$

Pick any $\xi \in D_x f_\omega^n E^u(x,\omega)$ with $\|\xi\| = 1$, and denote $\xi = D_x f_\omega^n \xi'$ for $\xi' \in E^u(x,\omega)$. Then there exists $\xi_1' \in T_x U(\omega)$ and $\xi_2' \in E^s(x,\omega)$ such that $\xi' = \xi_1' + \xi_2'$. Now,

$$\begin{aligned}
\inf_{\eta \in D_x f_\omega^n T_x U(\omega)} \|\xi - \eta\| &\le \|\xi - D_x f_\omega^n \xi_1'\| = \|D_x f_\omega^n \xi_2'\| \le e^{-\lambda n}\|\xi_2'\| \\
&= e^{-\lambda n}\|\xi' - \xi_1'\| = e^{-\lambda n}\left\| \frac{\xi'}{\|\xi'\|} - \frac{\xi_1'}{\|\xi'\|} \right\| \cdot \|\xi'\| \\
&\overset{(3.23)}{\le} e^{-\lambda n} \cdot \|\xi'\| \cdot \theta_0^{-1} \cdot \inf_{\zeta \in T_x U(\omega)} \left\| \frac{\xi'}{\|\xi'\|} - \zeta \right\|
\end{aligned}$$

$$\leq e^{-\lambda n} \cdot e^{-\lambda n} \|\xi\| \cdot \theta_0^{-1} \cdot \sup_{\beta \in E^u(\omega,x), \|\beta\|=1} \inf_{\zeta \in T_x U(\omega)} \|\beta - \zeta\|$$

$$\leq e^{-\lambda n} \cdot 1 \cdot \theta_0^{-1} \cdot \sup_{\beta \in E^u(\omega,x), \|\beta\|=1} \inf_{\zeta \in T_x U(\omega)} \|\beta - \zeta\| .$$

Therefore, we have

$$\sup_{\xi \in D_x f_\omega^n E^u(x,\omega), \|\xi\|=1} \inf_{\eta \in D_x f_\omega^n T_x U(\omega)} \|\xi - \eta\| \leq e^{-\lambda n} \cdot \theta_0^{-1} \cdot \sup_{\beta \in E^u(\omega,x), \|\beta\|=1} \inf_{\zeta \in T_x U(\omega)} \|\beta - \zeta\| . \tag{3.25}$$

By checking definitions and letting $C_6(\omega) = (C_4(\omega))^{-1} \max\{\gamma_1(\omega)^{-1}, \theta_0^{-1}\}$, then (3.20) is a corollary of (3.24), (3.25). The proof of Lemma 3.7 is complete. □

For any compact $A \subset U(\omega)$, let $\mathcal{O}$ be a small neighborhood of A in $U(\omega)$ such that

$$m_{U(\omega)}(\mathcal{O}) \leq 2 m_{U(\omega)}(A). \tag{3.26}$$

By (3.16), let $\delta_0 \in (0, \epsilon_0)$ be a sufficiently small number, then there exists a number $N_1(\omega)$ such that for any $n \geq N_1(\omega)$ and $\delta \in (0, \delta_0)$, we have

$$f_{\theta^n \omega}^{-n} B_{f_\omega^n U(\omega)}(f_\omega^n x, \delta) \subset \mathcal{O} \text{ for any } x \in A, \tag{3.27}$$

where $B_{f_\omega^n U(\omega)}(f_\omega^n x, \delta)$ is the δ-neighborhood of $f_\omega^n x$ on $f_\omega^n U(\omega)$.

Lemma 3.8. *For any $\delta \in (0, \delta_0)$ and any constant $C_7 > 1$, there exists $N_2(\omega) \in \mathbb{N}$ such that for any $n \geq N_2(\omega)$ and $x \in f_\omega^n U(\omega)$, we have*

$$\bar{\psi}_{\theta^n \omega}(B_{f_\omega^n U(\omega)}(x, \delta)) \subset B_{f_\omega^n V(\omega)}(\bar{\psi}_{\theta^n \omega}(x), 2\delta), \tag{3.28}$$

and

$$C_7^{-1} \leq \frac{m_{f_\omega^n U(\omega)}(B_{f_\omega^n U(\omega)}(x, \delta))}{m_{f_\omega^n V(\omega)}(\bar{\psi}_{\theta^n \omega}(B_{f_\omega^n U(\omega)}(x, \delta)))} \leq C_7, \tag{3.29}$$

where $\bar{\psi}_{\theta^n \omega} = f_\omega^n \circ \psi_\omega \circ f_{\theta^n \omega}^{-n} : f_\omega^n U(\omega) \to f_\omega^n V(\omega)$ is the holonomy map induced by the local stable manifolds.

Proof of Lemma 3.8. Notice that for any $x \in U(\omega)$, $\psi_\omega(x) = V(\omega) \cap W_\epsilon^s(x, \omega)$, so we have

$$d(f_\omega^n x, f_\omega^n \psi_\omega(x)) \leq e^{-\lambda n} d(x, \psi_\omega(x)). \tag{3.30}$$

By (3.20), (3.21) and (3.30), we know that $f_\omega^n U(\omega)$ and $f_\omega^n V(\omega)$ will be C^1-close to the unstable manifold uniformly for points on $f_\omega^n U(\omega)$ and $f_\omega^n V(\omega)$ as n goes to infinity. We let n be determined later. For any $x \in f_\omega^n U(\omega)$ and $\delta \in (0, \delta_0)$, by employing the exponential map, we may pretend that $B_{f_\omega^n U(\omega)}(x, \delta)$ and $\bar{\psi}_{\theta^n \omega}(B_{f_\omega^n U(\omega)}(x, \delta))$ lie in Euclidean space $E^u(x, \omega) \oplus E^s(x, \omega)$. Denote $P^u : E^u(x, \omega) \oplus E^s(x, \omega) \to E^u(x, \omega)$ to be the projection onto the first coordinate. Let

$$\Psi_n^1, \Psi_n^2 : E^u(x,\omega) \to E^s(x,\omega)$$

be C^2 maps, whose graph represent $B_{f_\omega^n U(\omega)}(x,\delta)$ and $\bar{\psi}_{\theta^n\omega}(B_{f_\omega^n U(\omega)}(x,\delta))$ respectively. For any $\eta \in (0,\delta)$, since $f_\omega^n U(\omega)$ and $f_\omega^n V(\omega)$ are C^1-close to the unstable manifolds, there exists $N_\eta(\omega) \in \mathbb{N}$ independent of $x \in U(\omega)$ such that for any $n \geq N_\eta(\omega)$,

$$\|D\Psi_n^i|_{E^u(x,\omega)(2\mathcal{P}\delta)}\| \leq \frac{1}{2}, \text{ and } Lip(D\Psi_n^i|_{E^u(x,\omega)(2\mathcal{P}\delta)}) \leq L+1, \text{ for } i=1,2, \tag{3.31}$$

and

$$\|(\Psi_n^1 - \Psi_n^2)|_{E^u(x,\omega)(2\mathcal{P}\delta)}\|_{C^1} + \mathcal{P} \sup_{z\in B_{f_\omega^n U(\omega)}(x,\delta)} \|z - \bar{\psi}_{\theta^n\omega}(z)\| < \eta, \tag{3.32}$$

where constant L is given in Lemma 3.2, and $\mathcal{P}$ is the constant in (3.3).

For any $n \geq N_\eta(\omega)$, $x \in f_\omega^n U(\omega)$ and $z \in B_{f_\omega^n U(\omega)}(x,\delta)$, we have

$$\|P^u(z) - P^u(\bar{\psi}_{\theta^n\omega}(z))\| \leq \mathcal{P}\|z - \bar{\psi}_{\theta^n\omega}(z)\| < \eta. \tag{3.33}$$

Combining (3.33) and the inequality $\|(\Psi_n^1 - \Psi_n^2)|_{E^u(x,\omega)(2\mathcal{P}\delta)}\|_{C^1} < \eta$, one has

$$B_{f_\omega^n V(\omega)}(\bar{\psi}_{\theta^n\omega}(x), \frac{1}{2}\delta) \subset \bar{\psi}_{\theta^n\omega}(B_{f_\omega^n U(\omega)}(x,\delta)) \subset B_{f_\omega^n V(\omega)}(\bar{\psi}_{\theta^n\omega}(x), 2\delta),$$

provided η sufficiently small and n correspondingly large. Inequality (3.33) also implies that the difference between the area of $P^u(B_{f_\omega^n U(\omega)}(x,\delta))$ and the area of $P^u(\bar{\psi}_{\theta^n\omega}(B_{f_\omega^n U(\omega)}(x,\delta)))$ is up to a scale of η. By letting η sufficiently small and pick n correspondingly large, and using the regularity of Lebesgue measure, the Lebesgue measure of

$$\begin{aligned} P^u(B_{f_\omega^n U(\omega)}(x,\delta)) \cup P^u(\bar{\psi}_{\theta^n\omega}(B_{f_\omega^n U(\omega)}(x,\delta))) \backslash \big(P^u(B_{f_\omega^n U(\omega)}(x,\delta)) \\ \cap P^u(\bar{\psi}_{\theta^n\omega}(B_{f_\omega^n U(\omega)}(x,\delta)))\big) \end{aligned}$$

is less than or equal to $C \cdot \eta$ for some constant C.

Moreover, for any $e \in P^u(B_{f_\omega^n U(\omega)}(x,\delta)) \cap P^u(\bar{\psi}_{\theta^n\omega}(B_{f_\omega^n U(\omega)}(x,\delta)))$, we have

$$\|D_e\Psi_n^1 - D_e\Psi_n^2\| \leq \|\Psi_n^1 - \Psi_n^2\|_{C^1} \overset{(3.32)}{<} \eta.$$

Denote $\tilde{\Psi}_n^i : E^u(x,\omega) \to E^u(x,\omega) \oplus E^s(x,\omega)$ by $e \mapsto (e, \Psi_n^i(e))$ for $i=1,2$. Then for any $e \in P^u(B_{f_\omega^n U(\omega)}(x,\delta)) \cap P^u(\bar{\psi}_{\theta^n\omega}(B_{f_\omega^n U(\omega)}(x,\delta)))$, we have

$$\begin{aligned} &\left\|(D_e\tilde{\Psi}_n^1)^*(D_e\tilde{\Psi}_n^1) - (D_e\tilde{\Psi}_n^2)^*(D_e\tilde{\Psi}_n^2)\right\| \\ &\leq \left\|(D_e\tilde{\Psi}_n^1)^*(D_e\tilde{\Psi}_n^1) - (D_e\tilde{\Psi}_n^1)^*(D_e\tilde{\Psi}_n^2)\right\| + \left\|(D_e\tilde{\Psi}_n^1)^*(D_e\tilde{\Psi}_n^2) - (D_e\tilde{\Psi}_n^2)^*(D_e\tilde{\Psi}_n^2)\right\| \\ &\overset{(3.31)}{\leq} \frac{3}{2}\eta + \frac{3}{2}\eta = 3\eta, \end{aligned}$$

where $*$ means the adjoint matrix. As a consequence, for such points e,

$$\left|Jac(\tilde{\Psi}_n^1)(e) - Jac(\tilde{\Psi}_n^2)(e)\right| = \left|\sqrt{\det((D_e\tilde{\Psi}_n^1)^*(D_e\tilde{\Psi}_n^1))} - \sqrt{\det((D_e\tilde{\Psi}_n^2)^*(D_e\tilde{\Psi}_n^{)}_n)}\right| \leq C\eta,$$

for some constant C. By (3.31), we have

$$(1+\frac{1}{2})^{\dim E^u(\omega,x)} > Jac(\tilde{\Psi}_n^i)(e) \geq \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2} \text{ for } i=1,2.$$

Therefore, for any $e \in P^u(B_{f_\omega^n U(\omega)}(x,\delta)) \cap P^u(\bar{\psi}_{\theta^n\omega}(B_{f_\omega^n U(\omega)}(x,\delta)))$, we have

$$\left|\log\frac{Jac(\tilde{\Psi}_n^1)(e)}{Jac(\tilde{\Psi}_n^2)(e)}\right| \leq C^*\cdot\eta, \tag{3.34}$$

where C^* is some constant. Now by area formula, we obtain

$$\begin{aligned}
&m_{f_\omega^n U(\omega)}((B_{f_\omega^n U(\omega)}(x,\delta))) \\
=&\int_{P^u(B_{f_\omega^n U(\omega)}(x,\delta))} Jac(\tilde{\Psi}_n^1)(e)dm(e) \\
\leq&\int_{P^u(B_{f_\omega^n U(\omega)}(x,\delta))\cap P^u(\bar{\psi}_{\theta^n\omega}(B_{f_\omega^n U(\omega)}(x,\delta)))} Jac(\tilde{\Psi}_n^1)(e)dm(e) + (3/2)^{\dim E^u}C\eta \\
\leq& e^{C^*\eta}\int_{P^u(B_{f_\omega^n U(\omega)}(x,\delta))\cap P^u(\bar{\psi}_{\theta^n\omega}(B_{f_\omega^n U(\omega)}(x,\delta)))} Jac(\tilde{\Psi}_n^2)(e)dm(e) + (3/2)^{\dim E^u}C\eta \\
\leq& e^{C^*\eta}\int_{P^u(\bar{\psi}_{\theta^n\omega}(B_{f_\omega^n U(\omega)}(x,\delta)))} Jac(\tilde{\Psi}_n^2)(e)dm(e) + (3/2)^{\dim E^u}C\eta \\
=& e^{C^*\eta}\cdot m_{f_\omega^n V(\omega)}(\bar{\psi}_{\theta^n\omega}(B_{f_\omega^n U(\omega)}(x,\delta))) + (3/2)^{\dim E^u}C\eta.
\end{aligned} \tag{3.35}$$

Symmetrically, we also have

$$m_{f_\omega^n V(\omega)}(\bar{\psi}_{\theta^n\omega}(B_{f_\omega^n U(\omega)}(x,\delta))) \leq e^{C^*\eta} m_{f_\omega^n U(\omega)}((B_{f_\omega^n U(\omega)}(x,\delta))) + (3/2)^{\dim E^u}C\eta. \tag{3.36}$$

Notice that $f_\omega^n U(\omega)$ and $f_\omega^n V(\omega)$ are $\dim(E^u(x,\omega))$-submanifold of M satisfying (3.31), therefore $m_{f_\omega^n U(\omega)}((B_{f_\omega^n U(\omega)}(x,\delta)))$ and $m_{f_\omega^n V(\omega)}(\bar{\psi}_{\theta^n\omega}(B_{f_\omega^n U(\omega)}(x,\delta))) > m_{f_\omega^n V(\omega)}((B_{f_\omega^n V(\omega)}(\bar{\psi}_{\theta^n\omega}(x), \frac{1}{2}\delta)))$ are bounded below uniformly by scale of δ. Finally, for any $C_7 > 1$, (3.29) is a consequence of (3.35) and (3.36) by letting η sufficiently small and n correspondingly large. This finishes the proof. □

Now we pick any $\delta \in (0, \delta_0)$, $C_7 > 1$, and fix

$$N = N(\omega) = \max\{N_1(\omega), N_2(\omega)\}, \tag{3.37}$$

where $N_1(\omega)$ is chosen satisfying (3.27), and $N_2(\omega)$ is chosen in Lemma 3.8 corresponding to $C_7 > 1$. Let $\{B_{f_\omega^N U(\omega)}(x,\delta)\}_{x\in f_\omega^N A}$ be a cover of $f_\omega^N A$ by δ-balls centered at points in $f_\omega^N A$. By the Besicovitch covering lemma (see, e.g., [17]), there exists a finite subcover $\{B_i\}_{i=1}^k \subset \{B_{f_\omega^N U(\omega)}(x,\delta)\}_{x\in f_\omega^N A}$ of $f_\omega^N A$ and a number $C' = C'(\dim(E^u))$ satisfying

$$\text{there is no point in } f_\omega^N A \text{ that lies in more than number } C' \text{ of the } B_i\text{'s.} \tag{3.38}$$

Next, we claim that there exists a number $C_8(\omega) > 1$ independent of $N(\omega)$ such that

$$C_8(\omega)^{-1} m_{V(\omega)}(\psi_\omega(f_{\theta^N\omega}^{-N} B_i)) \le m_{U(\omega)}(f_{\theta^N\omega}^{-N} B_i) \le C_8(\omega) m_{V(\omega)}(\psi_\omega(f_{\theta^N\omega}^{-N} B_i)). \tag{3.39}$$

If the above claim is true, then we arrive

$$\begin{aligned} m_{V(\omega)}(\psi_\omega(A)) &\le \sum_{i=1}^k m_{V(\omega)}(\psi_\omega(f_{\theta^N\omega}^{-N} B_i)) \le \sum_{i=1}^k C_8(\omega) m_{U(\omega)}(f_{\theta^N\omega}^{-N} B_i) \\ &\overset{(3.27),(3.38)}{\le} C' C_8(\omega) m_{U(\omega)}(\mathcal{O}) \\ &\overset{(3.26)}{\le} 2C' C_8(\omega) m_{U(\omega)}(A), \end{aligned}$$

i.e., (3.14) holds. Hence ψ_ω is absolutely continuous.

So it is left to prove the claim (3.39). We still need the following lemma.

Lemma 3.9. *For any $x \in U(\omega)$, denote*

$$H_\omega^n(x, \psi_\omega(x), T_x U(\omega), T_{\psi_\omega(x)} V(\omega)) := \frac{|\det(D_x f_\omega^n|_{T_x U(\omega)})|}{|\det(D_{\psi_\omega(x)} f_\omega^n|_{T_{\psi_\omega(x)} V(\omega)})|}. \tag{3.40}$$

There exists a number $C_9(\omega) > 0$ such that for any $n \in \mathbb{N}$, $x \in U(\omega)$,

$$C_9(\omega)^{-1} \le H_\omega^n(x, \psi_\omega(x), T_x U(\omega), T_{\psi_\omega(x)} V(\omega)) \le C_9(\omega).$$

As a consequence, the limit

$$H_\omega(x, \psi_\omega(x), T_x U(\omega), T_{\psi_\omega(x)} V(\omega)) := \lim_{n\to\infty} H_\omega^n(x, \psi_\omega(x), T_x U(\omega), T_{\psi_\omega(x)} V(\omega)) \tag{3.41}$$

exists and converges uniformly for all $x \in U(\omega)$.

Proof of Lemma 3.9. For any $j \in \mathbb{N}$, by Lemma 3.4 and Lemma 3.1, we have

$$\begin{aligned}
&\left|\det\left(D_{f^j_\omega x} f_{\theta^j\omega}|_{D_x f^j_\omega T_x U(\omega)}\right) - \det\left(D_{f^j_\omega \psi_\omega(x)} f_{\theta^j\omega}|_{D_{\psi_\omega(x)} f^j_\omega T_{\psi_\omega(x)} V(\omega)}\right)\right| \\
\le& \left|\det\left(D_{f^j_\omega x} f_{\theta^j\omega}|_{D_x f^j_\omega T_x U(\omega)}\right) - \det\left(D_{f^j_\omega x} f_{\theta^j\omega}|_{D_x f^j_\omega E^u(x,\omega)}\right)\right| \\
&+ \left|\det\left(D_{f^j_\omega x} f_{\theta^j\omega}|_{D_x f^j_\omega E^u(x,\omega)}\right) - \det\left(D_{f^j_\omega \psi_\omega(x)} f_{\theta^j\omega}|_{D_{\psi_\omega(x)} f^j_\omega E^u(\psi_\omega(x),\omega)}\right)\right| \\
&\quad + \left|\det\left(D_{f^j_\omega \psi_\omega(x)} f_{\theta^j\omega}|_{D_{\psi_\omega(x)} f^j_\omega E^u(\psi_\omega(x),\omega)}\right) - \det\left(D_{f^j_\omega \psi_\omega(x)} f_{\theta^j\omega}|_{D_{\psi_\omega(x)} f^j_\omega T_{\psi_\omega(x)} V(\omega)}\right)\right| \\
\overset{(3.4)}{\le}& C_2 d(D_x f^j_\omega T_x U(\omega), D_x f^j_\omega E^u(x,\omega)) \\
&+ C_2[d(f^j_\omega x, f^j_\omega \psi_\omega(x)) + d(E^u(f^j_\omega x, \theta^j\omega), E^u(f^j_\omega \psi_\omega(x), \theta^j\omega))] \\
&+ C_2 d(D_{\psi_\omega(x)} f^j_\omega E^u(\psi_\omega(x),\omega), D_{\psi_\omega(x)} f^j_\omega T_{\psi_\omega(x)} V(\omega)) \\
\overset{(3.20),(3.2),(3.21)}{\le}& C_6(\omega)C_2 e^{-\lambda j} d(T_x U(\omega), E^u(x,\omega)) + 2C_2C_1 d(f^j_\omega x, f^j_\omega \psi_\omega(x))^{\nu_1} \\
&\qquad + C_6(\omega)C_2 e^{-\lambda j} d(T_{\psi_\omega(x)} V(\omega), E^u(\psi_\omega(x), \theta^j\omega)) \\
\le& 2C_1C_2C_6(\omega)e^{-\lambda j\nu_1}[d(T_x U(\omega), E^u(x,\omega)) + d(T_{\psi_\omega(x)} V(\omega), E^u(\psi_\omega(x),\omega)) \\
&\qquad + d(x, \psi_\omega(x))^{\nu_1}] \\
\le& 6C_1C_2C_6(\omega)e^{-\lambda j\nu_1}.
\end{aligned}$$

Note that by the compactness of M and Ω and the continuity of $f_\omega \in$Diff$^2(M)$ on $\omega \in \Omega$, there exists a constant C_{10} such that for any $y \in M$ and $F(\omega) \subset T_y M$,

$$C_{10}^{-1} \le |\det(D_y f_\omega|_{F(\omega)})| \le C_{10}. \tag{3.42}$$

Denote $C_{11}(\omega) = 6C_6(\omega)C_1C_2C_{10}$, and we obtain that for $j \in \mathbb{N}$,

$$\frac{|\det(D_{f^j_\omega x} f_{\theta^j\omega}|_{D_x f^j_\omega T_x U(\omega)})|}{|\det(D_{f^j_\omega \psi_\omega(x)} f_{\theta^j\omega}|_{D_{\psi_\omega(x)} f^j_\omega T_{\psi_\omega(x)} V(\omega)})|} \le 1 + C_{11}(\omega)e^{-\lambda j\nu_1}. \tag{3.43}$$

Now (3.43) implies that for any $n \in \mathbb{N}$,

$$\begin{aligned}
H^n_\omega(x, \psi_\omega(x), T_x U(\omega), T_{\psi_\omega(x)} V(\omega)) &= \prod_{j=0}^{n-1} \frac{|\det(D_{f^j_\omega x} f_{\theta^j\omega}|_{D_x f^j_\omega T_x U(\omega)})|}{|\det(D_{f^j_\omega \psi_\omega(x)} f_{\theta^j\omega}|_{D_{\psi_\omega(x)} f^j_\omega T_{\psi_\omega(x)} V(\omega)})|} \\
&\le \prod_{j=0}^{n-1}(1 + C_{11}(\omega)e^{-\lambda j\nu_1}) \\
&\le \exp\left(\frac{C_{11}(\omega)}{1 - e^{\lambda\nu_1}}\right).
\end{aligned}$$

We denote $\exp(\frac{C_{11}(\omega)}{1-e^{\lambda\nu_1}})$ by $C_9(\omega)$. The same estimate holds for $H^n_\omega(\psi_\omega(x), x, T_{\psi_\omega(x)}V(\omega), T_xU(\omega))$ for any $n \in \mathbb{N}$. The proof of Lemma 3.9 is complete. $\square$

Lemma 3.10. *There exists $C_{13}(\omega) > 1$ independent of $N(\omega)$ such that for any $x, y \in f^{-N}_{\theta^N\omega}B_i$ and $p, q \in \psi_\omega(f^{-N}_{\theta^N\omega}B_i)$, we have*

$$C_{13}(\omega)^{-1} \leq \frac{|\det D_x f^N_\omega|_{T_xU(\omega)}|}{|\det D_y f^N_\omega|_{T_yU(\omega)}|} \leq C_{13}(\omega) \tag{3.44}$$

and

$$C_{13}(\omega)^{-1} \leq \frac{|\det D_p f^N_\omega|_{T_pV(\omega)}|}{|\det D_q f^N_\omega|_{T_qV(\omega)}|} \leq C_{13}(\omega). \tag{3.45}$$

Proof of Lemma 3.10. Denote $x_i = f^i_\omega(x)$ and $y_i = f^i_\omega(y)$ for $i = 0, ..., N$. We note that $x_N, y_N \in B_i$, therefore $d(x_N, y_N) < \delta$. By virtue of (3.16), we have

$$d(x_i, y_i) \leq C_4(\omega)^{-1}e^{-\lambda(N-i)}d(x_N, y_N) < C_4(\omega)^{-1}e^{-\lambda(N-i)}\delta \text{ for } i = 0, ..., N. \tag{3.46}$$

For $i = 0, ..., N$, by Lemma 3.4 and Lemma 3.1, we have

$$\begin{aligned}
&\left|\det\left(D_{x_i}f_{\theta^i\omega}|_{D_xf^i_\omega T_xU(\omega)}\right) - \det\left(D_{y_i}f_{\theta^i\omega}|_{D_yf^i_\omega T_yU(\omega)}\right)\right| \\
\leq &\left|\det\left(D_{x_i}f_{\theta^i\omega}|_{D_xf^i_\omega T_xU(\omega)}\right) - \det\left(D_{x_i}f_{\theta^i\omega}|_{E^u(x_i,\theta^i\omega)}\right)\right| \\
&\quad + \left|\det\left(D_{x_i}f_{\theta^i\omega}|_{E^u(x_i,\theta^i\omega)}\right) - \det\left(D_{y_i}f_{\theta^i\omega}|_{E^u(y_i,\theta^i\omega)}\right)\right| \\
&\qquad + \left|\det\left(D_{y_i}f_{\theta^i\omega}|_{E^u(y_i,\theta^i\omega)}\right) - \det\left(D_{y_i}f_{\theta^i\omega}|_{D_yf^i_\omega T_yU(\omega)}\right)\right| \\
\overset{(3.4)}{\leq} &C_2 d(D_xf^i_\omega T_xU(\omega), E^u(x_i,\theta^i\omega)) + C_2[d(x_i,y_i) + d(E^u(x_i,\theta^i\omega), E^u(y_i,\theta^i\omega))] \\
&\quad + C_2 d(E^u(y_i,\theta^i\omega), D_yf^i_\omega T_yU(\omega)) \\
\overset{(3.20),(3.21)}{\leq} &C_2C_6(\omega)e^{-\lambda i}(d(T_xU(\omega), E^u(x,\omega)) + d(T_yU(\omega), E^u(y,\omega))) + C_2(1+C_1)d(x_i,y_i)^{\nu_1} \\
\overset{(3.46)}{\leq} &2C_2C_6(\omega)e^{-\lambda i}\cdot\left(\sup_{x\in U(\omega)} d(T_xU(\omega), E^u(x,\omega))\right) + C_2(1+C_1)C_4(\omega)^{-1}e^{-\lambda(N-i)\nu_1}\delta^{\nu_1}.
\end{aligned}$$

By (3.42) and noticing $\sup_{x\in U(\omega)} d(T_xU(\omega), E^u(x,\omega)) < 1$, one has

$$\begin{aligned}
\frac{|\det\left(D_{x_i}f_{\theta^i\omega}|_{D_xf^i_\omega T_xU(\omega)}\right)|}{|\det\left(D_{y_i}f_{\theta^i\omega}|_{D_yf^i_\omega T_yU(\omega)}\right)|} &\leq 1 + C_{12}(\omega)e^{-\lambda i} + C_{12}(\omega)e^{-\lambda(N-i)\nu_1}\cdot(2\delta)^{\nu_1} \\
&\leq 1 + C_{12}(\omega)e^{-\lambda i} + C_{12}(\omega)e^{-\lambda(N-i)\nu_1},
\end{aligned} \tag{3.47}$$

where

$$C_{12}(\omega) := 2C_{10}C_2(1+C_1)C_4(\omega)^{-1}C_6(\omega). \tag{3.48}$$

Then (3.44) is a corollary of (3.47) by letting

$$C_{13}(\omega) := e^{C_{12}(\omega)\cdot\sum_{i=0}^{\infty} e^{-\lambda i} + C_{12}(\omega)\cdot\sum_{i=0}^{\infty} e^{-\lambda i \nu_1}}.$$

For (3.45), we notice that $f_\omega^N(p), f_\omega^N(q) \in f_\omega^N(\psi_\omega(f_{\theta^N\omega}^{-N}B_i)) = \bar{\psi}_{\theta^N\omega}(B_i)$ and $\bar{\psi}_{\theta^N\omega}(B_i)$ is contained in a ball in $f_\omega^N V(\omega)$ with radius less than 2δ by (3.28). Then (3.45) can be proved similarly as (3.44). $\square$

Now we are ready to prove the claim (3.39). Pick any $p_i \in B_i$, denote $q_i := \bar{\psi}_{\theta^N\omega}(p_i) \in f_\omega^N V(\omega)$ to be the holonomy image of p_i, by (3.44) and change of variable, we have

$$\begin{aligned} C_{13}(\omega)^{-1} \cdot \left|\det D_{p_i} f_{\theta^N\omega}^{-N}|_{T_{p_i} f_\omega^N U(\omega)}\right| \cdot m_{f_\omega^N U(\omega)}(B_i) &\le m_{U(\omega)}(f_{\theta^N\omega}^{-N}B_i) \\ &\le C_{13}(\omega) \cdot \left|\det D_{p_i} f_{\theta^N\omega}^{-N}|_{T_{p_i} f_\omega^N U(\omega)}\right| \cdot m_{f_\omega^N U(\omega)}(B_i). \end{aligned}$$

Recall that B_i is a δ-ball, then by Lemma 3.9 and (3.29),

$$\begin{aligned} C_7^{-1}C_9(\omega)^{-1}C_{13}(\omega)^{-1} \cdot \left|\det D_{q_i} f_{\theta^N\omega}^{-N}|_{T_{q_i} f_\omega^N V(\omega)}\right| \cdot m_{f_\omega^N V(\omega)}(\bar{\psi}_{\theta^N\omega}B_i) &\le m_{U(\omega)}(f_{\theta^N\omega}^{-N}B_i) \\ \le C_7C_9(\omega)C_{13}(\omega) \cdot \left|\det D_{q_i} f_{\theta^N\omega}^{-N}|_{T_{q_i} f_\omega^N V(\omega)}\right| \cdot m_{f_\omega^N V(\omega)}(\bar{\psi}_{\theta^N\omega}B_i). \end{aligned}$$

Again by (3.45) and change of variable, we obtain

$$\begin{aligned} C_7^{-1}C_9(\omega)^{-1}(C_{13}(\omega))^{-2} \cdot m_{V(\omega)}(\psi_\omega f_{\theta^N\omega}^{-N}B_i) &\le m_{U(\omega)}(f_{\theta^N\omega}^{-N}B_i) \\ &\le C_7C_9(\omega)(C_{13}(\omega))^2 \cdot m_{V(\omega)}(\psi_\omega f_{\theta^N\omega}^{-N}B_i). \end{aligned}$$

Denote $C_8(\omega) := C_7C_9(\omega)C_{13}(\omega)^2$, then the claim (3.39) is proved. The proof for absolute continuity is complete.

Next, we explore the Jacobian of the holonomy map. Recall that $\bar{\psi}_{\theta^n\omega} : f_\omega^n U(\omega) \to f_\omega^n V(\omega)$ is the holonomy map induced by the local stable manifolds. Notice that $\psi_\omega = f_{\theta^n\omega}^{-n} \circ \bar{\psi}_{\theta^n\omega} \circ f_\omega^n$, so for $m_{U(\omega)}$-a.s. $x \in U(\omega)$,

$$Jac(\psi_\omega)(x) = H_\omega^n(x, \psi_\omega(x), T_xU(\omega), T_{\psi_\omega(x)}V(\omega)) \cdot Jac(\bar{\psi}_{\theta^n\omega})(f_\omega^n(x)). \tag{3.49}$$

To derive the formula of $Jac(\psi_\omega)(x)$, we need the following lemma, whose condition is stronger than ones in [31, Theorem 3.3].

Lemma 3.11. *Let N and P be manifolds with $m_P(P) < \infty$ and $m_N(\partial N) = 0$, and $(\phi_n)_{n\ge 0}$, $\phi_n : N \to P$, be a sequence of absolutely continuous maps with Jacobian $Jac(\phi_n)$. If ϕ_n converges uniformly to an absolutely continuous and injective map $\phi : N \to P$ and $Jac(\phi_n)$ (by changing its value on zero measure set) converges uniformly to an integrable function $J : N \to \mathbb{R}$, then $Jac(\phi) = J$.*

Next, we will define a sequence of absolutely continuous maps $\pi_{\theta^n\omega} : f_\omega^n U(\omega) \to f_\omega^n V(\omega)$ such that the sequence of mapping

$$\psi_{n,\omega} = (f_{\theta^n\omega}^{-n}|_{f_\omega^n V(\omega)}) \circ \pi_{\theta^n\omega} \circ (f_\omega^n|_{U(\omega)}) : U(\omega) \to V(\omega) \tag{3.50}$$

converges to ψ_ω uniformly, and a version of $Jac(\psi_{n,\omega})$ converges uniformly to an integrable function. Here, a version of $Jac(\psi_{n,\omega})$ is defined by changing the value of it on a zero $m_{U(\omega)}$-measure set.

For $\delta > 0$ sufficiently small, we cover $f_\omega^n U(\omega)$ for any $n \in \mathbb{N}$ by balls $\{B_{f_\omega^n U(\omega)}(f_\omega^n(x), \delta)\}_{x \in U(\omega)}$. Since $U(\omega)$ is pre-compact, there exists a finite cover $\{B_{f_\omega^n U(\omega)}(f_\omega^n(x_i), \delta)\}_{i=1}^{k(\theta^n\omega)}$. This open cover produces a natural finite Borel partition of $f_\omega^n U(\omega)$, named $\{P_i(\theta^n\omega)\}_{i=1}^{k(\theta^n\omega)}$, in the following way:

$$P_1(\theta^n\omega) := (f_\omega^n U(\omega)) \cap B_{f_\omega^n U(\omega)}(f_\omega^n(x_i), \delta),$$

and

$$P_i(\theta^n\omega) := (f_\omega^n U(\omega)) \cap B_{f_\omega^n U(\omega)}(f_\omega^n(x_i), \delta) \backslash (\cup_{j=1}^{i-1} B_{f_\omega^n U(\omega)}(f_\omega^n(x_j), \delta))$$

for $i = 2, ..., k(\theta^n\omega)$. Without loss of generality, we assume $P_i(\theta^n\omega) \neq \emptyset$ for all i. This Borel partition satisfies that

$$P_i(\theta^n\omega) \subset B_{f_\omega^n U(\omega)}(f_\omega^n(x_i), \delta), \text{ and } m_{f_\omega^n U(\omega)}(\cup_i \partial P_i(\theta^n\omega)) = 0. \tag{3.51}$$

Denote $\Psi_n^{1,i}$ (resp. $\Psi_n^{2,i}$): $E^u(f_\omega^n(x_i), \theta^n\omega)(\mathcal{P}\delta) \to E^s(f_\omega^n(x_i), \theta^n\omega)$ to be the C^2 mapping such that $\exp_{x_i}(graph(\Psi_n^{1,i}))$ represents the component of $f_\omega^n U(\omega)$ (resp. $f_\omega^n V(\omega)$) passing through $f_\omega^n(x_i)$ (resp. $f_\omega^n(\psi_\omega(x_i))$). We define $\tilde{\pi}_{\theta^n\omega}$ restricted on $\exp_{x_i}^{-1}(P_i(\theta^n\omega))$ by

$$\tilde{\pi}_{\theta^n\omega}(e, \Psi_n^{1,i}(e)) = (e, \Psi_n^{2,i}(e)) \text{ for any } (e, \Psi_n^{1,i}(e)) \in \exp_{x_i}^{-1}(P_i(\theta^n\omega)),$$

i.e. the holonomy map induced by the flat foliation $\{e + E^s(f_\omega^n(x_i), \theta^n\omega) : \ e \in E^u(f_\omega^n(x_i), \theta^n\omega)(\mathcal{P}\delta)\}$. Then let

$$\pi_{\theta^n\omega}(y) = \exp_{x_i} \circ \tilde{\pi}_{\theta^n\omega} \circ \exp_{x_i}^{-1}(y), \text{ for } y \in P_i(\theta^n\omega).$$

This mapping $\pi_{\theta^n\omega}$ is naturally absolutely continuous since it is absolutely continuous on each set in the partition. Moreover, $\pi_{\theta^n\omega}$ is C^2 for $x \in$interior$(P_i(\theta^n\omega))$ for all i. Let $\psi_{n,\omega}$ be defined as (3.50) by using this $\pi_{\theta^n\omega}$.

Note that $f_\omega^n U(\omega)$ is C^1-close to $f_\omega^n V(\omega)$ as $n \to \infty$. By the local stable manifolds theorem, for any $\epsilon > 0$, there exists $N_\epsilon(\omega)$ such that

$$d_{f_\omega^n V(\omega)}(\bar{\psi}_{\theta^n\omega}(x), \pi_{\theta^n\omega}(x)) < \epsilon \text{ for } x \in f_\omega^n U(\omega),$$

where $\bar{\psi}_{\theta^n\omega} = f_\omega^n \circ \psi_\omega \circ f_{\theta^n\omega}^{-n}$. Therefore, for any $x \in U(\omega)$, $n \geq N_\epsilon(\omega)$, we have

$$d_{V(\omega)}(\psi_\omega(x),\psi_{n,\omega}(x))=d_{V(\omega)}(f_{\theta^n\omega}^{-n}\circ\bar{\psi}_{\theta^n\omega}(f_\omega^n(x)),f_{\theta^n\omega}^{-n}\circ\circ\pi_{\theta^n\omega}(f_\omega^n(x)))\overset{(3.16)}{\le}C_4(\omega)^{-1}e^{-\lambda n}\epsilon,$$

which goes to 0 uniformly for all $x\in U(\omega)$ as $n\to\infty$.

It is left to show that there exists a version of $Jac(\psi_{n,\omega})(x)$ converges uniformly for $x\in U(\omega)$. By (3.50), for $m_{U(\omega)}$-a.s. $x\in U(\omega)$, we have

$$Jac(\psi_{n,\omega})(x)=H_\omega^n(x,\psi_{n,\omega}(x),T_xU(\omega),T_{\psi_{n,\omega}(x)}V(\omega))\cdot Jac(\pi_{\theta^n\omega})(f_\omega^n(x)).$$

Note that (3.51) implies that $m_{U(\omega)}(f_{\theta^n\omega}^{-n}(\cup_i\partial P_i(\theta^n\omega)))=0$. For $x\in f_{\theta^n\omega}^{-n}(\cup_i\partial P_i(\theta^n\omega))$, we set $Jac(\pi_{\theta^n\omega})(f_\omega^n(x))=1$. For $x\in U(\omega)$ such that $f_\omega^n(x)\in$interior$(P_i)(\theta^n\omega)$, $\pi_{\theta^n\omega}$ is C^2 on $f_\omega^n x$. By the definition of $\tilde{\pi}_{\theta^n\omega}$, we have

$$Jac(\tilde{\pi}_{\theta^n})(\exp_{x_i}^{-1}(f_\omega^n(x)))=\frac{Jac(\tilde{\Psi}_n^{2,i})(e)}{Jac(\tilde{\Psi}_n^{1,i})(e)}\text{ for }e=(\tilde{\Psi}_n^{1,i})^{-1}(\exp_{x_i}^{-1}(f_\omega^n(x))),$$

where $\tilde{\Psi}_n^{\tau,i}(e)=(e,\Psi_n^{\tau,i}(e))$ for $\tau=1,2$. According to the proof of (3.34), for any $\eta>0$, there exists $N_\eta(\omega)\in\mathbb{N}$ such that for any $n>N_\eta(\omega)$, one has

$$Jac(\tilde{\pi}_{\theta^n})(f_\omega^n(x))=\frac{Jac(\tilde{\Psi}_n^{2,i})(e)}{Jac(\tilde{\Psi}_n^{1,i})(e)}\in(e^{-C^*\eta},e^{C^*\eta})$$

for some constant C^*. Moreover, we have

$$\frac{|\det D_{\tilde{\pi}_{\theta^n}(\exp_{x_i}^{-1}(f_\omega^n(x)))}\exp_{x_i}|}{|\det D_{\exp_{x_i}^{-1}(f_\omega^n(x))}\exp_{x_i}|}\to 1\text{ uniformly, as }n\to\infty$$

since $d(f_\omega^n(x),\pi_{\theta^n\omega}(f_\omega^n(x)))\to 0$ uniformly. Therefore, there exists a version of $Jac(\pi_{\theta^n})(f_\omega^n(x))$ converges to 1 uniformly as $n\to\infty$.

Finally, we show that

$$\frac{H_\omega^n(x,\psi_{n,\omega}(x),T_xU(\omega),T_{\psi_{n,\omega}(x)}V(\omega))}{H_\omega^n(x,\psi_\omega(x),T_xU(\omega),T_{\psi_\omega(x)}V(\omega))}=\frac{|\det D_{\psi_\omega(x)}f_\omega^n|_{T_{\psi_\omega(x)}V(\omega)}|}{|\det D_{\psi_{n,\omega}(x)}f_\omega^n|_{T_{\psi_{n,\omega}(x)}V(\omega)}|}\to 1 \tag{3.52}$$

uniformly as $n\to\infty$. For any $\epsilon>0$, we pick $N_1(\omega)\in\mathbb{N}$ such that for any $n\ge N$,

$$\sum_{n\ge N_1(\omega)}C_{12}(\omega)e^{-\lambda n}\le\epsilon,$$

where $C_{12}(\omega)$ is defined in (3.48). Then we pick $\eta>0$ sufficiently small such that if $y,z\in U(\omega)$ satisfying $\sup_{0\le i\le N_1(\omega)-1}d(f_\omega^i(y),f_\omega^i(z))<\eta$,

$$\frac{|\det D_yf_\omega^{N_1(\omega)}|_{T_yV(\omega)}|}{|\det D_zf_\omega^{N_1(\omega)}|_{T_zV(\omega)}|}\in(e^{-\epsilon},e^\epsilon),$$

and moreover,

$$C_{12}(\omega)\sum_{i=0}^{\infty} e^{-\lambda i v_1}\eta \le \epsilon.$$

By the constriction of $\psi_{n,\omega}$, we pick $N_2(\omega) > N_1(\omega)$ such that for any $n \ge \mathbb{N}_2(\omega)$,

$$\sup_{0\le i\le n-1} d(f_\omega^i(\psi_\omega(x)), f_\omega^i(\psi_{n,\omega}(x))) < \eta.$$

Now, according to the proof of (3.44), for any $n \ge N_2(\omega)$, we arrive

$$\begin{aligned}
&\frac{|\det D_{\psi_\omega(x)} f_\omega^n|_{T_{\psi_\omega(x)}V(\omega)}|}{|\det D_{\psi_{n,\omega}(x)} f_\omega^n|_{T_{\psi_{n,\omega}(x)}V(\omega)}|}\\
=&\frac{|\det D_{\psi_\omega(x)} f_\omega^{N_1(\omega)}|_{T_{\psi_\omega(x)}V(\omega)}|}{|\det D_{\psi_{n,\omega}(x)} f_\omega^{N_1(\omega)}|_{T_{\psi_{n,\omega}(x)}V(\omega)}|}\\
&\cdot\frac{|\det D_{f_\omega^{N_1(\omega)}(\psi_\omega(x))} f^{n-N_1(\omega)}_{\theta^{N_1(\omega)}\omega}|_{D_{\psi_\omega(x)}f_\omega^{N_1(\omega)}T_{\psi_\omega(x)}V(\omega)}|}{|\det D_{f_\omega^{N_1(\omega)}(\psi_{n,\omega}(x))} f^{n-N_1(\omega)}_{\theta^{N_1(\omega)}\omega}|_{D_{\psi_{n,\omega}(x)}f^{n-N_1(\omega)}_{\theta^{N_1(\omega)}\omega}T_xV(\omega)}|}\in(e^{-3\epsilon},e^{3\epsilon}).
\end{aligned}$$

So (3.52) holds. Therefore, by Lemma 3.11 and changing value on zero $m_{U(\omega)}$-measure set, we have

$$Jac(\psi_\omega)(x) = H_\omega(x, \psi_\omega(x), T_xU(\omega), T_{\psi_\omega(x)}V(\omega)) \text{ for } x \in U(\omega). \tag{3.53}$$

The proof of Proposition 3.1 is complete. □

3.5. Hölder continuity of the stable and unstable foliations on each fiber

In this subsection, we prove the Hölder continuity of the holonomy map between two local stable leaves (or unstable leaves). This result is known in deterministic hyperbolic systems (see, e.g., [34]), but we didn't find any reference to this result in RDS. We supply a proof in our settings, and the proof follows the idea given in Page 762 in [24]. Roughly speaking, the idea is that the action of the graph transform preserves a Hölder condition.

Proposition 3.2. *Suppose ϕ is C^2 Anosov on fibers, let $\varrho \in (0,1)$ satisfy*

$$\sup_{(p,\omega)\in M\times\Omega}\|D_pf_\omega|_{E^s(p,\omega)}\|t_{(p,\omega)}^{-\varrho} < 1,\quad \sup_{(p,\omega)\in M\times\Omega}\|D_pf_\omega^{-1}|_{E^u(p,\omega)}\|s_{(p,\omega)}^{-\varrho} < 1$$

where

$$\begin{aligned}
t_{(p,\omega)} &:= \inf\left\{\frac{d(f_\omega p, f_\omega q)}{d(p,q)} : q\in M,\ d(p,q)<\epsilon_0\right\} > 0,\\
s_{(p,\omega)} &:= \inf\left\{\frac{d(f_\omega^{-1}(p), f_\omega^{-1}(q))}{d(p,q)} : q\in M,\ d(p,q)<\epsilon_0\right\} > 0,
\end{aligned}$$

and ϵ_0 is the size of local stable and unstable manifolds. Then there exists a constant $\delta_0 > 0$ and $H = H(\delta_0, \varrho)$ such that for any $(q, \omega) \in M \times \Omega$,

$$\sup\left\{\frac{\sup_{x\in E^u(p,\omega)(\delta_0)} \|h^u_{(p,\omega)}(x) - \tilde{h}^u_{(q,\omega)}(x)\|}{d(p,q)^{\varrho}} : q \in M,\ d(p,q) < \delta_0\right\} \leq H < \infty, \tag{3.54}$$

$$\sup\left\{\frac{\sup_{x\in E^u(p,\omega)(\delta_0)} \|h^s_{(p,\omega)}(x) - \tilde{h}^s_{(q,\omega)}(x)\|}{d(p,q)^{\varrho}} : q \in M,\ d(p,q) < \delta_0\right\} \leq H < \infty, \tag{3.55}$$

where $h^u_{(p,\omega)}, \tilde{h}^u_{(q,\omega)} : E^u(p,\omega)(\delta_0) \to E^s(p,\omega)$ and $Exp_p(graph(h^u_{(p,\omega)}))$, $Exp_p(graph(\tilde{h}^u_{(q,\omega)}))$ represent the local unstable manifolds passing through p, q respectively, $h^s_{(p,\omega)}, \tilde{h}^s_{(q,\omega)}$: $E^s(p,\omega)(\delta_0) \to E^u(p,\omega)$ and $Exp_p(graph(h^s_{(p,\omega)}))$, $Exp_p(graph(\tilde{h}^s_{(q,\omega)}))$ represent the local stable manifolds passing through p, q respectively. Furthermore, the local product structure is Hölder continuous, i.e., there exists a constant $H' = H'(\delta_0, \varrho)$ such that for any $(x,\omega) \in M \times \Omega$, $y \in W^s_{\delta_0}(x,\omega)$, any $z \in M$ such that $W^s_\epsilon(z,\omega) \cap W^u_\epsilon(x,\omega) \neq \emptyset$, $W^s_\epsilon(z,\omega) \cap W^u_\epsilon(y,\omega) \neq \emptyset$, we have

$$d(W^s_\epsilon(z,\omega) \cap W^u_\epsilon(x,\omega), W^s_\epsilon(z,\omega) \cap W^u_\epsilon(y,\omega)) \leq H' d(x,y)^{\varrho};$$

for any $(x,\omega) \in M \times \Omega$, $y \in W^u_{\delta_0}(x,\omega)$, any $z \in M$ such that $W^u_\epsilon(z,\omega) \cap W^s_\epsilon(x,\omega) \neq \emptyset$, $W^u_\epsilon(z,\omega) \cap W^s_\epsilon(y,\omega) \neq \emptyset$, we have

$$d(W^u_\epsilon(z,\omega) \cap W^s_\epsilon(x,\omega), W^u_\epsilon(z,\omega) \cap W^s_\epsilon(y,\omega)) \leq H' d(x,y)^{\varrho}.$$

Proof. We first prove (3.54). Recall that for each point $p \in M$, there exist a neighborhood $N_p \subset M$ and constant ϵ such that the exponential map $Exp_p : B_\epsilon(0) \subset TpM \to M$ is a C^∞-diffeomorphism and $N_p \subset Exp_p(B_\epsilon(0))$. Now for all $p \in M$ and $\omega \in \Omega$, consider any continuous function $g_{(p,\omega)} : E^u(p,\omega)(\epsilon) \to E^s(p,\omega)$ with $g_{(p,\omega)}(0) = 0$, where $E^u(p,\omega)(\epsilon)$ is the ϵ-disk in $E^u(p,\omega)$ centered at the origin. Define the special norm by

$$\|g_{(p,\omega)}\|_* = \sup\left\{\frac{\|g_{(p,\omega)}(x)\|}{\|x\|} : x \in E^u(p,\omega)(\epsilon),\ x \neq 0\right\}.$$

Define

$$G^*_{(p,\omega)} := \{g_{(p,\omega)} : E^u(p,\omega)(\epsilon) \to E^s(p,\omega) |\ g_{(p,\omega)}(0) = 0 \text{ and } \|g_{(p,\omega)}\|_* < \infty\},$$

and

$$G_{(p,\omega)} := \left\{g_{(p,\omega)} \in G^*_{(p,\omega)} : Lip(g_{(p,\omega)}) \leq \frac{e^{-2\lambda}+1}{2}\right\}. \tag{3.56}$$

Sublemma 3.1. *$G^*_{(p,\omega)}$ equipped with $\|\cdot\|_*$ is a Banach space and $G_{(p,\omega)}$ is a closed subset.*

The above sublemma is a corollary of [37, Lemma III.3]. $\{G_{(p,\omega)}\}_{(p,\omega)\in M\times\Omega}$ gives a bundle G on $M \times \Omega$ with fiber $G_{(p,\omega)}$ on $(p,\omega) \in M \times \Omega$.

Now we define $f_{(p,\omega)} : T_pM(\epsilon) \to T_{f_\omega p}M$ by the local representation of f_ω with respect to exponential maps Exp_p and $Exp_{f_\omega p}$, i.e.,

$$f_{(p,\omega)}(v) = Exp^{-1}_{f_\omega p} \circ f_\omega \circ Exp_p(v),\ \forall v \in T_pM(\epsilon).$$

For any $(p,\omega) \in M \times \Omega$, we define a bundle map $\phi^*_{(p,\omega)}$ on $G_{(p,\omega)}$ satisfying for any $g_{(p,\omega)} \in G_{(p,\omega)}$,

$$graph(\phi^*_{(p,\omega)}g_{(p,\omega)}) = f_{(p,\omega)}(graph(g_{(p,\omega)})) \cap (E^u(f_\omega p, \theta\omega)(\epsilon) \oplus E^s(f_\omega p, \theta\omega)).$$

Note that the definition of $G_{(p,\omega)}$ is relying on ϵ. The next sublemma says that there exists $\epsilon > 0$ such that $\phi^*_{(p,\omega)}$ maps $G_{(p,\omega)}$ into $G_{(f_\omega p,\theta\omega)}$ and this map is a contraction.

Sublemma 3.2. *There exists a $\epsilon_0 > 0$ such that for any $\epsilon \in (0,\epsilon_0)$, $(p,\omega) \in M \times \Omega$, one has $\phi^*_{(p,\omega)}g_{(p,\omega)} \in G_{(f_\omega p,\theta\omega)}$, and it is a fiber contraction, i.e., for any $g_{(p,\omega)}$, $g'_{(p,\omega)} \in G_{(p,\omega)}$, we have*

$$\|\phi^*_{(p,\omega)}g_{(p,\omega)} - \phi^*_{(p,\omega)}g'_{(p,\omega)}\|_* \le \frac{e^{-2\lambda_0}+1}{2}\|g_{(p,\omega)} - g'_{(p,\omega)}\|_*,$$

Proof of Sublemma 3.2. Fix any $\epsilon' > 0$ such that

$$\frac{e^{-\lambda_0} + 2\mathcal{P}\epsilon'}{e^{\lambda_0} - 2\mathcal{P}\epsilon'} \le \frac{e^{-2\lambda_0}+1}{2}. \tag{3.57}$$

By compactness of Ω and M and the continuity of f_ω on ω, for the above ϵ', we can pick a $\epsilon_0 > 0$ sufficiently small such that for any $\epsilon < \epsilon_0$,

$$Lip((f_{(p,\omega)} - D_pf_\omega)|_{T_pM(\epsilon)}) < \epsilon'.$$

Pick any $g \in G_{(p,\omega)}$, let $f_{(p,\omega)}(x, g(x))$ have decomposition

$$f_{(p,\omega)}(x, g(x)) = (f_{(p,\omega),1}(x, g(x)), f_{(p,\omega),2}(x, g(x)))$$

with respect to $T_{f_\omega p}M = E^u(f_\omega p,\theta\omega) \oplus E^s(f_\omega p,\theta\omega)$, and we denote $h_{(p,\omega)} := f_{(p,\omega),1} \circ (id, g) : E^u(p,\omega)(\epsilon) \to E^u(f_\omega p,\theta\omega)$. Note that

$$\begin{aligned} f_{(p,\omega),1}(x, g(x)) &= P(E^u(f_\omega p,\theta\omega)) \circ f_{(p,\omega)} \circ (id, g)(x), \\ D_pf_\omega|_{E^u(p,\omega)}(x) &= P(E^u(f_\omega p,\theta\omega)) \circ D_pf_\omega \circ (id, g)(x), \end{aligned}$$

where $P(E^u(f_\omega p,\theta\omega)) : T_{f_\omega p}M \to E^u(f_\omega p,\theta\omega)$ is the projection with respect to the above decomposition. Then we have

$$\begin{aligned}
&Lip(f_{(p,\omega),1} \circ (id, g) - D_p f_\omega|_{E^u(p,\omega)(\epsilon)}) \\
&\le \mathcal{P} \cdot Lip((f_{(p,\omega)} - D_p f_\omega)|_{T_pM(\epsilon)}) \cdot Lip(id, g) \\
&< \mathcal{P}\epsilon',
\end{aligned} \tag{3.58}$$

where $\mathcal{P}$ is the constant in (3.3). By the Lipschitz Inverse function theorem (see, e.g., [37, Theorem I.2]), $h_{(p,\omega)}$ is a homeomorphism to its image and moreover,

$$\begin{aligned}
Lip(h^{-1}_{(p,\omega)}) &\le \frac{1}{\|D_p f_\omega|^{-1}_{E^u(p,\omega)}\|^{-1} - Lip(f_{(p,\omega),1} \circ (id, g) - D_p f_\omega|_{E^u(p,\omega)(\epsilon)})} \\
&\overset{(3.58)}{<} \frac{1}{m(D_p f_\omega|_{E^u(p,\omega)}) - \mathcal{P}\epsilon'},
\end{aligned} \tag{3.59}$$

where $m(D_p f_\omega|_{E^u(p,\omega)}) = \|D_p f_\omega|^{-1}_{E^u(p,\omega)}\|^{-1}$ denotes the co-norm. Then for any $g \in G_{(p,\omega)}$, we have

$$(\phi^*_{(p,\omega)} g)(y) = f_{(p,\omega),2}(h^{-1}_{(p,\omega)}(y), g(h^{-1}_{(p,\omega)}(y))) \text{ for } y \in E^u(f_\omega p, \theta\omega)(\epsilon). \tag{3.60}$$

Note that for $x \in E^u(p,\omega)(\epsilon)$,

$$\begin{aligned}
f_{(p,\omega),2}(x, g(x)) &= P(E^s(f_\omega p, \theta\omega)) \circ f_{(p,\omega)} \circ (id, g)(x), \\
D_p f_\omega|_{E^s(p,\omega)}(g(x)) &= P(E^s(f_\omega p, \theta\omega)) \circ D_p f_\omega \circ (id, g)(x),
\end{aligned}$$

then we have

$$\begin{aligned}
Lip(f_{(p,\omega),2} \circ (id, g)) &\le Lip(f_{(p,\omega),2} \circ (id, g) - D_p f_\omega|_{E^s(p,\omega)} \circ g) + Lip(D_p f_\omega|_{E^s(p,\omega)} \circ g) \\
&\le \|P(E^s(f_\omega p, \theta\omega))\| \cdot Lip(f_{(p,\omega)} - D_p f_\omega) \cdot Lip((id, g)) \\
&\quad + \|D_p f_\omega|_{E^s(p,\omega)}\| \\
&< \mathcal{P}\epsilon' + \|D_p f_\omega|_{E^s(p,\omega)}\|.
\end{aligned} \tag{3.61}$$

Combining (3.59), (3.60) and (3.61),

$$\begin{aligned}
Lip(\phi^*_{(p,\omega)} g) &\le Lip(f_{(p,\omega),2} \circ (id, g)) \cdot Lip(h^{-1}_{(p,\omega)}) \le \frac{\|D_p f_\omega|_{E^s(p,\omega)}\| + \mathcal{P}\epsilon'}{m(D_p f_\omega|_{E^u(p,\omega)}) - \mathcal{P}\epsilon'} \\
&\le \frac{e^{-\lambda_0} + \mathcal{P}\epsilon'}{e^{\lambda_0} - \mathcal{P}\epsilon'} \overset{(3.57)}{<} \frac{e^{-2\lambda_0} + 1}{2}.
\end{aligned}$$

Obviously that $(\phi^*_{(p,\omega)} g)(0) = 0$, so we have shown that $\phi^*_{(p,\omega)}$ maps $G_{(p,\omega)}$ to $G_{(f_\omega p, \theta\omega)}$.

Next, we show that $\phi^*_{(p,\omega)}$ is fiber-contraction. We note that

$$\frac{\|D_p f_\omega|_{E^s(p,\omega)}\| + 2\mathcal{P}\epsilon'}{m(D_p f_\omega|_{E^u(p,\omega)}) - 2\mathcal{P}\epsilon'} \le \frac{e^{-\lambda_0} + 2\mathcal{P}\epsilon'}{e^{\lambda_0} - 2\mathcal{P}\epsilon'} \overset{(3.57)}{\le} \frac{e^{-2\lambda_0} + 1}{2}.$$

Therefore, it is sufficient to show that for any $g, g' \in G_{(p,\omega)}$, for all $x \in E^u(p,\omega)(\epsilon)$,

$$\frac{\|(\phi^*_{(p,\omega)}g)(f_{(p,\omega),1}(x,g(x))) - (\phi^*_{(p,\omega)}g')(f_{(p,\omega),1}(x,g(x)))\|}{\|f_{(p,\omega),1}(x,g(x))\|} \leq \frac{\|D_p f_\omega|_{E^s(p,\omega)}\| + 2\mathcal{P}\epsilon'}{m(D_p f_\omega|_{E^u(p,\omega)}) - 2\mathcal{P}\epsilon'} \cdot \|g - g'\|_*, \tag{3.62}$$

since $h_{(p,\omega)} = f_{(p,\omega),1} \circ (id, g)$ is a homeomorphism.

Notice that

$$\begin{aligned}
&\|f_{(p,\omega),2}(x,g(x)) - f_{(p,\omega),2}(x,g'(x))\| \\
&\leq \big\|(f_{(p,\omega),2} - P(E^s(f_\omega p,\theta\omega))D_p f_\omega)(x,g(x)) - (f_{(p,\omega),2} \\
&\quad - P(E^s(f_\omega p,\theta\omega))D_p f_\omega)(x,g'(x))\big\| + \|P(E^s(f_\omega p,\theta\omega))D_p f_\omega(x,g'(x)) \\
&\quad - P(E^s(f_\omega p,\theta\omega))D_p f_\omega(x,g(x))\| \\
&\leq \big\{Lip(f_{(p,\omega),2} - P(E^s(f_\omega p,\theta\omega))D_p f_\omega) + \|D_p f_\omega|_{E^s(p,\omega)}\|\big\} \cdot \|g(x) - g'(x)\| \\
&= \big\{Lip(P(E^s(f_\omega p,\theta\omega))f_{(p,\omega)} - P(E^s(f_\omega p,\theta\omega))D_p f_\omega) + \|D_p f_\omega|_{E^s(p,\omega)}\|\big\} \cdot \|g(x) - g'(x)\| \\
&< \big(\mathcal{P}\epsilon' + \|D_p f_\omega|_{E^s(p,\omega)}\|\big)\,\|g(x) - g'(x)\|,
\end{aligned} \tag{3.63}$$

and

$$\begin{aligned}
&\|f_{(p,\omega),1}(x,g(x)) - f_{(p,\omega),1}(x,g'(x))\| \\
&\leq \|(f_{(p,\omega),1} - P(E^u(f_\omega p,\theta\omega))D_p f_\omega)(x,g(x)) - (f_{(p,\omega),1} \\
&\quad - P(E^u(f_\omega p,\theta\omega))D_p f_\omega)(x,g'(x))\| + \|P(E^u(f_\omega p,\theta\omega)D_p f_\omega)(x,g(x)) \\
&\quad - P(E^u(f_\omega p,\theta\omega)D_p f_\omega)(x,g'(x))\| \\
&\leq Lip(f_{(p,\omega),1} - P(E^u(f_\omega p,\theta\omega))D_p f_\omega)\|g(x) - g'(x)\| \\
&= Lip(P(E^u(f_\omega p,\theta\omega))f_{(p,\omega)} - P(E^u(f_\omega p,\theta\omega))D_p f_\omega)\|g(x) - g'(x)\| \\
&< \mathcal{P}\epsilon'\|g(x) - g'(x)\|.
\end{aligned} \tag{3.64}$$

Then (3.64) and (3.63) imply that

$$\begin{aligned}
&\|(\phi^*_{(p,\omega)}g)(f_{(p,\omega),1}(x,g(x))) - (\phi^*_{(p,\omega)}g')(f_{(p,\omega),1}(x,g(x)))\| \\
&= \|f_{(p,\omega),2}(x,g(x)) - (\phi^*_{(p,\omega)}g')(f_{(p,\omega),1}(x,g(x)))\| \\
&\leq \|f_{(p,\omega),2}(x,g(x)) - f_{(p,\omega),2}(x,g'(x))\| + \|f_{(p,\omega),2}(x,g'(x)) - (\phi^*_{(p,\omega)}g')(f_{(p,\omega),1}(x,g(x)))\| \\
&\overset{(3.63)}{\leq} \big(\mathcal{P}\epsilon' + \|D_p f_\omega|_{E^s(p,\omega)}\|\big)\,\|g(x) - g'(x)\| \\
&\quad + \|(\phi^*_{(p,\omega)}g')(f_{(p,\omega),1}(x,g'(x))) - (\phi^*_{(p,\omega)}g')(f_{(p,\omega),1}(x,g(x)))\| \\
&\leq \big(\mathcal{P}\epsilon' + \|D_p f_\omega|_{E^s(p,\omega)}\|\big)\,\|g(x) - g'(x)\| + Lip(\phi^*_{(p,\omega)}g')\|f_{(p,\omega),1}(x,g(x))
\end{aligned} \tag{3.65}$$

$$
\begin{aligned}
&- f_{(p,\omega),1}(x, g'(x))\| \\
&\overset{(3.64)}{<} \left(2\mathcal{P}\epsilon' + \|D_p f_\omega|_{E^s(p,\omega)}\|\right)\|g(x) - g'(x)\|. \qquad (3.66)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\|f_{(p,\omega),1}(x, g(x))\| &= \|(f_{(p,\omega),1} - P(E^u(f_\omega p, \theta\omega))D_p f_\omega)(x, g(x)) \\
&\quad + P(E^u(f_\omega p, \theta\omega))D_p f_\omega(x, g(x))\| \\
&\overset{(3.58)}{\geq} m(D_p f_\omega|_{E^u(p,\omega)})\|x\| - \mathcal{P}\epsilon'\|(x, g(x))\| \\
&\geq (m(D_p f_\omega|_{E^u(p,\omega)}) - 2\mathcal{P}\epsilon')\|x\|, \qquad (3.67)
\end{aligned}
$$

where we use $\|g(x)\| = \|g(x) - g(0)\| \leq \|x\|$ in the last step. Now (3.62) follows by (3.66), (3.67) and (3.57). □

By Sublemma 3.2, $\phi^* : G \to G$ defined by

$$
\begin{aligned}
&\{(p, \omega, g_{(p,\omega)}) : (p, \omega) \in M \times \Omega,\ g_{(p,\omega)} \in G_{(p,\omega)}\} \\
&\quad \mapsto \{(f_\omega(p), \theta\omega, \phi^*_{(p,\omega)} g_{(p,\omega)}) : (p, \omega) \in M \times \Omega\}
\end{aligned}
$$

is a fiber contraction. Then there exists a unique section $g^* : M \times \Omega \to G$ which is invariant under ϕ^* in the sense that for all $(p, \omega) \in M \times \Omega$,

$$
g^*_{(f_\omega(p),\theta\omega)} = \phi^*_{(p,\omega)} g^*_{(p,\omega)}.
$$

In fact, we can consider the space Σ of all sections $g : M \times \Omega \to G$. Note that

$$
\sup_{(p,\omega)\in M\times\Omega} \{\|g_{(p,\omega)} - g'_{(p,\omega)}\|_* : g_{(p,\omega)}, g'_{(p,\omega)} \in G_{(p,\omega)}\} \overset{(3.56)}{\leq} e^{-2\lambda} + 1 < \infty.
$$

Therefore, $\tilde{d}(g, g') = \sup\{\|g_{(p,\omega)} - g'_{(p,\omega)}\|_* : (p, \omega) \in M \times \Omega\}$ for any $g, g' \in \Sigma$ gives a metric on Σ. Moreover, Σ is complete since $G_{(p,\omega)}$ is closed. By sublemma 3.2, the action of ϕ^* on Σ is clearly a contraction mapping. The unique fixed point of ϕ^* is the unique invariant section. By virtue of contraction mapping theorem, one also can obtain the unique invariant section g^* by iterating any section $g \in G$, i.e.,

$$
g^*_{(p,\omega)} = \lim_{n\to\infty} ((\phi^*)^n g)_{(p,\omega)} = \lim_{n\to\infty} \phi^*_{(f_\omega^{-1} p, \theta^{-1}\omega)} \cdots \phi^*_{(f_\omega^{-n} p, \theta^{-n}\omega)} g_{(f_\omega^{-n} p, \theta^{-n}\omega)}, \qquad (3.68)
$$

for any $(p, \omega) \in M \times \Omega$.

By the stable and unstable manifolds theorem, we know that the local unstable manifold passing through p on the fiber $M \times \{\omega\}$ is exactly $Exp_p(graph(g^*_{(p,\omega)}))$. Next, we will show that the bundle map ϕ^* preserves a local Hölder property for an appropriate Hölder exponent.

Since $E^u(x, \omega)$ and $E^s(x, \omega)$ are uniformly continuous depending on $(x, \omega) \in M \times \Omega$, with the help of local coordinate charts, we may pick a sufficiently small $\delta_0 \in (0, \frac{\epsilon}{2\mathcal{P}})$ such that whenever $d(p, q) < \delta_0$, for any $\omega \in \Omega$, $g_{(q,\omega)} : E^u(q, \omega)(\epsilon) \to E^s(q, \omega)$ with $Lip(g_{(q,\omega)}) < \frac{e^{-2\lambda_0}+1}{2}$

can be viewed as a Lipschitz mapping from $E^u(p,\omega)(\delta_0)$ to $E^s(p,\omega)$ with Lipschtiz constant less than 1 with respect to the coordinate $E^u(p,\omega)\oplus E^s(p,\omega)$. We use notation $(g_{(q,\omega)})^{(p,\omega)}$ to represent $g_{(q,\omega)}$ in the coordinate $E^u(p,\omega)\oplus E^s(p,\omega)$.

From now on, we fix this δ_0. We pick $N>0$ depending on δ_0 such that

$$e^{-N\lambda} < \frac{\delta_0}{2\sup_{(x,\omega)\in M\times\Omega}\|D_x f_\omega^{-1}\|}. \tag{3.69}$$

For any constant $K>0$ and $\varrho\in(0,1)$, we define

$$G(\delta_0,\varrho,e^{-N\lambda},K) := \Bigg\{ g\in G : \sup_{x\in E^u(p,\omega)(\delta_0)} \|g_{(p,\omega)}(x) - (g_{(q,\omega)})^{(p,\omega)}(x)\| \le K d(p,q)^\varrho, \\ \text{whenever } e^{-N\lambda} < d(p,q) < \delta_0 \Bigg\}.$$

Sublemma 3.3. *There exists a constant $C=C(\delta_0)$ such that $G\subset G(\delta_0,\varrho,e^{-N\lambda},C(\delta_0)e^{N\lambda\varrho})$.*

Proof of Sublemma 3.3. Notice that both the Lipschitz constants of $g_{(p,\omega)}$ and $(g_{(q,\omega)})^{(p,\omega)}$ are less than 1, and $d(p,q)<\delta_0$, hence there exists a constant $C=C(\delta_0)>0$ such that

$$\sup_{x\in E^u(p,\omega)(\delta_0)} \|g_{(p,\omega)}(x) - (g_{(q,\omega)})^{(p,\omega)}(x)\| \le C.$$

Notice that $d(p,q)>e^{-N\lambda}$, so we have

$$\sup_{x\in E^u(p,\omega)(\delta_0)} \|g_{(p,\omega)}(x) - (g_{(q,\omega)})^{(p,\omega)}(x)\| \le C(\delta_0) \le C(\delta_0)e^{N\lambda\varrho} d(p,q)^\varrho. \quad \square$$

For any $g_{(q,\omega)}\in G_{(q,\omega)}$, the proof of Sublemma 3.2 implies that $\phi^*_{(q,\omega)}g_{(q,\omega)}\in G_{(f_\omega q,\theta\omega)}$ and $Lip(\phi^*_{(q,\omega)}g_{(q,\omega)}) \le \frac{e^{-2\lambda}+1}{2}$. If $d(f_\omega p, f_\omega q)<\delta_0$, then $Lip((\phi^*_{(q,\omega)}g_{(q,\omega)})^{(f_\omega p,\theta\omega)})<1$ by the choice of δ_0.

Sublemma 3.4. *Let $g\in G$, if $d(p,q)<\delta_0$, $d(f_\omega p, f_\omega q)<\delta_0$, and $\sup_{x\in E^u(p,\omega)(\delta_0)}\|g_{(p,\omega)}(x) - (g_{(q,\omega)})^{(p,\omega)}(x)\| \le K d(p,q)^\varrho$ for some constant K, then*

$$\sup_{x\in E^u(f_\omega p,\theta\omega)(\delta_0)} \left\| \left(\phi^*_{(p,\omega)}g_{(p,\omega)}\right)(x) - \left(\phi^*_{(q,\omega)}g_{(q,\omega)}\right)^{(f_\omega p,\theta\omega)}(x) \right\| \le K d(f_\omega p, f_\omega q)^\varrho \tag{3.70}$$

provided $\varrho\in(0,1)$ satisfying

$$\sup_{(p,\omega)\in M\times\Omega} \|D_p f_\omega|_{E^s(p,\omega)}\| t_{(p,\omega)}^{-\varrho} < 1, \tag{3.71}$$

where

$$t_{(p,\omega)} := \inf\left\{ \frac{d(f_\omega p, f_\omega q)}{d(p,q)} : \ q\in M,\ d(p,q)<\epsilon \right\} > 0. \tag{3.72}$$

Proof of Sublemma 3.4. We use the same notations as in the proof of Sublemma 3.2. By (3.71), we can pick a constant $\epsilon' > 0$ sufficiently small satisfying both (3.57) and

$$\sup_{(p,\omega)\in M\times\Omega} (2\mathcal{P}\epsilon' + \|D_p f_\omega\|_{E^s(p,\omega)}) t_{(p,\omega)}^{-\varrho} < 1. \tag{3.73}$$

Notice that following fact:

$$f_{(p,\omega),2}(x, (g_{(q,\omega)})^{(p,\omega)}(x)) = (\phi^*_{(q,\omega)} g_{(q,\omega)})^{(f_\omega p, \theta\omega)}(f_{(p,\omega),1}(x, (g_{(q,\omega)})^{(p,\omega)}(x))),$$

then we have

$$\begin{aligned}
&\|(\phi^*_{(p,\omega)} g_{(p,\omega)})(f_{(p,\omega),1}(x, g_{(p,\omega)}(x))) - (\phi^*_{(q,\omega)} g_{(q,\omega)})^{(f_\omega p,\theta\omega)}(f_{(p,\omega),1}(x, g_{(p,\omega)}(x)))\| \\
&= \|f_{(p,\omega),2}(x, g_{(p,\omega)}(x)) - (\phi^*_{(q,\omega)} g_{(q,\omega)})^{(f_\omega p,\theta\omega)}(f_{(p,\omega),1}(x, g_{(p,\omega)}(x)))\| \\
&\le \|f_{(p,\omega),2}(x, g_{(p,\omega)}(x)) - f_{(p,\omega),2}(x, (g_{(q,\omega)})^{(p,\omega)}(x))\| \\
&\quad + \|(\phi^*_{(q,\omega)} g_{(q,\omega)})^{(f_\omega p,\theta\omega)}(f_{(p,\omega),1}(x, (g_{(q,\omega)})^{(p,\omega)}(x))) \\
&\quad (\phi^*_{(q,\omega)} g_{(q,\omega)})^{(f_\omega p,\theta\omega)}(f_{(p,\omega),1}(x, g_{(p,\omega)}(x)))\| \\
&\overset{(3.63)}{\le} (\mathcal{P}\epsilon' + \|D_p f_\omega|_{E^s(p,\omega)}\|)\|g_{(p,\omega)}(x) - (g_{(q,\omega)})^{(p,\omega)}(x)\| \\
&\quad + Lip(\phi^*_{(q,\omega)} g_{(q,\omega)})^{(f_\omega p,\theta\omega)})\|f_{(p,\omega),1}(x, (g_{(q,\omega)})^{(p,\omega)}(x)) - f_{(p,\omega),1}(x, g_{(p,\omega)}(x))\| \\
&\overset{(3.64)}{\le} (2\mathcal{P}\epsilon' + \|D_p f_\omega|_{E^s(p,\omega)}\|) \sup_{x\in E^u(p,\omega)(\delta_0)} \|g_{(p,\omega)}(x) - (g_{(q,\omega)})^{(p,\omega)}(x)\| \\
&\le (2\mathcal{P}\epsilon' + \|D_p f_\omega|_{E^s(p,\omega)}\|) K d(p,q)^\varrho \\
&\le (2\mathcal{P}\epsilon' + \|D_p f_\omega|_{E^s(p,\omega)}\|) K t_{(p,\omega)}^{-\varrho} d(f_\omega p, f_\omega q)^\varrho \\
&\overset{(3.73)}{\le} K d(f_\omega p, f_\omega q)^\varrho,
\end{aligned}$$

where we note that in inequality (3.63) and (3.64), g' can be replaced by $(g_{(q,\omega)})^{(p,\omega)}$. Notice that $h_{(p,\omega)} = f_{(p,\omega),1} \circ (id, g_{(p,\omega)})$ is a homeomorphism, hence we get (3.70). □

Now consider

$$R_n(\omega) := \left\{ (f^n_{\theta^{-n}\omega} p, f^n_{\theta^{-n}\omega} q) \in M \times M \mid \max_{0\le k\le n} d(f^k_{\theta^{-n}\omega} p, f^k_{\theta^{-n}\omega} q) < \delta_0,\ e^{-N\lambda} < d(p,q) < \delta_0 \right\},$$

and let $S_n(\omega) = \cup_{i=0}^n R_i(\omega)$. Applying Sublemma 3.4 inductively and notice Sublemma 3.3, we see that for any $g \in G$, $(s,t) \in S_n(\omega)$,

$$\sup_{x\in E^u(s,\omega)(\delta_0)} \left\| ((\phi^*)^n g)_{(s,\omega)}(x) - (((\phi^*)^n g)_{(t,\omega)})^{(s,\omega)}(x) \right\| \le C(\delta_0) e^{N\lambda\varrho} d(p,q)^\varrho,$$

where $(\phi^*)^n$ is the n-th iteration of ϕ^*.

Sublemma 3.5. $\{(s,t) \in M \times M \mid t \notin W^u_{\delta_0}(s,\omega),\ d(s,t) < \delta_0\} \subset \cup_{n=0}^\infty S_n(\omega)$.

Proof of Sublemma 3.5. Now for any $(s,t) \in \{(s,t) \in M \times M | \, t \notin W^u_{\delta_0}(s,\omega), \, d(s,t) < \delta_0\}$. If $d(s,t) > e^{-N\lambda}$, then $(s,t) \in R_0(\omega)$. If $d(s,t) \le e^{-N\lambda}$. We note that $t \in W^s_{\delta_0}([t,s]_\omega, \omega)$, where $[t,s]_\omega = W^s_{loc}(t,\omega) \cap W^u_{loc}(s,\omega)$. Then

$$d(f_\omega^{-n}(s), f_\omega^{-n}(t)) \ge d(f_\omega^{-n}([t,s]_\omega), f_\omega^{-n}(t)) - d(f_\omega^{-n}([t,s]_\omega), f_\omega^{-n}(s)),$$

when tends to ∞ as $n \to \infty$. By (3.69), there exists a time $l \in \mathbb{N}$ such that $d(f_\omega^{-l}s, f_\omega^{-l}t) \in (e^{-N\lambda}, \delta_0)$ and $d(f_\omega^{-i}s, f_\omega^{-i}t) \le e^{-N\lambda} < \delta_0$ for $i \in \{0, 1, ..., l-1\}$. So $(s,t) \in R_l(\omega)$. In both cases, $(s,t) \in \cup_{n=0}^{\infty} S_n(\omega)$. This finishes the proof of Sublemma 3.5. $\square$

Hence the fix point obtained by (3.68) has the property that for any $p, q \in M$, $d(p,q) < \delta_0$ and $q \notin W^u_{\delta_0}(p,\omega)$,

$$\sup_{x \in E^u(p,\omega)(\delta_0)} \|g^*_{(p,\omega)}(x) - (g^*_{(q,\omega)})^{(p,\omega)}(x)\| \le C(\delta_0) e^{N\lambda\varrho} d(p,q)^\varrho := H(\delta_0,\varrho) d(p,q)^\varrho, \quad (3.74)$$

and we note that (3.74) is automatically true if $q \in W^u_{\delta_0}(p,\omega)$. Hence (3.54) holds. A similar proof can be applied to the stable manifold by reversing time.

Now let $y_0, y_1 \in graph(g^*_{(p,\omega)}|_{E^u(p,\omega)(\delta_0)})$, let $q \in M$ and $d(p,q) < \delta_0$. Let

$$z_0 := (P(E^u(p,\omega)y_0, (g^*_{(q,\omega)})^{(p,\omega)}(P(E^u(p,\omega)y_0)))$$

and

$$z_1 := (P(E^u(p,\omega)y_1, (g^*_{(q,\omega)})^{(p,\omega)}(P(E^u(p,\omega)y_1))).$$

Then by (3.54),

$$\|z_1 - y_1\| \le H(\delta_0,\varrho)\|z_0 - y_0\|^\varrho. \quad (3.75)$$

Denote

$$w_0 := Exp_p^{-1}(W^s_{loc}(Exp_p(y_0),\omega)) \cap graph((g^*_{(q,\omega)})^{(p,\omega)}),$$

and

$$w_1 := Exp_p^{-1}(W^s_{loc}(Exp_p(y_1),\omega)) \cap graph((g^*_{(q,\omega)})^{(p,\omega)}).$$

Sublemma 3.6. *When δ_0 sufficiently small, there exists a constant $\hat{C}$ such that*

$$\hat{C}^{-1} \le \frac{\|y_1 - z_1\|}{\|y_1 - w_1\|}, \frac{\|y_0 - z_0\|}{\|y_0 - w_0\|} \le \hat{C}.$$

Proof of Sublemma 3.6. Fix any number $L' > 0$. When $\delta_0 > 0$ is sufficiently small, then we have

$$l := \sup\{\|D_x(g^*_{(q,\omega)})^{(p,\omega)}\| : \, x \in E^u(p,\omega)(\delta_0), \, d(p,q) < \delta_0, \, \omega \in \Omega\} < \frac{1}{L'},$$

since $\|D_0(g^*_{p,\omega})^{(p,\omega)}\| = 0$ and C^1 continuity of local unstable manifolds as in (3) of Lemma 3.2. By the similar reason, the local stable manifold $Exp_p^{-1}(W^s_{\delta_0}(Exp_p(y), \omega))$ for any $y \in graph(g^*_{(p,\omega)}|_{E^u(p,\omega)(\delta_0)})$, as a graph of C^2 function from $E^s(p, \omega)$ to $E^u(p, \omega)$, has coslope$\leq L'$, provided $\delta_0 > 0$ sufficiently small.

Now for y_0, z_0 and w_0 defined as above, then w_0 lies in an area constrained by the following two cones:

$$z_0 + \{(x, y) \in E^u(p, \omega) \oplus E^s(p, \omega) : \ \|y\| \leq l\|x\|\}$$

and

$$y_0 + \{(x, y) \in E^u(p, \omega) \oplus E^s(p, \omega) : \ \|x\| \leq L'\|x\|\}.$$

By the knowledge of trigonometry, one can show that the following number satisfies the conclusion of Sublemma 3.6:

$$\hat{C} = \frac{2(1 + (L')^2)}{1 - lL'}$$

The case for y_1, z_1 and w_1 is similar. □

By (3.75) and Sublemma 3.6, we obtain

$$\|y_1 - w_1\| \leq \hat{C}^{1+\varrho} H(\delta_0, \varrho)\|y_0 - w_0\|^{\varrho},$$

i.e., the fiber holonomy map between local stable manifolds is uniformly ϱ-Hölder continuous at a small scale. A similar result holds for fiber holonomy map between local unstable manifolds. Let $H' = \hat{C}^{1+\varrho} H(\delta_0, \varrho)$, then the proof of Proposition 3.2 is complete. □

3.6. *Properties of the holonomy map between a pair of local stable leaves*

In this subsection, the properties of the Holonomy maps are further discussed.

For each $\omega \in \Omega$, $\tilde{\gamma}(\omega)$ and $\gamma(\omega)$ is said to be a pair of nearby local stable leaves if $\tilde{\gamma}(\omega)$ and $\gamma(\omega)$ are local stable manifolds and the fiber holonomy map $\psi_\omega : \tilde{\gamma}(\omega) \to \gamma(\omega)$ by $\psi_\omega(x) = W^u_\epsilon(x, \omega) \cap \gamma(\omega)$ for $x \in \tilde{\gamma}(\omega)$ is a homeomorphism. From the proof of Proposition 3.1, we know that ψ_ω is absolutely continuous and we have

$$Jac(\psi_\omega)(x) = \lim_{n\to\infty} \frac{|\det(D_x f_\omega^{-n}|_{E^s(x,\omega)})|}{|\det(D_{\psi_\omega(x)} f_\omega^{-n}|_{E^s(\psi_\omega(x),\omega)})|}, \tag{3.76}$$

where $Jac(\psi_\omega)$ denotes the Jacobian of ψ_ω, and f_ω^{-n} is defined in (2.1). The time in (3.76) goes backward since the holonomy map in this section is induced by unstable manifolds, while in the proof of Proposition 3.1, the holonomy map is induced by local stable manifolds.

In the following, we restrict the size of local stable and unstable manifolds $W^s_\epsilon(x, \omega)$, $W^u_\epsilon(x, \omega)$ satisfying $\epsilon \leq \min\{\epsilon_0, \delta_0\}$ to guarantee the Hölder continuity of the stable and unstable foliations, where δ_0 is the constant in Proposition 3.2.

Lemma 3.12. *There exist constants $a_0' > 0$ and $\nu_0 \in (0, 1)$ only depending on system ϕ such that for any $\psi_\omega : \tilde{\gamma}(\omega) \to \gamma(\omega)$ fiber holonomy map of a pair of nearby random local stable leaves, the followings hold:*

(1) ψ_ω and $\log Jac(\psi_\omega)$ are (a_0', ν_0)-Hölder continuous;
(2) $|\log Jac(\psi_\omega)(y)| \le a_0' d(y, \psi_\omega(y))^{\nu_0}$ for every $y \in \tilde{\gamma}(\omega)$;
(3) $d(f_\omega^{-1}x, f_\omega^{-1}\psi_\omega(x)) \le e^{-\lambda} d(x, \psi_\omega(x))$.

Proof of Lemma 3.12. In Proposition 3.2, we have proved that ψ_ω is (H', ϱ)-Hölder continuous for all $\omega \in \Omega$. To prove the other statements, we need the following fact: there exists a constant $C_{13} > 0$ such that for any $n \ge 0$, $x \in \tilde{\gamma}(\omega)$,

$$\frac{\left|det(D_{f_\omega^{-n}x} f_{\theta^{-n}\omega}^{-1}|_{E^s(f_\omega^{-n}x, \theta^{-n}\omega)})\right|}{\left|det(D_{f_\omega^{-n}\psi_\omega(x)} f_{\theta^{-n}\omega}^{-1}|_{E^s(f_\omega^{-n}\psi_\omega(x), \theta^{-n}\omega)})\right|} \le 1 + C_{13} e^{-\lambda n \nu_1} d(x, \psi_\omega(x))^{\nu_1}. \tag{3.77}$$

In fact, this is a consequence of (3.42) and the following inequality

$$\begin{aligned}
&\left|det(D_{f_\omega^{-n}x} f_{\theta^{-n}\omega}^{-1}|_{E^s(f_\omega^{-n}x, \theta^{-n}\omega)}) - det(D_{f_\omega^{-n}\psi_\omega(x)} f_{\theta^{-n}\omega}^{-1}|_{E^s(f_\omega^{-n}\psi_\omega(x), \theta^{-n}\omega)})\right| \\
&\overset{(3.5)}{\le} C_2[d(f_\omega^{-n}x, f_\omega^{-n}\psi_\omega(x)) + d(E^s(f_\omega^{-n}x, \theta^{-n}\omega), E^s(f_\omega^{-n}\psi_\omega(x), \theta^{-n}\omega))] \\
&\overset{(3.2)}{\le} C_2 e^{-\lambda n} d(x, \psi_\omega(x)) + C_2 C_1 e^{-\lambda n \nu_1} d(x, \psi_\omega(x))^{\nu_1} \\
&\le (C_2 + C_2 C_1) e^{-\lambda n \nu_1} d(x, \psi_\omega(x))^{\nu_1}.
\end{aligned}$$

We first prove statement (2). As a consequence of (3.76) and (3.77), we have

$$\begin{aligned}
Jac(\psi_\omega)(x) &\le \prod_{j=0}^{\infty} (1 + C_{13} e^{-\lambda j \nu_1} d(x, \psi_\omega(x))^{\nu_1}) \\
&\le \sum_{j=0}^{\infty} C_{13} e^{-\lambda j \nu_1} d(x, \psi_\omega(x))^{\nu_1} \text{ for any } x \in \tilde{\gamma}(\omega).
\end{aligned} \tag{3.78}$$

Symmetrically, we have

$$\frac{1}{Jac(\psi_\omega)(x)} = Jac(\psi_\omega^{-1})(\psi_\omega(x)) \le \sum_{j=0}^{\infty} C_{13} e^{-\lambda j \nu_1} d(x, \psi_\omega(x))^{\nu_1} \text{ for any } x \in \tilde{\gamma}(\omega).$$

Therefore,

$$|Jac(\psi_\omega)(x)| \le \sum_{j=0}^{\infty} C_{13} e^{-\lambda j \nu_1} d(x, \psi_\omega(x))^{\nu_1} \text{ for any } x \in \tilde{\gamma}(\omega).$$

Next, we prove the Hölder continuity of $\log Jac(\psi_\omega)$. Pick any $x, y \in \tilde{\gamma}(\omega)$, we consider two cases: (case 1) $d(x, \psi_\omega(x)) \le d(x, y)$ and (case 2) $d(x, \psi_\omega(x)) > d(x, y)$.

In (case 1), applying Proposition 3.2, we have

$$\begin{aligned}
&|\log Jac(\psi_\omega)(x) - \log Jac(\psi_\omega)(y)| \\
&\overset{(3.78)}{\le} \sum_{j=0}^{\infty} C_{13} e^{-\lambda j \nu_1} d(x, \psi_\omega(x))^{\nu_1} + \sum_{j=0}^{\infty} C_{13} e^{-\lambda j \nu_1} d(y, \psi_\omega(y))^{\nu_1} \\
&\le \sum_{j=0}^{\infty} C_{13} e^{-\lambda j \nu_1} d(x, \psi_\omega(x))^{\nu_1} + \sum_{j=0}^{\infty} C_{13} (H')^{\nu_1} e^{-\lambda j \nu_1} d(x, \psi_\omega(x))^{\nu_1 \varrho} \\
&\le \left(\sum_{j=0}^{\infty} C_{13} e^{-\lambda j \nu_1} + \sum_{j=0}^{\infty} C_{13} (H')^{\nu_1} e^{-\lambda j \nu_1} \right) d(x, y)^{\nu_1 \varrho} \\
&:= S_1 d(x, y)^{\nu_1 \varrho},
\end{aligned} \tag{3.79}$$

where $S_1 := \sum_{j=0}^{\infty} C_{13} e^{-\lambda j \nu_1} + \sum_{j=0}^{\infty} C_{13} (H')^{\nu_1} e^{-\lambda j \nu_1}$.

In (case 2), i.e., $d(x, \psi_\omega(x)) > d(x, y)$. Since $\psi_\omega(x) \in W^u_\epsilon(x, \omega)$, and x, y lie in local stable manifold $\tilde{\gamma}(\omega)$, there exists an integer $m > 0$ such that

$$d(f_\omega^{-k} x, f_\omega^{-k} \psi_\omega(x)) > d(f_\omega^{-k} x, f_\omega^{-k} y) \text{ for } 0 \le k \le m-1, \tag{3.80}$$

and

$$d(f_\omega^{-m} x, f_\omega^{-m} \psi_\omega(x)) \le d(f_\omega^{-m} x, f_\omega^{-m} y). \tag{3.81}$$

Note that in (3.80),

$$d(f_\omega^{-k} x, f_\omega^{-k} y) < d(f_\omega^{-k} x, f_\omega^{-k} \psi_\omega(x)) \le d(x, \psi_\omega(x)) \le \epsilon, \ \text{ for } 0 \le k \le m-1. \tag{3.82}$$

Denote

$$\beta := \sup \left\{ \frac{d(f_\omega^{-1} x, f_\omega^{-1} y)}{d(x, y)} : \ d(x, y) \le \epsilon, \ \omega \in \Omega \right\} \in (1, \infty),$$

and

$$\eta := \inf \left\{ \frac{d(f_\omega^{-1} x, f_\omega^{-1} y)}{d(x, y)} : \ d(x, y) \le \epsilon, \ \omega \in \Omega \right\} \in (0, 1),$$

then by (3.81), we have

$$\eta^m d(x, \psi_\omega(x)) \le \beta^m d(x, y).$$

As a consequence, $m \ge (\log \frac{d(x, \psi_\omega(x))}{d(x, y)}) / \log(\beta/\eta)$, and

$$e^{-m} \leq d(x,y)^{\frac{1}{\log(\beta/\eta)}} d(x,\psi_\omega(x))^{-\frac{1}{\log\beta/\eta}}. \tag{3.83}$$

Note that

$$\begin{aligned}\frac{Jac(\psi_\omega)(x)}{Jac(\psi_\omega)(y)} &= \frac{|\det D_x f_\omega^{-m}|_{E^s(x,\omega)}|}{|\det D_y f_\omega^{-m}|_{E^s(y,\omega)}|} \cdot \frac{|\det D_{\psi_\omega(y)} f_\omega^{-m}|_{E^s(\psi_\omega(y),\omega)}|}{|\det D_{\psi_\omega(x)} f_\omega^{-m}|_{E^s(\psi_\omega(x),\omega)}|} \\ &\quad \cdot \frac{Jac(f_\omega^{-m}\psi_\omega f_{\theta^{-m}\omega}^m)(f_\omega^{-m}x)}{Jac(f_\omega^{-m}\psi_\omega f_{\theta^{-m}\omega}^m)(f_\omega^{-m}y)},\end{aligned} \tag{3.84}$$

where $f_\omega^{-m}\psi_\omega f_{\theta^{-m}\omega}^m = f_\omega^{-m} \circ \psi_\omega \circ f_{\theta^{-m}\omega}^m$ is the holonomy map between $f_\omega^{-m}\tilde{\gamma}(\omega)$ and $f_\omega^{-m}\gamma(\omega)$, and we omit the composition notation for short. Therefore, we need to estimate the right side of (3.84). Similar as the proof of (3.77), we have

$$\left|\log|\det D_x f_\omega^{-m}|_{E^s(x,\omega)}| - \log|\det D_y f_\omega^{-m}|_{E^s(y,\omega)}|\right| \leq \sum_{k=0}^{m-1} C_{13} d(f_\omega^{-k}x, f_\omega^{-k}y)^{\nu_1},$$

and therefore,

$$\begin{aligned}&\left|\log|\det D_x f_\omega^{-m}|_{E^s(x,\omega)}| - \log|\det D_y f_\omega^{-m}|_{E^s(y,\omega)}|\right| \\ \leq& \sum_{k=0}^{m-1} C_{13} e^{-\lambda(m-1-k)\nu_1} d(f_\omega^{-(m-1)}x, f_\omega^{-(m-1)}y)^{\nu_1} \\ \overset{(3.80)}{\leq}& \left(\sum_{k=0}^{m-1} C_{13} e^{-\lambda(m-1-k)\nu_1}\right) d(f_\omega^{-(m-1)}x, f_\omega^{-(m-1)}\psi_\omega(x))^{\nu_1} \\ \leq& \left(\sum_{k=0}^{m-1} C_{13} e^{-\lambda(m-1-k)\nu_1}\right) e^{-\lambda(m-1)\nu_1} d(x,\psi_\omega(x))^{\nu_1}.\end{aligned}$$

Denote $S_2 := (\sum_{k=0}^{\infty} C_{13} e^{-k\lambda\nu_1}) e^{\lambda\nu_1}$, and note that $\frac{\lambda}{\log(\beta/\eta)} < 1$, then

$$\begin{aligned}&\left|\log|\det D_x f_\omega^{-m}|_{E^s(x,\omega)}| - \log|\det D_y f_\omega^{-m}|_{E^s(y,\omega)}|\right| \\ \leq& S_2 e^{-\lambda m\nu_1} d(x,\psi_\omega(x))^{\nu_1} \overset{(3.83)}{\leq} S_2 d(x,\psi_\omega(x))^{\nu_1 - \frac{\lambda\nu_1}{\log(\beta/\eta)}} d(x,y)^{\frac{\lambda\nu_1}{\log(\beta/\eta)}} \leq S_2 d(x,y)^{\frac{\lambda\nu_1}{\log(\beta/\eta)}}.\end{aligned}$$

Similar to above, we have

$$\begin{aligned}&\left|\log|\det D_{\psi_\omega(y)} f_\omega^{-m}|_{E^s(\psi_\omega(y),\omega)}| - \log|\det D_{\psi_\omega(x)} f_\omega^{-m}|_{E^s(\psi_\omega(x),\omega)}|\right| \\ \leq& \sum_{k=0}^{m-1} C_{13} d(f_\omega^{-k}\psi_\omega(x), f_\omega^{-k}\psi_\omega(y))^{\nu_1} \\ \leq& \sum_{k=0}^{m-1} C_{13} e^{-\lambda(m-1-k)\nu_1} d(f_\omega^{-(m-1)}\psi_\omega(x), f_\omega^{-(m-1)}\psi_\omega(y))^{\nu_1}\end{aligned}$$

By (3.82), both $d(f_\omega^{-(m-1)}(x), f_\omega^{-(m-1)}(y)) < \epsilon$ and $d(f_\omega^{-(m-1)}(x), f_\omega^{-(m-1)}(\psi_\omega(x))) < \epsilon$, which implies $d(f_\omega^{-(m-1)}(\psi_\omega(x)), f_\omega^{-(m-1)}(\psi_\omega(y))) < H' d(f_\omega^{-(m-1)}(x), f_\omega^{-(m-1)}(y))^\varrho$ by the Hölder continuity of local product structure. Therefore, we have

$$\begin{aligned}
&\left|\log|\det D_{\psi_\omega(y)} f_\omega^{-m}|_{E^s(\psi_\omega(y),\omega)}| - \log|\det D_{\psi_\omega(x)} f_\omega^{-m}|_{E^s(\psi_\omega(x),\omega)}|\right| \\
&\le \sum_{k=0}^{m-1} C_{13} e^{-\lambda(m-1-k)\nu_1} (H')^{\nu_1} d(f_\omega^{-(m-1)}x, f_\omega^{-(m-1)}y)^{\nu_1\varrho} \\
&\overset{(3.80)}{\le} \sum_{k=0}^{m-1} C_{13} e^{-\lambda(m-1-k)\nu_1} (H')^{\nu_1} d(f_\omega^{-(m-1)}x, f_\omega^{-(m-1)}\psi_\omega(x))^{\nu_1\varrho} \\
&\le \sum_{k=0}^{\infty} C_{13} e^{-\lambda k\nu_1} (H')^{\nu_1} e^{-\lambda(m-1)\nu_1\varrho} d(x, \psi_\omega(x))^{\nu_1\varrho} \\
&:= S_3 e^{-\lambda m\nu_1\varrho} d(x, \psi_\omega(x))^{\nu_1\varrho} \\
&\overset{(3.83)}{\le} S_3 d(x, \psi_\omega(x))^{\nu_1\varrho - \frac{\lambda\nu_1\varrho}{\log(\beta/\eta)}} d(x, y)^{\frac{\lambda\nu_1\varrho}{\log(\beta/\eta)}} \\
&\le S_3 d(x, y)^{\frac{\lambda\nu_1\varrho}{\log(\beta/\eta)}},
\end{aligned}$$

where $S_3 := \sum_{k=0}^{\infty} C_{13} e^{-k\lambda\nu_1} H^{\nu_1} e^{\lambda\nu_1\varrho}$. Note that $f_\omega^m \circ \psi_\omega \circ f_{\theta^{-m}\omega}^m$ is the holonomy map from $f_\omega^{-m}\tilde{\gamma}(\omega)$ to $f_\omega^{-m}\gamma(\omega)$ and satisfy (3.81), hence similar to (3.79), we have

$$\begin{aligned}
&\left|\log Jac(f_\omega^{-m}\psi_\omega f_{\theta^{-m}\omega}^m)(f_\omega^{-m}x) - \log Jac(f_\omega^{-m}\psi_\omega f_{\theta^{-m}\omega}^m)(f_\omega^{-m}y)\right| \\
&\le \sum_{j=0}^{\infty} C_{13} e^{-\lambda j\nu_1} d(f_\omega^{-m}x, f_\omega^{-m}\psi_\omega(x))^{\nu_1} + \sum_{j=0}^{\infty} C_{13} e^{-\lambda j\nu_1} d(f_\omega^{-m}y, f_\omega^{-m}\psi_\omega(y))^{\nu_1} \\
&\le \sum_{j=0}^{\infty} C_{13} e^{-\lambda j\nu_1} e^{-\lambda\nu_1} \left(d(f_\omega^{-(m-1)}x, f_\omega^{-(m-1)}\psi_\omega(x))^{\nu_1} + d(f_\omega^{-(m-1)}y, f_\omega^{-(m-1)}\psi_\omega(y))^{\nu_1} \right).
\end{aligned}$$

Notice (3.82), $f_\omega^{-(m-1)}x$ and $f_\omega^{-(m-1)}y$ are still close, and we can use the Hölder continuity of local product structure to obtain

$$d(f_\omega^{-(m-1)}y, f_\omega^{-(m-1)}\psi_\omega(y)) < H' d(f_\omega^{-(m-1)}x, f_\omega^{-(m-1)}\psi_\omega(x))^\varrho.$$

We continue the estimate

$$\begin{aligned}
&\left|\log Jac(f_\omega^{-m}\psi_\omega f_{\theta^{-m}\omega}^m)(f_\omega^{-m}x) - \log Jac(f_\omega^{-m}\psi_\omega f_{\theta^{-m}\omega}^m)(f_\omega^{-m}y)\right| \\
&\le \sum_{j=0}^{\infty} C_{13} e^{-\lambda j\nu_1} e^{-\lambda\nu_1} \Big(d(f_\omega^{-(m-1)}x, f_\omega^{-(m-1)}\psi_\omega(x))^{\nu_1} \\
&\quad + (H')^{\nu_1} d(f_\omega^{-(m-1)}x, f_\omega^{-(m-1)}\psi_\omega(x))^{\nu_1\varrho} \Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j=0}^{\infty} C_{13} e^{-\lambda(j+1)\nu_1}(1+(H')^{\nu_1}) e^{-(m-1)\lambda\nu_1\varrho} d(x,\psi_\omega(x))^{\nu_1\varrho} \\
&:= S_4 e^{-\lambda m \nu_1 \varrho} d(x,\psi_\omega(x))^{\nu_1\varrho} \\
&\overset{(3.83)}{\leq} S_4 d(x,\psi_\omega(x))^{\nu_1\varrho - \frac{\lambda\nu_1\varrho}{\log(\beta/\eta)}} d(x,y)^{\frac{\lambda\nu_1\varrho}{\log(\beta/\eta)}} \\
&\leq S_4 d(x,y)^{\frac{\lambda\nu_1\varrho}{\log(\beta/\eta)}},
\end{aligned}
$$

where $S_4 := \sum_{j=0}^{\infty} C_{13} e^{-\lambda(j+1)\nu_1}(1+(H')^{\nu_1}) e^{\lambda\nu_1\varrho}$. Hence by (3.84) and the above estimates, in (case 2), we obtain

$$
|\log Jac(\psi_\omega)(x) - \log Jac(\psi_\omega)(y)| \leq \max\{S_2, S_3, S_4\} d(x,y)^{\frac{\lambda\nu_1\varrho}{\log(\beta/\eta)}}.
$$

Now we define $a_0' := \max\{H', S_1, S_2, S_3, S_4, \sum_{j=0}^{\infty} C_{13} e^{-\lambda j \nu_1}\}$ and $\nu_0 := \min\{\frac{\lambda\nu_1\varrho}{\log(\beta/\eta)}, \nu_1\} = \frac{\lambda\nu_1\varrho}{\log(\beta/\eta)}$. Then property (1) and (2) are proved.

Property (3) follows the definition of holonomy map and contraction on local unstable manifolds when reverse time. The proof of Lemma 3.12 is complete. □

3.7. Measure disintegration on rectangles

We call $R(\omega) \subset M$ a rectangle if it is foliated by local stable manifolds and it has the local product structure. By the Lemma 3.12, for any rectangle $R(\omega)$ in small scale, the holonomy map between stable manifolds (resp. unstable manifolds) lying in $R(\omega)$ is absolutely continuous. As a consequence of this absolute continuity and Fubini's theorem, in [29, Chapter III section 6], the authors proved that on each local stable leaves, the disintegration of the Riemannian volume measure is equivalent to the inherited Riemannian measure. In here, we explore the corresponding Radon-Nikodym derivative of these two measures. The proof of this statement is parallel to the treatment of [33, Theorem 7.8].

Proposition 3.3. *There exist constant a_0'' only depending on the system such that for each $\omega \in \Omega$ and any rectangle $R(\omega) = [W_\epsilon^u(x_0,\omega), W_\epsilon^s(x_0,\omega)]$, there exists a measurable function $H(\omega) : R(\omega) \to \mathbb{R}^+$ such that for any bounded measurable function $\psi : M \to \mathbb{R}$, there is a disintegration*

$$
\int_{R(\omega)} \psi(x) dm(x) = \int \int_{\gamma(\omega)} \psi(x) H(\omega)|_{\gamma(\omega)}(x) dm_{\gamma(\omega)}(x) d\tilde{m}_{R(\omega)}(\gamma(\omega)), \tag{3.85}
$$

where $\gamma(\omega)$ denote the stable foliations in $R(\omega)$, and $\tilde{m}_{R(\omega)}$ is the quotient measure induced by Riemannian volume measure in the space of local stable leaves in $R(\omega)$. Moreover, for any local stable leaves $\gamma(\omega) \subset R(\omega)$, one has

$$
|\log H(\omega)(x) - \log H(\omega)(y)| \leq a_0'' d(x,y)^{\nu_0}, \ \forall x, y \in \gamma(\omega).
$$

Proof of Proposition 3.3. By using the exponential map, we can pretend $R(\omega)$ to be a subset of $T_{x_0}M$. Moreover, $R(\omega) \subset E^u(x_0,\omega)(r) \oplus E^u(x_0,\omega)^\perp(r)$ for some $r > 0$, where $E^u(x_0,\omega)(r)$

and $E^u(x_0,\omega)^\perp(r)$ are r-disks. We consider the following partition of $E^u(x_0,\omega)(r)\oplus E^u(x_0,\omega)^\perp(r)$

$$\mathcal{D}_\omega=\{D_\omega(\eta)=\eta+E^u(x_0,\omega)(r)|\ \eta\in E^u(x_0,\omega)^\perp(r)\}.$$

We denote $m_{D_\omega(\eta)}$ and $m_{E^u(x_0,\omega)^\perp}$ to be the Lebesgue measure on $D_\omega(\eta)$ and $E^u(x_0,\omega)^\perp$ respectively. Then for any Borel measurable set $A\subset R(\omega)$, we have

$$m(A)=\int\limits_{E^u(x_0,\omega)^\perp}\int\limits_{D_\omega(\eta)}1_A(\xi,\eta)dm_{D_\omega(\eta)}(\xi)dm_{E^u(x_0,\omega)^\perp}(\eta).$$

We may shrink ϵ and r to make sure that $D_\omega(\eta)$ are transverse to the local stable leaves for all $\eta\in E^u(x_0,\omega)^\perp(r)$. We denote $\psi^s_{\omega;0,\eta}:D_\omega(0)\to D_\omega(\eta)$ to be the holonomy map induced by the local stable manifolds. Denote $\xi=\psi^s_{\omega;0,\eta}(\xi')$ for $\xi'\in D_\omega(0)$, then we have

$$m(A)=\int\limits_{E^u(x_0,\omega)^\perp}\int\limits_{D_\omega(0)}1_A(\xi,\eta)\cdot Jac(\psi^s_{\omega;0,\eta})(\xi')dm_{D_\omega(0)}(\xi')dm_{E^u(x_0,\omega)^\perp}(\eta).$$

Applying Fubini's theorem, we obtain

$$m(A)=\int\limits_{D_\omega(0)}\int\limits_{E^u(x_0,\omega)^\perp}1_A(\xi,\eta)\cdot Jac(\psi^s_{\omega;0,\eta})(\xi')dm_{E^u(x_0,\omega)^\perp}(\eta)dm_{D_\omega(0)}(\xi').$$

By the local stable manifold theorem, there exists a C^2 function $h^s_{(x_0,\omega)}:E^u(x_0,\omega)^\perp(r)\to W^s_r(x_0,\omega)$. By changing of coordinate $\eta'=h^s_{(x_0,\omega)}(\eta)$ for $\eta\in E^u(x_0,\omega)^\perp(r)$, we have

$$m(A)=\int\limits_{D_\omega(0)}\int\limits_{W^s_r(x_0,\omega)}1_A(\xi,\eta)\cdot Jac(\psi^s_{\omega;0,\eta})(\xi')\cdot\frac{1}{Jac(h^s_{(x_0,\omega)})(\eta)}dm_{W^s_r(x_0,\omega)}(\eta')dm_{D_\omega(0)}(\xi').$$

Denote $\psi^u_{\omega;\xi',0}:W^s_r(\xi',\omega)\to W^s_r(x_0,\omega)$ to be the holonomy map induced by the local unstable manifolds, where $W^s_r(\xi',\omega)$ is the local stable manifold passing through $\xi'\in E^u(x_0,\omega)$. By change of coordinate $\zeta=(\psi^u_{\omega;\xi';0})^{-1}(\eta')$ for $\eta'\in W^s_r(x_0,\omega)$, we have

$$m(A)=\int\limits_{D_\omega(0)}\int\limits_{W^s_r(\xi,\omega)}1_A(\xi,\eta)\cdot Jac(\psi^s_{\omega;0,\eta})(\xi')\cdot\frac{Jac(\psi^u_{\omega;\xi',0})(\zeta)}{Jac(h^s_{(x_0,\omega)})(\eta)}dm_{W^s_r(\xi',\omega)}(\zeta)dm_{D_\omega(0)}(\xi'),$$

which implies that in (3.85),

- the space of local stable leaves is identified with $D_\omega(0)$ and $\tilde{m}_{R(\omega)}=m_{D(\omega)(0)}$.
- $H(\omega)(\zeta)=Jac(\psi^s_{\omega;0,\eta})(\xi')\cdot\frac{Jac(\psi^u_{\omega;\xi',0})(\zeta)}{Jac(h^s_{(x_0,\omega)})(\eta)}$ for $\zeta\in W^s_r(\xi',\omega)$, where $\eta=(h^s_{(x_0,\omega)})^{-1}(\psi^u_{\omega;\xi',0}(\zeta))$;

Next, let us verify the Hölder continuity of $\log H(\omega)$ on each local stable manifold $W^s_r(\xi', \omega)$. Pick any $\zeta_1, \zeta_2 \in W^s_r(\xi', \omega)$ and corresponding $\eta_1 = (h^s_{(x_0,\omega)})^{-1}(\psi^u_{\omega;\xi',0}(\zeta_1))$, $\eta_2 = (h^s_{(x_0,\omega)})^{-1}(\psi^u_{\omega;\xi',0}(\zeta_2))$. By (3.53), we have

$$Jac(\psi^s_{\omega;0,\eta})(\xi') = \lim_{n\to\infty} \frac{|\det D_{\xi'} f^n_\omega|_{E^u(x_0,\omega)}|}{|\det D_{\psi^s_{\omega;0,\eta}(\xi')} f^n_\omega|_{E^u(x_0,\omega)}|}.$$

We notice that $\zeta_1 = \psi^s_{\omega;0,\eta_1}(\xi')$ and $\zeta_2 = \psi^s_{\omega;0,\eta_2}(\xi')$. Therefore,

$$\begin{aligned}
\frac{Jac(\psi^s_{\omega;0,\eta_1})(\xi')}{Jac(\psi^s_{\omega;0,\eta_2})(\xi')} &= \lim_{n\to\infty} \frac{|\det D_{\psi^s_{\omega;0,\eta_2}(\xi')} f^n_\omega|_{E^u(x_0,\omega)}|}{|\det D_{\psi^s_{\omega;0,\eta_1}(\xi')} f^n_\omega|_{E^u(x_0,\omega)}|} = \lim_{n\to\infty} \frac{|\det D_{\zeta_2} f^n_\omega|_{E^u(x_0,\omega)}|}{|\det D_{\zeta_1} f^n_\omega|_{E^u(x_0,\omega)}|} \\
&\le \prod_{i=0}^{\infty} (1 + C_{10}C_2 d(f^i_\omega(\zeta_1), f^i_\omega(\zeta_2))) \\
&\le \prod_{i=0}^{\infty} (1 + C_{10}C_2 e^{-\lambda i} d(\zeta_1, \zeta_2)),
\end{aligned}$$

where C_2 and C_{10} comes from Lemma 3.4 and (3.42). Hence, we have

$$|\log Jac(\psi^s_{\omega;0,\eta_1})(\xi') - \log Jac(\psi^s_{\omega;0,\eta_2})(\xi')| \le \frac{C_{10}C_2}{1 - e^{-\lambda}} d(\zeta_1, \zeta_2). \tag{3.86}$$

Note that $\psi^u_{\omega;\xi',0} : W^s_r(\xi', \omega) \to W^s_r(x_0, \omega)$ is the holonomy map between local stable manifolds. By Lemma 3.12, we have

$$|\log Jac(\psi^u_{\omega;\xi',0})(\zeta_1) - \log Jac(\psi^u_{\omega;\xi',0})(\zeta_2)| \le a'_0 d(\zeta_1, \zeta_2)^{\nu_0}. \tag{3.87}$$

By Lemma 3.2, there exists a constant $C_s > 0$ such that

$$\begin{aligned}
&|\log Jac(h^s_{(x_0,\omega)})(\eta_1) - \log Jac(h^s_{(x_0,\omega)})(\eta_2)| \\
\le & C_s \|\eta_1 - \eta_2\| \le C_s d(\psi^u_{\omega;\xi',0}(\zeta_1), \psi^u_{\omega;\xi',0}(\zeta_2)) \le C_s a'_0 d(\zeta_1, \zeta_2)^{\nu_0}.
\end{aligned} \tag{3.88}$$

Inequalities (3.86), (3.87) and (3.88) imply that

$$|\log H(\omega)(\zeta_1) - \log H(\omega)(\zeta_2)| \le \max\left\{\frac{C_{10}C_2}{1 - e^{-\lambda}}, a'_0, C_s a'_0\right\} d(\zeta_1, \zeta_2)^{\nu_0} := a''_0 d(\zeta_1, \zeta_2)^{\nu_0}.$$

The proof of Proposition 3.3 is complete. □

4. Proof of main result

In this section, we prove Theorem 1. The proof is based on the study of the fiber transfer operator L_ω, which is defined by

$$L_\omega\varphi : M \to \mathbb{R},\ (L_\omega\varphi)(x) := \frac{\varphi((f_\omega)^{-1}x)}{|\det D_{(f_\omega)^{-1}(x)} f_\omega|} \tag{4.1}$$

for any bounded and measurable observable $\varphi : M \to \mathbb{R}$. We denote

$$L^n_\omega := L_{\theta^{n-1}\omega} \circ \cdots \circ L_{\theta\omega} \circ L_\omega \text{ for any } \omega \in \Omega \text{ and } n \in \mathbb{N}.$$

We first construct the suitable convex cone of observables C_ω on each fiber in Subsection 4.1. Then in Subsection 4.2, we prove that the transfer operator L_ω maps C_ω into $C_{\theta\omega}$. Moreover, $L^N_\omega C_\omega$ has finite diameter with respect to the projective metric on the cone $C_{\theta^N\omega}$, and this diameter is independent of ω, where N comes from the topological mixing on fibers property. Birkhoff's inequality implies the contraction of $L^N_\omega : C_\omega \to C_{\theta^N\omega}$ for all $\omega \in \Omega$. In Subsection 4.3, we explore the relationship between the unique random SRB measure and the operator $L^n_{\theta^{-n}\omega}$. We show the exponential decay of past and future correlations in Subsection 4.4 and Subsection 4.5 respectively by using the contraction of $L^n_{\theta^{-n}\omega}$ and L^n_ω for $n \geq N$.

Before starting the proof, we recall some constants that will be needed later on. Let K_1 be the constant in Lemma 3.5 such that for any $\omega \in \Omega$, $z \in M$,

$$\left|\log|\det(D_x f_\omega|_{E^s(x,\omega)})| - \log|\det(D_y f_\omega|_{E^s(y,\omega)})|\right| \leq K_1 d(x,y) \text{ for any } x, y \in W^s_\epsilon(z,\omega). \tag{4.2}$$

By the compactness of Ω and M and the continuity of $f_\omega \in \mathrm{Diff}^2(M)$ on ω, there exists a constant $K_2 > 0$ such that for any $x, y \in M$,

$$\left|\log|\det D_x f_\omega| - \log|\det D_y f_\omega|\right| \leq K_2 d(x,y). \tag{4.3}$$

Let $a_0 := \max\{a_0', a_0''\}$, then Lemma 3.12 and Proposition 3.3 hold for constants (a_0, ν_0). This ν_0 is the desired ν_0 in the statement of Theorem 1. Now Let's pick any $\kappa, \nu \in (0,1)$ satisfying $0 < \kappa + \nu < \nu_0$ as in the statement of Theorem 1, and pick $\kappa_1 \in (0,1)$ an auxiliary constant closing to 1 such that

$$0 < \kappa + \nu < \kappa_1 \nu_0. \tag{4.4}$$

Now we are going to prove Theorem 1 for fixed κ, ν.

4.1. Construction of Birkhoff cone

In this subsection, we will first construct convex cones of density functions on each local stable leaf. With the help of these convex cones of density functions on each local stable leaf, we can define our desired convex cone of observables on each fiber. Our construction of convex cone of observables is inspired by [30] and [39]. The definitions of the convex cone in a topological vector space, projective metric on the convex cone, and Birkhoff's inequality are recalled in the Appendix A.4, which also can be found in [11,30,39].

Denote m^s to be the inherited Riemannian volume measure on local stable manifolds. Then by the continuity of local stable manifolds, there exists a constant $A(\epsilon) > 0$ such that $A(\epsilon) \le m^s(W^s_\epsilon(x,\omega))$ for any $(x,\omega) \in M \times \Omega$. In the following, for any $\omega \in \Omega$, we say that the local stable leaf $\gamma(\omega) \subset M$ having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$ if $\gamma(\omega)$ is connected, and $m^s(\gamma(\omega)) \in (\frac{A(\epsilon)}{4J^2}, A(\epsilon))$, where the notation $\gamma(\omega)$ indicates that this local stable leaf is located on the fiber $M \times \{\omega\}$. Since $\gamma(\omega)$ is a connected curve, this means that the length of local stable manifold has length bounded from below and above. Applying the continuity of local stable manifolds again, there exists $\epsilon^* > 0$ such that for any $\gamma(\omega)$ of size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, there exists $(x,\omega) \in M \times \Omega$ satisfying

$$W^s_{\epsilon^*}(x,\omega) \subset \gamma(\omega). \tag{4.5}$$

For some constant $a > 0$, and a local stable leaf $\gamma(\omega)$ having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, we define $D(a,\kappa,\gamma(\omega))$ to be the collection of all bounded and measurable function $\rho(\cdot,\omega) : \gamma(\omega) \to \mathbb{R}$ satisfying the following conditions:

(D1) $\rho(x,\omega) > 0$ for $x \in \gamma(\omega)$;
(D2) for any $x, y \in \gamma(\omega)$, $|\log\rho(x,\omega) - \log\rho(y,\omega)| \le ad(x,y)^\kappa$.

We note that condition **(D2)** implies that any $\rho(\cdot,\omega) \in D(a,\kappa,\gamma(\omega))$ is continuous on $\gamma(\omega)$. In the left of this paper, we use the notation $\rho(\omega)$ to represent $\rho(\cdot,\omega) \in D(a,\kappa,\gamma(\omega))$ for convenience.

Lemma 4.1. *$D(a,\kappa,\gamma(\omega))$ is a convex cone (see Definition A.4 in the Appendix).*

Proof of Lemma 4.1. For any $\rho(\omega) \in D(a,\kappa,\gamma(\omega))$ and $t \in \mathbb{R}^+$, then $t\rho(x,\omega) > 0$ for $x \in \gamma(\omega)$ and

$$|\log t\rho(x,\omega) - \log t\rho(y,\omega)| = |\log\rho(x,\omega) - \log\rho(y,\omega)| \le ad(x,y)^\kappa.$$

Hence $t\rho(\omega) \in D(a,\kappa,\gamma(\omega))$.

For any $\rho_i(\omega) \in D(a,\kappa,\gamma(\omega))$ and $t_i \in \mathbb{R}^+$ for $i = 1,2$, it is clear that $t_1\rho_1(x,\omega) + t_2\rho_2(x,\omega) > 0$ for $x \in \gamma(\omega)$. By **(D2)**, we have

$$e^{-ad(x,y)^\kappa} \le \frac{\rho_i(x,\omega)}{\rho_i(y,\omega)} \le e^{ad(x,y)^\kappa} \text{ for } x, y \in \gamma(\omega) \text{ and } i = 1,2.$$

As a consequence, we have

$$e^{-ad(x,y)^\kappa} \le \frac{t_1\rho_1(x,\omega) + t_2\rho_2(x,\omega)}{t_1\rho_1(y,\omega) + t_2\rho_2(y,\omega)} \le e^{ad(x,y)^\kappa} \text{ for } x, y \in \gamma(\omega),$$

which is equivalent to

$$|\log(t_1\rho_1(x,\omega) + t_2\rho_2(x,\omega)) - \log(t_1\rho_1(y,\omega) + t_2\rho_2(y,\omega))| \le ad(x,y)^\kappa.$$

Hence $t_1\rho_1(\omega) + t_2\rho_2(\omega) \in D(a,\kappa,\gamma(\omega))$.

To check the last condition for convex cone, we pick any $\rho(\omega) \in \overline{D(a,\kappa,\gamma(\omega))} \cap -\overline{D(a,\kappa,\gamma(\omega))}$. Here the closure means the "integral closure" in a weaker sense, not the closure in the topological vector space, see Definition A.4. Then there exists $\rho_i(\omega) \in D(a,\kappa,\gamma(\omega))$ and $t_n^i \downarrow 0$ for $i = 1, 2$ such that $\rho(\omega) + t_n^1\rho_1(\omega) \in D(a,\kappa,\gamma(\omega))$ and $-\rho(\omega) + t_n^2\rho_2(\omega) \in D(a,\kappa,\gamma(\omega))$. By **(D1)**, for any $x \in \gamma(\omega)$, we have $\rho(x,\omega) + t_n^1\rho_1(x,\omega) > 0$ and $-\rho(x,\omega) + t_n^2\rho_2(x,\omega) > 0$. Letting $n \to \infty$, we arrive $\rho(x,\omega) \geq 0$ and $\rho(x,\omega) \leq 0$. Therefore, one must have $\rho(\omega) \equiv 0$. By definition, $D(a,\kappa,\gamma(\omega))$ is a convex cone. □

Next, we will introduce the projective metric $d^{a,\kappa}_{\gamma(\omega)}$ on $D(a,\kappa,\gamma(\omega))$ according to Definition A.5. For any $\rho_1(\omega), \rho_2(\omega) \in D(a,\kappa,\gamma(\omega))$, we define

$$\begin{aligned}\alpha^{a,\kappa}_{\gamma(\omega)}(\rho_1(\omega),\rho_2(\omega)) &:= \sup\{t > 0 : \ \rho_2(\omega) - t\rho_1(\omega) \in D(a,\kappa,\gamma(\omega))\};\\ \beta^{a,\kappa}_{\gamma(\omega)}(\rho_1(\omega),\rho_2(\omega)) &= \inf\{s > 0 : \ s\rho_1(\omega) - \rho_2(\omega) \in D(a,\kappa,\gamma(\omega))\},\end{aligned}$$

with the convention that $\sup\emptyset = 0$ and $\inf\emptyset = +\infty$. Now let us compute these two quantities. For any $t > 0$ such that $\rho_2(\omega) - t\rho_1(\omega) \in D(a,\kappa,\gamma(\omega))$, then by **(D1)**, one must have

$$\rho_2(x,\omega) - t\rho_1(x,\omega) > 0 \text{ for all } x \in \gamma(\omega),$$

which is equivalent to $t < \frac{\rho_2(x,\omega)}{\rho_1(x,\omega)}$ for all $x \in \gamma(\omega)$; and by **(D2)** one also must have

$$e^{-ad(x,y)^\kappa} \leq \frac{\rho_2(x,\omega) - t\rho_1(x,\omega)}{\rho_2(y,\omega) - t\rho_1(y,\omega)} \leq e^{ad(x,y)^\kappa} \text{ for all } x, y \in \gamma(\omega),$$

which is equivalent to

$$t \leq \inf_{x,y\in\gamma(\omega),x\neq y}\left\{\frac{\exp(ad(x,y)^\kappa)\rho_2(x,\omega) - \rho_2(y,\omega)}{\exp(ad(x,y)^\kappa)\rho_1(x,\omega) - \rho_1(y,\omega)}, \frac{\exp(ad(x,y)^\kappa)\rho_2(y,\omega) - \rho_2(x,\omega)}{\exp(ad(x,y)^\kappa)\rho_1(y,\omega) - \rho_1(x,\omega)}\right\}.$$

Therefore, we have

$$t \leq \inf\left\{\frac{\rho_2(x,\omega)}{\rho_1(x,\omega)}, \frac{\exp(ad(x,y)^\kappa)\rho_2(x,\omega) - \rho_2(y,\omega)}{\exp(ad(x,y)^\kappa)\rho_1(x,\omega) - \rho_1(y,\omega)} : \ x, y \in \gamma(\omega), x \neq y\right\}.$$

The case that $\{t > 0 : \rho_2(\omega) - t\rho_1(\omega) \in D(a,\kappa,\gamma(\omega))\} = \emptyset$ is equivalent to the right term of the above is equal to 0, and so $\alpha^{a,\kappa}_{\gamma(\omega)}(\rho_1(\omega),\rho_2(\omega)) = \sup\emptyset = 0$. Hence, we obtain

$$\begin{aligned}&\alpha^{a,\kappa}_{\gamma(\omega)}(\rho_1(\omega),\rho_2(\omega))\\ &\quad = \inf\left\{\frac{\rho_2(x,\omega)}{\rho_1(x,\omega)}, \frac{\exp(ad(x,y)^\kappa)\rho_2(x,\omega) - \rho_2(y,\omega)}{\exp(ad(x,y)^\kappa)\rho_1(x,\omega) - \rho_1(y,\omega)} : \ x, y \in \gamma(\omega), x \neq y\right\}.\end{aligned} \tag{4.6}$$

Similarly, we can obtain

$$\begin{aligned}&\beta^{a,\kappa}_{\gamma(\omega)}(\rho_1(\omega),\rho_2(\omega))\\ &\quad = \sup\left\{\frac{\rho_2(x,\omega)}{\rho_1(x,\omega)}, \frac{\exp(ad(x,y)^\kappa)\rho_2(x,\omega) - \rho_2(y,\omega)}{\exp(ad(x,y)^\kappa)\rho_1(x,\omega) - \rho_1(y,\omega)} : \ x, y \in \gamma(\omega), x \neq y\right\}.\end{aligned} \tag{4.7}$$

Now define

$$d^{a,\kappa}_{\gamma(\omega)}(\rho_1(\omega),\rho_2(\omega)) = \log \frac{\beta^{a,\kappa}_{\gamma(\omega)}(\rho_1(\omega),\rho_2(\omega))}{\alpha^{a,\kappa}_{\gamma(\omega)}(\rho_1(\omega),\rho_2(\omega))}, \tag{4.8}$$

with the convention that $d^{a,\kappa}_{\gamma(\omega)}(\rho_1(\omega),\rho_2(\omega)) = \infty$ if $\alpha^{a,\kappa}_{\gamma(\omega)}(\rho_1(\omega),\rho_2(\omega)) = 0$ or $\beta^{a,\kappa}_{\gamma(\omega)}(\rho_1(\omega), \rho_2(\omega)) = \infty$. By the property of projective metric (see Proposition A.2), the followings hold:

(P1) $d^{a,\kappa}_{\gamma(\omega)}(\rho_1(\omega),\rho_2(\omega)) = d^{a,\kappa}_{\gamma(\omega)}(\rho_2(\omega),\rho_1(\omega))$;
(P2) $d^{a,\kappa}_{\gamma(\omega)}(\rho_1(\omega),\rho_2(\omega)) \le d^{a,\kappa}_{\gamma(\omega)}(\rho_1(\omega),\rho_3(\omega)) + d^{a,\kappa}_{\gamma(\omega)}(\rho_3(\omega),\rho_2(\omega))$;
(P3) $d^{a,\kappa}_{\gamma(\omega)}(\rho_1(\omega),\rho_2(\omega)) = 0$ if and only if there exists a constant $t \in \mathbb{R}^+$ such that $\rho_1(\omega) = t\rho_2(\omega)$.

Note that **(P2)** and **(P3)** implies that

$$d^{a,\kappa}_{\gamma(\omega)}(\rho_1(\omega),\rho_2(\omega)) = d^{a,\kappa}_{\gamma(\omega)}(t_1\rho_1(\omega), t_2\rho_2(\omega)) \text{ for any } t_1, t_2 \in \mathbb{R}^+. \tag{4.9}$$

It is also convenient to introduce the following convex cone. Denote $D_+(\gamma(\omega))$ by the collection of all bounded and measurable functions $\zeta(\omega) : \gamma(\omega) \to \mathbb{R}$ such that $\zeta(x,\omega) > 0$ for $x \in \gamma(\omega)$. It is clear that $D_+(\gamma(\omega))$ is a convex cone. For any $\zeta_1(\omega), \zeta_2(\omega) \in D_+(\gamma(\omega))$, we define

$$\alpha_{+,\gamma(\omega)}(\zeta_1(\omega),\zeta_2(\omega)) = \sup\{t > 0 : \ \zeta_2(\omega) - t\zeta_1(\omega) \in D_+(\gamma(\omega))\}; \tag{4.10}$$

$$\beta_{+,\gamma(\omega)}(\zeta_1(\omega),\zeta_2(\omega)) = \inf\{s > 0 : \ s\zeta_1(\omega) - \zeta_2(\omega) \in D_+(\gamma(\omega))\}, \tag{4.11}$$

with the convention that $\sup\emptyset = 0$ and $\inf\emptyset = +\infty$, and the projective metric on $D_+(\gamma(\omega))$ by

$$d_{+,\gamma(\omega)}(\zeta_1(\omega),\zeta_2(\omega)) := \log \frac{\beta_{+,\gamma(\omega)}(\zeta_1(\omega),\zeta_2(\omega))}{\alpha_{+,\gamma(\omega)}(\zeta_1(\omega),\zeta_2(\omega))}, \tag{4.12}$$

with the convention that $d_{+,\gamma(\omega)}(\zeta_1(\omega),\zeta_2(\omega)) = \infty$ if $\alpha_{+,\gamma(\omega)}(\zeta_1(\omega),\zeta_2(\omega)) = 0$ or $\beta_{+,\gamma(\omega)}(\zeta_1(\omega),\zeta_2(\omega)) = \infty$. By computation, we have

$$\alpha_{+,\gamma(\omega)}(\zeta_1(\omega),\zeta_2(\omega)) = \inf\left\{\frac{\zeta_2(x,\omega)}{\zeta_1(x,\omega)} : \ x \in \gamma(\omega)\right\}, \tag{4.13}$$

$$\beta_{+,\gamma(\omega)}(\zeta_1(\omega),\zeta_2(\omega)) = \sup\left\{\frac{\zeta_2(x,\omega)}{\zeta_1(x,\omega)} : \ x \in \gamma(\omega)\right\}. \tag{4.14}$$

It is clear that $D(a,\kappa,\gamma(\omega)) \subset D_+(\gamma(\omega))$, and $d_{+,\gamma(\omega)}(\rho_1(\omega),\rho_2(\omega)) \le d^{a,\kappa}_{\gamma(\omega)}(\rho_1(\omega),\rho_2(\omega))$ for any $\rho_1(\omega), \rho_2(\omega) \in D(a,\kappa,\gamma(\omega))$.

By using density functions on $\gamma(\omega)$, we can define the corresponding density function on the pullback of $\gamma(\omega)$. For any $\omega \in \Omega$, and a local stable leaf $\gamma(\omega)$ having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, since the local stable manifold is a curve, we can divide $f_\omega^{-1}\gamma(\omega)$ into connected local stable manifolds with size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, named $\gamma_i(\theta^{-1}\omega)$ for i belonging to a finite index set such that $\gamma_i(\theta^{-1}\omega) \cap \gamma_j(\theta^{-1}\omega) = \partial\gamma_i(\theta^{-1}\omega) \cap \partial\gamma_j(\theta^{-1}\omega)$ for $i \neq j$. By using any continuous density functions $\rho(\omega)$ on $\gamma(\omega)$, we define the corresponding density function $\rho_i(\theta^{-1}\omega)$ on $\gamma_i(\theta^{-1}\omega)$ by

$$\rho_i(x,\theta^{-1}\omega) := \frac{|\det D_x f_{\theta^{-1}\omega}|_{E^s(x,\theta^{-1}\omega)}|}{|\det D_x f_{\theta^{-1}\omega}|}\rho(f_{\theta^{-1}\omega}x,\omega) \text{ for } x \in \gamma_i(\theta^{-1}\omega). \tag{4.15}$$

For any bounded and measurable function $\varphi : M \to \mathbb{R}$, by changing of variable, we have

$$\begin{aligned}
&\int_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\rho(y,\omega)dm_{\gamma(\omega)}(y)\\
&=\sum_i \int_{f_{\theta^{-1}\omega}\gamma_i(\theta^{-1}\omega)} \frac{\varphi((f_{\theta^{-1}\omega})^{-1}y)}{|\det D_{(f_{\theta^{-1}\omega})^{-1}y} f_{\theta^{-1}\omega}|}\cdot\rho(y,\omega)dm_{\gamma(\omega)}(y)\\
&=\sum_i \int_{\gamma_i(\theta^{-1}\omega)} \frac{\varphi(x)}{|\det D_x f_{\theta^{-1}\omega}|}\cdot\rho(f_{\theta^{-1}\omega}x,\omega)\cdot|\det D_x f_{\theta^{-1}\omega}|_{E^s(x,\theta^{-1}\omega)}|dm_{\gamma_i(\theta^{-1}\omega)}(x)\\
&=\sum_i \int_{\gamma_i(\theta^{-1}\omega)} \varphi(x)\rho_i(x,\theta^{-1}\omega)dm_{\gamma_i(\theta^{-1}\omega)}(x).
\end{aligned} \tag{4.16}$$

By using density function on $\gamma(\omega)$, we also can define the corresponding density function on the holonomy image of $\gamma(\omega)$. Given pair of local stable leaves $\gamma(\omega)$ and $\tilde{\gamma}(\omega)$, with the help of holonomy map $\psi^u_\omega : \tilde{\gamma}(\omega) \to \gamma(\omega)$ induced by the local unstable manifolds, for every continuous density function $\rho(\omega)$ on $\gamma(\omega)$, we associate the density $\tilde{\rho}(\omega)$ on $\tilde{\gamma}(\omega)$ by

$$\tilde{\rho}(x,\omega) = \rho(\psi_\omega(x),\omega)\cdot Jac(\psi^u_\omega)(x) \text{ for } x \in \tilde{\gamma}(\omega). \tag{4.17}$$

By the Radon-Nikodym theorem, we have

$$\int_{\tilde{\gamma}(\omega)} \tilde{\rho}(x,\omega)dm_{\tilde{\gamma}(\omega)}(x) = \int_{\gamma(\omega)} \rho(y,\omega)dm_{\gamma(\omega)}(y). \tag{4.18}$$

We can define the distance between $\tilde{\gamma}(\omega)$ and $\gamma(\omega)$ by

$$d_u(\tilde{\gamma}(\omega),\gamma(\omega)) := \sup\{d(x,\psi^u_\omega(x)) : \ x \in \tilde{\gamma}(\omega)\}, \tag{4.19}$$

where we use subscript u to indicate that the distance is induced by the local unstable manifolds.

Recall that constants $K_1, K_2, \kappa, \kappa_1, a_0$ and ν_0 are picked at the beginning of Section 4. Let a_1 be any number such that

$$\frac{K_1+K_2}{1-e^{-\lambda\kappa_1\nu_0}} < a_1. \tag{4.20}$$

Let $a \in \mathbb{R}$ be any number such that

$$a_1 a_0^{\kappa_1} + a_0 < \frac{a}{2}. \tag{4.21}$$

Let $D(a_1, \kappa, \gamma(\omega))$, $D(\frac{a}{2}, \kappa, \gamma(\omega))$, and $D(\frac{a}{2}, \kappa_1 \nu_0, \gamma(\omega))$ be convex cones defined just like $D(a, \kappa, \gamma(\omega))$. The relations between $\rho(\omega)$ and $\tilde{\rho}(\omega)$, $\rho(\omega)$ and $\rho_i(\theta^{-1}\omega)$ are given in the following lemma.

Lemma 4.2. *There are $\lambda_1 = \lambda_1(a_1, \kappa) > 0$ and $\Lambda_1 = \Lambda_1(\lambda_1, a) < 1$ such that*

(1) if $\rho(\omega) \in D(a_1, \kappa, \gamma(\omega))$, then $\rho_i(\theta^{-1}\omega) \in D(e^{-\lambda_1} a_1, \kappa, \gamma_i(\theta^{-1}\omega)) \subset D(a_1, \kappa, \gamma_i(\theta^{-1}\omega))$;
(2) if $\rho(\omega) \in D(\frac{a}{2}, \kappa, \gamma(\omega))$, then $\rho_i(\theta^{-1}\omega) \in D(e^{-\lambda_1} \frac{a}{2}, \kappa, \gamma_i(\theta^{-1}\omega)) \subset D(\frac{a}{2}, \kappa, \gamma_i(\theta^{-1}\omega))$;
(3) if $\rho(\omega) \in D(\frac{a}{2}, \kappa_1 \nu_0, \gamma(\omega))$, then $\rho_i(\theta^{-1}\omega) \in D(e^{-\lambda_1} \frac{a}{2}, \kappa_1 \nu_0, \gamma_i(\theta^{-1}\omega)) \subset D(\frac{a}{2}, \kappa_1 \nu_0, \gamma_i(\theta^{-1}\omega))$;
(4) if $\rho(\omega) \in D(a, \kappa, \gamma(\omega))$, then $\rho_i(\theta^{-1}\omega) \in D(e^{-\lambda_1} a, \kappa, \gamma_i(\theta^{-1}\omega)) \subset D(a, \kappa, \gamma_i(\theta^{-1}\omega))$;

furthermore,

(5) for any $\rho(\omega), \varsigma(\omega) \in D(a, \kappa, \gamma(\omega))$, we have

$$d^{a,\kappa}_{\gamma_i(\theta^{-1}\omega)}(\rho_i(\theta^{-1}\omega), \varsigma_i(\theta^{-1}\omega)) \le \Lambda_1 d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega), \varsigma(\omega)), \tag{4.22}$$

where $\rho_i(\theta^{-1}\omega)$ and $\varsigma_i(\theta^{-1}\omega)$ are density functions defined as in (4.15) on $\tilde{\gamma}_i(\theta^{-1}\omega)$, and $d^{a,\kappa}_{\gamma_i(\theta^{-1}\omega)}$ (resp. $d^{a,\kappa}_{\gamma(\omega)}$) is the projective metric on $D(a, \kappa, \gamma_i(\theta^{-1}\omega))$ (resp. $D(a, \kappa, \gamma(\omega))$).

Moreover,

$$\text{if } \rho(\omega) \in D(a_1, \kappa_1, \gamma(\omega)), \text{then } \tilde{\rho}(\omega) \in D(a/2, \kappa_1 \nu_0, \tilde{\gamma}(\omega)) \subset D(a/2, \kappa, \tilde{\gamma}(\omega)), \tag{4.23}$$

where $\tilde{\rho}(\omega)$ is defined in (4.17).

Proof of Lemma 4.2. We first prove (1). Let $\rho(\omega) \in D(a_1, \kappa, \gamma(\omega))$. Recall that $\rho_i(\theta^{-1}\omega)$ is defined in (4.15) on $\gamma_i(\theta^{-1}\omega) \subset f_\omega^{-1}\gamma(\omega)$. Clearly, $\rho_i(x, \theta^{-1}\omega) > 0$ for all $x \in \gamma_i(\theta^{-1}\omega)$. By (4.20), we can pick $\lambda_1 > 0$ closing to 0 so that

$$a_1 > \frac{K_1 + K_2}{e^{-\lambda_1} - e^{-\lambda \kappa_1 \nu_0}} \overset{(4.4)}{>} \frac{K_1 + K_2}{e^{-\lambda_1} - e^{-\lambda \kappa}} > 0. \tag{4.24}$$

Then for any $x, y \in \gamma_i(\theta^{-1}\omega)$

$$\begin{aligned}
&\left|\log \rho_i(x, \theta^{-1}\omega) - \log \rho_i(y, \theta^{-1}\omega)\right| \\
&\le \left|\log \rho(f_{\theta^{-1}\omega} x, \omega) - \log \rho(f_{\theta^{-1}\omega} y, \omega)\right| + \Big|\log |\det D_x f_{\theta^{-1}\omega}|_{E^s(x, \theta^{-1}\omega)}| \\
&\quad - \log |\det D_y f_{\theta^{-1}\omega}|_{E^s(y, \theta^{-1}\omega)}|\Big| + \Big|\log |\det D_x f_{\theta^{-1}\omega}| - \log |\det D_y f_{\theta^{-1}\omega}|\Big| \\
&\overset{(4.2),(4.3)}{\le} a_1 d(f_{\theta^{-1}\omega} x, f_{\theta^{-1}\omega} y)^\kappa + K_1 d(x, y) + K_2 d(x, y) \\
&\le a_1 e^{-\lambda \kappa} d(x, y)^\kappa + (K_1 + K_2) d(x, y)
\end{aligned}$$

$$\overset{(4.24)}{\le} a_1 e^{-\lambda_1} d(x,y)^\kappa.$$

This proves part (1). Similar proof can be applied to (2), (3) and (4) by using the fact

$$a > \frac{a}{2} > a_1 \overset{(4.24)}{>} \frac{K_1+K_2}{e^{-\lambda_1} - e^{-\lambda\kappa_1\nu_0}} > \frac{K_1+K_2}{e^{-\lambda_1}-e^{-\lambda\kappa}} > 0.$$

Next, we prove (5). By (4), we have a linear operator that maps from convex cone $D(a,\kappa,\gamma(\omega))$ to convex cone $D(a,\kappa,\gamma_i(\theta^{-1}\omega))$ given by $\rho(\omega) \mapsto \rho_i(\theta^{-1}\omega)$. By Birkhoff's inequality (Proposition A.3), if

$$R_i(\theta^{-1}\omega) := \sup\{d^{a,\kappa}_{\gamma_i(\theta^{-1}\omega)}(\rho_i(\theta^{-1}\omega), \varsigma_i(\theta^{-1}\omega)) :\ \rho(\omega), \varsigma(\omega) \in D(a,\kappa,\gamma(\omega))\} < \infty,$$

then

$$d^{a,\kappa}_{\gamma_i(\theta^{-1}\omega)}(\rho_i(\theta^{-1}\omega), \varsigma_i(\theta^{-1}\omega)) \le (1-e^{-R_i(\theta^{-1}\omega)}) d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega),\varsigma(\omega)). \tag{4.25}$$

To estimate $R_i(\theta^{-1}\omega)$, it suffices to estimate the diameter of $D(e^{-\lambda_1}a,\kappa,\gamma_i(\theta^{-1}\omega))$ in $D(a,\kappa,\gamma_i(\theta^{-1}\omega))$ under the projective metric $d^{a,\kappa}_{\gamma_i(\theta^{-1}\omega)}$, since $\rho_i(\theta^{-1}\omega), \varsigma_i(\theta^{-1}\omega) \in D(e^{-\lambda_1}a, \kappa, \gamma_i(\theta^{-1}\omega))$.

Pick any $\zeta_1(\theta^{-1}\omega)$, $\zeta_2(\theta^{-1}\omega) \in D(e^{-\lambda_1}a,\kappa,\gamma_i(\theta^{-1}\omega))$, then $\zeta_1(\theta^{-1}\omega), \zeta_2(\theta^{-1}\omega) \in D_+(\gamma_i(\theta^{-1}\omega))$ automatically. For any $x, y \in \gamma_i(\theta^{-1}\omega)$, $x \ne y$,

$$\begin{aligned}\frac{\exp(ad(x,y)^\kappa) - \zeta_2(y,\theta^{-1}\omega)/\zeta_2(x,\theta^{-1}\omega)}{\exp(ad(x,y)^\kappa) - \zeta_1(y,\theta^{-1}\omega)/\zeta_1(x,\theta^{-1}\omega)} &\ge \frac{\exp(ad(x,y)^\kappa) - \exp(e^{-\lambda_1}ad(x,y)^\kappa)}{\exp(ad(x,y)^\kappa) - \exp(-e^{-\lambda_1}ad(x,y)^\kappa)} \\ &\ge \tau_1,\end{aligned}$$

where

$$\tau_1 = \inf\left\{\frac{z - z^{\exp(-\lambda_1)}}{z - z^{-\exp(-\lambda_1)}} :\ z > 1\right\} = \lim_{z\to 1} \frac{z - z^{\exp(-\lambda_1)}}{z - z^{-\exp(-\lambda_1)}} = \frac{1-\exp(-\lambda_1)}{1+\exp(-\lambda_1)} \in (0,1).$$

Comparing

$$\begin{aligned}&\alpha^{a,\kappa}_{\gamma_i(\theta^{-1}\omega)}(\zeta_1(\theta^{-1}\omega), \zeta_2(\theta^{-1}\omega)) \\ &= \inf\left\{\frac{\zeta_2(x,\theta^{-1}\omega)}{\zeta_1(x,\theta^{-1}\omega)}, \frac{\exp(ad(x,y)^\kappa)\zeta_2(x,\theta^{-1}\omega) - \zeta_2(y,\theta^{-1}\omega)}{\exp(ad(x,y)^\kappa)\zeta_1(x,\theta^{-1}\omega) - \zeta_1(y,\theta^{-1}\omega)} :\ x,y \in \gamma_i(\theta^{-1}\omega), x \ne y\right\}\end{aligned}$$

and

$$\alpha_{+,\gamma_i(\theta^{-1}\omega)}(\zeta_1(\theta^{-1}\omega), \zeta_2(\theta^{-1}\omega)) \overset{(4.13)}{=} \inf\left\{\frac{\zeta_2(x,\theta^{-1}\omega)}{\zeta_1(x,\theta^{-1}\omega)} :\ x \in \gamma_i(\theta^{-1}\omega)\right\},$$

we have

$$\begin{aligned}&\frac{\alpha^{a,\kappa}_{\gamma_i(\theta^{-1}\omega)}(\zeta_1(\theta^{-1}\omega),\zeta_2(\theta^{-1}\omega))}{\alpha_{+,\gamma_i(\theta^{-1}\omega)}(\zeta_1(\theta^{-1}\omega),\zeta_2(\theta^{-1}\omega))}\\ &\geq \inf\left\{1,\frac{\exp(ad(x,y)^\kappa)-\zeta_2(y,\theta^{-1}\omega)/\zeta_2(x,\theta^{-1}\omega)}{\exp(ad(x,y)^\kappa)-\zeta_1(y,\theta^{-1}\omega)/\zeta_1(x,\theta^{-1}\omega)}:\ x,y\in\gamma_i(\theta^{-1}\omega),x\neq y\right\}\\ &\geq \tau_1.\end{aligned}$$

Similarly, let

$$\tau_2=\sup\left\{\frac{z-z^{-\exp(-\lambda_1)}}{z-z^{\exp(-\lambda_1)}}:\ z>1\right\}=\lim_{z\to 1}\frac{z-z^{-\exp(-\lambda_1)}}{z-z^{\exp(-\lambda_1)}}=\frac{1+\exp(-\lambda_1)}{1-\exp(-\lambda_1)}\in(1,\infty),$$

we have

$$\beta^{a,\kappa}_{\gamma_i(\theta^{-1}\omega)}(\zeta_1(\theta^{-1}\omega),\zeta_2(\theta^{-1}\omega))\leq \tau_2\beta_{+,\gamma_i(\theta^{-1}\omega)}(\zeta_1(\theta^{-1}\omega),\zeta_2(\theta^{-1}\omega)),$$

where $\beta^{a,\kappa}_{\gamma_i(\theta^{-1}\omega)}(\zeta_1(\theta^{-1}\omega),\zeta_2(\theta^{-1}\omega))$ and $\beta_{+,\gamma_i(\theta^{-1}\omega)}(\zeta_1(\theta^{-1}\omega),\zeta_2(\theta^{-1}\omega))$ are given as (4.7) and (4.14). Thus, we conclude

$$d^{a,\kappa}_{\gamma_i(\theta^{-1}\omega)}(\zeta_1(\theta^{-1}\omega),\zeta_2(\theta^{-1}\omega))\leq d_{+,\gamma_i(\theta^{-1}\omega)}(\zeta_1(\theta^{-1}\omega),\zeta_2(\theta^{-1}\omega))+\log(\tau_2/\tau_1). \tag{4.26}$$

Next, we estimate $d_{+,\gamma_i(\omega)}(\zeta_1(\theta^{-1}\omega),\zeta_2(\theta^{-1}\omega))$ for $\zeta_1(\theta^{-1}\omega),\zeta_2(\theta^{-1}\omega)\in D(e^{-\lambda_1}a,\alpha,\gamma_i(\theta^{-1}\omega))$. By the third property of projective metric **(P3)** and normalizing $\zeta_1(\theta^{-1}\omega),\zeta_2(\theta^{-1}\omega)$, we can assume that

$$\int\limits_{\gamma_i(\theta^{-1}\omega)}\zeta_1(x,\theta^{-1}\omega)dm_{\gamma_i(\theta^{-1}\omega)}(x)=\int\limits_{\gamma_i(\theta^{-1}\omega)}\zeta_2(x,\theta^{-1}\omega)dm_{\gamma_i(\theta^{-1}\omega)}(x)=1,$$

without changing $d_{+,\gamma_i(\omega)}(\zeta_1(\theta^{-1}\omega),\zeta_2(\theta^{-1}\omega))$. Therefore, there exists points $y_1,y_2\in\gamma_i(\theta^{-1}\omega)$ such that $\zeta_1(y_1,\theta^{-1}\omega)=1$ and $\zeta_2(y_2,\theta^{-1}\omega)=1$ by continuity. Then **(D2)** of $D(e^{-\lambda_1}a,\kappa,\gamma_i(\theta^{-1}\omega))$ implies for all $x\in\gamma_i(\theta^{-1}\omega)$

$$\frac{\zeta_2(x,\theta^{-1}\omega)}{\zeta_1(x,\theta^{-1}\omega)}=\frac{\zeta_2(x,\theta^{-1}\omega)/\zeta_1(y_1,\theta^{-1}\omega)}{\zeta_1(x,\theta^{-1}\omega)/\zeta_2(y_2,\theta^{-1}\omega)}\geq\frac{\exp(-e^{-\lambda_1}a(diam\gamma_i(\theta^{-1}\omega))^\kappa)}{\exp(e^{-\lambda_1}a(diam\gamma_i(\theta^{-1}\omega))^\kappa)}\geq e^{-2a},$$

where $diam$ means the diameter. It follows that $\alpha_{+,\gamma_i(\theta^{-1}\omega)}(\zeta_1(\theta^{-1}\omega),\zeta_2(\theta^{-1}\omega))\geq e^{-2a}$. Similarly, $\beta_{+,\gamma_i(\theta^{-1}\omega)}(\zeta_1(\theta^{-1}\omega),\zeta_2(\theta^{-1}\omega))\leq e^{2a}$. Therefore, $d_{+,\gamma_i(\theta^{-1}\omega)}(\zeta_1(\theta^{-1}\omega),\zeta_2(\theta^{-1}\omega))\leq 4a$. So we have

$$\begin{aligned}d^{a,\kappa}_{\gamma_i(\theta^{-1}\omega)}(\zeta_1(\theta^{-1}\omega),\zeta_2(\theta^{-1}\omega))&\overset{(4.26)}{\leq} d_{+,\gamma_i(\theta^{-1}\omega)}(\zeta_1(\theta^{-1}\omega),\zeta_2(\theta^{-1}\omega))+\log(\tau_2/\tau_1)\\ &\leq 4a+\log(\tau_2/\tau_1).\end{aligned}\tag{4.27}$$

Since $\zeta_1(\theta^{-1}\omega),\zeta_2(\theta^{-1}\omega)\in D(e^{-\lambda_1}a,\kappa,\gamma_i(\theta^{-1}\omega))$ are arbitrary chosen, $R_i(\theta^{-1}\omega)\leq 4a+\log(\tau_2/\tau_1)$. By (4.25), let $\Lambda_1=1-e^{-(4a+\log(\tau_2/\tau_1))}$. The proof of (5) is complete.

Finally, let's prove (4.23). Let $\rho(\omega) \in D(a_1, \kappa_1, \gamma(\omega))$ be arbitrarily chosen, and $\tilde{\rho}(\omega)$ on $\tilde{\gamma}(\omega)$ be defined by (4.17). It is clear that $\tilde{\rho}(x, \omega) > 0$. Moreover, for any $x, y \in \tilde{\gamma}(\omega)$, we have

$$\begin{aligned}
&|\log \tilde{\rho}(x, \omega) - \log \tilde{\rho}(y, \omega)| \\
\leq\, &|\log \rho(\psi_\omega(x), \omega) - \log \rho(\psi_\omega(y), \omega)| + |\log Jac(\psi_\omega)(x) - \log Jac(\psi_\omega)(y)| \\
\leq\, &a_1 d(\psi_\omega(x), \psi_\omega(y))^{\kappa_1} + a_0 d(x, y)^{\nu_0} \\
\leq\, &a_1 a_0^{\kappa_1} d(x, y)^{\kappa_1 \nu_0} + a_0 d(x, y)^{\nu_0} \\
\leq\, &(a_1 a_0^{\kappa_1} + a_0) d(x, y)^{\kappa_1 \nu_0} \\
\overset{(4.21)}{\leq} &\frac{a}{2} d(x, y)^{\kappa_1 \nu_0}.
\end{aligned}$$

So $\tilde{\rho}(\omega) \in D(a/2, \kappa_1 \nu_0, \tilde{\gamma}(\omega))$. The proof of Lemma 4.2 is complete. □

Remark 4.1. We can apply the proof of statement (1) inductively to show the following conclusion. For any local stable leaf $\gamma(\omega)$ having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, any $\rho(\omega) \in D(a_1, \kappa, \gamma(\omega))$, and any local stable leaf $\gamma^*(\theta^{-n}\omega) \subset f_\omega^{-n}\gamma(\omega)$, we can define $\rho^*(\theta^{-n}\omega)$ on $\gamma^*(\theta^{-n}\omega)$ by

$$\rho^*(x, \theta^{-n}\omega) = \frac{|\det D_x f^n_{\theta^{-n}\omega}|_{E^s(x,\theta^{-n}\omega)}|}{|\det D_x f^n_{\theta^{-n}\omega}|} \rho(f^n_{\theta^{-n}\omega} x, \omega) \text{ for } x \in \gamma^*(\theta^{-n}\omega).$$

Then $\rho^*(\theta^{-n}\omega) \in D(e^{-\lambda_1} a_1, \kappa, \gamma^*(\theta^{-n}\omega))$.

From now on, we fix parameters a_1 and a in (4.20) and (4.21), and λ_1, Λ_1 in Lemma 4.2. Now, we use convex cones $D(a_1, \kappa_1, \gamma(\omega))$, $D(\frac{a}{2}, \kappa, \gamma(\omega))$ and $D(a, \kappa, \gamma(\omega))$ to construct the convex cone of observables on the fiber $\{\omega\}$. Let $b > 1$ and $c > 0$ be parameters to be determined later. Recall that ν is the constant picked at the beginning of Sec. 4. For any $\omega \in \Omega$, we define $C_\omega(b, c, \nu)$ to be the collection of all bounded measurable functions $\varphi : M \to \mathbb{R}$ satisfying:

(C1) $\int_{\gamma(\omega)} \varphi(x)\rho(x, \omega) dm_{\gamma(\omega)}(x) > 0$ for every local stable submanifold $\gamma(\omega) \subset M$ size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, and every $\rho(\omega) \in D(a/2, \kappa, \gamma(\omega))$ satisfying $\int_{\gamma(\omega)} \rho(x, \omega) dm_{\gamma(\omega)}(x) = 1$;

(C2) $|\log \int_{\gamma(\omega)} \varphi(x)\rho(x, \omega) dm_{\gamma(\omega)}(x) - \log \int_{\gamma(\omega)} \varphi(x)\varsigma(x, \omega) dm_{\gamma(\omega)}(x)| \leq b \cdot d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega), \varsigma(\omega))$ for every local stable submanifold $\gamma(\omega) \subset M$ having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, any $\rho(\omega), \varsigma(\omega) \in D(a/2, \kappa, \gamma(\omega)) \subset D(a, \kappa, \gamma(\omega))$ satisfying $\int_{\gamma(\omega)} \rho(x, \omega) dm_{\gamma(\omega)}(x) = \int_{\gamma(\omega)} \varsigma(x, \omega) dm_{\gamma(\omega)}(x) = 1$.

(C3) $|\log \int_{\gamma(\omega)} \varphi(x)\rho(x, \omega) dm_{\gamma(\omega)}(x) - \log \int_{\tilde{\gamma}(\omega)} \varphi(x)\tilde{\rho}(x, \omega) dm_{\tilde{\gamma}(\omega)}(x)| \leq c d_u(\gamma(\omega), \tilde{\gamma}(\omega))^\nu$ for every pair of local stable leaves $\gamma(\omega), \tilde{\gamma}(\omega) \subset M$ having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, any $\rho(\omega) \in D(a_1, \kappa_1, \gamma(\omega))$ and $\tilde{\rho}(\omega)$ corresponding to $\rho(\omega)$ defined as (4.17).

Remark 4.2. Parameters a, a_1, κ_1, b and c are constructed to prove the contraction of the transfer operator on the convex cone of observables. We just need to guarantee that all auxiliary parameters only depend on κ and ν.

Remark 4.3. Note that **(C2)** is automatically fulfilled if φ is nonnegative. In fact, notice that

$$\int\limits_{\gamma(\omega)} \rho(x,\omega)dm_{\gamma(\omega)}(x) = \int\limits_{\gamma(\omega)} \varsigma(x,\omega)dm_{\gamma(\omega)}(x) = 1,$$

then we have

$$\inf_{y\in\gamma(\omega)} \left\{\frac{\rho(y,\omega)}{\varsigma(y,\omega)}\right\} \le 1,$$

otherwise $\int_{\gamma(\omega)} \rho(\omega)dm_{\gamma(\omega)} > \int_{\gamma(\omega)} \varsigma(\omega)dm_{\gamma(\omega)}$, a contradiction. So we have

$$\begin{aligned}\frac{\rho(x,\omega)}{\varsigma(x,\omega)} &\le \sup_{x\in\gamma(\omega)} \left\{\frac{\rho(x,\omega)}{\varsigma(x,\omega)}\right\} \Big/ \inf_{y\in\gamma(\omega)} \left\{\frac{\rho(y,\omega)}{\varsigma(y,\omega)}\right\} = \exp(d_{+,\gamma(\omega)}(\rho(\omega),\varsigma(\omega))) \\ &\le \exp(d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega),\varsigma(\omega))), \end{aligned} \tag{4.28}$$

where $d_{+,\gamma(\omega)}$ is the projective metric on $D_+(\gamma(\omega))$ defined as (4.12). Switch ρ and ς, we get $\frac{\varsigma(x,\omega)}{\rho(x,\omega)} \le \exp(d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega),\varsigma(\omega)))$. Therefore, we have

$$e^{-d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega),\varsigma(\omega))} \le \frac{\int_{\gamma(\omega)} \varphi(x)\rho(x,\omega)dm_{\gamma(\omega)}(x)}{\int_{\gamma(\omega)} \varphi(x)\varsigma(x,\omega)dm_{\gamma(\omega)}(x)} \le e^{d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega),\varsigma(\omega))}.$$

$(C2)$ is a consequence of the above inequality as long as $b > 1$.

Besides, it is clear that positive constant functions belong to $C_\omega(b,c,\nu)$ for all $\omega \in \Omega$.

Lemma 4.3. *For each $\omega \in \Omega$, $C_\omega(b,c,\nu)$ is a convex cone (see Definition A.4 in the Appendix).*

Proof of Lemma 4.3. For any $\varphi \in C_\omega(b,c,\nu)$, and $t > 0$, it is clear that $t\varphi \in C_\omega(b,c,\nu)$.

Now, we prove the convexity, i.e., $\varphi_1, \varphi_2 \in C_\omega(b,c,\nu)$ and $t_1, t_2 > 0$, we are going to prove $t_1\varphi + t_2\varphi_2 \in C_\omega(b,c,\nu)$. **(C1)** is clearly fulfilled. For every local stable submanifold $\gamma(\omega) \subset M$ having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, any $\rho(\omega), \varsigma(\omega) \in D(a/2,\kappa,\gamma(\omega)) \subset D(a,\kappa,\gamma(\omega))$ satisfying $\int_{\gamma(\omega)} \rho(x,\omega)dm_{\gamma(\omega)}(x) = \int_{\gamma(\omega)} \varsigma(x,\omega)dm_{\gamma(\omega)}(x) = 1$, by **(C2)**, we have

$$e^{-bd^{a,\kappa}_{\gamma(\omega)}(\rho(\omega),\varsigma(\omega))} \le \frac{\int_{\gamma(\omega)} \varphi_i(x)\rho(x,\omega)dm_{\gamma(\omega)}(x)}{\int_{\gamma(\omega)} \varphi_i(x)\varsigma(x,\omega)dm_{\gamma(\omega)}(x)} \le e^{bd^{a,\kappa}_{\gamma(\omega)}(\rho(\omega),\varsigma(\omega))}$$

The above implies that

$$e^{-bd^{a,\kappa}_{\gamma(\omega)}(\rho(\omega),\varsigma(\omega))} \le \frac{\int_{\gamma(\omega)} (t_1\varphi_1(x) + t_2\varphi_2(x))\,\rho(x,\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)} (t_1\varphi_1(x) + t_2\varphi_2(x))\,\varsigma(x,\omega)dm_{\gamma(\omega)}} \le e^{bd_{\gamma(\omega)}(\rho(\omega),\varsigma(\omega))}.$$

So **(C2)** is verified. Similarly, **(C3)** can also be verified. Therefore, $t_1\varphi + t_2\varphi_2 \in C_\omega(b,c,\nu)$.

Now, we prove that $-\overline{C_\omega(b,c,\nu)} \cap \overline{C_\omega(b,c,\nu)} = \{0\}$. Here, the closure means the "integral closure" in a weaker sense, see Definition A.4. Suppose $\varphi \in -\overline{C_\omega(b,c,\nu)} \cap \overline{C_\omega(b,c,\nu)}$, then there exists $\varphi_1, \varphi_2 \in C_\omega(b, c, \nu)$ and $t_n^1, t_n^2 \downarrow 0$ such that $\varphi + t_n^1\varphi_1 \in C_\omega(b, c, \nu)$ and $-\varphi + t_n^2\varphi_2 \in C_\omega(b, c, \nu)$ for each $n \in \mathbb{N}$. Hence, for any local stable leaf $\gamma(\omega)$ having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, and $\rho(\omega) \in D(a/2, \kappa, \gamma(\omega))$, we have

$$\int_{\gamma(\omega)} (\varphi + t_n^1\varphi_1)(x)\rho(x, \omega)dm_{\gamma(\omega)} > 0;$$

$$\int_{\gamma(\omega)} (-\varphi + t_n^2\varphi_2)(x)\rho(x, \omega)dm_{\gamma(\omega)} > 0.$$

Letting $n \to \infty$, by bounded convergence theorem, one must have

$$\int_{\gamma(\omega)} \varphi(x)\rho(x, \omega)dm_{\gamma(\omega)}(x) = 0. \tag{4.29}$$

Now pick $g \in C^{0,\kappa}(M)$ any κ-Hölder continuous function. Let

$$|g|_\kappa := \sup_{x \neq y \in M} \frac{|g(x) - g(y)|}{d(x, y)^\kappa}, \ g^+ := \frac{1}{2}(|g| + g), \ g^- := \frac{1}{2}(|g| - g),$$

and $B = \frac{2(|g|_\kappa + 1)}{a}$. Then it is clear that

$$\log(g^+ + B), \text{ and } \log(g^- + B)$$

are $(a/2, \kappa)$-Hölder continuous, i.e.,

$$\left|\log(g^\pm(x) + B) - \log(g^\pm(y) + B)\right| \le \frac{a}{2}d(x, y)^\kappa, \ \forall x, y \in M.$$

Then $(g^+(\cdot) + B)|_{\gamma(\omega)}, \ (g^-(\cdot) + B)|_{\gamma(\omega)}$ are in $D(a/2, \kappa, \gamma(\omega))$ by checking the definition. By (4.29) and the linearity of integration, we have

$$\int_{\gamma(\omega)} \varphi(x)g(x)dm_{\gamma(\omega)}(x) = 0.$$

We can pick $g \in C^{0,\kappa}(M)$ L^1-approximating φ since φ is bounded and measurable, hence we have $\int_{\gamma(\omega)} \varphi^2(x)dm_{\gamma(\omega)}(x) = 0$. So $\varphi(x) = 0$ for $x \in \gamma(\omega)$. Since $\gamma(\omega) \subset M$ is an arbitrary local stable leaf, $\varphi \equiv 0$. The proof of Lemma 4.3 is complete. □

Now $C_\omega(b, c, \nu)$ is a convex cone, so we can define the projective metric on $C_\omega(b, c, \nu)$. For any $\varphi_1, \varphi_2 \in C_\omega(b, c, \nu)$, define

$$\alpha_\omega(\varphi_1, \varphi_2) := \sup\{t > 0 : \ \varphi_2 - t\varphi_1 \in C_\omega(b, c, \nu)\},$$

$$\beta_\omega(\varphi_1, \varphi_2) := \inf\{s > 0 : \ s\varphi_1 - \varphi_2 \in C_\omega(b, c, \nu)\},$$

with the convention that $\sup\emptyset = 0$ and $\inf\emptyset = +\infty$, and let

$$d_\omega(\varphi_1,\varphi_2) := \log\frac{\beta_\omega(\varphi_1,\varphi_2)}{\alpha_\omega(\varphi_1,\varphi_2)},$$

with the convention that $d_\omega(\varphi_1,\varphi_2) = \infty$ if $\alpha_\omega(\varphi_1,\varphi_2) = 0$ or $\beta_\omega(\varphi_1,\varphi_2) = \infty$.

Lemma 4.4. *For any $\varphi_1,\varphi_2 \in C_\omega(b,c,\nu)$,*

$$\begin{aligned}\alpha_\omega(\varphi_1,\varphi_2) = \inf\Bigg\{&\frac{\int_{\gamma(\omega)}\varphi_2\rho'(\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\varphi_1\rho'(\omega)dm_{\gamma(\omega)}}, \frac{\int_{\gamma(\omega)}\varphi_2\rho'(\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\varphi_1\rho'(\omega)dm_{\gamma(\omega)}}\xi_\omega(\rho',\rho'',\varphi_1,\varphi_2)\\ &\frac{\int_{\gamma(\omega)}\varphi_2\rho(\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\varphi_1\rho(\omega)dm_{\gamma(\omega)}}\eta_\omega(\rho,\tilde\rho,\varphi_1,\varphi_2), \frac{\int_{\tilde\gamma(\omega)}\varphi_2\tilde\rho(\omega)dm_{\tilde\gamma(\omega)}}{\int_{\tilde\gamma(\omega)}\varphi_1\tilde\rho(\omega)dm_{\tilde\gamma(\omega)}}\eta_\omega(\tilde\rho,\rho,\varphi_1,\varphi_2)\Bigg\},\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}\beta_\omega(\varphi_1,\varphi_2) = \sup\Bigg\{&\frac{\int_{\gamma(\omega)}\varphi_2\rho'(\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\varphi_1\rho'(\omega)dm_{\gamma(\omega)}}, \frac{\int_{\gamma(\omega)}\varphi_2\rho'(\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\varphi_1\rho'(\omega)dm_{\gamma(\omega)}}\xi_\omega(\rho',\rho'',\varphi_1,\varphi_2)\\ &\frac{\int_{\gamma(\omega)}\varphi_2\rho(\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\varphi_1\rho(\omega)dm_{\gamma(\omega)}}\eta_\omega(\rho,\tilde\rho,\varphi_1,\varphi_2), \frac{\int_{\tilde\gamma(\omega)}\varphi_2\tilde\rho(\omega)dm_{\tilde\gamma(\omega)}}{\int_{\tilde\gamma(\omega)}\varphi_1\tilde\rho(\omega)dm_{\tilde\gamma(\omega)}}\eta_\omega(\tilde\rho,\rho,\varphi_1,\varphi_2)\Bigg\},\end{aligned} \tag{4.31}$$

where

$$\xi_\omega(\rho',\rho'',\varphi_1,\varphi_2) := \frac{\exp(bd^{a,\kappa}_{\gamma(\omega)}(\rho'(\omega),\rho''(\omega))) - \int_{\gamma(\omega)}\varphi_2\rho''(\omega)dm_{\gamma(\omega)}/\int_{\gamma(\omega)}\varphi_2\rho'(\omega)dm_{\gamma(\omega)}}{\exp(bd^{a,\kappa}_{\gamma(\omega)}(\rho'(\omega),\rho''(\omega))) - \int_{\gamma(\omega)}\varphi_1\rho''(\omega)dm_{\gamma(\omega)}/\int_{\gamma(\omega)}\varphi_1\rho'(\omega)dm_{\gamma(\omega)}}, \tag{4.32}$$

$$\eta_\omega(\rho,\tilde\rho,\varphi_1,\varphi_2) := \frac{\exp(cd_u(\gamma(\omega),\tilde\gamma(\omega))^\nu) - \int_{\tilde\gamma}\varphi_2\tilde\rho(\omega)dm_{\tilde\gamma(\omega)}/\int_{\gamma(\omega)}\varphi_2\rho(\omega)dm_{\gamma(\omega)}}{\exp(cd_u(\gamma(\omega),\tilde\gamma(\omega))^\nu) - \int_{\tilde\gamma}\varphi_1\tilde\rho(\omega)dm_{\tilde\gamma(\omega)}/\int_{\gamma(\omega)}\varphi_1\rho(\omega)dm_{\gamma(\omega)}}, \tag{4.33}$$

$$\eta_\omega(\tilde\rho,\rho,\varphi_1,\varphi_2) := \frac{\exp(cd_u(\gamma(\omega),\tilde\gamma(\omega))^\nu) - \int_{\gamma(\omega)}\varphi_2\rho(\omega)dm_{\gamma(\omega)}/\int_{\tilde\gamma}\varphi_2\tilde\rho(\omega)dm_{\tilde\gamma(\omega)}}{\exp(cd_u(\gamma(\omega),\tilde\gamma(\omega))^\nu) - \int_{\gamma(\omega)}\varphi_1\rho(\omega)dm_{\gamma(\omega)}/\int_{\tilde\gamma}\varphi_1\tilde\rho(\omega)dm_{\tilde\gamma(\omega)}}, \tag{4.34}$$

and the inf and sup runs over all $\rho'(\omega),\rho''(\omega) \in D(a/2,\kappa,\gamma(\omega))$ with $\int_{\gamma(\omega)}\rho^\tau(x,\omega)dm_{\gamma(\omega)}(x) = 1$ for $\tau = {}^\prime, {}^{\prime\prime}$, every pair of local stable leaves $\gamma(\omega)$ and $\tilde\gamma(\omega)$ having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, $\rho(\omega) \in D(a_1,\kappa_1,\gamma(\omega))$ and corresponding $\tilde\rho(\omega) \in D(a/2,\kappa,\tilde\gamma(\omega))$.

Proof of Lemma 4.4. We only compute α_ω, as the case for β_ω is similar. For any $t > 0$ such that $\varphi_2 - t\varphi_1 \in C_\omega(b,c,\nu)$, then $\varphi_2 - t\varphi_1$ must satisfy **(C1)**, **(C2)** and **(C3)**.

First, for every local stable submanifold $\gamma(\omega) \subset M$ having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, and every $\rho(\omega) \in D(a/2,\kappa,\gamma(\omega))$ satisfying $\int_{\gamma(\omega)}\rho(x,\omega)dm_{\gamma(\omega)}(x) = 1$, one must have $\int_{\gamma(\omega)}(\varphi_2 - t\varphi_1)\rho(\omega)dm_{\gamma(\omega)} > 0$, which implies

$$t < \frac{\int_{\gamma(\omega)} \varphi_2 \rho(\omega) dm_{\gamma(\omega)}}{\int_{\gamma(\omega)} \varphi_1 \rho(\omega) dm_{\gamma(\omega)}}.$$

Second, for every local stable submanifold $\gamma(\omega) \subset M$ having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, and $\rho(\omega)$, $\varsigma(\omega) \in D(a/2, \kappa, \gamma(\omega)) \subset D(a, \kappa, \gamma(\omega))$ satisfying $\int_{\gamma(\omega)} \rho(x, \omega) dm_{\gamma(\omega)}(x) = \int_{\gamma(\omega)} \varsigma(x, \omega) dm_{\gamma(\omega)}(x) = 1$, one must have

$$e^{-b \cdot d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega), \varsigma(\omega))} \leq \frac{\int_{\gamma(\omega)} (\varphi_2 - t\varphi_1) \rho(\omega) dm_{\gamma(\omega)}}{\int_{\gamma(\omega)} (\varphi_2 - t\varphi_1) \varsigma(\omega) dm_{\gamma(\omega)}}, \leq e^{b \cdot d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega), \varsigma(\omega))}$$

which implies

$$t \leq \frac{e^{b \cdot d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega), \varsigma(\omega))} \cdot \int_{\gamma(\omega)} \varphi_2 \varsigma(\omega) dm_{\gamma(\omega)} - \int_{\gamma(\omega)} \varphi_2 \rho(\omega) dm_{\gamma(\omega)}}{e^{b \cdot d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega), \varsigma(\omega))} \cdot \int_{\gamma(\omega)} \varphi_1 \varsigma(\omega) dm_{\gamma(\omega)} - \int_{\gamma(\omega)} \varphi_1 \rho(\omega) dm_{\gamma(\omega)}}$$

and

$$t \leq \frac{e^{b \cdot d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega), \varsigma(\omega))} \cdot \int_{\gamma(\omega)} \varphi_2 \rho(\omega) dm_{\gamma(\omega)} - \int_{\gamma(\omega)} \varphi_2 \varsigma(\omega) dm_{\gamma(\omega)}}{e^{b \cdot d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega), \varsigma(\omega))} \cdot \int_{\gamma(\omega)} \varphi_1 \rho(\omega) dm_{\gamma(\omega)} - \int_{\gamma(\omega)} \varphi_1 \varsigma(\omega) dm_{\gamma(\omega)}}.$$

Third, for every pair of local stable leaves $\gamma(\omega), \tilde{\gamma}(\omega) \subset M$ having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, and $\gamma(\omega)$ is the holonomy image of $\tilde{\gamma}(\omega)$, $\rho(\omega) \in D(a_1, \kappa_1, \gamma(\omega))$ and $\tilde{\rho}(\omega)$ corresponding to $\rho(\omega)$ defined as (4.17), one must have

$$e^{-c d_u(\gamma(\omega), \tilde{\gamma}(\omega))^\nu} \leq \frac{\int_{\gamma(\omega)} (\varphi_2 - t\varphi_1) \rho(\omega) dm_{\gamma(\omega)}}{\int_{\tilde{\gamma}(\omega)} (\varphi_2 - t\varphi_1) \tilde{\rho}(\omega) dm_{\tilde{\gamma}(\omega)}} \leq e^{c d_u(\gamma(\omega), \tilde{\gamma}(\omega))^\nu},$$

which implies

$$t \leq \frac{e^{c d_u(\gamma(\omega), \tilde{\gamma}(\omega))^\nu} \cdot \int_{\tilde{\gamma}(\omega)} \varphi_2 \tilde{\rho}(\omega) dm_{\tilde{\gamma}(\omega)} - \int_{\gamma(\omega)} \varphi_2 \rho(\omega) dm_{\gamma(\omega)}}{e^{c d_u(\gamma(\omega), \tilde{\gamma}(\omega))^\nu} \cdot \int_{\tilde{\gamma}(\omega)} \varphi_1 \tilde{\rho}(\omega) dm_{\tilde{\gamma}(\omega)} - \int_{\gamma(\omega)} \varphi_1 \rho(\omega) dm_{\gamma(\omega)}}$$

and

$$t \leq \frac{e^{c d_u(\gamma(\omega), \tilde{\gamma}(\omega))^\nu} \cdot \int_{\gamma(\omega)} \varphi_2 \rho(\omega) dm_{\gamma(\omega)} - \int_{\tilde{\gamma}(\omega)} \varphi_2 \tilde{\rho}(\omega) dm_{\tilde{\gamma}(\omega)}}{e^{c d_u(\gamma(\omega), \tilde{\gamma}(\omega))^\nu} \cdot \int_{\gamma(\omega)} \varphi_1 \rho(\omega) dm_{\gamma(\omega)} - \int_{\tilde{\gamma}(\omega)} \varphi_1 \tilde{\rho}(\omega) dm_{\tilde{\gamma}(\omega)}}.$$

Organizing, we obtain (4.30). □

4.2. Contraction of the fiber transfer operator

In this subsection, we will prove that the fiber transfer operator L_ω maps $C_\omega(b,c,\nu)$ into $C_{\theta\omega}(b,c,\nu)$ for all $\omega\in\Omega$. Moreover, the diameter of $L_\omega^N C_\omega(b,c,\nu)$ with respect to the projective metric on $C_{\theta^N\omega}(b,c,\nu)$ is finite, and this diameter is independent of $\omega\in\Omega$, where the number N comes from the topological mixing on fibers property and will be constructed in (4.54) and (4.55). Birkhoff's inequality (Proposition A.3 in the Appendix) implies the contraction of the fiber transfer operator $L_\omega^N : C_\omega(b,c,\nu)\to C_{\theta^N\omega}(b,c,\nu)$.

Lemma 4.5. *Recall that $\Lambda_1\in(0,1)$ is given in Lemma 4.2. Let $\lambda_2\in(\max\{\Lambda_1, e^{-\lambda\nu}\},1)$ be any number, for any $b>b_0(\lambda_2,\Lambda_1)=\frac{1}{\lambda_2-\Lambda_1}$, there exists $c_0=c_0(b,\nu)$ such that for any $c>c_0$ and for all $\omega\in\Omega$, we have $L_{\theta^{-1}\omega}(C_{\theta^{-1}\omega}(b,c,\nu))\subset C_\omega(\lambda_2 b,\lambda_2 c,\nu)\subset C_\omega(b,c,\nu)$. Recall that the fiber transfer operator $L_{\theta^{-1}\omega}$ is defined by*

$$(L_{\theta^{-1}\omega}\varphi)(x)=\frac{\varphi((f_{\theta^{-1}\omega})^{-1}x)}{|\det D_{(f_{\theta^{-1}\omega})^{-1}(x)}f_{\theta^{-1}\omega}|},$$

for any bounded and measurable $\varphi : M\to\mathbb{R}$.

Proof of Lemma 4.5. Pick any $\omega\in\Omega$ and any $\varphi\in C_{\theta^{-1}\omega}(b,c,\nu)$. It is clear that $L_{\theta^{-1}\omega}\varphi : M\to\mathbb{R}$ is bounded and measurable.

Let us first verify condition **(C1)** for $L_{\theta^{-1}\omega}\varphi$. Pick $\gamma(\omega)$ any local stable leaf having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, and any $\rho(\omega)\in D(a/2,\kappa,\gamma(\omega))$ satisfying $\int_{\gamma(\omega)}\rho(x,\omega)dm_{\gamma(\omega)}=1$. Since $(f_{\theta^{-1}\omega})^{-1}\gamma(\omega)$ is a curve, we can divide $(f_{\theta^{-1}\omega})^{-1}\gamma(\omega)$ into connected local stable submanifolds of size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, named $\gamma_i(\theta^{-1}\omega)$ for i belonging to a finite index set such that $\gamma_i(\theta^{-1}\omega)\cap\gamma_j(\theta^{-1}\omega)=\partial\gamma_i(\theta^{-1}\omega)\cap\partial\gamma_j(\theta^{-1}\omega)$ for $i\neq j$. Let $\rho_i(\theta^{-1}\omega)$ be defined as (4.15) on $\gamma_i(\theta^{-1}\omega)$. By Lemma 4.2,

$$\rho_i(\theta^{-1}\omega)\in D(e^{-\lambda_1}a/2,\kappa,\gamma_i(\theta^{-1}\omega))\subset D(a/2,\kappa,\gamma_i(\theta^{-1}\omega)).$$

Hence by (4.16), we have

$$\begin{aligned}
&\int\limits_{\gamma(\omega)}(L_{\theta^{-1}\omega}\varphi)(y)\rho(y,\omega)dm_{\gamma(\omega)}(y)\\
&=\sum_i\int\limits_{\gamma_i(\theta^{-1}\omega)}\varphi(x)\rho_i(x,\theta^{-1}\omega)dm_{\gamma_i(\theta^{-1}\omega)}(x)\\
&=\sum_i\int\limits_{\gamma_i(\theta^{-1}\omega)}\rho_i(\theta^{-1}\omega)dm_{\gamma_i(\theta^{-1}\omega)}\\
&\quad\cdot\int\limits_{\gamma_i(\theta^{-1}\omega)}\varphi(x)\frac{\rho_i(x,\theta^{-1}\omega)}{\int_{\gamma_i(\theta^{-1}\omega)}\rho_i(\theta^{-1}\omega)dm_{\gamma_i(\theta^{-1}\omega)}}dm_{\gamma_i(\theta^{-1}\omega)}(x)>0,
\end{aligned}$$

where the last inequality holds since $\varphi\in C_{\theta^{-1}\omega}(b,c,\nu)$ and using condition **(C1)** for φ.

Secondly, let us verify condition **(C2)** for $L_{\theta^{-1}\omega}\varphi$. For any $\rho(\omega), \zeta(\omega) \in D(a/2, \kappa, \gamma(\omega))$ such that

$$\int_{\gamma(\omega)} \rho(x,\omega) dm_{\gamma(\omega)}(x) = 1 \text{ and } \int_{\gamma(\omega)} \zeta(x,\omega) dm_{\gamma(\omega)}(x) = 1.$$

We divide $f_\omega^{-1}\gamma(\omega)$ into $\{\gamma_i(\theta^{-1}\omega)\}$ as before. Let $\rho_i(\theta^{-1}\omega)$ and $\zeta_i(\theta^{-1}\omega)$ be density functions defined as (4.15) on $\gamma_i(\theta^{-1}\omega)$ corresponding to $\rho(\omega)$ and $\zeta(\omega)$ respectively. We normalize these density functions by

$$\rho_i'(x,\theta^{-1}\omega) = \rho_i(x,\theta^{-1}\omega) / \int_{\gamma_i(\theta^{-1}\omega)} \rho_i(y,\theta^{-1}\omega) dm_{\gamma_i(\theta^{-1}\omega)}(y) \text{ for } x \in \gamma_i(\theta^{-1}\omega),$$

$$\zeta_i'(x,\theta^{-1}\omega) = \zeta_i(x,\theta^{-1}\omega) / \int_{\gamma_i(\theta^{-1}\omega)} \zeta_i(y,\theta^{-1}\omega) dm_{\gamma_i(\theta^{-1}\omega)}(y) \text{ for } x \in \gamma_i(\theta^{-1}\omega).$$

By Lemma 4.2, we have $\rho_i'(\theta^{-1}\omega), \zeta_i'(\theta^{-1}\omega) \in D(e^{-\lambda_1}\frac{a}{2}, \kappa, \gamma_i(\theta^{-1}\omega)) \subset D(\frac{a}{2}, \kappa, \gamma_i(\theta^{-1}\omega))$. Then

$$\begin{aligned}
&\int_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\zeta(y,\omega) dm_{\gamma(\omega)}(y) \\
&\overset{(4.16)}{=} \sum_i \int_{\gamma_i(\theta^{-1}\omega)} \varphi(x)\zeta_i(x,\theta^{-1}\omega) dm_{\gamma_i(\theta^{-1}\omega)}(x) \\
&= \sum_i \int_{\gamma_i(\theta^{-1}\omega)} \zeta_i(x,\theta^{-1}\omega) dm_{\gamma_i(\theta^{-1}\omega)}(x) \int_{\gamma_i(\theta^{-1}\omega)} \varphi(x)\zeta_i'(x,\theta^{-1}\omega) dm_{\gamma_i(\theta^{-1}\omega)}(x) \\
&\overset{\textbf{(C2)}}{\le} \sum_i \int_{\gamma_i(\theta^{-1}\omega)} \zeta_i(\theta^{-1}\omega) dm_{\gamma_i(\theta^{-1}\omega)} \cdot \exp(b d^{a,\kappa}_{\gamma_i(\theta^{-1}\omega)}(\rho_i'(\theta^{-1}\omega), \zeta_i'(\theta^{-1}\omega))) \\
&\quad \cdot \int_{\gamma_i(\theta^{-1}\omega)} \varphi\rho_i'(\theta^{-1}\omega) dm_{\gamma_i(\theta^{-1}\omega)} \\
&\overset{(4.9)}{=} \sum_i \frac{\int_{\gamma_i(\theta^{-1}\omega)} \zeta_i(\theta^{-1}\omega) dm_{\gamma_i(\theta^{-1}\omega)}}{\int_{\gamma_i(\theta^{-1}\omega)} \rho_i(\theta^{-1}\omega) dm_{\gamma_i(\theta^{-1}\omega)}} \cdot \exp(b d^{a,\kappa}_{\gamma_i(\theta^{-1}\omega)}(\rho_i(\theta^{-1}\omega), \zeta_i(\theta^{-1}\omega))) \\
&\quad \cdot \int_{\gamma_i(\theta^{-1}\omega)} \varphi\rho_i(\theta^{-1}\omega) dm_{\gamma_i(\theta^{-1}\omega)} \\
&\overset{(4.22)}{\le} \sum_i \frac{\int_{\gamma_i(\theta^{-1}\omega)} \zeta_i(\theta^{-1}\omega) dm_{\gamma_i(\theta^{-1}\omega)}}{\int_{\gamma_i(\theta^{-1}\omega)} \rho_i(\theta^{-1}\omega) dm_{\gamma_i(\theta^{-1}\omega)}} \cdot \exp(b\Lambda_1 d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega), \zeta(\omega)))
\end{aligned}$$

$$\cdot \int\limits_{\gamma_i(\theta^{-1}\omega)} \varphi\rho_i(\theta^{-1}\omega) dm_{\gamma_i(\theta^{-1}\omega)}.$$

Note that

$$\frac{\zeta_i(x,\theta^{-1}\omega)}{\rho_i(x,\theta^{-1}\omega)} \overset{(4.15)}{=} \frac{\zeta(f_{\theta^{-1}\omega}x,\omega)}{\rho(f_{\theta^{-1}\omega}x,\omega)} \overset{(4.28)}{\leq} \exp(d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega),\zeta(\omega))) \text{ for } x \in \gamma_i(\theta^{-1}\omega).$$

We continue the estimate

$$\begin{aligned}
&\int\limits_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\zeta(y,\omega) dm_{\gamma(\omega)}(y) \\
&\leq \exp\left(d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega),\zeta(\omega))\right) \exp\left(b\Lambda_1 d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega),\zeta(\omega))\right) \sum_i \int\limits_{\gamma_i(\theta^{-1}\omega)} \varphi\rho_i(\theta^{-1}\omega) dm_{\gamma_i(\theta^{-1}\omega)} \\
&= \exp\left((1+b\Lambda_1) d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega),\zeta(\omega))\right) \int\limits_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\rho(y,\omega) dm_{\gamma(\omega)}(y) \\
&\leq \exp\left(b\lambda_2 d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega),\zeta(\omega))\right) \int\limits_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\rho(y,\omega) dm_{\gamma(\omega)}(y),
\end{aligned}$$

provided $\lambda_2 \in (\Lambda_1, 1)$ and $b > \frac{1}{\lambda_2 - \Lambda_1} := b_0$. Switching $\rho(\omega)$ and $\zeta(\omega)$ in the above estimates, we get

$$\begin{aligned}
&\int\limits_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\rho(y,\omega) dm_{\gamma(\omega)}(y) \\
&\qquad \leq \exp(b\lambda_2 d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega),\zeta(\omega))) \int\limits_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\zeta(y,\omega) dm_{\gamma(\omega)}(y).
\end{aligned}$$

The above two estimates imply that

$$\begin{aligned}
&\left| \log \int\limits_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\rho(y,\omega) dm_{\gamma(\omega)} - \log \int\limits_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\zeta(y,\omega) dm_{\gamma(\omega)} \right| \\
&\leq b\lambda_2 d^{a,\kappa}_{\gamma(\omega)}(\rho(\omega),\zeta(\omega)).
\end{aligned}$$

Next, let us verify condition **(C3)** for $L_{\theta^{-1}\omega}\varphi$. Given any pair of local stable leaves $\gamma(\omega)$ and $\tilde{\gamma}(\omega)$. We first divide $(f_{\theta^{-1}\omega})^{-1}\gamma(\omega)$ into connected local stable manifolds of size between $\frac{A(\epsilon)}{4J}$ and $\frac{A(\epsilon)}{2J}$, named $\gamma_i(\theta^{-1}\omega)$, such that $\gamma(\omega) = \cup f_{\theta^{-1}\omega}\gamma_i(\theta^{-1}\omega)$. Let $\tilde{\gamma}_i(\theta^{-1}\omega)$ be the holonomy image of $\gamma_i(\theta^{-1}\omega)$ inside of $(f_{\theta^{-1}\omega})^{-1}\tilde{\gamma}(\omega)$. Naturally, we have $\tilde{\gamma}(\omega) = \cup f_{\theta^{-1}\omega}\tilde{\gamma}_i(\theta^{-1}\omega)$. Note that the Jacobian of holonomy map between local stable manifolds is bounded above by J and bounded below by J^{-1}. Therefore, $\tilde{\gamma}_i(\theta^{-1}\omega)$ have size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$. For any $\rho(\omega) \in D(a_1, \kappa_1, \gamma(\omega))$, let $\tilde{\rho}(\omega)$ be defined as (4.17), and we already see that $\tilde{\rho}(\omega) \in D(\frac{a}{2}, \kappa_1 \nu_0, \tilde{\gamma}(\omega)) \subset D(a, \kappa, \tilde{\gamma}(\omega))$. Let

$$\rho_i(x, \theta^{-1}\omega) = \frac{|\det D_x f_{\theta^{-1}\omega}|_{E^s(x,\theta^{-1}\omega)}|}{|\det D_x f_{\theta^{-1}\omega}|} \rho(f_{\theta^{-1}\omega} x, \omega) \text{ for } x \in \gamma_i(\theta^{-1}\omega)$$

and

$$(\tilde{\rho})_i(\theta^{-1}\omega) = \frac{|\det D_x f_{\theta^{-1}\omega}|_{E^s(x,\theta^{-1}\omega)}|}{|\det D_x f_{\theta^{-1}\omega}|} \tilde{\rho}(f_{\theta^{-1}\omega} x, \omega) \text{ for } x \in \gamma_i(\theta^{-1}\omega).$$

Then by (4.16), we have

$$\int_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\rho(y, \omega) dm_{\gamma(\omega)}(y) = \sum_i \int_{\gamma_i(\theta^{-1}\omega)} \varphi(x)\rho_i(x, \theta^{-1}\omega) dm_{\gamma_i(\theta^{-1}\omega)}(x),$$
$$\int_{\tilde{\gamma}(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\tilde{\rho}(y, \omega) dm_{\tilde{\gamma}(\omega)}(y) = \sum_i \int_{\tilde{\gamma}_i(\theta^{-1}\omega)} \varphi(x)(\tilde{\rho})_i(x, \theta^{-1}\omega) dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}(x).$$

By Lemma 4.2, $\rho_i(\theta^{-1}\omega) \in D(e^{-\lambda_1} a_1, \kappa_1, \gamma_i(\theta^{-1}\omega)) \subset D(a_1, \kappa_1, \gamma_i(\theta^{-1}\omega))$. Let $\psi^i_{\theta^{-1}\omega}$ be the holonomy map between $\tilde{\gamma}_i(\theta^{-1}\omega)$ and $\gamma_i(\theta^{-1}\omega)$, and define

$$\tilde{\rho}_i(x, \theta^{-1}\omega) = \rho_i(\psi^i_{\theta^{-1}\omega}(x), \theta^{-1}\omega) \cdot |Jac(\psi^i_{\theta^{-1}\omega})(x)| \text{ for } x \in \tilde{\gamma}_i(\theta^{-1}\omega)$$

to be the density function defined by the holonomy map corresponding to $\rho_i(\theta^{-1}\omega)$. Note that $(\tilde{\rho})_i(\theta^{-1}\omega)$ and $\tilde{\rho}_i(\theta^{-1}\omega)$ are different, and we keep in mind that $(\tilde{\rho})_i(\theta^{-1}\omega)$ is obtained by first applying the action of holonomy map then applying the action of pull back to $\rho(\omega)$, while $\tilde{\rho}_i(\theta^{-1}\omega)$ is obtained by reversing the order of these two actions. Since $\varphi \in C_{\theta^{-1}\omega}(b, c, \nu)$ and $\rho_i(\theta^{-1}\omega) \in D(a_1, \kappa_1, \gamma_i(\theta^{-1}\omega))$, by **(C3)**, we conclude for each i,

$$\begin{aligned} &\left| \log \int_{\gamma_i(\theta^{-1}\omega)} \varphi(x)\rho_i(x, \theta^{-1}\omega) dm_{\gamma_i(\theta^{-1}\omega)} - \log \int_{\tilde{\gamma}_i(\theta^{-1}\omega)} \varphi(x)\tilde{\rho}_i(x, \theta^{-1}\omega) dm_{\tilde{\gamma}_i(\theta^{-1}\omega)} \right| \\ &\leq c d_u(\gamma_i(\theta^{-1}\omega), \tilde{\gamma}_i(\theta^{-1}\omega))^\nu \\ &\overset{(4.19)}{\leq} c \cdot e^{-\lambda\nu} d_u(\gamma(\omega), \tilde{\gamma}(\omega))^\nu. \end{aligned} \tag{4.35}$$

To finish the proof of the condition **(C3)**, we need the following sublemma:

Sublemma 4.1. *There exists a number $K_0 = K_0(a, b) > 0$ depending on a, b and system constants such that for each i, the following inequality holds*

$$\left|\log \int\limits_{\tilde{\gamma}_i(\theta^{-1}\omega)} \varphi(x)\tilde{\rho}_i(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)} - \log \int\limits_{\tilde{\gamma}_i(\theta^{-1}\omega)} \varphi(x)(\tilde{\rho})_i(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}\right|$$
$$\leq K_0 d_u(\gamma(\omega),\tilde{\gamma}(\omega))^\nu. \tag{4.36}$$

We postpone the proof of Sublemma 4.1 and finish the proof of Lemma 4.5. We combine (4.35) and (4.36) to obtain

$$\left|\log \int\limits_{\gamma_i(\theta^{-1}\omega)} \varphi(x)\rho_i(x,\theta^{-1}\omega)dm_{\gamma_i(\theta^{-1}\omega)} - \log \int\limits_{\tilde{\gamma}_i(\theta^{-1}\omega)} \varphi(x)(\tilde{\rho})_i(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}\right|$$
$$\leq (c\cdot e^{-\lambda\nu}+K_0)d_u(\gamma(\omega),\tilde{\gamma}(\omega))^\nu.$$

As a consequence,

$$\begin{aligned}
&\left|\log \int\limits_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\rho(y,\omega)dm_{\gamma(\omega)}(y) - \log \int\limits_{\tilde{\gamma}(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\tilde{\rho}(y,\omega)dm_{\tilde{\gamma}(\omega)}(y)\right| \\
&= \Bigg|\log \sum_i \int\limits_{\gamma_i(\theta^{-1}\omega)} \varphi(x)\rho_i(x,\theta^{-1}\omega)dm_{\gamma_i(\theta^{-1}\omega)}(x) \\
&\quad - \log \sum_i \int\limits_{\tilde{\gamma}_i(\theta^{-1}\omega)} \varphi(x)(\tilde{\rho})_i(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}(x)\Bigg| \\
&\leq (c\cdot e^{-\lambda\nu}+K_0)d_u(\gamma(\omega),\tilde{\gamma}(\omega))^\nu \\
&\leq \lambda_2 c d_u(\gamma(\omega),\tilde{\gamma}(\omega))^\nu,
\end{aligned}$$

provided $\lambda_2 \in (e^{-\lambda\nu}, 1)$, and $c \geq \frac{K_0(a,b)}{\lambda_2 - \exp(-\lambda\nu)} := c_0$. The proof of Lemma 4.5 is complete. □

Proof of Sublemma 4.1. Applying Lemma 4.2 and (4.23) to $\rho(\omega) \in D(a_1,\kappa_1,\gamma(\omega))$, we see that $(\tilde{\rho})_i(\theta^{-1}\omega)$, $\tilde{\rho}_i(\theta^{-1}\omega)$ both belong to $D(a/2,\kappa_1\nu_0,\tilde{\gamma}_i(\theta^{-1}\omega)) \subset D(a/2,\kappa,\tilde{\gamma}_i(\theta^{-1}\omega))$. We fix i and prove Sublemma 4.1. Without ambiguity, we denote $\rho'(\theta^{-1}\omega) := (\tilde{\rho})_i(\theta^{-1}\omega)$ and $\rho''(\theta^{-1}\omega) := \tilde{\rho}_i(\theta^{-1}\omega)$ for short.

We normalize the random density $\rho'(\theta^{-1}\omega)$ and $\rho''(\theta^{-1}\omega)$ by letting

$$\bar{\rho}'(x,\theta^{-1}\omega) := \frac{\rho'(x,\theta^{-1}\omega)}{\int_{\tilde{\gamma}_i(\theta^{-1}\omega)} \rho'(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}(x)},$$
$$\bar{\rho}''(x,\theta^{-1}\omega) := \frac{\rho''(x,\theta^{-1}\omega)}{\int_{\tilde{\gamma}_i(\theta^{-1}\omega)} \rho''(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}(x)}.$$

Then by condition **(C2)**, we have

$$\begin{aligned}&\left|\log \int\limits_{\tilde{\gamma}_i(\theta^{-1}\omega)} \varphi(x)\rho'(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta)^{-1}\omega} - \log \int\limits_{\tilde{\gamma}_i(\theta^{-1}\omega)} \varphi(x)\rho''(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta)^{-1}\omega}\right| \\ &\le \left|\log \int\limits_{\tilde{\gamma}_i(\theta^{-1}\omega)} \rho'(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}(x) - \log \int\limits_{\tilde{\gamma}_i(\theta^{-1}\omega)} \rho''(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}(x)\right| \\ &\quad + bd^{a,\kappa}_{\tilde{\gamma}_i(\theta^{-1}\omega)}(\bar{\rho}'(\theta^{-1}\omega), \bar{\rho}''(\theta^{-1}\omega)).\end{aligned} \tag{4.37}$$

Next, we are going to estimate terms in the right hand of the above inequality. By definition, for $x \in \tilde{\gamma}_i(\theta^{-1}\omega)$, we have expressions

$$\rho'(x,\theta^{-1}\omega) = \frac{\left|\det D_x f_{\theta^{-1}\omega}|_{E^s(x,\theta^{-1}\omega)}\right|}{\left|\det D_x f_{\theta^{-1}\omega}\right|} \cdot \rho(\psi_\omega f_{\theta^{-1}\omega}(x),\omega)|Jac(\psi_\omega)(f_{\theta^{-1}\omega}x)|;$$

$$\rho''(x,\theta^{-1}\omega) = \frac{\left|\det D_{\psi^i_{\theta^{-1}\omega}(x)} f_{\theta^{-1}\omega}|_{E^s(\psi^i_{\theta^{-1}\omega}(x),\theta^{-1}\omega)}\right|}{\left|\det D_{\psi^i_{\theta^{-1}\omega}(x)} f_{\theta^{-1}\omega}\right|} \cdot \rho(f_{\theta^{-1}\omega}\psi^i_{\theta^{-1}\omega}(x),\omega)\cdot|Jac(\psi^i_{\theta^{-1}\omega})(x)|.$$

By definition of holonomy map, we have

$$\rho(\psi_\omega f_{\theta^{-1}\omega}(x),\omega) = \rho(f_{\theta^{-1}\omega}\psi^i_{\theta^{-1}\omega}(x),\omega) \text{ for } x \in \tilde{\gamma}_i(\theta^{-1}\omega). \tag{4.38}$$

By Lemma 3.12 (2), for $x \in \tilde{\gamma}_i(\theta^{-1}\omega)$, we have

$$\begin{aligned}&\left|\log|Jac(\psi_\omega)(f_{\theta^{-1}\omega}x)| - \log|Jac(\psi^i_{\theta^{-1}\omega})(x)|\right| \\ &\le a_0 d(f_{\theta^{-1}\omega}(x), \psi_\omega f_{\theta^{-1}\omega}(x))^{\nu_0} + a_0 d(x, \psi^i_{\theta^{-1}\omega}(x))^{\nu_0} \\ &\le a_0(1+e^{-\lambda\nu_0})d_u(\gamma(\omega),\tilde{\gamma}(\omega))^{\nu_0}.\end{aligned} \tag{4.39}$$

Combining Lemma 3.1 and Lemma 3.4, for all $x, y \in M$, $\omega \in \Omega$, we have

$$\left||\det D_x f_\omega|_{E^s(x,\omega)}| - |\det D_y f_\omega|_{E^s(y,\omega)}|\right| \le (C_2 + C_2C_1)d(x,y)^{\nu_0}. \tag{4.40}$$

Then, by (4.40) and (3.42), for $x \in \tilde{\gamma}_i(\theta^{-1}\omega)$, we have

$$\begin{aligned}&\left|\log|\det D_x f_{\theta^{-1}\omega}|_{E^s(x,\theta^{-1}\omega)}| - \log|\det D_{\psi^i_{\theta^{-1}\omega}(x)} f_{\theta^{-1}\omega}|_{E^s(\psi^i_{\theta^{-1}\omega}(x),\theta^{-1}\omega)}|\right| \\ &\le C_{10}(C_2 + C_2C_1)d(x,\psi^i_{\theta^{-1}\omega}(x))^{\nu_0} \\ &\le C_{10}(C_2 + C_2C_1)d_u(\gamma_i(\theta^{-1}\omega),\tilde{\gamma}_i(\theta^{-1}\omega))^{\nu_0} \\ &\le C_{10}(C_2 + C_2C_1)e^{-\lambda\nu_0}d_u(\gamma(\omega),\tilde{\gamma}(\omega))^{\nu_0}.\end{aligned} \tag{4.41}$$

Applying (4.3), we have

$$\left|\log|\det D_{\psi^i_{\theta^{-1}\omega}x} f_{\theta^{-1}\omega}| - \log|\det D_x f_{\theta^{-1}\omega}|\right| \le K_2 d(x, \psi^i_{\theta^{-1}\omega}x) \le K_2 e^{-\lambda} d_u(\gamma(\omega), \tilde{\gamma}(\omega)). \tag{4.42}$$

Then (4.38), (4.39), (4.41) and (4.42) imply

$$e^{-K_3 d_u(\gamma(\omega),\tilde{\gamma}(\omega))^{\nu_0}} \le \frac{\rho'(x,\theta^{-1}\omega)}{\rho''(x,\theta^{-1}\omega)} \le e^{K_3 d_u(\gamma(\omega),\tilde{\gamma}(\omega))^{\nu_0}} \text{ for any } x \in \tilde{\gamma}_i(\theta^{-1}\omega), \tag{4.43}$$

where $K_3 = \max\{a_0(1+e^{-\lambda\nu_0}), C_{10}(C_2 + C_2C_1)e^{-\lambda\nu_0}, K_2e^{-\lambda}\}$, and therefore,

$$\left|\log \int\limits_{\tilde{\gamma}_i(\theta^{-1}\omega)} \rho'(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}(x) - \log \int\limits_{\tilde{\gamma}_i(\theta^{-1}\omega)} \rho''(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}(x)\right|$$
$$\le K_3 d_u(\gamma(\omega), \tilde{\gamma}(\omega))^{\nu_0}. \tag{4.44}$$

Next, we estimate $d^{a,\kappa}_{\tilde{\gamma}_i(\theta^{-1}\omega)}(\bar{\rho}'(\theta^{-1}\omega), \bar{\rho}''(\theta^{-1}\omega))$. The inequality (4.43) also implies that

$$\begin{aligned} d_{+,\tilde{\gamma}_i(\theta^{-1}\omega)}(\bar{\rho}'(\theta^{-1}\omega), \bar{\rho}''(\theta^{-1}\omega)) &= d_{+,\tilde{\gamma}_i(\theta^{-1}\omega)}(\rho'(\theta^{-1}\omega), \rho''(\theta^{-1}\omega)) \\ &\overset{(4.12)}{\le} 2K_3 d_u(\gamma(\omega), \tilde{\gamma}(\omega))^{\nu_0}. \end{aligned} \tag{4.45}$$

Similar as the proof of (4.26), we have an estimate

$$\begin{aligned} d^{a,\kappa}_{\tilde{\gamma}_i(\theta^{-1}\omega)}(\bar{\rho}'(\theta^{-1}\omega), \bar{\rho}''(\theta^{-1}\omega)) &\le d_{+,\tilde{\gamma}_i(\theta^{-1}\omega)}(\bar{\rho}'(\theta^{-1}\omega), \bar{\rho}''(\theta^{-1}\omega)) \\ &\quad + \log\left(\hat{\tau}_2(\theta^{-1}\omega)/\hat{\tau}_1(\theta^{-1}\omega)\right), \end{aligned} \tag{4.46}$$

where

$$\hat{\tau}_1(\theta^{-1}\omega) = \inf_{x\neq y\in\tilde{\gamma}_i(\theta^{-1}\omega)} \left\{1, \frac{\exp(ad(x,y)^\kappa) - \rho''(y,\theta^{-1}\omega)/\rho''(x,\theta^{-1}\omega)}{\exp(ad(x,y)^\kappa) - \rho'(y,\theta^{-1}\omega)/\rho'(x,\theta^{-1}\omega)}\right\},$$

and

$$\hat{\tau}_2(\theta^{-1}\omega) = \sup_{x\neq y\in\tilde{\gamma}_i(\theta^{-1}\omega)} \left\{1, \frac{\exp(ad(x,y)^\kappa) - \rho''(y,\theta^{-1}\omega)/\rho''(x,\theta^{-1}\omega)}{\exp(ad(x,y)^\kappa) - \rho'(y,\theta^{-1}\omega)/\rho'(x,\theta^{-1}\omega)}\right\}.$$

So we need to estimate $\log\left(\hat{\tau}_2(\theta^{-1}\omega)/\hat{\tau}_1(\theta^{-1}\omega)\right)$. Denote

$$B_1(x,y,\theta^{-1}\omega) := \frac{\rho'(y,\theta^{-1}\omega)}{\rho'(x,\theta^{-1}\omega)} \cdot \exp(-ad(x,y)^\kappa) \text{ for } x, y \in \tilde{\gamma}_i(\theta^{-1}\omega), x \neq y,$$
$$B_2(x,y,\theta^{-1}\omega) := \frac{\rho''(y,\theta^{-1}\omega)}{\rho''(x,\theta^{-1}\omega)} \cdot \exp(-ad(x,y)^\kappa) \text{ for } x, y \in \tilde{\gamma}_i(\theta^{-1}\omega), x \neq y,$$

then

$$\frac{\exp(ad(x,y)^\kappa) - \rho''(y,\theta^{-1}\omega)/\rho''(x,\theta^{-1}\omega)}{\exp(ad(x,y)^\kappa) - \rho'(y,\theta^{-1}\omega)/\rho'(x,\theta^{-1}\omega)} = \frac{1 - B_2(x,y,\theta^{-1}\omega)}{1 - B_1(x,y,\theta^{-1}\omega)}. \tag{4.47}$$

Since $\rho'(\theta^{-1}\omega), \rho''(\theta^{-1}\omega) \in D(a/2, \kappa_1\nu_0, \tilde{\gamma}_i(\theta^{-1}\omega)) \subset D(a/2, \kappa, \tilde{\gamma}_i(\theta^{-1}\omega))$,

$$\begin{aligned}\log B_1(x,y,\theta^{-1}\omega) &= \log\rho'(y,\theta^{-1}\omega) - \log\rho'(x,\theta^{-1}\omega) - ad(x,y)^\kappa \\ &\le \frac{a}{2}d(x,y)^\kappa - ad(x,y)^\kappa \\ &\le -\frac{a}{2}d(x,y)^\kappa < 0. \end{aligned}\tag{4.48}$$

As a consequence, $B_1(x,y,\theta^{-1}\omega) \le e^{-\frac{a}{2}d(x,y)^\kappa} < 1$. Similarly, $B_2(x,y,\theta^{-1}\omega) \le e^{-\frac{a}{2}d(x,y)^\kappa} < 1$. Hence, on the one hand

$$\begin{aligned}&|B_1(x,y,\theta^{-1}\omega) - B_2(x,y,\theta^{-1}\omega)| \\ &\le \max\{B_1(x,y,\theta^{-1}\omega), B_2(x,y,\theta^{-1}\omega)\} \cdot |\log B_1(x,y,\theta^{-1}\omega) - \log B_2(x,y,\theta^{-1}\omega)| \\ &\le |\log\rho'(x,\theta^{-1}\omega) - \log\rho''(x,\theta^{-1}\omega)| + |\log\rho'(y,\theta^{-1}\omega) - \log\rho''(y,\theta^{-1}\omega)| \\ &\overset{(4.43)}{\le} 2K_3 d_u(\gamma(\omega), \tilde{\gamma}(\omega))^{\nu_0} \\ &\overset{(4.4)}{\le} 2K_3 d_u(\gamma(\omega), \tilde{\gamma}(\omega))^{\kappa+\nu}, \end{aligned}\tag{4.49}$$

where in the first inequality, we use the following inequality

$$|z_1 - z_2| \le \max\{z_1, z_2\}|\log z_1 - \log z_2| \text{ for any } z_1, z_2 \in (0,1).$$

On the other hand, recall that $\rho'(\theta^{-1}\omega)$, $\rho''(\theta^{-1}\omega)$ both belong to $D(a/2, \kappa_1\nu_0, \tilde{\gamma}_i(\theta^{-1}\omega))$, we also have

$$\begin{aligned}&|B_1(x,y,\theta^{-1}\omega) - B_2(x,y,\theta^{-1}\omega)| \\ &\le \max\{B_1(x,y,\theta^{-1}\omega), B_2(x,y,\theta^{-1}\omega)\} \cdot |\log B_1(x,y,\theta^{-1}\omega) - \log B_2(x,y,\theta^{-1}\omega)| \\ &\le |\log\rho'(x,\theta^{-1}\omega) - \log\rho'(y,\theta^{-1}\omega)| + |\log\rho''(x,\theta^{-1}\omega) - \log\rho''(y,\theta^{-1}\omega)| \\ &\le 2 \cdot \frac{a}{2} d(x,y)^{\kappa_1\nu_0} \\ &\le ad(x,y)^{\kappa+\nu}. \end{aligned}\tag{4.50}$$

Estimates (4.49) and (4.50) imply that

$$\begin{aligned}|B_1(x,y,\theta^{-1}\omega) - B_2(x,y,\theta^{-1}\omega)| &\le \max\{a, 2K_3\} d_u(\gamma(\omega), \tilde{\gamma}(\omega))^\nu \cdot d(x,y)^\kappa \\ &:= K_4 d_u(\gamma(\omega), \tilde{\gamma}(\omega))^\nu \cdot d(x,y)^\kappa. \end{aligned}\tag{4.51}$$

Then

$$\begin{aligned}\left|\log\frac{1-B_2(x,y,\theta^{-1}\omega)}{1-B_1(x,y,\theta^{-1}\omega)}\right| &\le \frac{|B_1(x,y,\theta^{-1}\omega)-B_2(x,y,\theta^{-1}\omega)|}{1-\max\{B_1(x,y,\theta^{-1}\omega),B_2(x,y,\theta^{-1}\omega)\}}\\ &\overset{(4.48)}{\le} \frac{K_4 d_u(\gamma(\omega),\tilde\gamma(\omega))^{\nu}\cdot d(x,y)^{\kappa}}{1-\exp(-\frac{a}{2}d(x,y)^{\kappa})}\\ &\le K_5 d_u(\gamma(\omega),\tilde\gamma(\omega))^{\nu},\end{aligned} \tag{4.52}$$

where $K_5 := K_4\cdot\sup_{z\in(0,1)}\frac{z^\kappa}{1-\exp(-\frac{a}{2}z^\kappa)}<\infty$, and in the first "$\le$", we use the following fact

$$|\log(1-z_1)-\log(1-z_2)|\le\frac{|z_1-z_2|}{1-\max\{z_1,z_2\}}\text{ for any } z_1,z_2\in(0,1).$$

Hence we have

$$|\log\hat\tau_2(\theta^{-1}\omega)/\hat\tau_1(\theta^{-1}\omega)| \overset{(4.47),(4.52)}{\le} 2K_5 d_u(\gamma(\omega),\tilde\gamma(\omega))^{\nu}. \tag{4.53}$$

Let $K_0=b(2K_3+2K_5)+K_3$, then by (4.37), (4.44), (4.45), (4.46) and (4.53), Sublemma 4.1 is proved. □

From now on, we fix any $\lambda_2\in(\max\{\Lambda_1,e^{-\lambda\nu}\},1)$, $b>b_0(\lambda_2,\Lambda_1)=\frac{1}{\lambda_2-\Lambda_1}$ and $c>c_0=c_0(b,\nu)$ as in Lemma 4.5. We have shown that

$$L_{\theta^{-1}\omega}(C_{\theta^{-1}\omega}(b,c,\nu))\subset C_\omega(\lambda_2 b,\lambda_2 c,\nu)\subset C_\omega(b,c,\nu),\ \forall\omega\in\Omega.$$

Recall that ϵ^* is a constant satisfying (4.5). Let $\delta\in(0,\epsilon^*/8)$ be the constant in Lemma 3.3 corresponding to $\epsilon^*/8$, i.e., for any $x,y\in M$, $d(x,y)<\delta$, for all $\omega\in\Omega$, we have

$$W^s_{\epsilon^*/8}(x,\omega)\cap W^u_{\epsilon^*/8}(y,\omega)\neq\emptyset.$$

Now let $\{B_{\delta/4}(x)\}_{x\in M}$ be an open cover of M. Pick a subcover $\{B_{\delta/4}(x_i)\}_{i=1}^l$ by the compactness of M. Now by the definition of topological mixing on fibers, there exists a $N\in\mathbb{N}$ such that for any $n\ge N$, $\omega\in\Omega$,

$$\phi^n(B_{\delta/4}(x_i)\times\{\omega\})\cap(B_{\delta/4}(x_j)\times\{\theta^n\omega\})\neq\emptyset \text{ for any } 1\le i,j\le l. \tag{4.54}$$

Moreover, we pick N large enough such that

$$e^{\lambda N}\epsilon^*\ge 24\epsilon,\ \text{and } e^{-\lambda N}\epsilon^*<2\delta. \tag{4.55}$$

From now on, we fix this constant N. Next, we are going to show that the diameter of image of $L^N_{\theta^{-N}\omega}:C_{\theta^{-N}\omega}(b,c,\nu)\to C_\omega(b,c,\nu)$ with respect to the projective metric on $C_\omega(b,c,\nu)$ is finite, and this finite diameter is independent of $\omega\in\Omega$, where we recall

$$L^N_{\theta^{-N}\omega}=L_{\theta^{-1}\omega}\circ\cdots\circ L_{\theta^{-(N-1)}\omega}\circ L_{\theta^{-N}\omega}.$$

Lemma 4.6. *There exists a constant $D_2 = D_2(\lambda_2, a, b, c, N) > 0$ such that for any $\omega \in \Omega$,*

$$\sup\left\{d_\omega(\varphi_1, \varphi_2) : \varphi_1, \varphi_2 \in L^N_{\theta^{-N}\omega} C_{\theta^{-N}\omega}(b, c, \nu)\right\} \le D_2 < \infty, \tag{4.56}$$

where d_ω is the projective metric on $C_\omega(b, c, \nu)$.

Proof of Lemma 4.6. By Lemma 4.5, $L_{\theta^{-1}\omega}(C_{\theta^{-1}\omega}(b, c, \nu)) \subset C_\omega(\lambda_2 b, \lambda_2 c, \nu) \subset C_\omega(b, c, \nu)$, for all $\omega \in \Omega$. For any $\varphi_1, \varphi_2 \in L^N_{\theta^{-N}\omega} C_{\theta^{-N}\omega}(b, c, \nu) \subset C_\omega(\lambda_2 b, \lambda_2 c, \nu)$, we need to estimate $\alpha_\omega(\varphi_1, \varphi_2)$ and $\beta_\omega(\varphi_1, \varphi_2)$ as in (4.30) and (4.31).

By (4.32) and condition **(C2)**, for any $\rho'(\omega), \rho''(\omega) \in D(a/2, \kappa, \gamma(\omega))$ with $\int_{\gamma(\omega)} \rho^\tau(\omega) dm_{\gamma(\omega)} = 1$ for $\tau = \prime, \prime\prime$, one has

$$\begin{aligned}
\xi_\omega(\rho', \rho'', \varphi_1, \varphi_2) &= \frac{\exp(bd^{a,\kappa}_{\gamma(\omega)}(\rho'(\omega), \rho''(\omega))) - \int_{\gamma(\omega)} \varphi_2 \rho'' dm_{\gamma(\omega)} / \int_{\gamma(\omega)} \varphi_2 \rho' dm_{\gamma(\omega)}}{\exp(bd^{a,\kappa}_{\gamma(\omega)}(\rho'(\omega), \rho''(\omega))) - \int_{\gamma(\omega)} \varphi_1 \rho'' dm_{\gamma(\omega)} / \int_{\gamma(\omega)} \varphi_1 \rho' dm_{\gamma(\omega)}} \\
&\ge \frac{\exp(bd^{a,\kappa}_{\gamma(\omega)}(\rho'(\omega), \rho''(\omega))) - \exp(b\lambda_2 d^{a,\kappa}_{\gamma(\omega)}(\rho'(\omega), \rho''(\omega)))}{\exp(bd^{a,\kappa}_{\gamma(\omega)}(\rho'(\omega), \rho''(\omega))) - \exp(-b\lambda_2 d^{a,\kappa}_{\gamma(\omega)}(\rho'(\omega), \rho''(\omega)))} \\
&\ge \tau_3,
\end{aligned}$$

where $\tau_3 := \inf\{\frac{z - z^{\lambda_2}}{z - z^{-\lambda_2}} : z > 1\} = \frac{1-\lambda_2}{1+\lambda_2} \in (0, 1)$. Similarly, we have

$$\xi_\omega(\rho', \rho'', \varphi_1, \varphi_2) \le \tau_4$$

for $\tau_4 = \sup\{\frac{z - z^{-\lambda_2}}{z - z^{\lambda_2}} : z > 1\} = \frac{1+\lambda_2}{1-\lambda_2} \in (1, \infty)$. Likewise, $\eta_\omega(\rho, \tilde{\rho}, \varphi_1, \varphi_2), \eta_\omega(\tilde{\rho}, \rho, \varphi_1, \varphi_2) \in [\tau_3, \tau_4]$ for any $\rho(\omega) \in D(a_1, \kappa_1, \gamma(\omega))$ and corresponding $\tilde{\rho}(\omega) \in D(a/2, \kappa, \tilde{\gamma}(\omega))$, where $\eta_\omega(\rho, \tilde{\rho}, \varphi_1, \varphi_2)$ and $\eta_\omega(\tilde{\rho}, \rho, \varphi_1, \varphi_2)$ are defined in (4.33) and (4.34) respectively.

Let $C_{+,\omega}$ be the collection of all bounded measurable functions $\varphi : M \to \mathbb{R}$ only satisfying condition **(C1)**, i.e., $\int_{\gamma(\omega)} \varphi(x)\rho(x, \omega) dm_{\gamma(\omega)}(x) > 0$ for every local stable submanifold $\gamma(\omega) \subset M$ having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, and every $\rho(\omega) \in D(a/2, \kappa, \gamma(\omega))$ satisfying $\int_{\gamma(\omega)} \rho(x, \omega) dm_{\gamma(\omega)}(x) = 1$. Next, we introduce the projective metric on $C_{+,\omega}$. For $\varphi_1, \varphi_2 \in C_{+,\omega}$, we define

$$\begin{aligned}
\alpha_{+,\omega}(\varphi_1, \varphi_2) &:= \sup\{t > 0 : \varphi_2 - t\varphi_1 \in C_{+,\omega}\}, \\
\beta_{+,\omega}(\varphi_1, \varphi_2) &:= \inf\{s > 0 : s\varphi_1 - \varphi_2 \in C_{+,\omega}\},
\end{aligned}$$

with the convention that $\sup \emptyset = 0$ and $\inf \emptyset = +\infty$, and let

$$d_{+,\omega}(\varphi_1, \varphi_2) := \log \frac{\beta_{+,\omega}(\varphi_1, \varphi_2)}{\alpha_{+,\omega}(\varphi_1, \varphi_2)} \tag{4.57}$$

with the convention that $d_{+,\omega}(\varphi_1, \varphi_2) = \infty$ if $\alpha_{+,\omega}(\varphi_1, \varphi_2) = 0$ or $\beta_{+,\omega}(\varphi_1, \varphi_2) = \infty$. By similar computation as before, we have

$$\alpha_{+,\omega}(\varphi_1,\varphi_2)=\inf\left\{\frac{\int_{\gamma(\omega)}\varphi_2\rho(\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\varphi_1\rho(\omega)dm_{\gamma(\omega)}}\right\}, \tag{4.58}$$

and

$$\beta_{+,\omega}(\varphi_1,\varphi_2)=\sup\left\{\frac{\int_{\gamma(\omega)}\varphi_2\rho(\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\varphi_1\rho(\omega)dm_{\gamma(\omega)}}\right\}, \tag{4.59}$$

where the supremum and infimum runs over all local stable leaf $\gamma(\omega)$ having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, and any $\rho(\omega)\in D(\frac{a}{2},\kappa,\gamma(\omega))$ with $\int_{\gamma(\omega)}\rho(x,\omega)dm_{\gamma(\omega)}=1$.

Comparing (4.30) with (4.58) and (4.31) with (4.59), and noticing that $D(a_1,\kappa_1,\gamma(\omega))\subset D(\frac{a}{2},\kappa,\gamma(\omega))$, we have for $\varphi_1,\varphi_2\in L^N_{\theta^{-N}\omega}C_{\theta^{-N}\omega}(b,c,\nu)\subset C_\omega(\lambda_2 b,\lambda_2 c,\nu)$,

$$\alpha_\omega(\varphi_1,\varphi_2)\geq\tau_3\alpha_{+,\omega}(\varphi_1,\varphi_2),\ \text{and}\ \beta_\omega(\varphi_1,\varphi_2)\leq\tau_4\beta_{+,\omega}(\varphi_1,\varphi_2).$$

As a consequence, we have

$$d_\omega(\varphi_1,\varphi_2)\leq d_{+,\omega}(\varphi_1,\varphi_2)+\log\frac{\tau_4}{\tau_3}. \tag{4.60}$$

Note that for $\varphi_1,\varphi_2\in L^N_{\theta^{-N}\omega}C_{\theta^{-N}\omega}(b,c,\nu)$, we have

$$\begin{aligned} d_{+,\omega}(\varphi_1,\varphi_2)&=\log\sup\left\{\frac{\int_{\gamma''(\omega)}\varphi_2\rho''(\omega)dm_{\gamma''(\omega)}/\int_{\gamma''(\omega)}\varphi_1\rho''(\omega)dm_{\gamma''(\omega)}}{\int_{\gamma'(\omega)}\varphi_2\rho'(\omega)dm_{\gamma'(\omega)}/\int_{\gamma'(\omega)}\varphi_1\rho'(\omega)dm_{\gamma'(\omega)}}\right\} \\ &=\log\sup\left\{\frac{\int_{\gamma''(\omega)}\varphi_2\rho''(\omega)dm_{\gamma''(\omega)}}{\int_{\gamma'(\omega)}\varphi_2\rho'(\omega)dm_{\gamma'(\omega)}}\cdot\frac{\int_{\gamma'(\omega)}\varphi_1\rho'(\omega)dm_{\gamma'(\omega)}}{\int_{\gamma''(\omega)}\varphi_1\rho''(\omega)dm_{\gamma''(\omega)}}\right\}, \end{aligned} \tag{4.61}$$

where the sup runs over all random local stable leaves $\gamma^\tau(\omega)$ having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, $\rho^\tau(\omega)\in D(\frac{a}{2},\kappa,\gamma^\tau(\omega))$ with $\int_{\gamma^\tau(\omega)}\rho^\tau(x,\omega)dm_{\gamma^\tau(\omega)}=1$ for $\tau={}',{}''$. Next, we are going to estimate

$$\frac{\int_{\gamma''(\omega)}\varphi(x)\rho''(x,\omega)dm_{\gamma''(\omega)}(x)}{\int_{\gamma'(\omega)}\varphi(x)\rho'(x,\omega)dm_{\gamma'(\omega)}(x)} \tag{4.62}$$

for $\varphi\in L^N_{\theta^{-N}\omega}C_{\theta^{-N}\omega}(b,c,\nu)$ and any $\rho'(\omega)\in D(a/2,\kappa,\gamma'(\omega))$, $\rho''(\omega)\in D(a/2,\kappa,\gamma''(\omega))$ with $\int_{\gamma''(\omega)}\rho''(\omega)dm_{\gamma''(\omega)}=\int_{\gamma'(\omega)}\rho'(\omega)dm_{\gamma'(\omega)}=1$. For $\varphi\in L^N_{\theta^{-N}\omega}C_{\theta^{-N}\omega}(b,c,\nu)$, let

$$\bar{k}_1(\omega)=\left(\int\limits_{\gamma'(\omega)}\varphi(x)dm_{\gamma'(\omega)}(x)\right)^{-1},\ \bar{k}_2(\omega)=\left(\int\limits_{\gamma''(\omega)}\varphi(x)dm_{\gamma''(\omega)}(x)\right)^{-1}, \tag{4.63}$$

and we define

$$k_1(x,\omega) := \bar{k}_1(\omega) / \int\limits_{\gamma'(\omega)} \bar{k}_1(\omega) dm_{\gamma'(\omega)} \text{ for } x \in \gamma'(\omega),$$

$$k_2(x,\omega) := \bar{k}_2(\omega) / \int\limits_{\gamma''(\omega)} \bar{k}_2(\omega) dm_{\gamma''(\omega)} \text{ for } x \in \gamma''(\omega).$$

By construction, the constant function $k_1(\omega) \in D(a/2, \kappa, \gamma'(\omega))$ and $k_2(\omega) \in D(a/2, \kappa, \gamma''(\omega))$ and $\int_{\gamma'(\omega)} k_1(x,\omega) dm_{\gamma'(\omega)} = 1$ and $\int_{\gamma''(\omega)} k_2(x,\omega) dm_{\gamma''(\omega)} = 1$. Now by **(C2)**,

$$\begin{aligned}
&\frac{\int_{\gamma''(\omega)} \varphi(x)\rho''(x,\omega) dm_{\gamma''(\omega)}(x)}{\int_{\gamma'(\omega)} \varphi(x)\rho'(x,\omega) dm_{\gamma'(\omega)}(x)} \\
&= \frac{\int_{\gamma''(\omega)} \varphi(x)\rho''(x,\omega) dm_{\gamma''(\omega)}(x)}{\int_{\gamma''(\omega)} \varphi(x) k_2(x,\omega) dm_{\gamma''(\omega)}(x)} \cdot \frac{\int_{\gamma'(\omega)} \varphi(x) k_1(x,\omega) dm_{\gamma'(\omega)}(x)}{\int_{\gamma'(\omega)} \varphi(x)\rho'(x,\omega) dm_{\gamma'(\omega)}(x)} \cdot \frac{\int_{\gamma'(\omega)} \bar{k}_1(\omega) dm_{\gamma'(\omega)}}{\int_{\gamma''(\omega)} \bar{k}_2(\omega) dm_{\gamma''(\omega)}} \\
&\le e^{\lambda_2 b d^{a,\kappa}_{\gamma''(\omega)}(\rho''(\cdot,\omega), k_2(\cdot,\omega))} \cdot e^{\lambda_2 b d^{a,\kappa}_{\gamma'(\omega)}(\rho'(\cdot,\omega), k_1(\cdot,\omega))} \cdot \frac{\int_{\gamma'(\omega)} \bar{k}_1(\omega) dm_{\gamma'(\omega)}}{\int_{\gamma''(\omega)} \bar{k}_2(\omega) dm_{\gamma''(\omega)}}.
\end{aligned} \tag{4.64}$$

Sublemma 4.2. *There exists a constant $D_1 = D_1(a,b,c,N) < \infty$ such that*

$$\frac{\int_{\gamma'(\omega)} \bar{k}_1(\omega) dm_{\gamma'(\omega)}}{\int_{\gamma''(\omega)} \bar{k}_2(\omega) dm_{\gamma''(\omega)}} \le D_1. \tag{4.65}$$

We postpone the proof of Sublemma 4.2 and finish the proof of Lemma 4.6. Now we let $\tau_5 = \sup\{\frac{z - z^{-1/2}}{z - z^{1/2}} : z > 1\} \in (1, \infty)$ and $\tau_6 = \inf\{\frac{z - z^{1/2}}{z - z^{-1/2}} : z > 1\} \in (0, 1)$. Similar to (4.27), the diameter of $D(a/2, \kappa, \gamma(\omega))$ with respect to the projective metric on $D(a, \kappa, \gamma(\omega))$ is bounded by $4a + \log \tau_5/\tau_6$. Therefore,

$$d^{a,\kappa}_{\gamma''(\omega)}(\rho''(\omega), \bar{k}_2(\omega)) \le 4a + \log \tau_5/\tau_6, \text{ and } d^{a,\kappa}_{\gamma'(\omega)}(\rho'(\omega), \bar{k}_1(\omega)) \le 4a + \log \tau_5/\tau_6.$$

Hence

$$\frac{\int_{\gamma''(\omega)} \varphi(x)\rho''(x,\omega) dm_{\gamma''(\omega)}(x)}{\int_{\gamma'(\omega)} \varphi(x)\rho'(x,\omega) dm_{\gamma'(\omega)}(x)} \overset{(4.64)}{\le} e^{2\lambda_2 b(4a + \log \tau_5/\tau_6)} D_1.$$

As a consequence, we have

$$d_{+,\omega}(\varphi_1, \varphi_2) \overset{(4.61)}{\le} \log(e^{4\lambda_2 b(4a + \log \tau_5/\tau_6)} D_1^2),$$

and $d_\omega(\varphi_1, \varphi_2) \overset{(4.60)}{\le} \log(e^{4\lambda_2 b(4a + \log \tau_5/\tau_6)} D_1^2) + \log \tau_4/\tau_3 := D_2$, which finishes the proof of Lemma 4.6. □

Proof of Sublemma 4.2. For any $\omega \in \Omega$ and $\psi \in C_\omega(b, c, \nu)$, we define

$$\|\psi\|_{\omega,+} = \sup \frac{\int_{\gamma(\omega)} \psi(x)\rho(x,\omega)dm_{\gamma(\omega)}(x)}{\int_{\gamma(\omega)} \rho(x,\omega)dm_{\gamma(\omega)}(x)}, \ \|\psi\|_{\omega,-} = \inf \frac{\int_{\gamma(\omega)} \psi(x)\rho(x,\omega)dm_{\gamma(\omega)}(x)}{\int_{\gamma(\omega)} \rho(x,\omega)dm_{\gamma(\omega)}(x)},$$

where the supremum and infimum runs over all local stable leaf $\gamma(\omega) \subset M$ having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, and any $\rho(\omega) \in D(a_1, \kappa, \gamma(\omega))$.

By noticing that constant function $\mathbf{1}$ belongs to $D(a_1, \kappa, \gamma(\omega))$, we have

$$\frac{\int_{\gamma'(\omega)} \bar{k}_1(\omega)dm_{\gamma'(\omega)}}{\int_{\gamma''(\omega)} \bar{k}_2(\omega)dm_{\gamma''(\omega)}} = \left(\frac{\int_{\gamma''(\omega)} \varphi \cdot \mathbf{1}dm_{\gamma''(\omega)}}{\int_{\gamma''(\omega)} \mathbf{1}dm_{\gamma''(\omega)}}\right) \Big/ \left(\frac{\int_{\gamma'(\omega)} \varphi \cdot \mathbf{1}dm_{\gamma'(\omega)}}{\int_{\gamma''(\omega)} \mathbf{1}dm_{\gamma'(\omega)}}\right) \leq \frac{\|\varphi\|_{\omega,+}}{\|\varphi\|_{\omega,-}},$$

and here $\bar{k}_1(\omega)$ and $\bar{k}_2(\omega)$ was defined in (4.63) by using $\varphi \in L^N_{\theta^{-N}\omega} C_{\theta^{-N}\omega}(b, c, \nu) \subset C_\omega(b, c, \nu)$. Therefore, Lemma 4.2 is proved if there exists $D_1 = D_1(a, b, c, N) > 0$ independent of $\omega \in \Omega$ such that

$$\frac{\|L^N_{\theta^{-N}\omega}\varphi\|_{\omega,+}}{\|L^N_{\theta^{-N}\omega}\varphi\|_{\omega,-}} \leq D_1 \text{ for any } \varphi \in C_{\theta^{-N}\omega}(b, c, \nu). \tag{4.66}$$

We need some preliminary inequalities before we start the proof of (4.66). By the continuity of f_ω in ω, there exists a constant $K_6 > 0$ such that

$$K_6^{-1} \leq |\det D_x f_\omega| \leq K_6 \text{ for all } (x, \omega) \in M \times \Omega. \tag{4.67}$$

We recall that for $\varphi \in C_{\theta^{-n}\omega}(b, c, \nu)$,

$$(L^n_{\theta^{-n}\omega}\varphi)(x) = \frac{\varphi(f_\omega^{-n}x)}{|\det D_{f_\omega^{-n}x} f^n_{\theta^{-n}\omega}|}.$$

Then (4.67) implies that

$$(K_6)^{-n} \leq \inf_{x\in M} |(L^n_{\theta^{-n}\omega}\mathbf{1})(x)| \leq \sup_{x\in M} |(L^n_{\theta^{-n}\omega}\mathbf{1})(x)| \leq (K_6)^n \text{ for any } \omega \in \Omega, \ n \in \mathbb{N}. \tag{4.68}$$

For any $n \geq 1$ and $\gamma(\omega) \subset M$ local stable leaf having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, we divide $f_\omega^{-n}\gamma(\omega)$ into finite pieces of connected local stable leaves having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, named $\gamma_i(\theta^{-n}\omega)$ for i belonging to a finite index set such that $\gamma_i(\theta^{-n}\omega) \cap \gamma_j(\theta^{-n}\omega) = \partial\gamma_i(\theta^{-n}\omega) \cap \partial\gamma_j(\theta^{-n}\omega)$ for $i \neq j$. For any $\rho(\omega) \in D(a_1, \kappa, \gamma(\omega))$, we define

$$\rho_i(x, \theta^{-n}\omega) = \frac{|\det D_x f^n_{\theta^{-n}\omega}|_{E^s(x,\theta^{-n}\omega)}|}{|\det D_x f^n_{\theta^{-n}\omega}|}\rho(f^n_{\theta^{-n}\omega}x, \omega) \text{ for } x \in \gamma_i(\theta^{-n}\omega).$$

By Remark 4.1, we have $\rho_i(\theta^{-n}\omega) \in D(a_1, \kappa, \gamma_i(\theta^{-n}\omega))$. Likewise (4.16), we have

$$\int_{\gamma(\omega)} (L^n_{\theta^{-n}\omega}\varphi)(x)\rho(x,\omega)dm_{\gamma(\omega)}(x) = \sum_i \int_{\gamma_i(\theta^{-n}\omega)} \varphi(x)\rho_i(x,\theta^{-n}\omega)dm_{\gamma_i(\theta^{-n}\omega)}(x).$$

Now we have

$$\begin{aligned}
&\frac{\int_{\gamma(\omega)}(L^n_{\theta^{-n}\omega}\varphi)(x)\rho(x,\omega)dm_{\gamma(\omega)}(x)}{\int_{\gamma(\omega)}\rho(x,\omega)dm_{\gamma(\omega)}(x)} = \frac{\sum_i \int_{\gamma_i(\theta^{-n}\omega)}\varphi(x)\rho_i(x,\theta^{-n}\omega)dm_{\gamma_i(\theta^{-n}\omega)}(x)}{\int_{\gamma(\omega)}\rho(x,\omega)dm_{\gamma(\omega)}(x)} \\
&\le \frac{\sum_i \int_{\gamma_i(\theta^{-n}\omega)}\rho_i(x,\theta^{-n}\omega)dm_{\gamma_i(\theta^{-n}\omega)}(x)\cdot\|\varphi\|_{\theta^{-n}\omega,+}}{\int_{\gamma(\omega)}\rho(x,\omega)dm_{\gamma(\omega)}(x)} \\
&= \frac{\int_{\gamma(\omega)}(L^n_{\theta^{-n}\omega}\mathbf{1})(x)\rho(x,\omega)dm_{\gamma(\omega)}(x)}{\int_{\gamma(\omega)}\rho(x,\omega)dm_{\gamma(\omega)}(x)}\cdot\|\varphi\|_{\theta^{-n}\omega,+} \\
&\overset{(4.68)}{\le} (K_6)^n\cdot\|\varphi\|_{\theta^{-n}\omega,+},
\end{aligned}$$

where the last inequality holds since $\rho(x,\omega) > 0$ for all $x \in \gamma(\omega)$ by **(D1)**. Since $\gamma(\omega)$ and $\rho(\omega) \in D(a_1,\kappa,\gamma(\omega))$ are arbitrary, we get

$$\|L^n_{\theta^{-n}\omega}\varphi\|_{\omega,+} \le (K_6)^n\|\varphi\|_{\theta^{-n}\omega,+}. \tag{4.69}$$

Next, there exists a constant $D_3 > 0$ such that for any $\omega \in \Omega$, any $\gamma(\omega) \subset M$ local stable leaf having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, $\rho_1(\omega), \rho_2(\omega) \in D(a/2,\kappa,\gamma(\omega))$, any $\varphi \in C_\omega(b,c,\nu)$, one has

$$\sup_{z\in\gamma(\omega)} \frac{\rho_2(z,\omega)\int_{\gamma(\omega)}\varphi(x)\rho_1(x,\omega)dm_{\gamma(\omega)}(x)}{\rho_1(z,\omega)\int_{\gamma(\omega)}\varphi(x)\rho_2(x,\omega)dm_{\gamma(\omega)}(x)} \le D_3 < \infty. \tag{4.70}$$

In fact, for any $z \in \gamma(\omega)$, then by condition **(C2)** and the fact that the diameter of $D(a/2,\kappa,\gamma(\omega))$ in $D(a,\kappa,\gamma(\omega))$ with respect to the projective metric $d^{a,\kappa}_{\gamma(\omega)}$ is bounded by $4a + \log(\tau_5/\tau_6)$, we have

$$\begin{aligned}
&\frac{\rho_2(z,\omega)}{\rho_1(z,\omega)}\cdot\frac{\int_{\gamma(\omega)}\varphi(x)\rho_1(x,\omega)dm_{\gamma(\omega)}(x)/\int_{\gamma(\omega)}\rho_1(\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\varphi(x)\rho_2(x,\omega)dm_{\gamma(\omega)}(x)/\int_{\gamma(\omega)}\rho_2(\omega)dm_{\gamma(\omega)}}\cdot\frac{\int_{\gamma(\omega)}\rho_1(\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\rho_2(\omega)dm_{\gamma(\omega)}} \\
&\le \frac{\rho_2(z,\omega)}{\int_{\gamma(\omega)}\rho_2(x,\omega)dm_{\gamma(\omega)}(x)}\cdot e^{bd^{a,\kappa}_{\gamma(\omega)}(\rho_1(\omega),\rho_2(\omega))}\cdot\frac{\int_{\gamma(\omega)}\rho_1(x,\omega)dm_{\gamma(\omega)}(x)}{\rho_1(z,\omega)} \\
&\le \frac{\rho_2(z,\omega)}{\int_{\gamma(\omega)}\rho_2(x,\omega)dm_{\gamma(\omega)}(x)}\cdot e^{(4a+\log\tau_5/\tau_6)b}\cdot\frac{\int_{\gamma(\omega)}\rho_1(x,\omega)dm_{\gamma(\omega)}(x)}{\rho_1(z,\omega)} \\
&\le e^{a/2(diam(\gamma(\omega)))^\kappa}\cdot e^{a/2(diam(\gamma(\omega)))^\kappa}\cdot e^{(4a+\log\tau_5/\tau_6)b} \\
&= e^{a+b(4a+\log\tau_5/\tau_6)} := D_3.
\end{aligned}$$

Now we are in the position to prove (4.66). Recall that $N \in \mathbb{N}$ satisfying (4.54) and (4.55). For any $\omega \in \Omega$ and $\varphi \in C_{\theta^{-N}\omega}(b, c, \nu)$, we choose $\gamma_*(\theta^{-N}\omega)$ a local stable leaf having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, and $\rho_*(\theta^{-N}\omega) \in D(a_1, \kappa, \gamma_*(\theta^{-N}\omega))$ such that

$$\frac{\int_{\gamma_*(\theta^{-N}\omega)} \varphi(x)\rho_*(x, \theta^{-N}\omega) dm_{\gamma_*(\theta^{-N}\omega)}}{\int_{\gamma_*(\theta^{-N}\omega)} \rho_*(x, \theta^{-N}\omega) dm_{\gamma_*(\theta^{-N}\omega)}} \geq \frac{1}{2}\|\varphi\|_{\theta^{-N}\omega,+}.$$

We pick $x_*(\theta^{-N}\omega) \in \gamma_*(\theta^{-N}\omega)$ such that

$$W^s_{\epsilon^*}(x_*(\theta^{-N}\omega), \theta^{-N}\omega) \subset \gamma_*(\theta^{-N}\omega) \subset W^s_{\epsilon}(x_*(\theta^{-N}\omega), \theta^{-N}\omega).$$

To avoid the size of the holonomy image of $\gamma_*(\theta^{-N}\omega)$ being too large, we break $\gamma_*(\theta^{-N}\omega)$ into pieces of size between $\frac{A(\epsilon)}{4J}$ and $\frac{A(\epsilon)}{J}$. The number of pieces is at most $\frac{A(\epsilon)}{A(\epsilon)/J} + 1 = J + 1$. We can pick one of pieces, named $\gamma^1_*(\theta^{-N}\omega)$, such that

$$\frac{\int_{\gamma^1_*(\theta^{-N}\omega)} \varphi(x)\rho_*(x, \theta^{-N}\omega) dm_{\gamma_*(\theta^{-N}\omega)}}{\int_{\gamma_*(\theta^{-N}\omega)} \rho_*(x, \theta^{-N}\omega) dm_{\gamma_*(\theta^{-N}\omega)}} \geq \frac{1}{2(J+1)}\|\varphi\|_{\theta^{-N}\omega,+}. \tag{4.71}$$

Pick any $\gamma(\omega) \subset M$ local stable leaf having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, and $x(\omega) \in \gamma(\omega)$ such that

$$W^s_{\epsilon^*}(x(\omega), \omega) \subset \gamma(\omega) \subset W^s_{\epsilon}(x(\omega), \omega).$$

Recall that $\{B_{\delta/4}(x_i)\}_{i=1}^l$ is a cover of M. Then there exists $i, j \in \{1, ..., l\}$ such that $x_*(\theta^{-N}\omega) \in B_{\delta/4}(x_i)$ and $x(\omega) \in B_{\delta/4}(x_j)$. Then by the choice of N, $\phi^N(B_{\delta/4}(x_i) \times \{\theta^{-N}\omega\}) \cap (B_{\delta/4}(x_j) \times \{\omega\}) \neq \emptyset$. Pick $y(\omega) \in f^N_{\theta^{-N}\omega} B_{\delta/4}(x_i) \cap B_{\delta/4}(x_j)$, then

$$d(y(\omega), x(\omega)) \leq d(y(\omega), x_j) + d(x_j, x(\omega)) \leq \delta/4 + \delta/4 < \delta.$$

Then

$$y_1(\omega) := W^u_{\epsilon^*/8}(y(\omega), \omega) \cap W^s_{\epsilon^*/8}(x(\omega), \omega) \subset W^u_{\epsilon^*/8}(y(\omega), \omega) \cap \gamma(\omega)$$

exists. Note that by (4.55), we have

$$d(f^{-N}_{\omega} y(\omega), f^{-N}_{\omega} y_1(\omega)) \leq e^{-\lambda N} \cdot \frac{\epsilon^*}{8} \leq \delta/4.$$

So

$$\begin{aligned} d(x_*(\theta^{-N}\omega), f^{-N}_{\omega} y_1(\omega)) &\leq d(x_*(\theta^{-N}\omega), x_i) + d(x_i, f^{-N}_{\omega} y(\omega)) + d(f^{-N}_{\omega} y(\omega), f^{-N}_{\omega} y_1(\omega)) \\ &\leq \delta/4 + \delta/4 + \delta/4 < \delta. \end{aligned}$$

As a consequence,

$$W^s_{\epsilon^*/8}(x_*(\theta^{-N}\omega),\theta^{-N}\omega)\cap W^u_{\epsilon^*/8}(f_\omega^{-N}y_1(\omega),\theta^{-N}\omega)\neq\emptyset. \tag{4.72}$$

Note that $f_\omega^{-N}y_1(\omega)\in f_\omega^{-N}\gamma(\omega)$, and $e^{\lambda N}\epsilon^*/8\overset{(4.55)}{\geq}3\epsilon$, therefore, $W^s_{3\epsilon}(f_\omega^{-N}y_1(\omega),\theta^{-N}\omega)\subset f_\omega^{-N}\gamma(\omega)$. Moreover, (4.72) implies that $\gamma_*(\omega)$ is sufficient close to $W^s_{3\epsilon}(f_\omega^{-N}y_1(\omega),\theta^{-N}\omega)\subset f_\omega^{-N}\gamma(\omega)$. As a consequence, $f_\omega^{-N}\gamma(\omega)$ contains a holonomy image of $\gamma_*^1(\theta^{-N}\omega)$, named $\gamma_1(\theta^{-N}\omega)$. Since the size of $\gamma_*^1(\theta^{-N}\omega)$ is between $\frac{A(\epsilon)}{4J}$ and $\frac{A(\epsilon)}{J}$, by holonomy, the size of $\gamma_1(\theta^{-N}\omega)$ is between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$.

Now for any $\rho(\omega)\in D(a_1,\kappa,\gamma(\omega))$, define

$$\rho_1(x,\theta^{-N}\omega)=\frac{|\det D_xf^N_{\theta^{-N}\omega}|_{E^s(x,\theta^{-N}\omega)}|}{|\det D_xf^N_{\theta^{-N}\omega}|}\rho(f^N_{\theta^{-N}\omega}x,\omega)\text{ for }x\in\gamma_1(\theta^{-N}\omega).$$

By Remark 4.1, $\rho_1(\theta^{-N}\omega)\in D(e^{-\lambda_1}a_1,\kappa,\gamma_1(\theta^{-N}\omega))\subset D(a_1,\kappa,\gamma_1(\theta^{-N}\omega))$. Let $\tilde\rho_1(\theta^{-N}\omega)$ be the density function on $\gamma_*^1(\theta^{-N}\omega)$ defined as (4.17) corresponding to $\rho_1(\theta^{-N}\omega)$, and therefore $\tilde\rho_1(\theta^{-N}\omega)\in D(a/2,\kappa,\gamma_*^1(\theta^{-N}\omega))$ by Lemma 4.2. Then

$$\begin{aligned}&\frac{\int_{\gamma(\omega)}(L^N_{\theta^{-N}\omega}\varphi)(x)\rho(x,\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\rho(x,\omega)dm_{\gamma(\omega)}}\\
&\geq\frac{\int_{\gamma_1(\theta^{-N}\omega)}\varphi(x)\rho_1(x,\theta^{-N}\omega)dm_{\gamma_1(\theta^{-N}\omega)}}{\int_{\gamma(\omega)}\rho(x,\omega)dm_{\gamma(\omega)}}\\
&\overset{\textbf{(C3)}}{\geq}\frac{\int_{\gamma_*^1(\theta^{-N}\omega)}\varphi(x)\tilde\rho_1(x,\theta^{-N}\omega)dm_{\gamma_*(\theta^{-N}\omega)}}{\int_{\gamma(\omega)}\rho(x,\omega)dm_{\gamma(\omega)}}\cdot e^{-cd_u(\gamma_1(\theta^{-N}\omega),\gamma_*(\theta^{-N}\omega))^\nu}.\end{aligned}$$

Pick any $z\in\gamma_*^1(\theta^{-N}\omega)$, and we note that $\rho_*(\theta^{-N}\omega)\in D(a_1,\kappa,\gamma_*^1(\theta^{-N}\omega))\subset D(a/2,\kappa,\gamma_*^1(\theta^{-N}\omega))$ and $\tilde\rho_1(\theta^{-N}\omega)\in D(a/2,\kappa,\gamma_*^1(\theta^{-N}\omega))$, then we continue the estimate

$$\begin{aligned}&\frac{\int_{\gamma(\omega)}(L^N_{\theta^{-N}\omega}\varphi)(x)\rho(x,\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\rho(x,\omega)dm_{\gamma(\omega)}}\\
&\overset{(4.70)}{\geq}\frac{\int_{\gamma_*^1(\theta^{-N}\omega)}\varphi(x)\rho_*(x,\theta^{-N}\omega)dm_{\gamma_*(\theta^{-N}\omega)}}{\int_{\gamma(\omega)}\rho(x,\omega)dm_{\gamma(\omega)}}\cdot e^{-cd_u(\gamma_1(\theta^{-N}\omega),\gamma_*^1(\theta^{-N}\omega))^\nu}\cdot D_3^{-1}\cdot\frac{\tilde\rho_1(z,\theta^{-N}\omega)}{\rho_*(z,\theta^{-N}\omega)}\\
&\overset{(4.71)}{\geq}\frac{\int_{\gamma_*^1(\theta^{-N}\omega)}\rho_*(x,\theta^{-N}\omega)dm_{\gamma_*(\theta^{-N}\omega)}\frac{1}{2(J+1)}\|\varphi\|_{\theta^{-N}\omega,+}}{\int_{\gamma(\omega)}\rho(x,\omega)dm_{\gamma(\omega)}}\cdot e^{-cd_u(\gamma_1(\theta^{-N}\omega),\gamma_*^1(\theta^{-N}\omega))^\nu}\cdot D_3^{-1}\\
&\quad\cdot\frac{\tilde\rho_1(z,\theta^{-N}\omega)}{\rho_*(z,\theta^{-N}\omega)}\\
&\overset{(4.70)}{\geq}\frac{\int_{\gamma_*^1(\theta^{-N}\omega)}\tilde\rho_1(x,\theta^{-N}\omega)dm_{\gamma_*(\theta^{-N}\omega)}\|\varphi\|_{\theta^{-N}\omega,+}}{2(J+1)\int_{\gamma(\omega)}\rho(x,\omega)dm_{\gamma(\omega)}}\cdot e^{-cd_u(\gamma_1(\theta^{-N}\omega),\gamma_*^1(\theta^{-N}\omega))^\nu}\cdot D_3^{-2}\\
&\overset{(4.18)}{=}\frac{1}{2(J+1)}e^{-cd_u(\gamma_1(\theta^{-N}\omega),\gamma_*^1(\theta^{-N}\omega))^\nu}\cdot D_3^{-2}\cdot\|\varphi\|_{\theta^{-N}\omega,+}\cdot\frac{\int_{\gamma_1(\theta^{-N}\omega)}\rho_1(x,\theta^{-N}\omega)dm_{\gamma_1(\theta^{-N}\omega)}}{\int_{\gamma(\omega)}\rho(x,\omega)dm_{\gamma(\omega)}}\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{2(J+1)}e^{-cd_u(\gamma_1(\theta^{-N}\omega),\gamma_*^1(\theta^{-N}\omega))^\nu}\cdot D_3^{-2}\cdot\|\varphi\|_{\theta^{-N}\omega,+}\\
&\cdot\frac{\int_{f^N_{\theta^{-N}\omega}\gamma_1(\theta^{-N}\omega)}(L^N_{\theta^{-N}\omega}1)(x)\rho(x,\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\rho(x,\omega)dm_{\gamma(\omega)}}\\
\overset{(4.68)}{\geq}&\frac{1}{2(J+1)}e^{-cd_u(\gamma_1(\theta^{-N}\omega),\gamma_*^1(\theta^{-N}\omega))^\nu}\cdot D_3^{-2}\cdot\|\varphi\|_{\theta^{-N}\omega,+}\cdot(K_6)^{-N}\\
&\cdot\frac{\int_{f^N_{\theta^{-N}\omega}\gamma_1(\theta^{-N}\omega)}\rho(x,\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\rho(x,\omega)dm_{\gamma(\omega)}}.
\end{aligned}
$$

Pick any $t\in f^N_{\theta^{-N}\omega}\gamma_1(\theta^{-N}\omega)\subset\gamma(\omega)$, then we continue the estimate

$$
\begin{aligned}
&\frac{\int_{\gamma(\omega)}(L^N_{\theta^{-N}\omega}\varphi)(x)\rho(x,\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\rho(x,\omega)dm_{\gamma(\omega)}}\\
\geq&\frac{e^{-cd_u(\gamma_1(\theta^{-N}\omega),\gamma_*^1(\theta^{-N}\omega))^\nu}}{2(J+1)}\cdot D_3^{-2}\cdot\|\varphi\|_{\theta^{-N}\omega,+}\cdot(K_6)^{-N}\\
&\cdot\frac{\int_{f^N_{\theta^{-N}\omega}\gamma_1(\theta^{-N}\omega)}\rho(x,\omega)/\rho(t,\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\rho(x,\omega)/\rho(t,\omega)dm_{\gamma(\omega)}}\\
\geq&\frac{e^{-cd_u(\gamma_1(\theta^{-N}\omega),\gamma_*^1(\theta^{-N}\omega))^\nu}}{2(J+1)}\cdot D_3^{-2}\cdot\|\varphi\|_{\theta^{-N}\omega,+}\cdot(K_6)^{-N}\cdot e^{-a_1(diam(\gamma(\omega)))^\kappa\cdot 2}\\
&\cdot\frac{\int_{f^N_{\theta^{-N}\omega}\gamma_1(\theta^{-N}\omega)}dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}dm_{\gamma(\omega)}}\\
\geq&\frac{e^{-c\epsilon^\nu-2a_1\epsilon^\kappa}}{2(J+1)}\cdot D_3^{-2}\cdot\|\varphi\|_{\theta^{-N}\omega,+}\cdot(K_6)^{-N}\cdot\left(\inf_{(x,\omega)\in M\times\Omega}m(D_xf_\omega|_{E^s(x,\omega)})\right)^N\\
&\cdot\frac{\int_{\gamma_1(\theta^{-N}\omega)}dm_{\gamma_1(\theta^{-N}\omega)}}{\int_{\gamma(\omega)}dm_{\gamma(\omega)}},
\end{aligned}
$$

where $m(D_xf_\omega|_{E^s(x,\omega)})=\|(D_xf_\omega|_{E^s(x,\omega)})^{-1}\|^{-1}$. Note that both $\gamma_1(\theta^{-N}\omega)\subset M$ and $\gamma(\omega)\subset M$ are local stable leaf having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$. Therefore,

$$
\frac{\int_{\gamma_1(\theta^{-N}\omega)}dm_{\gamma_1(\theta^{-N}\omega)}}{\int_{\gamma(\omega)}dm_{\gamma(\omega)}}\geq\frac{1}{4J^2}.
$$

We continue the estimate

$$
\frac{\int_{\gamma(\omega)}(L^N_{\theta^{-N}\omega}\varphi)(x)\rho(x,\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\rho(x,\omega)dm_{\gamma(\omega)}}
$$

$$\overset{(4.69)}{\geq} \frac{1}{8J^2(J+1)} e^{-c\epsilon^{\nu}-a\epsilon^{\kappa}} \cdot D_3^{-2} \cdot (K_6)^{-2N} \cdot \left(\inf_{(x,\omega)\in M\times\Omega} m(D_x f_\omega|_{E^s(p,\omega)}) \right)^N \cdot \|L^N_{\theta^{-N}\omega}\varphi\|_{\omega,+}$$
$$:= (D_1)^{-1}\|L^N_{\theta^{-N}\omega}\varphi\|_{\omega,+}.$$

Since $\gamma(\omega)$ and $\rho(\omega) \in D(a_1, \kappa, \gamma(\omega))$ are arbitrary, we have $\|L^N_{\theta^{-N}\omega}\varphi\|_{\omega,-} \geq (D_1)^{-1}\|L^N_{\theta^{-N}\omega}\varphi\|_{\omega,+}$. Hence (4.66) is proved. The proof of Sublemma 4.2 is complete. □

Note that the Lemma 4.6 is proved for all $\omega \in \Omega$, so we also have

$$\sup\{d_{\theta^N\omega}(L^N_\omega\varphi_1, L^N_\omega\varphi_2) : \varphi_1, \varphi_2 \in C_\omega(b, c, \nu)\} \leq D_2, \text{ for all } \omega \in \Omega, \tag{4.73}$$

where $d_{\theta^N\omega}$ is the projective metric on $C_{\theta^N\omega}(b, c, \nu)$.

Lemma 4.7. *There exist a number D_4 and a number $\Lambda \in (0, 1)$ both depending on D_2 and N such that for all $n \geq N$, for all $\omega \in \Omega$,*

$$d_\omega(L^n_{\theta^{-n}\omega}\varphi^1_{\theta^{-n}\omega}, L^n_{\theta^{-n}\omega}\varphi^2_{\theta^{-n}\omega}) \leq D_4\Lambda^n \textit{ for any } \varphi^1_{\theta^{-n}\omega}, \varphi^2_{\theta^{-n}\omega} \in C_{\theta^{-n}\omega}(b, c, \nu); \tag{4.74}$$
$$d_{\theta^n\omega}(L^n_\omega\varphi^1_\omega, L^n_\omega\varphi^2_\omega) \leq D_4\Lambda^n \textit{ for any } \varphi^1_\omega, \varphi^2_\omega \in C_\omega(b, c, \nu), \tag{4.75}$$

where d_ω and $d_{\theta^n\omega}$ are projective metric on $C_\omega(b, c, \nu)$ and $C_{\theta^n\omega}(b, c, \nu)$ respectively.

Proof. Now we have a linear operator $L^N_{\theta^{-N}\omega}$ maps cone $C_{\theta^{-N}\omega}(b, c, \nu)$ into cone $C_\omega(b, c, \nu)$ with finite diameter of $L^N_{\theta^{-N}\omega}(C_{\theta^{-N}\omega}(b, c, \nu))$ in $C_\omega(b, c, \nu)$, then we apply Birkhoff's inequality (Proposition A.3) to obtain that for all $\omega \in \Omega$,

$$d_\omega(L^N_{\theta^{-N}\omega}\varphi_1, L^N_{\theta^{-N}\omega}\varphi_2) \leq (1 - e^{-D_2})d_{\theta^{-N}\omega}(\varphi_1, \varphi_2), \text{ for all } \varphi_1, \varphi_2 \in C_{\theta^{-N}\omega}(b, c, \nu). \tag{4.76}$$

Now, for any $\omega \in \Omega$, $n \geq N$ and $\varphi^1_{\theta^{-n}\omega}, \varphi^2_{\theta^{-n}\omega} \in C_{\theta^{-n}\omega}(b, c, \nu)$, we write $n = (k+1)N + r$ for some $r\{0, ..., N-1\}$ and $k \geq 0$. Then

$$\begin{aligned}
&d_\omega(L^n_{\theta^{-n}\omega}\varphi^1_{\theta^{-n}\omega}, L^n_{\theta^{-n}\omega}\varphi^2_{\theta^{-n}\omega}) \\
&\overset{(4.76)}{\leq} (1 - e^{-D_2})^k d_{\theta^{-n+N+r}\omega}(L^{N+r}_{\theta^{-n}\omega}\varphi^1_{\theta^{-n}\omega}, L^{N+r}_{\theta^{-n}\omega}\varphi^2_{\theta^{-n}\omega}) \\
&=(1 - e^{-D_2})^{[\frac{n-N}{N}]} d_{\theta^{-n+N+r}\omega}(L^N_{\theta^{-n+r}\omega}L^r_{\theta^{-n}\omega}\varphi^1_{\theta^{-n}\omega}, L^N_{\theta^{-n+r}\omega}L^r_{\theta^{-n}\omega}\varphi^2_{\theta^{-n}\omega}) \\
&\leq \Lambda^{n-2N} D_2 = \frac{D_2}{\Lambda^{2N}}\Lambda^n := D_4\Lambda^n,
\end{aligned}$$

where $\Lambda = (1 - e^{-D_2})^{\frac{1}{N}} < 1$. Similarly, for all $\omega \in \Omega$, $n \geq N$ and $\varphi^1_\omega, \varphi^2_\omega \in C_\omega(b, c, \nu)$, one has

$$d_{\theta^n\omega}(L^n_\omega\varphi^1_\omega, L^n_\omega\varphi^2_\omega) \leq (1 - e^{-D_2})^{[\frac{n-N}{N}]} d_{\theta^{N+r}\omega}(L^N_{\theta^r\omega}L^r_\omega\varphi^1_\omega, L^N_{\theta^r\omega}L^r_\omega\varphi^2_\omega) \leq D_4\Lambda^n.$$

The proof of Lemma 4.7 is complete. □

4.3. Construction of the random SRB measure

In this subsection, we will prove that the sequence $(f^n_{\theta^{-n}\omega})_* m$ converges with respect to the weak* topology on $Pr(M)$ by using the contraction of $L^n_{\theta^{-n}\omega}$ when $n \geq N$. Moreover, we will prove that the random probability measure μ_ω defined by the weak* limit of $(f^n_{\theta^{-n}\omega})_* m$ is ϕ-invariant.

Before we introduce the next lemma, we need some preparations. For any $\omega \in \Omega$, since the local stable leaves form a partition in a neighborhood of a point on each M, we can cover M on the fiber $\{\omega\}$ by finite rectangles, i.e., $\mathcal{R}(\omega) = \{R_1(\omega), ..., R_i(\omega), ..., R_{k(\omega)}(\omega)\},\ k(\omega) < \infty$, satisfying

(A) each $R_i(\omega)$ is a proper rectangle, i.e., it is the closure of its interior;
(B) interior$(R_i(\omega)) \cap$ interior$(R_j(\omega)) = \emptyset$ if $i \neq j$;
(C) each $R_i(\omega)$ is foliated by local stable leaves having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$.

The conditions (A) and (B) of this cover is much weaker than the conditions of random Markov partition constructed in [20, Sec. 3]. We can obtain (C) by cutting and pasting some sets of the random Markov partition if necessary.

By Proposition 3.3, for any $i \in \{1, ..., k(\omega)\}$, there exists a function $H_i(\omega) : R_i(\omega) \to \mathbb{R}^+$ with $\log H_i(\omega)$ (a_0, ν_0)-Hölder continuous on each local stable leaf and for all bounded measurable functions $\psi : M \to \mathbb{R}$, we have disintegration

$$\int\limits_{R_i(\omega)} \psi(x)dm(x) = \int \int\limits_{\gamma^i(\omega)} \psi(x)H_i(\omega)(x)|_{\gamma^i(\omega)} dm_{\gamma^i(\omega)}(x)d\tilde{m}_{R_i(\omega)}(\gamma^i(\omega)), \tag{4.77}$$

where $\gamma^i(\omega)$ denotes the stable leaves in $R_i(\omega)$ and $\tilde{m}_{R_i(\omega)}$ is the quotient measure induced by Riemannian volume measure in the space of local stable leaves in $R_i(\omega)$.

Lemma 4.8. *For any $\omega \in \Omega$, given any positive function sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset C_\omega(b, c, \nu)$ satisfying*

$$\int\limits_M \varphi_n(x)dm(x) = 1 \textit{ for all } n \in \mathbb{N}, \tag{4.78}$$

and

$$d_{+,\omega}(\varphi_n, \varphi_m) \to 0 \textit{ exponentially as } n, m \to \infty,$$

where $d_{+,\omega}$ is the projective metric on $C_{+,\omega}$ defined in (4.57). Then for any continuous function $\psi : M \to \mathbb{R}$, the sequence $\left\{\int_M \varphi_n(x)\psi(x)dm(x)\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Proof. In the case that $\psi : M \to \mathbb{R}$ is a constant function, $\left\{\int_M \varphi_n(x)\psi(x)dm(x)\right\}_{n\in\mathbb{N}}$ is a constant sequence, therefore Cauchy. In the following, we only need to consider the case that ψ is nonconstant continuous function.

First, we consider any positive nonconstant continuous function $\psi : M \to \mathbb{R}$, satisfying

$$|\log \psi|_\kappa = \sup_{x,y\in M, x\neq y} \frac{|\log \psi(x) - \log \psi(y)|}{d(x,y)^\kappa} < \frac{a}{4}.$$

Let $R_i(\omega)$ and $H_i(\omega)$ be defined as above for $i \in \{1,...,k(\omega)\}$. Note that $\psi(\cdot)H_i(\omega)(\cdot)|_{\gamma^i(\omega)}$ is strictly positive on any local stable leaf $\gamma^i(\omega) \subset R_i(\omega)$. Moreover, $\log(\psi(\cdot)H_i(\omega)(\cdot))$ is $(a/2,\kappa)$-Hölder continuous on $\gamma^i(\omega)$ since both $\log\psi$ and $\log H_i(\omega)$ are $(a/2,\kappa)$-Hölder continuous on $\gamma^i(\omega)$. Hence $(\psi \cdot H_i(\omega))|_{\gamma^i(\omega)} \in D(\frac{a}{2},\kappa,\gamma^i(\omega))$. Therefore, by the representation of $\beta_{+,\omega}(\varphi_k,\varphi_l)$ and $\alpha_{+,\omega}(\varphi_k,\varphi_l)$ as in (4.59) and (4.58), for all $i \in \{1,...,k(\omega)\}$, any local stable leaf $\gamma^i(\omega) \subset R_i(\omega)$ and $k,l \in \mathbb{N}$, we have

$$\begin{aligned}
&\frac{\int_{\gamma^i(\omega)} \varphi_k(x)\psi(x)H_i(\omega)(x)|_{\gamma^i(\omega)} dm_{\gamma^i(\omega)}(x)}{\int_{\gamma^i(\omega)} \varphi_l(x)\psi(x)H_i(\omega)(x)|_{\gamma^i(\omega)} dm_{\gamma^i(\omega)}(x)} \\
&= \frac{\int_{\gamma^i(\omega)} \varphi_k(x)\psi(x)H_i(\omega)(x)|_{\gamma^i(\omega)} dm_{\gamma^i(\omega)}(x) / \int_{\gamma^i(\omega)} \psi(x)H_i(\omega)(x)|_{\gamma^i(\omega)} dm_{\gamma^i(\omega)}(x)}{\int_{\gamma^i(\omega)} \varphi_l(x)\psi(x)H_i(\omega)(x)|_{\gamma^i(\omega)} dm_{\gamma^i(\omega)}(x) / \int_{\gamma^i(\omega)} \psi(x)H_i(\omega)(x)|_{\gamma^i(\omega)} dm_{\gamma^i(\omega)}(x)} \qquad (4.79) \\
&\in [\alpha_{+,\omega}(\varphi_k,\varphi_l), \beta_{+,\omega}(\varphi_k,\varphi_l)].
\end{aligned}$$

By the assumption that $\int_M \varphi_k(x)dm(x) = \int_M \varphi_l(x)dm(x) = 1$ and (4.77), there exists a $\hat{i}$ and a local stable leaf $\gamma^{\hat{i}}(\omega) \subset R_{\hat{i}}(\omega)$ such that

$$\int\limits_{\gamma^{\hat{i}}(\omega)} \varphi_k(x)H_{\hat{i}}(\omega)(x)|_{\gamma^{\hat{i}}(\omega)} dm_{\gamma^i(\omega)}(x) \le \int\limits_{\gamma^{\hat{i}}(\omega)} \varphi_l(x)H_{\hat{i}}(\omega)(x)|_{\gamma^{\hat{i}}(\omega)} dm_{\gamma^i(\omega)}(x). \qquad (4.80)$$

Otherwise,

$$\begin{aligned}
\int\limits_M \varphi_k dm &= \sum_{i=1}^{k(\omega)} \int \int\limits_{\gamma^i(\omega)} \varphi_k(x)H_i(\omega)(x)|_{\gamma^i(\omega)} dm_{\gamma^i(\omega)}(x) d\tilde{m}_{R_i(\omega)} \\
&> \sum_{i=1}^{k(\omega)} \int \int\limits_{\gamma^i(\omega)} \varphi_l(x)H_i(\omega)(x)|_{\gamma^i(\omega)} dm_{\gamma^i(\omega)}(x) d\tilde{m}_{R_i(\omega)} = \int\limits_M \varphi_l dm,
\end{aligned}$$

a contradiction. Now for any i and local stable leaf $\gamma^i(\omega) \subset R_i(\omega)$, we have

$$\begin{aligned}
&\frac{\int_{\gamma^i(\omega)} \varphi_k(x)\psi(x)H_i(\omega)(x)|_{\gamma^i(\omega)} dm_{\gamma^i(\omega)}(x)}{\int_{\gamma^i(\omega)} \varphi_l(x)\psi(x)H_i(\omega)(x)|_{\gamma^i(\omega)} dm_{\gamma^i(\omega)}(x)} \\
&\overset{(4.79)}{\le} \beta_{+,\omega}(\varphi_k,\varphi_l) = \frac{\beta_{+,\omega}(\varphi_k,\varphi_l)}{\alpha_{+,\omega}(\varphi_k,\varphi_l)} \cdot \alpha_{+,\omega}(\varphi_k,\varphi_l) \\
&\overset{(4.79)}{\le} \frac{\beta_{+,\omega}(\varphi_k,\varphi_l)}{\alpha_{+,\omega}(\varphi_k,\varphi_l)} \cdot \frac{\int_{\gamma^{\hat{i}}(\omega)} \varphi_k(x)H_{\hat{i}}(\omega)(x)|_{\gamma^{\hat{i}}(\omega)} dm_{\gamma^i(\omega)}(x)}{\int_{\gamma^{\hat{i}}(\omega)} \varphi_l(x)H_{\hat{i}}(\omega)(x)|_{\gamma^{\hat{i}}(\omega)} dm_{\gamma^i(\omega)}(x)} \qquad (4.81)
\end{aligned}$$

$$\begin{aligned}
&\overset{(4.80)}{\leq} \frac{\beta_{+,\omega}(\varphi_k,\varphi_l)}{\alpha_{+,\omega}(\varphi_k,\varphi_l)}\cdot 1\\
&=\exp(d_{+,\omega}(\varphi_k,\varphi_l)),\ \text{for all } k,l\geq 1.
\end{aligned}$$

By assumption, $d_{+,\omega}(\varphi_k,\varphi_l)\to 0$ exponentially as $n,m\to\infty$. Now pick $N'>0$ such that for any $k,l>N'$, $d_{+,\omega}(\varphi_k,\varphi_l)<\frac{1}{2}$, then we have

$$\begin{aligned}
&\left|\int_M \varphi_k(x)\psi(x)dm(x)-\int_M \varphi_l(x)\psi(x)dm(x)\right|\\
&=\left|\int_M \varphi_l(x)\psi(x)dm(x)\right|\cdot\left|\frac{\int_M \varphi_k(x)\psi(x)dm(x)}{\int_M \varphi_l(x)\psi(x)dm(x)}-1\right|\\
&\leq \sup_{x\in M}|\psi(x)|\cdot\left|\frac{\sum_{i=1}^{k(\omega)}\int_{R_i(\omega)}\varphi_k(x)\psi(x)dm(x)}{\sum_{i=1}^{k(\omega)}\int_{R_i(\omega)}\varphi_l(x)\psi(x)dm(x)}-1\right| \qquad (4.82)\\
&=\|\psi\|_{C^0(M)}\cdot\left|\frac{\sum_{i=1}^{k(\omega)}\int\int_{\gamma^i(\omega)}\varphi_k(x)\psi(x)H_i(\omega)(x)|_{\gamma^i(\omega)}dm_{\gamma^i(\omega)}(x)d\tilde{m}_{R_i(\omega)}}{\sum_{i=1}^{k(\omega)}\int\int_{\gamma^i(\omega)}\varphi_l(x)\psi(x)H_i(\omega)(x)|_{\gamma^i(\omega)}dm_{\gamma^i(\omega)}(x)d\tilde{m}_{R_i(\omega)}}-1\right|\\
&\overset{(4.81)}{\leq}\|\psi\|_{C^0(M)}\cdot\left(e^{d_{+,\omega}(\varphi_k,\varphi_l)}-1\right)\\
&\leq 2\|\psi\|_{C^0(M)}\cdot d_{+,\omega}(\varphi_k,\varphi_l),
\end{aligned}$$

where in the first $\leq$ we use the positivity of φ_l and $\int\varphi_l dm=1$. Hence $\{\int_M\varphi_n(x)\psi(x)dm(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in this case.

Secondly, for any nonconstant function $\psi\in C^{0,\kappa}(M)$, let

$$B=\frac{5|\psi|_\kappa}{a}>0,$$

where $|\psi|_\kappa:=\sup_{x,y\in M,x\neq y}\frac{|\psi(x)-\psi(y)|}{d(x,y)^\kappa}$. We define

$$\psi_B^+:=\frac{1}{2}(|\psi|+\psi)+B,\ \psi_B^-:=\frac{1}{2}(|\psi|-\psi)+B.$$

Then for any $x,y\in M$, we have

$$|\psi_B^\pm(x)-\psi_B^\pm(y)|\leq|\psi|_\kappa d(x,y)^\kappa,$$

and therefore, we obtain

$$\left|\frac{\psi_B^\pm(x)}{\psi_B^\pm(y)}-1\right|\leq\frac{a}{5}d(x,y)^\kappa.$$

Switch x and y to get

$$|\log \psi_B^{\pm}|_\kappa = \sup_{x,y\in M, x\neq y} \frac{|\log \psi_B^{\pm}(x) - \log \psi_B^{\pm}(y)|}{d(x,y)^\kappa} < \frac{a}{4}. \tag{4.83}$$

Then we apply (4.82) and the linearity of integration, for any $k, l > N'$, one has

$$\begin{aligned}
&\left|\int_M \varphi_k(x)\psi(x)dm(x) - \int_M \varphi_l(x)\psi(x)dm(x)\right| \\
&\leq \left|\int_M \varphi_k(x)\psi_B^+(x)dm(x) - \int_M \varphi_l(x)\psi_B^+(x)dm(x)\right| \\
&\quad + \left|\int_M \varphi_k(x)\psi_B^-(x)dm(x) - \int_M \varphi_l(x)\psi_B^-(x)dm(x)\right| \\
&\leq 2(\|\psi_B^+\|_{C^0(M)} + \|\psi_B^-\|_{C^0(M)})d_{+,\omega}(\varphi_k, \varphi_l) \\
&\leq \left(4\|\psi\|_{C^0(M)} + \frac{20}{a}|\psi|_\kappa\right) d_{+,\omega}(\varphi_k, \varphi_l) \\
&\leq \max\{4, \frac{20}{a}\}\|\psi\|_{C^{0,\kappa}(M)} \cdot d_{+,\omega}(\varphi_k, \varphi_l).
\end{aligned}$$

Finally, for any nonconstant continuous function $\psi : M \to \mathbb{R}$, for any $\epsilon > 0$, we can pick a nonconstant function $\tilde{\psi} \in C^{0,\kappa}(M)$ such that

$$\sup_{x\in M} |\psi(x) - \tilde{\psi}(x)| < \epsilon/4.$$

Now, pick $N'' > N' > 0$ depending on $\tilde{\psi}$ and ϵ such that for all $k, l \geq N''$

$$\max\left\{4, \frac{20}{a}\right\} \cdot \|\tilde{\psi}\|_{C^{0,\kappa}(M)} \cdot d_{+,\omega}(\varphi_k, \varphi_l) < \epsilon/2.$$

Then for any $k, l \geq N''$, one has

$$\begin{aligned}
&\left|\int_M \varphi_k(x)\psi(x)dm(x) - \int_M \varphi_l(x)\psi(x)dm(x)\right| \\
&\leq \left|\int_M \varphi_k(x)\tilde{\psi}(x)dm(x) - \int_M \varphi_l(x)\tilde{\psi}(x)dm(x)\right| + \int_M \varphi_k|\psi - \tilde{\psi}|dm + \int_M \varphi_l|\psi - \tilde{\psi}|dm \\
&\overset{(4.78)}{\leq} \max\left\{4, \frac{20}{a}\right\} \cdot \|\tilde{\psi}\|_{C^{0,\kappa}(M)} \cdot d_{+,\omega}(\varphi_k, \varphi_l) + \epsilon/4 + \epsilon/4 \\
&\leq \epsilon,
\end{aligned}$$

where we note that φ_k and φ_l are positive functions. Hence, for any continuous function $\psi : M \to \mathbb{R}$, the sequence $\{\int_M \varphi_n(x)\psi(x)dm(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. The proof of Lemma 4.8 is complete. □

For any measurable function $\varphi : M \to \mathbb{R}$, we define the fiber Koopman operator

$$U_\omega\varphi : M \to \mathbb{R},\ (U_\omega\varphi)(x) := \varphi(f_{\theta^{-1}\omega}x). \tag{4.84}$$

We denote

$$U^n_\omega := U_{\theta^{-(n-1)}\omega} \circ \cdots \circ U_{\theta^{-1}\omega} \circ U_\omega \text{ for all } n \in \mathbb{N} \text{ and } \omega \in \Omega.$$

For any $\omega \in \Omega$ and any bounded measurable functions φ_1, φ_2, by changing variable, we have

$$\begin{aligned}\int_M (L_{\theta^{-1}\omega}\varphi_1)(y)\varphi_2(y)dm(y) &= \int_M \frac{\varphi_1((f_{\theta^{-1}\omega})^{-1}y)}{|\det D_{(f_{\theta^{-1}\omega})^{-1}(y)} f_{\theta^{-1}\omega}|}\varphi_2(y)dm(y)\\ &= \int_M \frac{\varphi_1(x)}{|\det D_x f_{\theta^{-1}\omega}|}\varphi_2(f_{\theta^{-1}\omega}x)|\det D_x f_{\theta^{-1}\omega}|dm(x)\\ &= \int_M \varphi_1(x)(U_\omega\varphi_2)(x)dm(x). \end{aligned}\tag{4.85}$$

Let $\mathbf{1}$ be the constant function $\mathbf{1}(x) \equiv 1$, then $\mathbf{1} \in C_\omega(b, c, \nu)$ for all $\omega \in \Omega$ by the Remark 4.3. Now consider $\varphi_n = L^n_{\theta^{-n}\omega}\mathbf{1}$ for $n \geq N$ and notice that for all $\omega \in \Omega$,

$$\int_M (L^n_{\theta^{-n}\omega}\mathbf{1})(x)dm(x) = \int_M \mathbf{1}(x)(U^n_\omega 1)(x)dm(x) = \int_M \mathbf{1}dm(x) = 1.$$

Moreover, by (4.74), we have

$$d_{+,\omega}(L^n_{\theta^{-n}\omega}\mathbf{1}, L^{n+k}_{\theta^{-(n+k)}\omega}\mathbf{1}) \leq d(L^n_{\theta^{-n}\omega}\mathbf{1}, L^n_{\theta^{-n}\omega}(L^k_{\theta^{-(n+k)}\omega}\mathbf{1})) \leq \Lambda^n \cdot D_4 \text{ for } n \geq N. \tag{4.86}$$

Hence the positive functions sequence $\{\varphi_n = L^n_{\theta^{-n}\omega}\mathbf{1}\}_{n\in\mathbb{N}} \subset C_\omega(b, c, \nu)$ satisfies conditions of Lemma 4.8. So for any $g \in C^0(M)$, $\{\int_M (L^n_{\theta^{-n}\omega}\mathbf{1})(x)g(x)dm\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Now define $\mathcal{F}_\omega : C^0(M) \to \mathbb{R}$ by $\mathcal{F}_\omega(g) = \lim_{n\to\infty} \int_M (L^n_{\theta^{-n}\omega}\mathbf{1})(x)g(x)dm(x)$. It is clear that $\mathcal{F}_\omega$ is a positive linear functional on $C^0(M)$. By the Riesz representation theorem, there exists a regular Borel measure μ_ω such that

$$\int_M g(x)d\mu_\omega(x) = \lim_{n\to\infty} \int_M (L^n_{\theta^{-n}\omega}\mathbf{1})(x)g(x)dm(x). \tag{4.87}$$

Moreover, μ_ω is a probability measure since $\mu_\omega(M) = \lim_{n\to\infty} \int_M (L^n_{\theta^{-n}\omega}\mathbf{1})(x)dm(x) = 1$.

Note that for each $g \in C^0(M)$, $\omega \mapsto \int_M g(x)d\mu_\omega(x)$ is measurable because of the measurability of $\omega \mapsto \int_M (L^n_{\theta^{-n}\omega}\mathbf{1})(x)g(x)dm(x)$. For any closed set $B \subset M$, let $g_k(x) := 1 -$

$\min\{kd(x,B),1\}$ for $k\in\mathbb{N}$ where $d(x,B):=\inf\{d(x,y):\ y\in B\}$, then $g_k(x)\in C^0(M)$ and $g_k(x)\searrow 1_B(x)$. Then by Monotone convergence theorem, we have

$$\mu_\omega(B)=\lim_{k\to\infty}\int\limits_M g_k(x)d\mu_\omega(x)=\lim_{k\to\infty}\lim_{n\to\infty}\int\limits_M (L^n_{\theta^{-n}\omega}\mathbf{1})(x,\omega)g_k(x)dm(x).$$

Hence $\omega\mapsto\mu_\omega(B)$ is measurable for any closed set $B\subset M$. By the definition in Section 2.2, $\omega\mapsto\mu_\omega$ defines a random probability measure.

Now for any continuous $g:M\to\mathbb{R}$,

$$\begin{aligned}\int\limits_M g(f_\omega x)d\mu_\omega&=\lim_{n\to\infty}\int\limits_M(L^n_{\theta^{-n}\omega}\mathbf{1})(x)g(f_\omega x)dm(x)\overset{(4.85)}{=}\lim_{n\to\infty}\int\limits_M g(f_\omega f^n_{\theta^{-n}\omega}x)dm(x)\\&=\lim_{n\to\infty}\int\limits_M g(f^{n+1}_{\theta^{-(n+1)}\theta\omega}x)dm(x)=\lim_{n\to\infty}\int\limits_M(L^{n+1}_{\theta^{-(n+1)}\theta\omega}\mathbf{1})(x)g(x)dm(x)\\&=\int\limits_M g(x)d\mu_{\theta\omega}.\end{aligned}$$

Thus, the random probability measure μ_ω is ϕ-invariant, i.e.,

$$(f_\omega)_*\mu_\omega=\mu_{\theta\omega}\text{ for all }\omega\in\Omega.\tag{4.88}$$

Notice that for any $g\in C^0(M)$, we have

$$\begin{aligned}\int\limits_M g(x)d\mu_\omega(x)&=\lim_{n\to\infty}\int\limits_M(L^n_{\theta^{-n}\omega}\mathbf{1})(x)g(x)dm(x)\overset{(4.85)}{=}\lim_{n\to\infty}\int\limits_M U^n_\omega g\,dm\\&=\lim_{n\to\infty}\int\limits_M g(f^n_{\theta^{-n}\omega}x)dm(x)=\lim_{n\to\infty}\int\limits_M g(y)d(f^n_{\theta^{-n}\omega})_*m(y).\end{aligned}$$

So μ_ω is actually the weak*-limit of $(f^n_{\theta^{-n}\omega})_*m$.

Remark 4.4. For each $\omega\in\Omega$, $k\in\mathbb{N}$ and any positive function $\varphi_{\theta^{-k}\omega}\in C_{\theta^{-k}\omega}(b,c,\nu)$ such that $\int_M\varphi_{\theta^{-k}\omega}(x)dm=1$, then we must have

$$\lim_{n\to\infty}\int\limits_M(L^n_{\theta^{-n}\omega}\varphi_{\theta^{-n}\omega})(x)g(x)dm(x)=\int\limits_M g(x)d\mu_\omega(x),\ \forall g\in C^0(M).\tag{4.89}$$

In fact, we define the sequence $\hat\varphi_n\in C_\omega(b,c,\nu)$ by $\hat\varphi_{2k}=L^k_{\theta^{-k}\omega}\mathbf{1}$, $\hat\varphi_{2k+1}=L^k_{\theta^{-k}\omega}\varphi_{\theta^{-k}\omega}$ for all $k\ge N$. By noticing

$$d_{+,\omega}(\hat\varphi_{2k},\hat\varphi_{2k+1})\le d_\omega(L^k_{\theta^{-k}\omega}\mathbf{1},L^k_{\theta^{-k}\omega}\varphi_{\theta^{-k}\omega})\overset{(4.74)}{\le}\Lambda^kD_4$$

and by (4.85),

$$\int_M (L^k_{\theta^{-k}\omega}\varphi_{\theta^{-k}\omega})(x)dm(x) = \int_M \varphi_{\theta^{-k}\omega}(x)(U^k_\omega \mathbf{1})(x)dm(x) = \int_M \varphi_{\theta^{-k}\omega}(x)dm(x) = 1.$$

So $\{\hat{\varphi}_n\}_{n\in\mathbb{N}} \subset C_\omega(b, c, \nu)$ satisfying the condition of Lemma 4.8. Thus, the sequence $\{\int_M \hat{\varphi}_n(x)g(x)dm(x)\}$ is a Cauchy sequence for all $g \in C^0(M)$. As a consequence, we have

$$\int_M g(x)d\mu_\omega(x) = \lim_{n\to\infty}\int_M (L^n_{\theta^{-n}\omega}\mathbf{1})(x)g(x)dm(x) = \lim_{n\to\infty}\int_M (L^n_{\theta^{-n}\omega}\varphi_{\theta^{-n}\omega})(x)g(x)dm(x).$$

4.4. Proof of the exponential decay of the (quenched) past random correlations

In this subsection, we prove the exponential decay of the past random correlations.

Lemma 4.9. *Let $\psi : M \to \mathbb{R}^+$ be a positive function such that $\log\psi$ is $(\frac{a}{4}, \kappa)$-Hölder continuous. Then there exists a constant $K(D_4) > 0$ depending on D_4 such that for any $\omega \in \Omega$, positive function $\varphi_{\theta^{-k}\omega} \in C_{\theta^{-k}\omega}(b, c, \nu)$ for $k \in \mathbb{N}$, and $n \geq N$, the following holds:*

$$\begin{aligned}&\left|\int_M \psi(f^n_{\theta^{-n}\omega}x)\varphi_{\theta^{-n}\omega}(x)dm(x) - \int_M \psi(x)d\mu_\omega(x)\int_M \varphi_{\theta^{-n}\omega}(x)dm(x)\right| \\ &\leq K(D_4)\cdot\|\psi\|_{C^0(M)}\cdot\int_M \varphi_{\theta^{-n}\omega}(x)dm(x)\cdot\Lambda^n.\end{aligned} \tag{4.90}$$

Recall that D_4 comes from Lemma 4.7, and N is constructed satisfying (4.54) and (4.55).

Proof. First, we consider that positive function $\varphi_{\theta^{-k}\omega} \in C_{\theta^{-k}\omega}(b, c, \nu)$ satisfying $\int_M \varphi_{\theta^{-k}\omega}(x)dm(x) = 1$ for all $k \in \mathbb{N}$. So

$$\int_M (L^n_{\theta^{-n}\omega}\varphi_{\theta^{-n}\omega})(x)dm(x) \overset{(4.85)}{=} \int_M \varphi_{\theta^{-n}\omega}\cdot U^n_\omega\mathbf{1}dm = 1, \text{ for all } n \in \mathbb{N}.$$

Then the proof of (4.82) indicates that for any $n \geq N,\ k \geq 0$, we have

$$\begin{aligned}&\left|\int_M \psi(x)(L^n_{\theta^{-n}\omega}\varphi_{\theta^{-n}\omega})(x)dm(x) - \int_M \psi(x)(L^{n+k}_{\theta^{-(n+k)}\omega}\varphi_{\theta^{-(n+k)}\omega})(x)dm(x)\right| \\ &\leq \|\psi\|_{C^0(M)}\left(e^{d_{+,\omega}(L^n_{\theta^{-n}\omega}\varphi_{\theta^{-n}\omega}, L^{n+k}_{\theta^{-(n+k)}\omega}\varphi_{\theta^{-(n+k)}\omega})} - 1\right) \\ &= \|\psi\|_{C^0(M)}\left(e^{d_{+,\omega}(L^n_{\theta^{-n}\omega}\varphi_{\theta^{-n}\omega}, L^n_{\theta^{-n}\omega}L^k_{\theta^{-(n+k)}\omega}\varphi_{\theta^{-(n+k)}\omega})} - 1\right) \\ &\leq \|\psi\|_{C^0(M)}\left(e^{d_\omega(L^n_{\theta^{-n}\omega}\varphi_{\theta^{-n}\omega}, L^n_{\theta^{-n}\omega}L^k_{\theta^{-(n+k)}\omega}\varphi_{\theta^{-(n+k)}\omega})} - 1\right)\end{aligned}$$

$$\overset{(4.74)}{\leq} \|\psi\|_{C^0(M)} \left(e^{D_4\Lambda^n} - 1\right)$$
$$\leq K(D_4)\|\psi\|_{C^0(M)}\Lambda^n$$

for some constant $K(D_4)$. Letting $k \to \infty$ in the above, by Remark 4.4, we have

$$\left| \int\limits_M \psi(x)(L^n_{\theta^{-n}\omega}\varphi_{\theta^{-n}\omega})(x)dm(x) - \int\limits_M \psi(x)d\mu_\omega(x) \right| \leq K(D_4)\cdot\|\psi\|_{C^0(M)}\Lambda^n. \tag{4.91}$$

Note that by (4.85), we have

$$\int\limits_M \psi(x)(L^n_{\theta^{-n}\omega}\varphi_{\theta^{-n}\omega})(x)dm(x) = \int\limits_M \psi(f^n_{\theta^{-n}\omega}x)\varphi_{\theta^{-n}\omega}(x)dm(x).$$

Hence (4.91) becomes

$$\left| \int\limits_M \psi(f^n_{\theta^{-n}\omega}x)\varphi_{\theta^{-n}\omega}(x)dm(x) - \int\limits_M \psi(x)d\mu_\omega(x) \right| \leq K(D_4)\cdot\|\psi\|_{C^0(M)}\Lambda^n. \tag{4.92}$$

Now for any positive function $\varphi_{\theta^{-n}\omega} \in C_{\theta^{-n}\omega}(b,c,\nu)$, let $\tilde{\varphi}_{\theta^{-n}\omega}(x) := \varphi_{\theta^{-n}\omega}(x)/\int_M \varphi_{\theta^{-n}\omega}(x)dm(x)$. We replace $\varphi_{\theta^{-n}\omega}$ by $\tilde{\varphi}_{\theta^{-n}\omega}$ in (4.92). Then (4.90) is proved. □

We still need the following lemma. Recall the constant K_2 in (4.3), and $C_2 + C_2C_1$ in (4.40).

Lemma 4.10. *Pick the number c in the definition of $C_\omega(b,c,\nu)$ sufficiently large satisfying not only $c > c_0(b,\nu)$ in Lemma 4.5, but also*

$$\max\left\{2a_0, \frac{K_2}{1-e^{-\lambda}}, \frac{C_2 + C_2C_1}{1-e^{-\lambda\nu_0}}\right\} < c. \tag{4.93}$$

Let c_1 be any constant such that

$$1 < c_1 < \max\left\{2a_0, \frac{K_2}{1-e^{-\lambda}}, \frac{C_2 + C_2C_1}{1-e^{-\lambda\nu_0}}\right\}. \tag{4.94}$$

Given any positive continuous function $\varphi : M \to \mathbb{R}^+$ with

$$|\log\varphi|_\nu = \sup_{x,y\in M,\ x\neq y} \frac{|\log\varphi(x) - \log\varphi(y)|}{d(x,y)^\nu} < c_1,$$

then $\varphi \cdot (L^l_{\theta^{-l}\omega}\mathbf{1}) \in C_\omega(b,c,\nu)$ for every $l \geq 1$ and all $\omega \in \Omega$.

Proof. We prove this lemma for each fixed $\omega \in \Omega$ and $l \geq 1$. First, $\varphi \cdot (L^l_{\theta^{-l}\omega}\mathbf{1})$ is obviously bounded and measurable function.

Let us verify **(C1)**. For every local stable manifold $\gamma(\omega)$ having size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$, and any $\rho(\omega) \in D(a/2, \kappa, \gamma(\omega))$ with $\int_{\gamma(\omega)} \rho(x,\omega) dm_{\gamma(\omega)}(x) = 1$, we have

$$\int\limits_{\gamma(\omega)} \varphi(x)(L^l_{\theta^{-l}\omega}\mathbf{1})(x)\rho(x,\omega)dm_{\gamma(\omega)}(x) \geq \inf_{x\in M} \varphi(x) \cdot \int\limits_{\gamma(\omega)} (L^l_{\theta^{-l}\omega}\mathbf{1})(x)\rho(x,\omega)dm_{\gamma(\omega)}(x) > 0$$

since $L^l_{\theta^{-l}\omega}\mathbf{1} \in C_\omega(b,c,\nu)$ and φ is positive and continuous.

By Remark 4.3, $\varphi \cdot L^l_{\theta^{-l}\omega}\mathbf{1}$ fulfills **(C2)** since $\varphi \cdot L^l_{\theta^{-l}\omega}\mathbf{1}$ is nonnegative. So it is left to verify **(C3)**.

Let $\gamma(\omega), \tilde{\gamma}(\omega)$ be any pair of local stable manifolds. Pick any $\rho(\omega) \in D(a_1, \kappa_1, \gamma(\omega))$, and let $\tilde{\rho}(\omega) \in D(a/2, \kappa, \tilde{\gamma}(\omega))$ be defined as (4.17) corresponding to $\rho(\omega)$. We divide $f^{-l}_\omega \gamma(\omega)$ into connected local stable manifolds of size between $\frac{A(\epsilon)}{4J}$ and $\frac{A(\epsilon)}{2J}$, named $\gamma_i(\theta^{-l}\omega)$, such that $\gamma(\omega) = \cup f^l_{\theta^{-l}\omega}\gamma_i(\theta^{-l}\omega)$. Let $\tilde{\gamma}_i(\theta^{-l}\omega)$ be the holonomy image of $\gamma_i(\theta^{-l}\omega)$ inside of $f^{-l}_\omega\tilde{\gamma}(\omega)$. Naturally, we have $\tilde{\gamma}(\omega) = \cup f_{\theta^{-1}\omega}\tilde{\gamma}_i(\theta^{-1}\omega)$. Note that the Jacobian of holonomy map between local stable manifolds is bounded above by J and bounded below by J^{-1}. Therefore, $\tilde{\gamma}_i(\theta^{-l}\omega)$ have size between $\frac{A(\epsilon)}{4J^2}$ and $A(\epsilon)$. Denote $\psi_\omega : \tilde{\gamma}(\omega) \to \gamma(\omega)$ to be the holonomy map between $\tilde{\gamma}(\omega)$ and $\gamma(\omega)$, and $\psi^i_{\theta^{-l}\omega} : \tilde{\gamma}_i(\theta^{-l}\omega) \to \gamma_i(\theta^{-l}\omega)$ the holonomy map induced by the local unstable manifolds. By using the definition of L_ω and changing of variables, we have

$$\begin{aligned}
&\int\limits_{\gamma(\omega)} (L^l_{\theta^{-l}\omega}\mathbf{1})(x)\varphi(x)\rho(x,\omega)dm_{\gamma(\omega)}(x) \\
&= \sum_i \int\limits_{\gamma_i(\theta^{-l}\omega)} \frac{\left|\det D_x f^l_{\theta^{-l}\omega}|_{E^s(x,\theta^{-l}\omega)}\right|}{\left|\det D_x f^l_{\theta^{-l}\omega}\right|} \rho(f^l_{\theta^{-l}\omega}x,\omega)\varphi(f^l_{\theta^{-l}\omega}x)dm_{\gamma_i(\theta^{-l}\omega)}(x) \\
&= \sum_i \int\limits_{\tilde{\gamma}_i(\theta^{-l}\omega)} \frac{\left|\det D_{\psi^i_{\theta^{-l}\omega}(x)} f^l_{\theta^{-l}\omega}|_{E^s(\psi^i_{\theta^{-l}\omega}(x),\theta^{-l}\omega)}\right|}{\left|\det D_{\psi^i_{\theta^{-l}\omega}(x)} f^l_{\theta^{-l}\omega}\right|} \cdot \rho(f^l_{\theta^{-l}\omega}\psi^i_{\theta^{-l}\omega}(x),\omega) \cdot \varphi(f^l_{\theta^{-l}\omega}\psi^i_{\theta^{-l}\omega}(x)) \\
&\qquad \cdot Jac(\psi^i_{\theta^{-l}\omega})(x)dm_{\tilde{\gamma}_i(\theta^{-l}\omega)}(x).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\int\limits_{\tilde{\gamma}(\omega)} (L^l_{\theta^{-l}\omega}\mathbf{1})(x)\varphi(x)\tilde{\rho}(x,\omega)dm_{\tilde{\gamma}(\omega)}(x) \\
&= \sum_i \int\limits_{\tilde{\gamma}_i(\theta^{-l}\omega)} \frac{\left|\det D_x f^l_{\theta^{-l}\omega}|_{E^s(x,\theta^{-l}\omega)}\right|}{\left|\det D_x f^l_{\theta^{-l}\omega}\right|} \tilde{\rho}(f^l_{\theta^{-l}\omega}x,\omega)\varphi(f^l_{\theta^{-l}\omega}x)dm_{\tilde{\gamma}_i(\theta^{-l}\omega)}(x)
\end{aligned}$$

$$= \sum_i \int\limits_{\tilde{\gamma}_i(\theta^{-l}\omega)} \frac{\left|\det D_x f^l_{\theta^{-l}\omega}|_{E^s(x,\theta^{-l}\omega)}\right|}{\left|\det D_x f^l_{\theta^{-l}\omega}\right|} \rho(\psi_\omega(f^l_{\theta^{-l}\omega}x),\omega)$$

$$\cdot Jac(\psi_\omega)(f^l_{\theta^{-l}\omega}x)\cdot \varphi(f^l_{\theta^{-l}\omega}x) dm_{\tilde{\gamma}_i(\theta^{-l}\omega)}(x).$$

Note that $f^l_{\theta^{-l}\omega}\psi^i_{\theta^{-l}\omega}(x) = \psi_\omega(f^l_{\theta^{-l}\omega}(x))$ for $x \in \tilde{\gamma}_i(\theta^{-l}\omega)$ by the invariance of stable and unstable manifolds, so we have

$$\rho(f^l_{\theta^{-l}\omega}\psi^i_{\theta^{-l}\omega}(x),\omega) = \rho(\psi_\omega(f^l_{\theta^{-l}\omega}\omega),\omega). \tag{4.95}$$

Since $\log\varphi$ is (c_1,ν)-Hölder continuous, for $x \in \tilde{\gamma}_i(\theta^{-l}\omega)$

$$\begin{aligned}\left|\log\varphi(f^l_{\theta^{-l}\omega}\psi^i_{\theta^{-l}\omega}(x)) - \log\varphi(f^l_{\theta^{-l}\omega}x)\right| &\le c_1 d(f^l_{\theta^{-l}\omega}\psi^i_{\theta^{-l}\omega}(x), f^l_{\theta^{-l}\omega}(x))^\nu \\ &\le c_1 d_u(\gamma(\omega),\tilde{\gamma}(\omega))^\nu. \end{aligned}\tag{4.96}$$

By Lemma 3.12 (2) and $a_0' \le a_0$, for $x \in \tilde{\gamma}_i(\theta^{-l}\omega)$, one has

$$\begin{aligned} &\left|\log Jac(\psi_\omega)(f^l_{\theta^{-l}\omega}x) - \log Jac(\psi^i_{\theta^{-l}\omega})(x)|\right| \\ &\le a_0 d(f^l_{\theta^{-l}\omega}x, \psi_\omega f^l_{\theta^{-l}\omega}x)^{\nu_0} + a_0 d(x,\psi^i_{\theta^{-l}\omega}x)^{\nu_0} \\ &\le a_0 d_u(\gamma(\omega),\tilde{\gamma}(\omega))^{\nu_0} + a_0 d_u(\gamma_i(\theta^{-l}\omega),\tilde{\gamma}_i(\theta^{-l}\omega))^{\nu_0} \\ &\le a_0 d_u(\gamma(\omega),\tilde{\gamma}(\omega))^{\nu_0} + a_0 e^{-\lambda l\nu_0} d_u(\gamma(\omega),\tilde{\gamma}(\omega))^{\nu_0} \\ &< 2a_0 d_u(\gamma(\omega),\tilde{\gamma}(\omega))^{\nu_0}. \end{aligned}\tag{4.97}$$

By (4.3), for $x \in \tilde{\gamma}_i(\theta^{-l}\omega)$, we deduce that

$$\begin{aligned} &\left|\log|\det D_{\psi^i_{\theta^{-l}\omega}(x)} f^l_{\theta^{-l}\omega}| - \log|\det D_x f^l_{\theta^{-l}\omega}|\right| \\ &\le K_2 d(x,\psi^i_{\theta^{-l}\omega}(x)) + K_2 d(f_{\theta^{-l}\omega}x, f_{\theta^{-l}\omega}\psi^i_{\theta^{-l}\omega}(x)) + \cdots + K_2 d(f^{l-1}_{\theta^{-l}\omega}x, f^{l-1}_{\theta^{-l}\omega}\psi^i_{\theta^{-l}\omega}(x)) \\ &\le K_2 e^{-l\lambda} d_u(\gamma(\omega),\tilde{\gamma}(\omega)) + K_2 e^{-(l-1)\lambda} d_u(\gamma(\omega),\tilde{\gamma}(\omega)) + \cdots + K_2 e^{-\lambda} d_u(\gamma(\omega),\tilde{\gamma}(\omega)) \\ &\le \frac{K_2}{1-e^{-\lambda}} \cdot d_u(\gamma(\omega),\tilde{\gamma}(\omega)). \end{aligned}\tag{4.98}$$

By applying (4.40), for $x \in \tilde{\gamma}_i(\theta^{-l}\omega)$,

$$\begin{aligned} &\left|\log|\det D_{\psi^i_{\theta^{-l}\omega}(x)} f^l_{\theta^{-l}\omega}|_{E^s(\psi^i_{\theta^{-l}\omega}(x),\theta^{-l}\omega)}| - \log|\det D_x f^l_{\theta^{-l}\omega}|_{E^s(x,\theta^{-l}\omega)}|\right| \\ &\le (C_2 + C_2C_1)\left[d(\psi^i_{\theta^{-l}\omega}(x),x)^{\nu_0} + \cdots + d(f^{l-1}_{\theta^{-l}\omega}\psi^i_{\theta^{-l}\omega}(x), f^{l-1}_{\theta^{-l}\omega}x)^{\nu_0}\right] \\ &\le \frac{C_2+C_2C_1}{1-e^{-\lambda\nu_0}} d_u(\gamma(\omega),\tilde{\gamma}(\omega))^{\nu_0}. \end{aligned}\tag{4.99}$$

Combining (4.95), (4.96), (4.97), (4.98) and (4.99), we conclude

$$\left|\log \int_{\gamma(\omega)} (L^l_{\theta^{-l}\omega}\mathbf{1})(x)\varphi(x)\rho(x,\omega)dm_{\gamma(\omega)}(x) - \log \int_{\tilde{\gamma}(\omega)} (L^l_{\theta^{-l}\omega}\mathbf{1})(x)\varphi(x)\tilde{\rho}(x,\omega)dm_{\tilde{\gamma}(\omega)}(x)\right|$$
$$\leq \max\left\{c_1, 2a_0, \frac{K_2}{1-e^{-\lambda}}, \frac{C_2+C_2C_1}{1-e^{-\lambda\nu_0}}\right\} d_u(\gamma(\omega),\tilde{\gamma}(\omega))^{\nu}$$
$$\overset{(4.93)(4.94)}{\leq} c d_u(\gamma(\omega),\tilde{\gamma}(\omega))^{\nu}.$$

Hence **(C3)** is verified. Therefore, $\varphi\cdot(L^l_{\theta^{-l}\omega}\mathbf{1}) \in C_\omega(b,c,\nu)$. The proof of Lemma 4.10 is complete. □

Now for any positive continuous function $\psi \in C^0(M)$ such that $\log\psi$ is $(a/4,\kappa)$-Hölder continuous, and for any positive continuous function $\varphi : M \to \mathbb{R}$ satisfying that $\log\varphi$ is (c_1,ν)-Hölder continuous, by Lemma 4.10, for each $n\in\mathbb{N}$ we have

$$\varphi\cdot L^l_{\theta^{-(l+n)}\omega}\mathbf{1} = \varphi\cdot L^l_{\theta^{-l}\theta^{-n}\omega}\mathbf{1} \in C_{\theta^{-n}\omega}(b,c,\nu) \text{ for all } l\in\mathbb{N},\ \omega\in\Omega.$$

Now, we apply Lemma 4.9 to obtain that for all $\omega\in\Omega$, $n\geq N$, we have

$$\left|\int_M \psi(f^n_{\theta^{-n}\omega}x)(\varphi\cdot L^l_{\theta^{-l}\theta^{-n}\omega}\mathbf{1})(x)dm(x) - \int_M \psi(x)d\mu_\omega(x)\int_M (\varphi\cdot L^l_{\theta^{-l}\theta^{-n}\omega}\mathbf{1})(x)dm(x)\right|$$
$$\leq K(D_4)\|\psi\|_{C^0(M)}\int_M \varphi\cdot(L^l_{\theta^{-l}\theta^{-n}\omega}\mathbf{1})dm\cdot\Lambda^n, \text{ for all } l\in\mathbb{N}. \tag{4.100}$$

Let $l\to\infty$, by (4.87), for all $\omega\in\Omega$, $n\geq N$, we have

$$\left|\int_M \psi(f^n_{\theta^{-n}\omega}x)\varphi(x)d\mu_{\theta^{-n}\omega}(x) - \int_M \psi(x)d\mu_\omega(x)\int_M \varphi(x)d\mu_{\theta^{-n}\omega}(x)\right|$$
$$\leq K(D_4)\|\psi\|_{C^0(M)}\int_M \varphi d\mu_{\theta^{-n}\omega}\cdot\Lambda^n$$
$$\leq K(D_4)\|\psi\|_{C^0(M)}\cdot\|\varphi\|_{C^0(M)}\cdot\Lambda^n. \tag{4.101}$$

Finally, given any $\psi\in C^{0,\kappa}(M)$ and $\varphi\in C^{0,\nu}(M)$. If ψ or φ is a constant function, then by (4.88),

$$\left|\int_M \psi(f^n_{\theta^{-n}\omega}x)\varphi(x)d\mu_{\theta^{-n}\omega}(x) - \int_M \psi(x)d\mu_\omega(x)\int_M \varphi(x)d\mu_{\theta^{-n}\omega}(x)\right| = 0.$$

Therefore, to prove the exponential decay of past random correlations, it is sufficient to consider the case that both $\psi \in C^{0,\kappa}(M)$ and $\varphi \in C^{0,\nu}(M)$ are nonconstant functions. Let

$$B_\psi = \frac{5|\psi|_\kappa}{a} > 0,\ B_\varphi = \frac{2|\varphi|_\nu}{c_1} > 0,$$

and define

$$\psi^+_{B_\psi} = \frac{1}{2}(|\psi| + \psi) + B_\psi,\ \psi^-_{B_\psi} = \frac{1}{2}(|\psi| - \psi) + B_\psi,$$
$$\varphi^+_{B_\varphi} = \frac{1}{2}(|\varphi| + \varphi) + B_\varphi,\ \varphi^-_{B_\varphi} = \frac{1}{2}(|\varphi| - \varphi) + B_\varphi.$$

Similar as (4.83), we can show that $\log \psi^\pm_{B_\psi}$ are $(a/4, \kappa)$-Hölder continuous, and $\log \varphi^\pm_{B_\varphi}$ are (c_1, ν)-Hölder continuous. By (4.101) and the linearity of integration, we conclude for all $\omega \in \Omega$ and $n \geq N$,

$$\begin{aligned}
&\left| \int_M \psi(f^n_{\theta^{-n}\omega}x)\varphi(x)d\mu_{\theta^{-n}\omega}(x) - \int_M \psi(x)d\mu_\omega(x) \int_M \varphi(x)d\mu_{\theta^{-n}\omega}(x) \right| \\
=&\left| \int_M \left(\psi^+_{B_\psi} - \psi^-_{B_\psi}\right)(f^n_{\theta^{-n}\omega}x)\left(\varphi^+_{B_\varphi} - \varphi^-_{B_\varphi}\right)(x)d\mu_{\theta^{-n}\omega}(x) \right. \\
&\left. - \int_M \left(\psi^+_{B_\psi} - \psi^-_{B_\psi}\right)(x)d\mu_\omega(x) \int_M \left(\varphi^+_{B_\varphi} - \varphi^-_{B_\varphi}\right)(x)d\mu_{\theta^{-n}\omega}(x) \right| \\
\leq& 4K(D_4) \cdot \max\left\{1, \frac{5}{a}\right\} \cdot \max\left\{1, \frac{2}{c_1}\right\} \cdot \|\psi\|_{C^{0,\kappa}(M)} \cdot \|\varphi\|_{C^{0,\nu}(M)} \cdot \Lambda^n.
\end{aligned}$$

Note that the above is true for all $n \geq N$. Next, for $n \in \{0, ..., N-1\}$, we have

$$\begin{aligned}
&\left| \int_M \psi(f^n_{\theta^{-n}\omega}x)\varphi(x)d\mu_{\theta^{-n}\omega}(x) - \int_M \psi(x)d\mu_\omega(x) \int_M \varphi(x)d\mu_{\theta^{-n}\omega}(x) \right| \\
\leq& 2\|\psi\|_{C^0(M)}\|\varphi\|_{C^0} \leq \frac{2}{\Lambda^N}\|\psi\|_{C^0(M)}\|\varphi\|_{C^0(M)}\Lambda^n.
\end{aligned}$$

Therefore, we let

$$K := \max\left\{4K(D_4) \cdot \max\left\{1, \frac{5}{a}\right\} \cdot \max\left\{1, \frac{2}{c_1}\right\}, 2\Lambda^{-N}\right\}, \tag{4.102}$$

then

$$\left|\int_M \psi(f^n_{\theta^{-n}\omega}x)\varphi(x)d\mu_{\theta^{-n}\omega}(x) - \int_M \psi(x)d\mu_\omega(x)\int_M \varphi(x)d\mu_{\theta^{-n}\omega}(x)\right|$$
$$\le K\|\psi\|_{C^{0,\kappa}(M)}\cdot\|\varphi\|_{C^{0,\nu}(M)}\cdot\Lambda^n,$$

for all $n\ge 0$. This finishes the proof for the past random correlations.

4.5. Proof of the exponential decay of the (quenched) future random correlation

In this subsection, we prove the exponential decay of the future random correlations.

Lemma 4.11. *Let $\psi : M \to \mathbb{R}^+$ be any positive continuous function satisfying that $\log\psi$ is $(\frac{a}{4},\kappa)$-Hölder continuous. Then for each $\omega\in\Omega$, positive function $\varphi_\omega\in C_\omega(b,c,\nu)$, for any $n\ge N$, the following holds:*

$$\left|\int_M \psi(f^n_\omega x)\varphi_\omega(x)dm - \int_M \psi(x)d\mu_{\theta^n\omega}\int_M \varphi_\omega(x)dm\right| \le K(D_4)\cdot\|\psi\|_{C^0(M)}\int_M \varphi_\omega(x)dm\cdot\Lambda^n. \tag{4.103}$$

Recall that $K(D_4)$ is defined in Lemma 4.9.

Proof. We first prove the case that positive function $\varphi_\omega\in C_\omega(b,c,\nu)$ satisfies $\int_M\varphi_\omega(x)dm(x)=1$. Note that by Lemma 4.5, for any $n\ge N,\ k\ge 0,\ L^n_\omega\varphi_\omega,\ L^{n+k}_{\theta^{-k}\omega}1\in C_{\theta^n\omega}(b,c,\nu)$ are positive functions. Similar proof as (4.82) can be applied on the fiber $\{\theta^n\omega\}$ to show that for $n\ge N$,

$$\begin{aligned}&\left|\int_M \psi(x)(L^n_\omega\varphi_\omega)(x)dm(x) - \int_M \psi(x)(L^{n+k}_{\theta^{-k}\omega}\mathbf{1})(x)dm(x)\right|\\
&\le \|\psi\|_{C^0(M)}\left(e^{d_{+,\theta^n\omega}(L^n_\omega\varphi_\omega,L^{n+k}_{\theta^{-k}\omega}\mathbf{1})}-1\right)\\
&\le \|\psi\|_{C^0(M)}\left(e^{d_{+,\theta^n\omega}(L^n_\omega\varphi_\omega,L^n_\omega L^k_{\theta^{-k}\omega}\mathbf{1})}-1\right)\\
&\overset{(4.75)}{\le} \|\psi\|_{C^0(M)}\left(e^{D_4\Lambda^n}-1\right)\\
&\le K(D_4)\cdot\|\psi\|_{C^0(M)}\Lambda^n.\end{aligned}\tag{4.104}$$

Notice that $L^{n+k}_{\theta^{-k}\omega}\mathbf{1}=L^k_{\theta^{-k}\theta^n\omega}L^n_{\theta^{-k}\omega}\mathbf{1}$, and $L^n_{\theta^{-k}\omega}\mathbf{1}\in C_{\theta^{n-k}\omega}(b,c,\nu)=C_{\theta^{-k}\theta^n\omega}(b,c,\nu)$. Moreover, $\int_M L^n_{\theta^{-k}\omega}\mathbf{1}dm=1$ for all $k\in\mathbb{N}$. By Remark 4.4, we have

$$\lim_{k\to\infty}\int_M \psi(x)(L^{n+k}_{\theta^{-k}\omega}\mathbf{1})(x)dm(x)=\lim_{k\to\infty}\int_M \psi\cdot L^k_{\theta^{-k}\theta^n\omega}(L^n_{\theta^{-k}\omega}\mathbf{1})dm=\int_M\psi(x)d\mu_{\theta^n\omega}(x).$$

Therefore, let $k\to\infty$ in (4.104), then for $n\ge N$,

$$\left|\int_M \psi(x)(L_\omega^n\varphi_\omega)(x)dm(x) - \int_M \psi(x)d\mu_{\theta^n\omega}(x)\right| \le K(D_4)\cdot\|\psi\|_{C^0(M)}\Lambda^n. \tag{4.105}$$

Note that by (4.85), we have

$$\int_M \psi(x)(L_\omega^n\varphi_\omega)(x)dm(x) = \int_M \psi(f_\omega^n x)\varphi_\omega(x)dm(x).$$

Hence (4.105) becomes that for any $n \ge N$,

$$\left|\int_M \psi(f_\omega^n x)\varphi_\omega(x)dm(x) - \int_M \psi(x)d\mu_{\theta^n\omega}(x)\right| \le K(D_4)\cdot\|\psi\|_{C^0(M)}\Lambda^n. \tag{4.106}$$

Now for any positive function $\varphi_\omega \in C_\omega(b,c,\nu)$, let $\tilde\varphi_\omega(x) := \varphi_\omega(x)/\int_M \varphi_\omega dm$. Then (4.103) can be proved by replacing φ_ω by $\tilde\varphi_\omega$ in (4.106). □

Now assume the function $\psi : M \to \mathbb{R}$ such that $\psi > 0$, and $\log\psi$ is $(a/4,\kappa)$-Hölder continuous. Let $\varphi : M \to \mathbb{R}^+$ with $\log\varphi$ is (c_1,ν)-Hölder continuous. Then by the Lemma 4.10, we have

$$\varphi\cdot L^l_{\theta^{-l}\omega}\mathbf{1} \in C_\omega(b,c,\nu) \text{ for all } l\in\mathbb{N} \text{ all } \omega\in\Omega.$$

Now, we apply Lemma 4.11 to obtain that for all $\omega\in\Omega$, $n \ge N$,

$$\left|\int_M \psi(f_\omega^n x)(\varphi\cdot L^l_{\theta^{-l}\omega}\mathbf{1})(x)dm(x) - \int_M \psi(x)d\mu_{\theta^n\omega}(x)\int_M (\varphi\cdot L^l_{\theta^{-l}\omega}\mathbf{1})(x)dm(x)\right|$$
$$\le K(D_4)\|\psi\|_{C^0(M)}\int_M \varphi\cdot(L^l_{\theta^{-l}\omega}\mathbf{1})dm\cdot\Lambda^n, \text{ for all } l\in\mathbb{N}.$$

Let $l\to\infty$, by (4.87), for all $\omega\in\Omega$, $n\ge N$,

$$\begin{aligned}&\left|\int_M \psi(f_\omega^n x)\varphi(x)d\mu_\omega(x) - \int_M \psi(x)d\mu_{\theta^n\omega}(x)\int_M \varphi(x)d\mu_\omega(x)\right|\\ \le& K(D_4)\|\psi\|_{C^0(M)}\int_M \varphi d\mu_\omega\cdot\Lambda^n \le K(D_4)\|\psi\|_{C^0(M)}\cdot\|\varphi\|_{C^0(M)}\cdot\Lambda^n.\end{aligned} \tag{4.107}$$

Finally, given $\psi\in C^{0,\kappa}(M)$ and $\varphi\in C^{0,\nu}(M)$. If ψ or φ is a constant function, then by (4.88),

$$\left|\int_M \psi(f_\omega^n x)\varphi(x)d\mu_\omega(x) - \int_M \psi(x)d\mu_{\theta^n\omega}(x)\int_M \varphi(x)d\mu_\omega(x)\right| = 0.$$

Therefore, to prove the exponential decay of past random correlations, it is sufficient to consider the case that both $\psi \in C^{0,\kappa}(M)$ and $\varphi \in C^{0,\nu}(M)$ are nonconstant functions. Let

$$B_\psi = \frac{5|\psi|_\kappa}{a} > 0,\ B_\varphi = \frac{2|\varphi|_\nu}{c_1} > 0,$$

and define

$$\psi^+_{B_\psi} = \frac{1}{2}(|\psi| + \psi) + B_\psi,\ \psi^-_{B_\psi} = \frac{1}{2}(|\psi| - \psi) + B_\psi,$$
$$\varphi^+_{B_\varphi} = \frac{1}{2}(|\varphi| + \varphi) + B_\varphi,\ \varphi^-_{B_\varphi} = \frac{1}{2}(|\varphi| - \varphi) + B_\varphi.$$

As before, $\log \psi^\pm_{B_\psi}$ are $(a/4, \kappa)$-Hölder continuous, and $\log \varphi^\pm_{B_\varphi}$ are (c_1, ν) Hölder continuous. By (4.107) and the linearity of integration, we conclude for $n \ge N$,

$$\begin{aligned}&\left| \int_M \psi(f^n_\omega x)\varphi(x)d\mu_\omega(x) - \int_M \psi(x)d\mu_{\theta^n\omega}(x) \int_M \varphi(x)d\mu_\omega(x) \right| \\ \le & 4K(D_4) \cdot \max\left\{1, \frac{5}{a}\right\} \cdot \max\left\{1, \frac{2}{c_1}\right\} \cdot \|\psi\|_{C^{0,\kappa}(M)} \cdot \|\varphi\|_{C^{0,\nu}(M)} \cdot \Lambda^n.\end{aligned}$$

Recall that K is defined in (4.102), then we arrive

$$\left| \int_M \psi(f^n_\omega x)\varphi(x)d\mu_\omega(x) - \int_M \psi(x)d\mu_{\theta^n\omega}(x) \int_M \varphi(x)d\mu_\omega(x) \right| \le K \cdot \|\psi\|_{C^{0,\kappa}(M)} \cdot \|\varphi\|_{C^{0,\nu}(M)} \cdot \Lambda^n,$$

for all $n \ge 0$. This finishes the proof for the future random correlations.

Data availability

No data was used for the research described in the article.

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Appendix A

A.1. Examples of Anosov and mixing on fibers system

Theorem 8.1 in [22] showed that the random dynamical systems satisfying the following conditions are topological mixing on fibers:

(A1) (Ω, θ) is a minimal irrational rotation on the compact torus;
(A2) ϕ is Anosov on fibers;
(A3) ϕ is topological transitive on $M \times \Omega$.

Remark A.1. This class of systems can't be topological mixing. In fact, it is well-known that a factor of a topological mixing system is also topological mixing. Notice that the irrational rotation is not topological mixing, hence ϕ can't be topological mixing.

A typical example satisfying (A1)-(A3) is given by following. Fiber Anosov maps on 2-d tori: Let θ be a homeomorphism on a compact metric space Ω and let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Define $\phi : \mathbb{T}^2 \times \Omega \to \mathbb{T}^2 \times \Omega$ by

$$\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}, \omega\right) = \left(A\begin{pmatrix} x \\ y \end{pmatrix} + h(\omega), \theta\omega\right) = \left(\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h_1(\omega) \\ h_2(\omega) \end{pmatrix}, \theta\omega\right),$$

where h is a continuous map from Ω to $\mathbb{T}^2$.

Theorem 8.2 in [22] showed that the systems satisfying the following conditions are also mixing on fibers:

(B1) (Ω, θ) is a homeomorphism on a compact metric space;
(B2) ϕ is Anosov on fibers;
(B3) There exists an f_ω-invariant Borel probability measure ν with full support (i.e. $\mathrm{supp}\nu = M$).

A typical example satisfying (B1)-(B3) is given by following. Random composition of 2×2 area-preserving positive matrices: let

$$\left\{B_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\right\}_{1 \le i \le p}$$

be 2×2 matrices with $a_i, b_i, c_i, d_i \in \mathbb{Z}^+$, and $|a_i d_i - c_i b_i| = 1$ for any $i \in \{1, ..., p\}$. Let $\Omega = \mathcal{S}_p := \{1, ..., p\}^{\mathbb{Z}}$ with the left shift operator θ be the symbolic dynamical system with p symbols. For any $\omega = (..., \omega_{-1}, \omega_0, \omega_1, ...) \in \Omega$, we define $g(\omega) = B_{\omega_0}$. Then the skew product $\tilde{\phi} : \Omega \times \mathbb{T}^2 \to \Omega \times \mathbb{T}^2$ defined by

$$\tilde{\phi}(x, \omega) = \big(g(\omega)x, \theta\omega\big)$$

is an Anosov on fibers system with continuous co-invariant splitting $\mathbb{R}^2 = E^u_\omega \oplus E^s_\omega$ for $\omega \in \mathcal{S}_p$, on which

$$\|Dg_\omega v\| \ge \kappa \|v\| \text{ for } v \in E^u_\omega;$$
$$\|Dg_\omega \eta\| \le \kappa^{-1} \|\eta\| \text{ for } \eta \in E^s_\omega$$

for $\kappa := \min_{1 \le i \le p} \min\{\sqrt{a_i^2 + c_i^2}, \sqrt{b_i^2 + d_i^2}\} \ge \sqrt{2}$ by Proposition 8.2 in [22].

A.2. Random SRB measure for Anosov and mixing on fibers systems

Let $F : \mathbb{Z} \times \Omega \times M \to M$ be a continuous random dynamical system over an invertible ergodic metric dynamical systems $(\Omega, \mathcal{B}, P, \theta)$.

Definition A.1. A map $K : \Omega \to 2^M$ is called a closed random set if $K(\omega)$ is closed for any $\omega \in \Omega$; and $\omega \mapsto d(x, K(\omega))$ is measurable for each fixed $x \in M$. $U : \Omega \to 2^M$ is called an open random set if U^c is a random closed set.

Definition A.2. F is called random topological transitive if for any given open random sets U and V with $U(\omega), V(\omega) \neq \emptyset$ for all $\omega \in \Omega$, there exists a random variable n taking values in $\mathbb{Z}$ such that the intersection $F(n(\omega), \theta^{-n(\omega)})U(\theta^{-n(\omega)}\omega) \cap V(\omega) \neq \emptyset$ $\mathbb{P}$-a.s.

The following Lemma is the Lemma A.1 in [22].

Lemma A.1. *If F is topological mixing on fibers, then F is random topological transitive.*

The following theorem is Theorem 4.3 in [20], which is the main result of the SRB measure for random hyperbolic systems.

Theorem 2. *Let F be a $C^{1+\alpha}$ RDS with a random topological transitive hyperbolic attractor $\Lambda \subset M \times \Omega$. Then there exists a unique F-invariant measure (SRB-measure) ν supported by Λ and characterized by each of the following:*

(i) $h_\nu(F) = \int \sum \lambda_i^+ d\nu$, *where λ_i are the Lyapunov exponents corresponding to ν;*
(ii) *$\mathbb{P}$-a.s. the conditional measure of ν_ω on the unstable manifolds are absolutely continuous with respect to the Riemannian volume on these submanifolds;*
(iii) $h_\nu(F) + \int f d\nu = \sup\{h_\mu(F) + \int f d\mu : F - \text{invariant } \mu\}$ *and the later is the topological pressure $\pi_F(f)$ of f which satisfies $\pi_F(f) = 0$;*
(iv) $\nu = \psi\tilde{\mu}$, *where ψ is the conjugation between F on Λ and two-sided shift σ on Σ_A, and $\tilde{\mu}$ is the equilibrium state for the σ and function $f \circ \psi$. The measure $\tilde{\mu}$ can be obtained as a natural extension of the probability measure μ which is invariant with respect to the one-sided shift on Σ_A^+ and such that $L_\eta^* \mu_{\theta\omega} = \mu_\omega$ $\mathbb{P}$-a.s. where $\eta - f \circ \psi = h - h \circ (\theta \times \sigma)$ for some random Hölder continuous function h;*
(v) *ν can be obtained as a weak* limit $\nu_\omega = \lim_{n\to} F(n, \theta^{-n}\omega)m_{\theta^{-n}\omega}$ $\mathbb{P}$-a.s. for any measure m_ω absolutely continuous with respect to the Riemannian volume such that $\mathrm{supp} m_\omega \subset U(\omega)$.*

A.3. Anosov on fibers system and partially hyperbolic systems

In this section, we show that Anosov on fibers system contains a class of partially hyperbolic systems.

Definition A.3. (M, f) is called partially hyperbolic in the narrow sense if the tangent bundle admits a splitting into three continuous vector subbundles $T_xM = E^1(x) \oplus E^2(x) \oplus E^3(x)$ which satisfy

(1) dominated splitting, i.e., $D_x f(E^i(x)) = E^i(f(x))$ for $i = 1, 2, 3$, and there exists a constant $c > 0$ and $\lambda \in (0, 1)$ such that $\|Df^n|_{E^i(x)}\| \le c\lambda^n \|Df^n|_{E^{i+1}(x)}\|$ for $i = 1, 2$,
(2) $E^1(x)$ is uniformly contracted and $E^3(x)$ is uniformly expanded under the action of $D_x f$.

We denote the dominated splitting by $T_x M = E^1(x) \oplus_< E^2(x) \oplus_< E^3(x)$.

Let Ω be a compact smooth manifold, and let $\theta : \Omega \to \Omega$ be a diffeomorphism. Denote $f(x, \omega) := f_\omega(x)$ and $\phi^{-1}(x, \omega) = (f_\omega^{-1}(x), \theta^{-1}\omega)$.

Proposition A.1. *Assume*

(a) $\phi : M \times \Omega \to M \times \Omega$ *is Anosov on fibers,*
(b) $f(x, \omega)$ *and* $f_\omega^{-1}(x)$ *are* C^1 *in* ω,
(c) The diffeomorphism θ *satisfying:*

$$\sup_{(x,\omega)\in M\times\Omega} \|D_x f_\omega|_{E^s(x,\omega)}\| < \inf_{\omega\in\Omega} \|D_\omega\theta^{-1}\|^{-1} := m_1$$
$$\le \sup_{\omega\in\Omega} \|D_\omega\theta\| := m_2 < \inf_{(x,\omega)\in M\times\Omega} \|D_x(f_{\theta^{-1}\omega})^{-1}|_{E^u(x,\omega)}\|^{-1}.$$

Then ϕ *is partially hyperbolic in the narrow sense.*

Proof. We first show the existence of a dominated splitting. Note that $T_{(x,\omega)}M \times \Omega = T_x M \times T_\omega\Omega$ already has a splitting $E^u(x, \omega) \times \{0\} \oplus E^s(x, \omega) \times \{0\} \oplus \{0\} \times T_\omega\Omega$, but this splitting may not be invariant. For any $v \in T_x M \times T_\omega\Omega$, then $v = v_1 + v_2 + v_3$ according to the above splitting. Notice that $\|D\phi(x, \omega)v_3\|$ only depends on $\|D_\omega f(x, \omega)\|$ and $\|D_\omega\theta\|$. $\|D\phi^{-1}(x, \omega)v_3\|$ only depends on $\|D_\omega(f_{\theta^{-1}\omega})^{-1}(x)\|$ and $\|D_\omega\theta^{-1}\|$, then by the compactness of M and Ω, there exists a number K such that

$$\|D\phi(x, \omega)v_3\| \le K\|v_3\|, \ \|D\phi^{-1}(x, \omega)v_3\| \le K\|v_3\|.$$

We let $P(E^u(x, \omega) \times \{0\})$ denote the projection map from $T_x M \times T_\omega\Omega$ to $E^u(x, \omega) \times \{0\}$ with respect to the splitting $E^u(x, \omega) \times \{0\} \oplus E^s(x, \omega) \times \{0\} \oplus \{0\} \times T_\omega\Omega$. $P(E^s(x, \omega) \times \{0\})$ and $P(\{0\} \times T_\omega\Omega)$ are similar notations. Since $E^s(x, \omega)$ and $E^u(x, \omega)$ are uniformly continuous on x and ω, there exists a number $\mathcal{P} > 1$ such that

$$\sup\{\|P(E^s(x, \omega) \times \{0\})\|, \|P(E^u(x, \omega) \times \{0\})\| : \ (x, \omega) \in M \times \Omega\} < \mathcal{P}.$$

Now consider the cone

$$C(x, \omega) := \{v \in T_x M \times T_\omega\Omega | \ \|v_2\| + b\|v_3\| \le \|v_1\|\},$$

where b is a number such that

$$b > \frac{2\mathcal{P}K}{e^{\lambda_0} - m_2}. \tag{A.1}$$

Denote

$$c_0 = \max\left\{\frac{2\mathcal{P}K + bm_2}{e^{\lambda_0}b}, e^{-2\lambda_0}\right\} \in (0,1).$$

For any $v \in C(x,\omega)$, we have

$$\begin{aligned}
D\phi(x,\omega)v &= D\phi(x,\omega)v_1 + D\phi(x,\omega)v_2 + D\phi(x,\omega)v_3 \\
&= D\phi(x,\omega)v_1 + P(E^u(\phi(x,\omega)) \times \{0\})D\phi(x,\omega)v_3 \\
&\qquad + D\phi(x,\omega)v_2 + P(E^s(\phi(x,\omega)) \times \{0\})D\phi(x,\omega)v_3 \\
&\qquad\quad + P(\{0\} \times T_{\theta\omega}\Omega)D\phi(x,\omega)v_3 \\
&= (D\phi(x,\omega)v)_1 + (D\phi(x,\omega)v)_2 + (D\phi(x,\omega)v)_3.
\end{aligned}$$

Then

$$\begin{aligned}
&\|(D\phi(x,\omega)v)_2\| + b\|(D\phi(x,\omega)v)_3\| \\
&\quad \le e^{-\lambda_0}\|v_2\| + \mathcal{P}K\|v_3\| + b\cdot m_2\|v_3\| \\
&\quad = e^{-\lambda_0}\|v_2\| + (\mathcal{P}K + bm_2)\|v_3\| \\
&\quad = e^{-\lambda_0}\|v_2\| + (2\mathcal{P}K + bm_2)\|v_3\| - \mathcal{P}K\|v_3\| \\
&\quad \le e^{\lambda_0}\Big(e^{-2\lambda_0}\|v_2\| + e^{-\lambda_0}(2\mathcal{P}K + bm_2)\|v_3\|\Big) - c_0\mathcal{P}K\|v_3\| \\
&\quad \overset{(A.1)}{<} e^{\lambda_0}(c_0\|v_2\| + c_0 b\|v_3\|) - c_0\mathcal{P}K\|v_3\| \\
&\quad \le c_0 e^{\lambda_0}\|v_1\| - c_0\mathcal{P}K\|v_3\| \\
&\quad \le c_0\|(D\phi(x,\omega)v)_1\|.
\end{aligned}$$

Hence $D\phi(x,\omega)C(x,\omega) \subset intC(\phi(x,\omega))$. By the cone-field criteria (see Theorem 2.6 in [16]), $T_xM \times T_\omega\Omega$ has a dominated splitting $S_1(x,\omega) \oplus_< S_2(x,\omega)$ with $\dim(S_2(x,\omega)) = \dim(E^u(x,\omega) \times \{0\})$. Notice that $E^u(x,\omega) \times \{0\}$ lies in $C(x,\omega)$ and it is invariant under $D\phi(x,\omega)$, so $S_2(x,\omega) = E^u(x,\omega) \times \{0\}$.

On the other hand, consider another cone

$$\mathcal{C}(x,\omega) = \{v \in T_xM \times T_\omega\Omega :\ \|v_1\| + d\|v_3\| \le \|v_2\|\},$$

where

$$d > \frac{2\mathcal{P}K}{e^{\lambda_0} - m_1^{-1}}. \tag{A.2}$$

Denote

$$c_1 = \max\left\{\frac{2\mathcal{P}K + m_1^{-1}d}{de^{\lambda_0}}, e^{-2\lambda_0}\right\} \in (0,1).$$

For any $v \in \mathcal{C}(x,\omega)$, we have

$$\begin{aligned}
D\phi^{-1}(x,\omega)v &= D\phi^{-1}(x,\omega)v_1 + D\phi^{-1}(x,\omega)v_2 + D\phi^{-1}(x,\omega)v_3\\
&= D\phi^{-1}(x,\omega)v_1 + P(E^u(\phi^{-1}(x,\omega))\times\{0\})D\phi^{-1}(x,\omega)v_3\\
&\quad + D\phi^{-1}(x,\omega)v_2 + P(E^s(\phi^{-1}(x,\omega))\times\{0\})D\phi^{-1}(x,\omega)v_3\\
&\quad + P(\{0\}\times T_\omega\Omega)D\phi^{-1}(x,\omega)v_3\\
&= (D\phi^{-1}(x,\omega)v)_1 + (D\phi^{-1}(x,\omega)v)_2 + (D\phi^{-1}(x,\omega)v)_3.
\end{aligned}$$

Then

$$\begin{aligned}
&\|(D\phi^{-1}(x,\omega)v)_1\| + d\|(D\phi^{-1}(x,\omega)v)_3\|\\
&\le e^{-\lambda_0}\|v_1\| + \mathcal{P}K\|v_3\| + d\cdot m_1^{-1}\|v_3\|\\
&\le e^{-\lambda_0}\|v_1\| + (2\mathcal{P}K + dm_1^{-1})\|v_3\| - c_1\mathcal{P}K\|v_3\|\\
&\le e^{\lambda_0}(e^{-2\lambda_0}\|v_1\| + e^{-\lambda_0}(2\mathcal{P}K + dm_1^{-1})\|v_3\|) - c_1\mathcal{P}K\|v_3\|\\
&\overset{(A.2)}{<} e^{\lambda_0}(c_1\|v_1\| + c_1 d\|v_3\|) - c_1\mathcal{P}K\|v_3\|\\
&\le c_1 e^{\lambda_0}\|v_2\| - c_1\mathcal{P}K\|v_3\|\\
&\le c_1\|(D\phi^{-1}(x,\omega)v)_2\|.
\end{aligned}$$

Hence $D\phi^{-1}(x,\omega)\mathcal{C}(x,\omega) \subset int(\mathcal{C}(\phi^{-1}(x,\omega)))$. By the cone-field criteria, $T_xM \times T_\omega\Omega$ has a dominated splitting $H_1(x,\omega) \oplus_< H_2(x,\omega)$ with $\dim H_1(x,\omega) = \dim(E^s(x,\omega)\times\{0\})$. Notice that $E^s(x,\omega)\times\{0\}$ lies in cone $\mathcal{C}(x,\omega)$ and it is invariant under $D\phi(x,\omega)$, so $H_1(x,\omega) = E^s(x,\omega)\times\{0\}$.

Now $T_xM \times T_\omega\Omega$ has two dominated splittings: $S_1(x,\omega)\oplus_<(E^u(x,\omega)\times\{0\})$ and $(E^s(x,\omega)\times\{0\})\oplus_< H_2(x,\omega)$. Then, by uniqueness of the dominated splitting (Proposition 2.2 in [16]), we have

$$T_xM \times T_\omega\Omega = (E^s(x,\omega)\times\{0\}) \oplus_< (S_1(x,\omega)\cap H_2(x,\omega)) \oplus_< (E^u(x,\omega)\times\{0\}).$$

Besides, we already know that $E^s(x,\omega)\times\{0\}$ is uniformly contracted under $D\phi(x,\omega)$ and $E^u(x,\omega)\times\{0\}$ is uniformly expanded under $D\phi(x,\omega)$. Hence ϕ is partially hyperbolic in the narrow sense. □

A.4. Convex cone, projective metric and Birkhoff's inequality

In this section, we introduce the notion of projective metric associated to a convex cone in a topological vector space. The following knowledge is borrowed from [39, Section 2.1], and also can be found in [30].

Definition A.4. Let E be a topological vector space. A subset $C \subset E$ is said to be a convex cone if

(1) $tv \in C$ for $v \in C$ and $t \in \mathbb{R}^+$;
(2) for any $t_1, t_2 \in \mathbb{R}^+$, $v_1, v_2 \in C$, then $t_1v_1 + t_2v_2 \in C$;

(3) $\bar{C} \cap -\bar{C} = \{0\}$, where $\bar{C}$ is the "integral closure" in a weaker sense, and it is defined by: $w \in \bar{C}$ if and only if there are $v \in C$ and $t_n \searrow 0$ such that $w + t_n v \in C$ for all $n \geq 1$.

Definition A.5. For a convex cone $C \subset E$, given any $v_1, v_2 \in C$, we define

$$\alpha(v_1, v_2) := \sup\{t > 0 : \ v_2 - t v_1 \in C\};$$
$$\beta(v_1, v_2) := \inf\{s > 0 : \ s v_1 - v_2 \in C\},$$

with the convention that $\sup \emptyset = 0$ and $\inf \emptyset = +\infty$. The projective metric between $v_1, v_2 \in C$ is defined by

$$d_C(v_1, v_2) = \log \frac{\beta(v_1, v_2)}{\alpha(v_1, v_2)}$$

with the convention that $d_C(v_1, v_2) = \infty$ if $\alpha(v_1, v_2) = 0$ or $\beta(v_1, v_2) = \infty$.

Proposition A.2. *d_C is a metric in the projective quotient of C, i.e.,*

(1) $d_C(v_1, v_2) \geq 0$ for all $v_1, v_2 \in C$ (guaranteed by (3) in Definition A.4);
(2) $d_C(v_1, v_2) = d_C(v_2, v_1)$ for all $v_1, v_2 \in C$;
(3) $d_C(v_1, v_3) \leq d_C(v_1, v_2) + d_C(v_2, v_3)$ for all $v_1, v_2, v_3 \in C$;
(4) $d_C(v_1, v_2) = 0$ if and only if there exists $t \in \mathbb{R}^+$ such that $v_1 = t v_2$.

Proposition A.3 *(Birkhoff's inequality). Let $E_1, \ E_2$ be two topological vector spaces, and $C_i \subset E_i$, for $i = 1, 2$ be convex cones. Let $L : E_1 \to E_2$ be a linear operator and assume that $L(C_1) \subset C_2$. Let $D = \sup\{d_{C_2}(L(v_1), L(v_2)) : \ v_1, v_2 \in C_1\}$. If $D < \infty$, then*

$$d_{C_2}(L(v_1), L(v_2)) \leq (1 - e^{-D}) d_{C_1}(v_1, v_2) \text{ for all } v_1, v_2 \in C_1.$$

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