Change of Basis

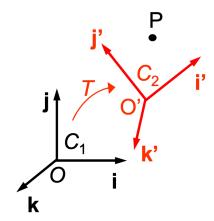
<u>Recall</u>: (Coordinate Systems) In homogeneous coordinate systems, a vector is denoted as:

$$oldsymbol{v} = v_1 oldsymbol{a} + v_2 oldsymbol{b} + v_3 oldsymbol{c}
ightarrow oldsymbol{v} = egin{bmatrix} oldsymbol{a} & oldsymbol{b} & oldsymbol{c} & O \end{bmatrix} egin{bmatrix} v_1 \ v_2 \ v_3 \ 0 \end{bmatrix}$$

A point is

$$P=p_1oldsymbol{a}+p_2oldsymbol{b}+p_3oldsymbol{c}+O o P=[oldsymbol{a} oldsymbol{b} oldsymbol{c} oldsymbol{c} O]egin{bmatrix} p_1\ p_2\ p_3\ 1 \end{bmatrix}$$

Transformations as a change of basis



Question: if we apply a transformation, M_1 , to coordinate system C_1 , and get C_2 , what is the new position of P in C_2 ?

We use $[x,y,z,1]^T$ to represent P's posotion in C_1 , and $[x',y',z',1]^T$ to represent P's posotion in C_2 .

Solution:

In coordinate system C_1 :

$$P = x oldsymbol{i} + y oldsymbol{j} + z oldsymbol{k} + O = \left[oldsymbol{i} \quad oldsymbol{j} \quad oldsymbol{k} \quad O
ight] egin{bmatrix} x \ y \ z \ 1 \end{bmatrix}$$

In coordinate system C_2 :

$$P = x'oldsymbol{i}' + y'oldsymbol{j}' + z'oldsymbol{k}' + O' = [oldsymbol{i}' \quad oldsymbol{j}' \quad oldsymbol{k}' \quad O'] egin{bmatrix} x' \ y' \ z' \ 1 \end{bmatrix}$$

Since C_2 is transformed from C_1 with M_1 ,

$$M_1 \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & O \end{bmatrix} = \begin{bmatrix} \mathbf{i'} & \mathbf{j'} & \mathbf{k'} & O' \end{bmatrix}$$
 (*1)

Hence

$$egin{bmatrix} \left[m{i} & m{j} & m{k} & O
ight] egin{bmatrix} x \ y \ z \ 1 \end{bmatrix} = M_1 \left[m{i} & m{j} & m{k} & O
ight] egin{bmatrix} x' \ y' \ z' \ 1 \end{bmatrix}$$

Since $\begin{bmatrix} i & j & k & O \end{bmatrix}$ is the <u>identity matrix</u>, we have

$$egin{bmatrix} x \ y \ z \ 1 \end{bmatrix} = M_1 egin{bmatrix} x' \ y' \ z' \ 1 \end{bmatrix}
ightarrow P_{C_1} = M_1 P_{C_2}$$

l.e.,
$$P_{C_2} = M_1^{-1} P_{C_1}$$

Question: what is M_1 with respect to the basis vectors?

Referring to Eq. (*1), and since $\begin{bmatrix} i & j & k & O \end{bmatrix}$ is the identity matrix:

$$M_1 = [m{i'} \quad m{j'} \quad m{k'} \quad O' \,] = egin{bmatrix} i'_x & j'_x & k'_x & O'_x \ i'_y & j'_y & k'_y & O'_y \ i'_z & j'_z & k'_z & O'_z \ 0 & 0 & 0 & 1 \ \end{bmatrix}$$

Successive transformations

If we transform C_2 to C_3 with M_2 . We cansider the $\begin{bmatrix} \mathbf{i}' & \mathbf{j}' & \mathbf{k}' & O' \end{bmatrix}$ as the identity matrix in the section transforamtion, then,

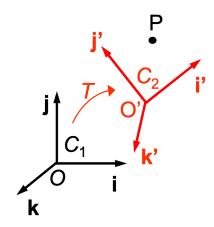
$$P_{C_2} = M_2 P_{C_2}$$

Therefore,

$$P_{C_1} = M_1 P_{C_2} = M_1 M_2 P_{C_3}$$

Note: $[i'' \quad j'' \quad k'' \quad O'']$ of C_3 is the matrix representation in the C_2 .

Coordinate system transformation with rotation and transformation



Since the i', j', and k' are orthogonal, the rotation from C_1 to C_2 is

$$R = egin{bmatrix} i'_x & j'_x & k'_x & 0 \ i'_y & j'_y & k'_y & 0 \ i'_z & j'_z & k'_z & 0 \ 0 & 0 & 0 & 1 \end{bmatrix}$$

The translation from O to O' is

$$T = egin{bmatrix} 1 & 0 & 0 & O_x' \ 0 & 1 & 0 & O_y' \ 0 & 0 & 1 & O_z' \ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus,

$$M_1 = TR = egin{bmatrix} i_x' & j_x' & k_x' & O_x' \ i_y' & j_y' & k_y' & O_y' \ i_z' & j_z' & k_z' & O_z' \ 0 & 0 & 0 & 1 \end{bmatrix}$$

P's posotion in C_2 , $[x',y',z',1]^T$, can be calculated with the inverse of the transformations

$$P_{C_2} = M_1^{-1} P_{C_1} = R^{-1} T^{-1} P_{C_1} = egin{bmatrix} i'_x & j'_x & k'_x & 0 \ i'_y & j'_y & k'_y & 0 \ i'_z & j'_z & k'_z & 0 \ 0 & 0 & 0 & 1 \end{bmatrix}^T egin{bmatrix} 1 & 0 & 0 & -O'_x \ 0 & 1 & 0 & -O'_y \ 0 & 0 & 1 & -O'_z \ 0 & 0 & 0 & 1 \end{bmatrix} P_{C_1}$$

Model-View Transformation

Given eye point P_{eye} , reference point P_{ref} , and up vector \mathbf{v}_{up} , build M_{cam} for model-view transformation.

$$egin{aligned} m{k}' &= rac{P_{eye} - P_{ref}}{|P_{eye} - P_{ref}|} \ m{i}' &= rac{m{v}_{up} imes m{k}'}{|m{v}_{up} imes m{k}'|} \ m{j}' &= m{k}' imes m{i}' \end{aligned}$$

The new origin $O^\prime=P_{eye}.$ Thus,

$$M_{cam} = egin{bmatrix} i'_x & j'_x & k'_x & O'_x \ i'_y & j'_y & k'_y & O'_y \ i'_z & j'_z & k'_z & O'_z \ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transform from WCS (word coordinate system) to VCS (view coordinate system)

$$P_{vcs} = M_{cam}^{-1} P_{wcs} = egin{bmatrix} i'_x & j'_x & k'_x & 0 \ i'_y & j'_y & k'_y & 0 \ i'_z & j'_z & k'_z & 0 \ 0 & 0 & 0 & 1 \end{bmatrix}^T egin{bmatrix} 1 & 0 & 0 & -O'_x \ 0 & 1 & 0 & -O'_y \ 0 & 0 & 1 & -O'_z \ 0 & 0 & 0 & 1 \end{bmatrix} P_{wcs}$$