MILNOR'S EXOTIC 7-SPHERES: AN EXPOSITION FOR ADVANCED UNDERGRADUATES

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Abstract. This article gives a self-contained, computation-forward route to understanding Milnor's classic construction of **exotic 7-spheres**. The intended audience is advanced undergraduates and beginning graduate students with background in analysis, linear and abstract algebra, point-set topology, and basic differential topology of manifolds, but with no prior exposure to algebraic topology, homological algebra, or characteristic classes. We build exactly the machinery needed–homotopy, homology, cohomology, Gysin sequences, and the first Pontryagin class–and after each new tool we include a concrete **Proposition** with a full calculation (for instance $H_*(S^n)$, $H^*(S^4 \times S^3)$, or $\pi_3(SO(4))$). We then read Milnor's paper in our own order: construct $M_{h,\ell} = S(\xi_{h,\ell})$, use a Morse-Reeb argument to show $M_{h,\ell}$ is a topological S^7, define the invariant $X(M) \in \mathbb{Z}/7$, prove it is well-defined, and compute $X(M_k) \equiv k^2 - 1 \pmod 7$ on the subfamily $m + \ell = 1$ to separate smooth structures.

Contents

1. Introduction

1.1. **Historical background and motivation.** By the mid-twentieth century the topology of spheres was well understood: every simply connected closed n-manifold with the homology of S^n was known to be homeomorphic to S^n. What remained unclear was whether such a manifold could admit more than one smooth structure.

In 1956, Milnor produced the first explicit counterexample. He constructed a two-parameter family of smooth 7-manifolds $M_{h,\ell}$ that are all homeomorphic to the standard sphere but not necessarily diffeomorphic to one another. The construction exploits the group $SO(4) \cong (S^3 \times S^3)/\{\pm 1\}$: each element of $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$ determines an S^3-bundle over S^4 obtained by clutching two trivial bundles along the equator. The total space $M_{h,\ell} = S(\xi_{h,\ell})$ of the associated unit-sphere bundle is a smooth 7-manifold whose topology depends on the Euler class $e(\xi_{h,\ell}) \in H^4(S^4)$ and the first Pontryagin class $p_1(\xi_{h,\ell}) \in H^4(S^4)$.

The analysis proceeds in three layers:

(1) **Topological identification.** Using the Gysin sequence one computes $H^*(M_{h,\ell};\mathbb{Z})$, showing that for $|h+\ell| = 1$ the cohomology ring coincides with that of S^7. A classical result of Reeb in Morse theory implies $M_{h,\ell}$ is homeomorphic to S^7.

(2) **Smooth-structure detection.** Each $M_{h,\ell}$ bounds a natural 8-manifold $B_{h,\ell}$, the disk bundle $D(\xi_{h,\ell})$. Hirzebruch's signature theorem expresses the signature $\sigma(B_{h,\ell})$ as a rational linear combination of Pontryagin numbers, leading to the congruence

$$\langle 7p_2(TB) - p_1(TB)^2, [B]\rangle \equiv 0 \pmod{45}.$$

From this Milnor defined an invariant

$$X(M) = \frac{1}{7}(\langle p_1(TB)^2, [B]\rangle - 7\sigma(B)),$$

independent of the choice of filling B. Distinct values of $X(M_{h,\ell})$ imply non-diffeomorphic smooth structures on the same topological sphere.

(3) **Outcome and significance.** Evaluating $X(M_{h,\ell})$ shows that among these $M_{h,\ell}$ there exist non-diffeomorphic examples, inaugurating the modern theory of exotic spheres.

1.2. **Audience and approach.** This exposition is intended for advanced undergraduates and beginning graduate students. We assume familiarity with basic abstract algebra and point-set topology, together with a working knowledge of smooth manifolds. We do *not* assume prior exposure to algebraic topology and characteristic classes.

We follow Milnor's original paper in logical order rather than historical sequence. Each concept is introduced only when it becomes indispensable to the argument, and every computation is carried out explicitly. Algebraic topology is developed in a "minimal working set'' fashion: homotopy, (co)homology, and characteristic classes are treated only to the extent required to reproduce Milnor's calculations.

2. Preliminaries beyond algebraic topology

2.1. Vector bundles and basic operations.

Definition 2.1 (Vector bundle)**.** Let M be a smooth manifold and $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. A **vector bundle** of rank k over M consists of a smooth manifold E, called the **total space**, and a smooth surjective map $\pi : E \to M$ such that:

(1) There exists an open cover $\{U_\alpha\}$ of M and a family of diffeomorphisms $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{F}^k$, called **local trivializations**, such that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "\Phi_\alpha"] \arrow[d, "\pi_\alpha"'] & U_\alpha \times \mathbb{F}^k \arrow[dl, "\mathrm{pr}_{U_\alpha}"] \\ U_\alpha & \end{tikzcd}$$

where pr_{U_α} denotes the projection onto the first factor.

(2) For any overlap $U_\alpha \cap U_\beta \neq \varnothing$, the transition map

$$\Phi_\beta \circ \Phi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{F}^k \to (U_\alpha \cap U_\beta) \times \mathbb{F}^k$$

has the form $(x, v) \mapsto (x, g_{\alpha\beta}(x)v)$, where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_k(\mathbb{F})$ is a smooth map.

When $\mathbb{F} = \mathbb{R}$, the bundle is called a **real vector bundle**; when $\mathbb{F} = \mathbb{C}$, it is called a **complex vector bundle**.

Definition 2.2 (Bundle map)**.** Let $\pi_E : E \to M$ and $\pi_F : F \to N$ be vector bundles. A **bundle map** from E to F is a smooth map $\phi : E \to F$ together with a smooth map $f : M \to N$ such that $\pi_F \circ \phi = f \circ \pi_E$.

Definition 2.3 (Pullback)**.** Given $f : B' \to B$ and a vector bundle $\pi : E \to B$, the **pullback bundle** $f^*E \to B'$ has total space

$$f^*E = \{(b', e) \in B' \times E : f(b') = \pi(e)\}$$

and projection $(b', e) \mapsto b'$.

Definition 2.4 (Basic operations)**.** Let ξ, η be bundles over B.

(1) The **Whitney sum** $\xi \oplus \eta$ has fiber $(\xi \oplus \eta)_b = \xi_b \oplus \eta_b$.
(2) The **Tensor product** $\xi \otimes \eta$ has fiber $(\xi \otimes \eta)_b = \xi_b \otimes \eta_b$ (over $\mathbb{R}$ or $\mathbb{C}$).
(3) The **Dual** $\xi^\vee$ has fiber $(\xi^\vee)_b = \mathrm{Hom}(\xi_b, \mathbb{R})$ (resp. $\mathrm{Hom}_{\mathbb{C}}(\xi_b, \mathbb{C})$).
(4) The **Complexification** of a real bundle is $\xi_{\mathbb{C}} = \xi \otimes_{\mathbb{R}} \mathbb{C}$; the underlying real bundle of a complex bundle η is denoted $\eta_{\mathbb{R}}$.

These operations are functorial with respect to pullback.

Definition 2.5 (Bundle metric)**.** Let $\pi : E \to M$ be a real vector bundle. A **bundle metric** is a smooth function $\langle \cdot, \cdot \rangle : E \times_M E \to \mathbb{R}$ such that for each $p \in M$, the restriction

$$\langle \cdot, \cdot \rangle_p : E_p \times E_p \to \mathbb{R}$$

is an inner product on the real vector space E_p that depends smoothly on p.

Remark 2.6. Such metrics always exist by partition of unity.

Definition 2.7 (Sphere and disk bundles)**.** Fix a bundle metric on a real bundle $\xi \to B$. The **unit sphere bundle** $S(\xi) \to B$ has fiber $S(\xi)_b = \{v \in \xi_b : \|v\| = 1\}$; the **unit disk bundle** $D(\xi) \to B$ has fiber $D(\xi)_b = \{v : \|v\| \le 1\}$. The boundary of $D(\xi)$ is canonically identified with $S(\xi)$, and collars identify a neighborhood of $S(\xi)$ in $D(\xi)$ with $S(\xi) \times [0, \varepsilon)$.

Definition 2.8 (Orientations for real bundles)**.** A real rank-k bundle $\xi \to B$ is **oriented** if its structure group reduces to $GL^+(k, \mathbb{R})$. Equivalently, $S(\xi) \to B$ is an S^{k-1}-bundle whose fibers inherit the standard orientation continuously. For oriented ξ, the associated $S(\xi)$ is an **oriented** sphere bundle.

Definition 2.9 (Metric and sphere/disk bundles)**.** A **bundle metric** is a smoothly varying inner product on fibers. The **unit sphere bundle** and **unit disk bundle** are

$$S(\xi) = \{v \in E : \|v\| = 1\}, \qquad D(\xi) = \{v \in E : \|v\| \le 1\}.$$

They are smooth manifolds and $\partial D(\xi) = S(\xi)$.

Proposition 2.10 (Pullback and sphere/disk)**.** *For $f : Y \to B$, there are natural diffeomorphisms*

$$S(f^*\xi) \cong f^*S(\xi), \qquad D(f^*\xi) \cong f^*D(\xi).$$

Proof. Both sides consist of pairs (y, v) with $f(y) = \pi(v)$ and a fiberwise norm constraint; local trivializations identify them. $\square$

2.2. Quaternions and a concrete model of $SO(4)$.

Definition 2.11 (Quaternions)**.** The **quaternion algebra** $\mathbb{H}$ is the 4-dimensional real associative algebra with basis $\{1, i, j, k\}$ and multiplication determined by

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ki = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Every element has the form $q = a + bi + cj + dk$ with $a, b, c, d \in \mathbb{R}$.

Proposition 2.12 (Unit quaternions)**.** *The subset*

$$\mathbb{H}_1 = \{q \in \mathbb{H} : a^2 + b^2 + c^2 + d^2 = 1\}$$

is a smooth 3–manifold diffeomorphic to the unit sphere $S^3 \subset \mathbb{R}^4$.

Proof. Identify $\mathbb{H} \cong \mathbb{R}^4$ via coordinates (a, b, c, d). The unit condition $a^2+b^2+c^2+d^2=1$ defines $S^3 \subset \mathbb{R}^4$. □

Proposition 2.13 (Left and right quaternionic actions)**.** *For $u, v \in S^3 \subset \mathbb{H}$, define*

$$L_u(x) = ux, \quad R_v(x) = xv^{-1} \quad (x \in \mathbb{H} \cong \mathbb{R}^4).$$

Then $L_u, R_v \in SO(4)$.

Proof. The quaternionic norm is multiplicative, so $\|ux\| = \|x\| = \|xv^{-1}\|$, and orientation is preserved by continuity from $u = v = 1$. □

Proposition 2.14 (Combined action and kernel)**.** *The map*

$$\Phi : S^3 \times S^3 \to SO(4), \quad \Phi(u, v)(x) = uxv^{-1},$$

is a smooth group homomorphism whose kernel is

$$\ker \Phi = \{(1, 1), (-1, -1)\}.$$

Proof. If $\Phi(u, v)$ is the identity, then $ux = xv$ for all x, forcing $u = v$ and u to commute with i, j, k, so $u = \pm 1$. □

2.3. **A working set of homological algebra.**

Definition 2.15 (Chain and cochain complexes)**.** A **chain complex** $(C_\bullet, \partial)$ over abelian groups is a sequence

$$\cdots \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots$$

with $\partial_k \circ \partial_{k+1} = 0$. The k-th **homology** is $H_k(C_\bullet) = \ker \partial_k / \operatorname{im} \partial_{k+1}$.

A **cochain complex** $(C^\bullet, d)$ is a sequence $C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots$ with $d^{k+1} \circ d^k = 0$ and $H^k(C^\bullet) = \ker d^k / \operatorname{im} d^{k-1}$.

Definition 2.16 (Exactness)**.** A sequence of abelian groups and homomorphisms

$$\cdots \to A_{k+1} \xrightarrow{f_{k+1}} A_k \xrightarrow{f_k} A_{k-1} \to \cdots$$

is **exact** at A_k if $\operatorname{im} f_{k+1} = \ker f_k$.

A **short exact sequence** is of the form $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ with u injective, v surjective, and $\operatorname{im} u = \ker v$. It **splits** if there exists $s : C \to B$ with $v \circ s = \mathrm{id}_C$, equivalently $B \cong A \oplus C$.

Definition 2.17 (Hom and $\otimes$)**.** For abelian groups A, B, $\operatorname{Hom}(A, B)$ is the group of homomorphisms $A \to B$, and the **tensor product** $A \otimes B$ is the abelian group corepresenting bilinear maps out of $A \times B$. Both are functorial in each variable; Hom is left exact in the first variable, $\otimes$ is right exact in each variable.

Definition 2.18 (The groups Tor and Ext in low degree)**.** For abelian groups A, B, the groups $\operatorname{Tor}_1(A, B)$ and $\operatorname{Ext}^1(A, B)$ measure how tensor product and Hom fail to be exact. They can be computed from a short exact presentation of A:

$$0 \to R \to F \to A \to 0,$$

where F is free. Applying $- \otimes B$ and $\operatorname{Hom}(-, B)$ gives

$$\operatorname{Tor}_1(A, B) = \ker(R \otimes B \to F \otimes B), \quad \operatorname{Ext}^1(A, B) = \operatorname{coker}(\operatorname{Hom}(F, B) \to \operatorname{Hom}(R, B)).$$

Example 2.19 (Basic computations)**.** For any abelian group G and integer $n \geq 1$,

$$\operatorname{Tor}_1(\mathbb{Z}, G) = 0, \quad \operatorname{Tor}_1(\mathbb{Z}/n, G) \cong G[n], \quad \operatorname{Ext}^1(\mathbb{Z}, G) = 0, \quad \operatorname{Ext}^1(\mathbb{Z}/n, G) \cong G/nG.$$

2.4. **Functoriality and naturality without category thory.**

Definition 2.20 (Functoriality)**.** A construction F on spaces is **functorial** if for every map $f : X \to Y$ there is an induced map $F(f) : F(X) \to F(Y)$ with $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$.

Definition 2.21 (Naturality)**.** A family of maps $\{\eta_X : F(X) \to G(X)\}$ is **natural** if for every $f : X \to Y$ the following diagram

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

commutes.

3. Necessary knowledge of homotopy and homology

3.1. **Basic notations of homotopy and the fundamental group.**

Definition 3.1 (Path and loop)**.** A **path** in a topological space X is a continuous map $\gamma : [0,1] \to X$. Its **initial** and **final** points are $\gamma(0)$ and $\gamma(1)$. A **loop** in X is a path $\gamma : [0,1] \to X$ such that $\gamma(0) = \gamma(1)$.

Definition 3.2 (Homotopy)**.** Let $f_0, f_1 : X \to Y$ be continuous. A **homotopy** from f_0 to f_1 is a continuous map $H : X \times [0,1] \to Y$ with $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$. We say that f_0 and f_1 are **homotopic** and write $f_0 \simeq f_1$. In particular, a **nullhomotopy** is a homotopy from a map to a constant map.

Remark 3.3. The homotopy relation $\simeq$ is an equivalence relation on the set of continuous maps $X \to Y$. The equivalence class of f is denoted $[f]$ and called the **homotopy class** of f.

Definition 3.4 (Based homotopy)**.** Fix $x_0 \in X$ and $y_0 \in Y$. $f_0, f_1 : (X, x_0) \to (Y, y_0)$ are **based homotopic** if there exists a homotopy H with $H(x_0, t) = y_0$ for all t.

Definition 3.5 (Concatenation and inversal)**.** If $\alpha, \beta : [0,1] \to X$ with $\alpha(1) = \beta(0)$, the **concatenation** of α and β is

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2t), & 0 \le t \le \frac{1}{2}, \\ \beta(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

The **inverse** of α is $\bar{\alpha}(t) = \alpha(1-t)$.

Lemma 3.6 (Reparametrization)**.** *Concatenation is associative up to path homotopy, the constant path is a two-sided unit up to homotopy, and $\alpha \cdot \bar{\alpha}$ and $\bar{\alpha} \cdot \alpha$ are homotopic to a constant path.*

Definition 3.7 (Fundamental group)**.** The **fundamental group** $\pi_1(X, x_0)$ is the set of path homotopy classes of loops $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$, with multiplication given by concatenation $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$.

Proposition 3.8. *The operation in Definition 3.7 is well-defined and makes $\pi_1(X, x_0)$ a group. The identity is the constant loop and each path has an inverse.*

Proof. Lemma 3.6 shows associativity, identity, and inverses up to homotopy; well-definedness follows because concatenation respects path homotopy classes. ☐

Proposition 3.9 (Functoriality). *A based map $f : (X, x_0) \to (Y, y_0)$ induces a homomorphism*

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\gamma] \mapsto [f \circ \gamma],$$

natural with respect to composition and identities.

Remark 3.10. This means that the fundamental group is a *functor* from based topological spaces to groups. Later, this functorial behavior ensures that the homomorphisms induced by bundle projections and inclusions interact predictably with our constructions.

Proposition 3.11 (Change of basepoint). *If X is path connected and α is a path from x_0 to x_1, the map*

$$\Phi_\alpha : \pi_1(X, x_0) \to \pi_1(X, x_1), \qquad [\gamma] \mapsto [\overline{\alpha} \cdot \gamma \cdot \alpha]$$

is an isomorphism.

Remark 3.12. Thus, for a path–connected space, $\pi_1(X)$ is essentially independent of the basepoint. We will later use this to simplify notation in topological spaces without fixing a particular basepoint each time.

3.2. **Coverings, lifting theorems, and a classical result.**

Definition 3.13 (Covering map). A continuous surjection $p : \widetilde{X} \to X$ is a **covering map** if for every $x \in X$ there exists an open neighborhood U such that $p^{-1}(U) = \bigsqcup_\alpha U_\alpha$ and $p|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism for each α.

Theorem 3.14 (Path lifting). *Given a covering map $p : \widetilde{X} \to X$, a path $\gamma : [0, 1] \to X$, and $\tilde{x}_0 \in p^{-1}(\gamma(0))$, there is a unique* ***lift*** *$\tilde{\gamma} : [0, 1] \to \widetilde{X}$ with $p \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = \tilde{x}_0$.*

Theorem 3.15 (Homotopy lifting). *If $H : Y \times [0, 1] \to X$ is a homotopy and $\tilde{f}_0 : Y \to \widetilde{X}$ is a lift of $H(\cdot, 0)$, then there is a unique lift $\widetilde{H} : Y \times [0, 1] \to \widetilde{X}$ with $\widetilde{H}(\cdot, 0) = \tilde{f}_0$ and $p \circ \widetilde{H} = H$.*

Lemma 3.16. *If $p : \widetilde{X} \to X$ is a covering map and $\tilde{x}_0 \in \widetilde{X}$, then*

$$p_* : \pi_1(\widetilde{X}, \tilde{x}_0) \to \pi_1(X, p(\tilde{x}_0))$$

is injective.

Proof. If a loop $\tilde{\gamma}$ projects to a nullhomotopic loop in X, lift the nullhomotopy to a homotopy in $\widetilde{X}$ from $\tilde{\gamma}$ to a constant loop. ☐

Proposition 3.17. *$\pi_1(S^1) \cong \mathbb{Z}$.*

Proof. Let $p : \mathbb{R} \to S^1$ be $p(t) = e^{2\pi i t}$. For a loop γ at $1 \in S^1$, lift to $\tilde{\gamma}$ with $\tilde{\gamma}(0) = 0$. Define the **degree**

$$\deg \gamma = \tilde{\gamma}(1) - \tilde{\gamma}(0) \in \mathbb{Z}.$$

This gives a surjective homomorphism $\deg : \pi_1(S^1, 1) \to \mathbb{Z}$. If $\deg \gamma = 0$ then $\tilde{\gamma}$ is a loop in $\mathbb{R}$ and hence nullhomotopic, so γ is nullhomotopic by Lemma 3.16. Therefore, deg is an isomorphism, and we obtain $\pi_1(S^1) \cong \mathbb{Z}$. ☐

3.3. **Higher homotopy groups and the key computation.**

Definition 3.18 (Homotopy group)**.** Fix a basepoint $x_0 \in X$. The n**th homotopy group** $\pi_n(X, x_0)$ is the set of based homotopy classes of based maps $f : (S^n, *) \to (X, x_0)$. If X is path–connected, we write simply $\pi_n(X)$.

Proposition 3.19 (Products)**.** *For any* $n \geq 1$*,* $\pi_n(X \times Y) \cong \pi_n(X) \oplus \pi_n(Y)$*.*

Idea. A based map $f : S^n \to X \times Y$ can be written uniquely as a pair (f_X, f_Y) of based maps $f_X : S^n \to X$ and $f_Y : S^n \to Y$. Two maps $f_0, f_1 : S^n \to X \times Y$ are based homotopic if and only if each component pair (f_{0X}, f_{1X}) and (f_{0Y}, f_{1Y}) are based homotopic in X and Y respectively. Thus, homotopy classes correspond bijectively to pairs $([f_X], [f_Y])$, and the group operation, i.e., concatenating along a sphere, is performed component by component. This yields the claimed direct sum isomorphism. □

Theorem 3.20. $\pi_3(S^3) \cong \mathbb{Z}$*, generated by the identity map on* S^3*.*

Proposition 3.21 (Coverings preserve higher homotopy)**.** *For a covering map* $p : \widetilde{X} \to X$ *between path–connected spaces and any* $k \geq 2$*, the induced map*

$$p_* : \pi_k(\widetilde{X}) \xrightarrow{\cong} \pi_k(X)$$

is an isomorphism.

Proof. **Surjectivity.** Given $f : S^k \to X$, pick a basepoint $\tilde{x}_0 \in p^{-1}(f(*))$. Since S^k is simply connected for $k \geq 2$, Theorem 3.14 allows f to lift uniquely to $\tilde{f} : S^k \to \widetilde{X}$ with $p \circ \tilde{f} = f$.

Injectivity. If two lifted maps $\tilde{f}_0, \tilde{f}_1$ satisfy $p \circ \tilde{f}_0 \simeq p \circ \tilde{f}_1$, lift the homotopy $H : S^k \times [0,1] \to X$ uniquely via Theorem 3.15. The lift connects $\tilde{f}_0$ and $\tilde{f}_1$ in $\widetilde{X}$. Thus, p_* is bijective. □

Lemma 3.22 (A concrete model for SO(4))**.** *Let* $\mathbb{H}$ *be the quaternions. Define*

$$\Psi : S^3_L \times S^3_R \to \mathrm{SO}(4), \quad \Psi(p, q)(x) = pxq^{-1},$$

where left and right multiplication act on $\mathbb{R}^4 \cong \mathbb{H}$*. Then* Ψ *is a covering map with kernel* $\{(1,1), (-1,-1)\}$*. Consequently,*

$$\mathrm{SO}(4) \cong (S^3_L \times S^3_R)/\{\pm(1,1)\}.$$

Proof. It's clear that $\|pxq^{-1}\| = \|x\|$ for any unit quaternions p, q. Also, both actions preserve orientation by continuity at $p = q = 1$. Thus, $\Psi(p, q) \in \mathrm{SO}(4)$. If $\Psi(p, q)$ acts trivially, then $px = xq$ for all $x \in \mathbb{H}$, forcing $p = q$ and p to commute with i, j, k, so $p = \pm 1$. Therefore, the kernel is exactly $\{(1,1), (-1,-1)\}$. Surjectivity follows since every element of SO(4) arises as a composition of a left and a right quaternionic rotation, unique up to this kernel, making Ψ a covering. □

Theorem 3.23. $\pi_3(\mathrm{SO}(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Proof. By Lemma 3.22, $\Psi : S^3 \times S^3 \to \mathrm{SO}(4)$ is a covering map. Proposition 3.21 with $k = 3$ gives $\pi_3(\mathrm{SO}(4)) \cong \pi_3(S^3 \times S^3)$. Then Proposition 3.19 and Theorem 3.20 yield $\pi_3(S^3 \times S^3) \cong \pi_3(S^3) \oplus \pi_3(S^3) \cong \mathbb{Z} \oplus \mathbb{Z}$. □

3.4. **Singular chains and homology.**

Definition 3.24 (Standard simplex and singular simplex)**.** The **standard n-simplex** is

$$\Delta^n = \Big\{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} : \ t_i \geq 0, \ \sum_{i=1}^{n} t_i = 1\Big\}.$$

A **singular n-simplex** in a space X is a continuous map $\sigma : \Delta^n \to X$.

Definition 3.25 (Singular chains)**.** The group of **singular n-chains** $C_n(X)$ is the free abelian group on all singular n-simplices in X. Elements of $C_n(X)$ are formal sums $\sum_i a_i\sigma_i$ with $a_i \in \mathbb{Z}$.

Definition 3.26 (Boundary)**.** The **boundary** $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined on generators by

$$\partial_n(\sigma) = \sum_{j=0}^{n} (-1)^j \, \sigma \circ \delta_j,$$

where $\delta_j : \Delta^{n-1} \hookrightarrow \Delta^n$ is the inclusion.

Proposition 3.27. *$\partial_{n-1} \circ \partial_n = 0$ for all n. Thus, $(C_*(X), \partial)$ is a chain complex.*

Definition 3.28 (Homology)**.** The group of **cycles** is $Z_n(X) = \ker \partial_n$. The group of **boundaries** is $B_n(X) = \operatorname{im} \partial_{n+1}$. The n-th **singular homology group** is

$$H_n(X) = Z_n(X)/B_n(X).$$

Definition 3.29 (Functoriality)**.** A continuous map $f : X \to Y$ induces a chain map $f_\# : C_*(X) \to C_*(Y)$ by composition $f_\#(\sigma) = f \circ \sigma$, hence homomorphisms $f_* : H_n(X) \to H_n(Y)$.

Proposition 3.30 (Homotopy invariance)**.** *If $f, g : X \to Y$ are homotopic then $f_* = g_*$ on H_n for all n.*

Corollary 3.31 (Homotopy equivalence)**.** *If $X \simeq Y$ then $H_n(X) \cong H_n(Y)$ for all n. In particular, deformation retracts preserve homology.*

3.5. **Relative homology and the long exact sequence of a pair.**

Definition 3.32 (Relative chains and homology)**.** For a subspace $A \subset X$, define $C_n(X, A) = C_n(X)/C_n(A)$ with the induced boundary. The **relative homology** is $H_n(X, A) = H_n(C_*(X, A))$.

Proposition 3.33 (Long exact sequence of a pair)**.** *The short exact sequence*

$$0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$$

of chain complexes induces a natural long exact sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to H_{n-1}(X) \to \cdots .$$

Proposition 3.34 (Sphere from the disk pair)**.** *For $n \geq 1$, the reduced homology of the sphere satisfies*

$$\widetilde{H}_k(S^n) \cong \begin{cases} \mathbb{Z}, & k = n, \\ 0, & k \neq n. \end{cases}$$

Proof. Consider the long exact sequence of the pair (D^n, S^{n-1}) and the fact that D^n is contractible, so $H_k(D^n) = 0$ for $k \geq 1$ and $H_0(D^n) \cong \mathbb{Z}$. The connecting map $H_n(D^n, S^{n-1}) \to \widetilde{H}_{n-1}(S^{n-1})$ is an isomorphism, inductively giving the result. $\square$

3.6. **Fundamental classes.**

Definition 3.35 (Local orientation class)**.** Let M be a connected, oriented n–manifold (with or without boundary). For $x \in M^\circ$ (the interior), the local homology group $H_n(M, M \setminus \{x\}) \cong \mathbb{Z}$ has a preferred generator o_x determined by the chosen orientation.

Definition 3.36 (Fundamental class of a closed manifold)**.** If M is closed, connected, and oriented of dimension n, the **fundamental class** is the unique element $[M] \in H_n(M)$ whose image in $H_n(M, M \setminus \{x\})$ under the canonical map $H_n(M) \to H_n(M, M \setminus \{x\})$ equals o_x for every $x \in M$.

Definition 3.37 (Relative fundamental class)**.** If M is compact, connected, oriented of dimension n with (possibly nonempty) boundary ∂M, the **relative fundamental class** is the unique element $[M, \partial M] \in H_n(M, \partial M)$ whose image in $H_n(M, M \setminus \{x\})$ equals o_x for every $x \in M^\circ$. The boundary carries the *outward–normal–first* orientation, and the connecting homomorphism satisfies

$$\partial[M, \partial M] = [\partial M] \in H_{n-1}(\partial M).$$

Proposition 3.38 (Naturality and degree)**.** *Let $f : (M, \partial M) \to (N, \partial N)$ be a continuous, proper map between oriented n–manifolds (with boundary allowed). There is an integer* $\deg(f)$ *(the **degree** of f) such that*

$$f_* [M, \partial M] = \deg(f)\, [N, \partial N].$$

In particular, for closed manifolds $f_*[M] = \deg(f)\, [N]$. If f is orientation–preserving of degree 1 *(e.g. a homeomorphism or diffeomorphism), then $f_*[M] = [N]$.*

Remark 3.39 (Pairing notation and use later). For a closed oriented M^n we write $\langle \alpha, [M] \rangle$ for the evaluation of $\alpha \in H^n(M)$ on the fundamental class. For a compact oriented pair $(M, \partial M)$ we similarly use $\langle \beta, [M, \partial M] \rangle$ for $\beta \in H^n(M, \partial M)$. These conventions are exactly what enter the statements of Poincaré duality and, later, the formulas defining and computing $q(B) = \langle \nu, \tilde{p} \smile \tilde{p} \rangle$ for an oriented filling $(B, \partial B = M)$.

4. Fundamentals of Cohomology and Characteristic Classes

Unless explicitly stated otherwise, all cohomology groups are taken with integer coefficients.

4.1. **Cochains, cocycles, and cohomology.**

Definition 4.1 (Singular cochains and coboundary)**.** Let X be a space and G an abelian group. The group of k–**cochains** is

$$C^k(X; G) = \mathrm{Hom}(C_k(X), G),$$

where $C_k(X)$ is the singular k–chain group with integer coefficients. The **coboundary** $\delta : C^k(X; G) \to C^{k+1}(X; G)$ is defined by precomposition with the boundary $\partial : C_{k+1}(X) \to C_k(X)$, so that $\delta^2 = 0$.

Definition 4.2 (Cohomology)**.** The group of **cocycles** is $Z^k(X; G) = \ker \delta$, the group of **coboundaries** is $B^k(X; G) = \operatorname{im} \delta$, and the k–**th cohomology group** is

$$H^k(X; G) = Z^k(X; G)/B^k(X; G).$$

When the coefficients are $\mathbb{Z}$ we write $H^k(X)$ for short.

Definition 4.3 (Kronecker pairing)**.** For $G = \mathbb{Z}$, there is a natural **Kronecker pairing**

$$\langle\ ,\ \rangle : H^k(X) \times H_k(X) \to \mathbb{Z}$$

induced by evaluation of cochains on chains. It satisfies naturality with respect to continuous maps.

4.2. **Cup product.**

Definition 4.4 (Cup product)**.** The **cup product** is a bilinear map

$$\smile : H^p(X) \times H^q(X) \to H^{p+q}(X)$$

defined at the cochain level. It makes $H^*(X) = \bigoplus_k H^k(X)$ a graded ring with unit $1 \in H^0(X)$.

Proposition 4.5 (Basic laws for the cup product)**.** *For any continuous $f : X \to Y$:*

- ***Functoriality**: $f^*(\alpha \smile \beta) = f^*\alpha \smile f^*\beta$.*
- ***Graded commutativity**: $\alpha \smile \beta = (-1)^{pq}\beta \smile \alpha$ for $\alpha \in H^p$, $\beta \in H^q$.*
- ***Associativity and unit**: $(\alpha \smile \beta) \smile \gamma = \alpha \smile (\beta \smile \gamma)$ and $1 \smile \alpha = \alpha = \alpha \smile 1$.*

4.3. **UCT and Künneth.**

Theorem 4.6 (Universal coefficient theorem for cohomology)**.** *For any space X and abelian group G, there is a natural short exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{k-1}(X), G) \to H^k(X; G) \to \mathrm{Hom}(H_k(X), G) \to 0.$$

which (noncanonically) splits. Therefore, $H^k(X; G)$ is determined by the homology of X together with the groups $\mathrm{Ext}^1(H_{k-1}(X), G)$.

Corollary 4.7 (Free case)**.** *If all $H_i(X)$ are free abelian, then*

$$H^k(X; G) \cong \mathrm{Hom}(H_k(X), G), \quad \textit{in particular } H^k(X) \cong H_k(X) \textit{ when } G = \mathbb{Z}.$$

Theorem 4.8 (Künneth theorem for cohomology)**.** *Let X, Y be CW complexes. There is a natural short exact sequence*

$$0 \longrightarrow \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y) \to H^n(X \times Y) \longrightarrow \bigoplus_{i+j=n+1} \mathrm{Tor}_1^{\mathbb{Z}}(H^i(X), H^j(Y)) \to 0,$$

which (noncanonically) splits.

Corollary 4.9 (Free case)**.** *If all cohomology groups of X and Y are free abelian, then* Tor *terms vanish and*

$$H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$$

as graded abelian groups, with the ring structure given by

$$(\alpha_1 \otimes \beta_1) \smile (\alpha_2 \otimes \beta_2) = (-1)^{|\beta_1||\alpha_2|}(\alpha_1 \smile \alpha_2) \otimes (\beta_1 \smile \beta_2).$$

4.4. **Cap product and Poincaré duality.**

Definition 4.10 (Cap product)**.** For a space X there is a bilinear **cap product**

$$\frown: H^k(X) \times H_\ell(X) \to H_{\ell-k}(X)$$

natural in both variables and compatible with the Kronecker pairing.

Theorem 4.11 (Poincaré duality, closed case)**.** *If M is a closed, connected, oriented n–manifold, capping with the fundamental class $[M] \in H_n(M)$ gives isomorphisms*

$$\mathrm{PD} : H^k(M) \xrightarrow{\cong} H_{n-k}(M), \quad \alpha \mapsto \alpha \frown [M].$$

Theorem 4.12 (Poincaré–Lefschetz duality, boundary case)**.** *If M is a compact, connected, oriented n–manifold with boundary, then*

$$H^k(M, \partial M) \cong H_{n-k}(M), \quad H^k(M) \cong H_{n-k}(M, \partial M).$$

Proposition 4.13 (Cup–cap compatibility)**.** *For $\alpha \in H^p(M)$ and $\beta \in H^q(M)$ on a closed oriented M,*

$$\mathrm{PD}(\alpha \smile \beta) = \mathrm{PD}(\beta) \frown \alpha,$$

and $\langle \alpha \smile \beta,\, [M] \rangle = \langle \alpha, \mathrm{PD}(\beta) \rangle$. The same formulas hold in the relative case with $(M, \partial M)$.

4.5. **Euler class of an oriented real vector bundle.**

Definition 4.14 (Thom class and Euler class)**.** Let $\pi : \xi \to B$ be an oriented real rank-n vector bundle with disk bundle $D(\xi)$ and sphere bundle $S(\xi)$. A **Thom class** is a class $U \in H^n(D(\xi), S(\xi))$ whose restriction to each fiber pair (D^n, S^{n-1}) is the positive generator of $H^n(D^n, S^{n-1}) \cong \mathbb{Z}$ determined by the orientation.

The **Euler class** of ξ is $e(\xi) := s^*(U) \in H^n(B)$, where $s : B \to D(\xi)$ is the zero section and $s^* : H^*(D(\xi), S(\xi)) \to H^*(B)$ is a pullback.

Proposition 4.15 (Basic properties of $e(\xi)$)**.** *For oriented real bundles ξ, η over B and any continuous $f : B' \to B$:*

- ***Naturality:*** *$e(f^*\xi) = f^*e(\xi)$.*
- ***Whitney multiplicativity in top degree:*** *If $\mathrm{rank}(\xi) + \mathrm{rank}(\eta) = \dim B$, then*
$$\langle e(\xi \oplus \eta), [B] \rangle = \langle e(\xi) \smile e(\eta), [B] \rangle.$$
- ***Zero counting:*** *If $\dim B = n = \mathrm{rank}(\xi)$ and $s : B \to \xi$ is a transverse section with isolated zeros, then*
$$\langle e(\xi), [B] \rangle = \sum_{s^{-1}(0)} \mathrm{ind}_x(s) \quad \textit{(signed count of zeros)}.$$

4.6. **Gysin sequence and a model computation.**

Definition 4.16 (Fiber integration)**.** Let $\pi : E \to B$ be an oriented S^r–bundle. There is a map (integration along the fiber)

$$\pi_* : H^{k+r}(E) \to H^k(B)$$

characterized by the property that for every $b \in B$ and the positive generator $u_b \in H^r(E_b) \cong \mathbb{Z}$ on the fiber $E_b \cong S^r$, we have

$$\langle \pi_*(\alpha), [B] \rangle = \langle \alpha, [E] \rangle \quad \text{and} \quad \pi_*(\alpha \smile \tilde{u}) = \langle \alpha|_{E_b}, [E_b] \rangle,$$

where $\tilde{u} \in H^r(E)$ restricts to u_b on each fiber. This map is natural in pullback squares of oriented sphere bundles.

Theorem 4.17 (Gysin long exact sequence)**.** *Let $\pi : E \to B$ be an oriented S^r–bundle (e.g. $E = S(\xi)$ for an oriented rank $r+1$ bundle ξ). There exists a class $e \in H^{r+1}(B)$ (the **Euler class**, Definition 4.14) such that we have a natural long exact sequence*

$$\cdots \to H^{k-r}(B) \xrightarrow{\smile e} H^{k+1}(B) \xrightarrow{\pi^*} H^{k+1}(E) \xrightarrow{\pi_*} H^{k+1-r}(B) \xrightarrow{\smile e} \cdots ,$$

where π_* is integration along the fiber.

Proposition 4.18 (Projection formula and naturality)**.** *For all $\alpha \in H^{k+r}(E)$ and $\beta \in H^*(B)$,*

$$\pi_*(\alpha \smile \pi^*\beta) = \pi_*(\alpha) \smile \beta.$$

Moreover, in a pullback square of oriented sphere bundles, pullback commutes with π^* and π_*.

4.7. **Pontryagin classes of real vector bundles.**

Definition 4.19 (Definition via complexification)**.** For a real vector bundle ξ, let $\xi_{\mathbb{C}} = \xi \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. The **Pontryagin classes** are defined by

$$p(\xi) = 1 + p_1(\xi) + p_2(\xi) + \cdots , \quad p_k(\xi) = (-1)^k c_{2k}(\xi_{\mathbb{C}}),$$

where c_i are the Chern classes. In what follows we only use p_1.

Proposition 4.20 (Basic properties)**.** *For real bundles ξ, η over B and $f : B' \to B$:*

- ***Naturality:*** *$p(f^*\xi) = f^*p(\xi)$.*
- ***Whitney product:*** *$p(\xi \oplus \eta) = p(\xi) \smile p(\eta)$.*

In particular, $p_1(\xi \oplus \eta) = p_1(\xi) + p_1(\eta)$.

Proposition 4.21 (Complex rank-2 case)**.** *If η is a complex rank-2 bundle and $\eta_{\mathbb{R}}$ its underlying real rank 4 bundle, then*

$$p_1(\eta_{\mathbb{R}}) = c_1(\eta)^2 - 2c_2(\eta).$$

Example 4.22 ($\mathbb{CP}^n$)**.** The Euler sequence gives $c(T\mathbb{CP}^n) = (1 + H)^{n+1}$, where $H \in H^2(\mathbb{CP}^n)$ is the hyperplane class. Thus

$$c_1 = (n+1)H, \quad c_2 = \binom{n+1}{2} H^2,$$

and by Proposition 4.21 (applied fiberwise to the complex tangent bundle),

$$p_1(T\mathbb{CP}^n) = (n+1)H^2.$$

For example, $p_1(T\mathbb{CP}^2) = 3H^2$, $p_1(T\mathbb{CP}^4) = 5H^2$.

4.8. **Vertical tangent and Pontryagin classes of sphere bundles.**

Definition 4.23 (Vertical tangent bundle)**.** For the unit sphere bundle $\pi : S(\xi) \to B$ of an oriented rank $r+1$ bundle ξ, the **vertical tangent bundle** T_{fib} is the kernel of $d\pi : T(S(\xi)) \to \pi^*TB$. It has rank r and fiber over $x \in S(\xi)$ equal to $T(S(\xi)_{\pi(x)})$.

Proposition 4.24 (Stable splitting)**.** *There is a canonical stable isomorphism*

$$T(S(\xi)) \oplus \varepsilon^1 \cong \pi^*TB \oplus \pi^*\xi.$$

Equivalently, $T_{\text{fib}} \oplus \varepsilon^1 \cong \pi^*\xi$ and $T\big(S(\xi)\big) \cong \pi^*TB \oplus T_{\text{fib}}$.

Corollary 4.25 (Pontryagin classes of the total space)**.** *From Proposition 4.24 and multiplicativity,*

$$p(T(S(\xi))) = \pi^*(p(TB)p(\xi)).$$

In particular, if $B = S^4$ then $p(TB) = 1$ (spheres are stably parallelizable), hence

$$p_1(T(S(\xi))) = \pi^* p_1(\xi).$$

4.9. **Signature and the L–class in dimension 8.**

Definition 4.26 (Intersection form and signature)**.** If M is a closed oriented $4k$–manifold, the bilinear form

$$H^{2k}(M) \times H^{2k}(M) \to \mathbb{Z}, \quad (\alpha, \beta) \mapsto \langle \alpha \smile \beta, [W] \rangle$$

is symmetric. Extending scalars to $\mathbb{R}$, it is diagonalizable with b_+ positive and b_- negative squares. The **signature** of M is $\tau(M) = b_+ - b_-$.

Theorem 4.27 (Hirzebruch signature theorem)**.** *For a closed oriented $4k$–manifold W, $\tau(W) = \langle L(W), [W] \rangle$, where $L(W) = 1 + L_1 + L_2 + \cdots$ is the L–class (a polynomial in the Pontryagin classes). The first terms are*

$$L_1 = \frac{1}{3} p_1, \quad L_2 = \frac{1}{45}(7p_2 - p_1^2).$$

In particular, for 8–manifolds,

$$\langle 7p_2 - p_1^2, [W] \rangle = 45\tau(W).$$

5. Reading Milnor, Part I: Building the Candidates

5.1. **From topology to construction.** The guiding idea is elementary: to produce closed 7–manifolds that might be spheres but with potentially different smooth structures, build them as total spaces of S^3–bundles over S^4. This keeps the geometry low–dimensional (fibers and base we can compute with) and puts virtually all complexity into a *gluing map* along the equator of the base. Two principles drive the section:

- **Classification on S^4 is discrete.** Oriented rank–4 real bundles over S^4 are classified by a pair of integers. These integers will later control the Euler and first Pontryagin classes.
- **Cohomology of the total space comes from Gysin.** Once the Euler class is known, the Gysin long exact sequence computes H^* of the total space almost at a glance. This yields a clean criterion for when the total space is a *homology* 7–sphere.

In this part we formalize these two points and conclude with the homology–sphere criterion. The next part will upgrade "homology sphere" to "topological sphere" by a two–critical–point Morse function, and the final part will separate smooth structures by a congruence of Pontryagin numbers.

5.2. **Rank–4 bundles over S^4 via clutching.** The classification of oriented rank–4 real vector bundles over S^4 follows directly from Theorem 3.23.

Remove the equator of S^4 to obtain two 4–disks whose intersection deformation–retracts to S^3. A rank–4 oriented bundle over S^4 is determined by trivial bundles over the two disks glued along the equator by a continuous map $g : S^3 \to \mathrm{SO}(4)$. Two such bundles are isomorphic exactly when the gluing maps are homotopic.

Thus, isomorphism classes of oriented rank–4 bundles over S^4 are in bijection with $\pi_3(\mathrm{SO}(4))$. Using the quaternionic model of SO(4), one has

$$\pi_3(\mathrm{SO}(4)) \cong \pi_3(S^3) \oplus \pi_3(S^3) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

We adopt the following uniform notation.

Definition 5.1 (The bundles $\xi_{h,\ell}$ and their sphere bundles)**.** For $(h,\ell) \in \mathbb{Z}^2$, let $\xi_{h,\ell} \to S^4$ be an oriented rank–4 real vector bundle classified by (h,ℓ), and set

$$E_{h,\ell} := S(\xi_{h,\ell}) \xrightarrow{\pi} S^4,$$

its unit S^3–sphere bundle with the fiber orientation induced by $\xi_{h,\ell}$. Fix once and for all $u \in H^4(S^4)$ with $\langle u, [S^4] \rangle = 1$.

Remark 5.2 (Quaternionic normalization and coordinates). Identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$. Left and right multiplication by unit quaternions define a covering

$$\Psi : S^3_L \times S^3_R \to \mathrm{SO}(4), \quad (p,q) \mapsto (x \mapsto pxq^{-1}).$$

Under this identification, a convenient set of representatives for the clutching maps is

$$f_{h,\ell} : S^3 \to \mathrm{SO}(4), \quad f_{h,\ell}(u) = \Psi(u^h, u^\ell),$$

so that (h,ℓ) literally record the "left" and "right" degrees. All sign conventions below are fixed so that the quaternionic Hopf fibration corresponds to $(0,1)$.

Definition 5.3 (Quaternionic Hopf fibration)**.** Left multiplication of unit quaternions on $\mathbb{H}$ restricts to a free action of S^3 on $S^7 \subset \mathbb{H}^2$:

$$q \cdot (x_1, x_2) = (qx_1, qx_2).$$

The orbit space is S^4, and the projection $S^3 \hookrightarrow S^7 \twoheadrightarrow S^4$ is the **quaternionic Hopf fibration**. It is the unit sphere bundle of a rank–4 quaternionic line bundle $\xi_{0,1} \to S^4$, whose characteristic classes satisfy $e(\xi_{0,1}) = -u$ and $p_1(\xi_{0,1}) = 2u$.

5.3. **Euler and Pontryagin classes on S^4: explicit linear formulas.** The Euler class and first Pontryagin class on S^4 are forced to depend *linearly* on the pair (h,ℓ). We record the precise statement and give a direct proof calibrated by the Hopf example.

Proposition 5.4 (Linearity and normalization on S^4)**.** *There exist four integers A, B, C, D such that for every $(h,\ell) \in \mathbb{Z}^2$,*

$$e(\xi_{h,\ell}) = (Ah + B\ell)u, \quad p_1(\xi_{h,\ell}) = (Ch + D\ell)u.$$

With the normalization in which the quaternionic Hopf fibration corresponds to $(0,1)$, one has

$$e(\xi_{h,\ell}) = (h - \ell)u, \quad p_1(\xi_{h,\ell}) = 2(h + \ell)u.$$

Proof. Linearity. The assignment $(h,\ell) \mapsto \xi_{h,\ell}$ is functorial in the clutching map, and characteristic classes are natural. Writing the clutching class additively in $\pi_3(\mathrm{SO}(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$, the map

$$\Phi_e : \pi_3(\mathrm{SO}(4)) \to H^4(S^4), \quad (h,\ell) \mapsto e(\xi_{h,\ell}),$$

is a group homomorphism because changing the clutching map by a generator in either factor corresponds to composing with a representative $S^3 \to \mathrm{SO}(4)$ supported near the equator, and e behaves additively under this modification (one may see this by restricting to a collar of the equator and using naturality of the Thom class).

Hence $e(\xi_{h,\ell})$ is an integer linear combination of h and ℓ. The same argument, using the multiplicativity of total Pontryagin class and naturality, shows $p_1(\xi_{h,\ell})$ is also linear in (h, ℓ).

Calibration. Consider the two basic bundles $\xi_{1,0}$ and $\xi_{0,1}$, obtained by restricting to the left and right S^3 factors.

The bundle $\xi_{0,1}$ is the underlying real rank–4 bundle of the quaternionic line bundle whose unit sphere bundle is the quaternionic Hopf fibration $S^7 \to S^4$. It is standard that

$$e(\xi_{0,1}) = -u, \quad p_1(\xi_{0,1}) = 2u,$$

(the first from the degree of the Hopf map, the second from $p_1 = c_1^2 - 2c_2$ applied fiberwise to a complex rank–2 model with $c_1 = 0$, $c_2 = -u$). By symmetry under left–right exchange, we have

$$e(\xi_{1,0}) = +u, \quad p_1(\xi_{1,0}) = 2u.$$

These two evaluations determine the linear formulas uniquely:

$$e(\xi_{h,\ell}) = (h - \ell)u, \quad p_1(\xi_{h,\ell}) = 2(h + \ell)u. \quad \square$$

5.4. **Cohomology of $E_{h,\ell}$ via the Gysin sequence.** We now compute $H^*(E_{h,\ell})$ using only the Gysin long exact sequence for the oriented S^3–bundle $\pi : E_{h,\ell} \to S^4$ together with the formula for $e(\xi_{h,\ell})$.

Proposition 5.5 (Cohomology groups of the total space)**.** *Let $a := h - \ell$. Then*

$$H^0(E_{h,\ell}) \cong \mathbb{Z}, \quad H^3(E_{h,\ell}) \cong \mathbb{Z}/|a|, \quad H^4(E_{h,\ell}) \cong \begin{cases} \mathbb{Z}, & a = 0, \\ 0, & a \neq 0, \end{cases} \quad H^7(E_{h,\ell}) \cong \mathbb{Z},$$

and $H^k(E_{h,\ell}) = 0$ for $k \notin \{0, 3, 4, 7\}$. In particular, $E_{h,\ell}$ has the same homology groups as S^7 if and only if $|a| = 1$.

Proof. Write $e = au \in H^4(S^4)$. The relevant part of the Gysin long exact sequence for an oriented S^3–bundle is

$$\cdots \to H^{k-3}(S^4) \xrightarrow{\smile e} H^{k+1}(S^4) \xrightarrow{\pi^*} H^{k+1}(E) \xrightarrow{\pi_*} H^{k-2}(S^4) \xrightarrow{\smile e} H^k(S^4) \to \cdots.$$

Since $H^*(S^4)$ is concentrated in degrees 0 and 4, only the windows around $k = 2, 3, 6$ contribute. Inspect the three windows explicitly:

Window around $k = 2$. We have

$$H^{-1}(S^4) = 0 \to H^3(S^4) = 0 \xrightarrow{\pi^*} H^3(E) \xrightarrow{\pi_*} H^0(S^4) \cong \mathbb{Z} \xrightarrow{\smile e} H^2(S^4) = 0.$$

Exactness gives $\pi_* : H^3(E) \twoheadrightarrow \mathbb{Z}$ and $\ker(\pi_*) = \mathrm{im}(\pi^*) = 0$, so $H^3(E)$ is cyclic and the succeeding window will determine its order.

Window around $k = 3$. We have

$$H^0(S^4) \cong \mathbb{Z} \xrightarrow{\smile e} H^4(S^4) \cong \mathbb{Z} \xrightarrow{\pi^*} H^4(E) \xrightarrow{\pi_*} H^1(S^4) = 0.$$

The first map is multiplication by a. Hence $\mathrm{coker}(\smile e) \cong \mathbb{Z}/|a|$, and exactness shows $H^4(E) \cong \mathbb{Z}/|a|$ if $a \neq 0$ *and* $H^4(E) \cong \mathbb{Z}$ if $a = 0$. However, recall from the previous window that $H^3(E)$ surjects onto $\mathbb{Z}$; comparing indices and exactness in the adjacent windows forces $H^4(E)$ to be 0 when $a \neq 0$ and $\mathbb{Z}$ when $a = 0$ (this is the standard “shift by one degree” phenomenon in the S^3–Gysin sequence).

Window around $k = 6$. Dually one obtains $H^7(E) \cong \mathbb{Z}$ with generator detected by $\pi^* u \smile v$ in the trivial case and by the fiberwise orientation class in general. Finally

$H^0(E) \cong \mathbb{Z}$ and all other groups vanish by degree reasons. Combining the windows yields the table in the statement. The homology–sphere criterion is immediate. □

Example 5.6 (The untwisted case)**.** If $h = \ell$ then $e = 0$ and the bundle is trivial, so $E_{h,h} \cong S^4 \times S^3$ with

$$H^*(E_{h,h}) \cong \mathbb{Z}[u, v]/(u^2, v^2), \quad |u| = 4, |v| = 3,$$

and $u \smile v$ generating H^7. This recovers the computation above and will serve as a reference model for ring–level statements later.

6. Reading Milnor, Part II: From Candidates to The Topological 7-Sphere

6.1. **From homology sphere to sphere: the strategy.** From Part I we know that $E_{h,\ell} = S(\xi_{h,\ell})$ is an integral homology 7–sphere exactly when $|h-\ell| = 1$. In this part we show that, for such pairs (h, ℓ), the manifold $E_{h,\ell}$ is in fact *homeomorphic* to S^7. The route is classical:

- introduce just enough Morse theory to speak about Morse functions and their indices;
- construct on $E_{h,\ell}$ a Morse function with exactly two nondegenerate critical points (one minimum, one maximum);
- invoke Reeb's sphere theorem to conclude that $E_{h,\ell}$ is a topological 7–sphere.

6.2. **A Morse mini–primer.**

Definition 6.1 (Critical point and index)**.** Let M be a smooth manifold and $f : M \to \mathbb{R}$ a smooth function. A point $p \in M$ is **critical** if $df_p = 0$; otherwise it is **regular**. The critical point is **nondegenerate** if its Hessian $Hf_p : T_pM \times T_pM \to \mathbb{R}$ is a nonsingular symmetric bilinear form. The number of negative eigenvalues of Hf_p is called the **index** of p.

Definition 6.2 (Morse function)**.** A smooth function $f : M \to \mathbb{R}$ is a **Morse function** if all its critical points are nondegenerate.

Morse theory studies how the topology of a manifold changes as one passes through critical values of such a function. In the simplest possible case, there are only two critical points:

Theorem 6.3 (Reeb's sphere theorem)**.** *If a closed smooth n–manifold admits a Morse function with exactly two nondegenerate critical points, then it is homeomorphic to S^n.*

Remark 6.4. Intuitively, there is a "bottom" and a "top", like the minimum and maximum on a round sphere, so the overall shape of this manifold must be spherical.

6.3. **A two–piece model for $E_{h,\ell}$ when $|h - \ell| = 1$.** The sphere bundle description from Part I yields a canonical *two–piece decomposition* of $E_{h,\ell}$ as a union along $S^3 \times S^3$:

$$E_{h,\ell} \cong (D^4 \times S^3) \cup_{\varphi_{h,\ell}} (S^3 \times D^4),$$

where $\varphi_{h,\ell} : S^3 \times S^3 \to S^3 \times S^3$ encodes the same left/right degrees (h, ℓ) as the bundle's clutching map. When $|h - \ell| = 1$ the map $\varphi_{h,\ell}$ has degree ± 1 on each factor; in particular, the two cores

$$\{0\} \times S^3 \subset D^4 \times S^3, \quad S^3 \times \{0\} \subset S^3 \times D^4$$

are glued so that the homology coming from each side cancels, consistent with $H^*(E_{h,\ell})$ from Part I.

6.4. **Constructing a Morse function on $E_{h,\ell}$ with two critical points.** We now build an explicit Morse function adapted to the two–piece decomposition above.

Proposition 6.5 (A height function with two critical points)**.** *Assume $|h-\ell|=1$. There exists a Morse function $f : E_{h,\ell} \to \mathbb{R}$ with exactly two nondegenerate critical points, one of index 0 and one of index 7.*

Construction and verification. On $D^4 \times S^3$ set

$$f_-(x,y) = -\|x\|^2 \quad (x \in D^4,\ y \in S^3),$$

so f_- has a unique nondegenerate maximum at $(0,y)$ for every fixed y; restricted to the boundary $S^3 \times S^3$ this becomes a *regular* value because df_- is nonzero along outward normals of D^4.

On $S^3 \times D^4$ set

$$f_+(u,v) = +\|v\|^2 \quad (u \in S^3,\ v \in D^4),$$

so f_+ has a unique nondegenerate minimum at $(u,0)$ for every fixed u; again df_+ is nonzero along inward normals of D^4 on the boundary.

Choose collar neighborhoods $S^3 \times S^3 \times (-\varepsilon,\varepsilon)$ of the common boundary inside both pieces and precompose with $\varphi_{h,\ell}$ so that on the overlap the boundary values match. Using a partition of unity on the collar, split f_- and f_+ into a smooth function f on the union:

$$f = \rho f_- + (1-\rho) f_+,$$

with ρ supported near the $D^4 \times S^3$ side and constant along the $S^3 \times S^3$ boundary. Since $\varphi_{h,\ell}$ has degree ± 1 on each S^3 factor when $|h-\ell|=1$, the boundary gradients of f_- and f_+ point in opposite normal directions and match transversely under $\varphi_{h,\ell}$. Thus, the splitted f has no new critical points on the collar (the gradient remains nonzero there).

Inside each piece, the only candidates for critical points are at the centers $(0,y)$ and $(u,0)$. After restriction to the boundary $E_{h,\ell}$ these collapse to two isolated points where the Hessians are, respectively, negative definite and positive definite in complementary directions. A standard local computation in product coordinates on the two pieces shows nondegeneracy with indices 7 and 0. Hence, f has exactly two nondegenerate critical points. □

Remark 6.6 (Why the degree condition matters)*.* If $|h-\ell| \neq 1$, the boundary gradients on $S^3 \times S^3$ need not glue to a nowhere–vanishing vector field along the collar; in fact the Gysin computation forces torsion in $H^3(E_{h,\ell})$ that obstructs cancelling all would–be saddle points. The case $|h-\ell|=1$ is exactly the situation in which the two halves can be glued so that no additional critical points are created.

6.5. **Conclusion: $E_{h,\ell}$ is a topological 7–sphere.** We can now finish the topological classification in the $|h-\ell|=1$ regime.

Theorem 6.7 (Topological type of $E_{h,\ell}$ for $|h-\ell|=1$)**.** *If $|h-\ell|=1$, then $E_{h,\ell}$ is homeomorphic to S^7.*

Proof. By the proposition above there exists a Morse function $f : E_{h,\ell} \to \mathbb{R}$ with exactly two nondegenerate critical points. Reeb's sphere theorem applies, yielding that $E_{h,\ell}$ is homeomorphic to S^7. □

We have shown that these manifolds are topological 7-sphere. The remaining question is whether they carry distinct smooth structures.

7. Reading Milnor, Part III: Distinguishing smooth structures

7.1. **Why a new invariant is needed.** By Part II, whenever $|h-\ell| = 1$ the manifold $E_{h,\ell} = S(\xi_{h,\ell})$ is a *topological* 7-manifold homeomorphic to S^7. To separate smooth structures we need a smooth invariant that

(i) is defined for any smooth 7-manifold M that bounds a compact oriented 8-manifold W of the type considered here,
(ii) is independent of the choice of such a bounding 8-manifold,
(iii) is effectively computable for $M = E_{h,\ell}$ using only the linear formulas for $e(\xi_{h,\ell})$ and $p_1(\xi_{h,\ell})$ from Part I.

Hirzebruch's signature theorem and Novikov additivity supply exactly the required "mod 45" rigidity in dimension 8.

7.2. **A canonical bounding model and basic identities.** Fix $(h,\ell) \in \mathbb{Z}^2$ with $|h-\ell| = 1$. Let $\pi : W_{h,\ell} \to S^4$ be the rank 4 disk bundle $D(\xi_{h,\ell})$, so that

$$\partial W_{h,\ell} = S(\xi_{h,\ell}) = E_{h,\ell}.$$

Two structural inputs used repeatedly are:

- **Stable splitting.** For the total space $W_{h,\ell}$ one has a stable isomorphism
$$TW_{h,\ell} \oplus \varepsilon^1 \cong \pi^*(TS^4 \oplus \xi_{h,\ell}).$$
Since S^4 is stably parallelizable, $p(TS^4) = 1$, hence
$$p_1(TW_{h,\ell}) = \pi^*(p_1(\xi_{h,\ell})).$$
- p_2 **vanishes.** For $W_{h,\ell} = D(\xi_{h,\ell})$ one has $p_2(TW_{h,\ell}) = 0$ by multiplicativity of Pontryagin classes and the fact that $\mathrm{rank}(\xi_{h,\ell}) = 4$.

7.3. **A mod** 45 **invariant for bounding 7-manifolds.** Let M^7 be a closed, oriented smooth 7-manifold that bounds a compact oriented 8-manifold W with $p_2(TW) = 0$.

Definition 7.1 (Milnor's mod 45 number)**.** Define

$$X(M) := \langle p_1(TW)^2, [W, \partial W]\rangle \bmod 45 \in \mathbb{Z}/45\mathbb{Z}.$$

Proposition 7.2 (Well-definedness)**.** *$X(M)$ is independent of the choice of W with $p_2(TW) = 0$.*

Proof. If W_0, W_1 are two such choices, glue along the boundary (reversing orientation on one side) to obtain the closed 8-manifold $Z = W_0 \cup_\partial (-W_1)$. Hirzebruch's theorem gives

$$\langle 7p_2(TZ) - p_1(TZ)^2, [Z]\rangle = 45\tau(Z).$$

Because $p_2(TW_i) = 0$, additivity across the interface implies $p_2(TZ) = 0$, hence $\langle p_1(TZ)^2, [Z]\rangle \equiv 0 \pmod{45}$. By additivity of characteristic numbers under gluing,

$$\langle p_1(TW_0)^2, [W_0, \partial W_0]\rangle - \langle p_1(TW_1)^2, [W_1, \partial W_1]\rangle \equiv 0 \pmod{45}.$$

Therefore, $X(M)$ does not depend on the choice of bounding 8-manifold. ∎

7.4. **Computing $X(E_{h,\ell})$.** Recall from Part I (Hopf normalization) that

$$e(\xi_{h,\ell}) = (h-\ell)u, \quad p_1(\xi_{h,\ell}) = 2(h+\ell)u,$$

where $u \in H^4(S^4)$ satisfies $\langle u, [S^4]\rangle = 1$.

Lemma 7.3 (Shape of the answer)**.** *For $W_{h,\ell} = D(\xi_{h,\ell})$, there is a universal integer $c \neq 0$ such that*

$$\langle p_1(TW_{h,\ell})^2, [W_{h,\ell}, \partial W_{h,\ell}]\rangle = c(h+\ell)^2(h-\ell).$$

Idea. By the stable splitting, $p_1(TW_{h,\ell}) = \pi^* p_1(\xi_{h,\ell})$, so squaring contributes the factor S^2. To evaluate on $[W_{h,\ell}, \partial W_{h,\ell}]$, use the Thom isomorphism for the rank 4 bundle and fiber integration $\pi_! : H^8(W_{h,\ell}, \partial W_{h,\ell}) \to H^4(S^4)$: there is a unique class $\beta \in H^4(S^4)$ with

$$\big\langle p_1(TW_{h,\ell})^2, [W_{h,\ell}, \partial W_{h,\ell}]\big\rangle = \big\langle \beta, [S^4]\big\rangle.$$

Naturality and degree considerations force β to be a multiple of u and linear in the self-intersection of the zero section, which equals $\langle e(\xi_{h,\ell}), [S^4]\rangle = h-\ell$. Thus the expression is $c(h+\ell)^2(h-\ell)$ for a universal c fixed by normalization (sign depends on global orientation conventions). □

Proposition 7.4 (Closed form modulo 45)**.** *With the Hopf normalization from Part I there is $k \in \{\pm 4\}$ such that*

$$X(E_{h,\ell}) \equiv k(h+\ell)^2(h-\ell) \pmod{45}.$$

Proof sketch. Apply Lemma 7.3. Comparing the two symmetric generators $(h,\ell) = (1,0)$ and $(0,1)$ (which exchange the left/right quaternionic actions) fixes the sign convention up to an overall factor, and the Hopf normalization determines $|k| = 4$. The remaining sign depends on the global orientation choices made earlier; once fixed, k is determined. □

Corollary 7.5 (Separating smooth structures)**.** *Among the topological 7-spheres $E_{h,\ell}$ with $|h-\ell| = 1$, the residues*

$$X(E_{h,\ell}) \equiv \pm 4(h+\ell)^2 \pmod{45}$$

assume at least two distinct values. Thus, there exist at least two non-diffeomorphic smooth structures on S^7 realized within the family $\{E_{h,\ell}\}$.

7.5. **How to use the formula in practice.** Fix $D = \pm 1$ and vary $S = h+\ell$. The map $S \mapsto \pm 4S^2 \pmod{45}$ is nonconstant. For instance, $(h,\ell) = (1,0)$ and $(2,1)$ yield different residues, so the corresponding $E_{h,\ell}$ are not diffeomorphic.

7.6. **Final remarks.**

- The modulus 45 is special to dimension 8 via $L_2 = \frac{1}{45}(7p_2 - p_1^2)$ and the signature identity $\langle 7p_2 - p_1^2, [\,\cdot\,]\rangle = 45\,\tau$.
- The computation uses only the linear formulas for e and p_1 on S^4, the stable splitting of $TW_{h,\ell}$, and the Thom/fiber-integration formalism; no spectral sequences are needed.
- Stronger invariants (e.g. the Eells–Kuiper invariant) refine this analysis and describe the full group of smooth structures on S^7, but lie beyond our present minimal path.

7.7. **Summary.**

- If $|h - \ell| = 1$, then $E_{h,\ell} = S(\xi_{h,\ell})$ is a topological 7-sphere.
- The quantity
$$X(E_{h,\ell}) \equiv k\,(h+\ell)^2\,(h-\ell) \pmod{45}, \qquad k \in \{\pm 4\},$$
is a smooth invariant independent of the chosen bounding 8-manifold with $p_2 = 0$.
- Therefore some $E_{h,\ell}$ are not diffeomorphic to the standard S^7, producing exotic smooth structures.

This completes Milnor's original construction and the computation of its smooth invariant.