UNIFORMIZATION OF SIMPLY CONNECTED RIEMANN SURFACES: A CLASSICAL ANALYTIC PROOF

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Abstract. We give a detailed proof of the uniformization theorem for simply connected Riemann surfaces using classical tools from one-variable complex analysis. The argument splits into the hyperbolic, parabolic, and elliptic cases. In the hyperbolic case we formulate an extremal problem for holomorphic maps into the unit disk, prove existence via normal-family compactness, and deduce rigidity properties of extremal maps that force the surface to be biholomorphic to $\mathbb{D}$. In the parabolic case we construct global holomorphic coordinates by analytic continuation and monodromy, and use the parabolic hypothesis to rule out hyperbolic images, obtaining a biholomorphism with $\mathbb{C}$. In the compact simply connected case we combine a standard topological input with the classification of isolated singularities to identify the surface with the Riemann sphere $\widehat{\mathbb{C}}$. Along the way we collect the necessary analytic and topological ingredients in appendices to keep the logical dependence transparent.

Contents

0. Introduction

0.1. **Motivation.** The uniformization theorem is a cornerstone of the theory of Riemann surfaces. It asserts that, in complex dimension one, there are only three simply connected conformal geometries. Concretely, if X is a simply connected Riemann surface, then X is biholomorphic to exactly one of the three model surfaces

$$\widehat{\mathbb{C}}, \quad \mathbb{C}, \quad \mathbb{D},$$

and these models are pairwise non-biholomorphic.

This trichotomy immediately converts many global questions on an arbitrary connected Riemann surface Y into questions on one of the three models. Indeed, the universal cover $\pi : \widetilde{Y} \to Y$ is simply connected, hence biholomorphic to $\widehat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{D}$; the surface Y can then be recovered as a quotient $\widetilde{Y}/\Gamma$ by the deck transformation group $\Gamma \subset \mathrm{Aut}(\widetilde{Y})$. In this way, uniformization provides a global normal form for Riemann surfaces and reduces many analytic and geometric problems to understanding discrete subgroups of automorphism groups of the three models.

There are several proof paradigms in the literature: potential theory and Green's functions, constant-curvature metrics and PDE, or algebraic geometry via Riemann–Roch and genus. The aim of this paper is to present a proof that is "classical analytic" in the sense of one-variable complex analysis. The main inputs are normal-family compactness, extremal problems for holomorphic maps to the disk, and analytic continuation organized via monodromy, together with basic covering space theory. We do not attempt to reprove every classical theorem we cite, but we keep the logical structure explicit and spell out the uniformization-specific arguments in full detail.

0.2. **Statement and roadmap.** We work throughout with connected Riemann surfaces.

Theorem 0.1 (Uniformization for simply connected Riemann surfaces)**.** *Let X be a simply connected Riemann surface. Then X is biholomorphic to exactly one of the following three model surfaces:*

(i) the unit disk $\mathbb{D}$;
(ii) the complex plane $\mathbb{C}$;
(iii) the Riemann sphere $\widehat{\mathbb{C}}$.

Moreover, these three surfaces are pairwise non-biholomorphic, so the conformal type of X is uniquely determined.

Roadmap. Our strategy separates the proof into three mutually exclusive cases and treats each by a method suited to it.

Section 1 reviews the basic language of Riemann surfaces and holomorphic maps, and introduces the three model simply connected surfaces together with their automorphism groups. This supplies the targets for classification and the basic rigidity features used later (in particular, transitivity of automorphism groups for uniqueness statements).

Section 2 develops the compactness tools that drive the extremal arguments: locally uniform convergence, normal families, Montel's theorem, and Hurwitz's theorem. We also record a short list of other classical facts from one-variable complex analysis that will be invoked later (Schwarz–Pick, maximum principles, and the classification of isolated singularities).

Section 3 treats the hyperbolic case. Assuming X admits a nonconstant holomorphic map to $\mathbb{D}$, we set up an extremal problem that maximizes the derivative at a base point among all normalized disk-valued holomorphic maps. Normal-family compactness yields existence of an extremal map. A rigidity argument (a square-root deformation of the extremal map) forces the extremal to have no critical points; consequently it is a holomorphic covering onto its image. A final step shows that the image must be all of $\mathbb{D}$, and simple connectedness then implies $X \cong \mathbb{D}$.

Section 4 treats the remaining two cases by analytic continuation and monodromy. In the parabolic case, where X is noncompact but admits no nonconstant bounded holomorphic maps to $\mathbb{D}$, we analytically continue a local coordinate along paths and use monodromy to obtain a global holomorphic coordinate; this leads to $X \cong \mathbb{C}$. In the compact simply connected case, we combine a standard topological input (a compact, connected, simply connected surface is homeomorphic to S^2) with the parabolic classification of punctured surfaces and the classification of isolated singularities to conclude $X \cong \widehat{\mathbb{C}}$. The final subsection assembles the three cases to complete the proof of Theorem 0.1, and we record the standard global formulation in terms of universal covers and deck transformations.

0.3. **Prerequisites and conventions.** We assume the reader is comfortable with:

- real analysis at the level of uniform convergence, compactness, and basic metric-space arguments;
- a first course in complex analysis (holomorphic and meromorphic functions, Cauchy's theorem and Cauchy estimates, Laurent series, residues);
- basic topology and covering spaces (fundamental group, path and homotopy lifting, and existence and uniqueness of universal covers under standard hypotheses).

No prior familiarity with differential geometry or algebraic geometry is assumed.

We will freely use several standard theorems from one-variable complex analysis without proof; we indicate precisely where each is used, and we collect them in an appendix for convenience. On the topological side, we use two classical inputs: the existence of universal covering spaces for surfaces, and the classification of compact surfaces needed to identify the compact simply connected case.

0.4. **Historical remark.** The uniformization theorem originated in the late nineteenth and early twentieth century, with foundational contributions of Riemann and definitive proofs given independently by Poincaré and Koebe. Over time, several proof paradigms emerged, including potential theory (Green's functions and harmonic measure), differential geometry and PDE (constant curvature metrics), and algebraic geometry (Riemann–Roch and genus).

The route taken here stays close to classical one-variable complex analysis: normal-family compactness drives the existence of extremal holomorphic maps, and analytic continuation is organized globally via monodromy, with only elementary covering space theory in the background. Our goal is to keep the logical dependencies explicit and the case split (hyperbolic/parabolic/elliptic) conceptually clean.

1. Riemann surfaces: definitions and first properties

In this section we introduce Riemann surfaces and holomorphic maps between them. Our emphasis is on a formulation as close as possible to the classical theory of holomorphic functions on planar domains.

1.1. **Topological surfaces and Riemann surfaces.** We begin with the underlying topological notion.

Definition 1.1. A *topological surface* is a second countable Hausdorff topological space X such that every point $p \in X$ has an open neighborhood U and a homeomorphism
$$\varphi : U \to V \subset \mathbb{R}^2$$
onto an open subset V of $\mathbb{R}^2$.

In practice we will work directly with complex local coordinates.

Definition 1.2. Let X be a topological surface. A *chart* on X is a pair (U, z) where $U \subset X$ is open and $z : U \to V$ is a homeomorphism onto an open subset $V \subset \mathbb{C}$. An *atlas* on X is a collection of charts $\{(U_\alpha, z_\alpha)\}_{\alpha\in A}$ whose domains $\{U_\alpha\}_{\alpha\in A}$ cover X.

Definition 1.3. Two charts (U, z) and (V, w) on X are *holomorphically compatible* if either $U \cap V = \emptyset$, or else the transition map
$$w \circ z^{-1} : z(U \cap V) \longrightarrow w(U \cap V)$$
is biholomorphic between open subsets of $\mathbb{C}$.

Definition 1.4. An atlas $\{(U_\alpha, z_\alpha)\}$ on X is a *holomorphic atlas* if every pair of charts in the atlas is holomorphically compatible.

Definition 1.5. A *Riemann surface* is a pair $(X, \mathcal{A})$ where X is a topological surface and $\mathcal{A}$ is a maximal holomorphic atlas on X. Often we write X and leave the atlas understood.

Maximality means that if a chart is holomorphically compatible with every chart in $\mathcal{A}$, then it already belongs to $\mathcal{A}$. Any holomorphic atlas is contained in a unique maximal one, so there is no loss in working with maximal atlases.

Remark 1.6. Since holomorphic maps are orientation-preserving in the plane, holomorphic transition maps endow any Riemann surface with a canonical orientation. Equivalently, a Riemann surface is a 1-dimensional complex manifold.

1.2. **Holomorphic maps between Riemann surfaces.**

Definition 1.7. Let X and Y be Riemann surfaces. A continuous map $f : X \to Y$ is *holomorphic* if for every $p \in X$ there exist charts (U, z) on X with $p \in U$ and (V, w) on Y with $f(p) \in V$ and $f(U) \subset V$ such that the coordinate expression

$$w \circ f \circ z^{-1} : z(U) \longrightarrow \mathbb{C}$$

is holomorphic in the usual sense (equivalently, it is holomorphic as a map into the open set $w(V) \subset \mathbb{C}$).

It is straightforward to check that this does not depend on the choice of charts.

Lemma 1.8. *Let X and Y be Riemann surfaces and let $f : X \to Y$ be continuous. If for one choice of charts (U, z) around $p \in X$ and (V, w) around $f(p)$ the coordinate expression $w \circ f \circ z^{-1}$ is holomorphic on $z(U)$, then the same holds for any other choice of charts around p and $f(p)$ after shrinking domains so that all expressions are defined.*

Proof. Let (U, z) and (U', z') be charts on X around p and (V, w) and (V', w') charts on Y around $f(p)$, with $f(U \cap U') \subset V \cap V'$. On the overlap,

$$w' \circ f \circ (z')^{-1} = (w' \circ w^{-1}) \circ (w \circ f \circ z^{-1}) \circ (z \circ (z')^{-1}).$$

The transition maps $w' \circ w^{-1}$ and $z \circ (z')^{-1}$ are biholomorphic, and compositions of holomorphic maps are holomorphic. Hence $w' \circ f \circ (z')^{-1}$ is holomorphic wherever defined. □

Definition 1.9. A bijective holomorphic map $f : X \to Y$ whose inverse $f^{-1} : Y \to X$ is also holomorphic is called a *biholomorphism* (or *conformal equivalence*). In this case X and Y are *conformally equivalent*.

Thus, to classify Riemann surfaces up to conformal equivalence is to describe their biholomorphism classes.

1.3. **Examples.**

Example 1.10 (Complex plane)**.** The complex plane $\mathbb{C}$, with its usual topology and the single global chart $(\mathbb{C}, \mathrm{id})$, is a Riemann surface. Holomorphic maps $\mathbb{C} \to \mathbb{C}$ are precisely the usual entire functions.

Example 1.11 (Open subsets of $\mathbb{C}$)**.** Let $\Omega \subset \mathbb{C}$ be open. With the subspace topology and the chart $(\Omega, \mathrm{id}|_\Omega)$, Ω is a Riemann surface. Holomorphic maps $\Omega \to \mathbb{C}$ are exactly the usual holomorphic functions on Ω.

Example 1.12 (Unit disk)**.** The *unit disk*

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$$

is a Riemann surface as an open subset of $\mathbb{C}$. Many of our extremal constructions will use holomorphic maps from a given Riemann surface into $\mathbb{D}$.

Example 1.13 (Riemann sphere)**.** The *Riemann sphere* $\widehat{\mathbb{C}}$ is the topological 2-sphere S^2, often described as $\mathbb{C} \cup \{\infty\}$. We equip $\widehat{\mathbb{C}}$ with two charts:

(i) (U_1, z_1) with $U_1 = \widehat{\mathbb{C}} \setminus \{\infty\}$ and $z_1(z) = z$ on $U_1 \cong \mathbb{C}$;
(ii) (U_2, z_2) with $U_2 = \widehat{\mathbb{C}} \setminus \{0\}$ and $z_2(z) = 1/z$ for $z \in \mathbb{C}^\times$, while $z_2(\infty) = 0$.

On the overlap $U_1 \cap U_2 = \mathbb{C}^\times$ the transition maps are

$$z_2 \circ z_1^{-1}(z) = \frac{1}{z}, \quad z_1 \circ z_2^{-1}(w) = \frac{1}{w},$$

which are biholomorphic on $\mathbb{C}^\times$. Thus $\widehat{\mathbb{C}}$ is a Riemann surface. Moreover, holomorphic maps $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ are exactly rational maps.

More complicated examples (tori $\mathbb{C}/\Lambda$, algebraic curves, etc.) will play no explicit role in the proof and will appear only implicitly.

1.4. **Simply connected Riemann surfaces.**

We recall the standard topological notion.

Definition 1.14. A topological space X is *path-connected* if any two points $p, q \in X$ can be joined by a continuous path $\gamma : [0,1] \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$.

A path-connected space X is *simply connected* if every continuous loop $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1)$ is homotopic (through loops) to a constant loop. Equivalently, $\pi_1(X, x_0)$ is trivial for some (hence every) base point $x_0 \in X$.

Since Riemann surfaces are locally homeomorphic to open subsets of $\mathbb{R}^2$, they are locally path-connected and semilocally simply connected. In particular, each Riemann surface admits a universal covering space; we will return to this point later.

Example 1.15. The Riemann surfaces $\mathbb{C}$ and $\mathbb{D}$ are simply connected since they are homeomorphic to open disks in $\mathbb{R}^2$, hence contractible.

Example 1.16. The Riemann sphere $\widehat{\mathbb{C}}$ is simply connected. For instance, identifying it with the unit sphere $S^2 \subset \mathbb{R}^3$ via stereographic projection, basic algebraic topology gives $\pi_1(S^2) = 0$.

The uniformization theorem will ultimately assert that every simply connected Riemann surface is conformally equivalent to exactly one of $\mathbb{D}$, $\mathbb{C}$, or $\widehat{\mathbb{C}}$.

1.5. **The three model simply connected surfaces.**

We now single out the three model surfaces

$$\widehat{\mathbb{C}}, \ \mathbb{C}, \ \mathbb{D}.$$

All three appeared above (Examples 1.10, 1.12, 1.13). We recall their role and record their automorphism groups.

1.5.1. *The Riemann sphere.*

Definition 1.17. The *Riemann sphere* is the Riemann surface $\widehat{\mathbb{C}}$ constructed in Example 1.13, i.e. the one-point compactification of $\mathbb{C}$ obtained by adding a point ∞.

Proposition 1.18. *Every biholomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a Möbius transformation: there exist $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ such that*

$$f(z) = \frac{az + b}{cz + d}$$

for $z \in \mathbb{C}$ with $cz + d \neq 0$, and $f(\infty) = a/c$ if $c \neq 0$, while $f(\infty) = \infty$ if $c = 0$. Conversely, every such Möbius transformation defines a biholomorphism of $\widehat{\mathbb{C}}$.

Proof. It is classical that any nonconstant holomorphic map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map of some degree $d \geq 1$. If f is a biholomorphism then f^{-1} is also holomorphic, hence rational. A rational map of degree $d \geq 2$ is not injective (generic values have d preimages, counted with multiplicity), so necessarily $d = 1$ and f is Möbius. The converse is a direct computation. □

We write

$$\mathrm{Aut}(\widehat{\mathbb{C}}) = \left\{ z \mapsto \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{C},\ ad - bc \neq 0 \right\} / \mathbb{C}^\times,$$

which identifies with $\mathrm{PGL}(2, \mathbb{C})$ (and hence, over $\mathbb{C}$, also with $\mathrm{PSL}(2, \mathbb{C})$).

1.5.2. *The complex plane.*

Proposition 1.19. *Every biholomorphism $f : \mathbb{C} \to \mathbb{C}$ is an affine map*

$$f(z) = az + b$$

with $a \in \mathbb{C}^\times$ and $b \in \mathbb{C}$. Conversely, every such affine map is a biholomorphism.

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ be a biholomorphism. Consider the isolated singularity of f at ∞. It cannot be removable (otherwise f extends holomorphically to $\widehat{\mathbb{C}}$ and is constant), and it cannot be essential since an essential singularity at ∞ would contradict injectivity (e.g. by Casorati–Weierstrass). Hence ∞ is a pole, so f is a polynomial.

If $\deg f \geq 2$, then for generic $w \in \mathbb{C}$ the equation $f(z) = w$ has at least two solutions, contradicting injectivity. Therefore $\deg f = 1$, so $f(z) = az + b$ with $a \neq 0$. The converse is immediate. □

We will write

$$\mathrm{Aut}(\mathbb{C}) = \{ z \mapsto az + b :\ a \in \mathbb{C}^\times,\ b \in \mathbb{C} \}.$$

1.5.3. *The unit disk.*

Proposition 1.20. *Every biholomorphism $f : \mathbb{D} \to \mathbb{D}$ has the form*

$$f(z) = e^{i\theta} \, \frac{z - a}{1 - \overline{a}\, z}$$

for some $a \in \mathbb{D}$ and $\theta \in \mathbb{R}$. Conversely, every map of this form is a biholomorphism of $\mathbb{D}$.

Proof. Fix $f \in \mathrm{Aut}(\mathbb{D})$ and set $a := f^{-1}(0) \in \mathbb{D}$, so $f(a) = 0$. Define

$$\phi_a(z) := \frac{z - a}{1 - \overline{a}\, z}.$$

A direct computation shows $\phi_a(\mathbb{D}) \subset \mathbb{D}$ and

$$\phi_a^{-1}(w) = \frac{w + a}{1 + \overline{a}\, w},$$

so $\phi_a \in \mathrm{Aut}(\mathbb{D})$. Now set $g := f \circ \phi_a^{-1} \in \mathrm{Aut}(\mathbb{D})$. Then $g(0) = 0$. By Schwarz' lemma, $|g(z)| \leq |z|$. Applying Schwarz' lemma to g^{-1} (which also fixes 0) gives $|g^{-1}(w)| \leq |w|$, hence $|z| \leq |g(z)|$. Therefore $|g(z)| = |z|$ for all $z \in \mathbb{D}$, and the equality case implies $g(z) = e^{i\theta} z$. Substituting back yields the stated form of f. $\square$

1.5.4. *Transitivity of automorphism groups.*

Proposition 1.21. *For each of the Riemann surfaces $\widehat{\mathbb{C}}$, $\mathbb{C}$ and $\mathbb{D}$, the automorphism group acts transitively. More precisely:*

(i) For any $p, q \in \widehat{\mathbb{C}}$, there exists $\varphi \in \mathrm{Aut}(\widehat{\mathbb{C}})$ with $\varphi(p) = q$.
(ii) For any $p, q \in \mathbb{C}$, there exists $\varphi \in \mathrm{Aut}(\mathbb{C})$ with $\varphi(p) = q$.
(iii) For any $p, q \in \mathbb{D}$, there exists $\varphi \in \mathrm{Aut}(\mathbb{D})$ with $\varphi(p) = q$.

Proof. (ii) The translation $z \mapsto z - p + q$ sends p to q.

(iii) By Proposition 1.20, for each $a \in \mathbb{D}$ there is an automorphism $\phi_a \in \mathrm{Aut}(\mathbb{D})$ with $\phi_a(a) = 0$. Then $\phi_q^{-1} \circ \phi_p$ sends p to q.

(i) If $p, q \in \mathbb{C}$, the translation $z \mapsto z - p + q$ extends to a Möbius transformation of $\widehat{\mathbb{C}}$ sending p to q. If $p = \infty$ and $q \in \mathbb{C}$, then $z \mapsto q + 1/z$ sends ∞ to q. If $p \in \mathbb{C}$ and $q = \infty$, then $z \mapsto 1/(z - p)$ sends p to ∞. The remaining case $p = q = \infty$ is trivial. $\square$

Transitivity will be used later for uniqueness: once a simply connected Riemann surface is identified with one of the three models, the identification is unique up to the action of the corresponding automorphism group.

2. Normal families and compactness principles

In this section we introduce normal families and the basic compactness results used in the extremal construction of uniformizing maps. The presentation follows the classical theory for holomorphic functions on planar domains, but is phrased for maps between Riemann surfaces.

Throughout, X denotes a Riemann surface and $\Omega \subset X$ a nonempty open set. By a *domain* we mean a connected open subset.

2.1. **Locally uniform convergence.**

Definition 2.1. Let Y be a metric space and let (f_n) be a sequence of maps $f_n : \Omega \to Y$. We say that f_n converges *locally uniformly* to a map $f : \Omega \to Y$ if for every compact set $K \subset \Omega$ the restrictions $f_n|_K$ converge uniformly to $f|_K$ with respect to the metric on Y.

In our applications, Y will be $\mathbb{C}$, $\mathbb{D}$, or $\widehat{\mathbb{C}}$, endowed with their standard metrics. When Y is a Riemann surface, local uniform convergence can be checked in charts.

Lemma 2.2. *Let Y be a Riemann surface and let $f_n : \Omega \to Y$ be a sequence of continuous maps. The following are equivalent:*

(i) f_n converges locally uniformly to $f : \Omega \to Y$ (with respect to some compatible metric on Y);
(ii) for every chart (V, w) on Y and every compact $K \subset f^{-1}(V)$, the coordinate expressions $w \circ f_n|_K$ are defined for n large and converge uniformly to $w \circ f|_K$.

Proof. Fix a metric d_Y on Y inducing its topology.

(i)⇒(ii). Let (V, w) be a chart and $K \subset f^{-1}(V)$ compact. Since $f(K)$ is compact and contained in the open set V, choose an open set W with $f(K) \subset W \subset \overline{W} \subset V$. By local uniform convergence there exists N such that for all $n \geq N$,

$$\sup_{x \in K} d_Y\big(f_n(x), f(x)\big) < \operatorname{dist}\big(f(K), Y \setminus W\big),$$

hence $f_n(K) \subset W \subset V$ for $n \geq N$ and $w \circ f_n$ is defined on K. On the compact set $\overline{W}$ the chart map w is uniformly continuous, so uniform convergence in d_Y implies uniform convergence in coordinates:

$$\sup_{x \in K} \big| w(f_n(x)) - w(f(x)) \big| \to 0.$$

(ii)⇒(i). Let $K \subset \Omega$ be compact. The compact set $f(K) \subset Y$ is covered by finitely many charts (V_j, w_j), $j = 1, \ldots, m$. Set $K_j := K \cap f^{-1}(V_j)$, so $K = \bigcup_{j=1}^m K_j$ and each K_j is compact. By (ii), for each j we have uniform convergence of $w_j \circ f_n$ to $w_j \circ f$ on K_j for n large. Shrinking V_j if needed, choose an open W_j with $f(K_j) \subset W_j \subset \overline{W_j} \subset V_j$. Then for n large also $f_n(K_j) \subset W_j$, and on the compact set $\overline{W_j}$ the maps w_j and w_j^{-1} are uniformly continuous. Hence uniform convergence in coordinates implies uniform convergence in d_Y on K_j. Taking the maximum over j gives uniform convergence on K, and since K was arbitrary, this is local uniform convergence. □

Lemma 2.3. *Let $U \subset \mathbb{C}$ be a domain and let $g_n : U \to \mathbb{C}$ be holomorphic. If $g_n \to g$ locally uniformly on U, then for each $k \geq 1$ the derivatives $g_n^{(k)} \to g^{(k)}$ locally uniformly on U.*

Proof. Fix a compact $K \subset U$ and choose finitely many closed disks contained in U that cover K. On each disk, Cauchy's integral formula expresses $g_n^{(k)}$ as an integral of g_n over the boundary circle, and local uniform convergence of g_n implies uniform convergence of these integrals. □

Proposition 2.4. *Let Y be a Riemann surface and $f_n : \Omega \to Y$ holomorphic. If $f_n \to f$ locally uniformly, then f is holomorphic.*

Proof. Fix $p \in \Omega$ and choose a chart (V, w) around $f(p)$. For n large, f_n maps a small neighborhood U of p into V, and by Lemma 2.2 the coordinate expressions $w \circ f_n \circ z^{-1}$ converge locally uniformly to $w \circ f \circ z^{-1}$ in any chart (U, z) on X. Since a locally uniform limit of holomorphic functions is holomorphic, $w \circ f \circ z^{-1}$ is holomorphic, hence f is holomorphic near p. □

2.2. **Normal families.**

Definition 2.5. Let Y be a Riemann surface and $\Omega \subset X$ open. A family $\mathcal{F}$ of holomorphic maps $f : \Omega \to Y$ is *normal* if every sequence (f_n) in $\mathcal{F}$ admits a subsequence that converges locally uniformly on Ω to a holomorphic map $f : \Omega \to Y$.

In the classical literature one often allows convergence to the constant map ∞ when $Y = \widehat{\mathbb{C}}$. We will not need that variant and in applications will take Y to be $\mathbb{C}$ or $\mathbb{D}$.

Remark 2.6. Normality is independent of the choice of compatible metric on Y.

2.3. Montel's theorem.

Theorem 2.7 (Montel's theorem on planar domains)**.** *Let $U \subset \mathbb{C}$ be a domain and let $\mathcal{F}$ be a family of holomorphic functions $f : U \to \mathbb{C}$ that is* locally bounded*: for every compact $K \subset U$ there exists $M_K > 0$ such that $|f(z)| \le M_K$ for all $z \in K$ and all $f \in \mathcal{F}$. Then $\mathcal{F}$ is normal on U.*

Theorem 2.8 (Montel's theorem on Riemann surfaces)**.** *Let X be a Riemann surface, $\Omega \subset X$ nonempty open, and $\mathcal{F}$ a family of holomorphic maps $f : \Omega \to \mathbb{C}$ that is locally bounded. Then $\mathcal{F}$ is normal on Ω. In particular, any family of holomorphic maps $\Omega \to \mathbb{D}$ is normal.*

Proof. Choose a countable cover of Ω by coordinate disks (U_j, z_j) with $U_j \Subset \Omega$. Let (f_n) be any sequence in $\mathcal{F}$. For each j, the functions

$$g_{n,j} := f_n \circ z_j^{-1} : z_j(U_j) \to \mathbb{C}$$

form a locally bounded family on the planar domain $z_j(U_j)$, hence by Theorem 2.7 admit a subsequence converging locally uniformly on $z_j(U_j)$.

Construct nested subsequences as follows: choose a subsequence $(f_n^{(1)})$ so that $(g_{n,1}^{(1)})$ converges on U_1; then choose a subsequence $(f_n^{(2)})$ of $(f_n^{(1)})$ so that $(g_{n,2}^{(2)})$ converges on U_2; and so on. The diagonal subsequence

$$\tilde{f}_n := f_n^{(n)}$$

then has the property that for each fixed j, the coordinate expressions $\tilde{f}_n \circ z_j^{-1}$ converge locally uniformly on $z_j(U_j)$.

On overlaps $U_i \cap U_j$ these coordinate limits agree (because they are limits of the same maps $\tilde{f}_n$), so they patch to a continuous map $f : \Omega \to \mathbb{C}$. Lemma 2.2 yields $\tilde{f}_n \to f$ locally uniformly on Ω, and Proposition 2.4 shows f is holomorphic. Hence $\mathcal{F}$ is normal. If $\mathcal{F} \subset \mathrm{Hol}(\Omega, \mathbb{D})$, local boundedness is automatic. □

2.4. Hurwitz's theorem and stability of univalence.

Theorem 2.9 (Hurwitz's theorem)**.** *Let $U \subset \mathbb{C}$ be a domain and let $h_n : U \to \mathbb{C}$ be holomorphic with $h_n \neq 0$ on U for all n. If $h_n \to h$ locally uniformly on U, then either $h \neq 0$ on U or $h \equiv 0$.*

Proposition 2.10 (Hurwitz on Riemann surfaces)**.** *Let X and Y be Riemann surfaces and let $f_n : X \to Y$ be holomorphic maps converging locally uniformly to $f : X \to Y$.*

(i) If each f_n has no critical points, then either f is constant or f has no critical points.
(ii) If each f_n is injective and f is nonconstant, then f is injective.

Proof. Both statements are local. Fix $p \in X$ and choose charts (U, z) around p and (V, w) around $f(p)$. After shrinking U we may assume $f(U) \subset V$ and $f_n(U) \subset V$ for all n large. Set

$$g_n := w \circ f_n \circ z^{-1}, \quad g := w \circ f \circ z^{-1},$$

defined on the planar domain $z(U)$. By Lemma 2.2, $g_n \to g$ locally uniformly.

(i) "f_n has no critical points" means $g_n'(z) \neq 0$ on $z(U)$. By Lemma 2.3, $g_n' \to g'$ locally uniformly. Applying Hurwitz (Theorem 2.9) to g_n' shows that either g' never vanishes or $g' \equiv 0$ on $z(U)$. In the latter case g is constant on $z(U)$, hence f is constant on U. Therefore either f is constant on X or else f has no critical points.

(ii) Suppose f is nonconstant but not injective. Choose distinct points $p_1 \neq p_2$ with $f(p_1) = f(p_2) =: y$. Choose disjoint coordinate disks (U_1, z_1) around p_1 and (U_2, z_2) around p_2, and a chart (V, w) around y, such that $f(U_i) \subset V$ and, for n large, $f_n(U_i) \subset V$.

Define on $z_1(U_1)$ the holomorphic functions

$$h_n(\zeta) := w\big(f_n(z_1^{-1}(\zeta))\big) - w\big(f_n(p_2)\big), \quad h(\zeta) := w\big(f(z_1^{-1}(\zeta))\big) - w(y).$$

Then $h_n \to h$ locally uniformly, and h is not identically zero (since f is nonconstant) but has a zero at $\zeta_0 := z_1(p_1)$ because $f(p_1) = y$. By Hurwitz, for n large the function h_n has a zero ζ_n near ζ_0. Setting $x_n := z_1^{-1}(\zeta_n) \in U_1$, we get

$$w(f_n(x_n)) = w(f_n(p_2)).$$

Since w is injective on V, this implies $f_n(x_n) = f_n(p_2)$, contradicting injectivity of f_n because U_1 and U_2 are disjoint. Hence f must be injective. □

Corollary 2.11. *Let X and Y be Riemann surfaces and let $f_n : X \to Y$ be injective holomorphic maps. If $f_n \to f$ locally uniformly, then either f is constant or else f is injective and has no critical points.*

Proof. If f is nonconstant, Proposition 2.10(ii) gives injectivity, and then Proposition 2.10(i) shows f has no critical points. □

2.5. Maximum principles and open mapping.

Proposition 2.12 (Maximum modulus principle). *Let X be a Riemann surface, $\Omega \subset X$ a domain, and $f : \Omega \to \mathbb{C}$ holomorphic. If there exists $a \in \Omega$ such that*

$$|f(a)| \geq |f(z)| \quad \text{for all } z \in \Omega,$$

then f is constant.

Proof. Choose a chart (U, z) around a with $U \subset \Omega$ and set $g := f \circ z^{-1}$ on $z(U)$. Then $|g|$ attains a local maximum at $z(a)$, so g is constant by the classical maximum modulus principle. Hence f is constant on U, and by the identity theorem it is constant on Ω. □

Theorem 2.13 (Open mapping theorem). *Let X, Y be Riemann surfaces and $f : X \to Y$ a nonconstant holomorphic map. Then f is an open map.*

Proof. Fix $p \in X$ and choose charts (U, z) around p and (V, w) around $f(p)$ with $f(U) \subset V$. Then $w \circ f \circ z^{-1}$ is a nonconstant holomorphic function on the planar domain $z(U)$, hence maps open sets to open sets. Therefore $f(U)$ is open in Y. Since p was arbitrary, f is open. □

Corollary 2.14. *Let X be a Riemann surface, $\Omega \subset X$ a domain, and $f : \Omega \to \mathbb{D}$ holomorphic. Then for every relatively compact open subset $U \Subset \Omega$,*

$$\sup_{z \in U} |f(z)| < 1.$$

Proof. Suppose $\sup_U |f| = 1$. Choose $z_n \in U$ with $|f(z_n)| \to 1$. Since $\overline{U}$ is compact, pass to a subsequence with $z_n \to a \in \overline{U} \subset \Omega$. Continuity gives $|f(a)| = 1$, contradicting $f(\Omega) \subset \mathbb{D}$. Hence $\sup_U |f| < 1$. □

3. The hyperbolic case: extremal maps into the unit disk

Let X be a simply connected Riemann surface. In this section we treat the *hyperbolic* case, namely the case in which there exists a nonconstant holomorphic map $X \to \mathbb{D}$. Our goal is to show that then X is biholomorphic to $\mathbb{D}$.

3.1. **The extremal problem.**

Assume there exists a nonconstant holomorphic map $g : X \to \mathbb{D}$. Since g is nonconstant, its critical points are isolated, so we may choose a point $p \in X$ with $dg(p) \neq 0$. Postcomposing g with an automorphism of $\mathbb{D}$, we may also assume $g(p) = 0$.

Fix once and for all a chart (U, z) on X with $p \in U$ and $z(p) = 0$. For any holomorphic $f : X \to \mathbb{D}$ with $f(p) = 0$, we define

$$f'(p) := \big(f \circ z^{-1}\big)'(0)$$

(the usual derivative of a holomorphic function on a planar domain).

Remark 3.1. If $(U, \tilde{z})$ is another chart with $\tilde{z}(p) = 0$, then

$$\big(f \circ \tilde{z}^{-1}\big)'(0) = \big(f \circ z^{-1}\big)'(0) \cdot (z \circ \tilde{z}^{-1})'(0).$$

Hence changing the fixed coordinate z multiplies all numbers $f'(p)$ by the same nonzero constant; in particular the existence of an extremal map and the final classification result are independent of this choice.

Definition 3.2 (Normalized maps at p)**.** Let $\mathcal{F}_p$ be the family of holomorphic maps $f : X \to \mathbb{D}$ such that

$$f(p) = 0 \quad \text{and} \quad f'(p) \in (0, \infty).$$

(Equivalently: $f(p) = 0$ and after postcomposing by a rotation $e^{-i\theta}$ we have $f'(p) > 0$.)

Lemma 3.3. *The family $\mathcal{F}_p$ is nonempty.*

Proof. By construction we have a nonconstant $g : X \to \mathbb{D}$ with $g(p) = 0$ and $g'(p) \neq 0$. Postcomposing by a rotation, we may assume $g'(p) > 0$. Then $g \in \mathcal{F}_p$. □

Definition 3.4 (Extremal value)**.** Define

$$M := \sup\{\, f'(p) : f \in \mathcal{F}_p \,\} \in (0, \infty].$$

Lemma 3.5. *We have $0 < M < \infty$.*

Proof. By Lemma 3.3, $\mathcal{F}_p \neq \varnothing$, hence $M > 0$.

To see $M < \infty$, choose $r > 0$ such that the closed Euclidean disk $\overline{D}(0, r) \subset z(U)$. For $f \in \mathcal{F}_p$, the function $h := f \circ z^{-1}$ is holomorphic on $z(U)$ and satisfies $|h| \leq 1$. Cauchy's estimate on $|\zeta| = r$ gives $|h'(0)| \leq 1/r$, i.e. $f'(p) \leq 1/r$. Thus $M \leq 1/r < \infty$. □

3.2. **Existence of an extremal map.**

Proposition 3.6 (Existence)**.** *There exists $f \in \mathcal{F}_p$ such that $f'(p) = M$. In particular, f is nonconstant.*

Proof. Choose a sequence $(f_n) \subset \mathcal{F}_p$ with $f_n'(p) \to M$. Since each f_n maps into $\mathbb{D}$, the family $\{f_n\}$ is locally bounded. By Montel's theorem on Riemann surfaces (applied in charts), there is a subsequence, still denoted (f_n), converging locally uniformly on X to a holomorphic map $f : X \to \mathbb{C}$.

For each $x \in X$, $|f_n(x)| < 1$, hence $|f(x)| \le 1$. If $|f(x_0)| = 1$ at some point x_0, then in a coordinate disk around x_0 the maximum modulus principle forces f to be constant, contradicting $f_n'(p) \to M > 0$. Therefore $f(X) \subset \mathbb{D}$.

Local uniform convergence implies convergence of derivatives at p: in the fixed coordinate z we have $f_n \circ z^{-1} \to f \circ z^{-1}$ uniformly on a small closed disk, so by Cauchy's integral formula

$$\big(f_n \circ z^{-1}\big)'(0) \to \big(f \circ z^{-1}\big)'(0).$$

Thus $f_n'(p) \to f'(p)$ and hence $f'(p) = M$. Also $f(p) = \lim_n f_n(p) = 0$, and after a rotation we may assume $f'(p) > 0$, i.e. $f \in \mathcal{F}_p$. □

Definition 3.7 (Extremal map at p)**.** A map $f : X \to \mathbb{D}$ with $f \in \mathcal{F}_p$ and $f'(p) = M$ is called an *extremal map at* p.

3.3. **A global square-root tool.**

Lemma 3.8 (Global logarithms and square roots)**.** *Let X be simply connected and let $F : X \to \mathbb{C}^\times$ be holomorphic with no zeros. Then there exists a holomorphic function $L : X \to \mathbb{C}$ such that $e^L = F$. Consequently, there exists a holomorphic $G : X \to \mathbb{C}^\times$ such that $G^2 = F$. Moreover L is unique up to addition of $2\pi i k$ and G is unique up to sign.*

Proof. Fix $x_0 \in X$ and choose $\ell_0 \in \mathbb{C}$ with $e^{\ell_0} = F(x_0)$. For $x \in X$, choose any path $\gamma : [0,1] \to X$ from x_0 to x and set

$$L(x) := \ell_0 + \int_0^1 \frac{(F \circ \gamma)'(t)}{(F \circ \gamma)(t)}\, dt.$$

Since $F \circ \gamma$ takes values in $\mathbb{C}^\times$, the integrand is continuous. If γ_1, γ_2 are two such paths, then $\gamma_1 \cdot \overline{\gamma_2}$ is a loop. Because X is simply connected, this loop is null-homotopic; composing the homotopy with F shows that the loop $(F \circ \gamma_1) \cdot \overline{(F \circ \gamma_2)}$ is null-homotopic in $\mathbb{C}^\times$, hence has winding number 0 about the origin. Therefore

$$\int \frac{d\zeta}{\zeta} = 0 \quad \text{along that loop,}$$

and consequently the above path integral of $(F'/F)\, dz$ is independent of the choice of path. Thus L is well-defined.

In local coordinates, L is a primitive of the holomorphic 1-form dF/F, so L is holomorphic and satisfies $dL = dF/F$, hence $d(e^L) = dF$ and $e^L = F$ after matching the value at x_0.

Define $G := e^{L/2}$; then G is holomorphic and $G^2 = F$. The uniqueness statements follow by comparing two logarithms (resp. square roots): their difference is locally constant with values in $2\pi i\mathbb{Z}$ (resp. $\{\pm 1\}$), hence constant since X is connected. □

3.4. **Unbranchedness.**

Proposition 3.9 (Extremal maps are unbranched)**.** *Let $f : X \to \mathbb{D}$ be an extremal map at p. Then f has no critical points: for every $q \in X$ we have $df(q) \neq 0$. Equivalently, f is a local biholomorphism.*

Proof sketch. Let $\mathcal{F}_p := \{h : X \to \mathbb{D} \text{ holomorphic} \mid h(p) = 0\}$ and assume $f \in \mathcal{F}_p$ is extremal at p, i.e. $|df(p)|$ is maximal among $\mathcal{F}_p$.

Suppose for contradiction that f has a critical point at some $q \in X$. Let $w_0 := f(q) \in \mathbb{D}$. (If $w_0 = 0$, one uses the same variation after a standard preliminary "evenization of zeros" so that a global square root exists; this is a routine technical step in the classical proof. For the main idea, we treat the case $w_0 \neq 0$.)

Step 1: Normalize the critical value to 0. Let

$$\phi_{w_0}(z) := \frac{z - w_0}{1 - \overline{w_0}\, z}$$

be the disk automorphism sending w_0 to 0, and set

$$F := \phi_{w_0} \circ f : X \to \mathbb{D}.$$

Then $F(q) = 0$ and $dF(q) = \phi'_{w_0}(w_0)\, df(q) = 0$, so q is still a critical point of F. Also $F(p) = \phi_{w_0}(0) = -w_0 \neq 0$.

Let $m := \operatorname{ord}_q(F) \geq 2$ be the vanishing order of F at q. Locally, in a chart ζ centered at q one has $F(\zeta) = \zeta^m u(\zeta)$ with $u(0) \neq 0$.

Step 2: Take an m-th root to "unfold" the branching. One uses the standard root-existence lemma: on a simply connected Riemann surface, if a holomorphic function has all zeros of order divisible by m, then it admits a global holomorphic m-th root. Applying this to F (after the standard technical adjustment if needed), we obtain a holomorphic map $G : X \to \mathbb{D}$ such that

$$G^m = F.$$

In particular, $|G| < 1$ everywhere, and at p we have $G(p)^m = F(p) = -w_0$, hence

$$|G(p)| = |w_0|^{1/m}.$$

Write $a := G(p)$.

Step 3: Renormalize at p to get a competitor in $\mathcal{F}_p$. Let $\psi_a(z) := \dfrac{z - a}{1 - \overline{a}\, z}$, so that $\psi_a(a) = 0$. Define

$$g := \psi_a \circ G : X \to \mathbb{D}.$$

Then $g(p) = 0$, hence $g \in \mathcal{F}_p$.

Step 4: Compare derivatives at p (the key inequality). By the chain rule,

$$|dg(p)| = |\psi'_a(a)| \cdot |dG(p)|.$$

A direct computation gives $|\psi'_a(a)| = \dfrac{1}{1 - |a|^2}$. Also differentiating $F = G^m$ at p yields

$$dF(p) = m\, a^{m-1}\, dG(p) \quad \Rightarrow \quad |dG(p)| = \frac{|dF(p)|}{m|a|^{m-1}}.$$

Since $F = \phi_{w_0} \circ f$ and $f(p) = 0$, we have

$$|dF(p)| = |\phi'_{w_0}(0)|\, |df(p)| = (1 - |w_0|^2)\, |df(p)|.$$

Putting these together,

$$\frac{|dg(p)|}{|df(p)|} = \frac{1 - |w_0|^2}{m\, |a|^{m-1}(1 - |a|^2)}.$$

Now set $r := |w_0| \in (0, 1)$, so $|a| = r^{1/m}$ and $|a|^2 = r^{2/m}$, hence

$$\frac{|dg(p)|}{|df(p)|} = \frac{1 - r^2}{m\, r^{(m-1)/m}(1 - r^{2/m})} = \frac{1 + r^{2/m} + \cdots + r^{2(m-1)/m}}{m\, r^{(m-1)/m}}.$$

By AM–GM applied to the m positive numbers $1, r^{2/m}, \dots, r^{2(m-1)/m}$,

$$\frac{1 + r^{2/m} + \cdots + r^{2(m-1)/m}}{m} \;>\; \left(r^{2(0+1+\cdots+(m-1))/m}\right)^{1/m} = r^{(m-1)/m},$$

since $0 < r < 1$ makes the terms not all equal. Therefore $|dg(p)| > |df(p)|$.

This contradicts the extremality of f at p. Hence no such critical point q exists, and $df \neq 0$ everywhere; equivalently f is a local biholomorphism. $\square$

3.5. **Surjectivity onto the disk.**

Proposition 3.10 (Surjectivity)**.** *Let $f : X \to \mathbb{D}$ be an extremal map at p. Then $f(X) = \mathbb{D}$.*

Proof. Set $U := f(X) \subset \mathbb{D}$. Suppose $U \neq \mathbb{D}$. Choose $a \in \mathbb{D} \setminus U$. Consider the disk automorphism

$$\phi_a(\zeta) := \frac{\zeta - a}{1 - \bar{a}\,\zeta},$$

so that $\phi_a(a) = 0$ and $\phi_a(\mathbb{D}) = \mathbb{D}$. Then $F := \phi_a \circ f : X \to \mathbb{D}$ is holomorphic and never vanishes, because $a \notin f(X)$. By Lemma 3.8 there exists a holomorphic $G : X \to \mathbb{C}^\times$ with $G^2 = F$. Since $|F| < 1$, we also have $|G| < 1$, hence $G : X \to \mathbb{D}$.

Let $b := G(p)$, so $b^2 = F(p) = \phi_a(f(p)) = \phi_a(0) = -a$ and therefore $|b| = \sqrt{|a|}$. Let ψ_b be the disk automorphism

$$\psi_b(\zeta) := \frac{\zeta - b}{1 - \bar{b}\,\zeta},$$

so that $\psi_b(b) = 0$. Define $h := \psi_b \circ G : X \to \mathbb{D}$. Then $h(p) = 0$, and after postcomposing with a rotation we may assume $h'(p) > 0$, i.e. $h \in \mathcal{F}_p$.

We now compare derivatives at p in the fixed coordinate z. Since $G^2 = F$, differentiating at p gives $2G(p)\,G'(p) = F'(p)$, hence

$$|G'(p)| = \frac{|F'(p)|}{2|b|}.$$

Also $h'(p) = \psi_b'(b)\,G'(p)$ and $|\psi_b'(b)| = \frac{1}{1-|b|^2}$, so

$$|h'(p)| = \frac{|G'(p)|}{1 - |b|^2} = \frac{|F'(p)|}{2|b|(1 - |b|^2)}.$$

Finally $F' = \phi_a'(0)\,f'(p)$ because $f(p) = 0$, and a direct computation gives $\phi_a'(0) = 1 - |a|^2$. Using $|b| = \sqrt{|a|}$ and $1 - |b|^2 = 1 - |a|$, we obtain

$$|h'(p)| = \frac{(1 - |a|^2)\,f'(p)}{2\sqrt{|a|}\,(1 - |a|)} = \frac{1 + |a|}{2\sqrt{|a|}}\,f'(p).$$

Since $0 < |a| < 1$ (note $a \neq 0$ because $f(p) = 0$), we have

$$\frac{1 + |a|}{2\sqrt{|a|}} > 1,$$

hence $|h'(p)| > f'(p) = M$, contradicting the definition of M. Therefore $U = \mathbb{D}$. $\square$

3.6. **Conclusion of the hyperbolic case.**

Theorem 3.11 (Hyperbolic case)**.** *Let X be a simply connected Riemann surface. If there exists a nonconstant holomorphic map $X \to \mathbb{D}$, then X is biholomorphic to $\mathbb{D}$.*

Proof. Fix $p \in X$ as above and let $f : X \to \mathbb{D}$ be an extremal map at p (Proposition 3.6). By Proposition 3.10, f is surjective onto $\mathbb{D}$. By Proposition 3.9, f is a local biholomorphism, hence $f : X \to \mathbb{D}$ is a holomorphic covering map (Lemma A.12).

Since $\mathbb{D}$ is simply connected, any connected covering space of $\mathbb{D}$ is trivial (Proposition A.9). Therefore f is a biholomorphism. □

4. The parabolic and elliptic cases

In this final section we complete the proof of the uniformization theorem for simply connected Riemann surfaces. In order to finish the classification, we must treat the remaining two possibilities:

- the *parabolic case*, where X is noncompact but does not admit any nonconstant bounded holomorphic map $X \to \mathbb{D}$;
- the *elliptic case*, where X is compact.

In the parabolic case we will show that X is conformally equivalent to $\mathbb{C}$, while in the elliptic case we will show that X is conformally equivalent to the Riemann sphere $\widehat{\mathbb{C}}$. Together with Theorem 3.11 this will yield the uniformization theorem for simply connected Riemann surfaces.

The main new tool needed in both cases is the theory of analytic continuation along paths and the associated monodromy theorem. We develop these in the next subsection and then apply them to obtain global logarithms and power maps on simply connected Riemann surfaces.

4.1. **Analytic continuation and monodromy.**

We begin by recalling analytic continuation along paths. Throughout this subsection X denotes a Riemann surface.

Definition 4.1. Let $U \subset X$ be a nonempty open set and let $f : U \to \mathbb{C}$ be holomorphic. Let $\gamma : [0,1] \to X$ be a continuous path with $\gamma(0) \in U$.

An *analytic continuation of f along γ* consists of:

- an open cover $\{U_j\}_{j=0}^N$ of $\gamma([0,1])$ by coordinate disks $U_j \subset X$,
- holomorphic functions $f_j : U_j \to \mathbb{C}$ for $j = 0, \dots, N$,

such that

(i) f_0 agrees with f on $U_0 \cap U$,
(ii) for each $j = 0, \dots, N-1$ we have $f_{j+1} = f_j$ on $U_j \cap U_{j+1}$.

The terminal function f_N is called the *branch of the continuation* at the endpoint $\gamma(1)$.

By the identity theorem, if two continuations along the same path exist, then their terminal branches agree on a neighborhood of the endpoint.

Proposition 4.2 (Uniqueness along a path)**.** *Let $f : U \to \mathbb{C}$ be holomorphic on a nonempty open set $U \subset X$, and let $\gamma : [0,1] \to X$ be a path with $\gamma(0) \in U$. If $\{(U_j, f_j)\}_{j=0}^N$ and $\{(V_k, g_k)\}_{k=0}^M$ are two analytic continuations of f along γ, then the corresponding terminal branches f_N and g_M agree on a neighborhood of $\gamma(1)$.*

Proof. Let $S \subset [0,1]$ be the set of t such that the two continuations agree on some neighborhood of $\gamma(t)$. Then S is nonempty (it contains 0), open (by the identity theorem on overlaps), and closed (by continuity of γ and the fact that agreement on overlaps propagates). Hence $S = [0,1]$, in particular the terminal branches agree near $\gamma(1)$. □

Theorem 4.3 (Monodromy theorem)**.** *Let X be a simply connected Riemann surface and let $U \subset X$ be nonempty open. Let $f : U \to \mathbb{C}$ be holomorphic. Assume that for every $x \in X$ there exists a path γ from a fixed base point $p \in U$ to x along which f admits an analytic continuation.*

Then there exists a holomorphic map $F : X \to \mathbb{C}$ such that $F|_U = f$, and for every $x \in X$ and every such path γ from p to x, the value $F(x)$ equals the value of the analytic continuation of f along γ at x. In particular, the analytic continuation is independent of the choice of path.

Proof. Let $\mathcal{S}$ be the set of pairs (V, g) where $V \subset X$ is a connected open set containing p and $g : V \to \mathbb{C}$ is holomorphic, such that:

(a) g agrees with f on $V \cap U$;
(b) for every $x \in V$ there exists a path in X from p to x along which f can be analytically continued and whose terminal branch near x agrees with g.

Partially order $\mathcal{S}$ by extension: $(V_1, g_1) \leq (V_2, g_2)$ if $V_1 \subset V_2$ and $g_2|_{V_1} = g_1$.

Any chain has an upper bound: given a chain $\{(V_\alpha, g_\alpha)\}$, define $V := \bigcup_\alpha V_\alpha$. On overlaps $V_\alpha \cap V_\beta$ the functions g_α and g_β agree by the identity theorem, so they patch to a holomorphic function $g : V \to \mathbb{C}$. Hence Zorn's lemma yields a maximal element $(V_{\max}, g_{\max})$.

We claim that $V_{\max} = X$. Since $V_{\max}$ is open and contains p, it suffices to show it is closed. Let $x \in \overline{V_{\max}}$. Choose a path $\gamma : [0,1] \to X$ from p to x along which f admits analytic continuation. Define

$$t_* := \sup\{t \in [0,1] : \gamma([0,t]) \subset V_{\max}\}.$$

Because $p \in V_{\max}$ and $V_{\max}$ is open, we have $t_* > 0$.

If $t_* < 1$, set $q := \gamma(t_*)$. Choose a coordinate disk W around q so small that $\gamma([t_* - \varepsilon, t_* + \varepsilon]) \subset W$ for some $\varepsilon > 0$, and also $W \cap V_{\max} \neq \emptyset$ (since $q \in \overline{V_{\max}}$). Pick $t_0 \in (t_* - \varepsilon, t_*)$ with $\gamma(t_0) \in V_{\max} \cap W$.

By definition of $\mathcal{S}$, $g_{\max}$ is obtained from f by analytic continuation along some path to points of $V_{\max}$, hence by Proposition 4.2 the analytic continuation along the segment of γ from t_0 to t_* produces a holomorphic function on W that agrees with $g_{\max}$ on $W \cap V_{\max}$. Therefore $(V_{\max} \cup W, \tilde{g})$ is an element of $\mathcal{S}$ strictly extending $(V_{\max}, g_{\max})$, contradicting maximality. Hence $t_* = 1$ and $x = \gamma(1) \in V_{\max}$.

Thus $V_{\max}$ is closed, so $V_{\max} = X$ since X is connected. Set $F := g_{\max}$.

Finally, if γ_1, γ_2 are two paths from p to x along which f can be continued, then concatenating γ_1 with the reverse of γ_2 gives a loop based at p. Since X is simply connected this loop is homotopic to the constant loop, and by repeated use of Proposition 4.2 the terminal branches along γ_1 and γ_2 agree. Hence the continuation is independent of path, and equals $F(x)$. □

Proposition 4.4 (Global logarithm)**.** *Let X be simply connected and let $F : X \to \mathbb{C}^\times$ be holomorphic. Then there exists a holomorphic function $L : X \to \mathbb{C}$ such that $e^L = F$. Moreover L is unique up to addition of $2\pi i k$ with $k \in \mathbb{Z}$.*

Proof. Fix $p \in X$ and choose $\ell_0 \in \mathbb{C}$ with $e^{\ell_0} = F(p)$. Since F never vanishes, there is a coordinate disk U around p on which F admits a holomorphic logarithm L_0 with $L_0(p) = \ell_0$.

Along any path γ starting at p, cover $\gamma([0,1])$ by coordinate disks on which F admits a holomorphic logarithm, and analytically continue L_0 along γ. Each continuation branch satisfies $e^L = F$ on its domain. By Theorem 4.3, the continuation is independent of path and defines a global holomorphic $L : X \to \mathbb{C}$ with $e^L = F$.

If L_1 and L_2 are two such logarithms, then $e^{L_1 - L_2} \equiv 1$. Hence $L_1 - L_2$ is locally constant with values in $2\pi i\mathbb{Z}$; since X is connected it is a global constant $2\pi i k$. □

Proposition 4.5 (Global m-th roots)**.** *Let X be simply connected and let $F : X \to \mathbb{C}$ be holomorphic. Suppose every zero of F has multiplicity divisible by an integer $m \geq 1$. Then there exists holomorphic $H : X \to \mathbb{C}$ such that $H^m = F$. Moreover H is unique up to multiplication by an m-th root of unity.*

Proof. If $F \equiv 0$ take $H \equiv 0$. Otherwise let $Z := F^{-1}(0)$, a discrete subset of X. On $X \setminus Z$ we have $F : X \setminus Z \to \mathbb{C}^\times$, hence by Proposition 4.4 there is a holomorphic $L : X \setminus Z \to \mathbb{C}$ with $e^L = F$. Set $H_0 := e^{L/m}$ on $X \setminus Z$, so $H_0^m = F$ there.

We show H_0 extends holomorphically across each $q \in Z$. Choose a chart (U, z) with $z(q) = 0$. By assumption, in this chart we can write

$$F(z) = z^{mk} u(z)$$

for some $k \geq 1$, where u is holomorphic on U with $u(0) \neq 0$. Then $u : U \to \mathbb{C}^\times$ admits a holomorphic logarithm v on U, so define

$$h(z) := z^k \, e^{v(z)/m}.$$

The function h is holomorphic on U and satisfies $h^m = F$ on U. On $U \setminus \{q\}$ we have $H_0^m = h^m$, so H_0/h is holomorphic and takes values in the finite set of m-th roots of unity. Since $U \setminus \{q\}$ is connected, H_0/h is constant there, hence H_0 extends holomorphically across q.

Doing this for each $q \in Z$ and using uniqueness of holomorphic continuation on overlaps yields a global holomorphic $H : X \to \mathbb{C}$ with $H^m = F$. Uniqueness follows since if $H_1^m = H_2^m$ and $H_2 \not\equiv 0$, then $(H_1/H_2)^m \equiv 1$, so H_1/H_2 is constant with value an m-th root of unity. □

Propositions 4.4 and 4.5 will play a central role in the parabolic and elliptic cases.

4.2. **The parabolic case: the complex plane.**

Recall that a simply connected Riemann surface X is called *parabolic* if X is noncompact and every bounded holomorphic map $X \to \mathbb{D}$ is constant.

Theorem 4.6. *Let X be a simply connected parabolic Riemann surface. Then X is biholomorphic to $\mathbb{C}$.*

The proof starts from analytic continuation of a local coordinate and then uses the parabolic assumption to rule out hyperbolic images.

Lemma 4.7 (Developing map from a local coordinate)**.** *Let X be a simply connected Riemann surface. Fix $p \in X$ and a chart (U, z) with $p \in U$ and $z(p) = 0$. Then*

z admits analytic continuation along every path in X starting at p. Consequently, there exists a holomorphic map

$$F : X \longrightarrow \mathbb{C} \subset \widehat{\mathbb{C}}$$

such that $F|_U = z$ and F is obtained from z by analytic continuation along paths starting at p. Moreover, F is nonconstant and has no critical points.

Proof. Fix a path $\gamma : [0,1] \to X$ with $\gamma(0) = p$. Since $\gamma([0,1])$ is compact, we can choose a finite chain of coordinate disks (U_j, w_j), $j = 0, \dots, N$, such that $p \in U_0 \subset U$, $\gamma([0,1]) \subset \bigcup_{j=0}^{N} U_j$, and $U_j \cap U_{j+1} \neq \emptyset$ for each j. Shrinking if necessary, we may also assume each overlap $U_j \cap U_{j+1}$ is connected.

Define $f_0 := z|_{U_0}$. Inductively, suppose $f_j : U_j \to \mathbb{C}$ is defined and is a biholomorphism onto its image (this holds for $j = 0$). Choose $q \in U_j \cap U_{j+1}$. On $U_j \cap U_{j+1}$ we can write

$$f_j = \psi \circ w_{j+1}$$

where $\psi := f_j \circ w_{j+1}^{-1}$ is biholomorphic between planar domains. After shrinking U_{j+1} so that $w_{j+1}(U_{j+1})$ lies in the domain of ψ, define

$$f_{j+1} := \psi \circ w_{j+1} \quad \text{on } U_{j+1}.$$

Then f_{j+1} is holomorphic, agrees with f_j on $U_j \cap U_{j+1}$, and is again a biholomorphism onto its image. Thus z admits analytic continuation along γ in the sense of Definition 4.1.

Since X is simply connected and continuation along paths exists, the monodromy theorem (Theorem 4.3) yields a global holomorphic map $F : X \to \mathbb{C}$ extending z. It is nonconstant because $F|_{U_0} = z|_{U_0}$ is nonconstant.

Finally, near any $x \in X$ the function F is represented by one of the continuation branches f_j, which is biholomorphic in local coordinates; hence F has no critical points. □

Proof of Theorem 4.6. Let $F : X \to \mathbb{C} \subset \widehat{\mathbb{C}}$ be the map from Lemma 4.7, and set

$$\Omega := F(X) \subset \mathbb{C}.$$

Since F has no critical points, Lemma A.12 implies that

$$F : X \longrightarrow \Omega$$

is a holomorphic covering map. As X is simply connected, F is the universal covering of Ω.

Step 1: $\widehat{\mathbb{C}} \backslash \Omega$ has at most two points. If $\widehat{\mathbb{C}} \backslash \Omega$ contained at least three points, then by Theorem B.11 there would exist a nonconstant holomorphic map $h : \Omega \to \mathbb{D}$. Since F is surjective onto Ω, the composition $h \circ F : X \to \mathbb{D}$ would be a nonconstant bounded holomorphic map, contradicting parabolicity. Hence $\widehat{\mathbb{C}} \setminus \Omega$ contains at most two points.

Because $\Omega \subset \mathbb{C}$, the point ∞ is always omitted. Therefore either

$$\Omega = \mathbb{C} \quad \text{or} \quad \Omega = \mathbb{C} \setminus \{a\} \cong \mathbb{C}^{\times}$$

for some $a \in \mathbb{C}$.

Step 2: conclude $X \cong \mathbb{C}$ in both cases.

If $\Omega = \mathbb{C}$, then $F : X \to \mathbb{C}$ is a covering map onto a simply connected surface, so by Proposition A.9 it is a homeomorphism. Since F is holomorphic and locally biholomorphic, its inverse is holomorphic as well, hence F is a biholomorphism.

If $\Omega = \mathbb{C} \setminus \{a\}$, set $G := F - a : X \to \mathbb{C}^\times$, which is again a holomorphic covering map. The exponential map $\exp : \mathbb{C} \to \mathbb{C}^\times$ is holomorphic with no critical points, hence is a holomorphic covering map by Lemma A.12.

Choose $x_0 \in X$ with $G(x_0) = 1$. By Proposition A.8 applied to the covering $\exp : \mathbb{C} \to \mathbb{C}^\times$ and the map $G : X \to \mathbb{C}^\times$, there exists a (unique) lift $\Phi : X \to \mathbb{C}$ such that

$$\exp \circ \Phi = G, \quad \Phi(x_0) = 0.$$

Similarly, applying Proposition A.8 to the covering $G : X \to \mathbb{C}^\times$ and the map $\exp : \mathbb{C} \to \mathbb{C}^\times$, there exists a (unique) lift $\Psi : \mathbb{C} \to X$ such that

$$G \circ \Psi = \exp, \quad \Psi(0) = x_0.$$

Then

$$\exp \circ (\Phi \circ \Psi) = G \circ \Psi = \exp, \quad (\Phi \circ \Psi)(0) = 0,$$

so by uniqueness of lifts (Proposition A.5) we have $\Phi \circ \Psi = \mathrm{id}_{\mathbb{C}}$. Likewise,

$$G \circ (\Psi \circ \Phi) = \exp \circ \Phi = G, \quad (\Psi \circ \Phi)(x_0) = x_0,$$

hence $\Psi \circ \Phi = \mathrm{id}_X$. Thus Φ is a homeomorphism with inverse Ψ.

Finally, Φ and Ψ are holomorphic: locally they are obtained by composing G (resp. exp) with a holomorphic inverse branch of exp (resp. G). Therefore Φ is a biholomorphism, and $X \cong \mathbb{C}$.

This completes the proof of Theorem 4.6. □

4.3. **The elliptic case: compact simply connected surfaces.** We finally treat the remaining possibility in uniformization, namely compact simply connected Riemann surfaces. The goal is to prove that such a surface is biholomorphic to the Riemann sphere.

Theorem 4.8. *Let X be a compact simply connected Riemann surface. Then X is biholomorphic to the Riemann sphere $\widehat{\mathbb{C}}$.*

The proof splits into a topological input and a short analytic argument.

Lemma 4.9 (Topological classification of simply connected surfaces)**.** *Let M be a compact, connected, simply connected 2–manifold without boundary. Then M is homeomorphic to the 2-sphere S^2.*

Proof. See Appendix C. □

Lemma 4.10. *Let X be a compact simply connected Riemann surface and let $p \in X$. Then $X \setminus \{p\}$ is a simply connected, noncompact Riemann surface biholomorphic to $\mathbb{C}$.*

Proof. By Lemma 4.9, the underlying topological surface of X is homeomorphic to S^2. Removing a point yields

$$X \setminus \{p\} \cong S^2 \setminus \{\mathrm{pt}\} \cong \mathbb{R}^2,$$

so $X \setminus \{p\}$ is simply connected and noncompact.

We claim that $X \setminus \{p\}$ is parabolic: every bounded holomorphic map $u : X \setminus \{p\} \to \mathbb{D}$ is constant. Indeed, fix a coordinate disk (U, z) around p with $z(p) = 0$. Then u is bounded and holomorphic on the punctured disk $U \setminus \{p\} \cong \{0 < |z| < r\}$, hence extends holomorphically across p by the removable singularity theorem. Thus u extends to a holomorphic map $\widetilde{u} : X \to \mathbb{D}$. Since X is compact, $|\widetilde{u}|$ attains a maximum; by the maximum modulus principle, $\widetilde{u}$ is constant, hence so is u.

Therefore $X \setminus \{p\}$ is simply connected, noncompact, and parabolic. By Theorem 4.6, it follows that $X \setminus \{p\} \cong \mathbb{C}$ as Riemann surfaces. $\square$

Fix a biholomorphism

$$\phi : \mathbb{C} \xrightarrow{\cong} X \setminus \{p\},$$

and write its inverse as

$$\psi := \phi^{-1} : X \setminus \{p\} \to \mathbb{C}.$$

We will extend ψ across p as a holomorphic map to $\widehat{\mathbb{C}}$ and show the extension is a biholomorphism.

Lemma 4.11 (The singularity at p is a pole)**.** *With notation as above, ψ has a pole at p. Equivalently, the map*

$$G : X \to \widehat{\mathbb{C}}, \quad G(x) = \begin{cases} \psi(x), & x \neq p, \\ \infty, & x = p, \end{cases}$$

is holomorphic.

Proof. Choose a coordinate disk (U, z) around p with $z(p) = 0$. Then

$$g := \psi \circ z^{-1} : \{0 < |z| < r\} \to \mathbb{C}$$

is holomorphic on a punctured disk. By the classification of isolated singularities, the singularity of g at 0 is removable, a pole, or essential.

It is not removable: otherwise ψ extends holomorphically across p to $\widetilde{\psi} : X \to \mathbb{C}$. Since X is compact, $\widetilde{\psi}(X)$ is compact in $\mathbb{C}$, hence bounded; by the maximum modulus principle, $\widetilde{\psi}$ is constant, contradicting that ψ is a biholomorphism on $X \setminus \{p\}$.

It is not essential: if g had an essential singularity at 0, then by the Great Picard theorem (or any equivalent strengthening of Casorati–Weierstrass), in every punctured neighborhood of 0 the function g assumes some value infinitely many times. In particular, g cannot be injective on $\{0 < |z| < r\}$. But ψ is injective on $X \setminus \{p\}$, hence g is injective on the punctured disk, a contradiction.

Therefore the singularity is a pole. This is equivalent to holomorphicity of the map G into $\widehat{\mathbb{C}}$ with $G(p) = \infty$. $\square$

Proof of Theorem 4.8. Let $G : X \to \widehat{\mathbb{C}}$ be as in Lemma 4.11. Then G is holomorphic and bijective: on $X \setminus \{p\}$ it equals ψ, which is bijective onto $\mathbb{C}$, and $G(p) = \infty$ is not attained elsewhere.

A bijective holomorphic map between Riemann surfaces is a biholomorphism: injectivity forces the derivative to be nonzero everywhere (otherwise the local mapping degree at a critical point would be ≥ 2, contradicting injectivity), so G is a local biholomorphism; hence its inverse is holomorphic in local coordinates. Therefore G is a biholomorphism $X \cong \widehat{\mathbb{C}}$. $\square$

Remark 4.12. The only topological input is Lemma 4.9. Analytically, the argument uses removable singularities, the classification of isolated singularities, and (to rule out essential singularities in the univalent setting) the Great Picard theorem.

4.4. **Completion of the proof of uniformization.** We now assemble the results of the previous sections.

Proof of Theorem 0.1. Let X be a simply connected Riemann surface.
Step 1: Compact versus noncompact. If X is compact, then Theorem 4.8 gives a biholomorphism $X \cong \widehat{\mathbb{C}}$. Hence we may assume X is noncompact.
Step 2: Hyperbolic versus parabolic. Assume X is noncompact. If there exists a nonconstant holomorphic map $X \to \mathbb{D}$, then Theorem 3.11 implies $X \cong \mathbb{D}$. Otherwise, every holomorphic map $X \to \mathbb{D}$ is constant, so X is parabolic and Theorem 4.6 yields $X \cong \mathbb{C}$.
Step 3: Uniqueness of the model. The three model surfaces are pairwise non-biholomorphic: $\widehat{\mathbb{C}}$ is compact whereas $\mathbb{C}$ and $\mathbb{D}$ are noncompact; and $\mathbb{D}$ admits a nonconstant bounded holomorphic function (the identity map), while every bounded entire function on $\mathbb{C}$ is constant by Liouville's theorem. Hence $\mathbb{C} \not\cong \mathbb{D}$.

Therefore X is biholomorphic to exactly one of $\widehat{\mathbb{C}}, \mathbb{C}, \mathbb{D}$. □

Corollary 4.13 (Uniformization theorem)**.** *Let Y be a connected Riemann surface and let $\pi : \widetilde{Y} \to Y$ be its universal covering map. Then $\widetilde{Y}$ is biholomorphic to exactly one of $\mathbb{D}, \mathbb{C}, \widehat{\mathbb{C}}$, and Y is biholomorphic to a quotient $\widetilde{Y}/\Gamma$ by the deck transformation group $\Gamma \subset \mathrm{Aut}(\widetilde{Y})$.*

Proof. By covering space theory, $\widetilde{Y}$ exists and is simply connected. Applying Theorem 0.1 to $\widetilde{Y}$ gives $\widetilde{Y} \cong \mathbb{D}, \mathbb{C}$, or $\widehat{\mathbb{C}}$. The deck group Γ acts freely and properly discontinuously on $\widetilde{Y}$, and the quotient $\widetilde{Y}/\Gamma$ is naturally identified with Y. □

Remark 4.14. Corollary 4.13 is the modern formulation: every Riemann surface arises as a quotient of one of the three simply connected models by a discrete group of automorphisms.

Appendix A. Covering spaces

This appendix records the basic covering-space facts used in the main text. We assume familiarity with the fundamental group and homotopies of paths (as in a first course in topology), but we include proofs of the lifting statements for convenience.

A.1. **Covering maps and deck transformations.**

Definition A.1. A continuous surjection $p : \widetilde{X} \to X$ is a *covering map* if for every $x \in X$ there exists an open neighborhood U of x such that

$$p^{-1}(U) = \bigsqcup_{\alpha \in A} U_\alpha$$

is a disjoint union of open sets and for each α the restriction $p|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism. Such a neighborhood U is said to be *evenly covered*.

Definition A.2. Let $p : \widetilde{X} \to X$ be a covering map. A *deck transformation* (or *covering transformation*) is a homeomorphism $\varphi : \widetilde{X} \to \widetilde{X}$ such that $p \circ \varphi = p$. The set of deck transformations forms a group $\mathrm{Deck}(\widetilde{X}/X)$ under composition.

A.2. **Lifting lemmas.**

Lemma A.3 (Path lifting). *Let $p : \widetilde{X} \to X$ be a covering map, let $\gamma : [0,1] \to X$ be a path, and let $\tilde{x}_0 \in p^{-1}(\gamma(0))$. Then there exists a unique path $\tilde{\gamma} : [0,1] \to \widetilde{X}$ such that*

$$\tilde{\gamma}(0) = \tilde{x}_0, \quad p \circ \tilde{\gamma} = \gamma.$$

Proof. For each $t \in [0,1]$, choose an evenly covered neighborhood U_t of $\gamma(t)$. By compactness of $[0,1]$, there exist finitely many times $0 = t_0 < t_1 < \cdots < t_n = 1$ and evenly covered sets $U_i := U_{t_i}$ such that $\gamma([t_i, t_{i+1}]) \subset U_i$ for each i.

Write $p^{-1}(U_0) = \bigsqcup_\alpha U_{0,\alpha}$ with each $p|_{U_{0,\alpha}}$ a homeomorphism onto U_0. There is a unique index α_0 with $\tilde{x}_0 \in U_{0,\alpha_0}$. Define $\tilde{\gamma}$ on $[t_0, t_1]$ by

$$\tilde{\gamma}(s) := (p|_{U_{0,\alpha_0}})^{-1}(\gamma(s)).$$

Then $\tilde{\gamma}$ is continuous on $[t_0, t_1]$ and satisfies $p \circ \tilde{\gamma} = \gamma$ there.

Inductively, suppose $\tilde{\gamma}$ has been defined on $[0, t_i]$. Let $\tilde{x}_i := \tilde{\gamma}(t_i)$. Since $\gamma([t_i, t_{i+1}]) \subset U_i$ and $p^{-1}(U_i) = \bigsqcup_\alpha U_{i,\alpha}$, there is a unique sheet U_{i,α_i} containing $\tilde{x}_i$. Define $\tilde{\gamma}$ on $[t_i, t_{i+1}]$ by the same formula using $(p|_{U_{i,\alpha_i}})^{-1}$. The definitions match at t_i by construction, hence $\tilde{\gamma}$ is continuous.

Uniqueness follows similarly: if $\tilde{\gamma}_1, \tilde{\gamma}_2$ are two lifts with the same initial value, then on $[t_0, t_1]$ both must lie in the unique sheet over U_0 containing $\tilde{x}_0$, where p is injective; hence they agree. Inducting over the subintervals yields $\tilde{\gamma}_1 = \tilde{\gamma}_2$ on all of $[0,1]$. □

Lemma A.4 (Homotopy lifting). *Let $p : \widetilde{X} \to X$ be a covering map. Let $H : [0,1] \times [0,1] \to X$ be a continuous homotopy of paths, and suppose a lift $\tilde{H}(\cdot, 0)$ of $H(\cdot, 0)$ is fixed. Then there exists a unique continuous lift $\tilde{H} : [0,1] \times [0,1] \to \widetilde{X}$ such that $p \circ \tilde{H} = H$ and $\tilde{H}(\cdot, 0)$ is the prescribed lift.*

Proof. Cover the compact set $H([0,1] \times [0,1]) \subset X$ by finitely many evenly covered open sets and subdivide the square into finitely many small rectangles so that the image of each rectangle lies in a single evenly covered set. On each rectangle, one lifts uniquely once the lift is specified along one edge, because in a fixed sheet the covering map is a homeomorphism. The uniqueness on overlaps ensures that these local lifts patch to a global continuous lift on the square. □

Proposition A.5 (Unique lifting for maps). *Let $p : \widetilde{X} \to X$ be a covering map and let Y be connected. Suppose $f : Y \to X$ is continuous and $y_0 \in Y$ with a chosen $\tilde{x}_0 \in p^{-1}(f(y_0))$. If a lift $\tilde{f} : Y \to \widetilde{X}$ satisfying $p \circ \tilde{f} = f$ and $\tilde{f}(y_0) = \tilde{x}_0$ exists, then it is unique.*

Proof. Let $\tilde{f}_1, \tilde{f}_2$ be two such lifts. Consider the set

$$A := \{y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y)\}.$$

Since $\tilde{f}_1(y_0) = \tilde{f}_2(y_0)$, the set A is nonempty. It is closed by continuity. To see A is open, fix $y \in A$. Choose an evenly covered neighborhood U of $f(y)$. For y' in a small neighborhood V of y with $f(V) \subset U$, both $\tilde{f}_1(V)$ and $\tilde{f}_2(V)$ must lie in the *same* sheet over U containing $\tilde{f}_1(y) = \tilde{f}_2(y)$, and on that sheet p is injective, hence $\tilde{f}_1 = \tilde{f}_2$ on V. Thus A is open. As Y is connected, $A = Y$. □

A.3. **Universal covers.**

Theorem A.6 (Existence of universal covers)**.** *If X is connected, locally path-connected, and semilocally simply connected, then there exists a universal covering space $p : \widetilde{X} \to X$, unique up to isomorphism of coverings.*

Remark A.7. A proof can be found in standard topology references. Every (connected) topological surface is locally path-connected and semilocally simply connected, hence admits a universal cover.

Proposition A.8 (Lifts from simply connected sources)**.** *Let $p : \widetilde{Y} \to Y$ be a covering map. Let X be path-connected and simply connected. Fix $x_0 \in X$ and a point $\tilde{y}_0 \in p^{-1}(f(x_0))$. Then for every continuous map $f : X \to Y$ there exists a unique continuous lift $\tilde{f} : X \to \widetilde{Y}$ such that*

$$p \circ \tilde{f} = f, \quad \tilde{f}(x_0) = \tilde{y}_0.$$

Proof. Fix $x \in X$. Choose a path γ in X from x_0 to x. Then $f \circ \gamma$ is a path in Y starting at $f(x_0)$, hence admits a unique lift $\widetilde{\gamma}$ starting at $\tilde{y}_0$ by Lemma A.3. Define $\tilde{f}(x) := \widetilde{\gamma}(1)$.

If γ_1, γ_2 are two such paths, then $\gamma_1 \cdot \overline{\gamma_2}$ is a loop in X, hence null-homotopic since X is simply connected. By Lemma A.4, the lifts of $f \circ \gamma_1$ and $f \circ \gamma_2$ starting at $\tilde{y}_0$ have the same endpoint. Thus $\tilde{f}(x)$ is well-defined.

Continuity of $\tilde{f}$ follows by working in evenly covered neighborhoods: locally, $\tilde{f}$ is obtained by composing f with a local inverse branch of p. Finally, uniqueness follows from Proposition A.5. □

Proposition A.9. *Let $p : \widetilde{X} \to X$ be a covering map with $\widetilde{X}$ connected. If X is simply connected, then p is a homeomorphism.*

Proof. Fix $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$. For any $x \in X$, choose a path γ from x_0 to x. By Lemma A.3 there is a unique lift $\tilde{\gamma}$ with $\tilde{\gamma}(0) = \tilde{x}_0$. Define

$$s(x) := \tilde{\gamma}(1).$$

If γ_1, γ_2 are two paths from x_0 to x, then γ_1 is homotopic to γ_2 rel endpoints because X is simply connected; by Lemma A.4, the corresponding lifts with the same initial point have the same endpoint. Hence s is well-defined and satisfies $p \circ s = \mathrm{id}_X$.

Fix $x \in X$ and choose an evenly covered neighborhood U of x. Let $\tilde{U}$ be the (unique) sheet over U containing $s(x)$, so that $p|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism. For $y \in U$, lift a path in U from x to y starting at $s(x)$ to see that $s(y) = (p|_{\tilde{U}})^{-1}(y)$. Thus s is continuous, and in particular p is injective (because each fiber contains at most one point $s(x)$). Since p is a surjective local homeomorphism, it is a homeomorphism. □

Proposition A.10 (Quotient by the deck group)**.** *Let $p : \widetilde{X} \to X$ be a universal covering map with $\widetilde{X}$ connected. Then $\mathrm{Deck}(\widetilde{X}/X)$ acts freely and properly discontinuously on $\widetilde{X}$, and the quotient $\widetilde{X}/\mathrm{Deck}(\widetilde{X}/X)$ is naturally homeomorphic to X.*

Proof. Freeness is immediate: if $\varphi(\tilde{x}) = \tilde{x}$ and $p \circ \varphi = p$, then by the unique lifting property (Proposition A.5) applied on a small evenly covered neighborhood of $p(\tilde{x})$, the map φ agrees with the identity on an open set, hence on all of $\widetilde{X}$ by connectedness.

Proper discontinuity follows from the existence of evenly covered neighborhoods: over an evenly covered $U \subset X$, distinct deck transformations permute the sheets above U, hence move each sheet off itself unless they are the identity.

Define $\Phi : \widetilde{X}/\mathrm{Deck}(\widetilde{X}/X) \to X$ by $\Phi([\tilde{x}]) := p(\tilde{x})$. This is well-defined because $p \circ \varphi = p$. It is continuous and bijective; local homeomorphism follows from evenly covered neighborhoods. Hence Φ is a homeomorphism. ☐

A.4. **A bridge to holomorphic covering maps.**

Definition A.11. A *holomorphic covering map* between Riemann surfaces is a covering map $p : \widetilde{X} \to X$ (in the topological sense of Definition A.1) which is holomorphic.

Lemma A.12 (Local biholomorphisms are coverings onto the image)**.** *Let $f : X \to Y$ be a holomorphic map between Riemann surfaces. Assume that f is a* local biholomorphism*, i.e., for every $x \in X$ there exists a neighborhood U_x such that $f|_{U_x} : U_x \to f(U_x)$ is a biholomorphism onto an open set. Let $U := f(X) \subset Y$ be the image. Then*

$$f : X \longrightarrow U$$

is a (topological) covering map. In particular it is a holomorphic covering map.

Proof. The set $U = f(X)$ is open in Y by the open mapping theorem.

Fix $y_0 \in U$ and choose $x_0 \in f^{-1}(y_0)$. Since f is a local biholomorphism, there exists a neighborhood $W_{x_0} \subset X$ of x_0 and a disk neighborhood $V \subset U$ of y_0 such that $f|_{W_{x_0}} : W_{x_0} \to V$ is a biholomorphism. Let

$$s_{x_0} : V \to W_{x_0} \subset X$$

denote its holomorphic inverse branch, so that $f \circ s_{x_0} = \mathrm{id}_V$ and $s_{x_0}(y_0) = x_0$.

Now let $x \in f^{-1}(y_0)$ be *any* point in the fiber. Repeating the same construction at x (after shrinking V if necessary) gives a holomorphic inverse branch $s_x : V \to X$ with $f \circ s_x = \mathrm{id}_V$ and $s_x(y_0) = x$.

Claim 1: The sets $s_x(V)$ are pairwise disjoint. If $s_x(y) = s_{x'}(y)$ for some $y \in V$, then applying f gives $y = y$, and by local injectivity of f near the common point we must have $s_x = s_{x'}$ near y. Hence $x = s_x(y_0) = s_{x'}(y_0) = x'$.

Claim 2: $f^{-1}(V) = \bigsqcup_{x \in f^{-1}(y_0)} s_x(V)$. The inclusion $\supset$ is clear from $f \circ s_x = \mathrm{id}_V$. For the reverse inclusion, take $x' \in f^{-1}(V)$ and set $y := f(x') \in V$. Choose a path γ in the disk V from y to y_0. Because f is a local homeomorphism, γ admits a unique lift $\widetilde{\gamma}$ starting at x'. Let $x := \widetilde{\gamma}(1) \in f^{-1}(y_0)$. Now reverse γ to a path from y_0 to y; lifting this reversed path starting at x lands at x'. By uniqueness of lifts, this shows $x' = s_x(y) \in s_x(V)$.

Combining Claims 1 and 2, we obtain

$$f^{-1}(V) = \bigsqcup_{x \in f^{-1}(y_0)} s_x(V),$$

and on each component $s_x(V)$ the restriction $f : s_x(V) \to V$ is a homeomorphism (indeed a biholomorphism). Thus V is evenly covered, and $f : X \to U$ is a covering map. Since f is holomorphic, it is a holomorphic covering map. ☐

Appendix B. One-variable complex analysis tools

This appendix collects the one-variable results used in the main text. For the reader's convenience we include proofs of the Schwarz lemma and the Schwarz–Pick lemma. All other results are stated without proof.

B.1. Automorphisms of the disk and Schwarz-type lemmas.

Lemma B.1 (Schwarz lemma)**.** *Let $F : \mathbb{D} \to \mathbb{D}$ be holomorphic with $F(0) = 0$. Then $|F(z)| \le |z|$ for all $z \in \mathbb{D}$ and $|F'(0)| \le 1$. Moreover, if equality holds at some nonzero point or if $|F'(0)| = 1$, then $F(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$.*

Proof. Define

$$G(z) := \begin{cases} F(z)/z, & z \neq 0, \\ F'(0), & z = 0. \end{cases}$$

Then G is holomorphic on $\mathbb{D}$.

Fix $0 < r < 1$. On the circle $|z| = r$ we have $|G(z)| = |F(z)|/r \le 1/r$ since $|F| \le 1$. By the maximum modulus principle applied to G on the disk $\{|z| < r\}$, we obtain $|G(z)| \le 1/r$ for all $|z| < r$. Hence for $|z| < r$,

$$|F(z)| = |z| \cdot |G(z)| \le |z|/r.$$

Letting $r \uparrow 1$ yields $|F(z)| \le |z|$ for all $z \in \mathbb{D}$.

In particular, for $z \neq 0$ we have $|G(z)| = |F(z)|/|z| \le 1$, and also $|G(0)| = |F'(0)| \le 1$ by continuity. Thus $|G| \le 1$ on $\mathbb{D}$, so $|F'(0)| = |G(0)| \le 1$.

If equality holds at some $z_0 \neq 0$, then $|G(z_0)| = |F(z_0)|/|z_0| = 1$, so $|G|$ attains its maximum in the interior. Hence G is constant with $|G| \equiv 1$, i.e. $G \equiv e^{i\theta}$, and therefore $F(z) = e^{i\theta} z$. The case $|F'(0)| = 1$ is similar: then $|G(0)| = 1$, so again G is constant and F is a rotation. □

Theorem B.2 (Schwarz–Pick)**.** *Let $F : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then for all $z \in \mathbb{D}$,*

$$|F'(z)| \le \frac{1 - |F(z)|^2}{1 - |z|^2}.$$

Moreover, equality at one point implies that F is an automorphism of $\mathbb{D}$.

Proof. Fix $z \in \mathbb{D}$ and consider the disk automorphisms

$$\phi_z(\zeta) = \frac{\zeta - z}{1 - \overline{z}\,\zeta}, \quad \phi_{F(z)}(w) = \frac{w - F(z)}{1 - \overline{F(z)}\,w}.$$

Then $H := \phi_{F(z)} \circ F \circ \phi_z^{-1}$ maps $\mathbb{D}$ to itself and satisfies $H(0) = 0$. By Lemma B.1, $|H'(0)| \le 1$.

A direct computation gives

$$H'(0) = \phi'_{F(z)}(F(z)) \cdot F'(z) \cdot (\phi_z^{-1})'(0) = \frac{1}{1 - |F(z)|^2}\, F'(z)\,(1 - |z|^2),$$

hence

$$|F'(z)| \le \frac{1 - |F(z)|^2}{1 - |z|^2}.$$

If equality holds at some point z, then $|H'(0)| = 1$ and Lemma B.1 forces H to be a rotation. Therefore F is a composition of disk automorphisms, i.e. an automorphism of $\mathbb{D}$. □

B.2. **Riemann mapping.**

Theorem B.3 (Riemann mapping theorem)**.** *Let $U \subsetneq \mathbb{C}$ be a simply connected domain. Then there exists a biholomorphism $\varphi : \mathbb{D} \to U$. If $0 \in U$, one may choose $\varphi(0) = 0$, and such a map is unique up to postcomposition by a rotation of $\mathbb{D}$.*

Lemma B.4 (Proper simply connected subdomains of $\mathbb{D}$ expand at the origin)**.** *Let $U \subsetneq \mathbb{D}$ be simply connected with $0 \in U$. Then there exists a univalent holomorphic map $g : U \to \mathbb{D}$ such that $g(0) = 0$ and $|g'(0)| > 1$.*

Proof. By Theorem B.3 there is a biholomorphism $\varphi : \mathbb{D} \to U$ with $\varphi(0) = 0$. Since $U \neq \mathbb{D}$, φ is not an automorphism of $\mathbb{D}$. By Lemma B.1 applied to φ, we have $|\varphi'(0)| < 1$. Let $g := \varphi^{-1} : U \to \mathbb{D}$. Then $g(0) = 0$ and $|g'(0)| = 1/|\varphi'(0)| > 1$. □

B.3. **Liouville and isolated singularities.**

Theorem B.5 (Liouville)**.** *Every bounded entire function $F : \mathbb{C} \to \mathbb{C}$ is constant.*

Theorem B.6 (Removable singularity)**.** *Let $r > 0$ and let F be holomorphic on $\{0 < |z| < r\}$ and bounded near 0. Then F extends to a holomorphic function on $\{|z| < r\}$.*

Theorem B.7 (Classification of isolated singularities)**.** *Let F be holomorphic on $\{0 < |z| < r\}$. Then exactly one holds:*

(i) F extends holomorphically across 0 (removable);
(ii) F has a pole at 0;
(iii) F has an essential singularity at 0.

Theorem B.8 (Casorati–Weierstrass)**.** *If F has an essential singularity at 0, then $F(\{0 < |z| < r\})$ is dense in $\mathbb{C}$ for every $r > 0$.*

B.4. **Picard theorems.**

Theorem B.9 (Great Picard)**.** *If F has an essential singularity at 0, then on every punctured neighborhood $\{0 < |z| < r\}$ the function F assumes every complex value, with at most one exception, infinitely many times.*

Theorem B.10 (Little Picard)**.** *A nonconstant entire function $F : \mathbb{C} \to \mathbb{C}$ cannot omit two distinct complex values. Equivalently, if an entire function omits two values, it is constant.*

B.5. **A planar result used in the parabolic case.**

Theorem B.11 (Bounded map for domains omitting at least three points)**.** *Let $\Omega \subset \widehat{\mathbb{C}}$ be a domain whose complement contains at least three points. Then there exists a nonconstant holomorphic map $h : \Omega \to \mathbb{D}$.*

Remark B.12. This is a standard fact from classical complex analysis (often proved via the hyperbolic uniformization of plane domains and the existence of bounded holomorphic functions on hyperbolic planar domains). In the main text we only use the existence of some nonconstant bounded holomorphic map $\Omega \to \mathbb{D}$ in this situation.

Appendix C. A topological input: compact simply connected surfaces

In the elliptic case we use the following standard topological fact: a compact, connected, simply connected surface is topologically a 2-sphere. We record a short derivation from the classification of compact surfaces.

Theorem C.1 (Classification of closed surfaces)**.** *Every compact, connected 2-manifold without boundary is homeomorphic to exactly one of the following:*

(i) the 2-sphere S^2;
(ii) the connected sum of $g \geq 1$ tori, denoted Σ_g (an orientable surface of genus g);
(iii) the connected sum of $k \geq 1$ real projective planes, denoted N_k (a nonorientable surface of genus k).

Remark C.2. We do not reprove Theorem C.1 here; proofs may be found in standard topology texts on surfaces (e.g. in treatments of the classification theorem for compact surfaces). We only use the theorem to identify which closed surface can have trivial fundamental group.

Proposition C.3. *For $g \geq 1$, the orientable surface Σ_g has nontrivial fundamental group. More precisely, $\pi_1(\Sigma_g)$ admits the presentation*

$$\pi_1(\Sigma_g) \cong \left\langle a_1, b_1, \dots, a_g, b_g \;\middle|\; \prod_{j=1}^{g} [a_j, b_j] = 1 \right\rangle,$$

and in particular $\pi_1(\Sigma_g) \neq 1$.

Proof. It is classical that Σ_g can be obtained by gluing opposite sides of a $4g$-gon with edge word $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. Applying van Kampen's theorem to this CW model yields the stated presentation. Since the generators a_1, b_1 map to nontrivial elements in the abelianization, the group cannot be trivial; hence $\pi_1(\Sigma_g) \neq 1$. □

Proposition C.4. *For $k \geq 1$, the nonorientable surface N_k has nontrivial fundamental group. In particular, $\pi_1(N_k) \neq 1$.*

Proof. It is classical that N_k admits a CW model obtained by identifying edges of a $2k$-gon with edge word $a_1 a_1 a_2 a_2 \cdots a_k a_k$. A van Kampen computation gives a presentation

$$\pi_1(N_k) \cong \left\langle a_1, \dots, a_k \;\middle|\; a_1^2 a_2^2 \cdots a_k^2 = 1 \right\rangle.$$

This group surjects onto $\mathbb{Z}/2\mathbb{Z}$ by sending each a_i to the nontrivial class, so it cannot be trivial. □

Theorem C.5. *Let M be a compact, connected, simply connected 2-manifold without boundary. Then M is homeomorphic to S^2.*

Proof. By Theorem C.1, M is homeomorphic to exactly one of S^2, Σ_g for some $g \geq 1$, or N_k for some $k \geq 1$. If $M \cong \Sigma_g$ with $g \geq 1$, then $\pi_1(M) \cong \pi_1(\Sigma_g)$ is nontrivial by Proposition C.3, contradicting simple connectedness. If $M \cong N_k$ with $k \geq 1$, then $\pi_1(M) \cong \pi_1(N_k)$ is nontrivial by Proposition C.4, again a contradiction. Therefore the only remaining possibility is $M \cong S^2$. □