EXOTIC SMOOTH STRUCTURES ON S^7: MILNOR'S CONSTRUCTION AND INVARIANTS WITH MINIMAL ALGEBRAIC TOPOLOGY

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Abstract. Milnor's 1956 discovery that the 7-sphere admits exotic smooth structures rests on a striking interaction between bundle theory, characteristic classes, and 8-dimensional signature phenomena. Starting from the clutching classification of oriented rank-4 bundles over S^4, one obtains a two-parameter family of S^3-bundles $E_{h,\ell} \to S^4$ whose topology is governed by $e(\xi_{h,\ell})$ and $p_1(\xi_{h,\ell})$. We compute the cohomology of $E_{h,\ell}$ via the Gysin sequence and show that for $|h-\ell| = 1$ these manifolds are topological 7-spheres. To distinguish smooth structures, we present Milnor's invariant $X(M) \in \mathbb{Z}/7\mathbb{Z}$, explain why it is well-defined, and evaluate it explicitly on the family $\{E_{h,\ell}\}$ using the Thom–Euler formalism together with the signature theorem. The resulting congruence exhibits concrete parameters for which $E_{h,\ell}$ is not diffeomorphic to the standard S^7.

Contents

1. Introduction

1.1. **Historical background and motivation.** A basic hope in early differential topology was that "being a sphere" might be a rigid notion: perhaps a manifold homotopy equivalent (or even homeomorphic) to S^n would automatically inherit the standard smooth structure. Milnor's 1956 discovery showed decisively that this is false in dimension 7. He produced manifolds that are topological 7-spheres but not diffeomorphic to the standard one, thereby separating topological and smooth classification in a concrete, computable way.

What makes Milnor's example especially instructive is that the construction is not ad hoc. It arises naturally from the classification of rank-4 bundles over S^4 and from the fact that, in dimension 8, the signature places strong arithmetic constraints on Pontryagin numbers. In modern language, the story is a first encounter with a general theme: bundle data controls topology, while subtle smooth information is detected by bordism-type invariants built from characteristic classes.

The goal of this paper is to make this mechanism transparent in the specific case of Milnor's family $E_{h,\ell} \to S^4$, and to explain how a purely smooth distinction emerges even when homotopy and homology have already stabilized.

1.2. **Audience and approach.** This paper is written for readers who are comfortable with smooth manifolds and basic point-set topology but have not necessarily studied algebraic topology or characteristic classes. Accordingly, we develop only a minimal set of tools: enough homotopy theory to interpret the clutching construction, enough (co)homology to run the Gysin sequence and Poincaré duality, and enough characteristic class theory to normalize e and relate p_1 to a concrete computation. A few standard global theorems are treated as black boxes (most notably Hirzebruch's signature theorem and Reeb's theorem), since their proofs would take us far from the main thread.

The exposition follows Milnor's original computations closely, but it reorganizes the prerequisites so that, once the reader reaches the "reading Milnor" sections, no new background notions are introduced on the fly.

2. Vector bundles and a concrete model of SO(4)

2.1. **Vector bundles, transition functions, and structure groups.**

Definition 2.1 (Vector bundle)**.** Let M be a smooth manifold and $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. A **vector bundle** of rank k over M consists of a smooth manifold E, called the **total space**, and a smooth surjective map $\pi : E \to M$ such that:

(1) There exists an open cover $\{U_\alpha\}$ of M and a family of diffeomorphisms $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{F}^k$, called **local trivializations**, such that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "\Phi_\alpha"] \arrow[d, "\pi_\alpha"'] & U_\alpha \times \mathbb{F}^k \arrow[dl, "\mathrm{pr}_{U_\alpha}"] \\ U_\alpha & \end{tikzcd}$$

where pr_{U_α} denotes the projection onto the first factor.

(2) For any overlap $U_\alpha \cap U_\beta \neq \varnothing$, the transition map

$$\Phi_\beta \circ \Phi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{F}^k \to (U_\alpha \cap U_\beta) \times \mathbb{F}^k$$

has the form $(x, v) \mapsto (x, g_{\alpha\beta}(x)v)$, where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_k(\mathbb{F})$ is a smooth map.

When $\mathbb{F} = \mathbb{R}$, the bundle is called a **real vector bundle**; when $\mathbb{F} = \mathbb{C}$, it is called a **complex vector bundle**.

Remark 2.2. Throughout, we write a vector bundle as $\pi : E \to B$ (or simply $\xi \to B$). For $b \in B$, its fiber is $E_b := \pi^{-1}(b)$.

Definition 2.3 (Trivial bundle and sections)**.** Let B be a smooth manifold and $k \geq 1$. The *trivial rank-k bundle* over B is $B \times \mathbb{R}^k \to B$. A *(smooth) section* of $\pi : E \to B$ is a smooth map $s : B \to E$ with $\pi \circ s = \mathrm{id}_B$.

Proposition 2.4 (Transition functions satisfy cocycle relations)**.** *Let* $\{(U_\alpha, \Phi_\alpha)\}$ *be a bundle atlas for* $\pi : E \to B$ *with transition functions* $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_k(\mathbb{F})$ *(where* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$*). Then on triple overlaps* $U_\alpha \cap U_\beta \cap U_\gamma$ *we have*

$$g_{\alpha\alpha} = \mathrm{id}, \quad g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}, \quad g_{\beta\alpha} = g_{\alpha\beta}^{-1}.$$

Proof. These are immediate from the identities $\Phi_\alpha \circ \Phi_\alpha^{-1} = \mathrm{id}$ and $\Phi_\gamma \circ \Phi_\alpha^{-1} = (\Phi_\gamma \circ \Phi_\beta^{-1}) \circ (\Phi_\beta \circ \Phi_\alpha^{-1})$ on the relevant overlaps, together with the defining form $(x, v) \mapsto (x, g_{\alpha\beta}(x)v)$ of transition maps. □

Definition 2.5 (Reduction of structure group)**.** Let $\xi \to B$ be a real rank-k vector bundle. We say the *structure group reduces to* a Lie subgroup $G \subset \mathrm{GL}_k(\mathbb{R})$ if there exists a bundle atlas whose transition functions take values in G.

Example 2.6. A real bundle is *oriented* iff its structure group reduces to $\mathrm{GL}^+(k, \mathbb{R})$. A choice of bundle metric (Definition 2.7 below) is equivalent to reducing the structure group to $\mathrm{O}(k)$, and an oriented bundle metric reduces further to $\mathrm{SO}(k)$.

2.2. **Bundle metrics and sphere/disk bundles.**

Definition 2.7 (Bundle metric)**.** Let $\pi : E \to B$ be a real vector bundle. A *bundle metric* is a smoothly varying inner product on fibers, i.e. a smooth assignment $b \mapsto \langle \cdot, \cdot \rangle_b$ where $\langle \cdot, \cdot \rangle_b$ is an inner product on E_b.

Proposition 2.8 (Existence of bundle metrics)**.** *Every real vector bundle admits a bundle metric.*

Proof. Choose a bundle atlas $\{(U_\alpha, \Phi_\alpha)\}$. Over each U_α, use Φ_α to identify $\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{R}^k$ and pull back the standard Euclidean inner product on $\mathbb{R}^k$ to get a local bundle metric $\langle \cdot, \cdot \rangle^{(\alpha)}$ on $\pi^{-1}(U_\alpha)$.

Let $\{\rho_\alpha\}$ be a smooth partition of unity subordinate to $\{U_\alpha\}$. Define a fiberwise inner product by

$$\langle v, w \rangle_b := \sum_\alpha \rho_\alpha(b) \, \langle v, w \rangle_b^{(\alpha)}.$$

For each fixed b, only finitely many terms contribute; the sum of inner products is an inner product; smoothness in b follows from smoothness of the ρ_α and the local metrics. □

Definition 2.9 (Sphere and disk bundles)**.** Fix a bundle metric on a real bundle $\xi : E \to B$. The *unit sphere bundle* $S(\xi) \to B$ and *unit disk bundle* $D(\xi) \to B$ are

$$S(\xi) := \{v \in E : \|v\| = 1\}, \quad D(\xi) := \{v \in E : \|v\| \le 1\}.$$

Proposition 2.10 (Smoothness and boundary)**.** *$S(\xi)$ and $D(\xi)$ are smooth manifolds, the projection maps to B are smooth, and $\partial D(\xi) = S(\xi)$ as manifolds with boundary.*

Proof. Work in a local trivialization $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$. In these coordinates the bundle metric is a smoothly varying positive definite quadratic form, so after choosing orthonormal frames (equivalently, reducing transition maps to $\mathrm{O}(k)$) we may assume the metric is standard in each trivialization. Then

$$S(\xi) \cap \pi^{-1}(U_\alpha) \cong U_\alpha \times S^{k-1}, \quad D(\xi) \cap \pi^{-1}(U_\alpha) \cong U_\alpha \times D^k,$$

which gives smooth atlases on $S(\xi)$ and $D(\xi)$ compatible on overlaps. In these local models, $\partial(U_\alpha \times D^k) = U_\alpha \times S^{k-1}$, hence globally $\partial D(\xi) = S(\xi)$. □

Proposition 2.11 (Pullback commutes with sphere/disk)**.** *Let $f : Y \to B$ be smooth and $\xi \to B$ a real bundle with a bundle metric. Then there are natural diffeomorphisms*

$$S(f^*\xi) \cong f^*S(\xi), \quad D(f^*\xi) \cong f^*D(\xi).$$

Proof. Both sides consist of pairs (y, v) with $f(y) = \pi(v)$ and the corresponding fiberwise norm constraint. Local trivializations identify the two constructions fiberwise and smoothly in y. □

2.3. Frame bundles and $\mathrm{SO}(4)$-structures.

Definition 2.12 (Frame bundle)**.** Let $\xi : E \to B$ be a real rank-k vector bundle. Its *frame bundle* $\mathrm{Fr}(\xi) \to B$ has fiber

$$\mathrm{Fr}(\xi)_b := \{(e_1, \ldots, e_k) : (e_1, \ldots, e_k) \text{ is an ordered basis of } E_b\}.$$

It carries a free right action of $\mathrm{GL}_k(\mathbb{R})$ by change of basis.

Definition 2.13 (Principal G-bundle)**.** Let B be a smooth manifold and G a Lie group. A principal G-bundle over B is a fiber bundle $\pi : P \to B$ equipped with a free right G-action $P \times G \to P$ such that $\pi(pg) = \pi(p)$, and locally $\pi^{-1}(U) \cong U \times G$ as G-spaces (with G acting on the second factor by right translation).

Remark 2.14. If $\xi \to B$ is an oriented rank–4 vector bundle with a metric, its oriented orthonormal frame bundle is a principal $SO(4)$–bundle. Conversely, associated bundles recover ξ from its frame bundle.

Proposition 2.15. $\mathrm{Fr}(\xi) \to B$ *is a smooth principal* $\mathrm{GL}_k(\mathbb{R})$*-bundle. If* ξ *is equipped with a bundle metric, the subbundle of orthonormal frames* $\mathrm{Fr}_{\mathrm{O}}(\xi) \to B$ *is a principal* $\mathrm{O}(k)$*-bundle. If in addition* ξ *is oriented, the oriented orthonormal frames form a principal* $\mathrm{SO}(k)$*-bundle* $\mathrm{Fr}_{\mathrm{SO}}(\xi) \to B$*.*

Proof. In a local trivialization $\pi^{-1}(U) \cong U \times \mathbb{R}^k$, a frame at $b \in U$ is identified with an invertible matrix in $\mathrm{GL}_k(\mathbb{R})$; hence $\mathrm{Fr}(\xi)|_U \cong U \times \mathrm{GL}_k(\mathbb{R})$ smoothly, with the standard right action. The metric (resp. orientation) is preserved exactly by $\mathrm{O}(k)$ (resp. $\mathrm{SO}(k)$), so the orthonormal (resp. oriented orthonormal) frames cut out principal subbundles. ☐

Remark 2.16. In particular, an *oriented* rank-4 real vector bundle with a bundle metric has structure group $\mathrm{SO}(4)$ and is encoded by its principal $\mathrm{SO}(4)$-bundle $\mathrm{Fr}_{\mathrm{SO}}(\xi) \to B$. This is needed for Milnor's construction.

2.4. **Quaternions and a concrete model of** $\mathrm{SO}(4)$**.**

Definition 2.17 (Quaternions)**.** Let $\mathbb{H}$ be the real associative algebra with basis $\{1, i, j, k\}$ and relations $i^2 = j^2 = k^2 = ijk = -1$. Write $q = a + bi + cj + dk$ with $a, b, c, d \in \mathbb{R}$. The conjugate is $\bar{q} := a - bi - cj - dk$ and the norm is $|q| := \sqrt{q\bar{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}$.

Proposition 2.18 (Unit quaternions)**.** *The set* $S^3 := \{q \in \mathbb{H} : |q| = 1\}$ *is a Lie group under multiplication and is (as a manifold) the unit sphere in* $\mathbb{R}^4$*.*

Proof. The condition $a^2+b^2+c^2+d^2 = 1$ cuts out the standard 3-sphere in $\mathbb{R}^4$, and the multiplicativity $|pq| = |p||q|$ implies closure under multiplication and inversion. Smoothness of the operations follows from polynomial formulas in coordinates. ☐

Notation 2.19. We identify $\mathbb{R}^4$ with $\mathbb{H}$ and equip it with the inner product $\langle x, y\rangle := \mathrm{Re}(x\bar{y})$. Then $|x|^2 = \langle x, x\rangle$.

Lemma 2.20 (Left/right multiplication is orthogonal)**.** *For any* $u, v \in S^3$*, the maps* $L_u(x) = ux$ *and* $R_v(x) = xv^{-1}$ *are elements of* $\mathrm{SO}(4)$*.*

Proof. Using $\langle x, y\rangle = \mathrm{Re}(x\bar{y})$ and $\overline{ux} = \bar{x}\,\bar{u}$, we compute

$$\langle ux, uy\rangle = \mathrm{Re}(ux\,\overline{uy}) = \mathrm{Re}(ux\,\bar{y}\,\bar{u}) = \mathrm{Re}(u(x\bar{y})\bar{u}) = \mathrm{Re}(x\bar{y}) = \langle x, y\rangle,$$

since conjugation by u preserves real part. Thus $L_u \in \mathrm{O}(4)$. Similarly $R_v \in \mathrm{O}(4)$. Because S^3 is connected and $L_1 = R_1 = \mathrm{id}$ has determinant $+1$, continuity forces $\det(L_u) = \det(R_v) = +1$ for all u, v, hence $L_u, R_v \in \mathrm{SO}(4)$. ☐

Proposition 2.21 (Quaternionic action and kernel)**.** *Define a smooth map*

$$\Phi : S^3 \times S^3 \longrightarrow \mathrm{SO}(4), \quad \Phi(u, v)(x) := uxv^{-1}.$$

Then Φ *is a Lie group homomorphism and*

$$\ker(\Phi) = \{(1, 1), (-1, -1)\}.$$

Proof. The homomorphism property follows from associativity in $\mathbb{H}$: $\Phi(u_1, v_1) \circ \Phi(u_2, v_2) = \Phi(u_1u_2, v_1v_2)$. If $\Phi(u, v)$ is the identity, then $ux = xv$ for all $x \in \mathbb{H}$. Setting $x = 1$ gives $u = v$, and then $ux = xu$ for all x, so u lies in the center of $\mathbb{H}$, hence $u = \pm 1$. Therefore $(u, v) = (1, 1)$ or $(-1, -1)$. ☐

Theorem 2.22 (A concrete model of SO(4))**.** *The homomorphism* $\Phi : S^3 \times S^3 \to \mathrm{SO}(4)$ *is surjective, hence induces an isomorphism of Lie groups*

$$\mathrm{SO}(4) \cong (S^3 \times S^3)/\{\pm(1,1)\}.$$

Equivalently, Φ *is a two-to-one covering map of Lie groups.*

Proof. Let $\mathfrak{s}^3 = \mathrm{Im}(\mathbb{H}) = \{ai + bj + ck\}$ be the Lie algebra of S^3. Differentiate Φ at $(1,1)$ to obtain a Lie algebra homomorphism $d\Phi_{(1,1)} : \mathfrak{s}^3 \oplus \mathfrak{s}^3 \to \mathfrak{so}(4)$. A tangent vector $(A,B) \in \mathfrak{s}^3 \oplus \mathfrak{s}^3$ acts on $x \in \mathbb{H}$ by

$$x \longmapsto Ax - xB.$$

Using $\langle x, y\rangle = \mathrm{Re}(x\bar{y})$ and the fact that A, B are purely imaginary, one checks this operator is skew-adjoint, hence lies in $\mathfrak{so}(4)$.

The map $(A,B) \mapsto (x \mapsto Ax - xB)$ is injective (a short computation using $x = 1, i, j, k$ forces $A = B = 0$), so $d\Phi_{(1,1)}$ is injective. Since both domain and codomain have dimension 6, it is an isomorphism. Therefore Φ is a local diffeomorphism at $(1,1)$, so its image is an open Lie subgroup of SO(4). Because $S^3 \times S^3$ is connected, the image is connected; since SO(4) is connected, the only connected open subgroup is all of SO(4). Thus Φ is surjective; together with Proposition 2.21 this yields the quotient description. Finally, a surjective Lie group homomorphism with discrete kernel is a covering map, so Φ is two-to-one. □

Remark 2.23. Later, when we introduce covering spaces and higher homotopy groups, Theorem 2.22 will let us compute $\pi_3(\mathrm{SO}(4))$ via $\pi_3(S^3 \times S^3)$.

3. Necessary knowledge of homotopy and homology

In this section we collect exactly the pieces of homotopy and homology theory that will be used later. On the homotopy side we introduce the fundamental group, covering spaces, and higher homotopy groups, and then compute $\pi_3(\mathrm{SO}(4))$ using the quaternionic model from Section 2. On the homology side we only go far enough to define singular homology, the homology of spheres, and fundamental classes with their degree interpretation. No prior knowledge of algebraic topology is assumed.

3.1. Basic notions of homotopy and the fundamental group.

Definition 3.1 (Path and loop)**.** A *path* in a topological space X is a continuous map $\gamma : [0,1] \to X$. Its initial and final points are $\gamma(0)$ and $\gamma(1)$. A *loop* in X is a path $\gamma : [0,1] \to X$ such that $\gamma(0) = \gamma(1)$.

Definition 3.2 (Homotopy)**.** Let $f_0, f_1 : X \to Y$ be continuous. A *homotopy* from f_0 to f_1 is a continuous map $H : X \times [0,1] \to Y$ with $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$. We say that f_0 and f_1 are *homotopic* and write $f_0 \simeq f_1$. A *null-homotopy* is a homotopy from a map to a constant map.

Remark 3.3. The relation $\simeq$ is an equivalence relation on the set of continuous maps $X \to Y$. The equivalence class of a map f is denoted $[f]$ and called the *homotopy class* of f.

Definition 3.4 (Based homotopy)**.** Fix basepoints $x_0 \in X$ and $y_0 \in Y$. Two maps $f_0, f_1 : (X, x_0) \to (Y, y_0)$ are *based homotopic* if there exists a homotopy H from f_0 to f_1 with $H(x_0, t) = y_0$ for all t.

Definition 3.5 (Concatenation and inverse of paths)**.** If $\alpha, \beta : [0,1] \to X$ with $\alpha(1) = \beta(0)$, their *concatenation* is

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2t), & 0 \le t \le \frac{1}{2}, \\ \beta(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

The *inverse* path is $\bar{\alpha}(t) = \alpha(1-t)$.

Lemma 3.6 (Reparametrization)**.** *Concatenation is associative up to path homotopy, the constant path is a two-sided unit up to homotopy, and $\alpha \cdot \bar{\alpha}$ and $\bar{\alpha} \cdot \alpha$ are homotopic to a constant path.*

Definition 3.7 (Fundamental group)**.** The *fundamental group* $\pi_1(X, x_0)$ is the set of path homotopy classes of loops $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$, with multiplication given by concatenation:

$$[\alpha] \cdot [\beta] := [\alpha \cdot \beta].$$

Proposition 3.8. *The operation in Definition 3.7 is well-defined and makes $\pi_1(X, x_0)$ a group. The identity is the constant loop and each loop has an inverse.*

Proof. Lemma 3.6 shows associativity, identity, and inverses up to homotopy; well-definedness follows because concatenation respects path homotopy classes. □

Proposition 3.9 (Functoriality)**.** *A based map $f : (X, x_0) \to (Y, y_0)$ induces a homomorphism*

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0), \quad [\gamma] \longmapsto [f \circ \gamma],$$

natural with respect to composition and identities.

Remark 3.10. This means that the fundamental group is a functor from based topological spaces to groups. Later, this functorial behavior ensures that the homomorphisms induced by bundle projections and inclusions interact predictably with our constructions.

Proposition 3.11 (Change of basepoint)**.** *If X is path-connected and α is a path from x_0 to x_1, the map*

$$\Phi_\alpha : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1), \quad [\gamma] \longmapsto [\alpha \cdot \gamma \cdot \bar{\alpha}]$$

is an isomorphism.

Remark 3.12. Thus, for a path-connected space, $\pi_1(X)$ is essentially independent of the basepoint. We will later use this to simplify notation without constantly mentioning basepoints.

3.2. Coverings, lifting theorems, and a classical example.

Definition 3.13 (Covering map)**.** A continuous surjection $p : \tilde{X} \to X$ is a *covering map* if for every $x \in X$ there exists an open neighborhood U such that $p^{-1}(U) = \bigsqcup_\alpha U_\alpha$ and $p|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism for each α.

Theorem 3.14 (Path lifting)**.** *Given a covering map $p : \tilde{X} \to X$, a path $\gamma : [0,1] \to X$, and $\tilde{x}_0 \in p^{-1}(\gamma(0))$, there is a unique lift $\tilde{\gamma} : [0,1] \to \tilde{X}$ with $p \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = \tilde{x}_0$.*

Theorem 3.15 (Homotopy lifting)**.** *If $H : Y \times [0,1] \to X$ is a homotopy and $\tilde{f}_0 : Y \to \tilde{X}$ is a lift of $H(\cdot, 0)$, then there is a unique lift $\tilde{H} : Y \times [0,1] \to \tilde{X}$ with $\tilde{H}(\cdot, 0) = \tilde{f}_0$ and $p \circ \tilde{H} = H$.*

Lemma 3.16. *If $p : \tilde{X} \to X$ is a covering map and $\tilde{x}_0 \in \tilde{X}$, then*

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, p(\tilde{x}_0))$$

is injective.

Proof. If a loop $\tilde{\gamma}$ at $\tilde{x}_0$ projects to a null-homotopic loop in X, lift the null-homotopy via Theorem 3.15 to a homotopy in $\tilde{X}$ from $\tilde{\gamma}$ to a constant loop. Thus $[\tilde{\gamma}] = 1$ in $\pi_1(\tilde{X}, \tilde{x}_0)$. □

Example 3.17 (First computation: $\pi_1(S^1) \cong \mathbb{Z}$)**.** Let $p : \mathbb{R} \to S^1$ be $p(t) = e^{2\pi i t}$. For a loop γ based at $1 \in S^1$, lift to $\tilde{\gamma}$ with $\tilde{\gamma}(0) = 0$. Define the *degree*

$$\deg \gamma := \tilde{\gamma}(1) - \tilde{\gamma}(0) \in \mathbb{Z}.$$

This gives a surjective homomorphism $\deg : \pi_1(S^1, 1) \to \mathbb{Z}$. If $\deg \gamma = 0$ then $\tilde{\gamma}$ is a loop in $\mathbb{R}$ and hence null-homotopic, so γ is null-homotopic by Lemma 3.16. Therefore deg is an isomorphism. We will not use $\pi_1(S^1)$ directly later, but this example calibrates the covering-space machinery.

3.3. **Higher homotopy groups and the key computation.**

Definition 3.18 (Higher homotopy groups)**.** Fix a basepoint $x_0 \in X$. The n-th homotopy group $\pi_n(X, x_0)$ is the set of based homotopy classes of based maps $f : (S^n, *) \to (X, x_0)$. If X is path-connected we write simply $\pi_n(X)$.

Proposition 3.19 (Products)**.** *For any $n \geq 1$,*

$$\pi_n(X \times Y) \cong \pi_n(X) \oplus \pi_n(Y).$$

Idea of proof. A based map $f : S^n \to X \times Y$ can be written uniquely as a pair (f_X, f_Y) of based maps $f_X : S^n \to X$ and $f_Y : S^n \to Y$. Two maps f_0, f_1 are based homotopic if and only if each component pair $(f_{0,X}, f_{1,X})$ and $(f_{0,Y}, f_{1,Y})$ are based homotopic in X and Y respectively. Thus, homotopy classes correspond bijectively to pairs $([f_X], [f_Y])$, and the group operation is performed componentwise. □

Theorem 3.20. *$\pi_3(S^3) \cong \mathbb{Z}$.*

Proof. It is classical that S^3 is 2-connected, i.e. $\pi_1(S^3) = \pi_2(S^3) = 0$. Therefore the Hurewicz theorem applies in degree 3, and the Hurewicz homomorphism

$$h : \pi_3(S^3) \longrightarrow H_3(S^3; \mathbb{Z})$$

is an isomorphism. Since $H_3(S^3; \mathbb{Z}) \cong \mathbb{Z}$, we conclude $\pi_3(S^3) \cong \mathbb{Z}$. □

Proposition 3.21 (Coverings preserve higher homotopy)**.** *For a covering map $p : \tilde{X} \to X$ between path-connected spaces and any $k \geq 2$, the induced map*

$$p_* : \pi_k(\tilde{X}) \xrightarrow{\cong} \pi_k(X)$$

is an isomorphism.

Proof. Surjectivity: given $f : S^k \to X$, pick a basepoint $\tilde{x}_0 \in p^{-1}(f(*))$. Since S^k is simply connected for $k \geq 2$, Theorem 3.14 allows f to lift uniquely to $\tilde{f} : S^k \to \tilde{X}$ with $p \circ \tilde{f} = f$.

Injectivity: if two lifted maps $\tilde{f}_0, \tilde{f}_1$ satisfy $p \circ \tilde{f}_0 \simeq p \circ \tilde{f}_1$, lift the homotopy $H : S^k \times [0,1] \to X$ uniquely via Theorem 3.15. The lift connects $\tilde{f}_0$ and $\tilde{f}_1$ in $\tilde{X}$, so they represent the same element of $\pi_k(\tilde{X})$. □

Lemma 3.22 (Quaternionic model for SO(4))**.** *Let* $\mathbb{H}$ *be the quaternions. Define*

$$\Psi : S^3_L \times S^3_R \longrightarrow \mathrm{SO}(4), \quad \Psi(p,q)(x) = pxq^{-1},$$

where left and right multiplication act on $\mathbb{R}^4 \cong \mathbb{H}$*. Then* Ψ *is a covering map with kernel* $\{(1,1),(-1,-1)\}$*. Consequently*

$$\mathrm{SO}(4) \cong (S^3_L \times S^3_R)/\{\pm(1,1)\}.$$

Proof. This is exactly Theorem 2.22 from Section 2, restated in the notation used here. We recall it for ease of reference. □

Theorem 3.23. $\pi_3(\mathrm{SO}(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$*.*

Proof. By Lemma 3.22, $\Psi : S^3 \times S^3 \to \mathrm{SO}(4)$ is a covering map. Proposition 3.21 with $k = 3$ gives $\pi_3(\mathrm{SO}(4)) \cong \pi_3(S^3 \times S^3)$. Then Proposition 3.19 and Theorem 3.20 yield

$$\pi_3(S^3 \times S^3) \cong \pi_3(S^3) \oplus \pi_3(S^3) \cong \mathbb{Z} \oplus \mathbb{Z}.$$ □

3.4. **Singular chains and homology.** We now introduce singular homology in just enough generality to treat spheres, manifolds, and degrees of maps. We will use homology only as a black box with these properties; full proofs can be found in any algebraic topology text.

Definition 3.24 (Standard simplex and singular simplex)**.** The standard n-simplex is

$$\Delta^n = \{(t_0,\dots,t_n) \in \mathbb{R}^{n+1} : t_i \geq 0,\ \sum_{i=0}^{n} t_i = 1\}.$$

A *singular n-simplex* in a space X is a continuous map $\sigma : \Delta^n \to X$.

Definition 3.25 (Singular chains)**.** The group of singular n-chains $C_n(X)$ is the free abelian group on all singular n-simplices in X. Elements of $C_n(X)$ are finite formal sums $\sum_i a_i \sigma_i$ with $a_i \in \mathbb{Z}$.

Definition 3.26 (Boundary)**.** The boundary $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined on generators by

$$\partial_n(\sigma) = \sum_{j=0}^{n} (-1)^j \, \sigma \circ \delta_j,$$

where $\delta_j : \Delta^{n-1} \hookrightarrow \Delta^n$ is the inclusion of the j-th face.

Proposition 3.27. $\partial_{n-1} \circ \partial_n = 0$ *for all* n*. Thus* $(C_*(X), \partial)$ *is a chain complex.*

Definition 3.28 (Homology)**.** The group of n-cycles is $Z_n(X) = \ker \partial_n$. The group of n-boundaries is $B_n(X) = \mathrm{im}\, \partial_{n+1}$. The n-th singular homology group is

$$H_n(X) = Z_n(X)/B_n(X).$$

Definition 3.29 (Functoriality)**.** A continuous map $f : X \to Y$ induces a chain map $f_\# : C_*(X) \to C_*(Y)$ by $f_\#(\sigma) = f \circ \sigma$, hence homomorphisms $f_* : H_n(X) \to H_n(Y)$.

Proposition 3.30 (Homotopy invariance)**.** *If $f, g : X \to Y$ are homotopic then $f_* = g_*$ on H_n for all n.*

Corollary 3.31. *If $X \simeq Y$ then $H_n(X) \cong H_n(Y)$ for all n. In particular, a deformation retract induces isomorphisms on homology.*

3.5. **Relative homology and fundamental classes.** This subsection introduces two notions that will be used throughout the remainder of the paper. Relative homology is the natural home for "chains in X whose boundary is allowed to lie in a subspace $A \subset X$". Fundamental classes package orientation into a canonical top-dimensional homology class, both for closed manifolds $[M] \in H_n(M)$ and for manifolds with boundary $[W, \partial W] \in H_n(W, \partial W)$. The pair (D^n, S^{n-1}) is the basic model to keep in mind.

Definition 3.32 (Relative chains and relative homology)**.** Let X be a space and $A \subset X$ a subspace. The *relative chain group* is

$$C_k(X, A) \;:=\; C_k(X)/C_k(A),$$

where $C_k(\,\cdot\,)$ denotes singular k-chains with integer coefficients. The singular boundary $\partial : C_k(X) \to C_{k-1}(X)$ preserves the subgroup $C_k(A)$, hence descends to a boundary operator $\partial : C_k(X, A) \to C_{k-1}(X, A)$. The *relative homology* is

$$H_k(X, A) \;:=\; H_k\big(C_*(X, A), \partial\big).$$

Equivalently, a relative cycle can be represented by a chain $c \in C_k(X)$ whose boundary lies in A, i.e. $\partial c \in C_{k-1}(A)$.

Proposition 3.33 (Long exact sequence of a pair)**.** *For each pair (X, A) there is a natural long exact sequence*

$$\cdots \longrightarrow H_k(A) \xrightarrow{i_*} H_k(X) \xrightarrow{j_*} H_k(X, A) \xrightarrow{\partial} H_{k-1}(A) \xrightarrow{i_*} H_{k-1}(X) \longrightarrow \cdots,$$

where $i : A \hookrightarrow X$ is inclusion, j is the quotient map on chains, and $\partial : H_k(X, A) \to H_{k-1}(A)$ is the connecting homomorphism.

Proof. This follows from the short exact sequence of chain complexes $0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$ by the standard snake-lemma construction. □

Proposition 3.34. *For $n \geq 1$,*

$$H_k(D^n, S^{n-1}) \cong \begin{cases} \mathbb{Z}, & k = n, \\ 0, & k \neq n, \end{cases} \quad \textit{and} \quad \partial : H_n(D^n, S^{n-1}) \to H_{n-1}(S^{n-1}) \textit{ is an isomorphism.}$$

Proof. Apply the long exact sequence of the pair (D^n, S^{n-1}):

$$\cdots \to H_k(S^{n-1}) \xrightarrow{i_*} H_k(D^n) \to H_k(D^n, S^{n-1}) \xrightarrow{\partial} H_{k-1}(S^{n-1}) \xrightarrow{i_*} H_{k-1}(D^n) \to \cdots.$$

We use the standard facts $H_0(D^n) \cong \mathbb{Z}$, $H_k(D^n) = 0$ for $k > 0$, and $H_0(S^{n-1}) \cong \mathbb{Z}$, $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$, $H_k(S^{n-1}) = 0$ otherwise.

If $k > n$, then $H_k(S^{n-1}) = H_k(D^n) = H_{k-1}(S^{n-1}) = 0$, so exactness forces $H_k(D^n, S^{n-1}) = 0$. If $1 \leq k \leq n-1$, then $H_k(D^n) = H_{k-1}(D^n) = 0$. Also

$H_k(S^{n-1}) = 0$, and $H_{k-1}(S^{n-1}) = 0$ except when $k = n$. Hence $H_k(D^n, S^{n-1}) = 0$ for $1 \le k \le n-1$. For $k = 0$, the segment

$$H_0(S^{n-1}) \xrightarrow{i_*} H_0(D^n) \to H_0(D^n, S^{n-1}) \to 0$$

shows i_* is an isomorphism (both spaces are connected), hence $H_0(D^n, S^{n-1}) = 0$.

Finally, for $k = n$ the relevant part becomes

$$0 = H_n(D^n) \longrightarrow H_n(D^n, S^{n-1}) \xrightarrow{\partial} H_{n-1}(S^{n-1}) \longrightarrow H_{n-1}(D^n) = 0,$$

so ∂ is an isomorphism and $H_n(D^n, S^{n-1}) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}$. □

We now turn to fundamental classes. For our purposes, the key point is that an orientation on an n-manifold produces a preferred generator in a *local* relative group, and these local generators determine a canonical global class.

Definition 3.35 (Local fundamental class)**.** Let M be a smooth n-manifold and $x \in M$. Choose an oriented coordinate chart $\varphi : U \to \mathbb{R}^n$ with $x \in U$, and shrink U so that $(U, U \setminus \{x\})$ is identified with $(D^n, D^n \setminus \{0\})$. Since $D^n \setminus \{0\}$ deformation retracts onto S^{n-1}, we have

$$H_n(U, U \setminus \{x\}) \;\cong\; H_n(D^n, D^n \setminus \{0\}) \;\cong\; H_n(D^n, S^{n-1}) \;\cong\; \mathbb{Z}.$$

The chosen orientation determines a preferred generator $\mu_x \in H_n(M, M \setminus \{x\}) \cong \mathbb{Z}$, called the *local fundamental class at x*.

Theorem 3.36 (Fundamental class of a closed oriented manifold)**.** *Let M be a connected, closed, oriented smooth n-manifold. There exists a unique class $[M] \in H_n(M; \mathbb{Z})$ such that for every $x \in M$, the image of $[M]$ under the natural map*

$$H_n(M) \longrightarrow H_n(M, M \setminus \{x\})$$

is the local generator μ_x. Moreover, $H_n(M; \mathbb{Z}) \cong \mathbb{Z}$ and $[M]$ is a generator.

Proof. (Existence.) Cover M by finitely many oriented coordinate balls $U_1, \dots, U_N$. On each U_i, the orientation picks a relative class $[U_i, \partial U_i] \in H_n(U_i, \partial U_i) \cong \mathbb{Z}$ by transporting the disk generator from Proposition 3.34. On overlaps $U_i \cap U_j$, these classes agree because the transition maps preserve orientation. A standard Mayer–Vietoris gluing argument then produces a global class $[M] \in H_n(M)$ restricting to the local generators μ_x.

(Uniqueness.) If $\alpha \in H_n(M)$ maps to 0 in every $H_n(M, M \setminus \{x\})$, then α vanishes on each coordinate ball and hence vanishes globally by Mayer–Vietoris. Therefore at most one class can restrict to the given family of local generators, and $[M]$ is unique. The last statement follows since $[M] \neq 0$ and $H_n(M)$ is free of rank 1 for connected closed oriented M. □

Definition 3.37 (Relative fundamental class)**.** Let W be a compact, connected, oriented smooth n-manifold with boundary ∂W. There exists a unique class

$$[W, \partial W] \in H_n(W, \partial W; \mathbb{Z})$$

whose image in $H_n(W, W \setminus \{x\})$ is the local generator μ_x for every interior point $x \in \mathrm{int}(W)$. This class is called the *relative fundamental class* of W.

Proposition 3.38 (Boundary compatibility)**.** *Let W be as above. Under the connecting homomorphism in the long exact sequence of the pair $(W, \partial W)$,*

$$\partial : H_n(W, \partial W) \longrightarrow H_{n-1}(\partial W),$$

one has

$$\partial\big([W, \partial W]\big) = [\partial W],$$

where $[\partial W] \in H_{n-1}(\partial W)$ is the fundamental class with its induced boundary orientation.

Proof. This is the global version of the disk model: locally near a boundary point, $(W, \partial W)$ looks like (D^n, S^{n-1}), and the long exact sequence is natural with respect to restriction to such coordinate charts. Tracing the preferred generators through Proposition 3.34 yields the identity. □

4. Fundamentals of Cohomology and Characteristic Classes

Unless explicitly stated otherwise, all (co)homology groups are taken with integer coefficients. The purpose of this section is to set up a small collection of constructions that will be used repeatedly in the reading of Milnor: the Kronecker pairing, the cup/cap products, Poincaré duality, the Thom–Euler formalism for oriented bundles, the Gysin sequence for sphere bundles, and the characteristic classes c_i and p_i. After this section, no further topological background will be introduced in the later parts.

4.1. **Cohomology, the Kronecker pairing, and the cup product.**

Definition 4.1 (Singular cochains and coboundary)**.** Let X be a space and G an abelian group. The group of k-cochains is

$$C^k(X; G) = \mathrm{Hom}(C_k(X), G),$$

where $C_k(X)$ is the singular chain group with integer coefficients. The coboundary $\delta : C^k(X; G) \to C^{k+1}(X; G)$ is defined by precomposition with the boundary $\partial : C_{k+1}(X) \to C_k(X)$. Then $\delta^2 = 0$.

Definition 4.2 (Cohomology)**.** The group of cocycles is $Z^k(X; G) = \ker \delta$ and the group of coboundaries is $B^k(X; G) = \operatorname{im} \delta$. The k-th cohomology group is

$$H^k(X; G) = Z^k(X; G)/B^k(X; G).$$

When $G = \mathbb{Z}$ we write $H^k(X)$ for short.

Definition 4.3 (Kronecker pairing)**.** For $G = \mathbb{Z}$, evaluation of cochains on chains induces a pairing

$$\langle\ ,\ \rangle : H^k(X) \times H_k(X) \longrightarrow \mathbb{Z},$$

called the *Kronecker pairing.* It is natural with respect to continuous maps.

Definition 4.4 (Relative cohomology)**.** For a pair (X, A) with $A \subset X$, define $C^k(X, A; G) \subset C^k(X; G)$ to be the subgroup of cochains vanishing on $C_k(A)$. This is preserved by δ, and we set

$$H^k(X, A; G) := H^k(C^*(X, A; G), \delta).$$

When $G = \mathbb{Z}$ we write $H^k(X, A)$.

Definition 4.5 (Cup product)**.** There is a bilinear pairing (defined at the cochain level)

$$\smile : H^p(X) \times H^q(X) \longrightarrow H^{p+q}(X),$$

called the *cup product*. It makes $H^*(X) = \bigoplus_k H^k(X)$ into a graded ring.

Proposition 4.6 (Basic laws)**.** *For any continuous map $f : X \to Y$ and classes $\alpha \in H^p(Y)$, $\beta \in H^q(Y)$,*

1. *(Naturality) $f^*(\alpha \smile \beta) = f^*\alpha \smile f^*\beta$.*
2. *(Graded commutativity) $\alpha \smile \beta = (-1)^{pq}\beta \smile \alpha$.*
3. *(Associativity and unit) $(\alpha \smile \beta) \smile \gamma = \alpha \smile (\beta \smile \gamma)$ and $1 \smile \alpha = \alpha = \alpha \smile 1$ for $1 \in H^0(X)$.*

Proposition 4.7 (Cohomology of spheres)**.** *For $n \geq 1$,*

$$H^k(S^n) \cong \begin{cases} \mathbb{Z}, & k = 0, n, \\ 0, & \textit{otherwise.} \end{cases}$$

In particular $H^4(S^4) \cong \mathbb{Z}$. Fix a generator $u \in H^4(S^4)$ such that $\langle u, [S^4] \rangle = 1$.

4.2. **Cap product, Poincaré duality, and the Thom–Euler formalism.** The bridge from cohomology classes to integers is provided by pairing with fundamental classes. Poincaré duality and the Thom isomorphism explain why characteristic classes are the natural input to such pairings.

Definition 4.8 (Cap product)**.** There is a bilinear pairing

$$\cap : H^p(X) \times H_n(X) \longrightarrow H_{n-p}(X),$$

called the *cap product*, natural in X, and compatible with the Kronecker pairing.

Theorem 4.9 (Poincaré duality)**.** *Let M be a closed, connected, oriented n-manifold with fundamental class $[M] \in H_n(M)$. Then the map*

$$\mathrm{PD} : H^p(M) \longrightarrow H_{n-p}(M), \quad \alpha \longmapsto \alpha \cap [M]$$

is an isomorphism for all p. More generally, if M is compact oriented with boundary, then capping with the relative fundamental class $[M, \partial M] \in H_n(M, \partial M)$ yields an isomorphism $H^p(M, \partial M) \cong H_{n-p}(M)$.

Definition 4.10 (Thom class)**.** Let $\pi : \xi \to B$ be an oriented real rank-n vector bundle with disk bundle $D(\xi)$ and sphere bundle $S(\xi) = \partial D(\xi)$. A *Thom class* is a class $U \in H^n(D(\xi), S(\xi))$ whose restriction to each fiber pair (D^n, S^{n-1}) is the generator determined by the orientation.

Theorem 4.11 (Thom isomorphism)**.** *With U as above, the map*

$$\Phi_U : H^k(B) \longrightarrow H^{k+n}(D(\xi), S(\xi)), \quad \alpha \longmapsto \pi^*\alpha \smile U$$

is an isomorphism for all k.

Definition 4.12 (Euler class)**.** Let $s : B \to D(\xi)$ be the zero section. The *Euler class* of ξ is

$$e(\xi) := s^*(U) \in H^n(B).$$

Proposition 4.13 (Naturality and Whitney product)**.** *Let $\xi \to B$ be an oriented rank-n bundle and let $f : B' \to B$ be continuous. Then $e(f^*\xi) = f^*e(\xi)$. If ξ, η are oriented bundles over B, then $e(\xi \oplus \eta) = e(\xi) \smile e(\eta)$ when $\mathrm{rank}(\xi) + \mathrm{rank}(\eta) = \dim B$ and the sum orientation is used.*

4.3. **Sphere bundles and the Gysin sequence.** For oriented sphere bundles, the Gysin sequence packages the Thom isomorphism into a long exact sequence that computes the cohomology of the total space from the base and the Euler class.

Definition 4.14 (Fiber integration)**.** Let $\pi : E \to B$ be an oriented S^r-bundle. There is a homomorphism

$$\pi_! : H^{k+r}(E) \longrightarrow H^k(B),$$

called *integration along the fiber*, characterized by the usual naturality and the projection formula.

Theorem 4.15 (Gysin long exact sequence)**.** *Let $\pi : E \to B$ be an oriented S^r-bundle (for instance $E = S(\xi)$ for an oriented rank $r+1$ bundle ξ). There exists a class $e \in H^{r+1}(B)$ such that we have a natural long exact sequence*

$$\cdots \to H^{k-r-1}(B) \xrightarrow{\smile e} H^k(B) \xrightarrow{\pi^*} H^k(E) \xrightarrow{\pi_!} H^{k-r}(B) \xrightarrow{\smile e} H^{k+1}(B) \to \cdots .$$

When $E = S(\xi)$, the class e is the Euler class $e(\xi)$.

Proposition 4.16 (Projection formula)**.** *For $\alpha \in H^{k+r}(E)$ and $\beta \in H^*(B)$, one has*

$$\pi_!(\alpha \smile \pi^*\beta) = \pi_!(\alpha) \smile \beta.$$

Proposition 4.17 (A model computation over S^4)**.** *Let $\pi : E \to S^4$ be an oriented S^3-bundle with Euler class $e = au \in H^4(S^4;\mathbb{Z})$. Then*

$$H^3(E;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & a = 0, \\ 0, & a \neq 0, \end{cases} \qquad H^4(E;\mathbb{Z}) \cong \mathbb{Z}/a\mathbb{Z},$$

and $H^0(E;\mathbb{Z}) \cong H^7(E;\mathbb{Z}) \cong \mathbb{Z}$ while all other cohomology groups vanish.

Proof. From the Gysin sequence, the only nontrivial part in degrees $0, 3, 4$ is

$$0 \to H^3(E) \to H^0(S^4) \xrightarrow{\cup e} H^4(S^4) \to H^4(E) \to 0,$$

and the middle map is multiplication by a under the identifications $H^0(S^4) \cong \mathbb{Z}$ and $H^4(S^4) \cong \mathbb{Z} \cdot u$. Exactness gives $H^3(E) \cong \ker(\times a)$ and $H^4(E) \cong \operatorname{coker}(\times a) \cong \mathbb{Z}/a\mathbb{Z}$. The remaining groups follow from the Gysin sequence. □

4.4. **Chern classes, Pontryagin classes, and the signature theorem.** We now introduce Chern classes for complex bundles, and Pontryagin classes for real bundles via complexification. These are the characteristic classes that appear in Milnor's computations. The only result in this subsection whose proof is not supplied here is the Hirzebruch signature theorem. It is quoted as a standard theorem.

Theorem 4.18 (Chern classes: existence and basic properties)**.** *To each complex vector bundle $E \to B$ of rank n there are associated classes $c_i(E) \in H^{2i}(B)$ for $0 \le i \le n$, called the* Chern classes*, with the following properties:*

(1) $c_0(E) = 1$ and $c_i(E) = 0$ for $i > n$.
(2) (Naturality) For any $f : B' \to B$, $c_i(f^*E) = f^*c_i(E)$.
(3) (Whitney sum formula) Writing the total Chern class $c(E) = 1 + c_1(E) + \cdots + c_n(E)$, one has $c(E \oplus F) = c(E) \smile c(F)$.

Moreover, these properties uniquely characterize the Chern classes.

Proposition 4.19 (Complexification and conjugation)**.** *Let $\xi \to B$ be a real vector bundle and $\xi_{\mathbb{C}} = \xi \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. Then $\xi_{\mathbb{C}}$ is naturally isomorphic to its conjugate bundle $\overline{\xi_{\mathbb{C}}}$.*

Definition 4.20 (Pontryagin classes via complexification)**.** For a real vector bundle $\xi \to B$, define its Pontryagin classes by

$$p(\xi) = 1 + p_1(\xi) + p_2(\xi) + \cdots, \quad p_k(\xi) := (-1)^k \, c_{2k}(\xi_{\mathbb{C}}) \in H^{4k}(B).$$

Proposition 4.21 (Basic properties of Pontryagin classes)**.** *For real bundles ξ, η over B and any $f : B' \to B$,*

1. *(Naturality) $p(f^*\xi) = f^*p(\xi)$.*
2. *(Whitney product) $p(\xi \oplus \eta) = p(\xi) \smile p(\eta)$. In particular, $p_1(\xi \oplus \eta) = p_1(\xi) + p_1(\eta)$.*

Proposition 4.22 (Underlying real bundle of a complex rank-2 bundle)**.** *Let $\eta \to B$ be a complex rank-2 bundle and $\eta_{\mathbb{R}}$ its underlying real rank-4 bundle. Then*

$$p_1(\eta_{\mathbb{R}}) = c_1(\eta)^2 - 2c_2(\eta) \in H^4(B).$$

Proof. As complex bundles one has $(\eta_{\mathbb{R}})_{\mathbb{C}} \cong \eta \oplus \overline{\eta}$. By the Whitney formula,

$$c((\eta_{\mathbb{R}})_{\mathbb{C}}) = c(\eta) \smile c(\overline{\eta}).$$

Using the standard relation $c_i(\overline{\eta}) = (-1)^i c_i(\eta)$ and writing $c(\eta) = 1 + c_1(\eta) + c_2(\eta)$, we get

$$c(\eta) \smile c(\overline{\eta}) = (1 + c_1 + c_2) \smile (1 - c_1 + c_2) = 1 + (2c_2 - c_1^2) + (\text{terms of degree } \geq 6).$$

Hence $c_2((\eta_{\mathbb{R}})_{\mathbb{C}}) = 2c_2(\eta) - c_1(\eta)^2$, and therefore

$$p_1(\eta_{\mathbb{R}}) = (-1)^1 c_2((\eta_{\mathbb{R}})_{\mathbb{C}}) = c_1(\eta)^2 - 2c_2(\eta).$$

□

Definition 4.23 (Intersection form and signature)**.** Let W be a closed oriented $4k$-manifold. The bilinear form

$$H^{2k}(W) \times H^{2k}(W) \longrightarrow \mathbb{Z}, \quad (\alpha, \beta) \longmapsto \langle \alpha \smile \beta, [W] \rangle$$

is symmetric. Extending scalars to $\mathbb{R}$, it diagonalizes with b^+ positive and b^- negative squares. The *signature* of W is $\tau(W) = b^+ - b^-$.

Theorem 4.24 (Hirzebruch signature theorem)**.** *For a closed oriented $4k$-manifold W,*

$$\tau(W) = \langle L(W), [W] \rangle,$$

where $L(W) = 1 + L_1 + L_2 + \cdots$ is the total L-class, a universal polynomial in the Pontryagin classes. In dimension 8 one has

$$L_2 = \frac{1}{45}(7p_2 - p_1^2), \quad \textit{hence} \quad \langle 7p_2 - p_1^2, [W] \rangle = 45 \, \tau(W).$$

5. Reading Milnor, Part I: Constructing the Family

5.1. From topology to construction. Milnor's construction starts from a very concrete source of smooth 7-manifolds: sphere bundles. If $\xi \to S^4$ is an oriented real vector bundle of rank 4, then after choosing a bundle metric (Section 2.2) we can form its unit sphere bundle

$$S(\xi) \xrightarrow{\pi} S^4,$$

which is an oriented S^3-bundle. The total space $S(\xi)$ is a closed smooth 7-manifold.

The philosophy is that bundles over S^4 are accessible because S^4 is obtained by gluing two 4-disks along the equator S^3, so a bundle over S^4 can be built by gluing two trivial bundles over the hemispheres using a single transition function on S^3 (the clutching map). In the present situation, the structure group is $SO(4)$ (Section 2.3), and Section 3.3 already computed the parameter space

$$\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

We now package this classification into notation and extract the characteristic classes and cohomology data that will drive the rest of the paper.

5.2. Rank-4 bundles over S^4 via clutching. Write $S^4 = D^4_+ \cup_{S^3} D^4_-$, where $D^4_\pm$ are the closed hemispheres and $D^4_+ \cap D^4_- = S^3$ is the equator. An oriented rank-4 vector bundle $\xi \to S^4$ is trivial over each hemisphere, so after choosing trivializations over $D^4_\pm$ the entire bundle is encoded by a single transition function (clutching map)

$$g : S^3 \longrightarrow SO(4),$$

describing how the two trivial bundles are identified along the equator. Changing trivializations modifies g by homotopy, and every map g produces a bundle by gluing. Thus isomorphism classes of oriented rank-4 bundles over S^4 are classified by $[S^3, SO(4)] \cong \pi_3(SO(4))$.

Using the quaternionic model of $SO(4)$, one has

$$\pi_3(SO(4)) \cong \pi_3(S^3) \oplus \pi_3(S^3) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

We adopt the following uniform notation.

Definition 5.1 (The bundles $\xi_{h,\ell}$ and their sphere bundles)**.** For $(h, \ell) \in \mathbb{Z}^2$, let $\xi_{h,\ell} \to S^4$ be an oriented rank-4 real vector bundle classified by (h, ℓ), and set

$$E_{h,\ell} := S(\xi_{h,\ell}) \xrightarrow{\pi} S^4,$$

its unit S^3-sphere bundle with the fiber orientation induced by $\xi_{h,\ell}$. Fix once and for all $u \in H^4(S^4; \mathbb{Z})$ with $\langle u, [S^4] \rangle = 1$.

Remark 5.2 (Quaternionic normalization and coordinates)*.* Identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$. Left and right multiplication by unit quaternions define a covering

$$\Psi : S^3_L \times S^3_R \longrightarrow SO(4), \quad (p, q) \longmapsto (x \longmapsto p\, x\, q^{-1}).$$

Under this identification, a convenient set of representatives for the clutching maps is

$$f_{h,\ell} : S^3 \longrightarrow SO(4), \quad f_{h,\ell}(v) = \Psi(v^h, v^\ell),$$

so that (h, ℓ) literally record the "left" and "right" degrees. All sign conventions below are fixed so that the quaternionic Hopf fibration corresponds to $(0, 1)$.

Definition 5.3 (Quaternionic Hopf fibration)**.** Left multiplication of unit quaternions on $\mathbb{H}$ restricts to a free action of S^3 on $S^7 \subset \mathbb{H}^2$:

$$q \cdot (x_1, x_2) = (qx_1, qx_2).$$

The orbit space is S^4, and the projection $S^3 \hookrightarrow S^7 \twoheadrightarrow S^4$ is the quaternionic Hopf fibration. It is the unit sphere bundle of a rank-4 quaternionic line bundle $\xi_{0,1} \to S^4$, whose characteristic classes satisfy $e(\xi_{0,1}) = -u$ and $p_1(\xi_{0,1}) = 2u$.

5.3. **Euler and Pontryagin classes on S^4: explicit linear formulas.** The Euler class and first Pontryagin class on S^4 are forced to depend linearly on the pair (h, ℓ). We record the precise statement and give a proof calibrated by the Hopf example.

Proposition 5.4 (Linearity and normalization on S^4)**.** *There exist four integers A, B, C, D such that for every $(h, \ell) \in \mathbb{Z}^2$,*

$$e(\xi_{h,\ell}) = (Ah + B\ell)\, u, \quad p_1(\xi_{h,\ell}) = (Ch + D\ell)\, u.$$

With the normalization in which the quaternionic Hopf fibration corresponds to $(0, 1)$, one has

$$e(\xi_{h,\ell}) = (h - \ell)\, u, \quad p_1(\xi_{h,\ell}) = 2(h + \ell)\, u.$$

Proof. Linearity. The clutching construction identifies $\pi_3(SO(4))$ with isomorphism classes of oriented rank-4 bundles over S^4, where the group law corresponds to composing clutching maps. Characteristic classes are natural, so the assignments

$$\Phi_e : \pi_3(SO(4)) \to H^4(S^4; \mathbb{Z}), \quad [g] \mapsto e(\xi_g),$$

$$\Phi_p : \pi_3(SO(4)) \to H^4(S^4; \mathbb{Z}), \quad [g] \mapsto p_1(\xi_g),$$

are group homomorphisms. Since $H^4(S^4; \mathbb{Z}) \cong \mathbb{Z} \cdot u$, both must be given by integer linear combinations of the coordinates $(h, \ell) \in \mathbb{Z} \oplus \mathbb{Z}$, yielding

$$e(\xi_{h,\ell}) = (Ah + B\ell)\, u, \quad p_1(\xi_{h,\ell}) = (Ch + D\ell)\, u$$

for some integers A, B, C, D.

Calibration. Consider the two basic bundles $\xi_{1,0}$ and $\xi_{0,1}$, obtained by restricting the clutching map to the left and right S^3 factors. By Definition 5.3, $\xi_{0,1}$ is the underlying real rank-4 bundle of the quaternionic line bundle over S^4, and by our normalization

$$e(\xi_{0,1}) = -u, \quad p_1(\xi_{0,1}) = 2u.$$

(For p_1, one may view $\xi_{0,1}$ as the underlying real bundle of a complex rank-2 bundle η with $c_1(\eta) = 0$ and $c_2(\eta) = -u$, and use the identity $p_1(\eta_{\mathbb{R}}) = c_1(\eta)^2 - 2c_2(\eta)$ from Section 4.4.)

By symmetry under exchanging the left and right S^3 factors in the covering $\Psi : S^3_L \times S^3_R \to SO(4)$, the Pontryagin normalization is the same for $\xi_{1,0}$, while the Euler class changes sign:

$$e(\xi_{1,0}) = +u, \quad p_1(\xi_{1,0}) = 2u.$$

These two evaluations determine A, B, C, D uniquely, hence

$$e(\xi_{h,\ell}) = (h - \ell)\, u, \quad p_1(\xi_{h,\ell}) = 2(h + \ell)\, u.$$

□

5.4. **Cohomology of $E_{h,\ell}$ via the Gysin sequence.** We now compute $H^*(E_{h,\ell};\mathbb{Z})$ using the Gysin long exact sequence for the oriented S^3-bundle $\pi : E_{h,\ell} \to S^4$, together with the formula for $e(\xi_{h,\ell})$.

Proposition 5.5 (Cohomology groups of the total space)**.** *Let $a := h - \ell$, so $e(\xi_{h,\ell}) = au \in H^4(S^4;\mathbb{Z})$. Then*

$$H^0(E_{h,\ell}) \cong \mathbb{Z}, \quad H^7(E_{h,\ell}) \cong \mathbb{Z},$$

and

$$H^3(E_{h,\ell}) \cong \begin{cases} \mathbb{Z}, & a = 0, \\ 0, & a \neq 0, \end{cases} \quad H^4(E_{h,\ell}) \cong \mathbb{Z}/a\mathbb{Z},$$

with $H^k(E_{h,\ell}) = 0$ for $k \notin \{0,3,4,7\}$. In particular, $E_{h,\ell}$ has the same integral cohomology as S^7 if and only if $|a| = 1$.

Proof. Write $E = E_{h,\ell}$ and $B = S^4$. For an oriented S^3-bundle, the Gysin sequence gives

$$\cdots \to H^{k-4}(B) \xrightarrow{\smile e} H^k(B) \xrightarrow{\pi^*} H^k(E) \xrightarrow{\pi_*} H^{k-3}(B) \xrightarrow{\smile e} H^{k+1}(B) \to \cdots .$$

Since $H^*(S^4;\mathbb{Z})$ is concentrated in degrees 0 and 4, we inspect the relevant degrees.

For $k = 3$, exactness yields

$$0 = H^{-1}(B) \to H^3(B) = 0 \to H^3(E) \to H^0(B) \xrightarrow{\smile e} H^4(B).$$

The last map is multiplication by a (since $e = au$ and $H^0(B) \cong \mathbb{Z}$), so $H^3(E) \cong \ker(\mathbb{Z} \xrightarrow{\times a} \mathbb{Z})$, giving $H^3(E) \cong 0$ if $a \neq 0$ and $H^3(E) \cong \mathbb{Z}$ if $a = 0$.

For $k = 4$, exactness yields

$$H^0(B) \xrightarrow{\smile e} H^4(B) \xrightarrow{\pi^*} H^4(E) \to H^1(B) = 0,$$

so $H^4(E) \cong \operatorname{coker}(\mathbb{Z} \xrightarrow{\times a} \mathbb{Z}) \cong \mathbb{Z}/a\mathbb{Z}$.

All remaining $H^k(E)$ for $1 \le k \le 6$, $k \neq 3,4$, vanish for degree reasons. Finally, for $k = 7$ one gets $H^7(E) \cong H^4(B) \cong \mathbb{Z}$, and $H^0(E) \cong \mathbb{Z}$ because E is connected. The last statement follows since $H^4(E) \cong \mathbb{Z}/a\mathbb{Z}$ is trivial exactly when $|a| = 1$. ☐

Example 5.6 (The untwisted case $a = 0$)**.** If $a = h - \ell = 0$, then $e(\xi_{h,\ell}) = 0$, and the Gysin sequence shows that $H^*(E_{h,\ell})$ agrees additively with the cohomology of $S^4 \times S^3$:

$$H^*(E_{h,\ell};\mathbb{Z}) \cong H^*(S^4 \times S^3;\mathbb{Z}).$$

In particular, there exist classes $u' \in H^4(E_{h,\ell};\mathbb{Z})$ and $v' \in H^3(E_{h,\ell};\mathbb{Z})$ whose degrees are $|u'| = 4$, $|v'| = 3$, such that additively $H^*(E_{h,\ell};\mathbb{Z}) \cong \mathbb{Z}\{1, v', u', u'v'\}$.

For the rest of the paper, the subfamily $|h - \ell| = 1$ is the one of interest: by the prop, these are exactly the $E_{h,\ell}$ whose integral cohomology agrees with that of S^7. Section 6 will show (using Morse theory rather than homotopy classification) that these manifolds are in fact *topological* 7-spheres; Section 7 will then distinguish their smooth structures.

6. Reading Milnor, Part II: Identifying the Underlying Topological Sphere

6.1. **Strategy.** In Section 5 we constructed the oriented S^3-bundle

$$S^3 \longrightarrow E_{h,\ell} \xrightarrow{\pi} S^4$$

as the unit sphere bundle of an oriented rank-4 vector bundle $\xi_{h,\ell} \to S^4$. Write $a := h - \ell$, so that

$$e(\xi_{h,\ell}) = au \in H^4(S^4; \mathbb{Z})$$

with $\langle u, [S^4] \rangle = 1$. The Gysin computation in Section 5.4 shows that $|a| = 1$ is exactly the condition for $E_{h,\ell}$ to have the same integral (co)homology as S^7. In this section we prove the stronger statement that for $|a| = 1$ the manifold $E_{h,\ell}$ is in fact a *topological* 7-sphere.

The argument is Morse-theoretic. First we construct a Morse function on $E_{h,\ell}$ with exactly four critical points, of indices $0, 3, 4, 7$. Then we compute the relevant differential in the Morse complex and show that when $|a| = 1$ the index 3 and index 4 critical points cancel, leaving a Morse function with exactly two critical points. Reeb's theorem, proved below, then implies $E_{h,\ell} \cong S^7$.

6.2. **Morse functions and Reeb's theorem.**

Definition 6.1 (Morse function and index)**.** Let M be a smooth n-manifold. A smooth function $f : M \to \mathbb{R}$ is a *Morse function* if every critical point p of f is nondegenerate, i.e. the Hessian $\mathrm{Hess}_p(f) : T_pM \times T_pM \to \mathbb{R}$ is a nondegenerate bilinear form. The *index* $\lambda(p)$ is the number of negative eigenvalues of $\mathrm{Hess}_p(f)$, counted with multiplicity.

Theorem 6.2 (Reeb)**.** *Let M be a closed, connected smooth n-manifold. If M admits a Morse function with exactly two critical points, necessarily of indices* 0 *and n, then M is homeomorphic to S^n.*

Proof. Let $f : M \to \mathbb{R}$ be Morse with exactly two critical points $p_{\min}, p_{\max}$, of indices 0 and n. Let $c_{\min} < c_{\max}$ be the corresponding critical values, and choose a regular value r with $c_{\min} < r < c_{\max}$.

Set $M_- := f^{-1}((-\infty, r])$ and $M_+ := f^{-1}([r, \infty))$. Then $M = M_- \cup M_+$ and $\partial M_- = \partial M_+ = f^{-1}(r)$. By the Morse lem, a neighborhood of $p_{\min}$ is modeled on a ball in $\mathbb{R}^n$ with f equal to $c_{\min} + \|x\|^2$; since there are no other critical points below r, the sublevel set M_- is obtained by flowing along a gradient-like vector field and is therefore a manifold with boundary diffeomorphic to a closed n-disk D^n. Similarly, a neighborhood of $p_{\max}$ is modeled on $c_{\max} - \|x\|^2$, and since there are no other critical points above r, the superlevel set M_+ is also diffeomorphic to a closed n-disk D^n.

Thus M is homeomorphic to $D^n \cup_\varphi D^n$ for some homeomorphism $\varphi : \partial D^n \to \partial D^n$, i.e. $\varphi : S^{n-1} \to S^{n-1}$. By the Alexander trick, φ extends to a homeomorphism $\Phi : D^n \to D^n$ defined by $\Phi(rx) = r\,\varphi(x)$ for $x \in S^{n-1}$, $r \in [0, 1]$. Therefore

$$D^n \cup_\varphi D^n \cong D^n \cup_{\mathrm{id}} D^n \cong S^n,$$

since gluing two copies of D^n along the identity gives the standard decomposition of S^n. $\square$

6.3. **A Morse function on $E_{h,\ell}$ with four critical points.** We first construct a Morse function on $E_{h,\ell}$ with four critical points. Conceptually, this is obtained by pulling back a height function on the base S^4, which yields a Morse–Bott function whose critical sets are exactly the fibers over the poles, and then perturbing it by a small Morse function along the S^3-fibers.

Lemma 6.3 (A two–critical–point Morse function on S^3)**.** *There exists a Morse function $g : S^3 \to \mathbb{R}$ with exactly two critical points, of indices* 0 *and* 3*.*

Proof. View $S^3 \subset \mathbb{R}^4$ and let g be the restriction of a generic linear functional $x \mapsto x_1$. Its critical points are the north and south poles, and the Hessian computation shows they have indices 0 and 3. □

Proposition 6.4 (A four–critical–point Morse function on $E_{h,\ell}$)**.** *For each (h, ℓ), the manifold $E_{h,\ell}$ admits a Morse function $F : E_{h,\ell} \to \mathbb{R}$ with exactly four critical points, of indices* $0, 3, 4, 7$*.*

Proof. Choose a Morse function $f : S^4 \to \mathbb{R}$ with exactly two critical points, of indices 0 and 4; for instance, the restriction of a generic linear functional on $\mathbb{R}^5$ to $S^4 \subset \mathbb{R}^5$. Consider the pullback $f \circ \pi : E_{h,\ell} \to \mathbb{R}$. Its critical locus is not discrete: the differential vanishes exactly on the fibers over the two critical points of f, hence the critical set is the disjoint union of two copies of S^3. Thus $f \circ \pi$ is Morse–Bott rather than Morse.

Fix a bundle metric and use local trivializations of $\pi : E_{h,\ell} \to S^4$ over neighborhoods of the two critical points of f. Let $g : S^3 \to \mathbb{R}$ be the Morse function from Lemma 6.3. Using the local product structures near the two special fibers, define a smooth function F on $E_{h,\ell}$ by

$$F = f \circ \pi + \varepsilon \widetilde{g},$$

where $\widetilde{g}$ agrees with g on each fiber in a neighborhood of each special fiber and is supported in small neighborhoods of those fibers (constructed via a cutoff function in the base). For $\varepsilon > 0$ sufficiently small, this perturbation breaks each critical S^3 into exactly two nondegenerate critical points, corresponding to the minimum and maximum of g. No new critical points appear away from those neighborhoods because df is bounded away from 0 there, and the perturbation is C^1-small.

Near the fiber over the index 0 critical point of f, the Hessian of F splits into a positive definite 4×4 block from the base directions and the Hessian of g in the fiber directions, so the resulting two critical points have indices 0 and 3. Near the fiber over the index 4 critical point of f, the base Hessian contributes 4 negative directions, so the two critical points have indices 4 and 7. Hence, F has exactly four critical points with indices $0, 3, 4, 7$. □

6.4. **The Morse complex and cancellation for $|h - \ell| = 1$.** We now explain how the Euler number $a = h - \ell$ governs the cancellation of the index 3 and index 4 critical points.

Definition 6.5 (Morse complex: the relevant special case)**.** Assume F is Morse–Smale with respect to some Riemannian metric on $E_{h,\ell}$. The Morse chain group $C_k(F)$ is the free abelian group generated by the critical points of index k, and the differential $\partial : C_k(F) \to C_{k-1}(F)$ counts oriented gradient flow lines between critical points of successive indices. The homology of this complex is canonically isomorphic to $H_*(E_{h,\ell}; \mathbb{Z})$.

Lemma 6.6 (Shape of the Morse complex for F)**.** *Let F be as in Proposition 6.4 and assume F is Morse–Smale. Then the Morse chain groups satisfy*

$$C_k(F) \cong \begin{cases} \mathbb{Z}, & k \in \{0,3,4,7\}, \\ 0, & \textit{otherwise}, \end{cases}$$

and the only potentially nonzero differential is

$$\partial_4 : C_4(F) \longrightarrow C_3(F),$$

which is multiplication by an integer $m \in \mathbb{Z}$ (well-defined up to sign, depending on orientation conventions).

Proof. There is exactly one critical point in each of the indices $0, 3, 4, 7$. Since ∂ lowers degree by 1, the only possible nontrivial map is from index 4 to index 3, and it is given by multiplication by some integer because both groups are free of rank one. $\square$

Proposition 6.7. *With notation as in Lemma 6.6, the integer m satisfies $m = \pm a$, where $a = h - \ell$.*

Proof. By Lemma 6.6, the Morse chain groups satisfy

$$C_k(F) \cong \begin{cases} \mathbb{Z}, & k \in \{0,3,4,7\}, \\ 0, & \text{otherwise}, \end{cases}$$

and the only potentially nonzero differential is

$$\partial_4 : C_4(F) \longrightarrow C_3(F),$$

which is multiplication by an integer $m \in \mathbb{Z}$. Hence the Morse homology in degree 3 is

$$H_3\big(C_*(F), \partial\big) \;\cong\; \ker(\partial_3)/\mathrm{im}(\partial_4) \;\cong\; \mathbb{Z}/|m|\mathbb{Z},$$

since $\partial_3 = 0$ and ∂_4 is multiplication by m.

On the other hand, Section 5.4 computed the integral cohomology of $E_{h,\ell}$:

$$H^4(E_{h,\ell}; \mathbb{Z}) \cong \mathbb{Z}/a\mathbb{Z} \quad \text{(with the convention } \mathbb{Z}/0\mathbb{Z} = \mathbb{Z}).$$

Since $E_{h,\ell}$ is a closed, connected, oriented 7-manifold, Poincaré duality gives

$$H_3(E_{h,\ell}; \mathbb{Z}) \cong H^4(E_{h,\ell}; \mathbb{Z}) \cong \mathbb{Z}/|a|\mathbb{Z}.$$

Because Morse homology agrees with singular homology, we must have

$$\mathbb{Z}/|m|\mathbb{Z} \;\cong\; H_3\big(C_*(F), \partial\big) \;\cong\; H_3(E_{h,\ell}; \mathbb{Z}) \;\cong\; \mathbb{Z}/|a|\mathbb{Z}.$$

Therefore $|m| = |a|$, i.e. $m = \pm a$. $\square$

Theorem 6.8 (Cancellation when $|h - \ell| = 1$)**.** *If $|h - \ell| = 1$, then $E_{h,\ell}$ admits a Morse function with exactly two critical points, of indices 0 and 7.*

Proof. Let F be as in Proposition 6.4. After a C^∞-small perturbation of the Riemannian metric we may assume (F, g) is Morse–Smale. Let p (resp. q) be the unique critical point of index 4 (resp. 3). By Proposition 6.7, the differential $\partial_4 : C_4(F) \to C_3(F)$ is multiplication by $m = \pm a$. Hence if $|a| = 1$ then $\partial_4 = \pm 1$, so the incidence number between p and q is a unit.

In Morse–Smale theory (equivalently, in the handle decomposition associated to F), the condition that the incidence number between an index k critical point and an index $k-1$ critical point is ± 1 is exactly the cancellation criterion: one can perform a standard cancellation deformation supported in a small region intersecting only this pair, thereby removing the critical points of indices 3 and 4 and creating no new critical points.

Therefore $E_{h,\ell}$ admits a Morse function with exactly two critical points, of indices 0 and 7. □

Corollary 6.9 (Topological 7-sphere)**.** *If $|h-\ell| = 1$, then $E_{h,\ell}$ is homeomorphic to S^7.*

Proof. By Theorem 6.8, $E_{h,\ell}$ admits a two–critical–point Morse function. The conclusion follows from Reeb's theorem (Theorem 6.2). □

Remark 6.10 (A modern shortcut via Smale). Once one knows that $E_{h,\ell}$ is a homology 7-sphere, there is a short modern route to conclude that $E_{h,\ell}$ is *homeomorphic* to S^7. Indeed, the bundle projection $S^3 \hookrightarrow E_{h,\ell} \to S^4$ is a (Serre) fibration with simply connected fiber and base. The long exact sequence of homotopy groups gives

$$\pi_1(S^3) \to \pi_1(E_{h,\ell}) \to \pi_1(S^4),$$

hence $\pi_1(E_{h,\ell}) = 0$. Therefore $E_{h,\ell}$ is a simply connected homology 7-sphere, and by Smale's solution of the generalized Poincaré conjecture in dimensions ≥ 5 it follows that $E_{h,\ell}$ is homeomorphic to S^7.

Historically, this shortcut was not available at the time of Milnor's work, since Smale's theorem had not yet been proved.

7. Reading Milnor, Part III: Distinguishing Smooth Structures

7.1. **Why a new invariant is needed.** By Part II, whenever $|h-\ell| = 1$ the manifold $E_{h,\ell} = S(\xi_{h,\ell})$ is homeomorphic to S^7. To decide whether different pairs (h,ℓ) yield *different smooth structures* on this underlying topological sphere, we need a smooth invariant. Milnor's original paper constructs an invariant

$$X(M) \in \mathbb{Z}/7\mathbb{Z}$$

defined for any closed oriented 7-manifold M satisfying the hypothesis

$$(*) \quad H^3(M;\mathbb{Z}) = H^4(M;\mathbb{Z}) = 0,$$

and then computes $X(E_{h,\ell})$ from the characteristic classes of the rank-4 bundle $\xi_{h,\ell} \to S^4$.

7.2. **The bounding disk bundle and two basic inputs.** Fix $(h,\ell) \in \mathbb{Z}^2$ with $|h-\ell| = 1$. Let

$$\pi : W_{h,\ell} := D(\xi_{h,\ell}) \longrightarrow S^4$$

be the rank-4 disk bundle, so that $\partial W_{h,\ell} = S(\xi_{h,\ell}) = E_{h,\ell}$.

Lemma 7.1 (A splitting of the tangent bundle)**.** *There is a (noncanonical) vector bundle isomorphism over $W_{h,\ell}$*

$$TW_{h,\ell} \cong \pi^*(TS^4) \oplus \pi^*(\xi_{h,\ell}).$$

Consequently,

$$p(TW_{h,\ell}) = \pi^*(p(TS^4))\, \pi^*(p(\xi_{h,\ell})).$$

In particular, since $\langle p_1(TS^4), [S^4]\rangle = 3\,\sigma(S^4) = 0$, one has

$$p_1(TW_{h,\ell}) = \pi^*(p_1(\xi_{h,\ell})) \in H^4(W_{h,\ell};\mathbb{Z}).$$

Proof. For any smooth vector bundle $\pi : E \to B$, the differential $d\pi : TE \to \pi^*TB$ exhibits TE as an extension

$$0 \longrightarrow V \longrightarrow TE \xrightarrow{d\pi} \pi^*TB \longrightarrow 0,$$

where $V = \ker(d\pi)$ is the vertical tangent bundle. In the case $E = W_{h,\ell} \subset \xi_{h,\ell}$, the vertical bundle is canonically identified with $\pi^*(\xi_{h,\ell})$ via fiberwise translation. Choosing any connection on $\xi_{h,\ell}$ splits the above exact sequence, yielding the claimed isomorphism $TW_{h,\ell} \cong \pi^*TS^4 \oplus \pi^*\xi_{h,\ell}$.

Multiplicativity of total Pontryagin classes under direct sum gives $p(TW_{h,\ell}) = p(\pi^*TS^4)\, p(\pi^*\xi_{h,\ell})$. Finally, the signature formula in dimension 4 yields $\langle p_1(TS^4), [S^4]\rangle = 3\,\sigma(S^4) = 0$, hence $p_1(TS^4) = 0$ in $H^4(S^4;\mathbb{Z}) \cong \mathbb{Z}$, and therefore $p_1(TW_{h,\ell}) = \pi^*p_1(\xi_{h,\ell})$. □

Lemma 7.2 (The relevant cohomology and the Thom generator)**.** *Let $W = W_{h,\ell}$ and $E = \partial W = E_{h,\ell}$. Then $H^4(W, E;\mathbb{Z}) \cong \mathbb{Z}$. Let $U \in H^4(W, E;\mathbb{Z})$ be the Thom class of $\xi_{h,\ell}$. The map*

$$i^* : H^4(W, E;\mathbb{Z}) \longrightarrow H^4(W;\mathbb{Z})$$

induced by inclusion $i : (W, \varnothing) \hookrightarrow (W, E)$ is an isomorphism (because $H^3(E) = H^4(E) = 0$), and satisfies

$$i^*(U) = \pi^*(e(\xi_{h,\ell})) \in H^4(W;\mathbb{Z}).$$

Moreover,

$$\langle U \smile U,\, [W, E]\rangle = \langle e(\xi_{h,\ell}),\, [S^4]\rangle = h - \ell.$$

Proof. Since $\xi_{h,\ell}$ has rank 4 over S^4, the Thom isomorphism gives $H^4(W, E) \cong H^0(S^4) \cong \mathbb{Z}$, generated by the Thom class U.

The long exact sequence for the pair (W, E) contains

$$H^3(E) \longrightarrow H^4(W, E) \xrightarrow{i^*} H^4(W) \longrightarrow H^4(E).$$

For $|h - \ell| = 1$, Part I and the Gysin computation in Part II show $E = E_{h,\ell}$ is a homology 7-sphere, hence $H^3(E) = H^4(E) = 0$. Thus i^* is an isomorphism.

The identity $i^*(U) = \pi^*(e(\xi_{h,\ell}))$ is the standard relationship between the Thom class and Euler class: under the Thom isomorphism, the square $U \smile U \in H^8(W, E)$ corresponds to $e(\xi_{h,\ell}) \in H^4(S^4)$, and evaluating on $[W, E]$ is the same as evaluating $e(\xi_{h,\ell})$ on $[S^4]$. Using the Hopf normalization from Part I, $\langle e(\xi_{h,\ell}), [S^4]\rangle = h - \ell$, hence $\langle U^2, [W, E]\rangle = h - \ell$. □

7.3. **Milnor's invariant $X(M) \in \mathbb{Z}/7\mathbb{Z}$.** Let M^7 be a closed oriented 7-manifold satisfying $(*)$, and let B^8 be any compact oriented 8-manifold with $\partial B = M$. Write $v \in H_8(B, M;\mathbb{Z})$ for the relative fundamental class determined by the chosen orientations.

Definition 7.3 (Intersection form and index)**.** The cup product induces a symmetric bilinear form on $H^4(B, M;\mathbb{Z})/\mathrm{tors}$ by

$$(a, b) \;:=\; \langle a \smile b,\, v\rangle.$$

Let $r(B)$ denote the index (signature) of this form, i.e. the number of positive eigenvalues minus the number of negative eigenvalues after extending scalars to $\mathbb{R}$.

The hypothesis $(*)$ implies that the inclusion-induced map $i^* : H^4(B, M) \to H^4(B)$ is an isomorphism (as in Lemma 7.2). Hence $p_1(TB) \in H^4(B)$ has a unique preimage $\widetilde{p}_1 \in H^4(B, M)$.

Definition 7.4 (Milnor's Pontryagin number)**.** Define

$$q(B) := \langle \widetilde{p}_1 \smile \widetilde{p}_1,\, v\rangle \in \mathbb{Z}.$$

Theorem 7.5 (Milnor)**.** *For any closed oriented 7-manifold M satisfying $(*)$, the residue class*

$$X(M) := 2q(B) - r(B) \in \mathbb{Z}/7\mathbb{Z}$$

does not depend on the choice of the bounding manifold B with $\partial B = M$.

Proof. Let B' and B'' be two compact oriented 8-manifolds with boundary M. Glue them along M (reversing the orientation on B'') to obtain a closed oriented 8-manifold

$$C := B' \cup_M (-B'').$$

Step 1: additivity of r and q. Under the hypothesis $(*)$, the natural maps $H^4(B', M) \to H^4(B')$ and $H^4(B'', M) \to H^4(B'')$ are isomorphisms, and the Mayer–Vietoris sequence for $C = B' \cup (-B'')$ yields a decomposition of the middle cohomology in which the intersection form of C is the orthogonal sum of the relative forms of B' and $-B''$. In particular,

$$r(C) = r(B') - r(B'').$$

Similarly, $p_1(TC)$ restricts to $p_1(TB')$ on B' and to $p_1(TB'')$ on B'', with the sign change coming only from the orientation reversal in the evaluation. One obtains

$$q(C) = q(B') - q(B'').$$

Step 2: a congruence for closed 8-manifolds. For any closed oriented 8-manifold C, Hirzebruch's signature theorem gives

$$\sigma(C) = \langle L_2(TC), [C]\rangle = \frac{1}{45}\langle 7p_2(TC) - p_1(TC)^2, [C]\rangle.$$

Reducing modulo 7 and using $45 \equiv 3 \pmod 7$, we find

$$\langle p_1(TC)^2, [C]\rangle \equiv -45\,\sigma(C) \equiv -3\,\sigma(C) \pmod 7.$$

Hence

$$2\,\langle p_1(TC)^2, [C]\rangle - \sigma(C) \equiv 2(-3\sigma(C)) - \sigma(C) = -7\sigma(C) \equiv 0 \pmod 7.$$

But by definition of $q(C)$ and $r(C)$, we have $q(C) = \langle p_1(TC)^2, [C]\rangle$ and $r(C) = \sigma(C)$. Therefore,

$$2q(C) - r(C) \equiv 0 \pmod 7.$$

Step 3: conclude independence. Using Step 1 and the congruence for C,

$$0 \equiv 2q(C) - r(C) = (2q(B') - r(B')) - (2q(B'') - r(B'')) \pmod 7,$$

so $2q(B') - r(B') \equiv 2q(B'') - r(B'') \pmod 7$. This shows $X(M)$ is well-defined. $\square$

7.4. **Computing $X(E_{h,\ell})$.** Assume $|h-\ell|=1$, and let $W=W_{h,\ell}=D(\xi_{h,\ell})$ with boundary $E=E_{h,\ell}$. Write

$$a := h-\ell \in \{\pm 1\}, \quad S := h+\ell \in \mathbb{Z}.$$

Recall from Part I (Hopf normalization) that, for the generator $u \in H^4(S^4;\mathbb{Z})$ with $\langle u, [S^4]\rangle = 1$,

$$e(\xi_{h,\ell}) = au, \quad p_1(\xi_{h,\ell}) = 2Su.$$

Proposition 7.6 (Closed form for $X(E_{h,\ell})$)**.** *With the conventions above,*

$$X(E_{h,\ell}) \equiv a(S^2-1) \equiv (h-\ell)((h+\ell)^2-1) \pmod 7.$$

Proof. We compute $r(W)$ and $q(W)$.

(1) The index $r(W)$. By Lemma 7.2, $H^4(W,E)\cong\mathbb{Z}$ with generator the Thom class U, and $\langle U^2,[W,E]\rangle = \langle e(\xi_{h,\ell}),[S^4]\rangle = a$. Thus the intersection form on $H^4(W,E)/\mathrm{tors}\cong\mathbb{Z}$ is the 1×1 matrix (a), so

$$r(W) = \operatorname{sign}(a) = a \quad (\text{since } a=\pm 1).$$

(2) The Pontryagin number $q(W)$. By Lemma 7.1, $p_1(TW) = \pi^* p_1(\xi_{h,\ell}) = 2S\,\pi^*(u)$. By Lemma 7.2, we have $i^*(U) = \pi^*(e(\xi_{h,\ell})) = a\,\pi^*(u)$. Since $i^* : H^4(W,E)\to H^4(W)$ is an isomorphism, the unique lift $\widetilde{p}_1 \in H^4(W,E)$ of $p_1(TW)$ is

$$\widetilde{p}_1 = \frac{2S}{a}\,U.$$

Therefore,

$$q(W) = \langle \widetilde{p}_1^{\,2},[W,E]\rangle = \frac{4S^2}{a^2}\langle U^2,[W,E]\rangle = \frac{4S^2}{a^2}\cdot a = \frac{4S^2}{a}.$$

Since $a=\pm 1$, this is $q(W)=4aS^2$.

(3) Assemble X. By definition,

$$X(E) = 2q(W) - r(W) = 2(4aS^2) - a = a(8S^2-1).$$

Reducing modulo 7, note that $8\equiv 1 \pmod 7$, hence

$$X(E) \equiv a(S^2-1) \pmod 7,$$

as claimed. ☐

Corollary 7.7 (Exotic smooth structures exist in the family $\{E_{h,\ell}\}$)**.** *There exist pairs (h,ℓ) with $|h-\ell|=1$ such that $E_{h,\ell}$ is not diffeomorphic to the standard sphere S^7.*

Proof. For the standard sphere $S^7=\partial D^8$, one has $H^4(D^8,S^7)=0$, hence $q(D^8)=r(D^8)=0$ and so $X(S^7)=0\in\mathbb{Z}/7\mathbb{Z}$.

On the other hand, take $(h,\ell)=(2,1)$. Then $|h-\ell|=1$, $S=h+\ell=3$, and Proposition 7.6 gives

$$X(E_{2,1}) \equiv (3^2-1) = 8 \equiv 1 \pmod 7,$$

which is nonzero. Hence, $E_{2,1}\not\cong S^7$ diffeomorphically. ☐

Remark 7.8. The congruence for $X(E_{h,\ell})$ shows that the family $\{E_{h,\ell}\}$ contains manifolds that are homeomorphic to S^7 but not diffeomorphic to it, hence exotic smooth structures on the 7-sphere exist. Milnor's invariant $X(M) \in \mathbb{Z}/7\mathbb{Z}$ detects some of these structures but does not classify them. In fact, the group of smooth structures on S^7 is $\Theta_7 \cong \mathbb{Z}/28\mathbb{Z}$ (Kervaire–Milnor), and refinements such as the Eells–Kuiper invariant distinguish all 28 classes.