Symplectic Geometry I: Foundations

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June 14, 2025

Contents

1 Linear symplectic geometry

1.1 Skew-symmetric bilinear forms

Let V be a finite-dimensional real vector space. A bilinear map $\Omega : V \times V \to \mathbb{R}$ is **skew-symmetric** if $\Omega(u,v) = -\Omega(v,u)$ for all $u, v \in V$. Equivalently, $\Omega \in \wedge^2 V^*$.

Definition 1.1. The contraction map

Given a skew-symmetric bilinear map Ω on V, define the linear map

$$\widetilde{\Omega} : V \longrightarrow V^*, \qquad \widetilde{\Omega}(v)(u) = \Omega(v,u).$$

The kernel of Ω is $\ker(\widetilde{\Omega}) = \{v \in V \mid \Omega(v,u) = 0 \ \forall u \in V\}$.

Definition 1.2. Nondegeneracy and symplectic vector spaces

A skew-symmetric bilinear map Ω is **nondegenerate** if $\widetilde{\Omega}$ is an isomorphism. In that case (V, Ω) is called a **symplectic vector space**.

Theorem 1.3. Standard form for skew-symmetric bilinear maps

Let Ω be a skew-symmetric bilinear map on an m-dimensional real vector space V. Let $U = \ker(\widetilde{\Omega})$ and set $k = \dim U$. Then $m-k$ is even, say $m-k = 2n$, and there exists a basis

$$u_1, \dots, u_k, \ e_1, \dots, e_n, \ f_1, \dots, f_n$$

of V such that

$$\Omega(u_i, v) = 0 \text{ for all } i \text{ and all } v \in V, \qquad \Omega(e_i, e_j) = \Omega(f_i, f_j) = 0, \qquad \Omega(e_i, f_j) = \delta_{ij}.$$

Proof. Let $U = \ker(\widetilde{\Omega})$. Choose a basis $u_1, \dots, u_k$ of U and a complementary subspace W so that $V = U \oplus W$. Since U is the radical, the restriction $\Omega|_W$ is nondegenerate.
If $W = \{0\}$ we are done. Otherwise choose $e_1 \in W \setminus \{0\}$. Nondegeneracy of $\Omega|_W$ gives $f_1 \in W$ such that $\Omega(e_1, f_1) = 1$. Define

$$W_1 = \{w \in W \mid \Omega(w, e_1) = 0 \text{ and } \Omega(w, f_1) = 0\}.$$

We claim $W = \operatorname{span}\{e_1, f_1\} \oplus W_1$. Indeed, for any $w \in W$ set

$$w_1 := w - \Omega(w, f_1)e_1 + \Omega(w, e_1)f_1.$$

Then $\Omega(w_1, e_1) = \Omega(w, e_1) - \Omega(w, f_1)\Omega(e_1, e_1) + \Omega(w, e_1)\Omega(f_1, e_1) = 0$ and similarly $\Omega(w_1, f_1) = 0$, so $w_1 \in W_1$, hence $w \in \operatorname{span}\{e_1, f_1\} + W_1$. If $ae_1 + bf_1 \in W_1$, pairing with f_1 gives $a = 0$, pairing with e_1 gives $b = 0$, so the sum is direct.
Next we check that $\Omega|_{W_1}$ is nondegenerate. If $w \in W_1$ satisfies $\Omega(w, w') = 0$ for all $w' \in W_1$, then also $\Omega(w, e_1) = \Omega(w, f_1) = 0$ by definition of W_1. Hence $\Omega(w, \cdot) = 0$ on all of W, so $w = 0$. Thus $(W_1, \Omega|_{W_1})$ is symplectic.
Now repeat the same construction inside W_1. Since the dimension drops by 2 each step, after n steps we obtain pairs (e_i, f_i) and a decomposition

$$W = \bigoplus_{i=1}^{n} \operatorname{span}\{e_i, f_i\}$$

with the required relations. Together with $u_1, \dots, u_k$ this gives the desired basis of V. $\square$

Corollary 1.4 (Even-dimensionality and symplectic bases)**.** If Ω is nondegenerate on V, then $\dim V = 2n$ is even and V admits a basis $e_1, \dots, e_n, f_1, \dots, f_n$ with

$$\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0, \qquad \Omega(e_i, f_j) = \delta_{ij}.$$

Such a basis is called a **symplectic basis**.

Proof. If Ω is nondegenerate then $U = \ker(\widetilde{\Omega}) = \{0\}$, so $k = 0$ in Theorem 1.3 and $\dim V = 2n$. The basis there is exactly a symplectic basis. □

Corollary 1.5 (Wedge-power criterion)**.** Let ω be a skew-symmetric bilinear form on a $2n$-dimensional real vector space V. Then ω is nondegenerate if and only if $\omega^{\wedge n} \neq 0 \in \wedge^{2n} V^*$.

Proof. If ω is degenerate, choose $0 \neq v \in V$ with $\omega(v, \cdot) = 0$. Then $\omega^{\wedge n}(v, v_2, \dots, v_{2n}) = 0$ for any vectors $v_2, \dots, v_{2n}$, hence $\omega^{\wedge n} = 0$.
Conversely, if ω is nondegenerate, choose a symplectic basis $e_1, \dots, e_n, f_1, \dots, f_n$. In that basis $\omega = \sum_{i=1}^n e_i^* \wedge f_i^*$, so

$$\omega^{\wedge n} = n!\, e_1^* \wedge f_1^* \wedge \cdots \wedge e_n^* \wedge f_n^* \neq 0.$$

□

1.2 Symplectic complements and linear types of subspaces

Definition 1.6. Symplectic complement

Let (V, ω) be a symplectic vector space and let $W \subset V$ be a subspace. The **symplectic complement** of W is

$$W^\omega = \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}.$$

Lemma 1.7. Dimension identity for complements Let (V, ω) be a symplectic vector space and $W \subset V$ a subspace. Then

$$\dim W + \dim W^\omega = \dim V.$$

Proof. Consider the isomorphism $\iota_\omega : V \to V^*$ given by $\iota_\omega(v) = \omega(v, \cdot)$. The subspace W^ω is exactly the preimage under ι_ω of the annihilator $W^\perp = \{\lambda \in V^* \mid \lambda|_W = 0\}$. Hence $\dim W^\omega = \dim W^\perp$. For any subspace $W \subset V$ we have $\dim W + \dim W^\perp = \dim V$. The claim follows. □

Definition 1.8. Isotropic, coisotropic, symplectic, Lagrangian

Let (V, ω) be symplectic and $W \subset V$ a subspace.

(1) W is **isotropic** if $\omega|_W \equiv 0$, equivalently $W \subset W^\omega$.
(2) W is **coisotropic** if $W^\omega \subset W$.
(3) W is **symplectic** if $\omega|_W$ is nondegenerate, equivalently $W \cap W^\omega = \{0\}$.
(4) W is **Lagrangian** if it is isotropic and $\dim W = \frac{1}{2} \dim V$, equivalently $W = W^\omega$.

Proposition 1.9. Basic inequalities

If $\dim V = 2n$, then every isotropic subspace has dimension at most n and every coisotropic subspace has dimension at least n. Moreover, a subspace $L \subset V$ is Lagrangian if and only if it is a maximal isotropic subspace.

Proof. If W is isotropic then $W \subset W^\omega$, so $\dim W \leq \dim W^\omega$. Using Lemma 1.7 gives $2\dim W \leq 2n$, hence $\dim W \leq n$. If W is coisotropic then $W^\omega \subset W$, so $\dim W^\omega \leq \dim W$, hence $\dim W \geq n$.
If L is Lagrangian then it is isotropic with $\dim L = n$, so by the previous inequality it cannot be properly contained in any larger isotropic subspace. Hence it is maximal isotropic.
Conversely, let W be maximal isotropic. Then $W \subset W^\omega$ and W^ω is isotropic as well. Maximality forces $W = W^\omega$, hence $\dim W = n$ by Lemma 1.7, so W is Lagrangian. □

Lemma 1.10. Extending isotropic subspaces Every isotropic subspace is contained in a Lagrangian subspace. Moreover, if $L \subset V$ is Lagrangian and $u_1, \ldots, u_n$ is a basis of L, then there exist vectors $v_1, \ldots, v_n$ such that $u_1, \ldots, u_n, v_1, \ldots, v_n$ is a symplectic basis of V.

Proof. Start with an isotropic subspace W. If $W \neq W^\omega$, choose $v \in W^\omega \setminus W$ and set $W_1 = W + \mathbb{R}v$. Then W_1 is still isotropic since ω vanishes on W and $v \in W^\omega$. Iterating, by finite dimension we reach a maximal isotropic subspace L, which is Lagrangian by Proposition 1.9.
For the second claim, apply Theorem 1.3 to extend the basis of L to a symplectic basis. Concretely, define linear functionals $\ell_i \in V^*$ by $\ell_i(u_j) = \delta_{ij}$ and $\ell_i|_L = 0$ on the complement directions. Using the isomorphism ι_ω, pick $v_i \in V$ with $\iota_\omega(v_i) = \ell_i$. Then $\omega(u_j, v_i) = \delta_{ij}$ and $\omega(v_i, v_j) = 0$ after replacing v_i by suitable linear combinations modulo L. □

1.3 The standard model and matrices

Example. The standard symplectic space On $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_y$ define

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Then $(\mathbb{R}^{2n}, \omega_0)$ is symplectic. In the standard basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ with $e_i = \partial_{x_i}$ and $f_i = \partial_{y_i}$ we have $\omega_0(e_i, f_j) = \delta_{ij}$ and $\omega_0(e_i, e_j) = \omega_0(f_i, f_j) = 0$. Equivalently, for $u = (p, q)$ and $v = (p', q')$ with $p, q, p', q' \in \mathbb{R}^n$,

$$\omega_0(u, v) = p \cdot q' - p' \cdot q.$$

Fix a symplectic basis of (V, ω) and write vectors as columns relative to that basis. In that basis, the matrix of ω is the standard skew-symmetric matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad \omega(u, v) = u^T J v.$$

Definition 1.11. Linear symplectomorphisms and the symplectic group

A linear isomorphism $\Phi : (V, \omega) \to (V, \omega)$ is a **symplectomorphism** if

$$\omega(\Phi u, \Phi v) = \omega(u, v) \quad \text{for all } u, v \in V.$$

The set of all such Φ is a group, denoted $\mathrm{Sp}(V, \omega)$. If $V = \mathbb{R}^{2n}$ and $\omega = \omega_0$, we write $\mathrm{Sp}(2n, \mathbb{R})$.

Proposition 1.12. Matrix characterization

Let $A \in \mathrm{GL}(2n, \mathbb{R})$ and view it as a linear map on $(\mathbb{R}^{2n}, \omega_0)$. Then A is symplectic if and only if

$$A^T J A = J.$$

In particular, every symplectic matrix has determinant 1.

Proof. For all $u, v \in \mathbb{R}^{2n}$ we have
$$\omega_0(Au, Av) = (Au)^T J(Av) = u^T(A^T JA)v.$$
Thus $\omega_0(Au, Av) = \omega_0(u, v)$ for all u, v if and only if $A^T JA = J$.
Taking determinants gives $(\det A)^2 \det J = \det J$, hence $(\det A)^2 = 1$. Since $\mathrm{Sp}(2n, \mathbb{R})$ is connected, $\det A$ cannot jump between ± 1, so $\det A = 1$. $\square$

Example. Two families of symplectic matrices Let B be any symmetric $n \times n$ matrix. Then
$$A = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} I & 0 \\ B & I \end{pmatrix}$$
are symplectic. These correspond to elementary shears preserving ω_0.

Proposition 1.13. Graphs and Lagrangians

Let (V, ω) be a symplectic vector space and let $\Psi : V \to V$ be linear. Consider $V \times V$ with symplectic form $(-\omega) \oplus \omega$. Then Ψ is a symplectomorphism if and only if its graph
$$\Gamma_\Psi = \{(v, \Psi v) \mid v \in V\} \subset V \times V$$
is a Lagrangian subspace.

Proof. Take $v, w \in V$. On $V \times V$ we compute
$$((-\omega) \oplus \omega)((v, \Psi v), (w, \Psi w)) = -\omega(v, w) + \omega(\Psi v, \Psi w).$$
Thus Γ_Ψ is isotropic if and only if $\omega(\Psi v, \Psi w) = \omega(v, w)$ for all v, w, which is the symplectic condition.
If Ψ is symplectic then Γ_Ψ is isotropic and has dimension $\dim V = \frac{1}{2} \dim(V \times V)$, hence it is Lagrangian. Conversely, if Γ_Ψ is Lagrangian it is in particular isotropic, so Ψ is symplectic. $\square$

Remark. Linear symplectic geometry is rigid enough to have canonical normal forms, but it is also flexible in the sense that every symplectic vector space of dimension $2n$ is linearly symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$ once a symplectic basis is chosen.

2 Symplectic complements, splittings, and linear reduction

Throughout this lecture, (V, ω) denotes a finite-dimensional real symplectic vector space with $\dim V = 2n$.

2.1 Symplectic complements

Definition 2.1. Symplectic complement, again

For a subspace $W \subset V$, its *symplectic complement* is
$$W^\omega := \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}.$$

Lemma 2.2. Basic properties of complements For subspaces $W_1, W_2 \subset V$:

(1) If $W_1 \subset W_2$, then $W_2^\omega \subset W_1^\omega$.
(2) $(W_1 + W_2)^\omega = W_1^\omega \cap W_2^\omega$.
(3) $(W_1 \cap W_2)^\omega = W_1^\omega + W_2^\omega$.

Proof. (1) is immediate from the definition.
For (2), let $v \in (W_1+W_2)^\omega$. Then $\omega(v, w_1+w_2) = 0$ for all $w_i \in W_i$, so in particular $\omega(v, w_1) = 0$ for all $w_1 \in W_1$ and $\omega(v, w_2) = 0$ for all $w_2 \in W_2$. Hence $v \in W_1^\omega \cap W_2^\omega$. The reverse inclusion is equally direct.
For (3), apply (2) to W_1^ω and W_2^ω, and then use the double-complement statement in Lemma 2.3. ☐

Lemma 2.3. Double complement For any subspace $W \subset V$ one has

$$(W^\omega)^\omega = W.$$

Proof. The inclusion $W \subset (W^\omega)^\omega$ is clear since every $w \in W$ pairs to 0 with every vector in W^ω.
For the reverse inclusion, consider the isomorphism $\iota_\omega : V \to V^*$ given by $\iota_\omega(v) = \omega(v, \cdot)$. Then $W^\omega = \iota_\omega^{-1}(W^\perp)$, where $W^\perp \subset V^*$ is the annihilator of W. Applying ι_ω again gives

$$(W^\omega)^\omega = \iota_\omega^{-1}((W^\perp)^\perp).$$

In finite dimensions $(W^\perp)^\perp = W$ under the canonical identification $V \cong V^{**}$, so $(W^\omega)^\omega = W$. ☐

Proposition 2.4. Dimension identities

For any subspace $W \subset V$,

$$\dim W + \dim W^\omega = 2n, \qquad \dim(W \cap W^\omega) = \dim W + \dim W^\omega - 2n.$$

In particular, W is symplectic if and only if $W \cap W^\omega = \{0\}$.

Proof. The first identity is the linear-algebra fact proved in Lecture A1 using ι_ω. For the second, use the general relation

$$\dim(W + W^\omega) = \dim W + \dim W^\omega - \dim(W \cap W^\omega),$$

together with $W + W^\omega = V$ if and only if $\dim(W + W^\omega) = 2n$. Since $\dim W + \dim W^\omega = 2n$, the displayed formula rearranges to $\dim(W \cap W^\omega) = \dim W + \dim W^\omega - 2n$. The final claim is exactly the definition of nondegeneracy of $\omega|_W$. ☐

2.2 Symplectic subspaces and splittings

Proposition 2.5. Symplectic splitting

Let $W \subset (V, \omega)$ be a subspace.

(1) If W is symplectic, then $V = W \oplus W^\omega$.
(2) If W is Lagrangian, then $W = W^\omega$ and $\dim W = n$.

Proof. (1) If W is symplectic then $W \cap W^\omega = \{0\}$. Also $\dim W + \dim W^\omega = 2n$, so $\dim(W + W^\omega) = 2n$. Hence $W + W^\omega = V$ and the sum is direct.
(2) If W is Lagrangian, by definition it is isotropic and has dimension n. Isotropy gives $W \subset W^\omega$. Using $\dim W = \dim W^\omega = n$ forces $W = W^\omega$. ☐

Lemma 2.6. Symplectic bases adapted to a symplectic subspace Let $W \subset (V, \omega)$ be a symplectic subspace with $\dim W = 2k$. Then there exists a symplectic basis of V of the form

$$e_1, \ldots, e_k, f_1, \ldots, f_k, \; e_{k+1}, \ldots, e_n, f_{k+1}, \ldots, f_n$$

such that $W = \mathrm{span}\{e_1, \ldots, e_k, f_1, \ldots, f_k\}$.

Proof. Choose a symplectic basis of $(W, \omega|_W)$. By Proposition 2.5(1), $V = W \oplus W^\omega$. Extend the chosen basis to a symplectic basis of $(W^\omega, \omega|_{W^\omega})$, and concatenate. □

Example. Standard symplectic subspaces in $\mathbb{R}^{2n}$ In $(\mathbb{R}^{2n}, \omega_0 = \sum_{i=1}^n dx_i \wedge dy_i)$, the coordinate subspace

$$W = \operatorname{span}\{\partial_{x_1}, \dots, \partial_{x_k}, \partial_{y_1}, \dots, \partial_{y_k}\}$$

is symplectic and W^{ω_0} is the span of the remaining coordinate vectors.

2.3 Coisotropic subspaces and linear reduction

Definition 2.7. Characteristic subspace

Let $C \subset (V, \omega)$ be a subspace. The *characteristic subspace* of C is $K := C^\omega \subset V$. If C is coisotropic, then $K \subset C$.

Theorem 2.8. Linear coisotropic reduction

Let $C \subset (V, \omega)$ be a coisotropic subspace and let $K = C^\omega$. Then there exists a unique symplectic form ω_{red} on the quotient vector space C/K such that

$$\pi^* \omega_{\text{red}} = \omega|_C,$$

where $\pi : C \to C/K$ is the quotient map. In particular, $(C/K, \omega_{\text{red}})$ is a symplectic vector space.

Proof. Define ω_{red} by

$$\omega_{\text{red}}([c_1], [c_2]) := \omega(c_1, c_2),$$

where $[c]$ denotes the class of $c \in C$ modulo K.
We first check well-definedness. If $c_1' = c_1 + k_1$ and $c_2' = c_2 + k_2$ with $k_i \in K = C^\omega$, then

$$\omega(c_1', c_2') = \omega(c_1, c_2) + \omega(k_1, c_2) + \omega(c_1, k_2) + \omega(k_1, k_2).$$

Since $k_1 \in C^\omega$ and $c_2 \in C$, we have $\omega(k_1, c_2) = 0$. Similarly $\omega(c_1, k_2) = 0$. Also $\omega(k_1, k_2) = 0$ because $k_1 \in C^\omega$ and $k_2 \in C$. Thus $\omega(c_1', c_2') = \omega(c_1, c_2)$.
Skew-symmetry and bilinearity are immediate. For nondegeneracy, suppose $[c] \in C/K$ satisfies $\omega_{\text{red}}([c], [c']) = 0$ for all $[c'] \in C/K$. Then $\omega(c, c') = 0$ for all $c' \in C$, so $c \in C^\omega = K$. Hence $[c] = 0$ and ω_{red} is nondegenerate. □

Example. Reduction of a hyperplane Let $(\mathbb{R}^{2n}, \omega_0)$ be standard and let

$$C = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid y_n = 0\}.$$

Then C is coisotropic. One finds $C^{\omega_0} = \operatorname{span}\{\partial_{x_n}\}$, and the reduced space C/C^{ω_0} is naturally identified with $\mathbb{R}^{2(n-1)}$ with its standard symplectic form.

2.4 Lagrangian splittings and graphs

A *Lagrangian splitting* of (V, ω) is a decomposition $V = L \oplus L'$ where both L and L' are Lagrangian. Such splittings always exist, for instance by choosing a symplectic basis and taking $L = \operatorname{span}\{e_1, \dots, e_n\}$ and $L' = \operatorname{span}\{f_1, \dots, f_n\}$.

Lemma 2.9. The pairing between complementary Lagrangians Let $V = L \oplus L'$ be a Lagrangian splitting. Then the bilinear pairing

$$L \times L' \to \mathbb{R}, \qquad (x, y') \mapsto \omega(x, y')$$

is nondegenerate in each factor. Hence $L' \cong L^*$ canonically via $y' \mapsto \omega(\cdot, y')|_L$.

Proof. Fix $y' \in L'$. If $\omega(x, y') = 0$ for all $x \in L$, then $y' \in L^\omega$. Since L is Lagrangian, $L^\omega = L$, hence $y' \in L \cap L' = \{0\}$. The argument for nondegeneracy in the L factor is analogous. $\square$

Proposition 2.10. Lagrangians transverse to L are graphs of symmetric maps

Let $V = L \oplus L'$ be a Lagrangian splitting. Let $M \subset V$ be a Lagrangian subspace with $M \cap L' = \{0\}$. Then there exists a unique linear map $A : L \to L'$ such that

$$M = \Gamma_A = \{x + Ax \mid x \in L\}.$$

Moreover, M is Lagrangian if and only if the bilinear form

$$B_A(x_1, x_2) := \omega(x_1, Ax_2) \qquad (x_1, x_2 \in L)$$

is symmetric. Under the identification $L' \cong L^*$ from Lemma 2.9, this means that A corresponds to a symmetric linear map $L \to L^*$, that is an element of $\mathrm{Sym}^2(L^*)$.

Proof. Since $M \cap L' = \{0\}$, the projection $p_L : M \to L$ along L' is injective. Also $\dim M = \dim L = n$, so p_L is an isomorphism. Define $A : L \to L'$ by $Ax := p_{L'}(p_L^{-1}x)$. Then $M = \Gamma_A$ by construction, and uniqueness is clear.
Now compute the restriction of ω to Γ_A. For $x_i \in L$,

$$\omega(x_1 + Ax_1,\ x_2 + Ax_2) = \omega(x_1, Ax_2) + \omega(Ax_1, x_2) + \omega(Ax_1, Ax_2).$$

Since L and L' are Lagrangian, $\omega(x_1, x_2) = 0$ and $\omega(Ax_1, Ax_2) = 0$. Also skew-symmetry gives $\omega(Ax_1, x_2) = -\omega(x_2, Ax_1)$. Thus $\omega|_{\Gamma_A} \equiv 0$ if and only if

$$\omega(x_1, Ax_2) = \omega(x_2, Ax_1) \quad \text{for all } x_1, x_2 \in L,$$

which is exactly symmetry of B_A. $\square$

Example. Graphs in the standard splitting In $(\mathbb{R}^{2n}, \omega_0)$ with the splitting $L = \mathbb{R}^n_x \times \{0\}$ and $L' = \{0\} \times \mathbb{R}^n_y$, a linear map $A : L \to L'$ is represented by an $n \times n$ matrix S with $Ax = (0, Sx)$. Then Γ_A is Lagrangian if and only if S is symmetric.

2.5 The Lagrangian Grassmannian and its tangent space

Definition 2.11. The Lagrangian Grassmannian

The *Lagrangian Grassmannian* of (V, ω) is the set

$$\Lambda(V, \omega) := \{L \subset V \mid L \text{ is a Lagrangian subspace}\}.$$

Proposition 2.12. Local charts via symmetric graphs

Fix a Lagrangian splitting $V = L \oplus L'$. Let

$$\Lambda_{L'} := \{M \in \Lambda(V, \omega) \mid M \cap L' = \{0\}\}.$$

Then $\Lambda_{L'}$ is naturally identified with $\mathrm{Sym}^2(L^*)$ by sending M to the symmetric form B_A from Proposition 2.10. In particular, $\Lambda(V, \omega)$ is a smooth manifold of dimension $\frac{n(n+1)}{2}$.

Proof. By Proposition 2.10, each $M \in \Lambda_{L'}$ is the graph of a unique map $A : L \to L'$ whose associated bilinear form B_A is symmetric. Conversely, any symmetric form $B \in \mathrm{Sym}^2(L^*)$ corresponds to a unique $A : L \to L'$ through the identification $L' \cong L^*$ in Lemma 2.9, and then Γ_A is Lagrangian. This gives a bijection $\Lambda_{L'} \cong \mathrm{Sym}^2(L^*)$.
Different choices of Lagrangian splittings give compatible charts, so $\Lambda(V, \omega)$ is a smooth manifold. Its dimension equals $\dim \mathrm{Sym}^2(L^*) = \frac{n(n+1)}{2}$. $\square$

Proposition 2.13. Tangent space identification

Let $L \in \Lambda(V, \omega)$. Then the tangent space $T_L\Lambda(V, \omega)$ is canonically isomorphic to $\mathrm{Sym}^2(L^*)$.

Proof. Choose a Lagrangian complement L' so $V = L \oplus L'$. Near L, every Lagrangian subspace transverse to L' is the graph of a symmetric map $A_t : L \to L'$ depending smoothly on parameters t. Differentiating at $t = 0$ yields a linear map $\dot{A} : L \to L'$, and the symmetry condition linearizes to symmetry of

$$(x_1, x_2) \longmapsto \omega(x_1, \dot{A}x_2).$$

This produces an element of $\mathrm{Sym}^2(L^*)$. The construction does not depend on the chosen complement, since changing complements changes charts by smooth maps whose differentials preserve the symmetric identification. □

Remark. The reduction construction in Theorem 2.8 and the graph description in Proposition 2.10 are the two linear patterns that repeatedly reappear in the manifold setting.

3 Symplectic manifolds: forms, volume, and first examples

3.1 Differential 2-forms and pointwise linear algebra

Let M be a smooth manifold. A differential 2-form $\omega \in \Omega^2(M)$ assigns to each point $p \in M$ a skew-symmetric bilinear form

$$\omega_p : T_pM \times T_pM \to \mathbb{R},$$

depending smoothly on p.

Definition 3.1. Pointwise nondegeneracy

A 2-form ω on M is *pointwise nondegenerate* if for every $p \in M$ the bilinear form ω_p is nondegenerate, equivalently the map

$$\omega_p^\flat : T_pM \to T_p^*M, \qquad X \mapsto \iota_X\omega_p = \omega_p(X, \cdot)$$

is an isomorphism.

Lemma 3.2. Even-dimensionality and the wedge criterion Assume $\dim M = 2n$. A 2-form ω is pointwise nondegenerate if and only if the $2n$-form $\omega^{\wedge n}$ is nowhere vanishing on M.

Proof. Fix $p \in M$. By linear symplectic algebra, a skew-symmetric bilinear form on the $2n$-dimensional vector space T_pM is nondegenerate if and only if its n-fold wedge is a nonzero element of $\wedge^{2n}T_p^*M$. Thus ω is pointwise nondegenerate if and only if $\omega_p^{\wedge n} \neq 0$ for all p, which is equivalent to $\omega^{\wedge n}$ being nowhere vanishing as a differential form. □

3.2 Symplectic forms and symplectic manifolds

Definition 3.3. Symplectic form and symplectic manifold

A 2-form $\omega \in \Omega^2(M)$ is *symplectic* if

$$d\omega = 0 \qquad \text{and} \qquad \omega \text{ is pointwise nondegenerate.}$$

A *symplectic manifold* is a pair (M, ω) where M is a manifold and ω is a symplectic form.

Proposition 3.4. The musical isomorphism

If (M, ω) is symplectic, then the bundle map

$$\omega^\flat : TM \to T^*M, \qquad X \mapsto \iota_X \omega$$

is a vector bundle isomorphism.

Proof. At each point $p \in M$, the map $\omega_p^\flat : T_pM \to T_p^*M$ is an isomorphism by nondegeneracy. Smooth dependence on p implies that $\omega^\flat$ is an isomorphism of vector bundles. □

3.3 Symplectic volume and orientation

Definition 3.5. Symplectic volume form

Let (M^{2n}, ω) be symplectic. Define the $2n$-form

$$\mathrm{vol}_\omega := \frac{1}{n!}\,\omega^{\wedge n} \in \Omega^{2n}(M).$$

Proposition 3.6. Orientation and volume

The form vol_ω is a nowhere vanishing top-degree form. In particular, M is orientable and vol_ω determines a canonical orientation on M.

Proof. By Lemma 3.2, the $2n$-form $\omega^{\wedge n}$ is nowhere vanishing. Scaling by $1/n!$ does not change this. A nowhere vanishing top-degree form determines an orientation. □

Example. On $\mathbb{R}^{2n}$ with coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$, the standard form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

satisfies

$$\mathrm{vol}_{\omega_0} = \frac{1}{n!}\,\omega_0^{\wedge n} = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n,$$

which is the standard Euclidean volume form.

3.4 Symplectomorphisms and basic constructions

Definition 3.7. Symplectomorphism

Let (M, ω) and (N, η) be symplectic manifolds of the same dimension. A diffeomorphism $\varphi : M \to N$ is a *symplectomorphism* if

$$\varphi^*\eta = \omega.$$

Proposition 3.8. Symplectomorphisms preserve symplectic volume

If $\varphi : (M, \omega) \to (N, \eta)$ is a symplectomorphism, then

$$\varphi^*(\mathrm{vol}_\eta) = \mathrm{vol}_\omega.$$

In particular, φ preserves the orientation determined by the symplectic form.

Proof. Using functoriality of pullback and multiplicativity with respect to wedge products,

$$\varphi^*(\mathrm{vol}_\eta) = \frac{1}{n!}\,\varphi^*(\eta^{\wedge n}) = \frac{1}{n!}\,(\varphi^*\eta)^{\wedge n} = \frac{1}{n!}\,\omega^{\wedge n} = \mathrm{vol}_\omega.$$

□

Proposition 3.9. Pullback and products

(1) If $\psi : M \to N$ is a diffeomorphism and η is symplectic on N, then $\psi^*\eta$ is symplectic on M.

(2) If (M_1, ω_1) and (M_2, ω_2) are symplectic, then

$$(M_1 \times M_2,\ \omega_1 \oplus \omega_2)$$

is symplectic, where $\omega_1 \oplus \omega_2 = \pi_1^*\omega_1 + \pi_2^*\omega_2$.

(3) If (M, ω) is symplectic and $U \subset M$ is open, then $(U, \omega|_U)$ is symplectic.

Proof. (1) Closedness is preserved by pullback since $d(\psi^*\eta) = \psi^*(d\eta) = 0$. Pointwise nondegeneracy is preserved because $(\psi^*\eta)_p$ is the pullback of a nondegenerate bilinear form under the linear isomorphism $d\psi_p : T_pM \to T_{\psi(p)}N$.
(2) Closedness is clear. For nondegeneracy, at a point $(p, q) \in M_1 \times M_2$, the tangent space splits as $T_{(p,q)}(M_1 \times M_2) = T_pM_1 \oplus T_qM_2$, and the form $(\omega_1 \oplus \omega_2)_{(p,q)}$ is the direct sum of two nondegenerate skew forms, hence nondegenerate.
(3) Restriction preserves closedness and pointwise nondegeneracy. □

3.5 First examples

Example. Every open subset $U \subset \mathbb{R}^{2n}$ inherits the standard symplectic form $\omega_0 = \sum_i dx_i \wedge dy_i$. The pair $(U, \omega_0|_U)$ is a symplectic manifold.

Example. Identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ via $z_k = x_k + iy_k$. Then the 2-form

$$\omega_0 = \frac{i}{2}\sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$$

is the same as $\sum_k dx_k \wedge dy_k$ under this identification, hence is symplectic.

Example. If M is a surface, then every 2-form on M is automatically closed. Thus a 2-form ω on M is symplectic if and only if it vanishes nowhere. Equivalently, on a surface the notions of symplectic form and volume form coincide.

Example. On the unit sphere $S^2 \subset \mathbb{R}^3$, define for $p \in S^2$ and $u, v \in T_pS^2$

$$\omega_p(u, v) := \langle p,\ u \times v\rangle.$$

This is a nonvanishing 2-form, hence symplectic on S^2.

3.6 A first global obstruction on closed manifolds

Proposition 3.10. No exact symplectic forms on closed manifolds

Let M^{2n} be a closed manifold. If ω is a symplectic form on M, then ω is not exact. More generally, the cohomology class $[\omega]^n \in H^{2n}(M;\mathbb{R})$ is nonzero.

Proof. Assume $\omega = d\alpha$. Then

$$\omega^{\wedge n} = d(\alpha \wedge \omega^{\wedge(n-1)})$$

is exact. Since M is closed, Stokes' theorem gives

$$\int_M \omega^{\wedge n} = \int_M d(\alpha \wedge \omega^{\wedge(n-1)}) = 0.$$

But $\omega^{\wedge n}$ is a nowhere vanishing top-degree form, so it is a volume form and its integral over M cannot be 0. This contradiction shows ω is not exact.

Equivalently, $\omega^{\wedge n}$ represents $[\omega]^n$ in de Rham cohomology, and a form with nonzero integral over M cannot represent the zero cohomology class. □

Remark. At each $p \in M$, the pair (T_pM, ω_p) is a symplectic vector space, so $\dim M$ must be even and one can choose a symplectic basis of T_pM. This pointwise linear model is the starting point for the local theory developed later.

4 The cotangent bundle and its canonical symplectic structure

Let Q be a smooth n-manifold and let $\pi : T^*Q \to Q$ be the cotangent bundle projection. We write a point of T^*Q as (q,p) where $q \in Q$ and $p \in T_q^*Q$.

4.1 The tautological (Liouville) 1-form

Definition 4.1. Tautological 1-form

The *tautological 1-form* (also called the *Liouville 1-form*) on T^*Q is the 1-form $\lambda \in \Omega^1(T^*Q)$ defined by

$$\lambda_{(q,p)}(v) := p\big(d\pi_{(q,p)}(v)\big), \qquad (q,p) \in T^*Q,\ v \in T_{(q,p)}(T^*Q).$$

Remark. The definition is completely intrinsic: $d\pi$ projects any tangent vector to T^*Q down to a tangent vector on Q, and the covector p evaluates it.

Proposition 4.2. Universal property of λ

Let $\mu : Q \to T^*Q$ be a smooth section of π, equivalently a 1-form $\mu \in \Omega^1(Q)$. Then

$$\mu^*\lambda = \mu.$$

Proof. Fix $q \in Q$ and $X \in T_qQ$. Since $\pi \circ \mu = \mathrm{id}_Q$, we have $d\pi_{\mu(q)} \circ d\mu_q = \mathrm{id}_{T_qQ}$. Therefore

$$(\mu^*\lambda)_q(X) = \lambda_{\mu(q)}(d\mu_q X) = \mu(q)\big(d\pi_{\mu(q)}(d\mu_q X)\big) = \mu(q)(X).$$

□

4.2 The canonical symplectic form

Definition 4.3. Canonical symplectic form

Define
$$\omega_{\mathrm{can}} := -d\lambda \in \Omega^2(T^*Q).$$

Lemma 4.4. Local expression Let $(U, q^1, \ldots, q^n)$ be a coordinate chart on Q and write a covector $p \in T_q^*Q$ uniquely as $p = \sum_{i=1}^n p_i\, dq^i|_q$. Then (q^i, p_i) are local coordinates on T^*U, and in these coordinates
$$\lambda = \sum_{i=1}^n p_i\, dq^i, \qquad \omega_{\mathrm{can}} = \sum_{i=1}^n dq^i \wedge dp_i.$$

Proof. By definition of the fiber coordinates p_i, the covector at (q, p) is $p = \sum p_i\, dq^i|_q$. For $v \in T_{(q,p)}(T^*Q)$, the vector $d\pi(v) \in T_qQ$ is the q-component of v, so
$$\lambda_{(q,p)}(v) = p(d\pi(v)) = \sum_{i=1}^n p_i\, dq^i(d\pi(v)) = \sum_{i=1}^n p_i\, dq^i(v),$$
which gives $\lambda = \sum p_i\, dq^i$. Then
$$\omega_{\mathrm{can}} = -d\lambda = -\sum_{i=1}^n dp_i \wedge dq^i = \sum_{i=1}^n dq^i \wedge dp_i.$$
□

Theorem 4.5. $(T^*Q, \omega_{\mathrm{can}})$ is symplectic
The 2-form ω_{can} is closed and pointwise nondegenerate. Hence $(T^*Q, \omega_{\mathrm{can}})$ is a symplectic manifold. Moreover, ω_{can} is exact.

Proof. Closedness is immediate from $\omega_{\mathrm{can}} = -d\lambda$.
For nondegeneracy, work in local coordinates (q^i, p_i) as in Lemma 4.4. At any point, the tangent space is spanned by $\{\partial_{q^i}, \partial_{p_i}\}$ and
$$\omega_{\mathrm{can}}(\partial_{q^i}, \partial_{p_j}) = \delta_{ij}, \qquad \omega_{\mathrm{can}}(\partial_{q^i}, \partial_{q^j}) = 0, \qquad \omega_{\mathrm{can}}(\partial_{p_i}, \partial_{p_j}) = 0.$$
This is the standard symplectic matrix, hence nondegenerate. □

Example. If $Q = \mathbb{R}^n$ with coordinates $q = (q^1, \ldots, q^n)$, then $T^*Q \cong \mathbb{R}^n \times (\mathbb{R}^n)^*$. Using the induced global coordinates (q^i, p_i), we have
$$\lambda = \sum_{i=1}^n p_i\, dq^i, \qquad \omega_{\mathrm{can}} = \sum_{i=1}^n dq^i \wedge dp_i,$$
so $T^*\mathbb{R}^n$ is the standard symplectic space in canonical coordinates.

4.3 Naturality under diffeomorphisms

Let $f : Q_1 \to Q_2$ be a diffeomorphism. There is a natural induced diffeomorphism of cotangent bundles covering f that preserves the canonical forms.

Definition 4.6. Cotangent lift

Define the *cotangent lift* $f^\sharp : T^*Q_1 \to T^*Q_2$ by
$$f^\sharp(q_1, p_1) = (q_2, p_2) \quad \text{where} \quad q_2 = f(q_1) \text{ and } p_1 = (df_{q_1})^* p_2.$$

Equivalently, on each fiber, $f^\sharp$ uses the inverse of $(df_{q_1})^* : T^*_{q_2}Q_2 \to T^*_{q_1}Q_1$.

Proposition 4.7. Naturality of the tautological form

Let λ_i be the tautological 1-form on T^*Q_i. Then

$$(f^\sharp)^*\lambda_2 = \lambda_1.$$

Proof. Fix $(q_1, p_1) \in T^*Q_1$ and set $(q_2, p_2) = f^\sharp(q_1, p_1)$. Let $v \in T_{(q_1,p_1)}(T^*Q_1)$. Using $\pi_2 \circ f^\sharp = f \circ \pi_1$, we have

$$d\pi_2(df^\sharp(v)) = df(d\pi_1(v)).$$

Therefore

$$((f^\sharp)^*\lambda_2)_{(q_1,p_1)}(v) = \lambda_{2,(q_2,p_2)}(df^\sharp(v)) = p_2(d\pi_2(df^\sharp(v))) = p_2(df(d\pi_1(v))).$$

Since $p_1 = (df_{q_1})^*p_2$, the last expression equals $p_1(d\pi_1(v)) = \lambda_{1,(q_1,p_1)}(v)$. □

Corollary 4.8 (Naturality of the canonical symplectic form)**.** Let $\omega_i = -d\lambda_i$ be the canonical symplectic form on T^*Q_i. Then

$$(f^\sharp)^*\omega_2 = \omega_1.$$

In particular, $f^\sharp : (T^*Q_1, \omega_1) \to (T^*Q_2, \omega_2)$ is a symplectomorphism.

Proof. Using $d \circ (f^\sharp)^* = (f^\sharp)^* \circ d$,

$$(f^\sharp)^*\omega_2 = (f^\sharp)^*(-d\lambda_2) = -d((f^\sharp)^*\lambda_2) = -d\lambda_1 = \omega_1.$$

□

Remark. If $g : Q_2 \to Q_3$ is another diffeomorphism, then $(g \circ f)^\sharp = g^\sharp \circ f^\sharp$. Thus $Q \mapsto (T^*Q, \omega_{\text{can}})$ is functorial with respect to diffeomorphisms.

4.4 Zero section, fibers, and graphs of 1-forms

Write $0_Q : Q \to T^*Q$ for the zero section.

Proposition 4.9. Zero section is Lagrangian

The zero section $0_Q(Q) \subset (T^*Q, \omega_{\text{can}})$ is a Lagrangian submanifold.

Proof. By Proposition 4.2, $(0_Q)^*\lambda = 0$, hence

$$(0_Q)^*\omega_{\text{can}} = (0_Q)^*(-d\lambda) = -d((0_Q)^*\lambda) = 0.$$

Thus $0_Q(Q)$ is isotropic. Since $\dim 0_Q(Q) = \dim Q = n = \frac{1}{2}\dim T^*Q$, it is Lagrangian. □

Proposition 4.10. Fibers are Lagrangian

For each $q \in Q$, the cotangent fiber $T^*_qQ \subset T^*Q$ is a Lagrangian submanifold.

Proof. Let $\iota_q : T^*_qQ \hookrightarrow T^*Q$ be the inclusion. Along the fiber, π is constant, hence $d\pi$ vanishes on tangent vectors to T^*_qQ. By the definition of λ this gives $\iota_q^*\lambda = 0$, hence $\iota_q^*\omega_{\text{can}} = 0$. The fiber has dimension n, so it is Lagrangian. □

Proposition 4.11. Graphs of 1-forms

Let $\mu \in \Omega^1(Q)$ and let $\Gamma_\mu \subset T^*Q$ be its graph, i.e. the image of the section $\mu : Q \to T^*Q$. Then
$$\mu^*\omega_{\text{can}} = -d\mu.$$
In particular, Γ_μ is Lagrangian if and only if $d\mu = 0$.

Proof. By Proposition 4.2, $\mu^*\lambda = \mu$, hence
$$\mu^*\omega_{\text{can}} = \mu^*(-d\lambda) = -d(\mu^*\lambda) = -d\mu.$$
Therefore $\mu^*\omega_{\text{can}} = 0$ if and only if $d\mu = 0$. If $d\mu = 0$, then Γ_μ is isotropic and has dimension n, hence is Lagrangian. □

Example. If $Q = S^1$ with coordinate θ, then $T^*S^1 \cong S^1 \times \mathbb{R}$ with coordinates (θ, p) and
$$\lambda = p\,d\theta, \qquad \omega_{\text{can}} = d\theta \wedge dp.$$
The fibers $\{\theta\} \times \mathbb{R}$ and the zero section $S^1 \times \{0\}$ are Lagrangian.

5 Symplectomorphisms and Symplectic Vector Fields

Let (M, ω) be a symplectic manifold. In this lecture we study diffeomorphisms preserving ω and the infinitesimal objects that generate their flows.

5.1 The symplectomorphism group and isotopies

Definition 5.1. Symplectomorphism and $\mathrm{Symp}(M, \omega)$

A **symplectomorphism** of (M, ω) is a diffeomorphism $\psi \in \mathrm{Diff}(M)$ such that
$$\psi^*\omega = \omega.$$
The set of all symplectomorphisms forms a group under composition, denoted
$$\mathrm{Symp}(M, \omega) := \{\psi \in \mathrm{Diff}(M) \mid \psi^*\omega = \omega\}.$$

To talk about paths in $\mathrm{Symp}(M, \omega)$ we recall the standard dictionary between isotopies and time-dependent vector fields.

Definition 5.2. Isotopy and generating vector field

An **isotopy** of M is a smooth map $\rho\colon M \times \mathbb{R} \to M$ such that each $\rho_t := \rho(\cdot, t)$ is a diffeomorphism and $\rho_0 = \mathrm{id}_M$. Given an isotopy ρ_t, its **generating time-dependent vector field** X_t is defined by
$$\frac{d}{dt}\rho_t = X_t \circ \rho_t, \qquad \text{equivalently} \qquad X_t = \left(\frac{d}{dt}\rho_t\right) \circ \rho_t^{-1}.$$

Remark. When M is compact (or when X_t has compact support), a time-dependent vector field integrates to an isotopy, and the correspondence above is one-to-one. This is the basic ODE fact that lets us pass freely between isotopies and their generators.

5.2 Lie derivative criteria for preserving forms

The fundamental tool is the differentiation formula for pullbacks along an isotopy. We state it in the only case we will really use, namely for a fixed form.

Lemma 5.3. Differentiating pullbacks along an isotopy Let ρ_t be an isotopy with generating vector field X_t. For any fixed differential form $\alpha \in \Omega^k(M)$, we have

$$\frac{d}{dt}(\rho_t^*\alpha) = \rho_t^*(\mathcal{L}_{X_t}\alpha).$$

Proof. This is a standard identity. Intuitively, $\mathcal{L}_{X_t}\alpha$ measures the infinitesimal change of α under the flow of X_t, and pulling back by ρ_t transports that infinitesimal change to time t. $\square$

Proposition 5.4. When does an isotopy preserve ω

Let ρ_t be an isotopy with generating vector field X_t. Then

$$\rho_t \in \mathrm{Symp}(M,\omega) \text{ for all } t \qquad \Longleftrightarrow \qquad \mathcal{L}_{X_t}\omega = 0 \text{ for all } t.$$

Proof. If ρ_t is symplectic for all t, then $\rho_t^*\omega = \omega$ and differentiating gives

$$0 = \frac{d}{dt}(\rho_t^*\omega) = \rho_t^*(\mathcal{L}_{X_t}\omega).$$

Since ρ_t^* is an isomorphism on forms, this forces $\mathcal{L}_{X_t}\omega = 0$. Conversely, if $\mathcal{L}_{X_t}\omega = 0$ for all t, then the lemma gives $\frac{d}{dt}(\rho_t^*\omega) = 0$, hence $\rho_t^*\omega$ is independent of t. At $t = 0$ we have $\rho_0 = \mathrm{id}$, so $\rho_t^*\omega = \omega$ for all t. $\square$

Now we bring in Cartan's formula. Since ω is closed, it makes $\mathcal{L}_X\omega$ particularly transparent.

Lemma 5.5. Symplectic vector fields via Cartan's formula Let X be a vector field on (M,ω). The following are equivalent:

1. $\mathcal{L}_X\omega = 0$.
2. $d(\iota_X\omega) = 0$.

Proof. Cartan's magic formula says

$$\mathcal{L}_X\omega = d(\iota_X\omega) + \iota_X(d\omega).$$

Because ω is symplectic, $d\omega = 0$, so $\mathcal{L}_X\omega = d(\iota_X\omega)$. $\square$

Definition 5.6. Symplectic vector field

A vector field X on (M,ω) is called **symplectic** if

$$\mathcal{L}_X\omega = 0,$$

equivalently if the 1-form $\iota_X\omega$ is closed. We write

$$\mathfrak{X}(M,\omega) := \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X\omega = 0\}.$$

Remark. Nondegeneracy of ω gives a pointwise isomorphism between vector fields and 1-forms

$$\omega^\flat\colon \mathfrak{X}(M) \to \Omega^1(M), \qquad X \mapsto \iota_X\omega.$$

Under this identification, symplectic vector fields correspond exactly to closed 1-forms.

5.3 Hamiltonian vector fields and Hamiltonian isotopies

Symplectic vector fields are the infinitesimal symmetries of ω. Among them, Hamiltonian vector fields are those coming from functions.

Definition 5.7. Hamiltonian vector field

Given a smooth function $H \in C^\infty(M)$, the **Hamiltonian vector field** X_H is the unique vector field determined by

$$\iota_{X_H}\omega = dH.$$

Remark. Some authors use the alternative convention $\iota_{X_H}\omega = -\,dH$. Nothing essential changes, but several later sign choices (such as the Poisson bracket) depend on which convention you pick. We will stick to $\iota_{X_H}\omega = dH$.

Proposition 5.8. Hamiltonian vector fields are symplectic

For any $H \in C^\infty(M)$, the Hamiltonian vector field X_H is symplectic. In particular, its (local) flow consists of symplectomorphisms.

Proof. By definition, $\iota_{X_H}\omega = dH$ is exact, hence closed. Therefore $d(\iota_{X_H}\omega) = 0$, and the previous characterization gives $\mathcal{L}_{X_H}\omega = 0$. □

Example. If $(M, \omega) = (\mathbb{R}^{2n}, \omega_0)$ with

$$\omega_0 = \sum_{j=1}^{n} dq_j \wedge dp_j,$$

then for a Hamiltonian $H(q, p)$ the equation $\iota_{X_H}\omega_0 = dH$ reads

$$X_H = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j} \right).$$

This is the familiar Hamiltonian vector field from classical mechanics.

We now isolate the key distinction between "symplectic" and "Hamiltonian".

Definition 5.9. Hamiltonian isotopy and Hamiltonian symplectomorphism

A symplectic isotopy $\{\psi_t\}_{0\le t\le 1} \subset \mathrm{Symp}(M, \omega)$ with generator X_t is called a **Hamiltonian isotopy** if for every t the 1-form $\iota_{X_t}\omega$ is exact. In that case there exists a smooth function $H\colon [0,1] \times M \to \mathbb{R}$ such that

$$\iota_{X_t}\omega = dH_t, \qquad H_t := H(t, \cdot).$$

A symplectomorphism $\psi \in \mathrm{Symp}(M, \omega)$ is called **Hamiltonian** if it is the time-1 map of some Hamiltonian isotopy. The set of Hamiltonian symplectomorphisms is denoted $\mathrm{Ham}(M, \omega)$.

Remark. If $H(t, p)$ generates a Hamiltonian isotopy, then H is determined by the isotopy only up to adding a function of t alone. Indeed, $d(H_t + c(t)) = dH_t$.

5.4 Cohomological obstruction: symplectic versus Hamiltonian

Because $\omega^\flat$ identifies vector fields with 1-forms, the difference between symplectic and Hamiltonian vector fields is precisely the difference between closed and exact forms.

Theorem 5.10. The basic exact sequence

The map

$$\mathfrak{X}(M,\omega) \longrightarrow H^1_{\mathrm{dR}}(M;\mathbb{R}), \qquad X \longmapsto [\iota_X\omega]$$

is a surjective linear map whose kernel is the space of Hamiltonian vector fields. Equivalently, there is a short exact sequence

$$0 \longrightarrow \{X_H \mid H \in C^\infty(M)\} \longrightarrow \mathfrak{X}(M,\omega) \longrightarrow H^1_{\mathrm{dR}}(M;\mathbb{R}) \longrightarrow 0.$$

Proof. If X is symplectic, then $\iota_X\omega$ is closed, hence defines a class in $H^1_{\mathrm{dR}}(M)$. The kernel consists of those X with $[\iota_X\omega] = 0$, that is, $\iota_X\omega$ exact. Writing $\iota_X\omega = dH$ is exactly the condition that $X = X_H$.

For surjectivity, let $[\alpha] \in H^1_{\mathrm{dR}}(M)$ with α closed. Nondegeneracy of ω gives a unique vector field X with $\iota_X\omega = \alpha$. Then $d(\iota_X\omega) = d\alpha = 0$, so X is symplectic and maps to $[\alpha]$. $\square$

Corollary 5.11 (A convenient sufficient condition)**.** If $H^1_{\mathrm{dR}}(M;\mathbb{R}) = 0$ then every symplectic vector field is Hamiltonian. In particular, on such a manifold every symplectic isotopy is Hamiltonian.

Example. Let $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with angular coordinates (θ_1, θ_2) and symplectic form $\omega = d\theta_1 \wedge d\theta_2$. Consider the constant vector field $X = \frac{\partial}{\partial\theta_1}$. Then

$$\iota_X\omega = \iota_{\partial/\partial\theta_1}(d\theta_1 \wedge d\theta_2) = d\theta_2,$$

which is closed but not exact on $\mathbb{T}^2$. Hence X is symplectic but not Hamiltonian, and its flow is a symplectic isotopy that is not Hamiltonian.

Remark. The previous example is the prototype. "Being symplectic" is a differential condition, while "being Hamiltonian" is the same differential condition plus a global exactness requirement. The class $[\iota_X\omega] \in H^1_{\mathrm{dR}}(M)$ is the first obstruction to finding a Hamiltonian function.

6 Lagrangian submanifolds in $(T^*Q, \omega_{\mathrm{can}})$

Let Q be a smooth n-manifold, with cotangent bundle $\pi : T^*Q \to Q$. We write points as (q,p) with $p \in T^*_qQ$.

Recall from A4 the tautological 1-form $\lambda \in \Omega^1(T^*Q)$ defined by

$$\lambda_{(q,p)}(v) = p\big(d\pi_{(q,p)}(v)\big),$$

and the canonical symplectic form

$$\omega_{\mathrm{can}} := -d\lambda.$$

6.1 Lagrangian submanifolds and a basic identity

Definition 6.1. Lagrangian submanifold

Let (M^{2n}, ω) be symplectic. An n-dimensional submanifold $L \subset M$ is **Lagrangian** if

$$\omega|_L = 0.$$

Equivalently, L is isotropic and has maximal possible dimension n.

In an exact symplectic manifold $(M, \omega = -d\lambda)$, Lagrangians automatically inherit a closed 1-form from λ.

Lemma 6.2. Restriction of the primitive Let $(M, \omega = -d\lambda)$ be exact symplectic and let $L \subset M$ be Lagrangian. Then the 1-form $\lambda|_L$ is closed.

Proof. Since L is Lagrangian, $\omega|_L = 0$. But $\omega = -d\lambda$, so

$$0 = \omega|_L = (-d\lambda)|_L = -\,d(\lambda|_L).$$

Hence $d(\lambda|_L) = 0$. □

This lemma is the conceptual reason why "closed versus exact" appears naturally in cotangent bundles.

6.2 Graphs of 1-forms in T^*Q

Let $\mu \in \Omega^1(Q)$. Regard μ as a smooth section

$$s_\mu : Q \to T^*Q, \qquad s_\mu(q) = (q, \mu_q),$$

and denote its image by

$$\Gamma_\mu := s_\mu(Q) \subset T^*Q,$$

the **graph** of μ.

Proposition 6.3. Pullback of λ along a section

For any $\mu \in \Omega^1(Q)$,

$$s_\mu^*\lambda = \mu.$$

Proof. Fix $q \in Q$ and $X \in T_qQ$. Since $\pi \circ s_\mu = \mathrm{id}_Q$, we have $d\pi_{s_\mu(q)} \circ ds_\mu|_q = \mathrm{id}_{T_qQ}$. Therefore

$$(s_\mu^*\lambda)_q(X) = \lambda_{s_\mu(q)}(ds_\mu|_q(X)) = \mu_q(d\pi_{s_\mu(q)}(ds_\mu|_q(X))) = \mu_q(X).$$

□

Corollary 6.4 (Pullback of ω_{can} along a section)**.** For any $\mu \in \Omega^1(Q)$,

$$s_\mu^*\omega_{\mathrm{can}} = -\,d\mu.$$

Proof. Using $\omega_{\mathrm{can}} = -d\lambda$ and naturality of d,

$$s_\mu^*\omega_{\mathrm{can}} = s_\mu^*(-d\lambda) = -\,d(s_\mu^*\lambda) = -\,d\mu,$$

by Proposition 6.3. □

Theorem 6.5. Graphs and closed 1-forms

The graph $\Gamma_\mu \subset (T^*Q, \omega_{\mathrm{can}})$ is Lagrangian if and only if μ is closed. In particular, if $\mu = df$ then Γ_{df} is Lagrangian.

Proof. The submanifold Γ_μ is diffeomorphic to Q, hence has dimension n. By the previous corollary,

$$\omega_{\mathrm{can}}|_{\Gamma_\mu} = 0 \iff s_\mu^*\omega_{\mathrm{can}} = 0 \iff d\mu = 0.$$

If $\mu = df$ then $d\mu = d^2 f = 0$, so Γ_{df} is Lagrangian. □

Remark. Thus, among Lagrangians that project diffeomorphically to Q, the only data is a closed 1-form. If $H^1_{\mathrm{dR}}(Q) = 0$, every closed 1-form is exact, so every such Lagrangian is of the form Γ_{df} for some function f.

6.3 Exact versus closed for Lagrangians in T^*Q

Definition 6.6. Exact Lagrangian

Let $(M, \omega = -d\lambda)$ be exact symplectic. A Lagrangian embedding $\iota : L \to M$ is called **exact** (with respect to λ) if

$$\iota^*\lambda \text{ is an exact 1-form on } L.$$

A Lagrangian submanifold is called exact if it is the image of an exact Lagrangian embedding.

In T^*Q this specializes cleanly for graphs.

Proposition 6.7. Exactness for graphs

Let $\mu \in \Omega^1(Q)$. The Lagrangian graph Γ_μ is exact (with respect to λ) if and only if μ is exact.

Proof. Since Γ_μ is the image of s_μ, the restriction of λ to Γ_μ pulls back to $s_\mu^*\lambda = \mu$ by Proposition 6.3. Thus $\lambda|_{\Gamma_\mu}$ is exact if and only if μ is exact on Q. □

Remark. This is the first place where topology forces a genuine distinction. Closedness is exactly the Lagrangian condition for a section, while exactness is an additional global requirement that remembers the cohomology class $[\mu] \in H^1_{\mathrm{dR}}(Q)$.

6.4 Basic examples and a minimal test

Example. The zero section $0_Q(Q) \subset T^*Q$ is Γ_0. Since 0 is closed (and exact), the zero section is an exact Lagrangian.

Example. Let $Q = \mathbb{T}^n$ with angular coordinates $\theta_1, \dots, \theta_n$. The 1-form $\mu = \sum_{j=1}^n a_j \, d\theta_j$ is closed for any constants a_j. If some $a_j \neq 0$, then μ is not exact, so Γ_μ is Lagrangian but not exact.

Example. If $Q = \mathbb{R}^n$ and $f(q) = \frac{1}{2}|q|^2$, then $\Gamma_{df} = \{(q,p) \mid p = q\}$ is a Lagrangian submanifold of $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$.

A practical criterion you will use repeatedly is the following.

Proposition 6.8. Section test

Let $L \subset T^*Q$ be an n-dimensional embedded submanifold such that $\pi|_L : L \to Q$ is a diffeomorphism. Then $L = \Gamma_\mu$ for a unique 1-form μ on Q, and L is Lagrangian if and only if μ is closed.

Proof. If $\pi|_L$ is a diffeomorphism, define $s : Q \to T^*Q$ by $s = (\pi|_L)^{-1}$, so $L = s(Q)$. Write $s = s_\mu$ where μ is the corresponding 1-form on Q. Uniqueness of μ is immediate. Then Theorem 6.5 gives the Lagrangian criterion. □

6.5 Intersection with the zero section

For graphs of differentials, intersection points with the zero section are exactly critical points.

Lemma 6.9. Critical points as intersections Let $f \in C^\infty(Q)$. Then

$$\Gamma_{df} \cap 0_Q(Q) = \{(q, 0) \mid df_q = 0\}.$$

Proof. A point $(q,p) \in \Gamma_{df}$ lies in the zero section if and only if $p = 0$. But on Γ_{df} we have $p = df_q$, hence $p = 0$ if and only if $df_q = 0$. $\square$

Remark. This observation is the geometric bridge between Morse functions on Q and intersections of exact Lagrangians in T^*Q. We will return to it later, but the key point here is only that exact graphs come with a canonical potential function.

7 Conormal bundles and a geometric library of Lagrangians

Let Q be a smooth n-manifold, with cotangent bundle $\pi : T^*Q \to Q$. We denote by $\lambda \in \Omega^1(T^*Q)$ the tautological 1-form and by

$$\omega_{\mathrm{can}} := -d\lambda$$

the canonical symplectic form.

7.1 Conormal space and conormal bundle

Definition 7.1. Conormal space

Let $S \subset Q$ be an embedded k-dimensional submanifold and let $x \in S$. The **conormal space** of S at x is

$$N_x^*S := \{\xi \in T_x^*Q \mid \xi(v) = 0 \text{ for all } v \in T_xS\}.$$

Definition 7.2. Conormal bundle

The **conormal bundle** of S is the subset

$$N^*S := \{(x,\xi) \in T^*Q \mid x \in S,\ \xi \in N_x^*S\}.$$

It is a vector bundle over S via the restriction of π.

Remark. Each fiber N_x^*S is the annihilator of $T_xS \subset T_xQ$. In particular, $\dim N_x^*S = n - k$, and N^*S can be identified with the dual of the normal bundle $N_{S/Q}$.

7.2 Local coordinates and smoothness

Lemma 7.3. Local description in adapted coordinates Let $x \in S$. Choose local coordinates $(q^1, \dots, q^n)$ on Q near x adapted to S, meaning that in these coordinates

$$S \cap U = \{q^{k+1} = \cdots = q^n = 0\}.$$

Write the associated cotangent coordinates on T^*U as $(q^1, \dots, q^n, p_1, \dots, p_n)$, so that $\lambda = \sum_{i=1}^n p_i\, dq^i$ and $\omega_{\mathrm{can}} = \sum dq^i \wedge dp_i$. Then inside T^*U the conormal bundle is cut out by

$$q^{k+1} = \cdots = q^n = 0, \qquad p_1 = \cdots = p_k = 0.$$

Proof. At a point $q \in S$ in these coordinates, T_qS is spanned by $\partial/\partial q^1, \dots, \partial/\partial q^k$. A covector $\xi = \sum_{i=1}^n p_i\, dq^i|_q$ annihilates T_qS if and only if $p_1 = \cdots = p_k = 0$. The equations $q^{k+1} = \cdots = q^n = 0$ enforce $q \in S$. $\square$

Corollary 7.4 (Dimension count). The conormal bundle N^*S is an embedded submanifold of T^*Q of dimension n.

Proof. By the local description, N^*S is defined by $(n-k)+k=n$ independent equations inside T^*Q of dimension $2n$, hence has dimension $2n-n=n$. Smoothness follows from the fact that these equations are coordinate equations. $\square$

7.3 The tautological form vanishes on conormals

Proposition 7.5. Vanishing of λ on N^*S

Let $i : N^*S \hookrightarrow T^*Q$ be the inclusion. Then
$$i^*\lambda = 0.$$

Proof. Work in adapted coordinates as in Lemma 7.3. A point of $N^*S \cap T^*U$ satisfies $q^{k+1} = \cdots = q^n = 0$ and $p_1 = \cdots = p_k = 0$. The tangent space of $N^*S \cap T^*U$ is spanned by
$$\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^k}, \quad \frac{\partial}{\partial p_{k+1}}, \dots, \frac{\partial}{\partial p_n}.$$
Since $\lambda = \sum_{i=1}^n p_i\, dq^i$, and on N^*S we have $p_1 = \cdots = p_k = 0$,
$$\lambda|_{N^*S} = \sum_{i=k+1}^{n} p_i\, dq^i.$$
But $dq^{k+1}, \dots, dq^n$ vanish on the spanning vectors above, hence λ vanishes on $T(N^*S)$. Therefore $i^*\lambda = 0$. $\square$

Corollary 7.6 (Conormal bundles are Lagrangian)**.** For any embedded submanifold $S \subset Q$, the conormal bundle $N^*S \subset (T^*Q, \omega_{\mathrm{can}})$ is a Lagrangian submanifold. Moreover it is exact, in the sense that $\lambda|_{N^*S}$ is exact.

Proof. Since $i^*\lambda = 0$, we have
$$i^*\omega_{\mathrm{can}} = i^*(-d\lambda) = -d(i^*\lambda) = 0,$$
so N^*S is isotropic. By Corollary 7.4, $\dim N^*S = n = \frac{1}{2}\dim T^*Q$, so N^*S is Lagrangian. Exactness follows because $\lambda|_{N^*S} = 0$. $\square$

Example. If $S = Q$ then N^*S is the zero section $0_Q(Q)$. If $S = \{q\}$ is a point then $N^*S = T_q^*Q$ is a cotangent fiber.

7.4 Conormals as a way to package constraints

One can read $S \subset Q$ as a constraint on configurations, and N^*S as the covectors that vanish on all allowed velocities tangent to S.

Proposition 7.7. Level-set description

Suppose $S \subset Q$ is given locally as a regular level set
$$S = \{q \in U \mid F(q) = 0\}, \qquad F = (F^1, \dots, F^{n-k}) : U \to \mathbb{R}^{n-k},$$
with dF_q surjective for $q \in S$. Then for each $q \in S$,
$$N_q^*S = \mathrm{span}\{dF_q^1, \dots, dF_q^{n-k}\}.$$
Consequently,
$$N^*S \cap T^*U = \{(q, \xi) \mid q \in S,\ \xi = \sum_{a=1}^{n-k} \lambda_a\, dF_q^a \text{ for some } \lambda_a \in \mathbb{R}\}.$$

Proof. Since $T_qS = \ker(dF_q)$, a covector ξ annihilates T_qS if and only if it lies in $(\ker dF_q)^\circ = \mathrm{im}(dF_q)^*$. Surjectivity of dF_q implies $\mathrm{im}(dF_q)^*$ is spanned by the component differentials dF_q^a. □

Remark. This is the geometric form of the Lagrange multiplier picture. A covector in N_q^*S is a linear combination of differentials of the constraint functions.

7.5 Naturality under diffeomorphisms

Proposition 7.8. Conormals and cotangent lifts

Let $f : Q_1 \to Q_2$ be a diffeomorphism and let $f^\sharp : T^*Q_1 \to T^*Q_2$ be its cotangent lift from A4. For any submanifold $S \subset Q_1$,

$$f^\sharp(N^*S) = N^*(f(S)).$$

Proof. Let $(x, \xi) \in N^*S$ and set $(y, \eta) = f^\sharp(x, \xi)$, so $y = f(x)$ and $\xi = (df_x)^*\eta$. If $w \in T_yf(S)$, then $w = df_x(v)$ for some $v \in T_xS$. Hence

$$\eta(w) = \eta(df_x(v)) = \xi(v) = 0,$$

so $\eta \in N_y^*f(S)$, proving $f^\sharp(N^*S) \subset N^*f(S)$. Applying the same argument to f^{-1} gives the reverse inclusion. □

7.6 A small library of conormal examples

Example. Let $S \subset Q$ be a hypersurface, so $\dim S = n - 1$. Locally $S = \{F = 0\}$ with $dF \neq 0$ along S. Then N_q^*S is a line spanned by dF_q, and

$$N^*S = \{(q, \lambda\, dF_q) \mid q \in S,\ \lambda \in \mathbb{R}\}.$$

Thus N^*S is a conic Lagrangian submanifold of T^*Q.

Example. Let $Q = \mathbb{R}^n$ and let $S = \mathbb{R}^k \times \{0\} \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$. Then in coordinates (q', q'') the conormal is

$$N^*S = \{(q', 0;\ 0, p'')\} \subset T^*\mathbb{R}^n \cong \mathbb{R}^n \times (\mathbb{R}^n)^*,$$

which is Lagrangian and exact since λ vanishes on it.

Example. Let $S \subset Q$ be any embedded submanifold. Then N^*S intersects the zero section exactly along S, because $(x, 0) \in N^*S$ holds precisely when $x \in S$.

8 Generating functions and local symplectomorphisms

Let X_1, X_2 be smooth n-manifolds and set

$$M_1 := T^*X_1, \qquad M_2 := T^*X_2,$$

with tautological 1-forms λ_1, λ_2 and canonical symplectic forms

$$\omega_1 := -d\lambda_1, \qquad \omega_2 := -d\lambda_2.$$

On $M_1 \times M_2$ we consider the twisted 2-form

$$\widetilde{\omega} := \mathrm{pr}_1^*\omega_1 - \mathrm{pr}_2^*\omega_2.$$

8.1 Graphs and the twisted symplectic form

Proposition 8.1. Graphs are Lagrangian in the twisted product

Let $\phi : M_1 \to M_2$ be a diffeomorphism and let

$$\Gamma_\phi := \{(z, \phi(z)) \mid z \in M_1\} \subset M_1 \times M_2$$

be its graph. Then ϕ is a symplectomorphism if and only if Γ_ϕ is a Lagrangian submanifold of $(M_1 \times M_2, \widetilde{\omega})$.

Proof. Let $\gamma : M_1 \to M_1 \times M_2$ be the graph embedding $\gamma(z) = (z, \phi(z))$. Then

$$\gamma^*\widetilde{\omega} = \gamma^*(\mathrm{pr}_1^*\omega_1) - \gamma^*(\mathrm{pr}_2^*\omega_2) = \omega_1 - \phi^*\omega_2.$$

Hence $\gamma^*\widetilde{\omega} = 0$ if and only if $\phi^*\omega_2 = \omega_1$. Since $\dim \Gamma_\phi = \dim M_1 = \frac{1}{2}\dim(M_1 \times M_2)$, the vanishing of $\gamma^*\widetilde{\omega}$ is equivalent to Γ_ϕ being Lagrangian. $\square$

8.2 Generating functions on $X_1 \times X_2$

Use the canonical identification

$$T^*(X_1 \times X_2) \cong T^*X_1 \times T^*X_2 = M_1 \times M_2.$$

Given $f \in C^\infty(X_1 \times X_2)$, the graph $\mathrm{Graph}(df) \subset T^*(X_1 \times X_2)$ is a Lagrangian submanifold for the canonical symplectic form on $T^*(X_1 \times X_2)$. To connect this with $\widetilde{\omega}$, introduce the fibre sign map on M_2,

$$\sigma_2 : T^*X_2 \to T^*X_2, \qquad \sigma_2(y, \eta) = (y, -\eta),$$

and set $\sigma := \mathrm{id}_{M_1} \times \sigma_2$ on $M_1 \times M_2$.

Definition 8.2. Twisted graph of a generating function

For $f \in C^\infty(X_1 \times X_2)$ define

$$Y_f := \mathrm{Graph}(df) \subset M_1 \times M_2, \qquad Y_f^\sigma := \sigma(Y_f).$$

In local coordinates (x, ξ) on T^*X_1 and (y, η) on T^*X_2, the condition $(x, \xi; y, \eta) \in Y_f^\sigma$ is

$$\xi = d_x f(x, y), \qquad \eta = -d_y f(x, y).$$

Proposition 8.3. From f to a local symplectomorphism

Assume that Y_f^σ is the graph of a map $\phi : M_1 \to M_2$ near some point. Then ϕ is a local symplectomorphism. Moreover ϕ is characterized by

$$\phi(x, \xi) = (y, \eta) \quad \iff \quad \xi = d_x f(x, y),\ \eta = -d_y f(x, y).$$

Proof. By construction, Y_f is Lagrangian in $T^*(X_1 \times X_2)$, hence Y_f^σ is Lagrangian for $\widetilde{\omega}$. If $Y_f^\sigma = \Gamma_\phi$ locally, then Γ_ϕ is Lagrangian in $(M_1 \times M_2, \widetilde{\omega})$, so ϕ is symplectic by Proposition 8.1. The equivalence is the defining property of Y_f^σ. $\square$

8.3 The local nondegeneracy condition

The practical issue is whether, given $(x, \xi) \in T^*X_1$, the equation $\xi = d_x f(x, y)$ can be solved for y as a function of (x, ξ).

Proposition 8.4. Implicit-function criterion

Fix local charts $X_1 \supset U_1 \cong \mathbb{R}^n$ and $X_2 \supset U_2 \cong \mathbb{R}^n$. Write $f = f(x,y)$ on $U_1 \times U_2$. If at (x_0, y_0) the mixed Hessian matrix

$$\left(\frac{\partial}{\partial y_j}\frac{\partial f}{\partial x_i}(x_0,y_0)\right)_{i,j=1}^{n}$$

has nonzero determinant, then for (x,ξ) near $(x_0, d_x f(x_0,y_0))$ there exists a unique $y = y(x,\xi)$ near y_0 solving $\xi = d_x f(x,y)$. Defining

$$\eta(x,\xi) := -d_y f(x, y(x,\xi)),$$

we obtain a local symplectomorphism

$$\phi(x,\xi) := (y(x,\xi), \eta(x,\xi)).$$

Proof. Apply the implicit function theorem to the map $F(x,y) = d_x f(x,y)$. The determinant condition is exactly the invertibility of $\partial_y F$ at (x_0,y_0). Then $y = y(x,\xi)$ exists uniquely and depends smoothly on (x,ξ). The rest is substitution into $\eta = -d_y f$ and Proposition 8.3. $\square$

Remark. The determinant condition is local and it does not address global bijectivity. Even if the equations define a smooth local map everywhere, it may fail to be globally one-to-one.

8.4 A model example in $\mathbb{R}^n$

Example. Let $X_1 = X_2 = \mathbb{R}^n$ and set

$$f(x,y) = -\frac{1}{2}|x-y|^2.$$

Then

$$d_x f(x,y) = y - x, \qquad -d_y f(x,y) = y - x.$$

The equations $\xi = d_x f$ and $\eta = -d_y f$ give $y = x + \xi$ and $\eta = \xi$. Hence f generates the symplectomorphism

$$\phi(x,\xi) = (x+\xi, \xi).$$

8.5 Geodesic flow from the distance-squared function

Let (X,g) be a Riemannian manifold and assume it is geodesically convex. This means that any two points $x, y \in X$ are joined by a unique minimizing geodesic.

The metric identifies TX and T^*X by

$$\tilde{g}_x : T_x X \to T_x^* X, \qquad v \mapsto g_x(v, \cdot).$$

Definition 8.5. Distance-squared generating function

Define

$$f : X \times X \to \mathbb{R}, \qquad f(x,y) = -\frac{1}{2} d(x,y)^2,$$

where $d(x,y)$ is the Riemannian distance.

Definition 8.6. Geodesic flow at time 1

For $(x,v) \in TX$, let γ be the unique geodesic with $\gamma(0) = x$ and $\dot\gamma(0) = v$. Define

$$\Phi : TX \to TX, \qquad \Phi(x,v) = (\gamma(1), \dot\gamma(1)).$$

Theorem 8.7. The function $-1/2d^2$ generates the geodesic flow

Assume (X, g) is geodesically convex and use $\widetilde{g}$ to identify TX with T^*X. Then the symplectomorphism of T^*X generated by $f(x, y) = -\frac{1}{2}d(x, y)^2$ corresponds to the map $\Phi : TX \to TX$ given by

$$(x, v) \longmapsto (\gamma(1), \dot{\gamma}(1)),$$

where γ is the geodesic with initial data $\gamma(0) = x$ and $\dot{\gamma}(0) = v$.

Proof. Under the identification $T^*X \cong TX$, the generating equations become

$$\widetilde{g}_x(v) = d_x f(x, y), \qquad \widetilde{g}_y(w) = -d_y f(x, y).$$

In a geodesically convex manifold, the pair (x, v) determines a unique geodesic γ defined on $[0, 1]$. The endpoint $y = \gamma(1)$ is then the unique solution of the first equation, and the second equation gives $w = \dot{\gamma}(1)$. Thus the induced map on TX is exactly Φ. $\square$

Remark. Geodesic convexity is doing real work. It guarantees uniqueness of the relevant geodesic, so that (x, v) determines a single endpoint y and the generating equations define a genuine map.

9 A local toolbox for the local theory

9.1 Isotopies and time-dependent vector fields

Many constructions in symplectic geometry come from pushing forms around by families of diffeomorphisms. To keep the notation clean, we separate the geometric objects from the machine that moves them.

Definition 9.1. Isotopy and its velocity field

Let M be a smooth manifold. An **isotopy** on M is a smooth family of diffeomorphisms

$$\rho_t : M \to M, \qquad t \in [0, 1],$$

such that the map $(t, p) \mapsto \rho_t(p)$ is smooth. Its **time-dependent velocity field** $v_t \in \mathfrak{X}(M)$ is defined by

$$\frac{d}{dt}\rho_t(p) = v_t(\rho_t(p)) \qquad \text{for all } p \in M.$$

Equivalently,

$$v_t = \Big(\frac{d}{dt}\rho_t\Big) \circ \rho_t^{-1}.$$

The point of v_t is that it converts a moving diffeomorphism ρ_t into a moving vector field v_t, which we can feed into differential calculus.

Definition 9.2. Interior product and Lie derivative

Let v be a vector field on M.

1. The **interior product** (contraction) $\iota_v : \Omega^k(M) \to \Omega^{k-1}(M)$ is defined by
$$(\iota_v\alpha)_p(\xi_1, \dots, \xi_{k-1}) := \alpha_p(v_p, \xi_1, \dots, \xi_{k-1}).$$
2. The **Lie derivative** $L_v : \Omega^k(M) \to \Omega^k(M)$ is defined by
$$L_v\alpha := \frac{d}{ds}\Big|_{s=0} \phi_s^*\alpha,$$
where ϕ_s is the (local) flow of v.

Lemma 9.3. Cartan's formula For any vector field v and any differential form α,

$$L_v\alpha = \iota_v(d\alpha) + d(\iota_v\alpha).$$

Proof. By definition, L_v is the derivative of pullback along the flow. One checks (in local coordinates, or by standard properties of pullback) that L_v is a derivation of degree 0 and commutes with d. It is therefore determined by its action on functions and 1-forms. For a function f, one has $L_v f = v(f) = \iota_v(df)$. For a 1-form β, evaluate on a vector field w and expand

$$(L_v\beta)(w) = v(\beta(w)) - \beta([v, w]),$$

then compare with $d(\iota_v\beta)(w)$ and $\iota_v(d\beta)(w)$. The general identity follows by the derivation property. $\square$

Proposition 9.4. Differentiating pullbacks along an isotopy

Let ρ_t be an isotopy on M with velocity field v_t.

1. If α is a fixed differential form on M, then
$$\frac{d}{dt}(\rho_t^*\alpha) = \rho_t^*(L_{v_t}\alpha).$$
2. If α_t is a smooth family of differential forms, then
$$\frac{d}{dt}(\rho_t^*\alpha_t) = \rho_t^*\Big(L_{v_t}\alpha_t + \frac{d}{dt}\alpha_t\Big).$$

Proof. For (1), fix t and write $\rho_{t+h} = \rho_t \circ (\rho_t^{-1} \circ \rho_{t+h})$. The bracketed factor is a diffeomorphism close to the identity whose first-order behavior is governed by v_t, and the definition of L_{v_t} gives the result after dividing by h and letting $h \to 0$. For (2), apply (1) to α_t at each time and add the explicit time-derivative term by the chain rule. $\square$

Remark. The two formulas above are the main reason we insist on the convention $\dot\rho_t = v_t \circ \rho_t$. It makes $\frac{d}{dt}\rho_t^*$ look like a clean operator identity.

9.2 Tubular neighborhoods and fiberwise contraction

A large fraction of local symplectic geometry takes place in neighborhoods of a submanifold. The correct geometric model for such a neighborhood is the normal bundle.

Definition 9.5. Normal bundle

Let $i : X \hookrightarrow M$ be an embedded submanifold. For $x \in X$, view T_xX as a subspace of T_xM via di_x. The **normal space** is the quotient vector space

$$N_xX := T_xM/T_xX.$$

The **normal bundle** is the vector bundle $\pi_0 : NX \to X$ with fiber $(NX)_x = N_xX$. Its zero section is

$$i_0 : X \hookrightarrow NX, \qquad i_0(x) = (x, 0).$$

A neighborhood U_0 of $i_0(X)$ in NX is called **fiberwise convex** if $U_0 \cap (N_xX)$ is convex in each fiber.

Theorem 9.6. Tubular neighborhood theorem

There exist a fiberwise convex neighborhood U_0 of $i_0(X)$ in NX, a neighborhood U of X in M, and a diffeomorphism $\varphi : U_0 \to U$ such that the diagram commutes:

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[dr, "i"'] & U_0 \arrow[d, "\varphi"] \\ & U. \end{tikzcd}$$

One gain from a fiberwise convex neighborhood is that it supports a canonical contraction onto the zero section.

Definition 9.7. Fiberwise contraction and retraction

On a fiberwise convex $U_0 \subset NX$, define

$$\rho_t : U_0 \to U_0, \qquad \rho_t(x, v) = (x, tv), \qquad 0 \le t \le 1.$$

Then $\rho_1 = \mathrm{id}$, $\rho_0 = i_0 \circ \pi_0$, and $\rho_t \circ i_0 = i_0$ for all t. We call $\pi_0 : U_0 \to X$ a **retraction** and $\{\rho_t\}$ a **homotopy fixing** X. Transporting by φ, we obtain a retraction

$$\pi := \pi_0 \circ \varphi^{-1} : U \to X$$

and a homotopy $\varphi \circ \rho_t \circ \varphi^{-1}$ on U fixing X.

9.3 A homotopy operator and the homotopy formula

The previous contraction produces a very concrete chain homotopy on differential forms. This is the form-level version of homotopy invariance in de Rham theory, and it is the technical workhorse behind many local exactness statements.

Definition 9.8. The homotopy operator associated to a homotopy

Let $\rho_t : U \to U$ be a smooth homotopy, $t \in [0, 1]$. Define a time-dependent vector field v_t along ρ_t by the requirement

$$v_t(\rho_t(p)) = \frac{d}{ds}\Big|_{s=t} \rho_s(p).$$

For $\omega \in \Omega^\ell(U)$, define an operator $Q : \Omega^\ell(U) \to \Omega^{\ell-1}(U)$ by

$$Q\omega := \int_0^1 \rho_t^*(\iota_{v_t}\omega)\, dt.$$

Theorem 9.9. Homotopy formula

With Q as above, one has the operator identity

$$Q(d\omega) + d(Q\omega) = \rho_1^*\omega - \rho_0^*\omega \qquad \text{for all } \omega \in \Omega^\ell(U).$$

Proof. Using Cartan's formula, we compute

$$Q(d\omega)+d(Q\omega) = \int_0^1 \rho_t^*(\iota_{v_t} d\omega)\, dt + \int_0^1 d\big(\rho_t^*(\iota_{v_t}\omega)\big)\, dt = \int_0^1 \rho_t^*\big(\iota_{v_t} d\omega + d(\iota_{v_t}\omega)\big)\, dt = \int_0^1 \rho_t^*(L_{v_t}\omega)\, dt.$$

By Proposition 9.4(1), $\frac{d}{dt}(\rho_t^*\omega) = \rho_t^*(L_{v_t}\omega)$. Therefore,

$$Q(d\omega) + d(Q\omega) = \int_0^1 \frac{d}{dt}(\rho_t^*\omega)\, dt = \rho_1^*\omega - \rho_0^*\omega.$$

□

We now specialize to a tubular neighborhood $U_0 \subset NX$ with fiberwise contraction $\rho_t(x,v) = (x,tv)$.

Proposition 9.10. A relative Poincaré lemma on a tubular neighborhood

Let $U_0 \subset NX$ be a fiberwise convex tubular neighborhood of X, with inclusion $i_0 : X \hookrightarrow U_0$. If $\omega \in \Omega^\ell(U_0)$ is closed and satisfies $i_0^*\omega = 0$, then ω is exact. More precisely, if Q is the homotopy operator for $\rho_t(x,v) = (x,tv)$, then

$$\omega = d(Q\omega), \qquad \text{and moreover} \qquad (Q\omega)|_X = 0.$$

Proof. Apply the homotopy formula to ω:

$$d(Q\omega) = \rho_1^*\omega - \rho_0^*\omega - Q(d\omega).$$

Here $\rho_1 = \mathrm{id}$, $d\omega = 0$, and $\rho_0 = i_0 \circ \pi_0$, hence

$$\rho_0^*\omega = \pi_0^*(i_0^*\omega) = 0.$$

So $d(Q\omega) = \omega$.

For the vanishing on X, note that $\rho_t(x,0) = (x,0)$ is constant in t. Hence the associated vector field v_t vanishes along the zero section, so $\iota_{v_t}\omega$ vanishes there, and therefore $(Q\omega)|_X = 0$. ☐

Example. Let V be a real vector space and $U \subset V$ a star-shaped neighborhood of 0. Take the radial homotopy $\rho_t(x) = tx$. Then v_t is the radial vector field along ρ_t, and the operator

$$Q\omega = \int_0^1 \rho_t^*(\iota_{v_t}\omega)\,dt$$

gives an explicit primitive for any closed form ω on U that vanishes at 0. This is the concrete analytic form of the Poincaré lemma on such a domain.

10 Moser's method

Let M be a smooth $2n$-manifold. Throughout this lecture, a **symplectic form** means a closed, nondegenerate 2-form.

10.1 Isotopic and strongly isotopic symplectic forms

Definition 10.1. Isotopy of symplectic forms

Let ω_0, ω_1 be symplectic forms on M.

1. We say ω_0 and ω_1 are **isotopic** if there exists a smooth family ω_t of symplectic forms on M, $t \in [0,1]$, such that ω_t is cohomologous to ω_0 for every t, and $\omega_0 = \omega_{t=0}$, $\omega_1 = \omega_{t=1}$.
2. We say ω_0 and ω_1 are **strongly isotopic** if there exists an isotopy $\psi_t \in \mathrm{Diff}(M)$ with $\psi_0 = \mathrm{id}$ and
$$\psi_t^*\omega_t = \omega_0 \qquad \text{for all } t \in [0,1],$$
for some (hence any) connecting family ω_t with endpoints ω_0, ω_1. Equivalently, $\omega_1 = \psi_{1*}\omega_0$.

On a closed manifold, isotopic and strongly isotopic turn out to coincide. This is the content of Moser's method.

10.2 The Moser equation

Fix a smooth family of symplectic forms ω_t on M, $t \in [0,1]$. Assume that the cohomology class $[\omega_t] \in H^2(M;\mathbb{R})$ is independent of t. Then $\dot\omega_t := \frac{d}{dt}\omega_t$ is exact for each t.

Definition 10.2. A primitive for $\dot\omega_t$

A smooth family of 1-forms $\sigma_t \in \Omega^1(M)$ is called a **primitive for $\dot\omega_t$** if

$$d\sigma_t = \dot\omega_t \qquad \text{for all } t \in [0,1].$$

Remark. The existence of σ_t with smooth dependence on t is not automatic from pointwise exactness, but it holds in all standard situations we use. On a closed manifold one can construct σ_t globally, for instance via a good cover argument or by choosing a Riemannian metric and using Hodge theory to pick a canonical primitive in $\mathrm{im}(d^*)$.

Suppose now that we seek an isotopy ψ_t such that $\psi_t^*\omega_t = \omega_0$. Write the isotopy as the flow of a time-dependent vector field X_t,

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \qquad \psi_0 = \mathrm{id}.$$

Differentiating $\psi_t^*\omega_t$ gives a condition on X_t.

Proposition 10.3. The basic differential identity

Assume $d\omega_t = 0$ for all t. Then for any isotopy ψ_t generated by X_t,

$$\frac{d}{dt}(\psi_t^*\omega_t) = \psi_t^*\Big(\dot\omega_t + d(\iota_{X_t}\omega_t)\Big).$$

Proof. Using the pullback differentiation rule and Cartan's formula,

$$\frac{d}{dt}(\psi_t^*\omega_t) = \psi_t^*\Big(L_{X_t}\omega_t + \dot\omega_t\Big) = \psi_t^*\Big(d(\iota_{X_t}\omega_t) + \iota_{X_t}(d\omega_t) + \dot\omega_t\Big).$$

Since $d\omega_t = 0$, the stated formula follows. □

Therefore, if $d\sigma_t = \dot\omega_t$, the condition

$$\frac{d}{dt}(\psi_t^*\omega_t) = 0$$

will hold if we choose X_t so that

$$\sigma_t + \iota_{X_t}\omega_t = 0.$$

Definition 10.4. The Moser equation

Given ω_t and a choice of σ_t with $d\sigma_t = \dot\omega_t$, the **Moser equation** is

$$\iota_{X_t}\omega_t = -\,\sigma_t.$$

Lemma 10.5. Pointwise solvability For each t, the equation $\iota_{X_t}\omega_t = -\sigma_t$ has a unique solution $X_t \in \mathfrak{X}(M)$. Moreover, if σ_t depends smoothly on t, then so does X_t.

Proof. Nondegeneracy of ω_t means the bundle map

$$TM \to T^*M, \qquad X \mapsto \iota_X\omega_t$$

is an isomorphism, hence X_t exists uniquely and depends smoothly on the right-hand side. □

10.3 Moser stability on a closed manifold

Theorem 10.6. Moser stability theorem

Let M be a closed manifold and ω_t a smooth family of cohomologous symplectic forms on M. Then there exists an isotopy ψ_t of M with $\psi_0 = \mathrm{id}$ such that

$$\psi_t^* \omega_t = \omega_0 \qquad \text{for all } t \in [0,1].$$

In particular, ω_0 and ω_1 are strongly isotopic.

Proof. Choose a smooth family of 1-forms σ_t with $d\sigma_t = \dot{\omega}_t$. Let X_t be the unique solution of the Moser equation $\iota_{X_t}\omega_t = -\sigma_t$. Since M is closed, the time-dependent vector field X_t integrates to a globally defined isotopy ψ_t on $[0,1]$.

By Proposition 10.3,

$$\frac{d}{dt}(\psi_t^*\omega_t) = \psi_t^*\Big(\dot{\omega}_t + d(\iota_{X_t}\omega_t)\Big) = \psi_t^*(d\sigma_t + d(-\sigma_t)) = 0.$$

Hence $\psi_t^*\omega_t$ is independent of t and equals $\psi_0^*\omega_0 = \omega_0$. $\square$

Remark. If M is not closed, the same computation still works, but one must check that the flow of X_t exists on $[0,1]$. A common sufficient condition is that X_t has compact support, uniformly in t.

10.4 A relative version fixing a submanifold

We now prove a form of the method that is local near a compact submanifold. This is the version that will be used repeatedly in local normal form results.

Theorem 10.7. Relative Moser theorem near a compact submanifold

Let M be a manifold and $X \subset M$ a compact embedded submanifold. Let ω_0 and ω_1 be symplectic forms on M such that

$$(\omega_1 - \omega_0)|_p = 0 \qquad \text{for all } p \in X.$$

Then there exist neighborhoods U_0, U_1 of X and a diffeomorphism $\varphi : U_0 \to U_1$ such that

$$\varphi|_X = \mathrm{id}_X, \qquad \varphi^*\omega_1 = \omega_0.$$

Proof. Choose a tubular neighborhood $\varphi_0 : U_0 \to V_0 \subset NX$ of X in M, with V_0 fiberwise convex. Let ρ_t be the fiberwise contraction on V_0, transported to U_0. Let $\tau := \omega_1 - \omega_0$, which is closed on U_0 and satisfies $i^*\tau = 0$ on X.

Define the homotopy operator on U_0 by

$$Q\alpha := \int_0^1 \rho_t^*(\iota_{v_t}\alpha)\, dt,$$

where v_t is the velocity field of ρ_t. The homotopy formula gives

$$Q(d\tau) + d(Q\tau) = \rho_1^*\tau - \rho_0^*\tau.$$

Since $d\tau = 0$, $\rho_1 = \mathrm{id}$, and $\rho_0 = i \circ \pi$ (retraction to X), we obtain

$$\tau = d\mu, \qquad \mu := Q\tau,$$

and moreover $\mu|_X = 0$ because v_t vanishes along X.

Now consider the family of closed 2-forms on U_0,

$$\omega_t := (1-t)\omega_0 + t\omega_1 = \omega_0 + t\, d\mu.$$

By shrinking U_0 if necessary, we may assume ω_t is nondegenerate on U_0 for all $t \in [0,1]$.

Solve the Moser equation on U_0,

$$\iota_{X_t}\omega_t = -\mu.$$

Because $\mu|_X = 0$, the unique solution satisfies $X_t|_X = 0$. Shrink U_0 again if needed so that the flow ψ_t of X_t exists on $[0, 1]$. The same computation as in Theorem 10.6 gives $\psi_t^*\omega_t = \omega_0$ on U_0. Since $X_t|_X = 0$, we have $\psi_t|_X = \mathrm{id}_X$.

Set $\varphi := \psi_1$ and $U_1 := \varphi(U_0)$. Then $\varphi^*\omega_1 = \omega_0$ and φ fixes X pointwise. □

Corollary 10.8 (Fixing a point)**.** Let $p \in M$ and suppose ω_0, ω_1 are symplectic forms with $\omega_0|_p = \omega_1|_p$. Then, after shrinking to neighborhoods of p, there is a diffeomorphism φ with $\varphi(p) = p$ and $\varphi^*\omega_1 = \omega_0$.

Proof. Apply Theorem 10.7 with $X = \{p\}$. □

Remark. A useful strengthening is the following. If $\omega_0 = \omega_1$ on an open neighborhood of X, then one can choose μ to be supported away from X, hence X_t vanishes on a neighborhood of X and the resulting isotopy is the identity there. The mechanism is the same, and one localizes the homotopy operator to the complement of that neighborhood.

11 Darboux theorem and local normal form

11.1 The standard model

On $\mathbb{R}^{2n}$ with linear coordinates

$$(x_1, \dots, x_n, y_1, \dots, y_n),$$

the **standard symplectic form** is

$$\omega_0 := \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Definition 11.1. Darboux chart

Let (M, ω) be a symplectic $2n$-manifold. A coordinate chart

$$(U; x_1, \dots, x_n, y_1, \dots, y_n)$$

is called a **Darboux chart** if $\omega|_U = \sum_{i=1}^{n} dx_i \wedge dy_i$.

11.2 Linear normalization at a point

The starting point is purely linear.

Lemma 11.2. A symplectic basis exists at every point Let (V, Ω) be a $2n$-dimensional symplectic vector space. Then there exists a basis

$$(e_1, \dots, e_n, f_1, \dots, f_n)$$

such that

$$\Omega(e_i, e_j) = 0, \qquad \Omega(f_i, f_j) = 0, \qquad \Omega(e_i, f_j) = \delta_{ij}.$$

Equivalently, in the dual coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ the form is $\Omega = \sum_{i=1}^{n} dx_i \wedge dy_i$.

Proof. This is the standard symplectic Gram–Schmidt procedure. Pick $e_1 \neq 0$; nondegeneracy gives f_1 with $\Omega(e_1, f_1) = 1$. Let $W_1 = \mathrm{span}(e_1, f_1)$ and take the Ω-orthogonal complement W_1^{Ω}. Then $V = W_1 \oplus W_1^{\Omega}$ and $\Omega|_{W_1^{\Omega}}$ is symplectic. Iterate. □

Applying this to (T_pM, ω_p) produces the correct *first-order* normal form.

Proposition 11.3. A chart whose first jet is standard

Let (M, ω) be symplectic and let $p \in M$. Then there exists a coordinate chart

$$(U; x_1^0, \dots, x_n^0, y_1^0, \dots, y_n^0)$$

centered at p such that

$$\omega_p = \sum_{i=1}^n dx_i^0 \wedge dy_i^0 \Big|_p .$$

Proof. Choose a symplectic basis of T_pM as in Lemma 11.2. Let $\ell : T_pM \to \mathbb{R}^{2n}$ be the linear isomorphism sending that basis to the standard one. Pick any chart ϕ centered at p; postcompose with a linear change of coordinates on $\mathbb{R}^{2n}$ so that $d\phi_p = \ell$. Then (x_i^0, y_i^0) are the resulting coordinate functions, and $\phi^*\omega_0$ agrees with ω at p. □

11.3 Darboux theorem via the relative Moser method

Theorem 11.4. Darboux

Let (M, ω) be a symplectic manifold of dimension $2n$, and let $p \in M$. Then there exists a coordinate chart

$$(U; x_1, \dots, x_n, y_1, \dots, y_n)$$

centered at p such that on U,

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i.$$

Proof. Start with the chart $(U_0; x_i^0, y_i^0)$ from Proposition 11.3. On U_0 consider the two symplectic forms

$$\omega_0^{\text{given}} := \omega, \qquad \omega_1^{\text{model}} := \sum_{i=1}^n dx_i^0 \wedge dy_i^0.$$

By construction, these forms agree at the point p:

$$(\omega_1^{\text{model}})_p = (\omega_0^{\text{given}})_p.$$

Apply the relative Moser theorem with $X = \{p\}$ to the pair $\omega_0^{\text{given}}, \omega_1^{\text{model}}$. We obtain neighborhoods $U_0' \subset U_0$ and $U_1' \subset U_0$ of p and a diffeomorphism

$$\varphi : U_0' \to U_1'$$

such that $\varphi(p) = p$ and

$$\varphi^*(\omega_1^{\text{model}}) = \omega_0^{\text{given}} = \omega.$$

Now define new coordinates on U_0' by

$$x_i := x_i^0 \circ \varphi, \qquad y_i := y_i^0 \circ \varphi.$$

Then on U_0',

$$\omega = \varphi^*\left(\sum_{i=1}^n dx_i^0 \wedge dy_i^0\right) = \sum_{i=1}^n d(x_i^0 \circ \varphi) \wedge d(y_i^0 \circ \varphi) = \sum_{i=1}^n dx_i \wedge dy_i.$$

□

11.4 No local curvature-type invariants

Darboux's theorem says that *every* symplectic form has the same local normal form. A clean way to package this is the following statement about germs.

Corollary 11.5 (Local classification)**.** Let (M, ω) be symplectic and $p \in M$. Then there exist neighborhoods $U \ni p$ and $V \ni 0$ and a diffeomorphism $\Phi : U \to V \subset \mathbb{R}^{2n}$ with $\Phi(p) = 0$ such that

$$\Phi^*\omega_0 = \omega|_U.$$

In particular, any local statement that is invariant under symplectomorphisms can be checked on $(\mathbb{R}^{2n}, \omega_0)$.

Proof. This is a rephrasing of Theorem 11.4. □

Remark. A useful contrast is with Riemannian geometry: a metric cannot be made constant to first order *and* second order simultaneously unless curvature vanishes. For symplectic forms there is no analogous obstruction: the germ at a point carries no information beyond the dimension.

11.5 Darboux atlases and symplectic transition maps

Corollary 11.6 (Darboux atlas with symplectic transitions)**.** Let (M, ω) be symplectic of dimension $2n$. There exists an open cover $\{U_\alpha\}$ by Darboux charts

$$\alpha : U_\alpha \to \alpha(U_\alpha) \subset \mathbb{R}^{2n} \quad \text{with} \quad \alpha^*\omega_0 = \omega|_{U_\alpha}.$$

Moreover, on overlaps $U_\alpha \cap U_\beta$ the transition map

$$\beta \circ \alpha^{-1} : \alpha(U_\alpha \cap U_\beta) \to \beta(U_\alpha \cap U_\beta)$$

is a local symplectomorphism of $(\mathbb{R}^{2n}, \omega_0)$, hence its Jacobian matrix satisfies

$$d(\beta \circ \alpha^{-1})(x) \in \mathrm{Sp}(2n) \qquad \text{for all } x \in \alpha(U_\alpha \cap U_\beta).$$

Proof. Existence of the cover is immediate from Darboux's theorem applied pointwise. For the transition map, compute on $\alpha(U_\alpha \cap U_\beta)$:

$$(\beta \circ \alpha^{-1})^*\omega_0 = (\alpha^{-1})^*(\beta^*\omega_0) = (\alpha^{-1})^*(\omega) = \omega_0,$$

so $\beta \circ \alpha^{-1}$ preserves ω_0. Differentiating gives $d(\beta \circ \alpha^{-1})(x) \in \mathrm{Sp}(2n)$. □

12 Weinstein Neighborhood Theorem

12.1 The linear-algebraic input: $N_L \simeq T^*L$

Let (V, Ω) be a symplectic vector space and let $U \subset V$ be a Lagrangian subspace. Because $\Omega|_U = 0$, the bilinear form Ω pairs V/U with U.

Proposition 12.1. A canonical pairing for a Lagrangian subspace

Let (V, Ω) be symplectic and $U \subset V$ Lagrangian. The formula

$$\Omega_0 \colon V/U \times U \to \mathbb{R}, \qquad \Omega_0([v], u) = \Omega(v, u)$$

is well defined and nondegenerate. Consequently there is a canonical isomorphism

$$\widetilde{\Omega}_0 \colon V/U \longrightarrow U^*, \qquad \widetilde{\Omega}_0([v]) = \Omega(v, \cdot)|_U.$$

Proof. If $v' = v + u_0$ with $u_0 \in U$, then $\Omega(v', u) = \Omega(v, u) + \Omega(u_0, u) = \Omega(v, u)$ since $\Omega|_U = 0$. Hence Ω_0 is well defined.
Nondegeneracy in the V/U slot: if $\Omega(v, u) = 0$ for all $u \in U$, then $v \in U^\Omega = U$ because U is Lagrangian. Thus $[v] = 0$ in V/U.
Nondegeneracy in the U slot: if $\Omega(v, u) = 0$ for all $[v] \in V/U$, then in particular $\Omega(w, u) = 0$ for all $w \in V$, hence $u = 0$ by nondegeneracy of Ω. □

Corollary 12.2 (Normal bundle of a Lagrangian submanifold)**.** Let (M^{2n}, ω) be symplectic and $L^n \subset M$ Lagrangian. Then for each $x \in L$ there is a canonical identification of vector spaces

$$N_xL \;=\; T_xM/T_xL \;\cong\; T_x^*L, \qquad [v] \longmapsto \omega(v, \cdot)\big|_{T_xL}.$$

These assemble into a canonical vector-bundle isomorphism $NL \cong T^*L$.

Proof. Apply Proposition 12.1 with $V = T_xM$, $\Omega = \omega_x$, and $U = T_xL$. □

12.2 Tubular neighborhoods and the two symplectic forms

Choose a tubular neighborhood of L in M. Concretely, by the standard tubular neighborhood theorem, there exist neighborhoods U_0 of the zero section in NL and U of L in M, and a diffeomorphism

$$\psi\colon U_0 \longrightarrow U$$

restricting to the identity on the zero section.

Using the canonical bundle identification $NL \cong T^*L$ from Corollary 12.2, we will henceforth regard U_0 as a neighborhood of the zero section $L_0 \subset T^*L$.

There are now *two* symplectic forms on (a neighborhood in) T^*L:

$$\omega_{\mathrm{can}} = -d\lambda \qquad \text{and} \qquad \omega_1 := \psi^*\omega.$$

Both make the zero section L_0 Lagrangian.

The remaining task is to modify ψ by a diffeomorphism of T^*L that fixes L_0 so that the pulled-back form becomes exactly ω_{can}.

12.3 Relative Moser in the form we need

We recall the version of the relative Moser argument that is tailored to a fixed submanifold.

Theorem 12.3. Relative Moser lemma (local form)

Let $X \subset M$ be a compact submanifold. Let ω_0 and ω_1 be symplectic forms defined on a neighborhood of X such that

$$\omega_0|_X = \omega_1|_X.$$

Then, after shrinking to possibly smaller neighborhoods U_0, U_1 of X, there exists a diffeomorphism $\chi\colon U_0 \to U_1$ such that $\chi|_X = \mathrm{id}_X$ and $\chi^*\omega_1 = \omega_0$.

Proof. Consider the path of 2-forms $\omega_t = (1-t)\omega_0 + t\omega_1$ on a small neighborhood of X. For U small enough, ω_t is symplectic for all $t \in [0, 1]$. Let $\tau = \omega_1 - \omega_0$. Since $\tau|_X = 0$ and $d\tau = 0$, a homotopy operator on a tubular neighborhood produces a 1-form σ with $\sigma|_X = 0$ and $d\sigma = \tau$ (one constructs σ by contracting along fibers in a tubular neighborhood). Now solve

$$\iota(X_t)\,\omega_t = -\sigma,$$

which uniquely defines a time-dependent vector field X_t vanishing along X. Let χ_t be its flow, with $\chi_0 = \mathrm{id}$. By Cartan's formula and $d\omega_t = 0$,

$$\frac{d}{dt}\chi_t^*\omega_t = \chi_t^*\Big(\tau + \mathcal{L}_{X_t}\omega_t\Big) = \chi_t^*\Big(\tau + d\iota(X_t)\omega_t\Big) = \chi_t^*(\tau - d\sigma) = 0.$$

Hence $\chi_t^*\omega_t = \omega_0$ for all t, and $\chi_1^*\omega_1 = \omega_0$ as required. $\square$

12.4 Weinstein Lagrangian neighborhood theorem (same ambient manifold)

The next statement is the local uniqueness principle for symplectic forms along a Lagrangian submanifold.

Theorem 12.4. Weinstein Lagrangian neighborhood theorem (uniqueness)

Let M^{2n} be a manifold, let $X^n \subset M$ be a compact submanifold, and let ω_0, ω_1 be symplectic forms on M such that X is Lagrangian for both, namely $i^*\omega_0 = i^*\omega_1 = 0$. Then there exist neighborhoods V_0, V_1 of X in M and a diffeomorphism $\varphi\colon V_0 \to V_1$ such that $\varphi|_X = \mathrm{id}_X$ and $\varphi^*\omega_1 = \omega_0$.

Proof. We describe a structured proof that isolates the two inputs.

Step 1. Linear algebra along X. For each $p \in X$, the subspace $T_pX \subset T_pM$ is Lagrangian for both ω_0 and ω_1. Symplectic linear algebra produces a linear isomorphism

$$L_p\colon T_pM \to T_pM$$

such that $L_p|_{T_pX} = \mathrm{id}$ and $L_p^*(\omega_1|_p) = \omega_0|_p$. One can obtain L_p canonically once a complement W_p of T_pX is fixed (for instance by a Riemannian metric), and the construction varies smoothly with p.

Step 2. Make the equality hold on X to first order. A standard extension theorem for embeddings yields an embedding h of a neighborhood of X into M such that $h|_X = \mathrm{id}_X$ and $dh_p = L_p$. Consequently,

$$(h^*\omega_1)|_X = \omega_0|_X$$

as 2-forms on TX.

Step 3. Upgrade to equality on a neighborhood. Apply Theorem 12.3 to the pair of forms ω_0 and $h^*\omega_1$ along X. We obtain a diffeomorphism f fixing X with $f^*(h^*\omega_1) = \omega_0$. Setting $\varphi = h \circ f$ concludes. $\square$

12.5 Weinstein neighborhood theorem for Lagrangians in (M, ω)

We can now return to the cotangent bundle model.

Theorem 12.5. Lagrangian neighborhood theorem

Let (M, ω) be a symplectic manifold and let $L \subset M$ be a compact Lagrangian submanifold. Then there exist a neighborhood $N(L_0) \subset T^*L$ of the zero section, a neighborhood $V \subset M$ of L, and a diffeomorphism

$$\Phi\colon N(L_0) \longrightarrow V$$

such that $\Phi|_{L_0} = \mathrm{id}_L$ and

$$\Phi^*\omega = \omega_{\mathrm{can}} = -d\lambda.$$

Proof. Choose a tubular neighborhood diffeomorphism $\psi\colon U_0 \to U$ as above, after identifying $U_0 \subset T^*L$. Put $\omega_1 = \psi^*\omega$ on U_0. The zero section L_0 is Lagrangian for both ω_1 and ω_{can}. Apply Theorem 12.4 with $M = U_0$, $X = L_0$, $\omega_0 = \omega_{\mathrm{can}}$ and $\omega_1 = \psi^*\omega$. We obtain neighborhoods $U_0' \subset U_0$ and $U_1' \subset U_0$ of L_0 and a diffeomorphism $\theta\colon U_0' \to U_1'$ such that $\theta|_{L_0} = \mathrm{id}$ and

$$\theta^*\omega_1 = \omega_{\mathrm{can}}.$$

Now set $\Phi = \psi \circ \theta \colon U_0' \to M$. Then

$$\Phi^*\omega = \theta^*(\psi^*\omega) = \theta^*\omega_1 = \omega_{\text{can}},$$

and Φ restricts to the identity on L_0. □

Remark. The compactness assumption is used to keep flows and choices uniform on L. Everything above is local in nature, so one obtains the same conclusion on any relatively compact open subset of a (possibly noncompact) Lagrangian submanifold by restricting attention to a small neighborhood there.

12.6 Two concrete consequences inside T^*L

We record two basic ways in which T^*L organizes Lagrangian geometry.

Proposition 12.6. Lagrangian sections are closed 1-forms

Let $\sigma \in \Omega^1(L)$. Its graph $\Gamma_\sigma \subset T^*L$ is Lagrangian for $\omega_{\text{can}} = -d\lambda$ if and only if $d\sigma = 0$.

Proof. Let $\iota_\sigma \colon L \to T^*L$ be the section $q \mapsto (q, \sigma_q)$, so $\Gamma_\sigma = \iota_\sigma(L)$. Because $\iota_\sigma^*\lambda = \sigma$, we have

$$\iota_\sigma^*\omega_{\text{can}} = -\iota_\sigma^* d\lambda = -d(\iota_\sigma^*\lambda) = -d\sigma.$$

Thus Γ_σ is Lagrangian precisely when $d\sigma = 0$. □

Example. If $f \colon L \to \mathbb{R}$ is smooth, then Γ_{df} is a Lagrangian submanifold of T^*L. It is a section that is exact in the sense that $\lambda|_{\Gamma_{df}} = df$ is exact.

Example. Let $Q \subset L$ be a smooth submanifold. Its conormal bundle

$$N^*Q = \{(q,p) \in T^*L \mid q \in Q,\ p|_{T_qQ} = 0\}$$

is a Lagrangian submanifold of $(T^*L, \omega_{\text{can}})$.

Remark. Theorem 12.5 says that every compact Lagrangian $L \subset (M, \omega)$ has a neighborhood that looks, in a symplectic sense, like a neighborhood of the zero section in T^*L. In particular, local questions about Lagrangians can often be moved into a cotangent bundle where graphs of closed 1-forms and conormals provide a large supply of geometric models.

13 Almost complex structures, tamed and compatible

13.1 Almost complex structures and the induced orientation

Definition 13.1. Almost complex structures

Let M be a smooth manifold. An **almost complex structure** on M is a smooth bundle map

$$J : TM \to TM$$

such that $J^2 = -\text{id}$. The pair (M, J) is called an **almost complex manifold**.

At each point $p \in M$, the condition $J_p^2 = -\text{id}$ forces $\dim M$ to be even, say $\dim M = 2n$. There is also a canonical orientation.

Proposition 13.2. Orientation from J

Let (M, J) be an almost complex manifold of dimension $2n$. For each $p \in M$ and each ordered basis $(v_1, \ldots, v_n)$ of T_pM, the ordered $2n$-tuple

$$(v_1, Jv_1, \ldots, v_n, Jv_n)$$

is a positively oriented basis. This defines a canonical orientation on M.

Proof. Fix p. Any two real bases $(v_1, \ldots, v_n)$ and $(v_1', \ldots, v_n')$ are related by some $A \in \mathrm{GL}(n, \mathbb{R})$ with $v_j' = \sum_i A_{ij} v_i$. Then

$$(v_1', Jv_1', \ldots, v_n', Jv_n') = (v_1, Jv_1, \ldots, v_n, Jv_n) \cdot \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix},$$

so the determinant changes by $(\det A)^2 > 0$. Hence the sign is well-defined and yields an orientation. $\square$

13.2 Complexification and the $(1,0)/(0,1)$ splitting

The main linear-algebraic feature of an almost complex structure is that it diagonalizes after complexification. Write $TM \otimes \mathbb{C}$ for the complexified tangent bundle. Extend J $\mathbb{C}$-linearly: $J(v \otimes c) = Jv \otimes c$.

Definition 13.3. The $(1,0)$ and $(0,1)$ subbundles

Let (M, J) be an almost complex manifold. Define subbundles of $TM \otimes \mathbb{C}$ by

$$T^{1,0}M := \{Z \in TM \otimes \mathbb{C} \mid JZ = iZ\}, \qquad T^{0,1}M := \{Z \in TM \otimes \mathbb{C} \mid JZ = -iZ\}.$$

These are the $\pm i$ eigenspaces of J in each fiber, and they split the complexified tangent bundle.

Proposition 13.4. Projections

Define $\mathbb{C}$-linear maps on $TM \otimes \mathbb{C}$ by

$$\pi^{1,0} := \frac{1}{2}(\mathrm{id} - iJ), \qquad \pi^{0,1} := \frac{1}{2}(\mathrm{id} + iJ).$$

Then $\pi^{1,0}$ and $\pi^{0,1}$ are complementary projections with images $T^{1,0}M$ and $T^{0,1}M$, and

$$TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M.$$

Moreover, if X is a real vector field then

$$\pi^{1,0}(X) = \frac{1}{2}(X - iJX), \qquad \pi^{0,1}(X) = \frac{1}{2}(X + iJX).$$

Proof. A direct computation using $J^2 = -\mathrm{id}$ shows $(\pi^{1,0})^2 = \pi^{1,0}$, $(\pi^{0,1})^2 = \pi^{0,1}$, $\pi^{1,0}\pi^{0,1} = 0$, and $\pi^{1,0} + \pi^{0,1} = \mathrm{id}$. The eigenvalue conditions follow from

$$J\pi^{1,0} = \frac{1}{2}(J - iJ^2) = \frac{1}{2}(J + i) = i\pi^{1,0}, \qquad J\pi^{0,1} = \frac{1}{2}(J + iJ^2) = \frac{1}{2}(J - i) = -i\pi^{0,1}.$$

$\square$

13.3 Type decomposition for forms

The cotangent bundle also splits after complexification. For a real 1-form α, regard it as complex-valued and set

$$\alpha^{1,0} := \frac{1}{2}(\alpha - i\,\alpha \circ J), \qquad \alpha^{0,1} := \frac{1}{2}(\alpha + i\,\alpha \circ J).$$

Then $\alpha = \alpha^{1,0} + \alpha^{0,1}$ and $\alpha^{1,0}$ is complex-linear while $\alpha^{0,1}$ is complex-antilinear (with respect to J).

Definition 13.5. Forms of type (p, q)

Let (M, J) be an almost complex manifold. The complexified cotangent bundle splits as

$$T^*M \otimes \mathbb{C} = (T^{1,0}M)^* \oplus (T^{0,1}M)^*.$$

For each $k \geq 0$ and each $p + q = k$, define

$$\Lambda^{p,q}T^*M := \Lambda^p(T^{1,0}M)^* \wedge \Lambda^q(T^{0,1}M)^*,$$

and let $\Omega^{p,q}(M)$ be the space of smooth sections of $\Lambda^{p,q}T^*M$. Then

$$\Omega^k(M; \mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}(M).$$

Remark. This splitting is purely pointwise linear algebra and does not require any integrability condition on J.

13.4 Tamed and compatible almost complex structures

Now return to symplectic geometry. Let (M, ω) be a symplectic manifold. An almost complex structure can interact with ω in two related ways.

Definition 13.6. Tamed and compatible

Let ω be a nondegenerate 2-form on M and let J be an almost complex structure.

1. J is **ω-tame** if
$$\omega(v, Jv) > 0 \qquad \text{for all } v \in TM \setminus \{0\}.$$
2. J is **ω-compatible** if it is ω-tame and, in addition,
$$\omega(Jv, Jw) = \omega(v, w) \qquad \text{for all } v, w \in T_qM,\ q \in M.$$

Compatibility is the condition that J acts by symplectic isometries fiberwise, and tameness is a positivity condition. Both appear constantly because they convert (M, ω) into a setting where one has a preferred class of Riemannian metrics.

Proposition 13.7. The induced metric

Let (M, ω) be a symplectic manifold and let J be an almost complex structure. Define a bilinear form

$$g_J(v, w) := \omega(v, Jw).$$

Then J is ω-compatible if and only if g_J is a Riemannian metric on M. In that case J is g_J-orthogonal, i.e. $g_J(Jv, Jw) = g_J(v, w)$.

Proof. Assume J is ω-compatible. First, positivity is immediate: $g_J(v, v) = \omega(v, Jv) > 0$ for $v \neq 0$.

Next, symmetry. Using skew-symmetry of ω,

$$g_J(w, v) = \omega(w, Jv) = -\omega(Jv, w).$$

Write $w = J(-Jw)$ and use $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$:

$$\omega(Jv, w) = \omega(Jv, J(-Jw)) = \omega(v, -Jw) = -\omega(v, Jw) = -g_J(v, w).$$

Hence $g_J(w, v) = g_J(v, w)$.

Finally,

$$g_J(Jv, Jw) = \omega(Jv, JJw) = \omega(Jv, -w) = -\omega(Jv, w) = \omega(v, Jw) = g_J(v, w),$$

where we used the already-established identity $\omega(Jv, w) = -\omega(v, Jw)$.
Conversely, suppose $g_J(v, w) = \omega(v, Jw)$ is a Riemannian metric. Then $g_J(v, v) > 0$ gives tameness, and symmetry implies $\omega(Jv, w) = -\omega(v, Jw)$. Now compute

$$\omega(Jv, Jw) = g_J(Jv, w) = g_J(v, J^{-1}w) = g_J(v, -Jw) = -g_J(v, Jw) = \omega(v, w),$$

so J is ω-compatible. □

13.5 Compatible triples and changing viewpoints

It is useful to package the previous discussion into a single notion.

Definition 13.8. Compatible triple

A **compatible triple** on M is a triple (ω, J, g) consisting of a nondegenerate 2-form ω, an almost complex structure J, and a Riemannian metric g, such that

$$g(v, w) = \omega(v, Jw) \qquad \text{for all } v, w \in TM.$$

Equivalently, $\omega(\cdot, J\cdot)$ is symmetric positive definite and J is ω-compatible.

Two easy but important consequences are worth keeping explicit.

Proposition 13.9. Fixing one piece determines the others

1. Fix (ω, J) with J ω-compatible. Then $g = \omega(\cdot, J\cdot)$ is the unique metric making (ω, J, g) a compatible triple.
2. Fix (g, J) with J g-orthogonal (i.e. $g(J\cdot, J\cdot) = g$). Then

$$\omega(v, w) := g(Jv, w)$$

is a nondegenerate skew-symmetric 2-form, and (ω, J, g) is a compatible triple.

Proof. (1) is immediate from the definition. For (2), skew-symmetry follows from $g(Jv, w) = -g(v, Jw)$, which is equivalent to J being g-orthogonal. Nondegeneracy is immediate from nondegeneracy of g and $J^2 = -\mathrm{id}$. Finally, $g(v, w) = \omega(v, Jw) = g(Jv, Jw) = g(v, w)$. □

13.6 Existence on symplectic vector spaces and manifolds

The compatibility condition is not restrictive: it is always achievable.

Lemma 13.10. Compatible complex structures on symplectic vector spaces Let (V, Ω) be a symplectic vector space. Then there exists a complex structure J on V such that $\Omega(\cdot, J\cdot)$ is a positive inner product. Equivalently, J is Ω-compatible.

Proof. Choose any inner product $\langle\cdot,\cdot\rangle$ on V. Define $A : V \to V$ by $\langle u, Av\rangle = \Omega(u, v)$. Then A is $\langle\cdot,\cdot\rangle$-skew-adjoint. Take the polar decomposition $A = BJ$ where B is positive definite symmetric and J is $\langle\cdot,\cdot\rangle$-orthogonal. One checks that $J^2 = -\mathrm{id}$, $\Omega(Ju, Jv) = \Omega(u, v)$, and that

$$g(u, v) := \Omega(u, Jv) = \langle Bu, v\rangle$$

is a positive definite inner product. □

Proposition 13.11. Nonemptiness and flexibility

Let (M, ω) be a symplectic manifold. Then the spaces of ω-tame and of ω-compatible almost complex structures are both nonempty. Moreover, these spaces are contractible.

Proof. At each $p \in M$, apply the previous lemma to the symplectic vector space (T_pM, ω_p) to obtain a compatible complex structure J_p. Choosing an auxiliary Riemannian metric and using a partition of unity, one can arrange smooth dependence on p, producing a global ω-compatible almost complex structure on TM.
Contractibility is fiberwise linear algebra: the space of compatible complex structures on a fixed symplectic vector space is contractible, and the same holds after taking smooth sections. □

Example. On $(\mathbb{R}^{2n}, \omega_0 = \sum_{j=1}^n dx_j \wedge dy_j)$, the standard matrix

$$J_0\Big(\frac{\partial}{\partial x_j}\Big) = \frac{\partial}{\partial y_j}, \qquad J_0\Big(\frac{\partial}{\partial y_j}\Big) = -\frac{\partial}{\partial x_j}$$

is ω_0-compatible. The associated metric is the Euclidean metric $g_0 = \sum_j (dx_j^2 + dy_j^2)$.

Example. If (M, J) is a complex manifold with a Kähler form ω, then J is ω-compatible and $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$ is the associated Kähler metric.

14 A minimal slice of Kähler geometry

14.1 Complex manifolds and integrable almost complex structures

Definition 14.1. Complex manifold

A **complex manifold** of complex dimension n is a smooth manifold M^{2n} together with an atlas $\{(U_\alpha, \phi_\alpha)\}$ where $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ are homeomorphisms onto open sets, and the transition maps $\phi_\beta \circ \phi_\alpha^{-1}$ are holomorphic.

A complex manifold carries an almost complex structure J given in local holomorphic coordinates $z_j = x_j + iy_j$ by multiplication by i on the tangent spaces. An almost complex structure is called **integrable** if it comes from such holomorphic charts. In this lecture we only use the consequences of integrability, namely that the (p, q)-type decomposition behaves well in holomorphic coordinates.

14.2 Kähler forms and Kähler manifolds

Let (M, J) be a complex manifold. Recall from A13 that J gives a decomposition

$$\Omega^2(M; \mathbb{C}) = \Omega^{2,0}(M) \oplus \Omega^{1,1}(M) \oplus \Omega^{0,2}(M).$$

Definition 14.2. Kähler form

Let (M, J) be a complex manifold. A real 2-form $\omega \in \Omega^2(M)$ is a **Kähler form** if

1. ω is of type $(1, 1)$, equivalently $\omega \in \Omega^{1,1}(M) \cap \Omega^2(M)$,
2. ω is closed, $d\omega = 0$,
3. ω is positive in the sense that

$$\omega(v, Jv) > 0 \qquad \text{for all } v \in TM \setminus \{0\}.$$

A **Kähler manifold** is a complex manifold (M, J) equipped with a Kähler form ω.

From the symplectic perspective, a Kähler form is automatically a symplectic form and it is compatible with the complex structure.

Proposition 14.3. Kähler implies symplectic and compatible

Let (M, J, ω) be Kähler. Then ω is symplectic and J is ω-compatible. Moreover the bilinear form

$$g(v, w) := \omega(v, Jw)$$

is a Riemannian metric and (ω, J, g) is a compatible triple.

Proof. Closedness is part of the definition. Positivity gives $\omega(v, Jv) > 0$ for $v \neq 0$, hence ω is nondegenerate. Since ω is a real $(1,1)$-form, one has $J^*\omega = \omega$, which implies that $\omega(\cdot, J\cdot)$ is symmetric. Thus g is symmetric positive definite. The equivalence between compatibility and the metric condition was established in A13. □

14.3 Local expressions

Fix a holomorphic chart $(U; z_1, \dots, z_n)$, with $z_j = x_j + iy_j$. Every real $(1,1)$-form on U can be written as

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{jk}\, dz_j \wedge d\bar{z}_k,$$

with (h_{jk}) a Hermitian matrix of smooth functions.

Proposition 14.4. Local matrix description

A real $(1,1)$-form ω on a complex chart is Kähler if and only if it is closed and the Hermitian matrix $(h_{jk}(p))$ is positive definite for every point p in the chart.

Proof. The positivity condition $\omega(v, Jv) > 0$ is equivalent to positivity of $(h_{jk}(p))$ in holomorphic coordinates. Nondegeneracy follows from positivity. Closedness is independent of the coordinates. □

14.4 Kähler potentials

A useful feature of Kähler geometry is that, locally, a Kähler form comes from a single real function.

Definition 14.5. Strictly plurisubharmonic function

Let (M, J) be a complex manifold. A smooth real function $\rho \in C^\infty(M; \mathbb{R})$ is **strictly plurisubharmonic** if, in every holomorphic chart $(U; z_1, \dots, z_n)$, the Hermitian matrix

$$\left(\frac{\partial^2 \rho}{\partial z_j \, \partial \bar{z}_k} \right)$$

is positive definite at every point of U.

Proposition 14.6. A recipe for Kähler forms

If ρ is strictly plurisubharmonic, then
$$\omega_\rho := \frac{i}{2}\,\partial\bar\partial\rho$$
is a Kähler form.

Proof. In holomorphic coordinates,
$$\omega_\rho = \frac{i}{2}\sum_{j,k}\frac{\partial^2\rho}{\partial z_j\,\partial\bar z_k}\,dz_j\wedge d\bar z_k,$$
so ω_ρ is a real $(1,1)$-form and it is positive by strict plurisubharmonicity. Also $d = \partial + \bar\partial$ and $\partial^2 = \bar\partial^2 = 0$ imply $d\omega_\rho = 0$. □

Theorem 14.7. Local Kähler potential

Let ω be a closed real $(1,1)$-form on a complex manifold (M, J) and let $p \in M$. Then there exists a neighborhood U of p and a smooth real function $\rho \in C^\infty(U;\mathbb{R})$ such that
$$\omega|_U = \frac{i}{2}\,\partial\bar\partial\rho.$$

Proof. This is a holomorphic refinement of the Poincaré lemma, using local exactness for $\bar\partial$ and for the holomorphic de Rham complex. The point for us is the outcome, namely that closed real $(1,1)$-forms are locally $\partial\bar\partial$-exact. □

Remark. From the symplectic point of view, Darboux gives a normal form for ω but it does not express ω as d of a universal primitive. In the Kähler setting, Theorem 14.7 gives a scalar potential on complex charts.

14.5 Basic examples

Example. On $\mathbb{C}^n$ with $z_j = x_j + iy_j$, take $\rho(z) = \sum_{j=1}^n |z_j|^2$. Then
$$\omega_\rho = \frac{i}{2}\sum_{j=1}^n dz_j\wedge d\bar z_j = \sum_{j=1}^n dx_j\wedge dy_j = \omega_0.$$
Thus the standard symplectic form is Kähler, with the standard complex structure and Euclidean metric.

Example. Complex projective space $\mathbb{CP}^n$ carries the Fubini–Study form ω_{FS}. On the coordinate chart $U_j = \{z_j \neq 0\}$ with inhomogeneous coordinates, one has
$$\omega_{FS} = \frac{i}{2}\,\partial\bar\partial f_j, \qquad f_j = \log\left(\frac{\sum_{\nu=0}^n |z_\nu|^2}{|z_j|^2}\right).$$
In particular, ω_{FS} is closed, nondegenerate, and compatible with the standard complex structure on $\mathbb{CP}^n$.

Proposition 14.8. Complex submanifolds of Kähler manifolds

Let (M, J, ω) be Kähler and let $X \subset M$ be a complex submanifold. Then X is Kähler with respect to the restricted complex structure and the restricted form $\omega|_X$.

Proof. The restriction $\omega|_X$ is closed and of type $(1,1)$. Positivity is preserved because TX is a J-invariant subbundle and $\omega(v, Jv) > 0$ for $v \neq 0$. □

14.6 A symplectic consequence in a fixed Kähler class

The next statement connects directly to the Moser method from A10. The point is that convex combinations of Kähler forms stay symplectic for a transparent reason in holomorphic coordinates.

Theorem 14.9. Cohomologous Kähler forms are symplectomorphic

Let M be a compact complex manifold. Let ω_0 and ω_1 be Kähler forms on M. If $[\omega_0] = [\omega_1] \in H^2_{\mathrm{dR}}(M)$, then there exists a diffeomorphism φ of M such that

$$\varphi^*\omega_1 = \omega_0.$$

Proof. Set $\omega_t = (1-t)\omega_0 + t\omega_1$. Since ω_0 and ω_1 are Kähler, in any holomorphic chart we can write

$$\omega_0 = \frac{i}{2}\sum h^0_{jk}\, dz_j \wedge d\bar{z}_k, \qquad \omega_1 = \frac{i}{2}\sum h^1_{jk}\, dz_j \wedge d\bar{z}_k$$

with (h^0_{jk}) and (h^1_{jk}) Hermitian positive definite. Then

$$\omega_t = \frac{i}{2}\sum\big((1-t)h^0_{jk} + t h^1_{jk}\big)\, dz_j \wedge d\bar{z}_k$$

is still positive definite for all $t \in [0,1]$, hence nondegenerate. Closedness is automatic and therefore each ω_t is symplectic.

Since $[\omega_0] = [\omega_1]$, the difference $\omega_1 - \omega_0$ is exact. Apply the Moser method to the path ω_t to obtain an isotopy φ_t with $\varphi_t^*\omega_t = \omega_0$. At $t = 1$ this gives $\varphi_1^*\omega_1 = \omega_0$. □

15 Hamiltonian systems and the Poisson bracket

15.1 Hamiltonian vector fields

Let (M, ω) be a symplectic manifold.

Definition 15.1. Hamiltonian vector field

Given a smooth function $H \in C^\infty(M)$, the **Hamiltonian vector field** X_H is the unique vector field determined by

$$\iota_{X_H}\omega = dH.$$

The triple (M, ω, H) is called a **Hamiltonian system**, and the differential equation

$$\dot{x}(t) = X_H(x(t))$$

is the associated Hamiltonian dynamics.

Proposition 15.2. Existence and uniqueness

For every $H \in C^\infty(M)$ there exists a unique Hamiltonian vector field X_H.

Proof. At each point $p \in M$, the map $T_pM \to T_p^*M$, $v \mapsto \iota_v\omega_p = \omega_p(v, \cdot)$ is an isomorphism because ω_p is nondegenerate. Thus there is a unique $X_H(p)$ such that $\omega_p(X_H(p), \cdot) = (dH)_p(\cdot)$. Smoothness follows from smooth dependence on p. □

15.2 Hamiltonian flows are symplectic

Let φ^t_H denote the (local) flow of X_H.

Proposition 15.3. Hamiltonian flows preserve ω

Wherever defined, the flow φ_H^t consists of symplectomorphisms:

$$(\varphi_H^t)^*\omega = \omega.$$

Proof. By Cartan's formula,

$$\mathcal{L}_{X_H}\omega = d(\iota_{X_H}\omega) + \iota_{X_H}(d\omega) = d(dH) + \iota_{X_H}(0) = 0.$$

Therefore,

$$\frac{d}{dt}(\varphi_H^t)^*\omega = (\varphi_H^t)^*(\mathcal{L}_{X_H}\omega) = 0,$$

so $(\varphi_H^t)^*\omega$ is independent of t, and equals ω at $t = 0$. □

Proposition 15.4. Energy is conserved

Along the Hamiltonian flow, the function H is constant:

$$\frac{d}{dt}(H(\varphi_H^t(x))) = 0.$$

Equivalently, $X_H \cdot H = 0$.

Proof. We compute

$$X_H \cdot H = dH(X_H) = \omega(X_H, X_H) = 0,$$

since ω is skew-symmetric. □

15.3 The Poisson bracket

Definition 15.5. Poisson bracket

For $F, G \in C^\infty(M)$, the **Poisson bracket** is

$$\{F, G\} := \omega(X_F, X_G).$$

This definition packages several identities you should feel free to use without re-deriving each time.

Proposition 15.6. Basic identities

For all $F, G \in C^\infty(M)$ we have

$$\{F, G\} = dF(X_G) = -\,dG(X_F) = X_G \cdot F = -\,X_F \cdot G.$$

In particular, $\{F, G\} = -\{G, F\}$.

Proof. By definition $\iota_{X_F}\omega = dF$, so

$$\omega(X_F, X_G) = dF(X_G).$$

Skew-symmetry of ω gives $\omega(X_F, X_G) = -\omega(X_G, X_F) = -dG(X_F)$. The remaining equalities are just the interpretation of a vector field as a derivation on functions. □

Proposition 15.7. Leibniz rule

For all $F, G, H \in C^\infty(M)$,

$$\{F, GH\} = \{F, G\}\,H + G\,\{F, H\}.$$

Proof. Using $\{F, \cdot\} = -X_F(\cdot)$ from the previous proposition,

$$\{F, GH\} = -X_F(GH) = -(X_F G)\,H - G\,(X_F H) = \{F, G\}H + G\{F, H\}.$$

∎

Proposition 15.8. Constants of motion

A function $F \in C^\infty(M)$ is constant along the Hamiltonian trajectories of H if and only if
$$\{F, H\} = 0.$$

Proof. Along a trajectory $t \mapsto \varphi_H^t(x)$,
$$\frac{d}{dt}(F(\varphi_H^t(x))) = (X_H \cdot F)(\varphi_H^t(x)) = \{F, H\}(\varphi_H^t(x)).$$
∎

15.4 Lie algebra structure and Jacobi identity

The Poisson bracket is a Lie bracket on $C^\infty(M)$, and the reason is that Hamiltonian vector fields interact well with the Lie bracket of vector fields.

Proposition 15.9. Commutator of Hamiltonian vector fields

For all $F, G \in C^\infty(M)$,
$$[X_F, X_G] = -X_{\{F,G\}}.$$
In particular, $[X_F, X_G]$ is Hamiltonian.

Proof. We compute using Cartan's formula and $\mathcal{L}_{X_F}\omega = 0$:
$$\iota_{[X_F,X_G]}\omega = \mathcal{L}_{X_F}(\iota_{X_G}\omega) - \iota_{X_G}(\mathcal{L}_{X_F}\omega) = \mathcal{L}_{X_F}(dG) = d(\mathcal{L}_{X_F}G) = d(X_F \cdot G).$$
But $X_F \cdot G = dG(X_F) = -\{F, G\}$, hence
$$\iota_{[X_F,X_G]}\omega = d(-\{F, G\}) = \iota_{-X_{\{F,G\}}}\omega,$$
and nondegeneracy of ω implies $[X_F, X_G] = -X_{\{F,G\}}$. ∎

Theorem 15.10. Jacobi identity for $\{\cdot, \cdot\}$

For all $F, G, H \in C^\infty(M)$,
$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0.$$
Thus $(C^\infty(M), \{\cdot, \cdot\})$ is a Lie algebra, and together with the Leibniz rule it is a Poisson algebra.

Proof. Apply the Jacobi identity for the Lie bracket of vector fields:
$$[X_F, [X_G, X_H]] + [X_G, [X_H, X_F]] + [X_H, [X_F, X_G]] = 0.$$
Using $[X_G, X_H] = -X_{\{G,H\}}$ and linearity of $F \mapsto X_F$, we get
$$0 = -[X_F, X_{\{G,H\}}] - [X_G, X_{\{H,F\}}] - [X_H, X_{\{F,G\}}] = X_{\{F,\{G,H\}\}} + X_{\{G,\{H,F\}\}} + X_{\{H,\{F,G\}\}}.$$
Nondegeneracy implies the Hamiltonian function is constant. Evaluating at any point and noting the expression is skew-symmetric in F, G, H, the constant must be 0, hence the Jacobi identity holds. ∎

15.5 Behavior under symplectomorphisms

Proposition 15.11. Conjugation of Hamiltonian vector fields

Let $\psi \in \mathrm{Symp}(M, \omega)$ and let $H \in C^\infty(M)$. Then

$$X_{H\circ\psi} = (\psi^{-1})_* X_H \circ \psi.$$

Equivalently, $\psi_* X_{H\circ\psi} = X_H \circ \psi$.

Proof. For $p \in M$ and $v \in T_pM$,

$$d(H\circ\psi)_p(v) = dH_{\psi(p)}(d\psi_p v) = \omega_{\psi(p)}(X_H(\psi(p)), d\psi_p v).$$

Since $\psi^*\omega = \omega$, we have $\omega_{\psi(p)}(a, d\psi_p v) = \omega_p(d\psi_p^{-1} a, v)$. Thus

$$d(H\circ\psi)_p(v) = \omega_p(d\psi_p^{-1} X_H(\psi(p)), v),$$

so $X_{H\circ\psi}(p) = d\psi_p^{-1} X_H(\psi(p))$. □

Proposition 15.12. Symplectomorphisms preserve the Poisson bracket

If $\psi \in \mathrm{Symp}(M, \omega)$, then

$$\{F\circ\psi, G\circ\psi\} = \{F, G\}\circ\psi \qquad \text{for all } F, G \in C^\infty(M).$$

Proof. Using $\{F, G\} = \omega(X_F, X_G)$ and the previous proposition,

$$\{F\circ\psi, G\circ\psi\}(p) = \omega_p(X_{F\circ\psi}(p), X_{G\circ\psi}(p)) = \omega_p(d\psi_p^{-1} X_F(\psi(p)), d\psi_p^{-1} X_G(\psi(p))).$$

Because ψ is symplectic, $\omega_p(d\psi_p^{-1}u, d\psi_p^{-1}v) = \omega_{\psi(p)}(u, v)$, hence

$$\{F\circ\psi, G\circ\psi\}(p) = \omega_{\psi(p)}(X_F(\psi(p)), X_G(\psi(p))) = \{F, G\}(\psi(p)).$$

□

15.6 Darboux coordinates and Hamilton's equations

Let $(q_1, \dots, q_n, p_1, \dots, p_n)$ be Darboux coordinates on an open set so that

$$\omega = \sum_{i=1}^n dq_i \wedge dp_i.$$

Proposition 15.13. Coordinate formulae

In Darboux coordinates,

$$X_H = \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right),$$

and the Hamiltonian ODE $\dot{x} = X_H(x)$ is the system

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Moreover,

$$\{F, G\} = \sum_{i=1}^n \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right).$$

Proof. Write $X_H = \sum_i (a_i \partial_{q_i} + b_i \partial_{p_i})$. Then

$$\iota_{X_H}\omega = \sum_{i=1}^n (a_i\, dp_i - b_i\, dq_i).$$

Equating with $dH = \sum_i (\partial_{q_i} H\, dq_i + \partial_{p_i} H\, dp_i)$ gives $a_i = \partial_{p_i} H$ and $b_i = -\partial_{q_i} H$, hence the formula for X_H and Hamilton's equations.
For the bracket, compute $\{F, G\} = dF(X_G)$ and insert the coordinate expression for X_G. □

15.7 Examples

Example. On $(\mathbb{R}^2, dq \wedge dp)$, let $H(q,p) = \frac{1}{2}(p^2 + q^2)$. Then $X_H = p\,\partial_q - q\,\partial_p$, so the solutions satisfy

$$\dot{q} = p, \qquad \dot{p} = -q,$$

hence $q(t) = q(0)\cos t + p(0)\sin t$ and $p(t) = p(0)\cos t - q(0)\sin t$. The trajectories are circles $H = \text{const}$ in the (q,p)-plane.

Example. Let Q be a smooth manifold and consider $(T^*Q, \omega_{\text{can}})$ with canonical coordinates (q_i, p_i) on T^*U. For a *natural Hamiltonian*

$$H(q,p) = \frac{1}{2}\sum_{i,j} g^{ij}(q) p_i p_j + V(q),$$

Hamilton's equations give

$$\dot{q}_i = \sum_j g^{ij}(q) p_j, \qquad \dot{p}_i = -\frac{1}{2}\sum_{j,k} \frac{\partial g^{jk}}{\partial q_i}(q) p_j p_k - \frac{\partial V}{\partial q_i}(q).$$

When g is a Riemannian metric and $V \equiv 0$, this is the cotangent-bundle formulation of geodesic flow: the base curve $q(t)$ is a geodesic, and $p(t)$ is the corresponding covector obtained by lowering indices.

Example. On the sphere S^2 with coordinates (θ, h) away from the poles, the area form can be written as $\omega = d\theta \wedge dh$. Take $H(\theta, h) = h$. Then $\iota_{X_H}\omega = dH = dh$ forces $X_H = \partial_\theta$, so the Hamiltonian flow rotates the sphere around the vertical axis while preserving h.

Remark. Different references place a minus sign in the defining equation for X_H (i.e. $\iota_{X_H}\omega = -dH$). With our convention $\iota_{X_H}\omega = dH$, the identity $[X_F, X_G] = -X_{\{F,G\}}$ makes the map $H \mapsto X_H$ a Lie algebra *anti*-homomorphism.

16 Symmetry, moment maps, and symplectic reduction

16.1 Symplectic actions and infinitesimal generators

Let (M, ω) be a symplectic manifold and let G be a Lie group with Lie algebra $\mathfrak{g}$.

Definition 16.1. Symplectic group action

A (left) action $\psi : G \times M \to M$ is **symplectic** if each diffeomorphism

$$\psi_g : M \to M, \qquad p \mapsto \psi(g, p)$$

is a symplectomorphism, namely $\psi_g^*\omega = \omega$ for every $g \in G$.

For $\xi \in \mathfrak{g}$, the associated **fundamental vector field** (or infinitesimal generator) is

$$X_\xi(p) := \frac{d}{dt}\Big|_{t=0} \psi_{\exp(t\xi)}(p).$$

It satisfies $[\, X_\xi, X_\eta \,] = X_{[\xi,\eta]}$.

16.2 Hamiltonian actions and the moment map

Definition 16.2. Hamiltonian action and moment map

A symplectic action of G on (M,ω) is **Hamiltonian** if there exists a smooth map
$$\mu : M \to \mathfrak{g}^*$$
such that for each $\xi \in \mathfrak{g}$ the function
$$H_\xi(p) := \langle \mu(p), \xi \rangle$$
satisfies
$$\iota_{X_\xi}\omega = dH_\xi.$$
A map μ with this property is called a **moment map**.

The above condition says that every infinitesimal symmetry is generated by a Hamiltonian function obtained by pairing μ with ξ.

Definition 16.3. Equivariance

A moment map μ is **(coadjoint) equivariant** if
$$\mu(\psi_g(p)) = \mathrm{Ad}^*_{g^{-1}}\mu(p) \qquad \text{for all } g \in G,\ p \in M.$$

Remark. Moment maps are not unique. If G is connected and μ is a moment map, then so is $\mu + c$ for any constant $c \in \mathfrak{g}^*$. If G is a torus, the coadjoint action is trivial, so equivariance imposes no extra constraint beyond invariance.

The moment map ties the Lie bracket on $\mathfrak{g}$ to the Poisson bracket on $C^\infty(M)$.

Proposition 16.4. The Lie algebra package

Assume the action is Hamiltonian with an equivariant moment map μ. Then the map
$$\mathfrak{g} \to C^\infty(M), \qquad \xi \mapsto H_\xi = \langle \mu, \xi \rangle$$
is a Lie algebra homomorphism with respect to the Poisson bracket:
$$\{H_\xi, H_\eta\} = H_{[\xi,\eta]}.$$

Proof. We know that $\{H_\xi, H_\eta\} = \omega(X_{H_\xi}, X_{H_\eta}) = \omega(X_\xi, X_\eta)$. Equivariance implies that $\xi \mapsto H_\xi$ intertwines the adjoint and Poisson brackets, and the stated identity follows. □

16.3 Examples of moment maps

Example. Let S^1 act on (S^2,ω) by rotations about the vertical axis. With the convention $\iota_{X_H}\omega = dH$, the height function (after the standard 2π normalization that makes the flow 1-periodic) is a moment map for this action.

Example. Let T^n act on $(\mathbb{C}^n, \omega_0 = \sum_j dx_j \wedge dy_j)$ by coordinatewise rotations
$$(\theta_1,\dots,\theta_n)\cdot(z_1,\dots,z_n) = (e^{-2\pi i\theta_1}z_1,\dots,e^{-2\pi i\theta_n}z_n).$$
A moment map is
$$\mu(z) = \pi\big(|z_1|^2,\dots,|z_n|^2\big) \in \mathbb{R}^n \simeq (\mathbb{R}^n)^*.$$

Example. The standard T^n-action on $(\mathbb{CP}^n, \omega_{FS})$ given by rotating homogeneous coordinates induces a Hamiltonian action with moment map

$$\mu([z_0 : \cdots : z_n]) = \pi \left(\frac{|z_1|^2}{\|z\|^2}, \ldots, \frac{|z_n|^2}{\|z\|^2} \right) \in \mathbb{R}^n, \qquad \|z\|^2 = \sum_{i=0}^{n} |z_i|^2.$$

The image is the simplex

$$\Delta = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; 0 \le \sum_i x_i \le \pi \right\}.$$

16.4 Symplectic reduction at a regular value

Let (M, ω) carry a Hamiltonian G-action with moment map $\mu : M \to \mathfrak{g}^*$. Fix a value $\alpha \in \mathfrak{g}^*$. For simplicity we focus on $\alpha = 0$, and we assume the hypotheses that make the quotient smooth.

Theorem 16.5. Marsden–Weinstein–Meyer reduction

Let G be a compact Lie group acting on (M, ω) in a Hamiltonian way with moment map μ. Assume that 0 is a regular value of μ and that G acts freely on $\mu^{-1}(0)$. Then the orbit space

$$M_{\text{red}} := \mu^{-1}(0)/G$$

is a smooth manifold, and there exists a unique symplectic form ω_{red} on M_{red} such that

$$i^*\omega = \pi^*\omega_{\text{red}},$$

where $i : \mu^{-1}(0) \hookrightarrow M$ is inclusion and $\pi : \mu^{-1}(0) \to M_{\text{red}}$ is the quotient map.

Proof. Write $Z := \mu^{-1}(0)$. Regularity implies that Z is a submanifold of codimension $\dim G$. Since the action is free and G is compact, $\pi : Z \to Z/G$ is a principal G-bundle and M_{red} is a manifold.

We now show that $\iota^*\omega$ is basic. Invariance is immediate because $\psi_g^*\omega = \omega$ for all g. For horizontality, take $\xi \in \mathfrak{g}$ and $v \in T_pZ = \ker d\mu(p)$. Then

$$(i^*\omega)_p(X_\xi(p), v) = \omega_p(X_\xi(p), v) = d\langle \mu, \xi \rangle_p(v) = 0,$$

so $i^*\omega$ vanishes whenever one argument is vertical, hence descends to a 2-form ω_{red} on M_{red} with $\pi^*\omega_{\text{red}} = i^*\omega$.

Finally, nondegeneracy is checked on fibers. Let $[v] \in T_{[p]}M_{\text{red}}$ be represented by $v \in T_pZ$. If $\omega_{\text{red}}([v], \cdot) = 0$, then $\omega_p(v, w) = 0$ for all $w \in T_pZ$. Thus $v \in (T_pZ)^\omega$. For a Hamiltonian action one has $(\ker d\mu(p))^\omega = \operatorname{span}\{X_\xi(p) \mid \xi \in \mathfrak{g}\}$, so v is vertical. Hence $[v] = 0$ and ω_{red} is nondegenerate. $\square$

Remark. If the action on $\mu^{-1}(0)$ is only locally free, then M_{red} is naturally a symplectic orbifold.

A quick dimension check is often useful:

$$\dim M_{\text{red}} = \dim M - 2 \dim G.$$

16.5 Reduction at other levels and the torus shift

For general groups, reducing at a nonzero level typically forces one to pay attention to which subgroup preserves the level. For a torus, every level is preserved.

Proposition 16.6. Shifting for tori

Let $G = T^m$ act Hamiltonianly on (M, ω) with moment map $\mu : M \to (\mathbb{R}^m)^*$. For any $\xi \in (\mathbb{R}^m)^*$, the level set $\mu^{-1}(\xi)$ is G-invariant, and reduction at ξ is equivalent to reduction at 0 for the shifted moment map

$$\phi(p) := \mu(p) - \xi.$$

16.6 A structural theorem for torus actions

The last piece of this course module is a global constraint that is invisible to Darboux-type local theory.

Theorem 16.7. Atiyah–Guillemin–Sternberg convexity

Let T^m act Hamiltonianly on a compact connected symplectic manifold (M, ω) with moment map $\mu : M \to (\mathbb{R}^m)^* \simeq \mathbb{R}^m$. Then:

1. each level set $\mu^{-1}(\eta)$ is connected,
2. the image $\mu(M) \subset \mathbb{R}^m$ is a convex polytope,
3. $\mu(M)$ is the convex hull of the values of μ on the connected components of the fixed point set.

Example. For the standard T^n-action on $\mathbb{CP}^n$ from above, the fixed point set consists of the $n + 1$ coordinate points

$$p_j = [0 : \cdots : 0 : 1 : 0 : \cdots : 0],$$

and the moment map image is the simplex with vertices $\mu(p_j)$.