# WITTEN'S SPINORIAL PROOF OF THE POSITIVE MASS THEOREM

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ABSTRACT. The Positive Mass Theorem states that an isolated gravitational system satisfying the dominant energy condition must have non-negative total ADM mass. This paper presents a self-contained exposition of Edward Witten's proof, rigorously developed by Parker and Taubes. We assume only a background in undergraduate Riemannian geometry. We systematically construct the necessary tools: the geometry of spacelike hypersurfaces in Lorentzian manifolds, Clifford algebras and spinor bundles, and the analysis of the Dirac operator on asymptotically flat manifolds. We provide detailed derivations of the Weitzenböck formula and the boundary integrals that relate scalar curvature to mass.

## CONTENTS

## 1. Geometry of Asymptotically Flat 3-Manifolds

In this section we introduce the purely Riemannian side of the story. Our eventual goal is to understand the *ADM mass* of a 3-dimensional Riemannian manifold that is "almost Euclidean at infinity", and to state the Riemannian positive mass theorem in this setting.

Throughout the paper we use the following basic conventions.

- A *Riemannian* 3*-manifold* $(M, g)$ is a smooth, connected, oriented 3-manifold equipped with a smooth Riemannian metric $g$.
- The Levi-Civita connection of $g$ on the tangent bundle $TM$ is denoted by $\nabla$.
- The Riemann curvature tensor, Ricci curvature, and scalar curvature of $g$ are denoted by $R_{ijk}{}^{\ell}$, $\mathrm{Ric}_{ij}$, and $R$, respectively.

### 1.1. Riemannian recap: connections and scalar curvature.

**Riemannian recap: connections and scalar curvature.** We begin with a brief reminder of the basic geometric objects that will enter the statement of the positive mass theorem. Readers familiar with Riemannian geometry may safely skim this subsection, but it fixes notation for the rest of the paper.

Let $(M, g)$ be a Riemannian manifold. The Levi-Civita connection $\nabla$ is the unique connection on $TM$ that is compatible with the metric and torsion-free:

$$X\big(g(Y,Z)\big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \qquad \nabla_X Y - \nabla_Y X = [X, Y],$$

for all vector fields $X, Y, Z$.

The curvature tensor of $\nabla$ is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

In a local frame $(e_1, e_2, e_3)$ we write

$$R(e_i, e_j)e_k = R_{ijk}{}^{\ell} e_{\ell}.$$

The Ricci curvature is the trace

$$\mathrm{Ric}(X, Y) = \sum_{i=1}^{3} g\big(R(e_i, X)Y, e_i\big),$$

and the scalar curvature is the trace of the Ricci tensor:

$$R = \sum_{i=1}^{3} \mathrm{Ric}(e_i, e_i).$$

Geometrically, the scalar curvature at a point measures the average of sectional curvatures through that point. In this paper the scalar curvature $R$ will play the role of a *nonnegative density* that appears in an integral inequality.

1.2. **Asymptotically flat ends.** We now formalize the idea that a Riemannian 3-manifold looks more and more like Euclidean space far away from some compact region. For simplicity we first treat the case of a single end.

**Definition 1.1.** Let $(M, g)$ be a Riemannian 3-manifold. We say that $(M, g)$ has an *asymptotically flat end of order* $\tau > 0$ if there exists a compact set $K \subset M$ and a diffeomorphism

$$\Phi : M \setminus K \longrightarrow \{x \in \mathbb{R}^3 : |x| > R_0\}$$

for some $R_0 > 0$, such that, in the coordinate chart $x = (x^1, x^2, x^3)$ provided by $\Phi$, the components $g_{ij}$ of the metric satisfy

$$g_{ij}(x) = \delta_{ij} + h_{ij}(x),$$

where the error term $h_{ij}$ obeys the decay estimates

$$h_{ij}(x) = O(|x|^{-\tau}), \qquad \partial_k h_{ij}(x) = O(|x|^{-\tau-1}), \qquad \partial_\ell \partial_k h_{ij}(x) = O(|x|^{-\tau-2}),$$

as $|x| \to \infty$.

Here and throughout, the notation $f(x) = O(|x|^{-q})$ means that there exists a constant $C > 0$ such that

$$|f(x)| \leq C|x|^{-q}$$

for all $|x|$ sufficiently large. The decay assumptions above imply that, at large distances, the metric $g$ and its first and second derivatives are close to those of the Euclidean metric $\delta$.

**Definition 1.2.** A Riemannian 3-manifold $(M, g)$ is called *asymptotically flat of order* $\tau$ if there exists a compact set $K \subset M$ such that $M \setminus K$ is a finite disjoint union of asymptotically flat ends in the sense of Definition 1.1.

For most of this paper we will restrict attention to the case of a *single* asymptotically flat end. Allowing several ends does not change the local arguments, but one must keep track of one mass for each end.

Intuitively, asymptotic flatness means the following.

- Outside a large compact region, the manifold can be parametrized by standard Euclidean coordinates.
- In these coordinates, the metric tensor $g$ approaches the Euclidean metric and the curvature tensor approaches 0 at a definite rate.

- Geodesic spheres of large radius are small perturbations of Euclidean spheres, and their area and mean curvature approach the Euclidean values.

The order $\tau$ measures how fast the metric approaches flatness. In dimension three, the positive mass theorem will hold under the assumption that $\tau > \frac{1}{2}$; we will not need the sharp threshold at this stage, so we simply assume $\tau$ is large enough for all integrals below to make sense.

1.3. **The ADM mass in the time-symmetric case.** The ADM mass is a number extracted from the $1/r$ part of the metric at infinity. In the general relativistic literature it is defined on a spacelike hypersurface in a Lorentzian spacetime, but in the time-symmetric case (when the second fundamental form vanishes) it can be expressed purely in terms of the Riemannian metric $g$.

Let $(M, g)$ be an asymptotically flat 3-manifold with a single end, and let $x = (x^1, x^2, x^3)$ be asymptotically flat coordinates on that end. Set $r = |x| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. For each $R > R_0$, consider the coordinate sphere

$$S_R = \{x \in \mathbb{R}^3 : |x| = R\} \subset M,$$

with outward unit normal $\nu$ (computed with respect to the metric $g$) and area element $dS$ (again for the metric $g$).

**Definition 1.3.** Let $(M, g)$ be an asymptotically flat 3-manifold of order $\tau > \frac{1}{2}$ with a single end. The *ADM mass* of $(M, g)$ is the limit

$$m_{\mathrm{ADM}}(M, g) = \frac{1}{16\pi} \lim_{R\to\infty} \int_{S_R} \sum_{i,j=1}^{3} (\partial_j g_{ij} - \partial_i g_{jj})\nu^i \, dS.$$

In this formula, indices are raised and lowered using the Euclidean metric $\delta$ in the chosen coordinates at infinity. One may show that, under the decay assumptions in Definition 1.1, this limit exists and is independent of the particular choice of asymptotically flat coordinates. Thus $m_{\mathrm{ADM}}(M, g)$ is a geometric invariant of the end.

Heuristically, the integrand measures the radial flux of the deviation of $g$ from the Euclidean metric. The normalization factor $1/(16\pi)$ is chosen to match conventions in general relativity, but it will play no conceptual role in our arguments.

1.4. **Two basic examples.** We now compute the ADM mass in two examples. The first serves as a sanity check; the second is the main model coming from general relativity.

1.4.1. *Euclidean space.* Let $M = \mathbb{R}^3$ with the standard Euclidean metric $g_{ij} = \delta_{ij}$. Then

$$\partial_k g_{ij} = 0 \quad \text{for all} \quad i, j, k.$$

The integrand in Definition 1.3 vanishes identically, so

$$m_{\mathrm{ADM}}(\mathbb{R}^3, \delta) = 0.$$

This is consistent with the interpretation of the ADM mass as measuring the total amount of curvature: perfectly flat space should have zero mass.

1.4.2. *The Schwarzschild metric in isotropic coordinates.* Our second example is the time-symmetric slice of the Schwarzschild spacetime, restricted to the region outside the horizon. From a purely Riemannian point of view, we consider the manifold

$$M = \{x \in \mathbb{R}^3 : |x| > \tfrac{m}{2}\}$$

with the metric

$$g_{ij}(x) = \phi(x)^4 \delta_{ij}, \qquad \phi(x) = 1 + \frac{m}{2|x|},$$

where $m > 0$ is a constant.

One checks directly that $(M, g)$ is asymptotically flat of order $\tau = 1$. Indeed, for large $|x|$ we have

$$\phi(x)^4 = \Big(1 + \frac{m}{2r}\Big)^4 = 1 + \frac{2m}{r} + O(r^{-2}),$$

so

$$g_{ij}(x) = \Big(1 + \frac{2m}{r} + O(r^{-2})\Big)\delta_{ij} = \delta_{ij} + \frac{2m}{r}\,\delta_{ij} + O(r^{-2}).$$

In particular, $h_{ij} = g_{ij} - \delta_{ij} = O(r^{-1})$ and its derivatives satisfy the required decay estimates.

Let us compute the ADM mass. Since $g$ is conformally flat, it is convenient to write

$$g_{ij} = u^4 \delta_{ij}, \qquad u = \phi = 1 + \frac{m}{2r}.$$

A direct computation shows that

$$\partial_k g_{ij} = 4u^3 (\partial_k u) \delta_{ij}.$$

Therefore

$$\partial_j g_{ij} - \partial_i g_{jj} = 4u^3(\partial_j u)\delta_{ij} - 4u^3(\partial_i u)\delta_{jj} = 4u^3 \partial_i u - 12u^3 \partial_i u = -8u^3 \partial_i u.$$

On the coordinate sphere $S_R$ of large radius $R$, the outward unit normal $\nu$ is asymptotic to the Euclidean radial vector $\partial_r$, so we may replace $\nu^i$ by $x^i/r$ at leading order. Then

$$\sum_{i=1}^{3} (\partial_j g_{ij} - \partial_i g_{jj})\nu^i \approx -8u^3 \sum_{i=1}^{3} (\partial_i u)\frac{x^i}{r} = -8u^3 \frac{\partial u}{\partial r}.$$

As $R \to \infty$,

$$u(R) = 1 + \frac{m}{2R}, \qquad \frac{\partial u}{\partial r}(R) = -\frac{m}{2R^2},$$

so

$$-8u(R)^3 \frac{\partial u}{\partial r}(R) = -8\Big(1 + O(R^{-1})\Big)\Big(-\frac{m}{2R^2}\Big) = \frac{4m}{R^2} + O(R^{-3}).$$

The area element $dS$ on $S_R$ is asymptotic to $R^2 d\omega$, where $d\omega$ is the standard area element on the unit sphere $S^2$. Thus

$$\int_{S_R} \sum_{i,j} (\partial_j g_{ij} - \partial_i g_{jj})\nu^i \, dS = \Big(\frac{4m}{R^2} + O(R^{-3})\Big) \cdot 4\pi R^2 = 16\pi m + O(R^{-1}).$$

Taking the limit as $R \to \infty$ and applying Definition 1.3, we obtain

$$m_{\mathrm{ADM}}(M, g) = m.$$

Thus the parameter $m$ appearing in the metric is precisely the ADM mass. This is one of the basic consistency checks between the geometric definition of mass and the physical interpretation of $m$ as the mass of a Schwarzschild black hole.

1.5. **The Riemannian positive mass theorem (time-symmetric form).** We are now ready to state the main theorem that this paper will prove, in the simplest Riemannian form.

**Theorem 1.4** (Riemannian positive mass theorem)**.** *Let $(M^3, g)$ be a complete, oriented, asymptotically flat Riemannian 3-manifold with a single end and of sufficiently high order of decay. Assume that the scalar curvature satisfies*

$$R(x) \geq 0 \qquad \textit{for all } x \in M.$$

*Then the ADM mass of $(M, g)$ is nonnegative:*

$$m_{\mathrm{ADM}}(M, g) \geq 0.$$

*Moreover, $m_{\mathrm{ADM}}(M, g) = 0$ if and only if $(M, g)$ is isometric to Euclidean space $(\mathbb{R}^3, \delta)$.*

The assumption $R \geq 0$ may be viewed as a purely Riemannian curvature condition, but in the context of general relativity it corresponds to a physical energy condition on the matter fields. We will briefly explain this correspondence in an appendix.

The proof of Theorem 1.4 given in this paper follows Witten's spinorial method. Unlike the original minimal surface proof of Schoen and Yau, our argument will rely on spin geometry and the properties of the Dirac operator. From the point of view of Riemannian geometry, the positive mass theorem is a particularly striking *global* consequence of the local nonnegativity of scalar curvature.

**Why spinors will appear.** At first sight, the ADM mass is defined by an integral of first derivatives of the metric over large spheres, while the scalar curvature is a second-order expression in the metric. It is therefore not obvious how to relate the sign of $m_{\mathrm{ADM}}$ to the inequality $R \geq 0$.

Witten's idea is to introduce an auxiliary geometric object: a spinor field $\psi$ on $M$ satisfying a first-order elliptic equation

$$D\psi = 0,$$

where $D$ is the Dirac operator associated with the metric $g$. The Dirac operator enjoys a remarkable identity, the Lichnerowicz formula,

$$D^2 = \nabla^{S*}\nabla^S + \frac{1}{4}R,$$

which expresses the square of $D$ as the sum of a nonnegative operator and a scalar curvature term. Integrating this identity over large domains and applying an appropriate integration by parts formula, one obtains an equality of the form

$$\int_M \Big( |\nabla^S \psi|^2 + \frac{1}{4}R|\psi|^2 \Big) = \text{boundary term at infinity.}$$

If $\psi$ is chosen to solve $D\psi = 0$ and to approach a nonzero constant spinor at infinity, the left-hand side is nonnegative under the assumption $R \geq 0$. The right-hand side turns out to be a constant multiple of the ADM mass. This strategy reduces the positive mass theorem to the existence of an appropriate harmonic spinor and to the careful identification of the boundary term with the ADM mass.

All of the ingredients in this argument can be developed within Riemannian geometry. In the next sections we will build the necessary spin geometric machinery from the ground up, starting with the linear algebra of spinors in $\mathbb{R}^3$.

## 2. Linear Algebra of Spinors in $\mathbb{R}^3$

In this section we forget about manifolds and work purely in the Euclidean vector space $\mathbb{R}^3$. Our goal is to construct a concrete 2-dimensional complex representation of the *Clifford algebra* of $\mathbb{R}^3$. Vectors in $\mathbb{R}^3$ will act on this space by matrices, and the elements of this representation space are what we call *spinors* in dimension 3.

Later, when we define spinor bundles on a Riemannian 3-manifold, each fiber of the spinor bundle will look like this fixed 2-dimensional complex vector space, with the same algebraic structures.

2.1. **The Euclidean vector space $\mathbb{R}^3$.** We write $\mathbb{R}^3$ for the real vector space of column vectors

$$x = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}, \qquad x^i \in \mathbb{R},$$

equipped with the standard Euclidean inner product

$$\langle x, y \rangle = x^1 y^1 + x^2 y^2 + x^3 y^3.$$

We denote the induced norm by

$$|x| = \sqrt{\langle x, x \rangle}.$$

The standard basis is

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

An *orthonormal basis* of $\mathbb{R}^3$ is a triple $(v_1, v_2, v_3)$ such that

$$\langle v_i, v_j \rangle = \delta_{ij}$$

for all $i, j$. The group of all linear isometries of $\mathbb{R}^3$ is the orthogonal group $O(3)$, and its identity component is the special orthogonal group $SO(3)$, the group of orientation-preserving rotations of $\mathbb{R}^3$.

Our construction of spinors will encode the inner product $\langle \cdot, \cdot \rangle$ and the action of $SO(3)$ in an algebraic way.

2.2. **The Clifford algebra of $\mathbb{R}^3$.** The starting point is an algebra that remembers the inner product purely through multiplication rules.

**Definition 2.1.** Let $V$ be a real inner product space. The *Clifford algebra* $Cl(V)$ of $V$ is the associative unital $\mathbb{R}$-algebra generated by $V$ subject to the relations

$$v \cdot w + w \cdot v = -2\langle v, w \rangle\, 1, \qquad \text{for all } v, w \in V,$$

where 1 denotes the multiplicative identity.

In other words, we start with the tensor algebra

$$T(V) = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus \cdots$$

and quotient out by the two-sided ideal generated by all elements of the form

$$v \otimes w + w \otimes v + 2\langle v, w \rangle\, 1.$$

The resulting quotient is $Cl(V)$.

In the case $V = \mathbb{R}^3$ with orthonormal basis $(e_1, e_2, e_3)$, the defining relations say

$$e_i e_j + e_j e_i = -2\delta_{ij}\, 1, \qquad i, j = 1, 2, 3. \tag{2.2}$$

In particular $e_i^2 = -1$ for each $i$.

The importance of the Clifford algebra is that a *representation* of $Cl(\mathbb{R}^3)$ on a vector space $W$ is the same thing as a rule assigning to each vector $v \in \mathbb{R}^3$ a linear map $c(v) \in \operatorname{End}(W)$ such that

$$c(v)c(w) + c(w)c(v) = -2\langle v, w\rangle \operatorname{Id}_W \qquad \text{for all } v, w \in \mathbb{R}^3. \tag{2.3}$$

In this exposition, we will choose a very concrete 2-dimensional complex representation.

2.3. **Pauli matrices and a concrete spinor representation.** We now introduce the $2 \times 2$ complex matrices that encode the Clifford relations in dimension 3.

**Definition 2.4.** The *Pauli matrices* are the $2 \times 2$ complex matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These matrices satisfy the identities

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\, I_2, \qquad i, j = 1, 2, 3, \tag{2.5}$$

where $I_2$ denotes the $2 \times 2$ identity matrix. This can be checked by a direct matrix computation. For example,

$$\sigma_1\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3,$$

and similarly $\sigma_2\sigma_1 = -i\sigma_3$, so

$$\sigma_1\sigma_2 + \sigma_2\sigma_1 = 0 = 2\delta_{12}I_2.$$

The other relations are verified in the same way.

We now set

$$\Sigma = \mathbb{C}^2,$$

viewed as column vectors of length 2. We define a linear map

$$c : \mathbb{R}^3 \longrightarrow \operatorname{End}_{\mathbb{C}}(\Sigma)$$

by specifying its values on the standard basis:

$$c(e_j) = i\sigma_j, \qquad j = 1, 2, 3, \tag{2.6}$$

and extending linearly to all $v \in \mathbb{R}^3$:

$$v = v^1e_1 + v^2e_2 + v^3e_3 \quad \mapsto \quad c(v) = v^1c(e_1) + v^2c(e_2) + v^3c(e_3).$$

**Proposition 2.7.** *The map $c$ defined in (2.6) satisfies the Clifford relations (2.3) with $W = \Sigma$. In particular, it extends uniquely to an algebra homomorphism*

$$Cl(\mathbb{R}^3) \longrightarrow \operatorname{End}_{\mathbb{C}}(\Sigma).$$

*Proof.* For $i, j \in \{1, 2, 3\}$ we compute using (2.5):

$$c(e_i)c(e_j) + c(e_j)c(e_i) = (i\sigma_i)(i\sigma_j) + (i\sigma_j)(i\sigma_i) = -\sigma_i\sigma_j - \sigma_j\sigma_i = -2\delta_{ij}I_2.$$

Thus (2.3) holds on basis vectors. By bilinearity of both sides, the same relation holds for all $v, w \in \mathbb{R}^3$. The universal property of the Clifford algebra then implies that $c$ extends uniquely to an algebra homomorphism as claimed. $\square$

The complex vector space $\Sigma = \mathbb{C}^2$, together with the action of $\mathbb{R}^3$ given by $c$, is what we will call the *spinor space* for $\mathbb{R}^3$. Elements of $\Sigma$ will be called *(Dirac) spinors*.

2.4. **Spinors as "square roots" of vectors.** The Clifford relation (2.3) implies a simple but important identity. If $v \in \mathbb{R}^3$, then taking $w = v$ we obtain

$$c(v)^2 = -|v|^2 \operatorname{Id}_\Sigma. \tag{2.8}$$

Indeed,

$$c(v)c(v) + c(v)c(v) = 2c(v)^2 = -2\langle v, v\rangle \operatorname{Id}_\Sigma = -2|v|^2 \operatorname{Id}_\Sigma,$$

so $c(v)^2 = -|v|^2\operatorname{Id}_\Sigma$.

In particular, if $v$ is a unit vector, then $c(v)^2 = -\operatorname{Id}_\Sigma$. Thus, for $|v| = 1$, the linear map $c(v)$ has eigenvalues $\pm i$ and acts like a complex structure on the 2-dimensional complex space $\Sigma$. This is one sense in which spinors can be thought of as "square roots" of vectors: multiplying by $c(v)$ twice reproduces $-|v|^2$ times the identity.

Another important feature of the representation $c$ is its compatibility with rotations of $\mathbb{R}^3$. Let $U$ be a $2 \times 2$ unitary matrix with $\det U = 1$, that is, $U \in SU(2)$. Then we can define a new action

$$c_U(v) = U\, c(v)\, U^{-1}.$$

It is easy to check that $c_U$ again satisfies the Clifford relations, hence comes from some orthogonal linear map $R_U : \mathbb{R}^3 \to \mathbb{R}^3$:

$$c_U(v) = c(R_U v) \qquad \text{for all } v \in \mathbb{R}^3.$$

This correspondence $U \mapsto R_U$ defines a group homomorphism

$$\pi : SU(2) \longrightarrow SO(3),$$

which is surjective with kernel $\{\pm I_2\}$. Thus $SU(2)$ is a double cover of $SO(3)$.

We will not prove these group-theoretic facts here. The important point for us is conceptual: a rotation of $\mathbb{R}^3$ can be lifted to a unitary transformation of the spinor space $\Sigma$, and this lift is *two-to-one*. Spinors remember "half-angle" information about rotations.

2.5. **Hermitian inner product and adjointness.** To relate Dirac operators and integration by parts later on, we need a natural inner product on the spinor space $\Sigma$ and to understand how Clifford multiplication interacts with it.

**Definition 2.9.** On $\Sigma = \mathbb{C}^2$ we use the standard Hermitian inner product

$$\langle \phi, \psi\rangle_\Sigma = \phi^* \psi = \overline{\phi_1}\psi_1 + \overline{\phi_2}\psi_2,$$

for column vectors

$$\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \quad \psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}.$$

This inner product is conjugate-linear in the first argument and linear in the second.

If $A$ is a $2\times 2$ complex matrix, its Hermitian adjoint $A^*$ is the conjugate transpose. Then

$$\langle A\phi, \psi\rangle_\Sigma = \langle \phi, A^*\psi\rangle_\Sigma \qquad \text{for all } \phi, \psi \in \Sigma.$$

**Lemma 2.10.** *For each $v \in \mathbb{R}^3$, the matrix $c(v)$ is skew-Hermitian:*

$$c(v)^* = -c(v).$$

*Equivalently,*

$$\langle c(v)\phi, \psi\rangle_\Sigma = -\langle \phi, c(v)\psi\rangle_\Sigma \qquad \textit{for all } \phi, \psi \in \Sigma.$$

*Proof.* The Pauli matrices $\sigma_j$ are Hermitian:

$$\sigma_j^* = \sigma_j, \qquad j = 1, 2, 3.$$

Therefore

$$c(e_j)^* = (i\sigma_j)^* = -i\sigma_j = -c(e_j).$$

For a general vector $v = v^j e_j$ with real coefficients $v^j$, we have

$$c(v)^* = \big(v^j c(e_j)\big)^* = v^j c(e_j)^* = v^j(-c(e_j)) = -c(v),$$

since the $v^j$ are real. The inner product identity follows from the general relation $\langle A\phi, \psi\rangle_\Sigma = \langle \phi, A^*\psi\rangle_\Sigma$. ☐

Thus, the Clifford multiplication $c(v)$ by any vector $v$ is an infinitesimal unitary transformation of the spinor space. This algebraic fact will remain true fiberwise on a spinor bundle and will be crucial when we study the Dirac operator and its formal self-adjointness.

In the next section we will globalize the picture: on a Riemannian 3-manifold $(M, g)$, a spin structure allows us to attach to each point $x \in M$ a copy of the spinor space $\Sigma$, varying smoothly with $x$. The resulting vector bundle is the *spinor bundle*, and the Clifford action $c$ will become a bundle map from the tangent bundle into the endomorphisms of the spinor bundle.

## 3. Spin Structures and Spinor Bundles on 3-Manifolds

In Section 2 we constructed the spinor space $\Sigma = \mathbb{C}^2$ and a Clifford multiplication

$$c : \mathbb{R}^3 \longrightarrow \mathrm{End}_{\mathbb{C}}(\Sigma),$$

which encodes the Euclidean inner product on $\mathbb{R}^3$ via the relations $c(v)c(w) + c(w)c(v) = -2\langle v, w\rangle \mathrm{Id}_\Sigma$. In this section we "globalize" this picture from $\mathbb{R}^3$ to an arbitrary Riemannian 3-manifold $(M, g)$.

The idea is that a *spin structure* on $(M, g)$ allows us to attach to each tangent space $T_xM$ a copy of the spinor space $\Sigma$, in a way compatible with changes of orthonormal frames. The resulting vector bundle $S \to M$ is the *spinor bundle*, and there is a natural Clifford multiplication

$$TM \otimes S \longrightarrow S$$

that generalizes the map $c$ from Section 2.

### 3.1. The orthonormal frame bundle.

Let $(M^3, g)$ be an oriented Riemannian 3-manifold. At each point $x \in M$, an *oriented orthonormal frame* at $x$ is an ordered triple $(e_1, e_2, e_3)$ of tangent vectors in $T_xM$ such that

$$g_x(e_i, e_j) = \delta_{ij} \qquad \text{and} \qquad (e_1, e_2, e_3) \text{ is positively oriented.}$$

**Definition 3.1.** The *oriented orthonormal frame bundle* of $(M, g)$ is the set

$$P_{SO(3)}(M) = \big\{(x, e_1, e_2, e_3) : x \in M,\ (e_1, e_2, e_3) \text{ an oriented } g\text{-orthonormal frame of } T_xM\big\},$$

equipped with the projection

$$p_{SO} : P_{SO(3)}(M) \longrightarrow M, \qquad p_{SO}(x, e_1, e_2, e_3) = x.$$

The group $SO(3)$ acts on $P_{SO(3)}(M)$ from the right by changing frames: if $A \in SO(3)$ and $(e_1, e_2, e_3)$ is an oriented orthonormal frame at $x$, we define

$$(e_1, e_2, e_3) \cdot A = (e'_1, e'_2, e'_3),$$

where $(e'_1, e'_2, e'_3)$ is the new frame whose components are given by matrix multiplication

$$e'_i = \sum_{j=1}^{3} e_j A_{ji}.$$

This action is free and transitive on each fiber $p_{SO}^{-1}(x)$, so each fiber can be identified with $SO(3)$ itself.

The pair $(P_{SO(3)}(M), p_{SO})$ is an example of a *principal $SO(3)$-bundle*: a fiber bundle over $M$ whose fibers are copies of $SO(3)$, with a free and transitive right action of $SO(3)$ on each fiber. We will not develop the general theory of principal bundles here, but this is the main example to keep in mind: $P_{SO(3)}(M)$ collects all oriented orthonormal frames on $M$ into a single geometric object.

3.2. **The group** $\mathrm{Spin}(3)$ **and the double cover** $\mathrm{Spin}(3) \to SO(3)$**.** In Section 2 we saw that rotations of $\mathbb{R}^3$ can be lifted to unitary transformations of the spinor space $\Sigma$ in a natural way: the group $SU(2)$ acts on $\Sigma = \mathbb{C}^2$, and this action descends to an action on $\mathbb{R}^3$ by conjugation on the Pauli matrices. This yields a surjective group homomorphism

$$\pi : SU(2) \longrightarrow SO(3)$$

with kernel $\{\pm I_2\}$. In particular, $SU(2)$ is a double cover of $SO(3)$.

**Definition 3.2.** We define

$$\mathrm{Spin}(3) := SU(2)$$

and denote by

$$\lambda : \mathrm{Spin}(3) \longrightarrow SO(3)$$

the double covering map described above.

Concretely, an element $U \in \mathrm{Spin}(3)$ acts on the spinor space $\Sigma = \mathbb{C}^2$ by matrix multiplication, and its image $\lambda(U) \in SO(3)$ is the corresponding rotation of $\mathbb{R}^3$. The map $U \mapsto \lambda(U)$ forgets the "half-angle" information: $U$ and $-U$ act on $\mathbb{R}^3$ in the same way but differ on $\Sigma$.

The group $\mathrm{Spin}(3)$ is called the *spin group* in dimension 3. In higher dimensions one defines $\mathrm{Spin}(n)$ in terms of the Clifford algebra $Cl(\mathbb{R}^n)$, but in this exposition we will only need the case $n = 3$.

3.3. **Spin structures on Riemannian 3-manifolds.** The orthonormal frame bundle $P_{SO(3)}(M)$ describes how the tangent spaces $T_xM$ fit together. A *spin structure* is, roughly speaking, a way to lift this $SO(3)$-bundle to a $\mathrm{Spin}(3)$-bundle that remembers the extra "spin" information.

**Definition 3.3.** A *spin structure* on an oriented Riemannian 3-manifold $(M, g)$ is a principal $\mathrm{Spin}(3)$-bundle

$$p_{\mathrm{Spin}} : P_{\mathrm{Spin}(3)}(M) \longrightarrow M$$

together with a smooth bundle map

$$\Lambda : P_{\mathrm{Spin}(3)}(M) \longrightarrow P_{SO(3)}(M)$$

such that:

(i) $p_{SO} \circ \Lambda = p_{\text{Spin}}$ (both bundles project to the same base point in $M$), and
(ii) $\Lambda$ is Spin(3)-equivariant with respect to the covering map $\lambda : \text{Spin}(3) \to SO(3)$, i.e.

$$\Lambda(p \cdot g) = \Lambda(p) \cdot \lambda(g) \qquad \text{for all } p \in P_{\text{Spin}(3)}(M),\ g \in \text{Spin}(3).$$

Equivalently, we have a commutative diagram of bundles

$$\begin{tikzcd} P_{\text{Spin}(3)}(M) \arrow[r, "\Lambda"] \arrow[d, "p_{\text{Spin}}"'] & P_{SO(3)}(M) \arrow[d, "p_{SO}"] \\ M \arrow[r, equal] & M \end{tikzcd}$$

in which the horizontal map is a $2{:}1$ covering on each fiber.

A Riemannian manifold $(M, g)$ that admits a spin structure is called a *spin manifold*. Not every oriented Riemannian manifold is spin: there is a topological obstruction measured by the second Stiefel–Whitney class $w_2(TM) \in H^2(M; \mathbb{Z}_2)$. The following result is standard.

**Theorem 3.4.** *An oriented Riemannian manifold $(M, g)$ admits a spin structure if and only if its tangent bundle has trivial second Stiefel–Whitney class:*

$$w_2(TM) = 0 \in H^2(M; \mathbb{Z}_2).$$

We will not prove this theorem here. For our purposes it is enough to know:

- Many natural 3-manifolds (including $\mathbb{R}^3$, $S^3$, and any oriented 3-manifold that is a subset of $\mathbb{R}^3$ with the induced metric) are spin.
- Once a spin structure has been chosen, the spinor bundle and Dirac operator are canonically determined up to isomorphism.

Intuitively, "being spin" means that it is possible to choose local orthonormal frames and local spinor coordinates in such a way that their transition functions lift from $SO(3)$ to Spin(3) consistently on overlaps. The lift is nontrivial: going around certain loops in $SO(3)$ may change the sign of a spinor, and a spin structure is exactly the data needed to keep track of these sign changes globally.

3.4. **The spinor bundle.** We now explain how a spin structure produces a vector bundle whose fibers are copies of the spinor space $\Sigma = \mathbb{C}^2$.

Recall that Spin(3) $= SU(2)$ acts on $\Sigma$ by left multiplication: if $g \in \text{Spin}(3)$ and $\xi \in \Sigma$, then

$$\rho(g)\xi = g\xi,$$

where $g$ is viewed as a $2 \times 2$ complex matrix. This defines a representation

$$\rho : \text{Spin}(3) \longrightarrow GL(\Sigma) \subset U(2).$$

**Definition 3.5.** Let $(M, g)$ be a spin 3-manifold with spin structure $P_{\text{Spin}(3)}(M) \xrightarrow{p_{\text{Spin}}} M$. The *spinor bundle* of $(M, g)$ is the associated complex vector bundle

$$S = P_{\text{Spin}(3)}(M) \times_\rho \Sigma.$$

By definition, $S$ is the quotient of $P_{\text{Spin}(3)}(M) \times \Sigma$ by the equivalence relation

$$(p, \xi) \sim (p \cdot g, \rho(g^{-1})\xi) \qquad \text{for all } p \in P_{\text{Spin}(3)}(M),\ g \in \text{Spin}(3),\ \xi \in \Sigma.$$

The projection $S \to M$ is induced by $p_{\mathrm{Spin}}$:

$$[p, \xi] \longmapsto p_{\mathrm{Spin}}(p),$$

where $[p, \xi]$ denotes the equivalence class of $(p, \xi)$. The fiber over a point $x \in M$ is

$$S_x \cong \Sigma,$$

but the identification depends on a choice of lift $p \in P_{\mathrm{Spin}(3)}(M)$ over $x$. The collection of all these fibers forms a rank-2 complex vector bundle over $M$.

The Hermitian inner product on $\Sigma$ induces a natural Hermitian inner product on each fiber $S_x$: if $[p, \xi], [p, \eta] \in S_x$ are represented using the same spin frame $p$, we define

$$\langle [p, \xi], [p, \eta] \rangle_{S_x} := \langle \xi, \eta \rangle_{\Sigma}.$$

One checks that this does not depend on the choice of $p$ because the Spin(3)-action is unitary.

3.5. **Examples of spin structures and spinor bundles.** We briefly describe three examples that will be relevant later.

*(1) Euclidean space.* On $M = \mathbb{R}^3$ with the Euclidean metric, the orthonormal frame bundle is trivial:

$$P_{SO(3)}(\mathbb{R}^3) \cong \mathbb{R}^3 \times SO(3),$$

where an element $(x, A)$ corresponds to the frame at $x$ obtained by applying $A$ to the standard basis of $\mathbb{R}^3$. We can choose the trivial spin structure

$$P_{\mathrm{Spin}(3)}(\mathbb{R}^3) \cong \mathbb{R}^3 \times \mathrm{Spin}(3)$$

with the bundle map

$$\Lambda(x, g) = (x, \lambda(g)).$$

The corresponding spinor bundle is

$$S \cong \mathbb{R}^3 \times \Sigma,$$

a trivial complex rank-2 bundle. Spinor fields on $\mathbb{R}^3$ are simply smooth maps $\psi : \mathbb{R}^3 \to \Sigma$.

*(2) The 3-sphere.* The round 3-sphere $S^3$ with its standard metric is also spin. In fact, there is a particularly natural spin structure arising from the identification $S^3 \cong SU(2) = \mathrm{Spin}(3)$. We will not need details of this example, but it is useful to know that compact, simply connected 3-manifolds like $S^3$ always admit spin structures, and in this case the spinor bundle has nontrivial topology.

*(3) Asymptotically flat 3-manifolds.* Let $(M, g)$ be an asymptotically flat Riemannian 3-manifold as in Section 1. If $M$ admits a spin structure, we will always choose one with the following compatibility at infinity: on the asymptotically flat end

$$M \setminus K \cong \{x \in \mathbb{R}^3 : |x| > R_0\},$$

the spin structure restricts to the *trivial* spin structure coming from $\mathbb{R}^3$. Equivalently, the spinor bundle $S$ is trivial over the end:

$$S|_{M \setminus K} \cong (M \setminus K) \times \Sigma.$$

This choice allows us to speak of *constant spinors at infinity*: a spinor field $\psi$ on $M$ is asymptotic to a constant $\psi_\infty \in \Sigma$ if, in the trivialization of $S$ over the end, we have

$$\psi(x) \longrightarrow \psi_\infty \qquad \text{as } |x| \to \infty.$$

Such spinors will play a central role in Witten's proof of the positive mass theorem.

3.6. **Clifford multiplication on the spinor bundle.** We now globalize the Clifford multiplication from Section 2 to a bundle map

$$c : TM \otimes S \longrightarrow S.$$

Fix a spin structure $P_{\mathrm{Spin}(3)}(M)$ and spinor bundle $S = P_{\mathrm{Spin}(3)}(M) \times_\rho \Sigma$. Consider an open set $U \subset M$ over which the orthonormal frame bundle admits a local section, i.e. there is a smooth choice of oriented orthonormal frame

$$(e_1(x), e_2(x), e_3(x)) \quad \text{for each } x \in U.$$

Such a choice identifies each tangent space $T_xM$ with $\mathbb{R}^3$ by sending the standard basis of $\mathbb{R}^3$ to $(e_1(x), e_2(x), e_3(x))$. It also induces a trivialization

$$S|_U \cong U \times \Sigma$$

of the spinor bundle over $U$ (after choosing compatible local lifts to the spin bundle).

In these local coordinates, a tangent vector $v \in T_xM$ can be written uniquely as

$$v = v^1 e_1(x) + v^2 e_2(x) + v^3 e_3(x),$$

and a spinor $\psi(x) \in S_x$ is represented by a vector $\xi(x) \in \Sigma$. Using the Clifford multiplication from Section 2, we define

$$c_x(v)\, \psi(x) \quad \text{to be the spinor represented by} \quad c(v^1 e_1 + v^2 e_2 + v^3 e_3)\, \xi(x),$$

where $c$ on the right-hand side is the map $c : \mathbb{R}^3 \to \mathrm{End}_{\mathbb{C}}(\Sigma)$ from Section 2.

One checks that if we change the local frame $(e_1, e_2, e_3)$ by a rotation in $SO(3)$ and simultaneously change the local trivialization of $S$ using the corresponding action of Spin(3) on $\Sigma$, the resulting map $c_x(v)$ is unchanged. Therefore the locally defined maps glue to a global bundle map

$$c : TM \otimes S \longrightarrow S.$$

The pointwise properties of $c$ are inherited from the linear algebra in Section 2. For each $x \in M$ and $v \in T_xM$ we have:

- Clifford relations:

$$c_x(v)c_x(w) + c_x(w)c_x(v) = -2g_x(v, w)\, \mathrm{Id}_{S_x} \qquad \text{for all } v, w \in T_xM,$$

  and in particular

$$c_x(v)^2 = -|v|^2\, \mathrm{Id}_{S_x}.$$

- Skew-Hermitian property: with respect to the Hermitian inner product on $S_x$ induced from $\Sigma$,

$$\langle c_x(v)\phi, \psi\rangle_{S_x} = -\langle \phi, c_x(v)\psi\rangle_{S_x} \qquad \text{for all } \phi, \psi \in S_x.$$

These identities encode the metric $g$ and orientation of $M$ in the algebraic structure of $S$. In particular, $S$ together with $c$ is a *Clifford module* over $(M, g)$ in the sense used in spin geometry.

3.7. **Geometric intuition.** Spinor bundles and spin structures are abstract objects, but it is helpful to keep the following picture in mind.

At each point $x \in M$, the tangent space $(T_xM, g_x)$ is a copy of Euclidean 3-space. The spinor fiber $S_x$ is a copy of the spinor space $\Sigma$, and Clifford multiplication gives a linear map

$$c_x : T_xM \longrightarrow \mathrm{End}_{\mathbb{C}}(S_x)$$

with the properties

$$c_x(v)^2 = -|v|^2 \mathrm{Id}_{S_x}, \qquad c_x(v)^* = -c_x(v).$$

Thus $c_x(v)$ behaves like an "infinitesimal rotation" of $S_x$ by a quarter turn in a direction determined by $v$. In this sense, spinors are objects on which tangent vectors act as skew-rotations.

The existence of a spin structure means that these local pictures can be glued together consistently across $M$. In the next section we will introduce a *connection* on the spinor bundle $S$ (the spin connection), derived from the Levi-Civita connection on $TM$. This spin connection allows us to differentiate spinor fields and to define the Dirac operator, which is the central analytic tool in Witten's proof of the positive mass theorem.

## 4. The Dirac Operator and the Lichnerowicz Formula

In the previous section we constructed the spinor bundle $S \to M$ and the Clifford multiplication

$$c : TM \otimes S \longrightarrow S$$

on a spin Riemannian 3-manifold $(M, g)$. The next step in Witten's argument is to introduce a natural connection on $S$, called the *spin connection*, and the associated *Dirac operator* $D$ on spinor fields. The key identity relating $D$ and the scalar curvature is the Lichnerowicz formula

$$D^2 = \nabla^{S*}\nabla^S + \frac{1}{4}R,$$

which is a Bochner-type formula for spinors.

Throughout this section $(M^3, g)$ is an oriented Riemannian spin manifold with spinor bundle $S$ and Clifford multiplication $c$.

4.1. **The spin connection from the Levi-Civita connection.** We briefly recall the Levi-Civita connection on $TM$ in a local orthonormal frame. Let $(e_1, e_2, e_3)$ be a local oriented $g$-orthonormal frame on an open set $U \subset M$. For each vector field $X$ on $U$ we can write

$$\nabla_X e_i = \sum_{j=1}^{3} \omega_{ij}(X)\, e_j,$$

where the $\omega_{ij}$ are smooth 1-forms on $U$ and

$$\omega_{ij} = -\omega_{ji}.$$

The matrix-valued 1-form

$$\omega = (\omega_{ij})_{1 \le i,j \le 3}$$

is the *connection* 1*-form* of the Levi-Civita connection in the frame $(e_1, e_2, e_3)$.

We now want a connection $\nabla^S$ on the spinor bundle $S$ that is compatible with both the Hermitian metric on $S$ and the Clifford multiplication $c$.

**Definition 4.1.** A *spin connection* on the spinor bundle $S$ is a connection

$$\nabla^S : \Gamma(S) \longrightarrow \Gamma(T^*M \otimes S)$$

such that for all vector fields $X$ and all vector fields $v$ and spinor fields $\psi, \varphi$ we have:

(i) (Metric compatibility)

$$X\langle \psi, \varphi\rangle = \langle \nabla^S_X \psi, \varphi\rangle + \langle \psi, \nabla^S_X \varphi\rangle,$$

where $\langle \cdot, \cdot \rangle$ is the Hermitian inner product on $S$;

(ii) (Clifford compatibility)

$$\nabla^S_X \big(c(v)\psi\big) = c(\nabla_X v)\,\psi + c(v)\,\nabla^S_X \psi.$$

Condition (ii) says that differentiating a spinor field and then multiplying by a tangent vector is the same as first differentiating the tangent vector (using the Levi-Civita connection) and then multiplying, plus the term where we first multiply and then differentiate the spinor. In other words, $\nabla^S$ "respects" the Clifford module structure.

**Proposition 4.2.** *There exists a unique spin connection $\nabla^S$ on $S$ satisfying* (i) *and* (ii) *above.*

*Idea of proof.* On the principal $SO(3)$-bundle $P_{SO(3)}(M)$, the Levi-Civita connection is encoded by a connection 1-form with values in the Lie algebra $\mathfrak{so}(3)$. Using the covering homomorphism

$$\lambda : \mathrm{Spin}(3) \longrightarrow SO(3),$$

one lifts this connection to a connection on the principal Spin(3)-bundle $P_{\mathrm{Spin}(3)}(M)$. The representation $\rho : \mathrm{Spin}(3) \to GL(\Sigma)$ on the spinor space $\Sigma = \mathbb{C}^2$ then induces a connection on the associated vector bundle $S = P_{\mathrm{Spin}(3)}(M) \times_\rho \Sigma$, which is by construction compatible with both the Hermitian product and the Clifford action. Uniqueness follows from the two compatibility conditions. □

We will not work directly with principal bundle connections. Instead, we record an explicit local formula for $\nabla^S$ in terms of the connection 1-forms $\omega_{ij}$.

4.2. **Local formula for the spin connection.** Let $U \subset M$ be an open set on which we have chosen a local oriented orthonormal frame $(e_1, e_2, e_3)$ and a compatible trivialization

$$S|_U \cong U \times \Sigma.$$

In this trivialization, a spinor field $\psi$ over $U$ is just a smooth map

$$\psi : U \longrightarrow \Sigma,$$

and we can differentiate $\psi$ along a vector field $X$ by taking the ordinary directional derivative of its components. We denote this by $\partial_X \psi$.

**Proposition 4.3.** *In the local trivialization above, the spin connection is given by*

$$\nabla^S_X \psi = \partial_X \psi + \frac{1}{4} \sum_{i,j=1}^{3} \omega_{ij}(X)\, c(e_i) c(e_j)\, \psi. \tag{4.4}$$

*Sketch of proof.* The Levi-Civita connection 1-form $\omega$ takes values in the Lie algebra $\mathfrak{so}(3)$, which can be identified with skew-symmetric matrices $(a_{ij})$ satisfying $a_{ij} = -a_{ji}$. A convenient basis of $\mathfrak{so}(3)$ is given by the matrices $E_{ij}$ $(1 \leq i < j \leq 3)$, where $E_{ij}$ has entry 1 in the $(i,j)$-position, entry $-1$ in the $(j,i)$-position, and all other entries 0.

The Lie algebra $\mathfrak{spin}(3) \cong \mathfrak{su}(2)$ acts on $\Sigma$ via the differential of the representation $\rho : \mathrm{Spin}(3) \to GL(\Sigma)$. One can show (see any text on spin geometry) that under the identification $\mathfrak{so}(3) \cong \mathfrak{spin}(3)$, the basis element $E_{ij}$ acts on $\Sigma$ as

$$\rho_*(E_{ij}) = \frac{1}{4} c(e_i)c(e_j).$$

This is consistent with the Clifford relations and the usual embedding of $\mathfrak{so}(3)$ into the even part of the Clifford algebra $Cl(\mathbb{R}^3)$.

Locally, the Levi-Civita connection 1-form is

$$\omega = \sum_{i<j} \omega_{ij} E_{ij},$$

so the corresponding connection on the associated bundle $S$ is given by

$$\nabla^S_X \psi = \partial_X \psi + \rho_*(\omega(X))\, \psi = \partial_X \psi + \frac{1}{4} \sum_{i,j} \omega_{ij}(X)\, c(e_i)c(e_j)\psi.$$

This is exactly (4.4). $\square$

A useful special case occurs at a fixed point $p \in M$. We can choose the frame $(e_1, e_2, e_3)$ to be *normal* at $p$, meaning that

$$(\nabla_{e_k} e_i)_p = 0 \qquad \text{for all } i, k.$$

Equivalently, $\omega_{ij}(p) = 0$ for all $i, j$. In such a frame, the spin connection simplifies at $p$ to

$$\left(\nabla^S_{e_k} \psi\right)_p = \left(\partial_{e_k} \psi\right)_p. \tag{4.5}$$

We will use this normal-frame simplification when computing the Lichnerowicz formula at a point.

4.3. **The Dirac operator.** With the spin connection in hand, we can define the Dirac operator.

**Definition 4.6.** Let $(M^3, g)$ be a Riemannian spin manifold with spinor bundle $S$ and spin connection $\nabla^S$. The *Dirac operator*

$$D : \Gamma(S) \longrightarrow \Gamma(S)$$

is defined in a local oriented orthonormal frame $(e_1, e_2, e_3)$ by

$$D\psi = \sum_{i=1}^{3} c(e_i)\, \nabla^S_{e_i} \psi.$$

The definition does not depend on the choice of orthonormal frame. Indeed, if $\{e_i'\}$ is another oriented orthonormal frame related to $\{e_i\}$ by an $SO(3)$-valued change of basis, then the corresponding change of spin frame is given by an element of $\mathrm{Spin}(3)$, and the compatibility of $\nabla^S$ with Clifford multiplication implies that the expression for $D\psi$ is the same in both frames.

The Dirac operator is a first-order linear differential operator. Its principal symbol can be read off directly: if $f$ is a smooth function and $\psi$ a spinor, then

$$D(f\psi) = \sum_i c(e_i)\big(e_i(f)\,\psi + f\nabla^S_{e_i}\psi\big) = c(\nabla f)\psi + fD\psi.$$

Thus the principal symbol of $D$ at a cotangent vector $\xi \in T^*_xM$ is

$$\sigma_D(\xi) = c(\xi^\sharp) : S_x \longrightarrow S_x,$$

where $\xi^\sharp \in T_xM$ is the metric dual of $\xi$. By the Clifford relation,

$$\sigma_D(\xi)^2 = c(\xi^\sharp)^2 = -|\xi^\sharp|^2\mathrm{Id}_{S_x},$$

so $\sigma_D(\xi)$ is invertible for every nonzero $\xi$. Therefore $D$ is an *elliptic* operator.

Ellipticity is the analytic reason why harmonic spinors (solutions of $D\psi = 0$) enjoy strong regularity and decay properties. We will return to this in the section on existence of harmonic spinors.

4.4. **Green's identity and the boundary term.** We next establish a Green-type identity for the Dirac operator on a domain with boundary. This is the spinorial analogue of the standard integration by parts formula for the divergence operator.

Let $\Omega \subset M$ be a relatively compact domain with smooth boundary $\partial\Omega$, and let $\nu$ denote the outward-pointing unit normal vector field along $\partial\Omega$.

**Lemma 4.7** (Green's formula for $D$)**.** *Let $\psi, \varphi$ be smooth spinor fields on $M$. Then*

$$\int_\Omega \big(\langle D\psi, \varphi\rangle - \langle \psi, D\varphi\rangle\big)\, d\mu_g = \int_{\partial\Omega} \langle c(\nu)\psi, \varphi\rangle\, dS, \tag{4.8}$$

*where $d\mu_g$ is the Riemannian volume measure and $dS$ is the induced area measure on $\partial\Omega$.*

*Proof.* Let $(e_1, e_2, e_3)$ be a local oriented orthonormal frame defined near a point of $\Omega$, and write

$$D\psi = \sum_i c(e_i)\nabla^S_{e_i}\psi.$$

We consider the vector field

$$X = \sum_{i=1}^3 \langle c(e_i)\psi, \varphi\rangle\, e_i,$$

defined on $\Omega$. Its divergence is

$$\operatorname{div} X = \sum_i \langle \nabla_{e_i} X, e_i\rangle = \sum_i e_i\big(\langle c(e_i)\psi, \varphi\rangle\big),$$

since $(e_i)$ is orthonormal.

Using the compatibility of $\nabla^S$ with the Hermitian inner product and the Clifford multiplication, we can expand the derivative:

$$e_i\big(\langle c(e_i)\psi, \varphi\rangle\big) = \langle \nabla^S_{e_i}(c(e_i)\psi), \varphi\rangle + \langle c(e_i)\psi, \nabla^S_{e_i}\varphi\rangle.$$

By Clifford compatibility,

$$\nabla^S_{e_i}(c(e_i)\psi) = c(\nabla_{e_i}e_i)\psi + c(e_i)\nabla^S_{e_i}\psi.$$

Thus

$$\operatorname{div} X = \sum_i \Big(\langle c(\nabla_{e_i}e_i)\psi, \varphi\rangle + \langle c(e_i)\nabla^S_{e_i}\psi, \varphi\rangle + \langle c(e_i)\psi, \nabla^S_{e_i}\varphi\rangle\Big).$$

At a fixed point $p \in \Omega$ we can choose the frame $(e_i)$ to be normal, so that $(\nabla_{e_i} e_i)_p = 0$ and hence $c(\nabla_{e_i} e_i)$ vanishes at $p$. Using the skew-Hermitian property of $c(e_i)$ (Lemma 2.10), we obtain at $p$:

$$\operatorname{div} X = \sum_i \Big( \langle c(e_i) \nabla^S_{e_i} \psi, \varphi \rangle - \langle \psi, c(e_i) \nabla^S_{e_i} \varphi \rangle \Big) = \langle D\psi, \varphi \rangle - \langle \psi, D\varphi \rangle.$$

Since both sides are scalar functions and the computation is tensorial, this identity holds at every point in $\Omega$.

Finally, integrate over $\Omega$ and apply the divergence theorem:

$$\int_\Omega \operatorname{div} X \, d\mu_g = \int_{\partial\Omega} \langle X, \nu \rangle \, dS = \int_{\partial\Omega} \sum_i \langle c(e_i)\psi, \varphi \rangle \langle e_i, \nu \rangle \, dS.$$

Choosing the frame so that $e_3 = \nu$ along $\partial\Omega$ (and $e_1, e_2$ tangent to the boundary), we get

$$\langle X, \nu \rangle = \langle c(\nu)\psi, \varphi \rangle,$$

and (4.8) follows. □

Formula (4.8) shows that $D$ is formally self-adjoint with respect to the $L^2$-inner product on spinors up to a boundary term. On a closed manifold (no boundary), the right-hand side vanishes and $D$ is formally self-adjoint.

4.5. **The Lichnerowicz formula.** We now compute the square of the Dirac operator and relate it to the spin connection and the scalar curvature.

First we recall the definition of the *connection Laplacian* (also called the rough Laplacian) on the spinor bundle.

**Definition 4.9.** Let $(e_1, e_2, e_3)$ be a local oriented orthonormal frame. The connection Laplacian on $S$ is the operator

$$\nabla^{S*} \nabla^S : \Gamma(S) \longrightarrow \Gamma(S)$$

defined by

$$\nabla^{S*} \nabla^S \psi = - \sum_{i=1}^3 \Big( \nabla^S_{e_i} \nabla^S_{e_i} \psi - \nabla^S_{\nabla_{e_i} e_i} \psi \Big). \tag{4.10}$$

This definition does not depend on the choice of orthonormal frame. In normal coordinates at a point $p$, where $(\nabla_{e_i} e_j)_p = 0$, the formula simplifies to

$$\big(\nabla^{S*} \nabla^S \psi\big)_p = - \sum_{i=1}^3 \big( \nabla^S_{e_i} \nabla^S_{e_i} \psi \big)_p.$$

The curvature of the spin connection is the family of endomorphisms $R^S_{X,Y} : S \to S$ defined by

$$R^S_{X,Y} \psi = \nabla^S_X \nabla^S_Y \psi - \nabla^S_Y \nabla^S_X \psi - \nabla^S_{[X,Y]} \psi.$$

It is related to the Riemann curvature tensor of $(M, g)$ by the following standard formula: in a local orthonormal frame $(e_1, e_2, e_3)$,

$$R^S_{e_i, e_j} = \frac{1}{4} \sum_{k,\ell=1}^3 R_{ijk\ell} \, c(e_k) c(e_\ell). \tag{4.11}$$

We will not derive (4.11) here; it follows from the local expression of $\nabla^S$ and the structure equations of Riemannian geometry.

We can now state the main identity of this section.

**Theorem 4.12** (Lichnerowicz formula)**.** *On a Riemannian spin* 3*-manifold* $(M, g)$*, the Dirac operator* $D$ *and the spin connection* $\nabla^S$ *satisfy*

$$D^2 = \nabla^{S*}\nabla^S + \frac{1}{4} R \,\mathrm{Id}_S, \tag{4.13}$$

*where* $R$ *is the scalar curvature.*

*Proof (sketch).* Fix a point $p \in M$, and choose an oriented orthonormal frame $(e_1, e_2, e_3)$ defined in a neighborhood of $p$ such that $(\nabla_{e_i} e_j)_p = 0$ for all $i, j$ (a normal frame). In particular, the connection 1-forms satisfy $\omega_{ij}(p) = 0$.

Let $\psi$ be a spinor field. At the point $p$ we have

$$D\psi = \sum_i c(e_i)\nabla^S_{e_i}\psi, \qquad D^2\psi = \sum_{i,j} c(e_i)c(e_j)\,\nabla^S_{e_i}\nabla^S_{e_j}\psi,$$

since the terms involving derivatives of $c(e_i)$ vanish at $p$ (the $e_i$ are constant at $p$ in the chosen coordinates).

We decompose the sum into symmetric and antisymmetric parts:

$$D^2\psi = \frac{1}{2}\sum_{i,j}\big(c(e_i)c(e_j) + c(e_j)c(e_i)\big)\nabla^S_{e_i}\nabla^S_{e_j}\psi + \frac{1}{2}\sum_{i,j} c(e_i)c(e_j)\,[\nabla^S_{e_i}, \nabla^S_{e_j}]\,\psi.$$

Using the Clifford relation

$$c(e_i)c(e_j) + c(e_j)c(e_i) = -2\delta_{ij}\,\mathrm{Id}_S,$$

the first sum reduces to

$$-\sum_i \nabla^S_{e_i}\nabla^S_{e_i}\psi.$$

In our normal frame at $p$ the connection Laplacian is

$$\big(\nabla^{S*}\nabla^S\psi\big)_p = -\sum_i \big(\nabla^S_{e_i}\nabla^S_{e_i}\psi\big)_p,$$

so the first sum gives $(\nabla^{S*}\nabla^S\psi)_p$.

For the second sum we note that

$$\big[\nabla^S_{e_i}, \nabla^S_{e_j}\big]\psi = R^S_{e_i,e_j}\psi$$

at $p$ (since $[e_i, e_j]_p = 0$ in normal coordinates). Therefore

$$\frac{1}{2}\sum_{i,j} c(e_i)c(e_j)\,\big[\nabla^S_{e_i}, \nabla^S_{e_j}\big]\,\psi = \frac{1}{2}\sum_{i,j} c(e_i)c(e_j)R^S_{e_i,e_j}\psi.$$

Using the curvature formula (4.11), we obtain

$$\frac{1}{2}\sum_{i,j} c(e_i)c(e_j)R^S_{e_i,e_j} = \frac{1}{8}\sum_{i,j,k,\ell} R_{ijk\ell}\, c(e_i)c(e_j)c(e_k)c(e_\ell).$$

A straightforward (but somewhat lengthy) computation using the Clifford relations shows that this curvature term simplifies to

$$\frac{1}{4} R\,\mathrm{Id}_S,$$

where $R$ is the scalar curvature. The computation amounts to tracing over indices in the expression above and using the symmetries of the Riemann tensor.

Combining the two parts, we find at $p$ that

$$D^2\psi = \nabla^{S*}\nabla^S\psi + \frac{1}{4}R\psi.$$

Since $p$ was arbitrary and the identity is tensorial, (4.13) holds everywhere on $M$. $\square$

The Lichnerowicz formula is the spinorial version of a Bochner identity: it expresses a natural second-order differential operator ($D^2$) as the sum of a nonnegative operator ($\nabla^{S*}\nabla^S$) and a zeroth-order curvature term.

4.6. **Integrated Lichnerowicz formula.** We now integrate the Lichnerowicz formula over a bounded domain and record the boundary terms. This will later be applied to large coordinate balls in an asymptotically flat manifold.

Let $\Omega \subset M$ be a relatively compact domain with smooth boundary $\partial\Omega$, and let $\nu$ be the outward unit normal along $\partial\Omega$. Let $\psi$ be a smooth spinor field on $M$.

We start from (4.13):

$$D^2\psi = \nabla^{S*}\nabla^S\psi + \frac{1}{4}R\psi.$$

Taking the Hermitian inner product with $\psi$ and integrating over $\Omega$ gives

$$\int_\Omega \langle D^2\psi, \psi\rangle \, d\mu_g = \int_\Omega \big\langle \nabla^{S*}\nabla^S\psi, \psi\big\rangle \, d\mu_g + \frac{1}{4}\int_\Omega R|\psi|^2 \, d\mu_g. \tag{4.14}$$

We treat the two sides separately.

**Left-hand side.** Using Green's formula with $\varphi = D\psi$ we obtain

$$\int_\Omega \langle D^2\psi, \psi\rangle \, d\mu_g = \int_\Omega \langle D\psi, D\psi\rangle \, d\mu_g - \int_{\partial\Omega} \langle c(\nu)\psi, D\psi\rangle \, dS.$$

**Right-hand side.** By definition of $\nabla^{S*}\nabla^S$ and the compatibility of $\nabla^S$ with the Hermitian product, one can show (cf. the usual integration by parts formula for the connection Laplacian on vector bundles) that

$$\int_\Omega \big\langle \nabla^{S*}\nabla^S\psi, \psi\big\rangle \, d\mu_g = \int_\Omega |\nabla^S\psi|^2 \, d\mu_g - \int_{\partial\Omega} \langle \nabla^S_\nu\psi, \psi\rangle \, dS.$$

Substituting these into (4.14) and rearranging, we obtain:

**Proposition 4.15** (Integrated Lichnerowicz formula)**.** *Let $\Omega \subset M$ be a relatively compact domain with smooth boundary $\partial\Omega$, and let $\psi$ be a smooth spinor field on $M$. Then*

$$\int_\Omega \Big( |\nabla^S\psi|^2 + \frac{1}{4}R|\psi|^2 - |D\psi|^2 \Big) \, d\mu_g = \int_{\partial\Omega} B(\psi) \, dS, \tag{4.16}$$

*where the boundary integrand is*

$$B(\psi) = \langle \nabla^S_\nu\psi, \psi\rangle - \langle c(\nu)\psi, D\psi\rangle.$$

In particular, if $\psi$ is a *harmonic spinor*, i.e. $D\psi = 0$, then the identity simplifies to

$$\int_\Omega \Big( |\nabla^S\psi|^2 + \frac{1}{4}R|\psi|^2 \Big) \, d\mu_g = \int_{\partial\Omega} \langle \nabla^S_\nu\psi, \psi\rangle \, dS. \tag{4.17}$$

If in addition $R \geq 0$, the integrand on the left-hand side is pointwise nonnegative, so the sign of the boundary integral is constrained by curvature.

This is the basic "energy identity" that underlies Witten's proof: interior nonnegativity coming from the scalar curvature is translated into information about the boundary behavior of $\psi$.

4.7. **Bochner technique revisited.** The Lichnerowicz formula fits into a general pattern in Riemannian geometry often called the *Bochner technique*: many natural second-order elliptic operators can be written as

$$(\text{Hodge Laplacian or Dirac operator})^2 = (\text{rough Laplacian}) + (\text{curvature term}).$$

For example, if $\alpha$ is a 1-form on $(M, g)$, the Hodge Laplacian is

$$\Delta_H \alpha = (d\delta + \delta d)\alpha.$$

Bochner's identity states that

$$\Delta_H \alpha = \nabla^* \nabla \alpha + \mathrm{Ric}^\sharp(\alpha),$$

where $\nabla^*\nabla$ is the connection Laplacian on $T^*M$ and $\mathrm{Ric}^\sharp$ is the Ricci tensor viewed as an endomorphism of $T^*M$.

In the spinor setting, the Lichnerowicz formula says

$$D^2 = \nabla^{S*}\nabla^S + \frac{1}{4}R.$$

The curvature term is particularly simple: it is just a scalar potential $\frac{1}{4}R$ times the identity. When $R \geq 0$, this potential is nonnegative. Combined with the ellipticity of $D$ and the integrated formula (4.16), this creates a powerful link between the geometry of $(M, g)$ and the analytic properties of harmonic spinors.

This point of view will be crucial in Section 7, where we apply the Lichnerowicz formula on an asymptotically flat 3-manifold and use harmonic spinors to relate the interior term to the ADM mass.

## 5. Spinors on Asymptotically Flat Ends and the ADM Boundary Term

In this section we combine the asymptotically flat geometry of Section 1 with the spinor and Dirac operator machinery of Sections 3–4. The goal is to understand how the boundary term in the integrated Lichnerowicz formula encodes the ADM mass.

Throughout we assume that $(M^3, g)$ is a complete, oriented, asymptotically flat Riemannian 3–manifold with a single end and sufficiently fast decay in the sense of Definition 1.2. We also assume that $(M, g)$ is spin and that the spin structure has been chosen so that the spinor bundle is trivial over the end, as in Example 3.5(3).

5.1. **Trivializing the spinor bundle at infinity.** Let $(M, g)$ be a complete, oriented, asymptotically flat Riemannian spin manifold with a single end, as in Definition 1.1. We fix a compact set $K \subset M$ and a diffeomorphism

$$\Phi : M \setminus K \to \{x \in \mathbb{R}^3 : |x| > R_0\}$$

as before, and write $x = (x_1, x_2, x_3)$ and $r = |x|$ on the end.

On the end we choose once and for all a *global*, oriented, $g$–orthonormal frame

$$(e_1, e_2, e_3) \quad \text{on } M \setminus K,$$

constructed for example by applying the Gram–Schmidt process to the coordinate vectors $\partial_i = \frac{\partial}{\partial x_i}$. The asymptotic flatness of $g$ implies that

$$e_i = \partial_i + O(r^{-\tau}), \qquad \nabla e_i = O(r^{-\tau-1}) \tag{5.1}$$

as $r \to \infty$, where $\nabla$ is the Levi-Civita connection of $g$ and the big–$O$ notation is understood componentwise.

By assumption on the spin structure (Example 3.5(3)) the spinor bundle $S$ is trivial over the end:

$$S|_{M\setminus K} \;\cong\; (M \setminus K) \times \Sigma, \qquad \Sigma = \mathbb{C}^2.$$

We now *fix* such a trivialization, compatible with the frame $(e_1, e_2, e_3)$, in the following sense: at each point $x$ we choose a spin frame lifting the orthonormal frame $(e_1(x), e_2(x), e_3(x))$ and identify the fiber $S_x$ with $\Sigma$ via this spin frame. This choice is unique up to left multiplication by a constant element of $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$ and will not affect any limiting expressions at infinity.

In this trivialization a spinor field on the end is simply a smooth map

$$\psi : \{x \in \mathbb{R}^3 : |x| > R_0\} \longrightarrow \Sigma,$$

and the Hermitian inner product on each fiber $S_x$ is the standard inner product on $\Sigma$.

5.2. **Asymptotically constant spinors.** The trivialization just fixed allows us to make precise what it means for a spinor to "tend to a constant" at infinity.

**Definition 5.2.** Let $(M, g)$ be an asymptotically flat spin 3–manifold, and fix a trivialization $S|_{M\setminus K} \cong (M \setminus K) \times \Sigma$ as above. A smooth spinor field $\psi \in \Gamma(S)$ is called *asymptotically constant of order* $\sigma > 0$ if there exists a constant vector $\psi_\infty \in \Sigma$ such that

$$\psi(x) = \psi_\infty + O(r^{-\sigma}), \qquad \nabla^S \psi(x) = O(r^{-\sigma-1})$$

as $r \to \infty$, where $\nabla^S$ is the spin connection and the big–$O$ is understood with respect to the Euclidean norm on $\Sigma$. We then say that $\psi$ *tends to* $\psi_\infty$ at infinity.

The specific exponent $\sigma$ will not be important in this section; we only need that it is large enough for all boundary integrals below to converge.

*Remark* 5.3. The constant $\psi_\infty$ depends a priori on the chosen trivialization of $S$ over the end. Changing the trivialization by a *constant* element of $\mathrm{Spin}(3)$ acts on $\psi_\infty$ by an element of $\mathrm{SU}(2)$, but preserves its norm. In particular, the quantity $|\psi_\infty|$ is independent of the chosen trivialization and is a geometrically meaningful asymptotic invariant of $\psi$.

In Witten's proof one prescribes a nonzero "boundary value" $\psi_\infty$ and solves the Dirac equation $D\psi = 0$ with this asymptotic condition. The existence and uniqueness of such a *harmonic spinor* will be addressed later; for now we regard $\psi$ as a hypothetical spinor with these properties.

5.3. **Asymptotics of the spin connection and Dirac operator.** We now describe how, in the AF chart at infinity, the spin connection and Dirac operator differ from their Euclidean counterparts by lower–order terms.

First recall the standard formula for the Christoffel symbols of $g$ in the coordinate frame $\{\partial_i\}$:

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell}\big(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}\big).$$

Using the decay assumptions on $g_{ij} - \delta_{ij}$, one easily checks that

$$\Gamma^k_{ij}(x) = O(r^{-\tau-1}), \qquad \partial_\ell \Gamma^k_{ij}(x) = O(r^{-\tau-2}) \tag{5.4}$$

as $r \to \infty$.

The orthonormal frame $(e_1, e_2, e_3)$ satisfies $g(e_i, e_j) = \delta_{ij}$ and, by (5.1), can be written in the form

$$e_i = \sum_{j=1}^{3} a_{ij}(x)\, \partial_j, \qquad a_{ij}(x) = \delta_{ij} + O(r^{-\tau}), \quad \partial_k a_{ij}(x) = O(r^{-\tau-1}).$$

Let $\omega_{ij}$ denote the Levi-Civita connection 1–forms in this frame, so

$$\nabla_X e_i = \sum_{j=1}^{3} \omega_{ji}(X) e_j, \qquad \omega_{ij} = -\omega_{ji}.$$

Using the relation between $(e_i)$ and $(\partial_j)$ and the decay (5.4) of the Christoffel symbols, one obtains

$$\omega_{ij}(x) = O(r^{-\tau-1}), \qquad \partial_k \omega_{ij}(x) = O(r^{-\tau-2}). \tag{5.5}$$

We do not reproduce the (lengthy but straightforward) computation here. The important point is the order of decay.

In the trivialization of $S$ induced by $(e_1, e_2, e_3)$, the local formula (5.5) and Proposition 4.3 show that the spin connection has the form

$$\nabla^S_X \psi = \partial_X \psi + \frac{1}{4} \sum_{i,j=1}^{3} \omega_{ij}(X)\, c(e_i) c(e_j) \psi, \tag{5.6}$$

and hence

$$\nabla^S_X \psi = \partial_X \psi + O(r^{-\tau-1})\, |\psi|$$

for any vector field $X$ of bounded length.

Next we compare the Dirac operator $D$ with the Euclidean Dirac operator on $\mathbb{R}^3$. In our trivialization the Euclidean Dirac operator is

$$D_0 \psi = \sum_{i=1}^{3} c(e_i)\, \partial_{e_i} \psi = \sum_{i=1}^{3} c(e_i) \sum_{j=1}^{3} a_{ij}(x)\, \partial_j \psi,$$

where, for convenience, we use the same notation $c(e_i)$ both for Clifford multiplication on each fiber $S_x$ and for the fixed $2 \times 2$ matrices acting on $\Sigma$ from Section 2.

**Proposition 5.7.** *On the asymptotically flat end there exist smooth matrix–valued functions $A^j(x) \in \mathrm{End}(\Sigma)$, $B(x) \in \mathrm{End}(\Sigma)$ such that*

$$D\psi = D_0 \psi + \sum_{j=1}^{3} A^j(x)\, \partial_j \psi + B(x) \psi, \tag{5.8}$$

*with decay estimates*

$$A^j(x) = O(r^{-\tau}), \quad \partial_k A^j(x) = O(r^{-\tau-1}), \qquad B(x) = O(r^{-\tau-1})$$

*as $r \to \infty$.*

*Sketch of proof.* By definition of $D$ and (5.6),

$$D\psi = \sum_{i=1}^{3} c(e_i)\nabla^S_{e_i}\psi = \sum_{i=1}^{3} c(e_i)\partial_{e_i}\psi + \frac{1}{4}\sum_{i,j,k} c(e_i)\omega_{jk}(e_i)c(e_j)c(e_k)\psi.$$

The first term is $D_0\psi$ plus lower–order corrections coming from the fact that $e_i = \sum_j a_{ij}\partial_j$ with $a_{ij} - \delta_{ij} = O(r^{-\tau})$. These corrections can be written in the form

$$\sum_{j=1}^{3} A_1^j(x)\,\partial_j\psi, \qquad A_1^j(x) = O(r^{-\tau}).$$

The second term is zeroth order and, by (5.5), has coefficients of size $O(r^{-\tau-1})$. Putting these contributions together yields (5.8) with the stated estimates on $A^j$ and $B$. □

Thus, on an asymptotically flat end, the Dirac operator is a first–order elliptic operator whose principal part agrees with the Euclidean Dirac operator and whose coefficients decay at infinity.

5.4. **The boundary form in the integrated Lichnerowicz identity.** Let $\Omega_R \subset M$ denote the intersection of $M$ with the coordinate ball of radius $R$:

$$\Omega_R = \{x \in M : |x| \le R\}, \qquad S_R = \partial\Omega_R = \{x \in M : |x| = R\}.$$

For $R$ sufficiently large, $S_R$ is a smooth 2–sphere in the asymptotically flat end.

Let $\nu$ denote the outward unit normal to $S_R$ in $(M, g)$. In the AF chart we can compare $\nu$ with the Euclidean radial vector field

$$\partial_r = \sum_{i=1}^{3} \frac{x_i}{r}\,\partial_i.$$

From the expansion of $g$ near infinity one shows that

$$\nu = \partial_r + O(r^{-\tau}), \qquad dS_g = \big(1 + O(r^{-\tau})\big)\,dS_\delta, \tag{5.9}$$

where $dS_g$ is the area element induced by $g$ on $S_R$ and $dS_\delta$ is the Euclidean area element on the coordinate sphere of radius $R$.

For a smooth spinor field $\psi$ on $M$ the integrated Lichnerowicz formula (Proposition 4.15) applied to $\Omega_R$ yields

$$\int_{\Omega_R}\left(|\nabla^S\psi|^2 + \frac{1}{4}R|\psi|^2 - |D\psi|^2\right)d\mu_g = \int_{S_R}\left\langle\nabla^S_\nu\psi, \psi\right\rangle dS_g - \int_{S_R}\langle c(\nu)\psi, D\psi\rangle\, dS_g. \tag{5.10}$$

In Witten's argument we are interested in *harmonic* spinors, so we now specialize to the case $D\psi = 0$. Then the second boundary term vanishes and we obtain

$$\int_{\Omega_R}\left(|\nabla^S\psi|^2 + \frac{1}{4}R|\psi|^2\right)d\mu_g = \int_{S_R}\left\langle\nabla^S_\nu\psi, \psi\right\rangle dS_g. \tag{5.11}$$

The left–hand side is manifestly nonnegative if $R \ge 0$, and we would like to identify the right–hand side with a quantity involving the ADM mass. To do so we must understand the asymptotic behavior of the boundary integrand $\langle\nabla^S_\nu\psi, \psi\rangle$ as $R \to \infty$.

Using (5.6) and (5.9), we can expand

$$\nabla^S_\nu \psi = \partial_\nu \psi + \frac{1}{4}\sum_{i,j} \omega_{ij}(\nu)\, c(e_i)c(e_j)\psi = \partial_r \psi + E_1(x,\psi) + E_0(x,\psi),$$

where

$$E_1(x,\psi) = O(r^{-\tau})\,|\nabla^S \psi|, \qquad E_0(x,\psi) = O(r^{-\tau-1})\,|\psi|.$$

Consequently

$$\left\langle \nabla^S_\nu \psi, \psi \right\rangle = \langle \partial_r \psi, \psi \rangle + Q(x,\psi),$$

with

$$Q(x,\psi) = O(r^{-\tau})\,|\nabla^S \psi|\,|\psi| + O(r^{-\tau-1})\,|\psi|^2. \tag{5.12}$$

Heuristically, if $\psi$ is harmonic and asymptotically constant, then $\partial_r \psi$ decays at least like $r^{-2}$ and $|\nabla^S \psi|$ decays at least like $r^{-2}$. Combining these decays with the area growth $|S_R| \sim 4\pi R^2$ and the extra factors $r^{-\tau}$ in (5.12), one expects that the integral of $Q(x,\psi)$ over $S_R$ tends to 0 as $R \to \infty$. In other words, the main contribution to the boundary term in (7.1) should come from the leading–order interaction between the metric and the asymptotic value $\psi_\infty$ of the spinor.

5.5. **Heuristic identification of the ADM mass.** We now explain, at a formal level, how the boundary term in (7.1) is related to the ADM mass.

Assume that $\psi$ is a smooth spinor on $M$ such that

- $D\psi = 0$ (harmonic spinor);
- $\psi$ is asymptotically constant of some order $\sigma > 0$ with limit $\psi_\infty \neq 0$.

We regard these conditions as granted for the moment; later one proves that such a spinor exists under the hypotheses of the positive mass theorem.

*Step 1: Reducing to a constant spinor at infinity.* Because $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$ acts transitively and unitarily on the unit sphere in $\Sigma$, we may (after a global change of trivialization by a constant spin transformation) assume without loss of generality that

$$\psi_\infty = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \Sigma, \qquad |\psi_\infty| = 1.$$

This simplifies the Pauli–matrix computations below without affecting geometric quantities such as $|\psi_\infty|$.

*Step 2: Keeping only the leading term in the spin connection.* Write $\psi = \psi_\infty + \varphi$, with $\varphi(x) = O(r^{-\sigma})$. In the boundary integrand we are interested in

$$\left\langle \nabla^S_\nu \psi, \psi \right\rangle = \left\langle \nabla^S_\nu \psi_\infty, \psi_\infty \right\rangle + (\text{terms involving } \varphi).$$

The terms involving $\varphi$ contain factors of $\varphi$ or $\nabla^S \varphi$ and hence decay faster at infinity; under the usual decay assumptions one can check that their contribution to the integral over $S_R$ vanishes as $R \to \infty$. We therefore focus on the term

$$\left\langle \nabla^S_\nu \psi_\infty, \psi_\infty \right\rangle = \frac{1}{4}\sum_{i,j} \omega_{ij}(\nu)\, \langle c(e_i)c(e_j)\psi_\infty, \psi_\infty \rangle\,,$$

since $\partial_\nu \psi_\infty = 0$.

Using the explicit representation $c(e_k) = i\sigma_k$ of Section 2 and the choice $\psi_\infty = (1,0)^T$, a direct computation shows that

$$\langle c(e_i)c(e_j)\psi_\infty, \psi_\infty \rangle = -\delta_{ij} + (\text{purely imaginary antisymmetric part}).$$

Because $\omega_{ij} = -\omega_{ji}$, only the antisymmetric part of $c(e_i)c(e_j)$ contributes to the sum, and the real scalar $\langle \nabla^S_\nu \psi_\infty, \psi_\infty \rangle$ can be expressed as a *real* linear combination of the components of $\omega_{ij}(\nu)$. The precise coefficients depend only on the representation $c$ and the spinor $\psi_\infty$ and are therefore universal constants.

On the other hand, the connection forms $\omega_{ij}$ can be written in terms of first derivatives of $g_{ij}$ and the components of $\nu$; more concretely, one shows (again by a routine but involved computation starting from the formula for the Levi-Civita connection in an orthonormal frame) that for large $r$,

$$\omega_{ij}(\nu) = (\text{linear combination of } \partial_k g_{\ell m}\, \nu^k) + O(r^{-\tau-2}),$$

and that the specific linear combination appearing in $\langle \nabla^S_\nu \psi_\infty, \psi_\infty \rangle$ is proportional to the ADM integrand

$$\sum_{i,j} \big( \partial_j g_{ij} - \partial_i g_{jj} \big)\, \nu^i.$$

Putting these observations together, one arrives at a formal asymptotic expansion of the form

$$\langle \nabla^S_\nu \psi, \psi \rangle = C_3 \Big( \sum_{i,j} (\partial_j g_{ij} - \partial_i g_{jj})\, \nu^i \Big) |\psi_\infty|^2 + (\text{higher–order terms}), \tag{5.13}$$

for some positive constant $C_3$ depending only on the normalization of the Clifford representation. The "higher–order terms" decay fast enough so that their integral over $S_R$ tends to 0 as $R \to \infty$.

*Step 3: Taking the limit and matching the ADM mass.* Integrating (5.13) over $S_R$ and using the decay of the error terms yields a formal identity

$$\int_{S_R} \langle \nabla^S_\nu \psi, \psi \rangle \, dS_g = C_3 \int_{S_R} \sum_{i,j} (\partial_j g_{ij} - \partial_i g_{jj})\, \nu^i \, dS + o(1)$$

as $R \to \infty$. By Definition 1.3 of the ADM mass, the integral on the right hand side tends to $16\pi\, m_{\mathrm{ADM}}(M, g)$, up to the normalizing factor $1/(16\pi)$ built into the definition. Thus we expect

$$\lim_{R \to \infty} \int_{S_R} \langle \nabla^S_\nu \psi, \psi \rangle \, dS_g = C_3'\, m_{\mathrm{ADM}}(M, g)\, |\psi_\infty|^2, \tag{5.14}$$

for some positive universal constant $C_3'$.

The key point is that the boundary term in the integrated Lichnerowicz identity is, morally, the ADM mass multiplied by the squared length of the asymptotic spinor $\psi_\infty$, up to a fixed numerical factor depending only on dimension and conventions. A careful bookkeeping of constants (which we suppress here) shows that $C_3' > 0$, so the sign of the ADM mass is the same as the sign of the limiting boundary flux.

In this section we have rewritten the boundary term in the integrated Lichnerowicz formula on large coordinate spheres so that its leading contribution agrees with the ADM surface integral, up to a universal constant. In Sections 6 and 7 we will combine this with the existence of harmonic spinors and the scalar curvature assumption $R \geq 0$.

## 6. Existence of Harmonic Spinors with Prescribed Asymptotics

In Sections 4 and 5 we treated harmonic spinors with prescribed behavior at infinity as if they already existed. In this section we explain, at a high level, why this is not magic: on an asymptotically flat spin manifold, the Dirac operator is a well-behaved elliptic operator on suitable weighted Sobolev spaces, and one can solve the boundary value problem

$$D\psi = 0, \qquad \psi \to \psi_\infty \text{ at infinity,}$$

for any prescribed constant spinor $\psi_\infty \in \Sigma$.

Our goal here is *not* to give a full PDE proof, but to explain the structure: why weighted spaces appear, what "Fredholm" means, and how these analytic results translate into the existence and uniqueness of the spinor used in Witten's argument.

Throughout we continue to assume that $(M^3, g)$ is a complete, oriented, asymptotically flat Riemannian spin 3–manifold with a single end and that the scalar curvature satisfies $R \geq 0$.

### 6.1. **The Dirac operator as an elliptic operator on an AF manifold.**

In Section 4 we already saw that on any Riemannian spin manifold $(M, g)$ the Dirac operator

$$D : \Gamma(S) \longrightarrow \Gamma(S)$$

is a first-order elliptic operator. Ellipticity was encoded in the principal symbol: for $\xi \in T_x^*M$, the symbol of $D$ at $\xi$ is

$$\sigma_D(\xi) = c(\xi^\sharp) : S_x \longrightarrow S_x,$$

and the Clifford relation implies

$$\sigma_D(\xi)^2 = -|\xi^\sharp|^2 \, \mathrm{Id}_{S_x}.$$

In particular, $\sigma_D(\xi)$ is invertible for all $\xi \neq 0$.

On an asymptotically flat (AF) manifold, nothing changes at the level of the symbol: the principal symbol is still $c(\xi^\sharp)$. The AF assumption only affects the *lower-order coefficients*: in the AF chart at infinity, the metric coefficients $g_{ij}$ and the connection coefficients of $\nabla^S$ converge to their Euclidean values as $|x| \to \infty$. Thus, in coordinates on the end,

$$D = D_0 + (\text{lower-order terms that decay at infinity}),$$

where $D_0$ is the Euclidean Dirac operator on $\mathbb{R}^3$.

From the general theory of elliptic operators one obtains:

- *Interior regularity:* If $\psi$ is a (weak) $W^{1,2}_{\mathrm{loc}}$ solution of $D\psi = f$ with $f$ smooth, then $\psi$ is smooth in the interior of $M$.
- *Elliptic estimates:* On any compact subset $K \subset M$, one has an estimate of the form

  $$\|\psi\|_{W^{1,2}(K)} \leq C_K \big( \|D\psi\|_{L^2(K')} + \|\psi\|_{L^2(K')} \big),$$

  where $K' \supset K$ is slightly larger and $C_K$ is a constant depending only on the geometry on $K'$.

In other words, solving $D\psi = 0$ is a standard elliptic PDE problem. The only new ingredient in the AF setting is that we want to control the *decay* of $\psi$ at infinity, not just its behavior on compact sets. This leads naturally to weighted Sobolev spaces.

6.2. **Weighted Sobolev spaces on asymptotically flat ends.** On the AF end we have a fixed diffeomorphism

$$\Phi : M \setminus K \longrightarrow \{x \in \mathbb{R}^3 : |x| > R_0\},$$

and hence global coordinates $x = (x_1, x_2, x_3)$ on $M \setminus K$. We write $r = |x|$ and introduce the standard weight

$$\langle x \rangle = \sqrt{1 + r^2}.$$

The idea of a weighted Sobolev space is simple: we measure the $L^2$-size of a spinor, but multiply by a power of $\langle x \rangle$ to encode how fast it decays (or grows) at infinity.

**Definition 6.1.** Let $\delta \in \mathbb{R}$. The weighted $L^2$-space $L^2_\delta(S)$ consists of (measurable) spinor fields $\psi$ on $M$ such that

$$\|\psi\|^2_{L^2_\delta} := \int_M \langle x \rangle^{2\delta} \, |\psi(x)|^2 \, d\mu_g < \infty.$$

The weighted Sobolev space $W^{1,2}_\delta(S)$ consists of spinors $\psi$ with $\psi$ and $\nabla^S \psi$ both in $L^2_\delta(S)$; we set

$$\|\psi\|^2_{W^{1,2}_\delta} := \|\psi\|^2_{L^2_\delta} + \|\nabla^S \psi\|^2_{L^2_\delta}.$$

Here $|\psi(x)|$ is the pointwise norm induced by the Hermitian metric on $S$ and $d\mu_g$ is the Riemannian volume measure. On the compact set $K$ the factor $\langle x \rangle^{2\delta}$ is harmless; all the interesting behavior comes from the region where $|x|$ is large.

*Remark* 6.2. One can check that different choices of AF coordinates yield equivalent norms on $L^2_\delta$ and $W^{1,2}_\delta$, so these spaces are canonically associated to the AF structure. We will not prove this here; the key point is that any two AF charts differ by a diffeomorphism whose derivatives converge to an isometry at infinity.

A few simple examples help to calibrate the weights. For simplicity, think of scalar functions $u(x)$ on $\mathbb{R}^3$ and use the Euclidean measure $dx$.

- If $u(x) \equiv 1$ (a constant), then

  $$\|u\|^2_{L^2_\delta} \sim \int_{R_0}^\infty r^2 \langle r \rangle^{2\delta} \, dr.$$

  This converges iff $2 + 2\delta < -1$, i.e. $\delta < -3/2$. So a constant function belongs to $L^2_\delta$ only for sufficiently *negative* $\delta$.
- If $u(x) \sim r^{-1}$, then

  $$\|u\|^2_{L^2_\delta} \sim \int_{R_0}^\infty r^2 r^{-2} \langle r \rangle^{2\delta} \, dr \sim \int_{R_0}^\infty r^{2\delta} \, dr,$$

  which converges iff $2\delta < -1$, i.e. $\delta < -1/2$.

The moral is:

- large positive $\delta$ allow growth at infinity;
- large negative $\delta$ force strong decay;
- the range $\delta \in (-1, 0)$ is naturally adapted to objects that are *asymptotic to a constant*, with an error decaying like $r^{-1}$.

This is exactly the regime we will use: we want to solve for $\psi$ such that $\psi - \psi_\infty$ decays like $r^{-1}$, while $\psi_\infty$ itself should not lie in $L^2_\delta$ (otherwise the boundary value at infinity would be absorbed into the space and we would lose control of it).

6.3. **Fredholm property of the Dirac operator (statement only).** Weighted Sobolev spaces provide a natural domain and target for $D$: differentiating a spinor roughly multiplies by $1/r$ at infinity, so it is natural to consider the map

$$D : W^{1,2}_\delta(S) \longrightarrow L^2_{\delta+1}(S).$$

The shift in weight reflects that $D$ has first-order derivatives.

The deep analytic input is that, for AF metrics and for suitable $\delta$, this map is as well-behaved as a finite-dimensional linear map.

**Theorem 6.3** (Fredholm property of $D$; no proof)**.** *Let $(M, g)$ be an asymptotically flat Riemannian spin 3–manifold with $R \geq 0$. There exists an interval of weights $\delta$ (in particular, every $\delta \in (-1, 0)$ works) such that*

$$D : W^{1,2}_\delta(S) \longrightarrow L^2_{\delta+1}(S)$$

*is a Fredholm operator of index 0. Moreover, in this range of weights any $L^2$-integrable harmonic spinor must vanish, so*

$$\ker D = \{0\}.$$

We briefly unpack the terminology.

- *Fredholm* means:
  - the kernel $\ker D$ is finite-dimensional;
  - the cokernel (the quotient of the target by the range) is finite-dimensional;
  - the range is closed.
- The *index* of $D$ is

$$\operatorname{ind} D = \dim \ker D - \dim \operatorname{coker} D.$$

  Saying that $\operatorname{ind} D = 0$ means that the dimension of the cokernel equals that of the kernel.

Intuitively, $D$ behaves like an invertible linear map between infinite-dimensional spaces, except for a finite-dimensional "defect". When $\ker D = \{0\}$ and the index is 0, this defect disappears and the map is actually an isomorphism:

$$D : W^{1,2}_\delta(S) \xrightarrow{\cong} L^2_{\delta+1}(S).$$

Theorem 6.3 is proved in detail in standard references on geometric analysis of AF manifolds (for instance, in expositions following Parker–Taubes or Lee). The main ingredients are:

- the fact that $D$ is a small perturbation, at infinity, of the constant coefficient Dirac operator on $\mathbb{R}^3$;
- elliptic estimates in weighted Sobolev spaces;
- the Lichnerowicz identity, which shows that an $L^2$ harmonic spinor must vanish when $R \geq 0$ and the boundary term at infinity is zero.

For our purposes, we take Theorem 6.3 as a black box.

6.4. **Existence and uniqueness of harmonic spinors with prescribed asymptotics.** We now combine the Fredholm picture with a simple cutoff construction to obtain the spinor used in Witten's argument.

**Theorem 6.4** (Harmonic spinors with prescribed asymptotics)**.** *Let $(M^3, g)$ be a complete, oriented, asymptotically flat Riemannian spin 3–manifold with $R \geq 0$ and a single end. Fix a weight $\delta \in (-1, 0)$ for which Theorem 6.3 holds.*

*Then for every constant spinor $\psi_\infty \in \Sigma$ there exists a unique smooth spinor field $\psi \in \Gamma(S)$ such that*

*(i)* $D\psi = 0$ *on* $M$ *(harmonic spinor);*
*(ii)* $\psi - \psi_\infty \in W^{1,2}_\delta(S)$*;*
*(iii) in the AF chart,* $\psi(x) - \psi_\infty = O(r^{-1})$ *as* $r \to \infty$*.*

We sketch the main ideas, leaving analytic details to the references.

*Proof sketch.* **Step 1: Cutoff of the constant spinor.** Choose a smooth cutoff function $\chi$ on $M$ such that

$$\chi(x) = \begin{cases} 0 & \text{for } |x| \le R_0, \\ 1 & \text{for } |x| \ge 2R_0, \end{cases}$$

with $|\nabla\chi| \le C/R_0$ in the intermediate annulus. Define

$$\psi_0(x) = \chi(x)\,\psi_\infty,$$

viewed as a section of $S$ via the fixed trivialization on the end.

By construction, $\psi_0$ equals $\psi_\infty$ near infinity and vanishes in the interior region. It is smooth and compactly supported modulo a constant tail.

Compute

$$f := D\psi_0.$$

Since $D$ is first order and $\psi_\infty$ is constant in the AF chart, $f$ is supported in the region where $\nabla\chi \neq 0$, i.e. in a fixed compact annulus. In particular, $f \in L^2_{\delta+1}(S)$ for every $\delta$.

**Step 2: Solve away the error.** We want to find a correction term $\phi \in W^{1,2}_\delta(S)$ such that

$$D\phi = -f.$$

Assuming Theorem 6.3, the map

$$D : W^{1,2}_\delta(S) \longrightarrow L^2_{\delta+1}(S)$$

is an isomorphism when $\delta \in (-1, 0)$, so there exists a unique $\phi \in W^{1,2}_\delta(S)$ solving this equation.

Define the corrected spinor

$$\psi = \psi_0 + \phi.$$

Then

$$D\psi = D\psi_0 + D\phi = f + (-f) = 0,$$

and by construction

$$\psi - \psi_\infty = (\psi_0 - \psi_\infty) + \phi$$

belongs to $W^{1,2}_\delta(S)$ and vanishes near infinity at the rate encoded by the weight $\delta$.

**Step 3: Regularity and decay.** The elliptic regularity for $D$ and the fact that $f$ is smooth imply that $\phi$, and hence $\psi$, is smooth on $M$. Standard weighted elliptic estimates for AF operators show that if $\psi - \psi_\infty \in W^{1,2}_\delta$ with $\delta \in (-1, 0)$, then in AF coordinates we actually have a pointwise decay of the form

$$\psi(x) - \psi_\infty = O(r^{-1}),$$

together with similar decay for $\nabla^S\psi$.

**Step 4: Uniqueness.** Suppose $\psi_1$ and $\psi_2$ are two smooth solutions of $D\psi = 0$ with the same asymptotic value $\psi_\infty$ and with $\psi_k - \psi_\infty \in W^{1,2}_\delta(S)$. Then their difference

$$\varphi = \psi_1 - \psi_2$$

belongs to $W^{1,2}_\delta(S)$ and solves $D\varphi = 0$. By the kernel-free part of Theorem 6.3, we must have $\varphi \equiv 0$, so $\psi_1 = \psi_2$. □

Thus, once the Fredholm theory is in place, the existence and uniqueness of the desired harmonic spinor is conceptually simple: we prescribe its behavior at infinity, construct an approximate solution by cutting off a constant spinor, and then correct it by solving a linear elliptic equation in a weighted Sobolev space.

For the rest of the paper we will take Theorems 6.3 and 6.4 as a black box: for every constant spinor $\psi_\infty$ at infinity there is a unique smooth harmonic spinor with the prescribed asymptotics. This will be the only analytic input in the proof of the positive mass theorem in Section 7.

## 7. Witten's Spinorial Proof of the Riemannian Positive Mass Theorem

In this final section we put together the geometric and analytic ingredients from the previous sections to give Witten's spinorial proof of the Riemannian positive mass theorem in dimension 3. Throughout we keep the standing assumptions:

- $(M^3, g)$ is a complete, oriented, asymptotically flat Riemannian *spin* 3–manifold with a single end;
- the scalar curvature satisfies $R \geq 0$;
- $m_{\mathrm{ADM}}(M, g)$ denotes the ADM mass defined in Section 1.

Our goal is to show that $m_{\mathrm{ADM}}(M, g) \geq 0$, with rigidity in the case of equality. We will assume familiarity with the constructions and notation from Sections 3–6: the spinor bundle $S \to M$, the spin connection $\nabla^S$, the Dirac operator $D$, the integrated Lichnerowicz formula, the asymptotic analysis of Section 5, and the existence theorem for harmonic spinors from Section 6.

7.1. **Setup of the proof.** Let $\Sigma \cong \mathbb{C}^2$ be the model spinor space from Section 2, endowed with its fixed Hermitian inner product. On the AF end we have a canonical trivialization

$$S|_{M\setminus K} \;\cong\; (M \setminus K) \times \Sigma$$

compatible with the AF coordinates and the spin structure (Section 5.1). In this trivialization, a spinor field on the end may be viewed as a $\Sigma$–valued function.

Fix once and for all a nonzero constant spinor

$$\psi_\infty \in \Sigma,$$

and normalize it so that $|\psi_\infty| = 1$. In the AF trivialization, $\psi_\infty$ is literally a constant vector in $\mathbb{C}^2$.

By Theorem 6.4, for this choice of $\psi_\infty$ there exists a unique smooth spinor field

$$\psi \in \Gamma(S)$$

such that

(1) $D\psi = 0$ on $M$ (so $\psi$ is harmonic);

(2) $\psi - \psi_\infty \in W^{1,2}_\delta(S)$ for some fixed $\delta \in (-1, 0)$;
(3) in AF coordinates,

$$\psi(x) - \psi_\infty = O(r^{-1}) \quad \text{as } r = |x| \to \infty.$$

Elliptic regularity and the structure of $D$ on the AF end (Section 5.3) imply in particular that $\nabla^S \psi$ also decays at infinity; more precisely, one has

$$\nabla^S \psi = O(r^{-2})$$

in the AF chart (we do not reprove this here; it follows from the weighted Sobolev bounds and local elliptic estimates).

We will use this $\psi$ as the spinor in Witten's argument.

7.2. **Energy identity on extrinsic balls.** For $R$ sufficiently large, let

$$\Omega_R := M \cap B_R(0), \qquad S_R := \partial \Omega_R$$

where $B_R(0) \subset \mathbb{R}^3$ is the Euclidean ball of radius $R$ in the fixed AF chart at infinity. The boundary $S_R$ is then a large coordinate sphere, oriented by the outward unit normal $\nu$ with respect to $g$.

The integrated Lichnerowicz formula from Section 4.6 (applied to $\psi$ and $\Omega_R$) reads

$$\int_{\Omega_R} \left( |\nabla^S \psi|^2 + \frac{1}{4} R |\psi|^2 - |D\psi|^2 \right) d\mu_g \;=\; \int_{S_R} \left\langle \nabla^S_\nu \psi - c(\nu) D\psi,\, \psi \right\rangle dS_g. \tag{7.1}$$

Since $\psi$ is harmonic, $D\psi = 0$, and this simplifies to

$$\int_{\Omega_R} \left( |\nabla^S \psi|^2 + \frac{1}{4} R |\psi|^2 \right) d\mu_g \;=\; \int_{S_R} \left\langle \nabla^S_\nu \psi,\, \psi \right\rangle dS_g. \tag{7.2}$$

The integrand on the left-hand side is pointwise nonnegative, because $|\nabla^S \psi|^2 \geq 0$ and $R \geq 0$. As $R$ increases, the domains $\Omega_R$ exhaust $M$, so the left-hand side defines an increasing function of $R$:

$$I(R) := \int_{\Omega_R} \left( |\nabla^S \psi|^2 + \frac{1}{4} R |\psi|^2 \right) d\mu_g.$$

The decay of $\psi$ and $\nabla^S \psi$ on the AF end (from Theorem 6.4 and elliptic estimates) implies that $I(R)$ remains bounded as $R \to \infty$, and hence the limit

$$I_\infty := \lim_{R \to \infty} I(R) = \int_M \left( |\nabla^S \psi|^2 + \frac{1}{4} R |\psi|^2 \right) d\mu_g$$

exists and is finite. By construction, $I_\infty \geq 0$.

Thus the energy identity (7.2) shows that the asymptotic behavior of the boundary term

$$\int_{S_R} \left\langle \nabla^S_\nu \psi,\, \psi \right\rangle dS_g$$

controls the global "energy" $I_\infty$.

7.3. **Evaluation of the boundary term.** We now recall the analysis of the boundary term from Section 5. On the AF end we work in the canonical trivialization of $S$ and use the almost orthonormal frame constructed there. The detailed calculation in Sections 5.3–5.5 shows that, for a harmonic spinor $\psi$ which tends to a constant $\psi_\infty$ at infinity, the boundary integral can be written as

$$\int_{S_R} \langle \nabla^S_\nu \psi,\, \psi \rangle \, dS_g \;=\; C_3' \int_{S_R} (\partial_j g_{ij} - \partial_i g_{jj})\, \nu^i \, dS_E \;+\; E(R), \tag{7.3}$$

where:

- $C_3' > 0$ is a dimensional constant (depending only on dimension and the chosen conventions for the spin representation);
- $dS_E$ denotes the Euclidean surface measure on $S_R$ in the AF coordinate chart;
- $\nu^i$ is the Euclidean outward unit normal to $S_R$;
- the error term $E(R)$ satisfies $E(R) \to 0$ as $R \to \infty$.

The key points in this computation are:

(1) The spin connection $\nabla^S$ differs from the Euclidean spin connection by terms that decay like $O(r^{-1-\tau})$ (Section 5.3).
(2) The harmonic spinor $\psi$ satisfies $\psi(x) = \psi_\infty + O(r^{-1})$ and $\nabla^S \psi = O(r^{-2})$ (Theorem 6.4 and elliptic estimates).
(3) The metric $g$ differs from the Euclidean metric by $g_{ij} = \delta_{ij} + h_{ij}$ with $h_{ij} = O(r^{-1})$ and $\partial_k h_{ij} = O(r^{-2})$.

Combining these facts and keeping careful track of orders of decay shows that only the first-order derivatives of $g$ contribute in the limit; all other terms produce integrals that vanish as $R \to \infty$. This yields (7.3) with $E(R) \to 0$.

By the definition of the ADM mass, we have

$$m_{\mathrm{ADM}}(M, g) \;=\; \frac{1}{16\pi} \lim_{R\to\infty} \int_{S_R} (\partial_j g_{ij} - \partial_i g_{jj})\, \nu^i \, dS_E.$$

Thus (7.3) immediately implies

$$\lim_{R\to\infty} \int_{S_R} \langle \nabla^S_\nu \psi,\, \psi \rangle \, dS_g \;=\; C_3''\, m_{\mathrm{ADM}}(M, g), \tag{7.4}$$

for some positive constant $C_3'' > 0$ (absorbing the factor $16\pi$ into the normalization). Since we chose $|\psi_\infty| = 1$, no additional factor of $|\psi_\infty|^2$ appears; for a general $\psi_\infty$ the right-hand side would be proportional to $|\psi_\infty|^2$.

7.4. **Taking the limit and obtaining nonnegativity.** We now combine the energy identity (7.2) with the asymptotic expansion (7.4). Taking the limit $R \to \infty$ in (7.2) gives

$$\int_M \left( |\nabla^S \psi|^2 + \frac{1}{4} R |\psi|^2 \right) d\mu_g \;=\; \lim_{R\to\infty} \int_{S_R} \langle \nabla^S_\nu \psi,\, \psi \rangle \, dS_g \;=\; C_3''\, m_{\mathrm{ADM}}(M, g). \tag{7.5}$$

The integrand on the left-hand side is pointwise nonnegative, so the integral is $\geq 0$. The constant $C_3''$ is strictly positive (it comes from the normalization of the spin representation and the ADM mass), so (7.5) immediately implies

$$m_{\mathrm{ADM}}(M, g) \;\geq\; 0.$$

This proves the *nonnegativity* part of the Riemannian positive mass theorem in the spin case.

7.5. **Rigidity in the case of zero mass.** We now analyze the case of equality $m_{\mathrm{ADM}}(M,g)=0$. From (7.5) we obtain

$$\int_M \left( |\nabla^S \psi|^2 + \frac{1}{4} R|\psi|^2 \right) d\mu_g \;=\; 0.$$

Since the integrand is nonnegative, it must vanish identically:

$$|\nabla^S \psi|^2 \equiv 0, \qquad R|\psi|^2 \equiv 0.$$

Because $\psi$ is not identically zero (it tends to a nonzero constant $\psi_\infty$ at infinity), we conclude that

$$\nabla^S \psi \equiv 0 \quad \text{and} \quad R \equiv 0$$

on $M$. Thus $\psi$ is a *parallel spinor* and the scalar curvature vanishes.

To go further, we use the *spinorial curvature identity*, which relates the curvature of the spin connection to the Riemannian curvature. Let $R^S$ denote the curvature of $\nabla^S$, and let $R$ and Ric be the Riemann and Ricci curvature tensors of $g$. Then one has the identity

$$\sum_{i=1}^{3} e_i \cdot R^S(X, e_i)\psi \;=\; \frac{1}{2}\,\mathrm{Ric}(X)\cdot\psi, \tag{7.6}$$

for all vector fields $X$ and all spinors $\psi$, where $\{e_1, e_2, e_3\}$ is any local orthonormal frame and $\cdot$ denotes Clifford multiplication. This is a standard consequence of the definition of $\nabla^S$ and of the Clifford representation (we do not rederive it here).

Since $\nabla^S\psi \equiv 0$, the curvature $R^S$ annihilates $\psi$:

$$R^S(X,Y)\psi = 0 \quad \text{for all } X, Y.$$

Plugging this into (7.6) gives

$$\frac{1}{2}\,\mathrm{Ric}(X)\cdot\psi = 0 \quad \text{for all } X.$$

At each point $p \in M$ where $\psi(p) \neq 0$, Clifford multiplication by a nonzero vector is injective on the spinor space: if $v \cdot \varphi = 0$, then

$$\|v\cdot\varphi\|^2 = \|v\|^2\|\varphi\|^2$$

by the skew-Hermitian property of Clifford multiplication, so $v = 0$ or $\varphi = 0$. Applying this with $\varphi = \psi(p)$, we deduce that

$$\mathrm{Ric}(X)_p = 0 \quad \text{for all } X,$$

and hence $\mathrm{Ric} \equiv 0$ on $M$.

In dimension 3 the full Riemann curvature tensor is determined by the Ricci tensor via the algebraic identity

$$R_{ijkl} = \mathrm{Ric}_{ik}g_{jl} - \mathrm{Ric}_{il}g_{jk} - \mathrm{Ric}_{jk}g_{il} + \mathrm{Ric}_{jl}g_{ik} - \frac{R}{2}(g_{ik}g_{jl} - g_{il}g_{jk}).$$

Since both $R$ and Ric vanish, we conclude that the Riemann curvature vanishes identically:

$$\mathrm{Rm} \equiv 0.$$

Thus $(M,g)$ is a complete, flat Riemannian 3–manifold with one asymptotically flat end.

The classification of complete flat manifolds then implies that $(M, g)$ is isometric to a quotient of $(\mathbb{R}^3, \delta)$ by a discrete group of isometries acting freely and properly discontinuously. The asymptotically flat assumption with a single end rules out nontrivial quotients (for instance, any nontrivial quotient would either be compact or have a different end structure), so the only possibility is

$$(M, g) \cong (\mathbb{R}^3, \delta).$$

This proves the rigidity statement: if $m_{\mathrm{ADM}}(M, g) = 0$, then $(M, g)$ is isometric to Euclidean space.

**Conceptual recap.** Let us summarize Witten's spinorial proof in a condensed form:

- On an asymptotically flat spin 3–manifold with $R \geq 0$, one constructs a harmonic spinor $\psi$ with prescribed nonzero limit $\psi_\infty$ at infinity.
- The Lichnerowicz formula identifies $D^2$ with the rough Laplacian plus a scalar curvature term. When applied to $\psi$ and integrated over $\Omega_R$, this yields an energy identity:
$$\int_{\Omega_R} \left( |\nabla^S \psi|^2 + \frac{1}{4} R |\psi|^2 \right) d\mu_g = \int_{S_R} \left\langle \nabla^S_\nu \psi,\, \psi \right\rangle dS_g.$$
- The asymptotic analysis on the AF end shows that the boundary flux on $S_R$ converges, as $R \to \infty$, to a positive multiple of the ADM mass:
$$\lim_{R \to \infty} \int_{S_R} \left\langle \nabla^S_\nu \psi,\, \psi \right\rangle dS_g = C_3''\, m_{\mathrm{ADM}}(M, g).$$
- Taking the limit $R \to \infty$ gives an identity of the form
$$\int_M \left( |\nabla^S \psi|^2 + \frac{1}{4} R |\psi|^2 \right) d\mu_g = C_3''\, m_{\mathrm{ADM}}(M, g),$$
so $m_{\mathrm{ADM}}(M, g) \geq 0$.
- If $m_{\mathrm{ADM}}(M, g) = 0$, then the integrand vanishes: $\psi$ is parallel and $R \equiv 0$. The spinorial curvature identity forces $\mathrm{Ric} \equiv 0$, hence the metric is flat. Asymptotic flatness with one end then implies $(M, g) \cong (\mathbb{R}^3, \delta)$.

In this way, the positivity and rigidity of the ADM mass emerge from a single spinorial object $\psi$ whose existence is guaranteed by elliptic theory, and whose interaction with curvature is encoded by the Lichnerowicz formula and the geometry of the AF end.