GRASSMANNIANS AS MODULI SPACES: FUNCTOR OF POINTS AND REPRESENTABILITY

YUXUAN FAN

Abstract. We study Grassmannians as moduli spaces through the functor-of-points formalism over an arbitrary base scheme S. Given a rank-n vector bundle E on S and an integer d, we consider the functor $G_{d,E}$ that assigns to an S-scheme T short exact sequences $0 \to U \to E_T \to Q \to 0$ with U and Q locally free of ranks d and $n-d$. We prove that $G_{d,E}$ is representable by a scheme $\mathrm{Gr}_S(d;E)$ by exhibiting the standard open subfunctors where a fixed minor is invertible and identifying each with an affine scheme via explicit matrix coordinates; the sheaf condition guarantees these charts glue uniquely. This construction yields the universal exact sequence on $\mathrm{Gr}_S(d;E)$ and its functoriality and base-change compatibility as formal consequences of representability. Over a field we recover the Plücker line bundle and show that the Plücker morphism is a closed immersion, and we compute the Zariski tangent space at $[U]$ as $\mathrm{Hom}(U, V/U)$, matching the linear-algebraic description with the moduli viewpoint.

Contents

1. Category-theoretic warm-up and functor of points

The goal of this section is to fix some categorical language and to explain what it means to look at a scheme "via its points." We will work in enough generality to apply later to schemes over a base scheme S, but we do not assume any higher category theory beyond basic notions of categories, functors, and natural transformations.

1.1. Categories, functors, and natural transformations.

We begin by recalling the basic language.

Definition 1.1. A *category* $\mathcal{C}$ consists of the following data:

(1) a class of *objects*, denoted $\mathrm{Ob}(\mathcal{C})$;
(2) for any pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set
$$\mathrm{Hom}_{\mathcal{C}}(X, Y)$$
of *morphisms* (or *arrows*) from X to Y;
(3) for any $X \in \mathrm{Ob}(\mathcal{C})$, a distinguished *identity morphism* $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$;
(4) for any composable pair of morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, a composite
$$g \circ f \in \mathrm{Hom}_{\mathcal{C}}(X, Z),$$

subject to the usual associativity and identity axioms:

(a) $(h \circ g) \circ f = h \circ (g \circ f)$ whenever both sides make sense;
(b) $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Example 1.2. Standard examples include:

(1) the category **Set** of sets and functions;
(2) the category **Ab** of abelian groups and homomorphisms;
(3) the category **Ring** of commutative rings and ring homomorphisms;
(4) the category Sch of schemes and morphisms of schemes;
(5) for a fixed scheme S, the category (Sch/S) of S-schemes (objects are morphisms $X \to S$, morphisms are commutative triangles over S).

Definition 1.3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ consists of

(1) for each object $X \in \mathrm{Ob}(\mathcal{C})$, an object $F(X) \in \mathrm{Ob}(\mathcal{D})$;
(2) for each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$,

such that

(a) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all X;
(b) $F(g \circ f) = F(g) \circ F(f)$ whenever $g \circ f$ is defined.

A *contravariant functor* $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is a covariant functor defined on the opposite category $\mathcal{C}^{\mathrm{op}}$, whose objects are those of $\mathcal{C}$ and whose morphisms are reversed.

Definition 1.4. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\eta : F \Rightarrow G$ consists of a morphism

$$\eta_X : F(X) \longrightarrow G(X)$$

in $\mathcal{D}$ for each object $X \in \mathcal{C}$, such that for every morphism $f : X \to Y$ in $\mathcal{C}$ the square

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

commutes. In this situation we also say that the family $\{\eta_X\}_{X \in \mathcal{C}}$ is *natural in* X.

Remark 1.5. In this article we will almost always consider *set-valued* contravariant functors $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, where $\mathcal{C}$ is either Sch or (Sch/S) for a fixed base scheme S. One should think of $F(X)$ as the set of "X-families" of some kind of geometric object.

1.2. **Representable functors and Yoneda's lemma.** We now single out a particularly important class of functors.

Definition 1.6. Let $\mathcal{C}$ be a category and let X be an object of $\mathcal{C}$. The *Hom-functor* (or *functor of points* of X) is the contravariant functor

$$h_X : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}, \qquad h_X(T) = \mathrm{Hom}_{\mathcal{C}}(T, X),$$

whose action on morphisms is given by precomposition: for $f : T' \to T$ and $\xi \in h_X(T)$,

$$h_X(f)(\xi) = \xi \circ f \in h_X(T').$$

A contravariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is said to be *representable* if there exists an object $X \in \mathcal{C}$ and a natural isomorphism

$$\varphi : h_X \xrightarrow{\sim} F.$$

In this case we say that X *represents* F, or that F is *represented* by X.

The following fundamental result explains why representable functors are so rigid.

Theorem 1.7 (Yoneda lemma)**.** *Let $\mathcal{C}$ be a locally small category (so that each $\mathrm{Hom}_{\mathcal{C}}(T, X)$ is a set), let $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be a contravariant functor, and let X be an object of $\mathcal{C}$. Then there is a natural bijection*

$$\Phi : \mathrm{Nat}(h_X, F) \xrightarrow{\sim} F(X)$$

between natural transformations $\eta : h_X \Rightarrow F$ and elements of $F(X)$. Explicitly, given η, the corresponding element is $\Phi(\eta) = \eta_X(\mathrm{id}_X) \in F(X)$; conversely, given $x \in F(X)$, the corresponding natural transformation $\eta^x : h_X \Rightarrow F$ has components

$$\eta^x_T : \mathrm{Hom}_{\mathcal{C}}(T, X) \longrightarrow F(T), \qquad \eta^x_T(f) = F(f)(x).$$

Proof. We first construct the map Φ. Let $\eta : h_X \Rightarrow F$ be a natural transformation. We set

$$\Phi(\eta) := \eta_X(\mathrm{id}_X) \in F(X).$$

Conversely, given an element $x \in F(X)$, define for each object T the map

$$\eta^x_T : \mathrm{Hom}_{\mathcal{C}}(T, X) \longrightarrow F(T), \qquad \eta^x_T(f) = F(f)(x).$$

We must check that η^x_T is natural in T. Let $g : T' \to T$ be a morphism in $\mathcal{C}$ and let $f : T \to X$. Then

$$F(g)\big(\eta^x_T(f)\big) = F(g)\big(F(f)(x)\big) = F(f \circ g)(x) = \eta^x_{T'}(f \circ g) = \eta^x_{T'}\big(h_X(g)(f)\big),$$

which is precisely the commutativity of the naturality square.

It is straightforward to verify that the assignments $\eta \mapsto \Phi(\eta)$ and $x \mapsto \eta^x$ are inverse to one another. Indeed, starting from $x \in F(X)$ we have

$$\Phi(\eta^x) = \eta^x_X(\mathrm{id}_X) = F(\mathrm{id}_X)(x) = x.$$

Conversely, starting from η and using naturality with respect to $f : T \to X$ and $\mathrm{id}_X : X \to X$ we obtain

$$\eta_T(f) = F(f)\big(\eta_X(\mathrm{id}_X)\big) = F(f)\big(\Phi(\eta)\big) = \eta^{\Phi(\eta)}_T(f),$$

for all T and all $f : T \to X$, hence $\eta = \eta^{\Phi(\eta)}$. This shows that Φ is a bijection. Naturality of Φ in both F and X is a routine diagram chase and will not be needed later. □

Corollary 1.8 (Yoneda embedding)**.** *For any locally small category $\mathcal{C}$, the assignment*

$$X \longmapsto h_X, \qquad (f : X \to Y) \longmapsto (h_f : h_X \to h_Y),$$

defines a fully faithful functor

$$h_{(-)} : \mathcal{C} \longrightarrow \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}),$$

where h_f is given by postcomposition $h_f(T) : \mathrm{Hom}(T, X) \to \mathrm{Hom}(T, Y)$, $h_f(T)(\phi) = f \circ \phi$. In particular, for any objects X, Y there is a natural bijection

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\sim} \mathrm{Nat}(h_X, h_Y).$$

Proof. Apply Theorem 1.7 to $F = h_Y$. Then $\mathrm{Nat}(h_X, h_Y) \cong h_Y(X) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$. It is easy to verify that the map sending $f : X \to Y$ to the natural transformation h_f described above realises this bijection. □

Corollary 1.9. *If $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is representable in two ways, say $F \cong h_X$ and $F \cong h_Y$, then X and Y are uniquely isomorphic in $\mathcal{C}$. More precisely, any isomorphism $h_X \cong h_Y$ of functors is induced by a unique isomorphism $X \cong Y$.*

Proof. An isomorphism $h_X \cong h_Y$ is in particular a natural transformation $h_X \Rightarrow h_Y$. By Corollary 1.8 it comes from a unique morphism $f : X \to Y$. The existence of an inverse natural transformation implies that f is an isomorphism. □

Thus, from the point of view of set-valued contravariant functors, an object X is completely determined (up to unique isomorphism) by its Hom-functor h_X. This is the formal underpinning of the *functor-of-points* philosophy.

1.3. **Functor of points for schemes.**

We now specialise to the categories relevant for algebraic geometry.

Notation 1.10. We write Sch for the category of schemes and morphisms of schemes. For a fixed base scheme S, we denote by (Sch/S) the category whose objects are morphisms $X \to S$ (called *S-schemes*) and whose morphisms are commutative triangles over S. A morphism in (Sch/S) will often be denoted simply by $f : X \to Y$ when the structure morphisms to S are understood.

Definition 1.11. Let S be a fixed scheme and let $X \to S$ be an S-scheme. The *functor of points* of X (over S) is the contravariant functor

$$h_X : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}, \qquad h_X(T) = \mathrm{Hom}_S(T, X),$$

where $\mathrm{Hom}_S(T, X)$ denotes morphisms $T \to X$ of S-schemes. Given a morphism $f : T' \to T$ of S-schemes, the map

$$h_X(f) : h_X(T) \longrightarrow h_X(T')$$

is given by precomposition: $h_X(f)(\xi) = \xi \circ f$.

Remark 1.12. Thus $h_X(T)$ is the set of all families of S-morphisms $T \to X$, and h_X records how such families restrict along morphisms of S-schemes. The Yoneda lemma tells us that X can be recovered (up to unique isomorphism) from the functor h_X. In practice, we will often *define* a functor $F : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathbf{Set}$ encoding a moduli problem, and then ask whether F is representable by some S-scheme M.

Example 1.13. Let $S = \mathrm{Spec}(R)$ be affine and let $\mathbb{A}^n_S$ denote the relative affine n-space over S, that is $\mathbb{A}^n_S = \mathrm{Spec}(R[x_1, \ldots, x_n])$. Then for any S-scheme T we have a natural bijection

$$h_{\mathbb{A}^n_S}(T) = \mathrm{Hom}_S(T, \mathbb{A}^n_S) \cong \Gamma(T, \mathcal{O}_T(T))^{\oplus n},$$

sending an S-morphism $T \to \mathbb{A}^n_S$ to the n-tuple of global functions obtained by pulling back the coordinate functions x_i. Equivalently, a T-point of $\mathbb{A}^n_S$ is the same thing as an n-tuple of global sections of $\mathcal{O}_T$.

Example 1.14. Let X, Y, S be schemes and suppose given morphisms $X \to S$ and $Y \to S$. The fiber product $X \times_S Y$ is characterised by the property that for every S-scheme T there is a natural bijection

$$\mathrm{Hom}_S(T, X \times_S Y) \;\cong\; \mathrm{Hom}_S(T, X) \times \mathrm{Hom}_S(T, Y),$$

where the right-hand side consists of pairs of morphisms $T \to X$ and $T \to Y$ over S. In other words, the functor of points of $X \times_S Y$ is the product of the functors of points of X and Y.

Proof. By the universal property of the fiber product, a morphism $T \to X \times_S Y$ over S is equivalent to a pair of S-morphisms $T \to X$ and $T \to Y$. This correspondence is functorial in T, which gives the claimed natural bijection. $\square$

Remark 1.15. Example 1.14 is typical: many geometric constructions on schemes (products, fiber products, open and closed subschemes, projective space, and eventually the Grassmannian) can be described completely in terms of their functor of points. In the next sections we will recast the basic language of moduli spaces in this style.

2. Representable functors, Zariski sheaves, and a representability criterion

In this section we specialise the general language of representable functors to the category of schemes. We also introduce a sheaf condition for functors on schemes, and formulate a representability criterion that will later be applied to the Grassmannian functor.

Throughout this section, S denotes a fixed base scheme, and (Sch/S) the category of schemes over S.

2.1. Representable functors on (Sch/S).

We begin by recalling the notion of representability in the concrete setting we care about.

Definition 2.1. A contravariant functor

$$F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

is said to be *representable* if there exists an S-scheme $X \to S$ and a natural isomorphism

$$\varphi : h_X \xrightarrow{\sim} F,$$

where $h_X(T) = \mathrm{Hom}_S(T, X)$ is the functor of points of X. In this situation we say that X *represents* F.

By Corollary 1.9, the representing S-scheme X is then uniquely determined up to unique isomorphism.

Example 2.2. Let E be a finite locally free $\mathcal{O}_S$-module of rank n and let $\mathbb{P}(E)$ denote the associated projective bundle over S. There is a functor F from $(\mathrm{Sch}/S)^{\mathrm{op}}$ to sets sending an S-scheme T to the set of isomorphism classes of pairs $(\mathcal{L}, \iota)$, where $\mathcal{L}$ is an invertible $\mathcal{O}_T$-module and $\iota : E_T \twoheadrightarrow \mathcal{L}$ is a surjection. One can show that F is representable by $\mathbb{P}(E)$. In other words, morphisms $T \to \mathbb{P}(E)$ correspond naturally to line subbundles of $E_T^\vee$, or equivalently to quotient line bundles of E_T.

Remark 2.3. This example is typical of what we will do later: we first formulate a *moduli functor* F, encoding families of objects over varying bases T, and then seek to prove that F is representable by some S-scheme M. In this case $M = \mathbb{P}(E)$, and for the Grassmannian M will be a scheme parameterising higher rank subbundles or quotient bundles of a fixed vector bundle.

2.2. Zariski sheaves on (Sch/S).

To turn representability into a practical criterion, it is convenient to isolate a sheaf condition on functors.

Definition 2.4. A contravariant functor

$$F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

is called a *Zariski sheaf* (on (Sch/S)) if the following gluing property holds.

For every S-scheme T and every Zariski open covering $\{U_i \subset T\}_{i\in I}$, the sequence

$$F(T) \longrightarrow \prod_{i\in I} F(U_i) \rightrightarrows \prod_{(i,j)\in I\times I} F(U_i\cap U_j)$$

is an equaliser. Concretely, this means:

(1) (*Locality*) If $x, y \in F(T)$ have the same restriction $x|_{U_i} = y|_{U_i} \in F(U_i)$ for all i, then $x = y$.

(2) (*Gluing*) Given a family of elements $x_i \in F(U_i)$ such that $x_i|_{U_i\cap U_j} = x_j|_{U_i\cap U_j}$ in $F(U_i \cap U_j)$ for all i, j, there exists a unique $x \in F(T)$ with $x|_{U_i} = x_i$ for all i.

Lemma 2.5. *Every representable functor $F : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathbf{Set}$ is a Zariski sheaf.*

Proof. Suppose $F \cong h_X$ for some S-scheme X. Let T be an S-scheme and let $\{U_i \subset T\}$ be a Zariski open covering.

For the locality statement, suppose $f, g : T \to X$ are morphisms of S-schemes such that $f|_{U_i} = g|_{U_i}$ for all i. Since morphisms of schemes are determined by their restrictions to an open cover, we have $f = g$. Thus locality holds.

For gluing, suppose we are given morphisms $f_i : U_i \to X$ of S-schemes such that for all i, j the restrictions to $U_i \cap U_j$ coincide:

$$f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}.$$

By the usual gluing lemma for morphisms of schemes, there exists a unique morphism $f : T \to X$ of S-schemes whose restriction to U_i is f_i for each i. This morphism f corresponds to the desired element of $F(T) = \mathrm{Hom}_S(T, X)$. Uniqueness is again guaranteed by the gluing lemma. Hence F is a Zariski sheaf. □

Remark 2.6. One can formulate this sheaf condition more systematically by viewing (Sch/S) as a site for the Zariski topology, but we will only need the concrete form given in Definition 2.4.

2.3. **Open and closed subfunctors.** In practice we will often encounter a functor F together with "subfunctors" that behave like open or closed subschemes of a representing object. We now formalise this notion.

Definition 2.7. Let $F, G : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathbf{Set}$ be functors and let $\alpha : F \Rightarrow G$ be a natural transformation. We say that α is *monomorphism* (or that F is a *subfunctor* of G) if for every S-scheme T the map

$$\alpha_T : F(T) \longrightarrow G(T)$$

is injective. In this case we often tacitly identify $F(T)$ with a subset of $G(T)$ and write $F \subset G$.

We now single out the case when G is representable.

Definition 2.8. Let X be an S-scheme and let $F \subset h_X$ be a subfunctor.

(1) We say that F is an *open subfunctor* of h_X if for every morphism $f : T \to X$ of S-schemes, the fiber product

$$F \times_{h_X} h_T$$

is representable by an open subscheme $U \subset T$. Equivalently, for each $T \to X$ there exists an open subscheme $U \subset T$ such that

$$\big(F \times_{h_X} h_T\big)(T') = \mathrm{Hom}_S(T', U)$$

for all $T' \to S$.

(2) We say that F is a *closed subfunctor* of h_X if for every morphism $f : T \to X$ of S-schemes, the fiber product $F \times_{h_X} h_T$ is representable by a closed subscheme $Z \subset T$.

Remark 2.9. If $U \subset X$ is an open subscheme, then the inclusion $h_U \hookrightarrow h_X$ is an open subfunctor in the above sense: indeed, for $f : T \to X$ the fiber product $h_U \times_{h_X} h_T$ is naturally isomorphic to $h_{U \times_X T}$, and $U \times_X T$ is an open subscheme of T. Similarly, closed subschemes give rise to closed subfunctors.

Remark 2.10. Definition 2.8 is a special case of the more general notion of a *representable morphism of functors*: a natural transformation $\alpha : F \Rightarrow G$ is called representable if for every S-scheme T and every element $\xi \in G(T)$ the fiber product $F \times_G h_T$ is representable by an S-scheme. In this language, $F \subset h_X$ is open (resp. closed) if the inclusion is a representable monomorphism and all base changes along $T \to X$ are open (resp. closed) immersions.

2.4. **A representability criterion via open covers.** We now record a useful criterion for recognising representable functors by means of an open cover by representable subfunctors. This will be the main tool in our later construction of the Grassmannian.

Theorem 2.11 (Representability via an open cover)**.** *Let*

$$F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

be a Zariski sheaf. Suppose there exists a family of subfunctors $\{F_i \subset F\}_{i \in I}$ such that:

*(1) (*Local cover*) For every S-scheme T and every element $x \in F(T)$ there exists a Zariski open covering $\{U_\alpha \subset T\}$ and indices $i(\alpha) \in I$ such that $x|_{U_\alpha} \in F_{i(\alpha)}(U_\alpha)$ for all α.*

*(2) (*Each F_i is representable*) For each $i \in I$ there exists an S-scheme X_i and a natural isomorphism*

$$\varphi_i : h_{X_i} \xrightarrow{\sim} F_i.$$

*(3) (*Intersections glue as opens*) For all $i, j \in I$, the intersection $F_i \cap F_j$ is representable, and under the identifications $F_i \cong h_{X_i}$ and $F_j \cong h_{X_j}$ it corresponds to open subschemes*

$$U_{ij} \subset X_i, \qquad U_{ji} \subset X_j$$

which are isomorphic via a unique isomorphism $U_{ij} \xrightarrow{\sim} U_{ji}$. These isomorphisms satisfy the evident cocycle condition on triple overlaps.

Then F is representable by an S-scheme X, obtained by gluing the X_i along the identifications $U_{ij} \cong U_{ji}$. Moreover, under the resulting identification $F \cong h_X$, each F_i corresponds to the subfunctor $h_{X_i} \subset h_X$ represented by the open subscheme $X_i \subset X$.

Proof (sketch). Using the data in (3), we may glue the schemes $\{X_i\}_{i \in I}$ along the isomorphisms $U_{ij} \xrightarrow{\sim} U_{ji}$ to obtain an S-scheme X together with open immersions $X_i \hookrightarrow X$ whose images cover X and such that $U_{ij} = X_i \cap X_j$.

We claim that $h_X \cong F$. For each i we have a natural isomorphism $\varphi_i : h_{X_i} \to F_i$; since X_i is an open subscheme of X, $m_i : h_{X_i} \to h_X$ is an open subfunctor, and we may view h_{X_i} and F_i as subfunctors of h_X and F respectively. The compatibility conditions on the overlaps imply that the φ_i agree on intersections, hence glue to a natural transformation

$$\varphi : h_X \longrightarrow F.$$

It remains to show that φ is an isomorphism. This can be checked Zariski locally on the source and the target. Condition (1) says that elements of $F(T)$ can be covered by pieces lying in various F_i, while the fact that $\{X_i\}$ cover X says the same for h_X. Since on each X_i the map φ restricts to the isomorphism $\varphi_i : h_{X_i} \to F_i$, and since both h_X and F are Zariski sheaves, one deduces that φ is a bijection on T-points for every T. The details are a straightforward diagram chase and are left to the reader.

Finally, by construction F_i corresponds to h_{X_i} under the isomorphism $F \cong h_X$. □

Remark 2.12. The theorem should be compared with the familiar fact that a scheme can be glued from an open cover by affine schemes whose overlaps agree in the obvious way. The difference here is that we start only with a functor F and enough local charts F_i to build a global representing scheme.

In the next section we will apply Theorem 2.11 to the Grassmannian functor: we will construct a Zariski cover by representable open subfunctors, and then deduce representability from the criterion above.

3. Moduli problems, fine moduli spaces, and universal families

We now recast representable functors in the language of moduli. The Grassmannian will later appear as the fine moduli space for a very concrete moduli problem involving subbundles or quotient bundles of a fixed vector bundle.

Throughout this section we fix a base scheme S and work over the category (Sch/S) as before.

3.1. **Moduli problems and families.**

The first step is to separate *objects* (such as vector bundles or subschemes) from the *families* in which they vary over different bases.

Definition 3.1. Let $\mathcal{C}$ be some class of geometric objects defined over schemes over S (for instance, vector bundles of fixed rank on S-schemes, or closed subschemes of a fixed ambient scheme). A *family of objects in $\mathcal{C}$ over an S-scheme T* is an object X_T in $\mathcal{C}$ "lying over" T, in a suitable sense.

In the examples we have in mind, a family over T will be, for instance, a vector bundle on T, or a closed subscheme of some fixed T-scheme. Two families X_T and X'_T over the same base T are *isomorphic* if there is an isomorphism between them over T.

Remark 3.2. The exact definition of a "family over T" depends on the moduli problem under consideration; we will make it explicit in each case. The common feature is that base change along a morphism $f : T' \to T$ yields a pullback family $X_{T'}$ over T', and that isomorphisms of families are required to be compatible with this base change.

Definition 3.3. A *moduli problem* on (Sch/S) is specified by giving, for each S-scheme T, a set $\mathcal{F}(T)$ of isomorphism classes of families of objects over T (in some fixed class $\mathcal{C}$), and for each morphism $f : T' \to T$ of S-schemes, a pullback map

$$f^* : \mathcal{F}(T) \longrightarrow \mathcal{F}(T')$$

satisfying the functoriality conditions

$$(\mathrm{id}_T)^* = \mathrm{id}_{\mathcal{F}(T)}, \qquad (g \circ f)^* = f^* \circ g^*$$

for composable morphisms $T'' \xrightarrow{f} T' \xrightarrow{g} T$. Equivalently, a moduli problem is a contravariant functor

$$\mathcal{F} : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}.$$

We call $\mathcal{F}$ the *moduli functor*.

Example 3.4. Let E be a fixed finite locally free $\mathcal{O}_S$-module of rank n. Fix an integer d with $0 < d < n$. We may define a moduli functor $\mathcal{F}$ sending an S-scheme T to the set of isomorphism classes of short exact sequences

$$0 \longrightarrow \mathcal{U} \longrightarrow E_T \xrightarrow{q} \mathcal{Q} \longrightarrow 0$$

on T, where $\mathcal{U}$ is finite locally free of rank d (equivalently, $\mathcal{Q}$ is finite locally free of rank $n-d$), and where isomorphisms between two such sequences are isomorphisms $\mathcal{U} \cong \mathcal{U}'$ and $\mathcal{Q} \cong \mathcal{Q}'$ making the evident diagram commute (in particular, inducing the identity on E_T). Base change along $f : T' \to T$ sends the sequence to its pullback

$$0 \longrightarrow f^*\mathcal{U} \longrightarrow E_{T'} \xrightarrow{f^*q} f^*\mathcal{Q} \longrightarrow 0.$$

This moduli functor is represented by the Grassmannian bundle $\mathrm{Gr}_S(d, E)$ of rank d subbundles of E (equivalently, of rank $n - d$ quotient bundles of E).

Remark 3.5. One could similarly define a moduli functor whose T-points are rank d subbundles $U \subset E_T$ with locally free quotient E_T/U. As we will see later, these two descriptions are equivalent; the choice between "subbundles" and "quotient bundles" is largely a matter of taste and convenience.

3.2. **Fine moduli spaces and universal families.** We now formulate the notion of a fine moduli space, which is simply a representing object for a given moduli functor.

Definition 3.6. Let $\mathcal{F} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathbf{Set}$ be a moduli functor. A *fine moduli space* for $\mathcal{F}$ consists of an S-scheme M together with an isomorphism of functors

$$\Phi : h_M \xrightarrow{\sim} \mathcal{F}.$$

Equivalently, M is a fine moduli space if $\mathcal{F}$ is representable and M is a representing S-scheme.

The associated *universal family* is the family over M corresponding to the identity element $\mathrm{id}_M \in h_M(M) = \mathrm{Hom}_S(M, M)$ under the bijection

$$h_M(M) \;\cong\; \mathcal{F}(M)$$

induced by Φ.

Proposition 3.7 (Universal families via Yoneda)**.** *Let $\mathcal{F}$ be a moduli functor on* (Sch/S) *and suppose that* (M, Φ) *is a fine moduli space for $\mathcal{F}$. Let $U_M \in \mathcal{F}(M)$ denote the universal family, that is, the element corresponding to* $\mathrm{id}_M \in h_M(M)$.

(1) For every S-scheme T and every element $x \in \mathcal{F}(T)$, there exists a unique morphism $f : T \to M$ of S-schemes such that

$$x \;=\; f^*U_M \;\in\; \mathcal{F}(T).$$

(2) More generally, giving a natural transformation $\eta : \mathcal{F} \Rightarrow h_N$ to the functor of points of some S-scheme N is equivalent to giving a unique morphism $g : M \to N$ of S-schemes.

Proof. The first statement is just the definition of representability unwound. Indeed, the isomorphism of functors $\Phi : h_M \to \mathcal{F}$ yields for each T a bijection

$$\Phi_T : \mathrm{Hom}_S(T, M) \xrightarrow{\sim} \mathcal{F}(T).$$

By construction, we have

$$\Phi_M(\mathrm{id}_M) = U_M \in \mathcal{F}(M).$$

Now let $x \in \mathcal{F}(T)$. By bijectivity of Φ_T, there is a unique morphism $f : T \to M$ such that $x = \Phi_T(f)$. Naturality of Φ implies that

$$x = \Phi_T(f) = \mathcal{F}(f)\big(\Phi_M(\mathrm{id}_M)\big) = f^*U_M.$$

For the second statement, compose η with Φ to obtain a natural transformation

$$\tilde{\eta} : h_M \xRightarrow{\eta\circ\Phi} h_N.$$

By Corollary 1.8, such a natural transformation is uniquely induced by a morphism $g : M \to N$ of S-schemes. Conversely, any morphism $g : M \to N$ yields a natural transformation $h_M \to h_N$, and hence (by composing with Φ^{-1}) a natural transformation $\mathcal{F} \to h_N$. □

Remark 3.8. Since $\mathcal{F} \cong h_M$, one has a canonical identification

$$\mathrm{Nat}(\mathcal{F}, h_N) \cong \mathrm{Nat}(h_M, h_N) \cong \mathrm{Hom}_S(M, N)$$

by Corollary 1.8. On the other hand, the *classifying map* $N \to M$ for a family over N is the morphism corresponding to the element of $\mathcal{F}(N)$ via the bijection $\mathcal{F}(N) \cong \mathrm{Hom}_S(N, M)$ in Proposition 3.7(1). In particular, any family over N is obtained by pulling back U_M along its classifying map $N \to M$.

3.3. **Coarse moduli spaces (optional).** In many interesting situations, the moduli functor $\mathcal{F}$ fails to be representable, typically because some objects have nontrivial automorphisms. In such cases one often settles for a weaker object, called a coarse moduli space.

Definition 3.9. Let $\mathcal{F} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathbf{Set}$ be a moduli functor. A *coarse moduli space* for $\mathcal{F}$ is an S-scheme M together with a natural transformation

$$\pi : \mathcal{F} \longrightarrow h_M$$

such that:

(1) For every algebraically closed field k with a morphism $\mathrm{Spec}(k) \to S$, the map

$$\mathcal{F}(\mathrm{Spec}(k)) \longrightarrow h_M(\mathrm{Spec}(k)) = M(k)$$

induces a bijection between isomorphism classes of objects over $\mathrm{Spec}(k)$ and k-points of M.

(2) (*Universal property*) For every S-scheme N and every natural transformation $\eta : \mathcal{F} \to h_N$, there exists a unique morphism $f : M \to N$ of S-schemes such that $\eta = h_f \circ \pi$, where $h_f : h_M \to h_N$ is induced by f.

Remark 3.10. In this language, a fine moduli space is precisely a coarse moduli space for which π is an isomorphism of functors. For our purposes the Grassmannian will always be a fine moduli space, so coarse moduli spaces will not play a direct role in the rest of the exposition; we have included them only for context.

3.4. **Linear moduli problems via short exact sequences.** We finish this section by specialising to the linear algebra setup that will be relevant for the Grassmannian. The point is that families of subbundles or quotient bundles are conveniently encoded by short exact sequences of locally free sheaves.

Definition 3.11. Let X be a scheme. A *vector bundle of rank r* (or a *locally free sheaf of rank r*) on X is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$ which is locally free of rank r, that is, for every $x \in X$ there exists an open neighbourhood $U \subset X$ of x and an isomorphism

$$\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus r}.$$

Definition 3.12. Let X be a scheme and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of finite rank.

(1) A *subbundle* of $\mathcal{E}$ of rank d is a subsheaf $\mathcal{U} \subset \mathcal{E}$ such that $\mathcal{U}$ is locally free of rank d and the quotient $\mathcal{E}/\mathcal{U}$ is locally free.
(2) A *quotient bundle* of $\mathcal{E}$ of rank d is a locally free sheaf $\mathcal{Q}$ of rank d together with a surjective morphism

$$q : \mathcal{E} \twoheadrightarrow \mathcal{Q}.$$

The two notions are essentially equivalent, as the following elementary lemma shows.

Lemma 3.13. *Let X be a scheme and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of finite rank.*

(1) Given a subbundle $\mathcal{U} \subset \mathcal{E}$, the quotient $\mathcal{Q} := \mathcal{E}/\mathcal{U}$ is locally free and fits into a short exact sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

(2) Conversely, given a short exact sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0$$

with $\mathcal{E}$ locally free and $\mathcal{U}, \mathcal{Q}$ locally free of finite rank, the subsheaf $\mathcal{U} \subset \mathcal{E}$ is a subbundle in the sense of Definition 3.12, and $\mathcal{Q}$ is a quotient bundle.

Proof. Both statements are local on X, so we may assume X is affine and $\mathcal{E} \cong \widetilde{E}$ for a free A-module E of finite rank. Then $\mathcal{U}$ corresponds to a submodule $U \subset E$, and the condition that $\mathcal{U}$ and $\mathcal{E}/\mathcal{U}$ are locally free corresponds to U and E/U being projective A-modules. Over a local ring, finitely generated projective modules are free, and hence $E \cong U \oplus (E/U)$ locally. This yields the desired short exact sequence and shows that a short exact sequence with projective terms corresponds to a direct summand. Translating back to sheaves, we obtain the assertions. ∎

Remark 3.14. The lemma shows that a rank d subbundle $\mathcal{U} \subset \mathcal{E}$ is equivalent data to a short exact sequence

$$0 \to \mathcal{U} \to \mathcal{E} \to \mathcal{Q} \to 0$$

with $\mathcal{Q}$ locally free of rank $\mathrm{rk}(\mathcal{E}) - d$. Thus, a moduli problem about subbundles can equally well be formulated as a moduli problem about quotient bundles, and vice versa.

Example 3.15. Let E be a finite locally free $\mathcal{O}_S$-module of rank n and let $0 < d < n$ be an integer. Consider the moduli functor

$$\mathcal{G} : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

defined as follows. For an S-scheme T, $\mathcal{G}(T)$ is the set of isomorphism classes of short exact sequences

$$0 \longrightarrow \mathcal{U} \longrightarrow E_T \longrightarrow \mathcal{Q} \longrightarrow 0$$

on T such that $\mathcal{U}$ and $\mathcal{Q}$ are locally free of ranks d and $n-d$ respectively. Morphisms $f : T' \to T$ of S-schemes act by pullback along f.

By Lemma 3.13, we may equally describe $\mathcal{G}(T)$ as the set of rank d subbundles $\mathcal{U} \subset E_T$ with locally free quotient, or as the set of rank $n-d$ quotient bundles of E_T. In later sections we will show that $\mathcal{G}$ is representable by the Grassmannian bundle $G(d, E) \to S$.

Remark 3.16. The Grassmannian moduli functor $\mathcal{G}$ is an example of a *linear* moduli problem: the objects parametrised are built from a fixed vector bundle E using only linear-algebraic data (subbundles, quotient bundles, and short exact sequences). This linearity is what ultimately makes $\mathcal{G}$ representable by a relatively explicit scheme, constructed by gluing affine charts.

4. Grassmannian moduli functors and basic properties

We now return to the moduli problem introduced in Example 3.15. Our goal in this section is to fix notation for the Grassmannian moduli functor and record several basic properties: equivalent descriptions of its T-points, behaviour under base change, and a Zariski sheaf property. In the next sections we will construct a representing scheme by gluing affine charts.

Throughout, S is a fixed base scheme and E is a finite locally free $\mathcal{O}_S$-module of rank n.

4.1. **Definition of the Grassmannian moduli functor.**

Fix an integer d with $0 < d < n$.

Definition 4.1. The *Grassmannian moduli functor* associated to (E, d) is the functor

$$G = G_{d,E} : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

defined as follows. For an S-scheme T, $G(T)$ is the set of isomorphism classes of short exact sequences

$$0 \longrightarrow \mathcal{U} \longrightarrow E_T \longrightarrow \mathcal{Q} \longrightarrow 0$$

of quasi-coherent $\mathcal{O}_T$-modules such that $\mathcal{U}$ and $\mathcal{Q}$ are locally free of ranks d and $n-d$ respectively. Morphisms $f : T' \to T$ of S-schemes act by pullback:

$$f^* : G(T) \longrightarrow G(T')$$

sends the isomorphism class of $0 \to \mathcal{U} \to E_T \to \mathcal{Q} \to 0$ to the class of the pulled back exact sequence

$$0 \longrightarrow \mathcal{U}_{T'} \longrightarrow E_{T'} \longrightarrow \mathcal{Q}_{T'} \longrightarrow 0.$$

Remark 4.2. This is just a restatement of Example 3.15 with slightly more systematic notation. It will be convenient to suppress the dependence on (d, E) from the notation whenever the context is clear.

4.2. **Equivalent descriptions of $G(T)$.** Using Lemma 3.13, we can reformulate the definition of $G(T)$ in two more geometric ways.

Proposition 4.3. *Let T be an S-scheme. There are natural bijections between the following three sets:*

(1) isomorphism classes of rank d subbundles $\mathcal{U} \subset E_T$ such that $E_T/\mathcal{U}$ is locally free;
(2) isomorphism classes of rank $n-d$ quotient bundles $q : E_T \twoheadrightarrow \mathcal{Q}$;
(3) the set $G(T)$ of isomorphism classes of short exact sequences

$$0 \longrightarrow \mathcal{U} \longrightarrow E_T \longrightarrow \mathcal{Q} \longrightarrow 0$$

with $\mathcal{U}$ and $\mathcal{Q}$ locally free of ranks d and $n-d$.

These bijections are functorial in T, and hence give equivalent descriptions of the Grassmannian moduli functor G.

Proof. The equivalence between (1) and (3), and between (2) and (3), is exactly Lemma 3.13 applied to the locally free sheaf E_T, together with the observation that isomorphisms of short exact sequences are the same thing as isomorphisms of subbundles (or quotient bundles) making the obvious diagrams commute. Functoriality in T is immediate from the functoriality of pullback for exact sequences, subbundles and quotient bundles. □

Remark 4.4. In later sections we will freely switch between these three points of view. For constructions involving local coordinates it is often convenient to work with quotient bundles (description (2)), while for geometric intuition it is often more natural to think in terms of subbundles (description (1)).

4.3. **Behaviour under base change.** We next record a simple but important compatibility of the functor G with change of base.

Lemma 4.5. *Let $f : S' \to S$ be a morphism of schemes and let $E' = E_{S'}$ denote the pullback of E to S'. Let*

$$G' = G_{d,E'} : (\mathrm{Sch}/S')^{\mathrm{op}} \to \mathbf{Set}$$

be the Grassmannian moduli functor associated to (E', d) over S'. Then the restriction of G to (Sch/S') (via the forgetful functor $(\mathrm{Sch}/S') \to (\mathrm{Sch}/S)$) is canonically isomorphic to G'. In other words, for every S'-scheme T there is a natural bijection

$$G(T) \;\cong\; G'(T)$$

identifying their pullback maps.

Proof. By definition, an S'-scheme T is in particular an S-scheme, and E_T can be identified with $(E_{S'})_T = E'_T$. A short exact sequence

$$0 \to \mathcal{U} \to E_T \to \mathcal{Q} \to 0$$

with locally free $\mathcal{U}$ and $\mathcal{Q}$ of the prescribed ranks is therefore the same datum whether we view it over S or over S'. This gives a natural identification $G(T) \cong G'(T)$ for each T, and functoriality in T is immediate from the definition of the pullback maps. □

Remark 4.6. Lemma 4.5 will later allow us to reduce some questions about $G_{d,E}$ for general S and E to the case where $S = \mathrm{Spec}(\mathbb{Z})$ and E is a free module of rank n, after which we can recover the general case by base change.

4.4. **The Grassmannian functor as a Zariski sheaf.** Finally, we show that the Grassmannian moduli functor G satisfies the Zariski sheaf condition introduced in Definition 2.4. This is essentially because vector bundles and short exact sequences of quasi-coherent sheaves glue over Zariski open covers.

Proposition 4.7. *The Grassmannian moduli functor*

$$G : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

is a Zariski sheaf on (Sch/S)*.*

Proof. Let T be an S-scheme and let $\{U_i \subset T\}_{i \in I}$ be a Zariski open covering.

By the equivalent description of the Grassmannian functor, we may identify $G(T)$ with the set of isomorphism classes of surjections

$$q : E_T \twoheadrightarrow \mathcal{Q}$$

where $\mathcal{Q}$ is locally free of rank $n - d$ (two such surjections $q : E_T \twoheadrightarrow \mathcal{Q}$ and $q' : E_T \twoheadrightarrow \mathcal{Q}'$ are equivalent if there exists an isomorphism $\varphi : \mathcal{Q} \xrightarrow{\sim} \mathcal{Q}'$ with $\varphi \circ q = q'$).

Locality. Suppose $x, y \in G(T)$ have the same restriction to each U_i. Choose representatives

$$q : E_T \twoheadrightarrow \mathcal{Q}, \qquad q' : E_T \twoheadrightarrow \mathcal{Q}'.$$

The hypothesis $x|_{U_i} = y|_{U_i}$ means that for each i there exists an isomorphism

$$\varphi_i : \mathcal{Q}|_{U_i} \xrightarrow{\sim} \mathcal{Q}'|_{U_i}$$

such that $\varphi_i \circ q|_{U_i} = q'|_{U_i}$. Since $q|_{U_i}$ is surjective, such a φ_i is necessarily *unique* (if it exists). Hence on overlaps $U_i \cap U_j$ we have $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$. By the gluing lemma for morphisms of sheaves, the φ_i glue to a global isomorphism $\varphi : \mathcal{Q} \xrightarrow{\sim} \mathcal{Q}'$ with $\varphi \circ q = q'$. Therefore $x = y$ in $G(T)$, and locality holds.

Gluing. Conversely, suppose we are given for each $i \in I$ an element $x_i \in G(U_i)$ represented by a surjection

$$q_i : E_{U_i} = E_T|_{U_i} \twoheadrightarrow \mathcal{Q}_i$$

with $\mathcal{Q}_i$ locally free of rank $n - d$, and assume these classes are compatible on overlaps: for all i, j the restrictions of x_i and x_j to $U_i \cap U_j$ coincide in $G(U_i \cap U_j)$.

Unwinding the equivalence relation, this means that for each i, j there exists an isomorphism

$$\varphi_{ij} : \mathcal{Q}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{Q}_j|_{U_i \cap U_j}$$

such that $\varphi_{ij} \circ q_i|_{U_i \cap U_j} = q_j|_{U_i \cap U_j}$. Again, surjectivity of $q_i|_{U_i \cap U_j}$ implies that each φ_{ij} is *unique*, and therefore the cocycle condition on triple overlaps holds automatically:

$$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik} \quad \text{on } U_i \cap U_j \cap U_k.$$

By Zariski descent for quasi-coherent sheaves, the $\mathcal{Q}_i$ glue to a quasi-coherent sheaf $\mathcal{Q}$ on T together with isomorphisms $\mathcal{Q}|_{U_i} \cong \mathcal{Q}_i$. By the gluing lemma for morphisms, the surjections q_i glue to a morphism

$$q : E_T \longrightarrow \mathcal{Q}$$

whose restriction to each U_i identifies with q_i; in particular, q is surjective since surjectivity is local on T. Moreover, $\mathcal{Q}$ is locally free of rank $n-d$ because this may be checked Zariski locally.

Thus $q : E_T \twoheadrightarrow \mathcal{Q}$ defines an element $x \in G(T)$ whose restriction to U_i is x_i. Uniqueness follows from the locality part (or directly from the uniqueness of the glued morphisms above). Hence G satisfies the sheaf gluing axiom, and therefore G is a Zariski sheaf. □

Remark 4.8. Proposition 4.7 shows that the Grassmannian moduli functor behaves like the functor of points of a scheme with respect to Zariski localisation on the base. In the next section we will use Theorem 2.11 to construct such a representing scheme explicitly by gluing together affine charts on which the universal short exact sequence takes a particularly simple form.

5. Affine charts and representability of the Grassmannian functor

In this section we construct the Grassmannian as a scheme by describing explicit affine charts for the Grassmannian moduli functor in the case of a trivial bundle. Throughout we fix a base scheme S and an integer $n \geq 1$.

5.1. **Reduction to the trivial bundle.** Let E be a finite locally free $\mathcal{O}_S$-module of rank n, and let $G_{d,E}$ be the Grassmannian moduli functor introduced in Definition 4.1. The following observation will allow us to reduce many questions to the case when E is a trivial bundle.

Lemma 5.1. *The question whether $G_{d,E}$ is representable by an S-scheme is Zariski local on S. More precisely, let $\{S_\alpha \subset S\}$ be a Zariski open cover such that $E|_{S_\alpha}$ is free of rank n on each S_α. If for every α the restricted functor $G_{d,E}|_{(\mathrm{Sch}/S_\alpha)}$ is representable by an S_α-scheme, then $G_{d,E}$ is representable by an S-scheme.*

Proof. By Lemma 4.5, the restriction of $G_{d,E}$ to (Sch/S_α) is canonically identified with the Grassmannian moduli functor $G_{d,E|_{S_\alpha}}$ associated to the restricted bundle $E|_{S_\alpha}$. By hypothesis, each of these is representable.

Since $G_{d,E}$ is a Zariski sheaf by Proposition 4.7, we may apply Theorem 2.11 to glue the representing schemes on the open cover $\{S_\alpha\}$ and obtain a global representing S-scheme. □

Remark 5.2. In view of Lemma 5.1, it suffices to treat the case when E is the trivial rank n bundle. We shall therefore specialise to $E = \mathcal{O}_S^{\oplus n}$ for the remainder of this section, and write $G_{d,n}$ for the corresponding moduli functor $G_{d,E}$.

5.2. **Standard open subfunctors.** We now define a family of subfunctors of $G_{d,n}$ that will serve as affine charts. These are the functorial analogues of the familiar "Schubert cells" in linear algebra.

Fix an integer d with $0 < d < n$ and write

$$\{1, \ldots, n\} = I \sqcup J$$

for a decomposition into a subset I of size d and its complement J of size $n-d$.

Definition 5.3. Let S and n, d, I be as above. The *standard open subfunctor*

$$G_I \subset G_{d,n}$$

is defined as follows. For an S-scheme T, an element of $G_I(T)$ is an isomorphism class of exact sequences

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{O}_T^{\oplus n} \longrightarrow \mathcal{Q} \longrightarrow 0$$

with $\mathcal{U}$ and $\mathcal{Q}$ locally free of ranks d and $n-d$, such that the composite map

$$\mathcal{U} \hookrightarrow \mathcal{O}_T^{\oplus n} \xrightarrow{\mathrm{pr}_I} \mathcal{O}_T^{\oplus d}$$

is an isomorphism of vector bundles. Here pr_I denotes the projection onto the direct summand corresponding to the coordinates in I.

Lemma 5.4. *For each subset $I \subset \{1, \dots, n\}$ of cardinality d, the assignment $T \mapsto G_I(T)$ defines a subfunctor $G_I \subset G_{d,n}$ which is an open subfunctor in the sense of Definition 2.8. Moreover, the family of subfunctors $\{G_I\}$ covers $G_{d,n}$, in the sense that for every T and every element $x \in G_{d,n}(T)$ there exists a Zariski open cover $\{U_\alpha \subset T\}$ and subsets I_α such that $x|_{U_\alpha} \in G_{I_\alpha}(U_\alpha)$ for all α.*

Proof. Let T be an S-scheme and let $x \in G_{d,n}(T)$ be represented by a short exact sequence

$$0 \longrightarrow \mathcal{U} \xrightarrow{\iota} \mathcal{O}_T^{\oplus n} \xrightarrow{q} \mathcal{Q} \longrightarrow 0$$

with $\mathcal{U}$ and $\mathcal{Q}$ locally free of ranks d and $n-d$. Locally on T we may choose a trivialisation $\mathcal{U}|_U \cong \mathcal{O}_U^{\oplus d}$ and view the inclusion $\iota|_U$ as a $n \times d$ matrix of functions on U. The condition that the composite

$$\mathcal{U}|_U \xrightarrow{\iota|_U} \mathcal{O}_U^{\oplus n} \xrightarrow{\mathrm{pr}_I} \mathcal{O}_U^{\oplus d}$$

be an isomorphism is equivalent to the condition that the $d \times d$ minor corresponding to the rows indexed by I has invertible determinant. In particular, the locus where this holds is an open subset of U, cut out by the nonvanishing of that determinant.

As we vary the local trivialisation and the open subsets U, these open loci glue to give a Zariski open subset $V_I(x) \subset T$ on which $x|_{V_I(x)}$ belongs to $G_I(V_I(x))$. This shows that for each x the set of points $t \in T$ such that the fiber of x at t lies in the fiber of G_I is open, and hence that G_I defines an open subfunctor of $G_{d,n}$.

To see that the $\{G_I\}$ cover $G_{d,n}$, let $x \in G_{d,n}(T)$ be as above. Zariski locally on T we may trivialise $\mathcal{U}$ and identify ι with a rank d matrix. At each point $t \in T$ there is at least one $d \times d$ minor which is invertible on some neighbourhood of t. This minor corresponds to some subset $I \subset \{1, \dots, n\}$ of cardinality d, and the corresponding open neighbourhood U_I of t is contained in $V_I(x)$. Covering T by such neighbourhoods yields the desired open cover. □

Remark 5.5. Intuitively, G_I is the locus where the subbundle $\mathcal{U}$ is spanned by d of the standard basis vectors of $\mathcal{O}_T^{\oplus n}$, with the remaining $(n-d) \times d$ entries of the injection matrix providing affine coordinates on the chart.

5.3. **Affine coordinate description of the standard charts.** We next show that each standard open subfunctor G_I is representable by an affine space of dimension $d(n-d)$ over S.

Let $m = d(n-d)$ and let $\mathbb{A}_S^m$ denote relative affine m-space over S, that is,

$$\mathbb{A}_S^m = \underline{\mathrm{Spec}}_S\big(\mathcal{O}_S[t_{jk}]_{1 \le j \le n-d,\ 1 \le k \le d}\big).$$

Proposition 5.6. *For each subset $I \subset \{1, \dots, n\}$ of cardinality d, the standard open subfunctor G_I is representable by $\mathbb{A}^m_S$. More precisely, there is a natural isomorphism of functors*

$$G_I \xrightarrow{\sim} h_{\mathbb{A}^m_S}.$$

Proof. Fix such a subset I and write J for its complement. For an S-scheme T, we will construct mutually inverse assignments between $G_I(T)$ and $h_{\mathbb{A}^m_S}(T) = \mathrm{Hom}_S(T, \mathbb{A}^m_S)$, naturally in T.

First, suppose given a morphism $\phi : T \to \mathbb{A}^m_S$. Under the usual identification

$$\mathrm{Hom}_S(T, \mathbb{A}^m_S) \cong \Gamma(T, \mathcal{O}_T)^{\oplus m},$$

ϕ corresponds to a family of functions

$$(a_{jk})_{1 \le j \le n-d,\ 1 \le k \le d} \in \Gamma(T, \mathcal{O}_T)^{\oplus m}.$$

We package these into an $(n-d) \times d$ matrix $A = (a_{jk})$ and consider the rank d subbundle $\mathcal{U}_\phi \subset \mathcal{O}_T^{\oplus n}$ defined as follows.

Write the direct sum decomposition

$$\mathcal{O}_T^{\oplus n} = \mathcal{O}_T^{\oplus d} \oplus \mathcal{O}_T^{\oplus (n-d)},$$

where by convention the first summand corresponds to the indices in I and the second to the indices in J. On this decomposition we define

$$\mathcal{U}_\phi := \left\{ \begin{pmatrix} u \\ Au \end{pmatrix} \;\middle|\; u \in \mathcal{O}_T^{\oplus d} \right\} \subset \mathcal{O}_T^{\oplus d} \oplus \mathcal{O}_T^{\oplus (n-d)} = \mathcal{O}_T^{\oplus n}.$$

Concretely, $\mathcal{U}_\phi$ is the image of the map

$$\mathcal{O}_T^{\oplus d} \longrightarrow \mathcal{O}_T^{\oplus n}, \qquad u \longmapsto \begin{pmatrix} u \\ Au \end{pmatrix}.$$

This is a rank d locally free subsheaf of $\mathcal{O}_T^{\oplus n}$, and the projection onto the first factor $\mathcal{O}_T^{\oplus n} \to \mathcal{O}_T^{\oplus d}$ restricts to an isomorphism $\mathcal{U}_\phi \cong \mathcal{O}_T^{\oplus d}$. Let $\mathcal{Q}_\phi$ be the quotient $\mathcal{O}_T^{\oplus n}/\mathcal{U}_\phi$, which is then locally free of rank $n-d$, and let

$$0 \longrightarrow \mathcal{U}_\phi \longrightarrow \mathcal{O}_T^{\oplus n} \longrightarrow \mathcal{Q}_\phi \longrightarrow 0$$

be the corresponding short exact sequence. By construction this is an object of $G_I(T)$, and we denote its isomorphism class by $\Psi_T(\phi)$.

Conversely, suppose given an element of $G_I(T)$, represented by an exact sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{O}_T^{\oplus n} \longrightarrow \mathcal{Q} \longrightarrow 0$$

with the property that

$$\mathcal{U} \xrightarrow{\cong} \mathcal{O}_T^{\oplus d}$$

via projection onto the coordinates indexed by I. Using this identification, we may view $\mathcal{U}$ as the image of a map

$$\mathcal{O}_T^{\oplus d} \longrightarrow \mathcal{O}_T^{\oplus d} \oplus \mathcal{O}_T^{\oplus (n-d)}$$

of the form $u \mapsto (u, Au)$ for a uniquely determined $(n-d) \times d$ matrix $A = (a_{jk})$ of functions on T. The entries $a_{jk} \in \Gamma(T, \mathcal{O}_T)$ define a morphism

$$\phi : T \longrightarrow \mathbb{A}^m_S,$$

and we set $\Phi_T([\mathcal{U} \subset \mathcal{O}_T^{\oplus n}]) := \phi$.

By construction, the assignments

$$\Psi_T : \mathrm{Hom}_S(T, \mathbb{A}^m_S) \to G_I(T), \qquad \Phi_T : G_I(T) \to \mathrm{Hom}_S(T, \mathbb{A}^m_S)$$

are inverse to one another, and the construction is clearly compatible with pullback along morphisms $T' \to T$. Hence they define mutually inverse natural transformations between $h_{\mathbb{A}^m_S}$ and G_I, giving the claimed isomorphism of functors. □

Remark 5.7. On global sections, Proposition 5.6 says exactly that the data of a rank d subbundle in "row-echelon form" with respect to the coordinates in I are parametrised by the $(n-d) \times d$ matrix of coefficients expressing the remaining coordinates in terms of those in I.

5.4. **Gluing the charts: representability of $G_{d,n}$.** We can now apply the representability criterion Theorem 2.11 to the family of open subfunctors $\{G_I\}$.

Theorem 5.8. *Let S be a scheme, $n \geq 1$ and $0 < d < n$ be integers. Then the Grassmannian moduli functor*

$$G_{d,n} : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

is representable by an S-scheme $\mathrm{Gr}_S(d,n)$, called the Grassmannian of d-planes in $\mathcal{O}_S^{\oplus n}$.

Moreover, $\mathrm{Gr}_S(d,n)$ is covered by open subschemes $\mathrm{Gr}_S(d,n)_I$ indexed by subsets $I \subset \{1, \ldots, n\}$ of cardinality d, each isomorphic to $\mathbb{A}_S^{d(n-d)}$, and such that the functor G_I corresponds to the functor of points of $\mathrm{Gr}_S(d,n)_I$.

Proof. By Lemma 5.4, the subfunctors G_I form an open cover of the Zariski sheaf $G_{d,n}$, and by Proposition 5.6 each G_I is representable by the S-scheme $\mathbb{A}_S^{d(n-d)}$. On overlaps $G_I \cap G_{I'}$ we obtain induced isomorphisms between suitable open subschemes of the corresponding affine spaces, arising from changes of basis between different row-echelon forms. These isomorphisms satisfy the evident cocycle condition on triple overlaps.

By Theorem 2.11, we may glue the affine schemes $\mathbb{A}_S^{d(n-d)}$ along these identifications to obtain an S-scheme $\mathrm{Gr}_S(d,n)$ representing $G_{d,n}$. The open immersion of each chart into $\mathrm{Gr}_S(d,n)$ has image an open subscheme $\mathrm{Gr}_S(d,n)_I$ isomorphic to $\mathbb{A}_S^{d(n-d)}$, and the family $\{\mathrm{Gr}_S(d,n)_I\}_I$ covers $\mathrm{Gr}_S(d,n)$. □

Remark 5.9. The transition maps between the charts $\mathrm{Gr}_S(d,n)_I$ and $\mathrm{Gr}_S(d,n)_{I'}$ can be described explicitly in terms of invertible $d \times d$ matrices and the usual row operations, but we will not need the precise formulas in what follows.

5.5. **Geometry of the Grassmannian: smoothness and dimension.** We conclude this section by recording basic geometric properties of the Grassmannian. For the moment we stay in the trivial-bundle case; in the next section we will extend these statements to Grassmann bundles of a general vector bundle E.

Corollary 5.10. *Let S be a scheme. Then the structure morphism*

$$\pi : \mathrm{Gr}_S(d,n) \longrightarrow S$$

is smooth of relative dimension $d(n-d)$. In particular, for any field k and any morphism $\mathrm{Spec}(k) \to S$, the fiber

$$\mathrm{Gr}_S(d,n) \times_S \mathrm{Spec}(k)$$

is a smooth k-scheme of dimension $d(n-d)$.

Proof. By Theorem 5.8, $\mathrm{Gr}_S(d,n)$ is covered by open subschemes each isomorphic to $\mathbb{A}_S^{d(n-d)}$. Since the morphism $\mathbb{A}_S^{d(n-d)} \to S$ is smooth of relative dimension $d(n-d)$ (it is the base change of the smooth morphism $\mathbb{A}_{\mathbb{Z}}^{d(n-d)} \to \mathrm{Spec}(\mathbb{Z})$), it follows that π is smooth of relative dimension $d(n-d)$.

Taking fibres over morphisms $\mathrm{Spec}(k) \to S$ yields that each fiber is a smooth k-scheme of dimension $d(n-d)$. Projectivity will be deduced in the next section from the Plücker embedding. □

Remark 5.11. Over an algebraically closed field k, the k-points of $\mathrm{Gr}_S(d,n)$ are in bijection with d-dimensional linear subspaces of k^n, and Corollary 5.10 recovers the familiar fact that the Grassmannian is a smooth projective variety of dimension $d(n-d)$. Our construction shows that the same object exists and varies smoothly over an arbitrary base scheme S.

6. Universal exact sequences and Grassmann bundles

In this section we globalise the Grassmannian construction from the trivial bundle case to an arbitrary vector bundle on a scheme. We first make the universal short exact sequence on the Grassmannian explicit, and then define Grassmann bundles and record their basic functorial properties, including compatibility with base change.

6.1. The universal short exact sequence on $\mathrm{Gr}_S(d,n)$.

We retain the notation from the previous section: S is a scheme, $n \geq 1$, $0 < d < n$, and $\mathrm{Gr}_S(d,n)$ denotes the Grassmannian constructed in Theorem 5.8, representing the moduli functor $G_{d,n}$ of rank d subbundles (or quotient bundles) of $\mathcal{O}_S^{\oplus n}$.

Let

$$\pi : \mathrm{Gr}_S(d,n) \longrightarrow S$$

be the structure morphism.

Definition 6.1. The *universal short exact sequence* on $\mathrm{Gr}_S(d,n)$ is the exact sequence of quasi-coherent sheaves

$$0 \longrightarrow \mathcal{U}_{\mathrm{univ}} \longrightarrow \mathcal{O}_{\mathrm{Gr}_S(d,n)}^{\oplus n} \longrightarrow \mathcal{Q}_{\mathrm{univ}} \longrightarrow 0 \tag{6.2}$$

corresponding to the element of $G_{d,n}(\mathrm{Gr}_S(d,n))$ associated to the identity morphism

$$\mathrm{id}_{\mathrm{Gr}_S(d,n)} \in \mathrm{Hom}_S\big(\mathrm{Gr}_S(d,n), \mathrm{Gr}_S(d,n)\big)$$

under the bijection

$$\mathrm{Hom}_S\big(\mathrm{Gr}_S(d,n), \mathrm{Gr}_S(d,n)\big) \;\cong\; G_{d,n}\big(\mathrm{Gr}_S(d,n)\big)$$

coming from the representability of $G_{d,n}$. The locally free sheaves $\mathcal{U}_{\mathrm{univ}}$ and $\mathcal{Q}_{\mathrm{univ}}$ are called the *universal subbundle* and *universal quotient bundle* on $\mathrm{Gr}_S(d,n)$.

Proposition 6.3 (Universal property)**.** *Let T be an S-scheme. Then:*

(1) For every morphism $f : T \to \mathrm{Gr}_S(d,n)$ of S-schemes, the pullback of (6.2) along f is a short exact sequence

$$0 \longrightarrow f^*\mathcal{U}_{\mathrm{univ}} \longrightarrow \mathcal{O}_T^{\oplus n} \longrightarrow f^*\mathcal{Q}_{\mathrm{univ}} \longrightarrow 0$$

with locally free kernel and cokernel of ranks d and $n-d$, representing an element of $G_{d,n}(T)$.

(2) Conversely, every element of $G_{d,n}(T)$, i.e. every short exact sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{O}_T^{\oplus n} \longrightarrow \mathcal{Q} \longrightarrow 0$$

with $\mathcal{U}$ and $\mathcal{Q}$ locally free of ranks d and $n-d$, arises uniquely as a pullback $f^*(\mathcal{U}_{\mathrm{univ}} \hookrightarrow \mathcal{O}_{\mathrm{Gr}_S(d,n)}^{\oplus n} \twoheadrightarrow \mathcal{Q}_{\mathrm{univ}})$ along a unique morphism $f : T \to \mathrm{Gr}_S(d,n)$.

Proof. By construction, the isomorphism of functors

$$h_{\mathrm{Gr}_S(d,n)} \;\cong\; G_{d,n}$$

sends a morphism $f : T \to \mathrm{Gr}_S(d,n)$ to the pullback of the universal element $[\mathcal{U}_{\mathrm{univ}} \to \mathcal{O}_{\mathrm{Gr}_S(d,n)}^{\oplus n} \to \mathcal{Q}_{\mathrm{univ}}]$ along f. This gives (1).

Conversely, given an element of $G_{d,n}(T)$, the same isomorphism provides a unique morphism $f : T \to \mathrm{Gr}_S(d,n)$ corresponding to it. By definition of the universal element, the associated short exact sequence on T is obtained as the pullback of (6.2). This is (2). □

Remark 6.4. Thus $\mathrm{Gr}_S(d,n)$ is a fine moduli space for rank d subbundles (or quotients) of the trivial bundle $\mathcal{O}_S^{\oplus n}$, and (6.2) is precisely the universal family in the sense of the general moduli discussion in Section 3.

6.2. **Grassmann bundles of a vector bundle.** We now generalise the Grassmannian construction from the trivial bundle $\mathcal{O}_S^{\oplus n}$ to an arbitrary vector bundle on a scheme.

Let X be a scheme and let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module of rank n.

Definition 6.5. Fix an integer d with $0 < d < n$. The *Grassmann moduli functor* associated to $(X, \mathcal{E}, d)$ is the functor

$$G_{d,\mathcal{E}} : (\mathrm{Sch}/X)^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

defined as follows. For an X-scheme $T \xrightarrow{g} X$, we set $G_{d,\mathcal{E}}(T)$ to be the set of isomorphism classes of short exact sequences

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{E}_T \longrightarrow \mathcal{Q} \longrightarrow 0 \tag{6.6}$$

of quasi-coherent $\mathcal{O}_T$-modules such that $\mathcal{U}$ and $\mathcal{Q}$ are locally free of ranks d and $n-d$ respectively. Morphisms of X-schemes $f : T' \to T$ act by pullback of (6.6).

Remark 6.7. As in Proposition 4.3, one may equivalently describe $G_{d,\mathcal{E}}$ in terms of rank d subbundles of $\mathcal{E}_T$ with locally free quotient, or rank $n-d$ quotient bundles of $\mathcal{E}_T$. We will freely pass between these descriptions.

Theorem 6.8 (Grassmann bundles)**.** *Let X be a scheme and $\mathcal{E}$ a finite locally free $\mathcal{O}_X$-module of rank n. Fix $0 < d < n$. Then:*

(1) The functor $G_{d,\mathcal{E}}$ is representable by an X-scheme

$$\pi : \mathrm{Gr}_X(d, \mathcal{E}) \longrightarrow X,$$

called the Grassmann bundle *(or Grassmannian bundle) of d-planes in $\mathcal{E}$.*

(2) There exists a universal short exact sequence on $\mathrm{Gr}_X(d,\mathcal{E})$

$$0 \longrightarrow \mathcal{U}_{\mathcal{E}} \longrightarrow \pi^*\mathcal{E} \longrightarrow \mathcal{Q}_{\mathcal{E}} \longrightarrow 0, \tag{6.9}$$

where $\mathcal{U}_{\mathcal{E}}$ and $\mathcal{Q}_{\mathcal{E}}$ are locally free of ranks d and $n-d$. This sequence is universal in the same sense as in Proposition 6.3: for any X-scheme T and any short exact sequence of the form (6.6) on T, there exists a unique morphism

$f : T \to \mathrm{Gr}_X(d, \mathcal{E})$ over X such that (6.6) is isomorphic to the pullback of (6.9) along f.

Proof. The representability statement is Zariski local on X. Cover X by open subschemes $\{U_\alpha\}$ on which $\mathcal{E}$ is trivial: $\mathcal{E}|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}^{\oplus n}$. On each U_α the restriction of $G_{d,\mathcal{E}}$ identifies with the Grassmannian moduli functor $G_{d,n}$ for the trivial bundle, which is representable by $\mathrm{Gr}_{U_\alpha}(d, n)$ by Theorem 5.8.

Using that $G_{d,\mathcal{E}}$ is a Zariski sheaf (the proof of Proposition 4.7 carries over verbatim), we may glue the schemes $\mathrm{Gr}_{U_\alpha}(d, n)$ along their canonical isomorphisms on overlaps to obtain an X-scheme $\mathrm{Gr}_X(d, \mathcal{E})$ representing $G_{d,\mathcal{E}}$. This proves (1).

For (2), apply the general "universal family" philosophy as in Section 3 to the representable moduli functor $G_{d,\mathcal{E}}$. The identity morphism $\mathrm{id}_{\mathrm{Gr}_X(d,\mathcal{E})}$ corresponds to a universal element of $G_{d,\mathcal{E}}\big(\mathrm{Gr}_X(d, \mathcal{E})\big)$, which is represented by a short exact sequence

$$0 \longrightarrow \mathcal{U}_{\mathcal{E}} \longrightarrow \pi^*\mathcal{E} \longrightarrow \mathcal{Q}_{\mathcal{E}} \longrightarrow 0$$

on $\mathrm{Gr}_X(d, \mathcal{E})$. The same argument as in Proposition 6.3 gives the desired classifying property. □

Remark 6.10. By construction, the fiber of $\pi : \mathrm{Gr}_X(d, \mathcal{E}) \to X$ over a point $x \in X$ is canonically isomorphic to the classical Grassmannian of d-dimensional subspaces of the n-dimensional vector space $\mathcal{E}(x)$.

6.3. **Base change for Grassmann bundles.** One of the key features of Grassmann bundles is their excellent compatibility with base change.

Proposition 6.11 (Base change)**.** *Let X be a scheme, $\mathcal{E}$ a finite locally free $\mathcal{O}_X$-module of rank n, and $0 < d < n$. Let $f : X' \to X$ be a morphism of schemes and set $\mathcal{E}' := f^*\mathcal{E}$ on X'. Then:*

(1) There is a canonical isomorphism of X'-schemes

$$\mathrm{Gr}_{X'}(d, \mathcal{E}') \;\cong\; \mathrm{Gr}_X(d, \mathcal{E}) \times_X X'.$$

(2) Under this identification, the universal short exact sequence

$$0 \longrightarrow \mathcal{U}_{\mathcal{E}'} \longrightarrow \pi'^*\mathcal{E}' \longrightarrow \mathcal{Q}_{\mathcal{E}'} \longrightarrow 0$$

on $\mathrm{Gr}_{X'}(d, \mathcal{E}')$ is obtained by pulling back the universal sequence (6.9) on $\mathrm{Gr}_X(d, \mathcal{E})$ along the projection $\mathrm{Gr}_X(d, \mathcal{E}) \times_X X' \to \mathrm{Gr}_X(d, \mathcal{E})$.

Proof. Consider the functor $G_{d,\mathcal{E}}$ on (Sch/X) and its restriction to (Sch/X') via the forgetful functor $(\mathrm{Sch}/X') \to (\mathrm{Sch}/X)$. For an X'-scheme $T \to X'$, viewed as an X-scheme via $f \circ (T \to X')$, the pullback of $\mathcal{E}$ to T is canonically isomorphic to the pullback of $\mathcal{E}'$ to T. Hence giving a short exact sequence of the form (6.6) on T with respect to $\mathcal{E}$ is equivalent to giving such a sequence with respect to $\mathcal{E}'$. This yields a natural identification of functors

$$G_{d,\mathcal{E}}|_{(\mathrm{Sch}/X')} \;\cong\; G_{d,\mathcal{E}'}.$$

On the other hand, the fiber product $\mathrm{Gr}_X(d, \mathcal{E}) \times_X X'$ represents the restriction of $G_{d,\mathcal{E}}$ to (Sch/X'), by the usual fiber-product description of the functor of points. Since $G_{d,\mathcal{E}'}$ is also representable, uniqueness of representing objects up to unique isomorphism gives the claimed canonical isomorphism

$$\mathrm{Gr}_{X'}(d, \mathcal{E}') \;\cong\; \mathrm{Gr}_X(d, \mathcal{E}) \times_X X'.$$

The compatibility of universal sequences follows from the naturality of the construction of the universal element with respect to restriction of the moduli functor, together with the functoriality of pullback for short exact sequences. □

Remark 6.12. Informally, Proposition 6.11 says that "Grassmann bundles commute with base change": forming $\mathrm{Gr}(d, \mathcal{E})$ is compatible with pulling back the base and the underlying vector bundle. This will be crucial later when we study constructions on Grassmannians that are defined fiberwise.

6.4. **Functoriality with respect to bundle maps.** Finally, we record a basic functoriality statement with respect to morphisms of vector bundles. The precise form needed in most applications is the case of isomorphisms of bundles; more general situations can be handled under suitable hypotheses on the map.

Proposition 6.13 (Functoriality in the bundle)**.** *Let X be a scheme, $0 < d < n$, and let $\varphi : \mathcal{E} \to \mathcal{E}'$ be a morphism of finite locally free $\mathcal{O}_X$-modules of rank n.*

(1) If φ is an isomorphism, then there is a unique isomorphism of X-schemes

$$\mathrm{Gr}_X(d, \mathcal{E}) \xrightarrow{\sim} \mathrm{Gr}_X(d, \mathcal{E}')$$

such that the pullback of the universal exact sequence for $\mathcal{E}'$ is identified with the pushforward of the universal exact sequence for $\mathcal{E}$ along φ.

(2) More generally:

(a) If φ is injective and $\mathrm{coker}(\varphi)$ *is finite locally free, then φ induces a natural morphism of Grassmann bundles*

$$\mathrm{Gr}_X(d, \mathcal{E}) \longrightarrow \mathrm{Gr}_X(d, \mathcal{E}'),$$

obtained by sending a rank d subbundle $\mathcal{U} \subset \mathcal{E}_T$ to its image $\varphi_T(\mathcal{U}) \subset \mathcal{E}'_T$.

(b) If φ is surjective and $\ker(\varphi)$ *is finite locally free, then φ induces a natural morphism of Grassmann bundles*

$$\mathrm{Gr}_X(d, \mathcal{E}') \longrightarrow \mathrm{Gr}_X(d, \mathcal{E}),$$

obtained (via the quotient-bundle description) by pulling back a locally free quotient of $\mathcal{E}'_T$ along φ_T.

Proof. For (1), note that an isomorphism $\varphi : \mathcal{E} \to \mathcal{E}'$ gives an isomorphism of moduli functors $G_{d,\mathcal{E}} \cong G_{d,\mathcal{E}'}$, by transporting short exact sequences along φ and its inverse. Representability then yields a unique isomorphism of representing schemes $\mathrm{Gr}_X(d, \mathcal{E}) \cong \mathrm{Gr}_X(d, \mathcal{E}')$, and the statement about universal sequences follows from functoriality of the universal element.

For (2)(a), fix T and a rank d subbundle $\mathcal{U} \subset \mathcal{E}_T$. Since $\mathrm{coker}(\varphi)$ is finite locally free on X, its pullback $\mathcal{C}_T := \mathrm{coker}(\varphi_T)$ is finite locally free on T. Consider the induced map $\overline{\varphi}_T : \mathcal{E}_T/\mathcal{U} \to \mathcal{E}'_T/\varphi_T(\mathcal{U})$. A diagram chase (or the snake lemma) applied to

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{U} \arrow[r] \arrow[d, "\varphi_T|_{\mathcal{U}}"] & \mathcal{E}_T \arrow[r] \arrow[d, "\varphi_T"] & \mathcal{E}_T/\mathcal{U} \arrow[r] \arrow[d, "\overline{\varphi}_T"] & 0 \\ 0 \arrow[r] & \varphi_T(\mathcal{U}) \arrow[r] & \mathcal{E}'_T \arrow[r] & \mathcal{E}'_T/\varphi_T(\mathcal{U}) \arrow[r] & 0 \end{tikzcd}$$

gives a short exact sequence

$$0 \longrightarrow \mathcal{E}_T/\mathcal{U} \xrightarrow{\overline{\varphi}_T} \mathcal{E}'_T/\varphi_T(\mathcal{U}) \longrightarrow \mathcal{C}_T \longrightarrow 0.$$

Both $\mathcal{E}_T/\mathcal{U}$ and $\mathcal{C}_T$ are finite locally free, hence $\mathcal{E}'_T/\varphi_T(\mathcal{U})$ is finite locally free as well. Therefore $\varphi_T(\mathcal{U}) \subset \mathcal{E}'_T$ is again a rank d subbundle with locally free quotient. This construction is functorial in T, so it defines a natural transformation of moduli functors, and hence a morphism of representing Grassmann bundles.

For (2)(b), use the quotient-bundle description of the Grassmannian. Given T and a locally free quotient $q' : \mathcal{E}'_T \twoheadrightarrow \mathcal{Q}$ of rank $n-d$, define $q := q' \circ \varphi_T : \mathcal{E}_T \twoheadrightarrow \mathcal{Q}$. We claim that q is again a locally free quotient of rank $n-d$. Let $\mathcal{K}_T := \ker(\varphi_T)$, which is finite locally free by assumption. Then there is an exact sequence

$$0 \longrightarrow \mathcal{K}_T \longrightarrow \ker(q) \longrightarrow \ker(q') \longrightarrow 0.$$

Here $\ker(q')$ is finite locally free of rank d, and $\mathcal{K}_T$ is finite locally free; since locally free sheaves are locally projective, the sequence splits Zariski-locally on T, so $\ker(q)$ is finite locally free. Hence q exhibits $\mathcal{Q}$ as a locally free quotient of $\mathcal{E}_T$ of rank $n-d$. Functoriality in T is clear, so representability yields the claimed morphism $\mathrm{Gr}_X(d, \mathcal{E}') \to \mathrm{Gr}_X(d, \mathcal{E})$. □

Remark 6.14. Proposition 6.13 shows that the construction $(X, \mathcal{E}) \mapsto \mathrm{Gr}_X(d, \mathcal{E})$ is natural both in the base (via pullback, as in Proposition 6.11) and, under mild hypotheses, in the vector bundle itself. This functoriality will be used implicitly when we construct various natural maps between Grassmannians and related moduli spaces in later sections.

7. Plücker embedding and basic geometry of the Grassmannian

In this section we discuss the Plücker line bundle on a Grassmann bundle, the Plücker embedding over a field, and the description of the Zariski tangent space. This provides a geometric perspective on the Grassmannian constructed in the previous sections.

7.1. **Determinant line bundles and the Plücker line bundle.** Let X be a scheme, let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module of rank n, and fix an integer d with $0 < d < n$. Recall that the Grassmann bundle

$$\pi : \mathrm{Gr}_X(d, \mathcal{E}) \longrightarrow X$$

represents the functor $G_{d,\mathcal{E}}$ of rank d subbundles (or rank $n-d$ quotient bundles) of $\mathcal{E}$, and comes equipped with a universal short exact sequence of vector bundles

$$0 \longrightarrow \mathcal{U}_\mathcal{E} \longrightarrow \pi^*\mathcal{E} \longrightarrow \mathcal{Q}_\mathcal{E} \longrightarrow 0, \tag{7.1}$$

where $\mathcal{U}_\mathcal{E}$ and $\mathcal{Q}_\mathcal{E}$ are locally free of ranks d and $n-d$.

Definition 7.2. The *determinant line bundles* associated to (7.1) are the line bundles

$$\det(\mathcal{U}_\mathcal{E}) := \bigwedge^d \mathcal{U}_\mathcal{E}, \qquad \det(\mathcal{Q}_\mathcal{E}) := \bigwedge^{n-d} \mathcal{Q}_\mathcal{E}$$

on $\mathrm{Gr}_X(d, \mathcal{E})$.

Taking determinants in (7.1) yields an isomorphism

$$\det(\mathcal{U}_\mathcal{E}) \otimes \det(\mathcal{Q}_\mathcal{E}) \;\cong\; \det(\pi^*\mathcal{E}) \;\cong\; \pi^* \det(\mathcal{E}),$$

and therefore

$$\det(\mathcal{Q}_\mathcal{E}) \;\cong\; \det(\mathcal{U}_\mathcal{E})^\vee \otimes \pi^* \det(\mathcal{E}). \tag{7.3}$$

Definition 7.4. The *Plücker line bundle* on $\mathrm{Gr}_X(d, \mathcal{E})$ is the line bundle

$$\mathcal{O}_{\mathrm{Gr}_X(d,\mathcal{E})}(1) := \det(\mathcal{Q}_{\mathcal{E}}).$$

Equivalently, by (7.3) it can be described as

$$\mathcal{O}_{\mathrm{Gr}_X(d,\mathcal{E})}(1) \;\cong\; \det(\mathcal{U}_{\mathcal{E}})^\vee \otimes \pi^* \det(\mathcal{E}).$$

Remark 7.5. If $\mathcal{E} \cong \mathcal{O}_X^{\oplus n}$ is a trivial bundle, then $\det(\mathcal{E}) \cong \mathcal{O}_X$ and the relation (7.3) simplifies to

$$\det(\mathcal{Q}_{\mathcal{E}}) \;\cong\; \det(\mathcal{U}_{\mathcal{E}})^\vee.$$

In this case one may equally well regard the Plücker line bundle as $\det(\mathcal{Q}_{\mathcal{E}})$ or as $\det(\mathcal{U}_{\mathcal{E}})^\vee$.

7.2. **The Plücker embedding over a field.** We now specialise to the case where the base is a field. Let k be a field, let $V = k^n$, and write

$$G = \mathrm{Gr}_k(d, V) = \mathrm{Gr}_k(d, n)$$

for the Grassmannian of d-dimensional subspaces of V. The universal exact sequence on G takes the form

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{O}_G^{\oplus n} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

and the Plücker line bundle is

$$\mathcal{O}_G(1) = \det(\mathcal{Q}) \cong \det(\mathcal{U})^\vee.$$

Set $W := \bigwedge^d V$. Then W is a k-vector space of dimension $\binom{n}{d}$.

Lemma 7.6. *The inclusion of bundles*

$$\mathcal{U} \hookrightarrow V \otimes_k \mathcal{O}_G \cong \mathcal{O}_G^{\oplus n}$$

induces an inclusion of line bundles

$$\det(\mathcal{U}) = \bigwedge^d \mathcal{U} \hookrightarrow \bigwedge^d (V \otimes_k \mathcal{O}_G) \cong W \otimes_k \mathcal{O}_G.$$

Dualising, we obtain a surjection of vector bundles

$$W^\vee \otimes_k \mathcal{O}_G \twoheadrightarrow \det(\mathcal{U})^\vee \cong \mathcal{O}_G(1).$$

Proof. The first statement is obtained by applying the d-th exterior power to the inclusion $\mathcal{U} \hookrightarrow V \otimes_k \mathcal{O}_G$. Since V is finite-dimensional, exterior powers commute with extension of scalars, giving

$$\bigwedge^d (V \otimes_k \mathcal{O}_G) \cong \bigwedge^d V \otimes_k \mathcal{O}_G = W \otimes_k \mathcal{O}_G.$$

The resulting inclusion of line bundles $\det(\mathcal{U}) \hookrightarrow W \otimes_k \mathcal{O}_G$ can then be dualised to a surjection $W^\vee \otimes_k \mathcal{O}_G \twoheadrightarrow \det(\mathcal{U})^\vee$. Finally, let

$$0 \to \mathcal{U} \to V \otimes_k \mathcal{O}_G \to \mathcal{Q} \to 0$$

be the universal exact sequence. By definition $\mathcal{O}_G(1) = \det(\mathcal{Q})$. Taking determinants gives

$$\det(\mathcal{U}) \otimes \det(\mathcal{Q}) \cong \det(V \otimes_k \mathcal{O}_G) \cong \mathcal{O}_G,$$

so $\det(\mathcal{U})^\vee \cong \det(\mathcal{Q}) = \mathcal{O}_G(1)$. □

We next recall the projective-space convention we adopt.

Definition 7.7. For a finite-dimensional k-vector space U, the projective space $\mathbb{P}(U)$ is the scheme representing the functor which associates to a k-scheme X the set of isomorphism classes of pairs $(\mathcal{L}, q)$, where $\mathcal{L}$ is a line bundle on X and

$$q : U \otimes_k \mathcal{O}_X \twoheadrightarrow \mathcal{L}$$

is a surjective $\mathcal{O}_X$-linear map. In other words, $\mathbb{P}(U)$ parametrises rank-one quotients of U.

With this convention, a surjection $U \otimes_k \mathcal{O}_X \twoheadrightarrow \mathcal{L}$ produces a unique morphism $X \to \mathbb{P}(U)$, and the pullback of the tautological line bundle $\mathcal{O}_{\mathbb{P}(U)}(1)$ along this morphism is isomorphic to $\mathcal{L}$.

Applying this to the surjection in Lemma 7.6 with $U = W^\vee$ gives the Plücker map.

Definition 7.8. The *Plücker embedding* is the morphism of k-schemes

$$\iota : G \longrightarrow \mathbb{P}(W^\vee) = \mathbb{P}\big((\bigwedge^d V)^\vee\big)$$

corresponding, via Definition 7.7, to the surjection

$$W^\vee \otimes_k \mathcal{O}_G \twoheadrightarrow \mathcal{O}_G(1)$$

of Lemma 7.6. By construction,

$$\iota^* \mathcal{O}_{\mathbb{P}(W^\vee)}(1) \;\cong\; \mathcal{O}_G(1).$$

Remark 7.9. On k-points, a point $[U] \in G(k)$ corresponds to a d-dimensional subspace $U \subset V$. Its top exterior power $\bigwedge^d U$ is a one-dimensional subspace of $W = \bigwedge^d V$, and the classical description of the Plücker embedding sends U to this line in W. In our quotient convention, this is encoded dually as a one-dimensional quotient of $W^\vee$, giving a point of $\mathbb{P}(W^\vee)$.

Theorem 7.10 (Plücker embedding is a closed immersion)**.** *The Plücker map*

$$\iota : \mathrm{Gr}_k(d, n) \longrightarrow \mathbb{P}\big((\bigwedge^d k^n)^\vee\big)$$

is a closed immersion. In particular, the Plücker line bundle $\mathcal{O}_G(1)$ is very ample, and the Grassmannian $\mathrm{Gr}_k(d, n)$ is a projective variety over k.

Sketch. Consider the standard affine charts $G_I \subset G$ indexed by subsets $I \subset \{1, \ldots, n\}$ of cardinality d, as in the construction of Section 5. Each G_I is isomorphic to an affine space $\mathbb{A}_k^{d(n-d)}$ and can be described by writing a d-plane in row-echelon form with respect to the coordinates in I.

Choose a basis of $W^\vee$ indexed by d-element subsets $J \subset \{1, \ldots, n\}$, and let $\{p_J\}$ be the corresponding homogeneous coordinates on $\mathbb{P}(W^\vee)$. These are the Plücker coordinates. On the chart G_I, the coordinate p_I is nowhere vanishing, and the ratios p_J/p_I can be expressed as polynomial functions in the affine coordinates on G_I, determined by the entries of the $(n-d) \times d$ matrix of parameters. This realises G_I as a closed subscheme of the standard affine chart $\{p_I \neq 0\} \subset \mathbb{P}(W^\vee)$.

One checks that these local descriptions glue compatibly to show that ι is an immersion whose image is closed, cut out by quadratic relations among the p_J (the Plücker relations). Since $\mathbb{P}(W^\vee)$ is projective and ι is a closed immersion, G is projective and $\mathcal{O}_G(1) \cong \iota^* \mathcal{O}(1)$ is very ample. $\square$

7.3. **Plücker coordinates and the example** $\mathrm{Gr}_k(2,4)$**.** To illustrate the Plücker coordinates more concretely, consider the case $n = 4$, $d = 2$. Then $G = \mathrm{Gr}_k(2,4)$ parametrises 2-dimensional subspaces of k^4. The vector space $\bigwedge^2 k^4$ has dimension 6, so its dual $W^\vee$ also has dimension 6, and

$$\mathbb{P}(W^\vee) \cong \mathbb{P}^5_k.$$

Choose a basis e_1, e_2, e_3, e_4 of k^4. Then the wedge products $e_i \wedge e_j$ with $1 \leq i < j \leq 4$ form a basis of $\bigwedge^2 k^4$, and the dual basis of $(\bigwedge^2 k^4)^\vee$ gives homogeneous coordinates

$$[p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34}]$$

on $\mathbb{P}^5_k$.

Proposition 7.11. *The image of* $\mathrm{Gr}_k(2,4)$ *under the Plücker embedding*

$$\iota : \mathrm{Gr}_k(2,4) \hookrightarrow \mathbb{P}^5_k$$

is the smooth quadric hypersurface defined by the single quadratic equation

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0. \tag{7.12}$$

Sketch. Let $U \subset k^4$ be a 2-dimensional subspace with basis vectors $v, w \in k^4$. Write

$$v = \sum_{i=1}^{4} v_i e_i, \qquad w = \sum_{i=1}^{4} w_i e_i.$$

Then

$$v \wedge w = \sum_{1 \leq i < j \leq 4} (v_i w_j - v_j w_i)\, e_i \wedge e_j,$$

so the Plücker coordinates of U are

$$p_{ij} = v_i w_j - v_j w_i, \qquad 1 \leq i < j \leq 4.$$

A direct computation shows that these coordinates satisfy (7.12). Conversely, one checks that any point of $\mathbb{P}^5_k$ with homogeneous coordinates (p_{ij}) satisfying (7.12) arises, up to overall scalar, from some pair of vectors $v, w \in k^4$ spanning a 2-plane, and that different planes produce different points. This shows that the image of $\mathrm{Gr}_k(2,4)$ is exactly the quadric hypersurface defined by (7.12). □

7.4. **Tangent space of the Grassmannian.** We now describe the Zariski tangent space of the Grassmannian at a k-point. We use the description of tangent vectors via dual numbers.

Definition 7.13. Let X be a k-scheme and $x \in X(k)$ a k-point. The *Zariski tangent space* $T_x X$ is the k-vector space of morphisms

$$\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2)) \longrightarrow X$$

sending the closed point to x.

In the functorial picture of Section 3, a tangent vector at $[U] \in \mathrm{Gr}_k(d, V)$ is thus a first-order deformation of the d-dimensional subspace $U \subset V$ over $k[\varepsilon]/(\varepsilon^2)$.

Theorem 7.14 (Tangent space of the Grassmannian)**.** *Let k be a field, $V = k^n$, and $G = \mathrm{Gr}_k(d, V)$. For a point $[U] \in G(k)$ corresponding to a d-dimensional subspace $U \subset V$, there is a natural isomorphism of k-vector spaces*

$$T_{[U]}G \;\cong\; \mathrm{Hom}_k(U, V/U).$$

In particular, $\dim_k T_{[U]}G = d(n-d)$ *for all* $[U]$*, so* G *is smooth of dimension* $d(n-d)$*.*

Proof. Let $D = \operatorname{Spec}(k[\varepsilon]/(\varepsilon^2))$. A tangent vector at $[U]$ is a morphism $f : D \to G$ sending the closed point to $[U]$. Pulling back the universal subbundle along f gives a rank d subbundle

$$\mathcal{U}_D \subset V \otimes_k \mathcal{O}_D \cong V \otimes_k k[\varepsilon]$$

whose reduction modulo (ε) is $U \subset V$.

Since $k[\varepsilon]$ is local and Artinian, every finitely generated projective $k[\varepsilon]$-module is free. Choose a basis $u_1, \ldots, u_d$ of U and lift it to elements

$$u_i + \varepsilon v_i \in V \otimes_k k[\varepsilon], \qquad v_i \in V,$$

which form a basis of $\mathcal{U}_D$ as a $k[\varepsilon]$-module. Two such choices define the same submodule $\mathcal{U}_D$ if and only if they differ by a change of basis on $k[\varepsilon]^d$, given by a matrix $A = 1 + \varepsilon B$ with $B \in M_d(k)$. This changes the tuple (v_i) by adding linear combinations of the u_j.

Modulo this ambiguity, the first-order deformation $\mathcal{U}_D$ is encoded by the class of the k-linear map

$$\varphi : U \longrightarrow V/U, \qquad u_i \longmapsto [v_i] \bmod U.$$

Conversely, given any $\varphi \in \operatorname{Hom}_k(U, V/U)$, choose representatives $v_i \in V$ of $\varphi(u_i)$ and define $\mathcal{U}_D$ to be the $k[\varepsilon]$-submodule of $V \otimes_k k[\varepsilon]$ generated by the elements $u_i + \varepsilon v_i$. This gives a deformation of U over D.

One checks that isomorphic deformations correspond to the same φ, and that this construction is natural in U. This yields a canonical isomorphism $T_{[U]}G \cong \operatorname{Hom}_k(U, V/U)$, and the dimension formula follows. $\square$

Remark 7.15. On a standard affine chart $G_I \cong \mathbb{A}_k^{d(n-d)}$, the tangent space at a point is identified with the space of first-order variations of the $(n-d) \times d$ matrix of coordinates, which is naturally isomorphic to $\operatorname{Hom}_k(U, V/U)$. This agrees with the intrinsic description above and shows that the dimension of G is $d(n-d)$.

7.5. **Basic geometric properties.** Finally we record some standard geometric properties of the Grassmannian over an algebraically closed field.

Proposition 7.16. *Let* k *be an algebraically closed field and* $G = \operatorname{Gr}_k(d, n)$*. Then:*

(1) G *is smooth, projective, connected, and irreducible of dimension* $d(n-d)$*.*
(2) The Picard group $\operatorname{Pic}(G)$ *is isomorphic to* $\mathbb{Z}$ *and is generated by the Plücker line bundle* $\mathcal{O}_G(1)$*.*

Proof. Smoothness and the dimension statement follow from Theorem 7.14. Projectivity and the fact that $\mathcal{O}_G(1)$ is very ample follow from the Plücker embedding Theorem 7.10.

Connectedness and irreducibility can be seen using the action of the algebraic group $\mathrm{GL}_n(k)$ on k^n: this induces a transitive action on d-planes, and G can be identified with a homogeneous space $\mathrm{GL}_n(k)/P$ for a suitable parabolic subgroup P. Since $\mathrm{GL}_n(k)$ is irreducible, the homogeneous space G is irreducible and in particular connected.

For the Picard group, one may use the homogeneous-space description together with the classification of line bundles on $\mathrm{GL}_n(k)/P$ to show that every line bundle on G is a tensor power of the Plücker line bundle, and that $\mathcal{O}_G(1)$ is not trivial. This gives $\operatorname{Pic}(G) \cong \mathbb{Z}$ with generator $[\mathcal{O}_G(1)]$. $\square$

Remark 7.17. From the moduli-theoretic point of view developed earlier, the Grassmannian $\mathrm{Gr}_X(d, \mathcal{E})$ is a smooth projective family of Grassmannians over X, with fibers $\mathrm{Gr}_k(d, n)$ as above. The Plücker line bundle on $\mathrm{Gr}_X(d, \mathcal{E})$ plays a central role in embedding these fibers into projective space and in studying intersection-theoretic properties via the geometry of Schubert varieties.