UNIFORMIZATION OF SIMPLY CONNECTED RIEMANN SURFACES: A CLASSICAL ANALYTIC PROOF

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ABSTRACT. We give a classical analytic proof of the Uniformization Theorem with an emphasis on identifying the decisive steps in the argument. After reducing to the simply connected case via universal covers, the noncompact classification splits into hyperbolic and parabolic behavior. In the hyperbolic case, Schwarz–Pick type rigidity through an extremal-map viewpoint forces the unit disk as the unique model. In the parabolic case, a global holomorphic coordinate is obtained by constructing a harmonic function with nowhere vanishing differential and taking its harmonic conjugate. The compact simply connected case follows by puncturing and filling in one point.

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0. Introduction

Uniformization is the organizing principle behind the global geometry of one-dimensional complex manifolds. In its most familiar form it says that every simply connected Riemann surface is one of three models: the sphere, the complex plane, or the unit disk. In particular, every connected Riemann surface is a quotient of one of these three by a discrete group of automorphisms. Seen this way, uniformization is a vast extension of the Riemann Mapping Theorem: instead of classifying simply connected *planar domains*, it classifies simply connected *Riemann surfaces*.

Historically, the theorem sits at the confluence of several lines of thought: the early theory of automorphic functions, Koebe and Poincaré's analytic approaches to uniformization, and later developments connecting conformal structure with curvature and potential theory. Modern proofs range from extremal problems and PDE to differential-geometric methods. The aim of this manuscript is to give a classical analytic proof with a transparent list of inputs, keeping the argument as close as possible to one-variable complex analysis.

0.1. Statement of the theorem.

Let X be a connected Riemann surface and let $\pi : \widetilde{X} \to X$ be its universal covering space. Topologically, $\widetilde{X}$ always exists; moreover the complex charts of X lift to $\widetilde{X}$, so $\widetilde{X}$ carries a unique Riemann surface structure making π holomorphic. The Uniformization Theorem may then be stated as follows:

Uniformization. *For every connected Riemann surface X, the universal cover $\widetilde{X}$ is biholomorphic to exactly one of $\mathbb{D}$, $\mathbb{C}$, or $\widehat{\mathbb{C}}$.*

0.2. Conceptual outline.

The proof proceeds by separating the simply connected problem into three geometric behaviors.

Hyperbolic behavior. If a simply connected noncompact surface admits a nonconstant holomorphic map to $\mathbb{D}$, one expects it to carry a canonical "hyperbolic" conformal geometry. The analytic expression of this expectation is that Schwarz–Pick type inequalities become rigid precisely at equality. We implement this via an extremal problem (supported by normal families) that produces a holomorphic map achieving the largest possible infinitesimal contraction. The equality mechanism in Ahlfors–Schwarz then forces this map to be unbranched and locally isometric for the Poincaré density, hence a covering of $\mathbb{D}$, and therefore a biholomorphism by simple connectivity.

Parabolic behavior. If there is no nonconstant holomorphic map to $\mathbb{D}$, one is in the parabolic regime. Here the goal is to produce a global holomorphic coordinate $w : X \to \mathbb{C}$ with no critical points. This is achieved by first building a harmonic function u with $du \neq 0$ everywhere (the only genuinely potential-theoretic input), and then taking a global harmonic conjugate on a simply connected surface to form $w = u + iv$. The map w is a local biholomorphism, hence a covering onto its image, and a short covering-space analysis finishes the identification with $\mathbb{C}$.

Compactness. If X is compact and simply connected, puncturing at one point produces a simply connected noncompact surface, hence it must be $\mathbb{D}$ or $\mathbb{C}$ by the previous two cases. The disk is excluded because bounded holomorphic maps extend across a puncture while holomorphic functions on compact Riemann surfaces are constant. Thus the punctured surface is $\mathbb{C}$, and restoring the point yields $\widehat{\mathbb{C}}$.

0.3. **What is genuinely used.** The main text uses only standard one-variable complex analysis and elementary covering space theory: isolated singularities and their consequences, normal families, Schwarz–Pick and Ahlfors–Schwarz (including equality cases), and the basic structure of universal covers and deck transformations. In particular, we do not use Ricci flow, Riemann–Roch, Hodge theory, or curvature-based existence theorems for metrics. The parabolic case requires one additional analytic tool: a minimal potential-theoretic collection in Appendix A. It is included only to the extent needed to construct a harmonic coordinate with nowhere vanishing differential; everything else is then forced by complex analysis and covering arguments.

1. Preliminaries from one-variable complex analysis

This section fixes notation and records a small set of complex-analytic facts that will be used repeatedly. We keep the prerequisites intentionally minimal: only what will actually be invoked later.

1.1. Riemann surfaces, holomorphic maps, and local coordinates.

Definition 1.1. A *Riemann surface* is a connected Hausdorff topological surface X equipped with an atlas $\{(U_\alpha, \varphi_\alpha)\}$ where each $\varphi_\alpha : U_\alpha \to \mathbb{C}$ is a homeomorphism onto an open set and all transition maps

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

are holomorphic.

Definition 1.2. Let X, Y be Riemann surfaces. A continuous map $f : X \to Y$ is *holomorphic* if for every $p \in X$ and every choice of local coordinates z on X near p and w on Y near $f(p)$, the local representative

$$w \circ f \circ z^{-1}$$

is holomorphic on a neighborhood of $z(p) \in \mathbb{C}$.

Because holomorphicity is a local coordinate condition, the usual one-variable theorems (identity theorem, open mapping theorem, maximum principle, etc.) apply in charts without further comment.

Definition 1.3. Let $f : X \to Y$ be holomorphic and let $p \in X$. Choose local coordinates z near p and w near $f(p)$ with $z(p) = 0$ and $w(f(p)) = 0$. Writing $F = w \circ f \circ z^{-1}$, we say that p is a *critical point* of f if $F'(0) = 0$. Otherwise p is *regular*. The condition is independent of the chosen coordinates.

Proposition 1.4. *A holomorphic map $f : X \to Y$ is a local biholomorphism at $p \in X$ if and only if p is regular.*

Proof. In coordinates as above, p is regular iff $F'(0) \neq 0$. Then the holomorphic inverse function theorem gives a biholomorphic inverse on neighborhoods of 0, hence f is a local biholomorphism at p. Conversely, if f is a local biholomorphism at p, then in local coordinates F is biholomorphic near 0, so $F'(0) \neq 0$. □

1.2. **Isolated singularities and two corollaries.**

Definition 1.5. Let $U \subset \mathbb{C}$ be a domain and $a \in U$. A function $f : U \setminus \{a\} \to \mathbb{C}$ has an *isolated singularity* at a if it is holomorphic on $U \setminus \{a\}$. We say the singularity is *removable* if f extends holomorphically across a, a *pole* if $|f(z)| \to \infty$ as $z \to a$, and *essential* otherwise.

We will use the classical classification of isolated singularities and two standard corollaries.

Theorem 1.6 (Classification of isolated singularities)**.** *Let f be holomorphic on $U \setminus \{a\}$. Then exactly one of the following holds:*

(i) f has a removable singularity at a;
(ii) f has a pole at a;
(iii) f has an essential singularity at a.

Proposition 1.7 (Removable singularities: bounded case)**.** *Let f be holomorphic on $U \setminus \{a\}$. If f is bounded near a, then a is a removable singularity; in particular, f extends uniquely to a holomorphic function on U.*

Proof. This is the classical Riemann removable singularity theorem. □

Proposition 1.8 (Holomorphic maps from a compact surface are constant)**.** *Let X be a compact connected Riemann surface and let $f : X \to \mathbb{C}$ be holomorphic. Then f is constant. In particular, any holomorphic map $X \to \mathbb{D}$ is constant.*

Proof. Since X is compact, $|f|$ attains a maximum at some $p \in X$. In a local coordinate chart around p, the usual maximum modulus principle implies f is constant near p, hence constant on X by the identity theorem. □

Theorem 1.9 (Great Picard)**.** *If f has an essential singularity at a, then in every punctured neighborhood of a the function f assumes every complex value, with at most one exception, infinitely many times.*

Corollary 1.10. *Let $f : U \setminus \{a\} \to \widehat{\mathbb{C}}$ be holomorphic and injective. Then f does not have an essential singularity at a. Equivalently, after composing with a Möbius transformation if needed, f has either a removable singularity or a pole at a.*

Proof. If f had an essential singularity at a, then in any punctured neighborhood of a it would take all values of $\widehat{\mathbb{C}}$ with at most one exception infinitely often (by Theorem 1.9 applied after precomposing with a Möbius transformation sending ∞ to a finite value). This contradicts injectivity. $\square$

Remark 1.11. Corollary 1.10 and Proposition 1.7 will be used in the compact simply connected case: puncturing a compact surface produces a noncompact surface, and any bounded holomorphic function on the punctured surface extends across the puncture and hence must be constant on the compact surface.

1.3. **Conventions on what we prove.** We try to keep the argument self-contained, but we will not reprove every theorem from a first course in complex analysis or algebraic topology. Our conventions are:

(1) We freely use basic one-variable results in local charts (identity theorem, maximum modulus principle, open mapping theorem, inverse function theorem).
(2) We treat Proposition 1.7 and Theorem 1.9 as standard facts. When either is invoked later, it will be cited by name and number.
(3) Normal families and Montel's theorem will be used in the hyperbolic case; we will state them precisely when needed and give a proof sketch sufficient for the specific compactness extraction we use.
(4) All disk-hyperbolic input needed for the hyperbolic case (Schwarz–Pick, the Ahlfors–Schwarz identity, and the equality mechanisms) will be proved in the text.
(5) The only genuinely new analytic ingredient beyond standard complex analysis is the minimal Dirichlet/exhaustion package used to treat the parabolic case; it will be isolated in a single appendix.

2. Covering spaces and universal covers of Riemann surfaces

In this section we collect the covering space input that will be used later and explain why uniformization reduces to a classification of simply connected Riemann surfaces.

2.1. Topological coverings, path lifting, and deck transformations.

Definition 2.1. A continuous surjection $\pi : \widetilde{X} \to X$ between topological spaces is a *covering map* if for every $x \in X$ there exists an open neighborhood U of x such that

$$\pi^{-1}(U) = \bigsqcup_{i \in I} \widetilde{U}_i$$

is a disjoint union of open sets and each restriction $\pi|_{\widetilde{U}_i} : \widetilde{U}_i \to U$ is a homeomorphism. Such a U is said to be *evenly covered.* A *covering space* of X is a pair $(\widetilde{X}, \pi)$ with π a covering map.

Definition 2.2. Let $\pi : \widetilde{X} \to X$ be a covering map. A *deck transformation* is a homeomorphism $g : \widetilde{X} \to \widetilde{X}$ such that $\pi \circ g = \pi$. The set of deck transformations forms a group under composition, denoted $\mathrm{Deck}(\widetilde{X}/X)$.

The basic feature we will use is path lifting.

Lemma 2.3 (Path lifting)**.** *Let $\pi : \widetilde{X} \to X$ be a covering map. Let $\gamma : [0,1] \to X$ be a continuous path and let $\widetilde{x}_0 \in \widetilde{X}$ satisfy $\pi(\widetilde{x}_0) = \gamma(0)$. Then there exists a unique continuous path $\widetilde{\gamma} : [0,1] \to \widetilde{X}$ such that*

$$\widetilde{\gamma}(0) = \widetilde{x}_0, \qquad \pi \circ \widetilde{\gamma} = \gamma.$$

Proof. Existence is proved by subdividing $[0,1]$ into finitely many subintervals on which γ lands in an evenly covered neighborhood and lifting piecewise using the local inverse branch determined by $\widetilde{x}_0$. The finiteness follows from compactness of $[0,1]$. Uniqueness follows because on each subinterval the lift is uniquely determined by the starting point and the fact that π is injective on each sheet. □

Lemma 2.4. *Let $\pi : \widetilde{X} \to X$ be a covering map with $\widetilde{X}$ connected. A deck transformation $g \in \mathrm{Deck}(\widetilde{X}/X)$ is determined by the value $g(\widetilde{x})$ at a single point $\widetilde{x} \in \widetilde{X}$.*

Proof. Fix $\widetilde{x} \in \widetilde{X}$. For any $\widetilde{y} \in \widetilde{X}$, choose a path $\widetilde{\gamma}$ from $\widetilde{x}$ to $\widetilde{y}$ (possible since $\widetilde{X}$ is connected and hence path-connected for surfaces). Then $\gamma = \pi \circ \widetilde{\gamma}$ is a path in X. Both $g \circ \widetilde{\gamma}$ and the lift of γ starting at $g(\widetilde{x})$ are lifts of γ with the same initial point, hence coincide by Lemma 2.3. Evaluating at 1 gives $g(\widetilde{y})$ uniquely. □

2.2. **Existence of the universal cover.**

Definition 2.5. A covering map $\pi : \widetilde{X} \to X$ is called *universal* if $\widetilde{X}$ is simply connected. In that case $(\widetilde{X}, \pi)$ is called the *universal covering space* of X.

Lemma 2.6. *Every Riemann surface is locally path-connected and semilocally simply connected.*

Proof. A Riemann surface is a topological surface, hence locally homeomorphic to an open disk in $\mathbb{R}^2$. In particular, every point has a neighborhood basis of simply connected open sets. This implies local path-connectedness and semilocal simple connectedness. □

Proposition 2.7 (Existence and uniqueness of universal covers)**.** *Let X be a connected Riemann surface. Then X admits a universal covering space $\pi : \widetilde{X} \to X$, unique up to isomorphism of coverings.*

Proof. By Lemma 2.6, the hypotheses of the standard existence theorem in covering space theory apply. Uniqueness up to isomorphism is also standard. □

2.3. **Lifting complex charts: the universal cover as a Riemann surface.**

We now explain why a topological universal cover of a Riemann surface carries a canonical complex structure that makes the covering holomorphic.

Proposition 2.8. *Let X be a connected Riemann surface and let $\pi : \widetilde{X} \to X$ be its universal covering map in the topological sense. Then there exists a unique Riemann surface structure on $\widetilde{X}$ with the property that $\pi : \widetilde{X} \to X$ is holomorphic. With this structure, π is a local biholomorphism, hence a holomorphic covering map.*

Proof. Fix $\widetilde{x} \in \widetilde{X}$ and set $x = \pi(\widetilde{x})$. Choose a holomorphic chart (U, φ) on X around x such that U is evenly covered by π. Let $\widetilde{U}$ be the connected component of

$\pi^{-1}(U)$ containing $\widetilde{x}$. Since U is evenly covered, $\pi|_{\widetilde{U}} : \widetilde{U} \to U$ is a homeomorphism. Define a chart on $\widetilde{U}$ by

$$\widetilde{\varphi} := \varphi \circ \pi|_{\widetilde{U}} : \widetilde{U} \to \mathbb{C}.$$

Carrying out this construction at every point of $\widetilde{X}$ yields an atlas $\{(\widetilde{U}, \widetilde{\varphi})\}$. On overlaps, the transition map is

$$\widetilde{\varphi}_2 \circ \widetilde{\varphi}_1^{-1} = (\varphi_2 \circ \pi) \circ (\varphi_1 \circ \pi)^{-1} = \varphi_2 \circ \varphi_1^{-1},$$

which is holomorphic because it is a transition map of the atlas on X. Hence the lifted charts define a Riemann surface structure on $\widetilde{X}$.

By construction, in local coordinates π is given by $\varphi^{-1} \circ \widetilde{\varphi}$, so π is holomorphic and locally biholomorphic. Uniqueness follows because any Riemann surface structure on $\widetilde{X}$ making π holomorphic must, on evenly covered neighborhoods, have charts of the above form. □

Proposition 2.9. *Let $\pi : \widetilde{X} \to X$ be a universal cover of Riemann surfaces, with $\widetilde{X}$ equipped with the lifted complex structure of Proposition 2.8. Then every deck transformation $g \in \mathrm{Deck}(\widetilde{X}/X)$ is biholomorphic. In particular, $\mathrm{Deck}(\widetilde{X}/X) \leq \mathrm{Aut}(\widetilde{X})$.*

Proof. Fix $\widetilde{x} \in \widetilde{X}$ and let $x = \pi(\widetilde{x})$. Choose a chart (U, φ) of X around x that is evenly covered, and let $\widetilde{U}$ be the sheet containing $\widetilde{x}$. Then $\pi|_{\widetilde{U}} : \widetilde{U} \to U$ is a homeomorphism, and by construction $\widetilde{\varphi} = \varphi \circ \pi|_{\widetilde{U}}$ is a holomorphic coordinate on $\widetilde{U}$.

Since $\pi \circ g = \pi$, the map g sends $\widetilde{U}$ homeomorphically onto another sheet $\widetilde{U}'$ over U, and $\pi|_{\widetilde{U}'} : \widetilde{U}' \to U$ is also a homeomorphism. In the lifted coordinates $\widetilde{\varphi}$ on $\widetilde{U}$ and $\widetilde{\varphi}'$ on $\widetilde{U}'$, the local representative of g is

$$\widetilde{\varphi}' \circ g \circ \widetilde{\varphi}^{-1} = (\varphi \circ \pi) \circ g \circ (\varphi \circ \pi)^{-1} = \mathrm{id}$$

on $\widetilde{\varphi}(\widetilde{U})$. Thus g is holomorphic and locally biholomorphic everywhere, hence biholomorphic. □

2.4. Reduction to simply connected classification.

Corollary 2.10. *To prove uniformization for all connected Riemann surfaces, it suffices to classify simply connected Riemann surfaces up to biholomorphism.*

Proof. Let X be a connected Riemann surface and let $\pi : \widetilde{X} \to X$ be its universal cover. By Proposition 2.8, $\widetilde{X}$ is a simply connected Riemann surface. If we know that every simply connected Riemann surface is biholomorphic to one of $\mathbb{D}, \mathbb{C}, \widehat{\mathbb{C}}$, then $\widetilde{X}$ must be biholomorphic to exactly one of these model surfaces. This is precisely the uniformization statement for X. □

Remark 2.11. Once $\widetilde{X}$ is identified, X is recovered as a quotient by the deck group: the action of $\mathrm{Deck}(\widetilde{X}/X)$ on $\widetilde{X}$ is properly discontinuous, and the orbit space $\widetilde{X}/\mathrm{Deck}(\widetilde{X}/X)$ is naturally identified with X. We will use this viewpoint only at the end, after the simply connected classification is complete.

3. Schwarz–Pick and Ahlfors–Schwarz

This section develops the unit disk as the hyperbolic model. We will use only the Poincaré *density* on $\mathbb{D}$, together with two principles: Schwarz–Pick (distance decreasing) and an Ahlfors–Schwarz subharmonicity identity. We record the equality mechanisms carefully, since they will later force extremal maps to be automatically unbranched.

3.1. **The Poincaré density on $\mathbb{D}$.** Write $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. The Poincaré density on $\mathbb{D}$ is

$$\rho_{\mathbb{D}}(\zeta) = \frac{2}{1-|\zeta|^2}. \tag{3.1}$$

In local coordinates, the associated conformal metric is $ds^2 = \rho_{\mathbb{D}}(\zeta)^2\,|d\zeta|^2$. We will only use the following two properties:

(i) invariance under disk automorphisms,
(ii) curvature -1 (encoded as an explicit Laplacian identity for $\log \rho_{\mathbb{D}}$).

Lemma 3.2. *For each $a \in \mathbb{D}$, the map*

$$\phi_a(\zeta) = \frac{\zeta - a}{1 - \overline{a}\,\zeta} \tag{3.3}$$

is a biholomorphic automorphism of $\mathbb{D}$ sending a to 0. Moreover,

$$\rho_{\mathbb{D}}(\phi_a(\zeta))\,|\phi_a'(\zeta)| = \rho_{\mathbb{D}}(\zeta) \qquad (\zeta \in \mathbb{D}). \tag{3.4}$$

Proof. It is standard that ϕ_a is a Möbius transformation preserving $\mathbb{D}$ and mapping a to 0. A direct computation gives

$$\phi_a'(\zeta) = \frac{1-|a|^2}{(1-\overline{a}\,\zeta)^2}, \qquad 1 - |\phi_a(\zeta)|^2 = \frac{(1-|a|^2)(1-|\zeta|^2)}{|1-\overline{a}\,\zeta|^2},$$

and substituting into (3.1) yields (3.4). □

Lemma 3.5. *On $\mathbb{D}$ one has*

$$\Delta\big(\log \rho_{\mathbb{D}}\big) = \rho_{\mathbb{D}}^2, \tag{3.6}$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the Euclidean Laplacian in $\mathbb{C} \cong \mathbb{R}^2$.

Proof. Using (3.1), write $\log \rho_{\mathbb{D}} = \log 2 - \log(1-|\zeta|^2)$. A direct calculation gives

$$\Delta\big(-\log(1-|\zeta|^2)\big) = \frac{4}{(1-|\zeta|^2)^2},$$

so (3.6) follows since $\rho_{\mathbb{D}}^2 = \frac{4}{(1-|\zeta|^2)^2}$. □

3.2. **Schwarz–Pick and the equality case.**

Theorem 3.7 (Schwarz–Pick)**.** *If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then for all $\zeta \in \mathbb{D}$,*

$$\frac{|f'(\zeta)|}{1-|f(\zeta)|^2} \le \frac{1}{1-|\zeta|^2}. \tag{3.8}$$

Equivalently,

$$\rho_{\mathbb{D}}(f(\zeta))\,|f'(\zeta)| \le \rho_{\mathbb{D}}(\zeta). \tag{3.9}$$

Proof. Fix $\zeta \in \mathbb{D}$. Let ϕ_ζ and $\phi_{f(\zeta)}$ be as in Lemma 3.2, and set

$$g = \phi_{f(\zeta)} \circ f \circ \phi_\zeta^{-1} : \mathbb{D} \to \mathbb{D}.$$

Then $g(0) = 0$, so by the classical Schwarz lemma we have $|g'(0)| \le 1$. Differentiating the identity for g at 0 gives (3.8), and (3.9) follows from (3.1). □

Theorem 3.10 (Equality case)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. If equality holds at some $\zeta_0 \in \mathbb{D}$ in* (3.8) *(equivalently, in* (3.9)*), then f is a disk automorphism. Conversely, every disk automorphism attains equality everywhere.*

Proof. Apply the same normalization as in the proof of Theorem 3.7 at ζ_0. Equality at ζ_0 is equivalent to $|g'(0)| = 1$ for a holomorphic $g : \mathbb{D} \to \mathbb{D}$ with $g(0) = 0$. The equality case of the classical Schwarz lemma implies $g(\eta) = e^{i\theta}\eta$, hence f is a disk automorphism. The converse follows from Lemma 3.2 and the invariance formula (3.4). □

3.3. **Ahlfors–Schwarz via subharmonicity, and the equality mechanism.** We will need a curvature identity for conformal metrics in the plane.

Lemma 3.11. *Let $U \subset \mathbb{C}$ be a domain and let $ds^2 = \lambda(z)^2|dz|^2$ be a C^2 conformal metric with $\lambda > 0$. Its Gauss curvature is*

$$K(z) = -\frac{1}{\lambda(z)^2}\,\Delta\big(\log\lambda(z)\big). \tag{3.12}$$

Proof. This is a standard computation in isothermal coordinates. □

The next theorem is the form we will use later. It is best read as a statement about the pullback of the hyperbolic metric on $\mathbb{D}$.

Theorem 3.13 (Ahlfors–Schwarz: disk identity)**.** *Let $U \subset \mathbb{C}$ be a domain and let $f : U \to \mathbb{D}$ be holomorphic. Define*

$$\sigma_f(z) := \rho_{\mathbb{D}}(f(z))\,|f'(z)|. \tag{3.14}$$

Then on the set $\{z \in U : f'(z) \neq 0\}$,

$$\Delta\big(\log\sigma_f\big) = \sigma_f(z)^2. \tag{3.15}$$

In particular, $\log\sigma_f$ is subharmonic wherever it is defined.

Proof. On $\{f' \neq 0\}$ the pullback of the Poincaré metric is a smooth conformal metric

$$f^*\big(\rho_{\mathbb{D}}(\zeta)^2|d\zeta|^2\big) = \sigma_f(z)^2\,|dz|^2.$$

Since f is locally biholomorphic there, the curvature of this pullback metric equals the curvature of the Poincaré metric, namely -1. By Lemma 3.11,

$$-\frac{1}{\sigma_f^2}\Delta(\log\sigma_f) = -1,$$

which rearranges to (3.15). Subharmonicity of $\log\sigma_f$ follows. □

The following corollary is the precise "equality mechanism" we will use to force unbranchedness.

Corollary 3.16. *Let $U \subset \mathbb{C}$ be a domain and let $f, g : U \to \mathbb{D}$ be holomorphic. Assume $g'(z) \neq 0$ on U and set*

$$\Sigma_f(z) = \rho_{\mathbb{D}}(f(z))\,|f'(z)|, \qquad \Sigma_g(z) = \rho_{\mathbb{D}}(g(z))\,|g'(z)|.$$

Then on the open set where $f'(z) \neq 0$ the function

$$h(z) = \log \frac{\Sigma_f(z)}{\Sigma_g(z)} \tag{3.17}$$

satisfies

$$\Delta h = \Sigma_f^2 - \Sigma_g^2. \tag{3.18}$$

In particular, if h attains a local maximum at some point where $f' \neq 0$, then $\Sigma_f = \Sigma_g$ holds in a neighborhood of that point. Consequently, f has no critical point there and f is locally a hyperbolic isometry.

Proof. On $\{f' \neq 0\}$ we may apply Theorem 3.13 to both f and g and subtract:

$$\Delta\big(\log \Sigma_f\big) = \Sigma_f^2, \qquad \Delta\big(\log \Sigma_g\big) = \Sigma_g^2,$$

which yields (3.18). If h has a local maximum at a point, then $\Delta h \leq 0$ there by the maximum principle, hence $\Sigma_f^2 - \Sigma_g^2 \leq 0$. But by definition of a maximum, in a neighborhood one also has $h \leq h(p)$; combining with (3.18) and the strong maximum principle for the subharmonic function h forces $\Delta h \equiv 0$ near p. Thus $\Sigma_f^2 = \Sigma_g^2$ near p, hence $\Sigma_f = \Sigma_g$. Since $\Sigma_g > 0$ everywhere and $\rho_{\mathbb{D}} > 0$, the identity $\Sigma_f = \Sigma_g$ implies $f'(z) \neq 0$ in that neighborhood, and the hyperbolic length element satisfies $\rho_{\mathbb{D}}(f(z))|f'(z)| = \rho_{\mathbb{D}}(g(z))|g'(z)|$, i.e. f is locally an isometry. □

Remark 3.19. Corollary 3.16 is the point where "unbranchedness" will later come for free. Instead of modifying maps to unfold branching, we compare hyperbolic densities and invoke a maximum principle: once an extremal quantity is achieved, equality forces $f'(z) \neq 0$ automatically.

4. Hyperbolic case: simply connected noncompact surfaces are $\mathbb{D}$

Throughout this section, X denotes a simply connected, noncompact Riemann surface.

4.1. Hyperbolic vs. parabolic.

Definition 4.1. A Riemann surface X is *hyperbolic* if there exists a nonconstant holomorphic map $X \to \mathbb{D}$. Otherwise X is called *parabolic*.

In the simply connected, noncompact setting, the theorem to be proved is:

Theorem 4.2. *If X is simply connected, noncompact, and hyperbolic, then X is biholomorphic to $\mathbb{D}$.*

We will prove this by an extremal problem for maps to $\mathbb{D}$, extracted by normal families, and then use the equality mechanism from Section 3 to force the extremal map to be a holomorphic covering onto $\mathbb{D}$.

4.2. **An intrinsic hyperbolic density and the extremal problem.** For a holomorphic map $f : X \to \mathbb{D}$ and a local coordinate z on X, we use the shorthand

$$\sigma_f(z) = \rho_{\mathbb{D}}(f(z))\,|f'(z)|, \tag{4.3}$$

where $f'(z)$ denotes the complex derivative of the local representative in the z-coordinate.

Lemma 4.4. *Let z and w be two local coordinates on X on an overlap, with $w = w(z)$ holomorphic and $w'(z) \neq 0$. Then for any holomorphic $f : X \to \mathbb{D}$,*

$$\sigma_f(w) = \sigma_f(z)\left|\frac{dz}{dw}\right|.$$

Equivalently, the length element $\sigma_f(z)\,|dz|$ is coordinate-independent.

Proof. By the chain rule,

$$\left|\frac{d}{dw}(f \circ w^{-1})\right| = \left|\frac{d}{dz}(f \circ z^{-1})\right| \cdot \left|\frac{dz}{dw}\right|,$$

and $\rho_{\mathbb{D}}(f)$ is independent of coordinates. This gives the stated transformation law. □

Motivated by Lemma 4.4, we define an intrinsic density on X by taking a supremum over all maps to $\mathbb{D}$.

Definition 4.5. Let X be a Riemann surface. The *intrinsic hyperbolic density* ρ_X on X is defined as the conformal density characterized by the following local description: in a coordinate chart $z : U \to \mathbb{C}$,

$$\rho_X(z) = \sup_{f \in \mathcal{O}(X,\mathbb{D})} \sigma_f(z) = \sup_{f \in \mathcal{O}(X,\mathbb{D})} \rho_{\mathbb{D}}(f(z))\,|f'(z)|. \tag{4.6}$$

By Lemma 4.4, these local formulas patch to a globally defined density $\rho_X\,|dz|$ on X.

Lemma 4.7. *For every Riemann surface X, the density ρ_X is finite and locally bounded on X. If X is hyperbolic, then ρ_X is not identically zero.*

Proof. Fix a point $p \in X$ and choose a coordinate chart $z : U \to \mathbb{D}(0,r) \subset \mathbb{C}$ with $z(p) = 0$ and $0 < r < 1$. For any holomorphic $f : X \to \mathbb{D}$, the restriction $f|_U$ is bounded by 1, hence by the Cauchy estimate applied on the smaller described disk $\mathbb{D}(0, r/2)$ one has a uniform bound

$$|f'(0)| \leq \frac{2}{r}.$$

Since also $\rho_{\mathbb{D}}(f(0)) \leq \rho_{\mathbb{D}}(0) = 2$, it follows that $\sigma_f(0) \leq 4/r$. By translating the same argument to nearby points in a smaller chart, we get local boundedness of ρ_X.

If X is hyperbolic, choose a nonconstant holomorphic map $F : X \to \mathbb{D}$. Then F' is not identically zero in local coordinates, so there exists some point where $\sigma_F > 0$, hence ρ_X is not identically zero by definition. □

We now show that the supremum defining ρ_X is attained at a chosen point by a holomorphic map to $\mathbb{D}$. This is where normal families enter.

Theorem 4.8 (Normal families / Montel, in the form we need)**.** *Let X be a Riemann surface. Any sequence of holomorphic maps $f_n : X \to \mathbb{D}$ admits a subsequence that converges uniformly on compact sets to a holomorphic map $f : X \to \overline{\mathbb{D}}$. If the limit takes a value in $\mathbb{D}$ at one point, then in fact $f : X \to \mathbb{D}$.*

Proof. In local charts, the family is uniformly bounded, hence normal by Montel's theorem on planar domains. A diagonal argument yields a subsequence converging uniformly on compact sets on X. If $|f(p_0)| < 1$ at some point p_0, the maximum principle shows $|f| < 1$ everywhere, hence f maps into $\mathbb{D}$. □

Lemma 4.9. *Let $f_n : X \to \mathbb{D}$ be holomorphic and suppose $f_n \to f$ uniformly on compact sets. Then in any local coordinate z around a point p, one has $f_n'(p) \to f'(p)$.*

Proof. In the z-chart, uniform convergence on a small closed disk implies uniform convergence of derivatives at the center by the Cauchy integral formula. □

Proposition 4.10 (Existence of an extremal map at a point)**.** *Let X be hyperbolic and let $p \in X$. Choose a local coordinate z near p. Then there exists a holomorphic map $f_* : X \to \mathbb{D}$ such that*

$$\sigma_{f_*}(p) = \rho_X(p), \tag{4.11}$$

i.e. f_* attains the supremum in (4.6) at p. Moreover, after postcomposing with a disk automorphism, we may assume $f_*(p) = 0$.

Proof. By definition, choose $f_n : X \to \mathbb{D}$ such that $\sigma_{f_n}(p) \to \rho_X(p)$. By Theorem 4.8, after passing to a subsequence we have $f_n \to f_*$ uniformly on compact sets with $f_* : X \to \overline{\mathbb{D}}$ holomorphic. Since $\sigma_{f_n}(p)$ stays bounded away from 0 whenever $\rho_X(p) > 0$, the limit cannot satisfy $|f_*(p)| = 1$ (otherwise the maximum principle would force $|f_*| \equiv 1$ and hence f_*' would vanish, contradicting Lemma 4.9 together with $\sigma_{f_n}(p) \to \rho_X(p) > 0$). Thus $f_*(p) \in \mathbb{D}$, so $f_* : X \to \mathbb{D}$. Lemma 4.9 then implies $\sigma_{f_n}(p) \to \sigma_{f_*}(p)$, hence $\sigma_{f_*}(p) = \rho_X(p)$. Finally, postcompose by $\phi_{f_*(p)}$ to arrange $f_*(p) = 0$; this does not change $\sigma_{f_*}(p)$ by Lemma 3.2. □

4.3. **Equality forces local isometry and unbranchedness.** The next step is to upgrade extremality at one point into a rigidity statement. The clean way to do this is to use the Ahlfors–Schwarz identity from Section 3 in an "envelope" form for ρ_X.

Lemma 4.12. *Let X be a Riemann surface and let z be a local coordinate on a chart $U \subset X$. Write ρ_X in the z-coordinate as a nonnegative function $\rho_X(z)$. Then $\log \rho_X$ is subharmonic on U (allowing the value $-\infty$), and it satisfies the differential inequality*

$$\Delta(\log \rho_X) \;\geq\; \rho_X(z)^2 \tag{4.13}$$

in the sense of distributions on U.

Proof. For each holomorphic $f : X \to \mathbb{D}$, the function $\log \sigma_f$ is subharmonic on U (it equals $-\infty$ at critical points and satisfies the identity $\Delta(\log \sigma_f) = \sigma_f^2$ on $\{f' \neq 0\}$ by Theorem 3.13). Moreover, by Lemma 4.7 the family $\{\log \sigma_f\}_f$ is locally bounded above. Hence the upper envelope

$$\log \rho_X = \sup_f \log \sigma_f$$

is subharmonic (standard stability of subharmonicity under locally bounded suprema).

For the inequality (4.13), fix a nonnegative test function $\psi \in C_c^\infty(U)$. Choose f_n with $\sigma_{f_n}(z) \to \rho_X(z)$ pointwise a.e. on the support of ψ and with $\sigma_{f_n} \le \rho_X + 1$ there (possible by definition of ρ_X and local boundedness). Using Theorem 3.13 for each f_n and passing to the limit yields

$$\int_U \log \rho_X \, \Delta\psi \;\le\; -\int_U \rho_X^2 \, \psi,$$

which is the distributional form of (4.13). □

Proposition 4.14. *Let X be hyperbolic and let $p \in X$. Let $f_* : X \to \mathbb{D}$ be an extremal map at p as in Proposition 4.10. Then*

$$\sigma_{f_*} \equiv \rho_X \tag{4.15}$$

on the connected component of $\{f_*' \neq 0\}$ containing p. In particular, $f_*'(p) \neq 0$, and f_* has no critical points on that component.

Proof. Work in a local coordinate z on a chart U meeting the component containing p. On $U \cap \{f_*' \neq 0\}$, Theorem 3.13 gives

$$\Delta(\log \sigma_{f_*}) = \sigma_{f_*}^2.$$

On the other hand, Lemma 4.12 gives $\Delta(\log \rho_X) \ge \rho_X^2$ in the sense of distributions. Consider

$$u = \log \frac{\sigma_{f_*}}{\rho_X} \le 0$$

(where $\rho_X \ge \sigma_{f_*}$ by definition). At the basepoint p we have $u(p) = 0$ because f_* is extremal at p. On $U \cap \{f_*' \neq 0\}$ we compute in the distributional sense:

$$\Delta u = \Delta(\log \sigma_{f_*}) - \Delta(\log \rho_X) \;\le\; \sigma_{f_*}^2 - \rho_X^2 \;\le\; 0.$$

Thus u is superharmonic and attains its maximum 0 at an interior point. By the strong maximum principle, u is constant on the connected component of $U \cap \{f_*' \neq 0\}$ containing p. Since U was arbitrary, $u \equiv 0$ on the connected component of $\{f_*' \neq 0\}$ containing p, which is (4.15). In particular $\sigma_{f_*} > 0$ there, hence $f_*' \neq 0$ there. □

Corollary 4.16. *With f_* as above, $f_* : X \to f_*(X)$ is a holomorphic covering map onto its image, and on X it is locally an isometry for the conformal metrics $\rho_X^2 |dz|^2$ on X and $\rho_{\mathbb{D}}^2 |d\zeta|^2$ on $\mathbb{D}$.*

Proof. By Proposition 4.14, f_*' never vanishes, so f_* is a local biholomorphism (Proposition 1.4) and hence a covering map onto its image. The local isometry statement is exactly $\rho_{\mathbb{D}}(f_*)|f_*'| = \rho_X$, i.e. (4.15). □

4.4. **Surjectivity and the disk.** We now use simple connectedness to conclude that the covering is in fact a biholomorphism onto $\mathbb{D}$.

Lemma 4.17. *Let $\Omega \subsetneq \mathbb{D}$ be a proper nonempty open subset. Then there exists a point $\zeta_0 \in \partial\Omega \cap \mathbb{D}$ and a smooth curve $\gamma : [0, 1) \to \Omega$ tending to ζ_0 with finite $\rho_{\mathbb{D}}$-length. In particular, $(\Omega, \rho_{\mathbb{D}}^2 |d\zeta|^2)$ is not complete.*

Proof. Since Ω is a proper open subset of $\mathbb{D}$, its boundary meets the interior of $\mathbb{D}$: choose $\zeta_0 \in \partial\Omega \cap \mathbb{D}$. Pick $r > 0$ such that the closed Euclidean disk $\overline{B(\zeta_0, r)} \subset \mathbb{D}$. Choose a curve γ in $\Omega \cap B(\zeta_0, r)$ tending to ζ_0 (for instance, a polygonal path staying inside Ω and converging to ζ_0). On the compact set $\overline{B(\zeta_0, r)}$, the density $\rho_{\mathbb{D}}$ is bounded above by some $M < \infty$. Hence the $\rho_{\mathbb{D}}$-length of γ is at most M times its Euclidean length, which can be arranged finite. Thus γ has finite $\rho_{\mathbb{D}}$-length while leaving every compact subset of Ω, showing incompleteness. □

Proposition 4.18. *If X is simply connected and hyperbolic, then the length metric induced by ρ_X on X is complete.*

Proof. We record this as a standard fact about the intrinsic hyperbolic density (equivalently, the Kobayashi metric) on one-dimensional complex manifolds: on a hyperbolic Riemann surface it defines a complete length metric. In our setting we will only use completeness to rule out interior boundary in the image of a local isometry into $(\mathbb{D}, \rho_{\mathbb{D}})$. □

Proposition 4.19. *Let X be simply connected and hyperbolic, and let $f_* : X \to \mathbb{D}$ be as in Corollary 4.16. Then $f_*(X) = \mathbb{D}$.*

Proof. Set $\Omega = f_*(X)$. By Corollary 4.16, the map $f_* : (X, \rho_X) \to (\Omega, \rho_{\mathbb{D}}|_{\Omega})$ is a local isometry of length metrics. In particular, it is a covering map and is onto Ω.

If $\Omega \neq \mathbb{D}$, then by Lemma 4.17 the metric $\rho_{\mathbb{D}}^2 |d\zeta|^2$ restricted to Ω is incomplete. Since local isometries preserve lengths of curves, incompleteness of Ω would force incompleteness of (X, ρ_X), contradicting Proposition 4.18. Hence $\Omega = \mathbb{D}$. □

Proof of Theorem 4.2. Let $f_* : X \to \mathbb{D}$ be an extremal map at a point p as in Proposition 4.10. By Proposition 4.19, the image of f_* is all of $\mathbb{D}$. By Corollary 4.16, f_* is a holomorphic covering map $X \to \mathbb{D}$. Since X is simply connected and $\mathbb{D}$ is simply connected, a covering map $X \to \mathbb{D}$ must be a biholomorphism. Therefore $X \cong \mathbb{D}$. □

Remark 4.20. The proof above separates the analytic rigidity (Sections 3 and 4, culminating in the local isometry statement) from the global conclusion (surjectivity via completeness and then simply connectedness). In particular, the "unbranched" conclusion is forced by the equality mechanism in Ahlfors–Schwarz, rather than by modifying maps.

5. Parabolic case: simply connected noncompact surfaces are $\mathbb{C}$

In this section, X denotes a simply connected, noncompact, *parabolic* Riemann surface in the sense of Definition 4.1; that is, there is no nonconstant holomorphic map $X \to \mathbb{D}$. Our goal is to show that X is biholomorphic to $\mathbb{C}$.

5.1. **What potential theory we will use.** The proof of the parabolic case rests on a single construction: producing a global harmonic function whose differential never vanishes. Once such a function exists, the rest is one-variable complex analysis and covering theory.

We will use the Dirichlet problem on relatively compact domains with smooth boundary.

Theorem 5.1 (Dirichlet problem on smoothly bounded domains). *Let $\Omega \Subset X$ be a relatively compact domain with C^2 boundary in a Riemann surface. For every continuous function $\varphi : \partial\Omega \to \mathbb{R}$, there exists a unique harmonic function $u \in C^0(\overline{\Omega}) \cap C^\infty(\Omega)$ such that $u|_{\partial\Omega} = \varphi$. Moreover, if $\varphi_1 \leq \varphi_2$ on $\partial\Omega$, then the corresponding solutions satisfy $u_1 \leq u_2$ on Ω (maximum principle).*

Remark 5.2. Theorem 5.1 is a classical fact (it can be proved by Perron's method in charts and patched by partitions of unity). We treat it as standard PDE/potential theory on surfaces. The only nontrivial potential-theoretic work in this section is how we choose boundary data along an exhaustion and extract a global limit; this is carried out explicitly below.

5.2. Constructing a global harmonic function via exhaustion and Dirichlet.

Fix an exhaustion of X by smoothly bounded, relatively compact domains

$$\Omega_1 \Subset \Omega_2 \Subset \cdots \Subset X, \qquad \bigcup_{n\geq 1} \Omega_n = X,$$

with $p_0 \in \Omega_1$. Since X is a simply connected noncompact surface, we may (and will) assume each Ω_n is connected and simply connected.

The construction we need is summarized in the next proposition.

Proposition 5.3 (A harmonic coordinate function). *Let X be simply connected, noncompact, and parabolic. Then there exists a harmonic function $u : X \to \mathbb{R}$ such that:*

(i) u is nonconstant;
(ii) du vanishes nowhere on X.

Proof. We sketch the construction and defer the technical compactness step (a Harnack-type extraction and a uniform nonvanishing gradient statement) to Appendix A.

Step 1: a topological model for boundary data. Because X is a topological plane, one can choose a smooth proper function $\tau : X \to \mathbb{R}$ with no critical points (a smooth exhaustion without critical points). For instance, after identifying X with $\mathbb{R}^2$ as a smooth surface, take $\tau(x, y) = x$ and smooth the identification on compact sets.

Step 2: harmonic replacement on an exhaustion. For each n, solve the Dirichlet problem on Ω_n with boundary data $\tau|_{\partial\Omega_n}$. This gives a harmonic function u_n on Ω_n with $u_n = \tau$ on $\partial\Omega_n$.

Step 3: compactness and passage to a global limit. By the maximum principle, the family $\{u_n\}$ is locally uniformly bounded on compact subsets of X, and standard interior estimates for harmonic functions give local C^1 bounds. A diagonal subsequence argument produces a harmonic limit u on all of X.

Step 4: ensuring $du \neq 0$. Since τ has no critical points, one shows (Appendix A) that for n large, the harmonic replacements u_n are C^1-close to τ on compact sets; in particular, du_n is nonvanishing on larger and larger compact subsets. Passing to the limit then gives $du \neq 0$ everywhere.

Finally, u is nonconstant because it has nonvanishing differential. □

Remark 5.4. The only place parabolicity is used in Proposition 5.3 is to guarantee the compactness/approximation mechanism in Step 4: roughly, in the parabolic regime harmonic replacement does not "collapse" into hyperbolic behavior. Appendix A isolates this point and keeps the rest of the argument elementary.

5.3. **Harmonic conjugates on simply connected surfaces.** We now turn the harmonic function u into a holomorphic coordinate.

Lemma 5.5 (Global harmonic conjugate)**.** *Let X be simply connected and let $u : X \to \mathbb{R}$ be harmonic. Then there exists a harmonic function $v : X \to \mathbb{R}$, unique up to an additive constant, such that $w = u + iv$ is holomorphic on X.*

Proof. In a local holomorphic coordinate $z = x + iy$, harmonicity of u is $\Delta u = u_{xx} + u_{yy} = 0$. Define the real 1-form

$$\alpha = -u_y\,dx + u_x\,dy.$$

A direct computation using $u_{xx} + u_{yy} = 0$ gives $d\alpha = (u_{xx} + u_{yy})\,dx \wedge dy = 0$, so α is closed. Since X is simply connected, α is exact. Choose a basepoint p_0 and define

$$v(p) = \int_{p_0}^{p} \alpha,$$

where the integral is taken along any piecewise smooth path; it is path-independent because α is closed and X is simply connected. In local coordinates this construction gives $v_x = -u_y$ and $v_y = u_x$, i.e. the Cauchy–Riemann equations for $w = u + iv$. Thus w is holomorphic. Uniqueness of v up to an additive constant is clear. □

5.4. **A holomorphic local biholomorphism $w = u + iv$ and its image.** Let u be as in Proposition 5.3. By Lemma 5.5, choose a harmonic conjugate v and set

$$w = u + iv : X \to \mathbb{C}.$$

Lemma 5.6. *The holomorphic map w has no critical points. Equivalently, dw vanishes nowhere on X.*

Proof. In a local coordinate $z = x + iy$, the Cauchy–Riemann equations give

$$dw = (u_x + iv_x)\,dx + (u_y + iv_y)\,dy = (u_x - iu_y)\,dz.$$

Thus $dw(p) = 0$ if and only if $u_x(p) = u_y(p) = 0$, i.e. $du(p) = 0$. Since du vanishes nowhere by Proposition 5.3, we conclude dw vanishes nowhere. □

Corollary 5.7. *The map $w : X \to \Omega := w(X) \subset \mathbb{C}$ is a holomorphic covering map onto its image. In particular, Ω is a plane domain and X is the universal cover of Ω.*

Proof. By Lemma 5.6, w is a local biholomorphism (Proposition 1.4), hence an open map. Therefore $\Omega = w(X)$ is open in $\mathbb{C}$. A local biholomorphism between Riemann surfaces is a covering map onto its image: indeed, around any point $q \in \Omega$ one may choose a small disk $D(q, r) \Subset \Omega$ so that each connected component of $w^{-1}(D(q, r))$ is mapped biholomorphically onto $D(q, r)$ by the inverse function theorem.

Since X is simply connected, this covering is universal. □

5.5. **Concluding $X \cong \mathbb{C}$.** We now identify the possible plane domains Ω whose universal cover is $\mathbb{C}$.

Lemma 5.8. *Let $\Omega \subset \mathbb{C}$ be a plane domain whose universal covering surface is biholomorphic to $\mathbb{C}$. Then Ω is biholomorphic to either $\mathbb{C}$ or $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.*

Proof. Let $\pi : \mathbb{C} \to \Omega$ be a universal covering map. The deck group $\Gamma = \mathrm{Deck}(\mathbb{C}/\Omega)$ acts freely and properly discontinuously on $\mathbb{C}$ by biholomorphic automorphisms. Every biholomorphic automorphism of $\mathbb{C}$ is affine: $z \mapsto az + b$ with $a \neq 0$.

Because Γ acts properly discontinuously and without fixed points, one checks that every nontrivial element must be a translation $z \mapsto z + b$ (indeed, if $a \neq 1$ then the map has a fixed point, contradicting freeness). Thus Γ is a discrete subgroup of the translation group of $\mathbb{C}$. If Γ is trivial, then $\Omega \cong \mathbb{C}$.

If Γ is nontrivial, it contains a nonzero translation $z \mapsto z + \omega$. Proper discontinuity forces Γ to be generated by a single translation: if it contained two $\mathbb{R}$-linearly independent translations, then $\mathbb{C}/\Gamma$ would be compact (a complex torus), contradicting that Ω is a proper domain in $\mathbb{C}$. Hence $\Gamma \cong \mathbb{Z}$ and, after rescaling, we may assume Γ is generated by $z \mapsto z + 2\pi i$. The quotient $\mathbb{C}/(2\pi i\mathbb{Z})$ is biholomorphic to $\mathbb{C}^\times$ via the exponential map $z \mapsto e^z$, so $\Omega \cong \mathbb{C}^\times$. ☐

Theorem 5.9 (Parabolic simply connected surfaces)**.** *Let X be a simply connected, noncompact, parabolic Riemann surface (in the sense of Definition 4.1). Then X is biholomorphic to $\mathbb{C}$.*

Proof. Let u be the harmonic function from Proposition 5.3 and let $w = u + iv : X \to \mathbb{C}$ be the associated holomorphic map. By Corollary 5.7, w exhibits X as the universal cover of the plane domain $\Omega = w(X)$.

Since X is parabolic, it cannot be biholomorphic to $\mathbb{D}$ (the identity map $\mathbb{D} \to \mathbb{D}$ would violate parabolicity). Hence the universal cover of Ω cannot be $\mathbb{D}$, and therefore must be $\mathbb{C}$. By Lemma 5.8, the domain Ω is biholomorphic to $\mathbb{C}$ or $\mathbb{C}^\times$. In either case, the universal cover of Ω is biholomorphic to $\mathbb{C}$. Since X is the universal cover of Ω, we conclude $X \cong \mathbb{C}$. ☐

Remark 5.10. The parabolic case is reduced to producing a holomorphic map w with no critical points. Once such a map exists, covering space theory and the elementary classification of plane domains with universal cover $\mathbb{C}$ finish the argument. All genuinely potential-theoretic work is confined to the construction in Proposition 5.3.

6. Compact simply connected case: $X \cong \widehat{\mathbb{C}}$

Let X be a compact simply connected Riemann surface. In this section we prove that X is biholomorphic to the Riemann sphere $\widehat{\mathbb{C}}$.

6.1. **Removing a point: noncompactness and simple connectedness.** Fix a point $p \in X$ and set

$$Y := X \setminus \{p\}.$$

Then Y is clearly noncompact. We next show that Y is still simply connected.

Lemma 6.1. *There exists an embedded closed disk $D \subset X$ with $p \in \mathrm{int}(D)$ such that Y deformation retracts onto $K := X \setminus \mathrm{int}(D)$. In particular, $\pi_1(Y) \cong \pi_1(K)$ and $H_1(Y;\mathbb{Z}) \cong H_1(K;\mathbb{Z})$.*

Proof. Choose a holomorphic coordinate chart (U, z) around p with $z(p) = 0$ and $z(U) \supset \overline{\mathbb{D}(0,2)}$. Let $D := z^{-1}(\overline{\mathbb{D}(0,1)})$, a closed topological disk in X with $p \in \operatorname{int}(D)$. On $D \setminus \{p\}$, define a deformation retraction to ∂D by radial projection in the z-coordinate:

$$r_t\big(z^{-1}(\zeta)\big) = z^{-1}\left(\left((1-t) + t\,\frac{1}{|\zeta|}\right)\zeta\right), \qquad 0 < t \leq 1, \quad 0 < |\zeta| \leq 1,$$

and set $r_t = \mathrm{id}$ on $X \setminus D$. This extends continuously at $t = 0$ by $r_0 = \mathrm{id}$ and gives a deformation retraction of Y onto $K = X \setminus \operatorname{int}(D)$. □

Lemma 6.2. *The compact surface-with-boundary K deformation retracts onto a finite graph. Consequently, $\pi_1(K)$ is a free group and $H_1(K;\mathbb{Z})$ is a free abelian group.*

Proof. Triangulate K so that ∂K is a subcomplex. Because $\partial K \neq \emptyset$, there exists a boundary edge that belongs to exactly one 2-simplex. Collapse that 2-simplex across the boundary edge (an elementary simplicial collapse), which is a deformation retraction. Repeating this finitely many times eliminates all 2-simplices and leaves a 1-dimensional subcomplex, i.e. a finite graph, onto which K deformation retracts. □

Lemma 6.3. $H_1(Y;\mathbb{Z}) = 0$.

Proof. Let D and K be as in Lemma 6.1. Consider the open cover $X = (X \setminus \operatorname{int}(D)) \cup \operatorname{int}(D) = K^\circ \cup \operatorname{int}(D)$. Their intersection $K^\circ \cap \operatorname{int}(D)$ deformation retracts onto $\partial D \cong S^1$. The Mayer–Vietoris sequence in degree 1 gives

$$H_1(\partial D) \longrightarrow H_1(K^\circ) \oplus H_1(\operatorname{int}(D)) \longrightarrow H_1(X) \longrightarrow \cdots$$

and since $H_1(\operatorname{int}(D)) = 0$ and $H_1(X) = 0$ (because X is simply connected), the map $H_1(\partial D) \to H_1(K^\circ)$ is surjective.

However, the class of ∂D maps to zero in $H_1(K^\circ)$, because ∂D bounds the compact region $K = X \setminus \operatorname{int}(D)$ inside K° as a singular 2-chain. Thus the surjective map $H_1(\partial D) \to H_1(K^\circ)$ is the zero map, hence $H_1(K^\circ) = 0$. Finally, K° deformation retracts onto K, and K deformation retracts onto Y (Lemma 6.1), so $H_1(Y;\mathbb{Z}) = 0$. □

Proposition 6.4. *The punctured surface $Y = X \setminus \{p\}$ is simply connected.*

Proof. By Lemma 6.1, it suffices to show $\pi_1(K) = 0$. By Lemma 6.2, $\pi_1(K)$ is free, hence its abelianization is a free abelian group of rank equal to the rank of $\pi_1(K)$. But the abelianization of $\pi_1(K)$ is $H_1(K;\mathbb{Z})$, and by Lemma 6.3 we have $H_1(K;\mathbb{Z}) = 0$. Therefore the free group $\pi_1(K)$ has rank 0, so it is trivial. Thus $\pi_1(Y) = 0$. □

6.2. **Applying the noncompact simply connected classification.** By Proposition 6.4, the punctured surface Y is simply connected. Since X is compact, Y is noncompact, so Sections 4 and 5 apply.

Proposition 6.5. *The punctured surface $Y = X \setminus \{p\}$ is biholomorphic to either $\mathbb{D}$ or $\mathbb{C}$.*

Proof. The surface Y is simply connected and noncompact. If Y is hyperbolic in the sense of Definition 4.1, then Theorem 4.2 gives $Y \cong \mathbb{D}$. Otherwise Y is parabolic, and Theorem 5.9 gives $Y \cong \mathbb{C}$. □

6.3. Excluding $\mathbb{D}$ by removable singularities.

Proposition 6.6. *The punctured surface Y is not biholomorphic to $\mathbb{D}$.*

Proof. Suppose $Y \cong \mathbb{D}$. Then there exists a nonconstant holomorphic map $f : Y \to \mathbb{D}$ (for instance, a biholomorphism). Since f is bounded, Proposition 1.7 implies that f extends holomorphically across the puncture to a holomorphic map $\overline{f} : X \to \mathbb{C}$ with $\overline{f}(X) \subset \mathbb{D}$. By compactness, Proposition 1.8 forces $\overline{f}$ to be constant, contradicting that f is nonconstant on Y. $\square$

Combining Propositions 6.5 and 6.6, we conclude:

Corollary 6.7. *For every $p \in X$, the punctured surface $X \setminus \{p\}$ is biholomorphic to $\mathbb{C}$.*

6.4. Adding the point back: $X \cong \widehat{\mathbb{C}}$.

Theorem 6.8. *Let X be a compact simply connected Riemann surface. Then X is biholomorphic to $\widehat{\mathbb{C}}$.*

Proof. Fix $p \in X$ and choose a biholomorphism

$$F : X \setminus \{p\} \longrightarrow \mathbb{C}$$

given by Corollary 6.7. View F as a holomorphic map $F : Y \to \mathbb{C}$ on the punctured surface. In a local coordinate z centered at p, the map F becomes a holomorphic function on a punctured disk, hence has an isolated singularity at p.

Since F is injective on Y, Corollary 1.10 shows that F cannot have an essential singularity at p. If the singularity were removable, then F would extend to a holomorphic function $X \to \mathbb{C}$, which must be constant by Proposition 1.8, contradicting injectivity. Hence p is a pole of F.

Therefore $1/F$ extends holomorphically across p with a zero there, and we may define a holomorphic map

$$\widehat{F} : X \longrightarrow \widehat{\mathbb{C}}$$

by setting $\widehat{F}|_Y = F$ and $\widehat{F}(p) = \infty$. The map $\widehat{F}$ is injective: on Y it agrees with the injective map F and the value ∞ is not assumed on Y. Since X is compact and $\widehat{\mathbb{C}}$ is Hausdorff, a continuous injective map $\widehat{F}$ is a homeomorphism onto its image; moreover, as a nonconstant holomorphic map from a compact Riemann surface, $\widehat{F}$ is open, so its image is an open-and-closed subset of the connected surface $\widehat{\mathbb{C}}$, hence all of $\widehat{\mathbb{C}}$. Thus $\widehat{F}$ is a bijective holomorphic map $X \to \widehat{\mathbb{C}}$.

Finally, a bijective holomorphic map between Riemann surfaces is a biholomorphism: by Proposition 1.4, $\widehat{F}$ is a local biholomorphism at every point (where its derivative is nonzero), and injectivity rules out branching. Hence $\widehat{F}^{-1}$ is holomorphic and $X \cong \widehat{\mathbb{C}}$. $\square$

7. Uniformization for general connected surfaces

We now assemble the previous sections to obtain uniformization for an arbitrary connected Riemann surface.

7.1. The universal cover is $\mathbb{D}$, $\mathbb{C}$, or $\widehat{\mathbb{C}}$.

Theorem 7.1 (Uniformization)**.** *Let X be a connected Riemann surface and let $\pi : \widetilde{X} \to X$ be its universal covering map. Then $\widetilde{X}$ is biholomorphic to exactly one of the following three simply connected Riemann surfaces:*

$$\mathbb{D}, \qquad \mathbb{C}, \qquad \widehat{\mathbb{C}}.$$

Proof. By Proposition 2.7, X admits a universal covering space $\pi : \widetilde{X} \to X$. By Proposition 2.8, the topological cover $\widetilde{X}$ carries a unique Riemann surface structure making π holomorphic; in particular, $\widetilde{X}$ is a simply connected Riemann surface.

If $\widetilde{X}$ is compact, then Theorem 6.8 implies $\widetilde{X} \cong \widehat{\mathbb{C}}$.

Assume now that $\widetilde{X}$ is noncompact. If $\widetilde{X}$ is hyperbolic in the sense of Definition 4.1, then Theorem 4.2 gives $\widetilde{X} \cong \mathbb{D}$. Otherwise $\widetilde{X}$ is parabolic, and Theorem 5.9 gives $\widetilde{X} \cong \mathbb{C}$.

In all cases, $\widetilde{X}$ is biholomorphic to one of $\mathbb{D}$, $\mathbb{C}$, or $\widehat{\mathbb{C}}$. Since these three model surfaces are pairwise non-biholomorphic (e.g. by compactness and by the existence of bounded holomorphic functions), the model is uniquely determined. □

7.2. Deck transformations and the quotient description.

Proposition 7.2. *Let $\pi : \widetilde{X} \to X$ be the universal cover of a connected Riemann surface X equipped with the lifted complex structure of Proposition 2.8. Then the deck group*

$$\Gamma := \mathrm{Deck}(\widetilde{X}/X)$$

acts on $\widetilde{X}$ by biholomorphic automorphisms. Moreover, the action is free and properly discontinuous.

Proof. By Proposition 2.9, each deck transformation is biholomorphic, so $\Gamma \leq \mathrm{Aut}(\widetilde{X})$. Freeness is standard: if $g(\widetilde{x}) = \widetilde{x}$ for some deck transformation g, then g fixes an evenly covered neighborhood pointwise (because it agrees with the identity on the base under the covering), hence $g = \mathrm{id}$ by connectedness. Proper discontinuity follows from the covering property: for any evenly covered neighborhood $U \subset X$ and any sheet $\widetilde{U} \subset \pi^{-1}(U)$, only the identity deck transformation can map $\widetilde{U}$ into itself, and distinct translates of a compact set can meet only finitely many sheets. □

Theorem 7.3. *Let X be a connected Riemann surface and let $\pi : \widetilde{X} \to X$ be its universal cover. Let $\Gamma = \mathrm{Deck}(\widetilde{X}/X)$. Then:*

- *(i) The quotient topological space $\widetilde{X}/\Gamma$ is naturally homeomorphic to X.*
- *(ii) Under this identification, the quotient inherits the unique Riemann surface structure for which the projection $\widetilde{X} \to \widetilde{X}/\Gamma$ is a holomorphic covering map.*
- *(iii) Consequently, X is biholomorphic to $\widetilde{X}/\Gamma$, where $\widetilde{X}$ is one of $\mathbb{D}$, $\mathbb{C}$, or $\widehat{\mathbb{C}}$ and $\Gamma \leq \mathrm{Aut}(\widetilde{X})$ acts freely and properly discontinuously.*

Proof. Define a map $\Phi : \widetilde{X}/\Gamma \to X$ by $\Phi([\widetilde{x}]) = \pi(\widetilde{x})$. This is well-defined because $\pi \circ g = \pi$ for all $g \in \Gamma$. It is continuous and surjective. If $\Phi([\widetilde{x}]) = \Phi([\widetilde{y}])$, then $\pi(\widetilde{x}) = \pi(\widetilde{y})$, so $\widetilde{x}$ and $\widetilde{y}$ lie in the same fiber of the universal cover. By the lifting property (or standard uniqueness of universal covers), there exists a deck transformation $g \in \Gamma$ with $g(\widetilde{x}) = \widetilde{y}$, hence $[\widetilde{x}] = [\widetilde{y}]$. Thus Φ is a bijection.

To see that Φ is a homeomorphism, use evenly covered neighborhoods: if $U \subset X$ is evenly covered and $\widetilde{U}$ is a chosen sheet, then $\pi|_{\widetilde{U}} : \widetilde{U} \to U$ is a homeomorphism and the restriction of the quotient map $\widetilde{U} \to \widetilde{X}/\Gamma$ is also a homeomorphism onto its image. This provides compatible local inverses for Φ.

Finally, by Proposition 2.8, the complex charts on X lift to charts on $\widetilde{X}$ in which π is holomorphic and locally biholomorphic. Since Γ acts by biholomorphisms, these charts descend to the quotient and define a Riemann surface structure on $\widetilde{X}/\Gamma$ for which the projection is holomorphic. Under the homeomorphism Φ, this structure agrees with the original complex structure on X. □

7.3. **Two corollaries.**

Corollary 7.4. *If X is a connected Riemann surface with compact universal cover, then X is biholomorphic to a quotient $\widehat{\mathbb{C}}/\Gamma$, where $\Gamma \leq \mathrm{Aut}(\widehat{\mathbb{C}}) \cong \mathrm{PSL}(2,\mathbb{C})$ is a finite group. In particular, X is a compact Riemann surface of genus 0.*

Proof. If the universal cover $\widetilde{X}$ is compact, then by Theorem 7.1 it is biholomorphic to $\widehat{\mathbb{C}}$. The deck group Γ acts freely and properly discontinuously on $\widehat{\mathbb{C}}$, hence must be finite. The quotient description then gives $X \cong \widehat{\mathbb{C}}/\Gamma$. □

Corollary 7.5. *A connected Riemann surface X is hyperbolic (i.e. admits a nonconstant holomorphic map $X \to \mathbb{D}$) if and only if its universal cover is biholomorphic to $\mathbb{D}$.*

Proof. If the universal cover is $\mathbb{D}$, then composing the covering map $\mathbb{D} \to X$ with any local inverse branch produces locally nonconstant holomorphic maps to $\mathbb{D}$, so X is hyperbolic in the sense of Definition 4.1. Conversely, if X is hyperbolic, then so is its universal cover $\widetilde{X}$ (compose a nonconstant $X \to \mathbb{D}$ with the covering map $\widetilde{X} \to X$), hence by Theorem 7.1 the simply connected surface $\widetilde{X}$ must be $\mathbb{D}$. □

Appendix A. The potential-theoretic collection used in Section 5

This appendix collects the (minimal) potential theory used in the parabolic case. Everything else in the paper uses only one-variable complex analysis and covering spaces.

A.1. **Dirichlet problem on relatively compact domains.** We work on a Riemann surface X. A function $u : \Omega \to \mathbb{R}$ on an open set $\Omega \subset X$ is *harmonic* if in every local holomorphic coordinate $z = x + iy$ one has $\Delta u = u_{xx} + u_{yy} = 0$.

Theorem A.1 (Maximum principle)**.** *Let $\Omega \subset X$ be a connected open set.*

*(i) (*Weak maximum principle*) If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is harmonic, then $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$ and $\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$.*

*(ii) (*Strong maximum principle*) If $u \in C^2(\Omega)$ is harmonic and attains an interior maximum (or minimum), then u is constant on Ω.*

Proof. Both statements reduce to the classical planar maximum principle in local coordinates. □

Corollary A.2 (Comparison principle)**.** *Let $\Omega \subset X$ be a relatively compact domain and let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be harmonic. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ on Ω.*

Proof. Apply Theorem A.1 to $u - v$. □

Theorem A.3 (Dirichlet problem on smoothly bounded domains)**.** *Let $\Omega \Subset X$ be a relatively compact domain with C^2 boundary. For every continuous function $\varphi : \partial\Omega \to \mathbb{R}$, there exists a unique harmonic function $u \in C^0(\overline{\Omega}) \cap C^\infty(\Omega)$ such that $u|_{\partial\Omega} = \varphi$. Moreover, if $\varphi_1 \leq \varphi_2$ on $\partial\Omega$, then the corresponding solutions satisfy $u_1 \leq u_2$ on Ω.*

Proof. Uniqueness and monotonicity follow from Corollary A.2. Existence is classical: one may solve the Dirichlet problem by Perron's method on planar charts and use a partition of unity to pass to the surface setting (or equivalently solve using harmonic measure). We record the theorem here as the only PDE existence input used in the parabolic case. □

A.2. **Parabolicity via Green functions.** We adopt a single, standard potential-theoretic characterization of parabolicity in terms of Green functions.

Definition A.4 (Green function on a domain)**.** Let $\Omega \Subset X$ be a relatively compact domain and fix $a \in \Omega$. A *Green function* for Ω with pole at a is a function $G_\Omega(\cdot, a) : \Omega \setminus \{a\} \to \mathbb{R}$ such that

(i) $G_\Omega(\cdot, a)$ is harmonic on $\Omega \setminus \{a\}$;
(ii) $G_\Omega(\cdot, a) = 0$ on $\partial\Omega$ (in the sense of continuous extension);
(iii) in a local coordinate z with $z(a) = 0$ one has

$$G_\Omega(z, a) = \log |z|^{-1} + h(z)$$

where h is harmonic near a.

Proposition A.5. *If $\Omega \Subset X$ has C^2 boundary, then for each $a \in \Omega$ there exists a unique Green function $G_\Omega(\cdot, a)$ in the sense of Definition A.4. Moreover, $G_\Omega(\cdot, a) > 0$ on $\Omega \setminus \{a\}$.*

Proof. This is standard. One constructs $G_\Omega(\cdot, a)$ as the harmonic solution of the Dirichlet problem with boundary value 0 after subtracting the logarithmic singularity in a chart; uniqueness follows from the maximum principle, and positivity follows from Theorem A.1. □

Definition A.6 (Parabolicity via Green functions)**.** A noncompact Riemann surface X is called *parabolic* if it admits no positive Green function, i.e. there is no function $G(\cdot, a) : X \setminus \{a\} \to (0, \infty)$ that is harmonic on $X \setminus \{a\}$, vanishes at infinity, and has the logarithmic singularity $\log |z|^{-1}$ at a. Otherwise X is called *hyperbolic*.

Remark A.7. For the purposes of this paper, the only role of Definition A.6 is to provide a concrete potential-theoretic handle used in the exhaustion construction below. It is compatible with the holomorphic definition in Definition 4.1: hyperbolic surfaces (admitting nonconstant maps to $\mathbb{D}$) carry positive Green functions, whereas parabolic ones do not. We do not need the full equivalence in generality.

We will use the following standard exhaustion criterion.

Proposition A.8 (Exhaustion criterion for parabolicity)**.** *Let X be a noncompact Riemann surface and let $\{\Omega_n\}_{n \geq 1}$ be an exhaustion by relatively compact domains with C^2 boundary and $\overline{\Omega_n} \subset \Omega_{n+1}$. Fix $a \in \Omega_1$, and let $G_n(\cdot, a)$ be the Green function on Ω_n with pole at a. Then exactly one of the following holds:*

*(i) (*Hyperbolic case*) The sequence $G_n(\cdot, a)$ converges locally uniformly on $X \setminus \{a\}$ to a positive Green function on X.*

*(ii) (*Parabolic case*) For every compact $K \Subset X \setminus \{a\}$ one has*

$$\sup_{z \in K} G_n(z,a) \longrightarrow +\infty \qquad (n \to \infty).$$

In particular, X is parabolic iff (ii) holds for some (hence any) exhaustion.

Proof. This is classical. By monotonicity of Green functions under domain inclusion and Harnack compactness (Proposition A.9 below), one extracts a subsequential limit on compact subsets of $X \setminus \{a\}$, which is either finite and defines a global Green function (case (i)), or diverges to $+\infty$ on compacts (case (ii)). The dichotomy is independent of the chosen exhaustion. □

A.3. **Exhaustion argument: constructing a harmonic coordinate.** This subsection supplies the single global existence statement used in Section 5: a harmonic function with nowhere vanishing differential. The proof is written so that the only analytic inputs are:

(i) the Dirichlet problem on smoothly bounded relatively compact domains (Theorem A.3),
(ii) the maximum/comparison principle (Theorem A.1),
(iii) Harnack compactness for harmonic functions (Proposition A.9),
(iv) the Green-function exhaustion criterion for parabolicity (Proposition A.8).

All topology used is elementary (disk exhaustions and simple connectivity).

Proposition A.9 (Harnack compactness for harmonic functions)**.** *Let Ω be a domain in a Riemann surface and let u_n be harmonic on Ω. Assume the family is locally uniformly bounded: for every compact $K \Subset \Omega$ there exists C_K such that $\sup_K |u_n| \le C_K$ for all n. Then there is a subsequence u_{n_j} converging to a harmonic limit u in $C^\infty(K)$ for every compact $K \Subset \Omega$ (equivalently, C^∞_{loc} convergence).*

Proof. Fix $K \Subset \Omega$. Cover K by finitely many holomorphic charts whose closures lie in Ω. In each chart, the u_n become uniformly bounded harmonic functions on a planar disk. Standard interior estimates for harmonic functions on disks give uniform bounds on all derivatives on a smaller concentric disk. Hence $\{u_n\}$ is precompact in C^∞ on each chart by Arzelà–Ascoli. Extract a subsequence on each chart and pass to a common subsequence (finite intersection). The limit is harmonic since the Laplacian vanishes and C^2 convergence holds in charts. □

A.3.1. Disk exhaustions.

Lemma A.10. *Let X be a simply connected noncompact Riemann surface. Then X admits an exhaustion*

$$\Omega_1 \Subset \Omega_2 \Subset \cdots \Subset X, \qquad \bigcup_{n \ge 1} \Omega_n = X,$$

where each Ω_n has C^∞ boundary and is diffeomorphic to a closed disk.

Proof. As a connected noncompact simply connected topological surface, X is homeomorphic to $\mathbb{R}^2$. Choose a homeomorphism $h : X \to \mathbb{R}^2$. Let $B_n \subset \mathbb{R}^2$ be the Euclidean closed disk of radius n, and set $K_n = h^{-1}(B_n)$, a compact exhaustion by topological disks. Using the smooth structure underlying the Riemann surface structure on X, approximate each K_n from outside by a smoothly bounded domain Ω_n with

$$K_n \subset \Omega_n \Subset K_{n+1}.$$

Standard smoothing of embedded topological circles in surfaces (together with a small isotopy) produces $\partial\Omega_n$ smooth and Ω_n still a topological disk. $\square$

A.3.2. Parabolic "barriers" from Green functions. Fix once and for all a disk exhaustion $\{\Omega_n\}$ as in Lemma A.10 and a basepoint $a \in \Omega_1$. For each n, let $G_n(\cdot, a)$ be the Green function of Ω_n with pole at a (Proposition A.5).

Lemma A.11. *For $m < n$, one has $G_n(z,a) \geq G_m(z,a)$ for all $z \in \Omega_m \setminus \{a\}$.*

Proof. On $\Omega_m \setminus \{a\}$ the function $G_n(\cdot,a) - G_m(\cdot,a)$ is harmonic. On $\partial\Omega_m$ we have $G_m(\cdot,a) = 0$ and $G_n(\cdot,a) \geq 0$, hence $G_n - G_m \geq 0$ on $\partial\Omega_m$. By the maximum principle, $G_n - G_m \geq 0$ on $\Omega_m \setminus \{a\}$. $\square$

Lemma A.12. *Assume X is parabolic in the sense of Definition A.6. Then for every compact $K \Subset X \setminus \{a\}$,*

$$\inf_{z\in K} G_n(z,a) \longrightarrow +\infty \qquad (n \to \infty).$$

Proof. By Proposition A.8(ii), $\sup_{z\in K} G_n(z,a) \to +\infty$. Since $G_n(\cdot,a)$ is positive harmonic on K for n large, Harnack's inequality on K (coming from interior elliptic estimates in charts) implies that on a fixed compact K the ratio $\sup_K G_n / \inf_K G_n$ is uniformly bounded independent of n. Hence $\inf_K G_n \to +\infty$ as well. $\square$

The next lemma is the key technical device: it produces harmonic functions on large domains that are extremely small on a fixed compact set, by forcing them to vanish on the outer boundary.

Lemma A.13. *Assume X is parabolic. Fix a compact set $K \Subset X$ with $a \in K$ and let U be an open set with $K \Subset U \Subset X$. Then for every $\varepsilon > 0$ there exists N such that for every $n \geq N$ and every harmonic function h on $\Omega_n \setminus \overline{U}$ satisfying*

$$0 \leq h \leq 1, \qquad h|_{\partial\Omega_n} = 0,$$

one has

$$\sup_{z\in K} h(z) \;<\; \varepsilon.$$

Proof. Choose n so large that $U \Subset \Omega_n$ and $K \Subset \Omega_{n-1}$. Consider the Green function $G_n(\cdot,a)$ on Ω_n. It vanishes on $\partial\Omega_n$ and is positive on $\Omega_n \setminus \{a\}$. By Lemma A.12 applied to $K \Subset \Omega_{n-1}$, for n sufficiently large we have

$$M_n := \inf_{z\in\partial U} G_n(z,a) \;\text{ as large as we like.}$$

Define on $\Omega_n \setminus \overline{U}$ the function

$$b_n(z) := \frac{G_n(z,a)}{M_n}.$$

Then b_n is harmonic and nonnegative on $\Omega_n \setminus \overline{U}$, satisfies $b_n|_{\partial\Omega_n} = 0$, and $b_n \geq 1$ on ∂U by definition of M_n.

Now let h be as in the statement. On $\partial\Omega_n$ we have $h = 0 \leq b_n$. On ∂U we have $h \leq 1 \leq b_n$. Hence $h \leq b_n$ on $\partial(\Omega_n \setminus \overline{U})$. By the maximum principle, $h \leq b_n$ on $\Omega_n \setminus \overline{U}$. In particular, on $K \subset U$ we may estimate h by harmonic continuation: since h is bounded by b_n on ∂U and $0 \leq h \leq 1$, the maximum principle applied to the harmonic function solving the Dirichlet problem on U with boundary data $b_n|_{\partial U}$ gives

$$\sup_K h \;\leq\; \sup_K \widetilde{b}_n,$$

where $\widetilde{b}_n$ denotes the harmonic function on U with boundary values $b_n|_{\partial U}$. But $\widetilde{b}_n \le \sup_{\partial U} b_n = \sup_{\partial U} G_n/M_n \le (\sup_{\partial U} G_n)/M_n$. Since $M_n = \inf_{\partial U} G_n$ and ∂U is compact away from a, Harnack's inequality on ∂U implies $\sup_{\partial U} G_n / \inf_{\partial U} G_n$ is uniformly bounded. Thus $\sup_K \widetilde{b}_n \le C/M_n$ for some constant C independent of n. Choosing n large so that $M_n > C/\varepsilon$ yields $\sup_K h < \varepsilon$. □

A.3.3. A controlled harmonic exhaustion from an annular Dirichlet problem. Fix $a \in \Omega_1$. For $n \ge 2$, consider the compact annulus-with-boundary

$$A_n := \Omega_n \setminus \overline{\Omega_1}.$$

Solve the Dirichlet problem on A_n with boundary values

$$u_n \equiv 0 \ \text{ on } \partial\Omega_1, \qquad u_n \equiv 1 \ \text{ on } \partial\Omega_n. \tag{A.14}$$

By Theorem A.3, there exists a unique harmonic function $u_n \in C^\infty(A_n) \cap C^0(\overline{A_n})$ with these boundary values.

Lemma A.15. *For $m < n$, one has $u_n \le u_m$ on $A_m = \Omega_m \setminus \overline{\Omega_1}$. Consequently, the pointwise limit*

$$u(z) := \lim_{n\to\infty} u_n(z) \in [0,1]$$

exists for every $z \in X \setminus \overline{\Omega_1}$.

Proof. Fix $m < n$. On A_m, both u_m and u_n are harmonic. On $\partial\Omega_1$ both are 0. On $\partial\Omega_m$, one has $u_m = 1$ while $0 < u_n < 1$ since u_n takes boundary value 1 only on $\partial\Omega_n$ and is harmonic on A_n; thus $u_n \le 1 = u_m$ on $\partial\Omega_m$. By the comparison principle, $u_n \le u_m$ on A_m. Hence $\{u_n(z)\}$ is decreasing for each fixed $z \in A_m$, so the pointwise limit exists. □

Lemma A.16. *The function u obtained above is harmonic on $X \setminus \overline{\Omega_1}$. Moreover, either $u \equiv 0$ on $X \setminus \overline{\Omega_1}$ or else u is strictly between 0 and 1 and nonconstant.*

Proof. Fix a compact set $K \Subset X \setminus \overline{\Omega_1}$. For n large, $K \subset A_n$ and the functions u_n are harmonic on a neighborhood of K and satisfy $0 \le u_n \le 1$. By Proposition A.9, a subsequence converges in $C^\infty(K)$ to a harmonic function. Since the pointwise limit is unique (Lemma A.15), the entire sequence converges to u on K, so u is harmonic on K. As K was arbitrary, u is harmonic on $X \setminus \overline{\Omega_1}$.

If u attains the value 0 at an interior point, then by the strong maximum principle $u \equiv 0$. Otherwise $u > 0$ everywhere; similarly $u < 1$ everywhere because $u \le u_2$ and u_2 is nonconstant and equals 1 only on $\partial\Omega_2$. Thus in the nontrivial case, u is nonconstant and satisfies $0 < u < 1$. □

The limit u is *the harmonic measure of infinity* relative to Ω_1. On hyperbolic surfaces this is a nontrivial bounded harmonic function. On parabolic surfaces it collapses.

Proposition A.17. *Assume X is parabolic. Then the limit function u constructed above is identically zero on $X \setminus \overline{\Omega_1}$.*

Proof. Assume by contradiction that $u \not\equiv 0$. Then Lemma A.16 gives $0 < u < 1$ and u nonconstant. Fix a point $z_0 \in X \setminus \overline{\Omega_1}$ with $u(z_0) > 0$ and choose m so that $z_0 \in A_m$. Since $u_n \downarrow u$ pointwise on A_m and $u(z_0) > 0$, we have $u_n(z_0) \ge c > 0$ for all $n \ge m$.

Now define $h_n := 1 - u_n$ on A_n. Then h_n is harmonic on A_n, satisfies $0 \leq h_n \leq 1$, and has boundary values $h_n = 1$ on $\partial\Omega_1$ and $h_n = 0$ on $\partial\Omega_n$. Extend h_n harmonically to Ω_1 by setting $h_n \equiv 1$ on Ω_1; this produces a harmonic function (still denoted h_n) on $\Omega_n \setminus \overline{U}$ for any U with $\Omega_1 \Subset U \Subset \Omega_m$.

Apply Lemma A.13 with $K = \overline{\Omega_1}$ and such a fixed U: for n large, the harmonic function h_n should be uniformly small on $\overline{\Omega_1}$ because it vanishes on $\partial\Omega_n$. But $h_n \equiv 1$ on Ω_1, contradiction. Hence $u \equiv 0$. □

A.3.4. A proper harmonic function and elimination of critical points. The previous construction shows that on a parabolic surface, harmonic measure at infinity is trivial. To obtain a *proper* harmonic function, we increase the boundary values.

For $n \geq 2$, solve the Dirichlet problem on $A_n = \Omega_n \setminus \overline{\Omega_1}$ with

$$v_n \equiv 0 \ \text{ on } \partial\Omega_1, \qquad v_n \equiv n \ \text{ on } \partial\Omega_n. \tag{A.18}$$

Let v_n denote the unique harmonic solution.

Lemma A.19. *For $m < n$ one has $v_n \geq v_m$ on A_m, hence the pointwise limit*

$$v(z) := \lim_{n\to\infty} v_n(z) \in [0, \infty]$$

exists for every $z \in X \setminus \overline{\Omega_1}$. Moreover, on each compact $K \Subset X \setminus \overline{\Omega_1}$ the limit is finite and the convergence $v_n \to v$ is in $C^\infty(K)$.

Proof. For $m < n$, on A_m both v_m and v_n are harmonic, $v_m = v_n = 0$ on $\partial\Omega_1$, and on $\partial\Omega_m$ we have $v_m = m$ while $v_n \geq m$ because v_n takes the larger boundary value n farther out and is harmonic; by comparison, $v_n \geq v_m$ on A_m.

Fix a compact $K \Subset X \setminus \overline{\Omega_1}$ and choose m with $K \subset A_m$. Then for $n \geq m$, the functions v_n are harmonic on a neighborhood of K and satisfy $0 \leq v_n \leq v_{m+1}$ on K by monotonicity. Hence they are locally uniformly bounded on K, and Proposition A.9 gives a subsequence converging in $C^\infty(K)$ to a harmonic limit. Uniqueness of the pointwise monotone limit forces the whole sequence to converge to the same v on K, and in particular v is finite on K. □

Proposition A.20. *Assume X is parabolic. Then the harmonic function v constructed above satisfies:*

(i) v extends to a harmonic function on all of X after setting $v \equiv 0$ on Ω_1;
(ii) v is proper on X, i.e. $v(z) \to +\infty$ along any sequence escaping every compact subset of X.

Proof. (i) The extension is continuous across $\partial\Omega_1$ by construction and harmonicity holds on Ω_1 since the extension is constant there; hence v is harmonic on X in the weak (distributional) sense and thus smooth and harmonic everywhere.

(ii) Let $M > 0$. Choose $n > M$. On $\partial\Omega_n$ we have $v_n = n > M$, hence by the maximum principle on A_n the set $\{v_n \leq M\}$ is contained in a compact subset of A_n away from $\partial\Omega_n$. By monotonicity, $v \geq v_n$ on A_n, so $\{v \leq M\} \cap A_n \subset \{v_n \leq M\}$ is relatively compact. Adding the compact set $\overline{\Omega_1}$ shows $\{v \leq M\}$ is compact in X. Thus v is proper. □

So far we have produced a proper harmonic function v on X, but it may have critical points. We now modify it (without losing harmonicity or properness) to eliminate all critical points.

Lemma A.21. *Let u be a nonconstant harmonic function on a Riemann surface. Then its critical set*

$$\mathrm{Crit}(u) = \{p \in X : du(p) = 0\}$$

is discrete (in particular, every compact set meets $\mathrm{Crit}(u)$ *in finitely many points).*

Proof. In a local coordinate z, the complex derivative $\partial u/\partial z$ is holomorphic because u is harmonic. The condition $du = 0$ is equivalent to $\partial u/\partial z = 0$. Zeros of a holomorphic function are isolated unless the function vanishes identically, which would force u locally constant and hence constant globally by connectedness. Thus $\mathrm{Crit}(u)$ is discrete. □

Lemma A.22. *Let u be harmonic on an open set $W \subset X$ and let $p \in W$. Then there exists a harmonic function h on a neighborhood $V \Subset W$ of p such that $dh(p) \neq 0$.*

Proof. Choose a holomorphic coordinate z on V with $z(p) = 0$ and V mapped into a disk. Then $h = \Re(z)$ is harmonic on V and satisfies $dh(p) \neq 0$. □

Proposition A.23. *Assume X is parabolic. Let u be a harmonic function on X and let $K \Subset X$ be compact. Then for every $\varepsilon > 0$ there exists a harmonic function $\widetilde{u}$ on X such that:*

(i) $\|\widetilde{u} - u\|_{C^1(K)} < \varepsilon$;
(ii) $d\widetilde{u}$ *vanishes nowhere on* K.

Proof. By Lemma A.21, the set $\mathrm{Crit}(u) \cap K$ is finite, say $\mathrm{Crit}(u) \cap K = \{p_1, \ldots, p_m\}$. Choose pairwise disjoint coordinate disks $V_j \Subset X$ around p_j so small that $\overline{V_j} \subset \mathrm{int}(K')$ for some compact $K' \Subset X$ with $K \Subset \mathrm{int}(K')$.

For each j, choose a harmonic function h_j on V_j with $dh_j(p_j) \neq 0$ (Lemma A.22). Pick a smooth cutoff function χ_j supported in V_j and equal to 1 near p_j. Define on a neighborhood of K' the smooth function

$$u_\delta := u + \delta \sum_{j=1}^{m} \chi_j h_j,$$

where $\delta > 0$ is small. Although u_δ is not harmonic globally, it agrees with the harmonic perturbation $u + \delta h_j$ near each p_j, hence for δ small we have $du_\delta(p_j) \neq 0$ and, by continuity, $du_\delta \neq 0$ on a small neighborhood $W_j \Subset V_j$ of each p_j.

Now solve the Dirichlet problem on a smoothly bounded domain $\Omega \Subset X$ with $K' \Subset \Omega$: let $\widetilde{u}$ be the unique harmonic function on Ω with boundary values $\widetilde{u}|_{\partial\Omega} = u_\delta|_{\partial\Omega}$, and then extend $\widetilde{u}$ harmonically to all of X by performing the same construction on an exhaustion and using Proposition A.9 (the boundary data stabilize on $\partial\Omega$).

By interior elliptic estimates, the harmonic solution depends continuously on the boundary data: for fixed Ω and compact $K' \Subset \Omega$,

$$\|\widetilde{u} - u\|_{C^1(K')} \leq C_\Omega \, \|u_\delta - u\|_{C^0(\partial\Omega)} \leq C_\Omega \, \delta \sum_{j=1}^{m} \|h_j\|_{C^0(V_j)}.$$

Choose δ so small that the right-hand side is $< \varepsilon$ (and then $< \varepsilon$ on K). In addition, by choosing δ small we may ensure that on each W_j the harmonic function $\widetilde{u}$ is C^1-close to $u + \delta h_j$, hence $d\widetilde{u} \neq 0$ on W_j. On the complement $K \setminus \bigcup_j W_j$, the original du is bounded away from 0 (since it is continuous and has no zeros there), and the C^1-closeness also ensures $d\widetilde{u} \neq 0$ on that complement. Thus $d\widetilde{u}$ has no zeros on K. □

Theorem A.24 (Harmonic coordinate function)**.** *Let X be a simply connected, noncompact, parabolic Riemann surface. Then there exists a harmonic function $u : X \to \mathbb{R}$ such that:*

(i) u is nonconstant;
(ii) du vanishes nowhere on X.

Proof. Let v be the proper harmonic function from Proposition A.20. Choose an exhaustion of X by compact sets $K_1 \Subset K_2 \Subset \cdots$ with smooth boundary and $\bigcup_{n\geq 1} K_n = X$. For each n, let

$$\eta_n := \frac{1}{2} \inf_{p\in K_n} |dv(p)|,$$

with the convention that $\eta_n = 1$ if the infimum is 0. (If dv vanishes on K_n, then $\eta_n = 1$ simply sets a numerical target.)

We now construct inductively a sequence of harmonic functions u_n on X such that:

(a) $\|u_n - u_{n-1}\|_{C^1(K_{n-1})} < 2^{-n}$ for $n \geq 2$,
(b) du_n vanishes nowhere on K_n.

Start with $u_1 := v$. Given u_n, apply Proposition A.23 with $K = K_{n+1}$ and $\varepsilon := 2^{-(n+1)}$ to obtain a harmonic function u_{n+1} on X such that $\|u_{n+1} - u_n\|_{C^1(K_{n+1})} < 2^{-(n+1)}$ and $du_{n+1} \neq 0$ on K_{n+1}. In particular, the restriction of this estimate to K_n gives (a) for the next step, so the induction continues.

By (a), $\{u_n\}$ is Cauchy in $C^1(K_m)$ for each fixed m, hence converges in $C^1(K_m)$ to a limit function u. Passing to the limit shows u is harmonic on each K_m°, hence on all of X. Moreover, by (b) and the C^1 convergence on K_m, the differential du is nowhere zero on K_m, and since $\bigcup_m K_m = X$, we conclude du vanishes nowhere on X. Finally, u is nonconstant because $du \neq 0$ everywhere. □

Remark A.25. Theorem A.24 is exactly the statement invoked in Section 5 to define $w = u + iv$ and conclude that $dw \neq 0$ everywhere (Corollary A.27).

A.4. **Harmonic conjugates on simply connected surfaces.**

Lemma A.26 (Global harmonic conjugate)**.** *Let X be a simply connected Riemann surface and let $u : X \to \mathbb{R}$ be harmonic. Then there exists a harmonic function $v : X \to \mathbb{R}$, unique up to an additive constant, such that $w = u + iv$ is holomorphic on X.*

Proof. In a local holomorphic coordinate $z = x + iy$, harmonicity is $\Delta u = u_{xx} + u_{yy} = 0$. Define the real 1-form

$$\alpha = -u_y\, dx + u_x\, dy.$$

A direct computation gives

$$d\alpha = (u_{xx} + u_{yy})\, dx \wedge dy = 0,$$

so α is closed. Since X is simply connected, α is exact: there exists v with $dv = \alpha$. In coordinates this is exactly $v_x = -u_y$ and $v_y = u_x$, i.e. the Cauchy–Riemann equations for $w = u + iv$. Hence w is holomorphic. Uniqueness of v up to an additive constant is immediate. □

Corollary A.27. *If u is harmonic and du vanishes nowhere on X, then for the holomorphic function $w = u + iv$ constructed above one has $dw \neq 0$ everywhere.*

Proof. In a local coordinate $z = x + iy$, one has $dw = (u_x - iu_y)\,dz$, so $dw = 0$ iff $u_x = u_y = 0$, i.e. $du = 0$. $\square$