Fun Template 2

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Contents

Conventions

Spacetime. A spacetime is a 4-dimensional smooth manifold $\mathcal{M}$ equipped with a Lorentzian metric g of signature $(-,+,+,+)$.

Indices. Abstract indices $a, b, c, \dots$ indicate tensor type. Coordinate indices $\mu, \nu, \rho, \dots \in \{0,1,2,3\}$ with $x^0 = t$. Spatial indices $i, j, k, \dots \in \{1,2,3\}$. The Einstein summation convention is used unless stated otherwise.

Causal character. For $v \neq 0$, timelike if $g(v,v) < 0$, null if $g(v,v) = 0$, spacelike if $g(v,v) > 0$. A time orientation is fixed, so future-directed is well-defined.

Raising/lowering. We use $v^\flat = g(v,\cdot)$ and $\alpha^\sharp$ for the inverse map. We write $v \cdot w := g(v,w)$.

Connection. ∇ always denotes the Levi-Civita connection of g (torsion-free and metric-compatible). In coordinates,

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}\, g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right).$$

Curvature sign convention. The Riemann tensor is defined by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) v^c = R^c{}_{dab}\, v^d, \qquad R_{ab} := R^c{}_{acb}, \qquad R := g^{ab} R_{ab},$$

and the Einstein tensor is $G_{ab} := R_{ab} - \frac{1}{2} R\, g_{ab}$.

Symmetrization. $T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba})$ and $T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba})$.

Volume form. We write $\sqrt{-g} := \sqrt{-\det(g_{\mu\nu})}$ in oriented local coordinates, so that $\mathrm{dVol}_g = \sqrt{-g}\, d^4x$.

Geodesics. Timelike geodesics parametrized by proper time satisfy $g(\dot\gamma, \dot\gamma) = -1$. Null geodesics are parametrized affinely and satisfy $\nabla_{\dot\gamma}\dot\gamma = 0$ with $g(\dot\gamma,\dot\gamma) = 0$.

Units. We use geometric units $c = G = 1$. Einsteins equation is written as

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}.$$

1 Minkowski Spacetime as a Lorentzian Manifold

1.1 Affine structure, inertial coordinates, and the Minkowski metric

Definition 1.1. Minkowski spacetime

Let V be a real 4-dimensional vector space equipped with a nondegenerate symmetric bilinear form

$$\eta\colon V \times V \to \mathbb{R}$$

of signature $(-,+,+,+)$. A **Minkowski spacetime** is an affine space $\mathbb{M}$ modeled on V, together with the metric η on the translation space, in the following sense. For each $p \in \mathbb{M}$ we identify $T_p\mathbb{M} \cong V$ canonically, and we use η to define an inner product on each $T_p\mathbb{M}$.

Concretely, choosing an origin $o \in \mathbb{M}$ identifies $\mathbb{M}$ with V via $p \mapsto \overrightarrow{op}$, but we will keep the affine viewpoint since it makes the isometry group transparent.

Definition 1.2. Inertial coordinates

Choose an origin $o \in \mathbb{M}$ and an η-orthonormal basis (e_0, e_1, e_2, e_3) of V such that

$$\eta(e_0, e_0) = -1, \qquad \eta(e_i, e_j) = \delta_{ij}, \qquad \eta(e_0, e_i) = 0.$$

The associated **inertial coordinates** (x^μ) on $\mathbb{M}$ are defined by writing each point $p \in \mathbb{M}$ uniquely as

$$p = o + x^\mu e_\mu,$$

and setting $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$.

In inertial coordinates, the metric components are constant,

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1), \qquad \eta = \eta_{\mu\nu}\, dx^\mu \otimes dx^\nu.$$

Definition 1.3. Lorentz group and Poincaré group

The **Lorentz group** is

$$O(1,3) = \{\Lambda \in GL(V) \mid \eta(\Lambda v, \Lambda w) = \eta(v, w) \text{ for all } v, w \in V\}.$$

The **Poincaré group** is the group of affine isometries of $\mathbb{M}$,

$$\mathrm{Iso}(\mathbb{M}, \eta) = \{\Phi\colon \mathbb{M} \to \mathbb{M} \text{ bijective affine} \mid \Phi^*\eta = \eta\}.$$

Proposition 1.4. Form of Minkowski isometries

Fix an origin $o \in \mathbb{M}$ so that $\mathbb{M}$ is identified with V as an affine space. Then every isometry $\Phi \in \mathrm{Iso}(\mathbb{M}, \eta)$ has the form

$$\Phi(p) = a + \Lambda p$$

for a unique translation vector $a \in V$ and a unique Lorentz transformation $\Lambda \in O(1,3)$. In particular,

$$\mathrm{Iso}(\mathbb{M}, \eta) \cong V \rtimes O(1,3).$$

Proof. Since Φ is affine, there exist a linear map $L\colon V \to V$ and a vector $a \in V$ such that $\Phi(p) = a + L(p)$ in the identification $\mathbb{M} \cong V$ determined by o. The differential of Φ at every point is L, hence for any $p \in \mathbb{M}$ and $v, w \in T_p\mathbb{M} \cong V$ we have

$$\eta(v, w) = \eta(d\Phi_p(v), d\Phi_p(w)) = \eta(Lv, Lw).$$

Thus $L \in O(1, 3)$. Uniqueness of (a, L) follows from uniqueness of the affine decomposition. $\square$

1.2 Causal cones and time orientation

Definition 1.5. Timelike, null, spacelike

Let $p \in \mathbb{M}$ and $v \in T_p\mathbb{M} \setminus \{0\}$. We say v is

$$\text{timelike if } \eta(v, v) < 0, \qquad \text{null if } \eta(v, v) = 0, \qquad \text{spacelike if } \eta(v, v) > 0.$$

The set of timelike vectors in $T_p\mathbb{M}$ has two connected components, called the **timelike cones**.

Definition 1.6. Time orientation and future direction

A **time orientation** is a continuous choice of one of the two timelike cones at each point. In Minkowski spacetime, fixing an inertial frame determines a distinguished timelike vector field ∂_t, and we declare a timelike vector v to be **future-directed** if $\eta(v, \partial_t) < 0$, equivalently $v^0 > 0$ in the associated inertial coordinates.

With this convention, the *future timelike cone* at p is

$$\mathcal{T}_p^+ = \{v \in T_p\mathbb{M} \mid \eta(v, v) < 0,\ v^0 > 0\},$$

and the *future null cone* is

$$\mathcal{N}_p^+ = \{v \in T_p\mathbb{M} \setminus \{0\} \mid \eta(v, v) = 0,\ v^0 > 0\}.$$

Proposition 1.7. Lorentz transformations preserve causal type and time orientation

Let $\Lambda \in O(1, 3)$. Then for any nonzero $v \in V$ the sign of $\eta(v, v)$ equals the sign of $\eta(\Lambda v, \Lambda v)$. Moreover, if Λ lies in the orthochronous component, then Λ maps the future timelike cone to itself.

Proof. The first statement is immediate from $\eta(\Lambda v, \Lambda v) = \eta(v, v)$. For the second statement, by definition of the orthochronous component one has $(\Lambda v)^0 > 0$ for all v with $v^0 > 0$ and $\eta(v, v) < 0$. $\square$

1.3 Proper time and proper length, the unit hyperboloid

Definition 1.8. Proper time and proper length

Let $\gamma\colon I \to \mathbb{M}$ be a piecewise C^1 curve with parameter λ.

(1) If γ is timelike, its **proper time** is

$$\tau(\gamma) = \int_I \sqrt{-\eta(\dot\gamma(\lambda), \dot\gamma(\lambda))}\, d\lambda.$$

(2) If γ is spacelike, its **proper length** is

$$L(\gamma) = \int_I \sqrt{\eta(\dot\gamma(\lambda), \dot\gamma(\lambda))}\, d\lambda.$$

Proposition 1.9. Reparametrization invariance

Let $\gamma\colon I \to \mathbb{M}$ be piecewise C^1 and timelike (or spacelike). If $\lambda = \lambda(s)$ is a C^1 monotone reparametrization, then the value of $\tau(\gamma)$ (or $L(\gamma)$) computed in the s-parameter agrees with the value computed in the λ-parameter.

Proof. Write $\tilde{\gamma}(s) = \gamma(\lambda(s))$. Then $\dot{\tilde{\gamma}}(s) = \dot{\gamma}(\lambda(s))\,\lambda'(s)$, hence

$$\sqrt{-\eta(\dot{\tilde{\gamma}}(s), \dot{\tilde{\gamma}}(s))} = \sqrt{-\eta(\dot{\gamma}(\lambda(s)), \dot{\gamma}(\lambda(s)))}\ |\lambda'(s)|.$$

Since λ is monotone, $|\lambda'(s)| = \pm\lambda'(s)$, and a change of variables gives equality of the integrals. $\square$

Definition 1.10. Future unit hyperboloid

The **future unit timelike hyperboloid** is

$$\mathcal{H}^3 = \{u \in V \mid \eta(u,u) = -1,\ u^0 > 0\}.$$

This set will be used as the space of possible four-velocities of massive particles. Note that $\mathcal{H}^3$ is a smooth embedded hypersurface in V.

1.4 Worldlines, four-velocity, and four-acceleration in the flat case

Definition 1.11. Worldline, four-velocity, and four-acceleration

A **worldline** of a massive particle is a future-directed timelike curve $\gamma\colon I \to \mathbb{M}$. If τ is the proper time parameter along γ, the **four-velocity** is

$$u = \frac{d\gamma}{d\tau},$$

and the **four-acceleration** is

$$a = \nabla_u u.$$

In inertial coordinates, the Levi-Civita connection is trivial, so $\nabla = \partial$ and

$$a^\mu = \frac{du^\mu}{d\tau}.$$

Proposition 1.12. Normalization and orthogonality

Let $\gamma(\tau)$ be a timelike worldline parametrized by proper time, with four-velocity u and four-acceleration a. Then

$$\eta(u,u) = -1, \qquad \eta(a,u) = 0.$$

Proof. By definition of proper time, $\eta(\dot{\gamma}, \dot{\gamma}) = -1$ when $\dot{\gamma} = d\gamma/d\tau = u$. This gives $\eta(u,u) = -1$. Differentiate the identity $\eta(u,u) = -1$ along the curve. Using metric-compatibility of ∇,

$$0 = \nabla_u(\eta(u,u)) = 2\,\eta(\nabla_u u, u) = 2\,\eta(a,u).$$

Hence $\eta(a,u) = 0$. $\square$

Let (t, x^i) be inertial coordinates. Define the ordinary spatial velocity by

$$v^i = \frac{dx^i}{dt}.$$

Then

$$u^0 = \frac{dt}{d\tau}, \qquad u^i = \frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau} = v^i\, u^0.$$

Imposing $\eta(u,u) = -1$, we obtain

$$-(u^0)^2 + \sum_{i=1}^{3}(u^i)^2 = -1 \quad\Longleftrightarrow\quad (u^0)^2\Big(1 - |v|^2\Big) = 1,$$

where $|v|^2 = \sum_i (v^i)^2$. Since $u^0 > 0$ for future-directed motion,

$$u^0 = \gamma := \frac{1}{\sqrt{1-|v|^2}}, \qquad u^i = \gamma v^i.$$

1.5 Two basic examples

Example. Inertial motion and proper time. Fix an inertial frame and let the particle move with constant spatial velocity $v \in \mathbb{R}^3$, $|v| < 1$. Consider the worldline

$$t(\lambda) = \lambda, \qquad x^i(\lambda) = x_0^i + v^i\lambda.$$

Then $\dot\gamma = (1, v)$ in components, hence

$$\eta(\dot\gamma,\dot\gamma) = -1 + |v|^2 = -(1-|v|^2).$$

The proper time between $\lambda = t_1$ and $\lambda = t_2$ is therefore

$$\Delta\tau = \int_{t_1}^{t_2} \sqrt{1-|v|^2}\, dt = (t_2 - t_1)\sqrt{1-|v|^2}.$$

Equivalently, $\Delta t = \gamma\,\Delta\tau$ with $\gamma = (1-|v|^2)^{-1/2}$.

Example. Constant proper acceleration in one spatial direction. Work in the (t,x)-plane and consider a future-directed timelike curve $\gamma(\tau) = (t(\tau), x(\tau))$ parametrized by proper time. Suppose its four-acceleration has constant magnitude

$$\eta(a,a) = \alpha^2, \qquad \alpha > 0,$$

and is always parallel to the x-direction in the instantaneous rest frame. The following curve realizes this:

$$t(\tau) = \frac{1}{\alpha}\sinh(\alpha\tau), \qquad x(\tau) = \frac{1}{\alpha}\cosh(\alpha\tau).$$

First compute the four-velocity,

$$u(\tau) = \dot\gamma(\tau) = \big(\cosh(\alpha\tau), \sinh(\alpha\tau)\big),$$

so

$$\eta(u,u) = -\cosh^2(\alpha\tau) + \sinh^2(\alpha\tau) = -1,$$

as required for proper time parametrization. Next compute the four-acceleration,

$$a(\tau) = \dot u(\tau) = \alpha\big(\sinh(\alpha\tau), \cosh(\alpha\tau)\big).$$

Then

$$\eta(a,a) = -\alpha^2\sinh^2(\alpha\tau) + \alpha^2\cosh^2(\alpha\tau) = \alpha^2,$$

so the proper acceleration is constant and equals α.
Eliminating τ gives the hyperbola

$$x^2 - t^2 = \frac{1}{\alpha^2}, \qquad x > 0,$$

so constant proper acceleration corresponds to a branch of a timelike hyperbola in Minkowski space.

2 Lorentzian Manifolds and Local Geometry

2.1 Lorentz metrics, signature, and local orthonormal frames

Definition 2.1. Semi-Riemannian metrics, index, and Lorentzian metrics

Let M be a smooth m-manifold. A **semi-Riemannian metric** on M is a smooth section

$$g \in \Gamma(\mathrm{Sym}^2 T^*M)$$

such that for every $p \in M$ the bilinear form g_p on T_pM is nondegenerate. The **index** of g_p is the number of negative eigenvalues of the matrix representing g_p in any basis. A semi-Riemannian metric g is **Lorentzian** if its index equals 1 at every point.

Nondegeneracy implies that the index is locally constant, hence constant on each connected component of M.

Proposition 2.2. Pointwise normal form

Let V be a real vector space of dimension 4, and let b be a nondegenerate symmetric bilinear form on V of index 1. Then there exists a basis (e_0, e_1, e_2, e_3) of V such that

$$b(e_0, e_0) = -1, \qquad b(e_i, e_j) = \delta_{ij}, \qquad b(e_0, e_i) = 0.$$

Equivalently, in that basis the matrix of b is $\mathrm{diag}(-1, 1, 1, 1)$.

Proof. By Sylvester's law of inertia, any two symmetric matrices representing b in different bases have the same numbers of positive and negative eigenvalues. Since the index is 1, there exists a basis in which the matrix is diagonal with exactly one negative and three positive entries. Rescaling each basis vector produces the diagonal entries $-1, 1, 1, 1$. □

Definition 2.3. Orthonormal frames

Let (M, g) be Lorentzian. A (local) frame (e_0, e_1, e_2, e_3) on an open set $U \subset M$ is called ***g*-orthonormal** if

$$g(e_0, e_0) = -1, \qquad g(e_i, e_j) = \delta_{ij}, \qquad g(e_0, e_i) = 0$$

on U. The vector field e_0 is timelike, and e_1, e_2, e_3 are spacelike.

Proposition 2.4. Existence of local orthonormal frames

Let (M, g) be Lorentzian and $p \in M$. There exists a neighborhood U of p and a smooth g-orthonormal frame (e_0, e_1, e_2, e_3) on U.

Proof. Choose a chart (U_0, x^μ) with $p \in U_0$ and coordinate vector fields $X_\mu = \partial_\mu$ on U_0. Since g_p has index 1, there exists some $v \in T_pM$ with $g_p(v, v) < 0$. Write $v = v^\mu X_\mu(p)$. Define a vector field on U_0 by

$$T = \sum_{\mu=0}^{3} v^\mu X_\mu.$$

By continuity of $q \mapsto g_q(T_q, T_q)$, after shrinking U_0 we may assume $g(T,T) < 0$ on U_0. Set

$$e_0 = \frac{T}{\sqrt{-g(T,T)}}.$$

Then $g(e_0, e_0) = -1$ on U_0.

Consider the rank-3 subbundle $E = e_0^{\perp} \subset TM|_{U_0}$. For each $q \in U_0$, the restriction $g_q|_{E_q}$ is positive definite. Choose any local smooth frame (Y_1, Y_2, Y_3) for E on a possibly smaller neighborhood $U \subset U_0$ (for instance, project X_i to E and shrink until they remain independent). Apply the usual Gram–Schmidt procedure to (Y_1, Y_2, Y_3) using the positive definite inner products $g|_{E_q}$. This yields smooth vector fields e_1, e_2, e_3 on U with

$$g(e_i, e_j) = \delta_{ij}, \qquad g(e_0, e_i) = 0.$$

Thus (e_0, e_1, e_2, e_3) is a smooth g-orthonormal frame on U. □

Remark. A g-orthonormal frame makes the metric *look* like the Minkowski metric in that frame, but this does not imply that the connection coefficients vanish. Vanishing of connection coefficients at a point requires a stronger normalization, which is not imposed here.

2.2 Time orientation and basic global examples

Definition 2.5. Time-orientable and time-oriented

Let (M, g) be Lorentzian. For each $p \in M$, the set of timelike vectors in T_pM has two connected components. The spacetime (M, g) is **time-orientable** if one can choose one of these two components continuously with p. A **time orientation** is such a choice. A spacetime equipped with a fixed time orientation is called **time-oriented**. A timelike or null vector is **future-pointing** if it lies in the chosen component or its closure.

Proposition 2.6. Time orientation via a timelike vector field

A Lorentzian manifold (M, g) is time-orientable if and only if there exists a smooth timelike vector field T on M such that $g(T, T) < 0$ everywhere. In this case, declaring a causal vector v to be future-pointing when $g(v, T) \leq 0$ defines a time orientation.

Proof. Assume (M, g) is time-orientable. Choose a covering $\{U_\alpha\}$ on which there exist smooth future-pointing timelike fields T_α (local existence follows from the openness of the timelike cone). Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then

$$T = \sum_\alpha \varphi_\alpha T_\alpha$$

is smooth. At each point p, the vectors $T_\alpha(p)$ that appear in the sum lie in the same open convex cone of future timelike vectors inside T_pM. A convex combination of vectors in this cone remains timelike and future-pointing, hence T is a global timelike vector field.

Conversely, if T is a global timelike vector field, then at each p the condition $g(v, T_p) < 0$ selects exactly one of the two timelike cones continuously in p, giving a time orientation. □

Example. A time-oriented spacetime with closed timelike curves. Let $M = S^1 \times \mathbb{R}^3$ with coordinates (t, x^1, x^2, x^3) where t is periodic, and let

$$g = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

The vector field ∂_t is smooth and timelike everywhere, so (M, g) is time-oriented. The curve $\gamma(\lambda) = (t = \lambda, x^i = 0)$ is a closed curve since t is periodic, and

$$g(\dot\gamma, \dot\gamma) = g(\partial_t, \partial_t) = -1,$$

so γ is a closed timelike curve.

Example. A Lorentzian quotient that is not time-orientable. Start with 2-dimensional Minkowski spacetime $(\mathbb{R}^2, \eta)$, $\eta = -dt^2 + dx^2$. Consider the isometry

$$f(t, x) = (-t, x + 1),$$

and form the quotient $M = \mathbb{R}^2/\langle f \rangle$ with the induced Lorentzian metric. The differential df flips the time direction. Any attempt to choose a continuous future cone on M would lift to an f-invariant choice on $\mathbb{R}^2$, but f identifies future-pointing vectors with past-pointing ones. Hence (M, g) is not time-orientable.

2.3 Coordinates, components, and transformation laws

Let (M, g) be Lorentzian and let (U, x^μ) be a chart. The coordinate vector fields ∂_μ form a local frame for TU, and the dual coframe is dx^μ. The metric is the tensor field

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu, \qquad g_{\mu\nu} = g(\partial_\mu, \partial_\nu).$$

Proposition 2.7. Change of coordinates for tensor components

Let (U, x^μ) and (U, y^α) be two charts on the same open set. Denote by $g_{\mu\nu}$ the components of g in the x-chart and by $g'_{\alpha\beta}$ the components in the y-chart. Then

$$g'_{\alpha\beta}(y) = \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta}\, g_{\mu\nu}(x(y)).$$

For the inverse metric components, writing $g^{\mu\nu}$ and $g'^{\alpha\beta}$ for the matrix inverses of $(g_{\mu\nu})$ and $(g'_{\alpha\beta})$, one has

$$g'^{\alpha\beta}(y) = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu}\, g^{\mu\nu}(x(y)).$$

Proof. Since g is a $(0, 2)$-tensor, for vector fields X, Y one has $g'(X, Y) = g(X, Y)$. In particular,

$$g'_{\alpha\beta} = g(\partial_{y^\alpha}, \partial_{y^\beta}).$$

By the chain rule,

$$\partial_{y^\alpha} = \frac{\partial x^\mu}{\partial y^\alpha}\, \partial_{x^\mu}.$$

Substituting into $g(\partial_{y^\alpha}, \partial_{y^\beta})$ and using bilinearity gives the first formula. The formula for $g^{\mu\nu}$ follows by applying the same argument to the induced bilinear form on T^*M and using that inverse matrices transform contravariantly. □

Remark. The array $(g_{\mu\nu})$ depends on the chosen coordinates. It is not a collection of "force components". A coordinate change can turn a constant matrix into a variable one, even when the spacetime is flat. What is invariant is the tensor g itself, and later the curvature derived from g.

Example. Minkowski metric in spherical coordinates. On $\mathbb{R}^4$ with inertial coordinates (t, x, y, z), introduce (r, θ, φ) on the spatial part so that

$$x = r \sin\theta \cos\varphi, \quad y = r \sin\theta \sin\varphi, \quad z = r \cos\theta.$$

Then

$$\eta = -dt^2 + dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2.$$

The components are not constant in these coordinates, but the spacetime is still flat since it is the same tensor field written in a different chart.

2.4 Coordinate-free tensors and abstract indices

Definition 2.8. Tensors and abstract indices

Let M be a smooth manifold. A (r,s)-**tensor field** is a smooth section of

$$T^r{}_sM := (TM)^{\otimes r} \otimes (T^*M)^{\otimes s}.$$

In abstract index notation, a $(0,2)$-tensor field is written T_{ab}, a vector field is X^a, and a 1-form is α_a. Contraction is written by repeating an upper and a lower index, for instance $T_{ab}X^b$.

Abstract indices do not refer to a coordinate chart. They are bookkeeping for tensor type and contractions. Coordinates enter only when we expand tensors in a chosen frame.

Let (U, x^μ) be a chart. Any vector field X on U can be written

$$X = X^\mu\, \partial_\mu, \qquad X^\mu = dx^\mu(X),$$

and any 1-form α can be written

$$\alpha = \alpha_\mu\, dx^\mu, \qquad \alpha_\mu = \alpha(\partial_\mu).$$

Proposition 2.9. Component formulas and invariance of contractions

Let T be a $(0,2)$-tensor field and X, Y vector fields on U. Writing

$$T = T_{\mu\nu}\, dx^\mu \otimes dx^\nu, \qquad X = X^\mu \partial_\mu, \qquad Y = Y^\nu \partial_\nu,$$

one has

$$T(X,Y) = T_{\mu\nu} X^\mu Y^\nu.$$

Under a change of coordinates, the components $T_{\mu\nu}$ and X^μ change, but the scalar $T(X,Y)$ is invariant.

Proof. The identity $T(X,Y) = T_{\mu\nu}X^\mu Y^\nu$ follows from multilinearity and the defining relations $dx^\mu(\partial_\nu) = \delta^\mu_\nu$. Invariance follows because $T(X,Y)$ is coordinate-free by definition of a tensor field, and the transformation laws for $T_{\mu\nu}, X^\mu, Y^\nu$ cancel exactly. □

2.5 Pointwise Minkowski model and causal classification

Proposition 2.10. Each tangent space is Minkowski space

Let (M,g) be a Lorentzian 4-manifold and $p \in M$. Then there exists a linear isomorphism

$$\ell_p \colon (T_pM, g_p) \longrightarrow (\mathbb{R}^4, \eta)$$

that is an isometry, meaning $g_p(v,w) = \eta(\ell_p v, \ell_p w)$ for all $v, w \in T_pM$. Consequently, for any $v \in T_pM$ the sign of $g_p(v,v)$ determines whether v is timelike, null, or spacelike, and this classification agrees with the corresponding classification in $(\mathbb{R}^4, \eta)$ after applying ℓ_p.

Proof. Choose a g-orthonormal basis (e_0, e_1, e_2, e_3) of T_pM, which exists by Proposition 2.2 applied to the bilinear form g_p. Define ℓ_p by sending e_μ to the standard basis of $\mathbb{R}^4$. In that basis, g_p has matrix $\mathrm{diag}(-1,1,1,1)$, so ℓ_p is an isometry. The causal classification is determined by the sign of $g_p(v,v)$, hence is preserved by ℓ_p. □

Remark. Because timelike and spacelike vectors are characterized by strict inequalities, the sets of timelike vectors and spacelike vectors in TM are open subsets. In particular, if $v \in T_pM$ is timelike, then any sufficiently small perturbation of v in TM remains timelike.

3 Tensor Calculus, Volume Forms, and Divergence

3.1 Raising and lowering indices, inner products, and orthogonal decompositions

Let (M, g) be a time-oriented Lorentzian 4-manifold of signature $(-,+,+,+)$.

Definition 3.1. Musical isomorphisms

For each $p \in M$, the metric g_p induces an isomorphism

$$\flat_p \colon T_pM \to T_p^*M, \qquad v \longmapsto v^\flat := g_p(v, \cdot).$$

Its inverse is denoted

$$\sharp_p \colon T_p^*M \to T_pM, \qquad \alpha \longmapsto \alpha^\sharp.$$

In abstract indices this is $v_a = g_{ab}v^b$ and $\alpha^a = g^{ab}\alpha_b$, where (g^{ab}) is the inverse matrix to (g_{ab}).

The metric also induces bilinear pairings on covectors and on general tensors by contracting indices with g^{ab} and g_{ab}.

Definition 3.2. Inner products induced by g

(1) For $\alpha, \beta \in T_p^*M$ define

$$\langle \alpha, \beta \rangle := g^{ab}\alpha_a\beta_b = g(\alpha^\sharp, \beta^\sharp).$$

(2) For p-forms $\omega, \sigma \in \Lambda^p T_p^*M$ define

$$\langle \omega, \sigma \rangle := \frac{1}{p!}\, \omega_{a_1 \cdots a_p}\, \sigma^{a_1 \cdots a_p},$$

where indices on σ are raised using g^{ab}.

Because g is Lorentzian, these pairings are nondegenerate but not positive definite.

Proposition 3.3. Orthogonal decomposition with respect to a timelike unit vector

Fix $p \in M$ and let $u \in T_pM$ be timelike with $g(u, u) = -1$. Then

$$T_pM = \operatorname{span}\{u\} \oplus u^\perp, \qquad u^\perp := \{w \in T_pM \mid g(w, u) = 0\},$$

and g restricts to a positive definite inner product on $u^\perp$.
Moreover, any $v \in T_pM$ decomposes uniquely as

$$v = -(g(v, u))\, u + v_\perp, \qquad v_\perp \in u^\perp,$$

and the projection onto $u^\perp$ is given by

$$(hv)^a = h^a{}_b v^b, \qquad h^a{}_b := \delta^a{}_b + u^a u_b.$$

Proof. Since $g(u, u) \neq 0$, the linear functional $w \mapsto g(w, u)$ is nonzero, hence $\dim u^\perp = 3$. The decomposition follows from linear algebra once we show $\operatorname{span}\{u\} \cap u^\perp = \{0\}$, which holds because $g(u, u) = -1$.
If $w \in u^\perp$ and $w \neq 0$, then w cannot be timelike or null, otherwise the 2-plane spanned by u and w would contain two linearly independent timelike directions, contradicting that g has index 1.

Hence $g(w,w) > 0$, so $g|_{u^\perp}$ is positive definite.
For the formula, define $v_\perp := v + (g(v,u))u$. Then

$$g(v_\perp, u) = g(v,u) + g(v,u)g(u,u) = g(v,u) - g(v,u) = 0,$$

so $v_\perp \in u^\perp$, and the decomposition is immediate. The projection tensor $h^a{}_b$ is the coordinate-free expression for $v \mapsto v_\perp$. □

3.2 The metric volume form and the appearance of $\sqrt{-g}$

Assume M is oriented. On an oriented pseudo-Riemannian manifold there is a canonical volume form associated to g.

Definition 3.4. Metric volume form

Let (M,g) be oriented Lorentzian. The **metric volume form** $\mathrm{dVol}_g \in \Omega^4(M)$ is the unique smooth 4-form such that for every positively oriented g-orthonormal frame (e_0, e_1, e_2, e_3) one has

$$\mathrm{dVol}_g(e_0, e_1, e_2, e_3) = 1.$$

In a positively oriented coordinate chart (U, x^μ), write $g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$ and $g = \det(g_{\mu\nu})$. For signature $(-,+,+,+)$ one has $g < 0$ at every point of U.

Proposition 3.5. Coordinate expression

In a positively oriented coordinate chart (U, x^μ),

$$\mathrm{dVol}_g = \sqrt{-g}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \qquad \sqrt{-g} > 0.$$

Proof. Both sides are smooth 4-forms on U, so it suffices to evaluate on the coordinate frame and use multilinearity. Let A be the 4×4 matrix taking the coordinate basis to a positively oriented g-orthonormal basis. Then $A^T(g_{\mu\nu})A = \eta$, hence

$$\det(g_{\mu\nu})\,(\det A)^2 = \det(\eta) = -1.$$

Since $\det A > 0$ by orientation, we have $\det A = \sqrt{-\det(g_{\mu\nu})} = \sqrt{-g}$. The wedge product transforms by $\det A$, which gives the stated formula. □

It is convenient to package dVol_g into the Levi-Civita tensor.

Definition 3.6. Levi-Civita tensor

Let ε_{abcd} be the $(0,4)$-tensor corresponding to dVol_g under the natural identification between alternating $(0,4)$-tensors and 4-forms. In local coordinates,

$$\varepsilon_{0123} = \sqrt{-g}, \qquad \varepsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\,[\mu\nu\rho\sigma],$$

where $[\mu\nu\rho\sigma] \in \{-1, 0, 1\}$ is the totally antisymmetric symbol with $[0123] = 1$.

3.3 Divergence and the Stokes formula

There are several equivalent ways to define divergence. The most robust one for integration is via the volume form.

Definition 3.7. Divergence

Let V be a smooth vector field on M. The **divergence** of V is the unique smooth function $\operatorname{div} V$ such that

$$\mathcal{L}_V(\mathrm{dVol}_g) = (\operatorname{div} V)\,\mathrm{dVol}_g.$$

Proposition 3.8. Divergence equals $\nabla_a V^a$

For the Levi-Civita connection ∇,

$$\operatorname{div} V = \nabla_a V^a.$$

Proof. Fix $p \in M$ and choose a local g-orthonormal frame (e_0, e_1, e_2, e_3) near p that is positively oriented and satisfies $(\nabla e_\mu)_p = 0$ at the point p (one can arrange this by choosing normal coordinates at p and orthonormalizing at p). Evaluate $\mathcal{L}_V(\mathrm{dVol}_g)$ at p on (e_0, e_1, e_2, e_3):

$$\mathcal{L}_V(\mathrm{dVol}_g)(e_0, e_1, e_2, e_3) = V\big(\mathrm{dVol}_g(e_0, e_1, e_2, e_3)\big) - \sum_{\mu=0}^{3} \mathrm{dVol}_g(e_0, \ldots, [V, e_\mu], \ldots, e_3).$$

The first term vanishes since $\mathrm{dVol}_g(e_0, e_1, e_2, e_3) = 1$ is constant. Using $[V, e_\mu] = \nabla_V e_\mu - \nabla_{e_\mu} V$ and $(\nabla_V e_\mu)_p = 0$ at p, we obtain

$$\mathcal{L}_V(\mathrm{dVol}_g)(e_0, e_1, e_2, e_3)\Big|_p = \sum_{\mu=0}^{3} \mathrm{dVol}_g(e_0, \ldots, \nabla_{e_\mu} V, \ldots, e_3)\Big|_p.$$

Since dVol_g agrees with the determinant in an orthonormal frame, the right-hand side equals the trace of the endomorphism $w \mapsto \nabla_w V$ at p, namely $\nabla_a V^a|_p$. This shows $\operatorname{div} V = \nabla_a V^a$. $\square$

Proposition 3.9. Stokes formula for divergence

Let $\Omega \subset M$ be a compact domain with smooth boundary $\partial\Omega$. Then

$$\int_\Omega \nabla_a V^a \,\mathrm{dVol}_g = \int_{\partial\Omega} \iota_V(\mathrm{dVol}_g),$$

where ι_V denotes interior product. If $\partial\Omega$ is non-null and n is the outward unit normal, then the boundary integral can be written as

$$\int_{\partial\Omega} \iota_V(\mathrm{dVol}_g) = \int_{\partial\Omega} g(V, n)\,\mathrm{dVol}_{\partial\Omega},$$

where $\mathrm{dVol}_{\partial\Omega}$ is the induced volume form on $\partial\Omega$.

Proof. Since $\mathcal{L}_V = d\iota_V + \iota_V d$ and $d(\mathrm{dVol}_g) = 0$, we have

$$(\nabla_a V^a)\,\mathrm{dVol}_g = \mathcal{L}_V(\mathrm{dVol}_g) = d\big(\iota_V(\mathrm{dVol}_g)\big).$$

Integrate over Ω and apply Stokes' theorem for differential forms to obtain the first identity. For the second, when $\partial\Omega$ is non-null, $\iota_n(\mathrm{dVol}_g)$ restricts to the induced volume form on $\partial\Omega$ up to the sign determined by the chosen outward normal, and $\iota_V(\mathrm{dVol}_g) = g(V, n)\,\iota_n(\mathrm{dVol}_g)$ on $\partial\Omega$. $\square$

3.4 Hodge star and a working dictionary for forms

The Hodge star is defined once we fix orientation and the metric volume form.

Definition 3.10. Hodge star

Let $\omega, \sigma \in \Omega^p(M)$. The **Hodge star** is the bundle map $*\colon \Omega^p(M) \to \Omega^{4-p}(M)$ characterized by

$$\omega \wedge *\sigma = \langle \omega, \sigma \rangle \, \mathrm{dVol}_g,$$

where $\langle \cdot, \cdot \rangle$ is the metric pairing on p-forms from Definition 3.2.

Lemma 3.11. Component formula Let $\omega \in \Omega^p(M)$. In local coordinates,

$$(*\omega)_{\mu_{p+1}\cdots\mu_4} = \frac{1}{p!}\, \omega^{\nu_1\cdots\nu_p}\, \varepsilon_{\nu_1\cdots\nu_p\mu_{p+1}\cdots\mu_4},$$

where $\varepsilon_{\dots}$ is the Levi-Civita tensor from Definition 3.6.

Proof. This is the standard computation obtained by imposing the defining identity $\omega \wedge *\sigma = \langle \omega, \sigma \rangle \, \mathrm{dVol}_g$ on basis forms $dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$ and using multilinearity and antisymmetry. □

Lemma 3.12. The $*^2$ sign in Lorentzian signature For $\omega \in \Omega^p(M)$,

$$* * \omega = (-1)^{p(4-p)+1}\, \omega,$$

in agreement with the convention stated in the Conventions box.

Proof. At a point p, choose a positively oriented g-orthonormal basis. In such a basis the computation reduces to the corresponding statement in $(\mathbb{R}^{1,3}, \eta)$, where the extra sign $(-1)^1$ comes from the single timelike direction. The exponent $p(4-p)$ is the usual Riemannian contribution from swapping basis elements in the wedge product. □

3.5 Two worked examples

Example. Divergence in Minkowski spacetime and a conserved current. In inertial coordinates on $(\mathbb{R}^4, \eta)$, one has $\mathrm{dVol}_\eta = dt \wedge dx \wedge dy \wedge dz$ and the connection is trivial, so for a vector field $J = J^\mu \partial_\mu$,

$$\nabla_\mu J^\mu = \partial_\mu J^\mu.$$

If $\partial_\mu J^\mu = 0$, then by Proposition 3.9 we get, for any compact domain Ω,

$$0 = \int_\Omega \partial_\mu J^\mu \, d^4x = \int_{\partial\Omega} \iota_J(d^4x).$$

Take Ω to be the slab between two constant-time slices $t = t_1$ and $t = t_2$ over a bounded spatial region $B \subset \mathbb{R}^3$. Then the boundary integral splits into flux through the two time slices plus flux through the timelike side boundary. If J decays so that the side flux vanishes, this gives equality of the integrals of J^0 over B at times t_1 and t_2.

Example. The $\sqrt{-g}$ formula for $\nabla_\mu V^\mu$. Let (U, x^μ) be a positively oriented coordinate chart and let $V = V^\mu \partial_\mu$. Start from

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\nu} V^\nu.$$

The trace of the Christoffel symbols satisfies

$$\Gamma^\mu_{\mu\nu} = \partial_\nu\big(\ln \sqrt{-g}\big),$$

so

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + (\partial_\nu \ln \sqrt{-g}) V^\nu = \frac{1}{\sqrt{-g}} \partial_\mu\big(\sqrt{-g}\, V^\mu\big).$$

A direct derivation of $\Gamma^\mu_{\mu\nu} = \partial_\nu(\ln \sqrt{-g})$ uses the determinant identity

$$\partial_\nu(\ln |g|) = g^{\mu\rho}\, \partial_\nu g_{\mu\rho},$$

together with the explicit Christoffel formula and the symmetry $g_{\mu\nu} = g_{\nu\mu}$.

4 Levi-Civita Connection and a Computation Template

4.1 Affine connections and covariant derivatives

Definition 4.1. Affine connection

Let M be a smooth manifold. An **affine connection** on TM is an $\mathbb{R}$-bilinear map

$$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \qquad (X, Y) \mapsto \nabla_X Y,$$

such that for all $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$,

$$\nabla_{fX} Y = f\,\nabla_X Y, \qquad \nabla_X(fY) = X(f)\,Y + f\,\nabla_X Y.$$

Definition 4.2. Torsion

The **torsion tensor** of an affine connection ∇ is the $(1,2)$-tensor

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y].$$

The connection is **torsion-free** if $T = 0$.

Given ∇ on TM, there is a unique extension to a covariant derivative on arbitrary tensor fields, determined by the Leibniz rule and by the requirement that $\nabla_X f = X(f)$ for functions.

Proposition 4.3. Coordinate expression for ∇ on vector fields

Let (U, x^μ) be a coordinate chart and write $\partial_\mu = \frac{\partial}{\partial x^\mu}$. There exist smooth functions $\Gamma^\rho_{\mu\nu}$ on U such that

$$\nabla_{\partial_\mu} \partial_\nu = \Gamma^\rho_{\mu\nu}\,\partial_\rho.$$

For a vector field $V = V^\rho \partial_\rho$ on U, one has

$$(\nabla_{\partial_\mu} V)^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\nu} V^\nu.$$

Proof. The first statement is the definition of the connection coefficients in the coordinate frame. For the second, use

$$\nabla_{\partial_\mu}(V^\rho \partial_\rho) = \partial_\mu(V^\rho)\partial_\rho + V^\rho \nabla_{\partial_\mu} \partial_\rho$$

and substitute $\nabla_{\partial_\mu} \partial_\rho = \Gamma^\sigma_{\mu\rho} \partial_\sigma$. □

Proposition 4.4. Coordinate expression for ∇ on covectors and general tensors

Let $\omega = \omega_\nu\, dx^\nu$ be a 1-form on U. Then

$$(\nabla_{\partial_\mu} \omega)_\nu = \partial_\mu \omega_\nu - \Gamma^\rho_{\mu\nu}\,\omega_\rho.$$

More generally, for a (r, s)-tensor $T^{\alpha_1 \cdots \alpha_r}{}_{\beta_1 \cdots \beta_s}$,

$$\nabla_\mu T^{\alpha_1 \cdots \alpha_r}{}_{\beta_1 \cdots \beta_s} = \partial_\mu T^{\alpha_1 \cdots \alpha_r}{}_{\beta_1 \cdots \beta_s} + \sum_{k=1}^{r} \Gamma^{\alpha_k}_{\mu\lambda}\, T^{\alpha_1 \cdots \lambda \cdots \alpha_r}{}_{\beta_1 \cdots \beta_s} - \sum_{\ell=1}^{s} \Gamma^{\lambda}_{\mu\beta_\ell}\, T^{\alpha_1 \cdots \alpha_r}{}_{\beta_1 \cdots \lambda \cdots \beta_s}.$$

Proof. For a 1-form, the defining property is

$$(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y).$$

Apply this with $X = \partial_\mu$ and $Y = \partial_\nu$ to obtain
$$(\nabla_\mu \omega)_\nu = \partial_\mu(\omega_\nu) - \omega(\nabla_{\partial_\mu}\partial_\nu) = \partial_\mu \omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho.$$
The general formula follows by applying the Leibniz rule repeatedly and using the vector and covector cases. $\square$

4.2 Levi-Civita connection, metric compatibility, and the Koszul formula

Definition 4.5. Metric compatibility

Let g be a pseudo-Riemannian metric on M and let ∇ be an affine connection. We say ∇ is **metric-compatible** if
$$\nabla g = 0,$$
meaning that for all vector fields X, Y, Z,
$$X\big(g(Y, Z)\big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Theorem 4.6. Levi-Civita connection

Let (M, g) be a pseudo-Riemannian manifold. There exists a unique affine connection ∇ on TM such that
$$T = 0 \qquad \text{and} \qquad \nabla g = 0.$$
This connection is called the **Levi-Civita connection** of g.

Proof. Uniqueness. Assume ∇ is torsion-free and metric-compatible. For vector fields X, Y, Z, write down metric-compatibility in three cyclic permutations:
$$\begin{aligned} X\, g(Y, Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \\ Y\, g(Z, X) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X), \\ Z\, g(X, Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \end{aligned}$$
Add the first two and subtract the third:
$$\begin{aligned} X\, g(Y, Z) + Y\, g(Z, X) - Z\, g(X, Y) = {}& g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ & + g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ & - g(\nabla_Z X, Y) - g(X, \nabla_Z Y). \end{aligned}$$
Now use torsion-freeness in the form
$$\nabla_Y X = \nabla_X Y - [X, Y], \qquad \nabla_Z X = \nabla_X Z - [Z, X], \qquad \nabla_Z Y = \nabla_Y Z - [Z, Y],$$
and substitute these into the right-hand side. After cancellations, one obtains the **Koszul formula**
$$\begin{aligned} 2\, g(\nabla_X Y, Z) = {}& X\, g(Y, Z) + Y\, g(Z, X) - Z\, g(X, Y) \\ & + g([X, Y], Z) - g([Y, Z], X) - g([X, Z], Y). \end{aligned} \tag{1}$$
The right-hand side depends only on g and the Lie brackets, hence determines $g(\nabla_X Y, Z)$ for every Z. Since g is nondegenerate, this determines $\nabla_X Y$ uniquely. Therefore ∇ is unique.

Existence. Define ∇ by requiring that $\nabla_X Y$ is the unique vector field satisfying (1) for all Z. This is well-defined because g is nondegenerate. One checks directly from (1) that ∇ is C^∞-linear in X and satisfies the Leibniz rule in Y, hence is an affine connection. Metric-compatibility follows by comparing the Koszul formula for (X, Y, Z) with that for (X, Z, Y), and torsion-freeness follows by comparing the formula for (X, Y, Z) with that for (Y, X, Z) and using antisymmetry of the bracket. $\square$

Proposition 4.7. Practical consequences of $\nabla g = 0$

Let ∇ be the Levi-Civita connection of g.

(1) Raising and lowering indices commute with ∇. In abstract indices,
$$\nabla_a(v_b) = \nabla_a(g_{bc}v^c) = g_{bc}\nabla_a v^c, \qquad \nabla_a(\alpha^b) = g^{bc}\nabla_a\alpha_c.$$

(2) Along any curve γ and vector fields Y, Z along γ,
$$\frac{d}{d\lambda}g(Y,Z) = g(\nabla_{\dot\gamma}Y, Z) + g(Y, \nabla_{\dot\gamma}Z).$$

Proof. (1) follows from $\nabla g = 0$ and the Leibniz rule. (2) is the same statement applied to $X = \dot\gamma$. $\square$

4.3 Derivation of the Christoffel formula in coordinates

Let (U, x^μ) be a coordinate chart. Since $[\partial_\mu, \partial_\nu] = 0$, the Koszul formula simplifies.

Proposition 4.8. Christoffel symbols from the Koszul formula

Let ∇ be the Levi-Civita connection of g. In local coordinates,
$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}\, g^{\rho\sigma}\big(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\big).$$

Proof. Apply (1) with $X = \partial_\mu$, $Y = \partial_\nu$, $Z = \partial_\sigma$. The bracket terms vanish, and we get
$$2\, g(\nabla_{\partial_\mu}\partial_\nu, \partial_\sigma) = \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}.$$
Write $\nabla_{\partial_\mu}\partial_\nu = \Gamma^\rho_{\mu\nu}\partial_\rho$. Then
$$g(\nabla_{\partial_\mu}\partial_\nu, \partial_\sigma) = \Gamma^\rho_{\mu\nu}\, g_{\rho\sigma}.$$
Therefore
$$2\,\Gamma^\rho_{\mu\nu}\, g_{\rho\sigma} = \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}.$$
Multiply by $g^{\sigma\lambda}$ and use $g_{\rho\sigma}g^{\sigma\lambda} = \delta^\lambda_\rho$ to obtain the stated formula. $\square$

Remark. The functions $\Gamma^\rho_{\mu\nu}$ are not the components of a tensor. They depend on the seelected coordinate chart, and their transformation law involves second derivatives of the coordinate change.

4.4 Covariant derivatives in coordinates and common pitfalls

Proposition 4.9. Comparing ∂_μ and ∇_μ

Let $V = V^\rho\partial_\rho$ and $\omega = \omega_\nu dx^\nu$ on U. Then
$$\nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\nu}V^\nu, \qquad \nabla_\mu\omega_\nu = \partial_\mu\omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho.$$
Consequently, $\partial_\mu V^\rho$ does not transform tensorially, while $\nabla_\mu V^\rho$ does.

Remark. Index position matters. For instance, even if $V_\nu = g_{\nu\rho}V^\rho$, one has
$$\partial_\mu V_\nu \neq g_{\nu\rho}\,\partial_\mu V^\rho$$
unless $\partial_\mu g_{\nu\rho} = 0$. By contrast, metric compatibility gives
$$\nabla_\mu V_\nu = g_{\nu\rho}\,\nabla_\mu V^\rho.$$

A component form of $\nabla g = 0$ is often used as a computational identity.

Proposition 4.10. Component form of $\nabla g = 0$

In local coordinates,

$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\sigma_{\mu\nu} g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} g_{\nu\sigma} = 0.$$

Equivalently,

$$\partial_\mu g_{\nu\rho} = \Gamma_{\mu\nu\rho} + \Gamma_{\mu\rho\nu}, \qquad \Gamma_{\mu\nu\rho} := g_{\rho\sigma}\Gamma^\sigma_{\mu\nu}.$$

Proof. Apply the general tensor formula from Proposition 4.4 to the $(0,2)$-tensor $g_{\nu\rho}$ and use $\nabla g = 0$. □

4.5 A worked computation template: Rindler coordinates on Minkowski spacetime

Example. Rindler chart and Christoffel symbols. Start with Minkowski spacetime in inertial coordinates (t, x, y, z) and metric

$$\eta = -dt^2 + dx^2 + dy^2 + dz^2.$$

On the region $\{x > |t|\}$ introduce coordinates (η, ρ, y, z) by

$$t = \rho \sinh \eta, \qquad x = \rho \cosh \eta, \qquad \rho > 0.$$

Compute the differentials:

$$dt = \sinh \eta \, d\rho + \rho \cosh \eta \, d\eta, \qquad dx = \cosh \eta \, d\rho + \rho \sinh \eta \, d\eta.$$

Then

$$\begin{aligned}
-dt^2 + dx^2 &= -(\sinh \eta \, d\rho + \rho \cosh \eta \, d\eta)^2 + (\cosh \eta \, d\rho + \rho \sinh \eta \, d\eta)^2 \\
&= -(\sinh^2 \eta)\, d\rho^2 - 2\rho \sinh \eta \cosh \eta \, d\rho \, d\eta - \rho^2 \cosh^2 \eta \, d\eta^2 \\
&\quad + (\cosh^2 \eta)\, d\rho^2 + 2\rho \sinh \eta \cosh \eta \, d\rho \, d\eta + \rho^2 \sinh^2 \eta \, d\eta^2 \\
&= (\cosh^2 \eta - \sinh^2 \eta)\, d\rho^2 - \rho^2 (\cosh^2 \eta - \sinh^2 \eta)\, d\eta^2 \\
&= d\rho^2 - \rho^2 \, d\eta^2.
\end{aligned}$$

Therefore the Minkowski metric becomes

$$g = -\rho^2 \, d\eta^2 + d\rho^2 + dy^2 + dz^2.$$

In the coordinate order $(x^0, x^1, x^2, x^3) = (\eta, \rho, y, z)$, the nonzero components are

$$g_{00} = -\rho^2, \qquad g_{11} = 1, \qquad g_{22} = 1, \qquad g_{33} = 1,$$

and the inverse metric has components

$$g^{00} = -\frac{1}{\rho^2}, \qquad g^{11} = 1, \qquad g^{22} = 1, \qquad g^{33} = 1.$$

Compute the partial derivatives of the metric components. The only nonzero derivative is

$$\partial_1 g_{00} = \frac{\partial}{\partial \rho}(-\rho^2) = -2\rho.$$

Now apply the Christoffel formula

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}).$$

Step 1: Γ^1_{00}. Set $\lambda = 1$, $\mu = \nu = 0$:

$$\Gamma^1_{00} = \frac{1}{2} g^{1\sigma} (\partial_0 g_{0\sigma} + \partial_0 g_{0\sigma} - \partial_\sigma g_{00}).$$

Since $g^{1\sigma} = 0$ unless $\sigma = 1$, this becomes

$$\Gamma^1_{00} = \frac{1}{2} g^{11}(2\partial_0 g_{01} - \partial_1 g_{00}).$$

Here $g_{01} = 0$ so $\partial_0 g_{01} = 0$, and $\partial_1 g_{00} = -2\rho$. Hence

$$\Gamma^1_{00} = \frac{1}{2}(0 - (-2\rho)) = \rho.$$

Step 2: Γ^0_{01} and Γ^0_{10}. Set $\lambda = 0$, $\mu = 0$, $\nu = 1$:

$$\Gamma^0_{01} = \frac{1}{2} g^{0\sigma}(\partial_0 g_{1\sigma} + \partial_1 g_{0\sigma} - \partial_\sigma g_{01}).$$

Since $g^{0\sigma} = 0$ unless $\sigma = 0$, this becomes

$$\Gamma^0_{01} = \frac{1}{2} g^{00}(\partial_0 g_{10} + \partial_1 g_{00} - \partial_0 g_{01}) = \frac{1}{2} g^{00}\, \partial_1 g_{00}.$$

Using $g^{00} = -1/\rho^2$ and $\partial_1 g_{00} = -2\rho$, we obtain

$$\Gamma^0_{01} = \frac{1}{2}\left(-\frac{1}{\rho^2}\right)(-2\rho) = \frac{1}{\rho}.$$

Because the Levi-Civita connection is torsion-free, $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ in any coordinate chart, so

$$\Gamma^0_{10} = \Gamma^0_{01} = \frac{1}{\rho}.$$

Step 3: all remaining components. If neither μ nor ν equals 1, then every derivative term $\partial_\mu g_{\nu\sigma}$ vanishes because the only ρ-dependence is in g_{00}. If one of μ, ν equals 1 but the relevant index pattern does not involve g_{00}, the same conclusion holds. A direct check shows that the only nonzero Christoffel symbols are

$$\Gamma^1_{00} = \rho, \qquad \Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{\rho}.$$

As a consistency check, note that $\det(g_{\mu\nu}) = -\rho^2$, hence $\sqrt{-g} = \rho$. The identity $\Gamma^\mu_{\mu\nu} = \partial_\nu(\ln\sqrt{-g})$ then gives $\Gamma^\mu_{\mu 1} = \partial_1(\ln\rho) = 1/\rho$ and $\Gamma^\mu_{\mu 0} = 0$, which matches the computed symbols.

5 Geodesics: Definition, ODE Theory, and the Exponential Map

5.1 Geodesics as autoparallels and their geometric meaning

Let (M, g) be a Lorentzian manifold and let ∇ be its Levi-Civita connection.

Definition 5.1. Geodesic

A smooth curve $\gamma\colon I \to M$ is a **geodesic** (with respect to ∇) if

$$\nabla_{\dot\gamma}\dot\gamma = 0$$

on I, where $\dot\gamma$ denotes the velocity vector field along γ.

The equation $\nabla_{\dot\gamma}\dot\gamma = 0$ says that the velocity vector is parallel transported along the curve itself. It does not depend on any chosen coordinates. In a general Lorentzian manifold, geodesics are the distinguished curves determined solely by the connection.

Remark. The definition of a geodesic uses only ∇, not the curvature. Curvature enters through how geodesics spread and how $\exp_p$ behaves, not through the geodesic equation itself.

5.2 Coordinate derivation of the geodesic equation

Let (U, x^μ) be a coordinate chart, and assume $\gamma(I) \subset U$. Write $x^\mu(\lambda) = x^\mu(\gamma(\lambda))$ for a parameter λ on I and

$$\dot\gamma = \frac{d\gamma}{d\lambda} = \dot x^\mu \, \partial_\mu, \qquad \dot x^\mu = \frac{dx^\mu}{d\lambda}.$$

Proposition 5.2. Coordinate form of the geodesic equation

The geodesic equation $\nabla_{\dot\gamma}\dot\gamma = 0$ is equivalent to the system of second-order ODEs

$$\ddot x^\rho + \Gamma^\rho_{\mu\nu}(x(\lambda))\, \dot x^\mu \dot x^\nu = 0,$$

where $\Gamma^\rho_{\mu\nu}$ are the Christoffel symbols of ∇ in the chart (U, x^μ), and $\ddot x^\rho = d^2 x^\rho / d\lambda^2$.

Proof. Compute using the connection rules. First,

$$\nabla_{\dot\gamma}\dot\gamma = \nabla_{\dot x^\mu \partial_\mu}(\dot x^\nu \partial_\nu) = \dot x^\mu \nabla_{\partial_\mu}(\dot x^\nu \partial_\nu).$$

By the Leibniz rule,

$$\nabla_{\partial_\mu}(\dot x^\nu \partial_\nu) = \partial_\mu(\dot x^\nu)\partial_\nu + \dot x^\nu \nabla_{\partial_\mu}\partial_\nu = \partial_\mu(\dot x^\nu)\partial_\nu + \dot x^\nu \Gamma^\rho_{\mu\nu}\partial_\rho.$$

Along the curve, $\partial_\mu(\dot x^\nu)\dot x^\mu = \frac{d}{d\lambda}\dot x^\nu = \ddot x^\nu$. Hence

$$\nabla_{\dot\gamma}\dot\gamma = (\ddot x^\rho + \Gamma^\rho_{\mu\nu}\dot x^\mu \dot x^\nu)\partial_\rho.$$

Therefore $\nabla_{\dot\gamma}\dot\gamma = 0$ if and only if each component satisfies the stated equation. □

5.3 Affine parameters and reparametrization

The geodesic equation is a second-order ODE in a chosen parameter. Not every parameter makes it look the same.

Definition 5.3. Affine parameter

Let $\gamma\colon I \to M$ be a geodesic with parameter λ. We call λ an **affine parameter** if γ satisfies $\nabla_{\dot\gamma}\dot\gamma = 0$ in that parameter. Equivalently, $\dot\gamma$ is parallel along γ.

Proposition 5.4. Reparametrization of geodesics

Let $\gamma\colon I \to M$ be a geodesic in affine parameter λ, and let $s \mapsto \lambda(s)$ be a smooth monotone reparametrization. Let $\tilde\gamma(s) = \gamma(\lambda(s))$. Then

$$\nabla_{\frac{d\tilde\gamma}{ds}} \frac{d\tilde\gamma}{ds} = \frac{d^2\lambda}{ds^2}\left(\frac{d\lambda}{ds}\right)^{-2} \frac{d\tilde\gamma}{ds}.$$

In particular, $\tilde\gamma$ is a geodesic in the strict sense $\nabla_{\tilde\gamma'}\tilde\gamma' = 0$ if and only if $\lambda(s)$ is affine, i.e. $\lambda(s) = as + b$ with $a \neq 0$.

Proof. Write $W = \frac{d\tilde\gamma}{ds}$ and $V = \frac{d\gamma}{d\lambda}$ so that $W = (d\lambda/ds)\, V$ along the curve. Then

$$\nabla_W W = \nabla_{(d\lambda/ds)V}((d\lambda/ds)V) = \left(\frac{d\lambda}{ds}\right)^2 \nabla_V V + \frac{d\lambda}{ds}\frac{d}{d\lambda}\left(\frac{d\lambda}{ds}\right) V.$$

Since γ is a geodesic in λ, $\nabla_V V = 0$. Also,

$$\frac{d}{d\lambda}\left(\frac{d\lambda}{ds}\right) = \frac{d}{ds}\left(\frac{d\lambda}{ds}\right)\frac{ds}{d\lambda} = \frac{d^2\lambda}{ds^2}\left(\frac{d\lambda}{ds}\right)^{-1}.$$

Substitute to get

$$\nabla_W W = \frac{d^2\lambda}{ds^2}\left(\frac{d\lambda}{ds}\right)^{-2} W,$$

which is the desired identity. The last statement follows because the prefactor vanishes exactly when $d^2\lambda/ds^2 = 0$. □

Remark. The proposition implies that the *image* of a geodesic curve is well-defined independent of parametrization: any smooth monotone reparametrization of an affinely parametrized geodesic produces a curve whose acceleration is everywhere tangent to itself. The strict equation $\nabla_{\dot\gamma}\dot\gamma = 0$ singles out the affine parameters.

5.4 Local existence, uniqueness, and the exponential map

Fix $p \in M$. In a coordinate chart (U, x^μ) about p, the geodesic equation becomes a smooth system of second-order ODEs. Standard ODE theory applies.

Theorem 5.5. Local existence and uniqueness of geodesics

Let (M, g) be a Lorentzian manifold with Levi-Civita connection ∇. For any $p \in M$ and any $v \in T_pM$, there exists $\varepsilon > 0$ and a unique geodesic $\gamma\colon (-\varepsilon, \varepsilon) \to M$ such that

$$\gamma(0) = p, \qquad \dot\gamma(0) = v.$$

Proof. Choose a chart (U, x^μ) with $p \in U$. Writing the geodesic equation in coordinates gives

$$\ddot x^\rho = F^\rho(x, \dot x), \qquad F^\rho(x, \dot x) := -\Gamma^\rho_{\mu\nu}(x)\dot x^\mu \dot x^\nu,$$

where F is smooth on $U \times \mathbb{R}^4$. Convert to a first-order system in (x, ξ) with $\xi = \dot x$:

$$\dot x^\rho = \xi^\rho, \qquad \dot\xi^\rho = F^\rho(x, \xi).$$

By the Picard–Lindelöf theorem, for given initial data $(x(0), \xi(0))$ there exists $\varepsilon > 0$ and a unique solution on $(-\varepsilon, \varepsilon)$. Translating back gives a unique geodesic with the prescribed initial conditions. □

Definition 5.6. Exponential map

For $p \in M$, define $\exp_p$ on a neighborhood $\mathcal{U} \subset T_pM$ of 0 as follows. For $v \in \mathcal{U}$, let γ_v be the unique geodesic with $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$, defined at least on $[0, 1]$. Then

$$\exp_p(v) := \gamma_v(1).$$

Proposition 5.7. Basic properties of $\exp_p$

There exists a star-shaped neighborhood $\mathcal{U} \subset T_pM$ of 0 on which $\exp_p$ is smooth. Moreover,

$$\exp_p(0) = p, \qquad (d\exp_p)_0 = \mathrm{id}_{T_pM}.$$

Proof. Smooth dependence of ODE solutions on initial conditions implies smoothness of $v \mapsto \gamma_v(1)$ on a sufficiently small star-shaped neighborhood. The identity $\exp_p(0) = p$ is immediate since γ_0 is constant. To compute the differential at 0, consider the curve $s \mapsto \exp_p(sv)$ for fixed v. This is the geodesic γ_v evaluated at time s up to reparametrization, hence

$$\left.\frac{d}{ds}\right|_{s=0} \exp_p(sv) = v,$$

which means $(d\exp_p)_0(v) = v$. □

5.5 A basic conserved quantity along geodesics

Proposition 5.8. Constancy of $g(\dot\gamma,\dot\gamma)$

Let γ be an affinely parametrized geodesic. Then the quantity

$$g(\dot\gamma,\dot\gamma)$$

is constant along γ. In particular, a geodesic that is timelike (resp. null, spacelike) at one point remains timelike (resp. null, spacelike) wherever it is defined.

Proof. Differentiate using metric compatibility:

$$\frac{d}{d\lambda}g(\dot\gamma,\dot\gamma) = \nabla_{\dot\gamma}(g(\dot\gamma,\dot\gamma)) = 2\,g(\nabla_{\dot\gamma}\dot\gamma,\dot\gamma).$$

If γ is a geodesic in affine parameter, then $\nabla_{\dot\gamma}\dot\gamma = 0$, hence the derivative vanishes and $g(\dot\gamma,\dot\gamma)$ is constant. $\square$

5.6 Two coordinate pictures of the same geodesics

Example. Geodesics in Minkowski spacetime are straight lines. On $(\mathbb{R}^4,\eta)$ in inertial coordinates, $\Gamma^\rho_{\mu\nu} = 0$. Hence the geodesic equation becomes

$$\ddot x^\rho = 0,$$

so every geodesic is of the form

$$x^\rho(\lambda) = a^\rho + b^\rho\lambda,$$

i.e. an affine line. The causal character is determined by $\eta_{\mu\nu}b^\mu b^\nu$, which is constant by Proposition 5.8.

Example. The same straight lines in Rindler coordinates. Use the Rindler chart from the previous lecture with coordinates (η,ρ,y,z) and metric

$$g = -\rho^2\,d\eta^2 + d\rho^2 + dy^2 + dz^2,$$

whose only nonzero Christoffel symbols are

$$\Gamma^1_{00} = \rho, \qquad \Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{\rho}.$$

Let λ be an affine parameter. The geodesic equations split into

$$\begin{aligned} \ddot\eta + 2\,\Gamma^0_{01}\dot\eta\dot\rho = 0 \qquad &\iff \qquad \ddot\eta + \frac{2}{\rho}\dot\eta\dot\rho = 0,\\ \ddot\rho + \Gamma^1_{00}(\dot\eta)^2 = 0 \qquad &\iff \qquad \ddot\rho + \rho(\dot\eta)^2 = 0,\\ \ddot y = 0, \qquad &\ddot z = 0. \end{aligned}$$

The first equation can be integrated once:

$$\frac{d}{d\lambda}(\rho^2\dot\eta) = 2\rho\dot\rho\dot\eta + \rho^2\ddot\eta = \rho^2\left(\ddot\eta + \frac{2}{\rho}\dot\eta\dot\rho\right) = 0,$$

so

$$\rho^2\dot\eta = C$$

for a constant C. Substituting $\dot\eta = C/\rho^2$ into the ρ-equation gives

$$\ddot\rho + \rho\left(\frac{C}{\rho^2}\right)^2 = \ddot\rho + \frac{C^2}{\rho^3} = 0.$$

Thus, even though Minkowski geodesics are straight lines in inertial coordinates, in Rindler coordinates they satisfy a nonlinear ODE with a first integral coming from the Killing field ∂_η (this Killing-field viewpoint will be formalized later).

6 The Free Particle Action and the Variational Derivation of the Geodesic Equation

6.1 Two action functionals and the class of variations

Fix a smooth Lorentzian manifold (M, g) with Levi-Civita connection ∇. Let $\gamma \colon [\lambda_0, \lambda_1] \to M$ be a C^∞ curve.

Definition 6.1. Variations with fixed endpoints

A **variation of γ with fixed endpoints** is a smooth map

$$\Gamma \colon (-\varepsilon, \varepsilon) \times [\lambda_0, \lambda_1] \to M$$

such that $\Gamma(0, \lambda) = \gamma(\lambda)$ and

$$\Gamma(s, \lambda_0) = \gamma(\lambda_0), \qquad \Gamma(s, \lambda_1) = \gamma(\lambda_1)$$

for all s with $|s| < \varepsilon$. The associated **variation vector field** along γ is

$$V(\lambda) \coloneqq \frac{\partial}{\partial} s\Big|_{s=0} \Gamma(s, \lambda) \in T_{\gamma(\lambda)} M,$$

which satisfies $V(\lambda_0) = V(\lambda_1) = 0$.

We will consider two standard Lagrangians built from the metric. Throughout, we adopt the signature convention $(-, +, +, +)$.

Definition 6.2. Length-type and energy-type actions

Assume γ is **timelike** so that $g(\dot\gamma, \dot\gamma) < 0$ everywhere. Define

$$S_{\text{len}}[\gamma] \coloneqq \int_{\lambda_0}^{\lambda_1} \sqrt{-g(\dot\gamma, \dot\gamma)}\, d\lambda.$$

For an arbitrary smooth curve (no causal assumption), define the **energy-type action**

$$S_{\text{en}}[\gamma] \coloneqq \frac{1}{2} \int_{\lambda_0}^{\lambda_1} g(\dot\gamma, \dot\gamma)\, d\lambda,$$

with the understanding that in this functional one keeps both endpoints and the parameter interval $[\lambda_0, \lambda_1]$ fixed.

Remark. For timelike curves, S_{len} is the proper time elapsed along the worldline, up to a sign convention if one multiplies by m. In contrast, S_{en} depends on the chosen parameter λ.

6.2 Euler–Lagrange for the energy Lagrangian and the Christoffel form

Work in a coordinate chart (U, x^μ) and assume $\gamma([\lambda_0, \lambda_1]) \subset U$. Write $x^\mu(\lambda) = x^\mu(\gamma(\lambda))$ and $\dot x^\mu = dx^\mu / d\lambda$.

Consider the Lagrangian

$$L_{\text{en}}(x, \dot x) = \frac{1}{2} g_{\mu\nu}(x)\, \dot x^\mu \dot x^\nu.$$

Proposition 6.3. Euler–Lagrange equation for L_{en}

The Euler–Lagrange equations

$$\frac{d}{d\lambda}\left(\frac{\partial L_{\text{en}}}{\partial \dot{x}^\rho}\right) - \frac{\partial L_{\text{en}}}{\partial x^\rho} = 0$$

are equivalent to

$$\ddot{x}^\rho + \Gamma^\rho_{\mu\nu}(x)\,\dot{x}^\mu \dot{x}^\nu = 0,$$

where $\Gamma^\rho_{\mu\nu}$ are the Levi-Civita Christoffel symbols of g in the chart.

Proof. Compute the derivatives:

$$\frac{\partial L_{\text{en}}}{\partial \dot{x}^\rho} = \frac{1}{2}(g_{\mu\nu}\,\delta^\mu_\rho\,\dot{x}^\nu + g_{\mu\nu}\,\dot{x}^\mu\,\delta^\nu_\rho) = g_{\rho\nu}\dot{x}^\nu,$$

using symmetry of $g_{\mu\nu}$. Also,

$$\frac{\partial L_{\text{en}}}{\partial x^\rho} = \frac{1}{2}\,\partial_\rho g_{\mu\nu}\,\dot{x}^\mu\dot{x}^\nu.$$

Therefore the Euler–Lagrange equation becomes

$$\frac{d}{d\lambda}(g_{\rho\nu}\dot{x}^\nu) - \frac{1}{2}\,\partial_\rho g_{\mu\nu}\,\dot{x}^\mu\dot{x}^\nu = 0.$$

Expand the total derivative:

$$\frac{d}{d\lambda}(g_{\rho\nu}\dot{x}^\nu) = (\partial_\mu g_{\rho\nu})\dot{x}^\mu\dot{x}^\nu + g_{\rho\nu}\ddot{x}^\nu,$$

so we obtain

$$g_{\rho\nu}\ddot{x}^\nu + (\partial_\mu g_{\rho\nu})\dot{x}^\mu\dot{x}^\nu - \frac{1}{2}\,\partial_\rho g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0.$$

Multiply by $g^{\rho\sigma}$ and relabel dummy indices to rewrite this as

$$\ddot{x}^\sigma + \frac{1}{2}\,g^{\sigma\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})\dot{x}^\mu\dot{x}^\nu = 0.$$

The term in parentheses is $2\Gamma^\sigma_{\mu\nu}$. This gives the claimed geodesic equation. □

Remark. The intermediate form

$$\frac{d}{d\lambda}(g_{\rho\nu}\dot{x}^\nu) - \frac{1}{2}\,\partial_\rho g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$$

is often computationally convenient because it postpones raising an index until the end.

There is also a coordinate-free first-variation identity, which is the clean geometric reason the Euler–Lagrange equations reduce to the geodesic equation.

Proposition 6.4. First variation of the energy functional

Let γ be a smooth curve and Γ a fixed-endpoint variation with variation field V. Then

$$\left.\frac{d}{ds}\right|_{s=0} S_{\text{en}}[\Gamma(s,\cdot)] = [g(\dot{\gamma}, V)]_{\lambda_0}^{\lambda_1} - \int_{\lambda_0}^{\lambda_1} g(\nabla_{\dot{\gamma}}\dot{\gamma}, V)\,d\lambda.$$

In particular, among fixed-endpoint variations, γ is a critical point of S_{en} if and only if

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0.$$

Proof. Differentiate under the integral sign:

$$\left.\frac{d}{ds}\right|_{s=0} S_{\text{en}}[\Gamma(s,\cdot)] = \frac{1}{2}\int_{\lambda_0}^{\lambda_1} \left.\frac{d}{ds}\right|_{s=0} g(\partial_\lambda\Gamma, \partial_\lambda\Gamma)\,d\lambda = \int_{\lambda_0}^{\lambda_1} g(\nabla_s\partial_\lambda\Gamma, \partial_\lambda\Gamma)\big|_{s=0}\,d\lambda.$$

Using the torsion-free property for the coordinate vector fields $\partial_s, \partial_\lambda$ on $(-\varepsilon, \varepsilon) \times [\lambda_0, \lambda_1]$, we have

$$\nabla_s\partial_\lambda\Gamma = \nabla_\lambda\partial_s\Gamma.$$

Hence the integrand becomes $g(\nabla_{\partial_\lambda} V, \dot\gamma)$. By metric compatibility,
$$\frac{d}{d\lambda} g(V,\dot\gamma) = g(\nabla_{\dot\gamma} V, \dot\gamma) + g(V, \nabla_{\dot\gamma}\dot\gamma).$$
Rearrange and integrate:
$$\int_{\lambda_0}^{\lambda_1} g(\nabla_{\dot\gamma} V, \dot\gamma)\, d\lambda = [g(V,\dot\gamma)]_{\lambda_0}^{\lambda_1} - \int_{\lambda_0}^{\lambda_1} g(V, \nabla_{\dot\gamma}\dot\gamma)\, d\lambda.$$
This is the claimed formula. If endpoints are fixed then $V(\lambda_0) = V(\lambda_1) = 0$ so the boundary term vanishes, and the vanishing of the integral for all such V forces $\nabla_{\dot\gamma}\dot\gamma = 0$. □

6.3 Reparametrization invariance and a convenient gauge choice

The two actions behave differently under reparametrization.

Proposition 6.5. Reparametrization behavior

Let $\lambda = \lambda(\sigma)$ be a smooth monotone change of parameter and set $\tilde\gamma(\sigma) = \gamma(\lambda(\sigma))$.

(1) The length-type action is invariant:
$$S_{\text{len}}[\tilde\gamma] = S_{\text{len}}[\gamma].$$
(2) The energy-type action is not invariant in general:
$$S_{\text{en}}[\tilde\gamma] = \frac{1}{2}\int g\left(\frac{d\gamma}{d\lambda}, \frac{d\gamma}{d\lambda}\right)\left(\frac{d\lambda}{d\sigma}\right) d\lambda,$$
so changing parameter changes the weight in the integral unless $d\lambda/d\sigma$ is constant.

Proof. For (1), note that $d\tilde\gamma/d\sigma = (d\lambda/d\sigma)\dot\gamma$, so
$$\sqrt{-g(\tilde\gamma', \tilde\gamma')}\, d\sigma = \sqrt{-g(\dot\gamma,\dot\gamma)}\, d\lambda.$$
For (2), substitute the same relation into S_{en} and simplify. □

Because S_{len} is invariant, its Euler–Lagrange equation cannot determine a preferred parameter. One sees this directly.

Proposition 6.6. Euler–Lagrange equation for the length Lagrangian

Assume γ is timelike and define
$$\ell(\lambda) := \sqrt{-g(\dot\gamma,\dot\gamma)}.$$
The Euler–Lagrange equations for the Lagrangian $L_{\text{len}} = \ell$ are equivalent to
$$\nabla_{\dot\gamma}\dot\gamma = \left(\frac{d}{d\lambda}\ln\ell\right)\dot\gamma.$$
In particular, if ℓ is constant along γ then γ satisfies the affine geodesic equation $\nabla_{\dot\gamma}\dot\gamma = 0$ in that parameter.

Proof. In coordinates, $L_{\text{len}}(x,\dot x) = \sqrt{-g_{\mu\nu}\dot x^\mu \dot x^\nu} = \ell$. Compute
$$\frac{\partial L_{\text{len}}}{\partial \dot x^\rho} = -\frac{g_{\rho\nu}\dot x^\nu}{\ell}, \qquad \frac{\partial L_{\text{len}}}{\partial x^\rho} = -\frac{1}{2\ell}\,\partial_\rho g_{\mu\nu}\,\dot x^\mu \dot x^\nu.$$
The Euler–Lagrange equation becomes
$$\frac{d}{d\lambda}\left(\frac{g_{\rho\nu}\dot x^\nu}{\ell}\right) - \frac{1}{2\ell}\,\partial_\rho g_{\mu\nu}\,\dot x^\mu \dot x^\nu = 0.$$
Multiply by ℓ and expand:
$$\frac{d}{d\lambda}(g_{\rho\nu}\dot x^\nu) - \frac{\dot\ell}{\ell}\, g_{\rho\nu}\dot x^\nu - \frac{1}{2}\,\partial_\rho g_{\mu\nu}\dot x^\mu \dot x^\nu = 0.$$

Raise an index exactly as in Proposition 6.3 to obtain

$$\ddot{x}^\rho + \Gamma^\rho_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \frac{\dot{\ell}}{\ell}\,\dot{x}^\rho.$$

This is the coordinate expression of

$$\nabla_{\dot\gamma}\dot\gamma = \frac{\dot{\ell}}{\ell}\,\dot\gamma,$$

which is the stated formula since $(\dot{\ell}/\ell) = \frac{d}{d\lambda}\ln \ell$. □

Remark. For a timelike worldline, choosing the parameter τ to be proper time forces $g(\dot\gamma, \dot\gamma) = -1$, hence $\ell \equiv 1$. In that gauge the length action gives the affine geodesic equation.

6.4 The massless case and null geodesics

For a null curve, $g(\dot\gamma, \dot\gamma) = 0$, hence the length-type integrand $\sqrt{-g(\dot\gamma, \dot\gamma)}$ vanishes. One must use a different variational formulation.

A convenient reparametrization-invariant action introduces an auxiliary positive function $e(\lambda)$ along the curve.

Definition 6.7. Reparametrization-invariant null action

Let $\gamma\colon [\lambda_0, \lambda_1] \to M$ be a smooth curve and let $e \in C^\infty([\lambda_0, \lambda_1])$ be everywhere positive. Define

$$S_0[\gamma, e] := \frac{1}{2}\int_{\lambda_0}^{\lambda_1} e(\lambda)^{-1}\, g(\dot\gamma, \dot\gamma)\, d\lambda.$$

Proposition 6.8. Field equations for the null action

Let (γ, e) be a critical point of S_0 under variations with fixed endpoints for γ and arbitrary variations of e. Then:

(1) The curve is null,
$$g(\dot\gamma, \dot\gamma) = 0.$$

(2) The curve satisfies
$$\nabla_{\dot\gamma}(e^{-1}\dot\gamma) = 0.$$

In particular, after choosing a parameter so that e is constant, γ satisfies the affine null geodesic equation

$$\nabla_{\dot\gamma}\dot\gamma = 0$$

together with $g(\dot\gamma, \dot\gamma) = 0$.

Proof. Varying e gives

$$\delta_e S_0 = -\frac{1}{2}\int_{\lambda_0}^{\lambda_1} e^{-2}\,\delta e\; g(\dot\gamma, \dot\gamma)\, d\lambda,$$

which forces $g(\dot\gamma, \dot\gamma) = 0$ since δe is arbitrary.

For the γ-variation, use the same computation as in Proposition 6.4 but with the weight e^{-1} inserted:

$$\delta_\gamma S_0 = \int_{\lambda_0}^{\lambda_1} e^{-1} g(\nabla_{\dot\gamma} V, \dot\gamma)\, d\lambda = \left[e^{-1} g(V, \dot\gamma)\right]_{\lambda_0}^{\lambda_1} - \int_{\lambda_0}^{\lambda_1} g\Big(V, \nabla_{\dot\gamma}(e^{-1}\dot\gamma)\Big)\, d\lambda.$$

The boundary term vanishes for fixed endpoints, and arbitrariness of V yields $\nabla_{\dot\gamma}(e^{-1}\dot\gamma) = 0$. If we reparametrize so that e becomes constant, then this reduces to $\nabla_{\dot\gamma}\dot\gamma = 0$. □

Remark. The parameter for a null geodesic is not determined by proper time, since proper time is identically zero along a null curve. The affine parameters are characterized by the condition that the geodesic equation takes the homogeneous form $\nabla_{\dot\gamma}\dot\gamma = 0$.

6.5 Interface with physical quantities

For massive particles, one typically uses proper time τ along a timelike worldline. Then $u = \dot\gamma = d\gamma/d\tau$ satisfies $g(u, u) = -1$.

Definition 6.9. Four-momentum

Let $\gamma(\tau)$ be a timelike worldline parametrized by proper time, with four-velocity u^a. The **four-momentum covector** is

$$p_a := m\, u_a = m\, g_{ab} u^b.$$

Equivalently, $p^\sharp = mu$ as a vector field along γ.

Proposition 6.10. Geodesic motion as covariant conservation of four-momentum

If γ is a timelike geodesic parametrized by proper time, then

$$\nabla_u p = 0.$$

Proof. Since m is constant and $p = mu^\flat$, we have $\nabla_u p = m\, \nabla_u(u^\flat)$. Metric compatibility gives $\nabla_u(u^\flat) = (\nabla_u u)^\flat$. For a geodesic in affine parameter τ, $\nabla_u u = 0$, hence $\nabla_u p = 0$. □

Observers are future-directed unit timelike vectors. Given an observer with four-velocity U^a at the event $\gamma(\tau)$, the energy measured by that observer is a contraction of momentum with U.

Definition 6.11. Energy measured by an observer

Let p_a be the four-momentum covector of a particle at an event, and let U^a be a future-directed unit timelike vector at the same event. The **observed energy** is

$$E := -p_a U^a.$$

Remark. With the sign convention $(-, +, +, +)$, a future-directed timelike momentum satisfies $p_a U^a < 0$ for any future-directed unit timelike U, hence $E > 0$.

For massless particles, one uses a null geodesic with an affine parameter λ and a null tangent $k^a = d\gamma^a/d\lambda$. The associated momentum is a null covector proportional to $k_a = g_{ab}k^b$, with the overall scaling reflecting the freedom to rescale the affine parameter. The measured energy is still $E = -p_a U^a$, and it is this quantity that later becomes the natural carrier of redshift.

7 Orthonormal Frames, Cartan Formalism, and Local Inertial Systems

7.1 Orthonormal frames and coframes (tetrads)

Let (M, g) be a time-oriented Lorentzian 4-manifold with signature $(-, +, +, +)$.

Definition 7.1. Orthonormal frame and coframe

An **orthonormal frame field** (tetrad) on an open set $U \subset M$ is a quadruple of vector fields

$$(e_0, e_1, e_2, e_3)$$

such that for all $p \in U$,

$$g(e_\alpha, e_\beta) = \eta_{\alpha\beta}, \qquad \eta = \mathrm{diag}(-1, 1, 1, 1).$$

The **dual coframe** is the quadruple of 1-forms $(\theta^0, \theta^1, \theta^2, \theta^3)$ uniquely determined by

$$\theta^\alpha(e_\beta) = \delta^\alpha_\beta.$$

In terms of the coframe, the metric takes the coordinate-free form

$$g = \eta_{\alpha\beta}\, \theta^\alpha \otimes \theta^\beta.$$

If (x^μ) is a coordinate chart, one writes

$$e_\alpha = e_\alpha{}^\mu\, \partial_\mu, \qquad \theta^\alpha = e^\alpha{}_\mu\, dx^\mu,$$

with the duality relations

$$e^\alpha{}_\mu e_\beta{}^\mu = \delta^\alpha_\beta, \qquad e^\alpha{}_\mu e_\alpha{}^\nu = \delta^\nu_\mu.$$

Then

$$g_{\mu\nu} = \eta_{\alpha\beta}\, e^\alpha{}_\mu e^\beta{}_\nu, \qquad \eta_{\alpha\beta} = g_{\mu\nu}\, e_\alpha{}^\mu e_\beta{}^\nu.$$

Remark. It is useful to keep two kinds of indices conceptually separate:

(1) World indices $\mu, \nu, \ldots$ refer to the coordinate basis ∂_μ and components in that basis.

(2) Frame indices $\alpha, \beta, \ldots$ refer to the orthonormal frame e_α and components measured in that frame.

A vector V has frame components $V^\alpha = \theta^\alpha(V)$ and world components $V^\mu = dx^\mu(V)$, related by $V^\alpha = e^\alpha{}_\mu V^\mu$.

A tetrad is not unique. If $\Lambda(x) \in SO^+(1,3)$ varies smoothly with $x \in U$, then

$$\tilde{e}_\alpha = \Lambda_\alpha{}^\beta\, e_\beta, \qquad \tilde{\theta}^\alpha = (\Lambda^{-1})^\alpha{}_\beta\, \theta^\beta$$

is another orthonormal frame/coframe giving the same metric.

7.2 Connection 1-forms and the first structure equation

Let ∇ be the Levi-Civita connection of g.

Definition 7.2. Connection 1-forms

Given an orthonormal frame (e_α) on U, the **connection 1-forms** $\omega^\alpha{}_\beta \in \Omega^1(U)$ are defined by

$$\nabla e_\beta = \omega^\alpha{}_\beta \otimes e_\alpha, \qquad \text{equivalently} \qquad \nabla_X e_\beta = \omega^\alpha{}_\beta(X)\, e_\alpha$$

for all vector fields X on U.

Metric compatibility forces antisymmetry in the orthonormal indices.

Proposition 7.3. Antisymmetry of $\omega_{\alpha\beta}$

Let $\omega_{\alpha\beta} := \eta_{\alpha\gamma}\, \omega^\gamma{}_\beta$. Then

$$\omega_{\alpha\beta} + \omega_{\beta\alpha} = 0.$$

Proof. Since $g(e_\alpha, e_\beta) = \eta_{\alpha\beta}$ is constant,

$$0 = X\big(g(e_\alpha, e_\beta)\big) = g(\nabla_X e_\alpha, e_\beta) + g(e_\alpha, \nabla_X e_\beta) = \omega_{\beta\alpha}(X) + \omega_{\alpha\beta}(X)$$

for all X, hence $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$. $\square$

The torsion-free condition is encoded in the first Cartan structure equation.

Theorem 7.4. First structure equation (torsion-free)

Let (θ^α) be the coframe dual to an orthonormal frame (e_α), and let $\omega^\alpha{}_\beta$ be the associated connection 1-forms. Then

$$d\theta^\alpha + \omega^\alpha{}_\beta \wedge \theta^\beta = 0.$$

Proof. Evaluate both sides on (X, Y). Using $d\theta^\alpha(X,Y) = X(\theta^\alpha(Y)) - Y(\theta^\alpha(X)) - \theta^\alpha([X,Y])$ and $\theta^\alpha(Y) = \eta^{\alpha\gamma} g(e_\gamma, Y)$, one checks the identity

$$d\theta^\alpha(X,Y) = \theta^\alpha(\nabla_X Y - \nabla_Y X - [X,Y]) - \omega^\alpha{}_\beta(X)\theta^\beta(Y) + \omega^\alpha{}_\beta(Y)\theta^\beta(X).$$

Since ∇ is torsion-free, $\nabla_X Y - \nabla_Y X - [X,Y] = 0$, so

$$d\theta^\alpha(X,Y) = -(\omega^\alpha{}_\beta \wedge \theta^\beta)(X,Y),$$

which is exactly the stated equation. □

For calculations, it is convenient to express $d\theta^\alpha$ in the coframe basis.

Definition 7.5. Structure coefficients of a coframe

Given a coframe (θ^α), define functions $C^\alpha{}_{\beta\gamma}$ by

$$d\theta^\alpha = -\frac{1}{2} C^\alpha{}_{\beta\gamma}\, \theta^\beta \wedge \theta^\gamma, \qquad C^\alpha{}_{\beta\gamma} = -C^\alpha{}_{\gamma\beta}.$$

Proposition 7.6. Computing ω from $d\theta$

Let $C_{\alpha\beta\gamma} := \eta_{\alpha\delta} C^\delta{}_{\beta\gamma}$. Then the Levi-Civita connection 1-forms in the orthonormal coframe satisfy

$$\omega_{\alpha\beta} = \frac{1}{2}\Big(C_{\alpha\beta\gamma} + C_{\gamma\alpha\beta} - C_{\beta\gamma\alpha}\Big)\,\theta^\gamma, \qquad \omega_{\alpha\beta} = -\omega_{\beta\alpha}.$$

Equivalently, the connection coefficients $\omega_{\alpha\beta\gamma} := \omega_{\alpha\beta}(e_\gamma)$ are

$$\omega_{\alpha\beta\gamma} = \frac{1}{2}\Big(C_{\alpha\beta\gamma} + C_{\gamma\alpha\beta} - C_{\beta\gamma\alpha}\Big).$$

Proof. Insert the expansion $\omega^\alpha{}_\beta = \omega^\alpha{}_{\beta\gamma}\theta^\gamma$ into the first structure equation:

$$d\theta^\alpha = -\omega^\alpha{}_{\beta\gamma}\, \theta^\gamma \wedge \theta^\beta = -\frac{1}{2}(\omega^\alpha{}_{\beta\gamma} - \omega^\alpha{}_{\gamma\beta})\theta^\beta \wedge \theta^\gamma.$$

Comparing with $d\theta^\alpha = -\frac{1}{2}C^\alpha{}_{\beta\gamma}\theta^\beta \wedge \theta^\gamma$ yields

$$C^\alpha{}_{\beta\gamma} = \omega^\alpha{}_{\beta\gamma} - \omega^\alpha{}_{\gamma\beta}.$$

Lower the first index with η and use antisymmetry $\omega_{\alpha\beta\gamma} = -\omega_{\beta\alpha\gamma}$ to solve for $\omega_{\alpha\beta\gamma}$ by cyclic permutations of (α, β, γ). The stated formula is the unique solution. □

Remark. If one wants to translate between the tetrad connection and the coordinate Christoffel symbols, one may use

$$\omega^\alpha{}_{\beta\mu} = e^\alpha{}_\nu\Big(\partial_\mu e_\beta{}^\nu + \Gamma^\nu_{\mu\rho} e_\beta{}^\rho\Big), \qquad \omega^\alpha{}_\beta = \omega^\alpha{}_{\beta\mu}\, dx^\mu.$$

This is an identity, not an extra definition.

7.3 Killing Christoffel symbols at a point and the meaning of "local inertial"

There are two closely related statements.

Theorem 7.7. Normal coordinates at a point

For any $p \in M$ there exists a coordinate chart (x^μ) centered at p such that

$$g_{\mu\nu}(p) = \eta_{\mu\nu}, \qquad \partial_\rho g_{\mu\nu}(p) = 0.$$

In particular, the Christoffel symbols in these coordinates satisfy

$$\Gamma^\rho_{\mu\nu}(p) = 0.$$

Proof. Use the exponential map $\exp_p$ to define a chart by identifying a neighborhood of $0 \in T_pM$ with a neighborhood of $p \in M$. Choose an orthonormal basis of T_pM to identify $T_pM \simeq \mathbb{R}^{1,3}$, giving coordinates x^μ near p. In these *Riemann normal coordinates*, geodesics through p are straight lines to first order, which implies $\Gamma^\rho_{\mu\nu}(p) = 0$. Metric compatibility then gives $\partial_\rho g_{\mu\nu}(p) = 0$ and the normalization $g_{\mu\nu}(p) = \eta_{\mu\nu}$ comes from the initial orthonormal choice at T_pM. $\square$

Remark. The point of $\Gamma(p) = 0$ is that the covariant derivative reduces to the ordinary derivative *at that single point*:

$$(\nabla_\mu V^\nu)(p) = (\partial_\mu V^\nu)(p)$$

in those coordinates. No coordinate choice can make $\Gamma^\rho_{\mu\nu}$ vanish on an open set unless the curvature vanishes there.

The same idea can be phrased in tetrad language.

Proposition 7.8. A freely falling orthonormal frame at a point

Fix $p \in M$. There exists an orthonormal frame (e_α) defined near p such that the associated connection 1-forms satisfy

$$\omega^\alpha{}_\beta(p) = 0.$$

Proof. Start with normal coordinates (x^μ) at p, so $\Gamma^\rho_{\mu\nu}(p) = 0$. Choose $e_\alpha(p)$ to be an orthonormal basis of T_pM and extend e_α to a local frame by parallel transport along the coordinate geodesics emanating from p (equivalently, along radial geodesics in the normal coordinate chart). By construction, ∇e_α vanishes at p, which is exactly $\omega^\alpha{}_\beta(p) = 0$. $\square$

Remark. The phrase **local inertial frame at** p can be read as either of the equivalent conditions:

(1) a coordinate chart with $g(p) = \eta$ and $\partial g(p) = 0$ (hence $\Gamma(p) = 0$);

(2) an orthonormal tetrad with $\omega(p) = 0$.

Both express the same geometric content: the Levi-Civita connection can be gauged away at a single point.

7.4 Fermi–Walker transport and a natural frame along a worldline

Let $\gamma(\tau)$ be a future-directed timelike curve parametrized by proper time, so $u = \dot\gamma$ satisfies $g(u,u) = -1$. Define the acceleration

$$a := \nabla_u u.$$

Definition 7.9. Fermi–Walker derivative

For a vector field X along γ, the **Fermi–Walker derivative** is

$$\frac{D_{\mathrm{FW}}X}{d\tau} := \nabla_u X + g(X,a)\,u - g(X,u)\,a.$$

We say that X is **Fermi–Walker transported** if $D_{\mathrm{FW}}X/d\tau = 0$.

Proposition 7.10. Basic properties of Fermi–Walker transport

Assume $D_{\mathrm{FW}}X/d\tau = 0$.

(1) The scalar $g(X,u)$ is constant along γ.

(2) If $g(X,u) = 0$ at one time, then $g(X,u) = 0$ for all time.

(3) If $g(X,u) = 0$, then $g(X,X)$ is constant along γ.

(4) If $a = 0$ (i.e. γ is a timelike geodesic), then Fermi–Walker transport reduces to parallel transport: $D_{\mathrm{FW}}X/d\tau = \nabla_u X$.

Proof. Using metric compatibility,

$$\frac{d}{d\tau}g(X,u) = g(\nabla_u X, u) + g(X, \nabla_u u) = g(\nabla_u X, u) + g(X,a).$$

If $D_{\mathrm{FW}}X/d\tau = 0$, then $\nabla_u X = -g(X,a)u + g(X,u)a$, hence

$$g(\nabla_u X, u) = -g(X,a)\,g(u,u) + g(X,u)\,g(a,u) = g(X,a),$$

since $g(u,u) = -1$ and $g(a,u) = \frac{1}{2}\nabla_u g(u,u) = 0$. Therefore $\frac{d}{d\tau}g(X,u) = 0$, proving (1) and (2). If additionally $g(X,u) = 0$, then $\nabla_u X = -g(X,a)u$ and

$$\frac{d}{d\tau}g(X,X) = 2g(\nabla_u X, X) = -2g(X,a)g(u,X) = 0,$$

giving (3). Finally, if $a = 0$ the defining formula gives $D_{\mathrm{FW}}X/d\tau = \nabla_u X$. □

A standard application is to build a distinguished orthonormal frame along an observer worldline.

Definition 7.11. Fermi–Walker tetrad along a worldline

Let $\gamma(\tau)$ be timelike with four-velocity u. A **Fermi–Walker tetrad** along γ is an orthonormal frame (e_0, e_1, e_2, e_3) along γ such that

$$e_0 = u, \qquad \frac{D_{\mathrm{FW}}e_i}{d\tau} = 0, \quad i = 1,2,3.$$

Proposition 7.12. Connection coefficients along a Fermi–Walker tetrad

Let (e_α) be a Fermi–Walker tetrad along γ and let $\omega^\alpha{}_\beta$ be the associated connection 1-forms. Then along γ,

$$\omega^i{}_j(u) = 0, \qquad \omega^i{}_0(u) = a^i,$$

where $a = a^i e_i$ is the spatial decomposition of the acceleration.

Proof. Write $\nabla_u e_\beta = \omega^\alpha{}_\beta(u)\, e_\alpha$. Since $e_0 = u$,

$$\nabla_u e_0 = \nabla_u u = a = a^i e_i,$$

so $\omega^i{}_0(u) = a^i$ and $\omega^0{}_0(u) = 0$. For $i \in \{1,2,3\}$, the condition $D_{\mathrm{FW}}e_i/d\tau = 0$ gives

$$\nabla_u e_i + g(e_i, a)u - g(e_i, u)a = 0.$$

Because $g(e_i, u) = 0$ and $g(e_i, a) = a^i$, this becomes $\nabla_u e_i = -a^i u = -a^i e_0$. Thus the spatial part of $\nabla_u e_i$ vanishes, meaning $\omega^j{}_i(u) = 0$ for all i,j. □

Remark. Given a tensor field T at a point of γ, an observer with tetrad (e_α) reads off the physically relevant components as the frame components $T_{\alpha\beta\cdots} = T(e_\alpha, e_\beta, \ldots)$. For four-momentum p, the energy measured by the observer is $E = -p \cdot u = -p \cdot e_0$.

7.5 Examples and calculation templates

Example. Minkowski tetrads in inertial coordinates. On $(\mathbb{R}^4, \eta)$ with standard inertial coordinates (t, x, y, z),

$$\theta^0 = dt, \quad \theta^1 = dx, \quad \theta^2 = dy, \quad \theta^3 = dz$$

is an orthonormal coframe, with dual frame

$$e_0 = \partial_t, \quad e_1 = \partial_x, \quad e_2 = \partial_y, \quad e_3 = \partial_z.$$

Since $d\theta^\alpha = 0$, the first structure equation gives $\omega^\alpha{}_\beta = 0$ in this frame.

Example. Template: compute $\omega^\alpha{}_\beta$ from a coframe in a non-inertial chart. Work on Minkowski spacetime but use spherical coordinates $(t, r, \vartheta, \varphi)$ on the spatial part, so

$$g = -dt^2 + dr^2 + r^2\, d\vartheta^2 + r^2 \sin^2 \vartheta\, d\varphi^2.$$

Choose the orthonormal coframe

$$\theta^0 = dt, \qquad \theta^1 = dr, \qquad \theta^2 = r\, d\vartheta, \qquad \theta^3 = r \sin \vartheta\, d\varphi.$$

Step 1: compute $d\theta^\alpha$ and rewrite in the θ-basis:

$$\begin{aligned}
&d\theta^0 = 0, \qquad d\theta^1 = 0,\\
&d\theta^2 = dr \wedge d\vartheta = \frac{1}{r}\, \theta^1 \wedge \theta^2,\\
&d\theta^3 = dr \wedge (\sin \vartheta\, d\varphi) + r \cos \vartheta\, d\vartheta \wedge d\varphi = \frac{1}{r}\, \theta^1 \wedge \theta^3 + \frac{\cot \vartheta}{r}\, \theta^2 \wedge \theta^3.
\end{aligned}$$

Step 2: solve $d\theta^\alpha + \omega^\alpha{}_\beta \wedge \theta^\beta = 0$. One convenient choice (and, after enforcing antisymmetry, the unique Levi-Civita solution) is

$$\omega^2{}_1 = -\frac{1}{r}\, \theta^2, \qquad \omega^3{}_1 = -\frac{1}{r}\, \theta^3, \qquad \omega^3{}_2 = -\frac{\cot \vartheta}{r}\, \theta^3,$$

with all other $\omega^\alpha{}_\beta$ determined by $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$ or equal to 0. Indeed, for example,

$$d\theta^2 + \omega^2{}_1 \wedge \theta^1 = \frac{1}{r}\, \theta^1 \wedge \theta^2 - \frac{1}{r}\, \theta^2 \wedge \theta^1 = 0,$$

and similarly for θ^3. This illustrates the practical point: even when the spacetime is flat, a nontrivial coframe in a non-inertial coordinate system produces nonzero connection 1-forms.

Example. Rindler coframe and a single nonzero connection component. In the Rindler chart (η, ρ, y, z) with metric

$$g = -\rho^2\, d\eta^2 + d\rho^2 + dy^2 + dz^2,$$

take the orthonormal coframe

$$\theta^0 = \rho\, d\eta, \qquad \theta^1 = d\rho, \qquad \theta^2 = dy, \qquad \theta^3 = dz.$$

Then $d\theta^0 = d\rho \wedge d\eta = \frac{1}{\rho}\, \theta^1 \wedge \theta^0$ and $d\theta^1 = d\theta^2 = d\theta^3 = 0$. The first structure equation gives

$$\omega^0{}_1 = \frac{1}{\rho}\, \theta^0, \qquad \omega^1{}_0 = -\frac{1}{\rho}\, \theta^0,$$

with all other $\omega^\alpha{}_\beta = 0$.

8 Observer Splitting and Basic Relativistic Effects

8.1 Observer fields and the spatial projector

Let (M, g) be a time-oriented Lorentzian 4-manifold with signature $(-, +, +, +)$. An *observer* is a future-directed unit timelike vector.

Definition 8.1. Observer four-velocity field

A smooth vector field $u \in \mathfrak{X}(M)$ is called an **observer four-velocity field** on an open set $U \subset M$ if

$$g(u, u) = -1$$

and u is future-directed at every point of U.

Given such a field, we obtain a canonical splitting of each tangent space into time and space relative to u.

Definition 8.2. Spatial subspace and projector

Let u be an observer field on U. At each $p \in U$ define the **spatial subspace**

$$\mathcal{S}_p := u(p)^{\perp} = \{v \in T_pM \mid g(v, u) = 0\}.$$

Define the **projection tensor**

$$h_{ab} := g_{ab} + u_a u_b.$$

Equivalently, as an endomorphism $h^a{}_b = \delta^a{}_b + u^a u_b$.

Proposition 8.3. Basic algebra of h

At each point:

(1) $h^a{}_b u^b = 0$ and $u_a h^a{}_b = 0$.

(2) $h^a{}_c h^c{}_b = h^a{}_b$ (idempotence).

(3) For any $v \in T_pM$, the decomposition

$$v = v_\parallel + v_\perp, \qquad v_\parallel := -(u \cdot v)\, u, \qquad v_\perp := h(v)$$

satisfies $v_\perp \in \mathcal{S}_p$ and $g(v_\parallel, v_\perp) = 0$.

(4) The bilinear form h_{ab} restricts to a positive definite inner product on $\mathcal{S}_p$.

Proof. (1) follows from $u^b u_b = -1$:

$$h^a{}_b u^b = \delta^a{}_b u^b + u^a u_b u^b = u^a - u^a = 0.$$

(2) is a direct computation:

$$h^a{}_c h^c{}_b = (\delta^a{}_c + u^a u_c)(\delta^c{}_b + u^c u_b) = \delta^a{}_b + u^a u_b + u^a u_b + u^a (u_c u^c) u_b = \delta^a{}_b + u^a u_b = h^a{}_b.$$

(3) and (4) are the orthogonal splitting statement from Lecture 3 with u fixed, and positivity of h on $u^\perp$ follows because g has index 1. ☐

8.2 Geometric definitions of time and space measurements

Fix an observer field u.

Let $\gamma(\tau)$ be a timelike worldline parametrized by proper time, so $u = \dot{\gamma}$ and $g(u, u) = -1$.

Proper time is the observer's intrinsic clock reading.

More generally, given any timelike curve $\gamma(\lambda)$ (not necessarily proper-time parametrized), the elapsed proper time between λ_0 and λ_1 is

$$\Delta\tau = \int_{\lambda_0}^{\lambda_1} \sqrt{-g(\dot\gamma,\dot\gamma)}\, d\lambda,$$

which is the length-type functional from the previous lecture.

Next we describe local simultaneity and space measurement. At each point p, the observer $u(p)$ singles out a distinguished local notion of simultaneity: directions orthogonal to $u(p)$.

Definition 8.4. Local simultaneity hyperplane

At p, the **local simultaneity hyperplane** of the observer u is $\mathcal{S}_p = u(p)^\perp \subset T_pM$.

This is purely pointwise. To integrate it to actual "simultaneous hypersurfaces" one needs an integrability condition, which will be revisited when discussing vorticity.

Definition 8.5. Spatial length measured by an observer

Let u be an observer field on U. Let $\sigma\colon [0,1] \to U$ be a smooth curve connecting two events. The **spatial length of σ as measured by u** is

$$L_u(\sigma) := \int_0^1 \sqrt{h(\dot\sigma,\dot\sigma)}\, d\lambda = \int_0^1 \sqrt{h_{ab}\dot\sigma^a\dot\sigma^b}\, d\lambda.$$

Remark. This definition depends on u. Even in flat Minkowski spacetime, different observer fields define different simultaneity hyperplanes and hence different spatial lengths between the same pair of events, because "space" is observer-dependent.

8.3 Decomposition of particle momentum: energy and spatial momentum

Let p^a be the four-momentum vector of a particle at some event, and let u be an observer at that event.

Definition 8.6. Energy and spatial momentum relative to an observer

Define

$$E := -p\cdot u = -g_{ab}p^a u^b.$$

Define the **spatial momentum** (as a vector) by projection:

$$p_\perp^a := h^a{}_b p^b.$$

Equivalently, as a covector,

$$(p_\perp)_a := h_a{}^b p_b.$$

Proposition 8.7. Momentum decomposition formula

One has

$$p^a = E\,u^a + p_\perp^a, \qquad u\cdot p_\perp = 0.$$

Moreover, for a massive particle with rest mass m and four-velocity v (so $p = mv$),

$$E = m\,\gamma, \qquad \gamma := -u\cdot v \geq 1,$$

and

$$p_\perp^2 := h_{ab} p_\perp^a p_\perp^b = E^2 - m^2.$$

For a massless particle ($p^2 = 0$),

$$p_\perp^2 = E^2.$$

Proof. Decompose p using Proposition 8.3:

$$p = -(u \cdot p)\, u + h(p) = E\, u + p_\perp,$$

and orthogonality is built into the definition. For a massive particle, $p = mv$ and $E = -m\, u \cdot v = m\gamma$. Compute

$$p^2 = g(p,p) = g(Eu + p_\perp, Eu + p_\perp) = -E^2 + h(p_\perp, p_\perp),$$

since $g(u,u) = -1$, $u \cdot p_\perp = 0$, and h coincides with g on the spatial subspace. Thus $p^2 = -m^2$ gives $h(p_\perp, p_\perp) = E^2 - m^2$. If $p^2 = 0$ (massless), the same identity gives $p_\perp^2 = E^2$. □

Remark. For a photon, one often writes the null tangent as k^a and momentum as $p^a = \hbar k^a$ or simply $p \propto k$ depending on conventions. The observed frequency is the contraction with u.

8.4 Minimal redshift/Doppler framework: frequency as $-k \cdot u$

Let γ be a null geodesic representing the worldline of light. Let k^a be its tangent vector field with respect to an affine parameter λ:

$$k^a = \frac{d\gamma^a}{d\lambda}, \qquad \nabla_k k = 0, \qquad g(k,k) = 0.$$

Definition 8.8. Observed frequency

Let u be an observer four-velocity at the event where the light ray meets the observer. The **observed angular frequency** is

$$\omega := -k \cdot u.$$

Remark. The affine parameter on a null geodesic is not unique: $\lambda \mapsto a\lambda + b$ rescales k by $1/a$, hence rescales ω by the same factor. Physical frequency is obtained once k is normalized by matching to an emitter's clock or by fixing k via a conserved quantity (typically through a Killing field).

The redshift between an emitter and a receiver is essentially the ratio of these contractions.

Proposition 8.9. Redshift ratio as a ratio of contractions

Suppose a light ray connects an emission event p and a reception event q. Let $u_{\rm em}$ be the emitter's four-velocity at p and $u_{\rm rec}$ the receiver's four-velocity at q. Let k be the tangent to the null geodesic, affinely parametrized, evaluated at the respective events. Then

$$\frac{\omega_{\rm rec}}{\omega_{\rm em}} = \frac{-k \cdot u_{\rm rec}}{-k \cdot u_{\rm em}}.$$

Equivalently, in terms of frequency $\nu = \omega/2\pi$,

$$\frac{\nu_{\rm rec}}{\nu_{\rm em}} = \frac{-k \cdot u_{\rm rec}}{-k \cdot u_{\rm em}}.$$

Proof. This is a definition-level statement: ω is defined as $-k \cdot u$ at the event of observation, with the same affine normalization of k used at both endpoints along the same geodesic. □

At this stage the formula is only a *framework*. To compute the ratio concretely one needs a way to relate k at p and q. Two standard mechanisms are:

(1) use symmetries (Killing fields), giving conserved quantities along the null geodesic;

(2) use transport equations for k along the geodesic in a chosen coordinate/frame.

We will formalize the first mechanism later.

8.5 Examples

Example. Special-relativistic Doppler in Minkowski spacetime. Work in Minkowski spacetime with inertial coordinates. Let the emitter be at rest in this frame, so

$$u_{\mathrm{em}} = \partial_t.$$

Let the receiver move with constant velocity v in the x-direction:

$$u_{\mathrm{rec}} = \gamma(\partial_t + v\,\partial_x), \qquad \gamma = \frac{1}{\sqrt{1-v^2}}.$$

Consider a photon propagating in the $+x$-direction. Choose an affine normalization so that its tangent is

$$k = \omega_{\mathrm{em}}(\partial_t + \partial_x),$$

which satisfies $g(k,k) = 0$ and gives $\omega_{\mathrm{em}} = -k \cdot u_{\mathrm{em}} = \omega_{\mathrm{em}}$.
Then the received frequency is

$$\begin{aligned} \omega_{\mathrm{rec}} &= -k \cdot u_{\mathrm{rec}} = -\omega_{\mathrm{em}}(\partial_t + \partial_x) \cdot \gamma(\partial_t + v\,\partial_x) \\ &= \omega_{\mathrm{em}}\gamma(1-v), \end{aligned}$$

since $g(\partial_t, \partial_t) = -1$, $g(\partial_x, \partial_x) = 1$, and cross terms vanish. Thus

$$\frac{\omega_{\mathrm{rec}}}{\omega_{\mathrm{em}}} = \gamma(1-v) = \sqrt{\frac{1-v}{1+v}}.$$

If the photon propagates in the $-x$-direction instead, one obtains $\omega_{\mathrm{rec}}/\omega_{\mathrm{em}} = \gamma(1+v) = \sqrt{\frac{1+v}{1-v}}$.

Example. A static observer family, acceleration, and the "gravitational field" preview. Let u be an observer field. Its **four-acceleration** is

$$a := \nabla_u u.$$

If $a = 0$, the observers move along timelike geodesics (freely falling). If $a \neq 0$, the observers must be supported by a force.
In the Rindler region of Minkowski spacetime with metric

$$g = -\rho^2\,d\eta^2 + d\rho^2 + dy^2 + dz^2,$$

consider the static observers at fixed (ρ, y, z), whose worldlines are η-lines. Their unit four-velocity is

$$u = \frac{1}{\rho}\,\partial_\eta,$$

since $g(\partial_\eta, \partial_\eta) = -\rho^2$.
Using the Christoffel symbols from Lecture 4,

$$\Gamma^1_{00} = \rho, \qquad \Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{\rho},$$

compute $a = \nabla_u u$ in coordinates. The only nonzero component comes from Γ^1_{00}:

$$a^\rho = u^\mu \partial_\mu u^\rho + \Gamma^\rho_{\mu\nu} u^\mu u^\nu = 0 + \Gamma^1_{00}\,(u^0)^2 = \rho\left(\frac{1}{\rho}\right)^2 = \frac{1}{\rho}.$$

Thus

$$a = \frac{1}{\rho}\,\partial_\rho, \qquad \sqrt{g(a,a)} = \frac{1}{\rho}.$$

Even though the spacetime is flat, the static observer family has nonzero proper acceleration. This is the clean geometric sense in which a non-inertial observer congruence experiences an effective "gravitational field". Later, in genuinely curved spacetimes, a for static observers will connect directly to gravitational redshift via $\omega = -k \cdot u$.

9 Symmetries and Conserved Quantities in Relativistic Geometry

9.1 Lie derivatives and the Killing equation

Let (M, g) be a Lorentzian manifold with Levi-Civita connection ∇.

Definition 9.1. Lie derivative (recall)

Let K be a vector field and T a tensor field. The **Lie derivative** $\mathcal{L}_K T$ is the derivative of T under the flow of K. In particular, the following hold.

(1) For a function f, $\mathcal{L}_K f = K(f)$.

(2) For a vector field X, $\mathcal{L}_K X = [K, X]$.

(3) For a 1-form α and vector field X,

$$(\mathcal{L}_K \alpha)(X) = K(\alpha(X)) - \alpha([K, X]).$$

It is uniquely determined on arbitrary tensors by the Leibniz rule and compatibility with contractions.

Definition 9.2. Killing vector field

A vector field K is called a **Killing vector field** if it generates isometries of g, i.e.

$$\mathcal{L}_K g = 0.$$

The metric-connection viewpoint gives a concrete PDE.

Proposition 9.3. Killing equation

A vector field K is Killing if and only if

$$\nabla_{(a} K_{b)} = 0,$$

equivalently,

$$\nabla_a K_b + \nabla_b K_a = 0.$$

Proof. Using metric compatibility $\nabla g = 0$ and torsion-freeness, one has for any vector fields X, Y,

$$\begin{aligned}(\mathcal{L}_K g)(X, Y) &= K(g(X, Y)) - g([K, X], Y) - g(X, [K, Y]) \\ &= g(\nabla_K X, Y) + g(X, \nabla_K Y) - g(\nabla_K X - \nabla_X K, Y) - g(X, \nabla_K Y - \nabla_Y K) \\ &= g(\nabla_X K, Y) + g(X, \nabla_Y K).\end{aligned}$$

Thus $\mathcal{L}_K g = 0$ is equivalent to

$$g(\nabla_X K, Y) + g(X, \nabla_Y K) = 0$$

for all X, Y, which in abstract indices is $\nabla_a K_b + \nabla_b K_a = 0$. $\square$

Note that the Killing equation is first-order and overdetermined. Its integrability conditions are governed by curvature, which is one reason symmetries strongly constrain geometry.

9.2 Core conservation law along geodesics

Theorem 9.4. A Killing field yields a conserved quantity along geodesics

Let K be a Killing vector field and let γ be an affinely parametrized geodesic with tangent $u = \dot{\gamma}$. Then

$$K \cdot u = g(K, u)$$

is constant along γ.

Proof. Define the function along γ,

$$F(\lambda) := g(K, u)\big|_{\gamma(\lambda)}.$$

Differentiate using the covariant derivative along γ and metric compatibility:

$$\frac{dF}{d\lambda} = \nabla_u\big(g(K, u)\big) = g(\nabla_u K, u) + g(K, \nabla_u u).$$

Since γ is an affinely parametrized geodesic, $\nabla_u u = 0$, so the second term vanishes:

$$\frac{dF}{d\lambda} = g(\nabla_u K, u).$$

Now apply the Killing equation in the form

$$\nabla_a K_b + \nabla_b K_a = 0.$$

Contract this with $u^a u^b$:

$$0 = u^a u^b(\nabla_a K_b + \nabla_b K_a) = u^a u^b \nabla_a K_b + u^a u^b \nabla_b K_a.$$

Rename dummy indices in the second term ($a \leftrightarrow b$) to see it equals the first term:

$$u^a u^b \nabla_b K_a = u^b u^a \nabla_a K_b = u^a u^b \nabla_a K_b.$$

Hence

$$0 = 2\, u^a u^b \nabla_a K_b, \qquad \text{so} \qquad u^a u^b \nabla_a K_b = 0.$$

But

$$g(\nabla_u K, u) = g_{bc}(\nabla_u K)^b u^c = (\nabla_u K)_c u^c = u^a \nabla_a K_c\, u^c = u^a u^c \nabla_a K_c,$$

which is exactly $u^a u^b \nabla_a K_b$ (rename c to b). Thus $g(\nabla_u K, u) = 0$, and hence

$$\frac{d}{d\lambda} g(K, u) = 0.$$

This proves $g(K, u)$ is constant along γ. $\square$

If γ is not affinely parametrized, the derivative contains an extra term $g(K, \nabla_u u)$. The affine geodesic equation is part of what makes the conservation law take the clean form above.

9.3 Two important families of Killing fields and their conserved quantities

First consider a Killing field K that is timelike in a region and whose flow corresponds to time translations there. For an affinely parametrized geodesic with tangent u (timelike for a massive particle or null for a photon), define

$$\mathcal{E} := -K \cdot u.$$

By Theorem 9.4, $\mathcal{E}$ is constant along γ. For a timelike particle with four-momentum $p = mu$, one often writes the conserved quantity as

$$\mathcal{E} = -K \cdot p,$$

and interprets it as energy associated with the symmetry K. For a null geodesic, one uses $p \propto k$ and obtains the conserved "frequency at infinity" type quantity once K is normalized.

Next assume the spacetime has rotational symmetry about some axis or, more generally, an $SO(3)$-action. Each infinitesimal rotation yields a Killing field. In axisymmetry, a distinguished one is the azimuthal Killing field Φ generating φ-translations in adapted coordinates. Define the conserved quantity along a geodesic:

$$\mathcal{L} := \Phi \cdot u = g(\Phi, u).$$

This is the component of angular momentum associated with the symmetry generated by Φ. In a spherically symmetric spacetime there are three independent rotational Killing fields, giving three conserved components which can be organized as an angular momentum vector in the standard way. In flat Minkowski spacetime, the ten Killing fields (four translations, three rotations, three boosts) recover conservation of energy-momentum and angular momentum about the origin. Boosts give conserved "center-of-mass" type quantities. We will not use the full Poincaré algebra, but the translation and rotation cases are the ones that repeatedly make geodesic motion and Einstein equations manageable.

9.4 From conserved quantities to first-order systems and a symmetry preview for Einstein's equation

The geodesic equation is second-order:

$$\ddot{x}^\rho + \Gamma^\rho_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0.$$

Killing fields provide first integrals. The standard reduction strategy is as follows.

(1) Choose coordinates adapted to the symmetry so that some coordinates are cyclic (do not appear in the metric components).

(2) For each Killing field K corresponding to a cyclic coordinate q, write the conserved quantity

$$C_K = K \cdot u = g_{\mu\nu} K^\mu \dot{x}^\nu.$$

In adapted coordinates, $K = \partial_q$, so

$$C_K = g_{q\nu} \dot{x}^\nu.$$

(3) Use these linear relations to solve for $\dot{q}$ (and possibly combinations of other $\dot{x}^\mu$).

(4) Use the normalization constraint

$$g(\dot{\gamma}, \dot{\gamma}) = \kappa, \qquad \kappa = \begin{cases} -1 & \text{timelike (proper time parameter)} \\ 0 & \text{null} \\ +1 & \text{spacelike} \end{cases}$$

to obtain a first-order equation for the remaining "radial" variable(s).

In practice, steps (2) and (4) often reduce the problem to a single effective one-dimensional equation that resembles an energy balance.

Example. Cyclic coordinate and a first integral. Let q be a coordinate such that $\partial_q g_{\mu\nu} = 0$. Then $K = \partial_q$ is Killing. Along a geodesic,

$$C = K \cdot u = g(\partial_q, \dot{\gamma}) = g_{q\nu} \dot{x}^\nu$$

is constant. If, additionally, the metric is block diagonal in the q direction, so $g_{q\nu} = 0$ for $\nu \neq q$, then the conserved quantity is simply

$$C = g_{qq}\dot{q}, \qquad \text{so} \qquad \dot{q} = \frac{C}{g_{qq}}.$$

Substituting into $g(\dot{\gamma}, \dot{\gamma}) = \kappa$ produces a first-order equation for the remaining coordinates.

The same mechanism that simplifies geodesic motion also simplifies Einstein's equation. Einstein's equation is a PDE for g. If a Lie group G acts by isometries, then g must be invariant under that action. In coordinates adapted to the symmetry, the number of independent metric components drops drastically, and many components become functions of fewer variables. This often turns the PDE system into a reduced system, sometimes ODEs (as in stationary, spherically symmetric vacuum leading to Schwarzschild), sometimes a simpler PDE system with fewer degrees of freedom.

For the next parts we will repeatedly use two symmetry-adapted ansätze. In a region where a timelike Killing field exists and the geometry is spherically symmetric, one looks for metrics of the form

$$g = -A(r)\,dt^2 + B(r)\,dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2),$$

where A, B are functions of a single variable r. The Killing field ∂_t leads to conserved energy along geodesics, and the rotational Killing fields lead to conserved angular momentum, so even test-particle motion is explicitly reducible.

For cosmology, spatial homogeneity and isotropy lead to the form

$$g = -dt^2 + a(t)^2\,\gamma_{ij}dx^i dx^j,$$

where γ_{ij} is a constant-curvature Riemannian metric on a 3-manifold (often $\mathbb{R}^3$, S^3, or H^3 up to quotients), and $a(t)$ is the scale factor. Here symmetry reduces Einstein's equation to ODEs for $a(t)$ (once matter content is specified), and redshift computations become clean because the observer field $u = \partial_t$ is built into the ansatz.

At the level of notation, the main preparations we need are:

(1) the Killing equation and conserved quantities along geodesics;
(2) the ability to compute with adapted coordinates and reduce second-order systems using first integrals;
(3) the observer formula for frequency $\omega = -k \cdot u$ from the previous lecture, now combined with time-translation symmetry to produce computable redshift factors.

These are exactly the tools we will use as soon as curvature and Einstein's equation enter.