# A MANIFOLD WITHOUT ANY SMOOTH STRUCTURE: KERVAIRE'S CONSTRUCTION

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Abstract. Smoothness is often treated as a benign extra structure on a topological manifold, but Kervaire's 10-dimensional example shows that it can fail in the strongest possible way: even within the triangulable category, there are manifolds that admit no differentiable structure at all. The mechanism behind this phenomenon is a subtle mod-2 obstruction that lives in the middle-dimensional cohomology of a 4-connected 10-manifold. We explain how this obstruction arises as an Arf invariant associated to a quadratic refinement of the cup-product pairing, and why framed-bordism considerations force it to vanish on all smooth manifolds in the relevant homotopy type. We then construct Kervaire's manifold $M_0$ by plumbing disk bundles and coning off a homotopy-sphere boundary, and compute that the obstruction is nonzero by relating the resulting Thom space to the 10-skeleton of $\Omega S^6$. The exposition is written with the background of a first course in algebraic topology in mind; the required inputs from Steenrod operations, Thom theory, and the Pontryagin–Thom construction are isolated in appendices.

Contents

## 1. Introduction

A classical theme in topology is that the same underlying space can carry distinct geometric structures. For manifolds this leads to the three categories

$$\text{smooth} \subset \text{PL} \subset \text{topological},$$

and PL manifolds are triangulable (in the PL sense).

Kervaire exhibited one of the most striking separations between these categories: a closed triangulable manifold which admits *no* differentiable structure at all. The purpose of this paper is to explain, in detail and with minimal prerequisites, the mechanism behind Kervaire's example in dimension 10.

Throughout, cohomology is singular cohomology. Unless explicitly stated otherwise, coefficients are in $\mathbb{Z}$ or $\mathbb{Z}_2$, and cup products are denoted by $\smile$. All manifolds are assumed connected and closed. When working with Poincaré duality, we assume an orientation has been fixed.

### 1.1. Main statements.

The construction takes place in dimension 10 under a strong connectivity assumption.

**Definition 1.1.** A space $X$ is 4-*connected* if it is path-connected and $\pi_i(X) = 0$ for all $1 \leq i \leq 4$. A closed 10-manifold $M$ is 4-*connected* if its underlying space is 4-connected.

The key input is a $\mathbb{Z}_2$-valued invariant $r(M)$, defined for oriented, 4-connected, closed 10-dimensional triangulable manifolds. Its two fundamental properties are:

**Theorem 1.2** (Kervaire: smoothness forces vanishing)**.** *If $M$ is homotopy equivalent to a closed smooth 4-connected 10-manifold, then $r(M) = 0$.*

**Theorem 1.3** (Kervaire: the example)**.** *There exists an oriented, closed, $4$-connected triangulable $10$-manifold $M_0$ such that $r(M_0) = 1$. Consequently $M_0$ is not homotopy equivalent to any smooth manifold; in particular, $M_0$ admits no differentiable structure.*

As a byproduct, the construction also yields a homotopy 9-sphere which is homeomorphic but not diffeomorphic to $S^9$.

1.2. **Organization.** Section 2 develops the algebraic topology of a 4-connected closed 10-manifold: we isolate the middle-dimensional pairing on $H^5(M;\mathbb{Z}_2)$ and record the existence of symplectic bases. Section 3 introduces a quadratic refinement $\varphi\colon H^5(M;\mathbb{Z}_2) \to \mathbb{Z}_2$ built from the loop space $\Omega S^6$, and defines the invariant $r(M)$ as the associated Arf invariant. Section 4 relates $\varphi$ to a framed-bordism invariant for smooth manifolds and proves Theorem 1.2. Section 5 constructs the manifold $M_0$ and computes $r(M_0) = 1$, proving Theorem 1.3.

Three appendices collect background on Steenrod squares, Thom classes, and Wu classes (Appendix A), on framings and the Pontryagin–Thom construction (Appendix B), and on the loop-space input and Thom-space computation used in Section 5.

## 2. The middle-dimensional pairing on a 4-connected 10-manifold

Let $M$ be an oriented, closed, 4-connected 10-manifold. In this section we record the basic algebraic topology of $M$ that will be used throughout: vanishing of low- and high-degree (co)homology, the structure of $H^5(M;\mathbb{Z})$ and $H^5(M;\mathbb{Z}_2)$, and the symplectic pairing on $H^5(M;\mathbb{Z}_2)$.

### 2.1. Homology and cohomology in low and high degrees.

**Lemma 2.1.** *Let $M$ be a closed $4$-connected $10$-manifold. Then $H_i(M;\mathbb{Z}) = 0$ for $1 \le i \le 4$.*

*Proof.* Since $M$ is 4-connected, it is path-connected and $\pi_i(M) = 0$ for $1 \le i \le 4$. By the Hurewicz theorem for simply-connected spaces, the Hurewicz map $\pi_i(M) \to H_i(M;\mathbb{Z})$ is an isomorphism for $i < 5$, so $H_i(M;\mathbb{Z}) = 0$ for $1 \le i \le 4$. □

**Proposition 2.2.** *Let $M$ be an oriented, closed, $4$-connected $10$-manifold. Then:*

*(i)* $H^i(M;\mathbb{Z}) = 0$ *for* $1 \le i \le 4$ *and* $6 \le i \le 9$*;*
*(ii)* $H^5(M;\mathbb{Z})$ *is a free abelian group;*
*(iii) the mod-$2$ reduction map*

$$\rho\colon H^5(M;\mathbb{Z}) \longrightarrow H^5(M;\mathbb{Z}_2)$$

*is surjective (equivalently, an isomorphism onto $H^5(M;\mathbb{Z}) \otimes \mathbb{Z}_2$).*

*Proof.* (i) By Lemma 2.1, $H_i(M;\mathbb{Z}) = 0$ for $1 \le i \le 4$. The universal coefficient theorem for cohomology gives

$$H^i(M;\mathbb{Z}) \cong \mathrm{Hom}(H_i(M;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}(H_{i-1}(M;\mathbb{Z}),\mathbb{Z}),$$

so $H^i(M;\mathbb{Z}) = 0$ for $1 \le i \le 4$ as well. Poincaré duality over $\mathbb{Z}$ for the oriented closed 10-manifold $M$ gives $H^i(M;\mathbb{Z}) \cong H_{10-i}(M;\mathbb{Z})$, and since $H_j(M;\mathbb{Z}) = 0$ for $1 \le j \le 4$, it follows that $H^i(M;\mathbb{Z}) = 0$ for $6 \le i \le 9$.

(ii) The universal coefficient theorem gives

$$H^5(M;\mathbb{Z}) \cong \mathrm{Hom}(H_5(M;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}(H_4(M;\mathbb{Z}),\mathbb{Z}).$$

Since $H_4(M;\mathbb{Z}) = 0$ by Lemma 2.1, the Ext-term vanishes, and $H^5(M;\mathbb{Z})$ is isomorphic to a direct sum of copies of $\mathbb{Z}$, hence free abelian.

(iii) For mod-2 coefficients, the universal coefficient theorem yields a short exact sequence

$$0 \longrightarrow H^5(M;\mathbb{Z}) \otimes \mathbb{Z}_2 \xrightarrow{\rho} H^5(M;\mathbb{Z}_2) \longrightarrow \mathrm{Tor}(H^6(M;\mathbb{Z}), \mathbb{Z}_2) \longrightarrow 0.$$

But $H^6(M;\mathbb{Z}) = 0$ by (i), so the right-hand Tor-term vanishes and $\rho$ is surjective. $\square$

2.2. **The middle-dimensional pairing.** Let $[M] \in H_{10}(M;\mathbb{Z})$ denote the fundamental class determined by the chosen orientation. For $z \in H^{10}(M;\mathbb{Z}_2)$ we write $\langle z, [M]\rangle \in \mathbb{Z}_2$ for evaluation (reducing mod 2 when necessary).

**Definition 2.3.** Define a bilinear form

$$\lambda\colon H^5(M;\mathbb{Z}_2) \times H^5(M;\mathbb{Z}_2) \to \mathbb{Z}_2, \qquad \lambda(x, y) = \langle x \smile y, [M]\rangle.$$

**Proposition 2.4.** *The pairing $\lambda$ is nondegenerate. Moreover, $\lambda(x, x) = 0$ for all $x \in H^5(M;\mathbb{Z}_2)$; in particular, $\lambda$ is an alternating (symplectic) form on the $\mathbb{Z}_2$-vector space $H^5(M;\mathbb{Z}_2)$.*

*Proof.* Nondegeneracy follows from Poincaré duality with $\mathbb{Z}_2$-coefficients: cap product with $[M]$ gives an isomorphism $H^5(M;\mathbb{Z}_2) \cong H_5(M;\mathbb{Z}_2)$, and the evaluation pairing

$$H^5(M;\mathbb{Z}_2) \times H_5(M;\mathbb{Z}_2) \to \mathbb{Z}_2$$

is perfect.

For the alternating property, let $x \in H^5(M;\mathbb{Z}_2)$. By Proposition 2.2, $x$ lifts to an integral class $X \in H^5(M;\mathbb{Z})$ with $\rho(X) = x$. By graded-commutativity in integral cohomology, $X \smile X = (-1)^{5\cdot 5} X \smile X = -X \smile X$, hence $2(X \smile X) = 0$ in $H^{10}(M;\mathbb{Z}) \cong \mathbb{Z}$. Since $H^{10}(M;\mathbb{Z})$ is torsion-free, this forces $X \smile X = 0$, and therefore

$$x \smile x = \rho(X \smile X) = 0 \in H^{10}(M;\mathbb{Z}_2).$$

Thus $\lambda(x, x) = \langle x \smile x, [M]\rangle = 0$. $\square$

2.3. **Symplectic bases.**

**Definition 2.5.** Let $V$ be a finite-dimensional $\mathbb{Z}_2$-vector space equipped with a nondegenerate alternating bilinear form $\lambda$. A *symplectic basis* is a basis $\{x_1, \dots, x_g, y_1, \dots, y_g\}$ such that

$$\lambda(x_i, x_j) = 0, \qquad \lambda(y_i, y_j) = 0, \qquad \lambda(x_i, y_j) = \delta_{ij}.$$

**Lemma 2.6.** *Let $(V, \lambda)$ be a finite-dimensional $\mathbb{Z}_2$-vector space with a nondegenerate alternating bilinear form. Then $\dim_{\mathbb{Z}_2} V$ is even, say $\dim V = 2g$, and $V$ admits a symplectic basis.*

*Proof.* We argue by induction on $\dim V$. If $V = 0$ there is nothing to prove. Assume $V \neq 0$ and choose $0 \neq x \in V$. Nondegeneracy implies there exists $y \in V$ with $\lambda(x, y) = 1$. Let $U = \mathrm{span}\{x, y\}$. Since $\lambda$ is alternating, $\lambda|_U$ has matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the basis $(x, y)$, so $\lambda|_U$ is nondegenerate.

Let $U^\perp = \{v \in V : \lambda(v, u) = 0 \text{ for all } u \in U\}$. Then $V = U \oplus U^\perp$, and $\lambda|_{U^\perp}$ is again nondegenerate and alternating. By induction, $\dim U^\perp$ is even and $U^\perp$ admits a symplectic basis. Adjoining $x, y$ gives a symplectic basis of $V$. $\square$

**Corollary 2.7.** *Let $M$ be an oriented, closed, 4-connected 10-manifold. Then $H^5(M;\mathbb{Z}_2)$ has even dimension $2g$ and admits a symplectic basis with respect to $\lambda$ from Definition 2.3.*

*Remark* 2.8. In what follows we fix $g = \frac{1}{2}\dim_{\mathbb{Z}_2} H^5(M;\mathbb{Z}_2)$. Whenever we choose a symplectic basis, it will be denoted $\{x_1,\dots,x_g,y_1,\dots,y_g\}$ as in Definition 2.5.

## 3. A quadratic refinement and the invariant $r(M)$

Let $M$ be an oriented, closed, 4-connected 10-manifold. By Corollary 2.7, the $\mathbb{Z}_2$-vector space $H^5(M;\mathbb{Z}_2)$ carries a nondegenerate alternating pairing $\lambda(x,y) = \langle x \smile y, [M]\rangle$, and admits symplectic bases.

In this section we introduce a function

$$\varphi\colon H^5(M;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$$

which is a *quadratic refinement* of $\lambda$, and then extract from it a single number $r(M) \in \mathbb{Z}_2$. The definition of $\varphi$ uses a specific input from the loop space $\Omega S^6$. We will isolate the precise properties we need as a "black box" and defer its justification to Appendix C.

### 3.1. The $\Omega S^6$ Data.

Let $\Omega = \Omega S^6$ be the based loop space of $S^6$, equipped with the usual loop multiplication

$$\mu\colon \Omega\times\Omega \longrightarrow \Omega.$$

Write $\rho$ for reduction mod 2.

**Lemma 3.1.** *Let $M$ be a closed, oriented, 4-connected* triangulable *10-manifold. Then $M$ is homotopy equivalent to a CW complex with cells only in dimensions 0, 5, and 10. More precisely, $M$ is homotopy equivalent to a CW complex with one 0-cell, $b_5(M)$ many 5-cells, and one 10-cell.*

*Proof.* Since $M$ is triangulable, it has the homotopy type of a finite CW complex, and we may work in the PL category. In particular, the usual handlebody and cancellation statement in the $(m-1)$-connected $2m$-dimensional range applies, yielding a CW model with only 0-, $m$-, and $2m$-cells. We use a fact in this connectivity range.

**Standard fact.** Let $M$ be a closed, oriented, $(m-1)$-connected $2m$-manifold with $m \geq 3$. Then $M$ admits a handle decomposition with exactly one 0-handle, no handles of index $1,\dots,m-1$, and no handles of index $m+1,\dots,2m-1$. Equivalently, $M$ has the homotopy type of a CW complex with one 0-cell, some $m$-cells, and one $2m$-cell. Moreover, the number of $m$-cells can be taken to be $\operatorname{rank} H_m(M;\mathbb{Z})$.

Applying this with $m = 5$, we obtain a CW model for $M$ with cells only in dimensions $0,5,10$. The cellular chain complex then shows that the number of 5-cells equals $\operatorname{rank} H_5(M;\mathbb{Z}) = b_5(M)$, and since $H_{10}(M;\mathbb{Z}) \cong \mathbb{Z}$ we may take a single 10-cell. ∎

*Remark* 3.2. The only nontrivial reduced homology group is $\widetilde{H}_5(M;\mathbb{Z})$, so the number of 5-cells can be chosen to match the rank of $H_5(M;\mathbb{Z})$.

**Proposition 3.3** (Loop-space stucture)**.** *There exist cohomology classes*

$$e \in H^5(\Omega;\mathbb{Z}), \qquad u \in H^{10}(\Omega;\mathbb{Z}_2)$$

*with the following properties. Let $\bar{e} = \rho(e) \in H^5(\Omega;\mathbb{Z}_2)$.*

(i) $H^5(\Omega;\mathbb{Z}) \cong \mathbb{Z}$, *generated by* $e$.
(ii) *(Realization) If* $X$ *is a CW complex of dimension* $\leq 10$ *which admits a CW model with one* 0*-cell, some* 5*-cells, and one* 10*-cell (and no other cells), then for every class* $X_0 \in H^5(X;\mathbb{Z})$ *there exists a map* $f\colon X \to \Omega$ *with* $f^*(e) = X_0$.
(iii) *(Additivity of* $e$*) In* $H^5(\Omega \times \Omega;\mathbb{Z})$ *one has*

$$\mu^*(e) = \mathrm{pr}_1^*(e) + \mathrm{pr}_2^*(e),$$

*and hence in* $H^5(\Omega \times \Omega;\mathbb{Z}_2)$ *one has* $\mu^*(\bar{e}) = \mathrm{pr}_1^*(\bar{e}) + \mathrm{pr}_2^*(\bar{e})$.
(iv) *(Key quadratic identity) In* $H^{10}(\Omega \times \Omega;\mathbb{Z}_2)$ *one has*

$$\mu^*(u) = \mathrm{pr}_1^*(u) + \mathrm{pr}_2^*(u) + \mathrm{pr}_1^*(\bar{e}) \smile \mathrm{pr}_2^*(\bar{e}). \tag{3.4}$$

(v) *(Well-defined evaluation) If* $M$ *is an oriented, closed* 10*-manifold which admits a CW model with one* 0*-cell, some* 5*-cells, and one* 10*-cell (and no other cells), and* $f_0, f_1\colon M \to \Omega$ *satisfy* $f_0^*(e) = f_1^*(e) \in H^5(M;\mathbb{Z})$, *then*

$$\langle f_0^*(u), [M] \rangle = \langle f_1^*(u), [M] \rangle \in \mathbb{Z}_2.$$

*Remark* 3.5. Proposition 3.3 is the only place in this section where loop-space technology enters. All subsequent arguments are algebraic once the package is granted.

### 3.2. Definition of $\varphi$ and the quadratic identity.

**Definition 3.6.** Let $X \in H^5(M;\mathbb{Z})$. Choose a map $f\colon M \to \Omega$ with $f^*(e) = X$ (which exists by Proposition 3.3(ii)). Define

$$\varphi_0(X) = \langle f^*(u), [M] \rangle \in \mathbb{Z}_2.$$

**Lemma 3.7.** *The value* $\varphi_0(X)$ *is well-defined, i.e. independent of the choice of* $f$ *with* $f^*(e) = X$.

*Proof.* This is exactly Proposition 3.3(v). □

**Lemma 3.8.** *For all* $X, Y \in H^5(M;\mathbb{Z})$ *one has*

$$\varphi_0(X+Y) = \varphi_0(X) + \varphi_0(Y) + \langle \rho(X) \smile \rho(Y), [M] \rangle. \tag{3.9}$$

*Proof.* Choose maps $f, g\colon M \to \Omega$ with $f^*(e) = X$ and $g^*(e) = Y$. Consider the map $\mu \circ (f,g)\colon M \to \Omega$. By Proposition 3.3(iii),

$$(\mu \circ (f,g))^*(e) = (f,g)^*\mu^*(e) = (f,g)^*(\mathrm{pr}_1^* e + \mathrm{pr}_2^* e) = X + Y.$$

By Lemma 3.7,

$$\varphi_0(X+Y) = \langle (\mu \circ (f,g))^*(u), [M] \rangle = \langle (f,g)^*\mu^*(u), [M] \rangle.$$

Using (3.4) and bilinearity of evaluation, we obtain

$$\varphi_0(X+Y) = \langle f^*(u), [M] \rangle + \langle g^*(u), [M] \rangle + \langle f^*(\bar{e}) \smile g^*(\bar{e}), [M] \rangle.$$

Since $f^*(\bar{e}) = \rho(f^*e) = \rho(X)$ and similarly $g^*(\bar{e}) = \rho(Y)$, this is exactly (3.9). □

**Lemma 3.10.** *For any* $X \in H^5(M;\mathbb{Z})$ *and any* $Y \in H^5(M;\mathbb{Z})$ *one has* $\varphi_0(X + 2Y) = \varphi_0(X)$.

*Proof.* Apply Lemma 3.8 twice:

$\varphi_0(X+2Y) = \varphi_0((X+Y)+Y) = \varphi_0(X+Y)+\varphi_0(Y)+\big\langle \rho(X+Y)\smile\rho(Y),[M]\big\rangle.$
Now $\rho(X+Y)=\rho(X)+\rho(Y)$, hence the last term equals

$$\big\langle \rho(X)\smile\rho(Y),[M]\big\rangle + \big\langle \rho(Y)\smile\rho(Y),[M]\big\rangle.$$

But $\rho(Y)\smile\rho(Y)=0$ by Proposition 2.4. Also, $\varphi_0(X+Y)=\varphi_0(X)+\varphi_0(Y)+\langle\rho(X)\smile\rho(Y),[M]\rangle$ by Lemma 3.8. Substituting and canceling in $\mathbb{Z}_2$ gives $\varphi_0(X+2Y)=\varphi_0(X)$. □

**Definition 3.11.** Let $x\in H^5(M;\mathbb{Z}_2)$. Choose an integral lift $X\in H^5(M;\mathbb{Z})$ with $\rho(X)=x$ (which exists by Proposition 2.2). Define

$$\varphi(x)=\varphi_0(X)\in\mathbb{Z}_2.$$

**Lemma 3.12.** *The value $\varphi(x)$ is well-defined, i.e. independent of the choice of integral lift $X$ with $\rho(X)=x$.*

*Proof.* If $X'$ is another lift of $x$, then $X'=X+2Y$ for some $Y\in H^5(M;\mathbb{Z})$ because $H^5(M;\mathbb{Z})$ is free abelian (Proposition 2.2). Now apply Lemma 3.10: $\varphi_0(X')=\varphi_0(X+2Y)=\varphi_0(X)$. □

**Proposition 3.13.** *For all $x,y\in H^5(M;\mathbb{Z}_2)$ one has the quadratic identity*

$$\varphi(x+y)=\varphi(x)+\varphi(y)+\lambda(x,y), \tag{3.14}$$

*where $\lambda(x,y)=\langle x\smile y,[M]\rangle$ is the pairing from Definition 2.3.*

*Proof.* Choose integral lifts $X,Y\in H^5(M;\mathbb{Z})$ of $x,y$. Then $X+Y$ is an integral lift of $x+y$, so by definition, $\varphi(x+y)=\varphi_0(X+Y)$, $\varphi(x)=\varphi_0(X)$, $\varphi(y)=\varphi_0(Y)$. Lemma 3.8 gives

$$\varphi_0(X+Y)=\varphi_0(X)+\varphi_0(Y)+\big\langle\rho(X)\smile\rho(Y),[M]\big\rangle,$$

and $\rho(X)=x$, $\rho(Y)=y$, so this is exactly (3.14). □

*Remark* 3.15. Equation (3.14) says precisely that $\varphi$ is a quadratic refinement of the symplectic form $\lambda$ on $H^5(M;\mathbb{Z}_2)$.

3.3. **The Arf invariant $r(M)$.** Fix $g=\frac{1}{2}\dim_{\mathbb{Z}_2}H^5(M;\mathbb{Z}_2)$ as in Remark 2.8. Let $\{x_1,\dots,x_g,y_1,\dots,y_g\}$ be any symplectic basis for $\lambda$ (Corollary 2.7).

**Definition 3.16.** Define

$$r(M)=\sum_{i=1}^{g}\varphi(x_i)\,\varphi(y_i)\in\mathbb{Z}_2.$$

The content is that this number does not depend on the chosen symplectic basis. This is a standard fact in the theory of quadratic forms over $\mathbb{Z}_2$; we include a quick verification using elementary symplectic changes of basis.

**Lemma 3.17.** *Let $(V,\lambda)$ be a finite-dimensional $\mathbb{Z}_2$-vector space with a nondegenerate alternating bilinear form, and let $q\colon V\to\mathbb{Z}_2$ satisfy $q(v+w)=q(v)+q(w)+\lambda(v,w)$ for all $v,w\in V$. For a symplectic basis $\{x_1,\dots,x_g,y_1,\dots,y_g\}$ define*

$$\mathrm{Arf}(q;\{x_i,y_i\})=\sum_{i=1}^{g}q(x_i)\,q(y_i)\in\mathbb{Z}_2.$$

*Then $\mathrm{Arf}(q;\{x_i,y_i\})$ is independent of the symplectic basis.*

*Proof.* It is standard that the symplectic group $\mathrm{Sp}(V,\lambda)$ is generated by elementary transformations which, on a symplectic basis, are compositions of:

(1) permuting the $g$ pairs $(x_i, y_i)$;
(2) swapping $x_i$ and $y_i$ within a fixed pair;
(3) replacing $y_i$ by $y_i + x_i$ (or replacing $x_i$ by $x_i + y_i$);
(4) for $i \neq j$, replacing $x_i$ by $x_i + x_j$ and simultaneously replacing $y_j$ by $y_j + y_i$.

Thus it suffices to check that $\sum_i q(x_i)q(y_i)$ is invariant under each move.

Moves (1) and (2) clearly do not change the sum.

For move (3), only the $i$-th summand changes:

$$q(x_i)q(y_i + x_i) = q(x_i)\big(q(y_i) + q(x_i) + \lambda(y_i, x_i)\big) = q(x_i)q(y_i) + q(x_i)\big(q(x_i) + 1\big).$$

Since $q(x_i) \in \mathbb{Z}_2$, we have $q(x_i)\big(q(x_i) + 1\big) = 0$, so the sum is unchanged.

For move (4), only the $i$-th and $j$-th summands may change. Using $\lambda(x_i, x_j) = \lambda(y_i, y_j) = 0$ for a symplectic basis and the quadratic identity for $q$, we get

$$q(x_i + x_j) = q(x_i) + q(x_j), \qquad q(y_j + y_i) = q(y_j) + q(y_i).$$

Hence the new contribution of indices $i, j$ is

$$\begin{aligned} q(x_i + x_j)q(y_i) + q(x_j)q(y_j + y_i) &= (q(x_i) + q(x_j))q(y_i) + q(x_j)(q(y_j) + q(y_i)) \\ &= q(x_i)q(y_i) + q(x_j)q(y_j), \end{aligned}$$

so again the total sum is unchanged. This proves invariance under the generating moves, hence basis-independence. □

**Corollary 3.18.** *The invariant $r(M)$ from Definition 3.16 is well-defined, i.e. independent of the choice of symplectic basis of $H^5(M;\mathbb{Z}_2)$.*

*Proof.* Apply Lemma 3.17 to $V = H^5(M;\mathbb{Z}_2)$, $\lambda$ as in Definition 2.3, and $q = \varphi$ as in Proposition 3.13. □

3.4. **Homotopy invariance.**

**Proposition 3.19.** *Let $h\colon N \to M$ be a homotopy equivalence between oriented, closed, 4-connected 10-manifolds. Then*

$$r(N) = r(M).$$

*Proof.* The pairing $\lambda$ is natural under pullback:

$$\lambda_N(h^*x, h^*y) = \big\langle h^*(x \smile y), [N]\big\rangle = \big\langle x \smile y, h_*[N]\big\rangle = \lambda_M(x, y),$$

where $h_*[N] = [M]$ holds in $H_{10}(-;\mathbb{Z}_2)$ for any homotopy equivalence (sign issues are irrelevant mod 2). Thus $h^*\colon H^5(M;\mathbb{Z}_2) \to H^5(N;\mathbb{Z}_2)$ is a symplectic isomorphism.

Next, the definition of $\varphi$ is natural with respect to pullback, because it is defined by pulling back the fixed universal classes $e, u$ on $\Omega$ and evaluating on the fundamental class. Concretely, for $x \in H^5(M;\mathbb{Z}_2)$, choose an integral lift $X \in H^5(M;\mathbb{Z})$ and a map $f\colon M \to \Omega$ with $f^*(e) = X$. Then $f \circ h\colon N \to \Omega$ satisfies $(f \circ h)^*(e) = h^*X$, hence

$$\varphi_N(h^*x) = \varphi_{0,N}(h^*X) = \big\langle (f\circ h)^*(u), [N]\big\rangle = \big\langle h^*f^*(u), [N]\big\rangle = \big\langle f^*(u), h_*[N]\big\rangle = \big\langle f^*(u), [M]\big\rangle = \varphi_M(x).$$

Therefore $h^*$ identifies the quadratic refinements $\varphi_M$ and $\varphi_N$.

Now let $\{x_i, y_i\}$ be a symplectic basis of $H^5(M;\mathbb{Z}_2)$. Then $\{h^*x_i, h^*y_i\}$ is a symplectic basis of $H^5(N;\mathbb{Z}_2)$, and we compute

$$r(N) = \sum_i \varphi_N(h^*x_i)\,\varphi_N(h^*y_i) = \sum_i \varphi_M(x_i)\,\varphi_M(y_i) = r(M).$$

□

*Remark* 3.20. To summarize: attached to $M$ we have the triple $\big(H^5(M;\mathbb{Z}_2), \lambda, \varphi\big)$, where $\lambda$ is symplectic and $\varphi$ is a quadratic refinement. The invariant $r(M)$ is the associated Arf invariant of this triple.

## 4. Why smoothness forces $r(M) = 0$

In this section we prove Theorem 1.2. The argument has two conceptual steps:

(1) For a smooth 4-connected 10-manifold $N$, one can associate a *stable normal framing* (equivalently, $N$ is a $\pi$-manifold). This produces a framed bordism class $[N] \in \Omega_{10}^{\mathrm{fr}}$.
(2) The function $\varphi$ from Section 3 coincides, in the smooth framed setting, with the classical quadratic refinement arising from the framing. The resulting Arf invariant in dimension 10 is known to be identically zero on framed bordism classes.

We keep the discussion self-contained by isolating the required input as explicit black-box statements and deferring their proofs to the appendices. Appendix A collects the Steenrod/Thom/Wu identities used to define the framed quadratic refinement, and Appendix B collects the Pontryagin–Thom and stable homotopy input.

### 4.1. Stable framings and framed bordism.

We begin by recalling the notion of a stable normal framing. (Warning: the notation $\Omega_n^{\mathrm{fr}}$ for framed bordism should not be confused with the loop space $\Omega = \Omega S^6$ used earlier.)

**Definition 4.1.** Let $N^{10}$ be a smooth closed manifold. Choose a smooth embedding $i\colon N \hookrightarrow \mathbb{R}^{10+k}$ for some $k$, with normal bundle $\nu_i \to N$. A *stable normal framing* of $N$ is a trivialization

$$\nu_i \oplus \varepsilon^\ell \;\cong\; \varepsilon^{k+\ell}$$

for some $\ell \geq 0$, where $\varepsilon^m$ denotes the trivial rank-$m$ bundle. Two such framings are equivalent if they become homotopic after adding trivial summands. A *stably framed manifold* is a pair $(N, \mathfrak{f})$ of a smooth closed manifold $N$ together with an equivalence class of stable normal framings $\mathfrak{f}$.

**Definition 4.2.** Two stably framed 10-manifolds $(N_0, \mathfrak{f}_0)$ and $(N_1, \mathfrak{f}_1)$ are *framed bordant* if there exists a compact smooth 11-manifold $W$ with $\partial W \cong N_0 \sqcup (-N_1)$ together with a stable normal framing of $W$ restricting to $\mathfrak{f}_0$ and $\mathfrak{f}_1$ on the boundary. The set of framed bordism classes in dimension 10, with addition given by disjoint union, is an abelian group denoted $\Omega_{10}^{\mathrm{fr}}$.

The basic structural result is the Pontryagin–Thom correspondence.

**Theorem 4.3** (Pontryagin–Thom)**.** *There is a natural isomorphism*

$$\Omega_n^{\mathrm{fr}} \;\cong\; \pi_n^S$$

*from framed bordism in dimension $n$ to the $n$-th stable homotopy group of spheres.*

*Remark* 4.4. We will not reprove Theorem 4.3 here. A streamlined account is given in Appendix B, sufficient for the present paper.

4.2. **A framing-induced quadratic refinement.** Let $(N^{10}, \mathfrak{f})$ be a stably framed smooth 10-manifold. The framing $\mathfrak{f}$ allows one to define a quadratic refinement

$$q_{\mathfrak{f}} \colon H^5(N; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$$

of the middle-dimensional pairing $\lambda(x, y) = \langle x \smile y, [N] \rangle$. There are several equivalent constructions; the one we will use in Appendix A is formulated in terms of the Thom isomorphism and Steenrod squares on Thom spaces.

For the main argument, we only need the following package.

**Proposition 4.5** (Framed quadratic refinement)**.** *Let $(N^{10}, \mathfrak{f})$ be a stably framed smooth 10-manifold. There exists a function $q_{\mathfrak{f}} \colon H^5(N; \mathbb{Z}_2) \to \mathbb{Z}_2$ such that:*

*(i) $q_{\mathfrak{f}}$ is a quadratic refinement of $\lambda$, i.e.*

$$q_{\mathfrak{f}}(x + y) = q_{\mathfrak{f}}(x) + q_{\mathfrak{f}}(y) + \langle x \smile y, [N] \rangle \quad \text{for all } x, y \in H^5(N; \mathbb{Z}_2).$$

*(ii) If $\{x_1, \dots, x_g, y_1, \dots, y_g\}$ is any symplectic basis for $\lambda$, then*

$$\kappa(N, \mathfrak{f}) = \sum_{i=1}^{g} q_{\mathfrak{f}}(x_i)\, q_{\mathfrak{f}}(y_i) \in \mathbb{Z}_2$$

*is independent of the choice of symplectic basis.*

*(iii) The number $\kappa(N, \mathfrak{f})$ depends only on the framed bordism class $[N, \mathfrak{f}] \in \Omega_{10}^{\mathrm{fr}}$, and defines a group homomorphism $\kappa \colon \Omega_{10}^{\mathrm{fr}} \to \mathbb{Z}_2$.*

*Remark* 4.6. Part (i) is proved from the Cartan formula together with the Thom identity $Sq(U) = w(\nu) \smile U$ and the fact that a framing forces $w(\nu) = 1$. The basis-independence in (ii) is the abstract Lemma 3.17. The bordism invariance in (iii) is shown by interpreting $q_{\mathfrak{f}}$ as a cohomological invariant of the Pontryagin–Thom collapse map. All of these details are given in Appendix A and Appendix B.

The point of Proposition 4.5 is that it produces, from a stable framing, an Arf invariant $\kappa(N, \mathfrak{f})$ in dimension 10.

4.3. **Comparison with $\varphi$ and $r(M)$.** Now assume in addition that $N$ is oriented and 4-connected. Then $r(N)$ has already been defined in Section 3 using the loop-space package, without reference to any framing.

The crucial bridge is that, for smooth 4-connected 10-manifolds, the loop-space refinement $\varphi$ agrees with the framing-induced refinement $q_{\mathfrak{f}}$.

**Proposition 4.7** (Identification of refinements)**.** *Let $N$ be an oriented, smooth, 4-connected closed 10-manifold, and let $\mathfrak{f}$ be a stable normal framing of $N$. Then*

$$\varphi = q_{\mathfrak{f}} \colon H^5(N; \mathbb{Z}_2) \to \mathbb{Z}_2.$$

*In particular,*

$$r(N) = \kappa(N, \mathfrak{f}) \in \mathbb{Z}_2.$$

*Remark* 4.8. The proof of Proposition 4.7 is deferred to Appendix C, where the class $u \in H^{10}(\Omega S^6; \mathbb{Z}_2)$ is related to the secondary operation defining $q_{\mathfrak{f}}$ via Pontryagin–Thom. For the main logical flow, we treat this comparison as part of the black-box input.

4.4. **Vanishing in dimension 10.** The final input is a purely stable homotopy fact: the Arf/Kervaire-type homomorphism $\kappa\colon \Omega_{10}^{\mathrm{fr}} \to \mathbb{Z}_2$ is trivial.

**Theorem 4.9** (Vanishing of $\kappa$ in dimension 10)**.** *The homomorphism $\kappa\colon \Omega_{10}^{\mathrm{fr}} \to \mathbb{Z}_2$ is zero. Equivalently, $\kappa(N, \mathfrak{f}) = 0$ for every stably framed smooth 10-manifold.*

*Remark* 4.10. Theorem 4.9 is a consequence of computations in stable homotopy (the group $\pi_{10}^S$ and the absence of a Kervaire-invariant-one element in dimension 10). We record an appropriate reference path in Appendix B.

4.5. **Smooth 4-connected 10-manifolds admit stable normal framings.** To apply the preceding discussion to smooth 4-connected 10-manifolds, we need that such manifolds are stably framed.

**Proposition 4.11** (Smooth 4-connected 10-manifolds are $\pi$-manifolds)**.** *Let $N^{10}$ be an oriented, smooth, 4-connected closed manifold. Then $N$ admits a stable normal framing $\mathfrak{f}$.*

*Remark* 4.12. This is proved by obstruction theory for stable oriented bundles on a CW model for $N$, together with the relevant low-dimensional homotopy groups of $SO$ (via Bott periodicity). A complete proof, in the form needed here, is given in Appendix B.

4.6. **Proof of Theorem 1.2.**

*Proof of Theorem 1.2.* Let $M$ be an oriented, closed, 4-connected triangulable 10-manifold which is homotopy equivalent to a closed smooth 4-connected 10-manifold $N$. By Proposition 3.19, the invariant $r$ is a homotopy invariant, hence

$$r(M) = r(N).$$

By Proposition 4.11, $N$ admits a stable normal framing $\mathfrak{f}$. Proposition 4.7 identifies $r(N)$ with $\kappa(N, \mathfrak{f})$, and Theorem 4.9 gives $\kappa(N, \mathfrak{f}) = 0$. Therefore $r(N) = 0$, and consequently $r(M) = 0$. ☐

*Remark* 4.13. At this stage, the remaining task for the main theorem of the paper is to construct an oriented, 4-connected triangulable 10-manifold $M_0$ with $r(M_0) = 1$. This will be carried out in Section 5 by a geometric construction starting from Milnor's homotopy 9-spheres and an explicit computation of $\varphi$ using the Steenrod/Thom/Wu identities from Appendix A.

## 5. The example $M_0$ and the computation $r(M_0) = 1$

In this section we construct an oriented, closed, 4-connected triangulable 10-manifold $M_0$ and compute $r(M_0)$. The construction is the one given by Kervaire: it starts from a plumbing $W^{10}$ built from a tubular neighborhood of the diagonal in $S^5 \times S^5$, and then forms $M_0$ by coning off $\partial W$. The computation of $\varphi$ on $M_0$ is reduced to a concrete Thom-collapse map and the 10-skeleton of $Q = \Omega S^6$; the needed loop-space input is collected in Appendix C.

5.1. **A tubular neighborhood of the diagonal.** Let $\Delta \subset S^5 \times S^5$ be the diagonal $\Delta = \{(x, x)\}$, and let $p\colon U \to S^5$ be a tubular neighborhood of $\Delta$, viewed as a $D^5$-bundle over $S^5$ via the identification $\Delta \cong S^5$.

**Lemma 5.1.** *The normal bundle $\nu_{\Delta/(S^5\times S^5)}$ is (canonically) isomorphic to the tangent bundle $\tau_{S^5}$. Consequently, the tubular neighborhood $U$ is the unit disk bundle $D(\tau_{S^5})$.*

*Proof.* At a point $(x,x)\in\Delta$, the tangent space splits as $T_{(x,x)}(S^5\times S^5)\cong T_xS^5\oplus T_xS^5$. The differential of the diagonal embedding identifies $T_xS^5$ with the *diagonal* subspace $\{(v,v)\}$. Using the product Riemannian metric, the orthogonal complement is the *anti-diagonal* $\{(v,-v)\}$, which is naturally isomorphic to $T_xS^5$ via $v\mapsto(v,-v)$. These identifications vary smoothly with $x$, giving $\nu_{\Delta/(S^5\times S^5)}\cong\tau_{S^5}$. □

Let $A\subset U$ denote the zero-section (equivalently, the diagonal), so $A\cong S^5$ and its normal bundle in $U$ is $p\colon U\to S^5$.

5.2. **The plumbing manifold $W^{10}$.** Take two copies $(U',A')$ and $(U'',A'')$ of $(U,A)$. Fix an embedded 5-disk $V\subset S^5$ and choose trivializations over $V$ so that

$$p'^{-1}(V)\cong D^5\times D^5,\qquad p''^{-1}(V)\cong D^5\times D^5,$$

where the first factor corresponds to the base $V$ and the second to the fiber.

Define an identification of these two 10-disks by swapping factors,

$$\Phi\colon p'^{-1}(V)\longrightarrow p''^{-1}(V),\qquad \Phi(u,v)=(v,u).$$

Let $W$ be the smooth 10-manifold obtained from the disjoint union $U'\sqcup U''$ by identifying $p'^{-1}(V)$ with $p''^{-1}(V)$ via $\Phi$, and then smoothing corners ("straightening the angles"). Write $\Sigma^9=\partial W$.

**Proposition 5.2.** *The images of the zero-sections $A'\subset U'$ and $A''\subset U''$ define smoothly embedded 5-spheres $A,B\subset W$ which intersect transversely in exactly one point. Moreover, the normal bundles $\nu_{A/W}$ and $\nu_{B/W}$ are both isomorphic to $p\colon U\to S^5$.*

*Proof.* Away from the gluing region $p'^{-1}(V)\cong D^5\times D^5$, the images of $A'$ and $A''$ remain disjoint. Inside the gluing region, the zero-section in $U'$ is $D^5\times\{0\}$ and in $U''$ is $D^5\times\{0\}$; under the swap $(u,0)\mapsto(0,u)$, these meet transversely at $(0,0)$ and nowhere else. The normal bundle statement follows because the plumbing identification is fiberwise linear (up to the chosen trivializations), hence it identifies the corresponding disk-bundle neighborhoods of the zero-sections. □

*Remark* 5.3. Kervaire's construction chooses the plumbing data so that $\Sigma^9=\partial W$ is a smooth homotopy 9-sphere. We will use only two consequences: $\Sigma^9$ is triangulable (since it is smooth), and in dimension $9\geq 5$ any homotopy 9-sphere is homeomorphic to $S^9$ (the generalized Poincaré conjecture).

5.3. **Coning off the boundary.** Let $C\Sigma^9$ be the cone on $\Sigma^9$, and define

$$M_0\ =\ W\cup_{\Sigma^9}C\Sigma^9.$$

Equivalently, $M_0$ is the quotient of $W$ obtained by collapsing $\Sigma^9$ to a point $*$.

**Proposition 5.4.** *The space $M_0$ is a closed, oriented, triangulable 10-manifold.*

*Proof.* Since $\Sigma^9$ is smooth, it is triangulable. As a homotopy 9-sphere in dimension $9\geq 5$, $\Sigma^9$ is homeomorphic to the standard sphere $S^9$. Consider the cone $C\Sigma^9$.

Away from the cone point, $C\Sigma^9$ is homeomorphic to $\Sigma^9\times(0,1]$, hence is a topological 10-manifold there. At the cone point, the link is $\Sigma^9\cong S^9$, so the cone point also has a neighborhood homeomorphic to $\mathbb{R}^{10}$. Thus $C\Sigma^9$ is a (topological) 10-manifold with boundary $\Sigma^9$.

Moreover, a fixed triangulation of $\Sigma^9$ cones to a triangulation of $C\Sigma^9$. Attaching $C\Sigma^9$ to $W$ along $\Sigma^9$ therefore produces a closed, oriented, triangulable 10-manifold

$$M_0 := W \cup_{\Sigma^9} C\Sigma^9.$$

The orientation on $W$ extends over the cone. Indeed, $C\Sigma^9 \setminus \{*\} \cong \Sigma^9 \times (0, 1]$ carries the product orientation extending the boundary orientation on $\Sigma^9 = \partial W$, and the cone point has a neighborhood homeomorphic to $\mathbb{R}^{10}$. □

*Remark* 5.5. In Proposition 5.4 we only use two facts: that $\Sigma^9$ is triangulable, and that $\Sigma^9$ is homeomorphic to $S^9$ in dimension $9 \geq 5$. The cone $C\Sigma^9$ is formed by coning a chosen triangulation of $\Sigma^9$, and no additional PL identification is needed.

**Proposition 5.6.** *The manifold $M_0$ is 4-connected.*

*Proof.* The plumbing description of $W$ gives a handle decomposition with one 0-handle, two 5-handles (whose cores are the spheres $A$ and $B$), and one 10-handle. Collapsing $\partial W$ to a point turns this into a CW structure on $M_0$ with one 0-cell (the cone point), two 5-cells, and one 10-cell. In particular, $M_0$ has no cells in dimensions $1, 2, 3, 4$, hence $\pi_i(M_0) = 0$ for $1 \leq i \leq 4$. Equivalently, the 4-skeleton of this CW structure is a point, so $\pi_i(M_0) = 0$ for $1 \leq i \leq 4$ by the cellular approximation theorem. □

5.4. **Middle homology and a symplectic basis.** The embedded spheres $A, B \subset W \subset M_0$ determine classes $[A], [B] \in H_5(M_0; \mathbb{Z})$.

**Proposition 5.7.** *Let $M$ be a closed, oriented* triangulable *10-manifold. Then the intersection pairing*

$$\lambda\colon H_5(M; \mathbb{Z}) \times H_5(M; \mathbb{Z}) \longrightarrow \mathbb{Z}$$

*is skew-symmetric. In particular, $\lambda(x, x) = 0$ for all $x \in H_5(M; \mathbb{Z})$, and hence $\lambda(x, x) \equiv 0 \pmod 2$.*

*Proof.* Write $[M] \in H_{10}(M; \mathbb{Z})$ for the fundamental class, and define

$$\lambda(x, y) = \langle \mathrm{PD}(x) \smile \mathrm{PD}(y), [M] \rangle,$$

where $\mathrm{PD}\colon H_5(M; \mathbb{Z}) \to H^5(M; \mathbb{Z})$ is Poincaré duality. Since cup product is graded-commutative, we have

$$\mathrm{PD}(x) \smile \mathrm{PD}(y) = (-1)^{5\cdot 5}\, \mathrm{PD}(y) \smile \mathrm{PD}(x) = -\,\mathrm{PD}(y) \smile \mathrm{PD}(x).$$

Pairing with $[M]$ gives $\lambda(x, y) = -\lambda(y, x)$, so $\lambda$ is skew-symmetric. Taking $x = y$ yields $\lambda(x, x) = 0$. □

Let $x, y \in H^5(M_0; \mathbb{Z}_2)$ be the mod-2 Poincaré duals of $[A]$ and $[B]$, characterized by

$$\langle x, [A] \rangle = 1, \quad \langle x, [B] \rangle = 0, \qquad \langle y, [A] \rangle = 0, \quad \langle y, [B] \rangle = 1.$$

**Lemma 5.8.** *With respect to $\lambda(\alpha, \beta) = \langle \alpha \smile \beta, [M_0] \rangle$, the classes $x, y$ satisfy*

$$\lambda(x, x) = 0, \qquad \lambda(y, y) = 0, \qquad \lambda(x, y) = 1.$$

*In particular, $\{x, y\}$ is a symplectic basis of $H^5(M_0; \mathbb{Z}_2)$, so $g = 1$ and $r(M_0) = \varphi(x)\varphi(y)$.*

*Proof.* By Poincaré duality, $\lambda(x, y)$ is the mod-2 reduction of $[A] \cdot [B]$, hence equals 1. The self-pairings vanish because the mod-2 reduction of $[A] \cdot [A]$ and $[B] \cdot [B]$ is 0 by Proposition 5.7. $\square$

5.5. **Computing $\varphi(x)$ and $\varphi(y)$ via Thom spaces.** Let $Q = \Omega S^6$. Appendix C recalls that $H^5(Q; \mathbb{Z}) \cong \mathbb{Z}$ and $H^{10}(Q; \mathbb{Z}) \cong \mathbb{Z}$, with preferred generators $e_1 \in H^5(Q; \mathbb{Z})$ and $e_2 \in H^{10}(Q; \mathbb{Z})$. Let $u_2 \in H^{10}(Q; \mathbb{Z}_2)$ be the mod-2 reduction of $e_2$.

We now compute $\varphi(x)$ and $\varphi(y)$ by producing maps $f_A, f_B \colon M_0 \to Q$ realizing the relevant liftings in $H^5$ and evaluating $u_2$ on $[M_0]$. The key point is that the normal bundles of $A$ and $B$ are the disk bundle $p \colon U \to S^5$, so we can use a Thom collapse map $M_0 \to \mathrm{Th}(p)$.

**Proposition 5.9.** *One has $\varphi(x) = 1$ and $\varphi(y) = 1$.*

*Proof.* We treat $x$; the case of $y$ is identical.

Since $H^5(M_0; \mathbb{Z})$ is free abelian, choose an integral class $X \in H^5(M_0; \mathbb{Z})$ whose mod-2 reduction is $x$ and which is Poincaré dual to $[A]$. Let $K = \mathrm{Th}(p) = U/\partial U$ be the Thom space of the disk bundle $p \colon U \to S^5$. Let $c_A \colon M_0 \to K$ be the Thom collapse map associated to the embedding $A \subset W \subset M_0$ and the tubular neighborhood $U$. Appendix C shows:

(i) $H^5(K; \mathbb{Z}) \cong \mathbb{Z}$ with generator (Thom class) $e_1$, and $c_A^*(e_1) = X$.
(ii) Writing $u_2 \in H^{10}(K; \mathbb{Z}_2)$ for the mod-2 reduction of the generator of $H^{10}(K; \mathbb{Z})$, one has $\langle c_A^*(u_2), [M_0] \rangle = 1$.
(iii) The 10-skeleton $Q^{(10)}$ of $Q = \Omega S^6$ is canonically homotopy equivalent to $K$, and the inclusion $Q^{(10)} \hookrightarrow Q$ identifies the corresponding generators $e_1$ and $u_2$.

Composing $c_A$ with $K \simeq Q^{(10)} \hookrightarrow Q$ yields a map $f_A \colon M_0 \to Q$ with $f_A^*(e_1) = X$ and $f_A^*(u_2)[M_0] = 1$. By the definition of $\varphi$ (Section 3), this implies $\varphi(x) = 1$. $\square$

5.6. **Conclusion.**

**Theorem 5.10.** *The manifold $M_0$ satisfies $r(M_0) = 1$.*

*Proof.* By Lemma 5.8, $\{x, y\}$ is a symplectic basis of $H^5(M_0; \mathbb{Z}_2)$ and $r(M_0) = \varphi(x)\varphi(y)$. By Proposition 5.9, $\varphi(x) = \varphi(y) = 1$, hence $r(M_0) = 1$. $\square$

**Corollary 5.11.** *The triangulable 10-manifold $M_0$ is not homotopy equivalent to any smooth manifold. In particular, $M_0$ admits no differentiable structure.*

*Proof.* This is immediate from Theorem 5.10 and Theorem 1.2. $\square$

**Corollary 5.12.** *Let $\Sigma^9 = \partial W$ be the homotopy sphere arising in the construction. Then $\Sigma^9$ is not diffeomorphic to $S^9$.*

*Proof.* We already know that $\Sigma^9$ is homeomorphic to $S^9$, hence a topological 9-sphere. Suppose for contradiction that there exists a diffeomorphism $\phi \colon \Sigma^9 \to S^9$. Gluing a standard 10-disk along $\phi$ produces a smooth closed 10-manifold

$$N := W \cup_\phi D^{10}.$$

On the other hand, coning off $\Sigma^9$ and using $\phi$ to identify the cone with the standard cone on $S^9$, we obtain a homeomorphism

$$M_0 = W \cup_{\Sigma^9} C\Sigma^9 \ \cong\ W \cup_\phi D^{10} = N.$$

Thus $M_0$ would be homeomorphic to the smooth manifold $N$. Pulling back the smooth atlas on $N$ along this homeomorphism equips $M_0$ with a smooth structure. This contradicts Corollary 5.11. □

## Appendix A. Steenrod squares, Thom classes, and Wu's formula

This appendix collects the mod-2 cohomology tools used implicitly in the main text, especially in Section 4. We keep the discussion to the minimal package we need: Steenrod squares, the Thom isomorphism, the Thom identity for Stiefel–Whitney classes, and Wu's formula on manifolds.

### A.1. **Steenrod squares.**

**Theorem A.1.** *For each $i \geq 0$ there is a natural cohomology operation*

$$Sq^i \colon H^n(X;\mathbb{Z}_2) \longrightarrow H^{n+i}(X;\mathbb{Z}_2)$$

*satisfying the following properties for all spaces $X$ and all classes $x \in H^n(X;\mathbb{Z}_2)$, $y \in H^m(X;\mathbb{Z}_2)$:*

*(i)* $Sq^0 = \mathrm{id}$.
*(ii) (Instability)* $Sq^i(x) = 0$ *for* $i > n$.
*(iii) (Top square)* $Sq^n(x) = x \smile x$.
*(iv) (Cartan formula)* $Sq^k(x \smile y) = \sum_{i+j=k} Sq^i(x) \smile Sq^j(y)$.
*(v) (Naturality) For every continuous* $f \colon X \to Y$, $f^* \circ Sq^i = Sq^i \circ f^*$.

*Remark* A.2. We will use only these formal properties. Standard references include Milnor–Stasheff and most algebraic topology texts treating cohomology operations.

**Lemma A.3.** *Let $M$ be an oriented, closed, 4-connected 10-manifold. Then for every $x \in H^5(M;\mathbb{Z}_2)$ one has $Sq^5(x) = 0$.*

*Proof.* By Theorem A.1(iii), $Sq^5(x) = x \smile x$. But Proposition 2.4 shows $x \smile x = 0$ for all $x \in H^5(M;\mathbb{Z}_2)$. Hence $Sq^5(x) = 0$. □

### A.2. **Thom spaces and the Thom isomorphism.**

Let $\xi \to B$ be a real vector bundle of rank $k$, with unit disk and sphere bundles $D(\xi)$ and $S(\xi)$. Its Thom space is

$$\mathrm{Th}(\xi) = D(\xi)/S(\xi).$$

**Theorem A.4** (Thom isomorphism)**.** *If $\xi \to B$ is a rank-$k$ vector bundle which is $\mathbb{Z}_2$-oriented, then there exists a class $U_\xi \in H^k(\mathrm{Th}(\xi);\mathbb{Z}_2)$ (a* Thom class*) such that cup product with $U_\xi$ induces an isomorphism*

$$\Phi_\xi \colon H^*(B;\mathbb{Z}_2) \xrightarrow{\cong} H^{*+k}(\mathrm{Th}(\xi);\mathbb{Z}_2), \qquad a \longmapsto \pi^*(a) \smile U_\xi,$$

*where $\pi \colon D(\xi) \to B$ is the bundle projection.*

*Proof sketch.* This is the usual Thom isomorphism in cohomology with $\mathbb{Z}_2$-coefficients, proved by excision and the fact that locally $(D^k, S^{k-1})$ has cohomology concentrated in degree $k$, with generator restricting compatibly on overlaps. □

### A.3. **Stiefel–Whitney classes via the Thom identity.**

**Theorem A.5** (Thom identity)**.** *Let $\xi \to B$ be a rank-$k$ bundle with Thom class $U_\xi \in H^k(\mathrm{Th}(\xi);\mathbb{Z}_2)$. There exist unique classes $w_i(\xi) \in H^i(B;\mathbb{Z}_2)$ for $0 \le i \le k$, called the Stiefel–Whitney classes of $\xi$, such that*

$$Sq^i(U_\xi) = \pi^*\big(w_i(\xi)\big) \smile U_\xi \qquad \text{in } H^{k+i}(\mathrm{Th}(\xi);\mathbb{Z}_2).$$

*Equivalently, for the total classes $Sq = \sum_i Sq^i$ and $w(\xi) = \sum_i w_i(\xi)$,*

$$Sq(U_\xi) = \pi^*\big(w(\xi)\big) \smile U_\xi.$$

*Proof.* Since $\Phi_\xi$ is an isomorphism, every class in $H^{k+i}(\mathrm{Th}(\xi);\mathbb{Z}_2)$ can be uniquely written as $\pi^*(a) \smile U_\xi$ for some $a \in H^i(B;\mathbb{Z}_2)$. Define $w_i(\xi)$ by requiring this equality for $Sq^i(U_\xi)$. Uniqueness is immediate from injectivity of $\Phi_\xi$. □

**Lemma A.6.** *If $\xi$ is stably trivial, i.e. $\xi \oplus \varepsilon^\ell \cong \varepsilon^{k+\ell}$, then $w(\xi) = 1$. In particular, if $\xi$ admits a stable framing, then $w_i(\xi) = 0$ for all $i > 0$.*

*Proof.* Stiefel–Whitney classes are stable and multiplicative under Whitney sum: $w(\xi \oplus \eta) = w(\xi) \smile w(\eta)$. Since $w(\varepsilon^m) = 1$, we get $w(\xi) = w(\xi \oplus \varepsilon^\ell) = w(\varepsilon^{k+\ell}) = 1$. □

### A.4. **Wu classes and the identity $w = Sq(v)$.**

Let $M^n$ be a closed $\mathbb{Z}_2$-oriented manifold with fundamental class $[M] \in H_n(M;\mathbb{Z}_2)$. Poincaré duality gives a perfect pairing $H^{n-i}(M;\mathbb{Z}_2) \times H^i(M;\mathbb{Z}_2) \to \mathbb{Z}_2$, so any linear functional on $H^{n-i}(M;\mathbb{Z}_2)$ is represented by a unique class in $H^i(M;\mathbb{Z}_2)$.

**Definition A.7.** For each $i \ge 0$, the *$i$-th Wu class* $v_i(M) \in H^i(M;\mathbb{Z}_2)$ is the unique class such that for all $x \in H^{n-i}(M;\mathbb{Z}_2)$,

$$\big\langle Sq^i(x), [M]\big\rangle = \big\langle v_i(M) \smile x, [M]\big\rangle.$$

Write $v(M) = \sum_i v_i(M)$ for the total Wu class.

**Theorem A.8** (Wu formula)**.** *For every closed $\mathbb{Z}_2$-oriented $n$-manifold $M$, the total Stiefel–Whitney class of the tangent bundle satisfies*

$$w(TM) = Sq\big(v(M)\big).$$

*Proof sketch.* Embed $M$ smoothly into some $\mathbb{R}^{n+k}$ with normal bundle $\nu$. Then

$$TM \oplus \nu \cong \varepsilon^{n+k},$$

so by multiplicativity of Stiefel–Whitney classes,

$$w(TM) \smile w(\nu) = 1 \quad \text{in } H^*(M;\mathbb{Z}_2).$$

Let $U \in H^k(\mathrm{Th}(\nu);\mathbb{Z}_2)$ be the Thom class of $\nu$, and let $\pi\colon \mathrm{Th}(\nu) \to M$ be the projection. The Thom identity (Theorem A.5) gives

$$Sq(U) = \pi^*\big(w(\nu)\big) \smile U.$$

Using the Thom isomorphism and Poincaré duality, the definition of the Wu classes $v_i(M)$ is equivalent to the statement that for all $x \in H^{n-i}(M;\mathbb{Z}_2)$, $\langle Sq^i(x), [M]\rangle = \langle v_i(M) \smile x, [M]\rangle$. Unwinding this through the Thom isomorphism and the relation $w(TM) \smile w(\nu) = 1$ shows that the total class $w(TM)$ is equal to $Sq(v(M))$. □

## Appendix B. Framings, Pontryagin–Thom, and the Kervaire invariant

This appendix supplies the background used in Section 4. We review stable normal framings, the Pontryagin–Thom correspondence, explain why smooth 4-connected 10-manifolds admit stable normal framings, and record the vanishing of the Kervaire invariant in dimension 10.

### B.1. Stable normal framings and the Pontryagin–Thom construction.

*Proof of Theorem 4.3 (sketch).* Let $(N^n, \mathfrak{f})$ be a stably framed manifold. Choose an embedding $i\colon N \hookrightarrow \mathbb{R}^{n+k}$ and represent $\mathfrak{f}$ by a trivialization $\nu_i \cong \varepsilon^k$ of the normal bundle. A tubular neighborhood identifies a neighborhood of $N$ with $N \times D^k$. Collapsing the complement of this neighborhood in $S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\}$ to the basepoint and using the framing to identify each fiber $D^k$ with the unit disk in $\mathbb{R}^k$, we obtain a continuous map

$$\mathrm{PT}(N, \mathfrak{f})\colon S^{n+k} \longrightarrow S^k,$$

well-defined up to homotopy after stabilizing in $k$. This gives a map $\Omega_n^{\mathrm{fr}} \to \pi_n^S$.

If $(N, \mathfrak{f}) = \partial(W, \mathfrak{F})$ is a framed boundary, the collapse maps for $N$ extend over the corresponding collapse for $W$, yielding a null-homotopy in the stable range; hence the construction factors through framed bordism.

Conversely, given a stable map $S^{n+k} \to S^k$, approximate it by a smooth map transverse to a regular value $p \in S^k$. The preimage $N = f^{-1}(p)$ is a smooth closed $n$-manifold, and the differential together with the chosen trivialization of $T_pS^k$ induces a framing of its normal bundle in $S^{n+k}$, hence a stable normal framing. These two constructions are inverse to each other up to the relevant equivalences, producing the natural isomorphism $\Omega_n^{\mathrm{fr}} \cong \pi_n^S$. $\square$

### B.2. Smooth 4-connected 10-manifolds admit stable normal framings.

*Proof of Proposition 4.11.* Let $N^{10}$ be an oriented, smooth, 4-connected closed manifold. Since $N$ is 4-connected, it admits a CW model with one 0-cell, some 5-cells, and one 10-cell (and no other cells).

Let $\tau : N \to BSO$ be the classifying map of the stable tangent bundle. Because $\pi_5(BSO) = \pi_4(SO) = 0$, the restriction of $\tau$ to the 5-skeleton is null-homotopic (the 5-skeleton is a wedge of 5-spheres). Since there are no cells in dimensions $6, 7, 8, 9$, the homotopy class of $\tau$ is determined by its obstruction on the top 10-cell, i.e. by a single element

$$o(\tau) \in H^{10}(N; \pi_{10}(BSO)) \;\cong\; \pi_{10}(BSO) \;\cong\; \pi_9(SO) \;\cong\; \mathbb{Z}_2.$$

Equivalently, collapsing the 5-skeleton gives a quotient map $q : N \to N/N^{(5)} \simeq S^{10}$, and $\tau$ is homotopic to the composite

$$N \xrightarrow{q} S^{10} \xrightarrow{\bar{\tau}} BSO,$$

where $[\bar{\tau}] \in \pi_{10}(BSO) \cong \mathbb{Z}_2$ corresponds to $o(\tau)$.

**Lemma B.1.** *In the above situation, the obstruction class $o(\tau)$ vanishes if and only if*

$$w_{10}(TN) = 0 \in H^{10}(N; \mathbb{Z}_2).$$

*Proof.* Since $\tau$ is null-homotopic on $N^{(5)}$, the only possible nontrivial obstruction to null-homotoping $\tau$ lies in degree 10 and is represented by $[\bar{\tau}] \in \pi_{10}(BSO) \cong \mathbb{Z}_2$. Because $H^i(S^{10};\mathbb{Z}_2) = 0$ for $0 < i < 10$, the pullback of any Stiefel–Whitney class $w_j$ with $j < 10$ under $\bar{\tau}$ must vanish; thus the only potentially nonzero class in degree 10 is $\bar{\tau}^*(w_{10}) \in H^{10}(S^{10};\mathbb{Z}_2) \cong \mathbb{Z}_2$. Moreover, in this cell range the primary $\mathbb{Z}_2$-obstruction to stably trivializing an oriented bundle is precisely the top Stiefel–Whitney class. Concretely, $o(\tau) = 0$ if and only if $\bar{\tau}$ is null-homotopic, which is equivalent to $\bar{\tau}^*(w_{10}) = 0$, and pulling back along $q$ gives

$$w_{10}(TN) = \tau^*(w_{10}) = q^*\big(\bar{\tau}^*(w_{10})\big).$$

Since $q^* : H^{10}(S^{10};\mathbb{Z}_2) \to H^{10}(N;\mathbb{Z}_2)$ is an isomorphism (for a CW model with one 10-cell), this shows $o(\tau) = 0$ if and only if $w_{10}(TN) = 0$. $\square$

**Lemma B.2.** *One has $w_{10}(TN) = 0$.*

*Proof.* Because $N$ is 4-connected and $\dim N = 10$, Poincaré duality implies that the only possibly nonzero rational cohomology groups are in degrees $0, 5, 10$. In particular, the middle pairing

$$H^5(N;\mathbb{Q}) \times H^5(N;\mathbb{Q}) \longrightarrow H^{10}(N;\mathbb{Q}) \cong \mathbb{Q}, \qquad (a, b) \mapsto \langle a \smile b, [N]\rangle$$

is nondegenerate and skew-symmetric (since 5 is odd, $a \smile b = -b \smile a$ over $\mathbb{Q}$). Hence $\dim_{\mathbb{Q}} H^5(N;\mathbb{Q})$ is even, so the Euler characteristic satisfies

$$\chi(N) = b_0 - b_1 + \cdots + b_{10} = 2 - b_5 \equiv 0 \pmod 2.$$

For any closed smooth 10-manifold, the evaluation of the top Stiefel–Whitney class agrees with the Euler characteristic mod 2:

$$\big\langle w_{10}(TN), [N]\big\rangle \equiv \chi(N) \pmod 2.$$

Since $H^{10}(N;\mathbb{Z}_2) \cong \mathbb{Z}_2$, this forces $w_{10}(TN) = 0$. $\square$

By Lemmas B.1 and B.2, the obstruction $o(\tau)$ is zero, so $\tau$ is null-homotopic. Equivalently, the stable tangent bundle is trivial:

$$TN \oplus \varepsilon^{\ell} \;\cong\; \varepsilon^{10+\ell}$$

for some $\ell$.

Finally, choose an embedding $i : N \hookrightarrow \mathbb{R}^{10+k}$ with normal bundle $\nu$. Then $TN \oplus \nu \cong \varepsilon^{10+k}$, and adding $\varepsilon^{\ell}$ gives

$$\nu \oplus \varepsilon^{10+\ell} \;\cong\; (TN \oplus \varepsilon^{\ell})^{\perp} \;\cong\; \varepsilon^{10+k+\ell}.$$

Thus $\nu$ is stably trivial, i.e. $N$ admits a stable normal framing. $\square$

*Remark* B.3. The argument above is tailored to our connectivity range: the CW model with cells only in dimensions $0, 5, 10$ forces all obstruction groups to vanish except the single $\mathbb{Z}_2$-class in degree 10. A more systematic treatment packages this in the Postnikov tower of $BSO$ and obstruction theory.

B.3. **The Kervaire invariant homomorphism and its vanishing in dimension 10.** For a stably framed $(4k+2)$-manifold $(N^{4k+2}, \mathfrak{f})$ there is a classical $\mathbb{Z}_2$-valued invariant, the *Kervaire invariant*, which can be defined as the Arf invariant of a quadratic refinement on the middle cohomology. In our case $4k+2 = 10$ and the middle degree is $2k+1 = 5$.

*Proof of Proposition 4.5 (outline).* Let $(N^{10}, \mathfrak{f})$ be stably framed. One defines a function $q_{\mathfrak{f}} \colon H^5(N; \mathbb{Z}_2) \to \mathbb{Z}_2$ as follows. Given $x \in H^5(N; \mathbb{Z}_2)$, choose an integral lift $X \in H^5(N; \mathbb{Z})$ (which exists and is unique modulo $2H^5(N; \mathbb{Z})$ since $H^5$ is free). Represent $X$ by a smooth map $g \colon N \to S^5$ (possible in this connectivity range by obstruction theory). For a regular value $p \in S^5$, the preimage $V = g^{-1}(p)$ is a smooth 5-manifold whose normal bundle in $N$ is identified with the pullback of $T_pS^5 \cong \mathbb{R}^5$. The stable framing $\mathfrak{f}$ on $N$ induces, together with this identification, a stable framing on $V$. From this framed data one extracts a $\mathbb{Z}_2$-number $q_{\mathfrak{f}}(x)$, which may be described equivalently via:

(a) a Pontryagin–Thom collapse map and a universal stable cohomology class, or
(b) a secondary cohomology operation associated to the Adem relation in degree 10.

The key output is that $q_{\mathfrak{f}}$ satisfies $q_{\mathfrak{f}}(x+y) = q_{\mathfrak{f}}(x) + q_{\mathfrak{f}}(y) + \langle x \smile y, [N] \rangle$, so it is a quadratic refinement of the symplectic form $\lambda$. Then Lemma 3.17 implies that $\kappa(N, \mathfrak{f}) = \sum_i q_{\mathfrak{f}}(x_i) q_{\mathfrak{f}}(y_i)$ is basis-independent.

Finally, bordism invariance follows because the construction factors through the framed bordism class $[N, \mathfrak{f}] \in \Omega_{10}^{\mathrm{fr}}$, and additivity under disjoint union shows $\kappa$ is a group homomorphism $\Omega_{10}^{\mathrm{fr}} \to \mathbb{Z}_2$. □

*Proof of Theorem 4.9 (by reference).* By Theorem 4.3, $\Omega_{10}^{\mathrm{fr}} \cong \pi_{10}^S$. Under this identification, $\kappa$ is the classical Kervaire invariant homomorphism $\pi_{10}^S \to \mathbb{Z}_2$. It is a theorem from the computation of stable homotopy groups of spheres and the analysis of Kervaire invariant one elements that the Kervaire invariant is zero in dimension 10. Equivalently, there is no element of Kervaire invariant one in $\pi_{10}^S$, hence $\kappa$ vanishes identically. □

*Remark* B.4. For the purposes of this exposition, the only needed consequence is the vanishing. A detailed account can be found in standard references on stable homotopy and the Kervaire invariant. In particular, one can consult treatments that compute $\pi_{10}^S$ and discuss the Kervaire invariant in low dimensions.

## Appendix C. The loop-space ingredients, the 10-skeleton of $\Omega S^6$, and the computation of $\varphi$ on $M_0$

This appendix supplies the loop-space ingredients used in Section 3 and the skeletal and Thom-space computations used in Section 5. It also explains, in the present 4-connected 10-dimensional range, why the loop-space refinement $\varphi$ agrees with the framing-induced refinement $q_f$ (Proposition 4.7).

Throughout let

$$Q = \Omega S^6$$

be the based loop space of $S^6$, equipped with loop multiplication

$$\mu : Q \times Q \longrightarrow Q.$$

We write $\rho$ for reduction mod 2. We will use preferred generators

$$e_1 \in H^5(Q;\mathbb{Z}), \qquad e_2 \in H^{10}(Q;\mathbb{Z}),$$

and denote by

$$u_2 = \rho(e_2) \in H^{10}(Q;\mathbb{Z}_2)$$

its mod-2 reduction. (Thus one may take $e = e_1$ and $u = u_2$ for Proposition 3.3.)

C.1. **The 10-skeleton of $Q$ and low-degree cohomology.**

**Proposition C.1.** *The loop space $Q = \Omega S^6$ admits a CW decomposition with one cell in each dimension $0, 5, 10$ and all remaining cells in dimensions $\geq 15$. In particular, the 10-skeleton $Q(10)$ has the homotopy type*

$$Q(10) \;\simeq\; S^5 \cup_{[\iota_5,\iota_5]} e^{10},$$

*where the attaching map of the 10-cell may be taken to represent the Whitehead square $[\iota_5, \iota_5] \in \pi_9(S^5)$.*

*Proof.* This is a standard consequence of the James construction and the Bott–Samelson description of $\Omega\Sigma S^5 \simeq \Omega S^6$: homology $H_*(Q;\mathbb{Z})$ is the tensor algebra on one generator in degree 5, so a CW model may be chosen with one cell in each multiple of 5. In degrees $\leq 10$ this gives cells in $0, 5, 10$ only. The identification of the 10-cell attaching map with the Whitehead square is the usual identification between the Samelson product in $\Omega S^6$ and the Whitehead product in $S^5$ under adjunction. $\square$

**Proposition C.2.** *There are canonical isomorphisms*

$$H^5(Q;\mathbb{Z}) \cong \mathbb{Z}, \qquad H^{10}(Q;\mathbb{Z}) \cong \mathbb{Z},$$

*with preferred generators $e_1$ and $e_2$ such that:*

*(i)* $\mu^*(e_1) = \mathrm{pr}_1^*(e_1) + \mathrm{pr}_2^*(e_1)$ *in* $H^5(Q \times Q;\mathbb{Z})$.

*(ii) Writing* $\bar{e}_1 = \rho(e_1) \in H^5(Q;\mathbb{Z}_2)$ *and* $u_2 = \rho(e_2) \in H^{10}(Q;\mathbb{Z}_2)$, *one has*

$$\mu^*(u_2) = \mathrm{pr}_1^*(u_2) + \mathrm{pr}_2^*(u_2) + \mathrm{pr}_1^*(\bar{e}_1) \smile \mathrm{pr}_2^*(\bar{e}_1) \tag{C.3}$$

*in* $H^{10}(Q \times Q;\mathbb{Z}_2)$.

*Proof.* In low degrees, $H^*(Q;\mathbb{Z})$ is the free graded-commutative algebra on a class in degree 5, so $H^5 \cong \mathbb{Z}$ and $H^{10} \cong \mathbb{Z}$. The multiplication $\mu$ makes $H^*(Q;\mathbb{Z})$ a graded Hopf algebra; the degree-5 generator can be chosen primitive, giving (i). Reducing mod 2 and using that $u_2$ is the mod-2 reduction of a degree-10 generator, the coproduct formula (ii) is the standard quadratic correction term arising from the Hopf algebra structure in degree 10 (equivalently, it is the identity already used as Proposition 3.3(iv)). $\square$

C.2. **Realization and well-defined evaluation in the 0-5-10 range.** The next two statements are exactly the parts of Proposition 3.3(ii)(v) that are actually used in this paper: all of our spaces are 4-connected 10-manifolds, and in our range we may take CW models with cells only in dimensions $0, 5, 10$.

**Proposition C.4.** *Let $X$ be a connected CW complex of dimension $\leq 10$ which admits a CW structure with one 0-cell, some 5-cells, and one 10-cell (and no other cells). Then for every class $X_0 \in H^5(X;\mathbb{Z})$ there exists a map $f : X \to Q$ such that $f^*(e_1) = X_0$.*

*Proof.* Let $X^{(5)}$ be the 5-skeleton. Then $X^{(5)} \simeq \bigvee_\alpha S^5$ is a wedge of 5-spheres, and the class $X_0$ is determined by its values on the 5-cells. Choose a map $f^{(5)} : X^{(5)} \to S^5$ of prescribed degrees on the wedge summands so that $(f^{(5)})^*(\iota) = X_0|_{X^{(5)}}$, where $\iota \in H^5(S^5;\mathbb{Z})$ is a generator. Compose with the adjoint (James) map $J : S^5 \to \Omega S^6 = Q$ to obtain a map still denoted $f^{(5)} : X^{(5)} \to Q$. By construction, $(f^{(5)})^*(e_1) = X_0|_{X^{(5)}}$.

To extend over the 10-cell, the only obstruction lies in

$$H^{10}(X;\pi_9(Q)).$$

But $\pi_9(Q) \cong \pi_{10}(S^6) = 0$ (by Freudenthal suspension in this range), hence the obstruction vanishes and $f^{(5)}$ extends to a map $f : X \to Q$. Since $e_1$ lives in degree 5, the equality $f^*(e_1) = X_0$ holds on $X$. □

**Proposition C.5.** *Let $M$ be an oriented, closed 10-manifold which admits a CW model with cells only in dimensions $0, 5, 10$. If $f_0, f_1 : M \to Q$ satisfy $f_0^*(e_1) = f_1^*(e_1) \in H^5(M;\mathbb{Z})$, then*

$$\langle f_0^*(u_2), [M]\rangle = \langle f_1^*(u_2), [M]\rangle \in \mathbb{Z}_2.$$

*Proof.* Since $f_0^*(e_1) = f_1^*(e_1)$, the maps $f_0$ and $f_1$ are homotopic on the 5-skeleton (this is the usual identification between $[\,\cdot\,, K(\mathbb{Z},5)]$ and $H^5$ in the absence of lower cells). Because $M$ has no cells in dimensions $6, 7, 8, 9$, we may choose a homotopy so that $f_0$ and $f_1$ agree on the 9-skeleton. Thus their difference is concentrated on the top 10-cell and determines an element

$$\delta(f_0, f_1) \in \pi_{10}(Q) \cong \pi_{11}(S^6).$$

Now $u_2 \in H^{10}(Q;\mathbb{Z}_2)$ restricts trivially to the 9-skeleton, so the difference $\langle f_0^*(u_2), [M]\rangle - \langle f_1^*(u_2), [M]\rangle$ depends only on the class $\delta(f_0, f_1)$ and is given by evaluating $u_2$ on the Hurewicz image of $\delta(f_0, f_1)$. Equivalently (and classically), this difference is the mod-2 Hopf invariant of the adjoint map $S^{11} \to S^6$ corresponding to $\delta(f_0, f_1)$.

By the Hopf invariant one theorem, there is no element of odd Hopf invariant in $\pi_{11}(S^6)$. Hence the mod-2 Hopf invariant is 0, and the two evaluations agree in $\mathbb{Z}_2$. □

C.3. **The Thom space $K = \mathrm{Th}(\tau_{S^5})$ and identification with $Q(10)$.** Let $\tau = \tau_{S^5}$ be the tangent bundle of $S^5$, and let

$$K = \mathrm{Th}(\tau) = D(\tau)/S(\tau)$$

be its Thom space. Let $U \in H^5(K;\mathbb{Z})$ be a (integral) Thom class.

**Proposition C.6.** *The Thom space $K$ has the homotopy type*

$$K \simeq S^5 \cup_{[\iota_5,\iota_5]} e^{10}.$$

*Moreover, under this identification the Thom class $U$ restricts to a generator of $H^5(S^5;\mathbb{Z})$, and the top class in $H^{10}(K;\mathbb{Z})$ corresponds to the 10-cell.*

*Proof.* The Thom isomorphism gives

$$H^5(K;\mathbb{Z}) \cong H^0(S^5;\mathbb{Z}) \cong \mathbb{Z}, \qquad H^{10}(K;\mathbb{Z}) \cong H^5(S^5;\mathbb{Z}) \cong \mathbb{Z},$$

and $H^i(K;\mathbb{Z}) = 0$ for $0 < i < 5$ and for $5 < i < 10$. Thus $K$ has the homotopy type of a CW complex with one cell in dimensions $0, 5, 10$. The attaching map of the 10-cell is determined (up to the usual identifications) by the characteristic class data of $\tau$; in this case it agrees with the Whitehead square $[\iota_5, \iota_5] \in \pi_9(S^5)$, yielding the claimed homotopy type. The statements about generators follow from the Thom isomorphism and the definition of the Thom class. □

**Corollary C.7.** *There is a canonical homotopy equivalence*

$$K \simeq Q(10),$$

*and under this equivalence the generators in degrees* 5 *and* 10 *match the restrictions of* $e_1$ *and* $e_2$ *from Proposition C.2.*

*Proof.* Both spaces have the homotopy type $S^5 \cup_{[\iota_5,\iota_5]} e^{10}$ by Propositions C.1 and C.6. Choosing the equivalence to identify the bottom $S^5$ and the top 10-cell fixes it uniquely up to homotopy, and the generator matching follows from the descriptions of the degree-5 and degree-10 classes. □

C.4. **Comparison with the framing-induced refinement.**

**Proposition C.8.** *Let* $N^{10}$ *be an oriented, smooth,* 4*-connected closed manifold and let* $f$ *be a stable normal framing of* $N$*. Then the loop-space refinement* $\varphi$ : $H^5(N; \mathbb{Z}_2) \to \mathbb{Z}_2$ *from Section 3 agrees with the framing-induced refinement* $q_f$ *from Proposition 4.5. In particular,*

$$r(N) = \kappa(N, f) \in \mathbb{Z}_2.$$

*Proof.* Let $x \in H^5(N; \mathbb{Z}_2)$ and choose an integral lift $X \in H^5(N; \mathbb{Z})$.

**Step 1: Choose a realizing map $F\colon N \to Q$ and a representing map $g\colon N \to S^5$.** Fix a CW model of $N$ with cells only in dimensions $0, 5, 10$. By Proposition C.4 there exists a map $F\colon N \to Q$ such that $F^*(e_1) = X$. Since $\dim N = 10$, by cellular approximation we may assume $F$ lands in the 10-skeleton $Q(10)$.

On the model $Q(10) \simeq S^5 \cup_{[\iota_5,\iota_5]} e^{10}$ there is a canonical cellular map

$$\pi\colon Q(10) \longrightarrow S^5$$

which is the identity on the bottom $S^5$ and collapses the 10-cell. Let $g = \pi \circ F\colon N \to S^5$. Then $g^*(\iota) = X$, where $\iota \in H^5(S^5; \mathbb{Z})$ is a chosen generator. After a small homotopy we may assume $g$ is smooth.

Choose a regular value $p \in S^5$. Then $V = g^{-1}(p)$ is a closed smooth 5-manifold and its normal bundle in $N$ is canonically identified with $g^*(T_p S^5) \cong \varepsilon^5$.

**Step 2: $q_f(x)$ as a Thom-collapse evaluation.** The stable normal framing $f$ on $N$ induces a stable normal framing on $V$ together with the above identification of the 5-dimensional normal bundle. In the classical construction of Kervaire, this framed data produces a number $q_f(x) \in \mathbb{Z}_2$. In the present 10-dimensional, 4-connected range, one can reorganize the construction as follows.

Let $U \subset N$ be a tubular neighborhood of $V$ identified with the unit disk bundle of the normal bundle $\nu_{V/N} \cong \varepsilon^5$, and let

$$c_V\colon N \longrightarrow \mathrm{Th}(\nu_{V/N})$$

be the Thom collapse map. Let $u \in H^5(\mathrm{Th}(\nu_{V/N}); \mathbb{Z})$ be an integral Thom class. The Pontryagin–Thom/secondary-operation description (Appendix B.3) associates to the framed data a preferred class

$$\widetilde{u}_2 \in H^{10}(\mathrm{Th}(\nu_{V/N}); \mathbb{Z}_2)$$

and the definition of $q_f(x)$ is equivalent, in this range, to the evaluation formula

$$q_f(x) \;=\; \big\langle c_V^*(\widetilde{u}_2), [N] \big\rangle \in \mathbb{Z}_2.$$

A detailed account of this equivalence can be found in classical treatments of the Kervaire invariant via Thom spaces and secondary operations.

**Step 3: Identify the relevant degree-10 class with the universal class $u_2$ on $Q = \Omega S^6$.** By Corollary C.7 we have a canonical identification

$$K = \mathrm{Th}(\tau S^5) \;\simeq\; Q(10),$$

and under this identification the degree-5 and degree-10 generators match the restrictions of $e_1$ and $e_2$ on $Q$. In particular, the mod-2 class $u_2 = \rho(e_2)$ restricts to the mod-2 generator on $Q(10)$.

With the choice of $F$ from Step 1, the evaluation of $u_2$ on $[N]$ is the number

$$\big\langle F^*(u_2), [N] \big\rangle \in \mathbb{Z}_2,$$

and in the present range the Pontryagin–Thom description identifies $\langle c_V^*(\widetilde{u}_2), [N]\rangle$ with $\langle F^*(u_2), [N]\rangle$. Therefore

$$q_f(x) \;=\; \langle F^*(u_2), [N]\rangle.$$

**Step 4: Compare with $\varphi$.** By definition of the loop-space refinement in Section 3,

$$\varphi(x) \;=\; \langle F^*(u_2), [N]\rangle.$$

Hence $\varphi(x) = q_f(x)$ for every $x \in H^5(N; \mathbb{Z}_2)$, so $\varphi = q_f$. The identity $r(N) = \kappa(N, f)$ then follows immediately from the definitions of $r$ and $\kappa$ as Arf invariants. ☐

C.5. **Thom collapse and the computation of $\varphi$ on $M_0$.** We keep the notation from Section 5: $A \subset M_0$ is the embedded $S^5$ with tubular neighborhood

$$p : U \to S^5$$

a disk bundle, and $K = \mathrm{Th}(p) = U/\partial U$.

**Lemma C.9.** *Let $c_A : M_0 \to K$ be the Thom collapse map associated to the embedding $A \subset M_0$ and the tubular neighborhood $U$. Let $e_1 \in H^5(K; \mathbb{Z})$ be the Thom class. If $X \in H^5(M_0; \mathbb{Z})$ is Poincaré dual to $[A]$, then*

$$c_A^*(e_1) = X.$$

*Proof.* This is the defining property of the Thom class and the Thom collapse: pulling back the Thom class gives the Poincaré dual class of the embedded submanifold. ☐

**Lemma C.10.** *Let $e_2 \in H^{10}(K; \mathbb{Z})$ be the generator corresponding under the Thom isomorphism to a generator of $H^5(S^5; \mathbb{Z})$, and let $u_2 = \rho(e_2) \in H^{10}(K; \mathbb{Z}_2)$. Then*

$$\big\langle c_A^*(u_2), [M_0] \big\rangle = 1 \in \mathbb{Z}_2.$$

*Proof.* Under the Thom isomorphism, $e_2$ corresponds to a generator in $H^5(S^5; \mathbb{Z})$, and evaluating $c_A^*(e_2)$ on $[M_0]$ computes the mod-2 self-intersection/Thom–Wu number of the associated disk bundle in $M_0$.

To make the evaluation explicit, note that under the Thom isomorphism the class $e_2 \in H^{10}(K; \mathbb{Z})$ corresponds to a generator of $H^5(S^5; \mathbb{Z})$, hence $c_A^*(e_2)$ evaluates on $[M_0]$ as the (integral) intersection number determined by the embedded sphere

$A$ and a transverse core sphere in the plumbing. In the notation of Section 5, this is exactly the intersection number $[A] \cdot [B]$, and Lemma 5.8 gives $[A] \cdot [B] = 1$. Reducing mod 2 yields $\langle c_A^*(u_2), [M_0] \rangle = 1 \in \mathbb{Z}_2$. $\square$

**Proposition C.11.** *Let $X \in H^5(M_0; \mathbb{Z})$ be Poincaré dual to $[A]$, and let $x = \rho(X) \in H^5(M_0; \mathbb{Z}_2)$. Then*

$$\varphi(x) = 1.$$

*The same holds for the class $y \in H^5(M_0; \mathbb{Z}_2)$ Poincaré dual to $[B]$.*

*Proof.* By Lemma C.9, $c_A^*(e_1) = X$. By Corollary C.7, we may identify $K \simeq Q(10)$ and then compose with the inclusion $Q(10) \hookrightarrow Q$ to obtain a map

$$f_A : M_0 \longrightarrow Q$$

with $f_A^*(e_1) = X$ and $f_A^*(u_2) = c_A^*(u_2)$. By Lemma C.10, $\langle f_A^*(u_2), [M_0] \rangle = 1$. By the definition of $\varphi$ in Section 3 (and Proposition C.5 for well-definedness),

$$\varphi(x) = \big\langle f_A^*(u_2), [M_0] \big\rangle = 1.$$

The argument for $y$ is identical using $B$ in place of $A$. $\square$