WITTEN–PARKER–TAUBES PROOF OF THE POSITIVE ENERGY THEOREM

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Abstract. We give a self-contained exposition of the spinorial proof of the positive energy theorem, following the argument introduced by Witten and its analytic completion by Parker and Taubes. Starting from an asymptotically flat initial data set (M^3, g, K) satisfying the dominant energy condition, we define the hypersurface Dirac operator and derive Witten's Weitzenböck identity. Its integrated form yields a nonnegative bulk term and a boundary flux at spatial infinity. The analytic input is the existence and uniqueness of solutions to the Witten equation $\mathcal{D}\psi = 0$ with prescribed asymptotics, obtained using weighted Sobolev spaces and Fredholm theory on asymptotically flat ends. Combining these ingredients gives $E \geq |P|$ on each end, together with rigidity in the equality case.

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1. Introduction

The positive energy theorem is a rigidity statement about asymptotically flat geometry. Roughly speaking, it says that an isolated gravitational system cannot have negative total energy, and that the only way to have vanishing mass is to be flat. In this exposition we follow the spinorial method introduced by Witten and developed in full geometric and analytic detail by Parker–Taubes.

1.1. Asymptotically flat initial data and total energy.

We work with an *initial data set* (M, g_{ij}, h_{ij}) arising from a spacelike hypersurface M in a 4-dimensional Lorentz manifold $(N, g_{\mu\nu})$ of signature $(-+++)$. Here g_{ij} is the induced Riemannian metric on M, and h_{ij} is the second fundamental form of the embedding $M \hookrightarrow N$ with respect to the future-pointing unit normal e_0.

Definition 1.1. An initial data set (M, g, h) is *asymptotically flat* if there exists a compact set $K \subset M$ such that $M \setminus K$ is a finite disjoint union of ends M_ℓ, and on each end there is a diffeomorphism

$$\Phi_\ell \colon M_\ell \longrightarrow \{x \in \mathbb{R}^3 : |x| > R\}$$

for some $R > 0$, in which the components of g_{ij} and h_{ij} satisfy, as $|x| \to \infty$,

$$g_{ij} = \delta_{ij} + O(|x|^{-\tau}), \qquad \partial_k g_{ij} = O(|x|^{-\tau-1}), \qquad h_{ij} = O(|x|^{-\tau-1}) \tag{1.2}$$

for some $\tau > \frac{1}{2}$, together with the corresponding decay for higher derivatives needed to justify integrations by parts on large coordinate spheres.

On an asymptotically flat end M_ℓ, the total energy and linear momentum are defined by boundary integrals over large coordinate spheres $S_r = \{|x| = r\} \subset \mathbb{R}^3$. Writing ν for the Euclidean outward unit normal to S_r and dS for the Euclidean surface measure, one sets

$$E_\ell = \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} \big(\partial_j g_{ij} - \partial_i g_{jj}\big)\, \nu^i \, dS, \tag{1.3}$$

and

$$(P_\ell)_k = \frac{1}{8\pi} \lim_{r\to\infty} \int_{S_r} \big(h_{ik} - (\mathrm{tr}_g h)\, g_{ik}\big)\, \nu^i \, dS. \tag{1.4}$$

The pair $(E_\ell, P_\ell) \in \mathbb{R} \times \mathbb{R}^3$ is called the ADM energy-momentum of the end M_ℓ. The Lorentz-invariant mass is

$$m_\ell = \sqrt{E_\ell^2 - |P_\ell|^2}. \tag{1.5}$$

The decay in (1.2) ensures that the limits (1.3) and (1.4) are finite.

1.2. **Constraint equations and the dominant energy condition.** Let D denote the Levi-Civita connection of (M, g), and let R be its scalar curvature. The Gauss–Codazzi equations express the Einstein tensor of $(N, g_{\mu\nu})$ in terms of (g, h) and their derivatives along M. If the spacetime satisfies the Einstein equation

$$G_{\mu\nu} = 8\pi T_{\mu\nu},$$

then (g, h) obey the constraint equations

$$R + (\mathrm{tr}_g h)^2 - |h|^2 = 16\pi\rho, \qquad D^j\big(h_{ij} - (\mathrm{tr}_g h)\, g_{ij}\big) = 8\pi J_i. \tag{1.6}$$

Here ρ and J are the energy and momentum densities measured by the unit normal e_0. Equivalently, in an orthonormal frame $\{e_0, e_1, e_2, e_3\}$ adapted to M,

$$\rho = T(e_0, e_0), \qquad J_i = T(e_0, e_i). \tag{1.7}$$

Definition 1.8. The *dominant energy condition* holds along M if

$$\rho \ge |J|_g \tag{1.9}$$

pointwise on M, where $|J|_g^2 = g^{ij} J_i J_j$.

The pointwise inequality (1.9) is the hypothesis that drives the nonnegativity in the spinorial argument.

1.3. **Main results.**

Theorem 1.10 (Positive energy theorem)**.** *Let (M, g, h) be a complete, oriented, asymptotically flat initial data set in the sense of Definition 1.1, arising as a spacelike hypersurface in a 4-dimensional Lorentz manifold satisfying the Einstein equation. Assume the dominant energy condition (1.9) holds. Then for each end M_ℓ,*

$$E_\ell \ge |P_\ell|. \tag{1.11}$$

Equivalently, $m_\ell \ge 0$.

Theorem 1.12 (Rigidity)**.** *In the situation of Theorem 1.10, suppose that $m_\ell = 0$ on some end M_ℓ. Then the spacetime is flat in a neighborhood of M, and (M, g, h) arises from a spacelike hyperplane in Minkowski space. In particular, after a global Lorentz transformation on the asymptotic end, g is Euclidean and $h \equiv 0$.*

1.4. **Spinorial strategy.** The spinor method replaces geometric positivity by an L^2 identity for a Dirac-type operator. Since M is 3-dimensional and oriented, it admits a spin structure, so one has a Dirac spinor bundle $S \to M$. The key point is that the relevant connection on S is the one induced from the ambient spacetime $(N, g_{\mu\nu})$, not the intrinsic Levi-Civita connection of (M, g). With respect to a local g-orthonormal frame $\{e_i\}_{i=1}^3$ on M, one defines the hypersurface Dirac operator

$$\mathscr{D}\psi = \sum_{i=1}^{3} e^i \cdot V_i \psi, \tag{1.13}$$

where V denotes the induced spin connection and $\cdot$ is Clifford multiplication.

A *Witten spinor* is a spinor ψ solving

$$\mathscr{D}\psi = 0 \tag{1.14}$$

and approaching a constant spinor on each asymptotically flat end. The central identity is a Weitzenböck-type formula which yields, after integration by parts on a large truncation of M,

$$\int_M \Big(|V\psi|^2 + \langle \psi, \mathcal{R}\psi \rangle \Big) \, d\mathrm{vol}_g = \lim_{r\to\infty} \int_{S_r} \mathcal{B}(\psi). \tag{1.15}$$

Here $\mathcal{R}$ is a pointwise self-adjoint endomorphism of S determined by the stress-energy tensor, and $\mathcal{B}(\psi)$ is an explicit boundary integrand. Under the dominant energy condition, $\langle \psi, \mathcal{R}\psi \rangle \geq 0$, so the left-hand side of (1.15) is nonnegative. The boundary term on the right-hand side can be expressed in terms of the ADM energy-momentum (E_ℓ, P_ℓ) and the chosen asymptotic constant spinor. Solving (1.14) with the appropriate boundary condition then gives (1.11). The existence and uniqueness of such solutions is an elliptic problem on a noncompact manifold and is obtained using weighted Sobolev spaces adapted to the decay (1.2).

2. Asymptotic flatness and ADM energy-momentum

This section records the asymptotic hypotheses used throughout the spinor proof and fixes the definitions of the ADM energy and momentum on each end.

2.1. **Asymptotically flat ends and decay.** Let (M, g_{ij}, h_{ij}) be an initial data set as in Section 1. Fix an end $M_\ell \subset M$, and choose an identification

$$\Phi_\ell \colon M_\ell \longrightarrow \{x \in \mathbb{R}^3 : |x| > R\}$$

for some $R > 0$, where $\{x^i\}$ are the standard coordinates on $\mathbb{R}^3$ and $r = |x|$. We write $S_r = \{|x| = r\}$ for the Euclidean coordinate spheres, ν for their outward Euclidean unit normal, and dS for Euclidean surface measure.

Definition 2.1. The end M_ℓ is *asymptotically flat of Parker–Taubes type* if, in the coordinates $\{x^i\}$,

$$g_{ij} = \delta_{ij} + a_{ij}, \qquad a_{ij} = O(r^{-1}), \quad \partial_k a_{ij} = O(r^{-2}), \quad \partial_\ell \partial_k a_{ij} = O(r^{-3}), \tag{2.2}$$

and the second fundamental form satisfies

$$h_{ij} = O(r^{-2}), \qquad \partial_k h_{ij} = O(r^{-3}). \tag{2.3}$$

The bounds (2.2)–(2.3) are chosen so that the boundary integrals defining energy and momentum have limits and so that later integrations by parts produce boundary terms with well-controlled errors.

Lemma 2.4. *Assume M_ℓ is asymptotically flat of Parker–Taubes type. Then on S_r,*

$$\big(\partial_j g_{ij} - \partial_i g_{jj}\big)\nu^i = O(r^{-2}), \tag{2.5}$$

and for each fixed k,

$$\big(h_{ik} - (\mathrm{tr}_\delta h)\,\delta_{ik}\big)\nu^i = O(r^{-2}), \tag{2.6}$$

where $\mathrm{tr}_\delta h = \delta^{ij} h_{ij}$.

Proof. By (2.2), $\partial g = O(r^{-2})$, which gives (2.5). By (2.3), $h = O(r^{-2})$, hence $\mathrm{tr}_\delta h = O(r^{-2})$, and (2.6) follows. □

Since $\mathrm{Area}(S_r) \sim 4\pi r^2$, Lemma 2.4 shows that the boundary integrals below are uniformly bounded in r. The existence of the limit requires a little more than boundedness, and is ensured by the additional derivative control in (2.2) and (2.3).

2.2. ADM energy.

Definition 2.7. Assume M_ℓ is asymptotically flat of Parker–Taubes type. The *ADM energy* of the end M_ℓ is

$$E_\ell = \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} \big(\partial_j g_{ij} - \partial_i g_{jj}\big)\,\nu^i\, dS, \tag{2.8}$$

provided the limit exists.

The integrand in (2.8) depends only on the asymptotic behavior of the metric in the chosen coordinates. Under the decay assumptions (2.2), the limit exists and is independent of the particular radius function used to exhaust the end by coordinate spheres. Moreover, changing to another asymptotically Euclidean coordinate system changes the integrand by a Euclidean divergence term whose flux tends to zero, so the resulting number E_ℓ depends only on the asymptotic end.

Remark 2.9. In a later section, the energy will arise as the boundary contribution of an integration by parts identity for the hypersurface Dirac operator. The decay in Definition 2.1 is tailored so that all error terms produced by replacing g with δ at infinity are integrable and do not affect the limiting boundary value.

2.3. ADM momentum.

The second fundamental form h_{ij} on M can be viewed as a tensor field on the end $M_\ell \simeq \{|x| > R\} \subset \mathbb{R}^3$. In the Parker–Taubes framework, the total momentum is defined directly from h using the Euclidean background of the asymptotic chart.

Definition 2.10. Assume M_ℓ is asymptotically flat of Parker–Taubes type. The *ADM momentum* of the end M_ℓ is the vector $P_\ell \in \mathbb{R}^3$ with components

$$(P_\ell)_k = \frac{1}{8\pi} \lim_{r\to\infty} \int_{S_r} \big(h_{ik} - (\mathrm{tr}_\delta h)\,\delta_{ik}\big)\,\nu^i\, dS, \qquad k = 1,2,3, \tag{2.11}$$

provided the limits exist.

Because $g_{ij} - \delta_{ij} = O(r^{-1})$ and $h_{ij} = O(r^{-2})$, one may replace $\mathrm{tr}_\delta h$ by $\mathrm{tr}_g h$ and δ_{ik} by g_{ik} in (2.11) without changing the limit. The form (2.11) is convenient since it is written purely in terms of the initial data on M in the chosen asymptotic chart.

Remark 2.12. There are several equivalent expressions for total linear momentum in the literature. For our purposes, any definition that agrees with (2.11) under the decay assumptions in Definition 2.1 would lead to the same inequality in Theorem 1.10.

2.4. **Mass and Lorentz form.** The pair (E_ℓ, P_ℓ) is the asymptotic energy-momentum of the end M_ℓ. It is natural to package these data using the Minkowski quadratic form

$$m_\ell^2 = E_\ell^2 - |P_\ell|^2, \tag{2.13}$$

where $|P_\ell|^2 = \sum_{k=1}^3 (P_\ell)_k^2$. The inequality $E_\ell \geq |P_\ell|$ is equivalent to $m_\ell^2 \geq 0$, and it is this nonnegativity that will emerge from the spinorial integration by parts identity.

It will also be useful to keep in mind that, once an asymptotic Lorentz frame is chosen at infinity, linear functionals of (E_ℓ, P_ℓ) appear naturally. Later, the boundary term produced by the Witten equation $\mathscr{D}\psi = 0$ will evaluate to such a linear functional determined by the asymptotic constant spinor on the end.

3. Spinor geometry and the hypersurface Dirac operator

Let $(N, g_{\mu\nu})$ be a 4-dimensional Lorentz manifold of signature $(-+++)$, and let $M \subset N$ be an oriented spacelike hypersurface with induced Riemannian metric g_{ij} and future-pointing unit normal e_0. We write $\{e_i\}_{i=1}^3$ for a local g-orthonormal frame of TM and $\{e^i\}$ for the dual coframe.

The spinorial argument uses Dirac spinors *along* M that are naturally associated to the Lorentz geometry of N restricted to M. We record here the basic structures and the Dirac operator that will enter the Weitzenböck identity in the next section. The Clifford and spin preliminaries are expanded in Appendix A, and the longer moving-frame computations are deferred to Appendix C.

3.1. **Clifford multiplication and inner products.** Let $S \to M$ denote the Dirac spinor bundle along M. Clifford multiplication by a covector $\alpha \in T^*N|_M$ is a bundle map

$$\alpha\cdot \ : \ S \longrightarrow S$$

satisfying the Clifford relations

$$\alpha \cdot \beta \cdot + \beta \cdot \alpha \cdot = -2\, g^{-1}(\alpha, \beta)\, \mathrm{Id}_S, \qquad \alpha, \beta \in T^*N|_M. \tag{3.1}$$

In particular, since $g(e^0, e^0) = -1$, we have $(e^0\cdot)^2 = \mathrm{Id}_S$, while for $\xi \in T^*M$ we have $(\xi\cdot)^2 = -|\xi|^2 \mathrm{Id}_S$.

There is a natural Hermitian form $(\,\cdot\,,\,\cdot\,)$ on S which is invariant under the $\mathrm{Spin}(3,1)$-action on the fibers and is not positive definite. The choice of the unit timelike covector e^0 produces a second Hermitian form.

Definition 3.2. Define a Hermitian inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ on S by

$$\langle \psi, \phi \rangle := (e^0 \cdot \psi, \phi), \qquad \psi, \phi \in \Gamma(S). \tag{3.3}$$

Lemma 3.4. *With respect to* $\langle\,\cdot\,,\,\cdot\,\rangle$*, Clifford multiplication satisfies:*

(1) $e^0\cdot$ *is Hermitian and* $\langle\,\cdot\,,\,\cdot\,\rangle$ *is positive definite.*

(2) If $\xi \in T^*M$ *is tangent to* M*, then* $\xi\cdot$ *is skew-Hermitian:*

$$\langle \xi \cdot \psi, \phi \rangle = -\langle \psi, \xi \cdot \phi \rangle.$$

(3) For $\xi, \eta \in T^*M$, the commutator $[\xi, \eta] := \xi \cdot \eta \cdot -\eta \cdot \xi \cdot$ is Hermitian.

Proof. These are standard consequences of the definition (3.3), the Clifford relations (3.1), and the fact that e^0 is timelike while ξ, η are spacelike. A detailed verification is given in Appendix A. □

The positive definiteness in Lemma 3.4 is essential. All coercive estimates and all nonnegativity statements in the spinor method are formulated using $\langle \cdot , \cdot \rangle$.

3.2. **Two connections and the second fundamental form.** Let V^N be the Levi-Civita connection of $(N, g_{\mu\nu})$. Restricting V^N to M and lifting to the spin bundle gives a connection on S, denoted

$$V \ = \ V^N|_M.$$

This connection is compatible with the indefinite form $(\cdot , \cdot)$, but it is not compatible with $\langle \cdot , \cdot \rangle$, since the latter depends on e^0, and e^0 varies along M.

On the other hand, the Levi-Civita connection D of (M, g) induces a Spin(3)-connection on the Spin(3)-bundle of M, hence a connection on S, denoted ∇. It is compatible with $\langle \cdot , \cdot \rangle$, but not with $(\cdot , \cdot)$.

The difference between V and ∇ is measured by the second fundamental form. Write $h_{ij} = g(V^N_{e_i} e_0, e_j)$. Equivalently, $V^N_{e_i} e_0 = h_{ij} e_j$, and one also has $V^N_{e_i} e_j$ has normal component $-h_{ij} e_0$.

Lemma 3.5. *For any spinor $\psi \in \Gamma(S)$ and any tangent vector e_i,*

$$V_i \psi \ = \ \nabla_i \psi + \frac{1}{2} h_{ij}\, e^j \cdot e^0 \cdot \psi, \tag{3.6}$$

where $V_i := V_{e_i}$ and $\nabla_i := \nabla_{e_i}$.

Proof. This is the standard hypersurface relation between the ambient spin connection and the intrinsic spin connection. A moving-frame derivation compatible with our sign conventions is given in Appendix C. □

Formula (3.6) is the point where the embedding data h_{ij} enters the spinorial argument. It explains why the operator used in the proof is not the intrinsic Dirac operator of (M, g), even though its principal symbol is the same.

3.3. **The hypersurface Dirac operator and the Witten equation.** Using the ambient-induced connection V, define the hypersurface Dirac operator

$$\mathscr{D}\psi \ = \ \sum_{i=1}^{3} e^i \cdot V_i \psi, \qquad \psi \in \Gamma(S). \tag{3.7}$$

This is a first-order elliptic operator on M. Its principal symbol agrees with that of the usual Dirac operator associated to (M, g), but it uses the connection V, hence depends on h_{ij} through (3.6).

The Witten equation is

$$\mathscr{D}\psi = 0. \tag{3.8}$$

On an asymptotically flat end, the spin structure is trivial, so there is a distinguished family of constant spinors on the end. Given such a constant spinor ψ_∞ on an end M_ℓ, we seek solutions ψ of (3.8) satisfying

$$\psi \to \psi_\infty \quad \text{as } r \to \infty \text{ on } M_\ell, \tag{3.9}$$

together with a quantitative decay condition that will be formulated using weighted Sobolev spaces in Section 5.

3.4. **Formal adjoint and integration by parts.** Let $\Omega \subset M$ be a compact domain with smooth boundary $\partial\Omega$, and let ν denote the outward unit normal to $\partial\Omega$ in (M, g). The basic integration by parts identity for $\mathscr{D}$ is the same as for the usual Dirac operator.

Lemma 3.10. *For $\psi, \phi \in \Gamma(S)$,*

$$\int_\Omega \big\langle \mathscr{D}\psi, \phi \big\rangle \, d\mathrm{vol}_g - \int_\Omega \big\langle \psi, \mathscr{D}\phi \big\rangle \, d\mathrm{vol}_g = \int_{\partial\Omega} \big\langle \nu \cdot \psi, \phi \big\rangle \, d\sigma_g. \tag{3.11}$$

In particular, $\mathscr{D}$ is formally self-adjoint with respect to $\langle \cdot\,, \cdot \rangle$.

Proof. In a local orthonormal frame one expands $\mathscr{D}$ as in (3.7), uses that $e^i \cdot$ is skew-Hermitian by Lemma 3.4, and applies the divergence theorem. The dependence of V on the ambient geometry does not change the boundary term in (3.11). A moving-frame verification is recorded in Appendix C. $\square$

Remark 3.12. The boundary integral in (3.11) is the origin of the ADM energy-momentum in the spinor proof. When Ω is a large truncation of an asymptotically flat end and ψ satisfies (3.9), the limit of the boundary expression can be rewritten as an explicit linear functional of (E_ℓ, P_ℓ) determined by ψ_∞.

4. Weitzenböck identity and the boundary term

In this section we record the Weitzenböck-type identity for the hypersurface Dirac operator $\mathscr{D}$, explain how the dominant energy condition enters as a pointwise positivity statement, and isolate the boundary term that will later be identified with the ADM energy-momentum.

Throughout, $\langle \cdot\,, \cdot \rangle$ denotes the positive definite Hermitian product from (3.3), V is the ambient-induced spin connection on S, and

$$\mathscr{D}\psi = \sum_{i=1}^{3} e^i \cdot V_i \psi.$$

We write μ for the Riemannian volume form on (M, g), and for covectors $\alpha, \beta \in T^*N|_M$ we use the Clifford commutator

$$[\alpha, \beta] := \alpha \cdot \beta \cdot - \beta \cdot \alpha \cdot .$$

4.1. **A Weitzenböck formula.** Let R be the scalar curvature of (M, g). Let Ric^N denote the Ricci tensor of $(N, g_{\mu\nu})$, and in an adapted orthonormal frame $\{e_0, e_1, e_2, e_3\}$ along M write

$$R_{00} := \mathrm{Ric}^N(e_0, e_0), \qquad R_{0i} := \mathrm{Ric}^N(e_0, e_i).$$

Proposition 4.1 (Weitzenböck formula)**.** *The hypersurface Dirac operator satisfies*

$$\mathscr{D}^*\mathscr{D} = \mathscr{D}^2 = V^*V + \mathcal{R}, \tag{4.2}$$

where $\mathscr{D}^*$ and V^* are the formal adjoints with respect to $\langle \cdot\,, \cdot \rangle$, and $\mathcal{R} \in \mathrm{End}(S)$ is the Hermitian endomorphism

$$\mathcal{R} = \frac{1}{4}\Big(R + 2R_{00} + 2R_{0i}\, e^0 \cdot e^i \cdot \Big). \tag{4.3}$$

Proof. The identity is verified in a local orthonormal frame adapted to $M \subset N$ by expanding $\mathscr{D}^2$, converting curvature terms to the spin curvature action, and tracking the contributions of the second fundamental form through the connection V. The resulting zeroth-order term is (4.3), while the remaining second-order part is the covariant Laplacian V^*V with respect to $\langle\,\cdot\,,\,\cdot\,\rangle$. A complete derivation with sign conventions is given in Appendix C. □

4.2. **Energy condition as pointwise positivity.** Assume the spacetime satisfies the Einstein equation

$$G_{\mu\nu} = 8\pi T_{\mu\nu}.$$

In an adapted orthonormal frame along M, set

$$T_{00} := T(e_0, e_0), \qquad T_{0i} := T(e_0, e_i).$$

The constraint equations identify T_{00} and T_{0i} with the energy and momentum densities ρ and J_i from (1.7).

Lemma 4.4. *Under the Einstein equation, the endomorphism $\mathcal{R}$ from* (4.3) *can be written as*

$$\mathcal{R} = 4\pi\Big(T_{00} + T_{0i}\, e^0 \cdot e^i \cdot\Big). \tag{4.5}$$

Proof. This is an algebraic consequence of the Gauss–Codazzi relations and the Einstein equation, expressing the combination $R + 2R_{00} + 2R_{0i}e^0 \cdot e^i\cdot$ in terms of T_{00} and T_{0i}. The computation is recorded in Appendix C. □

Lemma 4.6. *Assume the dominant energy condition $\rho \geq |J|_g$. Then $\mathcal{R}$ is pointwise nonnegative in the sense that*

$$\langle \psi, \mathcal{R}\psi \rangle \geq 0 \qquad \textit{for all } \psi \in S. \tag{4.7}$$

Proof. By Lemma 4.4, it suffices to consider the Hermitian endomorphism

$$A := T_{00} + T_{0i}\, e^0 \cdot e^i \cdot\,.$$

Let $b_i := T_{0i}$ and set $B := b_i\, e^0 \cdot e^i\cdot$. By Lemma 3.4, each $e^0 \cdot e^i\cdot$ is Hermitian, and using the Clifford relations one checks that

$$B^2 = |b|^2\,\mathrm{Id}_S, \qquad |b|^2 := \sum_{i=1}^{3} b_i^2.$$

Hence the eigenvalues of B are $\pm|b|$, and the eigenvalues of $A = T_{00}\mathrm{Id}_S + B$ are $T_{00} \pm |b|$. The dominant energy condition implies $T_{00} \geq |b|$, so $A \geq 0$, which gives (4.7) after multiplying by 4π. □

4.3. **Integral identity and the boundary term.** Let $\Omega \subset M$ be a compact domain with smooth boundary. Applying Proposition 4.1 to a spinor ψ, taking the $\langle\,\cdot\,,\,\cdot\,\rangle$-inner product with ψ, and integrating by parts yields the following identity.

Proposition 4.8 (Integral form)**.** *For any smooth spinor ψ on Ω,*

$$\int_\Omega \Big(|V\psi|^2 + \langle \psi, \mathcal{R}\psi\rangle - |\mathscr{D}\psi|^2\Big)\,\mathrm{dvol}_g = \int_{\partial\Omega} \mathcal{B}(\psi), \tag{4.9}$$

where

$$\mathcal{B}(\psi) := -\frac{1}{2}\,\big\langle \psi,\, [e^i, e^j] \cdot V_j\psi \big\rangle\,(e_i \lrcorner \mu). \tag{4.10}$$

Here $V_j := V_{e_j}$, and $e_i \lrcorner \mu$ is the contraction of the volume form with the vector e_i.

Proof. Using (4.2) we have

$$\langle \psi, \mathscr{D}^2\psi\rangle = \langle \psi, V^*V\psi\rangle + \langle \psi, \mathcal{R}\psi\rangle.$$

Integrating over Ω and using the defining property of the formal adjoints with respect to $\langle\cdot,\cdot\rangle$ gives the bulk terms $\int_\Omega |V\psi|^2$ and $\int_\Omega |\mathscr{D}\psi|^2$, together with the boundary contribution $\mathcal{B}(\psi)$. A moving-frame computation of the boundary expression (4.10), matching the conventions in Section 3, is given in Appendix C. □

When ψ satisfies the Witten equation $\mathscr{D}\psi = 0$, Proposition 4.8 simplifies to

$$\int_\Omega \Big(|V\psi|^2 + \langle \psi, \mathcal{R}\psi\rangle \Big)\, d\mathrm{vol}_g = \int_{\partial\Omega} \mathcal{B}(\psi). \tag{4.11}$$

By Lemma 4.6, the integrand on the left-hand side is pointwise nonnegative under the dominant energy condition.

4.4. **The boundary term at infinity.** Assume now that (M, g, h) is asymptotically flat in the sense of Definition 2.1 and has ends $M_1, \dots, M_k$. Let $M_R \subset M$ be the truncation obtained by removing the regions $\{r > R\}$ in each end, so that $\partial M_R = \bigsqcup_{\ell=1}^k S_R^{(\ell)}$ is a union of large coordinate spheres. On each end the spin structure is trivial, hence there is a distinguished space of constant spinors; let $\psi_{\infty,\ell}$ denote a constant spinor on M_ℓ.

Proposition 4.12 (Boundary limit)**.** *Let ψ be a smooth spinor on M such that on each end M_ℓ,*

$$\psi = \psi_{\infty,\ell} + o(1), \qquad V\psi \in L^2(M_\ell),$$

and the decay is sufficient to justify passing to the limit in (4.9) with $\Omega = M_R$. Then

$$\lim_{R\to\infty} \int_{\partial M_R} \mathcal{B}(\psi) = 4\pi \sum_{\ell=1}^k \Big(E_\ell \,\langle \psi_{\infty,\ell}, \psi_{\infty,\ell}\rangle + \big\langle \psi_{\infty,\ell},\, (P_\ell)_j\, e^0 \cdot dx^j \cdot \psi_{\infty,\ell}\big\rangle \Big), \tag{4.13}$$

where (E_ℓ, P_ℓ) are the ADM energy-momentum of M_ℓ from (2.8) and (2.11), and $\{dx^j\}$ denotes the standard coframe on the asymptotic chart of the end.

Proof. On each end one expresses $\mathcal{B}(\psi)$ in the asymptotic chart, replaces the orthonormal coframe by the coordinate coframe up to $O(r^{-1})$ errors, and uses the decay of g and h to identify the leading term of the flux with the ADM expressions. The detailed expansion, including the normalizations and the reduction to (E_ℓ, P_ℓ), is given in Appendix C. □

Corollary 4.14. *For each end M_ℓ, the endomorphism*

$$\mathcal{P}_\ell := (P_\ell)_j\, e^0 \cdot dx^j \cdot$$

is Hermitian and satisfies $\mathcal{P}_\ell^2 = |P_\ell|^2\, \mathrm{Id}_S$. In particular, its eigenvalues are $\pm|P_\ell|$, and for an eigenvector $\psi_{\infty,\ell}$ with eigenvalue $-|P_\ell|$,

$$E_\ell \,\langle \psi_{\infty,\ell}, \psi_{\infty,\ell}\rangle + \big\langle \psi_{\infty,\ell},\, \mathcal{P}_\ell \psi_{\infty,\ell}\big\rangle = (E_\ell - |P_\ell|)\, \langle \psi_{\infty,\ell}, \psi_{\infty,\ell}\rangle. \tag{4.15}$$

Proof. Hermitian symmetry follows from Lemma 3.4. Using the Clifford relations and that dx^j are tangent covectors, one computes

$$\mathcal{P}_\ell^2 = (P_\ell)_j (P_\ell)_m\, e^0 \cdot dx^j \cdot e^0 \cdot dx^m \cdot = (P_\ell)_j (P_\ell)_m\, dx^j \cdot dx^m \cdot = |P_\ell|^2\, \mathrm{Id}_S.$$

The remaining statements follow. □

Proposition 4.8, Lemma 4.6, and Proposition 4.12 together show that any solution of the Witten equation with prescribed asymptotic constants produces an inequality of the form

$$0 \leq 4\pi \sum_{\ell=1}^{k} \Big(E_\ell \langle \psi_{\infty,\ell}, \psi_{\infty,\ell} \rangle + \langle \psi_{\infty,\ell}, \mathcal{P}_\ell \psi_{\infty,\ell} \rangle \Big).$$

Choosing $\psi_{\infty,\ell}$ as in Corollary 4.14 isolates the factor $E_\ell - |P_\ell|$ on each end. The remaining step is to construct, for each prescribed $\psi_{\infty,\ell}$, a global spinor ψ solving $\mathscr{D}\psi = 0$ with $\psi \to \psi_{\infty,\ell}$ on M_ℓ.

5. Solving the Witten equation

In this section we prove the existence and uniqueness of solutions to the Witten equation

$$\mathscr{D}\psi = 0 \tag{5.1}$$

with prescribed asymptotic behavior on each asymptotically flat end. The organization follows Parker–Taubes: one works in weighted Sobolev spaces adapted to the ends, proves an a priori estimate for $\mathscr{D}$, and deduces that $\mathscr{D}$ is an isomorphism between the relevant weighted spaces. The asymptotically constant harmonic spinors are then obtained by a cutoff and correction argument.

5.1. **Basic analytic input.** We write μ for the Riemannian volume form on (M, g) and $\langle \cdot, \cdot \rangle$ for the positive definite inner product on the Dirac spinor bundle $S \to M$ fixed in Section 3. The associated pointwise norm is $|\psi| = \langle \psi, \psi \rangle^{1/2}$. For $1 \leq p < \infty$ we set

$$\|\psi\|_{L^p(M)} = \left(\int_M |\psi|^p \, \mu \right)^{1/p}, \qquad \|\psi\|_{L^\infty(M)} = \operatorname*{ess\,sup}_M |\psi|.$$

We use ∇ for the Riemannian spin connection on S which is compatible with $\langle \cdot, \cdot \rangle$. The hypersurface connection V and the hypersurface Dirac operator $\mathscr{D} = \sum_{i=1}^{3} e^i \cdot V_i$ were defined in Section 3.

Definition 5.2. The Sobolev space $H^1(M; S)$ is the completion of $C_0^\infty(M; S)$ in the norm

$$\|\psi\|_{H^1} = \|\psi\|_{L^2} + \|\nabla\psi\|_{L^2}.$$

More generally, for $p \geq 1$ we denote by $W^{1,p}(M; S)$ the completion of $C_0^\infty(M; S)$ in the norm $\|\psi\|_{W^{1,p}} = \|\psi\|_{L^p} + \|\nabla\psi\|_{L^p}$.

We record two standard tools that will be used repeatedly.

Lemma 5.3 (Sobolev and Morrey in dimension 3)**.** *There exists $C > 0$ such that for all $\psi \in C_0^\infty(M; S)$,*

$$\|\psi\|_{L^6} \leq C\big(\|\nabla\psi\|_{L^2} + \|\psi\|_{L^2}\big).$$

If $p > 3$ and $\psi \in W^{1,p}_{\mathrm{loc}}(M; S)$, then ψ has a locally Hölder continuous representative.

Theorem 5.4 (Lax–Milgram)**.** *Let $\mathbb{H}$ be a real Hilbert space with inner product $(\cdot, \cdot)_{\mathbb{H}}$ and norm $\|\cdot\|_{\mathbb{H}}$. Assume $B : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is a bounded bilinear form which is coercive, meaning that there exists $\alpha > 0$ with*

$$B(u, u) \geq \alpha \|u\|_{\mathbb{H}}^2 \quad \textit{for all } u \in \mathbb{H}.$$

Then for any bounded linear functional $\ell : \mathbb{H} \to \mathbb{R}$ there exists a unique $u \in \mathbb{H}$ such that

$$B(u, v) = \ell(v) \quad \text{for all } v \in \mathbb{H}.$$

We also use the basic elliptic regularity principle for Dirac-type operators.

Theorem 5.5 (Local elliptic regularity)**.** *Let $\Omega \subset M$ be relatively compact and let $\psi \in L^p_{\mathrm{loc}}(\Omega; S)$ satisfy $\mathscr{D}\psi = \eta$ in the distributional sense on Ω, with $\eta \in L^p_{\mathrm{loc}}(\Omega; S)$ and $1 < p < \infty$. Then $\psi \in W^{1,p}_{\mathrm{loc}}(\Omega; S)$. If η is smooth on Ω, then ψ is smooth on Ω.*

5.2. **Weighted Sobolev spaces and decay on the ends.** Fix $R \geq 1$ large enough so that each asymptotically flat end M_ℓ contains the exterior of a Euclidean ball of radius R under the chosen asymptotic chart. For each ℓ and each $r \geq R$ we write $M_{\ell,r} \subset M_\ell$ for the region corresponding to $\{x \in \mathbb{R}^3 : |x| > r\}$.

Choose a smooth function $\sigma : M \to [1, \infty)$ such that

(i) $\sigma \geq 1$ on M,
(ii) $\sigma = r$ on each $M_{\ell,2R}$ (here $r = |x|$ in the asymptotic coordinates),
(iii) $\sigma \equiv 1$ on the compact region $M \setminus \bigcup_\ell M_{\ell,R}$.

Definition 5.6 (Parker–Taubes weighted spaces)**.** Let $p \geq 2$, let $s \in \{0, 1\}$, and let δ satisfy

$$\frac{1}{2} - \frac{3}{p} \leq \delta \leq 2 - \frac{3}{p}.$$

Define $\mathcal{H}_{s,\delta,p}(M; S)$ to be the completion of $C^\infty_0(M; S)$ in the norm

$$\|\psi\|_{0,\delta,p} = \|\sigma^\delta \psi\|_{L^p}, \qquad \|\psi\|_{1,\delta,p} = \|\sigma^{1+\delta} \nabla\psi\|_{L^p} + \|\sigma^\delta \psi\|_{L^p}. \tag{5.7}$$

For notational convenience we write $\mathcal{H}_{s,\delta,p}$ when $(M; S)$ is clear.

The parameter δ measures decay: if $\psi \in \mathcal{H}_{0,\delta,p}$ with $\delta > 0$, then $\sigma^\delta \psi \in L^p$ forces ψ to be small along almost every ray in each end, and when $p > 3$ the Morrey embedding gives pointwise decay. This is the regime used for the correction term in the asymptotically constant construction below.

5.3. **Mapping properties and a basic identity.** Recall from Section 3 that the hypersurface connection V differs from ∇ by a zeroth order term involving the second fundamental form h_{ij}. In a local orthonormal coframe $\{e^i\}$ on M,

$$V_i = \nabla_i - \frac{1}{2} \sum_{j=1}^{3} h_{ij}\, e^0 \cdot e^j, \qquad \mathscr{D} = \sum_{i=1}^{3} e^i \cdot V_i.$$

Proposition 5.8. *Assume the asymptotic decay hypotheses from Section 2 and Section 3, so that $|h| = O(\sigma^{-2})$ on each end. If $p \geq 2$ and either $0 < \delta < 2 - \frac{3}{p}$ or $(p, \delta) = (2, -1)$, then the operators*

$$V : \mathcal{H}_{1,\delta,p} \longrightarrow \mathcal{H}_{0,\delta+1,p}, \qquad \mathscr{D} : \mathcal{H}_{1,\delta,p} \longrightarrow \mathcal{H}_{0,\delta+1,p}$$

are bounded linear maps.

Proof. Using the displayed formulas for V_i and $\mathscr{D}$, the weight $\sigma^{1+\delta}$ on $\nabla\psi$ in (5.7), and the estimate $\sigma^2|h| \leq C$ on the ends (with $\sigma \equiv 1$ on the compact part), one obtains

$$\|\sigma^{1+\delta} V\psi\|_{L^p} \leq C\big(\|\sigma^{1+\delta}\nabla\psi\|_{L^p} + \|\sigma^\delta\psi\|_{L^p}\big), \qquad \|\sigma^{1+\delta}\mathscr{D}\psi\|_{L^p} \leq C\big(\|\sigma^{1+\delta}\nabla\psi\|_{L^p} + \|\sigma^\delta\psi\|_{L^p}\big),$$

which is the claimed boundedness. □

The next identity is the analytic starting point. It is the L^2 form of the Weitzenböck formula from Section 4.

Lemma 5.9. *Assume the dominant energy condition so that the curvature endomorphism $\mathcal{R}$ from Section 4 satisfies $\langle \psi, \mathcal{R}\psi\rangle \geq 0$ pointwise. Then for every $\psi \in C_0^\infty(M; S)$,*

$$\|\mathscr{D}\psi\|_{L^2}^2 = \|V\psi\|_{L^2}^2 + \int_M \langle \psi, \mathcal{R}\psi\rangle\, \mu. \tag{5.10}$$

In particular, $\|\mathscr{D}\psi\|_{L^2} \geq \|V\psi\|_{L^2}$.

Proof. Apply the Weitzenböck identity $\mathscr{D}^*\mathscr{D} = V^*V + \mathcal{R}$ to ψ, take the L^2 inner product with ψ, and integrate by parts. Compact support eliminates boundary terms. □

Proposition 5.11 (Injectivity)**.** *Assume the dominant energy condition. If $p \geq 2$ and either $0 < \delta < 2 - \frac{3}{p}$ or $(p, \delta) = (2, -1)$, then*

$$\mathscr{D} : \mathcal{H}_{1,\delta,p} \longrightarrow \mathcal{H}_{0,\delta+1,p}$$

is injective.

Proof. Let $\psi \in \mathcal{H}_{1,\delta,p}$ satisfy $\mathscr{D}\psi = 0$. By local elliptic regularity (Theorem 5.5) ψ is smooth, and by density one may justify (5.10) for ψ by approximation. Hence $V\psi = 0$ and $\int_M \langle \psi, \mathcal{R}\psi\rangle\, \mu = 0$, so $\langle \psi, \mathcal{R}\psi\rangle \equiv 0$. The condition $\psi \in \mathcal{H}_{0,\delta,p}$ with $\delta > 0$ implies that $|\psi|$ has arbitrarily small values along almost every ray to infinity in each end. Along any smooth curve γ in an end one has the differential inequality

$$\frac{d}{ds}|\psi(\gamma(s))| \leq C\, \sigma(\gamma(s))^{-2}\, |\psi(\gamma(s))|$$

because $V\psi = 0$ and $V - \nabla$ is $O(\sigma^{-2})$. Integrating this inequality shows that if $|\psi|$ tends to 0 along some path to infinity, then $|\psi|$ vanishes identically. This gives $\psi \equiv 0$. The case $(p, \delta) = (2, -1)$ is similar and uses that $\sigma^{-1}\psi \in L^2$ forces smallness at infinity on almost every ray. □

5.4. **A coercive estimate in the Hilbert case.** Following Parker–Taubes, we single out the Hilbert space case $p = 2$, $\delta = -1$. For $\psi \in C_0^\infty(M; S)$ define

$$\|\psi\|_{\mathcal{H}}^2 = \|V\psi\|_{L^2}^2 + \|\sigma^{-1}\psi\|_{L^2}^2. \tag{5.12}$$

Let $\mathcal{H}$ denote the completion of $C_0^\infty(M; S)$ in this norm. By Proposition 5.8, $\mathcal{H}$ coincides with $\mathcal{H}_{1,-1,2}$ as a topological vector space, and (5.12) is equivalent to the weighted Sobolev norm $\|\cdot\|_{1,-1,2}$.

The key estimate is that $\mathscr{D}$ controls $\|\cdot\|_{\mathcal{H}}$.

Lemma 5.13 (Coercive estimate)**.** *Assume the dominant energy condition. There exists a constant $c > 0$ such that for all $\psi \in C_0^\infty(M; S)$,*

$$\|\psi\|_{\mathcal{H}}^2 \leq c\, \|\mathscr{D}\psi\|_{L^2}^2. \tag{5.14}$$

Consequently, $\mathscr{D} : \mathcal{H} \to L^2(M; S)$ has closed range and satisfies $\|\psi\|_{\mathcal{H}} \leq c^{1/2}\|\mathscr{D}\psi\|_{L^2}$ for all $\psi \in \mathcal{H}$.

Proof. The estimate is obtained by combining (5.10) with a Hardy-type inequality on each end and a cutoff decomposition into an interior part and an exterior part.

On an end, comparison with the Euclidean Hardy inequality gives a bound of the form

$$\|\sigma^{-1}\psi\|^2_{L^2(M_{\ell,2R})} \le C\,\|\nabla\psi\|^2_{L^2(M_{\ell,2R})}$$

for R sufficiently large. Since $V - \nabla$ is $O(\sigma^{-2})$, one also compares $\|\nabla\psi\|_{L^2}$ to $\|V\psi\|_{L^2}$ on the ends. A cutoff argument then yields a global inequality $\|\sigma^{-1}\psi\|^2_{L^2} \le C\|V\psi\|^2_{L^2}$. Using $\|\mathscr{D}\psi\|_{L^2} \ge \|V\psi\|_{L^2}$ from Lemma 5.9 gives (5.14). □

The Hilbert estimate gives existence for the square $\mathscr{D}^2$ by a direct variational argument.

Proposition 5.15 (Solving $\mathscr{D}^2 u = \eta$)**.** *Assume the dominant energy condition. For each $\eta \in C_0^\infty(M;S)$ there exists a unique $u \in \mathcal{H}$ such that*

$$\int_M \langle \mathscr{D}u, \mathscr{D}v\rangle\,\mu = \int_M \langle \eta, v\rangle\,\mu \quad \textit{for all } v \in \mathcal{H}. \tag{5.16}$$

Moreover u is smooth and satisfies $\mathscr{D}^2 u = \eta$ pointwise, and

$$\|u\|_{\mathcal{H}} \le C\|\eta\|_{L^2}$$

for a constant C independent of η.

Proof. Define a bilinear form $B(u,v) = \int_M \langle \mathscr{D}u, \mathscr{D}v\rangle\,\mu$ on $\mathcal{H}$. By Lemma 5.13, B is coercive on $\mathcal{H}$ because $B(u,u) = \|\mathscr{D}u\|^2_{L^2} \ge c^{-1}\|u\|^2_{\mathcal{H}}$. The functional $\ell(v) = \int_M \langle \eta, v\rangle\,\mu$ is bounded on $\mathcal{H}$ since $\sigma^{-1}v \in L^2$ and $\sigma \ge 1$. Apply Lax–Milgram (Theorem 5.4) to obtain a unique $u \in \mathcal{H}$ solving (5.16). Local elliptic regularity (Theorem 5.5) upgrades u to a smooth solution of $\mathscr{D}^2 u = \eta$. □

Corollary 5.17 (Solving $\mathscr{D}\psi = \eta$ for compactly supported η)**.** *Assume the dominant energy condition. For each $\eta \in C_0^\infty(M;S)$ there exists a unique $\psi \in \mathcal{H}$ such that*

$$\mathscr{D}\psi = \eta$$

and $\|\psi\|_{\mathcal{H}} \le C\|\eta\|_{L^2}$.

Proof. Let $u \in \mathcal{H}$ be the solution from Proposition 5.15 and set $\psi = \mathscr{D}u$. Then $\psi \in \mathcal{H}$ and $\mathscr{D}\psi = \mathscr{D}^2 u = \eta$. Uniqueness follows from injectivity (Proposition 5.11 in the case $(p,\delta) = (2,-1)$). □

5.5. **The weighted L^p a priori estimate and surjectivity.** We now return to general (p,δ) in the Parker–Taubes range. The core input is an a priori estimate controlling the weighted $W^{1,p}$ norm of ψ by the weighted L^p norm of $\mathscr{D}\psi$.

Proposition 5.18 (A priori estimate)**.** *Let $p \ge 2$ and $0 < \delta < 2 - \frac{3}{p}$. There exists a constant $c = c(\delta,p) > 0$ such that for all $\psi \in C_0^\infty(M;S)$,*

$$\|\psi\|_{1,\delta,p} \le c\,\|\mathscr{D}\psi\|_{0,\delta+1,p}. \tag{5.19}$$

Consequently, $\mathscr{D} : \mathcal{H}_{1,\delta,p} \to \mathcal{H}_{0,\delta+1,p}$ has closed range and satisfies (5.19) for all $\psi \in \mathcal{H}_{1,\delta,p}$.

Proof. The estimate is proved by a cutoff decomposition $\psi = \psi_{\text{in}} + \psi_{\text{ex}}$ into an interior part supported in a fixed compact set and an exterior part supported on the ends. On the compact part, standard elliptic estimates bound $\|\psi_{\text{in}}\|_{W^{1,p}}$ by $\|\mathscr{D}\psi_{\text{in}}\|_{L^p}$ plus a lower order term, and the weight is harmless because $\sigma \equiv 1$ there. On each end one extends the exterior part to $\mathbb{R}^3$ using the asymptotic trivialization of S, compares $\mathscr{D}$ with the Euclidean Dirac operator, and applies the

Euclidean weighted isomorphism estimate in $\mathbb{R}^3$ (proved in Appendix B). The decay assumptions ensure that the error terms are small for a large cutoff radius, and the pieces combine to give (5.19). $\square$

Proposition 5.20 (Surjectivity)**.** *Let $p \geq 2$ and $0 < \delta < 2 - \frac{3}{p}$. Then*

$$\mathscr{D} : \mathcal{H}_{1,\delta,p} \longrightarrow \mathcal{H}_{0,\delta+1,p}$$

is surjective. Moreover, for any $\eta \in \mathcal{H}_{0,\delta+1,p}$, the unique solution $\psi \in \mathcal{H}_{1,\delta,p}$ to $\mathscr{D}\psi = \eta$ satisfies

$$\|\psi\|_{1,\delta,p} \leq c(\delta, p)\, \|\eta\|_{0,\delta+1,p},$$

with $c(\delta, p)$ as in Proposition 5.18.

Proof. Let $\eta \in \mathcal{H}_{0,\delta+1,p}$ and choose $\eta_j \in C_0^\infty(M; S)$ with $\eta_j \to \eta$ in $\|\cdot\|_{0,\delta+1,p}$. By Corollary 5.17, for each j there is $\psi_j \in \mathcal{H}$ with $\mathscr{D}\psi_j = \eta_j$. Proposition 5.18 implies $\{\psi_j\}$ is Cauchy in $\mathcal{H}_{1,\delta,p}$ because

$$\|\psi_j - \psi_k\|_{1,\delta,p} \leq c(\delta, p)\, \|\eta_j - \eta_k\|_{0,\delta+1,p}.$$

Let ψ be the limit in $\mathcal{H}_{1,\delta,p}$. The boundedness of $\mathscr{D}$ (Proposition 5.8) gives $\mathscr{D}\psi = \eta$. Uniqueness follows from injectivity (Proposition 5.11). $\square$

Theorem 5.21 (Weighted isomorphism for $\mathscr{D}$)**.** *For $p \geq 2$ and $0 < \delta < 2 - \frac{3}{p}$, or for $(p, \delta) = (2, -1)$, the operator*

$$\mathscr{D} : \mathcal{H}_{1,\delta,p} \longrightarrow \mathcal{H}_{0,\delta+1,p}$$

is an isomorphism with bounded inverse. In particular, $\mathscr{D}^{-1}$ defines a Green operator for $\mathscr{D}$ on M in the Parker–Taubes weighted class.

Proof. Injectivity is Proposition 5.11. Surjectivity is Proposition 5.20 for $0 < \delta < 2 - \frac{3}{p}$ and Corollary 5.17 for $(p, \delta) = (2, -1)$. The a priori estimates (5.14) and (5.19) give boundedness of the inverse. $\square$

5.6. **Existence of asymptotically constant solutions.** We now construct the solutions to (5.1) which approach a prescribed constant spinor on each end. Fix constant spinors $\{\psi_{\infty,\ell}\}$ on the ends, defined using the asymptotic trivialization of S.

Proposition 5.22 (Existence and uniqueness of Witten spinors)**.** *Assume the dominant energy condition. Fix an exponent $p > 3$ and choose δ with*

$$0 < \delta < 1 - \frac{3}{p}.$$

Then there exists a unique smooth spinor ψ on M such that

(i) $\mathscr{D}\psi = 0$ *on* M,
(ii) for every $\varepsilon > 0$ *one has* $\lim_{r\to\infty} r^{1-\varepsilon}\, |\psi - \psi_{\infty,\ell}| = 0$ *in each end* M_ℓ,
(iii) $\psi - \psi_{\infty,\ell} \in \mathcal{H}_{1,\delta,p}$ *on each end, and hence* $\psi - \psi_{\infty,\ell}$ *decays in* C^0 *by Morrey's theorem.*

Proof. Choose a cutoff function $\beta = \beta_R$ on M with $0 \leq \beta \leq 1$, $\beta \equiv 1$ on each $M_{\ell,3R}$, and $\beta \equiv 0$ on $M \setminus \bigcup_\ell M_{\ell,2R}$. Define the approximate spinor

$$\psi_0 = \sum_\ell \beta\, \psi_{\infty,\ell},$$

viewed as a globally defined smooth spinor on M which agrees with $\psi_{\infty,\ell}$ near infinity in each end. Since $\psi_{\infty,\ell}$ is constant in the asymptotic trivialization and β is constant near infinity, the decay hypotheses imply

$$\mathscr{D}\psi_0 = O(r^{-2}) \quad \text{on each end.}$$

With the chosen (p,δ) this gives $\mathscr{D}\psi_0 \in \mathcal{H}_{0,\delta+1,p}$ because $\sigma^{\delta+1}r^{-2} \in L^p$ is equivalent to $\delta < 1 - \frac{3}{p}$. By Theorem 5.21 there exists a unique $\psi_1 \in \mathcal{H}_{1,\delta,p}$ solving

$$\mathscr{D}\psi_1 = -\mathscr{D}\psi_0.$$

Set $\psi = \psi_0 + \psi_1$. Then $\mathscr{D}\psi = 0$ and $\psi - \psi_{\infty,\ell} = \psi_1$ on each end outside a compact set. Since $p > 3$, Morrey's theorem implies that ψ_1 is continuous and decays at infinity, which yields the stated asymptotic property. Uniqueness follows because the difference of two such solutions lies in $\mathcal{H}_{1,\delta,p}$ and is annihilated by $\mathscr{D}$, hence is zero by Proposition 5.11. □

5.7. **Regularity and asymptotic control.** Finally we record the two consequences needed later in the boundary term computation.

Lemma 5.23. *Let ψ be the solution from Proposition 5.22. Then ψ is smooth on M and satisfies $V\psi \in L^2(M)$. Moreover, for any $p > 3$ and any $0 < \delta < 1 - \frac{3}{p}$, if $\psi - \psi_{\infty,\ell} \in \mathcal{H}_{1,\delta,p}$ on an end, then*

$$|\psi - \psi_{\infty,\ell}(x)| \to 0 \quad \textit{and} \quad |V\psi(x)| = o(r^{-1}) \quad \textit{as } r \to \infty$$

along that end.

Proof. Smoothness follows from Theorem 5.5 applied to $\mathscr{D}\psi = 0$. The inclusion $V\psi \in L^2$ follows from $\mathscr{D}\psi = 0$ together with the decay of $\psi - \psi_{\infty,\ell}$ and the relation between V and ∇ on the ends. The pointwise decay of $\psi - \psi_{\infty,\ell}$ follows from Morrey's theorem and the weighted norm. The decay of $V\psi$ follows from the weighted control of $\nabla(\psi - \psi_{\infty,\ell})$ and the $O(r^{-2})$ size of $V - \nabla$. □

6. Proof of the main theorem and rigidity

6.1. **Completion of the proof of $E \geq |P|$.** Fix an end M_ℓ of an asymptotically flat initial data set (M, g, h) satisfying the dominant energy condition. Choose asymptotic constant spinors $\psi_{\infty,m}$ on each end M_m so that

$$\psi_{\infty,m} = 0 \quad (m \neq \ell), \qquad \psi_{\infty,\ell} \neq 0.$$

Let ψ be the corresponding Witten spinor given by Proposition 5.22, so that $\mathscr{D}\psi = 0$ and $\psi \to \psi_{\infty,m}$ on M_m.

Apply Proposition 4.8 with $\Omega = M_R$ and use $\mathscr{D}\psi = 0$ to obtain

$$\int_{M_R} \Big(|V\psi|^2 + \langle \psi, \mathcal{R}\psi \rangle \Big) \, \mathrm{dvol}_g = \int_{\partial M_R} \mathcal{B}(\psi). \tag{6.1}$$

By Lemma 4.6, the integrand on the left is pointwise nonnegative. Letting $R \to \infty$ and using Proposition 4.12 yields

$$0 \leq 4\pi \Big(E_\ell \, \langle \psi_{\infty,\ell}, \psi_{\infty,\ell} \rangle + \big\langle \psi_{\infty,\ell}, \, (P_\ell)_j \, e^0 \cdot dx^j \cdot \psi_{\infty,\ell} \big\rangle \Big). \tag{6.2}$$

Set $\mathcal{P}_\ell = (P_\ell)_j \, e^0 \cdot dx^j \cdot$. By Corollary 4.14, $\mathcal{P}_\ell$ is Hermitian with eigenvalues $\pm|P_\ell|$. Choose $\psi_{\infty,\ell}$ to be an eigenvector for the eigenvalue $-|P_\ell|$. Then (6.2) becomes

$$0 \leq 4\pi \left(E_\ell - |P_\ell| \right) \langle \psi_{\infty,\ell}, \psi_{\infty,\ell} \rangle,$$

and since $\psi_{\infty,\ell} \neq 0$ we conclude $E_\ell \geq |P_\ell|$. As ℓ was arbitrary, this proves Theorem 1.10.

6.2. **The equality case.** Fix an end M_ℓ and assume $m_\ell = 0$, so $E_\ell = |P_\ell|$. Choose $\psi_{\infty,m}$ as above with $\psi_{\infty,m} = 0$ for $m \neq \ell$, and choose $\psi_{\infty,\ell}$ to be an eigenvector of $\mathcal{P}_\ell$ with eigenvalue $-|P_\ell|$. Let ψ be the corresponding Witten spinor.

With this choice, the right-hand side of (6.2) vanishes, hence taking $R \to \infty$ in (6.1) gives

$$\int_M \Big(|V\psi|^2 + \langle \psi, \mathcal{R}\psi \rangle \Big)\, d\mathrm{vol}_g = 0. \tag{6.3}$$

Since both terms in the integrand are pointwise nonnegative, (6.3) implies

$$V\psi \equiv 0, \qquad \langle \psi, \mathcal{R}\psi \rangle \equiv 0. \tag{6.4}$$

In particular, ψ is V-parallel and tends to the nonzero constant $\psi_{\infty,\ell}$ on M_ℓ, so ψ is nowhere zero on M.

Using the Einstein equation, Lemma 4.4 identifies

$$\mathcal{R} = 4\pi \Big(T_{00} + T_{0i}\, e^0 \cdot e^i \cdot \Big).$$

Together with (6.4) and the fact that ψ is nowhere zero, this forces the endomorphism $\mathcal{R}$ to vanish along M. Consequently $T_{00} = 0$ and $T_{0i} = 0$ along M, hence $\rho \equiv 0$ and $J \equiv 0$.

Finally, the condition $V\psi \equiv 0$ implies that the curvature of the connection V annihilates ψ. Choosing two linearly independent asymptotic constants in the $-|P_\ell|$ eigenspace produces two linearly independent V-parallel spinors on M. The resulting holonomy reduction forces the curvature of V to vanish, and the curvature identities relating the V-curvature to the spacetime curvature and the second fundamental form then yield that the spacetime is flat in a neighborhood of M and $h \equiv 0$. In particular, (M, g) is Euclidean and (M, g, h) arises from a spacelike hyperplane in Minkowski space. This proves Theorem 1.12.

6.3. **A brief comment.** There is a second proof of the positive energy theorem using minimal hypersurfaces, due to Schoen and Yau. The spinor method used here has a different character: the inequality is obtained from the positivity in (6.1) together with the construction of Witten spinors, and the energy-momentum enters through the boundary term at infinity. We do not discuss extensions to other asymptotic geometries or related mass invariants.

Appendix A. A quick guide to Clifford algebras and spinors

This appendix records the linear algebra behind the spinor bundle $S \to M$ used in Section 3. Our conventions follow Parker–Taubes.

A.1. **Clifford algebras and a 2×2 matrix model.** Let $(\mathbb{R}^{3,1}, g)$ be Minkowski space with signature $(-,+,+,+)$. Write $\mathrm{Cl}(\mathbb{R}^{3,1})$ for the real Clifford algebra generated by $\mathbb{R}^{3,1}$ with relations

$$x \cdot y \cdot + y \cdot x \cdot = -2\, g(x, y)\, \mathrm{Id}. \tag{A.1}$$

In particular, $x \cdot x \cdot = -g(x, x)\mathrm{Id}$.

A concrete model convenient for Parker–Taubes starts from the fundamental representation $V = \mathbb{C}^2$ of $SL(2,\mathbb{C})$. Let $\bar{V}$ denote the complex-conjugate representation, and identify $V \otimes \bar{V}$ with the space of complex-linear maps $V^* \to V$. Conjugate transpose gives an involution $x \mapsto x^*$ on $V \otimes \bar{V}$, and its fixed set is a real 4-dimensional subspace W. Choosing a basis of V identifies W with the Hermitian 2×2 matrices, hence with $\mathbb{R}^{3,1}$ via

(A.2)

$$x = (x^0, x^1, x^2, x^3) \longleftrightarrow X(x) = \begin{pmatrix} x^0 + x^1 & x^2 + ix^3 \\ x^2 - ix^3 & x^0 - x^1 \end{pmatrix}, \qquad \det X(x) = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2.$$

Thus the determinant on W recovers the Minkowski quadratic form up to the sign convention in (A.2).

A.2. **Spin groups in dimension 3 and 3 + 1.** The identification (A.2) makes the standard double cover $\mathrm{Spin}^+(3,1) \to SO^+(3,1)$ very explicit. For $A \in SL(2,\mathbb{C})$ and a Hermitian matrix $X \in W$, define

$$X \longmapsto AXA^*.$$

This action preserves $\det X$, hence induces a homomorphism $SL(2,\mathbb{C}) \to SO^+(3,1)$. Its kernel is $\{\pm I\}$, so one obtains the standard isomorphism

$$\mathrm{Spin}^+(3,1) \cong SL(2,\mathbb{C}).$$

A choice of a unit future timelike covector e^0 (equivalently a time orientation) picks out the subgroup $\mathrm{Spin}(3) \subset \mathrm{Spin}^+(3,1)$ stabilizing e^0. Under the model above this subgroup is $SU(2) \subset SL(2,\mathbb{C})$, giving

$$\mathrm{Spin}(3) \cong SU(2).$$

This is the inclusion used when one views the spinor bundle along a spacelike hypersurface as a bundle associated to a Spin(3)-structure on M together with extra Lorentzian data.

A.3. **Spin structures on oriented 3-manifolds.** Let (M^3, g) be an oriented Riemannian 3-manifold. Let $F(M) \to M$ be the principal $SO(3)$ bundle of oriented g-orthonormal frames. A *spin structure* on M is a principal Spin(3) bundle $P(M) \to M$ together with a double covering $\mathrm{Spin}(3) \to SO(3)$-equivariant map $P(M) \to F(M)$.

Proposition A.3. *If M is oriented, then M admits a spin structure. The set of isomorphism classes of spin structures is a torsor for $H^1(M;\mathbb{Z}_2)$.*

In the asymptotically flat situation of Sections 2–6, each end M_ℓ is diffeomorphic to $\mathbb{R}^3 \setminus K$ for a compact K. Since $H^1(\mathbb{R}^3 \setminus K; \mathbb{Z}_2) = 0$, the spin structure restricts canonically to the end, and the associated spinor bundle becomes canonically trivial there. This yields the distinguished family of *constant spinors at infinity* used to pose the boundary condition $\psi \to \psi_\infty$ in the Witten equation.

A.4. **Dirac spinors, Clifford multiplication, and two Hermitian forms.** We now record the fiberwise algebra that underlies Definition 3.2 and Lemma 3.4.

The Dirac spinor representation. Let $V = \mathbb{C}^2$ be the fundamental representation of $SL(2,\mathbb{C})$, and let V^* be its dual. Parker–Taubes realize the (complex) Dirac spinor representation as

$$S \cong V \oplus V^*,$$

with the natural $SL(2,\mathbb{C})$-action on each summand. Via (A.2), a vector $x \in \mathbb{R}^{3,1}$ is identified with a Hermitian map $x : V^* \to V$. Using the $SL(2,\mathbb{C})$-invariant symplectic form on V, one defines the corresponding adjoint $x' : V \to V^*$. Clifford multiplication $x\cdot : S \to S$ is then given by

$$x \cdot (\xi, \eta) = (x\,\eta,\ x'\,\xi), \qquad (\xi, \eta) \in V \oplus V^*. \tag{A.4}$$

A determinant computation in this model gives the Clifford relations (A.1).

An invariant indefinite form and the e^0*-twist.* There is an $SL(2,\mathbb{C})$-invariant Hermitian form $(\cdot,\cdot)$ on S which is not positive definite. With respect to $(\cdot,\cdot)$, Clifford multiplication by $x \in \mathbb{R}^{3,1}$ is Hermitian in the sense that

$$(x \cdot \psi, \phi) = (\psi, x \cdot \phi). \tag{A.5}$$

Fix a unit future timelike covector e^0. As in Section 3, define a second Hermitian form on S by

$$\langle \psi, \phi \rangle := (e^0 \cdot \psi, \phi). \tag{A.6}$$

This form is Spin(3)-invariant and is the one used throughout the analytic part of the argument.

Lemma A.7. *With $\langle\cdot,\cdot\rangle$ defined by (A.6), the following hold.*

(1) $e^0\cdot$ is Hermitian with respect to $\langle\cdot,\cdot\rangle$, and $\langle\cdot,\cdot\rangle$ is positive definite.
(2) If $x \in \mathbb{R}^{3,1}$ satisfies $g(x, e^0) = 0$ (spacelike and orthogonal to e^0), then $x\cdot$ is skew-Hermitian:

$$\langle x \cdot \psi, \phi \rangle = -\langle \psi, x \cdot \phi \rangle.$$

(3) If $x, y \in \mathbb{R}^{3,1}$ satisfy $g(x, e^0) = g(y, e^0) = 0$, then the commutator $[x, y] := x \cdot y \cdot - y \cdot x\cdot$ is Hermitian with respect to $\langle\cdot,\cdot\rangle$.

Proof. (1) Since $g(e^0, e^0) = -1$, (A.1) gives $(e^0\cdot)^2 = \mathrm{Id}$. Choose a fiberwise basis adapted to the splitting $S \cong V \oplus V^*$ in which $e^0\cdot$ acts as the identity on each summand after identifying $V^* \cong V$ via $e^0\cdot$. In such a basis the right-hand side of (A.6) becomes a sum of squared moduli of components, hence $\langle \psi, \psi \rangle > 0$ for $\psi \neq 0$. Hermiticity of $e^0\cdot$ with respect to $\langle\cdot,\cdot\rangle$ follows from $(e^0\cdot)^2 = \mathrm{Id}$ and the definition.

(2) Use the Clifford relation in the form

$$e^0 \cdot x \cdot + x \cdot e^0 \cdot = -2g(e^0, x)\mathrm{Id}.$$

If $g(e^0, x) = 0$, then $e^0 \cdot x \cdot = -x \cdot e^0 \cdot$. Using (A.6) and (A.5),

$$\langle x \cdot \psi, \phi \rangle = (e^0 \cdot x \cdot \psi, \phi) = (-x \cdot e^0 \cdot \psi, \phi) = -(e^0 \cdot \psi, x \cdot \phi) = -\langle \psi, x \cdot \phi \rangle.$$

(3) If $x\cdot$ and $y\cdot$ are skew-Hermitian with respect to $\langle\cdot,\cdot\rangle$, then $(x \cdot y\cdot)^* = y \cdot x\cdot$. Hence $[x, y]^* = (x \cdot y \cdot - y \cdot x\cdot)^* = x \cdot y \cdot - y \cdot x\cdot$. □

Appendix B. Sobolev and weighted spaces: minimal tools

This appendix collects the analytic facts used in Section 5. Throughout, M is a 3-manifold with finitely many asymptotically flat ends, $S \to M$ is the Dirac spinor bundle used in the main text, and $\sigma : M \to [1, \infty)$ is the weight function fixed there. We write $\mathcal{H}_{s,\delta,p}(M; S)$ for the Parker–Taubes weighted spaces from Definition 5.6.

B.1. $H^1(\mathbb{R}^3)$ **and the Sobolev embedding.** We record the two embeddings used in the body. Proofs are standard, but we include short arguments tailored to $\mathbb{R}^3$.

Lemma B.1 (Sobolev in $\mathbb{R}^3$)**.** *There exists $C > 0$ such that for every $u \in C_c^\infty(\mathbb{R}^3)$,*

$$\|u\|_{L^6(\mathbb{R}^3)} \leq C\|\nabla u\|_{L^2(\mathbb{R}^3)}.$$

Equivalently, $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ continuously.

Proof. Let $G(x) = \frac{1}{4\pi|x|}$ be the fundamental solution of $-\Delta$ on $\mathbb{R}^3$. For $u \in C_c^\infty(\mathbb{R}^3)$ one has $u = G * (-\Delta u) = \nabla G * \nabla u$, so

$$|u(x)| \leq \int_{\mathbb{R}^3} |\nabla G(x-y)|\,|\nabla u(y)|\,dy \leq C \int_{\mathbb{R}^3} \frac{|\nabla u(y)|}{|x-y|^2}\,dy.$$

The operator $I_1(f)(x) = \int |x-y|^{-2} f(y)\,dy$ is the Riesz potential of order 1. In $\mathbb{R}^3$ it maps L^2 to L^6 boundedly. Applying this to $f = |\nabla u|$ gives the claim. ☐

Lemma B.2 (Morrey in $\mathbb{R}^3$)**.** *If $p > 3$ and $u \in W^{1,p}(\mathbb{R}^3)$, then u has a Hölder continuous representative and*

$$\|u\|_{C^{0,\alpha}(\mathbb{R}^3)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^3)}, \qquad \alpha = 1 - \frac{3}{p}.$$

In particular, $W^{1,p}(\mathbb{R}^3) \hookrightarrow C^0(\mathbb{R}^3)$.

Proof. Fix $x \in \mathbb{R}^3$ and $0 < \rho \leq 1$. By the mean value estimate and Hölder,

$$|u(x) - u_{B_\rho(x)}| \leq C\rho \left(\frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |\nabla u|^p \right)^{1/p} \leq C\rho^{1-3/p} \|\nabla u\|_{L^p(\mathbb{R}^3)}$$

A similar estimate for $u_{B_\rho(x)} - u_{B_\rho(x')}$ with $\rho \sim |x - x'|$ gives the Hölder bound. ☐

B.2. **Weighted spaces and basic lemmas.** We now record the elementary properties of $\mathcal{H}_{s,\delta,p}$ that are used implicitly in Section 5.

Lemma B.3 (Cutoffs)**.** *Let $\chi \in C^\infty(M)$ have bounded derivatives up to order* 1*. Then multiplication by χ defines bounded linear maps*

$$\chi : \mathcal{H}_{0,\delta,p}(M;S) \to \mathcal{H}_{0,\delta,p}(M;S), \qquad \chi : \mathcal{H}_{1,\delta,p}(M;S) \to \mathcal{H}_{1,\delta,p}(M;S).$$

Proof. For $s = 0$ this is immediate from $\|\sigma^\delta \chi\psi\|_{L^p} \leq \|\chi\|_{L^\infty} \|\sigma^\delta \psi\|_{L^p}$. For $s = 1$,

$$\nabla(\chi\psi) = (d\chi) \otimes \psi + \chi\nabla\psi,$$

so $\|\sigma^{\delta+1}\nabla(\chi\psi)\|_{L^p}$ is bounded by the sum of $\|\sigma^{\delta+1}(d\chi)\psi\|_{L^p}$ and $\|\sigma^{\delta+1}\chi\nabla\psi\|_{L^p}$. Since $\sigma \geq 1$ and $d\chi$ is bounded, $\sigma^{\delta+1}|d\chi| \leq C\sigma^\delta$. ☐

Lemma B.4 (Decay from positive weight)**.** *Assume $\delta > 0$ and $p \geq 2$. If $\psi \in \mathcal{H}_{0,\delta,p}(M;S)$, then on each end $\sigma^\delta\psi \in L^p$ implies that $|\psi|$ becomes arbitrarily small along almost every ray to infinity. If moreover $p > 3$ and $\psi \in \mathcal{H}_{1,\delta,p}(M;S)$, then $|\psi(x)| = o(\sigma(x)^{-\delta})$ as $\sigma(x) \to \infty$ on each end.*

Proof. On an end, write the integral in polar coordinates. Since $\sigma = r$ for r large, $\sigma^\delta\psi \in L^p$ gives

$$\int_R^\infty \int_{S^2} r^{p\delta} |\psi(r,\omega)|^p\, r^2\, d\omega\, dr < \infty,$$

so by Fubini the inner integral is finite for almost every ω and thus $r^\delta|\psi(r,\omega)| \to 0$ along a subsequence $r \to \infty$. This is the first claim.

If $\psi \in \mathcal{H}_{1,\delta,p}$ with $p > 3$, apply Lemma B.2 on each annulus $\{2^k \le r \le 2^{k+1}\}$ after rescaling to unit size. The weighted $W^{1,p}$ control then gives a uniform Hölder bound for $r^\delta \psi$ on each annulus, while the L^p smallness forces the supremum to tend to 0 as $k \to \infty$. □

B.3. **Partition of unity and end reduction.** Let $M = \big(M \setminus \cup_\ell M_{\ell,R}\big) \cup \bigcup_\ell M_{\ell,R}$ be the decomposition into a compact core and exterior regions in the ends. Choose smooth functions $\chi_0, \chi_1, \dots, \chi_k$ with

$$\chi_0 + \chi_1 + \cdots + \chi_k \equiv 1,$$

where χ_0 is supported in the compact core and χ_ℓ is supported in the ℓ-th end, equal to 1 on $M_{\ell,2R}$, with $|d\chi_\ell| \le C/R$.

Lemma B.5 (Norm comparison)**.** *For fixed R as above there exists $C = C(R)$ such that for $s \in \{0,1\}$ and all smooth compactly supported ψ,*

$$C^{-1} \sum_{\ell=0}^{k} \|\chi_\ell \psi\|_{s,\delta,p} \le \|\psi\|_{s,\delta,p} \le C \sum_{\ell=0}^{k} \|\chi_\ell \psi\|_{s,\delta,p}.$$

Proof. The upper bound uses the triangle inequality and Lemma B.3. For the lower bound, use $\psi = \sum \chi_\ell \psi$ and observe that on the support of χ_ℓ only finitely many cutoff derivatives appear, with bounds independent of ψ. □

B.4. **Lax–Milgram.** We use the Hilbert case $p = 2$ when discussing weak solutions.

Theorem B.6 (Lax–Milgram)**.** *Let H be a real or complex Hilbert space. Suppose $a : H \times H \to \mathbb{C}$ is a bounded sesquilinear form and there exists $\lambda > 0$ such that*

$$\Re\, a(u,u) \ge \lambda \|u\|_H^2 \qquad \text{for all } u \in H.$$

Then for every bounded linear functional $L \in H^*$ there exists a unique $u \in H$ such that

$$a(u,v) = L(v) \qquad \text{for all } v \in H,$$

and $\|u\|_H \le \lambda^{-1} \|L\|_{H^*}$.

Proof. Define $A : H \to H^*$ by $Au(v) = a(u,v)$. Boundedness gives $\|Au\| \le C\|u\|$. Coercivity implies $\|Au\| \ge \lambda\|u\|$ by taking $v = u$ and using Cauchy–Schwarz in H^*. Thus A is injective with closed range. Since H identifies with H^* by Riesz, the range is all of H^*, so $Au = L$ has a unique solution and the estimate follows from $\|Au\| \ge \lambda\|u\|$. □

B.5. **Dirac-type regularity and decay improvement.** We state the local elliptic estimate used to pass from weak solutions to smooth ones.

Proposition B.7 (Local $W^{1,p}$ estimate)**.** *Let $U \subset \mathbb{R}^3$ be open and let*

$$L = \sum_{i=1}^{3} a^i(x)\partial_i + b(x)$$

be a first-order elliptic operator on $\mathbb{C}^m$ with $a^i, b \in C^\infty(U)$. If $u \in L^p_{\mathrm{loc}}(U)$ and $Lu = f$ in the sense of distributions with $f \in L^p_{\mathrm{loc}}(U)$, then $u \in W^{1,p}_{\mathrm{loc}}(U)$. Moreover, for $U' \Subset U$,

$$\|u\|_{W^{1,p}(U')} \le C\big(\|f\|_{L^p(U)} + \|u\|_{L^p(U)}\big).$$

If f is smooth, then u is smooth.

Proof. This is the standard interior estimate for first-order elliptic systems. One proves it by freezing coefficients, comparing with the constant-coefficient operator on small balls, and absorbing the error. The bootstrap to smoothness uses the estimate iteratively after differentiating the equation. □

Lemma B.8 (Decay improvement)**.** *Assume $\delta > 0$ and $p > 3$. If $\psi \in \mathcal{H}_{1,\delta,p}(M;S)$ satisfies $\mathscr{D}\psi = f$ with $f \in \mathcal{H}_{0,\delta+1+\varepsilon,p}(M;S)$ for some $\varepsilon > 0$, then on each end $\psi = o(\sigma^{-\delta-\varepsilon})$.*

Proof. On an end, write the equation in asymptotic coordinates as the Euclidean Dirac operator plus lower order terms that decay. Apply Proposition B.7 on dyadic annuli and use that the weighted norms of f gain ε powers of σ. Combining with Lemma B.2 gives the pointwise improvement. □

B.6. **Fredholm property and index zero.** This subsection explains the structural steps behind the Fredholm statements in Section 5. The only nontrivial global input is the Euclidean model theorem, stated below in a form that matches the Parker–Taubes setup.

Theorem B.9 (Euclidean model isomorphism)**.** *Let $\mathscr{D}_0 = \sum_{i=1}^{3} e^i \cdot \partial_i$ be the Euclidean Dirac operator on $\mathbb{R}^3$ acting on $\mathbb{C}^2$-valued spinors. Fix $p \geq 2$ and $0 < \delta < 2 - \frac{3}{p}$. Then*

$$\mathscr{D}_0 : \mathcal{H}_{1,\delta,p}(\mathbb{R}^3;\mathbb{C}^2) \longrightarrow \mathcal{H}_{0,\delta+1,p}(\mathbb{R}^3;\mathbb{C}^2)$$

is an isomorphism with bounded inverse.

Proof. This is a standard result for first-order elliptic operators on $\mathbb{R}^3$ with power weights in the stated range. One route is to use the constant-coefficient theory together with weighted singular integral estimates. Another route is to deduce the Fredholm property and index 0 from a parametrix construction on dyadic annuli and then rule out kernel and cokernel in the positive-weight regime. □

Lemma B.10 (Smallness of the perturbation on the ends)**.** *Fix $p \geq 2$ and $0 < \delta < 2 - \frac{3}{p}$. For R sufficiently large, on each end one has*

$$\|(\mathscr{D} - \mathscr{D}_0)\psi\|_{0,\delta+1,p} \leq \varepsilon(R)\, \|\psi\|_{1,\delta,p}$$

for all ψ supported in $\{\sigma \geq R\}$, where $\varepsilon(R) \to 0$ as $R \to \infty$.

Proof. In asymptotic coordinates, $\mathscr{D}$ has the form

$$\mathscr{D} = \sum_{i=1}^{3} (e^i \cdot)\big(\partial_i + A_i(x)\big),$$

where $e^i\cdot$ differs from the constant Euclidean Clifford matrices by $O(r^{-1})$ and $A_i(x) = O(r^{-2})$ together with one derivative, by the decay assumptions on g and h. Expanding $\mathscr{D} - \mathscr{D}_0$ yields terms of the form $a^i(x)\partial_i\psi + b(x)\psi$ with $a^i = O(r^{-1})$ and $b = O(r^{-2})$. Multiplying by $\sigma^{\delta+1}$ and using that $\sigma = r$ for large r gives a bound by $\varepsilon(R)\|\psi\|_{1,\delta,p}$. □

Proposition B.11 (Exterior a priori estimate)**.** *Fix $p \geq 2$ and $0 < \delta < 2 - \frac{3}{p}$. For R sufficiently large there exists $C = C(R,p,\delta)$ such that for all ψ supported in $\{\sigma \geq R\}$,*

$$\|\psi\|_{1,\delta,p} \leq C\, \|\mathscr{D}\psi\|_{0,\delta+1,p}.$$

Proof. By Theorem B.9,

$$\|\psi\|_{1,\delta,p} \le C_0\|\mathscr{D}_0\psi\|_{0,\delta+1,p}.$$

Write $\mathscr{D}_0\psi = \mathscr{D}\psi - (\mathscr{D}-\mathscr{D}_0)\psi$ and apply Lemma B.10. For R large, $\varepsilon(R)$ is small enough to absorb the perturbation term into the left-hand side. □

Lemma B.12 (Compact embedding)**.** *Let $p \ge 1$, $0 < \delta < \delta'$, and $s \in \{0,1\}$. The inclusion*

$$\mathcal{H}_{1,\delta,p}(M;S) \hookrightarrow \mathcal{H}_{0,\delta',p}(M;S)$$

is compact.

Proof. Use Lemma B.5 to reduce to the compact core and finitely many ends. On the core, this is Rellich compactness. On an end, decompose into dyadic annuli and rescale each annulus to unit size. The gain $\delta'-\delta > 0$ forces uniform smallness in the tail, while Rellich gives compactness on each fixed annulus. A diagonal argument yields compactness globally. □

Proposition B.13 (Fredholm and index)**.** *Fix $p \ge 2$ and $0 < \delta < 2 - \frac{3}{p}$. Then*

$$\mathscr{D} : \mathcal{H}_{1,\delta,p}(M;S) \to \mathcal{H}_{0,\delta+1,p}(M;S)$$

is Fredholm of index 0*.*

Proof. Choose R large and cut off $\psi = \psi_{\rm in} + \psi_{\rm ex}$ using the partition of unity from (B.3). On the exterior part, Proposition B.11 gives an estimate with no lower-order term. On the compact core, standard elliptic theory gives

$$\|\psi_{\rm in}\|_{W^{1,p}} \le C\big(\|\mathscr{D}\psi_{\rm in}\|_{L^p} + \|\psi_{\rm in}\|_{L^p}\big).$$

Combining these yields a global estimate

$$\|\psi\|_{1,\delta,p} \le C\big(\|\mathscr{D}\psi\|_{0,\delta+1,p} + \|\psi\|_{0,\delta',p}\big)$$

for some $\delta' > \delta$. By Lemma B.12, the second term is compact relative to the $\mathcal{H}_{1,\delta,p}$ norm. This implies that $\mathscr{D}$ has closed range and finite-dimensional kernel and cokernel, hence is Fredholm.

To see that the index is 0, consider the formal adjoint acting between dual weighted spaces. The Fredholm index is stable under compact perturbations and agrees with the index of the Euclidean model on each end, which is 0 by Theorem B.9. □

Appendix C. Long computations and normalizations

This appendix supplies the moving-frame computations used in Section 3 and Section 4. Indices $a, b, c, \dots$ range over $\{0,1,2,3\}$ and $i, j, k, \dots$ over $\{1,2,3\}$. We work along a spacelike hypersurface $M^3 \subset (N^{3,1}, g_{\mu\nu})$ with future unit normal e_0, and write $\{e_a\}$ for a local g-orthonormal frame of $TN|_M$ adapted so that $e_i \in TM$. Let $\{e^a\}$ be the dual coframe. Clifford multiplication satisfies

$$e^a \cdot e^b \cdot + e^b \cdot e^a \cdot = -2\, g^{ab}\,\mathrm{Id}.$$

C.1. **The relation between V and ∇.** Let ω_{ab} be the Levi-Civita connection 1-forms of N in the frame $\{e_a\}$, so

$$\nabla^N e_a = \omega_{ab}\, e_b, \qquad \omega_{ab} = -\omega_{ba}.$$

The induced spin connection on the Dirac spinor bundle $S \to M$ determined by ∇^N is

$$V_X\psi = X(\psi) + \frac{1}{4}\,\omega_{ab}(X)\, e^a \cdot e^b \cdot \psi, \qquad X \in TM. \tag{C.1}$$

Let θ_{ij} be the Levi-Civita connection 1-forms of (M, g) in the tangential frame $\{e_i\}$, so $\nabla^M e_i = \theta_{ij} e_j$ and $\theta_{ij} = -\theta_{ji}$. The Riemannian spin connection on S induced by ∇^M is

$$\nabla_X\psi = X(\psi) + \frac{1}{4}\,\theta_{ij}(X)\, e^i \cdot e^j \cdot \psi. \tag{C.2}$$

The second fundamental form is

$$h_{ij} = g(\nabla^N_{e_i} e_0, e_j),$$

so $\nabla^N_{e_i} e_0 = h_{ij} e_j$ and $\nabla^N_{e_i} e_j$ has normal component $-h_{ij} e_0$. Equivalently,

$$\omega_{0j}(e_i) = h_{ij}, \qquad \omega_{j0}(e_i) = -h_{ij}. \tag{C.3}$$

Restricting ω_{ij} to TM gives $\omega_{ij}|_{TM} = \theta_{ij}$. Substituting these relations into (C.1) and comparing with (C.2) gives the desired difference formula.

Lemma C.4. *For $X = e_i$,*

$$V_i\psi = \nabla_i\psi + \frac{1}{2}\, h_{ij}\, e^j \cdot e^0 \cdot \psi. \tag{C.5}$$

Equivalently, using $e^j \cdot e^0 = -e^0 \cdot e^j$,

$$V_i\psi = \nabla_i\psi - \frac{1}{2}\, h_{ij}\, e^0 \cdot e^j \cdot \psi.$$

Proof. From (C.1) and $\omega_{ij}|_{TM} = \theta_{ij}$,

$$V_i\psi = e_i(\psi) + \frac{1}{4}\,\theta_{jk}(e_i)\, e^j \cdot e^k \cdot \psi + \frac{1}{4}\,\omega_{0j}(e_i)\, e^0 \cdot e^j \cdot \psi + \frac{1}{4}\,\omega_{j0}(e_i)\, e^j \cdot e^0 \cdot \psi.$$

The first two terms are $\nabla_i\psi$ by (C.2). Using (C.3) and $\omega_{0j} = -\omega_{j0}$,

$$\frac{1}{4}\,\omega_{0j}(e_i)\, e^0 \cdot e^j + \frac{1}{4}\,\omega_{j0}(e_i)\, e^j \cdot e^0 = \frac{1}{4}\, h_{ij}\,(e^0 \cdot e^j - e^j \cdot e^0) = \frac{1}{2}\, h_{ij}\, e^0 \cdot e^j.$$

Rewriting with $e^0 \cdot e^j = -e^j \cdot e^0$ gives (C.5). □

C.2. **Step-by-step verification of the Weitzenböck formula.** Recall $\mathscr{D} = \sum_{i=1}^3 e^i \cdot V_i$. Fix a point $p \in M$ and choose an adapted orthonormal frame $\{e_a\}$ in a neighborhood of p so that

(i) $[e_i, e_j]_p = 0$ for $1 \le i, j \le 3$,
(ii) $(\nabla^M_{e_i} e_j)_p = 0$ for $1 \le i, j \le 3$,
(iii) $(\nabla^N_{e_0} e_i)_p = 0$ for $1 \le i \le 3$.

In the dual coframe this implies, at p,

$$(V_i e^j)_p = -h_{ij}\, e^0, \qquad (V_i e^0)_p = -h_{ij}\, e^j, \qquad (V_0 e^j)_p = 0. \tag{C.6}$$

Computing $\mathscr{D}^2$. At p,

$$\begin{aligned}\mathscr{D}^2\psi &= \sum_{i,j} e^i \cdot V_i\big(e^j \cdot V_j\psi\big) \\ &= \sum_{i,j} e^i \cdot e^j \cdot V_iV_j\psi + \sum_{i,j} e^i \cdot (V_ie^j) \cdot V_j\psi. \end{aligned} \tag{C.7}$$

Using (C.6) gives

$$\sum_{i,j} e^i \cdot (V_ie^j) \cdot V_j\psi = -\sum_{i,j} h_{ij}\, e^i \cdot e^0 \cdot V_j\psi. \tag{C.8}$$

For the first term in (C.7), split into $i = j$ and $i \neq j$. Since $(e^i\cdot)^2 = -\mathrm{Id}$,

$$\sum_{i} e^i \cdot e^i \cdot V_iV_i\psi = -\sum_i V_iV_i\psi. \tag{C.9}$$

For $i \neq j$, use $e^i \cdot e^j = -e^j \cdot e^i$ to write

$$\sum_{i\neq j} e^i \cdot e^j \cdot V_iV_j\psi = \frac{1}{2}\sum_{i\neq j} e^i \cdot e^j \cdot (V_iV_j - V_jV_i)\psi = \sum_{i<j} e^i \cdot e^j \cdot \Omega_{ij}\psi, \tag{C.10}$$

where $\Omega_{ij} := [V_i, V_j]$ is the spin curvature endomorphism. As usual, Ω_{ij} is the image of the Riemann curvature tensor of N under the spin representation. Writing R_{abij} for the Riemann tensor components in the orthonormal frame,

$$\Omega_{ij} = \frac{1}{4}\, R_{abij}\, e^a \cdot e^b. \tag{C.11}$$

Substituting (C.11) into (C.10),

$$\sum_{i<j} e^i \cdot e^j \cdot \Omega_{ij}\psi = \frac{1}{4}\sum_{i<j} R_{abij}\, e^i \cdot e^j \cdot e^a \cdot e^b \cdot \psi. \tag{C.12}$$

A standard Clifford simplification, using the Bianchi identity and the symmetries of R_{abij}, yields

$$\sum_{i<j} e^i \cdot e^j \cdot \Omega_{ij} = \frac{1}{4}\Big(R + 2R_{00} + 2R_{0j}\, e^0 \cdot e^j\Big), \tag{C.13}$$

where R is the scalar curvature of (M, g) and $R_{00} = \mathrm{Ric}^N(e_0, e_0)$, $R_{0j} = \mathrm{Ric}^N(e_0, e_j)$. Combining (C.8), (C.9), and (C.13) gives, at p,

$$\mathscr{D}^2\psi = -\sum_i V_iV_i\psi - \sum_{i,j} h_{ij}\, e^i \cdot e^0 \cdot V_j\psi + \frac{1}{4}\Big(R + 2R_{00} + 2R_{0j}\, e^0 \cdot e^j\Big)\psi. \tag{C.14}$$

Since the computation is tensorial, (C.14) holds at every point.

Formal adjoints and the Weitzenböck identity. Let $\langle\cdot,\cdot\rangle$ be the positive definite inner product on S from (3.3). Let $\mu = e^1 \wedge e^2 \wedge e^3$ be the Riemannian volume form on M. A direct computation in the frame above gives the integration by parts formula

$$d\big[\langle\phi, V_i\psi\rangle\, e_i \lrcorner \mu\big] = \big(\langle V_i\phi, V_i\psi\rangle + \langle\phi, V_i^*V_i\psi\rangle\big)\mu, \tag{C.15}$$

where the formal adjoint of V_i is

$$V_i^* = -V_i - h_{ij}\, e^j \cdot e^0. \tag{C.16}$$

From (C.16) one gets

$$V^*V = -\sum_i V_i V_i - \sum_{i,j} h_{ij}\, e^j \cdot e^0 \cdot V_i. \tag{C.17}$$

Using $e^j \cdot e^0 = -e^0 \cdot e^j$ and relabeling dummy indices, the lower-order term in (C.17) matches the second term in (C.14). Define

$$\mathcal{R} := \frac{1}{4}\Big(R + 2R_{00} + 2R_{0j}\, e^0 \cdot e^j\Big). \tag{C.18}$$

Then (C.14) and (C.17) give

$$\mathscr{D}^2 = V^*V + \mathcal{R}. \tag{C.19}$$

This is Proposition 4.1.

The boundary form. A second computation gives the divergence identity for $\mathscr{D}$,

$$d\big[\langle \phi, e^i \cdot \psi\rangle\, e_i \lrcorner \mu\big] = \big(\langle \phi, \mathscr{D}\psi\rangle - \langle \mathscr{D}\phi, \psi\rangle\big)\mu. \tag{C.20}$$

Combining (C.15) and (C.20) with (C.19) yields the integral identity from Section 4 after simplifying the boundary expression using

$$V_i\psi + e^i \cdot \mathscr{D}\psi = (\delta^{ij} + e^i \cdot e^j)\, V_j\psi = \frac{1}{2}\,[e^i, e^j] \cdot V_j\psi. \tag{C.21}$$

Substituting (C.21) into the boundary terms coming from (C.15) and (C.20) gives the 2-form

$$\mathcal{B}(\psi) = -\frac{1}{2}\,\big\langle \psi, [e^i, e^j] \cdot V_j\psi\big\rangle\, (e_i \lrcorner \mu),$$

which is (4.10).

C.3. **Expanding the boundary term and matching ADM data.** Fix an end M_ℓ with asymptotic coordinates $x = (x^1, x^2, x^3)$ and $r = |x|$. Let S_r be the coordinate sphere $\{r = \text{const}\}$ and let $d\Omega^i$ denote the standard Euclidean area 2-forms

$$d\Omega^i = \frac{\partial}{\partial x^i} \lrcorner (dx^1 \wedge dx^2 \wedge dx^3).$$

Let ψ_0 be a constant spinor in the asymptotic trivialization of S on the end. Write Γ_{kji} for the Christoffel symbols of g in these coordinates, with indices lowered using δ_{ij}.

On the end, the orthonormal coframe satisfies $e^i = dx^i + O(r^{-1})$ and the connection coefficients satisfy $\Gamma_{kji} = O(r^{-2})$ and $h_{ij} = O(r^{-2})$. These facts allow replacement of e^i by dx^i in the boundary integral without changing its limit.

The leading expression for $V\psi_0$. Using the asymptotic trivialization, the connection V can be compared with the flat spin connection ∂. Since $\partial\psi_0 = 0$,

$$V_j\psi_0 = -\frac{1}{4}\,\Gamma_{kj\ell}\, dx^k \cdot dx^\ell \cdot \psi_0 - \frac{1}{2}\, h_{jk}\, dx^0 \cdot dx^k \cdot \psi_0 + O(r^{-3})\,|\psi_0|. \tag{C.22}$$

Here dx^0 denotes the unit timelike covector corresponding to the Hermitian structure at infinity, so $(dx^0\cdot)^2 = \mathrm{Id}$ and $(dx^k\cdot)^2 = -\mathrm{Id}$.

Reducing the boundary form. Insert (C.22) into the boundary integrand

$$-\frac{1}{2}\big\langle \psi_0, [dx^i, dx^j] \cdot V_j\psi_0 \big\rangle \, d\Omega^i.$$

The term involving $\Gamma_{kj\ell}$ contributes a scalar multiple of $\langle \psi_0, \psi_0\rangle$. A Clifford algebra calculation gives

(C.23)
$$-\frac{1}{2}\big\langle \psi_0, [dx^i, dx^j] \cdot \Big(-\frac{1}{4}\,\Gamma_{kj\ell}\, dx^k \cdot dx^\ell\Big)\psi_0 \big\rangle = \frac{1}{4}\,(\partial_j g_{ij} - \partial_i g_{jj})\,\langle \psi_0, \psi_0\rangle + O(r^{-3})|\psi_0|^2.$$

The term involving h_{jk} contributes the Hermitian endomorphism $dx^0 \cdot dx^k$. Using $[dx^i, dx^j] \cdot dx^0 \cdot dx^k$ and the symmetry of h_{jk} yields

(C.24)
$$-\frac{1}{2}\big\langle \psi_0, [dx^i, dx^j] \cdot \Big(-\frac{1}{2}\, h_{jk}\, dx^0 \cdot dx^k\Big)\psi_0 \big\rangle = (h_{ik} - \delta_{ik} h_{jj})\,\big\langle \psi_0, dx^0 \cdot dx^k \cdot \psi_0\big\rangle + O(r^{-3})|\psi_0|^2.$$

Combining (C.23) and (C.24), and integrating over S_r, gives

$$\begin{aligned} -\frac{1}{2}\int_{S_r} \big\langle \psi_0, [dx^i, dx^j] \cdot V_j\psi_0 \big\rangle \, d\Omega^i &= \frac{1}{4}\int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj})\,\langle \psi_0, \psi_0\rangle \, d\Omega^i \\ &\quad + \int_{S_r} (h_{ik} - \delta_{ik} h_{jj})\,\big\langle \psi_0, dx^0 \cdot dx^k \cdot \psi_0\big\rangle \, d\Omega^i + o(1). \end{aligned} \tag{C.25}$$

Identification with ADM energy and momentum. By the definitions in Section 2, the ADM energy and momentum of the end are

$$E_\ell = \frac{1}{16\pi}\lim_{r\to\infty}\int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj})\, d\Omega^i, \qquad (P_\ell)_k = \frac{1}{8\pi}\lim_{r\to\infty}\int_{S_r} (h_{ik} - \delta_{ik} h_{jj})\, d\Omega^i.$$

Taking $r \to \infty$ in (C.25) yields

(C.26)
$$\lim_{r\to\infty}\Big(-\frac{1}{2}\int_{S_r} \big\langle \psi_0, [dx^i, dx^j] \cdot V_j\psi_0 \big\rangle \, d\Omega^i\Big) = 4\pi\Big(E_\ell\,\langle \psi_0, \psi_0\rangle + \big\langle \psi_0, (P_\ell)_k\, dx^0 \cdot dx^k \cdot \psi_0\big\rangle\Big),$$

which is the boundary limit used in Proposition 4.12.

C.4. **Stability under changes of frames and coordinates.** This subsection explains why the limits in (C.26) are independent of auxiliary choices.

Replacing e^i by dx^i. Let $\{e^i\}$ be an orthonormal coframe on the end and write $e^i = dx^i + \eta^i$ with $\eta^i = O(r^{-1})$. In the boundary integrand, the coefficient $[e^i, e^j]$ differs from $[dx^i, dx^j]$ by $O(r^{-1})$, while $V\psi_0 = O(r^{-2})$ by (C.22). Thus the difference between using e^i and using dx^i in the boundary integral is

$$\int_{S_r} O(r^{-1}) \cdot O(r^{-2})\, dA = O(r^{-1}),$$

which vanishes as $r \to \infty$. This justifies working with the coordinate coframe in the limit.

Changing asymptotic coordinates. Let x and $\tilde{x}$ be two asymptotic coordinate systems on the same end satisfying the decay hypotheses of Section 2. The transition map has the form $\tilde{x} = Ax+b+f(x)$ where $A \in SO(3)$, $b \in \mathbb{R}^3$, and $f(x) = o(1)$ with $\partial f = O(r^{-1-\varepsilon})$ for some $\varepsilon > 0$. One checks directly that the difference between the flux integrals defining E_ℓ and $(P_\ell)_k$ in the two coordinate systems can be written as the flux of a divergence of a tensor field whose components decay like $O(r^{-2-\varepsilon})$. Integrating over S_r gives a difference $O(r^{-\varepsilon})$, hence the limits agree.

In particular, the right-hand side of (C.26) depends only on the asymptotic end and the choice of asymptotic constant spinor ψ_0, and not on the specific asymptotic chart.