ICT Theory
The Set Theory Basics

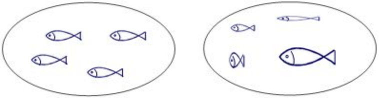
R. Pergl J. Vanicek

Department of Information Engineering
The Faculty of Economics and Management
Czech University of Life Sciences

October 19, 2009

What is a set?

- Well-defined collection of items (objects, events and abstract constructions) that can be differentiated from each other: "The set of outstanding people" is not suitable, for outstanding is very subjective.



- The question, whether an element belongs to the set or not, must be clearly resolved before the construction of the whole set.
- If x belongs to the set A, we shall say that x is an element of A, or x is a member of A and write $\mathbf{x} \in \mathbf{A}$; if not, we shall write $\mathbf{x} \notin \mathbf{A}$.
- The element can be a member of the set, or not. To use the expression "multiple element" of the set is not meaningful.
- The sets with the same members are equal.
- The set with no elements is called an **empty set** and it is usually represented by the symbol $\emptyset$.
- **Subset:** $A \subseteq B \Leftrightarrow \forall x[x \in A \Rightarrow x \in B]$
- **Proper subset:** if $A \neq B$: $A \subset B$

How to describe a set?

1. By enumeration of the symbols of the set members:
   - $A = \{1, 3, 5, 7, 9\}$
   - $C = \{1, 3, 5, 7, 9, ...\}$
   - $D = \{"John", "Peter", "Thomas"\}$
   - $\{1, 2, 3, 4, 5\} = \{2, 1, 5, 4, 3\} = \{1, 1, 1, 2, 3, 4, 5\} = \{1, 2, 3, 4, 5, 5, 4, 3, 2, 1\}$
2. By description of the set using distinctive predicate: $\{x : P(x)\}$ or $\{x \mid P(x)\}$
   - $B = \{x \mid x \textit{ is odd}\}$
3. By specifying a set, which is already defined, and an additional property which characterizes the necessary and sufficient condition for a element to belong to the current set:
   - $D = \{n \in \mathbb{N} : n \leq 5\} = \{1, 2, 3, 4, 5\}$

Common set notations (1/2)

- $\mathbb{Z}$ ... the set of integers $\{0, 1, -1, 2, -1, 3, -3, \ldots\}$
- $\mathbb{N}$ ... the set of nonnegative integers or natural numbers
- $\mathbb{Z}^+$ ... the set of positive integers
- $\mathbb{Q}$ ... the set of rational numbers $\{a/b \mid a, b \text{ is integer, } b \text{ not zero}\}$
- $\mathbb{Q}^+$ ... the set of positive rational numbers
- $\mathbb{Q}^*$ ... the set of nonzero rational numbers
- $\mathbb{R}$ ... the set of real numbers
- $\mathbb{R}^+$ ... the set of positive real numbers
- $\mathbb{R}^*$ ... the set of nonzero real numbers
- $\mathbb{C}$ ... the set of complex numbers

Common set notations (2/2)

Closed interval	$\langle a, b \rangle = \{x \in \mathbb{R} \mid a \leq x \leq b\}$
Open interval	$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$
Half closed interval	$\langle a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$
	$(a, b\rangle = \{x \in \mathbb{R} \mid a < x \leq b\}$

- Example: Filled circle with center in (x_1, y_1) and radius r: $\{x, y \in \mathbb{R} : (x - x_1)^2 + (y - y_1)^2 \leq r^2\}$

Set operations

Union:	$A \cup B = \{x \mid x \in A \vee x \in B\}$
Intersection:	$A \cap B = \{x \mid x \in A \wedge x \in B\}$
Difference:	$A \div B = \{x \mid x \in A \wedge x \notin B\}$
Symmetric difference:	$A \oplus B = \triangle(A, B) =$
	$= \{x \mid (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$
	$A \oplus B = (A \cup B) \div (A \cap B)$
Potency set:	$2^A = \{B \mid B \subseteq A\}$ (contains 2^A elements)
Complement of the set A:	$\overline{A} = U \div A$, i.e. all Universe except the set A.

Other set concepts

- **Ordered pair** (a, b) is a set $\{\{a, b\}, a\}$. a is here assigned as being the first (and thus b is the second). Example: ordered pairs of x and y coordinates.
- **n-tuple** of of elements from the sets $A_1, A_2, \ldots, A_n$ is roughly defined as a set of n elements $a_1, a_2, \ldots, a_n$, where $a_j \in A_j$ indicates the position of the item.
- **Cartesian product** $A \times B$ is the set of all the ordered pairs (a, b) (all possible combinations).
- The Cartesian product $A_1 \times A_2 \times \ldots \times A_n$ of the sets $A_1, A_2, \ldots A_n$ is the set of all n-tuples from these sets

Problems of the Naive set theory

- The theory presented so far is called a **naive set theory**.
- It is not always easy to decide if the definition of the set is correct or not (the collection of all the members of the defined set must be fully specified before the set definition).
- For example the notion "a set of all the possible sets" is not acceptable definition from this point of view: It contains a hidden logical ring. If we would follow this term, some paradoxes can appear.
- **Russel's Paradox**:
  - $S = \{A \mid \text{is a set and } A \notin A\}$
    1. If $S \in S$ then $S \notin S$.
    2. If $S \notin S$ then $S \in S$.

Axiomatic set theories

- The main concepts are **set** and **to be a member** $\in$
- Not clearly defined sets are called **classes**.
- Axiomatic sets are defined "on their own" and they do not build upon any external entities.
- **Zermelo-Frankel's** set theory: defined by 9 axioms.
- **Gödel-Bernays** set theory: based on the concept of "class".