Relations

Two main objectives:

1. For a description (record) of some set of objects or events, which can be characterized by their attributes.
   - E.g. house of inhabitants: (first name, surname, date of birth, flat number) $S \subseteq N \times S \times D \times F$,.
   - (Jan, Novák, 2. 2. 1970, 7) $\in S$
   - Relations are here to formally describe a selected part of the world: database relations
1. To formally describe the attributes of a given set and the relation between two or more objects or events (entities).
   - The relation „is bigger“ between numbers: (3, 2) $\in$ >.

Relation „being a father“ on the set of all the people: George is a father of Thomas (George, Thomas) $\in F$ ... $F \subseteq P \times P$, where P is the set of all people.

Relation with Generally Different Domains

- Let's have some finite sequence of sets $A_1, A_2, \ldots, A_n$, then we can say the **relation between domains** $A_1, A_2, \ldots, A_n$ is the arbitrary subset of the Cartesian product $A_1 \times A_2 \times \ldots \times A_n$.

  The relation R then contains only *some* (selected) n-tuples $(a_1, a_2, \ldots, a_n)$, where

  $$a_j \in A_j, \text{ for } j = 1, 2, \ldots, n. \; R \subseteq A_1 \times A_2 \times \ldots A_n = \mathop{\times}_{j=1}^{n} A_j .$$

- A relation is principally represented by a table. The rows of this table are individual entities (objects or events). The columns indicate the individual domains (the attributes which we are interested in). The value in the relevant row and column formally describes the attribute of the relevant domain:

first name	**surname**	**date of birth**	**flat number**
John	Dean	02/12/1970	3
Patricia	Smith	06/07/1963	2
Bob	Jones	11/09/1950	6
Mirinda	Stage	12/11/1980	10
…	…	…	…

Example Relation *A:* House of inhabitants

- To work with relations, common set operations are naturally available: **union of the relations**, **intersection of the relations**, **difference of relations** or **symmetric difference of the relations**, defined as the relevant set operation on the Cartesian product of the domains.

- We may introduce additional operations as well.
- Query languages are used for working with relational tables: most used: **SQL** (Structured Query Language).

Most common additional opereations:

- **Projection:**
  1. We select some of the attributes of the examined entities $A_{k_1}, A_{k_2}, ..., A_{k_m}$, where $m \le n$.
  2. The set of all m-tuple $(a_{k_1}, a_{k_2}, ..., a_{k_m})$, to which exist at least one $(a_1, a_2, \ldots, a_n) \in R$ is called a **projection of the relation** R to the domains $A_{k_1}, A_{k_2}, ..., A_{k_m}$.

  - In the SQL language, projection is specified after the SELECT statement.
  - Example: (surname, flat number)

surname	**flat number**
Dean	3
Smith	2
Jones	6
Stage	10
…	…

Example Relation *A2:* Projection of the house of inhabitants

- **Selection**
  - We specify a constraint using a logic formula that selects just some rows.
  - Example: all rows with flat numbers higher than 5 inhabited by inhabitants older than 40 years.

Bob	Jones	11/09/1950	6

Example Relation *A3:* Selection of the house of inhabitants

- In the SQL language, selection is specified in the "where" clause.
- Selection and projection may be combined.

o **Union of relations** (called usually "joining" in relational databases):

- We have multiple relations defined generally on various domains.
- Some domains are present in multiple relations.
- In the union, there are all the domains. The rows are connected via identical attribute values.
- Example: Let us have another relation specifying jobs of people:

surname	**job**
Dean	IT specialist
Smith	Nurse
Jones	Bus driver
Stage	Teacher
…	…

Example Relation *B:* Jobs of people

Now $A \cup B$ is:

first name	**surname**	**date of birth**	**flat number**	**job**
John	Dean	02/12/1970	3	IT specialist
Patricia	Smith	06/07/1963	2	Nurse
Bob	Jones	11/09/1950	6	Bus driver
Mirinda	Stage	12/11/1980	10	Teacher
…	…	…	…	…

Example Relation $A \cup B$*:* House of inhabitants

o The **set** of relations together with the mentioned operations, and possibly with other derived operations, is called the r**elation algebra** or also **Codd algebra**.

Example 2:

A set of all the names,

B set of all the permissible passport numbers,

C set of all the cities and villages,

D set of all the country,

E set of all the continents.

- The relation then will be the Cartesian product subset $A \times B \times C \times D \times E$, meaning the set of some ordered 5-tuple (a, b, c, d, e), where $a \in A$, $b \in B$, … and so on.
- If there exists an ordered quintuple (a, b, c, d, e) that belongs to the relation, it is interpreted as the existence of the object (student) with given attributes, meaning with the name a, passport number b, living in the city c, country d and continent e.

Name	**Passport Number**	**City**	**Country**	**Continent**
John Smith	6702180000	London	UK	Europe
Peter Dole	7008180000	Manchester	UK	Europe
Anna Slaba	5612310000	Prague	Czech Republic	Europe
Zhu Tiancai	9612310000	Chen-Tia-Gou	China	Asia

Relations on a Set

- All the domains $A_1, A_2, \dots, A_n$, on which a relation is defined are identical: ***n*-ary relation of the set** A: $\underbrace{A \times A \times \ldots \times A}_{n-times} = A^n$.
- n ... **relation degree** or the **relation arity**.
- **An unary relation** $n = 1$: specifies a subset of set A with some distinct feature.
  - Example: unary relation “to be a foreign student” on a set of students.
- **A binary relation** $n = 2$: relations between two items of the set A
  - Examples: „being an ancestor“ on a set of people, “being bigger“ on the set of real numbers.
  - The most common type, we will speak about this type of relations further and we will call “binary relation” just “relation”.
- **The ternary relation** $n = 3$: relations between three items from the set A (Mother – Father – Child on a set of people, Dividend –Divisor – Quotient on some number set).
  - Operations ... a special example of ternary relations: the first two items of the ternary determine the third item unambiguously.
- **The quaternary relations** $n = 4$: relation between four items in a set (Dividend – Divisor – Integer part Quotient – Remainder).

Attributes of Binary Relations

The (binary) relation on the set A is called:

- **Reflexive relation**: for all $a \in A$ the relation $a\ R\ a$ can be applied.
    - That means that every (every!) element is in relation with itself.
    - Example: The relation “to be greater or equal” on the set of natural numbers: every number is in the relation “greater or equal” with itself.
    - If there is at least one element that is not in relation with itself, we say that the relation is “not reflexive”.
        - Example: {(1,1), (1,2), (2,2), (1,3)} – (3,3) is missing
    - If there is no element that is in relation with itself, we call the relation **irreflexive** or **antireflexive**.
        - Example: “to be a father” on the set of all people: no one can be father to themselves.
- **Symmetric relation**: for all a, $b \in A$ applies the relation

$$a\ R\ b \Rightarrow b\ R\ a \quad \ldots\ldots \quad a\ R\ b \Leftrightarrow b\ R\ a.$$

- Symmetric relation is something like a “mirror”
- Example: relation “to be a neighbour” on the set of city inhabitants. If Paul is a neighbour to Peter, than Peter is a neighbour to Paul. This holds for **all** pairs of neighbours!
- If there is at least one pair of elements where the condition does not hold, we call the relation “not symmetric”.

Antisymmetric relation: for all a, $b \in A$ applies that if $a\ R\ b$ and in the same time $b\ R\ a$, then it means $a = b$. Or, equivalently, if $a\ R\ b$ with $a \neq b$, then $b\ R\ a$ must not hold.

- Example: The relation "x is even, y is odd" for pairs (x, y) of integers is antisymmetric.
    - Be careful, that the symmetric and antisymmetric are not opposites
    - Examples of relations, which are neither symmetric nor anti-symmetric:
        - on the set of numbers {1,2,3} let us consider the relation containing pairs {(1,2), (2,1), (1,3)})

        - The existence of the items (1,2) and (2,1) breaks the anti-symmetry
        - The presence of (1,3) and absence of (3,1) breaks the symmetry.
- **The asymmetric relation**: $a\ R\ b \Rightarrow \neg(b\ R\ a)$, i.e. antisymmetric and irreflexive.
- **Transitive relation**: for all a, b, $c \in A$ there applies the relation

$$(a\ R\ b \wedge b\ R\ c) \Rightarrow a\ R\ c.$$

    - For this attribute we need three elements of the set.
    - Example: Arabic saying "A friend of my friend is also my friend".
    - If there is at least one 3-tuple that breaks the rule we say that the relation is "not transitive".
    - If there is no 3-tuple where the constraint holds, we call the relation **negatively transitive.**
- **Weak complete relation**: if for every items a, $b \in A$ that differ, i.e., $\boldsymbol{a \neq b}$, is either $a\ R\ b$ or $b\ R\ a$ (i.e. the relation is not reflexive) (Example: mutual love of all humans, but without self-love)
- **The complete relation**: for two arbitrary items $a, b \in A$ applies either $a\ R\ b$ or $b\ R\ a$. (Example: "Heaven on Earth! Mutual love of all humans and also self-love")

Equivalence

- Very common relation that is reflexive, symmetric and transitive.
- Items, which are equivalent, can be, from a certain view-point, interchangeable.
- The set of people with the relation „having the same parents“ or „having the same blood group“, or even „having the same number of hair on their head“
- Common symbols: ≈, ≅, ≊, ≐, ≏, ...
- There is a difference between equivalence as a binary relation **vs.** equivalence as a logical connective (⇔), which is an operation between logic values in the Boolean algebra.
- The division of the set on which is the equivalence defined specifies subsets of equivalence items that is called **category of equivalence** or **equivalence category.**
    - Each pair is disjoint (i.e. the intersection is an empty set).
    - Any item from the equivalence set then can be used as a representative of this set.

The Relation of Preference and Order

- Relations are useful for expressing preferences and superiority:
    - „is greater“ or „is a multiple of some integer “ on the set of the integer numbers,
    - “being a subset” on the set of subsets
    - „being older“, „being a father“ or „being an ancestor“ on the set of people.
- We use symbols like $\succ$ or $\succsim$,
- The preference naturally requires **transitivity**.
- If we do not apply any other constraint, we get the **the quasiorder relation**.
- Other types of orders deal with the situation when for two elements of the set, there applies neither $a \succ b$ nor $b \succ a$ so they are “unsortable”. There may be generally due to one of the following:
- *a* and *b* are completely **interchangeable**, i.e. *equivalent* (what is better than *a*, is also better than *b*, and opposite),
    - Example: two computers with the same processor type if we compare computers according to the processor speed.

a and *b* are **not comparable**

- Example: We are byuing a product and specify functionality, safety and reliability as the criteria.
    - Incomparable products are products where one product is better in the function complexity but it is worse concerning the safety.
    - Note that in this example there are equal elements as well: i.e. two products with the same characteristics.

Types of orders

Non-strict orders

- **The quasiorder** of the sets A is the (binary) relation $\succsim$ on A, which is *transitive and reflexive*.
    - *Incomparable* and *equivalent* items.
    - Example: a selection of a job applicant according to 3 subject average results (math, physics, informatics).
- **The partial order** of the sets A is the quasiorder of the set A, which is in addition *anti-symmetric* (meaning $(a \succsim b \wedge b \succsim a) \Rightarrow a = b$).
    - *Incomparable items* and no equal pairs of different items.
    - Example: The set inclusion on the set of all subsets.
- **The weak order** of the sets A is the quasiorder of the set A, which is in addition *complete* (meaning always is true that $a \succsim b$, or $b \succsim a$).
    - *Equal items* and no incomparable pairs.
    - Example: a selection of a job applicant according to leaving exam grade.
- **The order** of the set A is quasiorder, which is in the same time *partial order* and *weak order*.
    - No incomparable nor equal items.
    - Example: sorting of natural numbers.

Strict orders

- If we do not allow the element to be compared with itself, we speak about **strict orders**. They are asymmetric, i.e. anti-reflexive.
- **Strict partial order**: transitive and asymmetric; evidently also anti-reflexive (for all $a \in A$ does not apply $a \succsim a$).
- **Strict weak order**: asymmetric and negatively transitive (then it is already evidently also transitive).
- **Strict order**: transitive, asymmetric and complete.

Relation property \ Relation name	Quasiorder	Weak order	Order (linear)	Strict order	Strict weak order	Partial order	Strict partial order	Equivalence
Reflexive	✓✓	✓	✓			✓✓		✓✓
Symmetric								✓✓
Transitive	✓✓	✓✓	✓✓	✓✓	✓	✓✓	✓✓	✓✓
Asymmetric				✓✓	✓✓		✓✓	
Anti-symmetric			✓✓	✓	✓	✓✓		
Negatively transitive			✓	✓	✓✓			
Complete		✓✓	✓✓					
Weakly complete		✓	✓	✓✓				

The symbol ✓✓ is used at the attributes which were used in the definition:
The attributes marked by one symbol ✓ are already their consequence

Methods of Relations Specification

Example: the set of computers and the set of test tasks. The test task and computer are in relation if the test task is performed on it.

- **Items listing:** ordered pairs from the set which are in relation

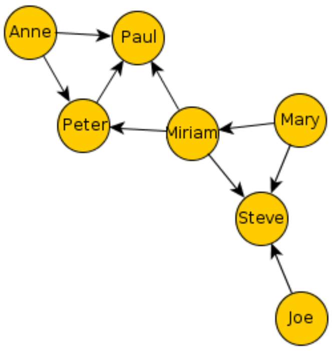
  {(Spreadsheet, Gorgon), (Spreadsheet, Centaur), (Spreadsheet Cerberus), (2D Graphics, Hydra), (2D Graphics, Typhon), …}

- the **Relation matrix**

		Computer					
		Hydra	Gorgon	Typhon	Centaur	Pegasus	Cerberus
Test Task	Spreadsheet		□		□	□	□
	2D Graphics	□		□		□	□
	3D Graphics	□		□		□	
	Database	□	□		□	□	

  - We may use ticks/true-false/yes-no/0-1 in the cells.
  - Usable just for binary relations.
  - For relations on a set it may become very expressive:
    - The reflexive relations will have the diagonal of ones. The anti-reflexive relations will have the diagonal of zeros.
    - The symmetry implies the matrix symmetry according to the diagonal; the relation may be then defined just using a „triangular“ matrix.

- **Directed graph**: plane with the set of points - *vertices* (nodes), directed edges.



This graph may for example specify the relation “owes money to”.

- The reflexive relation: knots on **all** of its vertices.
- The symmetric relation graph: arrows in both directions (or no arrow)
- The transitive relation: the following condition is true: It is possible to get from the vertex *u* to the vertex *v* by the journey which all leads in direction of the arrows, then there exists direct arrow from *u* to *v*.