Questions:

1. (10 points) Find the sale price of any piece of art on artsy.com (or another art website). You can paste a small image of the art here if you want (if it is school-appropriate). Assume the art increases in value by 10% per year, and you pay for insurance that costs of 1% of the value per year. Assume the insurance is charged each year *before* the art value increases that year (i.e. if the art was valued at $100 at the start of a year, then the insurance would be $1 that year). Calculate the *total* insurance paid over the first 40 years of owning the art. You must use a summation formula. Show your work clearly.

Art -> Babs Bleeker (€1,000)

insurane price = x

$x_1 = 1000 \cdot 0.01 = 10$ for the first year

$x_2 = (1000 + 1000 \cdot 0.1) \cdot 0.01 = 11$

$x_3 = ((1000 + 1000 \cdot 0.1) + (1000 + 1000 \cdot 0.1) \cdot 0.1) \cdot 0.01 = 12.1$

So the formula would be $x_n = (1000 \cdot 1.1^{n-1}) \cdot 0.01$

$$S = \sum_{x=1}^{40} (1000 \cdot 1.1^{n-1}) \cdot 0.01 \quad \text{or} \quad S = \sum_{x=1}^{40} (10 \cdot 1.1^{n-1}$$

this is gometric progression with initial term $\cancel{x}_1 = 10$ and $r = 1.1$

$\cancel{S_n = 10 \quad n(1}$ $S_n = \dfrac{x(1-r^n)}{1-r}$

$$S_{40} = \frac{10 \cdot (1 - 1.1^{40})}{1 - 1.1} = 4425.93$$

2. Consider these 6 functions: A. $n \log n$ B. $n!$ C. 2^n D. $100n$ E. $n^8 - 10n^9$ F. n^3

(15 points) List them in increasing order so that each function is big-O of (is bounded by) the next function. For example, you would list n before n^2. You need only write the letters (e.g. A, B, C, D, E, F).

D -> A -> F -> E -> C -> B

3. Determine whether each of the following functions is $O(x^2)$. If yes, state witnesses C and k such that $f(x) \leq Cx^2$ for all $x > k$. If no, provide a brief explanation showing that for any chosen C and k, there is an $x > k$ with $f(x) > Cx^2$.

a. (10 points) $f(x)=25x+50$

$f(x) \leq Cx^2$ for all $x > k$

$25x + 50 \leq Cx^2$

$$\frac{25x+50}{x^2} \leq C$$

$$\frac{25}{x} + \frac{50}{x^2} \leq C$$

let $k = 10$

$$\frac{25}{10} + \frac{50}{100} = 2.5 + 0.5 = 3 \leq C$$

$C = 3$ for $x > 10$

For $x = 20$: $(25 \cdot 20 + 50 = 550) \leq (3 \cdot 400 = 1200)$

Answer: $f(x) = 25x + 50$ is $O(x^2)$

Witnesses: $C = 3$, $k = 10$

b. (5 points) $f(x)= 10x^2+x^3$

$f(x) = 10x^2 + x^3$

$f(x) \leq Cx^2$ for all $x > k$

$10 + x \leq C$

$10 + x \to \infty$, thus it will eventually exceed C.

So, $10x^2 + x^3 \leq Cx^2$ is not true for all $x > k$.

4. (10 points) Suppose that a and b are integers, $a \equiv 5 \pmod{11}$, and $b \equiv 9 \pmod{11}$. Find the integer c with $0 \leq c \leq 10$ to satisfy the following congruence. You must show your work.

$c \equiv a^2 - b \pmod{11}$.

$a \equiv 5 \pmod{11}$

$b \equiv 9 \pmod{11}$

$0 \leq c \leq 10$

$a \equiv 5 \pmod{11}$, so $a^2 \equiv 5^2 \pmod{11}$,

$b \equiv 9$

$a^2 \equiv 25 \pmod{11}$ $\quad$ $25 = 11 \cdot 2 + 3$

so $a^2 \equiv 3 \pmod{11}$

$c \equiv 3 - 9 \pmod{11}$

$c \equiv -6 \pmod{11}$

$-6 = 11 \cdot (-1) + 5$

so $c \equiv 5 \pmod{11}$

5. (15 points) Choose two 4-digit numbers that do not have any repeating digits (e.g. don't choose 1,000 because the digit "0" repeats). Use the Euclidean algorithm to find their greatest common divisor. Show your work.

Numbers: 1240 and 2456

$2456 = 1240 \cdot 1 + 1216$

$gcd(1240, 2456) = gcd(1240, 1216)$

$1240 = 1216 \cdot 1 + 24$

$gcd(1240, 1216) = gcd(1216, 24)$

$1216 = 24 \cdot 50 + 16$

$gcd(1216, 24) = gcd(24, 16)$

$24 = 16 \cdot 1 + \boxed{8}$

$gcd(24, 16) = gcd(16, 8)$

$\underline{16 = 8 \cdot 2 + 0}$

So, $gcd(1240, 2456) = 8$

6. (10 points) Choose an *odd* 3-digit number > 200 that is not divisible by 3. Determine whether or not it is prime. You may use any valid analytical or computational method, but you may not simply look it up or use someone else's software. You must show your work. If you wrote code, include your code (paste in the code itself or a screen capture of the code).

Number: 211

Using Trial division $\sqrt{211} \approx 14.53$

~~So.~~

So prime numbers less than 14.53:

2, 3, 5, 7, 11, 13

2: It's odd, so not divisible

3: Remainder is 1 when divided by 3, not divisible

5: doesn't end with 5 or 0, not divisible

7: $211 = 30 \cdot 7 + 1$, remainder is 1, not divisible

11: $211 : 11 \approx 19.18$, not divisible

13: $211 : \cancel{16.2}\ 11 \approx 16.23$, not divisible

Answer: 211 is prime number

7. Choose two different numbers a and m between 10 and 20 (inclusive) that are relatively prime.

a. (10 points) Find the inverse x of a modulo m. You may use any valid analytical or computational method, including brute force, but you may not simply look it up or use someone else's software. You must show your work. If you wrote code, include your code (paste in the code itself or a screen capture of the code).

Numbers: $a = 15$ $m = 16$

a and m are ~~r~~elatively prime since $\gcd(15,6)$ is 1.

$$15x \equiv 1 \pmod{16}$$

$$15x = 16k + 1$$

Using Extended Euclidean Algorithm

$$16 = 1 \cdot 15 + 1$$

$$1 = 16 - 1 \cdot 15$$

$$1 = (-1) \cdot 15 + 1 \cdot 16$$

Expressing modular inverse with positive itteger

$$x = -1 + 16 = 15$$

Answer: 15

b. (5 points) Explain, in simple terms, why x is the inverse of a modulo m.

$$15 \cdot 15 \equiv 1$$

$$15 \cdot 15 = 225$$

$225 : 16 = 14$, remainder is 1

Thus $15 \cdot 15 \equiv 1 \pmod{16}$ is correct.

$$15x \equiv 1 \pmod{16}$$

8. (10 points) Choose a number n between 501 and 999, and a number a between 8 and 13. Use Fermat's little theorem to calculate a^n **mod** 7 efficiently.

$n = 523$

$a = 9$

$a^{p-1} \equiv 1 \pmod{p}$

$a^6 \equiv 1 \pmod{7}$

$523 : 6 = 87$, remainder is 1.

$523 \equiv 1 \pmod{6}$

$9^{523} \equiv 9^1 \pmod{7}$

$9 : 7 = 1$, remainder 2.

Answer: $9 \equiv 2 \pmod{7}$

$9^{523} \bmod 7 = 2$