Reflections and Refractions in Ray Tracing

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When you're writing a ray tracer, soon or late you'll stumble on the problem of reflection and refraction. When you want to visualize mirror-like objects, you need to reflect our viewing ray. To simulate a lens, you need refraction. While most people have heard of the law of reflection[1] and Snell's law[2], they often have difficulties how to actually calculate the direction vectors of the reflected and refracted rays. And when googling around for info, you find a lot of theory but only a little of practice. In this tutorial, I'll try to show you how to apply the theory in practice.

In the first section, we'll describe the setup. We'll name the used variables and put some conditions on them. In the two sections following on that, we'll derive equations for the reflected and refracted direction vectors based on the law of reflection and Snell's law. I know it's a bit of theory, but it isn't too hard. If you're a bit scared, you can skip to the conclusions though.

1 The Beginnings

In Figure 1 we have an interface (= surface) between two materials with different indexes of refraction η_1 and η_2. These two materials could be air ($\eta \approx 1$) and water ($20°C$: $\eta \approx 1.33$). It does not matter which of η_1 and η_2 is the greater one. η_1 could be air and η_2 water, or vice versa. All that counts is that η_1 is the index of refraction of the material you come from, and η_2 of the material you go to (in case of refraction). This is a very important concept and sometimes misunderstood.

The direction vector of the incident ray (= incoming ray) is $\overrightarrow{i}$, and we assume this vector is normalized. The direction vectors of the reflected and transmitted rays are $\overrightarrow{r}$ and $\overrightarrow{t}$ and will be calculated in this tutorial. These vectors are (or will be) normalized as well. We also have the normal vector $\overrightarrow{n}$, orthogonal to the interface and points towards the first material η_1. Again, $\overrightarrow{n}$ is normalized.

$$\left|\overrightarrow{i}\right| = \left|\overrightarrow{r}\right| = \left|\overrightarrow{t}\right| = \left|\overrightarrow{n}\right| = 1 \tag{1}$$

The direction vectors of the rays can be split in components orthogonal and parallel to the interface. We call these the normal part $\overrightarrow{v_\perp}$ and the tangent part $\overrightarrow{v_\parallel}$ of a vector $\overrightarrow{v}$ (in this paragraph I'll use the generic vector $\overrightarrow{v}$, but the story really is for $\overrightarrow{i}$, $\overrightarrow{r}$ and $\overrightarrow{t}$). The normal part $\overrightarrow{v_\perp}$ can be found by orthogonal projection[3] on $\overrightarrow{n}$. Taking in account (1), this is:

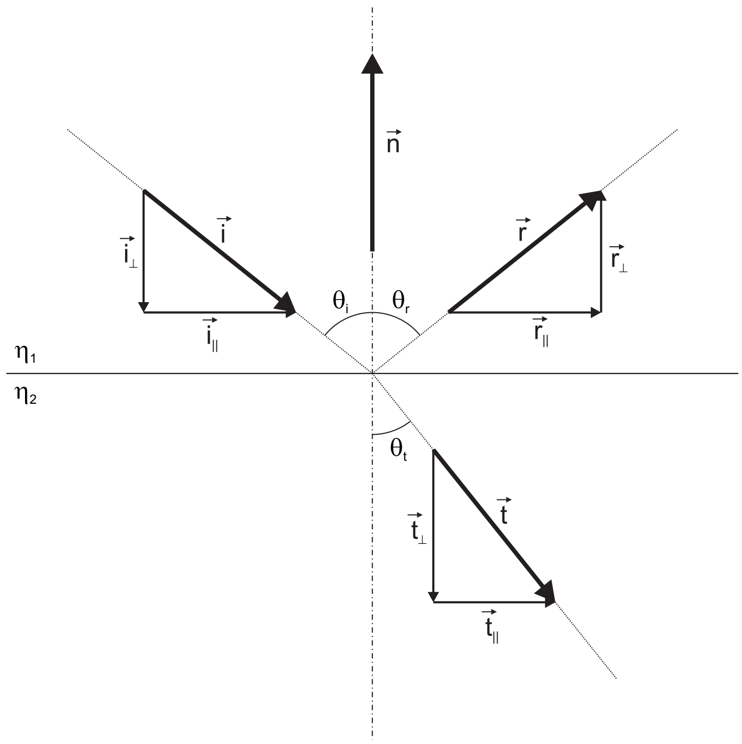

Figure 1: the situation

$$\overrightarrow{v_\perp} = \frac{\overrightarrow{v} \cdot \overrightarrow{n}}{|\overrightarrow{n}|^2}\overrightarrow{n} = (\overrightarrow{v} \cdot \overrightarrow{n})\,\overrightarrow{n} \tag{2}$$

The tangent part $\overrightarrow{v_\parallel}$ is the difference between $\overrightarrow{v}$ and $\overrightarrow{v_\perp}$:

$$\overrightarrow{v_\parallel} = \overrightarrow{v} - \overrightarrow{v_\perp} \tag{3}$$

The dot product between $\overrightarrow{v_\parallel}$ and $\overrightarrow{v_\perp}$ is zero:

$$\begin{aligned} \overrightarrow{v_\parallel} \cdot \overrightarrow{v_\perp} &= \overrightarrow{v} \cdot \overrightarrow{v_\perp} - \overrightarrow{v_\perp} \cdot \overrightarrow{v_\perp} \\ &= \overrightarrow{v} \cdot (\overrightarrow{v} \cdot \overrightarrow{n})\,\overrightarrow{n} - (\overrightarrow{v} \cdot \overrightarrow{n})^2\,\overrightarrow{n} \cdot \overrightarrow{n} \\ &= (\overrightarrow{v} \cdot \overrightarrow{n})^2 - (\overrightarrow{v} \cdot \overrightarrow{n})^2\,|\overrightarrow{n}|^2 \\ &= 0 \end{aligned} \tag{4}$$

That proves two things: $\overrightarrow{v_\perp}$ and $\overrightarrow{v_\parallel}$ are orthogonal, and $\overrightarrow{v_\perp}$ is indeed an orthogonal projection on $\overrightarrow{n}$.

$$\overrightarrow{v_\perp} \perp \overrightarrow{v_\parallel} \tag{5}$$

Hence, we can apply Pythagoras:

$$|\overrightarrow{v}|^2 = \left|\overrightarrow{v_\parallel}\right|^2 + |\overrightarrow{v_\perp}|^2 \tag{6}$$

Furthemore, all normal parts are parallel to each other and $\overrightarrow{n}$. The tangent parts are parallel as well.

$$\overrightarrow{i_\perp} \quad \parallel \quad \overrightarrow{r_\perp} \parallel \overrightarrow{t_\perp} \parallel \overrightarrow{n} \tag{7}$$

$$\overrightarrow{i_\parallel} \parallel \overrightarrow{r_\parallel} \parallel \overrightarrow{t_\parallel} \tag{8}$$

The angles of incidence, reflection and refraction are θ_i , θ_r and θ_t and they are the smallest positive angles between the respective rays and the normal vector $\overrightarrow{n}$. Basic trigonometry and equations (1) and (5) tell us for any of this angles θ the following properties apply:

$$\cos\theta = \frac{|\overrightarrow{v_\perp}|}{|\overrightarrow{v}|} = |\overrightarrow{v_\perp}| \tag{9}$$

$$\sin\theta = \frac{\left|\overrightarrow{v_\parallel}\right|}{|\overrightarrow{v}|} = \left|\overrightarrow{v_\parallel}\right| \tag{10}$$

In fact, (9) says nothing else than[1]:

$$\cos\theta = \pm\overrightarrow{v} \cdot \overrightarrow{n} \tag{11}$$

Now we had all that, we can start with the *fun* stuff.

[1]The used sign in (11) depends on the relative orientation of $\overrightarrow{v}$ and $\overrightarrow{n}$.

2 Reflection

We start with the easiest of both problems: the calculation of the reflected ray. In case of specular reflection, this is very easy: the law of reflection says that the angle of incidence θ_i is equal to the angle of reflection θ_r [1].

$$\theta_r = \theta_i \tag{12}$$

Using equations (9) and (10), this law tells us:

$$|\overrightarrow{r_\perp}| = \cos\theta_r = \cos\theta_i = \left|\overrightarrow{i_\perp}\right| \tag{13}$$

$$\left|\overrightarrow{r_\parallel}\right| = \sin\theta_r = \sin\theta_i = \left|\overrightarrow{i_\parallel}\right| \tag{14}$$

Because of (7) and (8) and Figure 1, we can figure out that both parts are:

$$\overrightarrow{r_\perp} = -\overrightarrow{i_\perp} \tag{15}$$
$$\overrightarrow{r_\parallel} = \overrightarrow{i_\parallel} \tag{16}$$

After summation, we get the desired direction of reflection:

$$\overrightarrow{r} = \overrightarrow{r_\parallel} + \overrightarrow{r_\perp} = \overrightarrow{i_\parallel} - \overrightarrow{i_\perp} \tag{17}$$

By using equations (2) and (3), we can work this out to:

$$\begin{aligned} \overrightarrow{r} &= \overrightarrow{i_\parallel} - \overrightarrow{i_\perp} \\ &= \left[\overrightarrow{i} - \left(\overrightarrow{i} \cdot \overrightarrow{n}\right)\overrightarrow{n}\right] - \left(\overrightarrow{i} \cdot \overrightarrow{n}\right)\overrightarrow{n} \\ &= \overrightarrow{i} - 2\left(\overrightarrow{i} \cdot \overrightarrow{n}\right)\overrightarrow{n} \end{aligned} \tag{18}$$

That's it! But can we be sure $\overrightarrow{r}$ calculated in (18) is normalized as requested? Yes, using (6), (13), (14) and (1) we have:

$$|\overrightarrow{r}|^2 = \left|\overrightarrow{r_\parallel}\right|^2 + |\overrightarrow{r_\perp}|^2 = \left|\overrightarrow{i_\parallel}\right|^2 + \left|\overrightarrow{i_\perp}\right|^2 = \left|\overrightarrow{i}\right|^2 = 1 \tag{19}$$

And that's very cool, since we can avoid a costly normalisation because of this.

3 Refraction

The calculation of the refracted ray starts with Snell's law [2] which tells that the products of the index of refraction and sines of the angles must be equal:

$$\eta_1 \sin\theta_i = \eta_2 \sin\theta_t \tag{20}$$

You can write this as:

$$\sin\theta_t = \frac{\eta_1}{\eta_2}\sin\theta_i \tag{21}$$

With this equation you can already see there will be a bit of a problem if $\sin\theta_1 > \frac{\eta_2}{\eta_1}$. If that's the case, $\sin\theta_2$ would have to be greater than 1. BANG! That's not possible. What we have here is *total internal reflection* or TIR. What exactly this TIR is will be addressed later. For now, we'll just add a condition to our law:

$$\sin\theta_t = \frac{\eta_1}{\eta_2}\sin\theta_i \Leftrightarrow \sin\theta_i \leq \frac{\eta_2}{\eta_1} \tag{22}$$

In what follows, we'll assume this condition is fullfilled, so we don't have to worry about it.

Fine, now we know the theory, we should try to find a formula for $\overrightarrow{t}$. This first thing we'll do is to split it up in a tangent and normal part:

$$\overrightarrow{t} = \overrightarrow{t_\parallel} + \overrightarrow{t_\perp} \tag{23}$$

Of both parts, we'll do $\overrightarrow{t_\parallel}$ first, because Snell's law tells us something about sines, and the norms of the tangent parts happen to be equal to sines. Hence, because of 10 and 22, we can write:

$$\left|\overrightarrow{t_\parallel}\right| = \frac{\eta_1}{\eta_2}\left|\overrightarrow{i_\parallel}\right| \tag{24}$$

Since $\overrightarrow{t_\parallel}$ and $\overrightarrow{i_\parallel}$ are parallel and point in the same direction, this becomes:

$$\overrightarrow{t_\parallel} = \frac{\eta_1}{\eta_2}\overrightarrow{i_\parallel} = \frac{\eta_1}{\eta_2}\left[\overrightarrow{i} - \cos\theta_i\overrightarrow{n}\right] \tag{25}$$

Don't worry about the $\cos\theta_i$, later on it will make things easier if we just leave it there. If you need it, you can easily calculate it with (11), it equals $\overrightarrow{i}\cdot\overrightarrow{n}$.

Great, so we have one part already (it's not that bad, is it? ;-) Now the other one, and that's quite simple if you use Pythagoras (6) and the knowledge that we're dealing with normalized vectors (1):

$$\overrightarrow{t_\perp} = -\sqrt{1-\left|\overrightarrow{t_\parallel}\right|^2}\,\overrightarrow{n} \tag{26}$$

Now we have both parts. It's time to substitute (25) and (26) in (23) to get the refracted direction vector. If we do that and we regroup a little so we get only one term in $\overrightarrow{n}$, we get (hold on!):

$$\overrightarrow{t} = \frac{\eta_1}{\eta_2}\overrightarrow{i} - \left(\frac{\eta_1}{\eta_2}\cos\theta_i + \sqrt{1-\left|\overrightarrow{t_\parallel}\right|^2}\right)\overrightarrow{n} \tag{27}$$

It's a bit unfortunate we still need $\overrightarrow{t_\parallel}$ under the square root. Luckely, we don't really need the vector $\overrightarrow{t_\parallel}$, but its norm. And this norm equals to $\sin\theta_t$ (10). We get:

$$\overrightarrow{t} = \frac{\eta_1}{\eta_2}\overrightarrow{i} - \left(\frac{\eta_1}{\eta_2}\cos\theta_i + \sqrt{1-\sin^2\theta_t}\right)\overrightarrow{n} \tag{28}$$

Now instead, we need $\sin^2\theta_t$, but we know that's given by Snell's law (20):

$$\sin^2\theta_t = \left(\frac{\eta_1}{\eta_2}\right)^2 \sin^2\theta_i = \left(\frac{\eta_1}{\eta_2}\right)^2 \left(1-\cos^2\theta_i\right) \tag{29}$$

The last two equations are all we need to calculate the refracted direction vector.

4 Total Internal Reflection

If we take a closer look to equation (28), you will notice it's only valid if the value under the square root isn't negative. At first glance, that seems to be a second condition next to the one in Snell's law (22). However, that changes if you take a closer look to what this *new* condition means:

$$1 - \sin^2\theta_t \geq 0 \tag{30}$$

That's exactly the same als the original condition:

$$\sin^2\theta_t \leq 1 \tag{31}$$

Isn't this beautiful? In two completely different situations, we have noticed restrictions on the equations: the value of a sine that shouldn't be greater than one, and the value under a square root that shouldn't be negative. And yet, it turns out they're one and the same.

Anyway, if this condition is not fullfilled, we can't find a refracted direction vector. That means we can't do transmission: *if the condition isn't fullfilled, there's no transmission*.

Why exactly is this called *total internal reflection*, and why is it called a reflection while it's a restriction for refraction? First of all, it can only happen if you go from a denser material to a less dense material (e.g. from glass to air). This is because $\sin\theta_t$ can only become greater than one if $\frac{\eta_1}{\eta_2} > 1$. In other words: it can only happen if you're *inside* a denser material. Hence the internal. Secondly, physics says that each photon (= a packet of light) that arrives at the interface, but go through it (= transmission) or be reflected. Well, it can be absorbed too, but let's ignore that :-). If no photons can go through because the condition in Snell's law isn't fullfilled, then they all must be reflected: total reflection. Together, this gives *total internal reflection*.

5 Conclusion

By now, we have derived two equations to calculate the reflected and refracted direction vectors by using vector arithmetic only. Here they are again:

$$\begin{aligned}
\overrightarrow{r} &= \overrightarrow{i} - 2\cos\theta_i \overrightarrow{n} \\
\overrightarrow{t} &= \frac{\eta_1}{\eta_2}\overrightarrow{i} - \left(\frac{\eta_1}{\eta_2}\cos\theta_i + \sqrt{1-\sin^2\theta_t}\right)\overrightarrow{n}
\end{aligned}$$

with

$$\begin{aligned}
\cos\theta_i &= \overrightarrow{i} \cdot \overrightarrow{n} \\
\sin^2\theta_t &= \left(\frac{\eta_1}{\eta_2}\right)^2 \left(1-\cos^2\theta_i\right)
\end{aligned}$$

In case of refraction, there's a condition that limits the range of incoming angles θ_i. Outside this range, the refracted direction vector does not exists. Hence, there's no transmission. This is called *total internal reflection*. The condition is:

$$\sin^2\theta_t \leq 1$$

Algorithm 1 the code

```
Vector reflect(const Vector& normal, const Vector& incident)
{
    const double cosI = dot(normal, incident);
    return incident - 2 * cosI * normal;
}

Vector refract(const Vector& normal, const Vector& incident,
               double n1, double n2)
{
    const double n = n1 / n2;
    const double cosI = dot(normal, incident);
    const double sinT2 = n * n * (1.0 - cosI * cosI);
    if (sinT2 > 1.0)
    {
        return Vector::invalid;
    }
    return n * incident - (n + sqrt(1.0 - sinT2)) * normal;
}
```

References

[1] Eric Weisstein's World of Physics, *Law of Reflection*, http://scienceworld.wolfram.com

[2] Eric Weisstein's World of Physics, *Snell's Law*, http://scienceworld.wolfram.com

[3] 3D Geometry Primer, *More On Vector Arithmetic*, http://www.flipcode.com/geometry