Draft version September 19, 2018
Typeset using LaTeX **twocolumn** style in AASTeX61

ON THE DEFINITION OF A MASS OPERATOR FROM THE PENTA-DIMENSIONAL PERSPECTIVE

Noé Aubin-Cadot[1]

[1] *Université de Montréal, Département de Mathématiques et Statistiques, C.P. 6128 Succ. Centre-ville, Montréal, QC H3C 3J7, Canada*

ABSTRACT

Proper time τ (equivalently $s = c\tau$) is hypothesized to be a real spatial dimension. Such a hypothesis is natural once Einstein's special relativity train is revisited. Mass is defined as its conjugate variable. A mass operator $\hat{m}$ follows, generalizing Wigner's definition of mass. The massive Dirac equation becomes even more elegant. A penta-dimensional decomposition of the Yang-Mills equations is given, out of which the Proca equation emerges. Then, because τ is a physical dimension, lightlike and timelike geodesics in 4D space-time are seen as lightlike geodesics in 5D space-time. These are then seen as geodesics in a 4D Riemannian setting. Advantages and drawbacks of a possible conformally-Euclidean gravitational theory are quickly glimpsed at. Anti-gravity predicted by the Schwarzschild metric is revisited. An electronic plasma should start to levitate at 13.4 MeV. It follows that the solar corona could float in anti-gravity and that anti-gravity could be a source of nuclear fusion disruptions inside a tokamak or similar device.

1. PROPER TIME, AS A PHYSICAL DIMENSION

If you are reading this, you might know what *special relativity* is. Let yourself be back in those days where you first heard of it. There was someone sitting beside some bushes looking at a fast moving train passing by. Inside that train was a mirror glued at the ceiling and one glued to the floor. Between those two mirrors was a point particle moving vertically or diagonally (depending on the reference frame) at the speed of light c (independently of the reference frame). A cartesian coordinate system was drawn where s was the coordinate along the vertical of the train and x was a coordinate along the horizontal train's track. Because the point particle was moving at the speed of light, infinitesimal time intervals dt were given by the Pythagorean theorem:

$$c^2dt^2 = dx^2 + ds^2 \tag{1}$$

Then ds^2 was isolated and promoted to a so-called *Minkowski metric*:

$$ds^2 = c^2dt^2 - dx^2 \tag{2}$$

From now on, special relativity was a mere ritornello consisting of transforming space-time (ct, x) while preserving ds^2. Then time went on, the moving train went away and the person beside the bushes was long gone. Eventually, to describe gravity, the Minkowski metric ds^2 was promoted, once again, to a *pseudo-Riemannian metric*:

$$ds^2 = \sum_{i,j} g_{i,j}\, dx^i \otimes dx^j \tag{3}$$

Now, in a standard general relativistic (GR) context, s has a double role: a metric ds^2 role and a nice parameter $\tau = s/c$, the so-called *proper time*, for timelike geodesics. But thinking of it not only as some abstract *internal clock* but as a real tangible dimension as physical as it was inside Einstein's train seems to be both heresy and natural. After all, invoking a physical *fifth dimension* to space-time (ct, x, y, z) is not only historically banal (Weyl, Kaluza, Klein, Souriau, etc.) but fairly mundane compared to the number of *vertical* dimensions invoked in gauge theoretic particle physics. Let's hypothesize that s *is a real physical dimension*. It is now time to hop back inside Einstein's train and play with our "brand new" dimension s.

2. MASS, AS IT'S CONJUGATE VARIABLE

Here we are, back inside Einstein's train $\mathbb{R}^3 \times S$ where S could be $\mathbb{R}$, S^1, $[0, 1]$ or $]0, 1[$. Fix a flat Euclidean metric:

$$c^2dt^2 = dx^2 + dy^2 + dz^2 + ds^2 \tag{4}$$

Instead of a point particle bouncing at the speed of light c between the two mirrors, consider instead a time-dependent complex scalar wave function $\psi(t, x, y, z, s)$ propagating according to the wave equation:

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial s^2} \tag{5}$$

As usual, let's define those four standard operators:

$$\hat{H} = i\hbar\frac{\partial}{\partial t} \qquad \hat{p}_x = -i\hbar\frac{\partial}{\partial x}$$
$$\hat{p}_y = -i\hbar\frac{\partial}{\partial y} \qquad \hat{p}_z = -i\hbar\frac{\partial}{\partial z}$$

It is then unavoidable to define a *mass operator*:

$$\hat{m} = -i\frac{\hbar}{c}\frac{\partial}{\partial s} \tag{6}$$

It can equivalently be written as:

$$\hat{m} = -i\frac{\hbar}{c^2}\frac{\partial}{\partial \tau} \tag{7}$$

Now, the wave equation (5) can be reformulated as:

$$\hat{H}^2\psi = c^2(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 + \hat{m}^2c^2)\psi \tag{8}$$

If ψ is a *penta-dimensional de Brolie wave*:

$$\psi(t,x,y,z,s) = e^{i(-Et+p_x x+p_y y+p_z z+mcs)/\hbar} \tag{9}$$

equation (8) applied to (9) implies:

$$E^2 = p^2c^2 + m^2c^4 \tag{10}$$

where $p := p_x^2 + p_y^2 + p_z^2$. From this follows other famous equalities:

$$\nu = E/h \qquad \lambda_B = h/p \qquad \lambda_C = h/(mc)$$

where $h = 2\pi\hbar$ as usual. Here λ_B is the de Broglie wavelength and λ_C is the Compton wavelength. If you ever wondered what is the physical interpretation of the Compton wavelength, here is it: *the Compton wavelength is the de Broglie wavelength in the s direction.* Now, letting $\lambda_E := hc/E$, equation (10) reformulates geometrically as the Pythagorean theorem for the reciprocals:

$$1/\lambda_E^2 = 1/\lambda_B^2 + 1/\lambda_C^2$$

Nowadays, mass is defined in a slightly more convoluted way than (6). According to the so-called *Wigner classification*, mass is defined via the Klein-Gordon (KG) equation:

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{m^2c^2}{\hbar^2}\psi \tag{11}$$

Or, in algebraic jargon, mass squared is defined as the first Casimir invariant of the Poincaré group. Because equation (5) generalizes the KG equation, it is worth investigating how $\hat{m}$ fits in modern physics. Although the Dirac and the Proca equations will be dealt with below, a full generalization of the Higgs mechanism will not because it would involve dismantling a delicate watchmaking where charges, hypercharges, Weinberg angle, etc., are notions that cannot be naively left under the rug.

Before moving on, let's emphasize that the $(+,+,+,+)$ Riemannian space (x,y,z,s) is the ideal place to consider anti-self-dual (ASD) Yang-Mills (YM) instantons without the need to invoke the algebraic obscurantism of a *Wick rotation* $t \mapsto \sqrt{-1}\cdot t$. After all, maybe Einstein's train is indeed a cobordism between two mirrors.

3. UNCERTAINTY ON MASS

Analogously to the canonical commutation relations:

$$-[t,\hat{H}] = [x,\hat{p}_x] = [y,\hat{p}_y] = [z,\hat{p}_z] = i\hbar$$

one finds:

$$[s,\hat{m}c] = i\hbar$$

This implies an uncertainty principle on mass:

$$(\Delta s)(\Delta mc) \geq \hbar/2$$

Sending the c term to the right hand side one gets:

$$(\Delta s)(\Delta m) \geq \hbar/(2c) \tag{12}$$

The added c term to the right, making $\hbar/(2c) \ll \hbar/2$, might be the reason why a quantum physics with a fixed known classical mass is a good approximation of reality. Inequality (12) can be written in terms of proper time $\tau = s/c$ as:

$$(\Delta\tau)(\Delta m) \geq \hbar/(2c^2)$$

making the right hand side even smaller and negligible for most situations. But what about neutrinos?

4. SCHRÖDINGER EQUATION REVISITED

For $x, y \in \mathbb{R}$, $x \neq 0$, we have:

$$|y| \ll |x| \implies \sqrt{x^2+y^2} \approx |x| + y^2/(2|x|)$$

Using this approximation on equation (10), written as:

$$E = \sqrt{p^2c^2 + m^2c^4}$$

gives two approximations, one of which is famous for its use in non-relativistic mechanics:

$$p \ll |m|c \implies E \approx |m|c^2 + p^2/(2|m|) \tag{13}$$
$$|m|c \ll p \implies E \approx pc + (mc)^2c/(2p) \tag{14}$$

where, again, $p = p_x^2 + p_y^2 + p_z^2$. Those two approximations, reformulated in terms of operators, yield the

Schrödinger equation (without potential V) and another equation:

$$p \ll |m|c \implies i\hbar\partial_t \approx |m|c^2 - (\hbar^2/(2|m|))\Delta \tag{15}$$
$$|m|c \ll p \implies i\hbar\partial_t \approx pc - (\hbar^2 c/(2p))\partial_s^2 \tag{16}$$

The reader is invited to work out the quantum harmonic oscillator in this last equation by adding a $ks^2/2$ potential. Another exercice would be to see the interior of Einstein's train between the two mirrors as an infinite well potential. In both these exercices, the average mass $\langle\hat{m}\rangle_\psi = \langle\psi|\hat{m}|\psi\rangle$ vanishes on any eigenstate ψ of the Hamiltonian. Only superpositions of $\hat{H}$'s eigenstates can have non zero mass. Letting such a superposition of eigenstates evolve in time, the average mass will oscillate. Again, what about neutrinos?

5. NEGATIVE MASS

Because mass is now a momentum, it can be negative. This corresponds to nothing esoteric: a wavefront can go upward or downward inside Einstein's train. Because mass is usually known to be a non-negative real number, let's define:

$$m_{\text{usual}} := |\langle\hat{m}\rangle_\psi| \in \mathbb{R}_{\geq 0}$$

6. A "QUANTIZATION" OF FIELDS

Have you ever wondered why fields are quantized in quantum field theory? Here it is: *QFT's fields are quantized because of the two mirrors in Einstein's train.* Although such an explanation seems both highly far-fetched and somewhat childish, it links two 1905 seminal paper's by Einstein where, in one, Albert hypothesizes that light is quantized and where, in the other, a light particle bounces in Einstein's train to explain special relativity. Now that it's getting hot inside Einstein's train, let's revisit the Dirac equation.

7. DIRAC EQUATION REVISITED

Let's fix some notation:

$$x^i = (x^0, x^1, x^2, x^3, x^4) = (ct, x, y, z, s)$$
$$p_i = (p_0, p_1, p_2, p_3, p_4) = (E/c, p_x, p_y, p_z, mc)$$

In terms of *penta-momentum* operator:

$$\begin{aligned}\hat{p}_i &= (\hat{p}_0, \hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4)\\ &= (c^{-1}\hat{H}, \hat{p}_x, \hat{p}_y, \hat{p}_z, \hat{m}c)\\ &= i\hbar(c^{-1}\partial_t, -\partial_x, -\partial_y, -\partial_z, -\partial_s)\\ &= i\hbar(\partial_0, -\partial_1, -\partial_2, -\partial_3, -\partial_4)\end{aligned}$$

where $\partial_i := \partial/\partial x^i$. Suppose that there exists five algebraic entities a^0, a^1, a^2, a^3, a^4, whose algebraic properties are yet to be determined, such that:

$$\left(\sum_{i=0}^{4} a^i \partial_i\right)^2 = (\partial_0)^2 - (\partial_1)^2 - (\partial_2)^2 - (\partial_3)^2 - (\partial_4)^2$$

It follows that a^i must verify the relations of a Clifford algebra:

$$a^i a^j + a^j a^i = 2\eta^{i,j}, \quad \forall i, j = 0, 1, 2, 3, 4$$

where $\eta^{i,j}$ is a penta-dimensional $(+,-,-,-,-)$ Minkowski metric. Define a *penta-dimensional Dirac operator*:

$$D := \sum_{i=0}^{4} a^i \partial_i \tag{17}$$

Promote the scalar field ψ to a $\mathbb{C}^n$-valued field on which the a^i terms act (i.e. fix a representation of the Clifford algebra). Then, the penta-dimensional Dirac equation reads:

$$D\psi = 0 \tag{18}$$

This "new" Dirac equation generalizes in a compact way Dirac's original equation:

$$i\hbar\partial_t\psi = \left(mc^2\alpha_0 - i\hbar c\sum_{j=1}^{3}\alpha_j\partial_j\right)\psi$$

where mass was a distinguished classical fixed quantity. If ψ is a solution of the Dirac equation (18), then each vectorial component of ψ is a solution to the wave equation (5), itself a generalization of the KG equation (11).

The distinguished mass term of a given massive fermion field in a given QFT Lagrangian is now embedded in the penta-dimensional Dirac operator (17) without the need to invoke a Yukawa coupling between the fermionic field and the Higgs field (and one could suppose that a given massive fermion field is of de Broglie type in the s direction). But what about gauge boson fields?

8. PENTA YANG-MILLS DECOMPOSITION

Let G be a Lie group among U(1), SU(2), SU(3) or any product of these and let $\mathfrak{g}$ be G's Lie algebra. Let P be a trivial G-principal bundle over the penta-dimensional base space (ct, x, y, z, s). Let $\mathcal{G} < \text{Aut}(P) < \text{Diff}(P)$ be the gauge group (a subgroup of the G-bundle automorphisms group acting fiberwise). Let $\mathcal{A}$ be the space of connection forms over P. The $\mathcal{G}$-action on $\mathcal{A}$ is chosen to be the *right* group action $A \cdot \Lambda = \Lambda^* A$ acting via pull-backs (and not the *left* group action $\Lambda \cdot A = (\Lambda^{-1})^* A$).

Let A be a connexion form over P. Pull-back A down to the base space via any global section of P. Now the connexion form lives on the base space (ct, x, y, z, s). Denote it again by A. Although this last A depends on a choice of section of P to pull it down to the base space, we don't care because we could always gauge transform

A. The $\mathcal{G}$-action on $\mathcal{A}$ corresponds on the base space to this celebrated gauge transformation equation:

$$A \mapsto \mathrm{Ad}_{\lambda^{-1}} A + \lambda^{-1} d\lambda$$

where $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is the usual adjoint G-action on its Lie algebra $\mathfrak{g}$.

From the connection A on the 5D space (ct, x, y, z, s) let's define:

$$\begin{aligned} \phi_s &:= A(\ell \partial_s) \\ A_s &:= A - \phi_s \ell^{-1} ds \end{aligned}$$

Here, the *Planck length* $\ell := \sqrt{\hbar G / c^3}$ plays only a currency exchange role between unit-free differential geometric objects ϕ_s and A_s and the fact that s has units of physical length ($[ds] = dist.$, $[\partial_s] = 1/dist.$). Both ϕ_s and A_s are $\mathfrak{g}$-valued and are defined on each space-time slice $(ct, x, y, z, s = const.)$. This gives a decomposition of the connection A:

$$A = A_s + \phi_s \ell^{-1} ds \tag{19}$$

as a path of connections A_s and of fields ϕ_s on space-time (ct, x, y, z). Such a decomposition from a penta-dimensional gauge theory over (ct, x, y, z, s) to a quadri-dimensional theory over (ct, x, y, z) is the analog of a famous connection form decomposition going from quadri-dimensional space-time (ct, x, y, z) to t-parametrized fields lying on a tri-dimensional space (x, y, z) which is known by physicists as the electromagnetic four-potential decomposition $(\phi_t, \vec{A}_t)$, by theoretical physicists as a way to go from 4D-YM connexions to 3D magnetic monopoles and by mathematicians as the very first step to understand the yet unsolved *Atiyah-Floer conjecture*.

To a gauge transformation $\Lambda \in \mathcal{G}$ over the 5D space corresponds a path Λ_s of gauge transformations over the 4D space. The corresponding group action on the decomposition (19) is:

$$\begin{aligned} A_s &\mapsto \mathrm{Ad}_{\lambda_s^{-1}} A_s + \lambda_s^{-1} d\lambda_s \\ \phi_s &\mapsto \mathrm{Ad}_{\lambda_s^{-1}} \phi_s + \lambda_s^{-1} \ell \partial_s \lambda_s \end{aligned}$$

The curvature form $F_A = dA + \frac{1}{2}[A \wedge A]$ on the 5D space decomposes as:

$$F_A = F_s + J_s \wedge (\ell^{-1} ds)$$

where F_s and J_s are $\mathfrak{g}$-valued differential forms defined on each independent 4D slice $(ct, x, y, z, s = const.)$ as:

$$F_s := F_{A_s} \tag{20}$$

$$J_s := d_{A_s} \phi_s - \ell \partial_s A_s \tag{21}$$

This decomposition is preserved under gauge transformations over the 5D space. If one works without trivializing P, then F_s and J_s would take values in the adjoint bundle $\mathrm{Ad}P := P \times_{\mathrm{Ad}} \mathfrak{g}$.

Now, recall that the Yang-Mills equations are:

$$0 = d_A F_A \qquad \text{(i.e. Bianchi identity)} \tag{22}$$

$$0 = d_A^* F_A \tag{23}$$

Here, d_A (resp. d_A^*) denotes the exterior covariant (resp. co-)derivative on the 5D space. Let's fix a unit-free metric on the (ct, x, y, z, s) space:

$$g = \ell^{-2} \left(c^2 dt^2 - dx^2 - dy^2 - dz^2 - ds^2 \right) \tag{24}$$

Without this ℓ^{-2} in g, one could loose track of physical units when using Hodge duality. Working out the decomposition of the YM equations from the 5D space (ct, x, y, z, s) to the 4D space (ct, x, y, z), one gets:

$$0 = d_{A_s}^* J_s \tag{25}$$

$$d_{A_s}^* F_s = -\ell \partial_s J_s + [\phi_s, J_s] \tag{26}$$

$$d_{A_s} F_s = 0 \qquad \text{(i.e. Bianchi, again)} \tag{27}$$

$$\partial_s F_s = d_{A_s} \ell \partial_s A_s \tag{28}$$

Equation (25) is the current conservation equation, equation (26) is the YM inhomogeneous equation and equation (27) is the YM homogeneous equation. Here, d_{A_s} and $d_{A_s}^*$ denote the ones on each 4D slice and not on the 5D space as in equations (22, 23).

9. THE PROCA EQUATION

If one *gauges away* ϕ_s, equations (25, 26) become:

$$0 = d_{A_s}^* (\partial_s A_s) \tag{29}$$

$$d_{A_s}^* F_s = \ell^2 \partial_s \partial_s A_s \tag{30}$$

Using definition (6) of the mass operator $\hat{m}$, equation (30) becomes:

$$d_{A_s}^* F_s = -\ell^2 \left(\frac{\hat{m} c}{\hbar} \right)^2 A_s \tag{31}$$

If A_s is sinusoidally de Broglie in the s direction, this reformulates as:

$$d_{A_s}^* F_s = -\ell^2 \left(\frac{m c}{\hbar} \right)^2 A_s \tag{32}$$

This equation is known as the *Proca equation*. Because the Proca equation is a special case of the *Stueckelberg equation* and because the Stueckelberg equation is a special case of the Higgs mechanism, equation (30) seems to indicate a path to unify, at least philosophically, the mass operator $\hat{m} = -ic^{-1} \hbar \partial_s$ and the Higgs mechanism.

In our present setting, it seems that an eventual Higgs field φ_s should not be a mere auxiliary $\mathbb{C}^2$-valued field but rather an appropriate $\mathrm{End}(\mathfrak{g})$-valued field φ_s such that:

$$\ell \partial_s A_s = \varphi_s \circ A_s \tag{33}$$

In the electroweak model, an s-independent φ_s should suffice. However, for quark confinement questions related to YM theory, decomposition (33) should be used with parsimony because of the complexity of ASD YM instantons on the (x, y, z, s) space.

10. ELECTROMAGNETISM

As a toy model, let's consider electromagnetism, i.e. $G = \mathrm{U}(1)$. Because G is abelian, F_s and J_s from equations (20, 21) become:

$$F_s = dA_s \tag{34}$$

$$J_s = d\phi_s - \ell \partial_s A_s \tag{35}$$

Thus, equations (25,26,27,28) become:

$$0 = d^* J_s \tag{36}$$

$$d^* F_s = -\ell \partial_s J_s \tag{37}$$

$$dF_s = 0 \tag{38}$$

$$\partial_s F_s = d(\ell \partial_s A_s) \tag{39}$$

Gauge transformations over the 5D space (ct, x, y, z, s) act on A_s and ϕ_s as:

$$A_s \mapsto A_s + d\ln(\lambda_s)$$

$$\phi_s \mapsto \phi_s + \ell \partial_s \ln(\lambda_s)$$

Because such a transformation leaves invariant F_s and J_s, both F_s and J_s are physically tangible. The F_s term is already known as Maxwell's electromagnetic field tensor. What about J_s? There are three possibilities:

1. J_s is *unphysical*, there is no fifth dimension, this document should end up in a shredder;
2. J_s is a *matter current*;
3. J_s is a *photonic current*, whatever that means.

Although the first option is the safest bet, the second option seems at first somewhat natural. Indeed, once one fixes $\partial_s A_s = 0$ (i.e. massless electromagnetism), equation (37) looks a lot like Maxwell's inhomogeneous equation where charged matter interacts with the electromagnetic field. One could then factorize J_s as a contraction of a spinor with itself to take account of fermions such as electrons. But why factorize J_s as a spinor contraction but not F_s? This is where it gets mathematically either unnatural either very complicated. Because of this, the third option might be worth a thought or two.

Instead of gauging away ϕ_s one could gauge away $d^* A_s$. This last gauge fixing is famously known as the *Lorenz gauge*. In such a gauge, equation (36) becomes:

$$0 = d^* d\phi_s$$

This equation tells us that for each independent s, the ϕ_s field propagates "as a massless field would do" according to the usual 4D wave equation:

$$\frac{1}{c^2} \frac{\partial^2 \phi_s}{\partial t^2} = \frac{\partial^2 \phi_s}{\partial x^2} + \frac{\partial^2 \phi_s}{\partial y^2} + \frac{\partial^2 \phi_s}{\partial z^2} \tag{40}$$

Wigner (i.e. KG) tells us that this implies that ϕ_s is a massless field. However, $\partial_s \phi_s$ does not need to vanish. This is where Wigner's definition of mass is not equivalent to the definition (6) of the mass operator $\hat{m}$. Even when the field ϕ_s is massive (according to $\hat{m}$), it always propagates *at the speed of light* (hence the idea that J_s could be a "photonic current"). Although making a clear distinction between the electromagnetic field F_s and a photonic current J_s, interacting through equation (37), seems good for mental health (e.g. picturing the Earth's magnetic field without invoking "virtual photons"), such a distinction between electromagnetism and photonism would need to rewrite too much physics for today.

It is now time to leave the quantum world and go to a much bigger scale: gravity.

11. RIEMANNIANIZED GRAVITY

Einstein's general relativity is a theory where a $(+, -, -, -)$ pseudo-Riemannian metric g plays two roles. First, g is the dynamical variable of the Einstein equation. Second, g tells the geodesic equation which trajectories are possible for falling objects. Here, it will only be question of the geodesic equation.

Suppose we are given a t-independent $(+, -, -, -)$ pseudo-riemannian metric of this kind:

$$ds^2 = g_{t,t} cdt \otimes cdt - \sum_{i,j=1}^{3} g_{i,j} dx^i \otimes dx^j \tag{41}$$

Consider a geodesic $\gamma : \mathbb{R} \to \mathbb{R}^4$ of this metric. Unless one is interested in science-fiction, there are two physically meaningful possibilities: either γ it is *timelike* and represents massive matter, either γ is *lightlike* and represents massless light. If γ is timelike, it can be parametrized by its proper time $\tau = s/c$. If γ is lightlike, it cannot be parametrized by τ because $ds = 0$. Our goal here is to put lightlike geodesics and timelike geodesics on the same footing. This can be done now that we know that lightlike geodesics represent waves propagating horizontally inside Einstein's train and timelike geodesics represent waves propagating mainly vertically inside that same train. Let's recall that a geodesic depends on a particular parametrization while *pre-geodesics*, being the image set of a geodesic, does not depend on a choice of parametrization. Pre-geodesics of the metric (41) correspond to lightlike pre-geodesics of this penta-dimensional pseudo-Riemannian metric:

$$g_{t,t}cdt \otimes cdt - \sum_{i,j=1}^{3} g_{i,j}dx^i \otimes dx^j - ds \otimes ds \tag{42}$$

Because lightlike pre-geodesics are invariant under conformal transformations of the metric, the lightlike pre-geodesics of (42) correspond to lightlike pre-geodesics of this other pseudo-Riemannian metric:

$$cdt \otimes cdt - (g_{t,t})^{-1} \sum_{i,j=1}^{3} g_{i,j}dx^i \otimes dx^j - (g_{t,t})^{-1}ds \otimes ds$$

Lightlike pre-geodesics of this metric correspond to pre-geodesics of this Riemannian metric:

$$(cdt)^2 = (g_{t,t})^{-1} \sum_{i,j=1}^{3} g_{i,j}dx^i \otimes dx^j + (g_{t,t})^{-1}ds \otimes ds \tag{43}$$

Let's call (43) the *Riemannianized metric* of (41). Thus, in many physically significant pseudo-Riemannian metrics, one can remove the *pseudo* out of it. Now, lightlike and timelike geodesics can be both parametrized by t with ease.

12. THE GEODESIC EQUATION (A SPECIFIC CASE)

Suppose that, given some coordinate system (q^0, q^1, q^2), we have a diagonal metric:

$$c^2dt^2 = g_{0,0}dq^0 \otimes dq^0 + g_{1,1}dq^1 \otimes dq^1 + g_{2,2}dq^2 \otimes dq^2$$

whose coefficients $g_{i,j}$ depend only on the q^1 parameter but are constant on q^0 and q^2. Then, its Christoffel symbols:

$$\Gamma^i_{j,k} = \sum_m (1/2)g^{i,m}(\partial_j g_{m,k} + \partial_k g_{m,j} - \partial_m g_{j,k})$$

are explicitly given by:

$$\begin{aligned}
\Gamma^0_{0,1} &= \Gamma^0_{1,0} = (1/2)\partial_1 \ln(g_{0,0}) \\
\Gamma^1_{0,0} &= -(1/2)(g_{1,1})^{-1}\partial_1 g_{0,0} \\
\Gamma^1_{1,1} &= (1/2)\partial_1 \ln(g_{1,1}) \\
\Gamma^1_{2,2} &= -(1/2)(g_{1,1})^{-1}\partial_1 g_{2,2} \\
\Gamma^2_{1,2} &= \Gamma^2_{2,1} = (1/2)\partial_1 \ln(g_{2,2})
\end{aligned}$$

All other symbols $\Gamma^i_{j,k}$ vanish. It follows that the geodesic equation

$$\ddot{q}^i = -\sum_{j,k=0}^{2} \Gamma^i_{j,k}\dot{q}^j\dot{q}^k$$

becomes explicitly:

$$\begin{aligned}
\ddot{q}^0 =& -(\partial_1 \ln(g_{0,0}))\dot{q}^0\dot{q}^1 \\
\ddot{q}^1 =& (1/2)(g_{1,1})^{-1}(\partial_1 g_{0,0})(\dot{q}^0)^2 \\
& - (1/2)(\partial_1 \ln(g_{1,1}))(\dot{q}^1)^2 \\
& + (1/2)(g_{1,1})^{-1}(\partial_1 g_{2,2})(\dot{q}^2)^2 \\
\ddot{q}^2 =& -(\partial_1 \ln(g_{2,2}))\dot{q}^1\dot{q}^2
\end{aligned}$$

Here the dots denote d/dt.

13. THE RIEMANNIANIZED SCHWARZSCHILD METRIC

The *Schwarzschild metric* on the equator $\theta = \pi/2$ is:

$$ds^2 = (1 - R/r)c^2dt^2 - (1 - R/r)^{-1}dr^2 - r^2d\varphi^2 \tag{44}$$

where $R = 2GM/c^2$ is the Schwarzschild radius of some massive object (e.g. Earth or Sun). Its corresponding Riemannianized metric is:

$$c^2dt^2 = (1-R/r)^{-2}dr^2 + (1-R/r)^{-1}(r^2d\varphi^2 + ds^2) \tag{45}$$

This metric is of the kind seen in section §12 where:

$$(q^0, q^1, q^2) = (s, r, \varphi)$$

and where:

$$\begin{aligned}
g_{0,0} &= (1 - R/r)^{-1} \\
g_{1,1} &= (1 - R/r)^{-2} \\
g_{2,2} &= (1 - R/r)^{-1}r^2
\end{aligned}$$

Clearly, $g_{0,0}$, $g_{1,1}$ and $g_{2,2}$ only depend on $q^1 = r$. So, the geodesic equation can be explicitly written as:

$$\ddot{s} = \frac{R\dot{s}\dot{r}}{r^2 - rR} \tag{46}$$

$$\ddot{r} = -\frac{R\dot{s}^2}{2r^2} + \frac{R\dot{r}^2}{r^2 - rR} + \frac{(2r - 3R)\dot{\varphi}^2}{2} \tag{47}$$

$$\ddot{\varphi} = -\frac{\dot{r}\dot{\varphi}(2r - 3R)}{r^2 - rR} \tag{48}$$

If one wishes to rewrite this in terms of proper time $\tau = s/c$, one obviously gets:

$$\ddot{\tau} = \frac{R\dot{\tau}\dot{r}}{r^2 - rR} \tag{49}$$

$$\ddot{r} = -\frac{Rc^2\dot{\tau}^2}{2r^2} + \frac{R\dot{r}^2}{r^2 - rR} + \frac{(2r - 3R)\dot{\varphi}^2}{2} \tag{50}$$

$$\ddot{\varphi} = -\frac{\dot{r}\dot{\varphi}(2r - 3R)}{r^2 - rR} \tag{51}$$

14. RIEMANNIANIZED SCHWARZSCHILD METRIC (ISOTROPIC COORDINATES)

The Schwarzschild metric can also be written in so-called *isotropic coordinates* $(ct, \rho, \theta, \varphi)$. Here, the radius r was replaced by another radius ρ which is related to r by:

$$r = \rho \cdot (1 + R/4\rho)^2$$

The radius ρ does not completely cover the whole range of r but only $r \geq R$. Also, ρ covers this range twice. However, despite these oddities, ρ is by far closer to a typical "Euclidean" radius than r. Indeed, in isotropic coordinates the Schwarzschild metric (again at $\theta = \pi/2$) reads:

$$ds^2 = \frac{(1 - R/(4\rho))^2}{(1 + R/(4\rho))^2} c^2 dt^2 - (1 + R/(4\rho))^4 d\sigma^2 \tag{52}$$

where:

$$d\sigma^2 = dx^2 + dy^2 = d\rho^2 + \rho^2 d\varphi^2$$

One should be aware that when $r \to \infty$, both radius r and ρ are not asymptotically equal but are slightly shifted by half a Schwarzschild radius:

$$r \approx \rho + R/2$$

This $R/2$ shift and the double covering of r by ρ should, I suspect, be linked to spin 1/2 representations. The Riemannianized version of the Schwarzschild metric in isotropic coordinates is:

$$c^2 dt^2 = \frac{(1 + R/(4\rho))^2}{(1 - R/(4\rho))^2} ds^2 + \frac{(1 + R/(4\rho))^6}{(1 - R/(4\rho))^2} d\sigma^2 \tag{53}$$

where $d\sigma^2$ is the same as above. Again, the Riemannianized isotropic Schwarzschild metric is of the kind seen in section §12. So, the geodesic equation can be explicitly written as:

$$\begin{aligned}
\ddot{s} =& \frac{R\dot{s}\dot{\rho}}{\rho^2} \frac{1}{1 - X^2} \\
\ddot{\rho} =& - \frac{\dot{s}^2 R}{2\rho^2} \frac{1}{(1 + X)^4 (1 - X^2)} \\
& + \frac{\dot{\rho}^2 R}{2\rho^2} \frac{2 - X}{1 - X^2} \\
& + \rho\dot{\varphi}^2 \frac{1 - 4X + X^2}{1 - X^2} \\
\ddot{\varphi} =& - \frac{2\dot{\rho}\dot{\varphi}}{\rho} \frac{1 - 4X + X^2}{1 - X^2}
\end{aligned}$$

where $X := R/(4\rho)$ is used to simplify the presentation. As before, as a matter of taste, one could rewrite these equations in terms of proper time $\tau = s/c$:

$$\begin{aligned}
\ddot{\tau} =& \frac{R\dot{\tau}\dot{\rho}}{\rho^2} \frac{1}{1 - X^2} \\
\ddot{\rho} =& - \frac{\dot{\tau}^2 c^2 R}{2\rho^2} \frac{1}{(1 + X)^4 (1 - X^2)} \\
& + \frac{\dot{\rho}^2 R}{2\rho^2} \frac{2 - X}{1 - X^2} \\
& + \rho\dot{\varphi}^2 \frac{1 - 4X + X^2}{1 - X^2} \\
\ddot{\varphi} =& - \frac{2\dot{\rho}\dot{\varphi}}{\rho} \frac{1 - 4X + X^2}{1 - X^2}
\end{aligned}$$

Before moving on to the weak field approximation $R \ll r$ of these equations, let's quickly visit another interesting metric.

15. CONFORMALLY-EUCLIDEAN GRAVITY

It would be nice if all the $(1 - R/r)$ terms in the Riemannianized Schwarzschild metric (45) were to the same power. However, Nature is not that simple. One term has a power of -2 and the other has a power of -1. In fact, this two-to-one ratio is important: it is responsible for light being deflected twice what Newtonian gravity predicts. It doesn't mean that we shouldn't quickly glimpse at what a *conformally-Euclidean based gravitational theory* could look like. Suppose we are given such a metric:

$$c^2 dt^2 = n(\vec{x})^2 \cdot (dx^2 + dy^2 + dz^2 + ds^2) \tag{54}$$

where $\vec{x}$ denotes (x, y, z, s) and where $n(\vec{x})$ is some $\mathbb{R}_{>0}$-valued function. Here, gravity would be nothing more than refraction as described by Fermat's geometrical optics in the (x, y, z, s) space. Such a gravitational theory is appealing for its mathematical simplicity because it is based on the eikonal equation:

$$\ddot{\vec{x}} = n^{-3} c^2 \vec{\nabla} n - 2n^{-1} (\dot{\vec{x}} \cdot \vec{\nabla} n) \dot{\vec{x}}$$

instead of the full geodesic equation. Such a theory also seems appealing for its physical naturalness (e.g. if one wants to think of gravity as a mirage of refracted quantum matter waves propagating inside Einstein's train).

Now, as a toy model, one should try a Riemannian metric of this kind:

$$n(r)^2 (dx^2 + dy^2 + dz^2 + ds^2)$$

where $r = x^2 + y^2 + z^2$. From such a metric, two critical tests need to be addressed: the deflection angle of light by the sun (DALS) and Mercury's perihelion's precession angle over a year (MPPA). If one likes to do numerical simulations, Runge-Kutta 7 with a fixed step-size of Δt close to 0.1s $\sim$ 0.05s works well for MPPA and close to 0.00001s works well for DALS. In such numerical simulations, one will find that $n(r)^2 = e^{R/r}$ gives roughly a 70% relative error on MPPA. If one tries instead $n(r)^2 = (1 - R/r)^{-1}$, one finds a roughly 35% relative error on MPPA. Because of those Taylor expansions:

$$e^{R/r} \approx 1 + R/r + (1/2)(R/r)^2 + ...$$
$$(1 - R/r)^{-1} \approx 1 + R/r + (2/2)(R/r)^2 + ...$$

one is lead to try

$$n(r)^2 = 1 + R/r + (3/2)(R/r)^2 \tag{55}$$

Such a "gravitational index of refraction" $n(r)$ gives a striking 1% relative error on MPPA (about the same precision as a Schwarzschild metric would get). Although this is a somewhat unexpected result, it fails the other test by giving the same DALS as Newton's gravitational theory, i.e. half the Einsteinian DALS. One then cannot go further without invoking that photons have a gravitational constant G twice that of matter. A doubled gravitational constant would mean that g loses its special status of describing *the* background. If gravity is indeed described by a conformally-Euclidean metric (54), where n is given by (55), then the $\approx$ 88 days year duration of Parker Solar Probe in the 2024 orbit should be off by roughly 40 seconds compared to what Einstein's GR predicts.

A conformally-Euclidean based gravitational theory is probably mere science-fiction. However, the geodesic equations from the Schwarzschild metric (44), from the Riemannianized Schwarzschild metric (45) and from the Riemannianized isotropic Schwarzschild metric (53) were all equivalently successful at both tests MPPA and DALS. So, *Riemannianizing* a pseudo-Riemannian metric is not just algebraic sorcery, it does work.

Now, before going on to our last dish, we need to talk about something serious: Einstein's train's height.

16. THE TRAIN'S HEIGHT

How tall is Einstein's train? How distant are both mirrors? How thick is our Universe? This is an old question dating back from the 1920's where Kaluza, Klein, Einstein and other theoretical physicists were wondering how to deal with the fact that we do not see a fifth spacetime dimension. Back in those days, the fifth dimension was supposed to be *rolled up* as a tiny imperceptible circle S^1. Its radius is commonly thought to be roughly the Planck length ℓ. In section §8, the Planck length only had a role of giving physical length units to differential geometric objects where physical units don't "glue" well to coordinate-free manifolds. Other than erudite numerology, there is no physical justification to suppose that Einstein's train's height is equal or even close to the Planck length except from the facts that it is very small, that it has units of a length and that it is a fundamental constant of our Universe.

Now, our Universe is filled with an omnipresent ambiant bath of photons called the *cosmic microwave background* (CMB). This background radiation has the spectrum of a fairly cold radiating black body at $T = 2.725$ kelvins. Using Wien's displacement law $\lambda = b/T$, where $b \approx 2.898 \times 10^{-3}$ m $\cdot$ K is Wien's constant, the CMB's peak wavelength λ is roughly 1.063 mm long. Although the CMB is supposed to come from a distant past, suppose instead it is just where useless bits of photons pile up at the end of various processes where thermodynamical energy decay is unavoidable. Appart from tiny multipolar fluctuations, the CMB looks to be pretty much at a thermodynamical rest, i.e. seems to be in a sort of standing wave mode (photons come as much as they go). Assuming the equipartition theorem applies to pentadimensional Yang-Mills energy, there should be roughly the same amount of energy in the (x, y, z) directions as in the s direction. Although photons do not propagate vertically inside Einstein's train but only horizontally, it could be that somehow the CMB's spectrum is the same horizontally and vertically inside the train. Because horizontally the CMB's peak wavelength is roughly 1mm, this indicates that Einstein's train could be tall of approximately a millimetre or so. If we do not perceive this 1mm thickness, it might be that light propagates only horizontally and not vertically. Moreover, because of gravitational Doppler shift and space-time deformations, the train's height should not be constant but dynamical, as is the radius of the small circle in Kaluza-Klein's theory.

Before the train's height question derails this document (which is already flirting with the shredder), let's go on with our final topic: Schwarzschild anti-gravity and its possible physical manifestations.

17. SCHWARZSCHILD ANTI-GRAVITY

Suppose that we are in a low gravity approximation $R \ll r$. This approximation is valid both at the surface of the Earth and of the Sun. The geodesic equation corresponding to the Riemannianized Schwarzschild metric

(45) is then approximated by:

$$\ddot{\tau} = \frac{R\dot{\tau}\dot{r}}{r^2} \quad (56)$$

$$\ddot{r} = -\frac{Rc^2\dot{\tau}^2}{2r^2} + \frac{R\dot{r}^2}{r^2} + r\dot{\varphi}^2 \quad (57)$$

$$\ddot{\varphi} = -\frac{2\dot{r}\dot{\varphi}}{r} \quad (58)$$

Because relativistic speed are to be considered, there might be ambiguity in the last $r\dot{\varphi}^2$ term as to what is centripetal acceleration and what is not. Remember that the real ”Euclidean” radius is the isotropic radius ρ. This is where the isotropic version of the Schwarzschild metric comes handy. Let's consider the low gravity approximation $R \ll \rho$ (i.e. $X \ll 1$) of the geodesic equation of the Riemaniannized isotropic Schwarzschild metric (53):

$$\ddot{\tau} = \frac{R\dot{\tau}\dot{\rho}}{\rho^2} \quad (59)$$

$$\ddot{\rho} = -\frac{\dot{\tau}^2 c^2 R}{2\rho^2} + \frac{\dot{\rho}^2 R}{\rho^2} + \rho\dot{\varphi}^2 \quad (60)$$

$$\ddot{\varphi} = -\frac{2\dot{\rho}\dot{\varphi}}{\rho} \quad (61)$$

Now, we are certain that the last term $\rho\dot{\varphi}^2$ is purely centripetal and nothing else. Suppose that a point particle moves in a purely radial motion, i.e. that $\dot{\varphi} = 0$. Equations (59, 60, 61) then become:

$$\ddot{\tau} = \frac{R\dot{\tau}\dot{\rho}}{\rho^2} \quad (62)$$

$$\ddot{\rho} = -\frac{\dot{\tau}^2 c^2 R}{2\rho^2} + \frac{\dot{\rho}^2 R}{\rho^2} \quad (63)$$

$$\ddot{\varphi} = 0 \quad (64)$$

In low gravity $R \ll \rho$, the present metric (53) is approximated by:

$$c^2 dt^2 = ds^2 + d\rho^2 + \rho^2 d\varphi^2 \quad (65)$$

So, in our $\dot{\varphi} = 0$ scenario, we have a speed condition of $1 = \dot{\tau}^2 + \beta^2$, where $\beta := \dot{\rho}/c$. Substituting this inside equation (63), one gets:

$$\ddot{\rho} = \frac{Rc^2}{2\rho^2}(-1 + 3\beta^2)$$

Using $R = 2GM/c^2$ and letting $g := GM/\rho^2$ implies:

$$\ddot{\rho} = g \cdot (-1 + 3\beta^2)$$

Here, g is the usual Newtonian gravitational acceleration. At the surface of planet Earth, $g \approx 9.81\text{m/s}^2$. If β is zero, $\ddot{\rho} = -g$ is the same acceleration predicted by Newton, which is great. But, if $|\beta| > 1/\sqrt{3} \approx 57.7\%$, gravity seems to disappear independently of β's sign. Worse, when $|\beta| = 1$, gravity points radially *outward* at a $2g$ rate.

This phenomenon was observed in the early days of Einstein's GR by D. Hilbert and subsequently rediscovered by various physicists. Actually, this anti-gravitational push is unsurprising. Think of gravity as a refraction. When a light ray enters a higher index of refraction its speed goes down (which corresponds to $\ddot{\rho} > 0$). Then when it leaves towards a lower index of refraction its speed goes up again (which corresponds, once more, to $\ddot{\rho} > 0$). This, combined with the fact that the final aesthetic of the geodesic equation is highly sensitive to a particular choice of coordinate chart, might be the reason why this *anti-gravity phenomenon* predicted by the Schwarzschild metric is perpetually rediscovered then forgotten again as a mathematical curiosity.

In fact, the question is not if the equations are right or if the physics involved here is real, but rather if it manifests itself somewhere in the Universe in the way one would naively think about how anti-gravity should look like.

18. ENERGY CONDITION FOR ANTI-GRAVITY

Define $\gamma := 1/\dot{\tau} = 1/\sqrt{1-\beta^2}$. Because γ is a monotonically increasing function of $|\beta|$, the anti-gravity condition $|\beta| > 1/\sqrt{3}$ translates as $\gamma > \sqrt{3/2}$. Since the energy of a relativistic mass m particle is $E = mc^2\gamma$, the kinetic energy of that same particle is:

$$K = E - mc^2 = mc^2(\gamma - 1) \quad (66)$$

This, in return, gives us a rough kinetic energy scale at which anti-gravity should manifest itself:

$$K > mc^2(\sqrt{3/2} - 1) \quad (67)$$

This formula does not take into account kinetic energy due to motions in the two other usual spatial dimensions, so one might want to multiply it by some factor.

Since $h = 6.626 \times 10^{-34}\text{J}\cdot\text{s}$, $c = 2.998 \times 10^8\text{m/s}$, $m_e = 9.109 \times 10^{-31}\text{kg}$ and $m_p = 1.673 \times 10^{-27}\text{kg}$, the anti-gravity kinetic energy condition yields respectively $K_e > 13.4$ MeV for the electron and $K_p > 7.3$ GeV for the proton.

Suppose that some dust, gas or plasma bathes in hard enough radiations so that the particles *shake* at $|\beta| > 1/\sqrt{3}$. Then, anti-gravity should emerge as a macroscopic phenomenon.

X rays detected from the solar corona are of the order of K_e and electrons out there have been conjectured to have energies over 50 MeV's. This is more than enough for anti-gravity to manifests itself. Now, even if that doesn't explain why the solar corona is so incredibly hot, it could explain why it doesn't fall back to the Sun. Does the solar corona float in anti-gravity?

In a tokamak (or other similar device), where one wishes to maintain deuterium fusion, one finds a plasma consisting of two beams: one of ions and one of electrons. The energies involved are also of the order of MeV's. Moreover, the electronic beam generally follows a twisted path along a torus, hence is prone to relativistic oscillations in the radial ρ direction. Nuclear fusion is constantly plagued with disruptions, electron runaways and dynamical instabilities. Even if the electronic beam goes mostly horizontally and not vertically, could it be that anti-gravity be a source of disruptions?

REFERENCES

Atiyah, M. F., New invariants of 3- and 4-dimensional manifolds, Proceedings of Symposia in Pure Mathematics, 48.

Bott, R., 1985, On some recent interactions between mathematics and physics, Canadian Mathematical Society, No. 2, Vol., 28, p.129-164.

Candela, A. M. & Sánchez, M., 2008, Geodesics in semi-Riemannian Manifolds: Geometric Properties and Variational Tools, Recent developments in pseudo-Riemannian Geometry, ESI-Series on Math. and Phys., EMS Publishing House, p.359-418.

Carmeli, M., 1972, Behaviour of fast particles in the Schwarzschild field, Nuovo Cimento Lettere, Vol. 3, Ser. 2, p. 379 - 383

Cartan, É., 1937, The Theory of Spinors, M.I.T. Press, 1966.

Donaldson, S. K., Kronheimer, P. B., 1990, The geometry of four-manifolds, Oxford Mathematical Monographs.

Einstein, A., 1916, La relativité, Petite Bibliothèque Payot, Éditions Payot & Rivages, 2001.

Evans, J. & Nandi K. K. & Islam A., 1996, The optical-mechanical analogy in general relativity - exact newtonian forms for the equations of motion of particles and photons, General Relativity and Gravitation, Plenum Publishing Corporation, Vol. 28, No. 4.

Felber, F., 2010, Test of relativistic gravity for propulsion at the Large Hadron Collider, Franklin AIP Conf. Proc. 1208, 247-260.

Fixsen, D. J., 2009, The Temperature of the Cosmic Microwave Background, Astrophysical Journal. vol. 707, no. 2, p.916-920.

Folland, G. B., 2008, Quantum Field Theory: A Tourist Guide for Mathematicians, AMS, Mathematical Surveys and Monographs, Vol. 149.

Hamilton, M.J.D., 2015, The Higgs boson for mathematicians - lecture notes on gauge theory and symmetry breaking, arXiv:1512.02632.

Hilbert, D., 1917, Die Grundlagen der Physik (Zweite Mitteilung), Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Math.-Phys. Kl. (1917), Vol. 1917, p.53-76.

Jaffe, A., Taubes, C., 1980, Vortices and Monopoles: Structure of Static Gauge Theories, Birkhäuser, Progress in Physics - 2.

Kalaydzhyan, T., 2015, Testing general relativity on accelerators, Physics Letters B, Volume 750, p.112-116.

Kaluza, Th., 1921, On the unification problem in physics, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.), p.966-972.

Klein, O., 1926, Quantum theory and five-dimensional relativity theory, Z.Phys. 37, (1926), p.895-906.

Krucker, S., Battaglia, M., Cargill, P.J. et al., Hard X-ray emission from the solar corona, Astron Astrophys Rev (2008) 16: 155.

Misner, C. W. & Thorne, K. & Wheeler, J., 1973, Gravitation, W. H. Freeman and Company.

Penzias, A. A. & Wilson, R. W., 1965, A Measurement of Excess Antenna Temperature at 4080 Mc/s, Astrophysical Journal, vol. 142, p.419-421.

Planck, M., 1900, On the Theory of the Energy Distribution Law of the Normal Spectrum, Verhandl. Dtsch. Phys. Ges., 2, 237.

Souriau, J.-M., 1962, Équations d'onde à 5 dimensions, Séminaire Janet, Mécanique analytique et mécanique céleste, tome 6 (1962-1963), no. 2, p.1-11.

Weyl, H., 1918, Gravitation and Electricity, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.), (1918), 465.

Wigner, E., 1937, On Unitary Representations Of The Inhomogeneous Lorentz Group, Annals of Mathematics, Second Series, Vol. 40, No. 1 (Jan., 1939), p.149-204.