Algorithms Complexity

Fabrizio Ferrandi

Contacts

- Abednego Kambale Wamuhindo
- abednegowamuhindo.kambale@polimi.it

Algorithm as a recipe

Ingredients:

- spaghetti:450g (1 pound)
- bacon: 225g (½ pound)
- egg yolks: 5
- Pecorino or Parmigiano-Reggiano cheese, grated: 360ml (1½ cups)
- olive oil, extra-virgin: 3-4 tablespoons
- pepper, freshly ground: ½ tablespoon
- Salt

Utensils:

- large pot
- large skillet
- bowl
- measuring cups and spoons
- fork

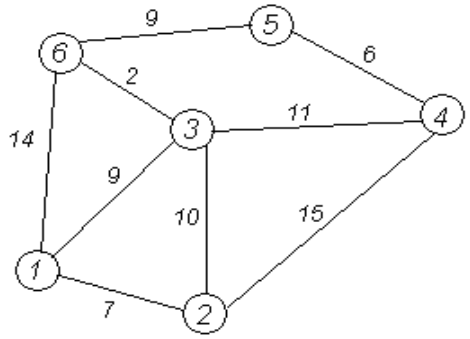
Procedure

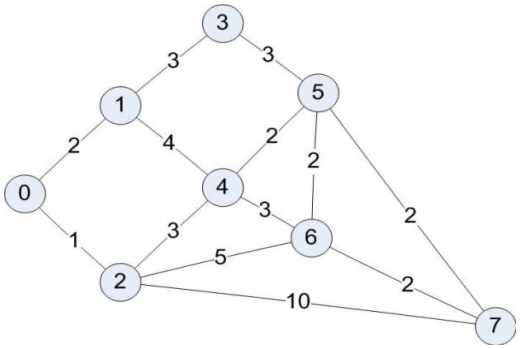
1. Dice the bacon into small pieces (1 inch [2.5cm] will do).
2. The Pot: Bring a big pot of water to a boil and add salt when it begins to simmer.
3. The Pot: Cook the spaghetti until it is al dente and drain it, reserving ½ cup (118 ml) of water.
4. The Skillet: As spaghetti is cooking, heat the olive oil in a large skillet over a medium-high heat. When the oil is hot, add the pancetta and cook for about 10 minutes over a low flame until the pancetta has rendered most of its fat but is still chewy and barely browned.
5. The Bowl: In a bowl, slowly whisk about ½ cup of the pasta water into the egg yolks, using a fork. Add the Parmesan cheese and pepper. Mix with a fork.
6. The Skillet: Transfer the spaghetti immediately to the skillet with the pancetta. Toss it and turn off the heat. Add the egg mixture to the skillet with the pasta and toss all the ingredients to coat the pasta. Taste the pasta and add salt and black pepper, if necessary.

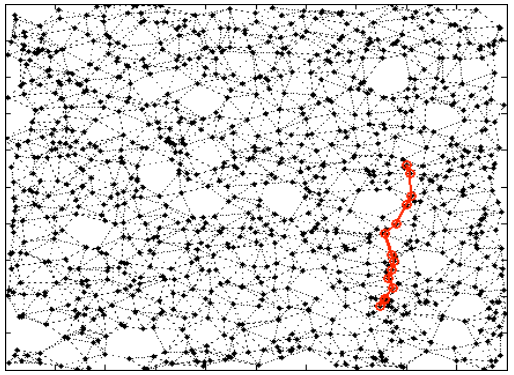
Algorithm Complexity

- How long it takes to prepare *carbonara* for 2 persons?
  - How long for 10?
    - How long for 100?

Shortest path computation







A lot of problems can be translated
↕ to shortest path pb.

Is there any algorithm computing the shortest path between two vertices in a graph? How long does it take?

Analysis of algorithms

The theoretical study of computer-program performance and resource usage

what we want to study TODAY.

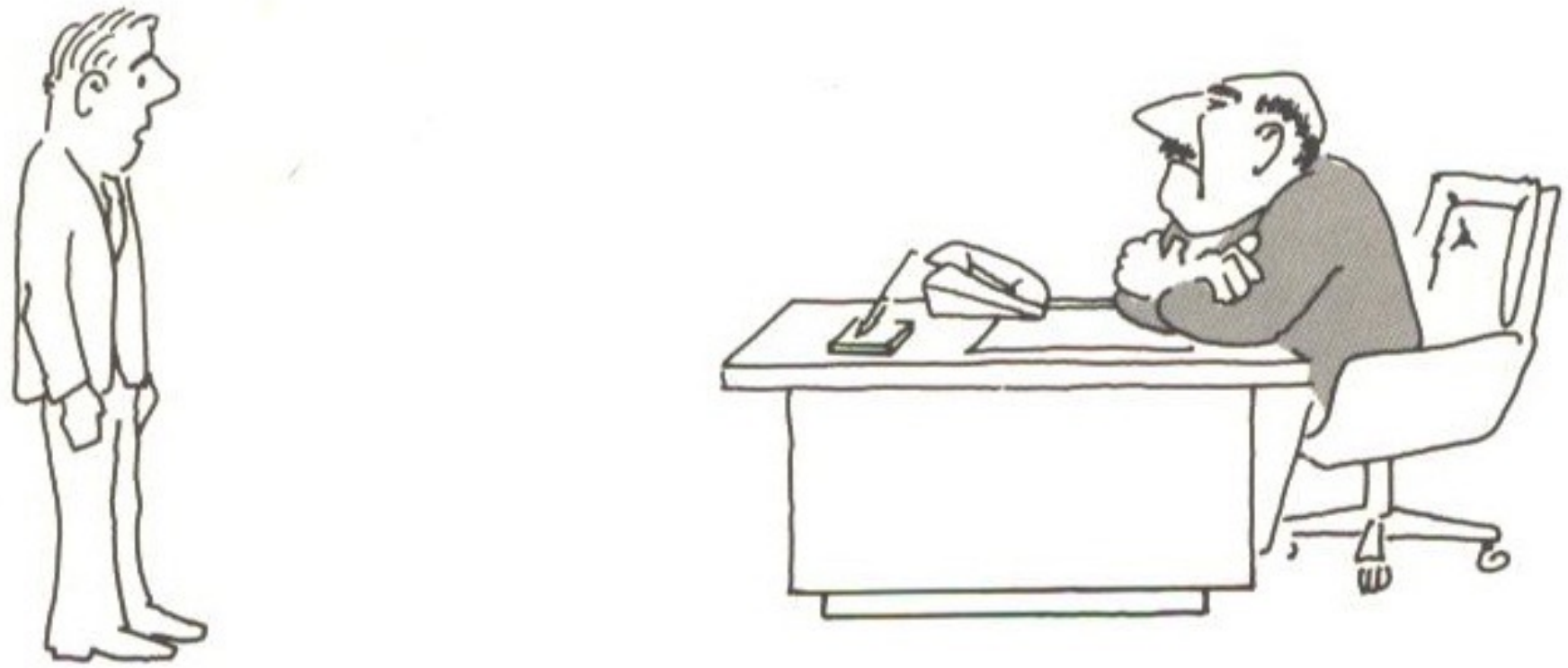
What's usually more important than performance?

- **modularity**
- **correctness**
- **maintainability**
- **functionality**
- **robustness**
- **user-friendliness**
- **programmer time**
- **simplicity**
- **extensibility**
- **reliability**

Why study algorithms and performance?

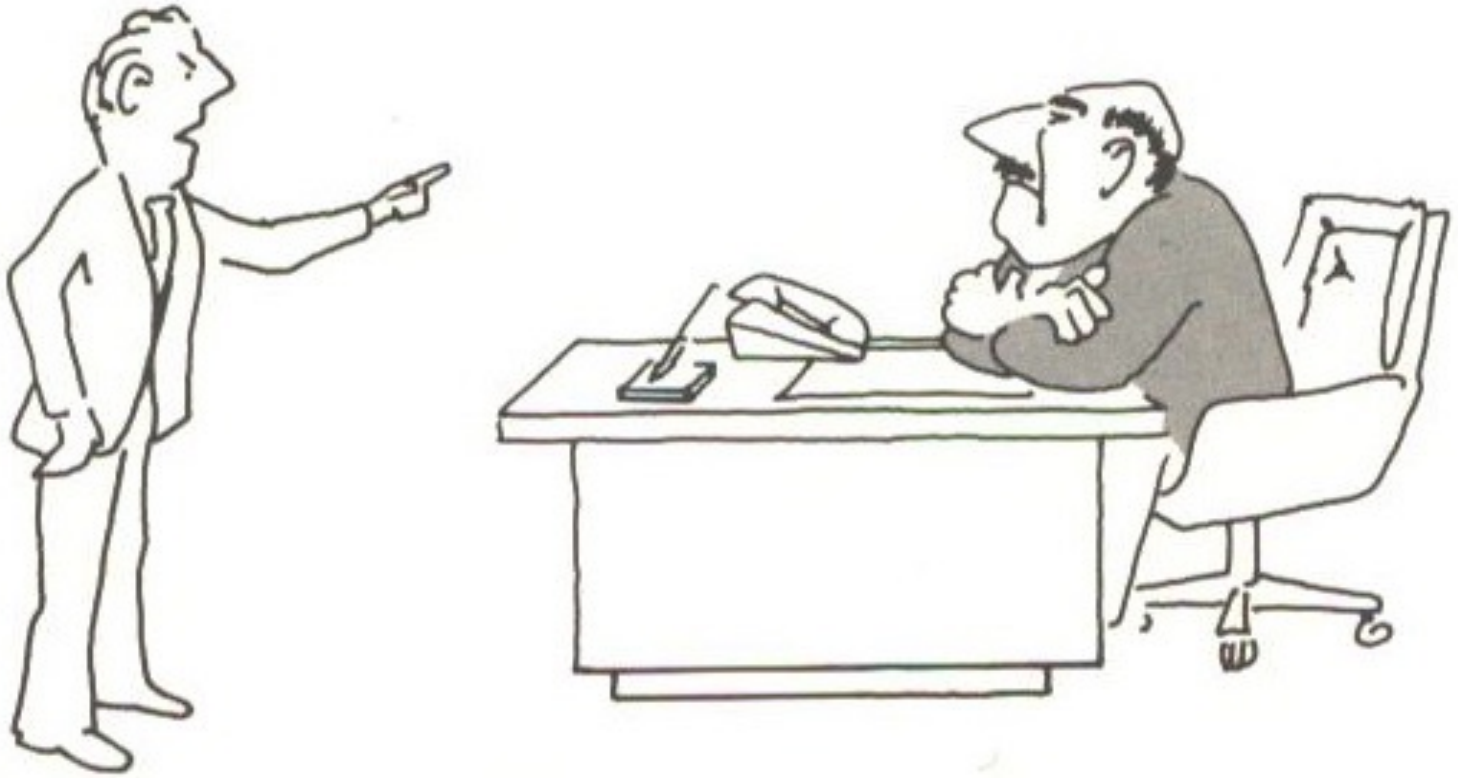
- Algorithms analysis help us to understand *scalability*
- Performance often draws the line between **what is feasible and what is impossible**
- Algorithmic mathematics provides a *language* for talking about program behavior
- Performance is the *currency* of computing
- The lessons of program performance generalize to other computing resources
- Speed is fun!

without taking into consideration the hardware specs.



"I can't find an efficient algorithm, I guess I'm just too dumb."

Comic from Computers and *Intractability*, Garey and Johnson



"I can't find an efficient algorithm, because no such algorithm is possible!"

Comic from Computers and *Intractability*, Garey and Johnson

"I can't find an efficient algorithm, but neither can all these famous people."

Comic from Computers and *Intractability*, Garey and Johnson

The problem of sorting

Input: **sequence** $\langle a_1, a_2, \ldots, a_n \rangle$ **of numbers.**

Output: **permutation** $\langle a'_1, a'_2, \ldots, a'_n \rangle$ **such that** $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

↑ ordered

Example:

Input: 8 2 4 9 3 6

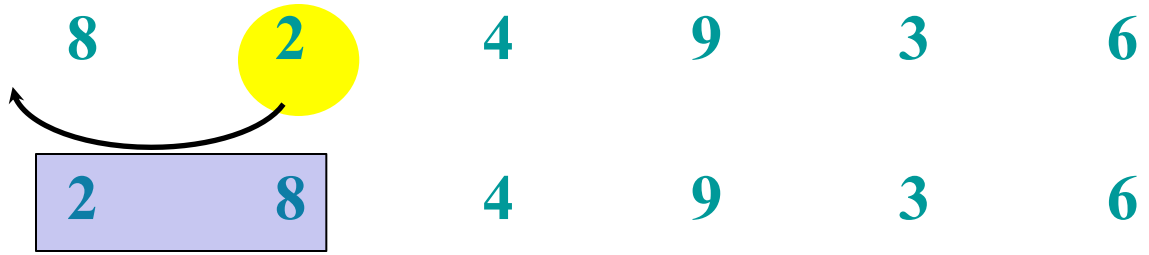
Output: 2 3 4 6 8 9

Example of insertion sort

8	2	4	9	3	6

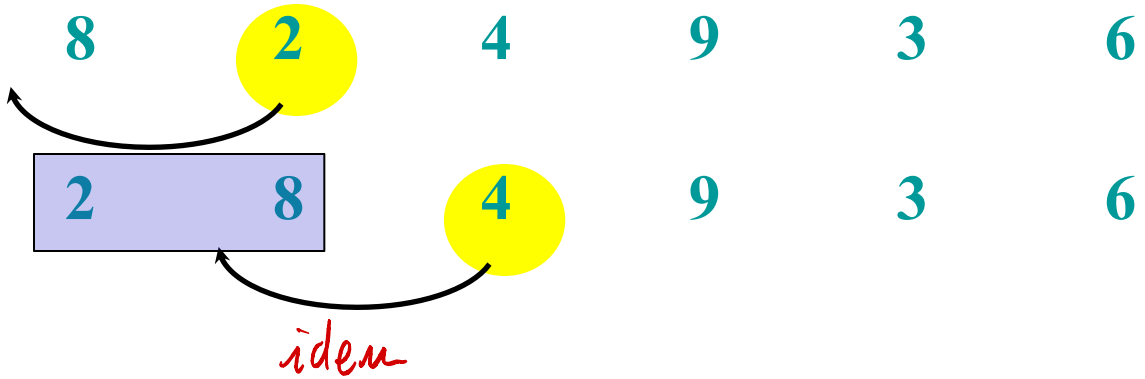
Example of insertion sort

Example of insertion sort

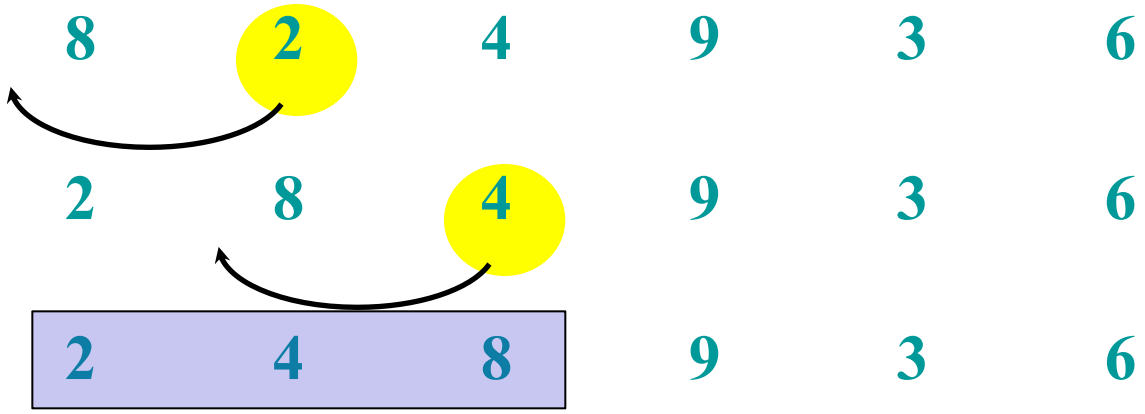


Yes, 2 is smaller than 8, so we put it before 8.

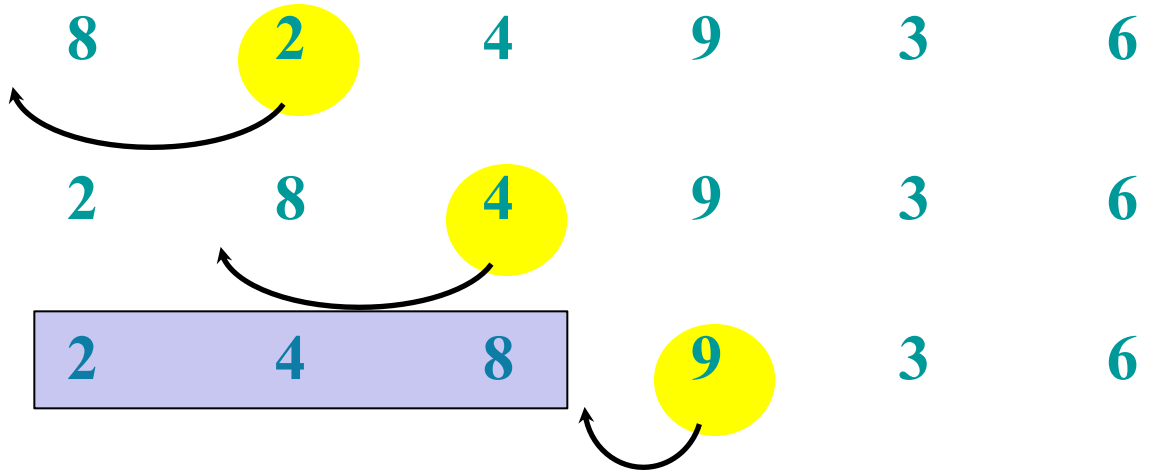
Example of insertion sort



Example of insertion sort

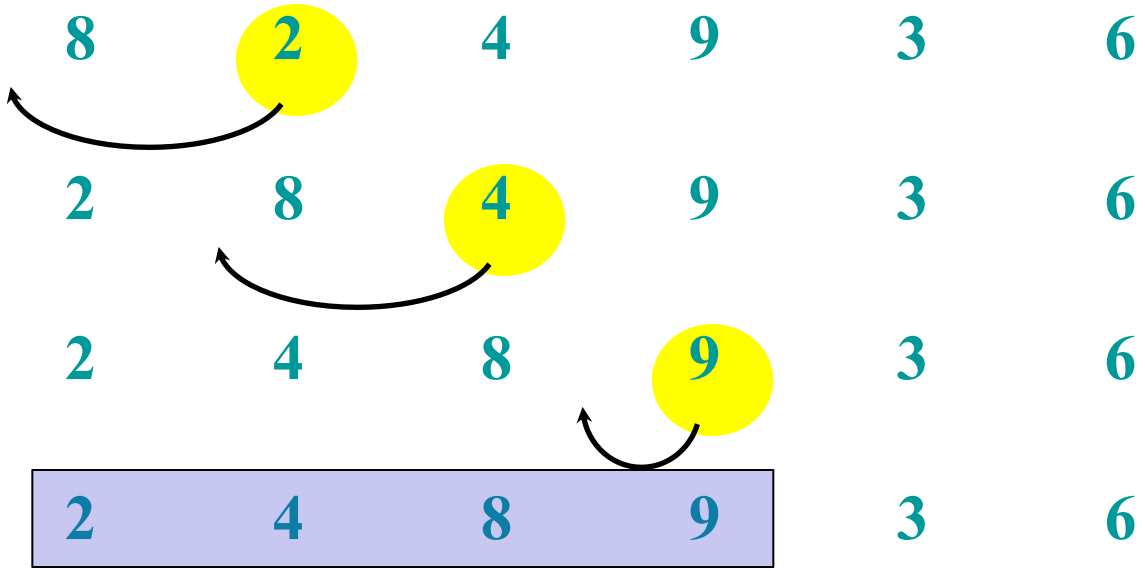


Example of insertion sort

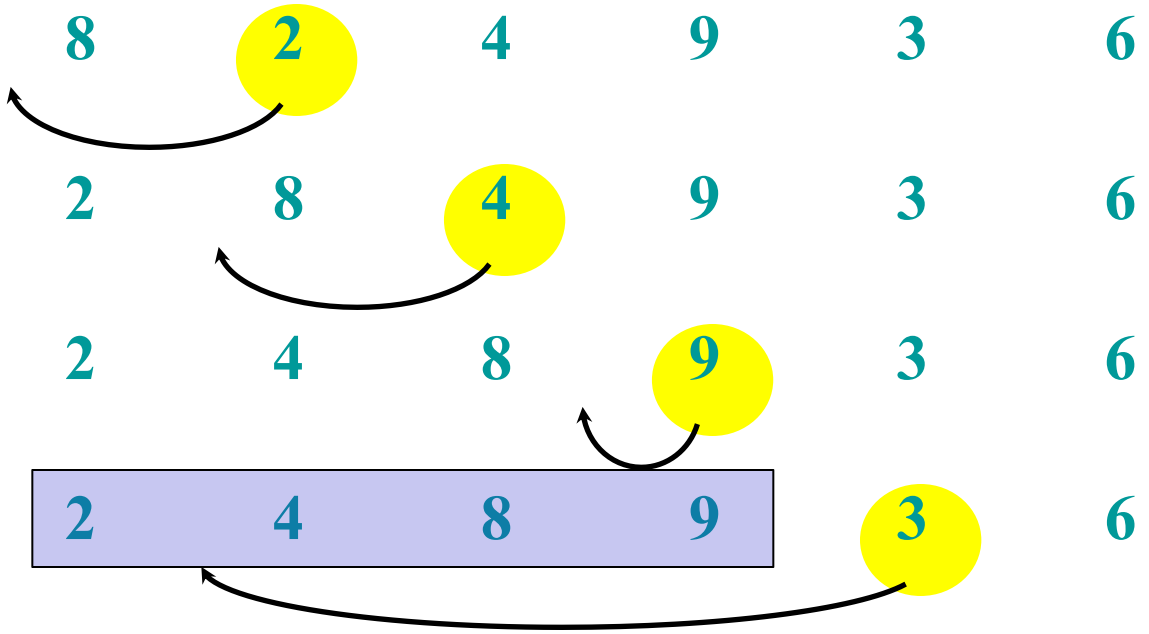


9 is bigger than 8 so we keep it on the right.

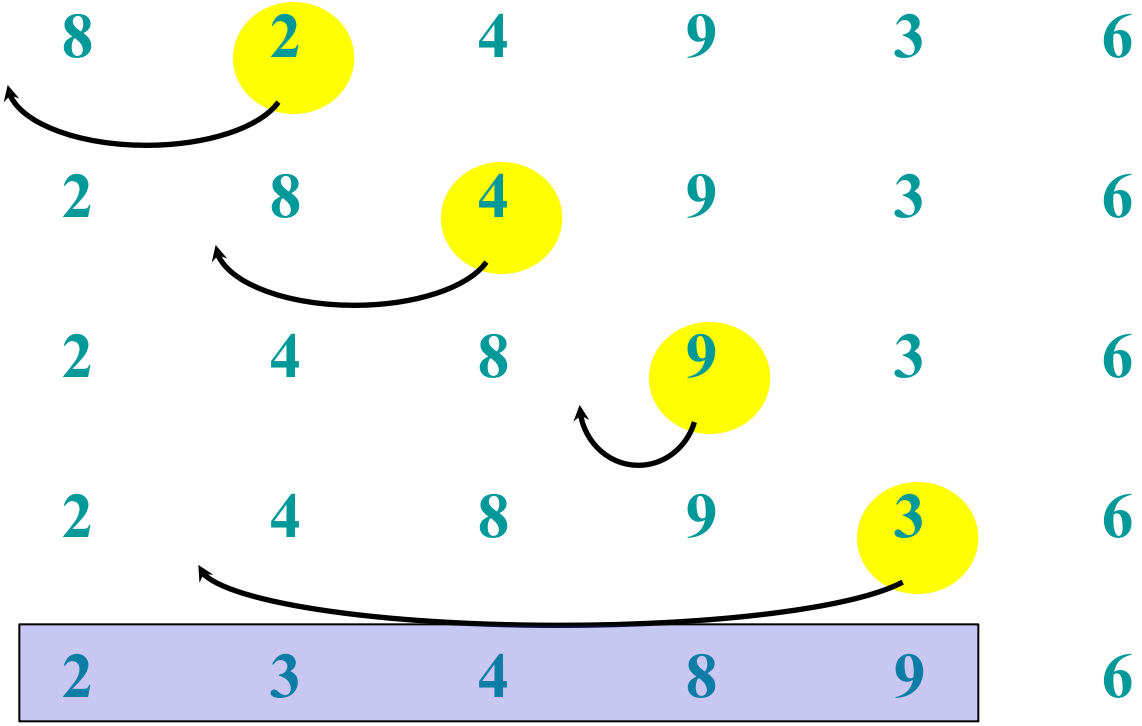
Example of insertion sort



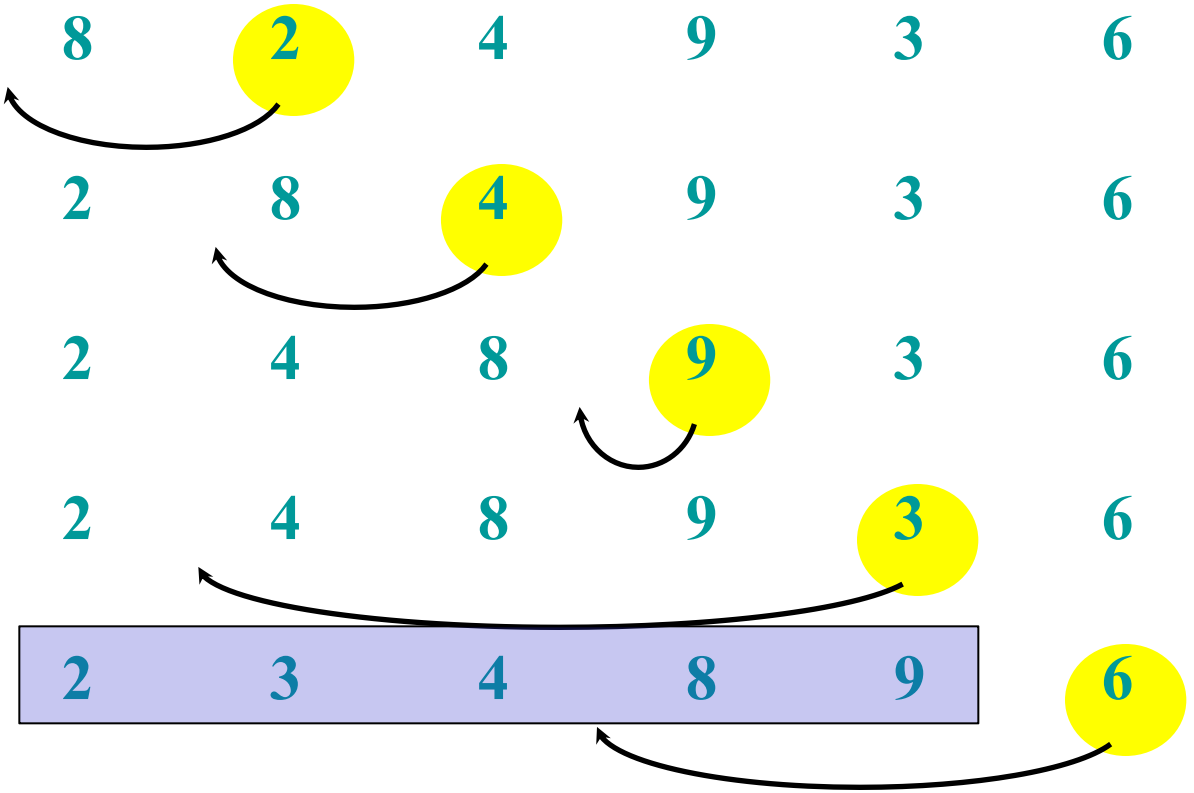
Example of insertion sort



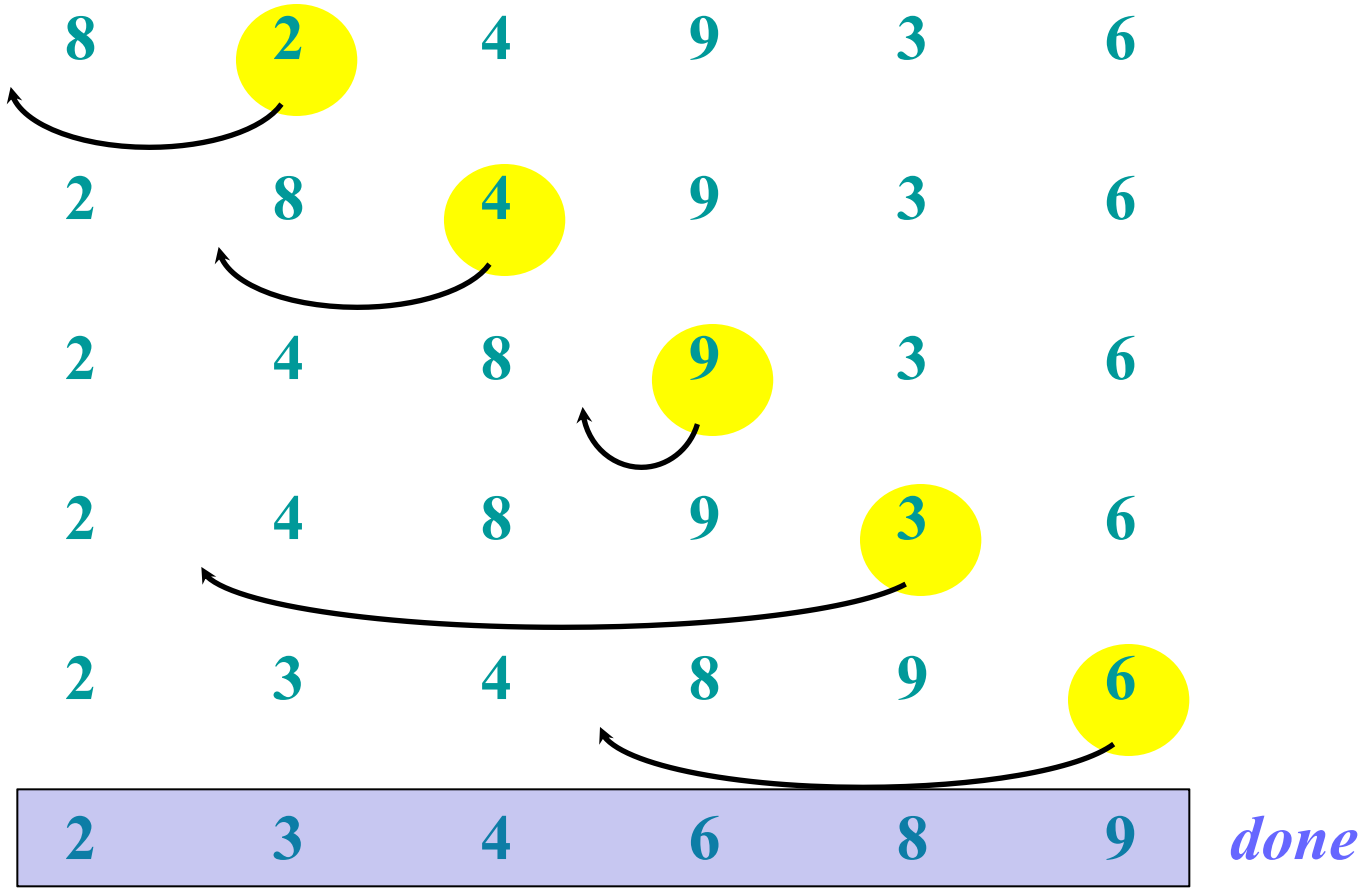
Example of insertion sort



Example of insertion sort

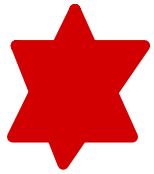


Example of insertion sort



Insertion sort

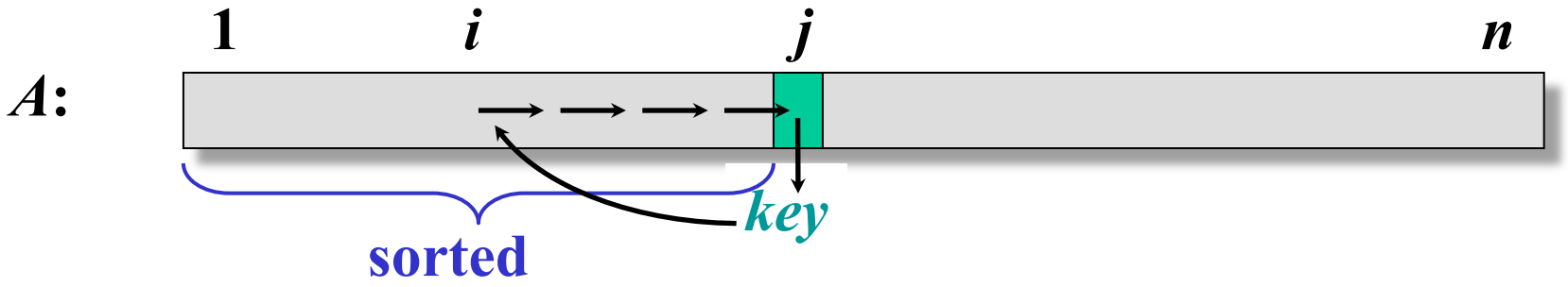
Pseudo code



```
INSERTION-SORT (A, n)    ⊳ A[1 . . n]
    for j ← 2 to n do
        key ← A[ j]
        i ← j – 1
        while i > 0 and A[i] > key do
            A[i+1] ← A[i]
            i ← i – 1
        A[i+1] = key
```

we haven't reached the left end.

our elt is still smaller.



What is the complexity of this program?

Insertion sort C++ implementation

```
void insertion_sort(std::vector<int> &A)
{
    int i;
    int j;
    int key;
    for(j=1; j<A.size(); j++)
    {
       key = A[j];
       i = j-1;
       while(i>=0 && A[i]>key)
       {
          A[i+1] = A[i];
          i = i-1;
       }
       A[i+1] = key;
    }
} // http://ideone.com/siE9V
```

Running time

- **The running time depends on the input: an already sorted sequence is easier to sort**
- **Parameterize the running time by the size of the input, since short sequences are easier to sort than long ones**
- **Generally, we seek upper bounds on the running time, because everybody likes a guarantee**

Kinds of analyses

Worst-case: (usually)

- $T(n)$ = maximum time of algorithm on any input of size n

Average-case: (sometimes)

- $T(n)$ = expected time of algorithm over all inputs of size n
- Need assumption of statistical distribution of inputs

Best-case: (bogus)

- Cheat with a slow algorithm that works fast on *some* input

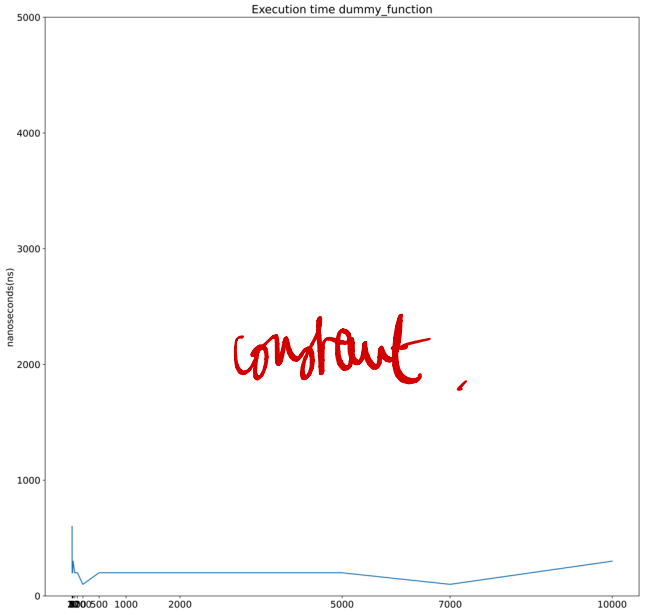
Example

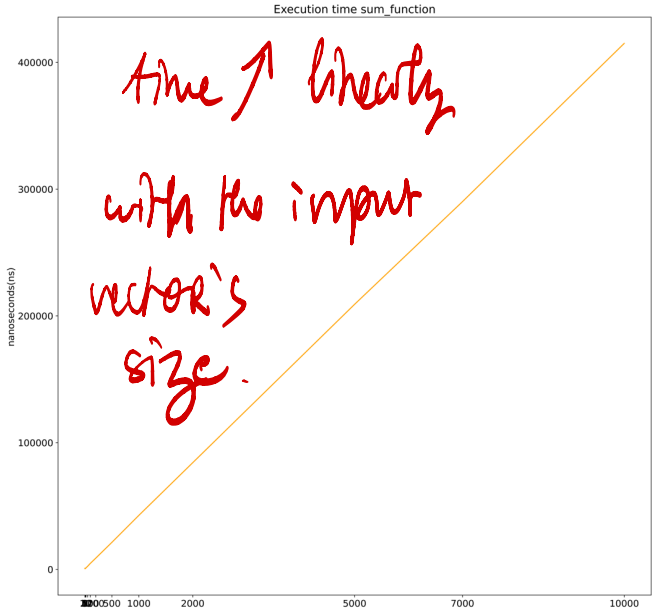
n in [1, 2, 3, 4, 5 ,10, 20, 50, 100, 200, 500, 1000, 2000, 5000,7000, 10000]

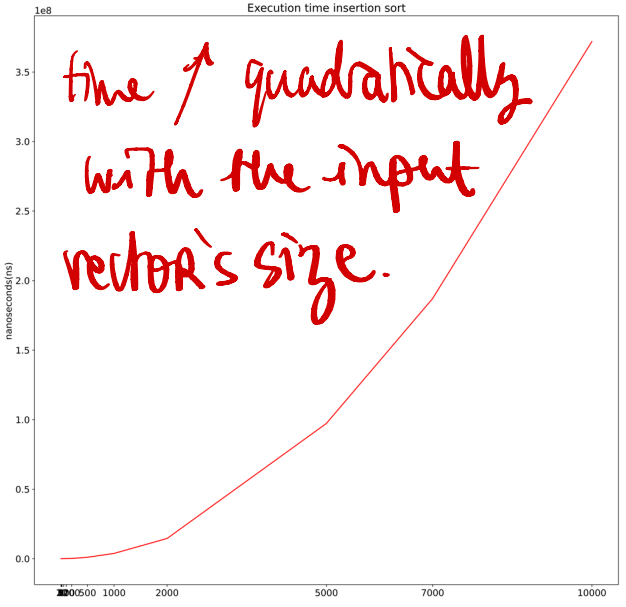
```
void dummy(std::vector<int> &A)
{
        int total = 0;
        int key = A[0];
        total += key;
        return total;
}
```

```
void sum(std::vector<int> &A)
{
        int total = 0;
        for(int i=0; i<A.size(); i++)
                total += A[i];
        return total;
}
```

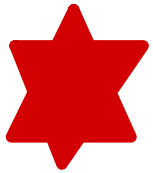
```
Void Insertion_sort(std::vector<int> &A)
```







Machine-independent time



What is insertion sort's worst-case time?

- **It depends on the speed of our computer**

BIG IDEA:

- **Ignore machine-dependent constants**
- **Look at *growth* of $T(n)$ as $n \to \infty$**

"Asymptotic Analysis"

Θ-notation

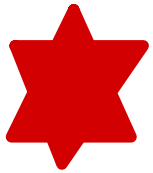
Math:

$\Theta(g(n)) = \{ f(n)$: there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0 \}$

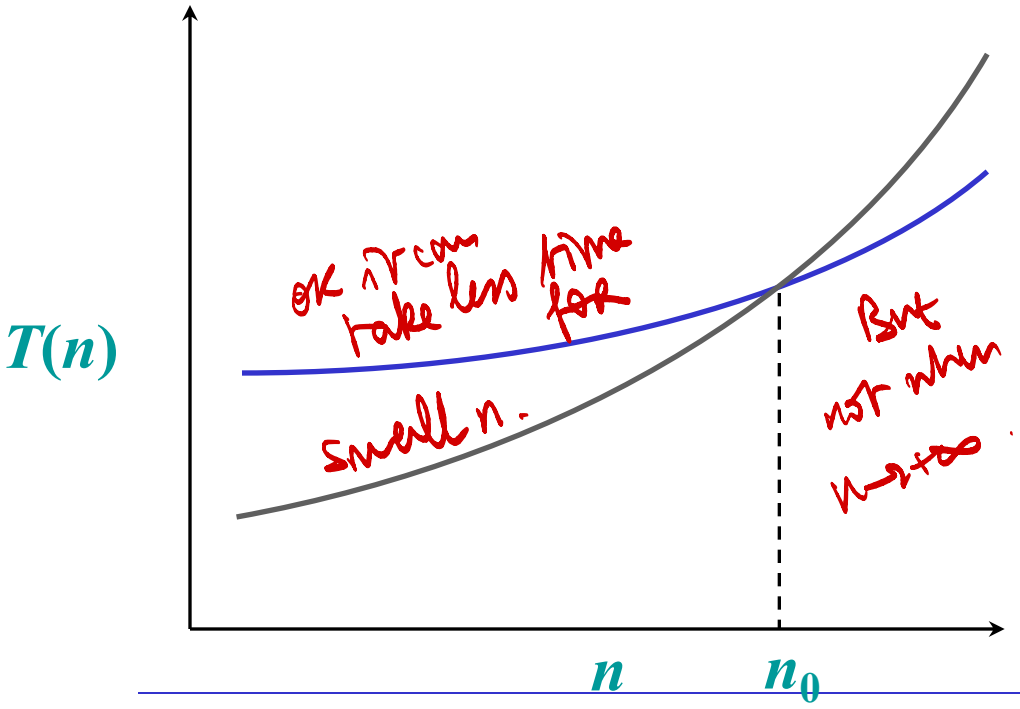
Engineering:

- Drop low-order terms; ignore leading constants
- Example: $3n^3 + 90n^2 - 5n + 6046 = \Theta(n^3)$

Asymptotic performance



When n gets large enough, a $\Theta(n^2)$ algorithm *always* beats a $\Theta(n^3)$ algorithm



- **We shouldn't ignore asymptotically slower algorithms, however**
- **Real-world design situations often call for a careful balancing of engineering objectives**
- **Asymptotic analysis is a useful tool to help to structure our thinking**

Insertion sort analysis

Worst case: **Input reverse sorted**

$$T(n) = \sum_{j=2}^{n} \Theta(j) = \Theta(n^2)$$

[arithmetic series]

Average case: **All permutations equally likely**

$$T(n) = \sum_{j=2}^{n} \Theta(j/2) = \Theta(n^2)$$

Is insertion sort a fast sorting algorithm?

- **Moderately so, for small *n***
- **Not at all, for large *n***

Merge sort

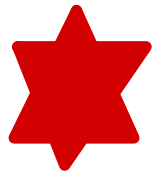
Another sorting algorithm. RECURSIVE.

MERGE-SORT $A[1 \ldots n]$

1. If $n = 1$, done
2. Recursively sort $A[\,1 \ldots \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \ldots n\,]$
3. "*Merge*" the 2 sorted lists

Key subroutine: **MERGE**

Merge Sort in C++



```
void merge_sort(std::vector<int> &A)
{
        std::vector<int> A1;
        std::vector<int> A2;

        if (A.size()==1) return;

        for(int i=0; i<A.size()/2; i++)
                A1.push_back(A[i]);

        for(int i=A.size()/2; i<A.size(); i++)
                A2.push_back(A[i]);

        merge_sort(A1);
        merge_sort(A2);
        A = merge(A1,A2);
}
```

Merging two sorted arrays

20 **12**

13 **11**

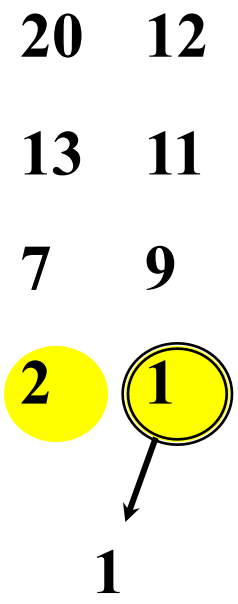
7 **9**

2 **1** : 1 small then 2

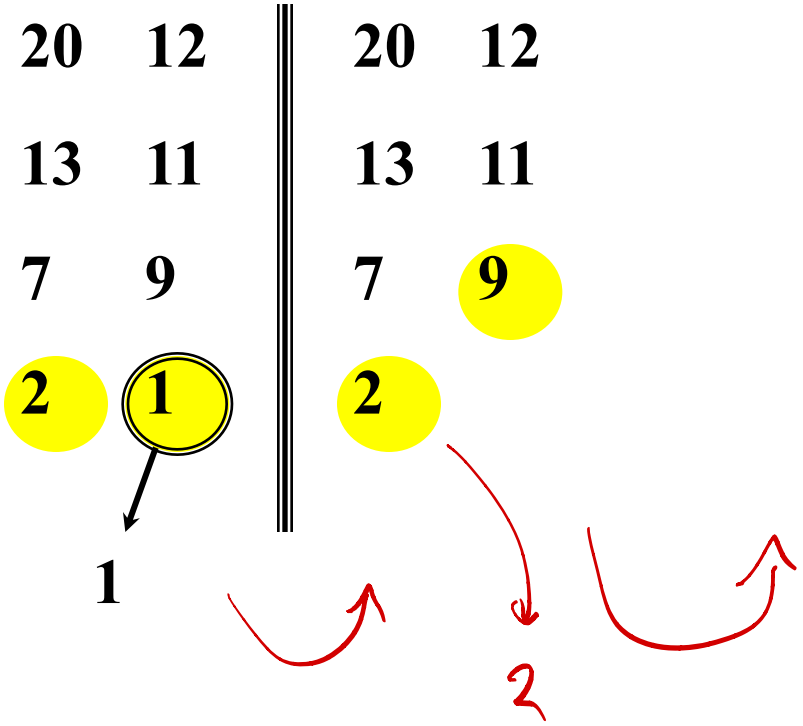
↓

1

Merging two sorted arrays



Merging two sorted arrays



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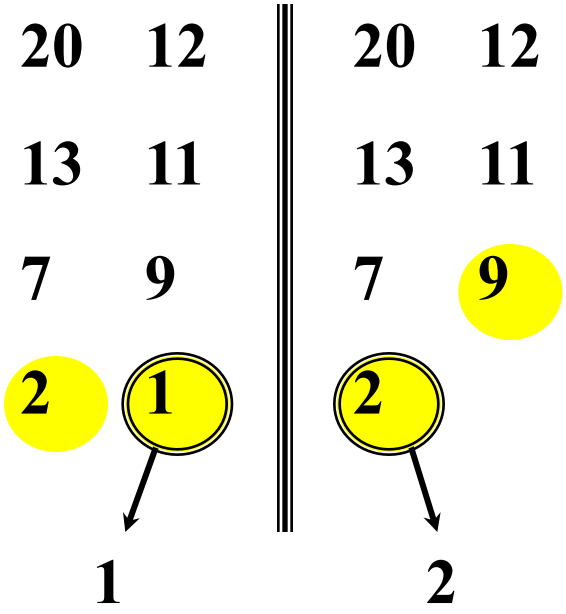

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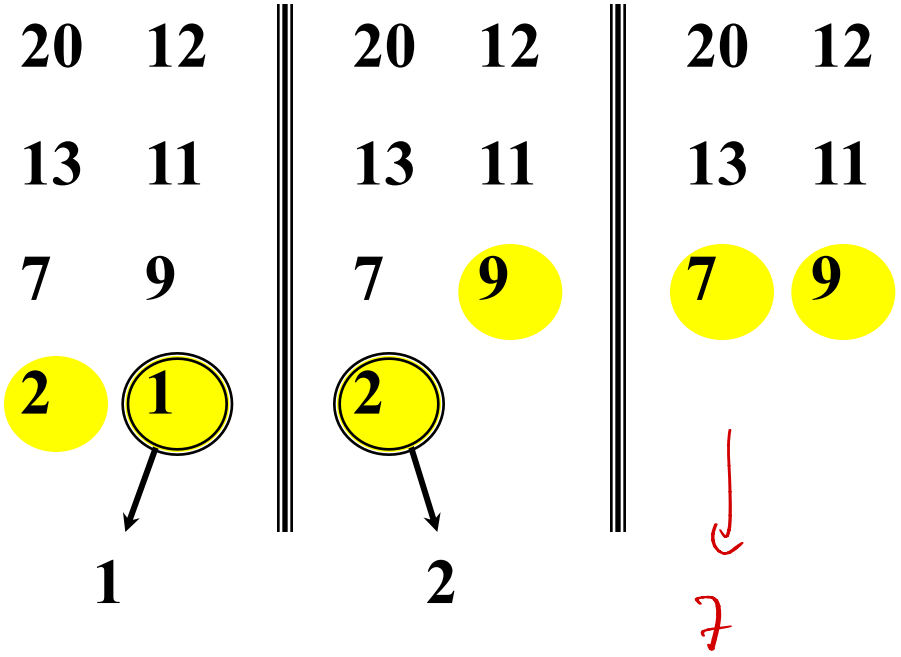

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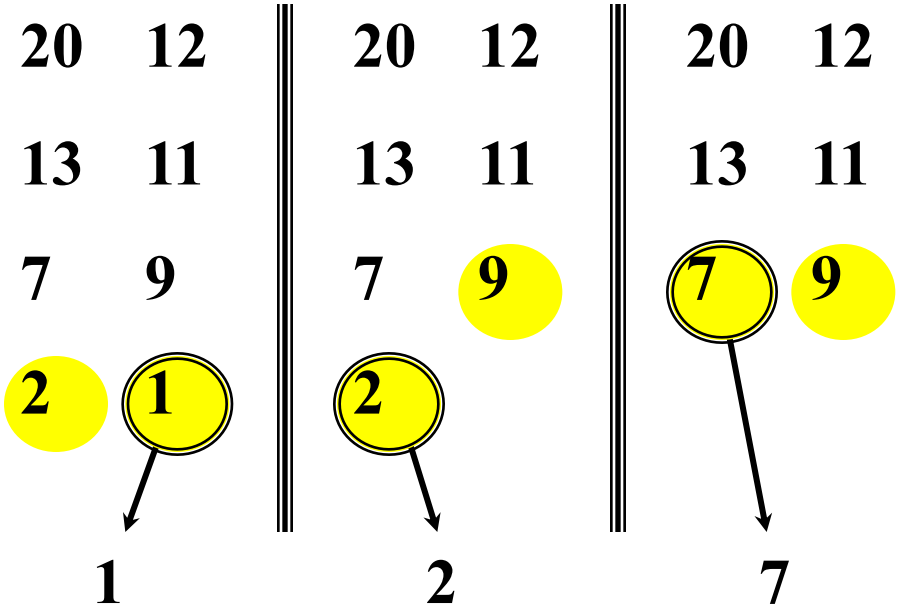

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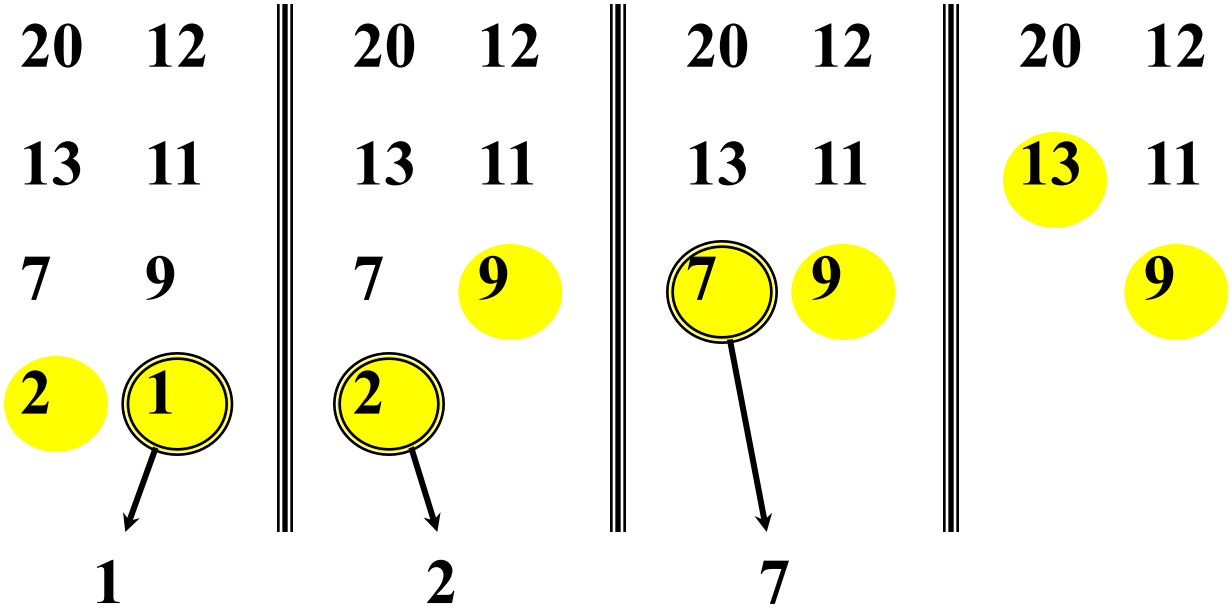

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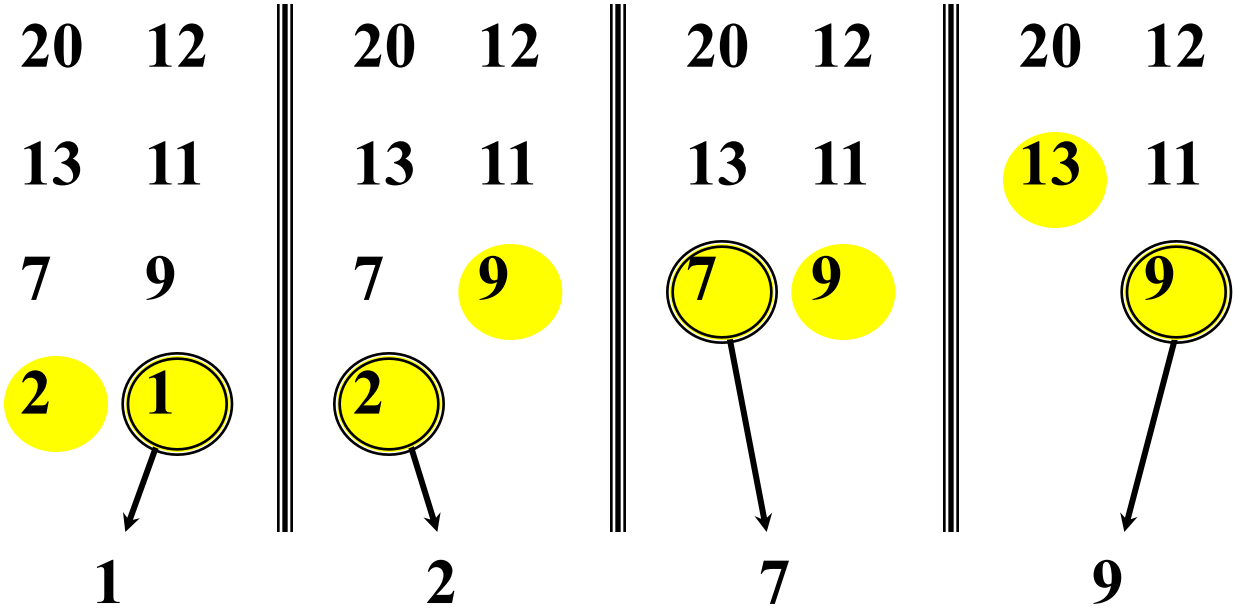

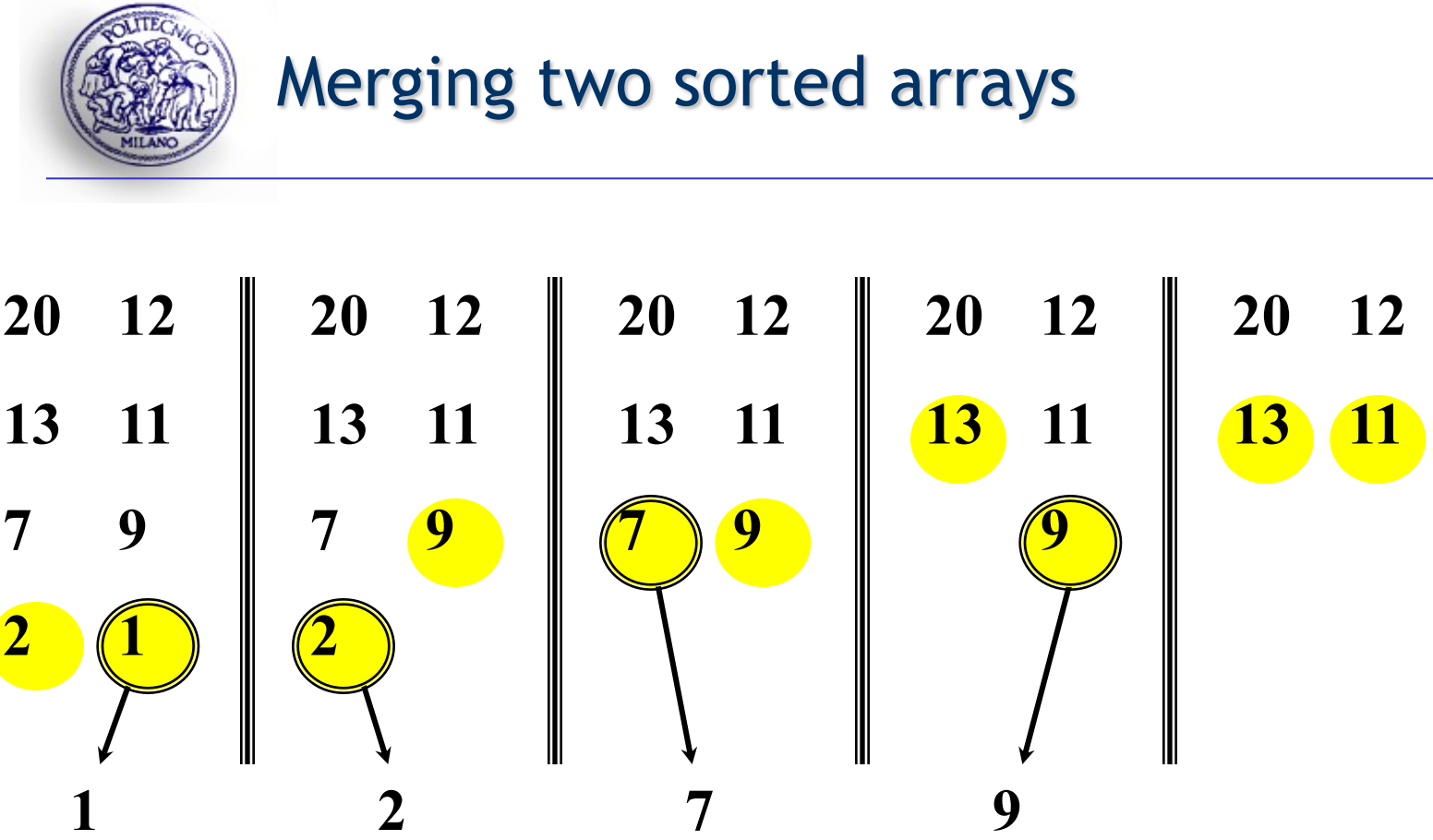

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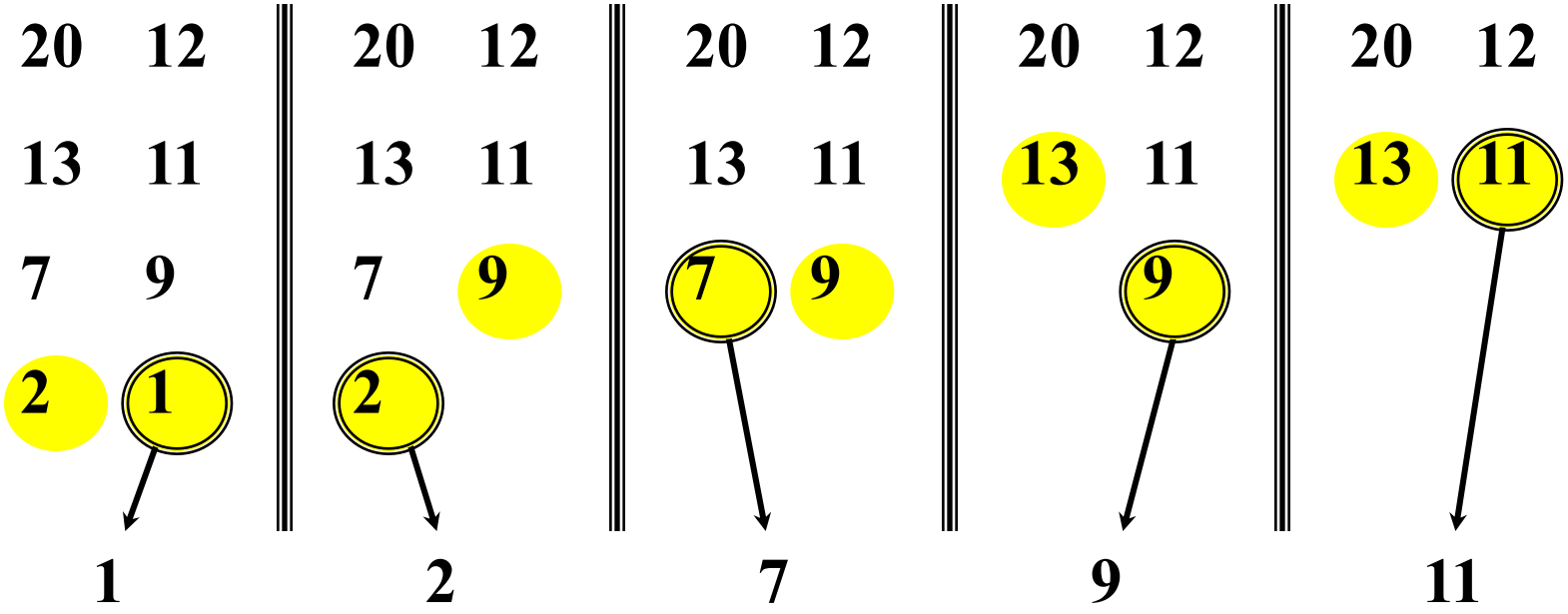

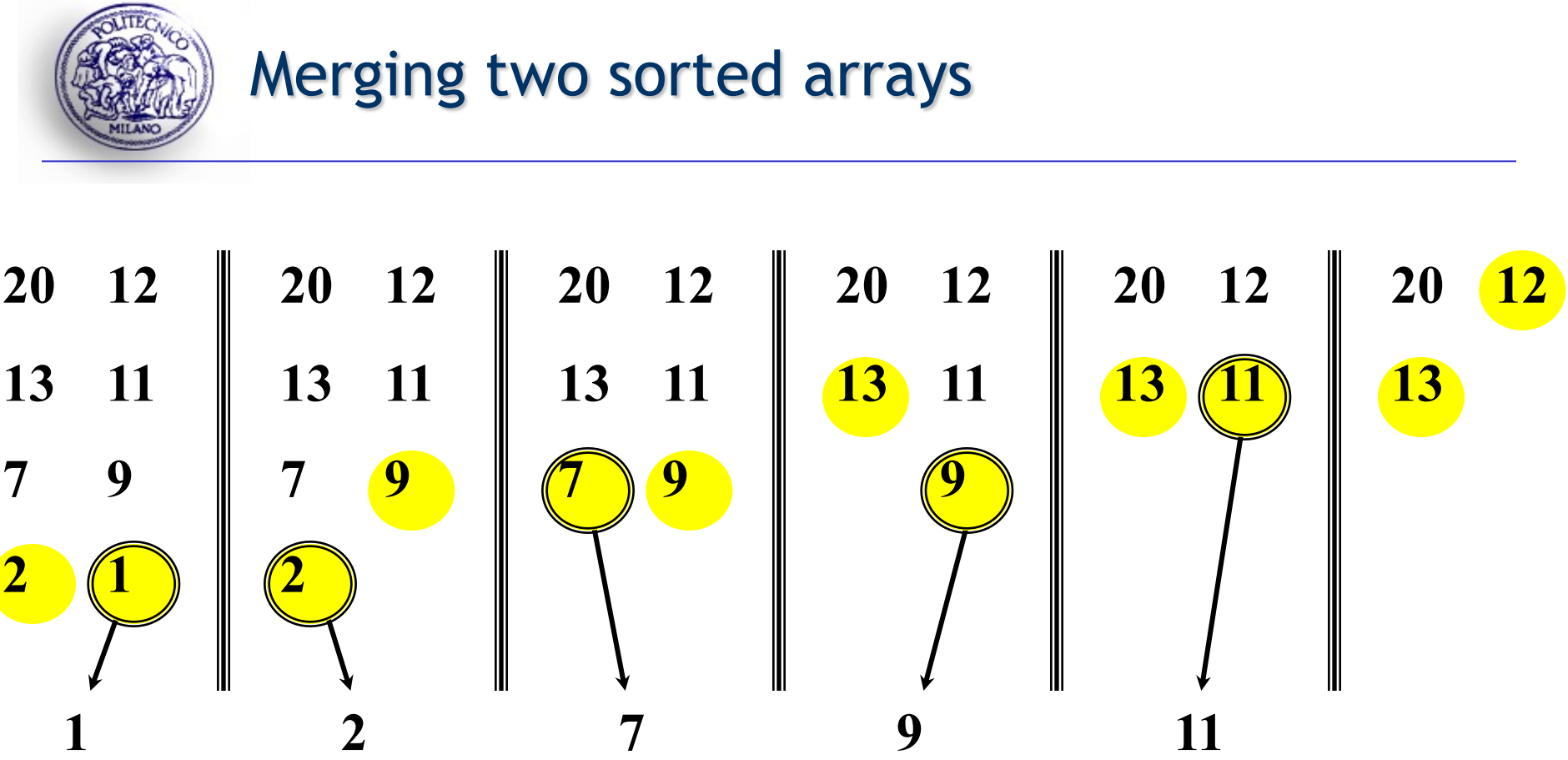

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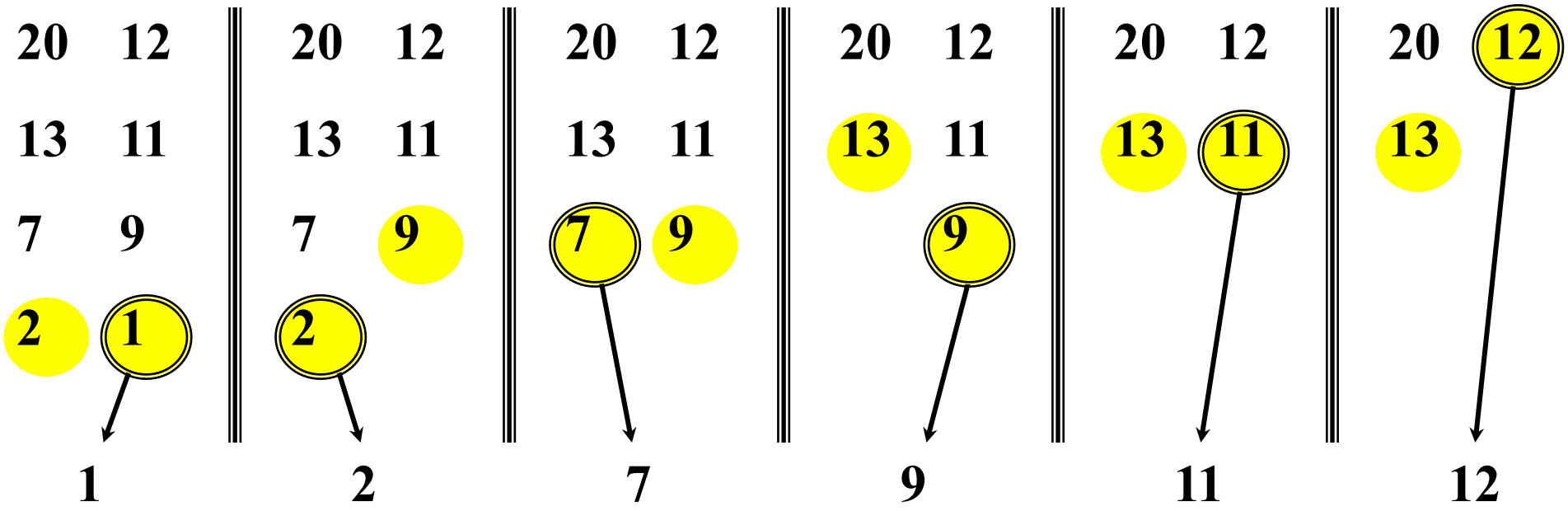

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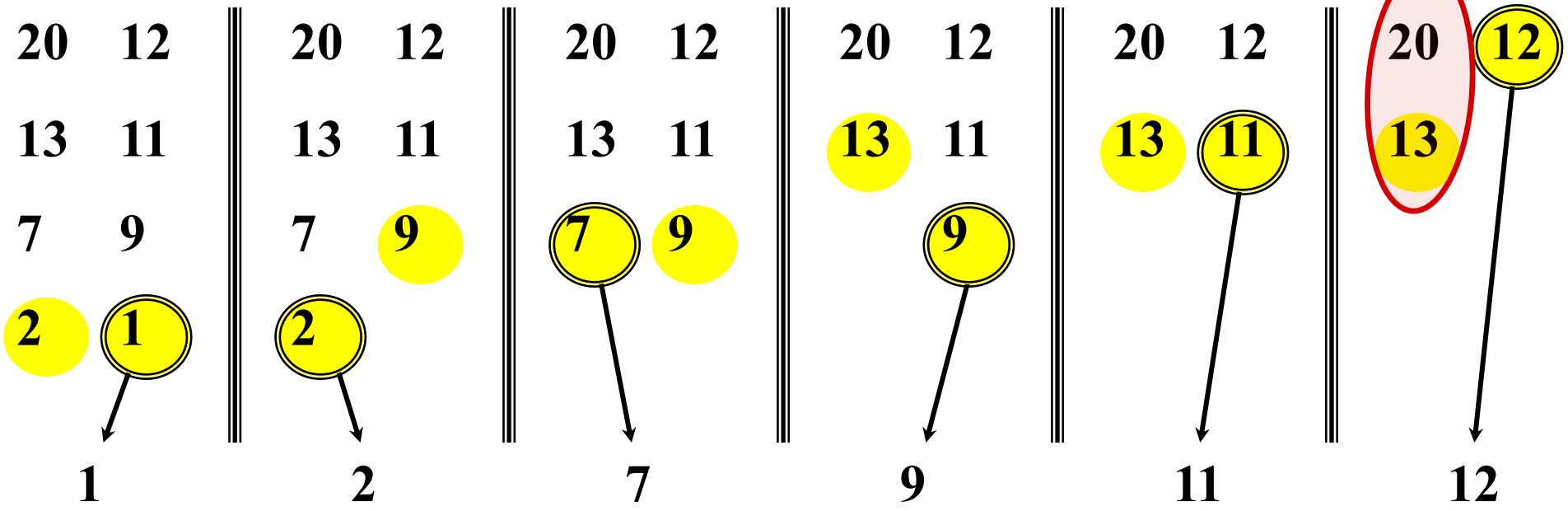
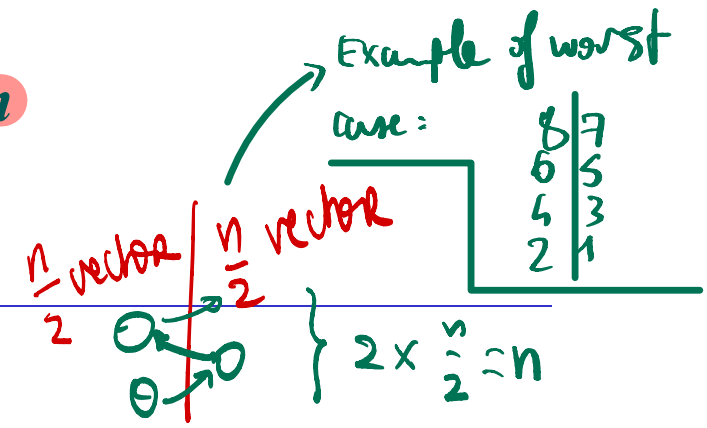


one array left only so you can put it directly be already sorted.

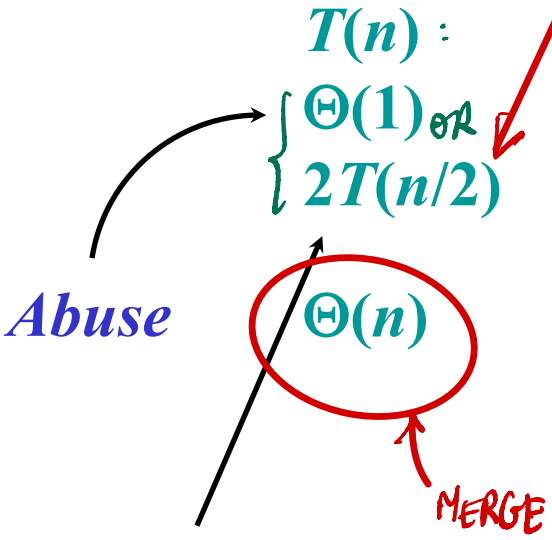
Time = $\Theta(n)$ to merge a total of n elements (linear time)

worse case scenario :



Analyzing merge sort

2 calls to mergesort (& we divide by 2 the size each time).



MERGE-SORT $A[1 \ldots n]$

1. **If** $n = 1$, **done**
2. **Recursively sort** $A[\,1 \ldots \lceil n/2 \rceil\,]$ **and** $A[\,\lceil n/2 \rceil + 1 \ldots n\,]$
3. ***"Merge"*** **the 2 sorted lists**

Sloppiness: **Should be** $T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor)$ **, but it turns out not to matter asymptotically**

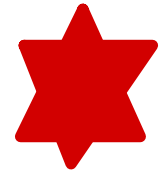
Recurrence for merge sort

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1 \\ 2T(n/2) + \Theta(n) \text{ if } n > 1 \end{cases}$$

↑ divide by 2 the size & sort both subarrays.

↖ MERGE step @ the end.

Recursion tree



Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant

$\Theta(n)$

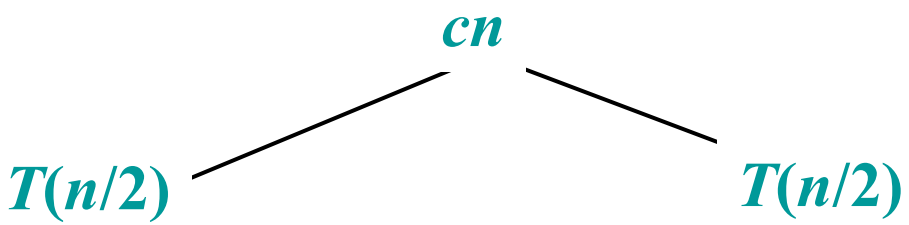
Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant

$T(n)$

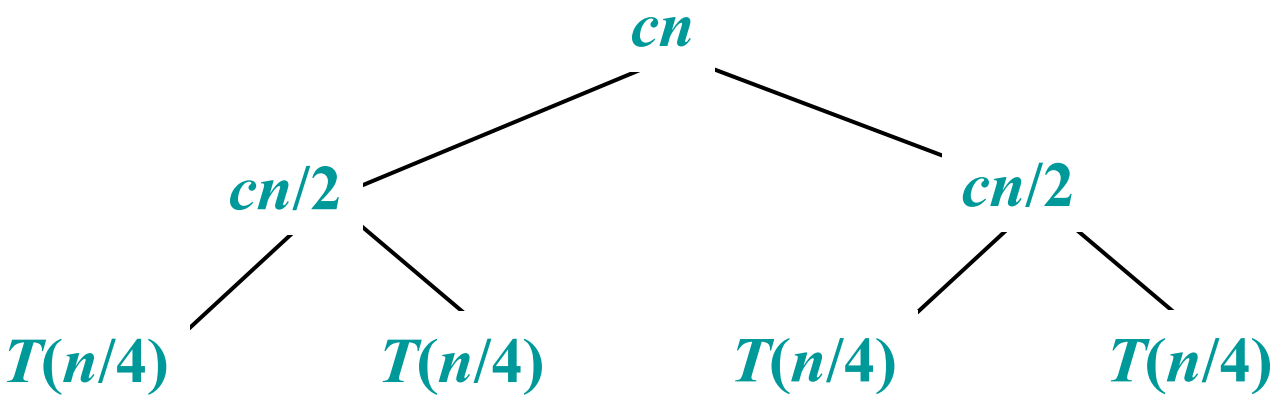
Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant



Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant



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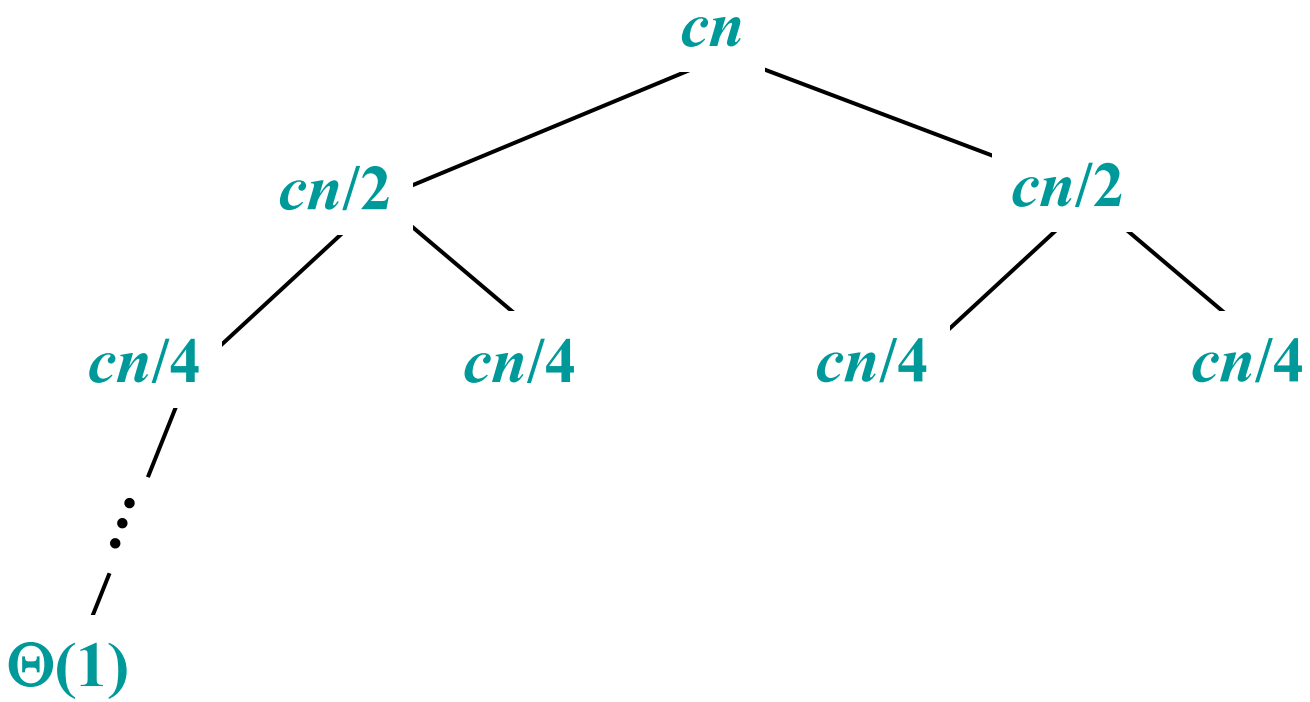

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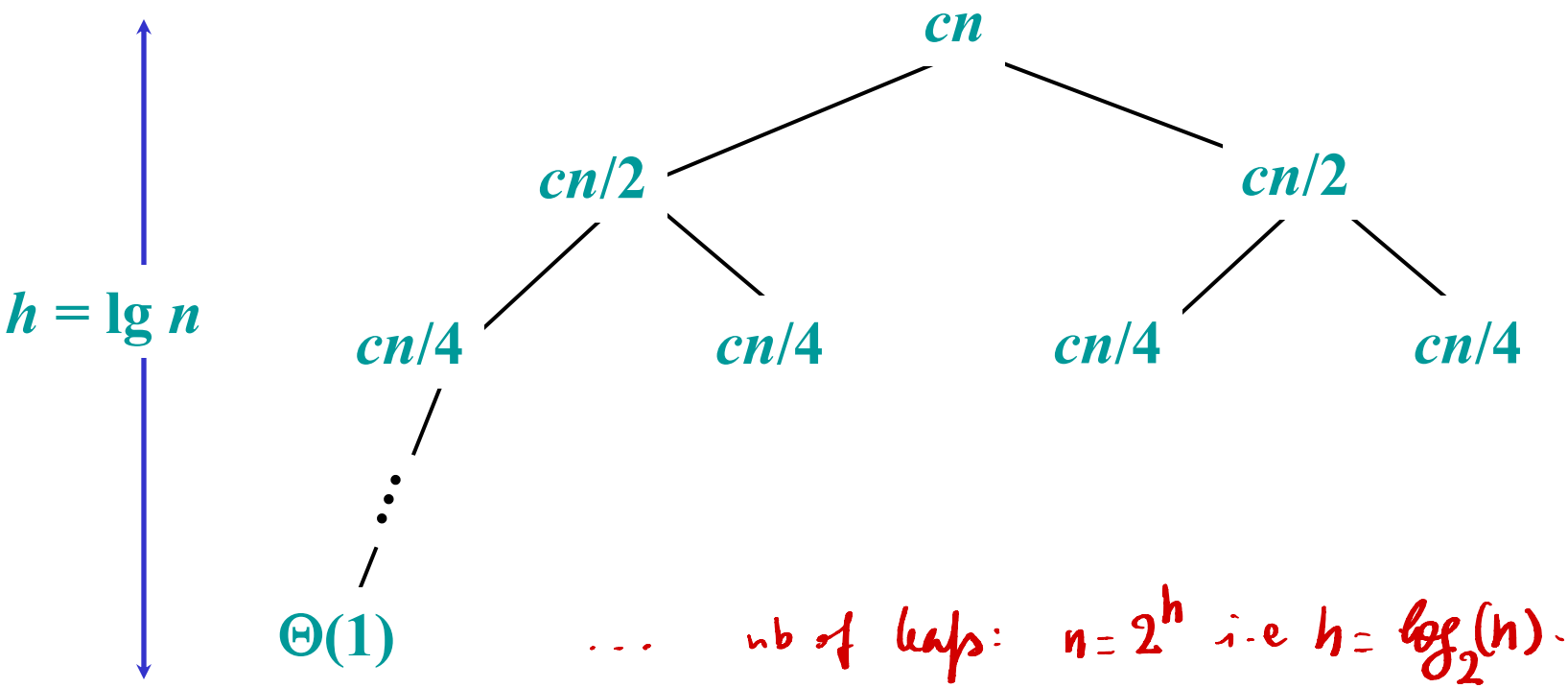

Recursion tree

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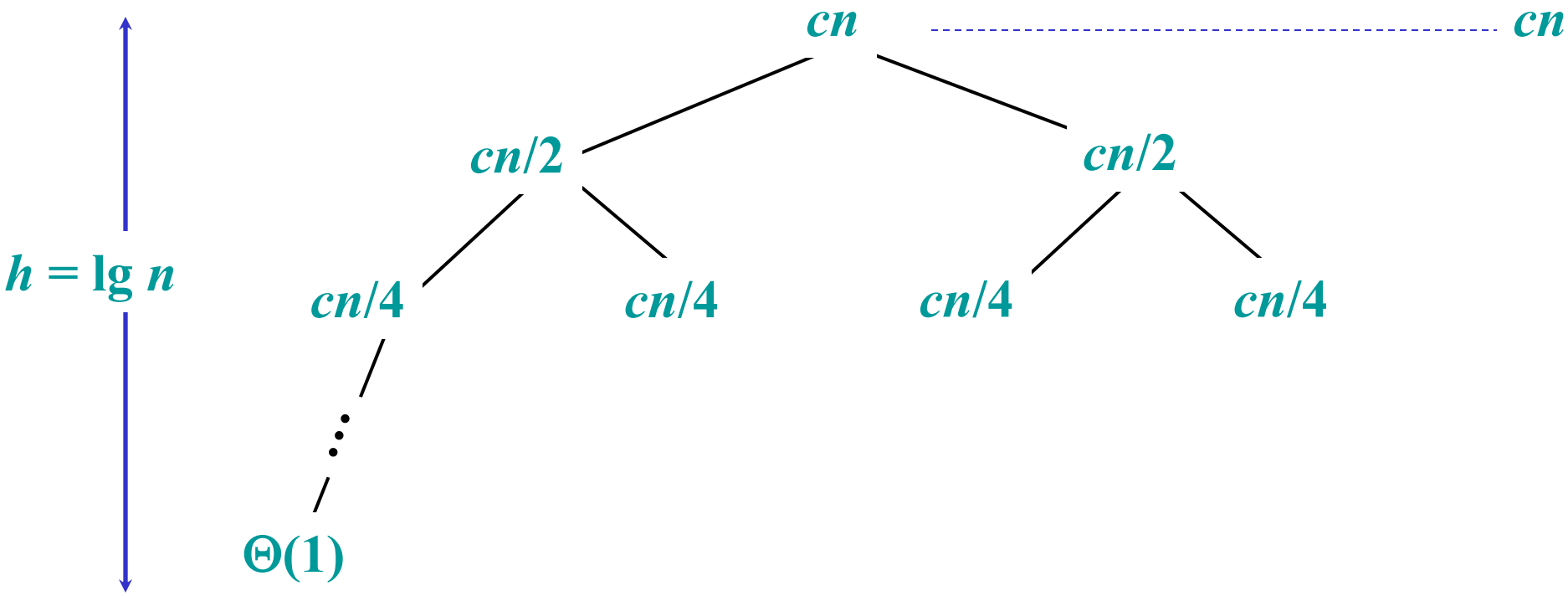

Recursion tree

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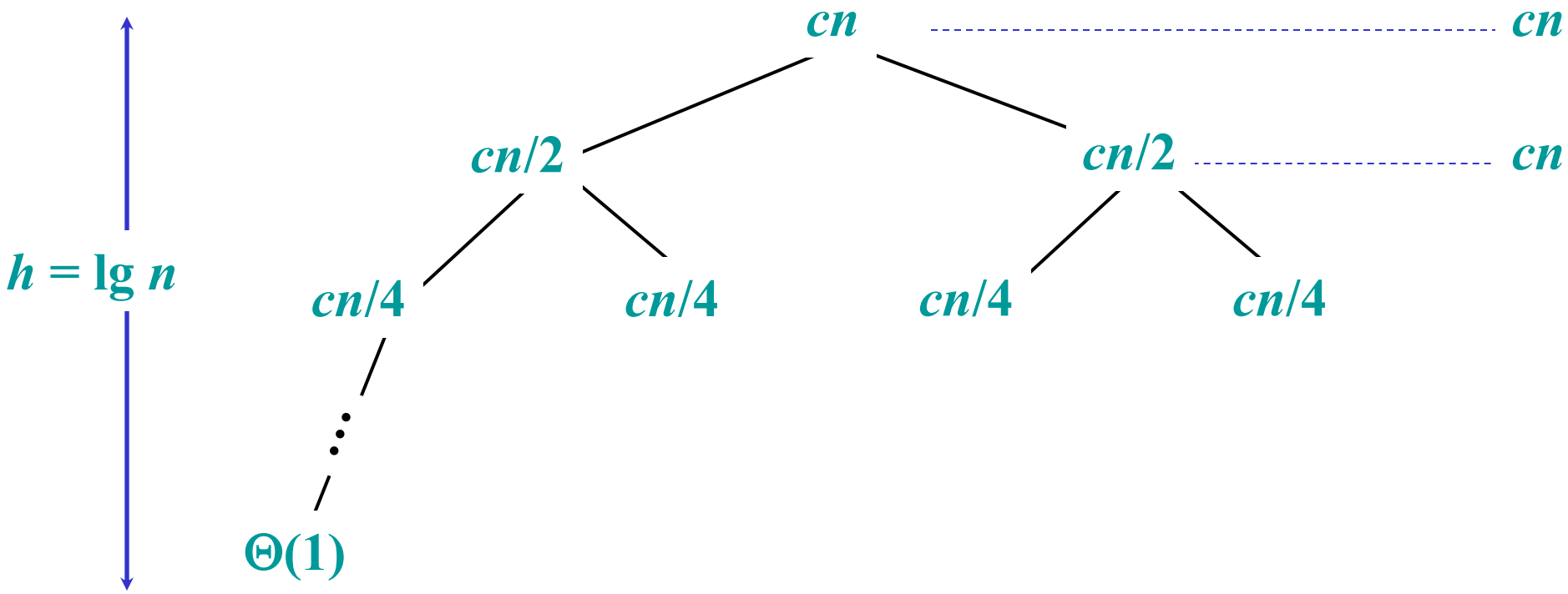

Recursion tree

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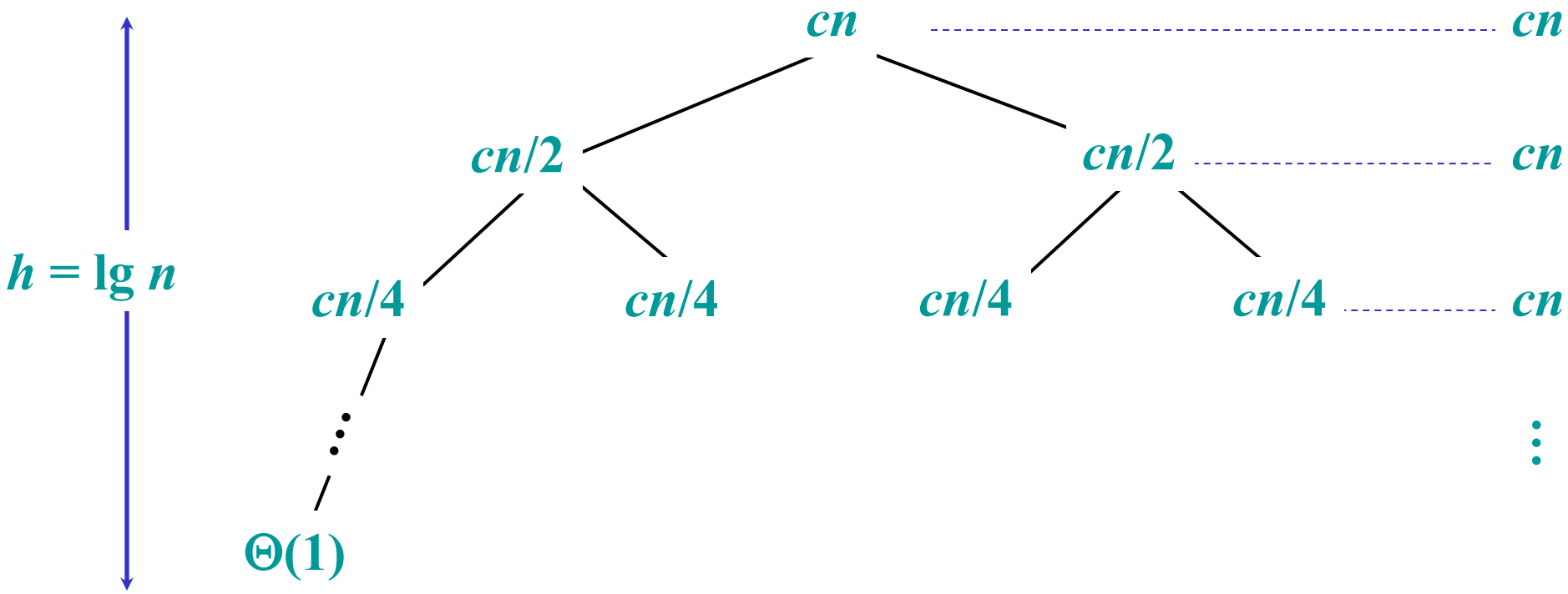

Recursion tree

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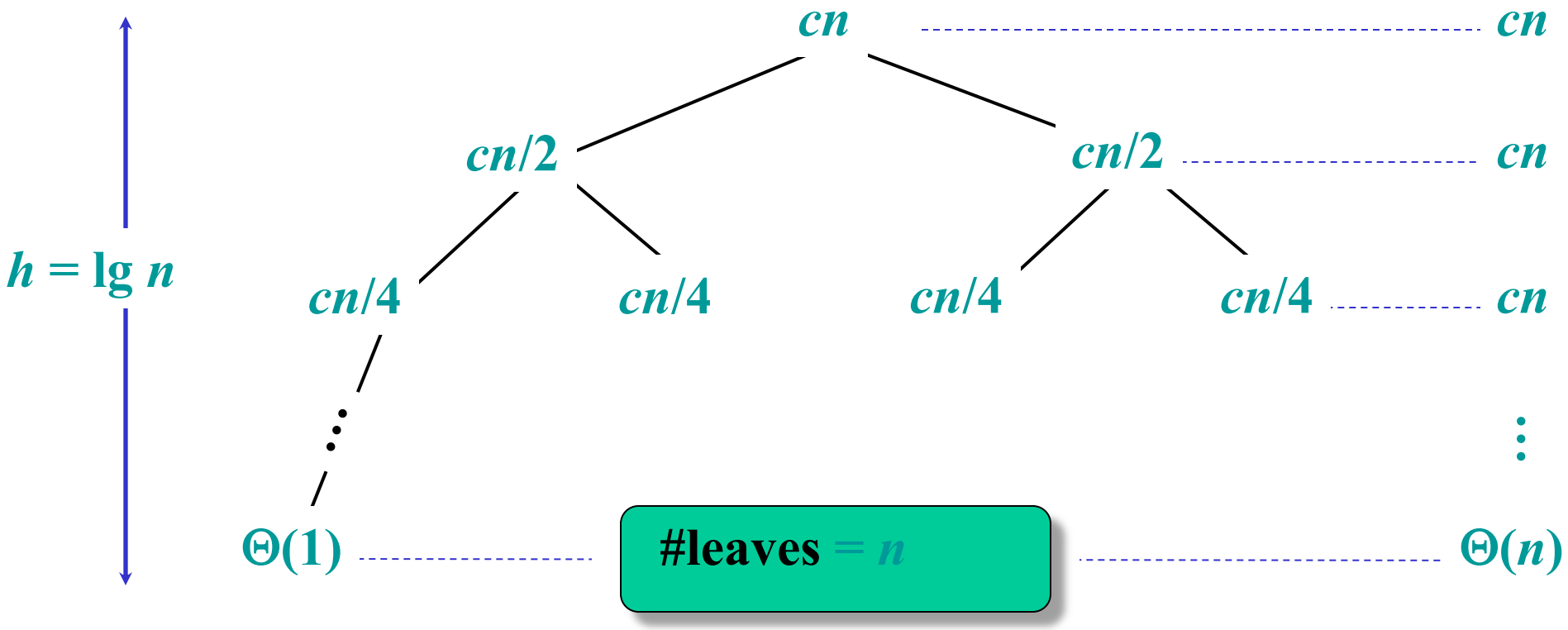
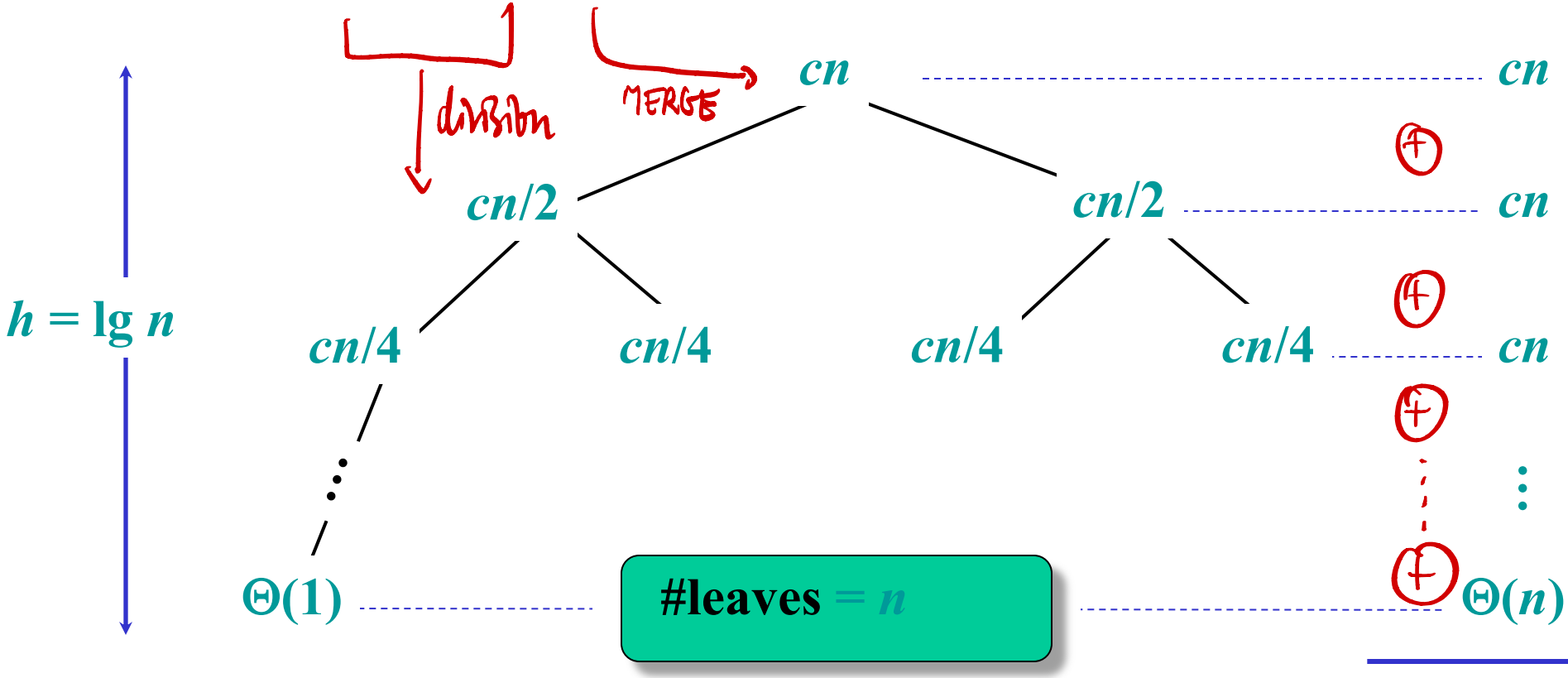


Recursion tree

$T(n) = 2T(n/2) + cn = 2\left(T\left(\frac{n}{2}\right) + c\frac{n}{2}\right) + cn$

$= \dots \cong \log(n) \times cn = \Theta(n \log(n))$

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant



Total = $\Theta(n \lg n)$

at each step: $\Theta(n)$;
There are $h = \log(n)$ so: $\Theta(n)\,\Theta(\log(n)) =$

Conclusions

vs $\Theta(n)$ for merge sort.

↑

spatial complexity of insertion sort: $\Theta(1)$.

- $\Theta(n \lg n)$ grows more slowly than $\Theta(n^2)$
- Therefore, merge sort asymptotically beats insertion sort in the worst case
- In practice, merge sort beats insertion sort for $n > 30$ or so
- Go test it out for yourself!

space ↑

always a trade off between memory & time. Depending on the hardware!

↳ if you have a lot of memory: use MERGESORT. If you are running by memory & n small → use insertion sort.

Asymptotic notation

O-notation (upper bounds):

We write $f(n) = O(g(n))$ if there exist constants $c > 0$, $n_0 > 0$ such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$.

Asymptotic notation

***O*-notation (upper bounds):**

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EXAMPLE: $2n^2 = O(n^3)$ $(c = 1, n_0 = 2)$

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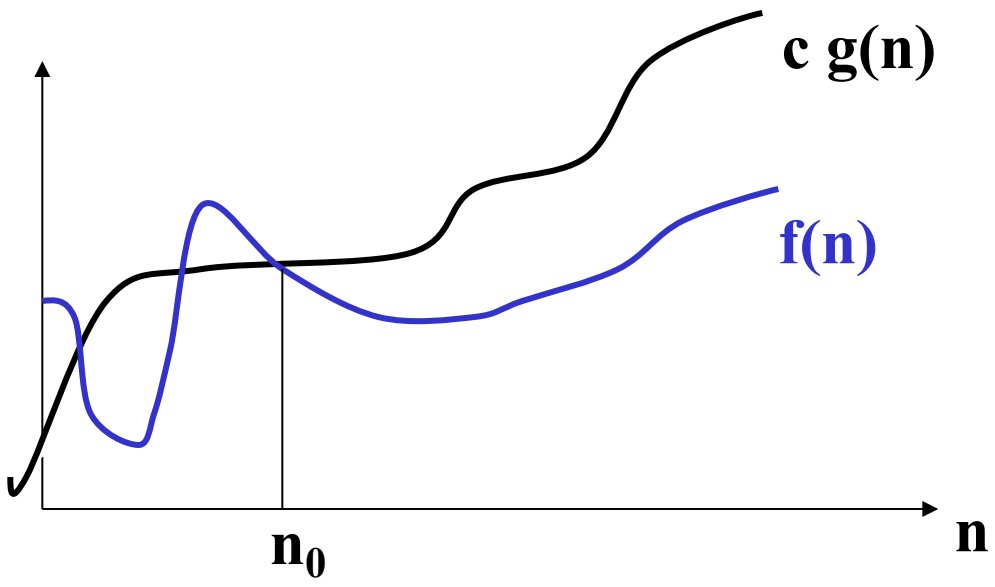
functions, not values

Asymptotic notation



Examples:

$2n^2 = O(n^3)$

$n^2+1000n = O(n^2)$

$n^{1.99999} = O(n^2)$

$n^{2.1}/\lg \lg n = O(n^3)$

Set definition of O-notation

$O(g(n)) = \{ f(n)$: **there exist constants** $c > 0$, $n_0 > 0$ **such that** $0 \le f(n) \le cg(n)$ **for all** $n \ge n_0 \}$

EXAMPLE: $2n^2 \in O(n^3)$

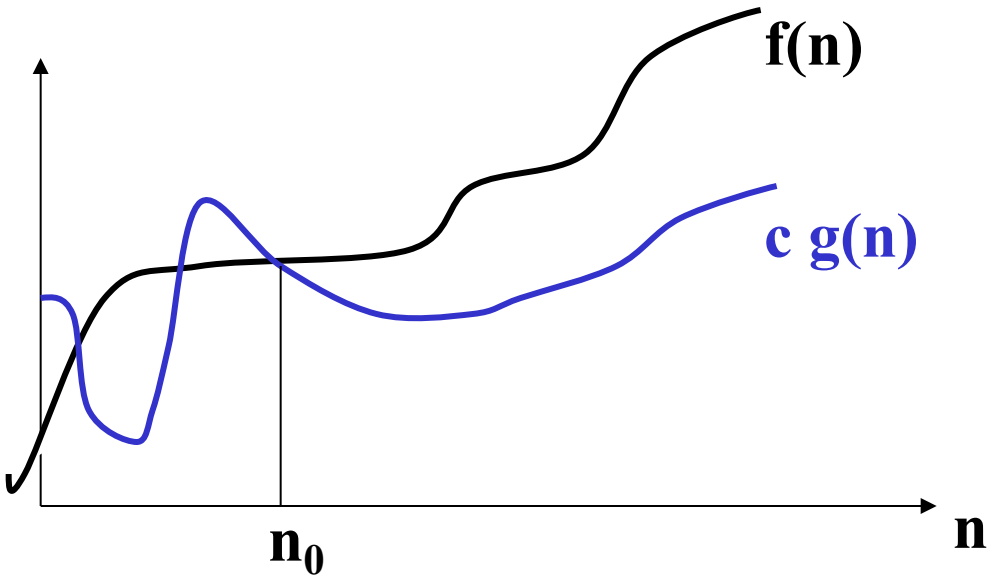
Ω-notation (lower bounds)

***O*-notation is an *upper-bound* notation. It makes no sense to say $f(n)$ is at least $O(n^2)$**

$\Omega(g(n)) = \{ f(n)$ **: there exist constants** $c > 0$, $n_0 > 0$ **such that** $0 \le cg(n) \le f(n)$ **for all** $n \ge n_0 \}$

EXAMPLE: $\sqrt{n} = \Omega(\log n)$ $(c = 1, n_0 = 16)$

Ω-notation (lower bounds)



Examples:

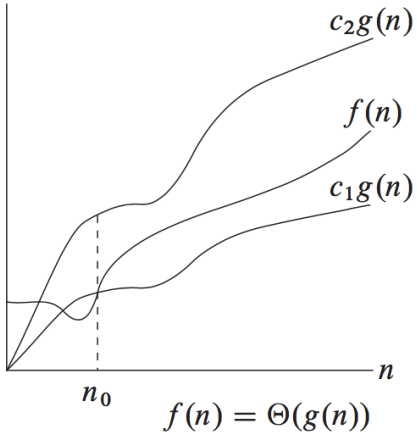
$sqrt(n) = \Omega(\lg n)$

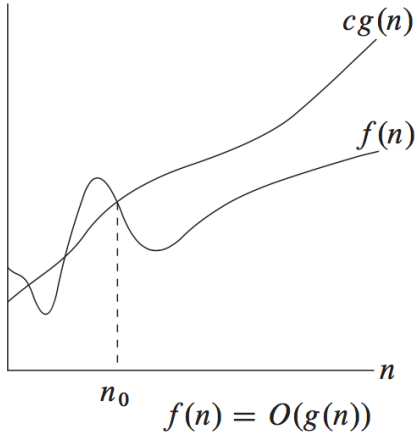
$n^2+1000n = \Omega(n^2)$

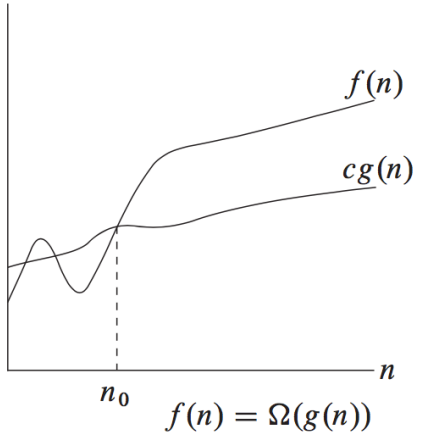
$n^2 \lg \lg n = \Omega(n^2)$

Θ-notation (tight bounds)

$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$$





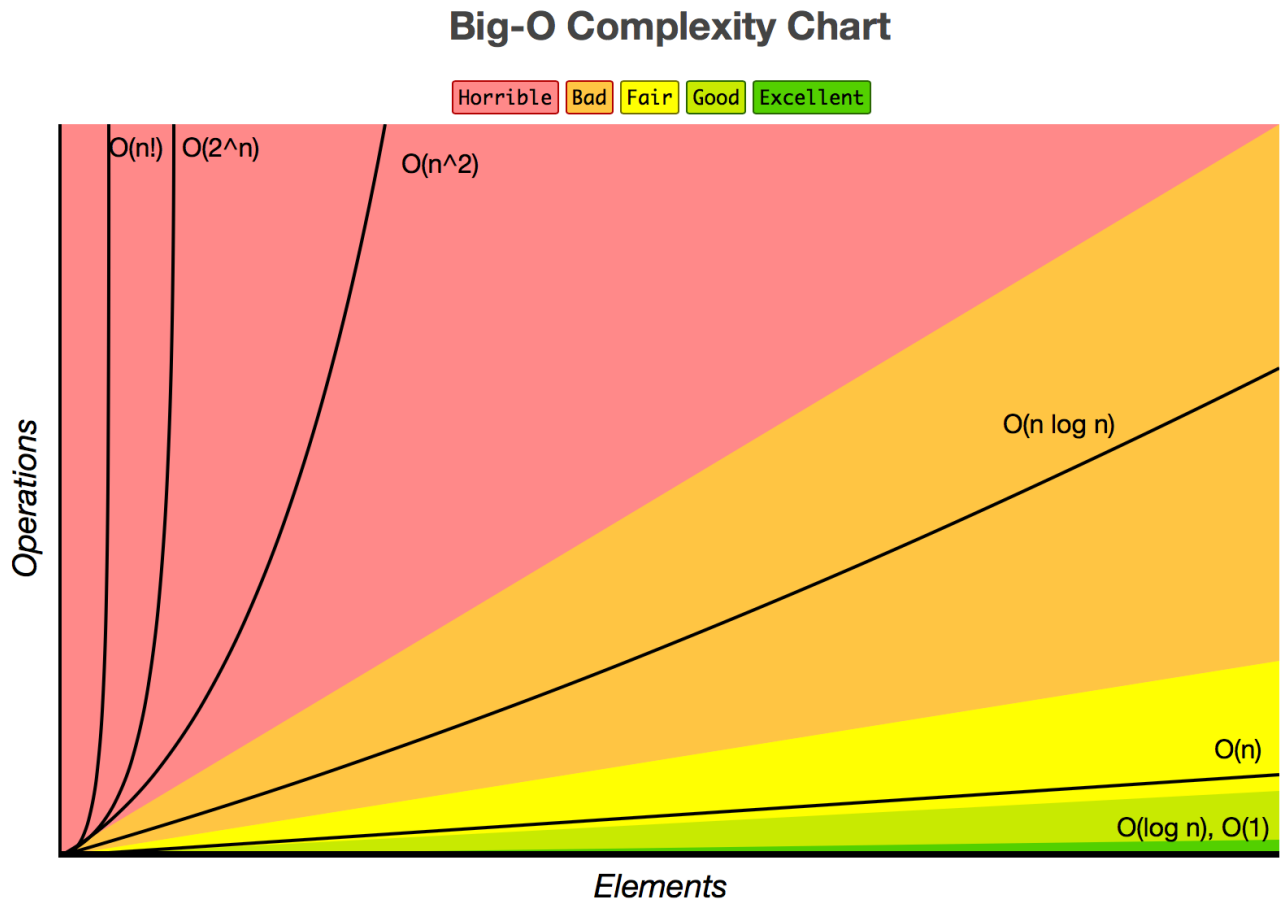


T. Cormen, C. R. Leiserson, R. L. Rivest, C. Stein. Introduction to Algorithms

O vs. Θ

- In the literature, we sometimes find O-notation informally describing asymptotically tight bounds, that is, what we have defined using Θ-notation
- In algorithms analysis, when we write f(n)= O(g(n)), we are merely claiming that some constant multiple of g(n) is an asymptotic upper bound on f (n), with no claim about how tight an upper bound it is
- Using O-notation, we can often describe the running time of an algorithm merely by inspecting the algorithm overall structure

Resources



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Resources

Common Data Structure Operations

Data Structure	Time Complexity								Space Complexity
	Average				Worst				Worst
	Access	Search	Insertion	Deletion	Access	Search	Insertion	Deletion	
Array	Θ(1)	Θ(n)	Θ(n)	Θ(n)	O(1)	O(n)	O(n)	O(n)	O(n)
Stack	Θ(n)	Θ(n)	Θ(1)	Θ(1)	O(n)	O(n)	O(1)	O(1)	O(n)
Queue	Θ(n)	Θ(n)	Θ(1)	Θ(1)	O(n)	O(n)	O(1)	O(1)	O(n)
Singly-Linked List	Θ(n)	Θ(n)	Θ(1)	Θ(1)	O(n)	O(n)	O(1)	O(1)	O(n)
Doubly-Linked List	Θ(n)	Θ(n)	Θ(1)	Θ(1)	O(n)	O(n)	O(1)	O(1)	O(n)
Skip List	Θ(log(n))	Θ(log(n))	Θ(log(n))	Θ(log(n))	O(n)	O(n)	O(n)	O(n)	O(n log(n))
Hash Table	N/A	Θ(1)	Θ(1)	Θ(1)	N/A	O(n)	O(n)	O(n)	O(n)
Binary Search Tree	Θ(log(n))	Θ(log(n))	Θ(log(n))	Θ(log(n))	O(n)	O(n)	O(n)	O(n)	O(n)
Cartesian Tree	N/A	Θ(log(n))	Θ(log(n))	Θ(log(n))	N/A	O(n)	O(n)	O(n)	O(n)
B-Tree	Θ(log(n))	Θ(log(n))	Θ(log(n))	Θ(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(n)
Red-Black Tree	Θ(log(n))	Θ(log(n))	Θ(log(n))	Θ(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(n)
Splay Tree	N/A	Θ(log(n))	Θ(log(n))	Θ(log(n))	N/A	O(log(n))	O(log(n))	O(log(n))	O(n)
AVL Tree	Θ(log(n))	Θ(log(n))	Θ(log(n))	Θ(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(n)
KD Tree	Θ(log(n))	Θ(log(n))	Θ(log(n))	Θ(log(n))	O(n)	O(n)	O(n)	O(n)	O(n)

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Resources

Array Sorting Algorithms

Algorithm	Time Complexity			Space Complexity
	Best	Average	Worst	Worst
Quicksort	Ω(n log(n))	θ(n log(n))	O(n^2)	O(log(n))
Mergesort	Ω(n log(n))	θ(n log(n))	O(n log(n))	O(n)
Timsort	Ω(n)	θ(n log(n))	O(n log(n))	O(n)
Heapsort	Ω(n log(n))	θ(n log(n))	O(n log(n))	O(1)
Bubble Sort	Ω(n)	θ(n^2)	O(n^2)	O(1)
Insertion Sort	Ω(n)	θ(n^2)	O(n^2)	O(1)
Selection Sort	Ω(n^2)	θ(n^2)	O(n^2)	O(1)
Tree Sort	Ω(n log(n))	θ(n log(n))	O(n^2)	O(n)
Shell Sort	Ω(n log(n))	θ(n(log(n))^2)	O(n(log(n))^2)	O(1)
Bucket Sort	Ω(n+k)	θ(n+k)	O(n^2)	O(n)
Radix Sort	Ω(nk)	θ(nk)	O(nk)	O(n+k)
Counting Sort	Ω(n+k)	θ(n+k)	O(n+k)	O(k)
Cubesort	Ω(n)	θ(n log(n))	O(n log(n))	O(n)

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