• Barrier option : Up & Out Call Option.

Log-transform - θ Method :

$\boxed{1}$ $$\frac{\partial V}{\partial t} + \left(r - \frac{r^2}{2}\right)\frac{\partial V}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} - rV = 0$$

$\forall t \in [0, T]$

$\forall x \in (-\infty, u]$

We need to truncate $[x_{MIN}, u]$

$u = \log(U/S_0)$

Rmk: with PDE, you just price "out" options. For "in" option, there is a "parity"... Only with MC we can directly price both.

EU option
"OUT" option

"In" option price.

$\boxed{2}$ $V(T, x) = (S_0 e^x - K)^+ \mathbb{1}_{\{x<u\}}$: otherwise we go "out".

$\boxed{3}$ $V(t, u) = 0 \rightarrow$ BARRIER.

$\boxed{4}$ $V(t, x_{MIN}) = 0 \rightarrow$ CALL.

We can use theta method to price it: cf Thetamethod_UpOut_Call.m.

- Add the barrier.
- Change only payoff @ maturity & BCs.

The matrices don't change because they depend on the EQ but it is the same equation.

Crank-Nicholson & Stability:

θ method $\longrightarrow$ $\boxed{\text{unconditionally stable}}$ if $\theta \leq \frac{1}{2}$.

Under which hypothesis?

$V(T, S) = f(S)$

$\boxed{\text{Theory requires } f \in \mathcal{C}^2([S_{min}, S_{max}]).}$

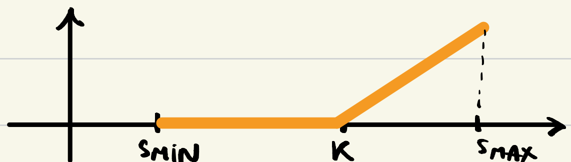
Example:

$\mathcal{C}^0$ ✓ & a.e differentiable (pb in K).

- EU

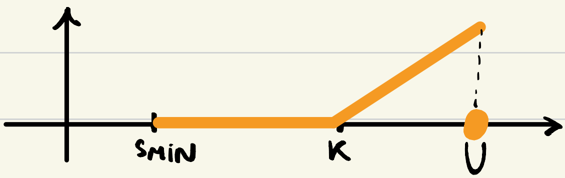
$f \in \mathcal{C}^2([S_{min}, K-\varepsilon] \cup [K+\varepsilon, S_{MAX}])$

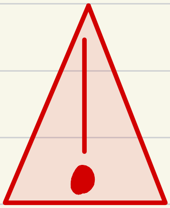
$\forall \varepsilon > 0$.



• UP & Out :

NOT CONTINUOUS
in point U.





⊕ Not differentiable
in K & in U.

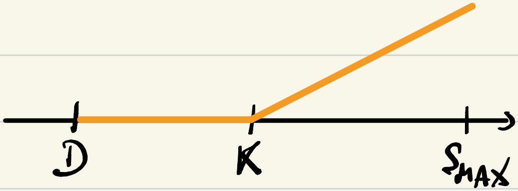
What happens when we move θ?

$$\boxed{\theta \to 0 \; : \; \text{Algorithm is MORE stable}}$$

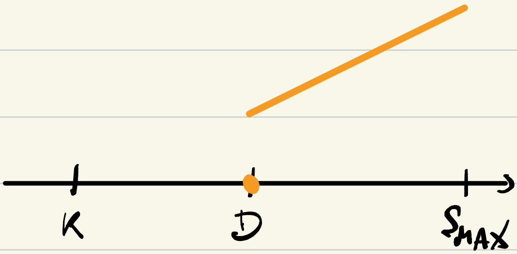
• Barrier Option : Down & Out Call Option.

- Down barrier
- Modify the grid
- Modify payoff at maturity : "$(x' > d)$".
- Modify the BCs.

$D < K$:
continuous

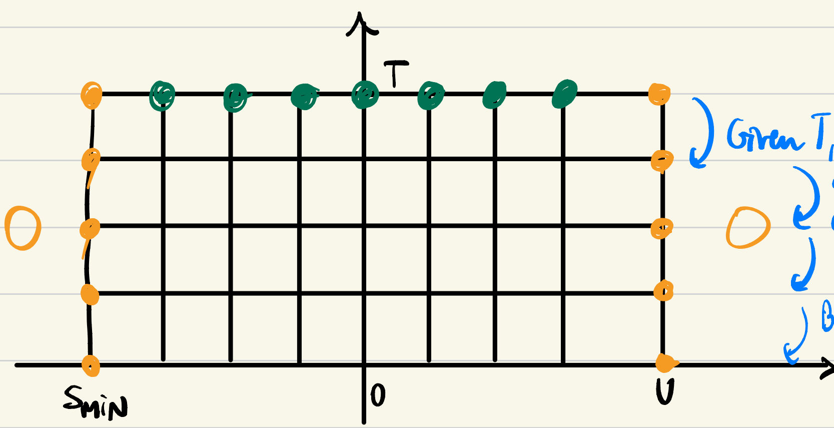


$D > K$: NOT
continuous



One last thing ...

Let's take an Up&Out Call.



Given T, compute $T-\Delta t$
given $T-\Delta t$
compute $T-2\Delta t$
...
BACKWARD IN TIME ALGORITHM.

what we do in PDE! (*)

Continuous monitoring $(S_{MAX}-K)^+ \mathbb{1}_{\{\max_{t\in[0,T]} S(t)<U\}}$

Discrete monitoring $(S_{MAX}-K)^+ \mathbb{1}_{\{\max_{j\in[[1,M]]} S(j\Delta t)<U\}}$

in MC (for ex).

(*) Because we started from this PDE:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \left(r-\dfrac{\sigma^2}{2}\right)\dfrac{\partial V}{\partial x} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 V}{\partial x^2} - rV = 0 \quad \forall x\in[x_{min}, u]\ \forall t\in[0,T]; \\ V(T,x) = (S_0 e^x - K)^+ \mathbb{1}_{\{x<u\}}; \\ V(t,u) = 0; \\ V(t,x_{MIN}) = 0. \end{cases}$$

This is:

TRUNCATED PDE FOR CONTINUOUS MONITORING.

And we discretize it by FD: but by doing this we APPROXIMATE the CONTINUOUS MONITORING SOL.

Which is $\boxed{\neq}$ from computing a discrete monitoring solution.

So here,

if $T=1$, $\Delta t = \frac{1}{12}$ $\rightsquigarrow$ It is NOT ~~monthly monitoring~~.

$\boxed{\text{Conclusion: MC always relies on discrete monitoring \& PDE always relies on } \mathcal{C}^0 \text{ monitoring.}}$

04/11/2024

Pricing American Options using PDE:

→ if interested about the theoretical framework, see the work on "obstacle problem". We will more deeply address the algorithmic point of view.

$V(t,s)$: American Option price @ time t when the underlying asset price is s.

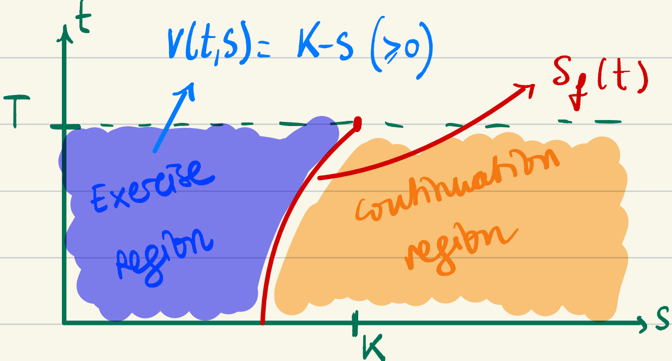
Under B&S, we have, for a PUT :

bc AM Call = EU Call.

$$\begin{cases} \dfrac{\partial V}{\partial t} + rS\dfrac{\partial V}{\partial S} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} - rV = 0 \quad \forall t \in [0,T], \forall S \in \ldots ? \\ V(T,S) = (K-S)^+ \\ V(t,S) \geq (K-S)^+ \quad \forall t \in [0,T],\ S \in \mathbb{R}^+ . \end{cases}$$

Exactly like EU case (unique sol !).

New condition.



with :

- $S_F(T) = K$.
- $S_f(t) \leq K \quad \forall t \in [0,T]$