Computational Finance - Lesson 1

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General Informations

- Ten lessons together:
  - Theoretical lessons about portfolio management
  - Matlab exercises
  - Python & Machine Learning

- For attending students:
  - The exam consists in some exercises that you have to resolve using matlab and then you have to write a report where you have to show and discuss your results.
  - You have to work in groups of 3-4 people
  - I will give you the text of the exam at the beginning of November 2024, and you have to submit to me your work before the Christmas holidays (later I will give you the exact date)
  - The grade that I will give to you (maximum of 5 points) will contribute to the final grade of the course

- For not attending students, you will have to answer to some theoretical questions about the topics we will see together during the oral exam conducted by Prof.Marazzina

Portfolio Management – An introduction

- Let’s suppose that your parents give you some allowance and you manage to save 1000 euros on the side. You can decide to put into investments, buying stocks or lottery tickets or whatever you want. But you have to break it down in percentage following some criteria.

- So how do you want to use this money on day one? Which is the **criteria**?

- Before you have to answer another question, what is your **goal**?

- That is what portfolio management is about!

Portfolio Management – An introduction

- **Portfolio**: a set of financial instruments belonging to the market.
- The **market** for an investor is represented as a set of N securities and an investment horizon τ. Ex: a market could be the S&P500 index with $\tau = 1\ week$.
- These securities can be any tradable asset: bonds, commodities, equities, currencies and so on.
- **Portfolio management**: building a portfolio and making it evolve to reach the investor's objectives while respecting the investor's constraints.
- In general in the asset allocation problem:
  - The investor seeks the combination of securities that best suit their needs in an uncertain environment
  - In order to determine the optimal allocation, the investor needs to model, estimate and manage uncertainty or risk
  - The investor focuses on a function of his portfolio’s value at the end of the investment horizon.

Financial Instruments

- Financial instruments are monetary contracts between parties. They can be created, traded, modified and settled

- They may be divided into two types:

  - ***Cash instruments***: The values of cash instruments are directly influenced and determined by the markets. Stocks and bonds are common examples of such instruments. They can be also instruments like loans and deposits.

  - ***Derivative instruments***: They are instruments which derive their value from the value and characteristics of one or more underlining entities such as an asset, index, or interest rate. Options, forwards, futures are common examples of derivatives.

Financial Instruments

- Financial instruments may also be divided according to an asset class, which depends on whether they are debt-based, equity-based or currency-based. In particular:

  - ***Debt-based instruments*** are essentially loans made by an investor to the owner of the asset. They comprehend Treasury Bills, interest-rate futures, deposits (for short-term debt) and Bonds, Loans, Bond futures (or options on bond futures), interest rate options (for long-term debt)

  - ***Equity-based instruments*** represent ownership of an asset. They comprehend stocks, shares, stock options, equity futures, but also ETFs and mutual funds

  - currency based (***Foreign exchange - based instruments***, include derivatives such as forwards, futures, and options on currency pairs

Financial Instruments - definitions

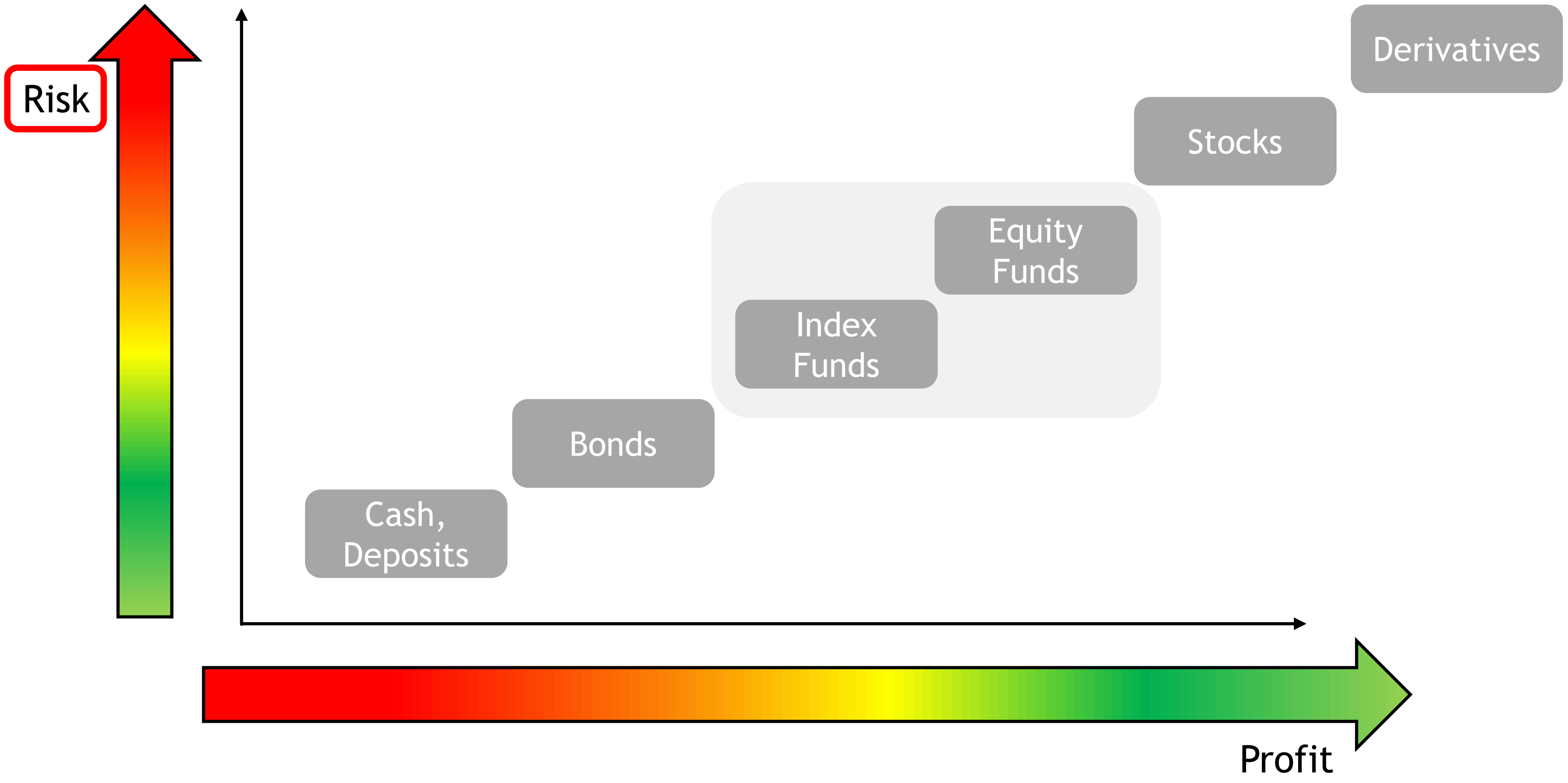
- *Bond*: it is a contract that gives the investor (bondholder) the right to receive, at predefined maturities, the repayment of the amount borrowed and a remuneration in the form of interest (the coupon). For the issuing entity, which may be a state or other public sector entity, a supranational institution, a bank or other corporation, the obligation is a debt.

- *Stocks*: they are contracts representing a fraction of a company's share capital and giving the investor the status of a shareholder, its associated payment rights (e.g. profit participation) and/or control rights (e.g. voting right in meetings). The return on the shares is linked to the financial performance of the issuing company.

- *Futures* : Standardized contracts whereby the parties agree to exchange at a predetermined price on a future date, currencies, securities or commodities.

- *Options*: An options contract offers the buyer the opportunity right to buy or sell—depending on the type of contract they hold—the underlying asset. Unlike futures, the holder is not required to buy or sell the asset if they decide against it.

Financial Instruments



Risk

- Risk is a very difficult factor to model/estimate when operating on the market
- The general risk factors are:
  - ***Economic risk***: The price of a security is generally strongly correlated with economic fluctuations (economic cycles ecc)
  - ***Inflation risk***: Investments must be assessed with a view to the inflation rate and inflation outlook at any given time
  - ***Leveraging risk***: Leveraged investments are considerably more sensitive to movements in the price of the financial instruments purchased than are investments which do not involve borrowing. The potential profit increases, the risk of loss grows accordingly
  - ***Currency risk***: Exchange rates of individual currencies can fluctuate considerably
  - ***Liquidity risk***: Low market liquidity can make it difficult for investors to sell financial instruments at market value

taux de change

Risk

- In addition each financial instrument have some specific linked risks, some example are reported below.

Bonds

- ***Solvency risk***: The issuer of bonds may become temporarily or permanently unable to pay interest or to repay the loan.
- ***Interest rate risk***: The future development of the interest rate level is uncertain, meaning that the purchaser of a bond with a fixed interest rate bears the risk that the price for the bond may drop if interest rates increase.

that's what happened with some bonds owned by Allianz ~ 2022 (central bank increased interest rate).

Equity

- ***Price volatility***: The value of shares fluctuates greatly, increasing the risk that investors could suffer financial loss on their investments. Fluctuations in the value of shares in the shorter or longer term can be unforeseeable.
- ***Investment risk***: No claims rights exist for shareholders against the share issuer, the investor is at risk of losing the entire investment if the company's operations fail.

Risk – Volatility

- The first way in which we can define risk is using the concept of ***volatility***
- The ***volatility*** (usually denoted by σ) is the degree of variation of a trading price series over time.
- A higher volatility means that a security's value can potentially be spread out over a larger range of values. This means that the price of the security can change dramatically over a short time period in either direction.
- A lower volatility means that a security's value does not fluctuate dramatically, and tends to be more steady.
- Usually volatility is measured by the ***standard deviation*** of ***the returns*** of the security.
- There exits multiple types of volatility. One of the most important is the ***historical volatility***.
- Historical volatility refers to the volatility of a financial instrument over a specified period in the past. It is calculated by the standard deviation of past returns.

Much more to learn about risk (in the following lessons)!

Portfolio Management – An introduction

- The financial environment is a *stochastic domain*. The portfolio’s value is a *random variable*, the price of a security is a random variable and so on.

- So how can we choose wisely or ‘predict’ the future if the evolution of the market is not deterministic?

- We are in the domain of *probabilities*.

- The best that we can do is to find some ways to use probability theory, statistics, math, technology and economic knowledge in order to have the greater probability that our allocation choices are good (efficient) for us.

- Because of the uncertainty of the future, a starting point is to find ***invariance***. If we find some quantities that are invariant, i.e. they have the same behavior through time, then we can learn from the past.

Portfolio Management – An introduction

- In order to solve a generic asset allocation problem we need to go through the following steps:
  - **Detecting invariance**, we have to find quantities that show the same behavior trough time in order to model them.
  - **Estimating the market**, we estimate the distribution of the market invariants from a time series of observations.
  - **Modelling the market**, we map the distribution of the invariants into the distribution of the market in a generic time in the future.
  - **Defining optimality**, analysis of the investor's profile (investor's objective, constraints ecc)
  - **Computing optimal allocation**, determine the allocation that maximizes the valuable features of the investor's objective given his constraints
  - Accounting for **estimation risk** in the optimization process

The quest of invariance

- In view of making the best possible asset allocation decision, the investor is interested in *modeling the value of the securities* in his market at his investment horizon.
- The prices at the investment horizon P_τ are a multivariate random variable: therefore modeling the market means determining the distribution of P_τ.
- In a stochastic environment apparently almost any distribution seems suitable to describe the market. If something unexpected happens, one might always blame the non-zero probability of that specific event.
- Nevertheless, a rational approach should link the market model, i.e. the distribution of the prices at the investment horizon, with the observations, i.e. the past realizations of some market observables.
- The market displays some phenomena that repeat themselves identically throughout history: we call these phenomena ***invariants***. The first step consists in detecting the invariants, i.e. the market variables that can be modeled as the realization of a set of ***independent and identically distributed*** random variables. Invariants $\longrightarrow$ iid RV.

The quest of invariance

- We need a more precise definition of the concept of invariant.
- Consider a starting point $\tilde{t}$ and a time interval $\tilde{\tau}$, which we call the estimation interval. Consider the set of equally-spaced dates:

$$\mathcal{D}_{\widetilde{t},\widetilde{\tau}} \equiv \{\widetilde{t}, \widetilde{t}+\widetilde{\tau}, \widetilde{t}+2\widetilde{\tau}, \ldots\}$$

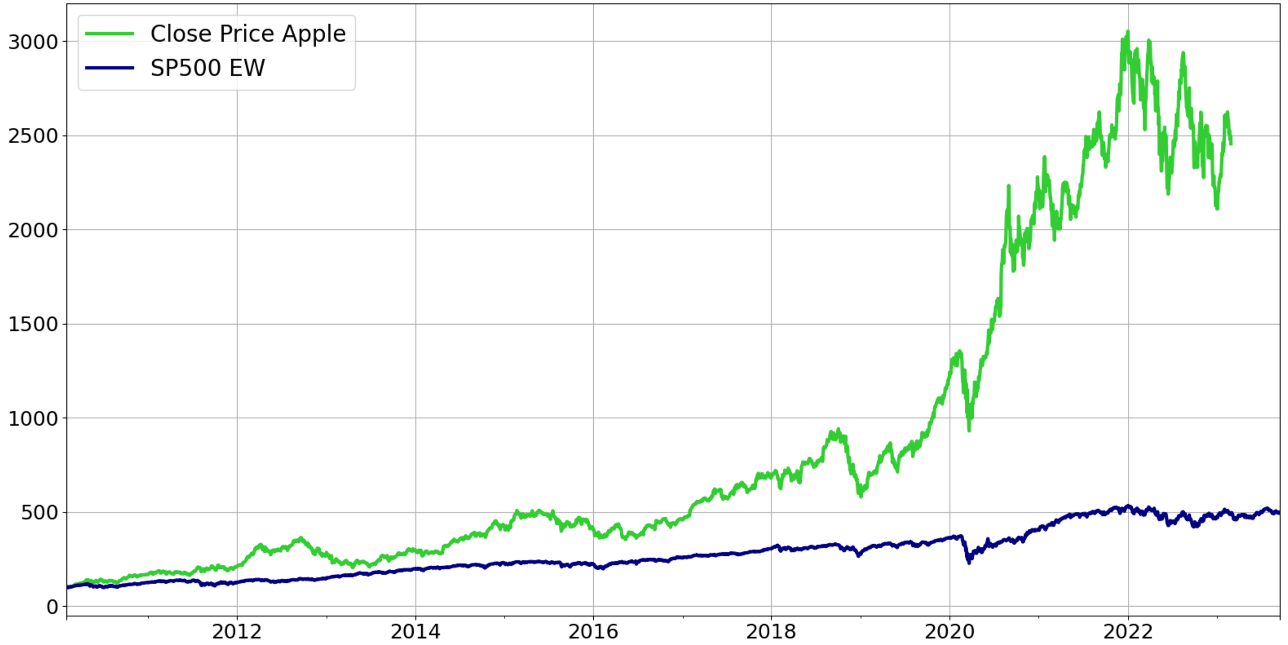
- Consider a set of random variables $X_t, t \in D_{\tilde{t},\tilde{\tau}}$
- The random variables X_t are market invariants for the starting point $\tilde{t}$ and the estimation interval $\tilde{\tau}$ if they are ***independent and identically distributed*** and if the realization x_t of X_t becomes available at time t.
- A *time homogenous invariant* is an invariant whose distribution does not depend on the reference time. In our quest for invariance, we will always look for time-homogeneous invariants.

The quest of invariance

- To detect invariance, we look into the time series of the financial data available.
- The **time series** of a generic set of random variables is the *set of past realizations* of those random variables. Denoting as T the current time, the time series is the set

$$x_t, \quad t = \widetilde{t}, \widetilde{t} + \widetilde{\tau}, \ldots, T,$$

where the lower case notation indicates that x_t is the specific realization of the random variable X_t occurred at time t in the past.



The quest of invariance

- In order to detect invariance, we can perform two simple graphical tests:

1. The first test consists in splitting the time series into two series, then we compare the respective histograms. If X_t is an invariant, in particular all the terms in the series are identically distributed, then the two histograms should look very similar to each other.

2. The second test consists of the scatter-plot of the time series on one axis against its lagged values on the other axis. In other words, we compare the following two series:

$$x_t \text{ versus } x_{t-\tilde{\tau}},$$

If X_t is an invariant, in particular all the terms in the series are independent of each other, the scatter plot must be symmetrical with respect to the reference axes.

The quest of invariance - Prices

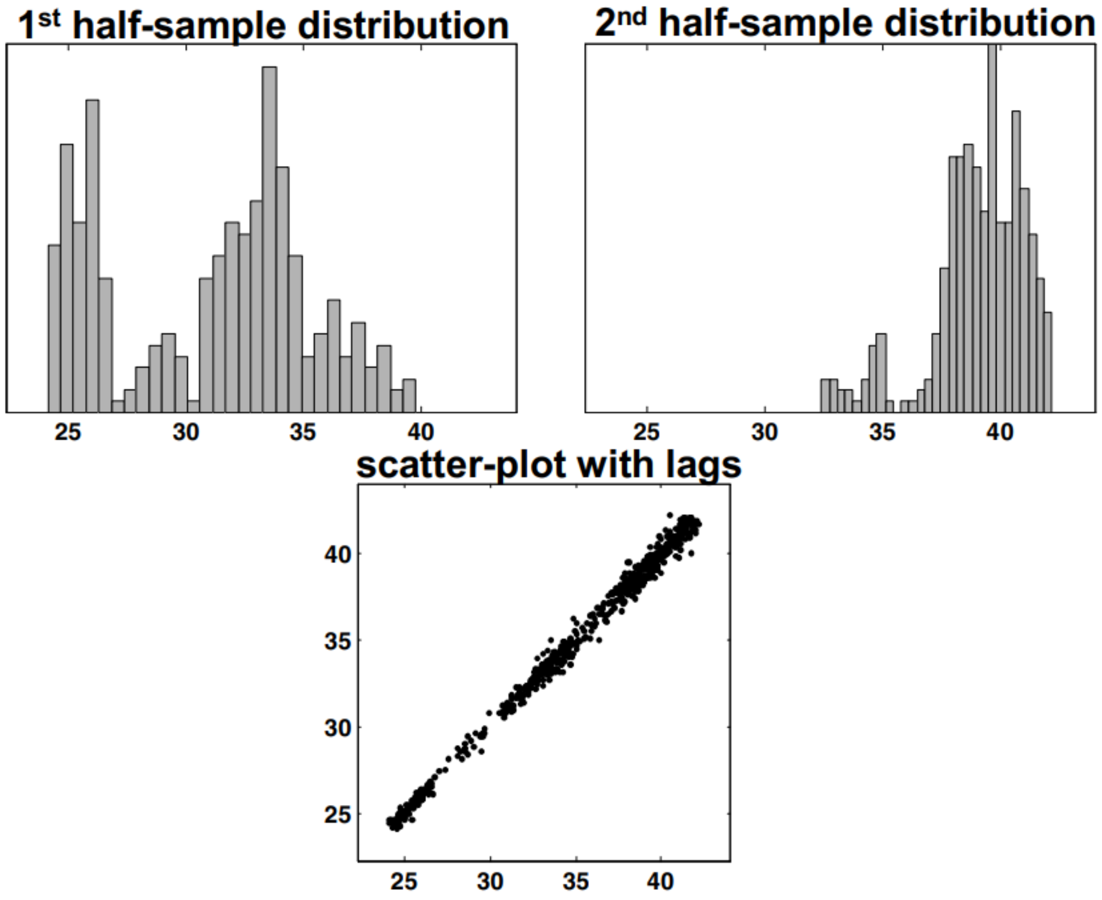
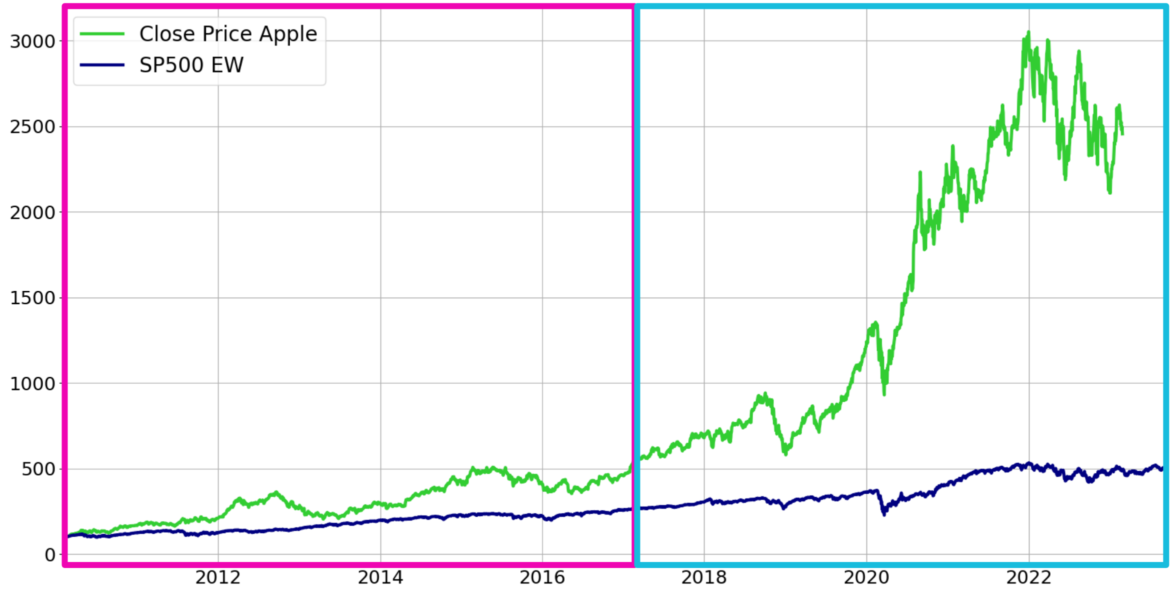


Fig. 3.1. Stock prices are not market invariants



The quest of invariance

- Let's introduce a really important variable in finance: the returns
- The ***total return*** at time t for a horizon τ on any asset (equity, etc.) that trades at the price P_t at the generic time t is defined as the following multiplicative factor between two subsequent prices:

$$\boxed{H_{t,\tau} \equiv \frac{P_t}{P_{t-\tau}}.}$$

- The ***linear return*** at time t for a horizon τ is defined as follows:

$$\boxed{L_{t,\tau} \equiv \frac{P_t}{P_{t-\tau}} - 1.}$$

- The ***compounded return*** at time t for a horizon τ is defined as follows:

$$\boxed{C_{t,\tau} \equiv \ln\left(\frac{P_t}{P_{t-\tau}}\right)}$$

The quest of invariance

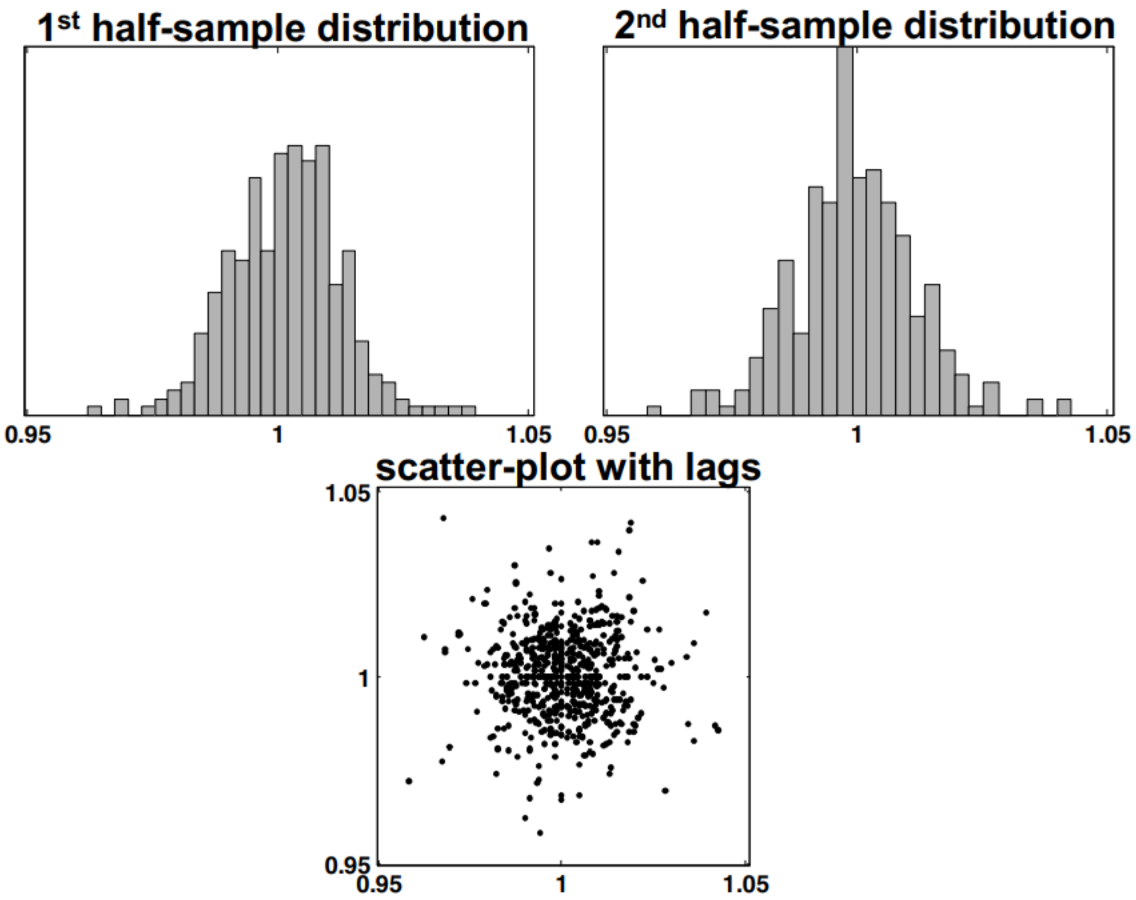
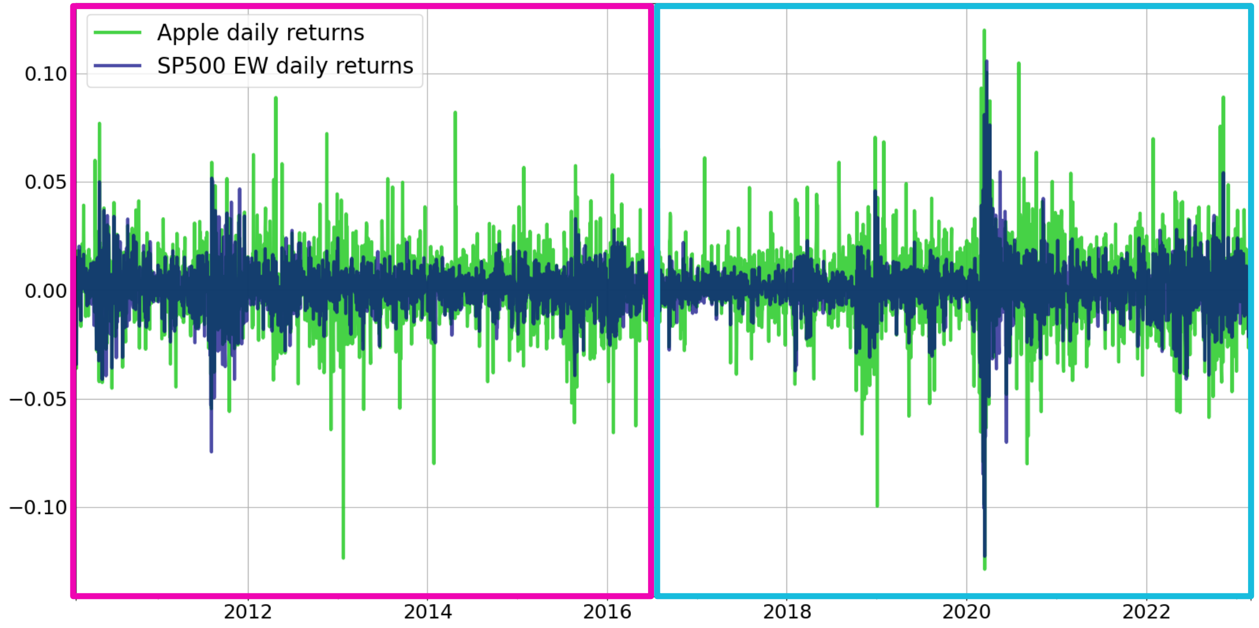


Fig. 3.2. Stock returns are market invariants



- ✓ Therefore we accept the set of total returns as invariants for the equity market.
- ✓ More in general, any function g of the total returns defines new invariants for the equity market
- ✓ NB: Each financial instrument has its own invariant!

Estimating the distribution of market invariants

- In order to estimate the distribution of the market, we have to actually construct estimators for the market invariants.
- We aim at inferring the "truth", as represented by a generic number S of features of the distribution of the market invariants. These features can be expressed as an S-dimensional vector of functionals of the probability density function $G[f(X]$
- An *estimator* is *a vector-valued function that associates a vector in* $\mathbb{R}^S$ *, i.e. a set of S numbers, with available information*:

$$\boxed{\text{estimator: information } i_T \mapsto \text{number } \widehat{\mathrm{G}}}$$

- Notice that the definition of estimator is not related to the goal of estimation. Again, an estimator is simply a function of currently available information.
- Some examples:

$$\widehat{G}[i_T] \equiv \frac{1}{T}\sum_{t=1}^{T} x_t. \qquad \widehat{G}[i_T] \equiv x_1 x_T. \qquad \widehat{G}[i_T] \equiv 3.$$

Estimating the distribution of market invariants

- Although the definition of estimator is very general, an estimator serves its purpose only if its value is close to the true, unknown value that we are interested in:

$$\widehat{\mathbf{G}}\left[i_T\right] \approx \mathbf{G}\left[f_{\mathbf{X}}\right]$$

True, unknown distribution of mkt invariants.

- Therefore we need a criterion to evaluate estimators.
- The main requirement is its ***replicability***: an estimator is good not only if the result of the estimation is close to the true unknown value, but also if this does not happen by chance
- The distribution of the information is fully determined by the true, unknown distribution $f(X)$ of the market invariants. Therefore, the distribution of the estimator $\hat{G}$ is also determined by the true, unknown distribution $f(X)$ of the market invariants
- For example, if the invariants are normally distributed with the following unknown parameters:

$$X_t \sim \mathrm{N}\left(\mu, \sigma^2\right)$$

then the following estimator is normally distributed with the following parameters: $\widehat{G}\left[I_T\right] \equiv \frac{1}{T}\sum_{t=1}^{T} X_t \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{T}\right)$

Great example to understand what it means: how $f(X)$ influences $\hat{G}$.

Estimating the distribution of market invariants

- Under fairly general conditions, sample averages computed over the whole time series approximate the expectation computed with the true distribution, and the approximation improves with the number of observations in the time series.

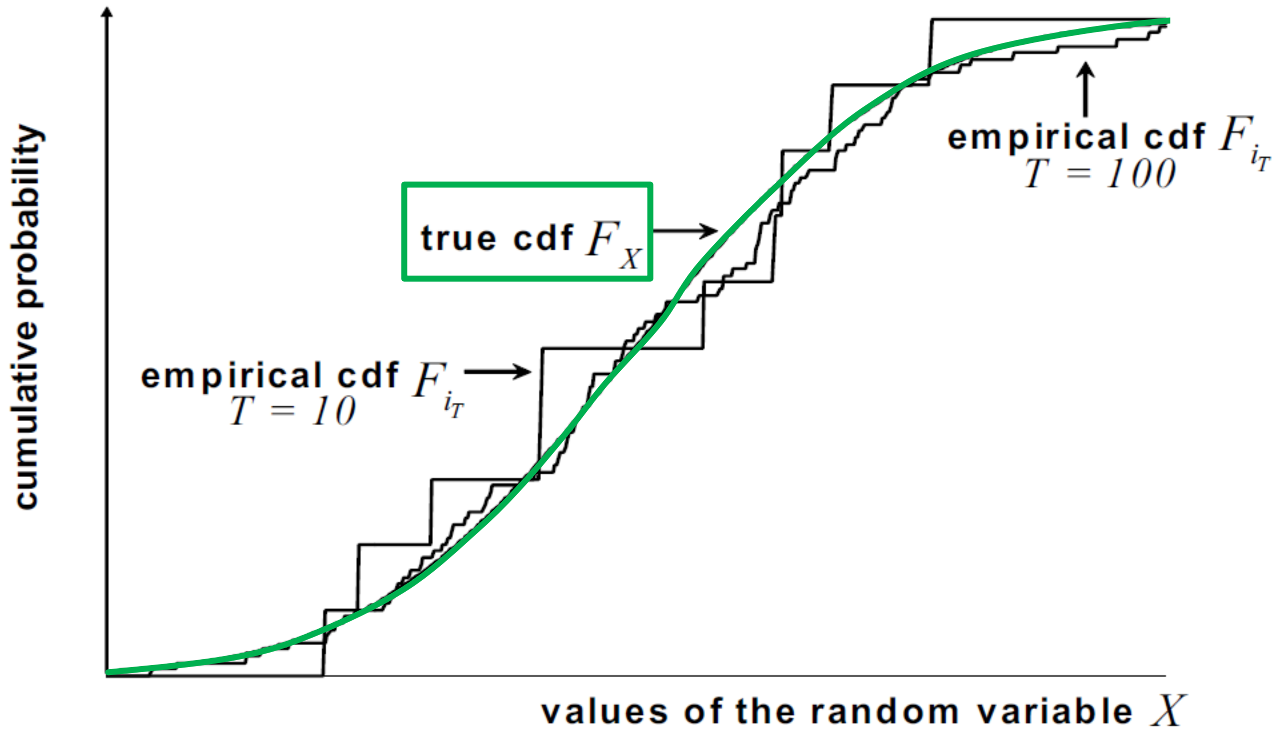
- This result is known as the ***law of large numbers (LLN)***:

$$\frac{1}{T}\sum_{t=1}^{T}\{\text{past}\} \underset{T\to\infty}{\approx} \mathrm{E}\{\text{future}\}$$

- The Law of Large Numbers implies the *Glivenko-Cantelli theorem*. This theorem states that the empirical distribution of a set of independent and identically distributed variables, as represented for example by its cumulative distribution function, tends to the true distribution as the number of observations goes to infinity:

$$\lim_{T\to\infty} F_{i_T}(\mathbf{x}) = F_{\mathbf{X}}(\mathbf{x})$$

Estimating the distribution of market invariants



Estimating the distribution of market invariants

- There are different methods to calculate an estimator: parametric, non-parametric, maximum likelihood etc..

- The main estimators for a generic distribution describe

    - Location: mean, median and mode
    - Dispersion: standard deviation (univariate case), covariance matrix (multivariate case)
    - Shape: Skewness & Kurtosis

Estimators – The normal case

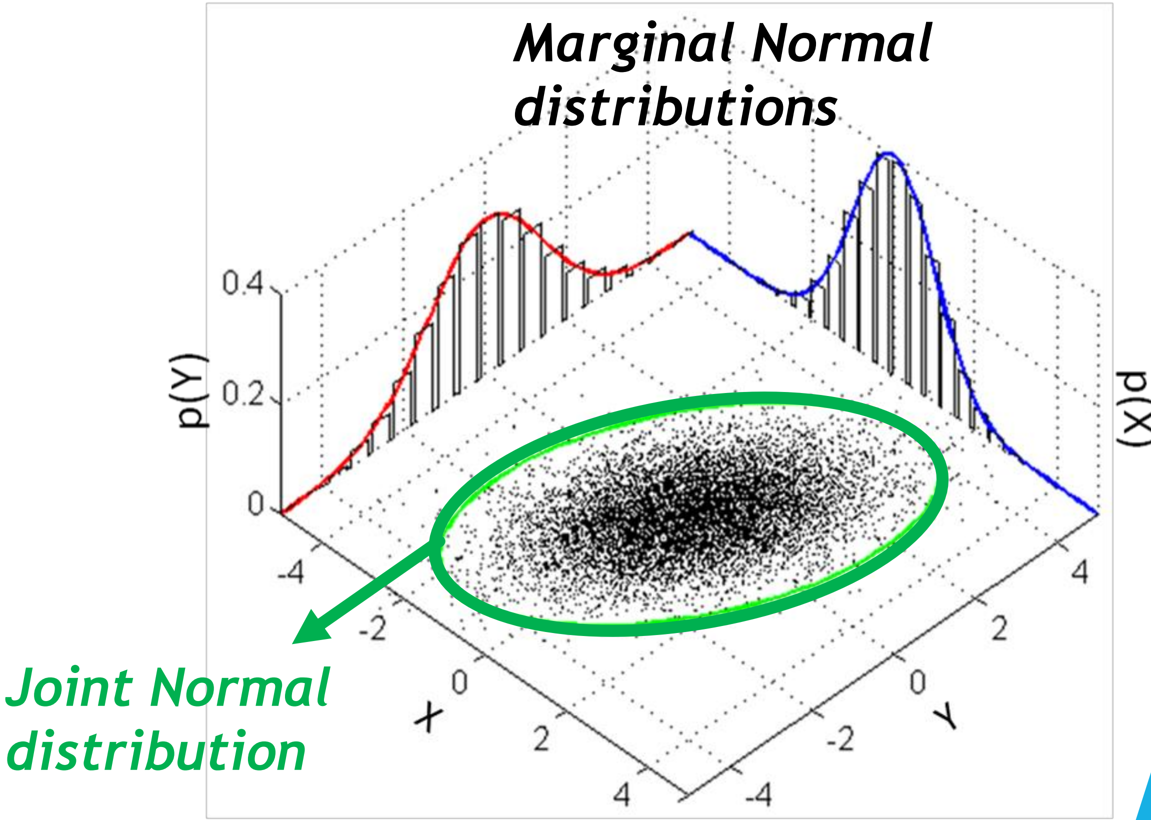
- Let's suppose that our invariants X_t are distributed following a ***normal distribution***, $\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where μ is the *expected value* of the distribution and Σ is the *variance-covariance matrix*

- The location estimator is the ***sample mean***:

$$\widehat{\boldsymbol{\mu}}[i_T] = \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}_t$$

- The dispersion estimator is the sample ***covariance matrix***:

$$\widehat{\boldsymbol{\Sigma}}[i_T] = \frac{1}{T}\sum_{t=1}^{T} (\mathbf{x}_t - \widehat{\boldsymbol{\mu}})(\mathbf{x}_t - \widehat{\boldsymbol{\mu}})'$$



Evaluating allocations – Investor’s objectives

- Consider a market of N securities.
- At the time T when the investment is made the investor can purchase α_n units of the generic n-th security. Therefore, the allocation is represented by the N-dimensional vector α
- With the allocation α the investor forms a portfolio whose value at the time the investment decision is made is:

$$w_T(\boldsymbol{\alpha}) \equiv \boldsymbol{\alpha}'\mathbf{p}_T.$$

- At the investment horizon τ the market prices of the securities are a multivariate random variable. Therefore at the investment horizon the portfolio is a one-dimensional random variable, namely the following simple function of the market prices:

$$W_{T+\tau}(\boldsymbol{\alpha}) \equiv \boldsymbol{\alpha}'\mathbf{P}_{T+\tau}$$

- The investor has one or more ***objectives*** Ψ, namely quantities that the investor perceives as beneficial and therefore he desires in the largest possible amounts.

Evaluating allocations – Investor's objectives

- **Absolute Wealth**: The investor focuses on the value at the horizon of the portfolio

$$\Psi_{\boldsymbol{\alpha}} \equiv W_{T+\tau}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}'\mathbf{P}_{T+\tau}$$

- **Relative Wealth**: The investor is concerned with overperforming a reference portfolio, whose allocation we denote as β.

$$\Psi_{\boldsymbol{\alpha}} \equiv W_{T+\tau}(\boldsymbol{\alpha}) - \gamma(\boldsymbol{\alpha}) W_{T+\tau}(\boldsymbol{\beta})$$

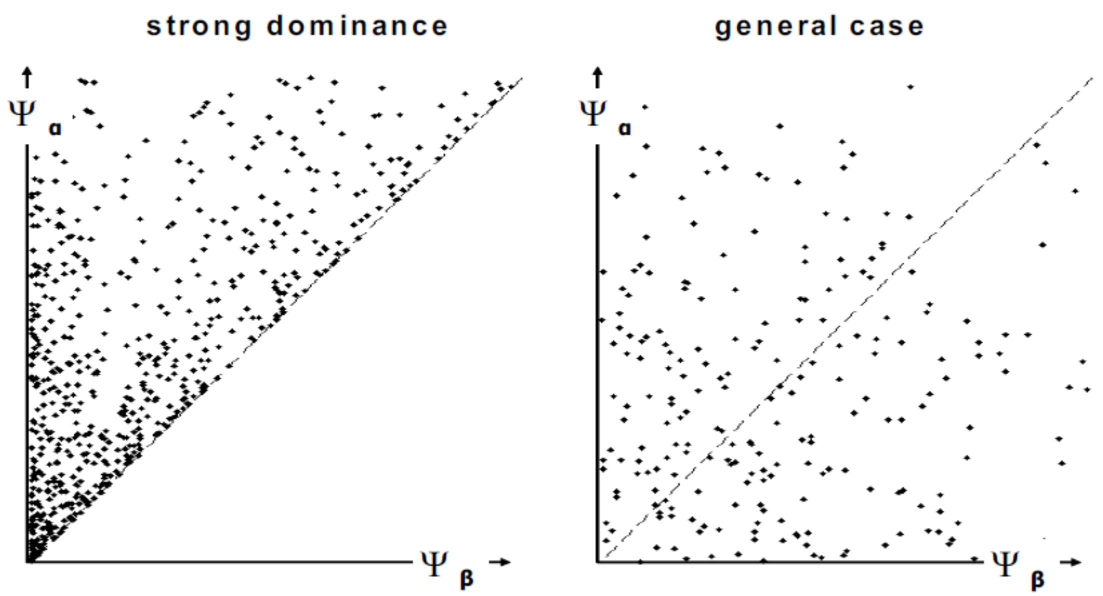
The function γ is a normalization factor such that at the time the investment decision is made the reference portfolio and the allocation have the same value:

$$\gamma(\boldsymbol{\alpha}) \equiv \frac{w_T(\boldsymbol{\alpha})}{w_T(\boldsymbol{\beta})}$$

- If the *markets were deterministic*, the investor could compute the objective relative to a given allocation as a deterministic function of that allocation, and thus he would choose the allocation that gives rise to the largest value of the objective.
- Instead, the *market prices* at the investment horizon *are stochastic* and therefore the market vector is a random variable, and so is the investor's objective. Since the objective is a random variable we need some tools to figure out in which sense a random variable is "larger" or is "better" than another one.

Estimating allocations – Stochastic Dominance

- Suppose that the investor can choose between an allocation $\boldsymbol{\alpha}$ that gives rise to the objective Ψ_α and an allocation $\boldsymbol{\beta}$ that gives rise to the objective Ψ_β.
- When confronted with two different objectives Ψ_α and Ψ_β, it is natural to first check whether in all possible scenarios one objective is larger than the other.



When this happens, the objective Ψ_α, or the allocation $\boldsymbol{\alpha}$, is said to **strongly dominate** the objective Ψ_β, or the allocation $\boldsymbol{\beta}$:

$$\Psi_\alpha \geq \Psi_\beta \text{ in all scenarios}$$

That is said to be dominance of zero-order

Estimating allocations – Stochastic Dominance

- An equivalent definition reads as follows in terms of the ***cumulative distribution function*** of the difference of the objectives:

"in all scenarios"

$$F_{\Psi_\alpha - \Psi_\beta}(0) \equiv \mathbb{P}\left\{\Psi_\alpha - \Psi_\beta \leq 0\right\} = 0$$

since, in the previous slide: $\mathbb{P}(\Psi_\alpha \geq \Psi_\beta) = 1$.

- Nevertheless, strong dominance cannot be a general criterion to evaluate allocations:

   I. Strong dominance never takes place. Instead, in general an allocation could give rise to an objective that in some scenarios is larger and in some scenarios is smaller than another.

- So let's introduce the ***weak dominance***.

IDEA

- We would be prone to choose an allocation α over another allocation β if the probability density function of the ensuing objective were concentrated around larger values than for the other allocation.

- Alternatively we can use the cumulative distribution function of the objectives

Estimating allocations – Stochastic Dominance

- The objective Ψ_α is said to ***weakly dominate*** the objective Ψ_β, if the following condition holds true:

$$F_{\Psi_\alpha}(\psi) \leq F_{\Psi_\beta}(\psi) \text{ for all } \psi \in (-\infty, +\infty)$$

where F_Ψ is the cumulative distribution function of the objective Ψ. That condition is said as ***first - order dominance***

($\mathcal{G}F_{\Psi_\alpha}$ is under $\mathcal{G}F_{\Psi_\beta}$).

- First - order dominance could be too restrictive, so we can go on and calculate the ***second-order dominance***. We can say that the objective Ψ_α dominates at second-order Ψ_β if

$$\int_{-\infty}^{x} F_{\Psi_\alpha}(t)\,dx \leq \int_{-\infty}^{x} F_{\Psi_\beta}(t)\,dx \text{ for all } x \in \mathbb{R}$$

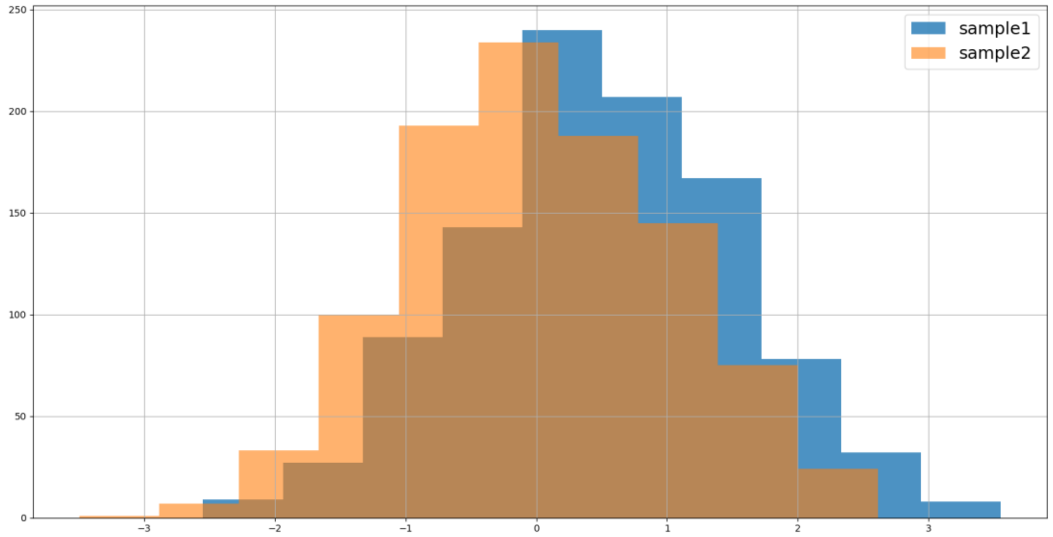
- If even second-order dominance does not take place, we must pursue weaker and weaker criteria. More in general, we say that the objective Ψ_α ***order-q dominates*** the objective Ψ_β, or the allocation, if for all $\psi \in (-\infty, \infty)$ the following inequality holds:

$$\mathcal{I}^q\left[f_{\Psi_\alpha}\right](\psi) \leq \mathcal{I}^q\left[f_{\Psi_\beta}\right](\psi)$$

density: coherent bc 1st order is $F_{\Psi_\alpha} \leq F_{\Psi_\beta}$ ...

where $\mathcal{I}^q$ is the q-iterated integral of the pdf.

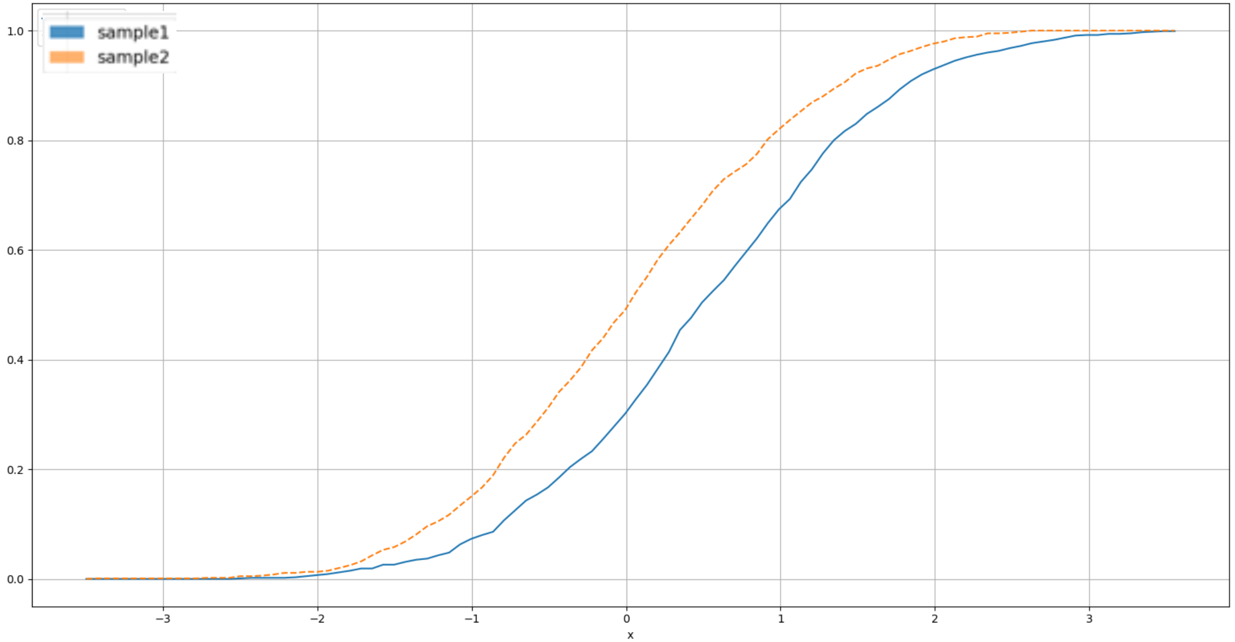
1st order dominance, *in practice…*



- Let's suppose that we have two allocations ***1*** and ***2*** that give rise to objectives:

$$\begin{cases} \Psi_1 = \text{sample } 1 \\ \Psi_2 = \text{sample } 2 \end{cases}$$

- We perform an hypothesis test to find if Ψ_1 weakly dominates Ψ_2

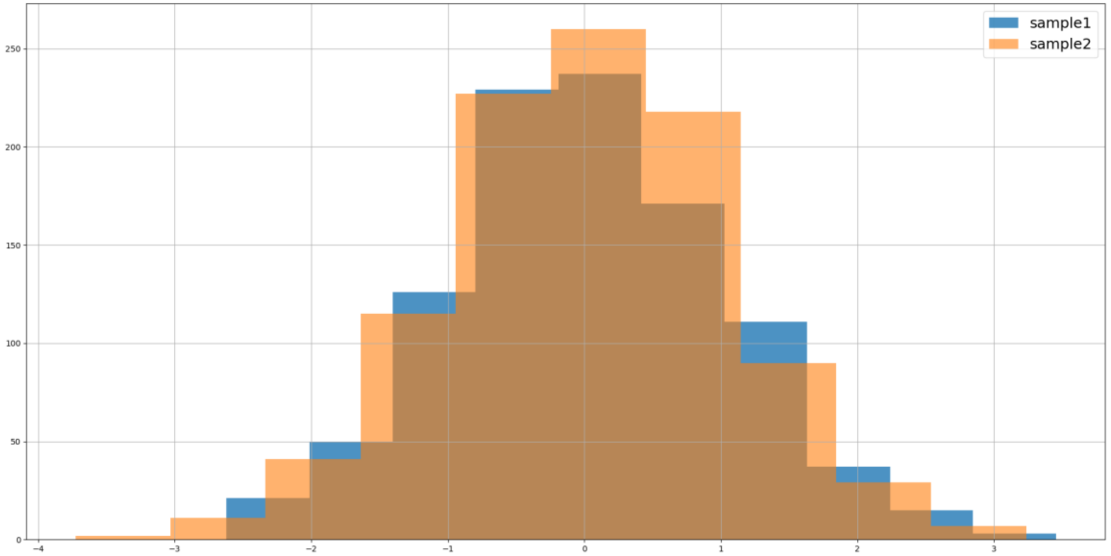


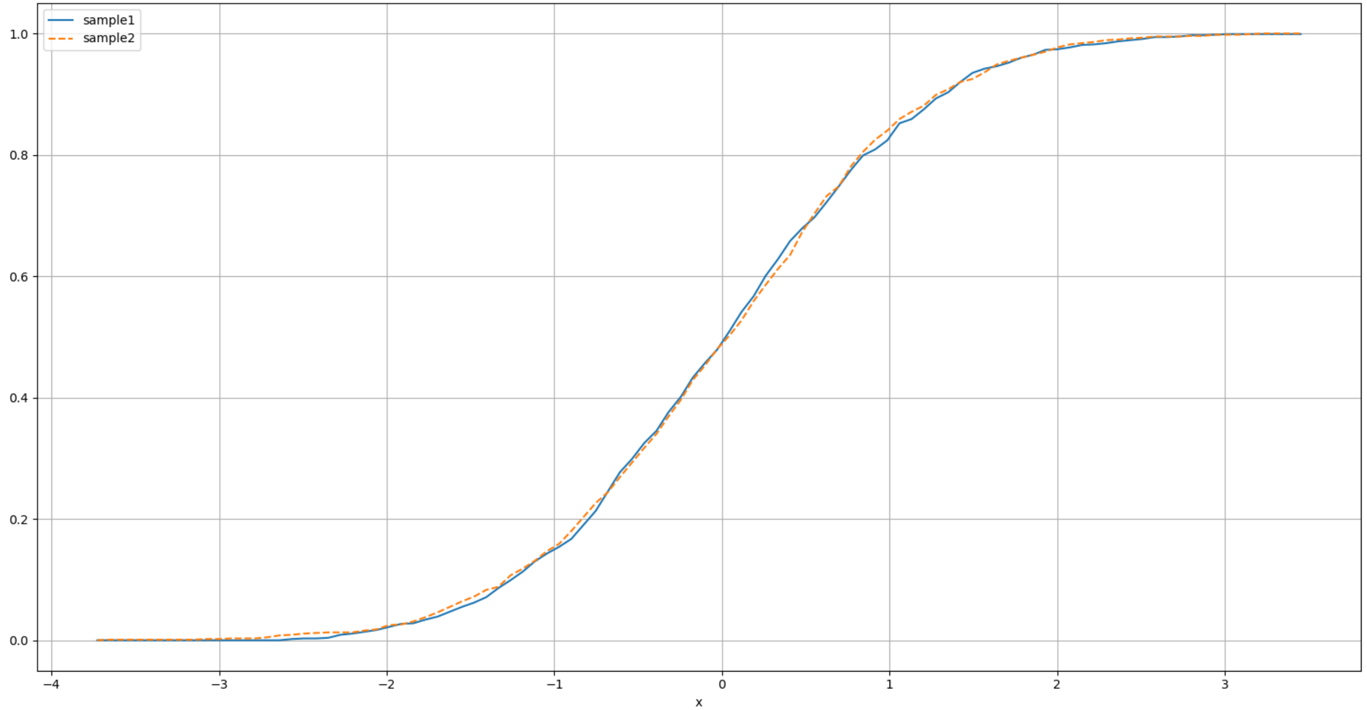
- *P-value = 1*, means that the null hypothesis is true

- In fact we know, from the plot in the right hand, that:

$$CDF_1 \leq CDF_2 \quad \text{everywhere}$$

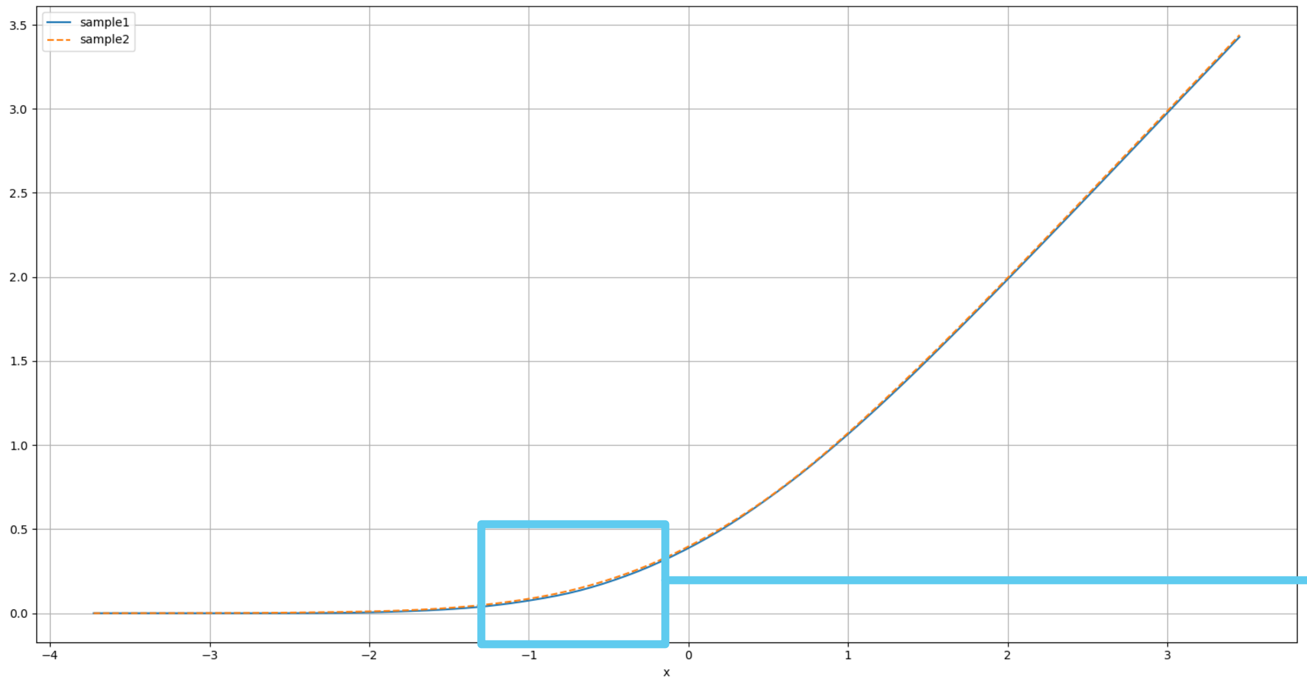
1st order dominance

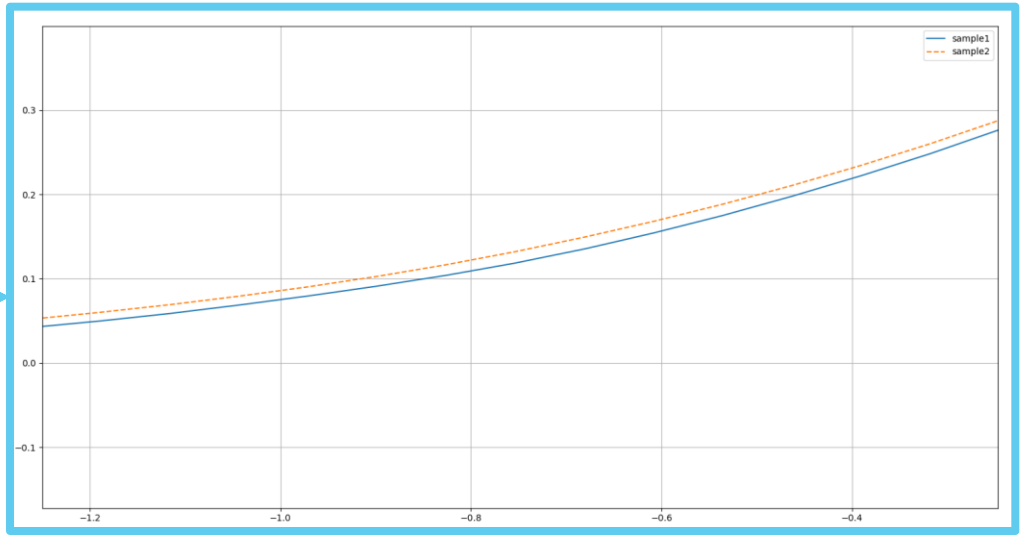




- *P-value = 0.45*, we can not say anything about which sample dominates the other at the first order.
- So let’s proceed with the hypothesis test at the second order

2nd order dominance





- *P-value =1*, telling us that the sample1 dominates sample2 at the second order

Stochastic Dominance - Drawbacks

- The intuitive meaning behind dominance of orders higher than two is not evident.
- The computation of the generic ***q-th cumulative distribution*** is not practically feasible in most situations.
- There is no guarantee that there exists an order such that a portfolio stochastically dominates or is dominated by another: consequently, the investor might not be able to rank his potential investments and thus choose an allocation.

To solve this problem, we can summarize all the features of a given allocation α into one single number S that indicates the respective degree of satisfaction

$$\boldsymbol{\alpha} \mapsto \mathcal{S}(\boldsymbol{\alpha})$$

The investor will then choose the allocation that corresponds to the highest degree of satisfaction.

For example, the expected value of the investor's objective is a number that depends on the allocation:

$$\boldsymbol{\alpha} \mapsto \mathcal{S}(\boldsymbol{\alpha}) \equiv \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\}.$$

As such, it is an index of satisfaction.

Index of satisfaction

- A generic index of satisfaction can be characterized by the following properties:
  - **Money equivalence** (naturally measured in units of money)
  - **Estimability**, satisfaction associated with a generic allocation α is fully determined by the marginal distribution of the investor's objective Ψ_α
  - **Sensibility**, $\Psi_{\boldsymbol{\alpha}} \geq \Psi_{\boldsymbol{\beta}}$ in all scenarios $\Rightarrow \mathcal{S}(\boldsymbol{\alpha}) \geq \mathcal{S}(\boldsymbol{\beta})$
  - **Consistence with stochastic dominance**
  - **Constancy**, if there exists an allocation ***b*** that yields a deterministic objective ψ_b, it is reasonable to require that the index coincide with the objective.
  - **Positive homogeneity**, if we rescale the allocation by a given positive factor the objective is rescaled by the same factor: $S(\lambda\alpha) = \lambda S(\alpha)$ for all $\lambda \geq 0$
  - **Translation invariance**, $S(\alpha + \beta) = S(\alpha) + \Psi_\beta$ (the allocation β gives a deterministic objective Ψ_β)
  - **Sub/super additivity**, if it states $S(\alpha + \beta) \leq S(\alpha) + S(\beta)$/ $S(\alpha + \beta) \geq S(\alpha) + S(\beta)$
  - **Risk aversion/propensity/neutrality**

Index of satisfaction

- Risk aversion/propensity/neutrality
- An index of satisfaction is ***risk averse*** if the risk-free allocation ***b*** is preferred to the risky joint allocation ***b+f*** for any level of the risk-free outcome Ψ_b and any fair game ***f*:**

$E[\Psi_f]=0$.

i.e if: $$\mathrm{E}\{\Psi_{\mathbf{f}}\} \equiv 0 \Rightarrow \mathcal{S}(\mathrm{b}) \geq \mathcal{S}(\mathrm{b}+\mathrm{f})$$

- The ***risk premium*** is the dissatisfaction due to the uncertainty of a risky allocation:

$$\mathrm{RP} \equiv \mathcal{S}(\mathrm{b}) - \mathcal{S}(\mathrm{b}+\mathrm{f})$$

- Any random variable Ψ, for which the expected value is defined, can be factored into the sum of a deterministic component $E[\Psi]$ and a fair game $\Psi - E[\Psi]$. Therefore the risk premium can be expressed as the difference between the satisfaction arising from the expected objective and that arising from the risky allocation.

?

$$\mathrm{RP}(\boldsymbol{\alpha}) \equiv \mathrm{E}\{\Psi_{\boldsymbol{\alpha}}\} - \mathcal{S}(\boldsymbol{\alpha})$$

Index of satisfaction

- ✓ An index of satisfaction is risk averse if the risk premium is positive for any allocation:

$$\text{risk aversion:}\ \ \mathrm{RP}(\boldsymbol{\alpha}) \geq 0,$$

- ✓ An index of satisfaction is risk seeking if a risky allocation is preferred to a risk-free allocation with the same expected value, i.e. the investor is willing to pay a positive amount to play a risky game:

$$\text{risk propensity:}\ \ \mathrm{RP}(\boldsymbol{\alpha}) \leq 0.$$

- ✓ Finally, an index of satisfaction is risk neutral if a risky allocation is perceived as equivalent to a risk-free allocation with the same expected value.

$$\text{risk neutrality:}\ \ \mathrm{RP}(\boldsymbol{\alpha}) \equiv 0.$$

Index of Satisfaction - Expected Utility

- Consider an investor with a given **objective $\boldsymbol{\Psi}$** such as absolute wealth, or relative wealth.
- Consider a generic allocation $\boldsymbol{\alpha}$ that gives rise to the objective Ψ_{α}.
- Let's recall the ***utility theory***, based on the assumption of **rationality** and describes all decision outcomes in terms of the ***utility*** (or ***value***) placed on them by individuals according to their preferences.
- Thus a utility function $u(\psi)$ describes the extent to which the investor enjoys the generic outcome $\Psi_{\alpha} = \psi$ of the objective, in case that realization takes place.
- To build an index of satisfaction we can weight the utility from every possible outcome by the probability of that outcome. In other words, we consider the ***expected utility*** from the given allocation:

$$\boxed{\boldsymbol{\alpha} \mapsto \mathrm{E}\left\{u\left(\Psi_{\boldsymbol{\alpha}}\right)\right\} \equiv \int_{\mathbb{R}} u\left(\psi\right) f_{\Psi_{\boldsymbol{\alpha}}}\left(\psi\right) d\psi,} \longrightarrow \text{Expected utility hypothesis}$$

where $f(\Psi)$ is the probability density function of the objective.

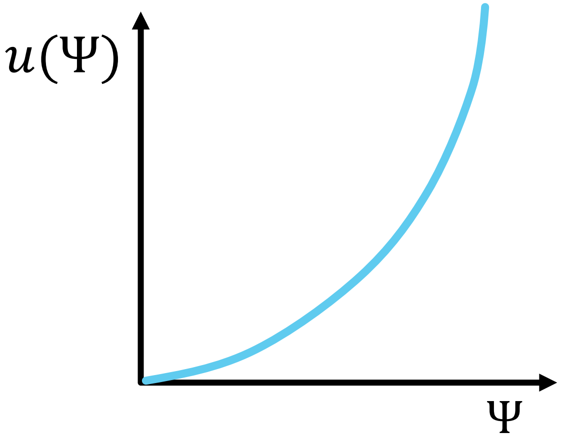
Expected Utility

- Thus means that if we have two allocations α and β, with the respective objectives Ψ_α and Ψ_β, and a utility function $u(\cdot)$ describing our preferences, we will chose allocation α over β, if

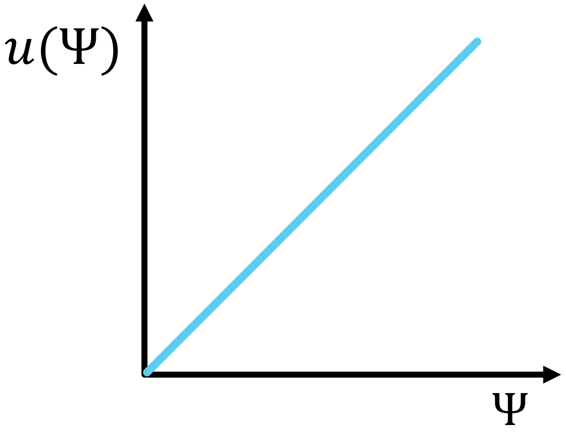
$$E[u(\Psi_\alpha)] \geq E\left[u\left(\Psi_\beta\right)\right]$$

- If a utility function describes the investor's preferences, then they describe also his attitude through risk. In particular based on the concavity/convexity of the utility function, we can tell if an investor is risk averse/lover/neutral.

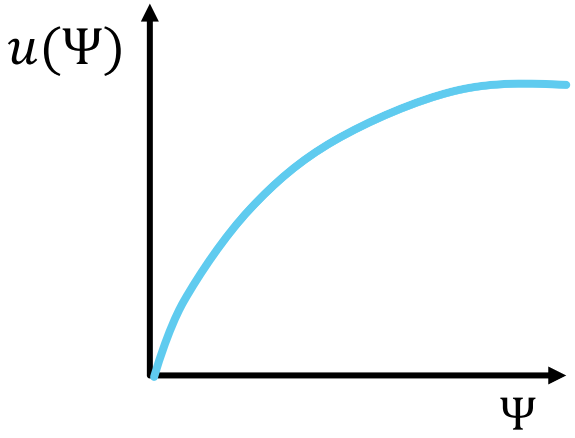
Risk lover $\Leftrightarrow u''(\cdot) > 0$



Risk neutral $\Leftrightarrow u''(\cdot) = 0$



Risk averse $\Leftrightarrow u''(\cdot) < 0$



Expected Utility - Certainty Equivalent

- For an index of satisfaction we want to measure satisfaction in terms of money.
- In order to satisfy this requirement, we consider the ***certainty-equivalent of an allocation***, which is the risk-free amount of money that would make the investor as satisfied as the risky allocation:

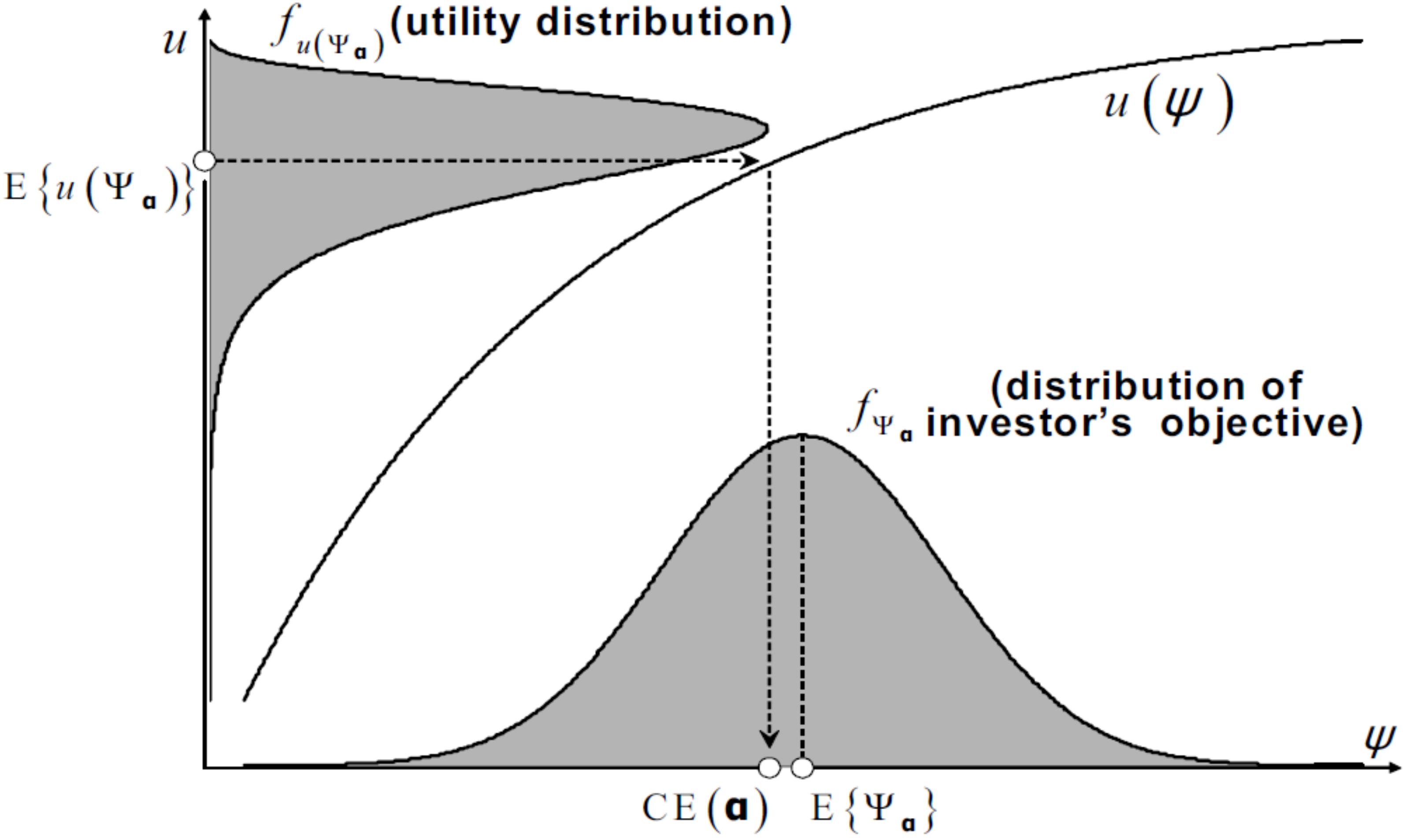
$$\boldsymbol{\alpha} \mapsto \mathrm{CE}(\boldsymbol{\alpha}) \equiv u^{-1}\left(\mathrm{E}\left\{u\left(\Psi_{\boldsymbol{\alpha}}\right)\right\}\right)$$

- <u>Proof</u>:
  - Let's recall the definition of risk - premium: $RP = E[\Psi_\alpha] - S(\alpha) \quad \Rightarrow \quad S(\alpha) = E[\Psi_\alpha] - RP$
  - Let's recall the expected utility criteria: $u\big(S(\alpha)\big) = E[u(\Psi_\alpha)]$
  - So substituting $S(\alpha) \; \Rightarrow u(E[\Psi_\alpha] - RP) = E[u(\Psi_\alpha)]$
  - Let's define the certainty equivalent as $\mathrm{CE}(\alpha) = \; E[\Psi_\alpha] - RP \quad \Rightarrow \quad u\big(CE(\alpha)\big) = E[u(\Psi_\alpha)]$
  - Then applying on both sides $u^{-1}(\cdot)$, we obtain

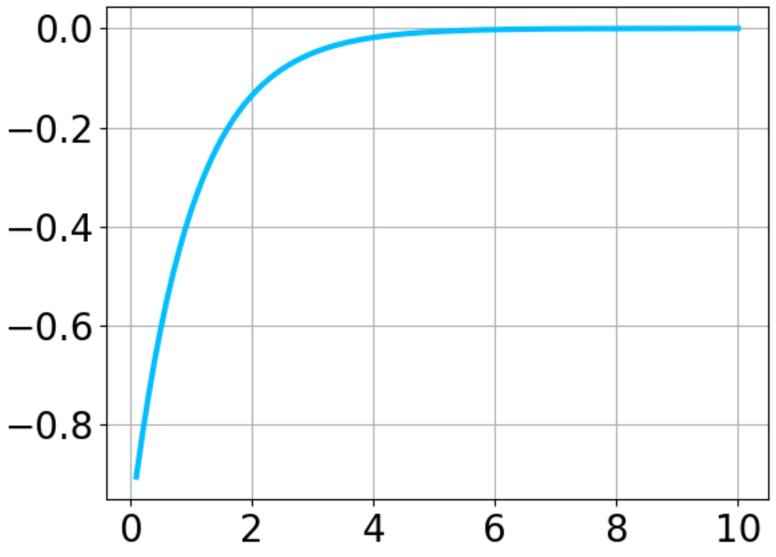
$$\boxed{CE(\alpha) = \; u^{-1}(E[u(\Psi_\alpha)])}$$

Expected Utility - Certainty Equivalent

Graphical intuition :

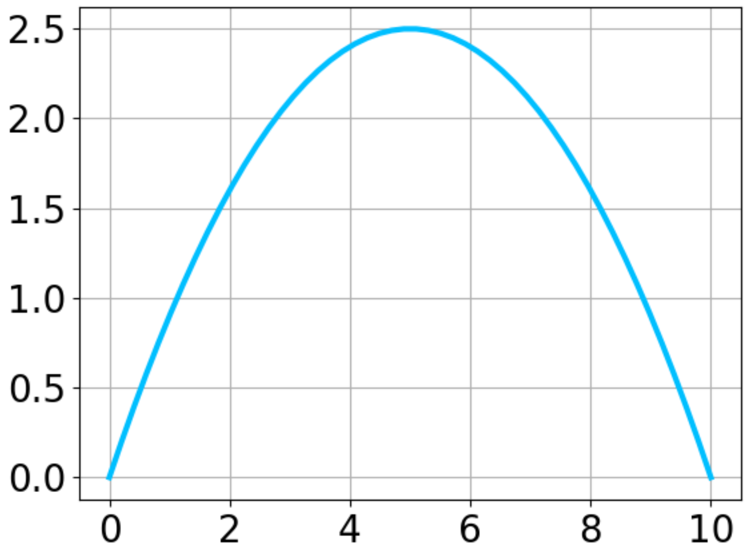


Certainty Equivalent & Utility Functions



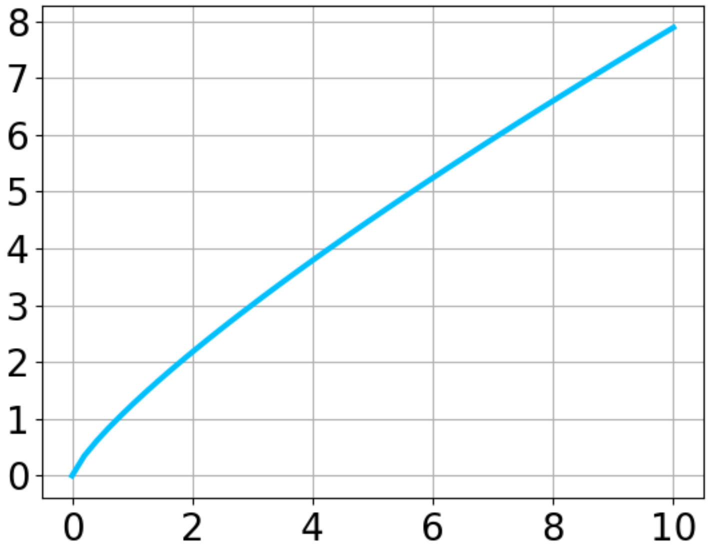
Exponential

$$u(x) = -\frac{1}{a}e^{-ax}$$



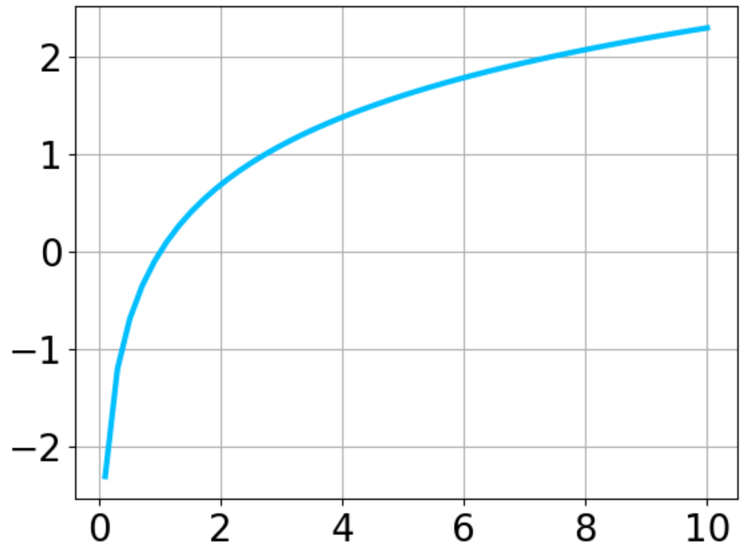
Quadratic

$$u(x) = x - \frac{1}{b}x^2$$



Power

$$u(x) = \frac{1}{1-\gamma}x^{1-\gamma}$$



Logarithmic

$$u(x) = \ln(x)$$

Building Strategies

- We have seen the main building blocks of portfolio management: the financial instruments, the investor's objectives and constraints and the risk .
- It's time to focus on the investment strategies, i.e. investment choices, in order to build the most efficient portfolio we can ,

- A strategy is a set of investments choices based on a determined information set I_t, a function of the information set:

$$S(t) = f(I(t))$$

- Let's assume N is the number of possible investment assets, a single choice could be defined as a signal s_t^i:

$$S(t) = (s_t^1, s_t^2, .., s_t^N)$$

- If N=1, the strategy is a timing-strategy
- If N $>$ 1 the strategy is cross-assets

Building Strategies – Market Positions

- **Long position** -> buy an asset with the perspective of an increase of value

Asset Price

Pay-off of long position

- P_0: asset price at t = 0 (buying date)
- P_T: asset price at t = T (selling date)

$$r = \frac{P_T}{P_0} - 1$$

- **Short position** -> sell an asset with the perspective of a decrease of value

Pay-off of short position

- P_0: asset price at t= 0 (loan + selling asset)
- P_T: asset price at t = T (buying)

$$r = 1 - \frac{P_T}{P_0}$$

Building Strategies – Market Positions

- Long-Only Strategy → Take a long position on one or more assets (ex. N = 1)

:Opening Long Position

Closing Long Position

Strategy

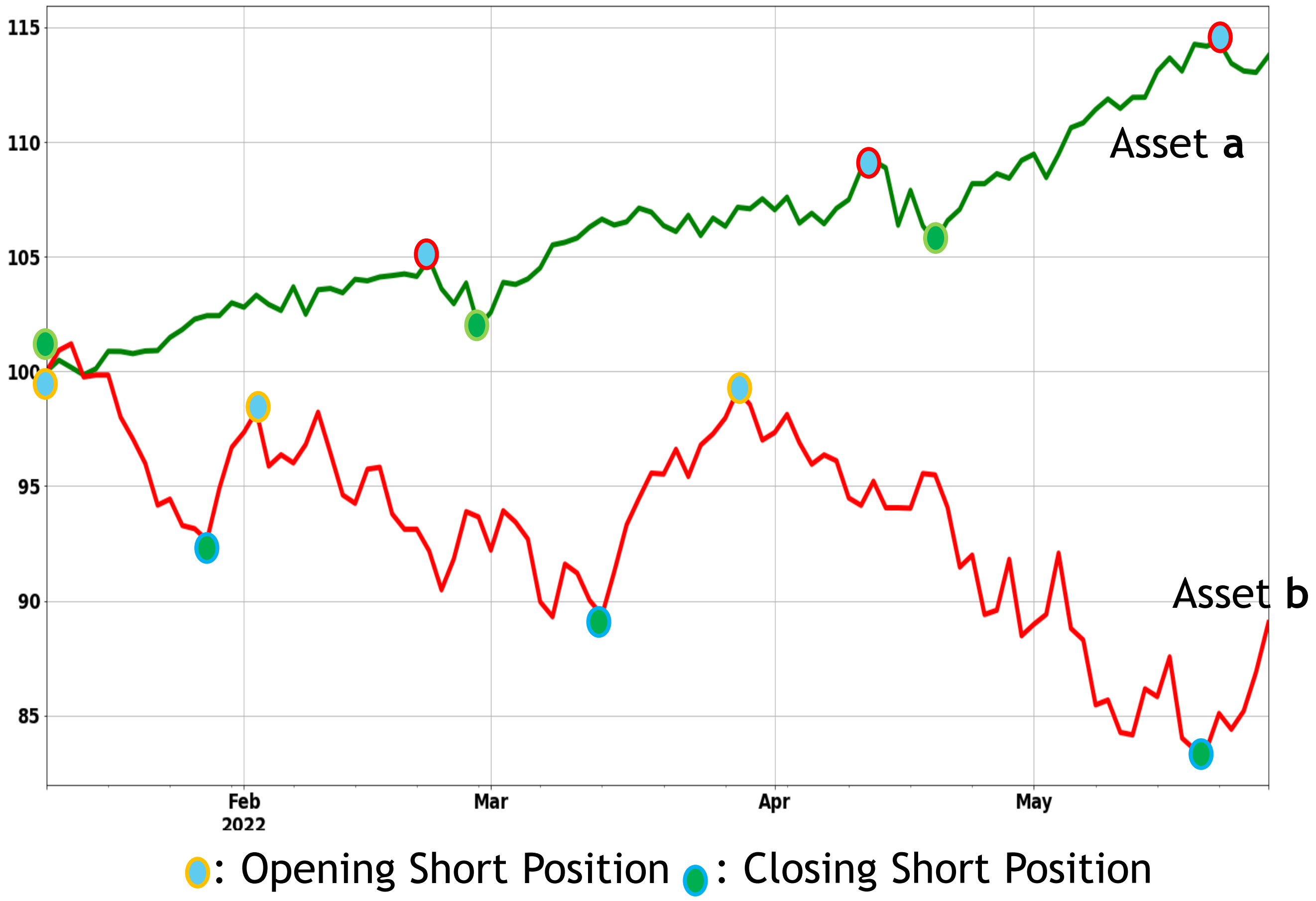
$$\begin{cases} s_t = 1 & \text{Open} < \text{t} < \text{Close} \\ s_t = 0 & \text{otherwise} \end{cases}$$

Payoff

$$X_\tau = \prod_{t=1}^{\tau} (1 + s_{t-1} r_t)$$

Building Strategies – Market Positions

- Long-Short Strategy → Taking simultaneously a long position on one or more asset and a short position on one or more asset (ex N = 2)



Long – Short Strategy

$$
\begin{cases}
s_t^a = 1 & \text{OpenLongPos} < t < \text{CloseLongPos} \\
s_t^b = -1 & \text{OpenShortPos} < t < \text{CloseShortPos} \\
s_t^b = s_t^a = 0 & \textit{Otherwise}
\end{cases}
$$

Payoff

$$X_\tau = \prod_{t=1}^{\tau} (1 + s_{t-1}^a r_t^a + s_{t-1}^b r_t^b)$$

Performance evaluation

- Once the strategy is defined, one can calculate the ***Equity Curve***

$$X_\tau = \prod_{t=1}^{\tau} \left(1 + \sum_{i=1}^{N} s_{t-1}^{i} r_{t}^{i} \right)$$

- $X_0 = 100$
- s_t^i: signal at time t, where i = (1, …, N) represent the i-th asset
- r_t^i is the return of the i-th asset in the interval (t-1,t)
- N, is the number of the assets

Performance evaluation

- In order to evaluate the performance of the strategy, we can calculate some performance metrics, i.e. functions of the equity curve.
- These metrics can be divided in three groups:

  I. ***Reward metrics***: Annualized returns
  II. ***Risk metrics***: Annualized Volatility, Maximum Drawdown
  III. ***Risk/Reward metrics***: Sharpe Ratio, Calmar Ratio

- **Annualized Returns:** Annualized return is the rate of return on an investment over a period of one year taking into account the effects of compounding. It allows investors to compare the performance of different investments over various time periods on a standardized basis.

$$AnnRet = \sqrt[T/250]{\frac{X_T}{X_0}} - 1$$

Performance evaluation

- **Annualized Volatility**: it is a statistical measure of the ***dispersion of returns*** of a financial instrument over a given period, expressed in terms of an annualized standard deviation. It is used to quantify the risk of an investment or a portfolio by indicating how much the value of an investment is likely to fluctuate over a given period.

$$AnnVol = Std\left(\frac{X_t}{X_{t-1}} - 1\right)$$

- **Maximum Drawdown:** Maximum drawdown is a specific measure of drawdown that looks for the greatest movement from a high point to a low point, before a new peak is achieved. It is an indicator of downside risk over a specified time period.

$$DD = min\left\{\frac{X_t}{X_{max}} - 1 : t = 1, \dots, T\right\}$$

Performance evaluation

- ***Sharpe Ratio***: it measures the return of an investment such as a security or portfolio compared to a risk-free asset, after adjusting for its risk. The ratio describes how much excess return you receive for the extra volatility you endure for holding a riskier asset.

$$Sharpe = \frac{AnnRet - RiskFree}{AnnVol}$$

- ***Calmar Ratio:*** it is a measure of risk-adjusted returns, like Sharpe ratio. But instead of using volatility to asses risk, it uses the maximum drawdown.

$$Calmar = \frac{AnnRet \; - RiskFree}{MaxDD}$$

Example

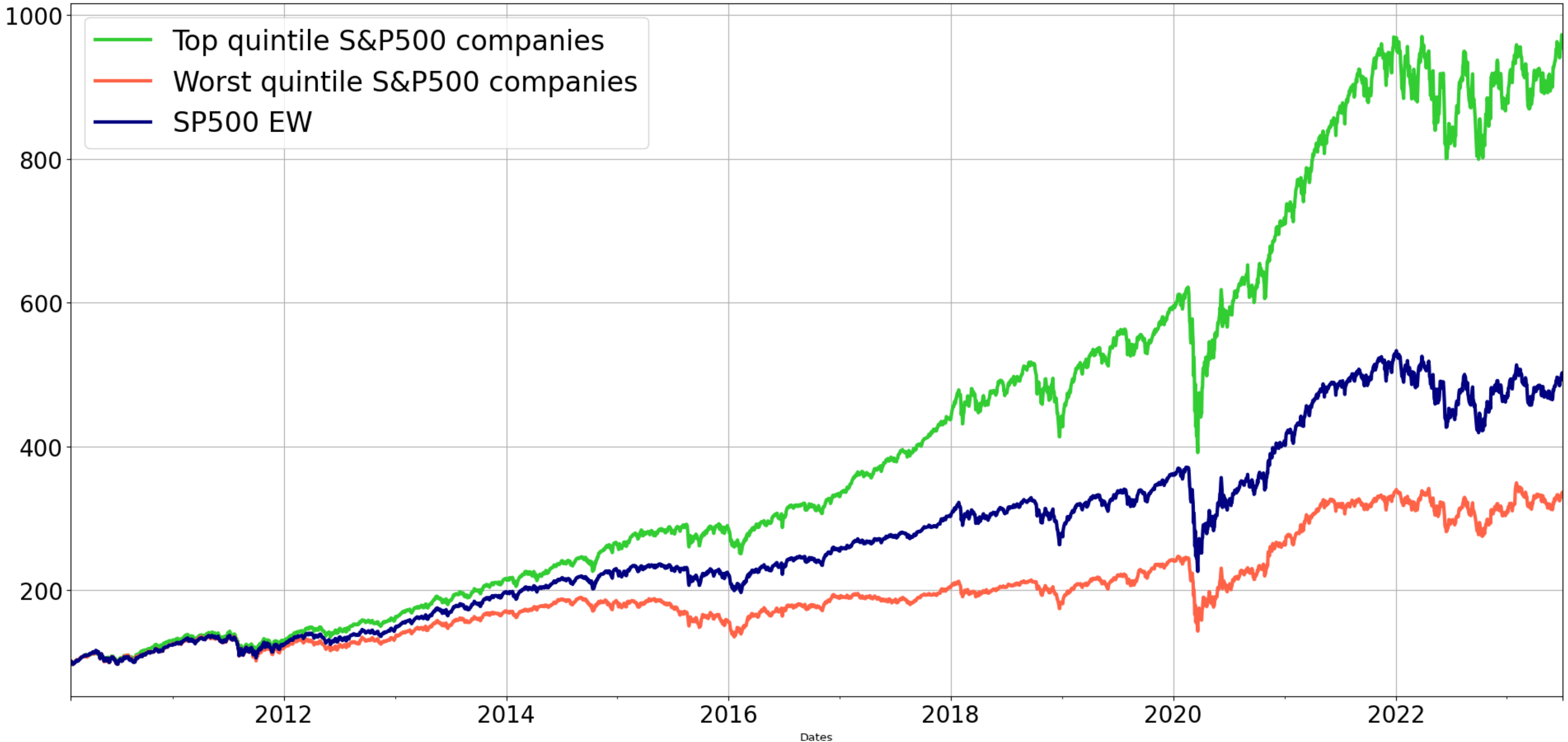
- Universe: Constituents of SP500
- Total wealth = 100
- At the end of each month, we can rank the 500 companies from the best to the worst (following some criteria). Thus we can create 5 quintiles (each containing 100 companies)
- We take the top quintile and the worst quintile and we create two portfolios.
- In particular we build two equally weighted portfolios. That means that we invest the same amount of wealth in each company, i.e. they all have the same weight in the portfolio.

$$R(P) = \sum_{i=1}^{100} w_i r_i \quad with\, w_i = \frac{1}{100}$$

- From the daily returns we can calculate the equity curve

Example

	Top quintile	Worst quintile	SP500 EW
Sharpe Ratio	0,94	0,46	0,65
Drawdown	-37,01%	-42,05%	-38,99%
Calmar Ratio	0,46	0,21	0,30
AnnRet	16,86%	8,66%	11,70%
AnnVol	17,92%	18,81%	17,96%



Example

	Strategy A	Strategy B
Sharpe Ratio	1,02	0,83
Drawdown	-4,48%	-8,27%
Calmar Ratio	0,51	0,43
AnnRet	2,27%	3,54%
AnnVol	2,22%	4,25%

- Now we construct two long-short strategies:
- Strategy A: Long position on the top quintile, short position on the SP500 equally weighted index.
- Strategy B: Long position on the top quintile, short position on the worst quintile.

Example

	Strategy A	Strategy B
Sharpe Ratio	1,02	0,83
Drawdown	-4,48%	-8,27%
Calmar Ratio	0.51	0.43
AnnRet	2,27%	3,54%
AnnVol	2,22%	4,25%

?

- Very different values respect to previous example. Why?
- There exist another block to use when building a strategy, that is the ***leverage***.
- The leverage controls the exposition to the market.
- The exposition to the market controls the amount of the portfolio exposed to the market risk (systemical).

- Before we had the portfolio exposed at 100% $R(P) = 1 * \sum_{i=1}^{100} w_i r_i$
- Now, instead we have exposed the long leg at the 50% and the short leg (that counterbalance the long) at 50%, resulting a final exposure at 0%.

$$R(P) = 0.5 * \sum_{l=1}^{100} w_l r_l \; - 0.5 * \sum_{s=1}^{100} w_s r_s$$

- That is why we have lower returns and lower volatilities.

cf Matlab code "Strategy_lesson1.m"

Timing Strategy with Moving Averages – Example B

- **Objective:** Buy SPX Index when it shows a positive trend and sell it when its trend is negative, using moving averages crossover.

- Why **Moving Average crossover**?

  - We need **two moving averages**, one that reflects a **short-term movement** of the asset (30/50 days) and one that reflects a **long-term movement of the asset** (90/200 days)

cf Matlab code "Strateg_lesson1.m"

  - The short-term moving average reflects the average price of the last 50 days. If this is higher than the long-term moving average, it means that more recent prices are higher compared to those from the more distant past (last 200 days)

  - In other words, the market has shown an upward trend in the last 50 days compared to the previous 200 days

  - The strategy is based on the idea that markets tend to follow trends, and that an upward trend is more likely to continue when recent forces (prices over the last 50 days) are stronger than long-term trends (last 200 days). This is often described as **momentum**.

Timing Strategy with Moving Averages

- Universe: S&P500 Index

- Total wealth = 100

  - Buy Signal: If MA(50d) > MA(200d)
  - Out Signal: If MA(50d) < MA(200d)

  "implementation of the strategy"

- The strategy is daily, thus each day we are going to check if the MA(50d) is greater or lower than the MA(200d) defining the signal s and we can compute the return of the strategy as:

$$R(P) = \sum_{i=1}^{T} s_i r_i \quad \begin{cases} s_i = 1 \text{ if MA(50d)} > \text{MA(200d)} \\ s_i = 0 \text{ if MA(50d)} < \text{MA(200d)} \end{cases}$$

- From the daily returns we can calculate the equity curve