4.3 Models of jump-diffusion type

A Lévy process of jump-diffusion type has the following form:

$$X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \tag{4.11}$$

where $(N_t)_{t\geq 0}$ is the Poisson process counting the jumps of X and Y_i are jump sizes (i.i.d. variables). To define the parametric model completely, we must now specify the distribution of jump sizes $\nu_0(x)$. It is especially important to specify the tail behavior of ν_0 correctly depending on one's beliefs about behavior of extremal events, because as we have seen in Chapter 3, the tail behavior of the jump measure determines to a large extent the tail behavior of probability density of the process (cf. Propositions 3.13 and 3.14).

In the *Merton model* [291], jumps in the log-price X_t are assumed to have a Gaussian distribution: $Y_i \sim N(\mu, \delta^2)$. This allows to obtain the probability density of X_t as a quickly converging series. Indeed,

$$\mathbb{P}\{X_t \in A\} = \sum_{k=0}^{\infty} \mathbb{P}\{X_t \in A \big| N_t = k\}\mathbb{P}\{N_t = k\},$$

which entails that the probability density of X_t satisfies

$$p_t(x) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k \exp\left\{-\frac{(x-\gamma t - k\mu)^2}{2(\sigma^2 t + k\delta^2)}\right\}}{k!\sqrt{2\pi(\sigma^2 t + k\delta^2)}}. \tag{4.12}$$

In a similar way, prices of European options in the Merton model can be obtained as a series where each term involves a Black-Scholes formula.

In the *Kou model* [238], the distribution of jump sizes is an asymmetric exponential with a density of the form

$$\nu_0(dx) = [p\lambda_+ e^{-\lambda_+ x} 1_{x>0} + (1-p)\lambda_- e^{-\lambda_-|x|} 1_{x<0}]dx \tag{4.13}$$

with $\lambda_+ > 0$, $\lambda_- > 0$ governing the decay of the tails for the distribution of positive and negative jump sizes and $p \in [0,1]$ representing the probability of an upward jump. The probability distribution of returns in this model has semi-heavy (exponential) tails. The advantage of this model compared to the previous one is that due to the memoryless property of exponential random variables, analytical expressions for expectations involving first passage times may be obtained [239]. Key properties of Merton and Kou models are summarized in Table 4.3.

TABLE 4.3: Two jump-diffusion models: the Merton model and the Kou model

	Merton model	Kou model
Model type	Compound Poisson jumps + Brownian motion	
Parameters (excluding drift)	4 parameters: σ — diffusion volatility, λ — jump intensity, μ — mean jump size and δ — standard deviation of jump size	5 parameters: σ — diffusion volatility, λ — jump intensity, λ_+, λ_-, p — parameters of jump size distribution
Lévy density	$\nu(x) = \frac{\lambda}{\delta\sqrt{2\pi}} \exp\{-\frac{(x-\mu)^2}{2\delta^2}\}$	$\nu(x) = p\lambda\lambda_+ e^{-\lambda_+ x} 1_{x>0} + (1-p)\lambda\lambda_- e^{-\lambda_- \lvert x\rvert} 1_{x<0}$
Characteristic exponent	$\Psi(u) = -\frac{\sigma^2 u^2}{2} + ibu + \lambda\{e^{-\delta^2 u^2/2 + i\mu u} - 1\}$	$\Psi(u) = -\frac{\sigma^2 u^2}{2} + ibu + iu\lambda\{\frac{p}{\lambda_+ - iu} - \frac{1-p}{\lambda_- + iu}\}$
Probability density	Admits a series expansion (4.12)	Not available in closed form
Cumulants:		
$E[X_t]$	$t(b + \lambda\mu)$	$t(b + \lambda p/\lambda_+ - \lambda(1-p)/\lambda_-)$
Var X_t	$t(\sigma^2 + \lambda\delta^2 + \lambda\mu^2)$	$t(\sigma^2 + \lambda p/\lambda_+^2 + \lambda(1-p)/\lambda_-^2)$
c_3	$t\lambda(3\delta^2\mu + \mu^3)$	$t\lambda(p/\lambda_+^3 - (1-p)/\lambda_-^3)$
c_4	$t\lambda\{3\delta^3 + 6\mu^2\delta^2 + \mu^4\}$	$t\lambda(p/\lambda_+^4 + (1-p)/\lambda_-^4)$
Tail behavior of probability density	Tails are heavier than Gaussian but all exponential moments are finite	Semi-heavy (exponential) tails: $p(x) \sim e^{-\lambda_+ x}$ when $x \to +\infty$ and $p(x) \sim e^{-\lambda_- \lvert x\rvert}$ when $x \to -\infty$