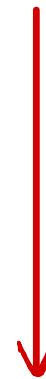


Matlab for Carr-Madan : see Lecture 7.

Carr-Madan Method

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Goal:

To price EU options
exploiting Fourier
Transform.

- Framework :
cf Notes, page 71.

$$S_t = S_0 \exp(rt + X_t) \quad \text{where } (X_t)_t \text{ is Lévy st.}$$

$$\int e^x \nu(dx) < +\infty, \quad |x| > 1$$

Requirement on a bound
for the exp. of large jumps.

(We know that in Lévy,
large jumps are limited,
but we also need
something about the exp.
of the large jumps).

$$\Psi_X(-i) = 0$$

We are risk-
neutral (under \mathbb{Q})

iff $\Psi_X(-i) = 0$.

Then the discounted
stock price is a
Martingale .

$$\mathbb{E}^{\mathbb{Q}} \left[e^{-r(T-t)} S(T) \mid \mathcal{F}_t \right] = S_t$$

- We know that :
for $t < T$.

$$\text{price}_{\text{call}}(t) = \text{price}(t) = \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T-t)} (S_T - K)^+ \mid \mathcal{F}_t \right]$$

- In 1998, Carr & Madan formula :

DEF:

- F.T of a smooth enough function f : $F(f)(v) = \int_{-\infty}^{+\infty} e^{ixv} f(x) dx$.
- Inverse Fourier Transform (I.F.T): $F^{-1}(f)(v) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ixv} f(x) dx$.
- Φ_T is the F.T of X at T : $\Phi_T(u) = E[e^{iuX_T}] = \int_{-\infty}^{+\infty} e^{iux} \rho_T(x) dx$
where ρ_T is the density of X_T . So: $\Phi_T(u) = F(\rho_T)(u)$.

⚠ In Lévy, we don't know ρ , but Φ is known, ie the FT of ρ .

So the idea of Carr & Madan is to look in the Fourier Space.

- $\Phi_T(-i) = 1$ because $\Psi_X(-i) = 0$ [ASSUMPTION] & we know
that: $\Phi_T(u) = e^{\Psi_X(u)^T}$ so $\Phi_T(-i) = e^0 = 1$.
- $\Phi_T(0) = 1$ because $\Phi_T(0) = \int_{-\infty}^{+\infty} e^0 \rho_T(x) dx = \int_{-\infty}^{+\infty} \rho_T(x) dx = 1$
 ρ_T is a density

So, using all the information of the previous slide, we want to compute the price, given by: $\mathbb{E}^Q[e^{-r(T-t)}(S_T - K)^+] | \mathcal{F}_t$ (cf slide 2).

- Let's define: $k = \log(K/S_0)$ so that, at $t=0$:

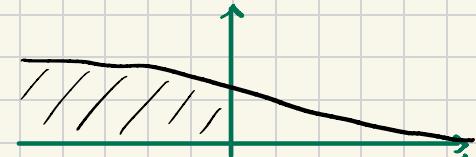
$$\begin{aligned} c(k) &= \mathbb{E}^Q \left[e^{-rT} (S_0 e^{rT + X(T)} - S_0 e^k)^+ \right] \\ &= S_0 \mathbb{E}^Q \left[e^{-rT} (e^{rT + X(T)} - e^k)^+ \right] \end{aligned}$$

$$(K = \exp(k) S_0)$$

(we can forget S_0 , take $S_0=1$ & then $\times S_0$ if it is $\neq 1$).

The goal is to compute that $\mathbb{E}[]$.

- Let's forget S_0 ($S_0=1$): $c(k) = \mathbb{E}^Q \left[e^{-rT} (e^{rT + X(T)} - e^k)^+ \right]$
- $c(k) \xrightarrow[k \rightarrow +\infty]{} 0$ and $c(k) \xrightarrow[k \rightarrow -\infty]{} \mathbb{E}^Q [e^{X(T)}] = \Phi_T(-i) = 1$ (cf slide 3).
- So this is a non-integrable function since:



The idea now is to define a new function, $c(k)$ -smoothing, such that it is integrable (in F.E we probably use $\exp(-\cdot) \times c(k)$ to do idem).

- Let's define: $\zeta(k) = c(k) - \underbrace{(1 - e^{k-rT})^+}_{\text{Nothing stochastic here: we can compute the price}} \quad \boxed{\text{C}(k) \text{ once we know } \zeta: C(k) = \zeta(k) + (1 - e^{k-rT})^+}$
- $\zeta(k) \xrightarrow{k \rightarrow +\infty} 0$
- $\zeta(k) \xrightarrow{k \rightarrow -\infty} 0$

Now we will apply the F.T of ζ .

- Let: $g(v) = \mathcal{F}(\zeta)(v) = \int_{-\infty}^{+\infty} e^{ivk} \zeta(k) dk$

Let's prove: $\zeta(k) = e^{-rT} \int_{-\infty}^{+\infty} (e^{rT+x} - e^k) (\mathbb{1}_{\{k \leq x+rT\}} - \mathbb{1}_{\{k \leq rT\}}) p_T(dx)$

$$\begin{aligned}
 &= e^{-rT} \int_{-\infty}^{+\infty} (e^{rT+x} - e^k) \mathbb{1}_{\{k \leq x+rT\}} p_T(dx) - \left(\int_{-\infty}^{+\infty} e^x - e^k e^{-rT} p_T(dx) \right) \mathbb{1}_{\{k \leq rT\}} \\
 &= e^{-rT} \int_{-\infty}^{+\infty} (e^{rT+x} - e^k)^+ p_T(dx) - \left(\underbrace{\int_{-\infty}^{+\infty} e^x p_T(dx) - e^{rT} \int_{-\infty}^{+\infty} p_T(dx)}_{=1 \text{ since we are under } Q. (\mathbb{E}[e^{X_T}] = \phi_T(-i) = 1)} \right) \mathbb{1}_{\{k \leq rT\}} \\
 &= c(k)
 \end{aligned}$$

If it's not L^1 , we should write:
 $\underbrace{p_T(dx)}$

$$\begin{aligned}
 &= c(k) - (1 - e^{k-rT})^+ \quad \blacksquare \\
 &= \zeta(k) \quad \checkmark
 \end{aligned}$$

• Let's compute the Fourier Transform using the previous formula:

$$q_T(v) = e^{-rT} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{ivk} (e^{rT+x} - e^k) (\mathbb{1}_{\{k \leq x+rT\}} - \mathbb{1}_{\{k \leq rT\}}) q_T(dx) dk$$

We switch ∫

$$= e^{-rT} \int_{-\infty}^{+\infty} \int_{rT}^{rT+x} e^{ivk} (e^{rT+x} - e^k) dk q_T(dx)$$

$$= e^{-rT} \int_{-\infty}^{+\infty} \int_{rT}^{rT+x} \left(e^{ivk + rT + x} - e^{(iv+1)k} \right) dk q_T(dx)$$

$$= \left[\frac{e^{ivk + rT + x}}{iv} - \frac{e^{(iv+1)k}}{iv+1} \right]_{rT}^{rT+x}$$

$$= \frac{e^{iv(rT+x) + rT + x}}{iv} - \frac{e^{(iv+1)(rT+x)}}{iv+1} - \frac{e^{ivrT + rT + x}}{iv} + \frac{e^{(iv+1)rT}}{iv+1}$$

$$= \frac{e^{(iv+1)(rT+x)}}{iv} - \frac{e^{(iv+1)(rT+x)}}{iv+1} - \frac{e^{(iv+1)rT+x}}{iv} + \frac{e^{(iv+1)rT}}{iv+1}$$

$$= e^{(iv+1)(rT+x)} \underbrace{\left(\frac{1}{iv} - \frac{1}{iv+1} \right)}_{\frac{1}{iv(iv+1)}} + \dots$$

So let's come back to $g_T(v)$:

$$\begin{aligned}
 g_T(v) &= \int_{-\infty}^{+\infty} e^{-rT} \left(\frac{e^{(iv+1)(rT+x)}}{iv(iv+1)} - \frac{e^{(iv+1)rT+x}}{iv} + \frac{e^{(iv+1)rT}}{iv+1} \right) \rho_T(dx) \\
 &= \int_{-\infty}^{+\infty} \left(\frac{e^{ivrT} + (iv+1)x}{iv(iv+1)} - \frac{e^{ivrT+x}}{iv} + \frac{e^{ivrT}}{iv+1} \right) \rho_T(dx) \\
 &= \frac{e^{ivrT}}{iv(iv+1)} \int_{-\infty}^{+\infty} e^{(iv+1)x} \rho_T(dx) - \frac{e^{ivrT}}{iv} \int_{-\infty}^{+\infty} e^x \rho_T(dx) + \frac{e^{ivrT}}{iv+1} \int_{-\infty}^{+\infty} \rho_T(dx) \\
 &= \frac{e^{ivrT}}{iv(iv+1)} \left(\int_{-\infty}^{+\infty} e^{i(v-1)x} \rho_T(dx) \right) \stackrel{=1}{=} + \frac{e^{ivrT}}{iv(iv+1)} \left(-iv-1 + \stackrel{=1}{i}v \right) \stackrel{=-1}{=}
 \end{aligned}$$

$$g_T(v) = \frac{e^{ivrT}}{iv(iv+1)} \left[\phi_{X_T}(v-i) - 1 \right]$$

Care Madam Formula

Then we can collect $\zeta(k)$ by inserting the formula using the IFT:

$$\zeta(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikv} g(v) dv$$

↓

$$C(k) = \zeta(k) + (1 - e^{k-rT})^+$$

RMK: it is a very general framework that can be applied to other models: 1) you compute $g(\omega)$ analytically;
we cannot compute $\int g(\omega) d\omega$ analytically,
this is analytically,
we need to approximate it!
2) you invert the formula by IFT, numerically;
3) you get the price of the EU call option $C(K)$.

End of 11/10/2022 recording

Characteristic Function

(Recap on what we did last time)

Let us consider a process X_t such that the underlying asset S_t is given by

$$S_t = S_0 e^{rt+X_t}$$

and for which the characteristic function

$$\phi_{X_t}(v) = E[e^{ivX_t}] = \int_{\mathbb{R}} e^{ivx} f_{X_t}(x) dx = \mathcal{F}f_{X_t}$$

FT of the
 pdf of X_t .

is well-known. Here f_{X_t} is the pdf, and \mathcal{F} is the Fourier transform operator.



We just assume that there exists a characteristic fn.

In the following we will use ϕ_t as a short for ϕ_{X_t} .

Moreover, in the risk-neutral measure, it must be $\phi_t(-i) = 1$. $\phi_t(-i)$

Proof: under \mathbb{Q} , $E[S_t | \mathcal{F}_0] = E[S_t] = S_0$. But $E[S_t] = E[S_0 e^{X_t}] = S_0 E[e^{i(-i)X_t}]$. ■

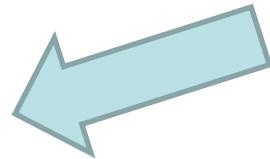
For example, considering the risk-neutral measure, for the GBM we have

$$\phi_{X_t}(v) = e^{-\frac{\sigma^2}{2}ivT - \frac{\sigma^2}{2}v^2T}$$

Even for the Heston model, the characteristic function is known analytically

Another example of a more complex model.

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^1$$



$$dV_t = \kappa(\theta - V_t)dt + \sigma \sqrt{V_t} dW_t^2$$

$$\phi_{X_T}(\omega) = e^{-B(\omega)+C(\omega)}$$

$$B(\omega) = \frac{2\zeta(\omega)(1 - e^{-\psi(\omega)T})V_0}{2\psi(\omega) - (\psi(\omega) - \gamma(\omega))(1 - e^{-\psi(\omega)T})}$$

$$C(\omega) = -\frac{2\kappa\theta}{\sigma^2} \left[2 \log \left(\frac{2\psi(\omega) - (\psi(\omega) - \gamma(\omega))(1 - e^{-\psi(\omega)T})}{2\psi(\omega)} \right) + (\psi(\omega) - \gamma(\omega))T \right]$$

$$\zeta(\omega) = -\frac{1}{2}(\omega^2 + i\omega)$$

$$\psi(\omega) = \sqrt{\gamma(\omega)^2 - 2\sigma^2\zeta(\omega)}$$

$$\gamma(\omega) = \kappa - \rho\sigma\omega i$$

Carr-Madan

- Peter Carr and Dilip B. Madan, *Option Valuation Using the Fast Fourier Transform*, 1999 → This is the initial article.
- Rama Cont and Peter Tankov, *Financial Modelling with Jump Processes*, Chapman & Hall / CRC Press, 2003 → This is the formulation that we are going to use.

Idea: compute the option price of european options (in our case, call options) moving to the Fourier space, where an analytical formula is provided.

The following notes are based on Cont-Tankov (2003), Section 11.1.3

DEFINITIONS:

Recall the definition of the Fourier transform of a function f :

$$\mathbf{F}f(v) = \int_{-\infty}^{\infty} e^{ixv} f(x) dx$$

Usually v is real but it can also be taken to be a complex number. The inverse Fourier transform is given by:

$$\mathbf{F}^{-1}f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixv} f(v) dv$$

For $f \in L^2(\mathbb{R})$, $\mathbf{F}^{-1}\mathbf{F}f = f$, but this inversion formula holds in other cases as well. In what follows we denote by $k = \ln K$ the log strike and assume without loss of generality that $t = 0$.

In order to compute the price of a call option

$$\underline{C(k) = e^{-rT} E[(e^{rT+X_T} - e^k)^+]}$$

Idea: we would like to express its Fourier transform in strike in terms of the characteristic function $\Phi_T(v)$ of X_T and then find the prices for a range of strikes by Fourier inversion. However we cannot do this directly because $C(k)$ is not integrable (it tends to a positive constant as $k \rightarrow -\infty$). The key idea of the method is to instead compute the Fourier transform of the (modified) time value of the option, that is, the function

$$\underline{z_T(k) = e^{-rT} E[(e^{rT+X_T} - e^k)^+] - (1 - e^{k-rT})^+}. \quad (11.17)$$


 $C(k)$

Let $\zeta_T(v)$ denote the Fourier transform of the time value:

$$\zeta_T(v) = \mathbf{F}z_T(v) = \int_{-\infty}^{+\infty} e^{ivk} z_T(k) dk. \quad (11.18)$$

It can be expressed in terms of characteristic function of X_T in the following way. First, we note that since the discounted price process is a martingale, we can write

$$z_T(k) = e^{-rT} \int_{-\infty}^{\infty} \rho_T(x) dx (e^{rT+x} - e^k) (1_{k \leq x+rT} - 1_{k \leq rT}).$$

Now we can compute $\zeta_T(v) = \mathcal{F}z_T(v)$.

$$\begin{aligned}
\zeta_T(v) &= e^{-rT} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx e^{ivk} \rho_T(x) (e^{rT+x} - e^k) (1_{k \leq x+rT} - 1_{k \leq rT}) \\
&= e^{-rT} \int_{-\infty}^{\infty} \rho_T(x) dx \int_{x+rT}^{rT} e^{ivk} (e^k - e^{rT+x}) dk \\
&= \int_{-\infty}^{\infty} \rho_T(x) dx \left\{ \frac{e^{ivrT}(1-e^x)}{iv+1} - \frac{e^{x+ivrT}}{iv(iv+1)} + \frac{e^{(iv+1)x+ivrT}}{iv(iv+1)} \right\}
\end{aligned}$$

The first term in braces disappears due to martingale condition and, after computing the other two, we conclude that

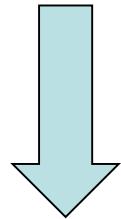
$$\boxed{\zeta_T(v) = e^{ivrT} \frac{\Phi_T(v-i) - 1}{iv(1+iv)}} \quad (11.19)$$

The martingale condition guarantees that the numerator is equal to zero for $v = 0$ and the fraction has a finite limit for $v \rightarrow 0$.

So the above formula is well defined, even for $v = 0$.

Option prices can now be found by inverting the Fourier transform:

$$z_T(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ivk} \zeta_T(v) dv \quad (11.20)$$



once $z_T(k)$ is obtained, one can get $C(k)$, call price.

$$C(k) = z_T(k) + (1 - e^{k-rT})^+.$$

Now the missing point is how to compute
11.20 in an accurate & fast way: FFT.

Using the FFT

fft Discrete Fourier transform.

fft(X) is the discrete Fourier transform (DFT) of vector X.

For length N input vector x, the DFT is a length N vector X, with elements

$$X(k) = \sum_{n=1}^N x(n) * \exp(-1i * 2 * \pi * (k-1) * (n-1) / N), \quad 1 \leq k \leq N.$$

The inverse DFT (computed by IFFT) is given by

$$x(n) = (1/N) \sum_{k=1}^N X(k) * \exp(1i * 2 * \pi * (k-1) * (n-1) / N), \quad 1 \leq n \leq N.$$

We have $\mathcal{Z}_T(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ivk} g_T(v) dv$. And we want to compute:

DFT $F_n = \sum_{k=0}^{N-1} F_k e^{-2\pi i n \frac{k}{N}} \quad (n = k+1, k = n-1)$

1st step : to truncate the integral :

$$\mathcal{Z}_T(k) \approx \frac{1}{2\pi} \int_{-A/2}^{A/2} e^{-ivk} g_T(v) dv$$

2nd step : quadrature :

$$\mathcal{Z}_T(k) \approx \frac{1}{2\pi} \frac{A}{N-1} \sum_{j=0}^{N-1} w_j e^{-iv_j k} g_T(v_j)$$

where : $v_j = -\frac{A}{2} + j \times \underbrace{\frac{A}{N-1}}_{=A}$ and w_j : weights.

Pourquoi ?? d'un coup c'est $g_T(v_j)$??

$$g_T(v_j) = g_j = \frac{1}{2\pi} \frac{A}{N-1} \sum_{j=0}^{N-1} w_j g_j e^{-ik(-\frac{A}{2} + j \frac{A}{N-1})}$$

$$= \frac{1}{2\pi} \frac{A}{N-1} e^{ikA/2} \sum_{j=0}^{N-1} w_j g_j e^{-ikj} \frac{A}{N-1}$$

Let's define : $k = \frac{2\pi n}{N \frac{A}{N-1}}$ so that :

$$F_n = \frac{1}{2\pi} \Delta e^{i \frac{\pi n(N-1)}{N}} \sum_{j=0}^{N-1} w_j g_j e^{-i \frac{2\pi n}{N} j}$$

So we find the same expression as DFT :

$$F_n = \underbrace{\frac{1}{2\pi} \Delta e^{i \frac{\pi n(N-1)}{N}}}_{\text{Known quantity}} \cdot \text{DFT}(w_j g_j)$$

We have to compute

$$z_T(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ivk} \zeta_T(v) dv.$$

Since $z_T(k)$ must be real, then it holds

) Trick of Functional Analysis.

$$z_T(k) = \frac{1}{\pi} \int_0^{+\infty} e^{-ivk} \zeta_T(v) dv$$

To compute this integral, we can use a quadrature formula, i.e.,

$$\begin{aligned} z_T(k) &\approx \frac{1}{\pi} \int_0^{A(N-1)/N} e^{-ivk} \zeta_T(v) dv \\ &\approx \frac{1}{\pi} \sum_{j=0}^{N-1} w_j \eta e^{-i\eta j k} \zeta_T(\eta j). \end{aligned}$$

If we set $w_0 = w_{N-1} = 0.5$, 1 otherwise, we are using a trapezoidal formula with nodes $j\eta$, with $\eta = A/N$.

If we now consider the following grid for the log-strike $k_l = -\lambda N/2 + \lambda l$, with $\lambda = 2\pi/(N\eta)$ and $l = 0, \dots, N-1$, we obtain

$$\begin{aligned}
 z_T(k_l) &\approx \frac{1}{\pi} \sum_{j=0}^{N-1} w_j \eta e^{-i\eta j(-\lambda N/2 + \lambda l)} \zeta_T(\eta j) \\
 &= \frac{1}{\pi} \sum_{j=0}^{N-1} w_j \eta e^{i\eta j \lambda N/2} e^{-i\eta j \lambda l} \zeta_T(\eta j) \\
 &= \frac{1}{\pi} \sum_{j=0}^{N-1} w_j \eta e^{ij\pi} e^{-ijl2\pi/N} \zeta_T(\eta j) \\
 &= \frac{1}{\pi} FFT(\{w_j \eta e^{ij\pi} \zeta_T(\eta j)\}_{j=0}^{N-1})
 \end{aligned}$$

*Not exactly the same formula, but in
the end it is the same result.*

Matlab code

```
function fii = trasf_fourier_BS (r,sig,T,v)
fii = (exp(1i*r*v*T)).*...
((characteristic_func_BS(sig,T,v-1i) ...
-1)./(1i*v.* (1+1i*v))) ;
```

```
function f = characteristic_func_BS(sig,T,y)
f = exp(1i* (-sig^(2)/2*T).*y ...
-(T*sig.^ (2)*y.^ (2))/2) ;
```

```

function [P_i] = FFT_BS(K_i)
%
% European Call - BS model
%
% [P_i] = FFT_BS(K_i) - could be a vector
% INPUT: K_i = strikes - could be a vector
% OUTPUT: P_i = prices
%
%--- model parameters
S = 100; % spot price
T = 1; % maturity
r = 0.0367; % risk-free interest rate
sig = 0.17801; % volatility
Npow = 15;
N = 2^(Npow); % grid point
A = 600; % upper bound
eta = A/N;
lambda = 2*pi/(N*eta);
k = -lambda * N/2 + lambda * (0:N-1); % log-strike grid
K = S * exp(k); % strike
v = eta*(0:N-1);
v(1)=1e-22; %correction term: could not be equal to zero
% (otherwise NaN)

```

```

%PRICING
tic
% Fourier transform of z_k
tr_fou = trasf_fourier_BS(r,sig,T,v);
% Trapezoidal rule
w = [0.5 ones(1,N-2) 0.5];
h = exp(1i*(0:N-1)*pi).*tr_fou.*w*eta;
P = S * real( fft(h)/pi + max(1-exp(k-r*T),0)); % prices
time=toc

% delete too small and too big strikes
index=find( (K>0.1*S & K<3*S) );
K=K(index); P=P(index);

```

```
% PLOT
figure
plot(K, P, 'r') ;
hold on
axis([0 2*S 0 S]) ;
xlabel('strike') ;
ylabel('option price') ;

% INTERPOLATION
P_i = interp1(K, P, K_i, 'spline');
```

22/10/2024

Topic of today : FFT & other derivatives.

14/10/2022 Recording

We want to see if we can use FFT technique

when we deal with other kinds of derivatives, like path-dependent ones.

↳ Answer: we can do something, but not as much as Carr Madan. We are not able to get closed formulas.

Let's consider for example a Barrier Option :

$$\begin{cases} M_t = \max_{0 \leq s \leq t} X_s \\ m_t = \min_{0 \leq s \leq t} X_s \end{cases}; \quad \begin{cases} S_t = S_0 e^{rt} \\ k = \log\left(\frac{B}{S_0}\right) \end{cases}; \quad b = \log\left(\frac{B}{S_0}\right) \text{ where } B \text{ is a barrier.}$$

→ let's consider an up & out call:

$$C(T; r, b) = S_0 \left(e^{rT} - \underbrace{\frac{e^{rk}}{K/S_0}}_{\text{Payoff of EU Call style}} \right)^+ \mathbb{1}_{\{M_T \leq b\}}$$

]

Payoff of our upout call, i.e., "Price @ Maturity T".

Payoff of EU Call style Barrier

So as we can see from the expression above: now, we are not only interested in the distribution of X_t anymore. We are now interested in the joint distribution of X_t and M_t .

For simplicity, we assume:

- $r = 0 \rightarrow$ NO DISCOUNT,
- $\psi_X(-i) = 0 \rightarrow$ DISCOUNTED STOCK PRICE IS A MG.

Let's move to $t < T$:

$$\bullet C(t; k, b) = \mathbb{E}^Q \left[S_0 (e^{X_t - k})^+ \mathbb{1}_{\{M_t \leq b\}} \mid \mathcal{F}_t \right],$$

WE ARE UNDER Q (cf assumptions above)

P_T : joint density of (X_T, M_T) .

$$\bullet C(0; k, b) = \int \int_{\mathbb{R}^2} S_0 (e^x - e^k)^+ \mathbb{1}_{\{y < b\}} P_T(dx, dy).$$

So if we want to make something close to CM, we need to compute this integral, but now we have joint density.

↳ Can we define a similar "bidimensional F.T"? The answer is No.

Hint (out of our scope): apply a "double FT" in k and in b :

$$\iint_{\mathbb{R}^2} e^{iuk} e^{iwb} C(0; k, b) dk db, \text{ not surprisingly we arrive to a similar}$$

result as Carr Madan:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} e^{iuk} e^{ivb} C(0; k, b) dk db = \frac{F(u-i, v)}{uv(1+...u)}, \text{ with } F \text{ the characteristic function of } (X_T, M_T).$$

MAJOR PROBLEM: we don't know F . So even if we are able to get a formula, it is not useful since we don't know F .

However, researchers made computations and discovered:

$$q \int_0^{+\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{iuk + ivb} e^{-qt} C(t; k, b) dt \quad (\text{Laplace transform})$$

||

$$\frac{\phi_q^+(v+u-i) \phi_q^-(u-i)}{uv(1+iu)}$$

with ϕ_q^\pm : "Wiener Hopf Factors".

No more
the joint
density!

$$\left\{ \begin{array}{l} \phi_q^+(u) \phi_q^-(u) = \frac{q}{q - \psi_X(u)} \\ \text{This is KNOWN.} \end{array} \right. \quad (\phi_q^+ \text{ has support in } \mathbb{R}^+, \phi_q^- \text{ in } \mathbb{R}^-)$$

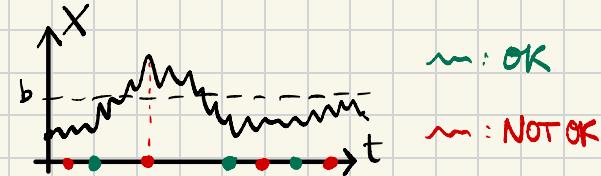
So the idea is : 1 compute $\frac{q}{q - \Psi_X(u)}$ then 2 take the inverse Laplace & two times the inverse Fourier transforms. SIMILAR IDEA AS CM.

What is the problem ? It is very hard, numerically, to compute $\frac{q}{q - \Psi_X(u)}$.

→ That's why we won't go further: It's complex & not used.

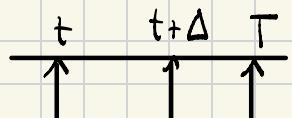
Clearly, what we have shown in ... was not the correct way to perform: so we have to change our POV.

So, our idea will be: apply FOURIER TRANSFORM TO BARRIER OPTIONS
BUT IN THE DISCRETE MONITORING CASE.
 (we check for the barrier only on some dates)



→ cf "conv"/"convolution" on WEBEEP. What we'll see now is a convolution method:

Let's consider first a EU option: $T > t$, $t + \Delta < T$.



We know that: $C(x, t) = e^{-r(T-t)} E^Q [C(X_T, T) | \mathcal{F}_t]$

we always need to be under Q .

$$X_t = x$$

And this is TRUE for any $t < T$, so it means that:

$$C(x, t) = e^{-r\Delta} \mathbb{E}^Q \left[C(X_{t+\Delta}, \underbrace{t+\Delta}_{< T}) \mid X_t = x \right]$$

$$= e^{-r\Delta} \int_{\mathbb{R}} C(y, t+\Delta) P_{x,t,\Delta}(y) dy$$

$$\uparrow$$

$$\mathbb{P}(X_{t+\Delta} = y \mid X_t = x)$$

conditional!

Indeed, this is
"conditional
expectation".

Now, in the Lévy framework: $\mathbb{P}(X_{t+\Delta} = y \mid X_t = x) = \mathbb{P}(X_\Delta = y - x)$
using the stationary and identically distributed increments.

So: $C(x, t) = e^{-r\Delta} \int_{\mathbb{R}} C(y, t+\Delta) f_\Delta(y-x) dy \quad (E) \quad [\text{Lévy framework}]$

\uparrow
density of the increments of X .

\uparrow
density of X_Δ .

DEF: [CONVOLUTION]
$$g_1(x) = \int_{\mathbb{R}} g_2(x-y) g_3(y) dy \quad (g_1 = g_2 * g_3)$$

PROP: $F(g_1) = F(g_2) \times F(g_3)$ ($* \xrightarrow{F} x$)

Let's define : $f_\Delta^b(x) = f_\Delta(-x)$ such that (E)

becomes : $c(x, t) = e^{-r\Delta} \int_{\mathbb{R}} f_\Delta^b(x-y) c(y, t+\Delta) dy$

This is written as a convolution. Now we use the above property :

$$F(c(\cdot, t)) = e^{-r\Delta} F(f_\Delta^b) \times F(c(\cdot, t+\Delta))$$

i.e : $c(\cdot, t) = F^{-1} \left[e^{-r\Delta} F(c(\cdot, t+\Delta)) \underbrace{F(f_\Delta^b)} \right]$

? Key point: what is it?

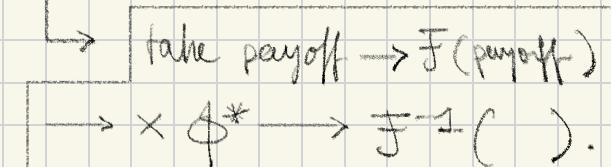
$$\mathcal{F}(f_\Delta) = \phi_{X_\Delta} \quad \leftarrow \quad " \mathcal{F}(\text{density}) = \phi ".$$

$$\mathcal{F}(f_\Delta^b) = \mathcal{F}(f_\Delta(-\cdot)) = \phi_{X_\Delta}^* \quad \leftarrow \text{conjugate (by C analysis).}$$

we know it so we can use it.

THIS IS THE SO CALLED "CONV METHOD".

CONV METHOD FOR EUROPEAN OPTION:



$$C(x, T) = S_0 (e^x - e^k)^+ . \text{ Compute :}$$

$\mathcal{F}(S_0(e^x - e^k)^+) \times \phi_{X_T}^*(u)$ and then take \mathcal{F}^{-1} .

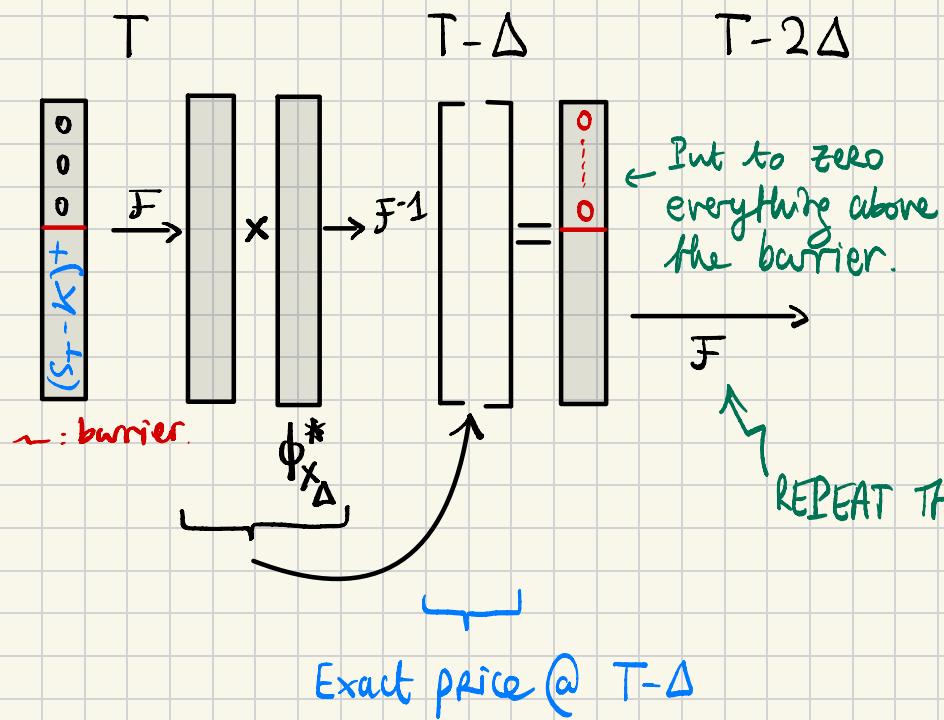
$$C(x, 0) = \mathcal{F}^{-1} \underset{u \rightarrow x}{\underset{x \rightarrow u}{\left[\mathcal{F}(S_0(e^x - e^k)^+) \times \phi_{X_T}^*(u) \right]}}$$

↑ Price @ time 0.

However, use CM if you can (more accurate).

We can exploit this method for BARRIER OPTIONS w/ DISCRETE MONITORING.

CONV METHOD FOR BARRIER OPTION:



Monitoring dates.

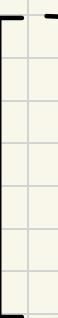
T $T-\Delta$ $T-2\Delta$... 2Δ Δ 0

$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ (S_T - K)^+ \end{bmatrix}$

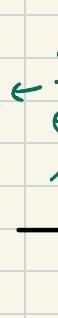
$\xrightarrow{\mathcal{F}}$



\times



$\xrightarrow{\mathcal{F}^{-1}}$



$= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$

Put to zero
everything above
the barrier.

$\xrightarrow{\mathcal{F}}$



REPEAT THE PROCESS ...

\downarrow

Exact price @ $T-\Delta$

→ The idea is that between two monitoring dates, the evaluation is the same as European option. So we repeat the process till $t=0$, taking each time into account the barrier.

of CONV-Method.pdf → fully explained method.

Now, let's look @ the code:

- we need a function to compute characteristic fct : charfunction.m

↳ with a flag if we want the complex conjugate.

↳ there is a correction to take into account the risk free rate & the potential dividends, and also to have $\phi_x(-i) = 1$ (i.e. $\Psi_x(-i) = 0$ under Q).

- main.m:

↳ $N_{date} = 12 \rightarrow$ Monthly monitoring.

↳ $N = 2112$: grid of the log returns : $\log(S_t/S_0)$.

- kernel.m: we need two grids : one for log price and one
for Fourier space.

↓
 $d\omega, \omega$ (with 2π)

N, dx, x

see why
in the CM
video.

↳ we invert the characteristic function using "fft" (⚠ not ifft)

↳ we need to correct the shift induced by MATLAB:

$h = \text{real}(\text{fftsift}(\text{fft}(\text{iifftsift}(H))))$ (shifts of the GRID)

This corrects the fact that x is $-N \rightarrow N-1$.
correction: the w grid which doesn't go from 0 to ... but from -N to N-1.

h is the density of $(-x)$ (since it was $\phi_x^* = H$).

→ see the comments that explain the shifts in grids.

- CONV.m: new algorithm that uses kernel & goes backward in times.

↳ once again we have to transform H to be in MATLAB style: $H = \text{fftshift}(H)$.

↳ if we remove the line " $V(S \leq \text{Barrier}) = 0$ " we are basically pricing a EU option.

As a conclusion, this code can be used to price a barrier option with discrete monitoring under any Lévy, exactly like we previously did with MC. And it's a general method: the only thing we have to change if we want to use another kind of Lévy is the computation of the characteristic function. Nothing else.

⚠ Important to understand why we shift: MATLAB wants $0 \rightarrow 2N$, we want $-N \rightarrow N$.

