12/11/2024

Stochastic Volatility :

Why ?

- Lévy has independent increments.
  ↳ WRONG in an econometric pov.

- Implied Volatility : in Lévy, it's deterministic (not flat as in B&S).
  ↳ Some tests suggest that it should be stochastic.

Lévy → SATO processes (1^{st} generalisation).
Lévy → Stochastic Volatility (2^{nd} generalisation).

↓

we don't want to lose the jumps :

stochastic Volatility w/ jumps

↓

Main drawback { of course we'll have more params : like 9 for Heston Model !

1) without jumps :

$$dS(t) = r S(t)\,dt + \sigma(t) S(t)\,dW(t)$$

with $(\sigma(t))_{t \geq 0}$ a process. So it's not like B&S ! $\sigma(t) = f(y(t))$ with

$$dy = (\dots)\,dt + (\dots)\,d\hat{W}(t) \quad , \text{ and}$$

$$\rho = dW\,d\hat{W} .$$

- The function form $f(\,)$.
- The dynamic of y.
- The value of ρ.

$\Big\}$ We have to decide 3 things

Let's create a table with the most famous choice.

Model :	ρ :	f :	dynamic of y : $\boxed{y(0)>0}$
Hull & White (1997)	0	$f(y)=\sqrt{y}$	$dy = C_1 y\,dt + C_2 y\,d\hat{W}_t$
Scott	0	$f(y)=e^{y}$	$dy = \lambda(\eta - y_t)dt + \beta\,d\hat{W}_t$
Heston (1993)	$\neq 0$ (usually <0).	$f(y)=\sqrt{y}$	$dy = \lambda(\eta - y_t)dt + \beta\sqrt{y}\,d\hat{W}_t$ C.I.R

ABM with mean reversion.
→ ⚠ It can become <0.

Hence...

The most (and only) used stochastic model is

HESTON MODEL.

C.I.R.: Let's recall its dynamics:

$$\begin{cases} dy_t = \lambda(\eta - y_t)\,dt + \beta\sqrt{y_t}\,d\hat{W}_t \\ y_0 > 0 \end{cases}$$

- $\mathbb{E}[y_t] = \eta + (y_0 - \eta)e^{-\lambda t}$
- $Var(y_t) = \dfrac{\beta^2\eta}{2\lambda} + \dfrac{\beta^2(y_0-\eta)}{\lambda}e^{-\lambda t} + \dfrac{\beta^2(\eta - 2y_0)}{2\lambda}e^{-2\lambda t}$

$$\xrightarrow[t\to+\infty]{} \frac{\beta^2\eta}{2\lambda}$$

That's why β is called "vol-of-vol", or "vol-of-var":

$(y_t)_t$ is the volatility since $(f(y_t))_t = (\sigma(t))_t$ and β is the vol. in y_t's SDE.

$$\boxed{\text{Feller Condition:}}$$

- if $y_0 > 0$ & $\beta^2 \leq 2\lambda\eta$ then:

$$\mathbb{P}(y_t = 0) = 0 \quad \& \quad \mathbb{P}(y_t < 0) = 0;$$

- if $y_0 > 0$ & $\beta^2 > 2\lambda\eta$ then:

$\mathbb{P}(y_t = 0) > 0$ & $\mathbb{P}(y_t < 0) = 0$.

Integrated C.I.R: we'll use it in the last lecture.

$Y_t = \int_0^t y_s \, ds$; $(X_t)_t$ is a Lévy.

Time Change: $Z_t = X_{Y_t}$ $\forall t \geq 0$. ⚠ NOT A LÉVY. It's a stochastic volatility process.

- $\mathbb{E}[Y_t] = \eta t + \dfrac{(y_0 - \eta)(1 - e^{-\lambda t})}{\lambda}$

(Not mean-reverting since $\xrightarrow[t \to +\infty]{} +\infty$).

DM uses his computer (2023 → Heston).

He copy-pastes 2023 HESTON to the 2024 folder.

DM uses paper: "Efficient simulation of the Heston Stoch. Vol. model" (Andersen)

Let's write a code to sample CIR using Euler scheme. This way:

$$t_i = i\Delta t \qquad \int_{t_i}^{t_{i+1}} dy_t = \int_{t_i}^{t_{i+1}} \lambda(\eta - y_t)\,dt + \int_{t_i}^{t_{i+1}} \beta\sqrt{y_t}\,d\hat{W}_t$$

so:

$$y_{t_{i+1}} \hat{=} y_{t_i} + \lambda(\eta - y_{t_i})\Delta t + \beta\sqrt{y_{t_i}}\sqrt{\Delta t}\, Z$$

where $Z \sim \mathcal{N}(0,1)$.

cf "CIR_simulation.m" + "Heston.m"

↳ What happened if:
- $\eta \nearrow$: increases;
- $\lambda \nearrow$: more rapidly reaches the value of η and remains around.

↳ if we zoom we can see a non-null proba to go <0, whereas the theory says it's impossible. ⚠ Discretization error. ⟶ See the warning in the console "Imaginary part..".

↳ we can try to fix this: via a $|\cdot|$ or $\max(\cdot, 0)$.

Back to the slides : "the Heston model is defined by the coupled ... SDE :".

↳ without drift, but it's okay !

↳ Risk Neutral slide.

↳ Euler Scheme slide.

⇝ The slides recap everything.

↳ Advanced Scheme : we'll use the article from Leif Andersen, 2007.

↳ cf slides where we have screenshots.

↳ cf "CIR_Andersen.m"

See also "Heston_Pricing.m"

↓

"FFT_Heston.m" we can implement CM for Heston since we know its characteristic function (cf slides on CM).

Let the price of a call $C(t, S, v)$. Heston:

$$\begin{cases} dS(t) = \mu S(t)\,dt + \sqrt{v(t)}\, S(t)\, dZ_1(t) \\ d\left(\sqrt{v(t)}\right) = -\beta \sqrt{v(t)}\,dt + \delta\, dZ_2(t) \end{cases}$$

with $\quad dZ_1(t)\, dZ_2(t) = \rho\, dt$.

By applying Itô's Lemma

$$\begin{cases} dv(t) = \left(\delta^2 - 2\beta v(t)\right) dt + 2\delta \sqrt{v(t)}\, dZ_2(t) \\ \qquad\;\; = \underbrace{2\beta}_{\lambda}\Big(\underbrace{\tfrac{\delta^2}{2\beta}}_{\eta} - v(t)\Big) dt + \underbrace{2\delta}_{\theta} \sqrt{v(t)}\, dZ_2(t) \\ \text{with} \quad \lambda = 2\beta, \quad \eta = \dfrac{\delta^2}{2\beta}, \quad \theta = 2\delta. \end{cases}$$

Given this, due to Itô's Lemma, we can show that $C(t, S, v)$ satisfies the following PDE.

$$\frac{\partial C}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 C}{\partial S^2} + \rho \sigma v S \frac{\partial^2 C}{\partial S \partial v} + \frac{1}{2} \varepsilon^2 v \frac{\partial^2 C}{\partial v^2} + rS \frac{\partial C}{\partial S} +$$

$$\Big[\lambda(\eta - v) - \underbrace{\hat{\lambda}(t, S, v)}_{\lambda v}\Big] \frac{\partial C}{\partial v} - rC = 0$$

$\boxed{\text{why } \hat{\lambda}\text{?}}$

λv : ASSUMPTION on the shape of $\hat{\lambda}$.

We can hedge against S, but not against v! $\Big\{$ MARKET RISK PREMIUM

⇝ Some people might tell you that it is possible to hedge against the variance. If you think it is possible, then remove the term $\hat{\lambda}(t,S,v)$: there is no risk premium.

& If it's not possible → then we need to estimate $\hat{\lambda}(t,S,v) = \lambda v$ → what is λ ?

In our market, we will consider that there are enough instruments to hedge against the variance v. Therefore, there is no MARKET RISK PREMIUM.

EXAMPLE: Risk Neutral Valuation in Incomplete Market:

Assume R : underlying asset, w/ simple dynamic:

$$dR = \mu dt + \sigma dW_t$$

Take a derivative : $F(R(T),T) = \Phi(R(T))$
(at maturity, price = payoff).

Assume that you can buy/sell the derivative, but not R (it is not liquid). So you have no possibility to fully hedge against R. Hence "incomplete market".

How to build a "Δ-hedging" strategy here? Let's assume $\exists$ another derivative written on R: $F_2(R(T),T) = \Phi(R(T))$.

Let's build the strategy:

$$\Pi = F(R(t),t) - \Delta F_2(R(t),t)$$

Then: $$d\Pi = \left[\frac{\partial F}{\partial t} + \mu \frac{\partial F}{\partial R} + \sigma^2 \frac{\partial^2 F}{\partial R^2}\right] dt + \sigma \frac{\partial F}{\partial R} dW_t$$

$$- \Delta \left(\frac{\partial F_2}{\partial t} + \mu \frac{\partial F_2}{\partial R} + \frac{\sigma^2}{2} \frac{\partial^2 F_2}{\partial R^2} \right) dt - \Delta \sigma \frac{\partial F_2}{\partial R} dW_t$$

- If R was on the market:

$\Pi = F(R(t),t) - \Delta R$ so

$$d\Pi = \left[(\dots)dt + \sigma \frac{\partial F}{\partial R} dW_t\right] - \Delta (\mu dt + \sigma dW_t)$$

I want a non-risky portfolio, i.e.,

$$\boxed{\Delta = \frac{\partial F}{\partial R}}$$

→ VERY SIMPLE.

But what happens when R NOT in the mkt:

$$\boxed{\Delta = \frac{\frac{\partial F}{\partial R}}{\frac{\partial F_2}{\partial R}}}$$

Then $d\Pi = \left[\frac{\partial F}{\partial t} + \cancel{\mu\frac{\partial F}{\partial R}} + \frac{\sigma^2}{2}\frac{\partial^2 F}{\partial R^2} - \underbrace{\cancel{\Delta\mu\frac{\partial F_2}{\partial R}}}_{\cancel{\mu\frac{\partial F}{\partial R}}}\right]dt + \dots$

Given that it's risk-free, now:

$$d\Pi = r\Pi\,dt$$

it should have this evolution otherwise there is an arbitrage.

So $$\underbrace{Fr\cancel{dt} - \frac{\partial F}{\partial R} rR\cancel{dt}} = \left(\frac{\partial F}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 F}{\partial R^2}\right)\cancel{dt}$$

$$d\Pi = r\Pi\, dt = r(F - \Delta R)\, dt$$

So in case where R is on the mkt:

$$\boxed{\frac{\partial F}{\partial t} + rR\frac{\partial F}{\partial R} + \frac{\sigma^2}{2}\frac{\partial^2 F}{\partial R^2} - Fr = 0}$$

<u>scheme :</u> "Δ hedging"

- write Π
- write $d\Pi$
- ensure $d\Pi$ is not risky :

  $$? \Delta \text{ s.t } \underbrace{(\ldots)}_{=0} dW_t$$

- $d\Pi = r\Pi\, dt$ (to avoid arbitrage)
  and with previous expression it gives us the PDE.

But we are <u>not</u> where R is on the market:

Let's take $\Delta = \frac{\partial F}{\partial R} \Big/ \frac{\partial F_2}{\partial R}$, we get that:

$$d\Pi = \left[\left(\frac{\partial F}{\partial t} + \cancel{\mu \frac{\partial F}{\partial R}} + \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial R^2} \right) - \frac{\frac{\partial F}{\partial R}}{\frac{\partial F_2}{\partial R}} \left(\frac{\partial F_2}{\partial t} + \cancel{\mu \frac{\partial F_2}{\partial R}} + \frac{\sigma^2}{2} \frac{\partial^2 F_2}{\partial R^2} \right) \right] dt$$

No arbitrage condition ($d\Pi = r\Pi dt$)

$$= rF\,dt - \frac{\partial F}{\partial R} \Big/ \frac{\partial F_2}{\partial R} (rF_2)\, dt$$

Let's arange the terms:

$$\frac{\frac{\partial F}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial R^2} - rF}{\frac{\partial F}{\partial R}} = \frac{\frac{\partial F_2}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial R^2} - rF_2}{\frac{\partial F_2}{\partial R}}$$

So this is constant :

$$\frac{\dfrac{\partial F}{\partial t} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 F}{\partial R^2} - rF}{\dfrac{\partial F}{\partial R}} = \hat{\lambda} \qquad \text{i.e}$$

hard task to calibrate

$$\frac{\partial F}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 F}{\partial R^2} - rF = \hat{\lambda}\frac{\partial F}{\partial R}.$$

$q - r = \hat{\lambda}$

Sometimes it is written :

$$\frac{\partial F}{\partial t} + (r - q)\frac{\partial F}{\partial R} + \frac{\sigma^2}{2}\frac{\partial^2 F}{\partial R^2} - rF = 0.$$

"Market risk premium" (because we cannot do a perfect hedging).

→ It was just to understand where $\hat{\lambda}$ comes from

We start from Heston ... (1993)

↳ We get a PDE* → Price EU $C(T, S, v) = (S-K)^+$

→ get a PDE in logprice $F(t, x, v)$

* $\hat{\lambda} \neq 0$ incomplete

$\hat{\lambda} = 0$ complete

Call : $F(T, x, v) = (S_0 e^{x} - K)^+$

$F(T, x, v) = e^{iux}$: what happens if I do this ?

We will show that we get the CHARACTERISTIC FUNCTION.

(for us : $F(t, x, v) = \mathbb{E}^{\mathbb{Q}}\left[e^{iuX_T} \middle| X_t = x\right]$,

$F(0, x, v) = \mathbb{E}^{\mathbb{Q}}\left[e^{iuX_T} \middle| X_0 = x\right] = \phi_{X_T}(u)$

and : $F(T, x, v) = \mathbb{E}^{\mathbb{Q}}\left[e^{iuX_T} \middle| X_T = x\right] = e^{iux}$).