Brownian subordination:

Let $(S_t)_{t\geq 0}$ be a subordinator with Laplace exponent $\ell(u)$, $(W_t)_{t\geq 0}$ a Wiener process, independent of $(S_t)_t$.

We define:

$$X_t = \mu S_t + \sigma W_{S_t}$$

$\mu t + W_t$ $(\mu, 1, 0)$- Lévy

$\big\downarrow S_t$

$\mu S_t + W_{S_t}$

This Lévy process has characteristic exponent: $\Psi_X(u) = \ell\Big(\underbrace{i\mu u - u^2\frac{\sigma^2}{2}}_{\text{charac. exponent of}}\Big)$.

Now we have to choose $(S_t)_{t\geq 0}$? We will take an α-stable subordinator.

We construct the class of:

"NORMAL TEMPERED STABLE PROCESSES."

NORMAL	TEMPERED	STABLE
We start by a BM: $\mu t + \sigma W_t$	We use tempered to model tails	$(S_t)_t$ used as an α-stable subordinator

BROWNIAN SUBORDINATION.

This is the class of processes that we are going to study.

- **α stable subordinator:**

$$\rho(x) = \frac{A}{x^{1+\alpha}} \mathbb{1}_{\{x>0\}}, \quad \alpha \in [0,1). \qquad S_t$$

- **tempering:**

$$\tilde{\rho}(x) = \frac{Ae^{-\lambda_+ x}}{x^{1+\alpha}} \mathbb{1}_{\{x>0\}}, \quad A, \lambda_+ > 0. \qquad \tilde{S}_t$$

→ Putting things together, we obtain the process: $\mu \tilde{S}_t + \sigma W_{\tilde{S}_t}$ N.T.S. Process, which is again Lévy.

N: normal, T: tempered, S: stable

What are the meanings of the parameters:

A : number impacting on the intensity of all jumps.

λ_+ : impacts on large jumps.

α : impacts on small jumps.

What is the Laplace exponent of $(\tilde{S}_t)_{t\geq 0}$?

$$\tilde{\ell}(u) = A\Gamma(-\alpha)\left[(\lambda-u)^{\alpha} - 1\right] \text{ where}$$

$$\boxed{\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t}\,dt \quad \text{(GAMMA FCT)}}.$$

Now, let's take : $\mu\tilde{S}_t + W_{\tilde{S}_t}$.

What if want to make a MC simulation ?

(*) Simulate $(\tilde{S}_t)_t$: we need pdf $\tilde{S}_t$.

(*) Simulate $(W_{\tilde{S}_t})_t$.

NOT ALWAYS KNOWN.

It is known only for $\alpha = 0$, $\alpha = \frac{1}{2}$

Fix $t>0$:

$\alpha = 0$: $\tilde{S}_t \sim$ GAMMA

$$p_t^{\tilde{S}}(x) = \frac{\lambda_+^{At}}{\Gamma(At)} x^{At-1} e^{-\lambda_+ x}$$

icdf('gamma', rand, ...) (*)

pdf known (only 2 cases)

Fix $t>0$:

$\alpha = \frac{1}{2}$: $\tilde{S}_t \sim$ INVERSE GAUSSIAN

$$p_t^{\tilde{S}}(x) = \frac{At}{x^{3/2}} \exp\left(2At\sqrt{\pi\lambda_+} - \lambda_+ x - \pi\frac{A^2t^2}{x}\right).$$

icdf ('inverse gaussian', rand, ...) (*)

MATLAB { (*) As homework, look in MATLAB to the help page, to understand the meaning of this last parameter. How to sample these 2 pdfs.

DEF:

- **VARIANCE GAMMA:** $\alpha = 0$

$$\left(\mu \tilde{S}_t + \sigma W_{\tilde{S}_t}\right)_t \quad \text{with} \quad \forall t,\ \tilde{S}_t \sim \text{GAMMA}.$$

- **NORMAL INVERSE GAUSSIAN:** $\alpha = 1/2$

$$\left(\mu \tilde{S}_t + \sigma W_{\tilde{S}_t}\right)_t \quad \text{with} \quad \forall t,\ \tilde{S}_t \sim IG.$$

These are the 2 most famous examples of: INFINITE ACTIVITY LÉVY PROCESSES.

$(\gamma, 0, \nu)$ ↑ NO BM.

But we can add a BM:

$$X_t = \mu \tilde{S}_t + \sigma W_{\tilde{S}_t} + \tilde{\sigma} \tilde{W}_t \quad \text{where}$$

$$dW_t \, d\tilde{W}_t = 0 \, dt \longrightarrow (\gamma, \tilde{\sigma}, \nu)$$

↑ W/ BM.

Extended VG. Extended NIG.

Do we really need a Brownian motion?
NO: because the infinitesimal small jumps "mimic" a Brownian Motion.
⟶ We will prove this in a few weeks.

★ $\left(\mu \tilde{S}_t + \sigma W_{\tilde{S}_t}\right)_t$ is RICH ENOUGH to model the evolution of the log-price.

Algorithm for MC of $(\mu\tilde{S}_t + \sigma W_{\tilde{S}_t})_t \overset{\text{"}}{=} (X_t)_t$:

- N: # of timesteps
- $t_1 = i\Delta t$, $i \in \{0, \dots, N\}$ and $\Delta t = T/N$.
- $X_0 = 0$

$\longrightarrow$ simulate $\tilde{S}_{\Delta t}$

$\longrightarrow$ simulate $z \sim \mathcal{N}(0,1)$

$\longrightarrow$ $X_{\Delta t} = X_0 + \mu\tilde{S}_{\Delta t} + \sigma\sqrt{\tilde{S}_{\Delta t}}\, z$

$W_{\tilde{S}_t} \sim \mathcal{N}(0, \tilde{S}_t)$
$\overset{(d)}{=} \sqrt{\tilde{S}_t}\, \mathcal{N}(0,1)$.

Then, once we have that, we want:

$\longrightarrow$ Simulate $\tilde{S}_{\Delta t}$ (*)

$\longrightarrow$ Simulate $z \sim \mathcal{N}(0,1)$

$\longrightarrow$ $X_{2\Delta t} = X_{\Delta t} + \mu\underline{(\tilde{S}_{2\Delta t} - \tilde{S}_{\Delta t})} + \sigma\sqrt{\tilde{S}_{\Delta t}}\, z$

(*) $\sim \tilde{S}_{\Delta t}$ because it's LEVY !

$\boxed{\text{etc.}}$

of above [So the KEY point is to be able to simulate $\tilde{S}_{\Delta t}$: which is one HOMEWORK.

RECAP OF WHERE WE ARE :

So, in our framework we have:

$$\left.\begin{aligned} S_t &= S_0 e^{X_t} \\ (X_t)_t &\ \text{LEVY} \end{aligned}\right\}$$ And we have seen, for $(X_t)_t$:

LEVY Proc. that we usually consider to model log. prices of a stock

- $\longrightarrow \mu t + \sigma W_t$ (LEVY) $\longrightarrow$ B&S Model.
- $\longrightarrow \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$: JUMP DIFFUSION → Merton Model. / Kou Model.
- $\longrightarrow \mu \tilde{S}_t + \sigma W_{\tilde{S}_t}$: INFINITE ACTIVITY → NIG Model. / VG Model.
- $\longrightarrow \mu \tilde{S}_t + \sigma W_{\tilde{S}_t} + \tilde{\sigma} \tilde{W}_t$: INFINITE ACTIVITY → Extended NIG. / Extended VG.
  $(dW_t \, d\tilde{W}_t = 0 dt)$
- $\longrightarrow$ α-stable process (generalized, tempering).

impacts both positive & negative small jumps.

Now, let's take $\alpha \in [0,1)$;

$$\nu(x) = \frac{C_-}{|x|^{1+\alpha}} e^{-\lambda_-|x|} \mathbb{1}_{\{x<0\}} + \frac{C_+}{x^{1+\alpha}} e^{-\lambda_+ x} \mathbb{1}_{\{x>0\}}.$$

1D lévy process which comes from an α-stable + tempering process. Where $C_-, C_+, \lambda_-, \lambda_+ > 0$.

DEF:

WE CAN GENERALIZE EASILY:

$\alpha_-, \alpha_+ \in [0,1)$.

$$\nu(x) = \frac{C_-}{|x|^{1+\alpha_-}} e^{-\lambda_-|x|} \mathbb{1}_{\{x<0\}} + \frac{C_+}{x^{1+\alpha_+}} e^{-\lambda_+ x} \mathbb{1}_{\{x>0\}}$$

Generalized Tempered α-stable PROCESS.

$\downarrow$ 1D Lévy process: generalized α-stable + tempering.

Now we can construct a Lévy (γ, σ, ν) process and use it with that ν Lévy measure in order to work in a Lévy framework.

Infinite number of infinitesimal jumps mimic Wiener process ?

Infinite-activity (I.A) Lévy processes can be "well approximated" by a J.D process ?

- Let's take $(X_t)_{t\geq 0}$ Lévy $(\gamma, 0, \nu)$. The Lévy Itô decomposition tells us that:

$$X_t = \gamma t + \sum_{s\leq t} \Delta X_s \mathbb{1}_{\{|\Delta X_s|>1\}} + \lim_{\varepsilon \to 0^+} N_t^{\varepsilon}$$

where $N_t^{\varepsilon} = \sum_{s\leq t} \Delta X_s \mathbb{1}_{\{\varepsilon \leq |\Delta X_s| \leq 1\}} - t\int_{\varepsilon \leq |x| \leq 1} x\,\nu(dx)$.

- Let's take $\bar{\varepsilon} > 0$,

Compound Poisson + drift.

$$X_t^{\bar{\varepsilon}} = \gamma t + \sum_{s\leq t} \Delta X_s \mathbb{1}_{\{|\Delta X_s|>1\}} + N_t^{\bar{\varepsilon}}$$

It's a FA process because it has not an ∞ nb of infinitesimal jumps.

Is there a theory which tells us what is the error when considering the second process instead of the first ?

DEF: $R_t^{\bar{\varepsilon}} = \lim_{\varepsilon \to 0^+} N_t^{\varepsilon} - N_t^{\bar{\varepsilon}} = X_t - X_t^{\bar{\varepsilon}}$.

$$\underbrace{\underset{\text{Lévy}}{\underset{\uparrow}{X_t}} \quad - \quad \underset{\text{Lévy}}{\underset{\uparrow}{X_t^{\bar{\varepsilon}}}}}_{\text{linear transfo. of Lévy} \Rightarrow \text{LÉVY.}}$$

$(R_t^{\bar{\varepsilon}})_t$ is a Lévy $\left(0, 0, \mathbb{1}_{0<|x|<\bar{\varepsilon}} \, \nu(dx)\right)$.

THM: Let $(X_t)_t$ (γ, A, ν) Lévy. $\forall t \geq 0$,

$$\begin{cases} \mathbb{E}[X_t] = t\left(\gamma + \int_{|x|\geq 1} x \, \nu(dx)\right); \\ \mathrm{Var}(X_t) = t\left(A + \int_{\mathbb{R}} x^2 \, \nu(dx)\right). \end{cases}$$

Therefore, using the previous thm, we have:

$$\mathbb{E}[R_t^{\bar{\varepsilon}}] = 0 \quad \& \quad \mathrm{Var}(R_t^{\bar{\varepsilon}}) = t\underbrace{\left(\int_{0<|x|<\bar{\varepsilon}} x^2 \, \nu(dx)\right)}_{= \sigma^2(\bar{\varepsilon})}$$

$(\bar{\varepsilon} \in (0,1))$.

For an α-stable process: $\sigma(\bar{\varepsilon}) \sim (\bar{\varepsilon})^{1-\alpha/2}$.