The 1st implementation on Matlab (Kou & Merton) → It's jump diffusion.

- Let's start with jump diffusion :

$$X_t = \mu t + \sigma W_t + \sum_{s\in[0,t)} \Delta X_s = \mu t + \sigma W_t + \sum_{i=1}^{N(t)} Y_i \text{, where}$$

$N(t) \sim \text{Poiss}(\lambda t)$ and $(Y_i)_i$ iid RV.

The most used Lévy processes in finance are:

- Merton: $Y_i \sim \mathcal{N}(\mu_J, \delta_J^2)$, so defined by: $(\sigma, \lambda, \mu_J, \delta_J)$.

For this process we have the following results:

- $$\nu(x) = \lambda f(x) = \lambda \frac{1}{\delta_J\sqrt{2\pi}} e^{-\frac{(x-\mu_J)^2}{2\delta_J^2}}.$$

f is the pdf of std normal.

- $$\psi(u) = i\mu u - \sigma^2 \frac{u^2}{2} + \lambda\left(e^{-\delta_J^2 u^2/2 + i\mu_J u} - 1\right).$$

Let's take $(X_t)_{t\geq 0}$. What about its density:

$$p_{X_t}(x) = e^{-\lambda t} \sum_{k=0}^{+\infty} (\lambda t)^k \frac{e^{-\frac{x-\gamma t - k\mu_J}{2(\sigma^2 t + k\delta_J^2)}}}{k \times 2\pi(\sigma^2 t + k\delta_J^2)}$$ (pdf of X_t)

→ We will never use this formula: it's true that we have a closed formula for the pdf, but we cannot use it since there is an ∞-sum: we must troncate the sum...

→ So NOT USEFUL for pricing.

- **Kou**: (γ, σ^2, ν)

- Lévy density: $p \in [0,1]$, $\lambda^+, \lambda^- > 0$,

$$\nu(dx) = \lambda \underbrace{\left[p \lambda^+ e^{-\lambda^+ x} \mathbb{1}_{x>0} + (1-p) \lambda^- e^{-\lambda^-|x|} \mathbb{1}_{x<0} \right]}_{\text{density of the jump size: } f(x)} dx.$$

- $$\psi(u) = i\mu u - \frac{\sigma^2 u^2}{2} + iu\lambda \left\{ \frac{p}{\lambda^+ - iu} - \frac{1-p}{\lambda^- + iu} \right\}.$$

Starting from next time, we are going to do some coding on the computer. We'll have an introduction about MC and then we will implement MC.

END OF RECORDING 16/09/2022

Let's simulate these models on MATLAB (IT'S ONLY JUMP DIFFUSION).

↳ FOR now...

01/10/2024

Recording of 19/09/2022.

Simulation of Merton & Kou: MATLAB

How to simulate jumptimes? 2 Algorithms

1) Conditional simulation:

1st step: we simulate $N(t)$ $\longrightarrow$ nb of jumps.
2nd step: conditional to $N(t)$, we simulate when jumps occurs via $\mathcal{U}$ distribution: $T_1, \dots, T_{N(t)} \sim \mathcal{U}([0,T])$.

2) Countdown simulation:

1st step: we know that inter arrival times $T_i \sim Exp(\lambda)$.

$\longrightarrow$ see folder "Lecture04_JD_MC" for Matlab sims.

Now what we have to do is:
- To introduce other kinds of Lévy processes;
  ↳ Not all the processes are jump diffusion.
- To see in which case we are under the risk neutral measure because to price things, we need to be able to define a Lévy proc. s.t we have a risk neutral measure.
  ↳ in B&S it's very simple: $\mu = r$, here it's not so simple (even if it's not too difficult).
- To do some MC to price derivatives under the risk neutral measure, for Lévy models.

We continue on the construction of Lévy processes:

Let's see other kinds of Lévy processes (not jump diffusion).

Building a Lévy process:

How can we construct a Lévy process starting from a Lévy process?

3 rules: Lévy → Lévy

(1) Linear transformation;
(2) Subordination;
(3) Exponential tilting.

(1) THM: Let $(X_t)_{t\geq 0}$ be a Lévy proc. in $\mathbb{R}^d$ with (γ, A, ν). Let M be a $n\times d$ matrix.

Linear Transformat°

Then: $Y_t = MX_t \in \mathbb{R}^n$ is a Lévy process $(\tilde{\gamma}, \tilde{A}, \tilde{\nu})$ where:

transfo. by M.

- $\tilde{A} = MAM^T$;
- $\tilde{\gamma} = M\gamma + \underbrace{\int_{\mathbb{R}^n} y\left(\mathbb{1}_{\{|y|\leq 1\}} - \mathbb{1}_{S_1}\right)\tilde{\nu}(dy)}_{\text{Extra term.}}$

where $S_1 = \{Mx : |x| \leq 1\}$;

• $\tilde{\nu}(B) = \nu(\{x : Mx \in B\})$, $\forall B$.

transformation by M.

Rmk: The extra term comes from : separation between small & large jumps. Maybe a small jump for X is transformed to a big one for Y by M. It's a "correction integral" for the small jumps. Let's take an example to understand ...

Example: $d = n = 1$;

$M = 2$; $(X_t)_{t \geq 0}$ Lévy ;

$(Y_t)_t = (2X_t)_t$ is Lévy.

BUT : only jumps smaller than $1/2$ are "small jumps" for both X and Y !

PROP: If: $(X_t)_t$ is Lévy (γ_1, A_1, ν_1);
$(Y_t)_t$ is Lévy (γ_2, A_2, ν_2);

Then:

$$(X_t + Y_t)_t \text{ is Lévy } (\gamma, A, \nu)$$

where:

- $A = A_1 + A_2$;
- $\nu(B) = \nu_1(B) + \nu_2(B)$;
- $\gamma = \gamma_1 + \gamma_2 - \int_{[-\sqrt{2};-1] \cup [1, \sqrt{2}]} y \, \nu(dy)$.

Proof: $X = \begin{bmatrix} X_t \\ Y_t \end{bmatrix}$ & $M = \begin{bmatrix} 1 & 1 \end{bmatrix}$ & use the previous theorem. ■

07/10/2024

---

(2) **THM:** Let $(S_t)_{t \geq 0}$ a subordinator. The moment-generating function of the subordinator is:

subordination

$$E\left[e^{uS_t}\right] = e^{t\ell(u)} \quad \text{where: } \ell(u) = bu + \int_0^{+\infty} (e^{ux} - 1) \rho(dx)$$

with $b \geq 0$, $\rho((-\infty, 0]) = 0$, $(S_t)_{t \geq 0}$ FINITE VARIATION

(Lévy measure)

Let $(X_t)_{t\geq 0}$ Lévy process in $\mathbb{R}^d$ (γ, A, ν),

let ψ_X be the Lévy exponent. Then:

$Y_t = X_{S_t}$ is a Lévy, and:

$\psi_Y(u) = \ell(\psi_X(u))$, (γ^Y, A^Y, ν^Y) with:

$$\begin{cases} A^Y = bA \\ \nu^Y(B) = b\nu(B) + \displaystyle\int_0^{+\infty} \underbrace{p_s^X}_{\text{pdf of } X_s}(B)\, \rho(ds) \\ \gamma^Y = b\gamma + \displaystyle\int_0^{+\infty}\int_{|x|\leq 1} x\, p_s^X(dx)\, \rho(ds). \end{cases}$$

Rmk: pdf_s^X is sometimes UNKNOWN!

Example: Processes that are used in FINANCE:

- STABLE PROCESS:

$(X_t)_{t\geq 0}$ is an α-stable process, $0 < \alpha < 2$, if:

NO WIENER PROCESS!

it is Lévy $(\gamma, 0, \nu)$ & $\exists\, \lambda$ on $S = \{x \in \mathbb{R}^d : |x| < 1\}$

a measure

s.t: $\nu(B) = \displaystyle\int_S \lambda(d\xi) \int_0^{+\infty} \mathbb{1}_B(r\xi) \frac{dr}{r^{1+\alpha}}$.

HORRIBLE DEF! Fortunately, we'll always work in 1D & it is WAY simpler:

DEF: In 1D, $(X_t)_{t\geq 0}$ is a (real) α-Stable process,

$0 < \alpha < 2$ if it is Lévy $(\gamma, \overset{\text{NO WIENER}}{\overset{\downarrow}{0}}, \nu)$ and

$\exists A, B > 0$, $\nu(x) = \frac{A}{x^{1+\alpha}} \mathbb{1}_{\{x>0\}} + \frac{B}{|x|^{1+\alpha}} \mathbb{1}_{\{x<0\}}$.

<u>DEF:</u> $(X_t)_{t\geq 0}$ is an <u>α-stable subordinator</u> if:

- $(X_t)_{t\geq 0}$ is a real Lévy proc. $(\gamma, 0, \rho)$;
- $\rho(x) = \frac{A}{x^{1+\alpha}} \mathbb{1}_{x>0}$;
- $\alpha \in (0,1)$;
- $b = \gamma - \int_{|x|<1} x\rho(dx) > 0$.

And: $\ell(u) = C_1 \int_0^{+\infty} \frac{e^{-ux} - 1}{x^{1+\alpha}} dx$,

$C_1(A, \alpha, \gamma)$.

(In finance, we will take a Wiener and define $(W_{X_t})_{t\geq 0}$. We will talk about it after (3).

(3) THM: $(X_t)_{t\geq 0}$ Lévy (γ, A, ν) s.t $\begin{cases} \int_{|x|\leq 1} x^2 \nu(dx) < +\infty; \\ \int_{|x|} \nu(dx) < +\infty. \end{cases}$

Exponential tilting

Exponential tilting:

$$\tilde{\nu}(dx) = e^{\theta \cdot x} \nu(dx).$$ If:

$\exists \theta \in \mathbb{R}^d$ s.t: $\begin{cases} \int_{|x|\leq 1} |x|^2 \tilde{\nu}(dx) < +\infty, \\ \int_{|x|>1} \tilde{\nu}(dx) < +\infty \end{cases}$

Then $(\gamma, A, \tilde{\nu})$ defines a new Lévy proc.

1D Tempering:

$$\tilde{\nu}(dx) = \left(\mathbb{1}_{\{x>0\}} e^{-\lambda_+ x} + \mathbb{1}_{\{x<0\}} e^{-\lambda_- |x|} \right) \nu(dx)$$

$\lambda_\pm > 0$

Tempering & subordination give rise to our infinite activity Lévy proc. that we need in Finance!