Computational Finance - Lesson 5

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Beyond MPT

Diversification

- It has been studied that making incorrect assumptions in the portfolio construction can lead to **unefficient portfolio allocation**.
- The will to reduce estimation risk has led researchers and practitioners to investigate more robust portfolio construction schemes (i.e. Black-Litterman).
- In recent years the financial industry has moved further, building investment strategies that can completely **ignore assumptions on expected returns**, thus being based only on the risk model alone.
- Diversification of investments and thus of risks is a possible solution. Obviously '*Diversification is not an issue for the intelligent investor who is able to perfectly forecast returns*.'
- But we have learnt that this is not an easy task.

Diversification Measures (DM)

- A portfolio is diversified if it is not heavily exposed to individual shocks.

- The notion of portfolio diversification is closely related to that of asset segmentation. Assets may be divided based on:
  - Their asset class (equity, bonds,..)
  - Risk arguments on security level (volatility, correlation and other risk metrics)
  - Their similarity of sensitivity to macroeconomic factors

- Thus diversification can be expressed in terms of:

  i. Portfolio weights
  ii. Risk contributions
  iii. Economic scenario dependence: Asset price behaviour is considered as a mere manifestation of the underlying (hidden) macroeconomic factors, in this case prior information might be particularly useful in identifying relevant factors

DM – Portfolio weights

- They do not require informations on risk properties. Depending only on the weights they do not suffer from estimation risk

- They are suitable for quantifying diversification when dealing with assets belonging to the same asset class, thus similar in terms of risk.

- Some of the most important metrics are: **Herfindahl index**, the **Lorentz curve**, the **Gini coefficient** and the **entropy.**

- The ***Herfindahl Index*** is used to quantify the concentration of investments within a portfolio. It provides a way to measure how spread out or concentrated the investments are across different assets or asset classes. A higher Herfindahl Index indicates a more concentrated portfolio, while a lower value suggests a more diversified one.

- Let N be the number of assets in the portfolio and w_i the weight of the i-th asset, the Herfindahl Index is calculated as follows:

$$HI = \sum_{i}^{N} (w_i)^2$$

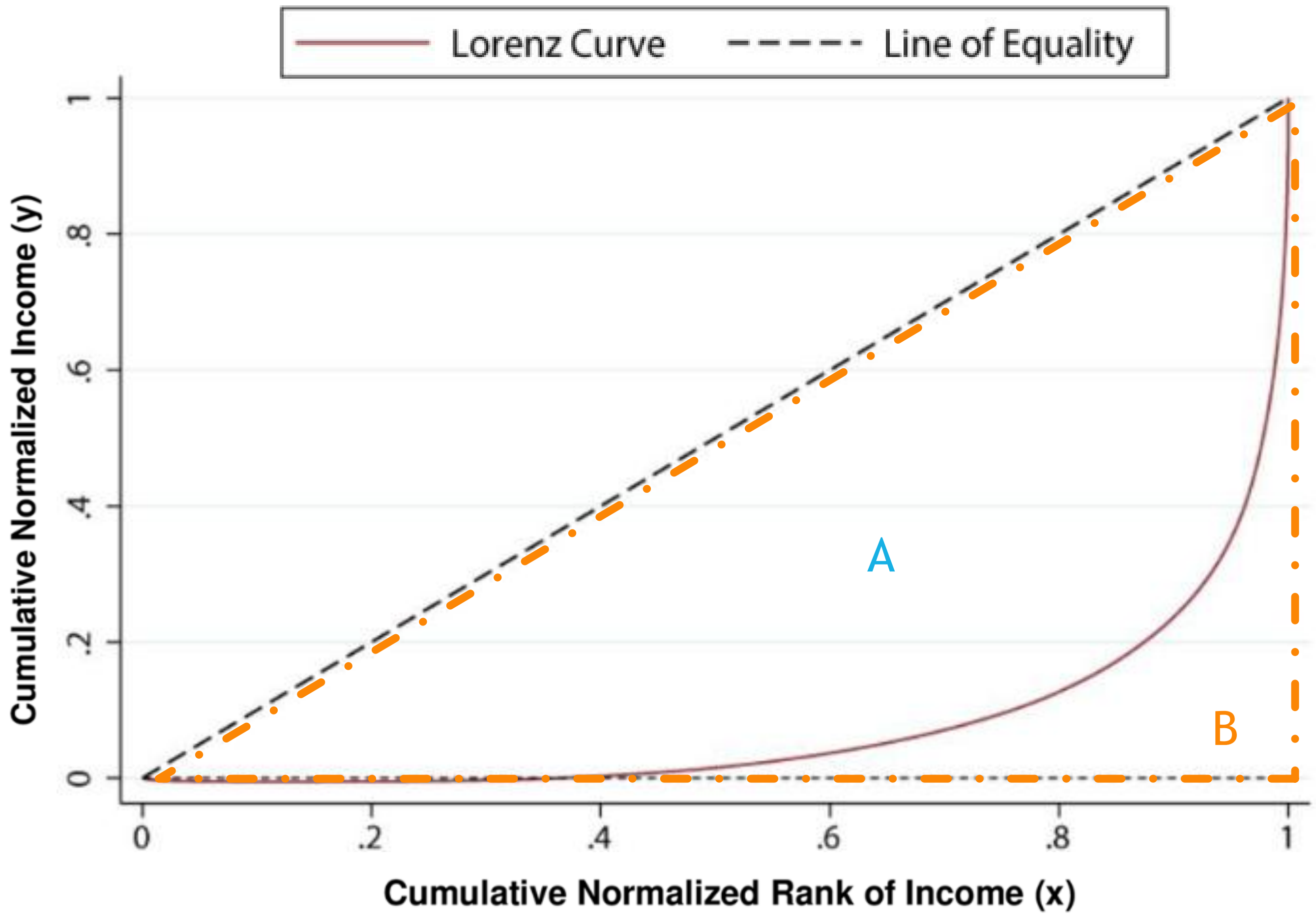
DM – Portfolio weights

- The **Lorenz Curve** is a graphical representation that helps visualize income or wealth distribution within a population. In the context of portfolio management, it can be used to represent the distribution of returns or gains across different investments within a portfolio.

- Here's how the Lorenz Curve is constructed:

  I. **Rank the Investments:** Arrange the investments in the portfolio in ascending order based on their returns or gains. The investment with the lowest return is on the left, and the one with the highest return is on the right.
  II. **Calculate Cumulative Proportions:** For each investment, calculate the cumulative proportion of the total returns. This is done by dividing the sum of the returns of the ranked investments up to a given point by the total sum of returns.
  III. **Plot the Curve:** Plot the cumulative proportion of returns on the y-axis and the cumulative proportion of investments on the x-axis.

DM – The Lorenz Curve



- If the curve is close to the perfect equality line, it suggests a well-diversified portfolio.
- Conversely, if the curve deviates significantly from the line, it indicates that a few investments are contributing disproportionately to the returns.

DM – The Gini Coefficient

- The **Gini coefficient** is a statistical measure of inequality. It is calculated starting from the Lorenz curve.

- We define as A the area between the Lorenz curve and the equity line, and B as the area under the equality line.

- Thus the Gini coefficient is calculated as: $G = \frac{A}{A+B}$

- The Gini coefficient produces a value between 0 and 1.

- A Gini coefficient of 0 indicates perfect equality (i.e., all investments contribute equally to the returns), while a coefficient of 1 implies maximum inequality (i.e., one investment contributes all the returns)

DM - Entropy measures

- The link between portfolio diversification and entropy lies in the notion of **uncertainty**.

- In Information Theory, entropy indicates the degree of predictability of a stochastic dynamical system the higher the entropy, the less predictable the system.

- Let us suppose that a physical system can be described by N discrete states and p_i represent the probability associated to each state. The so called ***Shannon Entropy*** is defined as:

$$H = -\sum_{i=1}^{N} p_i \ln p_i$$

- The average number of relevant states is defined as $\eta = e^H$
- The entropy is maximal, $\eta = N$, where all states are equally likely, $p_i = \frac{1}{N}$, i.e. the system is unpredictable
- The entropy is minimal, $\eta = 1$, where the probability of one state is equal to 1, i.e. the system is fully deterministic

DM - Entropy measures

- The ***entropy measure in portfolio weights*** H_w is computed simply by replacing the probability set $\{p_i\}$ with the portfolio weights $\{w_i\}$

$$H_w = -\sum_{i=1}^{N} w_i \ln w_i$$

- The entropy is maximal for the equally weighted portfolio, $w_i = \frac{1}{N} \;\; \forall i \in [1, N]$ and minimal for $w_i = 1$ and $w_j = 0 \;\; \forall j \neq i$. η_w ranges from 1 (maximal concentration) to N (maximal diversification).

- The metric is well defined only for long – portfolios with weights that sum to 1.

DM – Risk Contributions

- They rely on **historical evaluation of informations about asset risk** (for example using the covariance matrix) thus they suffer from the estimation risk.
- Those measures provide a good indication of diversification in an investment universe characterized by a rather large volatility spectrum (for example, diversified portfolios invested in assets that range from short-term bonds to highly volatile stocks)
- Some of the most famous measures belonging to this category are **entropy-based** (expressed in terms of asset volatility, asset risk contribution and principal components) and the **diversification ratio.**
- One can use also the metrics we have shown before with the caution of adding a risk contribution in their definition.

DM – The diversification Ratio

- The ***Diversification Ratio (DR)*****,** also known as the diversification benefit or diversification effect, is a measure of the effectiveness of diversifying investments within a portfolio to reduce risk. It quantifies the extent to which adding new assets to a portfolio contributes to risk reduction.

- The diversification ratio is calculated using the following formula:

$$DR = \frac{\sigma_P - \sum_i^N w_i \sigma_i}{\sigma_P}$$

where σ_P is the volatility of the portfolio and the term $\sum_i^N w_i \sigma_i$ represents a weighted average volatility, i.e. the average risk of the individual assets in the portfolio weighted by their respective contributions to the portfolio's total value.

- This formulation implies that $0 \leq DR \leq 1$. A higher diversification ratio indicates a greater reduction in risk due to diversification. A ratio of 1 implies perfect diversification, where the addition of new assets has completely eliminated all **unsystematic** (specific to individual assets) risk.

- It's important to note that the diversification ratio is a measure of the effectiveness of diversification in reducing unsystematic risk.

DM - Entropy Measures

- The ***entropy measure in asset volatility*** H_{vol} is a function of asset volatilities.

- The measure is defined by replacing the set of probabilities $\{p_i\}$ with the quantities $\{w_i^2\sigma_i^2 Z\}$ where σ_i represents the volatility of the i-th asset and Z is a normalization constant to ensure that all adds to 1.

- In particular $Z = \left(\sum_i^N w_i^2\sigma_i^2\right)^{-1}$

- Thus the entropy H_{vol} is defined as:

$$H_{vol} = -\sum_{i=1}^{N}\left(\frac{w_i^2\sigma_i^2}{\sum_i^N w_i^2\sigma_i^2}\right)\ln\left(\frac{w_i^2\sigma_i^2}{\sum_i^N w_i^2\sigma_i^2}\right)$$

- In this case $\eta_{vol} = \exp(H_{vol})$ represents the average number of relevant assets in the risk space, assuming zero correlation among assets. η_{vol} ranges from 1 to N.

DM - Entropy Measures

- The ***entropy measure in risk contribution*** H_w^{risk} is a function of asset *risk contributions*.
- In this case the set of probabilities $\{p_i\}$ is replaced by the set of risk contributions $\{\theta_i\}$ (that have to be normalized).
- Thus $\theta_i = \frac{|\sum_{j=1}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij}|}{\sum_h^N |\sum_{j=1}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij}|}$, the absolute value has to be added in order to take into account possible negative correlations.

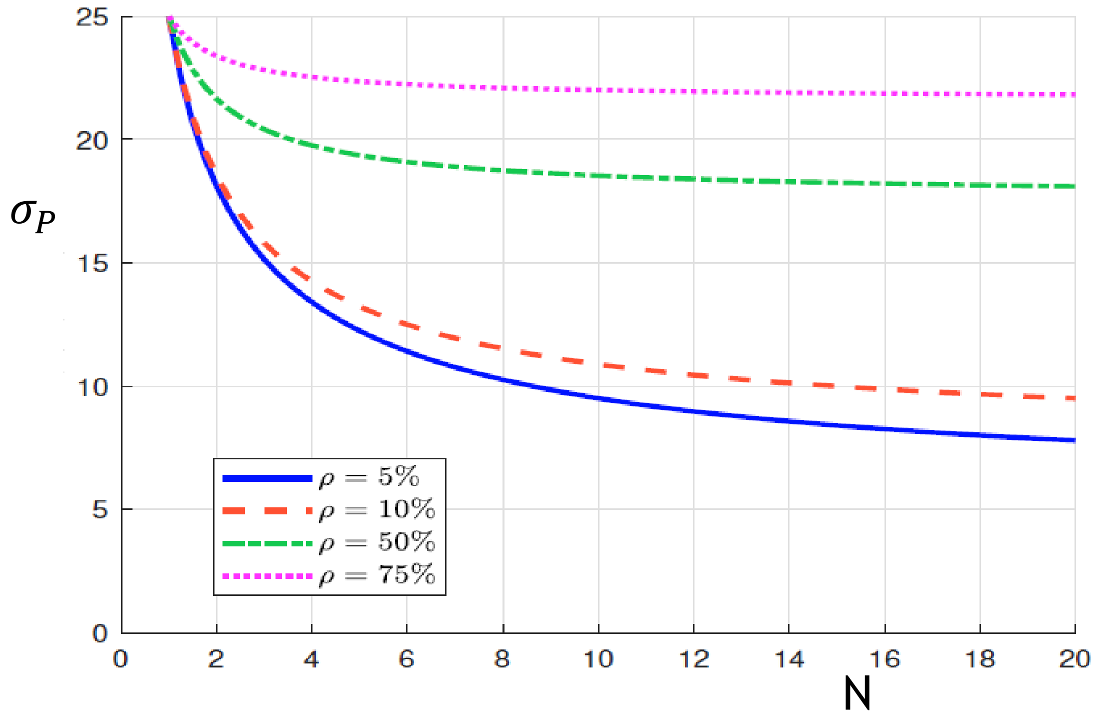
$$H_w^{risk} = -\sum_{i=1}^{N} \theta_i \ln \theta_i$$

Diversification Measures and Portfolio Allocation

Equally Weighted Portfolio

- The main idea of using the equally weighted portfolio is to define a portfolio independently from the estimated statistics.

- In this case correlations between assets are more important than volatilities to benefit from diversification (= risk reduction)



$$\sigma_P^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i>j} w_i w_j \sigma_i \sigma_j \rho_{i,j}$$

Risk Parity Portfolio

- Risk parity is an investment strategy that focuses on balancing the risk contributions of different assets in a portfolio. The goal is to create a portfolio where ***each asset contributes equally to the overall risk***, rather than being weighted by their market value or expected return.

- After the estimation of asset risks, it in necessary to compute the **risk contribution** of each asset, i.e. the proportion of total portfolio risk that each asset contributes.

- Let Σ be the covariance matrix and $\boldsymbol{w}$ the vector of assets weights in the portfolio P. The volatility of the portfolio is then:

$$\sigma_P = \sqrt{w^T \Sigma w}$$

- If we compute the gradient of σ_P respect to $\boldsymbol{w}$, we obtain:

$$\begin{aligned} \frac{\partial \sigma_P}{\partial w} &= \frac{\partial}{\partial w} \left(w^T \Sigma w\right)^{1/2} \\ &= \frac{1}{2}\left(w^T \Sigma w\right)^{\frac{1}{2}-1}(2\Sigma w) = \frac{\Sigma w}{\sqrt{w^T \Sigma w}} \end{aligned}$$

Risk Parity Portfolio

- It follows that the ***marginal volatility*** of the i-th asset, also called the ***marginal risk contribution*** is computed as:

$$MR_i = \frac{\partial \sigma_P}{\partial w_i} = \frac{(\Sigma w)_i}{\sqrt{w^T \Sigma w}} = \sigma_i \sum_j^N w_j \frac{\rho_{i,j}\sigma_j}{\sigma_P}$$

- The ***Absolute Risk Contribution*** of the i-th asset is defined as

$$RC_i = w_i \frac{\partial \sigma_P}{\partial w_i} = w_i \sigma_i \sum_j^N w_j \frac{\rho_{i,j}\sigma_j}{\sigma_P}$$

- And the ***Relative Risk Contribution*** of the i-th asset as:

$$RC_i^{rel} = \frac{RC_i}{\sigma_P}$$

Risk Parity Portfolio

- Then we have to adjust the weights of the assets so that each one contributes equally to the overall portfolio risk. This means allocating more capital to assets with lower risk and less capital to assets with higher risk.

Example

- 3 assets
- Volatilities are respectively 30%, 20% and 15%
- Correlations are set to 80% between the 1^{st} asset and the 2^{nd} asset, 50% between the 1^{st} asset and the 3^{rd} asset and 30% between the 2^{nd} asset and the 3^{rd} asset

Starting Portfolio

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	50.00%	29.40%	14.70%	70.43%
2	20.00%	16.63%	3.33%	15.93%
3	30.00%	9.49%	2.85%	13.64%
Volatility			20.87%	

Risk Parity Portfolio

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	19.69%	27.31%	5.38%	33.33%
2	32.44%	16.57%	5.38%	33.33%
3	47.87%	11.23%	5.38%	33.33%
Volatility			16.13%	

Most Diversified Portfolio

- To introduce the concept of the most diversified portfolio, we have to give an alternative definition of the ***Diversification Ratio***, that is:

$$DR = \frac{\sum_i^N w_i \sigma_i}{\sigma_P} = \frac{w^T \sigma}{\sigma_P}$$

- Thus the DR of a portfolio fully invested in one asset is equal to 1,i.e. the volatility of the single asset is by definition the volatility of the portfolio.

- In the general case, $DR \geq 1$

- Therefore the **Most Diversified Portfolio** is defined as the portfolio that maximizes the diversification ratio:

$$w^* = \max \ln(DR)$$

with the following constraints, $\sum_i w_i = 1, \;\; 0 \leq w_i \leq 1$

Most Diversified Portfolio

- Under the assumption that all the assets have the same Sharpe Ratio, that is

$$SR = \frac{\mu_i - r}{\sigma_i} = s$$

It can be demonstrated that the DR of portfolio P is proportional to its Sharpe Ratio:

$$DR = \frac{1}{s}\frac{\sum_i^N w_i\,(\mu_i - r)}{\sigma_P}$$

$$= \frac{1}{s}\frac{w^T\mu \; - r}{\sigma_P}$$

$$= \frac{1}{s}\;SR(P)$$

- That means that maximizing the DR is the equivalent to maximizing the Sharpe Ratio

Maximization of Entropy Metrics

- Another way to achieve an optimal diversified portfolio allocation is to **maximize the entropy metrics**, i.e. find the optimal weights that maximize the entropy of the portfolio.

- The process is the following:

  i. **Define the Objective Function:** The objective is to maximize the entropy of the portfolio's returns. We can chose here to use all the possible definition of Portfolio entropy (i.e. entropy in portfolio weights, in asset volatilities and in risk contributions)

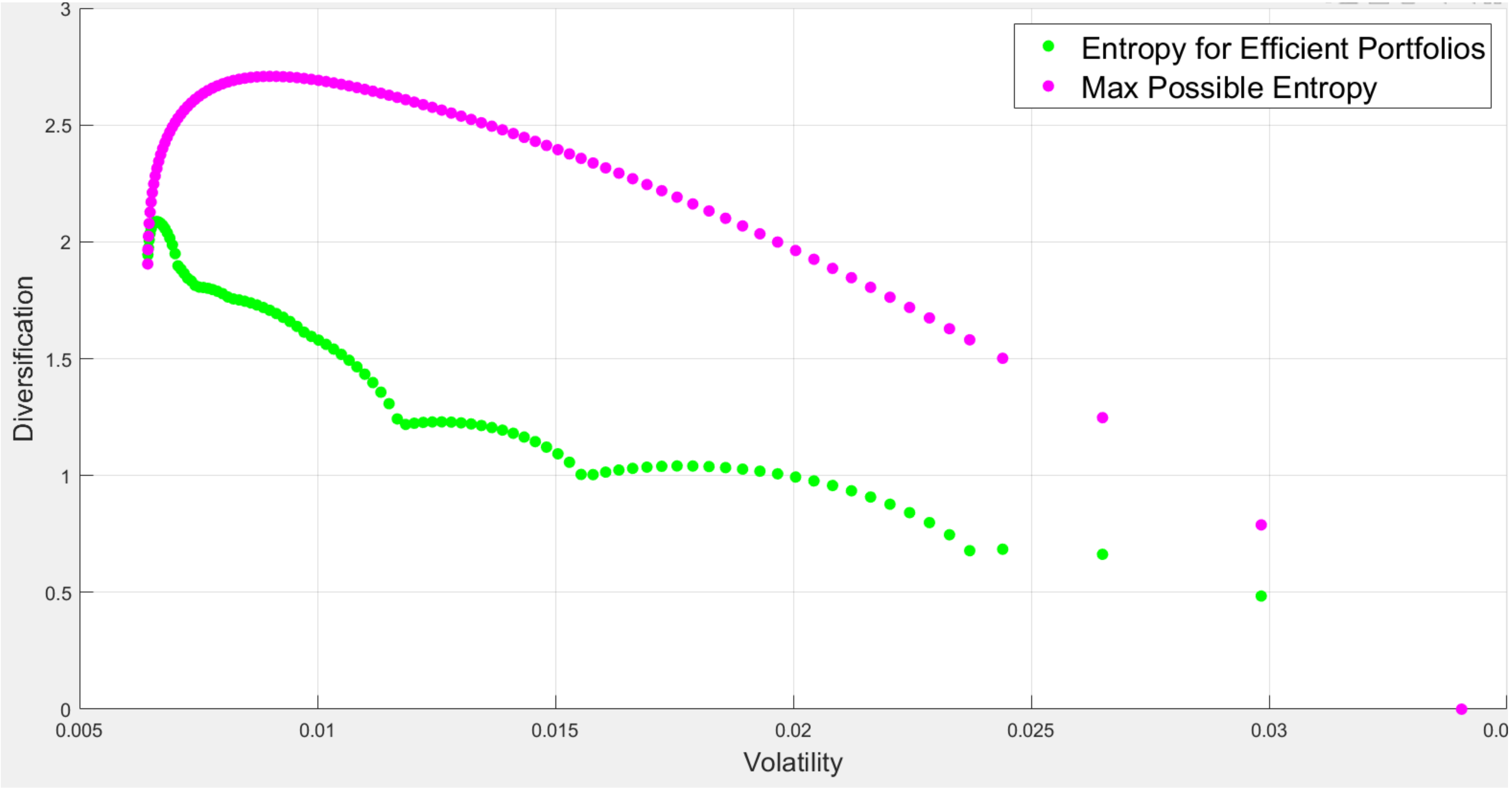
  ii. **Define constraints:** budget constraints (i.e. weights) and risk constraints (additional constraints on risk measures of the portfolio)

  iii. **Solve the optimization** problem in order to obtain the optimal weights:

$$w^* = \max_{w}(-\sum_{i} p_i \ln(p_i))$$

Diversification curves

EX: Computation of the effective entropy (in weights) for the 100 portfolios composing the efficient portfolio frontier and their maximal possible entropy (for the same level of risk)



Diversification & portfolio allocation

Equally weighted Portfolio

- Enhanced Diversification
- Do not depend on estimation procedures
- Simplicity

- Potential for Suboptimal Risk Management
- Potential for Inefficiency

Risk Parity Portfolio, Most Diversified Portfolio, Max Entropy Portfolio

- Enhanced Diversification
- Adaptability to Market Conditions
- Focus on risk modelling

- Sensitivity to Estimation Errors
- Assuming stationarity
- Asset Class Selection