or another way to see it is :

- $v_t = v_0 - \lambda \int_0^t v_s\, ds + Z_t$: just integrate the SDE.

___ : we have to deal with this integral.

It requires a theoretical result.

Lemma: $f : [0,T] \longrightarrow \mathbb{R}$ left continuous.

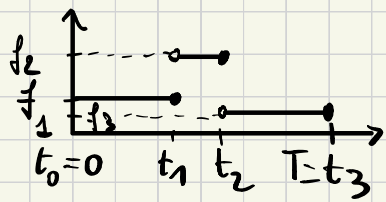
$(Z_t)_t$ Lévy.

$$\mathbb{E}\left[\exp\left(i \int_0^T f(t)\, dZ(t)\right)\right] = \exp\left(\int_0^t \Psi(f(t))\, dt\right) \text{ where}$$

Ψ is the characteristic exponent of Z.

Proof: Assume f piecewise constant.

$$f(t) = \sum_{i=1}^{N} f_i \mathbb{1}_{]t_{i-1}, t_i]}(t)$$



$$\mathbb{E}\left[\exp\left(i \int_0^T f(t)\, dZ(t)\right)\right] = \mathbb{E}\left[\exp\left(i \int_0^T \sum_{j=1}^{N} f_j \mathbb{1}_{]t_{j-1}, t_j]}\, dZ_t\right)\right]$$

$$= \mathbb{E}\left[\exp\left(i \sum_{j=1}^{N} f_j \left(Z_{t_j} - Z_{t_{j-1}}\right)\right)\right]$$

Lévy $\left\{\text{independent increments}\right.$ $= \prod_{j=1}^{N} \mathbb{E}\left[e^{i f_j (Z_{t_j} - Z_{t_{j-1}})}\right]$

Lévy $\left\{\begin{array}{l}\text{Because } Z_{t_j - t_{j-1}} \\ = Z_{t_j} - Z_{t_{j-1}}\end{array}\right.$ $= \prod_{j=1}^{N} \underbrace{\mathbb{E}\left[e^{i f_j Z_{(t_j - t_{j-1})}}\right]}$

$$= e^{(t_j - t_{j-1}) \psi(f_j)} = \phi_{t_j - t_{j-1}}(f_j).$$

$$= e^{\sum_{j=1}^{N} \psi(f_j)(t_j - t_{j-1})}$$

$$e^{\int_0^T \psi(f(t)) dt} \qquad = e^{\int_0^T \psi\left(\sum_{j=1}^{N} f_j \mathbb{1}_{(t_{j-1}, t_j]}\right) dt}$$

These two are equal. ■

Let's compute $\mathbb{E}[e^{iu v_t}]$ using the lemma:

$$\mathbb{E}[e^{iu v_t}] = \exp\left(iu v_0 e^{-\lambda t} + \int_0^t \psi(u e^{\lambda(s-t)}) ds\right)$$

$$v_t = v_0 \exp(-\lambda t) + \int_0^t e^{\lambda(s-t)} ds$$

⚠ It is not Lévy, since if it was the case, the log of the C.F would be linear in time($\phi_{X_T} = e^{t\psi_X} \rightarrow \log(\phi_{X_T}) = t\,\psi_X$ ⫫ of t).

Clearly here it's not the case.

But some behaviours are close to Levy.

<u>Proposition</u>: $(Z_t)_t$ Levy w/ triplet (γ, A, ν).

Let $(v_t)_{t\geq 0}$ s.t $dv_t = -\lambda v_t\,dt + dZ_t$.

Then:

$\rightarrow (v_t)_t$ is ∞-divisible process

$\rightarrow (\gamma_t, A_t, \nu_t)$

$$A_t = \frac{A}{2\lambda}\left(1-e^{-2\lambda t}\right);$$

$$\gamma_t = \frac{\gamma}{\lambda}\left(1-e^{-\lambda t}\right) + v_0 e^{-\lambda t};$$

$$\forall B \in \mathcal{B}(\mathbb{R}),\quad \nu_t(B) = \int_1^{e^{\lambda t}} \frac{\nu(\xi B)}{\xi\lambda}\,d\xi.$$

<u>Proof</u>: (Important proof)

(I) <u>ν_t is a Levy measure</u>:

$\rightarrow$ positive.

$$\rightarrow \nu_t([1,+\infty)) = \int_1^{\lambda t} \frac{\nu([\xi,+\infty))}{\lambda \xi} d\xi$$

$$\leq \frac{\nu([1,+\infty))}{\lambda} \left(e^{-\lambda t} - 1\right).$$

$$< +\infty \; .$$

$$\rightarrow \nu_t((-\infty,1]) < +\infty \quad \text{too}.$$

$$\rightarrow \int_{-1}^{1} z^2 \nu_t(dz) = \int_{-1}^{1} z^2 \int_1^{e^{\lambda t}} \frac{\nu(\xi dz)}{\lambda \xi} d\xi$$

$$= \int_1^{e^{\lambda t}} \frac{1}{\lambda \xi^3} \underbrace{\int_{-\xi}^{\xi} z^2 \nu(dz)}_{<+\infty} d\xi < +\infty$$

II Lévy Khinchin Formula:

$$\mathbb{E}\left[e^{iu v_t}\right] = \exp\Big(iu v_0 e^{-\lambda t} + \frac{i\gamma u}{\lambda}\left(1-e^{-\lambda t}\right) -$$

Another way to write the CF.

$$\frac{A u^2}{4\lambda}\left(1-e^{-2\lambda t}\right) + \underline{\int_1^{e^{-\lambda t}} \frac{1}{d\xi} \int_{-\infty}^{+\infty} \nu(\xi dz)\Big(e^{iuz}}$$

$$\underline{-1 - iuz \mathbb{1}_{|z|\leq 1}\Big) d\xi}\Big)$$

↑ the CF stops before this if Gaussian OU.

but $\displaystyle\int_{-\infty}^{+\infty}\left(e^{iuz}-1-iuz\,\mathbb{1}_{|z|\leq 1}\right)\underbrace{\nu_t(dz)}_{=\int_1^{e^{-\lambda t}}\frac{\nu(\xi dz)}{d\xi}d\xi}$

+ change the order of integration & we obtain the above result. ■

We have to move to POSITIVE OU.

So in our application in finance, we will use <u>positive non-gaussian</u> OU to construct time change:

$$\left.\begin{array}{l} X_t, \\ V_t = \int_0^t v_s\,ds \ , \ v \geq 0 \ \text{OU} \end{array}\right\} X_{V_t}.$$

↳ BNS.

Positive OU :

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Positive OU :

(I)

$$dy_t = -\lambda y_t\,dt + \sigma\, dZ_t \quad \text{where } (Z_t)_t \text{ is Lévy}$$

and with : - $(Z_t)_t$ a subordinator ;

- $y_0 \geq 0$.

We can also define the integrated process :

$$v_t = \int_0^t y_s\,ds$$

Prop: • $v_t = \frac{y_0}{\lambda}\left(1 - e^{-\lambda t}\right) + \frac{1}{\lambda}\int_0^t \left(1 - e^{\lambda(s-t)}\right) dZ_s$.

• **Laplace Transform :**

$$\mathbb{E}\left[e^{u v_t}\right] = \int_{\mathbb{R}} e^{us} \underbrace{f(s)}_{\text{pdf of } v}\,ds$$

we are not in $\cancel{\neq e^{t\mathcal{L}(u)}}$ the Lévy world.

$$= \exp\left(\frac{u y_0}{\lambda}\left(1 - e^{-\lambda t}\right) + \int_0^t \ell\left(\frac{u}{\lambda}\left(1 - e^{\lambda(s-t)}\right)\right) ds\right)$$

where $e^{t\ell(u)} = \mathbb{E}\left[e^{u Z_t}\right]$ (Z is Lévy !).

N.B: $\ell(u)$ is like a charac. exponent but for LT.

Why $\mathcal{LT}$? And not $\mathcal{FT}$?

Because $(v_t)_t$ is the integrated process of y and therefore it is non-negative. And we need to have the charac. exp. of the $\mathcal{F}T$ and the $\mathcal{L}$. exp. of the subordinator. Here v plays the role of the subordinator. Therefore we are interested in its $\mathcal{L}$. exp.

We can use (I) in two ways:

- Time change
- BNS

$$\left.\begin{array}{l} X_T \perp\!\!\!\perp v_t \\ X_t \rightsquigarrow X_{v_t} \\ y_t = \sigma_t^2. \\ dX_t = (\mu + \beta y_t)dt + \underbrace{\sigma_t}_{\sqrt{y_t}} dW_t + \rho\, dZ_t \end{array}\right\} S_t = S_0 e^{X_t}.$$

Rmk: usually, correlation ρ is negative.

In the mkt, when variance $\nearrow$, prices $\searrow$.

BNS

$\sigma_t^2 = y_t \qquad \lambda > 0 \qquad y_0 > 0$

$\rho \leq 0 \qquad \sigma > 0 .$

$(Z_t)_t$ is Lévy subordinator : $(\gamma, \underbrace{A}_{=0 \text{ for a subordinator}}, \nu)$.

for example : $\boxed{\text{KOU}}$

needed for a subordinator.

$(\gamma, 0, \nu)$ with : $\nu = \lambda^L p \lambda_+ e^{-\lambda_+ x} + \cancel{(1-p)\lambda^L \lambda_- e^{-\lambda_- |x|}}$

$\gamma \geq 0$

$\gamma = 0$: suggestion.

$\boxed{\lambda^L, \lambda_+}$

remaining parameters.

$p = 1$ bc if not, we have <0 jumps & it's not a subordinator. Then $\cancel{\lambda^-}$.

So this is nothing else than:

a Compound Poisson with jumps which are only positive, jumpsize $\sim$ Exp(λ^+).

$\rightarrow \rho, \lambda, \sigma, y_0, \lambda^L, \lambda^+$: 6 parameters. $\oplus$ μ, β.

THEOREM:

$$\mathbb{E}\left[e^{iuX_t}\right] = \exp\Big(iu\mu t + y_0\left(iu\beta - \frac{u^2}{2}\right)\varepsilon(\lambda, t)$$

$$+ \int_0^t \ell\left(i\rho u + \left(iu\beta - \frac{u^2}{2}\right)\varepsilon(\lambda, t-s)\right) ds$$

with ℓ the $\mathcal{L}$. exp. of Z : $\mathbb{E}[e^{uZ_t}] = e^{t\ell(u)}$;

$$\varepsilon(\lambda, t) = \frac{1-e^{-\lambda t}}{\lambda}.$$

LEMMA :

$$\mathbb{E}\left[e^{v\nu_t + wZ_t}\right] = \exp\left(v\frac{y_0}{\lambda}\left(1-e^{-\lambda t}\right) + \int_0^t \ell\left(w + \frac{v}{\lambda}\left(1-e^{\lambda(s-t)}\right)\right)ds\right.$$

Proof of the theorem :

$$\mathbb{E}\left[e^{iuX_t}\right] = \mathbb{E}_{\mathcal{F}_0}\left[\exp\left(iu\left(\overbrace{\mu t + \beta\int_0^t y_s\,ds + \int_0^t \sqrt{y_s}\,dW_s + \rho Z_t}^{\text{integration of } dX_t}\right)\right)\right]$$

$\rightsquigarrow$: sources of randomness : Z Z W Z

$\mathcal{F}_0$: information @ time 0 (filtration).

$\mathcal{G} = \mathcal{F}_0 \cup Z_s$, $0 \leq s \leq t$: we know also Z ...

So, by Tower Property :

$$\mathbb{E}\left[e^{iuX_t}\right] = \dots = \mathbb{E}\left[\mathbb{E}\left[\dots \mid \mathcal{G}\right] \mid \mathcal{F}_0\right]$$

under $\mathcal{G}$, only W is "unknown" $= \mathbb{E}\left[e^{iu(\mu t + \beta v_t + \rho Z_t)} \mathbb{E}\left[e^{iu\int_0^t \sqrt{y_s}\, dW_s} \,\middle|\, \mathcal{G}\right] \middle| \mathcal{F}_0\right]$

under $\mathcal{G}$, this is a Lévy Integral.

$$\mathbb{E}\left[e^{iu\int_0^t \sigma_s dW_s}\right] \underset{\text{when } \sigma \text{ cst.}}{=} \mathbb{E}\left[e^{iu\sigma W_t}\right] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{iu\sigma x} e^{-\frac{x^2}{2t}} dx$$

$$= e^{-\frac{u^2}{2}\sigma^2 t}$$

When $\sigma \neq$ cst : $\mathbb{E}\left[e^{iu\int_0^t \sigma_s dW_s}\right] = e^{-\frac{u^2}{2}\underbrace{\int_0^t y_s ds}_{v_t}}$ $\quad (y = \sigma^2)$

So :
$$\mathbb{E}\left[e^{iuX_t}\right] = e^{iu\mu t}\, \mathbb{E}\left[e^{(iu\beta - \frac{u^2}{2})v_t + \rho Z_t}\right]$$

Then we can apply the above lemma to compute the $\mathbb{E}[.]$ and it's done. ■

So, we have the characteristic fct.

Therefore we can find μ s.t.

$\mathbb{E}\left[e^{-rt + X_t}\right] = 1$ <u>RISK NEUTRAL</u>.

And then <u>CALIBRATE</u> the model on Euro. Options with B&S. (+ obviously MC).

Time Change

Let $(Y_t)_t$ a Lévy.

$X_t = Y_{v_t}$ where $(v_t)_t$

is a non-decreasing process.

Characteristic function:

$$\mathbb{E}\left[e^{iuX_t}\right] = e^{\varphi_v(t, \varphi_Y(u))}$$

, φ_Y characteristic exponent of Y_t ; $\varphi_v(t,v)$ Laplace exponent of v_t : $\mathbb{E}[e^{v v_t}] = e^{\varphi_v(t,u)}$.

What are the common $(Y_t)_t$ in Finance ?

- Kou Model (F.A);
- NIG (I.A) or VG (I.A).

LEMMA:

Let $(M_t)_t$ a positive Martingale, $(v_t)_t$ a non-decreasing process. Then $(M_{v_t})_t$ is a Martingale in the filtration generated by v_t - and - M_{v_t}.

$$M_t = \quad \Big| \quad Y_t = e^{rt + \hat{Z}_t} \quad \text{where } \hat{Z} \text{ Lévy.}$$

$$X_t = M_{v_t} \quad \Big| \quad X_t = Y_{v_t} = e^{r v_t + \hat{Z}_{v_t}}$$

In this framework, Y_t is RISK NEUTRAL if $\psi_{\hat{Z}}(-i) = 0$.

Risk Neutral condition for Y, also for X.

This is what the lemma tells us. (Then, "for free", X_t is Risk-Neutral (No condition on v).

Remark: @ t, to know Y_{v_t}, we need to know v_s with s from 0 to t, and Y_s with s from 0 to v_t. Hence the filtration generated by "v_t -&- M_{v_t}".

$$M_{v_t} = Y_{v_t}$$

Proof of the lemma:

$\mathcal{F}_t^M$: filtration generated by M_s, $0 \le s \le t$.

$\mathcal{F}_t^v$: " " v_s, $0 \le s \le t$.

$\mathcal{F}_t$: "full" filtration generated by $v_s, M_{v_s}, 0 \le s \le t$.

We want to prove that: $\mathbb{E}[M_{v_t} | \mathcal{F}_s] = M_{v_s}, \forall s \le t$.

$$\mathbb{E}[M_{v_t} | \mathcal{F}_s] \underset{\text{TOWER}}{=} \mathbb{E}\left[\mathbb{E}[M_{v_t} | \mathcal{F}_s \cup \mathcal{F}_t^v] | \mathcal{F}_s\right] = \mathbb{E}\Big[\underbrace{\mathbb{E}[M_{v_t} | \mathcal{F}_{v_s}^M \cup \mathcal{F}_t^v]}_{= M_{v_s} \text{ (M MG!)}} | \mathcal{F}_s\Big]$$

to use MG property

to make v_t deterministic

So: $\mathbb{E}[M_{v_t} | \mathcal{F}_s] = \mathbb{E}[M_{v_s} | \mathcal{F}_s] = M_{v_s}$. ■

RECAP: WHAT WE SHOULD BE ABLE TO DO IN PRICING:

Models

Lévy

Stoch. Vol → without jumps → Heston, Hull & White

Stoch. Vol → with jumps → Bates, BNS, Time Change

Algorithms

(↗ +2D PDE → Basket Options (Not in exam))

FFT

- Carr Madan
  - Lévy
  - Stoch Vol.

  for EUROPEAN.
- Conv → Lévy.

  For EU BARRIER.

PDE

Lévy

For EUROPEAN, BARRIER, AMERICAN (PSOR).

MC

we should be able to code it for:

Every Model

for ANY contracts.

(even, we should be able to write it for BNS). cf next page

MC for BNS

BNS: $dx_t = (\mu + \beta y_t)\,dt + \sqrt{y_t}\,dW_t + \rho\, dZ_t$

$$dy_t = -\lambda y_t\,dt + dZ_t$$

1) Simulate $(y_t)_t$: one way $\rightarrow$ Euler scheme :

$$y_{t+\Delta} = y_t - \lambda y_t \Delta + \underbrace{(Z_{t+\Delta} - Z_t)}_{\sim Z_\Delta}$$

2) Given y :

$$X_{t+\Delta} = X_t + (\mu + \beta y_t)\Delta + \sqrt{y_t}\,\sqrt{\Delta}\,\mathcal{N}(0,1) + \rho(Z_{t+\Delta} - Z_t)$$

Known using 1).

END OF COMPUTATIONAL FINANCE COURSE.

Pr. Daniele Marazzina.

Noé DEBROIS

AY 2024/2025