Tratto da: ‘Spitzer identity, Wiener-Hopf factorization and pricing of discretely monitored exotic options’

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Online Supplementary Material

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In this section we discuss other numerical methods presented in the literature which are based on Fourier and Hilbert transforms. We will not try to be exhaustive, but limit ourselves to those approaches that are most related to our own, and thus we will not cover e.g. the Cos method [2], as well as different approaches like the ones based on advanced quadrature [e.g. 1].

The general recursion pricing equation to compute the price (or cost) c of a plain vanilla derivative, such as an European call option, at time t given its value at time $t+\Delta$ can be computed from its price at time $t+\Delta$ using the backward-in-time density

$$c(x,t) = e^{-r\Delta} \int_{-\infty}^{+\infty} f_{\mathrm{b}}(x-x',\Delta) c(x',t+\Delta) dx'.$$

Here the derivative price is a function of the log-price x of the underlying asset and of the time t. This function $x \to c(x,t)$ is, in general, not square integrable and thus its Fourier transform does not exist. However, this problem can be worked around introducing the damped call price $C(x,t) = e^{\alpha x} c(x,t)$, $\alpha < 0$ being the so-called damping factor. The Fourier transform of the backward-in-time transition density $f_{\mathrm{b}}(x,\Delta) := f(-x,\Delta)$ is the conjugate $\Psi^*(\xi,\Delta)$ of the characteristic function. Therefore, in Fourier space the above equation becomes

$$\widehat{C}(\xi,t) = e^{-r\Delta} \Psi^*(\xi - i\alpha, \Delta) \widehat{C}(\xi, t+\Delta),$$

since

$$\begin{aligned}
\widehat{C}(\xi,t) &= \mathcal{F}_{x\to\xi}[e^{\alpha x} c(x,t)] = \int_{-\infty}^{+\infty} c(x,t) e^{ix(\xi - i\alpha)} dx \\
&= e^{-r\Delta} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{\mathrm{b}}(x-x',\Delta) c(x',t+\Delta) e^{ix(\xi-i\alpha)} dx' dx \\
&= e^{-r\Delta} \int_{-\infty}^{+\infty} f_{\mathrm{b}}(z,\Delta) e^{iz(\xi-i\alpha)} dz \int_{-\infty}^{+\infty} e^{i\xi x'} C(x',t+\Delta) dx' \\
&= e^{-r\Delta} \Psi^*(\xi - i\alpha,\Delta) \widehat{C}(\xi,t+\Delta),
\end{aligned}$$

changing the order of integration and defining $z = x - x'$; see Lord et al. [9] for further details. Therefore we have

$$c(\xi,t) = e^{-r\Delta} e^{-\alpha x} \mathcal{F}^{-1}_{\xi\to x}[\Psi^*(\xi - i\alpha,\Delta)\widehat{C}(\xi,t+\Delta)].$$

The methods considered in the following sections for pricing path dependent derivatives are based on the above described backward recursion from time $t+\Delta$ to time t. For ease of exposition, we will consider only a down-and-out put barrier option and we neglect the damping factor; for a down-and-out put option the damping factor is not necessary anyway, since the payoff and the option price are square integrable functions.

A. Dampen Payoff

In pricing derivatives, we are interested in the truncated damped payoff for a call and a put option

$$\phi(x) = e^{\alpha x} S_0 (e^x - e^k)^+ \mathbf{1}_{x \le u} \quad \text{and} \quad \phi(x) = e^{\alpha x} S_0 (e^k - e^x)^+ \mathbf{1}_{x \ge l}, \tag{1}$$

respectively, where $k = \log(K/S_0)$ is the rescaled log-strike of the option, and $l = \log(L/S_0)$ and $u = \log(U/S_0)$ are the rescaled lower and upper log-barriers.[1] The damping factor $e^{\alpha x}$ with a suitable choice of the parameter α makes the Fourier transform of the payoff well defined.

The Fourier transform of the truncated damped payoff for a barrier option is

$$\widehat{\phi}(\xi) = K e^{k(\alpha + i\xi)} \left(\frac{1 - e^{b(\alpha+i\xi)}}{\alpha + i\xi} - \frac{1 - e^{b(1+\alpha+i\xi)}}{1 + \alpha + i\xi} \right) \tag{2}$$

with $b = \log(U/K)$ for a call option and $b = \log(L/K)$ for a put option [3], or equivalently

$$\widehat{\phi}(\xi) = S_0 \left(\frac{e^{b(1+\alpha+i\xi)} - e^{a(1+\alpha+i\xi)}}{1 + \alpha + i\xi} - \frac{e^{k+b(\alpha+i\xi)} - e^{k+a(\alpha+i\xi)}}{\alpha + i\xi} \right) \tag{3}$$

with $a = \max(l, k)$, $b = u$ for a call option and $a = \min(k, u)$, $b = l$ for a put option [7, Equation (3.26)].

Therefore

$$\widehat{C}(\xi, T) = \widehat{\phi}(\xi)$$

is the starting point of the backward-in-time recursion.

B. Convolution and Hilbert transform

First of all, we briefly describe the convolution method [8, 9], as well as the method based on the Hilbert transform due to Feng and Linetsky [3]. Both are based on obtaining the option price recursively via

$$v(x, j) = e^{-r\Delta} \int_l^{+\infty} f_b(x - x', \Delta) v(x', j-1) dx' \, \mathbf{1}_{x>l}, \tag{4}$$

where $v(x, j)$ is the value of the option for the log-price x at time $(N - j)\Delta$. Therefore

$$v(x, j) = e^{-r\Delta} \mathcal{P}_\Omega \big(f(-x, \Delta) * v(x, j-1) \big) = e^{-r\Delta} \mathcal{P}_\Omega \mathcal{F}^{-1}_{\xi \to x} \big(\Psi^*(\xi, \Delta) \widehat{v}(\xi, j-1) \big), \tag{5}$$

[1] In the European case, we truncate the domain $\Omega = \mathbb{R}$ to the interval $[l, u]$, thus L and U are the truncating levels. In the Barrier case, L and U are the lower and upper barrier, respectively, if defined, otherwise the domain truncating levels.

where we recall that $*$ is the convolution operator and $\mathcal{P}_\Omega$ is the projector operator on $\Omega := (l, +\infty)$, i.e., $\mathcal{P}_\Omega f(x) = \mathbf{1}_{x\in\Omega} f(x)$. The indicator function $\mathbf{1}_{x\in\Omega}$ can be replaced by the Heaviside step function centered on l: it is 1 if $x > l$ and 0 if $x < l$, while for $x = l$ it can be assigned the values 0 (left-continuous choice), 1 (right-continuous choice) or 1/2 (symmetric choice). The value for $x = l$ matters only from a numerical point of view, as the measure of this point is zero.

At each time step the convolution method proceeds by moving from the real to the Fourier space and backward trough the iteration

$$v_{j-1} \xrightarrow{\mathcal{F}} \widehat{v}_{j-1} \xrightarrow{*} \Psi^* \widehat{v}_{j-1} \xrightarrow{\mathcal{P}\mathcal{F}^{-1}} v_j, \quad j = 1, \dots, N.$$

This method has been used, among others, by Jackson et al. [8] and Lord et al. [9]. Lord et al. improved this numerical methods in order to have a monotonic convergence to zero of the discretization error.

The method of Feng and Linetsky [3] is based on the Hilbert transform, Equation (26) in the article. In fact, considering the generalized Plemelj-Sokhotsky relation

$$\mathcal{F}\mathcal{P}_\Omega h = \frac{1}{2}\big[\mathcal{F}h + ie^{i\xi l}\mathcal{H}_\xi(e^{-i\xi l}\mathcal{F}h)\big],$$

the Fourier transform of Equation (5) yields

$$\widehat{v}(\xi, j) = \frac{1}{2}e^{-r\Delta}(\Psi^*(\xi,\Delta)\widehat{v}(\xi, j-1) + ie^{i\xi l}\mathcal{H}_\xi(e^{-i\xi l}\Psi^*(\xi,\Delta)\widehat{v}(\xi, j-1))).$$

Thus all the computations are in Fourier space:

$$v_0 \xrightarrow{\mathcal{F}} \widehat{v}_0 \longrightarrow \dots \longrightarrow \widehat{v}_{j-1} \xrightarrow{*} \Psi^*\widehat{v}_{j-1} \xrightarrow{\mathcal{H}} \widehat{v}_j \longrightarrow \dots \longrightarrow \widehat{v}_N \xrightarrow{\mathcal{F}^{-1}} v_N.$$

The Hilbert transform is computed in Fourier space via a sinc function expansion which provides an exponentially decaying error, as explained in Section 3.1 of the article. Therefore the Hilbert method is preferable to the convolution approach. The computational cost of both methods is $\mathcal{O}(NM \log M)$.

C. Quadrature methods

The recursion given by Equation (4) has been solved using quadrature [4, 6]. If the domain is truncated as in Fusai et al. [5], the quadrature nodes are x_i, $i = 1, \dots, M$, $\mathbf{K}$ is an $M \times M$ square matrix with elements $K_{ij} = e^{-r\Delta} f(x_j - x_i, \Delta)$, $\mathbf{D}$ is an $M \times M$ diagonal matrix which contains the quadrature weights, and $(\mathbf{v}_j)_i = v(x_i, j),\ j = 0, \ldots, N$, then Equation (4) becomes

$$\mathbf{v}_j = \mathbf{K}\mathbf{D}\mathbf{v}_{j-1} \tag{6}$$

for $j = 1, \ldots, N$. Thus, in order to compute the option price, one only has to perform N matrix-vector multiplications.

This approach can be efficiently implemented using the FFT, provided Newton-Cotes quadrature rules are considered. In fact, if the quadrature formula is characterized by equidistant nodes, $\mathbf{K}$ is a Toeplitz matrix and the matrix-vector multiplication in Equation (6) can be performed using the FFT as follows.

We recall that an $M \times M$ Toeplitz matrix $\mathbf{T}$ can be embedded in a $2M \times 2M$ circulant matrix $\mathbf{C}$, i.e. a special kind of Toeplitz matrix where each row vector is rotated one element to the right relative to the preceding row vector. Thus, given an $M \times 1$ vector $\mathbf{x}$, we can compute the component i of $\mathbf{Tx}$, $i = 1, \ldots, M$, as

$$(\mathbf{Tx})_i = \left(\mathrm{FFT}^{-1}\left(\mathrm{FFT}(\mathbf{c})\,\mathrm{FFT}(\mathbf{x}^*)\right)\right)_i,$$

$\mathbf{c}$ being the first column of the circulant matrix $\mathbf{C}$ and $\mathbf{x}^*$ being the extension of the vector $\mathbf{x}$ obtained padding $\mathbf{x}$ with M zeros. Thus Equation (6) becomes

$$(\mathbf{v}_j)_i = \left(\mathrm{FFT}^{-1}\left(\mathrm{FFT}(\mathbf{c})\,\mathrm{FFT}\left((\mathbf{D}\mathbf{v}_{j-1})^*\right)\right)\right)_i,$$

$i = 1, \ldots, M$, $\mathbf{c}$ being the first column of the circulant matrix embedding $\mathbf{K}$. Since $(\mathbf{K})_{i,j} = e^{-r\Delta} f(x_j - x_i, \Delta) = e^{-r\Delta} f((j-i)h, \Delta)$, h being the distance between the quadrature nodes, and f is computed with an inverse Fourier transform of the characteristic function Ψ, it follows that $\widehat{\mathbf{c}} := \mathrm{FFT}(\mathbf{c})$ can be computed directly by using Ψ, avoiding one FFT. At the end the computational cost of this pricing procedure becomes $2NM \log M$, since for each iteration of the pricing recursion we have to compute one FFT and one inverse FFT. We also have to compute the matrix-vector multiplication $\mathbf{D}\mathbf{v}_{j-1}$, but as $\mathbf{D}$ is a diagonal matrix, the computational cost consists of M operations. Thus the scheme of the quadrature-FFT based approach is

$$\mathbf{v}_{j-1} \longrightarrow \mathbf{D}\mathbf{v}_{j-1} \xrightarrow{\mathcal{F}} \mathcal{F}[\mathbf{D}\mathbf{v}_{j-1}] \xrightarrow{*} \widehat{\mathbf{c}}\mathcal{F}[\mathbf{D}\mathbf{v}_{j-1}] \xrightarrow{\mathcal{F}^{-1}} \mathbf{v}_j.$$

D. Bibliography

[1] Andricopoulos, A. D., M. Widdicks, D. P. Newton, P. W. Duck. 2007. Extending quadrature methods to value multi-asset and complex path dependent options. *Journal of Financial Economics* **83**(2) 471–499.

[2] Fang, F., C. W. Oosterlee. 2009. Pricing early-exercise and discrete barrier options by Fourier-cosine series expansions. *Numerische Mathematik* **114**(1) 27–62.

[3] Feng, L., V. Linetsky. 2008. Pricing discretely monitored barrier options and defaultable bonds in Lévy process models: a Hilbert transform approach. *Mathematical Finance* **18**(3) 337–384.

[4] Fusai, G., G. Longo, M. Marena, M. C. Recchioni. 2009. Lévy processes and option pricing by recursive quadrature. C. W. Hurlington, ed., *Economic Dynamics: Theory, Games and Empirical Studies*. Nova Publishers, New York, 31–57.

[5] Fusai, G., D. Marazzina, M. Marena, M. Ng. 2012. z-transform and preconditioning techniques for option pricing. *Quantitative Finance* **12**(9) 1381–1394.

[6] Fusai, G., M. C. Recchioni. 2007. Analysis of quadrature methods for pricing discrete barrier options. *Journal of Economic Dynamics and Control* **31**(3) 826–860.

[7] Green, R., G. Fusai, I. D. Abrahams. 2010. The Wiener-Hopf technique and discretely monitored path-dependent option pricing. *Mathematical Finance* **20**(2) 259–288.

[8] Jackson, K., S. Jaimungal, V. Surkov. 2008. Fourier space time-stepping for option pricing with Lévy models. *Journal of Computational Finance* **12**(2) 1–29.

[9] Lord, R., F. Fang, F. Bervoets, C. W. Oosterlee. 2008. A fast and accurate FFT-based method for pricing early-exercise options under Lévy processes. *SIAM Journal on Scientific Computing* **30** 1678–1705.