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B&S PDE:

European Call: $V(t,S)$

$$\begin{cases} \dfrac{\partial V}{\partial t} + rS\dfrac{\partial V}{\partial S} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} - rV = 0 \\ \forall S \in \mathbb{R}^+,\ \forall t \in [0,T]; \\ V(T,S) = (S-K)^+. \end{cases}$$

OR, $x = \log(S/S_0)$,

$v(t,x) := V(t, S_0 e^x)$:

$$\begin{cases} \dfrac{\partial v}{\partial t} + \left(r - \dfrac{\sigma^2}{2}\right)\dfrac{\partial v}{\partial x} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 v}{\partial x^2} - rV = 0 \\ \forall x \in \mathbb{R},\ \forall t \in [0,T], \\ v(T,x) = (S_0 e^x - K)^+ \end{cases}$$

Finite Difference:

Relics of Taylor Formula:

$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + \frac{\Delta x^3}{6} f'''(x) + \frac{\Delta x^4}{24} f''''(x) \ldots \quad (A)$$

$$f(x-\Delta x) = f(x) - \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) - \frac{\Delta x^3}{6} f'''(x) + \frac{\Delta x^4}{24} f''''(x) \ldots \quad (B)$$

f' approximation

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

From (A) :

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} \underbrace{- \frac{\Delta x}{2} f''(x) - \frac{\Delta x^2}{6} f'''(x) + \dots}_{= \mathcal{O}(\Delta x)}$$

so :

$$\boxed{f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} + \mathcal{O}_{\Delta x \to 0}(\Delta x)} \quad \text{(i) Forward}$$

From (B) :

$$\boxed{f''(x) = \frac{f(x) - f(x-\Delta x)}{\Delta x} + \mathcal{O}_{\Delta x \to 0}(\Delta x)} \quad \text{(ii) Backward}$$

(A) - (B) :

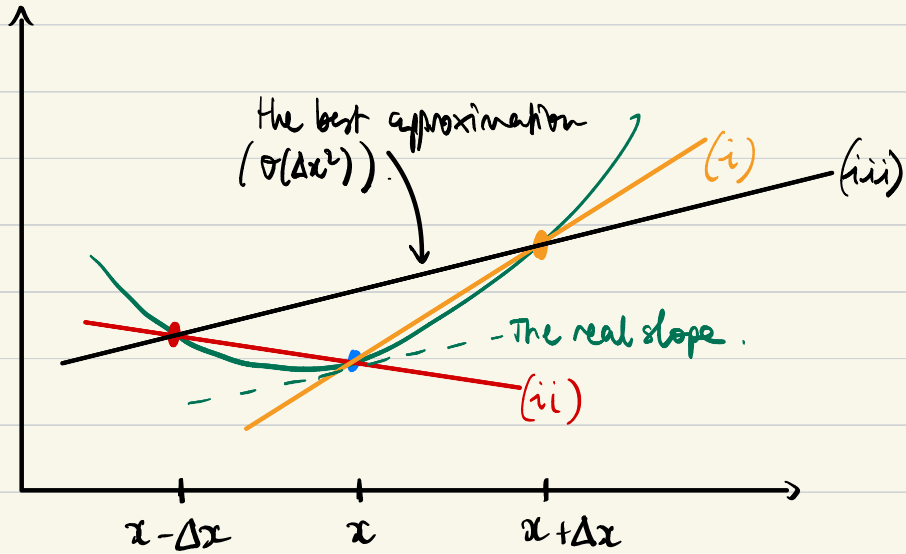
$$f(x+\Delta x) - f(x-\Delta x) = 2\Delta x f'(x) + \frac{\Delta x^3}{3} f'''(x) + \dots$$

So :

$$\boxed{f'(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} + \mathcal{O}(\Delta x^2)} \quad \text{(iii) Centered}$$

(i), (ii) & (iii) are 3 ways to <u>approximate</u> f'.

Geometrical point of view :



A + B : $f(x+\Delta x) + f(x-\Delta x) = 2f(x) + \cancel{\Delta x f'(x)}$

$$\cancel{-\Delta x f'(x)} + \cancel{2}\frac{\Delta x^2}{\cancel{2}} f''(x) + 2\frac{\Delta x^4}{24} f''''(x) + \dots$$

from A + B we can get :

$$\boxed{f''(x) = \frac{f(x+\Delta x) + f(x-\Delta x) - 2f(x)}{\Delta x^2} + \underset{\Delta x \to 0}{\theta(\Delta x^2)}}$$ (iv)

We can use these schemes (approx.) to deal with PDEs, like B&S' PDE.

29/10/2024 • We are going to price $\begin{cases} \text{EUROPEAN} \\ \text{BARRIER} \\ \text{AMERICAN} \end{cases}$

continuous-monitoring, under B&S, using PDE.

• Then we will move to Lévy, using PIDE: European, Barrier, American.

Finite Difference (scheme):

$$\left.\begin{array}{l} \left.\begin{array}{l} \dfrac{f(x+\Delta x)-f(x)}{\Delta x} \\[2ex] \dfrac{f(x)-f(x-\Delta x)}{\Delta x} \end{array}\right\} \text{1st order.} \\[4ex] \dfrac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x} \quad \text{2nd order.} \end{array}\right\} \simeq f'(x).$$

$$\frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2} \simeq f''(x).$$ 2nd order.

Let $x = \log(S/S_0)$. Under these assumptions, a European Call Option under B&S satisfies the following PDE: ($V = V(t,x)$: price of EU Call)

$$\begin{cases} \dfrac{\partial V}{\partial t} + \left(r - \dfrac{\sigma^2}{2}\right)\dfrac{\partial V}{\partial x} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 V}{\partial x^2} - rV = 0, \quad \forall x \in \mathbb{R}, \forall t \in [0,T] \\ V(T,x) = (S_0 e^x - K)^+ \end{cases}$$

We want to solve numerically ...

I) TRUNCATION :

$\forall x \in \mathbb{R} \longrightarrow \forall x \in [x_{MIN}, x_{MAX}]$.

$\forall S \in \mathbb{R} \longrightarrow \forall S \in [S_{MIN}, S_{MAX}]$.

$$x_{MIN} = \log\left(\frac{S_{MIN}}{S_0}\right) \quad \& \quad x_{MAX} = \log\left(\frac{S_{MAX}}{S_0}\right).$$

How to choose MIN/MAX ?

1) By experience : $S_{MIN} = 0.2 \times S_0$, $S_{MAX} = 3S_0$,
i.e, we assume the spot price has a negligible probability of falling below $20\% \, S_0$, and rising above $3S_0$.

⚠ If the interval [min, max] is large, I will need a very large N to have an acceptable Δx ...

2) By theory : we will do it under B&S.

Let $Z \sim \mathcal{N}(0,1)$, then :

$$\mathbb{P}(Z < -6) = \mathbb{P}(Z > 6) \simeq 10^{-8} \quad : \text{NEGLIGIBLE}.$$

↳ since $S_T = S_0 \, e^{(r - \sigma^2/2)T + \sigma\sqrt{T} Z}$, I can set :

$$\begin{cases} S_{MIN} = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)T - 6\sigma\sqrt{T}\right) ; \\ S_{MAX} = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + 6\sigma\sqrt{T}\right) . \end{cases}$$