32' of 21/09/2024 RECORDING

01/10/2024

RECAP: • Lévy-Itô decomposition: (γ, A, ν) gives $X_t = \dots$

1st way to describe Lévy proc.: via triplet.

⚠ For all Lévy:

$$\int_{|x|>1} \nu(dx) < +\infty$$

• 2 cases:

- Finite Activity: $\int_{|x|<1} \nu(dx) < +\infty$;
- Infinite Activity:

$$\int_{|x|<1} |x|^2 \nu(dx) < +\infty$$

but $\int_{|x|<1} |x| \nu(dx) = +\infty$. ⚠

Finite variation: $\int_{|x|<1} |x| \nu(dx) < +\infty$

In these two cases, we can write:

$$\lim_{\varepsilon \to 0^+} \int_{\substack{[0,t]\times\mathbb{R}^d \\ \varepsilon<|x|<1}} x\left(J_X(dx \times ds) - \nu(dx)ds\right)$$

bc everything converges

$$= \overbrace{\int_{\substack{[0,t]\times\mathbb{R}^d \\ |x|<1}} x J_X(dx\times ds)}^{CV} - \overbrace{t\int_{|x|<1} x\nu(dx)}^{CV}$$

2nd way to describe Lévy proc.: via caracteristic exponent.

• Lévy-Khincin Formula: which gives us the caracteristic exponent.

Subordinator :

- We will use subordinator for variance and construction of Lévy processes.
- How will we construct Lévy proc.? Time change.
  Assume we have a process $(X_t)_t$ (ex: BM), and another one : $(S_t)_{t\geq 0}$.
  We can build $(X_{S_t})_{t\geq 0}$. ⚠ if time goes back...
  So we need a positive & non-decreasing time process.

THM: Let $(X_t)_{t\geq 0}$ be a Lévy process (γ, A, ν). We have 4 equivalent conditions :

(i) $X_t \geq 0$ a.s $\forall t > 0$;

(ii) $\exists t > 0$: $X_t \geq 0$ a.s ;

(iii) $(X_t)_t$ non-decreasing ;

(iv) X_t is a finite variation process with $\nu([-\infty, 0]) = 0$ (non-neg. jumps) &

$$\mu = \left(\gamma - \int_{|x|<1} x \, \nu(dx)\right) > 0 .$$

Proof:
- (i) => (ii) is trivial.
- (iii) => (i) is trivial ($\nearrow$ & $X_0 = 0$).
- Let's prove $\boxed{(ii) \Rightarrow (iii)}$:

Let $\bar{t} > 0$ s.t $X_{\bar{t}} \geqslant 0$ a.s.

Let $n \in \mathbb{N}^*$, $\Delta = \frac{\bar{t}}{n}$. So:

$$0 \leqslant X_{\bar{t}} = X_{n\Delta} = \left(X_{n\Delta} - X_{(n-1)\Delta}\right) + \left(X_{(n-1)\Delta} - \dots + \left(X_{2\Delta} - X_{\Delta}\right) + \left(X_{\Delta} - X_0\right)\right)$$

[summation of n increments]

Because all incr. are $\perp\!\!\!\perp$ and their sum $\geqslant 0$.

So what we have is that:

$$\left(X_{j\Delta} - X_{(j-1)\Delta}\right) \geqslant 0 \quad \forall j \in \{1, \dots, n\}.$$

Basically we can prove that:

$$\left(X_{p\bar{t}} - X_{q\bar{t}}\right) \geqslant 0 \quad \forall p, q \in \mathbb{Q}, \ p > q.$$

And given the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$,

$(X_t - X_s) \geqslant 0$, $\forall t > s$. $\boxed{\text{(iii) is proven}}$.

So (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). Now we want to prove that (iii) $\Leftrightarrow$ (iv).

- $\boxed{(iv) \Rightarrow (iii)}$:

$$X_t = \mu t + \underset{\text{no BM because FINITE VARIATION}}{0} + \int_{[0,t] \times \mathbb{R}^d} x \, J_X(dx \times ds) = \mu t + \sum_{s \in [0,t]} \Delta X_s$$

$$X_t = \underbrace{\mu t}_{\substack{>0 \\ \text{increasing}}} + \underbrace{\sum_{\substack{s\in[0,t] \\ |\Delta X_s|>0}} \Delta X_s}_{>0,\ \text{non decreasing}}$$

Because by assumpt°, there is no negative jump.

Therefore, (iii) holds true.

- $\boxed{(iii) \Rightarrow (iv):}$

[thm: if $(X_t)_t$ is non-decreasing and right-c° process, then $(X_t)_t$ is a finite Variation process.]

$$(iii) \Rightarrow \underbrace{\mu t}_{>0} + \underbrace{\sum_{s\in[0,t]} \Delta X_s}_{\text{non-decreasing}} .$$

$$\Updownarrow$$

$$\nu((-\infty,0]) = 0$$

Therefore (iv) holds. ■

Constructing a subordinator:

THM: Let $(X_t)_{t\geq 0}$ be a Lévy Process in $\mathbb{R}^d$. Let $f : \mathbb{R}^d \longrightarrow [0,+\infty)$ be a positive function such that $f(x) = O(|x|^2)$ in a neighborhood of 0. Then $S_t = \sum_{\substack{s\leq t \\ \Delta X_s \neq 0}} f(\Delta X_s)$ is a subordinator.

Rmk: X_t (γ, A, ν). Lévy Khincin Representation:

$$\Psi(z) = i\gamma\cdot z - \frac{1}{2} z^T A z + \int_{\mathbb{R}^d} \left(e^{iz\cdot x} - 1 - iz\cdot x \mathbb{1}_{\{|x|<1\}}\right)\nu(dx)$$

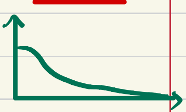
we chose that "small jumps" $\Leftrightarrow$ $|x| < 1$

Let $g : x \mapsto \mathbb{1}_{\{|x|<1\}}$: we could do the same with $|x| < 1/2$, ... in fact we can choose

any g of the form:

$g : \mathbb{R}^d \longrightarrow \mathbb{R}$ s.t $g(x) \underset{x\to 0}{=} 1 + o(|x|)$ &

$g(x) \underset{x\to+\infty}{=} O\left(\frac{1}{|x|}\right)$



$$\Psi(z) = i\gamma \cdot z - \frac{1}{2} z^T A z + \int_{\mathbb{R}^d} e^{izx} - 1 - izx \mathbb{1}_{\{|x|<1\}} + \underbrace{(izx\, g(x) - izx\, g(x))}_{+0} \nu(dx)$$

$$\Psi(z) = \underset{\text{drift}}{i\gamma \cdot z} - \underset{\text{BM}}{\frac{1}{2} z^T A z} + \underset{\text{integral over jumps}}{\int_{\mathbb{R}^d} e^{izx} - 1 - izx\, g(x)\, \nu(x)}$$

$$+ \underbrace{\int_{\mathbb{R}^d} i(z \cdot x)\left(g(x) - \mathbb{1}_{\{|x|<1\}}\right) \nu(dx)}$$

$$= iz \cdot \int_{\mathbb{R}^d} x \left(g(x) - \mathbb{1}_{\{|x|<1\}}\right) \nu(dx)$$

$$\Psi(z) = i z^T \tilde{\gamma} - \frac{1}{2} z^T A z + \int_{\mathbb{R}} \left(e^{iz \cdot x} - 1 - iz \cdot x\, g(x)\right) \nu(dx)$$

with $\tilde{\gamma} = \gamma - \int_{\mathbb{R}^d} x \left(\mathbb{1}_{\{|x|>1\}} - g(x)\right) \nu(dx)$

So we obtain an analogous formula to Lévy Khincin but with a separation between SMALL & LARGE jumps which is g.

- $\mathbb{1}_{\{|x|<1\}}$ $\quad (\gamma, A, \nu)$

- $g(x)$ $\quad (\tilde{\gamma}, A, \nu)$ $\quad$ **EXAMPLE:** $g(x) = \mathbb{1}_{\{|x|<\frac{1}{2}\}}$.

Then we get: $\tilde{\gamma} = \gamma - \int_{\mathbb{R}^d} x \, \mathbb{1}_{\{\frac{1}{2} \leq |x| < 1\}} \, \nu(dx)$

$$= \gamma - \int_{\frac{1}{2} \leq |x| < 1} x \, \nu(dx) \in \mathbb{R}.$$

So the message is that all the theory that we developed using a separation of 1 can be redeveloped if we change the separation: it's not a problem! Only γ changes, but not A and not ν.

Why do we need this Remark for the previous THM? Because f can change a jump of $3/4$ to a jump of 1.5 for ex. According to the previous remark, it's not a pb.

Proof: $(X_t)_{t\geq 0}$ is a Lévy. $S_t = \sum_{\substack{s\leq t \\ \Delta X_s \neq 0}} f(\Delta X_s)$.

$f(x) \underset{x\to 0}{=} O(|x|^2) \iff \exists \varepsilon > 0, \exists C > 0 :$

$$\forall x \in (-\varepsilon, \varepsilon),\ f(x) \leq Cx^2.$$

So : $f(\Delta X_s) \leq C(\Delta X_s)^2 \quad \forall \Delta X_s \neq 0$ s.t $|\Delta X_s| \leq \varepsilon$.

$(X_t)_t$ is Lévy : $\int_{|x|<1} x^2 \nu(dx) < +\infty$ i.e :

$$\sum_{\substack{s\in[0,t) \\ \Delta X_s \neq 0 \\ |\Delta X_s|<1}} |\Delta X_s|^2 < +\infty.$$

$\mathbb{1}_{\{|x|<1\}}$

So :

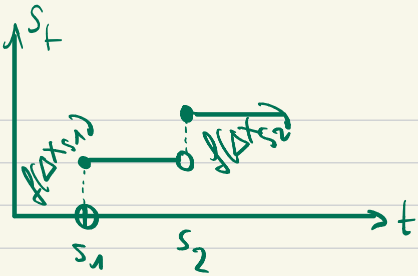
$$\sum_{\substack{s\in[0,t) \\ \Delta X_s \neq 0 \\ |\Delta X_s|<\varepsilon}} f(\Delta X_s) \leq C \sum_{\substack{s\in[0,t) \\ \Delta X_s \neq 0 \\ |\Delta X_s|<\varepsilon}} |\Delta X_s|^2 < +\infty$$

$\mathbb{1}_{\{|x|<\varepsilon\}}$

Using the previous Rmk :

NOT A PROBLEM.

$$S_t = \sum_{\substack{s \leq t \\ \Delta X_s \neq 0}} f(\Delta X_s)$$



⚠ ΔX_s iid $\Rightarrow f(\Delta X_s)$ iid.

Proposition: if $(X_t)_t$ is s.t $\int_{|x|<1} x^2 \nu(dx) < +\infty$ then:

$(S_t)_t$ is s.t $\int_{|x|<1} x^2 \hat{\nu}(dx) < +\infty$.

Where ν is the Lévy measure for X, $\hat{\nu}$ for S.

So, now, let's "meet" some Lévy processes:

Lévy Process → Jump Diffusion : $X_t = \mu t + \sigma W_t + \sum_{s \in [0,t)} \Delta X_s$.

→ Pure jumps / Finite activity : $\cancel{X_t = \mu t + \sum_{s \in [0,t]} \Delta X_s}$.

→ Infinite activity.

⚠ Keep in mind that for us, $X_t = \log\left(\frac{S_t}{S_0}\right)$.

so the 2nd option is not interesting.