Think about this!

$T_i(\omega)$: i^{th} jump occurs given ω.

$\forall A$, $M(\omega, A) = \#\{i \geq 1 : T_i(\omega) \in A\}$.

INTEGER-VALUED RANDOM MEASURE.

We can write : $N_t(\omega) = M(\omega, [0,t]) = \int_0^t M(\omega, ds)$.

DEF: Compensated Random measure :

$$\hat{M}(\omega, A) = M(\omega, A) - \int_A \lambda \, dt = M(\omega, A) - \lambda|A|.$$

Now we are ready to introduce "Levy processes".

Lévy Processes

DEF: Let $(\Omega, \mathcal{F}, \mathbb{P})$ a proba. space, $\Omega \subseteq \mathbb{R}^d$.
A CADLAG process $(X_t)_{t \geq 0}$ s.t $X_0 = 0$ is a Lévy if:

i) Increments are independent: $0 \leq t_0 \leq t_1 \leq \dots \leq t_n$
$X_{t_0} \perp\!\!\!\perp X_{t_1} - X_{t_0} \perp\!\!\!\perp \dots \perp\!\!\!\perp X_{t_n} - X_{t_{n-1}}$;

ii) Increments are stationary:
$\forall t, h > 0$, $X_{t+h} - X_t \sim X_h$

Continuité en proba. cf wiki

iii) Stochastic continuity: $\forall t > 0$,
$\forall \varepsilon > 0$, $\mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) \xrightarrow[h \to 0]{} 0$.

Example: $\cdot (N_t)_{t \geq 0}$; $\cdot (W_t)_{t \geq 0}$: only lèvy proc. which is $\mathcal{C}^0$.

PROP: $\cdot$ We have infinite divisibility: $t > 0, n \geq 1$,
$\Delta = t/n$, $X_t = X_{n\Delta} = (X_{n\Delta} - X_{(n-1)\Delta}) + X_{(n-1)\Delta} \dots$
Y_n $\quad$ Y_2 $\quad + (X_{2\Delta} - X_{\Delta}) + X_{\Delta}$

So : $X_t = \sum_{i=1}^{n} Y_i$, where $Y_i = X_{i\Delta} - X_{(i-1)\Delta}$

and : • Y_i are <u>independent</u>;

• $Y_i \sim X_\Delta$ $\forall i$: <u>stationary</u>;

i.e : $(Y_i)_i$ are iid.

<u>Rmk</u>: The infinite divisibility is precisely the ability to write X_t as a sum of iid terms.

• <u>Characteristic function of a Levy process</u>:

$\forall u \in \mathbb{R}^d$, $\phi_{X_t}(u) = \mathbb{E}\left[e^{iuX_t}\right]$.

– ϕ_{X_t} is <u>multiplicative</u>:

$$\phi_{X_{t+s}}(u) = \mathbb{E}\left[e^{iuX_{t+s}}\right] = \mathbb{E}\left[e^{iu(X_{t+s}-X_t+X_t)}\right]$$

$$= \mathbb{E}\left[e^{iu(X_{t+s}-X_t)}\, e^{iuX_t}\right]$$

$$\overset{\perp\!\!\!\perp}{=} \mathbb{E}\left[e^{iu\underbrace{(X_{t+s}-X_t)}_{\sim X_s \text{ by def.}}}\right] \mathbb{E}\left[e^{iuX_t}\right]$$

$$= \phi_{X_s}(u)\,\phi_{X_t}(u).$$

- **characteristic exponent for Lévy proc.:**

$\exists \, \Psi_X : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that :

$$\phi_{X_t}(u) = e^{t \Psi_X(u)} .$$

t is "outside", which is really useful.
That's the reason why we will never write the characteristic fct of a Lévy proc.
Instead, we'll use the characteristic exponent, from which we can easily get ϕ_{X_t} .

Example: Poisson:

$\phi_{X_t}(u) = e^{\lambda t(e^{iu} - 1)}$ so that : $\Psi_X(u) = \lambda(e^{iu} - 1)$.

So we are able to count the jumps when they occur. Now we would like to capture the size of the jumps.

Compounded Poisson Process:

DEF: A compounded Poisson Process (CPP) $(X_t)_{t\geq 0}$ with: $\begin{cases} \bullet \text{ Intensity } \lambda; \\ \bullet \text{ Jumps size distribut° } f; \end{cases}$ is s.t

$X_t = \sum_{i=1}^{N_t} Y_i$ where $(N_t)_{t\geq 0}$ is a counting process with intensity λ, and Y_i iid R.V with distribution f.

PROP:
- $(X_t)_{t\geq 0}$ is for sure CADLAG,
- Poisson proc. is a CPP where f is s.t $\mathbb{P}(Y_i = 1) = 1$.
- $(X_t)_{t\geq 0}$ is a CPP iff $(X_t)_{t\geq 0}$ is a lévy process with piecewise constant paths.

How to compute ϕ of a CPP?

PROP:

$$\phi_{X_t}(u) = e^{t \underbrace{\left\{ \lambda \int_{\mathbb{R}^d} (e^{iu\cdot x} - 1) f(dx) \right\}}_{\psi_X(u)}}.$$

Proof:

$$\phi_{X_t}(u) = \mathbb{E}[e^{iuX_t}] = \mathbb{E}[\mathbb{E}[e^{iuX_t} | N_t]]$$

$$= \mathbb{E}[\mathbb{E}[e^{iu\sum_{i=1}^{N_t} Y_i} | N_t]] \underset{\text{measurable wrt } N_t.}{=} \mathbb{E}\left[\prod_{i=1}^{N_t} \mathbb{E}[e^{iuY_i}]\right]$$

$Y_i \perp\!\!\!\perp N_t$ — we don't condition anymore because N_t does not appear.

Y_i iid

$$= \mathbb{E}\Big[\underbrace{\mathbb{E}[e^{iuY_1}]}_{=\hat{f}(u)}{}^{N_t}\Big] = \mathbb{E}[\hat{f}(u)^{N_t}]$$

$= \hat{f}(u)$: we'll compute it afterwards.

Transfer thm

$$= \sum_{n=0}^{+\infty} \underbrace{e^{-\lambda t}\frac{(\lambda t)^n}{n!}}_{\mathbb{P}(N_t=n)}(\hat{f}(u))^n \underset{\text{CV. series}}{=} e^{\lambda t(\hat{f}(u)-1)}$$

Transfer thm

$$\hat{f}(u) = \mathbb{E}[e^{iuY_1}] = \int_{\mathbb{R}^d} e^{ius} f(ds)$$ so putting

everything together, we get :

$1 = \int_{\mathbb{R}^d} f(ds)$

$$\phi_{X_t}(u) = e^{\lambda t\left(\int_{\mathbb{R}^d} e^{ius} f(ds) - 1\right)}$$

$$= e^{\lambda t\left(\int_{\mathbb{R}^d}(e^{ius}-1)f(ds)\right)} \quad \blacksquare$$

<u>DEF</u>: The Lévy measure of a CPP is :

$$\nu(A) = \lambda f(A) \quad , \forall A.$$ ⚠ Not a proba measure.

So that the characteristic exponent is :

$$\Psi_X(u) = \int_{\mathbb{R}^d} (e^{iux}-1)\,\nu(dx)$$

<u>Rmk</u>: In most of the case : $\exists K$ s.t $f(dx) = K(x)dx$: easier...

$\mathbb{E}[\text{nb de saut}, \neq 0, \in A, @\, t\in[0,1]] = \nu(A) = \underline{\text{levy measure}(A)}$.

PROP: About the <u>Lévy measure</u> of a CPP:

- $\nu(A) = \mathbb{E}\left[\#\{t\in[0,1] : \Delta X_t \neq 0, \Delta X_t \in A\}\right]$

where $\Delta X_t = X_t - X_{t^-} = X_t - \lim_{h\to 0^+} X_{t-h}$.

1 réalisation nous donne une mesure

- $J_X(B) = \#\{(t, \Delta X_t) \in B\}$, $B \in \underset{\text{time}}{[0,+\infty)} \times \underset{\text{jump}}{\mathbb{R}^d}$

Multidim. random measure.

How do we get this $\circled{=}$?

$$X_t = \underbrace{\sum_{i=1}^{N_t} Y_i}_{\sum \text{jumps}} = \underbrace{\sum_{s\in[0,t]} \Delta X_s}_{\sum \text{jumps}} \overset{?}{=} \int_{[0,t]\times\mathbb{R}^d} x\, J_X(ds \times dx).$$

30/09/24 Now we want to write Lévy processes.

The first possibility is:

DEF: <u>FINITE ACTIVITY LÉVY:</u> $X_t = \mu t + \sigma W_t + \underbrace{\sum_{i=1}^{N(t)} Y_i}_{\text{CPP part (cf before)}}$ (1).

Remember: $X_t = \log\frac{S_t}{S_0}$

This writing is ok for finite activity.

LÉVY:

This is equivalent to write:

$$X_t = \mu t + \sigma W_t + \underbrace{\sum_{s\in[0,t]} \Delta X_s}_{\text{F.A}: \in\mathbb{R}} \quad (2).$$

Main problem:

$$\sum_{s\in[0,t]} \Delta X_s$$

And we know that we can

Mesure aléatoire : $\Omega \to M(E)$ où $M(E)$ = espace des mesures.

also exploit the Random measure :

All the Lévy proc. can be written in this way! (Not the other forms)

the other 2 forms are only for FA Lévy.

$$X_t = \mu t + \sigma W_t + \int_{[0,t]\times\mathbb{R}^d} x \, J_X(ds \times dx) \quad (3).$$

J_X is the Poisson random measure with intensity $\nu(dx)\,dt$.

Lévy measure (cf def 2 pages above).

Now let's consider General Lévy Processes :

We can ALWAYS define $\nu(.)$: Lévy measure.

$\longrightarrow$ $\nu(A) < +\infty$ $\forall$ compact set $\begin{cases} A \subseteq \mathbb{R}^d, \\ 0 \notin A. \end{cases}$

(Any compact set from $\mathbb{R}^d$ not containing 0)

$\longrightarrow$ Remember previously :

$$\nu(A) = \mathbb{E}\left[\#\left\{ s \in [0,1], \Delta X_s \neq 0 \,,\, \Delta X_s \in A \right\} \right].$$

$\hookrightarrow$ $\cancel{\nu([1,2]) = +\infty}$ NOT POSSIBLE.

DEF : INFINITE ACTIVITY LÉVY :

- Finite number of "large jumps" ;

For the moment we choose the jumps of size ≥ 1.

- Infinite number of "infinitesimal jumps".

PROP: **Lévy Ito Decomposition:** (of infinite activity lévy)

Let $(X_t)_t$ be a Lévy process in $\mathbb{R}^d$, ν its Lévy measure. Then:

- ν is a Radon measure on $\mathbb{R}^d \setminus \{0\}$;
- $\int_{|x|\geq 1} \nu(dx) < +\infty$;
- $\int_{|x|<1} |x|^2 \nu(dx) < +\infty$; $\quad$ Finite Activity: $\left[\Leftarrow \int_{|x|<1} |x| \nu(dx) < +\infty \Leftarrow \int_{|x|<1} \nu(dx) < +\infty \right]$
- $\exists \gamma \in \mathbb{R}^d$, $A \in \mathbb{R}^{d \times d}$ (A is a VARIANCE COVARIANCE MATRIX) and let $(B_t)_t$ a Brownian Motion (BM) with A as VARIANCE-COV. MATRIX, such that:

$$X_t = \gamma t + B_t + X_t^{\ell} + \lim_{\varepsilon \to 0} \tilde{X}_t^{\varepsilon}$$

where

$$X_t^{\ell} = \int_{\substack{|x|\geq 1 \\ s\in[0,t]}} x \, J_X(dx \times ds) = \underbrace{\sum_{\substack{s\in[0,t] \\ |\Delta X_s|\geq 1}} \Delta X_s}$$

We can use that w/o any pb bc. the large jumps are in finite nb.

Compound Poisson

and:

$$X_t^{\varepsilon} = \int_{\substack{\varepsilon<|x|<1 \\ s\in[0,t]}} x \, J_X(dx \times ds)$$

$\hookrightarrow \overset{\wedge}{=} \sum_{s\in[0,t], \varepsilon<|\Delta X_s|<1} \Delta X_s \, {}^{\nearrow}$: but we cannot use that writing ⚠