COMPUTATIONAL FINANCE

ACADEMIC YEAR 2024/2025

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A bit of history...

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- We want to model a stock w/ maths.
  ↳ PhD from Louis Bachelier: 1900

$$\underbrace{S_t}_{\text{stock @ } t} = \underbrace{S_0}_{\text{stock today}} + \sigma \underbrace{W_t}_{\text{Wiener}}$$

ABM

⚠ $\neq 0$ probability that $S_t < 0$.

- Nothing changed till 1973 with Black, Scholes & Merton:

$$S_t = S_0 e^{\mu t + \sigma W_t}$$

. Equivalently: $\frac{dS_t}{S_t} = (\mu + \frac{\sigma^2}{2})dt + \sigma dW_t$.

GBM

- Then, with Dupire in 1994, came the "Local Volatility Model":

$$dS_t = \hat{\mu} S_t dt + \sigma(t, S_t) dW_t.$$

- But concurrently, there was the idea of modeling the stock' price not continuously, but with jumps.

Why are these models important? Bc. there are analytical formulas for a large nb. of derivatives, using GBM.

Example:

European Call: (T, K), S

$$\begin{cases} T > 0 & \text{maturity} \\ K & \text{strike} \end{cases}, \quad \begin{cases} -\ t = 0: \text{ purchase of the contract} \\ -\ T > 0: \max(S_T - K, 0) \end{cases}$$

Question: What is the price of this contract?

It is not correct, but was used for a long time!

$\Big\{$ Under **B&S**: $P = S_0 \mathcal{N}(d_1) - K e^{-rT} \mathcal{N}(d_2)$.

closed formula — cdf of $\mathcal{N}(0,1)$

[But when we will introduce jumps, we won't we able to use closed formula anymore.

We will need:

- M.C.
- Methods based on F.T.
- Methods based on PDE.

That's why this course exists!

B&S formula → **Econometric way**: $dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$ i.e with Itô:

$d(\log(S_t)) = \hat{\mu}\,dt + \sigma\,dW_t$ i.e $\log(S_t) = \log(S_0) + \hat{\mu} t + \sigma W_t$

i.e $\log\left(\frac{S_t}{S_0}\right) = \hat{\mu} t + \sigma W_t$ i.e $\int_{\hat{s}}^{\hat{s}+\Delta} d\log(S_t) = \int_{\hat{s}}^{\hat{s}+\Delta} \hat{\mu}\,dt + \int_{\hat{s}}^{\hat{s}+\Delta} \sigma\,dW_t$

Let's see why it's not correct.

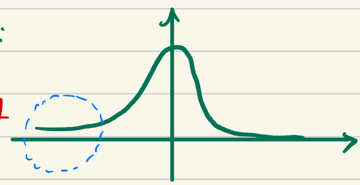
i.e $\boxed{\log\left(\frac{S_{\hat{s}+\Delta}}{S_{\hat{s}}}\right) = \hat{\mu}\Delta + \sigma\left(W_{\hat{s}+\Delta} - W_{\hat{s}}\right)}$

usually $\Delta = 1/252$ (252 trading days).

DEF: $\overbrace{\log\left(\frac{S_{\hat{s}+\Delta}}{S_{\hat{s}}}\right)}^{D_{\hat{s}}}$: "daily log returns" $\sim \mathcal{N}(\hat{\mu}\Delta, \sigma^2\Delta)$ bc. $W_{\hat{s}+\Delta} - W_{\hat{s}} \sim \mathcal{N}(0,\Delta)$

Rmk: You can get market prices and take the log returns to see if it agrees with the normal distribution. If you do this, you'll see that in fact there is NO SYMMETRY:

!! bigger tail on the left



"Daily log returns"

$$\left.\begin{aligned} \Pr\left(D_t > \hat{\mu}\Delta + 6\sigma\sqrt{\Delta}\right) &\simeq 10^{-8} \\ < \hat{\mu}\Delta - 6\sigma\sqrt{\Delta} &\simeq 10^{-8} \end{aligned}\right\} \text{under B\&S...}$$

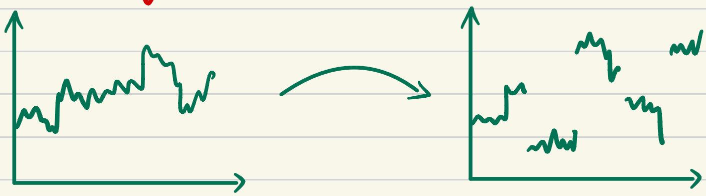
If you look at the data you'll see that the probability of $D_t < \hat{\mu}\Delta - 6\sigma\sqrt{\Delta}$ is bigger.

⚠ So the B&S model, based on GBM, under-estimates the probability of having large values (in |.|) in daily log returns.

+ we don't have symmetry IRL

How to solve this problem?

One way is the introduction of jumps.



Another assumption.

It does not exist a "100%-correct" assumption.

B&S formula $\longrightarrow$ Formula itself:

Let's see it is not correct by another way.

CALIBRATION:
$$dS_t = \underbrace{\hat{\mu}}_{} S_t\,dt + \sigma S_t\,dW_t \quad \text{under } \mathbb{P}$$
$$\hat{\mu} = \mu + \frac{\sigma^2}{2}$$

$\Big\downarrow$ cf MF2.

PRICE OF THE CALL OPTION WE SAW EARLIER:
$$P = e^{-rT}\,\mathbb{E}_0^{\mathbb{Q}}\left[(S_T - K)^+\right]$$

$\mathbb{Q}$ measure: riskfree measure

i.e the measure under which:
$$\mathbb{E}_0^{\mathbb{Q}}\left[S(T)\right] = \mathbb{E}^{\mathbb{Q}}\left[S(T)\,|\,\mathcal{F}_0\right] = S_0$$

Under $\mathbb{Q}$, we can easily compute:
$$dS_t = \underline{\underline{r}}\,S_t\,dt + \sigma\,S_t\,dW_t \qquad (\text{GBM}, r: \text{riskfree IR}).$$

only parameter we have to calibrate.

$$P = S_0 N(d_1) - Ke^{-rT} N(d_2).$$

? Let's find a σ such that $P_{MODEL} = P_{MARKET}$, for any derivatives written on the same underlying asset.

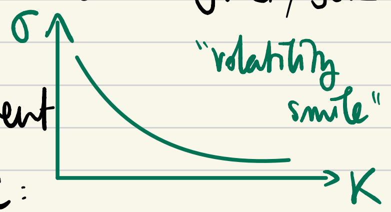
B&S formula is invertible!

$(K_1, T_1) \longrightarrow$ find the mkt price $P_1 \longrightarrow$ Invert B&S: $\sigma_1 \Rightarrow P_1 = P^{BS}_{\sigma_1}$

$(K_2, T_2) \longrightarrow$ " " $P_2 \longrightarrow \sigma_2 \Rightarrow P_2 = P^{BS}_{\sigma_2}$

$(K_3, T_3) \longrightarrow$ " " $P_3 \longrightarrow \sigma_3 \Rightarrow P_3 = P^{BS}_{\sigma_3}$

This is a proof that B&S formula is not correct. Why? Because if we look @ the market, we will find, for a given maturity T:



This graph is a disagreement with the B&S formula bc:

we had $\neq$ sigmas $\sigma_1, \sigma_2, \sigma_3, \dots$ for $\neq$ call options written on the same underlying

But our assumption was: $dS_t = rS_t dt + \sigma S_t dW_t$

σ was assumed constant for ANY derivatives written on the same underlying.

If it was correct, we would find the same sigmas $\sigma_1 = \sigma_2 = \sigma_3 \ldots$

This is why the sigmas $\sigma_1, \sigma_2, \sigma_3, \ldots$ are called the Implied Volatilities which are defined as the wrong numbers which when put in the wrong formula give the correct price !

This is why, instead of "σ s.t $P_{MOD} = P_{MKT}$", the real calibration that we do is the following:

N: nb of derivatives.

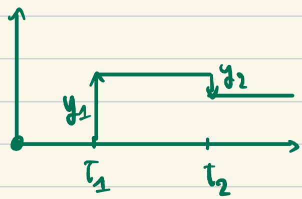
?

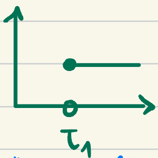
$$\text{Find } \sigma \text{ s.t } \quad \min \sum_{i=1}^{N} \underbrace{\left(P^i_{model} - P^i_{MKT}\right)^2}_{\text{"pricing error"}}$$

Conclusion: We have seen in 2 ways the limitations of B&S model. One way to overcome these limitations is the introduction of jumps.

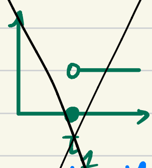
Jumps :

we want to define "stochastic jump processes". We need 2 things : when the jump occurs, and what is the size of the jump.



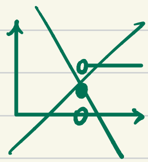


"optional"



~~"predictable"~~

~~left limit is = to the value of the fct~~



we will use this one because the size of the jump cannot be predicted !

⚠ The jump cannot go to $+\infty$.

So, basically, we will work with :

RIGHT CONTINUOUS LEFT LIMITED PROCESSES.

"CADLAG" from french.

"continu à droite, (admet une) limite à gauche".

Let's define the tools we need to work in this framework:

We need an ACCURATE & FAST algorithm to price european call.

why? ↳ We will fit the model prices to the mkt prices (CALIBRATION). Then we can price anything with the calibrated parameters.

Rmk: We need it to be fast: imagine you calibrate on 20 call options and your opti. algo. converges in 1000 iterations, then you have to price 20 000 call options....

↳ That's why B&S formula was nice: immediate.

KEY: to use a Fourier Based Method: FFT...

- $\log\left(S_t/S_0\right) \sim \mathcal{N}(\cancel{\mu t, \sigma^2 t})$ NO Monte Carlo!

?? we don't know the distribution anymore.

DEF: Characteristic function. Let X be a R.V in $\mathbb{R}^d$.

$$\phi_X : \mathbb{R}^d \longrightarrow \mathbb{R}$$

$$z \longmapsto \phi_X(z) = \mathbb{E}\left[e^{iz\cdot X}\right] = \int_{\mathbb{R}^d} e^{iz\cdot x}\, d\mu_X(x)$$

where μ_X is the distribution of X; i.e $\mu_X(A) = \mathbb{P}(X\in A)$ $(\forall A\in \mathcal{B}(\mathbb{R}^d))$