Matlab for Carr-Madan : see Lecture 7.

Carr-Madan Method

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Goal: [To price EU options exploiting Fourier Transform.]

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• **Framework** : $S_t = S_0 \exp(rt + X_t)$ where $(X_t)_t$ is Lévy s.t :

cf Notes, page 71.

$$\underbrace{\int_{|x|>1} e^x \nu(dx) < +\infty}_{}, \qquad \underbrace{\Psi_X(-i) = 0}_{}.$$

Requirement on a bound for the exp. of large jumps.

(we know that in Lévy, large jumps are limited, but we also need smthg about the exp. of the large jumps).

We are risk-neutral (under $\mathbb{Q}$) iff $\Psi_X(-i) = 0$.

↓

Then the discounted stock price is a Martingale. $\Big\}$ $\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} S(T) \,\middle|\, \mathcal{F}_t\right] = S_t$.

• We know that : $\text{price}_{\text{CALL}}(t) = \text{price}(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)^+ \,\middle|\, \mathcal{F}_t\right]$

for $t < T$.

- In 1998, Carr & Madan Formula :

DEF:

- F.T of a smooth enough function f : $\mathcal{F}(f)(v) = \int_{-\infty}^{+\infty} e^{ixv} f(x)\, dx$.
- Inverse Fourier Transform (I.F.T): $\mathcal{F}^{-1}(f)(v) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ixv} f(x)\, dx$.
- ϕ_T is the F.T of X at T : $\phi_T(u) = \mathbb{E}\left[e^{iuX_T}\right] = \int_{-\infty}^{+\infty} e^{iux} \rho_T(x)\, dx$

where ρ_T is the density of X_T. So : $\phi_T(u) = \mathcal{F}(\rho_T)(u)$.

⚠ In Lévy, we don't know ρ, but ϕ is known, i.e the FT of ρ.
So the idea of Carr & Madan is to look in the Fourier Space.

- $\phi_T(-i) = 1$ because $\Psi_X(-i) = 0$ [ASSUMPTION] & we know that : $\phi_T(u) = e^{\Psi_X(u)T}$ so $\phi_T(-i) = e^0 = 1$.
- $\phi_T(0) = 1$ because $\phi_T(0) = \int_{-\infty}^{+\infty} e^0 \rho_T(x)\, dx = \int_{-\infty}^{+\infty} \rho_T(x)\, dx \overset{\rho_T \text{ is a density}}{=} 1$

So, using all the information of the previous slide, we want to compute the price, given by: $\mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}(S_T-K)^+ \mid \mathcal{F}_t]$ (cf slide 2).

- Let's define : $k = \log(K/S_0)$ so that, at $t=0$:

$$c(k) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\left(S_0 e^{rT+X(T)} - S_0 e^{k}\right)^+\right]$$

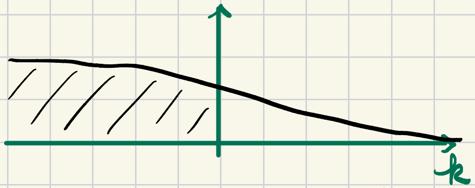
$$= S_0\, \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\left(e^{rT+X(T)} - e^{k}\right)^+\right]$$

($K = \exp(k) S_0$)

(we can forget S_0, take $S_0 = 1$ & then $\times S_0$ if it is $\neq 1$).

The goal is to compute that $\mathbb{E}[\;\;]$.

- Let's forget S_0 ($S_0 = 1$): $c(k) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\left(e^{rT+X(T)} - e^{k}\right)^+\right]$.
  - $c(k) \xrightarrow[k \to +\infty]{} 0$ and $c(k) \xrightarrow[k \to -\infty]{} \mathbb{E}^{\mathbb{Q}}[e^{X(T)}] = \phi_T(-i) = 1$ (cf slide 3).
  - So this is a non-integrable function since:



The idea now is to define a new function, $c(k)$-smthg, such that it is integrable (in F.E we probably use $\exp(---) \times c(k)$ to do idem).

- Let's define : $z(k) = c(k) - \underbrace{\left(1 - e^{k - rT}\right)^+}$.
  - $z(k) \xrightarrow[k \to +\infty]{} 0$.
  - $z(k) \xrightarrow[k \to +\infty]{} 0$.

[Nothing stochastic here: we can compute the price $c(k)$ once we know z: $c(k) = z(k) + (1 - e^{k-rT})^+$.

Now we will apply the F.T of z.

- Let : $g(v) = \mathcal{F}(z)(v) = \int_{-\infty}^{+\infty} e^{ivk}\, z(k)\, dk$.

⚠ Normally the FT has a "-"

If it's not $\mathcal{L}$-$\int$, we should write: $\underbrace{\rho_T(dx)}$.

Let's proove:

$$z(k) = e^{-rT} \int_{-\infty}^{+\infty} \left(e^{rT+x} - e^{k}\right)\left(\mathbb{1}_{\{k \leq x + rT\}} - \mathbb{1}_{\{k \leq rT\}}\right) \rho_T(x)\, dx$$

$$= e^{-rT} \int_{-\infty}^{+\infty} \underbrace{\left(e^{rT+x} - e^{k}\right)\mathbb{1}_{\{k \leq x+rT\}}}_{=1 \text{ if } e^{rT+x} - e^{k} \geq 0,\ 0 \text{ otherwise.}} \rho_T(dx) - \left(\int_{-\infty}^{+\infty} e^{x} - e^{k-rT} \rho_T(dx)\right)\mathbb{1}_{\{k \leq rT\}}$$

$$= \underbrace{e^{-rT} \int_{-\infty}^{+\infty} \left(e^{rT+x} - e^{k}\right)^+ \rho_T(dx)}_{= c(k)} - \left(\underbrace{\int_{-\infty}^{+\infty} e^{x} \rho_T(dx)}_{= 1 \text{ since we are under } \mathbb{Q}.\ (\mathbb{E}[e^{X_T}] = \phi_T(-i) = 1)} - e^{k-rT} \underbrace{\int_{-\infty}^{+\infty} \rho_T(dx)}_{=1.}\right)\mathbb{1}_{\{k \leq rT\}}$$

$$= c(k) - \left(1 - e^{k-rT}\right)\mathbb{1}_{\{k - rT \leq 0\}} = \underbrace{c(k) - \left(1 - e^{k-rT}\right)^+}_{= z(k)\ \checkmark} \quad \blacksquare$$

• Let's compute the Fourier Transform using the previous formula:

$$g_T(v) = e^{-rT}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{ivk}\left(e^{rT+x}-e^{k}\right)\left(\mathbb{1}_{\{k\leq x+rT\}}-\mathbb{1}_{\{k\leq rT\}}\right)\rho_T(dx)\,dk$$

Normally, the F.T has a "–" in the exp.

We switch $\int$

$$= e^{-rT}\int_{-\infty}^{+\infty}\int_{rT}^{rT+x} e^{ivk}\left(e^{rT+x}-e^{k}\right)dk\,\rho_T(dx)$$

$$= e^{-rT}\int_{-\infty}^{+\infty}\int_{rT}^{rT+x}\left(e^{ivk+rT+x}-e^{(iv+1)k}\right)dk\,\rho_T(dx)$$

$$= \left[\frac{e^{ivk+rT+x}}{iv}-\frac{e^{(iv+1)k}}{iv+1}\right]_{rT}^{rT+x}$$

$$= \frac{e^{iv(rT+x)+rT+x}}{iv}-\frac{e^{(iv+1)(rT+x)}}{iv+1}-\frac{e^{ivrT+rT+x}}{iv}+\frac{e^{(iv+1)rT}}{iv+1}$$

$$= \frac{e^{(iv+1)(rT+x)}}{iv}-\frac{e^{(iv+1)(rT+x)}}{iv+1}-\frac{e^{(iv+1)rT+x}}{iv}+\frac{e^{(iv+1)rT}}{iv+1}$$

$$= e^{(iv+1)(rT+x)}\underbrace{\left(\frac{1}{iv}-\frac{1}{iv+1}\right)}_{\frac{1}{iv(iv+1)}} + \dots$$

So let's come back to $g_T(v)$:

$$g_T(v) = \int_{-\infty}^{+\infty} e^{-rT}\left(\frac{e^{(iv+1)(rT+x)}}{iv(iv+1)} - \frac{e^{(iv+1)rT+x}}{iv} + \frac{e^{(iv+1)rT}}{iv+1}\right)\rho_T(dx)$$

$$= \int_{-\infty}^{+\infty}\left(\frac{e^{ivrT+(iv+1)x}}{iv(iv+1)} - \frac{e^{ivrT+x}}{iv} + \frac{e^{ivrT}}{iv+1}\right)\rho_T(dx)$$

$$= \frac{e^{ivrT}}{iv(iv+1)}\int_{-\infty}^{+\infty} e^{(iv+1)x}\rho_T(dx) - \frac{e^{ivrT}}{iv}\underbrace{\int_{-\infty}^{+\infty} e^{x}\rho_T(dx)}_{=1} + \frac{e^{ivrT}}{iv+1}\underbrace{\int_{-\infty}^{+\infty}\rho_T\, dx}_{=1}$$

$$= \frac{e^{ivrT}}{iv(iv+1)}\left(\int_{-\infty}^{+\infty} e^{i(v-i)x}\rho_T(dx)\right) + \frac{e^{ivrT}}{iv(iv+1)}\underbrace{\left(-iv-1+iv\right)}_{=-1}$$

$$\boxed{g_T(v) = \frac{e^{ivrT}}{iv(iv+1)}\left[\phi_{X_T}(v-i) - 1\right]}$$

<u>Carr Madan Formula.</u>

Then we can collect $z(k)$ by inverting the formula using the IFT:

$$z(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikv} g(v)\, dv$$

and finally get the price:

$$\downarrow$$

$$c(k) = z(k) + \left(1 - e^{k-rT}\right)^+ .$$

RMK: it is a very general framework that can be applied to other models: 1) you compute $g(v)$ analytically;

we cannot compute this $\int$ analytically, we need to approximate it! ← 2) you invert the formula by IFT, numerically;

3) you get the price of the EU call option $C(k)$.

End of 11/10/2022 recording

21/10/2024

12/10/2022 RECORDING
04/10/2021 RECORDING

Characteristic Function

(Recap of what we did last time)

Let us consider a process X_t such that the underlying asset S_t is given by

$$S_t = S_0\, e^{rt+X_t}$$

and for which the characteristic function

FT of the pdf of X_t.

$$\phi_{X_t}(v) = E\left[e^{ivX_t}\right] = \int_{\mathbb{R}} e^{ivx} f_{X_t}(x)dx = \mathcal{F} f_{X_t}$$

is well-known. Here f_{X_t} is the pdf, and $\mathcal{F}$ is the Fourier transform operator.

we just assume that there exists a characteristic fct.

In the following we will use ϕ_t as a short for ϕ_{X_t}.
Moreover, in the risk-neutral measure, it must be $\phi_t(-i) = 1$.

<u>Proof:</u> under $\mathbb{Q}$, $\mathbb{E}[S_t|\mathcal{F}_0] = \mathbb{E}[S_t] = S_0$. But $\mathbb{E}[S_t] = \mathbb{E}[S_0 e^{X_t}] = S_0 \underbrace{\mathbb{E}[e^{i(-i)X_t}]}_{\phi_t(-i)}$. ∎

For example, considering the risk-neutral measure, for the GBM we have

$$\phi_{X_t}(v) = e^{-\frac{\sigma^2}{2} ivT - \frac{\sigma^2}{2} v^2 T}$$

Even for the Heston model, the characteristic function is known analytically

Another example of a more complex model.

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^1$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^2$$

HESTON

$$\phi_{X_T}(\omega) = e^{\,B(\omega)+C(\omega)}$$

$$B(\omega) = \frac{2\zeta(\omega)(1 - e^{-\psi(\omega)T})V_0}{2\psi(\omega) - (\psi(\omega) - \gamma(\omega))(1 - e^{-\psi(\omega)T})}$$

$$C(\omega) = -\frac{2\kappa\theta}{\sigma^2}\left[2\log\left(\frac{2\psi(\omega) - (\psi(\omega) - \gamma(\omega))(1 - e^{-\psi(\omega)T}}{2\psi(\omega)}\right) + (\psi(\omega) - \gamma(\omega))T\right]$$

$$\zeta(\omega) = -\frac{1}{2}(\omega^2 + i\omega)$$

$$\psi(\omega) = \sqrt{\gamma(\omega)^2 - 2\sigma^2\zeta(\omega)}$$

$$\gamma(\omega) = \kappa - \rho\sigma\omega i$$

Carr-Madan

- Peter Carr and Dilip B. Madan, *Option Valuation Using the Fast Fourier Transform*, 1999 → This is the initial article.
- Rama Cont and Peter Tankov, *Financial Modelling with Jump Processes*, Chapman & Hall / CRC Press, 2003 → This is the formulation that we are going to use.

Idea: compute the option price of european options (in our case, call options) moving to the Fourier space, where an analitycal formula is provided.

The following notes are based on Cont-Tankov (2003), Section 11.1.3

DEFINITIONS:

F.T of the probabilists!
(so that "$\phi = \mathcal{F}(\text{pdf})$")

Recall the definition of the Fourier transform of a function f:

$$\mathbf{F}f(v) = \int_{-\infty}^{\infty} e^{ixv} f(x)dx$$

"Fourier-Stieltjes"

Usually v is real but it can also be taken to be a complex number. The inverse Fourier transform is given by:

$$\mathbf{F}^{-1}f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixv} f(v)dx$$

For $f \in L^2(\mathbb{R})$, $\mathbf{F}^{-1}\mathbf{F}f = f$, but this inversion formula holds in other cases as well. In what follows we denote by $k = \ln K$ the log strike and assume without loss of generality that $t = 0$.

implicitely, we take $S_0 = 1$ and @ the end we can multiply by S_0.