Computational Finance - Lesson 8

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Value-at-Risk (VaR)

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- Risk is a very difficult factor to model/estimate when dealing with market operations.
- Up until now we have defined risk using the concept of **volatility,** but this is not the only choice.
- Another, very popular, measure of risk is named **Value-at-Risk (VaR).**
- In particular **Value at Risk (VaR)** is a statistical measure used in finance and risk management to estimate the potential loss on an investment or portfolio of investments over a specific time period h, at a certain confidence level p.
- In simpler terms, VaR quantifies the **maximum amount of loss** that a financial portfolio may experience within a given time frame and with a certain level of confidence.
- VaR's introduction into asset allocation marked a shift toward a more disciplined and quantifiable approach to risk management, aligning portfolio construction with measurable risk thresholds while aiding decision-making and compliance

Value-at-Risk (VaR)

- Let's consider a portfolio at time t with its market value $V(t)$, then the ***portfolio loss*** in the interval $[t, t+h]$ is defined as

$$L(t+h) = -\big(V(t+h) - V(t)\big)$$

  And the percentage loss is

$$L(t+h)_{\%} = \frac{L(t+h)}{V(t)}$$

- $VaR^p(t+h)$ at time t with confidence level $1-p$ on the horizon $[t, t+h]$ is defined as

$$\Pr\big(L(t+h) > VaR^p(t+h)\big) = p$$

- Therefore in practice the VaR is a ***quintile of the distribution of losses***, that represents the maximum ***expected loss*** for an asset, given the ***horizon h***, a confidence ***level 1-p*** and the ***volatility*** of the asset.

- Set $F_L^{-1}(q)$ the quantile corresponding to the level of probability q of the distribution of $L(t+h)$

$$VaR^p(t+h) = F_L^{-1}(1-p)$$

  And the **percentage VaR** is

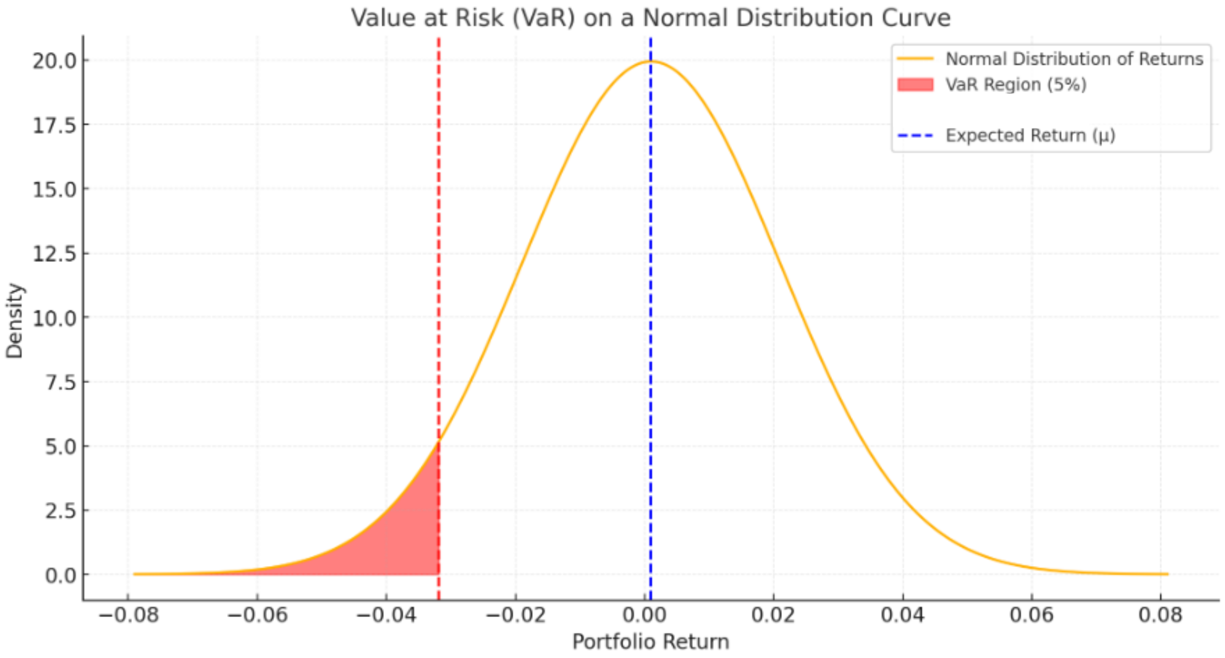
$$VaR^p(t+h)_{\%} = \frac{VaR^p(t+h)}{V(t)}$$

Value-at-Risk (VaR)

- For example, a financial firm may determine an asset has a 3% one-month VaR of 2%, representing a 3% chance of the asset declining in value by 2% during the one-month time frame.

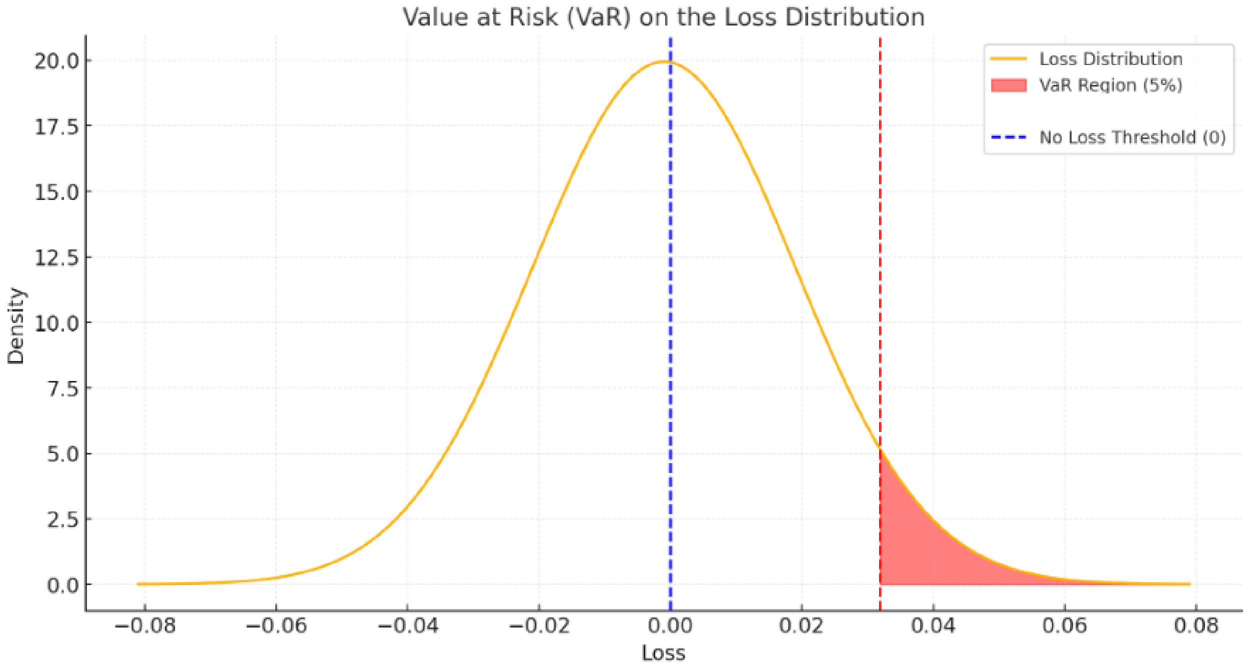
- The left handed plot illustrates the Value at Risk (VaR) for a portfolio with returns following a normal distribution, while the right handed plot represents the VaR for the same portfolio computed on the Loss distribution

Profit-Loss Distribution



- The blue dashed line represents the expected return of portfolio
- The red dashed line indicates the VaR threshold at the 95% confidence level
- The shaded red region represents the left tail of the distribution, capturing the 5% worst-case losses

Loss Distribution



- The x-axis shows the losses, with 0 representing no loss
- The red dashed line indicates the VaR, which is the loss threshold that will be exceeded with a 5% probability
- The red shaded area represents the worst 5% of loss scenarios

Value-at-Risk (VaR)

- There are two principal methods to estimate VaR:

  - ***Parametric Models:***

    - **Method:** Assumes a specific distribution for the returns (e.g., normal distribution) and estimates the parameters of that distribution (mean and standard deviation).
    - **Procedure:** Once the distribution is assumed, the VaR can be calculated analytically using statistical properties of the distribution.

  - ***Historical simulation:***

    - **Method:** VaR is calculated based on historical price changes. The historical returns of the portfolio or asset are used to simulate the distribution of future returns.
    - **Procedure:** Historical returns are sorted, and the VaR is determined based on the historical returns at the chosen confidence level. For example, the 95% VaR corresponds to the loss level exceeded only 5% of the time in the historical data

VaR – Normal distribution of returns

- Let's consider a portfolio with a single asset with horizon h = 1. Let $VaR^{0.01}(t+1)$ be the VaR at a confidence level of 99% of the loss, and if the loss has mean $\mu = 0$, and standard deviation $\sigma(t+1)$:

$$L(t+1) = \sigma(t+1)z(t+1)$$

Where $z(t+1)$ is a random variable distributed as a standard normal distribution.

- Then the VaR can be expressed as

$$VaR^{0.01}(t+1) = \sigma(t+1)\Phi^{-1}(0.99)$$

Where Φ^{-1} is the **cumulative distribution function of the standard normal distribution.**

- In general with $\mu \neq 0$, the loss of the portfolio will be described as a normal distribution $N(\mu, \sigma)$, then

$$VaR^{p}(t+1) = \mu + \sigma\Phi^{-1}(1-p)$$

VaR – Normal distribution of returns

- If $h \neq 1$, the expression becomes:

$$VaR^p(t+h) = h\mu + \sqrt{h}\mu\Phi^{-1}(1-p)$$

- If we consider in the calculation the returns distribution, instead of the losses distribution, the formula above becomes:

$$VaR^p(t+h) = h\mu - \sqrt{h}\mu\Phi^{-1}(1-p)$$

- Pros:
  - **Simplicity:** The method is straightforward and easy to implement, making it accessible for quick risk assessments.
  - **Analytical Solution:** With a normal distribution, VaR can be calculated analytically using mathematical formulas.
- Cons:
  - **Normality Assumption:** The assumption of a normal distribution may not accurately represent the true distribution of financial returns, diminishing the accuracy of the VaR estimate.
  - **Sensitive to Outliers**: Outliers or extreme events in the historical data can significantly impact the estimated mean and standard deviation, leading to inaccurate VaR estimates.
  - **Non-Stationarity**: Assumes that the statistical properties of returns (mean and standard deviation) are constant over time, which may not hold true during periods of market stress.

VaR – Historical Simulation

- In this procedure we *assume that the distribution of the loss of an asset is approximated with its historical distribution*. In particular we consider the losses of the portfolio, built with the asset allocation in t, that could have been occurred in the past and then we assume that this distribution is equal to the future's one.

- Let T be the sample of the percentage losses until time t: $\{L^{\%}(t+1-\tau)\}_{\tau=1}^{T}$. The historical losses can be represented as an histogram and the VaR is calculated numerically: $VaR_{\%}^{p}(t+1)$ is the (1-p) – percentile.

- Pros:
  - **Non-Parametric**: Does not rely on assumptions about the distribution of returns, making it robust in capturing complex and changing market conditions.
  - **Simple and Intuitive:** Easy to understand and implement, requiring minimal assumptions and parameters.
- Cons:
  - **Data Sensitivity:** Highly sensitive to the time period chosen for historical data. VaR estimates can vary significantly depending on the period selected.
  - **Assumes Stationarity:** Like many statistical methods, historical simulation assumes that the statistical properties of the data remain constant over time, which may not hold during periods of market stress.
  - **Inefficiency with Large Data Sets:** When dealing with a large dataset, historical simulation may lead to wrong estimates, weightening recent data as the oldest.

Marginal VaR

- It is possible to consider not only the risk of a portfolio as a whole, but also the risk associated at each single component of the portfolio. The loss of a portfolio is the linear combination of the loss of its assets.

$$L^{\%}(t+1) = \sum_{n=1}^{N} w_n L_n^{\%}(t+1)$$

Using Eulero's Identity it can be demonstrated that:

$$L^{\%}(t+1) = \sum_{n=1}^{N} \frac{\partial L_n^{\%}(t+1)}{\partial w_n} w_n$$

Therefore the VaR of the portfolio is:

$$VaR_{\%}^{p}(t+1) = \sum_{n=1}^{N} \frac{\partial VaR_{\%}^{p}(t+1)}{\partial w_n} w_n$$

Where $x_n = w_n V(t)$

$$VaR^{p}(t+1) = \sum_{n=1}^{N} \frac{\partial VaR^{p}\ (t+1)}{\partial x_n} x_n$$

VaR

- So it can be defined as the ***marginal VaR***, the variation of the VaR respect to a marginal investment on asset n belonging to the portfolio

$$MVaR_n^p = \frac{\partial VaR^p}{\partial x_n}$$

- and the ***component VaR*** as the marginal VaR multiplied for the investment

$$CVaR_n^p = MVaR_n^p x_n$$

- VaR has been introduced in order to capture different aspects of risk. In particular:

    - While VaR provides a measure of the **potential loss magnitude** over a specified time period at a given confidence level, volatility indicates the degree of variation in the price of an asset, but it doesn't explicitly convey information about the potential extent of losses or the probability of extreme events.
    - VaR can be designed to capture **extreme tail risks** by selecting an appropriate confidence level (e.g., 99% VaR captures extreme events with 1% probability). This is important in risk management, especially for institutions that need to consider potential outlier events. While volatility gives a sense of how much prices may fluctuate, it doesn't explicitly quantify the likelihood or impact of extreme events.

VaR - Drawbacks

- VaR has been a very popular measure of risk because of its intuitive nature, but has one big limitation. In particular:

  - VaR may not adequately capture the ***severity of extreme events*** because it only considers the magnitude of the loss at a specific quantile (confidence level). Once the VaR threshold is breached, it provides no information about the severity of losses beyond that point.

- Therefore recently there have been proposed different *coherent* measures of risk. The most important is the **Expected Shortfall (ES)**, also known as **Conditional Value at Risk (CVaR).**

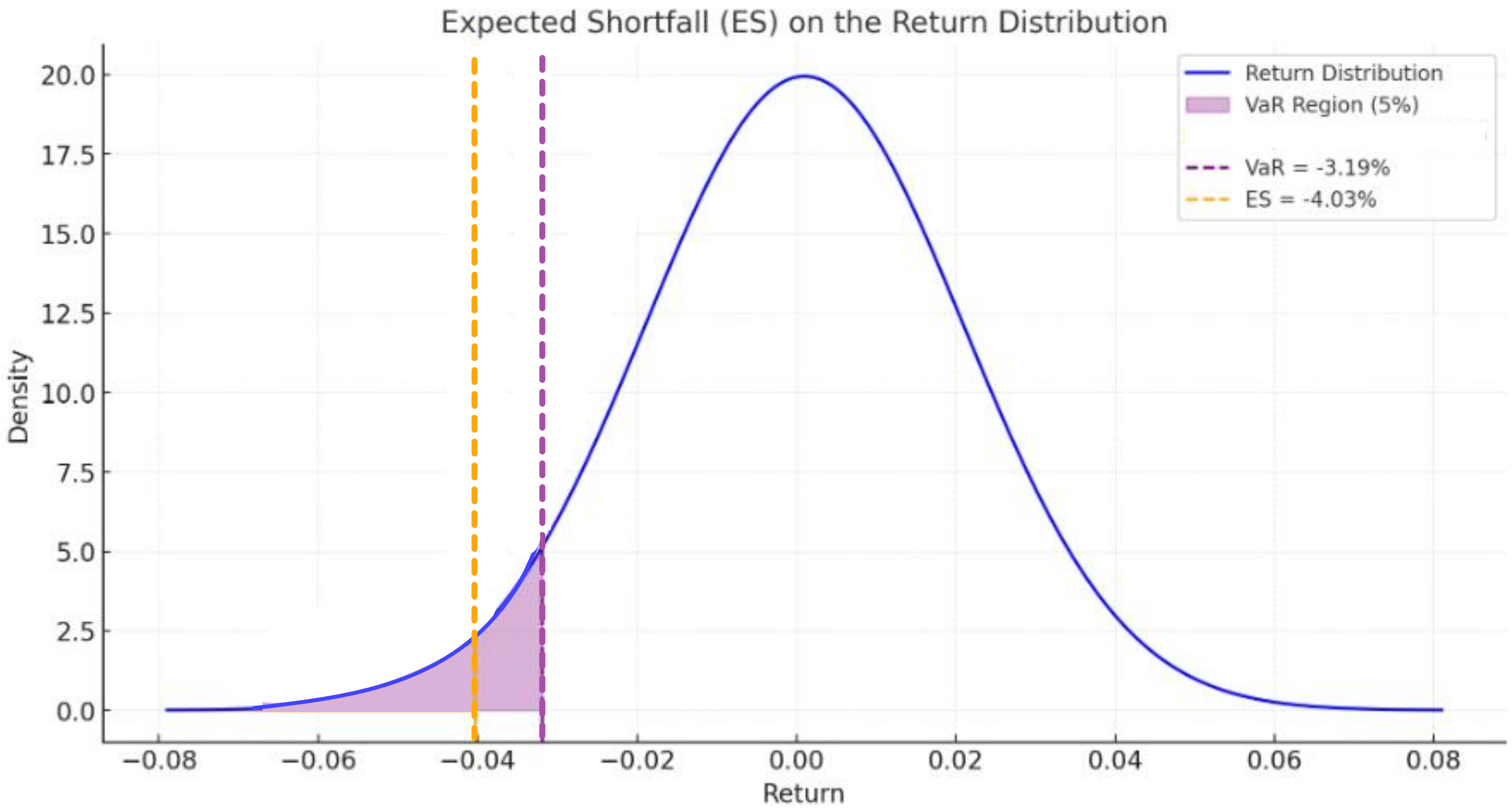
  - **Captures Tail Risk Better:** ES is more sensitive to the severity of extreme events and tail risk
  - **Considers the Entire Tail:** It incorporates information about the entire distribution of losses beyond the VaR threshold
  - **Better Reflects Systemic Risk:** ES is considered a more coherent measure of risk, particularly in capturing systemic risk and extreme market conditions

Expected Shortfall (ES)

Expected Shortfall

- **Expected Shortfall (ES),** also known as *Conditional Value at Risk (CVaR)*, is a risk measure that captures the average loss in the tail of the loss distribution beyond the Value at Risk (VaR) threshold

- It addresses a key limitation of VaR by considering not only the threshold loss but also the ***severity of losses in extreme scenarios*** giving a more comprehensive measure of tail risk



- Blue curve: represents the return distribution

- Purple dashed line: marks the VaR (the lowest return that will be exceeded with a 95% probability)

- Purple region: represents all returns below the VaR threshold. This is the entire area used to compute the Expected Shortfall (ES)

- Orange dashed line: marks the ES, which is a single value: the average of all returns within the purple region

Expected Shortfall

- In general, a coherent measure of risk $\rho(\cdot)$ must satisfy the following properties (L_1, L_2 be two random variables describing the losses):

  - Sub-additivity $\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2)$
  - Omogeneity: $\rho(\alpha L) = \alpha\rho(L) \quad \forall\alpha > 0$
  - Traslation invertibility: $\rho(\alpha + L) = \rho(L) + \alpha \quad \forall\alpha \in R$
  - Monotony: L_1, L_2: $L_1 \leq L_2 \quad \rightarrow \rho(L_1) \leq \rho(L_2)$

- Let L be a loss such that $E[L] < \infty$ with probability distribution F_L, the ***Expected Shortfall*** is defined as:

$$ES^p = \frac{1}{p}\int_0^p F_L^{-1}(1-u)\,du$$

With confidence level $1 - p$. This means that ES is the mean of VaR^u for $0 \leq u \leq p$

$$ES^p = \frac{1}{p}\int_0^p VaR^u\,du$$

Expected Shortfall

- If the Loss is described by a normal distribution, ES can be expressed by

$$ES^p(t+1) = \mu + \sigma \frac{\phi\left(\Phi^{-1}(1-p)\right)}{p}$$

Where ϕ, Φ are respectively the probability density function and the cumulative of the standard normal distribution.

- The ES is more sensitive to extreme losses respect to the VaR. In fact it is valid that:

$$\frac{ES^p}{VaR^p} \geq 1$$

- It is also valid that:

$$\lim_{p \to 0} \frac{ES^p}{VaR^p} = 1$$

meaning that the difference between ES and VaR is neglectable when the distribution is normal and the probability p is small, while it is increasing when the distribution has fat tails and p is large.

Comparison between different risk-measures

Aspect	Value at Risk (VaR)	Volatility	Expected Shortfall (ES)
Definition	Measures the maximum potential loss at a given confidence level over a specified time horizon	Measures the dispersion of returns around their mean	Measures the average loss beyond the VaR threshold, focusing on tail risk
Type of Risk	Focuses on **downside risk** under a confidence level	Captures **total risk**, including both upside and downside variability	Focuses on **extreme downside risk**, providing insights into the severity of losses
Interpretation	"There is a 5% chance of losing more than $X in a day."	"The portfolio has an annualized volatility of Y%."	"If losses exceed the VaR, the average loss will be $Z."
Focus	Concentrates on **threshold losses** in the lower tail of the return distribution	Reflects **average variability**, agnostic to direction	Highlights **tail losses** with an emphasis on their severity
Directional Bias	Provides insight into **loss potential only**; does not consider gains	Neutral, capturing both positive and negative deviations	Tail-focused; considers only the most severe losses
Assumptions	Relies on assumptions about return distribution (e.g., normality or empirical data)	Assumes normality or symmetric variability around the mean	Assumes return distribution but better accounts for tail behavior
Applicability	Common for regulatory reporting and setting **risk limits**	Used for general risk estimation and theoretical models like **Mean-Variance Optimization**	Popular in risk management and **stress testing** for understanding worst-case scenarios
Ease of Communication	Easier to interpret for non-technical audiences due to monetary terms	Abstract for non-specialists; requires explanation to relate to potential losses	Clear for downside risks but less intuitive for general stakeholders than VaR
Sensitivity to Extremes	Considers losses up to a threshold but ignores the size of losses beyond it	Does not distinguish between mild and extreme deviations	Guides **robust optimization**, especially for tail-risk-averse investors
Usage in Asset Allocation	Often used as a **constraint** to cap portfolio risk at acceptable loss levels	Used in **volatility-based models** and portfolio diversification	Guides **robust optimization**, especially for tail-risk-averse investors
Calculation Complexity	Simpler to calculate than ES; relies on distribution assumptions or historical data	Straightforward; often calculated as standard deviation	More complex; requires detailed modeling of tail behavior
Focus on Tail Risk	Partial focus; considers only losses up to the VaR threshold	None; does not differentiate between normal and extreme deviations	Full focus; emphasizes **severity** of losses in the tail
Limitations	- Ignores the magnitude of losses beyond the threshold - Sensitive to assumptions	- Treats upside and downside risk equally - Ignores tail risk	- More computationally intensive - Requires a more accurate model of tails

VaR and ES in Portfolio Optimization

VaR & ES in Portfolio Optimization

- Value at Risk (VaR) and Expected Shortfall (ES) can be used in portfolio optimization as a risk measure to guide investment decisions and construct portfolios that align with an investor's risk tolerance and objectives.

- Basically, instead of using volatility as risk measure, we use VaR or ES as risk measures in the possible objective functions.

- Let ***e*** be the array of expected return and ***w*** the array of weights. Thus the generic formulation of portfolio optimization, under standard constraints, is given by:

$$\max(w^T e \ - VaR^p)$$

- Or equivalently:

$$\max(w^T e \ - ES^p)$$

- Another way to include VaR or ES in Portfolio Optimization is to **maximize portfolio return for a given VaR(or ES).** We have to specify a target VaR level (or a target ES level) and aim to maximize the portfolio's expected return subject to the constraint that VaR (ES) does not exceed the specified threshold.

VaR & ES in Portfolio Optimization

- Benefits of Using VaR (ES) in Portfolio Optimization:

  - **Explicit Risk Measure:** VaR (ES) provides a clear and concise measure of downside risk, allowing investors to explicitly incorporate risk considerations into the optimization process.
  - **Tail Risk Management:** By optimizing with respect to VaR (ES), investors can explicitly manage tail risk and avoid extreme losses, which may be crucial during periods of market stress.
  - **Customized Risk Profiles:** Different investors have varying risk tolerances. Using VaR (ES) allows for the customization of portfolios based on specific risk thresholds.

- Final Considerations:

  - **Limitations of VaR** (ES) **:** While VaR (ES) is a widely used risk measure, it has limitations, such as assuming a particular distribution of returns. Investors should be aware of these limitations and consider using multiple risk measures for a more comprehensive risk assessment.

  - **Dynamic Nature of Markets:** Market conditions change over time, and historical data used in VaR (ES) calculations may not fully capture future risks. Regularly reassessing and adjusting portfolios is important.