$S_{MIN} = S_0 \exp(x_{MIN})$

So :
$$\begin{cases} x_{MIN} = \left(r - \frac{\sigma^2}{2}\right)T - 6\sigma\sqrt{T} \ ; \\ x_{MAX} = \left(r - \frac{\sigma^2}{2}\right)T + 6\sigma\sqrt{T} \ . \end{cases}$$

Using these 2 techniques, I'm able to truncate the domain.

II) DISCRETIZATION :

- $t \in [0, T]$ $\rightsquigarrow$ $\boxed{M \in \mathbb{N}, \ t_j = j\Delta t, \ j \in \{0, \dots, M\} \text{ and } \Delta t = \frac{T}{M}}$

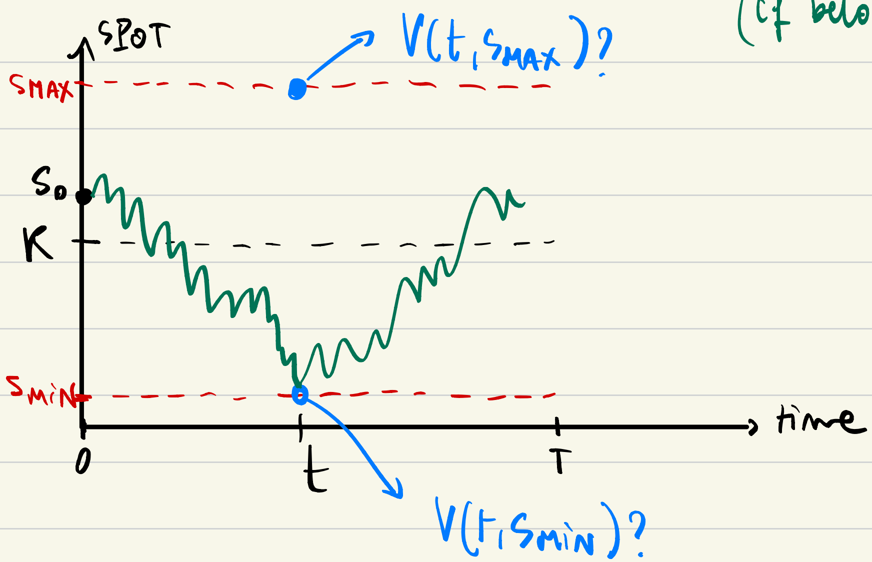
- $x \in [x_{MIN}, x_{MAX}]$ $\rightsquigarrow$ $\boxed{N \in \mathbb{N}, \ x_i = x_{min} + i\Delta x, \ i \in \{0, \dots, N\} \ \& : \ \Delta x = \frac{x_{MAX} - x_{MIN}}{N}}$

II*) BOUNDARY CONDITIONS :

Now that we have a PDE on a bounded domain, we have a pb. We need boundary conditions.

$$\begin{cases} \dfrac{\partial V}{\partial t} + \left(r - \dfrac{\sigma^2}{2}\right)\dfrac{\partial V}{\partial x} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 V}{\partial x^2} - rV = 0, & \forall x \in [x_{MIN}, x_{MAX}] \\ & \forall t \in [0,T]; \\ V(T,x) = \left(S_0 e^{x} - K\right)^+; & \\ V(t, x_{MIN}) = 0 \quad \text{(cf below)} & \forall t \in [0,T]; \\ V(t, x_{MAX}) = S_0 e^{x_{MAX}} - K e^{-r(T-t)} & \forall t \in [0,T]. \end{cases}$$

(cf below)



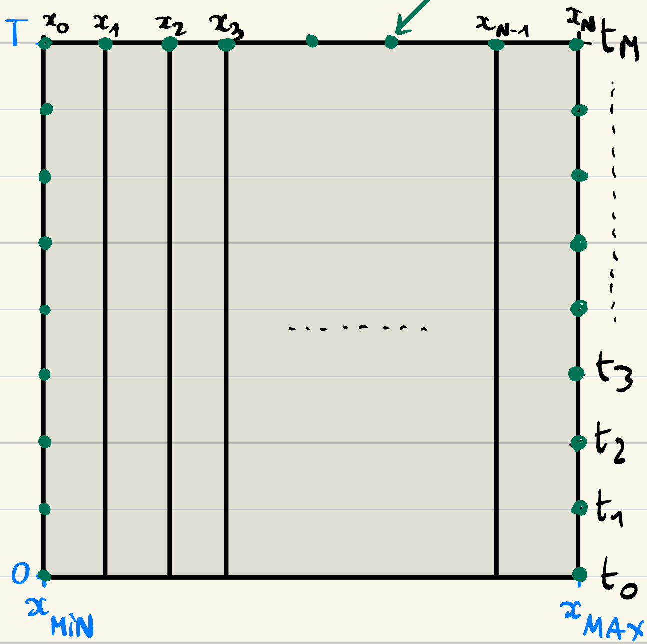
- We fix $V(t, S_{MIN}) = 0$ because we are pretty sure that if the spot price reaches S_{MIN}, the option will not be exercised, since $\mathbb{P}(S_T > K \mid S_t = S_{min}) \simeq 0$.

- $\mathbb{P}(S_T > K \mid S_t = S_{max}) \simeq 1$. Same reasoning so we fix:

$$
\begin{aligned}
V(t, S_{MAX}) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)^+ \,\middle|\, S_t = S_{max}\right] \\
&\simeq \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K) \,\middle|\, S_t = S_{max}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} S_T \,\middle|\, S_t = S_{max}\right] - Ke^{-r(T-t)} \\
&\overset{MG}{=} S_{MAX} - Ke^{-r(T-t)}
\end{aligned}
$$

$$V(t, S_{MAX}) = S_{MAX} - Ke^{-r(T-t)}.$$

Known Solution: terminal condition + boundary conditions.



Now we want to find an algorithm to compute the missing values.

Let's take t_j, x_i with : $j \in \{0, \dots, M-1\}$ and $i \in \{1, \dots, N-1\}$ (Interior point).

$\downarrow$

$$\frac{\partial V}{\partial t}(t_j, x_i) + \left(r - \frac{\sigma^2}{2}\right) \overbrace{\frac{V(t_j, x_i + \Delta x) - V(t_j, x_i - \Delta x)}{2\Delta x}}^{\text{the most accurate approx. of } \frac{\partial V}{\partial x}.} +$$

$$\frac{\sigma^2}{2} \frac{V(t_j, x_i + \Delta x) - 2V(t_j, x_i) + V(t_j, x_i - \Delta x)}{\Delta x^2}$$

approx. of $\frac{\partial^2 V}{\partial x^2}$.

$$-rV(t_j, x_i) = \mathcal{O}(\Delta x^2)$$

[For simplicity, we'll use the following notation:
$$\forall i \in \{0, \dots, N\}, \quad V_i(t) = V(t, x_i).$$]

- So we get:

$$\frac{\partial V_i}{\partial t}(t_j) + \left(r - \frac{\sigma^2}{2}\right) \frac{V_{i+1}(t_j) - V_{i-1}(t_j)}{2\Delta x} + \frac{\sigma^2}{2} \frac{V_{i+1}(t_j) - 2V_i(t_j) + V_{i-1}(t_j)}{\Delta x^2}$$
$$- r V_i(t_j) = 0$$
← we solve this, with an $O(\Delta x^2)$ error.

We do not have derivative wrt x anymore.

- How to deal with $\frac{\partial V_i}{\partial t}(t_j)$?

BWD 1) $$\frac{\partial V_i}{\partial t}(t_j) = \frac{V_i(t_j) - V_i(t_{j-1})}{\Delta t}$$ or:

FWD 2) $$\frac{\partial V_i}{\partial t}(t_j) = \frac{V_i(t_{j+1}) - V_i(t_j)}{\Delta t}.$$

If we do this, our error will be $O(\Delta t + \Delta x^2)$ in total.

Let's denote $V_{j,i} = V(t_j, x_i)$ and use the two schemes:

BACKWARD SCHEME:

$$\frac{V_{j,i} - V_{j-1,i}}{\Delta t} + \left(r - \frac{\sigma^2}{2}\right)\frac{V_{j,i+1} - V_{j,i-1}}{2\Delta x} + \frac{\sigma^2}{2}\frac{V_{j,i+1} - 2V_{j,i} + V_{j,i-1}}{\Delta x^2} - rV_{j,i} = 0$$

FORWARD SCHEME:

$$\frac{V_{j+1,i} - V_{j,i}}{\Delta t} + \left(r - \frac{\sigma^2}{2}\right)\frac{V_{j,i+1} - V_{j,i-1}}{2\Delta x} + \frac{\sigma^2}{2}\frac{V_{j,i+1} - 2V_{j,i} + V_{j,i-1}}{\Delta x^2} - rV_{j,i} = 0$$

We know value @ time t_j and we want to compute it @ time t_{j-1}.

BACKWARD: move $V_{j-1,i}$ on the left, everything else on the right:

$$-\frac{1}{\Delta t} \underbrace{V_{j-1,i}}_{\text{UNKNOWN}} = A \underbrace{V_{j,i-1}}_{\text{KNOWN}} + B \underbrace{V_{j,i}}_{\text{KNOWN}} + C \underbrace{V_{j,i+1}}_{\text{KNOWN}}$$

$$\begin{cases} A = +\left(r - \frac{\sigma^2}{2}\right) \times \frac{1}{2\Delta x} - \frac{\sigma^2}{2} \times \frac{1}{\Delta x^2} \\ B = -\frac{1}{\Delta t} + \frac{\sigma^2}{\Delta x^2} + r \\ C = -\left(r - \frac{\sigma^2}{2}\right) \times \frac{1}{2\Delta x} - \frac{\sigma^2}{2} \times \frac{1}{\Delta x^2} \end{cases}$$

This is called Explicit Euler Scheme.

↓

Because the solution is given explicitely.