and: $\tilde{X}_t^{\varepsilon} = \int_{\substack{\varepsilon<|x|<1 \\ s\in[0,t]}} x\left(J_X(dx\times ds) - \nu(dx)\,ds\right)$

CCP

CP

We cannot split in 2 with IA!

$$\left(= \sum_{\substack{s\in[0,t] \\ \varepsilon<|\Delta X_s|<1}} \Delta X_s - \overbrace{\int_{\substack{\varepsilon<|x|<1 \\ s\in[0,t]}} \nu(dx)\,ds}^{= t\int_{\varepsilon<|x|<1} \nu(dx)} \right)$$

DEF: In this case, we define the **LÉVY TRIPLET:** (γ, A, ν).

Rmk:

- $\gamma t + B_t$ is the "continuous part" of the process.
- X_t^{ε}, X_t^{ℓ} are COMPOUND POISSON.
- $\tilde{X}_t^{\varepsilon}$ is a COMPENSATED COMPOUND POISSON.
  ↳ it's a MG so $\mathbb{E}_0[\tilde{X}_t^{\varepsilon}] = \tilde{X}_0^{\varepsilon} = 0$ a.s.

What happens when $\varepsilon \to 0$?

$$\left| \lim_{\varepsilon\to 0} X_t^{\varepsilon} \right| = +\infty$$ for some Lévy processes;

↳ "infinite activity Lévy proc."

$$\left| \lim_{\varepsilon\to 0} \tilde{X}_t^{\varepsilon} \right| < +\infty$$ for all Lévy processes.

$\in \mathbb{R}$

MG + CENTRAL LIMIT THEOREM

↳ That's why we need $\tilde{X}_t^{\varepsilon}$!

Because $(\tilde{X}_t^{\varepsilon})_t$ is a MG but NOT $(X_t^{\varepsilon})_t$.

⚠ Moral of the story:

So the Ito-Lévy Decomposition tells us that we can only consider the "full integral" $\tilde{X}_t^\varepsilon = \int_{\substack{\varepsilon<|x|<1 \\ s\in[0,t]}} x\left(J_X(dx \times ds) - \nu(dx)ds\right)$

but we cannot "split the integral" in two parts : $X_t^\varepsilon = \int_{\substack{\varepsilon<|x|<1 \\ s\in[0,t]}} x\, J_X(dx \times ds)$ & $\underset{\substack{s\in[0,t] \\ \varepsilon<|\Delta X_s|<1}}{\text{“}\sum\text{”}} \Delta X_s = X_t^\varepsilon$

$\int_{\substack{\varepsilon<|x|<1 \\ s\in[0,t]}} x\, \nu(dx)\, ds$ because even though "$X_t^{\varepsilon,b} = t\int_{\varepsilon<|x|<1} x\,\nu(dx)$"

we know that $\forall$ Lévy $\lim_{\varepsilon\to 0} \tilde{X}_t^\varepsilon \in \mathbb{R}$,

we have for some Lévy proc. (the Inf. activity Lévy) that $\lim_{\varepsilon\to 0} X_t^\varepsilon$, $\lim_{\varepsilon\to 0} X_t^{\varepsilon,b} = +\infty$.

we have two things that go to $+\infty$ so we have to treat them as 1 block. In fact, it is a "$+\infty - +\infty = c \in \mathbb{R}$"...

So, we were able to write a GENERAL LÉVY PROCESS (γ, A, ν):

$$X_t = \gamma t + B_t + X_t^{\ell} + \lim_{\varepsilon \to 0^+} \hat{X}_t^{\varepsilon}.$$

Rmk: $X_t = \mu t + B_t + \sum_{s \in [0,t]} \Delta X_s$ is a "finite activity" or "jump diffusion" process. But now, we wrote the general expression above.

So the question is: can we derive the FA/JD expression starting from the general one?

If $\int_{|x| \le 1} \nu(dx) < +\infty$ [Finite Activity], then, if we start from the above expression:

$$X_t = \gamma t + B_t + \sum_{\substack{s \in [0,t] \\ |\Delta X_s| \ge 1}} \Delta X_s + \lim_{\varepsilon \to 0} \underbrace{\int_{\substack{[0,t] \times \mathbb{R}^d \\ \varepsilon < |x| < 1}} x \left(J_x(dx\,ds) - \nu(dx)ds \right)}$$

$$= \int_{[0,t] \times \mathbb{R}^d, \, 0 < |x| < 1} x J_X(dx, ds) - \int_0^t \int_{0 < |x| < 1} x \, \nu(dx) \, ds$$

$$= \sum_{\substack{s\in[0,t] \\ |\Delta X_s|<1}} \Delta X_s - t \int_{0<|x|<1} x\,\nu(dx)$$

So, putting everything together ($\sim$ & $\sim$):

$$X_t = \underbrace{\left(\gamma - \int_{0<|x|<1} x\,\nu(dx)\right)}_{\mu} t + B_t + \sum_{s\in[0,t]} \Delta X_s$$

And we obtain the same expression!

FINITE ACTIVITY

$$\left\{ X_t = \mu t + B_t + \sum_{s\in[0,t]} \Delta X_s \quad , \mu = \gamma - \int_{0<|x|<1} x\,\nu(dx) \qquad \int_{|x|<1} \nu(dx) < +\infty \right.$$

FINITE VARIATION

$$\left\{ X_t = \mu t + B_t + \int_{[0,t]\times\mathbb{R}^d} x\, J_X(dx, ds) \qquad \int_{|x|<1} |x|\,\nu(dx) < +\infty \right.$$

What does FINITE VARIATION mean?

DEF: • TOTAL VARIATION: $f : [a,b] \longrightarrow \mathbb{R}^d$,

$$TV = \sup_{\substack{a=t_0<t_1<\dots<t_n=b \\ n\in\mathbb{N}}} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| .$$

- **FINITE VARIATION:** $TV < +\infty$.

$$\Updownarrow$$

$$\begin{cases} A = 0 ; & (\text{bc. BM has } \infty \text{ TV}) \\ \displaystyle\int_{|x|<1} |x| \, \nu(dx) < +\infty . & \end{cases}$$

Finite variation Lévy is really between FA & IA:

F.A: $\left| \int_{\mathbb{R}^d} \nu(dx) \right| < +\infty$

F.V Lévy: $\int_{\mathbb{R}^d} \nu(dx) = +\infty$ but $\int_{\mathbb{R}^d} |x| \nu(dx) < +\infty$.

I.A: $\int_{\mathbb{R}^d} \nu(dx) = +\infty$

THM: Lévy Khincin Formula. Let $(X_t)_{t \geq 0}$ be a Lévy process (γ, A, ν).

General Characteristic Exponent.

$$\mathbb{E}\left[e^{iz X_t}\right] = e^{t \Psi(z)}$$

where:

$$\Psi(z) = i \gamma \cdot z - \frac{1}{2} z^T A z + \int_{\mathbb{R}^d} \left(e^{iz \cdot x} - 1 - i z \cdot x \mathbb{1}_{|x| \leq 1} \right) \nu(dx)$$

Proof:

$$X_t = \underbrace{\gamma t + B_t}_{X_t^c} + X_t^{\ell} + \lim_{\varepsilon \to 0^+} \tilde{X}_t^{\varepsilon}$$

$$\lim_{\varepsilon \to 0} X_t^{\varepsilon} = X_t$$

$$X_t^{\varepsilon} = X_t^c + X_t^{\ell} + \tilde{X}_t^{\varepsilon}$$

$$\mathbb{E}\left[e^{iz \cdot X_t}\right] = \mathbb{E}\left[e^{iz \cdot X_t^c} e^{iz \cdot X_t^{\ell}} \cdot e^{iz \cdot \tilde{X}_t^{\varepsilon}}\right]$$

$$= \mathbb{E}\left[e^{iz\cdot X_t^c}\right]\mathbb{E}\left[e^{iz\cdot X_t^\ell}\right]\mathbb{E}\left[e^{iz\cdot \tilde{X}_t^\varepsilon}\right].$$

- $$\mathbb{E}\left[e^{iz\cdot(\gamma t + B_t)}\right] = e^{iz\cdot\gamma t}\,\mathbb{E}\left[e^{iz\cdot B_t}\right] = e^{\left(i\gamma\cdot z - \frac{1}{2}z^T A z\right)t};$$

- $$\mathbb{E}\left[e^{iz\cdot X_t^\ell}\right] = e^{t\int_{|x|\geq 1}\left(e^{iz\cdot x}-1\right)\nu(dx)};$$

- $$\mathbb{E}\left[e^{iz\cdot \tilde{X}_t^\varepsilon}\right] = e^{t\int_{\varepsilon<|x|<1}\left(e^{iz\cdot x}-1-iz\cdot x\right)\nu(dx)}.$$

So :
$$\psi_{X^\varepsilon} = i\gamma\cdot z - \frac{1}{2}z^T A z + \int_{|x|>\varepsilon}\left(e^{iz x}-1\right)\nu(dx) + \int_{\varepsilon<|x|<1} iz\cdot x\,\nu(dx)$$

$|x|\geq 1$ & $1>|x|>\varepsilon$

$$\begin{cases} X^\varepsilon \xrightarrow[\varepsilon\to 0]{} X \\ \psi_{X^\varepsilon} \xrightarrow[\varepsilon\to 0]{} \psi_X \end{cases}$$

$\blacksquare$

Now we need "subordinator" (cf page 38)

END OF RECORDING 14/09/2022.
1h11mn24s OF RECORDING 21/09/2021.

RECORDING
21/09/2021
LEVY PROC.

So I can split the 2 $\int$.

WITH FINITE ACTIVITY

Let's deal with a Levy! Using Lévy Ito Decomposition we can write:

large jumps:

$$X_t = \gamma t + \sigma W_t + \int_{\substack{|x|\geq 1 \\ s\in[0,t]}} x\, J_X(dx \times ds)$$

$$+ \lim_{\varepsilon \to 0} \int_{\substack{\varepsilon<|x|<1 \\ s\in[0,t]}} x\left(J_X(dx \times ds) - \underbrace{\nu(dx)}_{=\nu(x)dx \,=\, \lambda f(x)dx}ds\right)$$

(usually, $\nu(\cdot) = \lambda f(\cdot)$)

$$= \sum_{s\in[0,t]} \Delta X_s \mathbb{1}_{\{|\Delta X_s| \geq 1\}}$$

(*)
$$= \sum_{i=1}^{N_t} Y_i \mathbb{1}_{\{|Y_i|\geq 1\}}$$

$(N_t)_t$: Poisson process with intensity λ,
$(Y_i)_i$: iid $\sim$ pdf f.

Compound Poisson Process.

Here we can split bc we are w/ F.A PROC.

$$= \lim_{\varepsilon \to 0} \sum_{i=0}^{N_t} Y_i \mathbb{1}_{\{\varepsilon < |Y_i| < 1\}} - \lim_{\varepsilon \to 0} \int_{\varepsilon < |x| < 1} x \nu(dx)\, t$$

1st lim : OK, bounded $\quad$ 2nd lim : OK, bounded.

25'23'' 21/09/2021

$$= \sum_{i=0}^{N_t} Y_i \mathbb{1}_{\{|Y_i| < 1\}} - \int_{0<|x|<1} x \nu(dx) \times t \quad (**)$$

let's compute the limit, when $\varepsilon \to 0$.

By putting everything together, we get:

$$X_t = \gamma t + \sigma W_t + \underbrace{\sum_{i=1}^{N_t} Y_i \mathbb{1}_{\{|Y_i| \geq 1\}} + \sum_{i=0}^{N_t} Y_i \mathbb{1}_{\{|Y_i| < 1\}}}_{= \sum_{i=1}^{N_t} Y_i} - t \int_{0<|x|<1} x \nu(dx).$$

$$= t \lambda \int_{0<|x|<1} x f(x) dx$$

$\nu(.) = \lambda f(.)$ + Lebesgue Measure.

i.e : $$X_t = \underbrace{\left(\gamma - \lambda \int_{0<|x|<1} x f(x)\, dx \right)}_{\mu} t + \sigma W_t + \sum_{i=1}^{N_t} Y_i.$$

RECORDING 21/09/2021 LEVY PROC.

Remember that, for a FA/JD process, we wrote:

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$$. So it's the same!

Summing up: we start from a General Lévy Process (γ, σ^2, ν): It can be FA or IA!

$$X_t = \gamma t + \sigma W_t + X_\ell^t + \lim_{\varepsilon \to 0} \tilde{X}_t^\varepsilon .$$

ANY LÉVY CAN BE WRITTEN LIKE THAT.

But if we know that our process is FINITE ACTIVITY we can write it

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$$

BUT the drift $\mu = \gamma - t \int_{0<|x|<1} x \, \nu(dx) \neq \gamma .$

So, what we did is:

Jump Diffusion (expression with a Σ ν) $\xrightarrow{1}$ Lévy Ito Decomposition (for IA Lévy) triplet: (γ, σ^2, ν)

Lévy Ito Decomposition $\xrightarrow{2}$ Jump Diffusion

$$\mu = \gamma - t \int_{0<|x|<1} x \, \nu(dx) = \gamma - \lambda t \int_{0<|x|<1} x f(x) \, dx .$$