TABLE 4.5: Two models based on Brownian subordination: variance gamma process and normal inverse Gaussian process

Model name	**Variance gamma**	**Normal inverse Gaussian**
Model type	Finite variation process with infinite but relatively low activity of small jumps	Infinite variation process with stable-like ($\alpha = 1$) behavior of small jumps
Parameters (excluding drift)	3 parameters: σ and θ — volatility and drift of Brownian motion and κ — variance of the subordinator	
Lévy measure	$\nu(x) = \frac{1}{\kappa\lvert x\rvert}e^{Ax-B\lvert x\rvert}$ with $A = \frac{\theta}{\sigma^2}$ and $B = \frac{\sqrt{\theta^2+2\sigma^2/\kappa}}{\sigma^2}$	$\nu(x) = \frac{C}{\lvert x\rvert}e^{Ax}K_1(B\lvert x\rvert)$ with $C = \frac{\sqrt{\theta^2+\sigma^2/\kappa}}{\cancel{2}\pi\sigma\sqrt{\kappa}}$ and denoting $A = \frac{\theta}{\sigma^2}$ and $B = \frac{\sqrt{\theta^2+\sigma^2/\kappa}}{\sigma^2}$
Characteristic exponent	$\Psi(u) = -\frac{1}{\kappa}\log(1 + \frac{u^2\sigma^2\kappa}{2} - i\theta\kappa u)$	$\Psi(u) = \frac{1}{\kappa} - \frac{1}{\kappa}\sqrt{1 + u^2\sigma^2\kappa - 2i\theta u\kappa}$
Probability density	$p_t(x) = C\lvert x\rvert^{\frac{t}{\kappa}-\frac{1}{2}}e^{Ax}K_{\frac{t}{\kappa}-\frac{1}{2}}(B\lvert x\rvert)$ with $C = \sqrt{\frac{\sigma^2\kappa}{2\pi}}\frac{(\theta^2\kappa+2\sigma^2)^{\frac{1}{4}-\frac{\theta}{2\kappa}}}{\Gamma(t/\kappa)}$	$p_t(x) = Ce^{Ax}\frac{K_1(B\sqrt{x^2+t^2\sigma^2/\kappa})}{\sqrt{x^2+t^2\sigma^2/\kappa}}$ with $C = \frac{t}{\pi}e^{t/\kappa}\sqrt{\frac{\theta^2}{\kappa\sigma^2} + \frac{1}{\kappa^2}}$
Cumulants:		
$E[X_t]$	θt	θt
$\mathrm{Var}\, X_t$	$\sigma^2 t + \theta^2\kappa t$	$\sigma^2 t + \theta^2\kappa t$
c_3	$3\sigma^2\theta\kappa t + 2\theta^3\kappa^2 t$	$3\sigma^2\theta\kappa t + 3\theta^3\kappa^2 t$
c_4	$3\sigma^4\kappa t + 6\theta^4\kappa^3 t + 12\sigma^2\theta^2\kappa^2 t$	$3\sigma^4\kappa t + 15\theta^4\kappa^3 t + 18\sigma^2\theta^2\kappa^2$
Tail behavior	Both Lévy density and probability density have exponential tails with decay rates $\lambda_+ = B - A$ and $\lambda_- = B + A$.	

resulting from Brownian subordination by adding the word "normal" to the name of subordinator). Its characteristic exponent is

$$\Psi(u) = \frac{1-\alpha}{\kappa\alpha}\left\{1 - \left(1 + \frac{\kappa(u^2\sigma^2/2 - i\theta u)}{1-\alpha}\right)^{\alpha}\right\} \tag{4.22}$$

in the general case and

$$\Psi(u) = -\frac{1}{\kappa}\log\{1 + \frac{u^2\sigma^2\kappa}{2} - i\theta\kappa u\} \tag{4.23}$$

in the variance gamma case ($\alpha = 0$).

The Lévy measure of a normal tempered stable process can be computed