This way, we are able to price an American Option.

<u>Summarizing, to price an AM option, we have 3 steps:</u>

1) don't need to discount K: it's given @ t_0 not @ maturity. We are sure that we are in the "EARLY EXERCISE ZONE".

2) have to take into account in the solution that we have a payoff in the arg. of SOR: "max(K-So exp(x), 0)". When computing x_new, we have to make sure that it is $\geq (K-S)^+$. Hence x_new = max(payoff, ...).

$\longrightarrow$ SOR is slower than backslash "\", but using it we can price AM options.

$A \backslash b \longrightarrow SOR(A,b)$: Increase Complexity, Decrease Accuracy

PSOR(A, b, payoff), i.e: solves

$Ax = b$, under the condition that $x \geq$ payoff.

Price AMERICAN OPTION

(3 sources of) ERROR:

- $M \longrightarrow +\infty$
- $N \longrightarrow +\infty$
- (MaxIter, Tol) TOL $\longrightarrow 0$

  Then MaxIter $\longrightarrow +\infty$.

NEXT TIME : PDEs for LÉVY PROCESSES

05/11/2024

PIDE under Lévy:

We did PDE, under B&S, for :

- EU options ;
- Barrier options;
- American options.

$\downarrow$

Now we want to work on PIDE, under Lévy, for the same derivatives.

Finite Activity (FA) | Infinite Activity (IA)

1) Finite Activity Lévy :

Let's work with Merton, which has the following Lévy-Measure :

$$\nu(y) = \lambda \frac{e^{\frac{-(y-\mu)^2}{2\delta^2}}}{\sqrt{2\pi\delta^2}}$$

1.1) European Call:

Let $x = \log\left(\frac{S}{S_0}\right)$, let $v(t,x)$ the price of our EU call. Then:

$$\begin{cases} \dfrac{\partial v}{\partial t} + \left(r - \dfrac{\sigma^2}{2}\right)\dfrac{\partial v}{\partial x} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 v}{\partial x^2} - rv + \\ \qquad \displaystyle\int_{\mathbb{R}} v(t,x+y) - v(t,x) - (e^y - 1)\frac{\partial v}{\partial x}(t,x)\,\nu(y)\,dy = 0 \\ \qquad\qquad \forall t \in [0,T],\ \forall x \in \mathbb{R}; \\ v(T,x) = (S_0 e^x - K)^+ \end{cases}$$

Rmk: This is the PDE that we have to deal with. DERIVATION OF THIS EQUATION ? Notice that the first part of the equation is B&S PDE, but then we also have the integral term.

(1) F.A Lévy :

so $\displaystyle\int_{\mathbb{R}} \left(v(t,x+y) - v(t,x) - (e^y - 1)\frac{\partial v}{\partial x} \right) \nu(y)\,dy$

$$= \underbrace{\int_{\mathbb{R}} v(t,x+y)\,\nu(y)\,dy}_{= I(t,x)} - v(t,x) \underbrace{\boxed{\int_{\mathbb{R}} \nu(y)\,dy}}_{=\lambda} - \frac{\partial v}{\partial x} \underbrace{\boxed{\int_{\mathbb{R}} (e^y - 1)\,\nu(y)\,dy}}_{=\alpha}$$

λ is known. α is unknown : we will compute it numerically.

Therefore the PDE becomes :

$$\underbrace{\frac{\partial v}{\partial t} + \left(r - \frac{\sigma^2}{2} - \alpha \right)\frac{\partial v}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2} - (r + \lambda) v}_{\text{close to B\&S.}} + I(t,x) = 0 .$$

(2) Truncation :

$I(t,x)$ is an integral on an unbounded domain so to compute it numerically, we need to approximate it by truncation of its domain:

$$I(t,x) = \int_{\mathbb{R}} v(t,x+y)\,\nu(y)\,dy$$

Truncation

$$\hat{I}(t,x) = \int_{l_b}^{u_b} v(t,x+y)\,\nu(y)\,dy$$

Quadrature with weights $(w_{ii})_{ii}$

$$\tilde{I}(x) = \sum_{ii=0}^{N_q} \Delta w_{ii}\, v(t,x+y_{ii})\,\nu(y_{ii})$$

where $y_{ii} = l_b + ii\,\Delta$, $\Delta = \dfrac{u_b - l_b}{N_q}$.

For example, in the trapezoidal quadrature rule: $w_0 = w_{N_q} = \frac{1}{2}$ & $w_{ii} = 1$ otherwise.

Therefore we will not study IDE from previous page, but the following ($I \leftarrow \tilde{I}$):

$$\frac{\partial v}{\partial t} + \left(r - \frac{\sigma^2}{2} - \alpha\right)\frac{\partial v}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2} - (r+\lambda)\,v + \tilde{I}(t,x) = 0.$$

(computable)

$$
\begin{cases}
\dfrac{\partial v}{\partial t} + \left(r - \dfrac{\sigma^2}{2} - \alpha\right)\dfrac{\partial v}{\partial x} + \dfrac{\sigma^2}{2}\dfrac{\partial^2 v}{\partial x^2} - (r+\lambda)v + \tilde{I}(t,x) = 0 \\
\qquad\qquad \forall t \in [0,T],\ \forall x \in [x_{MIN}, x_{MAX}]; \\
v(T,x) = (S_0 e^x - K)^+ ; \quad \text{(@ maturity)} \\
v(t,x) = 0 \quad \forall x \leq x_{MIN} ; \quad \text{(BC)} \\
v(t,x) = S_0 e^x - K e^{-r(T-t)} \quad \forall x \geq x_{MAX} . \quad \text{(BC)}
\end{cases}
$$

(3) Finite Difference (F.D) ⊕ Operator splitting:

$\boxed{\text{THETA METHOD}}$ $\quad v_{j,i} = v(t_j, x_i)$, $j \in [\![0,M]\!]$, $i \in [\![0,N]\!]$,

$t_j = j\,\Delta t$, $\Delta t = \dfrac{T}{M}$, $\quad x_i = x_{MIN} + i\,\Delta x$, $\Delta x = \dfrac{x_{MAX} - x_{MIN}}{N}$

$$
0 = \frac{v_{j+1,i} - v_{j,i}}{\Delta t} + \theta\left[\left(r - \frac{\sigma^2}{2} - \alpha\right)\frac{v_{j+1,i+1} - v_{j+1,i-1}}{2\Delta x} + \frac{\sigma^2}{2}\,\frac{v_{j+1,i+1} - 2v_{j+1,i} + v_{j+1,i-1}}{\Delta x^2} - (r+\lambda)v_{j+1,i}\right] + (1-\theta) \times
$$
$$
\left[\left(r - \frac{\sigma^2}{2} - \alpha\right)\frac{v_{j,i+1} - v_{j,i-1}}{2\Delta x} + \frac{\sigma^2}{2}\,\frac{v_{j,i+1} - 2v_{j,i} + v_{j,i-1}}{\Delta x^2} - (r+\lambda)v_{j,i}\right] + \tilde{I}_{j+1,i}
$$