DEF: **Moments**. Let $n \in \mathbb{N}$,

$$m_n(X) = \mathbb{E}[X^n] \quad \text{"moment"};$$

$$\mu_n(X) = \mathbb{E}[(X-\mathbb{E}[X])^n] \quad \text{"centered moment"}.$$

PROP:
- If $\mathbb{E}[|X|^n] < +\infty$, then $\phi_X \in \mathcal{C}^n(I)$, I open set containing 0, and:

$$m_k = \frac{1}{i^k} \cdot \frac{\partial^k}{\partial z^k} \phi_X(0), \quad k = 1, \dots, n.$$

- If ϕ_X has n continuous derivatives in 0, then:

$$m_k = \frac{1}{i^k} \frac{\partial^k}{\partial z^k} \phi_X(0), \quad k = 1, 2, \dots, n.$$

DEF: **Moment generating function**.

$$M_X(u) = \mathbb{E}[e^{u \cdot X}]$$

$$m_n = \frac{\partial^n}{\partial u^n} M_X(0).$$

Rmk: $M_X(u) = \phi_X(-iu)$: easy to go $M_X \leftrightarrow \phi_X$.

DEF: **Characteristic exponent**.

? Ψ_X s.t $\phi_X(u) = e^{\Psi_X(u)}$.

$$\phi_X(0) = \mathbb{E}[e^{i \cdot 0 \cdot X}] = \mathbb{E}[e^0] = 1$$

Assume that ϕ_X is $\mathcal{C}^0$ in 0. Then:

$\boxed{\Psi_X(u) = \log(\phi_X(u))}$ in a neighborhood of 0

$(\Psi_X(0) = 0)$.

$$\begin{tikzcd}
 & \cancel{\text{Wiener}} \text{ (but } \mathcal{C}^0\text{)} & \\
\text{Processes} \arrow[ur] \arrow[r] & \underline{\text{Poisson process}} \arrow[dl] \arrow[dr] & \\
\text{exponential RV} & & \text{Poisson distribution}
\end{tikzcd}$$

DEF: Exponential R.V. Y RV $\sim Exp(\lambda)$ iff:

$$P_r(Y \leq y) = \underbrace{1 - e^{-\lambda y}}_{\text{cdf}}, \quad \forall y \in \mathbb{R}^+.$$

$$\text{pdf} = \lambda e^{-\lambda y} \mathbb{1}_{\{y \geq 0\}}.$$

THM: Absence of memory. Let $T \geq 0$ a RV s.t

$$\mathbb{P}(T > t+s \mid T > t) = \mathbb{P}(T > s), \quad \forall t, s > 0.$$

This is equivalent to : $T \sim \text{Exp}$.

Proof:

- Let $T \sim \text{Exp}(\lambda)$.

$$\mathbb{P}(T > t+s \mid T > t) = \frac{\mathbb{P}(T > t+s)}{\mathbb{P}(T > t)} = \frac{1 - \mathbb{P}(T \leq t+s)}{1 - \mathbb{P}(T \leq t)}$$

$$= \frac{\cancel{1} - (\cancel{1} - e^{-\lambda(t+s)})}{\cancel{1} - (\cancel{1} - e^{-\lambda t})} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s}$$

$$= 1 - (1 - e^{-\lambda s}) = 1 - \mathbb{P}(T \leq s) = \mathbb{P}(T > s).$$

- Let $g(t) = \mathbb{P}(T > t)$. We assume that

$$\mathbb{P}(T > t+s \mid T > t) = \mathbb{P}(T > s) \quad \forall t, s > 0.$$

$$\mathbb{P}(T > t+s \mid T > t) = \frac{\mathbb{P}(T > t+s)}{\mathbb{P}(T > t)}$$

So :

$$g(t+s) = \mathbb{P}(T > t+s) = \mathbb{P}(T > t)\,\mathbb{P}(T > s)$$
$$= g(t)\,g(s)$$

$g(t) = \mathbb{P}(T > t)$ is $\left\{\begin{array}{l}\text{RIGHT } \mathcal{C}^0, \\ \text{NON-INCREASING}\end{array}\right.$ and

$g(t+s) = g(t)\,g(s)$ $\forall t, s > 0$. So : $g(t) = e^{\lambda t}$ for

any t. So $T \sim \text{Exp}(\lambda)$. $\blacksquare$

DEF: Poisson distribution. N R.V with values in $\mathbb{N}$.

$N \sim \text{Poiss}(\lambda)$ iff: $\mathbb{P}(N=n) = e^{-\lambda} \dfrac{\lambda^n}{n!}, \forall n \in \mathbb{N}$.

23/09/2024 $M(u) = e^{\lambda(e^u - 1)}$.

PROP: $(\tau_i)_{i \geq 1}$ R.V iid $\sim \text{Exp}(\lambda)$;

$$N_t = \inf\left\{ n \geq 0 : \sum_{i=1}^{n+1} \tau_i > t \right\}, \forall t > 0.$$

We have that $N_t \sim \text{Poiss}(\lambda t)$, that is:

$$\forall n \in \mathbb{N}, \quad \mathbb{P}(N_t = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

pdf of a Σ of iid RV

Proof: Let: $T_n = \sum_{i=1}^{n} \tau_i$.

this comes from basic theory.

$$P_n(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \mathbb{1}_{[0,+\infty)}(t).$$

This is the pdf of T_n.

$$\mathbb{P}(T_{n+1} > t) = 1 - \underbrace{\mathbb{P}(T_{n+1} \leq t)}_{\text{cdf of } T_{n+1}}$$

$$= 1 - \int_0^t \underbrace{\lambda e^{-\lambda s} \frac{(\lambda s)^n}{n!}}_{P_{n+1}(s) \leftarrow \text{pdf of } T_{n+1}} ds$$

$$= 1 + \int_0^t \underbrace{-\lambda e^{-\lambda s}}_{f'} \underbrace{\frac{(\lambda s)^n}{n!}}_{g} ds$$

$$f = e^{-\lambda s} \quad , \quad g' = \frac{(\lambda s)^{n-1}}{(n-1)!} \times \lambda$$

So :

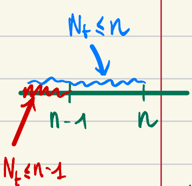
$$\mathbb{P}(T_{n+1} > t) = 1 + \left[e^{-\lambda s} \frac{(\lambda s)^n}{n!} \right]_0^t - \int_0^t \underbrace{\lambda e^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!}}_{P_n(s)} ds$$

$$= 1 + e^{-\lambda t} \frac{(\lambda t)^n}{n!} - \mathbb{P}(T_n < t)$$

$$= e^{-\lambda t} \frac{(\lambda t)^n}{n!} + \mathbb{P}(T_n > t)$$

It means :

$N_t \leq n-1$



$$\underbrace{\mathbb{P}(T_{n+1} > t) - \mathbb{P}(T_n > t)}_{\mathbb{P}(N_t = n)} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

$N_t \leq n$

we recognize a Poisson distribution.

So finally : $\quad N_t \sim \text{Poisson}(\lambda t)$. ■

PROP: • $Y_1 \sim \text{Poisson}(\lambda_1)$, $Y_2 \sim \text{Poisson}(\lambda_2)$, & $Y_1 \perp\!\!\!\perp Y_2$.

Then : $Y_1 + Y_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

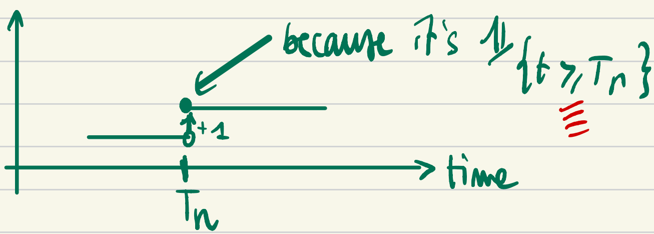
• $Y \sim \text{Poisson}(\lambda)$. Then : $Y = \sum_{i=1}^{n} Y_i$ where $Y_1, Y_2, \dots, Y_n$ iid $\sim \text{Exp}(\lambda/n)$, $\forall n$ [INFINITE DIVISIBILITY]

Poisson Process :

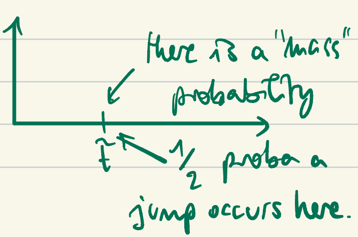
DEF : $(\tau_i)_{i\geq 1}$ sequence of iid RV $\sim Exp(\lambda)$.
$T_n = \sum_{i=1}^{n} \tau_i$. $N_t = \sum_{n=1}^{+\infty} \mathbb{1}_{\{t\geq T_n\}}$ is a Poisson Process with intensity λ.

Rmk: This definition is exactly equivalent to the previous one with the inf.
If we fix time, N_t RV $\sim$ Poisson(λt).
$(N_t)_{t\geq 0}$ Poisson Process w/ intensity λ.

Rmk: $\rightarrow (N_t)_{t\geq 0}$ is a CADLAG process :



$\rightarrow$ $N_{t^-} = N_t$ with probability 1. Let's write things about that : assume $\exists \tilde{t}$ s.t
$\mathbb{P}(N_{\tilde{t}^-} \neq N_{\tilde{t}}) = 1/2$ ($\neq 1$) :
We don't want such mass.



Fortunately, with Poisson Process we don't have mass !

PROP: • $\mathbb{P}(N_t = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \quad \forall t, \forall n.$

• $\mathbb{E}[e^{iuN_t}] = e^{\lambda t(e^{iu} - 1)}$.

• $(N_t)_{t \geq 0}$ has independent increments:

$$\forall t_1 < t_2 < \dots < t_n, \quad N_{t_n} - N_{t_{n-1}} \perp\!\!\!\perp N_{t_{n-1}} - N_{t_{n-2}}, \dots, N_{t_2} - N_{t_1}, N_{t_1}$$

$(N_0 = 0)$

• $\mathbb{E}[N_t] = \lambda t, \quad \forall t \geq 0.$

⚠ We don't have martingale property with (N_t).

So we need to define the "compensated Poisson Process".

DEF: **Compensated Poisson Process:**

$$\tilde{N}_t = N_t - \lambda t.$$

$$\tilde{N}_0 = 0 \quad \text{and} \quad \mathbb{E}[\tilde{N}_t] = \mathbb{E}[N_t] - \lambda t = 0.$$

Clearly, $(\tilde{N}_t)_t$ is not a Poisson Process since it doesn't take only $\mathbb{N}$ values.

PROP: For the compensated Poisson Process:

$$\phi(z) = e^{\lambda t(e^{iz} - 1 - iz)}.$$

Proof:

$$\phi(z) = \mathbb{E}\left[e^{iz\tilde{N}_t}\right] = \mathbb{E}\left[e^{iz(N_t - \lambda t)}\right]$$

$$= \mathbb{E}\left[e^{izN_t} e^{-iz\lambda t}\right] = e^{-iz\lambda t}\,\phi_{N_t}(z)$$

$$= e^{-iz\lambda t}\, e^{\lambda t(e^{iz} - 1)} = e^{\lambda t(e^{iz} - 1 - iz)} \quad \blacksquare$$

PROP: $(\tilde{N}_t)_{t \geq 0}$ is a martingale:

$$\forall t > s, \quad \mathbb{E}[\tilde{N}_t \mid \tilde{N}_s] = \tilde{N}_s.$$

Proof:

$$\mathbb{E}[\tilde{N}_t \mid \tilde{N}_s] = \mathbb{E}[\tilde{N}_t - \tilde{N}_s + \tilde{N}_s \mid \tilde{N}_s]$$

$$= \mathbb{E}[\tilde{N}_s \mid \tilde{N}_s] + \mathbb{E}[\underbrace{\tilde{N}_t - \tilde{N}_s}_{\perp\!\!\!\perp \tilde{N}_s} \mid \tilde{N}_s]$$

$$= \tilde{N}_s + \underbrace{\mathbb{E}[\tilde{N}_t - \tilde{N}_s]}_{=0} = \tilde{N}_s. \quad \blacksquare$$

THM: Let $(X_t)_{t\geq 0}$ be a counting process with independent & stationary increments. Then $(X_t)_{t\geq 0}$ is a Poisson Process.

Rmk: "Stationary increments" means that
$\forall t>s,\ h>0,\quad X_{t+h} - X_{s+h} \sim X_t - X_s$.

The previous theorem tells us that:
Poisson Process is our only "counting process" choice if we want to work w/ independent & stationary increments: this will be the framework of this course.

Rmk: $T_1, \dots, T_n$ sequence of random jump times.

$$N_t = \#\{i\geq 1 : T_i \in [0,t]\} ;$$
$$N_t - N_s = \#\{i \geq 1 : T_i \in]s,t]\} .$$
intuitive way of thinking it.

DEF: Let's introduce a probability space:
$\mathbb{P}$, ω event and: