PROP: $\forall t \geq 0,\ \exists C > 0,$

$$\left| \mathbb{E}\left[F(X_t) - F(X_t^{\bar{\varepsilon}}) \right] \right| \leq C\, \sigma(\bar{\varepsilon})\, t .$$

for example, if F is a payoff:

$$\left| \mathbb{E}[F(X_T)] - \mathbb{E}[F(X_T^{\bar{\varepsilon}})] \right| \leq C \sigma(\bar{\varepsilon}) \sqrt{T} .$$

↑ we can bound the error we make when pricing with NO $<\bar{\varepsilon}$ jump.

THM: Let $\tilde{X}_t^{\bar{\varepsilon}} = X_t^{\bar{\varepsilon}} + \sigma(\bar{\varepsilon})\, W_t .$

$$\sigma^{-1}(\varepsilon)\, R_t^{\varepsilon} \xrightarrow{(d)} W_t \quad \text{iff} \quad \forall k > 0,$$

$\min(k\sigma(\varepsilon), \varepsilon)$

$$\frac{\sigma\left(k\sigma(\varepsilon) \wedge \varepsilon\right)}{\sigma(\varepsilon)} \xrightarrow[\varepsilon \to 0]{} 1 .$$

(if ε small enough $\sigma^{-1}(\varepsilon) R_t^{\varepsilon} \sim W_t$ &

$R_t^{\varepsilon} \sim \sigma(\varepsilon) W_t$.)

Idea of the proof: $R_t^{\bar{\varepsilon}}$ $\quad \mathbb{E}\left[R_t^{\bar{\varepsilon}}\right] = 0, \quad Var(R_t^{\bar{\varepsilon}}) = t\, \sigma^2(\bar{\varepsilon}).$

$$Y_1^{\bar{\varepsilon}} = \frac{R_t^{\bar{\varepsilon}}}{\sigma(\bar{\varepsilon})} \longrightarrow \begin{cases} \mathbb{E}[Y_1^{\bar{\varepsilon}}] = 0 \\ \operatorname{var}(Y_1^{\bar{\varepsilon}}) = 1 \end{cases}$$

$$\hookrightarrow \text{CADLAG} \xrightarrow{\bar{\varepsilon} \to 0} Y_1^{\bar{\varepsilon}} \xrightarrow{d} W_1 .$$

PROP: $\forall t > 0, \ \exists C > 0,$

$$\left| \mathbb{E}[F(X_t)] - \mathbb{E}[F(\tilde{X}_t^{\bar{\varepsilon}})] \right| \leq C \sigma(\bar{\varepsilon}) \rho(\bar{\varepsilon}) t$$

where $\rho(\bar{\varepsilon}) = \dfrac{\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} |x|^3 \nu(dx)}{\sigma^3(\bar{\varepsilon})}$.

What is hard here is to prove the THM.
But, there is a simpler condition which implies it:

if $\dfrac{\sigma(\varepsilon)}{\varepsilon} \xrightarrow[\varepsilon \to 0]{} +\infty$ then the THM

holds true.

End of 20/09/2022 recording.

Beginning of recording 21/09/2022 recording

08/10/2024

So now, what we still need are:

- Itō lemma, but for our processes w/ jumps;

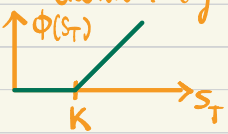
classical world
$$\begin{cases} dX_t = \mu dt + \sigma dW(t), \quad f \in \mathcal{C}^2, \\ df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)\overbrace{(dX_t)^2}^{=\sigma^2 dt^2} \end{cases}$$

$\times$	dt	dW_t
dt	0	0
dW_t	0	dt

- What does it mean to be risk neutral?
  - First we need to define the "stock price": $S_t = S_0 \exp(X_t)$.
  - ? Risk-neutral measure $\mathbb{Q}$?

Fundamental Theorem of Asset Pricing (FTAP):

If we have a T-claim: e.g. EU CALL, payoff: $\Phi(S_T) = \max(S_T - K, 0)$



Then, under $\mathbb{Q}$ (risk neutral):

$$\text{price} = \mathbb{E}_0^{\mathbb{Q}}\Big[\underbrace{\Phi(S_T)e^{-rT}}_{\text{discounted payoff.}}\Big] = \mathbb{E}_0^{\mathbb{Q}}\left[e^{-rT}\max(S_T - K, 0)\right]$$

→ In the classical world, we are under $\mathbb{Q}$ if the following is true: $\mathbb{E}_0^{\mathbb{Q}}\left[e^{-rT}S_T\right] = S_0$ i.e:

$\mathbb{E}_0^{\mathbb{Q}}\left[e^{-rT}S_0 e^{X_T}\right] = S_0$ i.e: $\mathbb{E}_0^{\mathbb{Q}}\left[e^{-rT + X_T}\right] = 1$ i.e:

$\mathbb{E}_0^{\mathbb{Q}}\left[e^{-rT + \mu T + \sigma W_T}\right] = 1$ i.e: $e^{(\mu - r)T}\,\mathbb{E}_0^{\mathbb{Q}}\left[e^{\sigma W_T}\right] = 1$

i.e $e^{(\mu - r)T} e^{+\frac{\sigma^2}{2}T} = 1$ i.e $e^{(\mu - r + \frac{\sigma^2}{2})T} = 1$

i.e $\mu = r - \frac{\sigma^2}{2}$: $\rightsquigarrow$ $dX_t = \underbrace{\left(r - \frac{\sigma^2}{2}\right)}_{\mu \text{ under } \mathbb{Q}}dt + \sigma dW_t$.

This is for GBM!

↓

How to do it for our Lévy proc.?

Check how to recompute: $\mathbb{E}_0^{\mathbb{Q}}\left[e^{\sigma W_T}\right] = e^{\frac{\sigma^2}{2}T}$

So let's do both points, but for our processes.

~~THM~~: **Lévy Itô Formula.**

J.D $$dX_t = \underbrace{\mu dt + \sigma dW_t}_{dX_t^c} + d\overbrace{\left(\sum_{i=1}^{N(t)} Y_i\right)}^{dJ_t} = dX_t^c + dJ_t$$

- Let $f \in \mathcal{C}^2(\mathbb{R})$ and $Y_t = f(X_t)$.
- Let $T > 0$, $(T_i)_{i \in \{1, \dots, N(T)\}}$ the jump times.
- Let $t \in]T_i, T_{i+1}[$,

$$\int_{T_i}^t dX_s = \int_{T_i}^t dX_s^c + \underbrace{\int_{T_i}^t dJ_s}_{=0} = \int_{T_i}^t \mu ds + \sigma dW_s .$$

So : $X_t - X_{T_i} = \mu(t - T_i) + \sigma(W(t) - W(T_i))$.

Let's do it for $Y_t = f(X_t)$:

$$\int_{T_i}^t dY_s = \int_{T_i}^t df(X_s) = \int_{T_i}^t f'(X_s) ds + \frac{1}{2}\int_{T_i}^t f''(X_s)(dX_s)^2$$

$t \in]T_i, T_{i+1}[$: so no jump.
↳ no discontinuity.

classical Itô

$$\forall t \in]T_i, T_{i+1}[, f(X_t) = f(X_{T_i}) + \frac{\sigma^2}{2}\int_{T_i}^t f''(X_s) ds + \mu\int_{T_i}^t f'(X_s) ds + \sigma\int_{T_i}^t f'(X_s) dW_s$$

$$\int_{T_i}^t f'(X_s) dX_s^c = \mu\int_{T_i}^t f'(X_s) ds + \sigma\int_{T_i}^t f'(X_s) dW_s$$

Let $t \to T_{i+1}$:

$$f(X_{T_{i+1}^-}) = f(X_{T_i}) + \frac{\sigma^2}{2}\int_{T_i}^{T_{i+1}} f''(X_s)\,ds + \int_{T_i}^{T_{i+1}} f'(X_s)\,dX_s^c$$

$$f(X_{T_{i+1}}) \underset{\text{right } \mathcal{C}^0}{=} f(X_{T_{i+1}^+}) = f(X_{T_{i+1}^-} + \Delta X_{i+1}) \pm f(X_{T_{i+1}^-})$$

$$= f(X_{T_{i+1}^-}) + \left(f(X_{T_{i+1}^-} + \Delta X_{i+1}) - f(X_{T_{i+1}^-})\right)$$

$$= f(X_{T_i}) + \int_{T_i}^{T_{i+1}} f''(X_s)\,ds + \int_{T_i}^{T_{i+1}} f'(X_s)\,dX_s^c + \Big(f(\underbrace{X_{T_{i+1}^-} + \Delta X_{i+1}}_{= X_{T_{i+1}^+} \underset{r\text{-}\mathcal{C}^0}{=} X_{T_{i+1}}}) - f(X_{T_{i+1}^-})\Big)$$

Let $t > 0$:

$$\underbrace{f(X_T) = f(\underset{\overset{\shortparallel}{0}}{X_0}) + \int_0^t f''(X_s)\,ds + \int_0^t f'(X_s)\,dX_s^c}_{\text{classical continuous part}} + \underbrace{\sum_{\substack{s \le t \\ \Delta X_s \neq 0}} \Big[\overbrace{f(X_s)}^{f(X_{s^-} + \Delta X_s)} - f(X_{s^-})\Big]}_{\text{jump part!}}$$

$$\underset{\mathscr{C}^0\text{ part}}{dX_s = dX_s^c} + \underset{\text{jump}}{\Delta X_s}$$ so we can replace:

$$(*)\ f(X_T) = f(X_0) + \int_0^t f''(X_s)ds + \int_0^t f'(X_s)dX_s + \sum_{\substack{s\leq t \\ \Delta X_s \neq 0}} \left[f(X_{s^-} + \Delta X_s) - f(X_{s^-}) \right] - \Delta X_s f'(X_{s^-})$$

This formula can be used for any Lévy (the sum CV if you have ∞ activity lévy).

THM: Let $(X_t)_{t\geq 0}$ be a Lévy process (γ, σ^2, ν), $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f \in \mathscr{C}^2(\mathbb{R})$.

Then formula $(*)$ holds true.

The key point is to understand why the Σ is finite. For F.A it's ok, but the question is when we work with infinitesimal jumps: bc they can be in infinite number.

But we can use <u>Taylor Formula</u>:

$$f(X_{s^-} + \Delta X_s) = f(X_{s^-}) + \Delta X_s f'(X_{s^-}) + \frac{1}{2}\Delta X_s^2 f''(X_s) + \dots$$

$$\sum_{\substack{s\leq t \\ \Delta X_s \neq 0}} \underbrace{|f(X_{s^-} + \Delta X_s) - f(X_{s^-}) - \Delta X_s f'(X_{s^-})|}_{\text{IS } \frac{f''(X_s)}{2}\Delta X_s^2} \leq C \underbrace{\sum_{\substack{s\leq t \\ \Delta X_s \neq 0}} \Delta X_s^2}_{\text{It's a Lévy (subordinator). Therefore } <+\infty}$$

bc it's Lévy.

(X_t^ε) is J.D (cf previous course) : therefore the previous formula holds true :

$$(**) \quad f(X_t^\varepsilon) = f(X_0^\varepsilon) + \int_0^t f''(X_s^\varepsilon)\,ds + \int_0^t f'(X_s^\varepsilon)\,dX_s^\varepsilon + \sum_{\substack{s\le t \\ |\Delta X_s^\varepsilon|>\varepsilon}} \Big(f(X_{s-}^\varepsilon + \Delta X_s^\varepsilon) - f(X_s^\varepsilon)\Big) - \Delta X_s f'(X_{s-}^\varepsilon)$$

$\varepsilon \longrightarrow 0$:

$$X_t = X_t^\varepsilon + \underbrace{R_t^\varepsilon}_{\substack{\text{mean } 0 \\ \text{var} \xrightarrow[\varepsilon\to 0]{} 0}} \; ; \qquad \lim_{\varepsilon\to 0} f(X_t^\varepsilon) = f(X_t) \; ;$$

$\varepsilon \to 0$ in $(**)$ gives $(*)$. Therefore $(*)$ holds true also for general processes.

PROP: Let $(X_t)_t$ be a Lévy (γ, σ^2, ν), $f \in \mathcal{C}^2(\mathbb{R})$.

Then: $\quad f(X_t) = M_t + V_t$, where :

$$M_t = f(X_0) + \int_0^t f'(X_{s-})\,\sigma\,dW_s + \int_{[0,t)\times\mathbb{R}} \hat{J}_X(ds,dy)\Big(f(X_{s-}+y) - f(X_{s-})\Big)$$

is a MARTINGALE ; & :

$$V_t = \int_0^t \frac{\sigma^2}{2} f''(X_s)\,ds + \int_0^t \gamma f'(X_s)\,ds + \int_{[0,t]\times\mathbb{R}} ds\,\nu(dy)\Big(f(X_{s-}+y) - f(X_{s-}) - y f'(X_{s-})\mathbb{1}_{\{|y|\le 1\}}\Big)$$

is a Finite Variation (FV) process.

Let $S_t = S_0 \exp(X_t)$ & $\hat{S}_t = e^{-rt} S_t$.

$$f(X_t) = S_0 \exp(X_t) = f'(X_t) = f''(X_t)$$

$$Y_t = \underset{\substack{\| \\ S_0}}{Y_0} + \int_0^t Y_s \, dX_s + \frac{\sigma^2}{2} \int_0^t Y_s \, ds + \sum_{\substack{s \leq t \\ \Delta X_s \neq 0}} \Big(\underbrace{f(X_{s-} + \Delta X_s)}_{= S_0 e^{X_{s-}} e^{\Delta X_s}} - S_0 e^{X_{s-}} - \Delta X_s S_0 e^{X_{s-}} \Big)$$

$$Y_t = \underline{\hspace{6cm}} + \sum_{\substack{s \leq t \\ \Delta X_s \neq 0}} S_0 e^{X_{s-}} \left(e^{\Delta X_s} - 1 - \Delta X_s \right)$$

$$Y_t = \underline{\hspace{6cm}} + \int_{[0,t] \times \mathbb{R}} Y_{s-} \left(e^{z} - 1 - z \right) J_X(ds, dz)$$

What is $(V_t)_t$ here?

$$V_t = \int_0^t \frac{\sigma^2}{2} Y_s \, ds + \int_0^t \gamma Y_s \, ds + \int_{[0,t] \times \mathbb{R}} Y_{s-} \left(e^{y} - 1 - y \mathbb{1}_{\{|y| \leq 1\}} \right)$$

$$= \int_0^t Y_{s-} \left(\gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} e^{y} - 1 - y \mathbb{1}_{\{|y| \leq 1\}} \, \nu(dy) \right) ds = 0$$

Requirement in order to be under $\mathbb{Q}$.

$$\boxed{\gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left(e^{y} - 1 - y \mathbb{1}_{\{|y| \leq 1\}} \right) \nu(dy) = 0}$$

i.e : For $(S_t)_t$: $\gamma = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} \left(e^y - 1 - y\,\mathbb{1}_{\{|y|\leq 1\}}\right)\nu(dy)$.

$= \Psi_X(-i)$

For $(\hat{S}_t)_t$: $\gamma = r - \frac{\sigma^2}{2} - \int_{\mathbb{R}} \left(e^y - 1 - y\,\mathbb{1}_{\{|y|\leq 1\}}\right)\nu(dy)$.

So, the γ is FULLY DEFINED by the risk-neutral measure.

So, the framework is the following:

Let $S_t = S_0 e^{rt + X_t}$

$e^{-rt} S_t = S_0 e^{X_t}$ & we are under $\mathbb{Q}$

iff $\Psi_X(-i) = 0$.

Usually you start working with $\hat{X}\ (0, \sigma^2, \nu)$

& then you take $X\ (-\Psi_{\hat{X}}(-i), \sigma^2, \nu)$:

by construction, X has a zero characteristic exponent: $\Psi_X(-i) = 0$.

Usually you calibrate the process without drift & then you add the exact drift.

END OF 21/09/2022 RECORDING