Computational Finance - Lesson 4

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Black-Litterman Model

Black - Litterman Model

- The Black-Litterman model is a **portfolio allocation technique** developed by Fischer Black and Robert Litterman in the early 1990s. It was designed to address some limitations of the traditional mean-variance optimization approach.

- The BL Model has been created in order to resolve some of the drawbacks of the MPT (extreme sensitivity to the inputs and the estimation errors).

- The Black-Litterman model addresses these issues by introducing a structured approach that combines subjective views with market equilibrium:

  i. **Incorporating Subjective Views:** Investors and analysts often have subjective views about expected returns, expected volatilities, or correlations between assets. The Black-Litterman model allows for the explicit incorporation of these views into the portfolio allocation process.

Black-Litterman Model

ii. **Combining Views with Market Equilibrium:** The model combines the subjective views with the *market equilibrium implied by the market capitalization-weighted index*. This is done in a way that ensures the resulting portfolio is consistent with the investor's beliefs while also respecting market realities.

iii. **Estimating More Stable Expected Returns:** The Black-Litterman model provides a way to estimate expected returns and covariances that are less sensitive to small changes in input assumptions compared to traditional mean-variance optimization. This is achieved by blending the subjective views with the market equilibrium.

iv. **Output:** The output of the Black-Litterman model is an adjusted expected return vector and covariance matrix, which can be used for portfolio allocation.

Market Equilibrium

- **Market Equilibrium** is a state in which assets are priced correctly based on their risk and expected returns. It represents a situation where there are no opportunities for arbitrage (i.e., risk-free profits) in the market.

- In particular a market equilibrium state is characterized by the following properties:
  - **Prices Reflect Information:** The prices of assets in the market accurately reflect all available information about those assets.
  - **No Arbitrage Opportunities:** There are no opportunities for investors to make risk-free profits by exploiting mispriced assets.
  - **Investors are Rational:** Investors are assumed to be rational and make decisions based on their assessment of risk and return.
  - **Supply Equals Demand:** There is a balance between the supply and demand for each asset in the market.

  - **Market Capitalization-Weighted Index:** The commonly used benchmark for market equilibrium is a market capitalization-weighted index. This index reflects the total market value of all assets and is considered a representation of the overall market.

Market Capitalization-Weighted Index

- A market capitalization-weighted index is used as a reference point in the Black-Litterman model for several reasons:
  - **Reflects Market Reality:** A market capitalization-weighted index provides a snapshot of the actual market, reflecting the collective decisions and valuations of all market participants. It represents the current state of the market, including the weights assigned to each asset based on its market capitalization.
  - **Benchmark for Market Equilibrium:** It serves as a benchmark for what the market considers as the optimal allocation of assets. This is important because the model aims to adjust the portfolio based on the investor's views while still respecting market realities.
  - **Neutral Starting Point:** Starting with a market capitalization-weighted index is a neutral and widely accepted starting point. It doesn't assume any particular view about expected returns or risks, avoiding possible biases.
  - **Simple and Transparent:** Market capitalization-weighted indices are straightforward and transparent, making them easy to understand and use as a starting point for portfolio allocation.

Market Capitalization-Weighted Index

- A generic market index is a hypothetical portfolio of investment holdings that represents a segment of the financial market. The calculation of the index value comes from the prices of the underlying holdings.

- In a Market Capitalization-Weighted Index, **each company's weight** in the index is determined by its **market capitalization**, which is the total value of all its outstanding shares.

- Therefore, the largest companies in terms of market capitalization have a greater impact on the movements of the index. This reflects the relative importance of these companies in the overall U.S. stock market.

- As companies grow or shrink in terms of market value, their influence on the index changes accordingly. This makes the index dynamic and responsive to market trends.

- Thus such an index, like the S&P 500, aims to provide a comprehensive representation of the U.S. economy by including companies from various sectors, though it is skewed towards larger corporations.

Market Capitalization-Weighted Index: S&P500

1. Apple: 7.05%
2. Microsoft: 6.54%
3. Amazon: 3.24%
4. NVIDIA: 2.79%
5. Alphabet: 2.13%
6. Tesla: 1.95%

⋮

9. Meta: 1.81%

⋮

13. JPMorgan: 1.18%

Black – Litterman Model

- How it Works:

  i. **Market Equilibrium:** Begin with the market capitalization-weighted index as a representation of market equilibrium.

  ii. **Investor Views:** The investor provides his/her subjective views on expected returns, volatilities, or correlations for certain assets.

  iii. **Combining Views:** The model combines these subjective views with the market equilibrium to obtain an updated expected return vector and covariance matrix.

  iv. **Portfolio Allocation:** These updated estimates are then used in the portfolio optimization process to determine the optimal allocation of assets.

- How to do it? Using Bayesian Statistics

Bayesian Statistics

Bayesian statistics

- Bayesian statistics is a theory in the field of statistics based on the **Bayesian interpretation of probability** where probability expresses a *degree of belief in an event*. The degree of belief may be based on prior knowledge about the event, such as the results of previous experiments, or on personal beliefs about the event.

- The different interpretation of probability between the Bayesian inference and the frequentist inference lead to a different procedure of parameter's estimation.

Frequentist Inference	Bayesian Inference
Parameters (like means, variances, etc.) are considered **fixed and unknown constants**. The goal is to estimate these parameters using sample data.	Parameters are seen as **random variables** with probability distributions. Combining prior information with the likelihood of the observed data to update our beliefs about the parameters, we obtain a posterior distribution.

Bayes Theorem

- At the heart of Bayesian statistics is **Bayes' theorem**, which is a fundamental principle in probability theory. It mathematically describes how our beliefs (expressed as probabilities) should be updated in the face of new evidence. Bayes' theorem relates the conditional and marginal probabilities of events. The theorem is expressed as:

$$\boxed{P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}}$$

  - $P(A)$ is the **prior probability.** This represents our initial belief about event A before considering any new evidence. It's based on previous information, subjective opinions, or historical data.

  - $P(B)$ is the **marginal probability** of evidence B. This is the overall probability of observing the evidence B, regardless of whether event A is true or not.

  - $P(B|A)$ is the **likelihood.** This represents the probability of observing the evidence B given that event A is true. It quantifies how well the evidence supports the hypothesis.

  - $P(A|B)$ is the **posterior probability**. This is the updated belief about event A after considering the evidence. It combines the prior belief with the new information provided by the likelihood function.

Bayesian Inference

- So let's suppose we have some data X with a pdf $f_\theta(\boldsymbol{X})$, in the frequentist approach the estimator $\hat{\theta}$ is a number which is supposed to be close to the true, unknown parameter θ_{true}.

$$\mathbf{X} \rightarrow \hat{\theta}$$

- In a Bayesian context an estimator yields a random variable θ, which takes values in a given range Θ. The distribution of θ is called the **posterior distribution**, which can be represented for instance in terms of its probability density function $f_\theta^{po}(\boldsymbol{X})$.

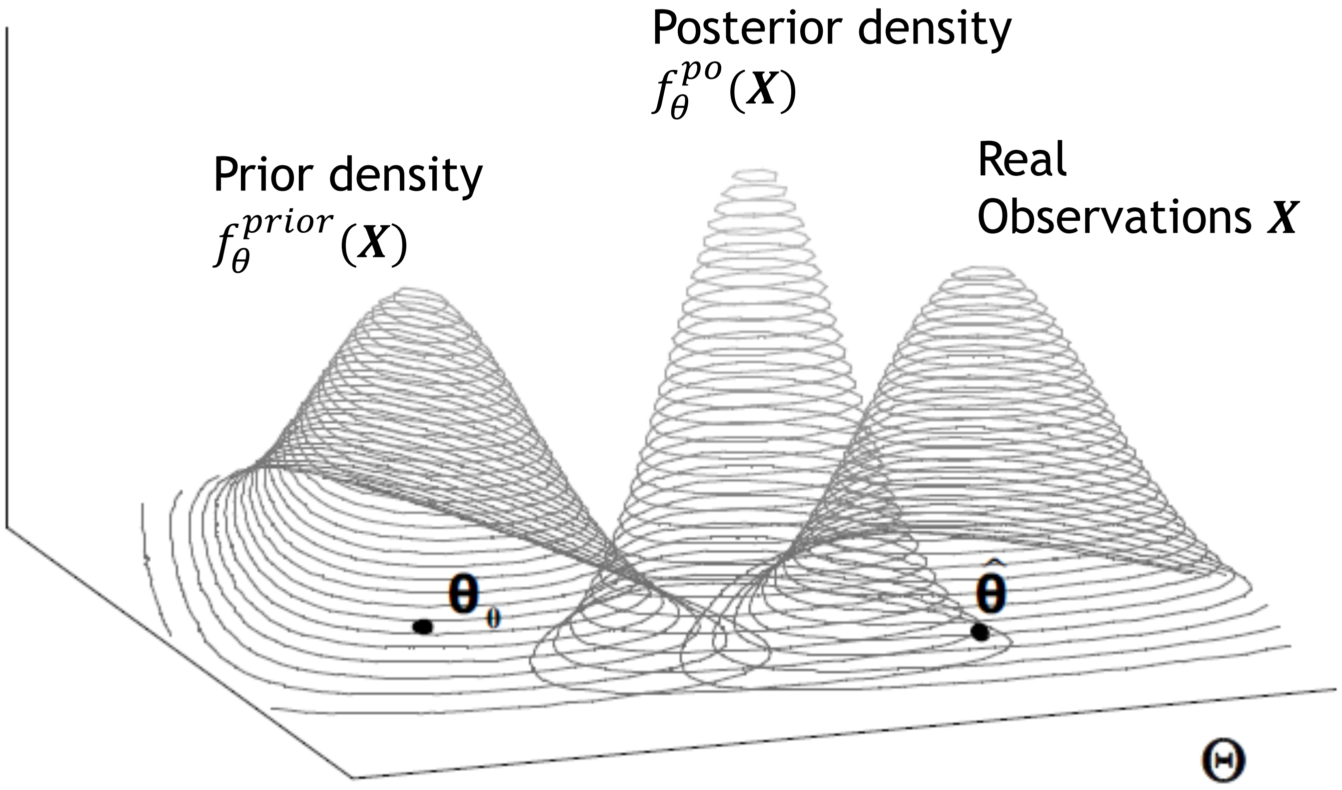
posterior pdf

- Secondly, in a Bayesian context an estimator does not depend only on backward-looking historical information **X**. Indeed, the investor/statistician typically has some *prior knowledge* of the unknown value θ_{true} based on his experience e_C, where *C* denotes the level of confidence in his experience.

- Thus summarizing:

$$\{X, e_C\} \rightarrow f_\theta^{po}(\boldsymbol{X})$$

Bayesian Inference



Bayesian Inference

- **Real Observations Distribution:** the purely classical estimator based on historical information X gives rise to a distribution of the market parameters θ that is peaked around the classical estimate $\hat{\theta}$.

- **Prior Distribution:** the investor equates his experience e_C to a number C of pseudo-observations, that only he sees, located in a "prior" value θ_0. These observations give rise to a distribution of the market parameters θ which is called the prior distribution $f_\theta^{prior}(\boldsymbol{X})$. The larger the number of these pseudo-observations, the higher the investor's confidence in his own experience and thus the more concentrated the prior distribution around θ_0.

- **Posterior Distribution:** It provides a theoretical way to blend the above two distributions into a third distribution, i.e. a spectrum of values and respective probabilities for the parameters θ.

  - In particular, when the confidence C in the investor's experience is large the posterior becomes peaked around the prior value θ_0. On the other hand, when the number of observations T in the time series is large the posterior becomes peaked around the classical estimate $\hat{\theta}$

Black-Litterman Allocation

Black-Litterman Allocation

- We present first the theory for the general case, where the market is described by a generic distribution and the investor can express views on any function of the market.

- Consider a market represented by the multivariate random variable X with a pdf f_X (*prior distribution*).

- The investor has a personal view on the outcome of the market **X**, **V**, i.e. a random variable that could be larger or smaller than **X** (*Views distribution*).

- Therefore the view V is a perturbation of the "official" outcome, and as such it is expressed as a conditional distribution **V|x**, that can be represented for instance by the respective conditional probability density function $f_{V|x}$ (*likelihood distribution*).

- Thus we can compute the posterior distribution, i.e. *distribution of the market conditioned on the investor's opinion* **X|v**, represented by $f_{X|v}$ using Bayes Theorem:

Posterior distribution:

$$\boxed{f_{X|v} = \frac{f_{V|x}\, f_X}{f_V}}$$

!

- Black and Litterman computed an analytical solution in a specific yet quite general case, namely when there are linear views on Normal markets.

Linear Views on Normal Market

- A "***linear view***" refers to an investor's subjective belief or assumption about the expected returns or relationships between assets. These views are expressed in a linear form and are used to adjust the market equilibrium to better reflect the investor's beliefs.

  " V " in the previous slide

  EX: an investor might express the view that one asset is expected to outperform another by a certain percentage -> $R_1 - R_2 > 0.02$

- Reference Model (**Normal Market**):
  - The starting point is normally distributed expected returns, $r \sim N(\mu_0, \Sigma)$
  - We define μ_0, the unknown mean return, as a random variable itself distributed as $\mu_0 \sim N(\mu_r, \Sigma_0)$, μ_r is our estimate of the mean and Σ_0 is the variance of the unknown mean, μ_0, about our estimate.
  - Defining Σ_r as the variance of the returns about our estimate μ_r, it can be demonstrated that $\Sigma_r = \Sigma + \Sigma_0$.
  - Thus the canonical reference model for the Black-Litterman model expected return is: $r \sim N(\mu_r, \Sigma_r)$

Black-Litterman Allocation

- Building Blocks:

1) **Prior Distribution:** The market variables **X** are assumed to be Normal, i.e. $X \sim N(\mu, \Sigma)$

μ here represents the ***Implied Equilibrium Return Vector***, i.e. the expected returns that would make the current market prices in "equilibrium" under certain assumptions. The equilibrium returns are derived using a ***reverse optimization method*** starting from the maximization of the quadratic utility function:

$$U = w^T\mu - \frac{\lambda}{2} w^T \Sigma w$$

where λ is the risk aversion coefficient, Σ is the covariance matrix of returns. If we maximize the utility with no constraints, there is a closed form solution. We find the exact solution by taking the first derivative of U with respect to the weights (w) and setting it to 0.

$$\frac{\partial U}{\partial w} = \mu - \lambda\Sigma w = 0$$

Thus the vector of implied equilibrium returns is extracted from known information using the following:

$$\mu = \lambda\Sigma w_{mkt}$$

w_{mkt} is the *market capitalization weight vector of assets*.

Black-Litterman Allocation

<u>Other building blocks:</u>

2) **Views Distribution:** The investor's views are mathematically represented as a linear function of the market $\mathbf{Q} = \boldsymbol{P}\,\boldsymbol{\mu} + \boldsymbol{\varepsilon},\ \boldsymbol{\varepsilon} \sim \boldsymbol{N}(\mathbf{0}, \boldsymbol{\Omega})$, where **P** is a matrix of dimensions KxN, in which each row represents a linear view.

"V|x 3 slides above"

3) **Likelihood distribution:** The conditional distribution of the investor's views given the outcome of the market is assumed normal, $V|P\mu \sim N(P\mu, \Omega)$, where the symmetric and positive matrix Ω represents the variances of the views and it is inversely related to the ***confidence on the views***. There are several ways to calculate Ω, a particularly convenient choice for the uncertainty matrix is:

confidence on the views ⇒ <u>small</u> variance of the views.

<u>computation of Ω:</u>

$$\Omega = \tau P \Sigma P'$$

$$\Omega = \begin{bmatrix} (p_1 \Sigma p_1{}')\tau & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & (p_k \Sigma p_k{}')\tau \end{bmatrix}$$

Variances of view portfolios

τ is a scalar. One of the most used interpretation of τ is the following: $\tau = 1/N_{obs}$

Black-Litterman Allocation

4) By Bayes theorem it is possible to compute the ***posterior distribution***, i.e. the distribution of the market conditioned on the investor's views. It is found that:

Posterior distribution:

$$\boxed{X|v \sim N(\mu_{BL}, \Sigma_{BL})}$$

Where the expected values read:

$$\mu_{BL} = [\underbrace{(\tau\Sigma)}_{C}^{-1} + P'\Omega^{-1}P]^{-1}[\underbrace{(\tau\Sigma)}_{C}^{-1}\mu + P'\Omega^{-1}Q]$$

And the covariance matrix reads:

$$\Sigma_{BL} = [\overbrace{(\tau\Sigma)}^{C}{}^{-1} + P'\Omega^{-1}P]^{-1}$$

Σ_{BL} is the variance of the posterior mean estimate about the actual mean. It is the uncertainty in the posterior mean estimate, and is not the variance of the returns.

5) **Portfolio Optimization:** By combining the Bayesian posterior distribution of μ_{BL} and the model of asset returns $r \sim N(\mu_0, \Sigma)$, you then have the posterior prediction of asset returns as $r \sim N(\mu_{BL}, \Sigma + \Sigma_{BL})$

%% Black-Litterman PTF { See in MATLAB, in opti. port, we indeed use $\Sigma + \Sigma_{BL}$.

~ : in Matlab

Black Litterman Model

Risk aversion coefficient
$\lambda = (E[r] - r_f)/\sigma^2$

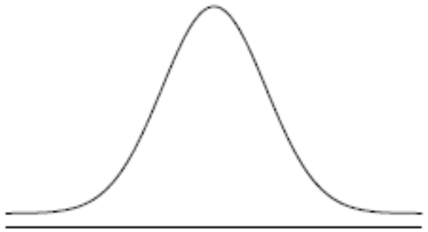
Covariance Matrix
Σ

Market Capitalization Weight
w_{mkt}

Implied Equilibrium Return Vector
$\mu = \lambda \Sigma w_{mkt}$

mu_mkt = lambda.* CovMatrix *wMKT

Prior Distribution



$N(\mu, \tau\Sigma)$

P(..., asset=...)
Views
Q
q(...)

Uncertainty of Views
Ω

Omega(1,1)= ...
Omega(2,2)= ...
...

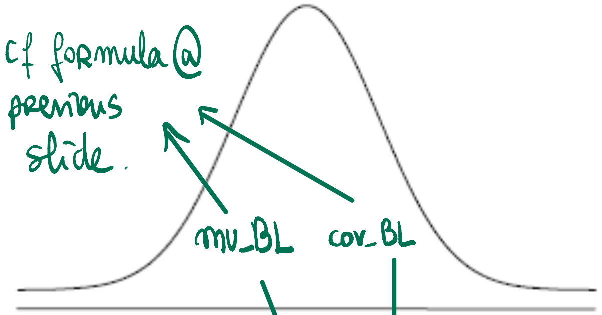
⚠ Pass from the annual view to the daily view

~252 days of business in 1 year

View Distribution

$N\sim(Q, \Omega)$

Posterior Distribution



cf formula @ previous slide.

mu_BL

cov_BL

$N\sim(\mu_{BL}, \Sigma_{BL})$

Portfolio Optimization