Hull & White Model (1997) :

$$\begin{cases} dS(t) = \phi S(t)\,dt + \sigma(t) S(t)\,dW_t \\ \sigma(t) = \sqrt{V(t)} \\ dV(t) = \mu V(t)\,dt + \xi V(t)\,dZ_t , \quad dW_t\,dZ_t = 0 . \end{cases}$$

GBM so ≥ 0

No correlation

- **F T-claim :**

$$F(t, S(t), \sigma^2(t)) = e^{-r(T-t)} \int_0^{+\infty} F(T, S(T))\, p\big(S(T) \mid S(t), \sigma^2(t)\big)\, dS(T)$$

This is exactly like writing it with $\mathbb{E}^{\mathbb{Q}}$:

$$\mathbb{E}^{\mathbb{Q}}\Big[e^{-r(T-t)} \underbrace{F(T, S(T))}_{\hat{\Phi}(S(T)) \,:\, \text{payoff.}} \,\Big|\, S(t), \sigma^2(t) \Big]$$

- Let $\bar{V}_t$ be the integrated variance @ t :

$$\bar{V}_t = \frac{1}{T-t} \int_t^T \sigma^2(\tau)\, d\tau$$

We know that we can always write :

$$p(x|y) = \int g(x|z)\, h(z|y)\, dz$$

law which connects x & y.

law $x \mapsto z$.

law $z \mapsto y$.

So :

$$F(t, S(t), \sigma^2(t)) = e^{-r(T-t)} \int_0^{+\infty}\int_0^{+\infty} (S(T)-K)^+ \times g(S(T)|\bar{V}_t) \times h(\bar{V}_t | S_t, \sigma_t^2)\, dS(T)\, d\bar{V}_t$$

we wrote $p(S(T)|S(t), \sigma^2(t))$ using the above formula.

We did this because we have a thm :

THM: $\log\left(\frac{S(T)}{S(0)} \middle| \bar{V}_0\right) \overset{(d)}{=} \mathcal{N}\left(\left(rT - \bar{V}\frac{T}{2}\right); \bar{V}\,T\right)$.

So let's use it (with $t=0$) :

$$F(0, S(0), \sigma^2(0)) = \int_0^{+\infty} e^{-r(T-t)} \int_0^{+\infty} g(S(T)|\bar{V}(0))\,(S(T)-K)^+\, dS(T)\; h(\bar{V}(0)|S_0, \sigma_0^2)\, d\bar{V}(0)$$

⚠ $\bar{V}(0) = \frac{1}{T}\int_0^T \sigma^2(\tau)d\tau$ so $\bar{V}(0) \in \mathcal{F}_T$.

$$C(\bar{V}(0)) = S(0)\mathcal{N}(d1) - Ke^{-rT}\mathcal{N}(d2)$$ Like in BS

with $d_1 = \dfrac{\log(S/K) + (r - \frac{\bar{V}(0)}{2})T}{\sqrt{V(0)T}}$, $d_2 = d_1 - \sqrt{\bar{V}(0)T}$.

So our problem is finally to compute:

$$\int_0^{+\infty} C(\bar{V}(0))\, h(\bar{V}(0) \mid S(0), \sigma^2(0))\, d\bar{V}_{(0)}$$

|| simplification of notation

$$\int_0^{+\infty} C(\bar{V})\, h(\bar{V} \mid S_0, \sigma_0^2)\, d\bar{V}.$$

we know this term analytically

But we don't know how to compute the whole integral.

What do we know about h?

If we look @ the H&W paper, we

learn that what we know are the moments of h :

$$\mathbb{E}[\bar{V}] = \frac{e^{\mu T} - 1}{\mu T} V_0$$

$$\mathbb{E}[\bar{V}^2] = \left[\frac{2\, e^{(2\mu+\xi^2)T}}{(\mu+\xi^2)(2\mu+\xi^2)T^2} + \frac{2}{\mu T^2}\left(\frac{1}{2\mu+\xi^2} - \frac{e^{\mu T}}{\mu+\xi^2} \right) \right] V_0^2 .$$

And the idea is to exploit a Taylor expansion :

$$\bar{\bar{V}} = \mathbb{E}[\bar{V}]$$

$$F(0, S(0), \sigma_0^2) = C(\bar{\bar{V}}) + \frac{1}{2} \frac{\partial^2 C}{\partial \bar{V}^2}(\bar{\bar{V}}) \underbrace{\int \underbrace{(\bar{V} - \bar{\bar{V}})}_{\Delta} h(\bar{V}) d\bar{V}}_{Var(\bar{V}) = \mathbb{E}[\bar{V}^2] - \mathbb{E}[\bar{V}]^2} \underbrace{+ \cdots}_{\text{extra term}}$$

I.e we wrote $C(\bar{V}) = C(\bar{\bar{V}} + \overbrace{(\bar{V} - \bar{\bar{V}})}^{\Delta})$ and

Then :

$$c(\bar{\bar{v}}+h) = c(\bar{\bar{v}}) + \Delta \frac{\partial c}{\partial \bar{v}}(\bar{\bar{v}}) + \frac{\Delta^2}{2}\frac{\partial^2 c(\bar{\bar{v}})}{\partial \bar{v}^2} + \dots$$

$$\int_0^{+\infty} \left(c(\bar{\bar{v}}) + \Delta \frac{\partial c}{\partial \bar{v}}(\bar{\bar{v}}) + \frac{1}{2}\Delta^2 \frac{\partial^2 c(\bar{v})}{\partial \bar{v}^2} + \dots \right) h(\bar{v})\, d\bar{v} .$$

$$= c(\bar{\bar{v}}) \int_0^{+\infty} h(\bar{v})\, d\bar{v} + \frac{\partial c(\bar{\bar{v}})}{\partial \bar{v}} \underbrace{\int_0^{+\infty} (\bar{v} - \bar{\bar{v}})\, h(\bar{v})\, d\bar{v}} + \dots$$

$$= \int_0^{+\infty} \bar{v}\, h(\bar{v})\, d\bar{v} - \bar{\bar{v}} \int_0^{+\infty} h(\bar{v})\, d\bar{v}$$

$$= \underbrace{\mathbb{E}[\bar{v}]}_{\bar{\bar{v}}} - \bar{\bar{v}} = 0$$

Stochastic Volatility Models w/ Jumps:

$\boxed{\text{BATES:}}$

Bates = Heston + lognormal jumps
(~ avg between Heston & Merton)

$$\begin{cases} dS_t = \mu S_t\, dt + \sqrt{V(t)}\, S_t\, dW_t^S + S_t\, dZ_t \\ dV_t = \xi(\eta - V_t)\, dt + \theta\sqrt{V_t}\, dW_t^V \\ dW_t^S\, dW_t^V = \rho\, dt, \quad Z_t \perp W_t^S,\ Z_t \perp W_t^V, \\ (Z_t)_t \text{ Compound Poisson} \end{cases}$$

$$Z_t = \sum_{i=1}^{N_t} K_i, \quad N_t \sim \text{Poiss}(\lambda t) \quad \forall t \geq 0.$$

K = jumpsize, $\ln(1+K) \sim \mathcal{N}\left(\ln(1+\bar{K}) - \frac{1}{2}\delta^2, \delta^2\right)$

Parameters of Bates Model: 8

$$\mu, \underbrace{\lambda, \bar{K}, \delta}_{CPP}, \underbrace{\xi, \eta, \theta}_{V(t)}, \rho$$

Under risk-neutrality, drift is: $\mu = r - \lambda\bar{K}$.

So, under $\mathbb{Q}$ (RN) :

$$X_t = \log(S_t / K) \rightsquigarrow X_0 = 0 .$$

$$dX_t = \left(r - \lambda \bar{K} - \frac{1}{2} V_t\right) dt + \sqrt{V_t}\, dW_t^S + d\tilde{Z}_t$$

The dynamic of the variance doesn't change.

Rmk: Bates is like Heston + Merton, but we could do Heston + Any Lévy proc. !

Goal: to find the characteristic function.

$$\mathbb{E}\left[e^{iuX_T}\right] = \mathbb{E}\left[e^{iu\int_0^T (r-\lambda\bar{K}-\frac{1}{2}V_t)dt} \times e^{iu\int_0^T \sqrt{V_t}\,dW_t^S} \times e^{iu\underbrace{\int_0^T dZ_t}_{\tilde{Z}_T}}\right]$$

Key point $\tilde{Z}_T \perp\!\!\!\perp W^S, W^V$ so :

$$\mathbb{E}\left[e^{iuX_T}\right] = \mathbb{E}\left[e^{iu\left(\int_0^T (r-\lambda\bar{K}-\frac{1}{2}V_t)dt + \int_0^T \sqrt{V_t}\,dW_t^S\right)}\right] \times$$

Characteristic fct of Heston model with the only difference being the term $\lambda \bar{k}$.

$$\underbrace{\mathbb{E}\left[e^{iu\tilde{Z}_T}\right]}_{\text{Characteristic fct of a CPP: we know how to compute it.}}$$

J_0 :

$$\underset{\text{Bates}}{\Phi_B} = \underset{\text{Heston}}{\Phi_H} \times \Phi_{CPP}$$

Characteristic function Bates/"Heston":

$$f_B(x, v, t) = \mathbb{E}\left[e^{iuX_T} \middle| \begin{matrix} X_t = x \\ V_t = v \end{matrix}\right] = \mathbb{E}\left[e^{iuX_T^c}\right] \mathbb{E}\left[e^{iuJ_T}\right]$$

classical Heston if $\lambda = 0$.

"Merton": CF w/ jumpsize ~ lognormal.

$$\underbrace{e^{t\lambda\left(e^{iu\left(\ln(1+\bar{K}) - \frac{1}{2}\delta^2\right) - \frac{\delta^2 u^2}{2}} - 1\right)}}_{\text{KNOWN}}$$

We already computed the term coming from Merton. The new part is the one coming from Heston. Let's work in a "pricing" POV:

$$C(x, v, t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\left(S_0 e^{X_T} - K\right)^+ \middle| X_t = x, V_t = v\right]$$

The two — terms are quite similar: one of them is an expected value, the other one is the expected value (under $\mathbb{Q}$) of the discounted payoff.

Same form + same depending variables: their value will be given by the same

PDE not surprisingly, but the main difference will be the terminal condition.

$$dX_t^c = \left(r - \lambda\bar{K} - \frac{1}{2}V_t\right)dt + \sqrt{V_t}\, dW_t^S$$

because we just want to compute the "Heston" part.

$$f(x,v,t) = \mathbb{E}\left[e^{iuX_T^c} \,\middle|\, X_t = x,\ V_t = v\right]$$

Exactly as when we want to price an option, we start by writing the dynamic:

$$df = \left[\frac{\partial f}{\partial t} + \frac{1}{2}v\frac{\partial^2 f}{\partial x^2} + \rho\theta v\frac{\partial^2 f}{\partial x \partial v} + \frac{1}{2}\theta^2 v\frac{\partial^2 f}{\partial v^2} + \left(r - \lambda\bar{K} - \frac{v}{2}\right)\frac{\partial f}{\partial x} + \xi(\eta - v)\frac{\partial f}{\partial v}\right]dt + \sqrt{v}\frac{\partial f}{\partial x}dW_t^S + \theta\sqrt{v}\frac{\partial f}{\partial v}dW_t^v$$

Application of Ito's Lemma.

(Because we are in F.A! So between one jump and the other, the dynamic is driven by the $\mathcal{C}^0$ part: so we don't see the jump part appear here!)

Thus, since f is an expected value, it is a Martingale. Hence its drift should be zero : THINK ABOUT THAT

$$\begin{cases} \dfrac{\partial f}{\partial t} + \xi(\eta - v)\dfrac{\partial f}{\partial v} + \left(r - \lambda\bar{K} - \frac{v}{2}\right)\dfrac{\partial f}{\partial x} + \frac{1}{2} v \dfrac{\partial^2 f}{\partial x^2} + \\ \qquad \rho\theta v \dfrac{\partial^2 f}{\partial x \partial v} + \frac{1}{2} v \theta^2 \dfrac{\partial^2 f}{\partial v^2} = 0 \qquad \forall x \in \mathbb{R}, \forall v \in \mathbb{R}^+, \forall t \in [0,T]; \\ f(x, v, T) = e^{iux}. \end{cases}$$

We have to solve this PDE: luckily we can even solve it analytically. Then we'll have the C.F for Bates (and Heston) & then we'll be able to perform Carr Madan.

<u>Guess solution:</u> $f(x,v,t) = \exp\left(C(T-t) + v D(T-t) + iux\right)$

- According to the terminal solution:

$$C(0) + v D(0) + iux = iux$$

i.e $\boxed{C(0) + v D(0) = 0}$

- Insert f inside the PDE :

$$\frac{\partial f}{\partial x} = iuf \quad ; \quad \frac{\partial^2 f}{\partial x^2} = -u^2 f \quad ; \quad \frac{\partial f}{\partial v} = D(T-t) f \; ;$$

$$\frac{\partial^2 f}{\partial v^2} = D^2(T-t) f \quad ; \quad \frac{\partial^2 f}{\partial x \partial v} = iu D(T-t) f \; ;$$

$$\frac{\partial f}{\partial t} = \left(-C'(T-t) - D'(T-t) v\right) f .$$ So we get :

$$-C'(T-t) - D'(T-t) v + \xi(\eta - v) D(T-t) +$$
$$\left(r - \lambda \bar{K} - \frac{v}{2}\right) iu - \frac{1}{2} v u^2 + \rho \theta v iu D(T-t) +$$
$$\frac{1}{2} v \theta^2 D^2(T-t) = 0 , \quad \forall t \in [0,T], \underbrace{\cancel{\forall x \in \mathbb{R}}}_{\text{doesn't appear anymore.}}, \forall v \in \mathbb{R}^+$$

i.e :

$$\overbrace{-C'(T-t) + \xi\eta D(T-t) + iu(r - \lambda \bar{K})}^{\text{independent of } v} + v\Big(-D'(T-t) -$$
$$\xi D(T-t) - \frac{1}{2} iu - \frac{1}{2} u^2 + \rho \theta iu D(T-t) + \frac{1}{2}\theta^2 D^2(T-t)\Big) = 0$$

This is true $\forall v \in \mathbb{R}^+$ iff :

$$\begin{cases} -D'(T-t) = \frac{1}{2}\theta^2 D^2(T-t) + (i\rho\theta u - \xi) D(T-t) - \frac{u^2 + iu}{2} \quad \forall t \in [0,T] \\ D(0) = 0 \end{cases} \quad \boxed{\text{ODE}}$$

We can solve this ODE and obtain D :

$$\boxed{\begin{aligned} D(s) &= \frac{u^2 + iu}{\gamma \coth\left(\frac{\gamma s}{2}\right) + \xi - i\rho\theta u} \\ \text{where} \quad \gamma &= \sqrt{\theta^2(u^2 + iv) + (\xi - i\rho\theta v)^2} \end{aligned}}$$

Then we can solve the other part (independent of v) :

$$\begin{cases} C'(T-t) = \xi\eta D(T-t) + iv(r - \lambda\bar{K}) \\ C(0) = 0 \end{cases}$$

We can get $C(s)$ too.

This is the way we compute the characteristic function of the Heston Model / Bates Model.

$\longrightarrow$ Then we can apply CARR-MADAN for Heston / Bates.

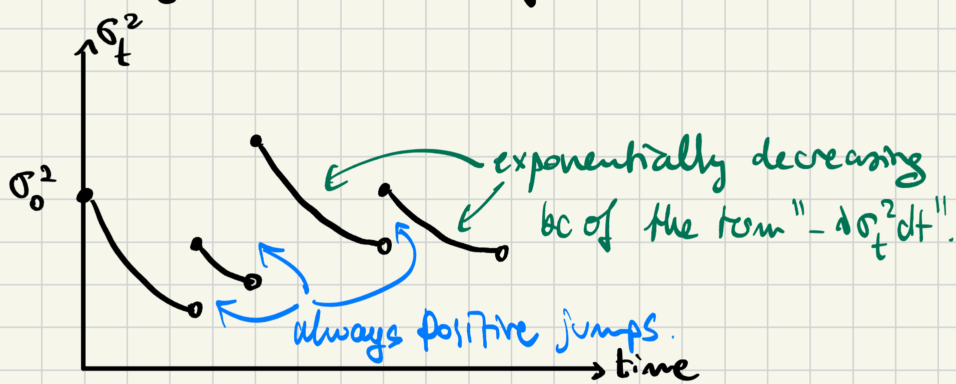
Ornstein-Uhlenbeck (OU) Stochastic volatility model:

BNS Barndorff-Nielsen & Shepard:

$$\begin{cases} S_t = S_0 \exp(X_t) \quad \text{and} \\ dX_t = (\mu + \beta\sigma_t^2)\,dt + \sigma_t\,dW_t + \rho\,dZ_t \\ d\sigma_t^2 = -\lambda\sigma_t^2\,dt + dZ_t, \quad (Z_t)_t \text{ Lévy subordinator}^* . \end{cases}$$

No BM in the variance: the stochasticity of the variance is fully defined by a Lévy process, which appears both in the dynamics of the logprice & of the variance.

* to avoid negative values of σ^2.



N.B: BNS is called OU because its variance has a non-gaussian Lévy-driven OU dynamics.
→ $(Z_t)_t$ is called "background driving Lévy process of σ_t^2".

Before arriving to BNS, we have to discuss about the OU family.

$\boxed{\text{OU}}$

· Gaussian OU : $dv_t = -\lambda v_t \, dt + \gamma dW_t$

we cannot use it : non-zero proba that the process goes to negative values.

· Non-Gaussian OU : $dv_t = -\lambda v_t \, dt + dZ_t$
where $(Z_t)_t$ is Lévy, with jumps.

We then have :

- $v_t = v_0 e^{-\lambda t} + \int_0^t e^{\lambda(s-t)} dZ_s$, $Z(\gamma, \sigma^2, \nu)$.