```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% PLOT
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
figure
plot(K,P, 'r') ;
hold on
axis([0  2*S  0  S]) ;
xlabel('strike') ;
ylabel('option price') ;


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% INTERPOLATION
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
P_i = interp1(K,P,K_i,'spline');
```

22/10/2024 Topic of today : FFT & other derivatives. 14/10/2022 Recording

We want to see if we can use FFT technique when we deal with other kinds of derivatives, like path-dependent ones.

↳ $\boxed{\text{Answer:}}$ we can do something, but not as much as Carr Madan. We are not able to get closed formulas.

Let's consider for example a Barrier Option :

$$\begin{cases} M_t = \max\limits_{0\le s\le t} X_s \\ m_t = \min\limits_{0\le s\le t} X_s \end{cases} ; \quad \begin{cases} S_t = S_0 e^{X_t} \\ k = \log\left(\frac{K}{S_0}\right) \end{cases} ; \quad b = \log\left(B/S_0\right) \text{ where } B \text{ is a barrier.}$$

→ let's consider an up & out call :

$$\underbrace{C(T; k, b) = S_0 \left(e^{X_T} - \underbrace{e^{k}}_{K/S_0}\right)^+}_{\text{Payoff of EU Call style}} \underbrace{\mathbb{1}_{\underbrace{\{M_T \le b\}}_{\text{Up \& Out}}}}_{\text{Barrier}}$$

] Payoff of our up&out Call, i.e, "Price @ Maturity T".

So as we can see from the expression above: now, we are not only interested in the distribution of X_t anymore. We are now interested in the **joint distribution of X_t and M_t**.

For simplicity, we assume:
- $r = 0 \longrightarrow$ NO DISCOUNT;
- $\psi_X(-i) = 0 \longrightarrow$ DISCOUNTED STOCK PRICE IS A MG.

Let's move to $t < T$:

- $C(t; k, b) = \mathbb{E}^{\mathbb{Q}}\left[S_0 (e^{X_t} - e^k)^+ \mathbb{1}_{\{M_t \leq b\}} \middle| \mathcal{F}_t \right]$;

WE ARE UNDER $\mathbb{Q}$ (cf assumptions above)

ρ_T: joint density of (X_T, M_T).

- $C(0; k, b) = \int_{\mathbb{R}} \int_{\mathbb{R}} S_0 (e^x - e^k)^+ \mathbb{1}_{\{y < b\}} \rho_T(dx, dy)$.

So if we want to make something close to CM, we need to compute this integral, but now we have joint density.

↳ Can we define a similar "bidimensional F.T"? The answer is NO.

Hint (out of our scope): apply a "double FT" in k and in b:

$\int_{\mathbb{R}} \int_{\mathbb{R}} e^{iuk} e^{ivb} C(0; k, b) \, dk \, db$, not surprisingly we arrive to a similar result as Carr Madan :

$$\int_{\mathbb{R}} \int_{\mathbb{R}} e^{iuk} e^{ivb} \, C(0;k,b) \, dk \, db = \frac{F(u-i,v)}{uv(1+\dots u)} ,$$ with F the

characteristic function of (X_T, M_T).

MAJOR PROBLEM : we don't know F. So even if we are able to get a formula, it is not useful since we don't know F.

However, researchers made computations and discovered :

$$q \int_0^{+\infty} \iint_{\mathbb{R}\,\mathbb{R}} e^{iuk + ivb} e^{-qt} \, C(t;k,b) \, dt \qquad \text{(Laplace transform)}$$

$$\| $$

$$\frac{\phi_q^+(v+u-i)\,\phi_q^-(u-i)}{uv(1+iu)}$$ with $\phi_q^{\pm}$: "Wiener Hopf Factors".

No more the joint density!

$$\left\{ \phi_q^+(u)\,\phi_q^-(u) = \frac{q}{\underbrace{q - \psi_X(u)}_{\text{This is KNOWN.}}} \right.$$ (ϕ_q^+ has support in $\mathbb{R}^+$, ϕ_q^- in $\mathbb{R}^-$)

So the idea is : $\boxed{1}$ compute $\frac{q}{q-\psi_X(u)}$ then $\boxed{2}$ take the inverse Laplace & two times the inverse Fourier transforms. SIMILAR IDEA AS CM.

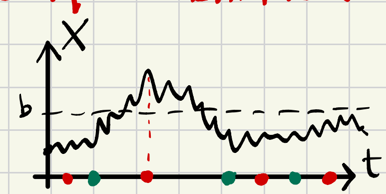
What is the problem ? It is very hard, numerically, to compute $\frac{q}{q-\psi_X(u)}$.

→ That's why we won't go further : it's complex & not used.

Clearly, what we have shown in ~~ was not the correct way to perform : so we have to change our POV.

So, our idea will be : apply FOURIER TRANSFORM TO BARRIER OPTIONS BUT IN THE DISCRETE MONITORING CASE.

(we check for the barrier only on some dates)

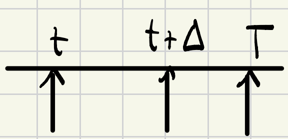


~~ : OK

~~ : NOT OK

→ cf "conv"/"convolution" on WEBEEP. What we'll see now is a convolution method:

Let's consider first a EU option : $T > t$, $t + \Delta < T$.



$X_t = x$

we know that : $C(x,t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\left[C(X_T, T) \mid \mathcal{F}_t \right]$

we always need to be under $\mathbb{Q}$.

And this is TRUE for any $t < T$, so it means that:

$$C(x,t) = e^{-r\Delta} \mathbb{E}^{\mathbb{Q}}\Big[C(X_{t+\Delta}, \underbrace{t+\Delta}_{<T}) \,\Big|\, X_t = x \Big]$$

$$= e^{-r\Delta} \int_{\mathbb{R}} C(y, t+\Delta)\, p_{x,t,\Delta}(y)\, dy$$

$\mathbb{P}(X_{t+\Delta} = y \mid X_t = x)$

conditional!

Indeed, this is "conditional expectation".

Now, in the Lévy framework: $\mathbb{P}(X_{t+\Delta} = y \mid X_t = x) = \mathbb{P}(X_\Delta = y - x)$ using the stationary and identically distributed increments.

So: $$C(x,t) = e^{-r\Delta} \int_{\mathbb{R}} C(y, t+\Delta)\, f_\Delta(y-x)\, dy \quad (E) \quad [\text{Lévy framework}]$$

density of the increments of X.

density of X_Δ.

DEF : [CONVOLUTION] $$g_1(x) = \int_{\mathbb{R}} g_2(x-y)\, g_3(y)\, dy \qquad (g_1 = g_2 * g_3)$$

PROP : $$\mathcal{F}(g_1) = \mathcal{F}(g_2) \times \mathcal{F}(g_3) \qquad \left(* \xrightarrow{\mathcal{F}} \times \right).$$

Let's define : $\boxed{f_\Delta^b(x) = f_\Delta(-x)}$ such that (E)

becomes : $$c(x,t) = e^{-r\Delta} \int_{\mathbb{R}} f_\Delta^b(x-y)\, c(y, t+\Delta)\, dy.$$

This is written as a convolution. Now we use the above property :

$$\mathcal{F}(c(\cdot,t)) = e^{-r\Delta}\, \mathcal{F}(f_\Delta^b) \times \mathcal{F}(c(\cdot, t+\Delta))$$

i.e : $$c(\cdot,t) = \mathcal{F}^{-1}\left[e^{-r\Delta} \mathcal{F}(c(\cdot, t+\Delta))\, \underbrace{\mathcal{F}(f_\Delta^b)}_{?} \right]$$

Key point : what is it ?

$$\mathcal{F}(f_\Delta) = \phi_{X_\Delta} \longleftarrow \text{"}\mathcal{F}(\text{density}) = \phi\text{"}.$$

$$\mathcal{F}(f_\Delta^b) = \mathcal{F}(f_\Delta(-\,\cdot\,)) = \underbrace{\phi^*_{X_\Delta}}_{\text{we know it so we can use it}} \longleftarrow \text{conjugate (by } \mathbb{C} \text{ analysis)}.$$

THIS IS THE SO CALLED "CONV METHOD".

↳ take payoff $\to \mathcal{F}(\text{payoff})$ $\to \times\, \phi^* \to \mathcal{F}^{-1}(\ )$.

CONV METHOD FOR EUROPEAN OPTION:

$c(x,T) = S_0(e^x - e^k)^+$. Compute:

$\mathcal{F}(S_0(e^x - e^k)^+) \times \phi^*_{X_T}(u)$ and then take $\mathcal{F}^{-1}$:

$$C(x,0) = \mathcal{F}^{-1}_{u\to x}\left[\mathcal{F}_{x\to u}\left(S_0(e^x - e^k)^+\right) \times \phi^*_{X_T}(u)\right].$$

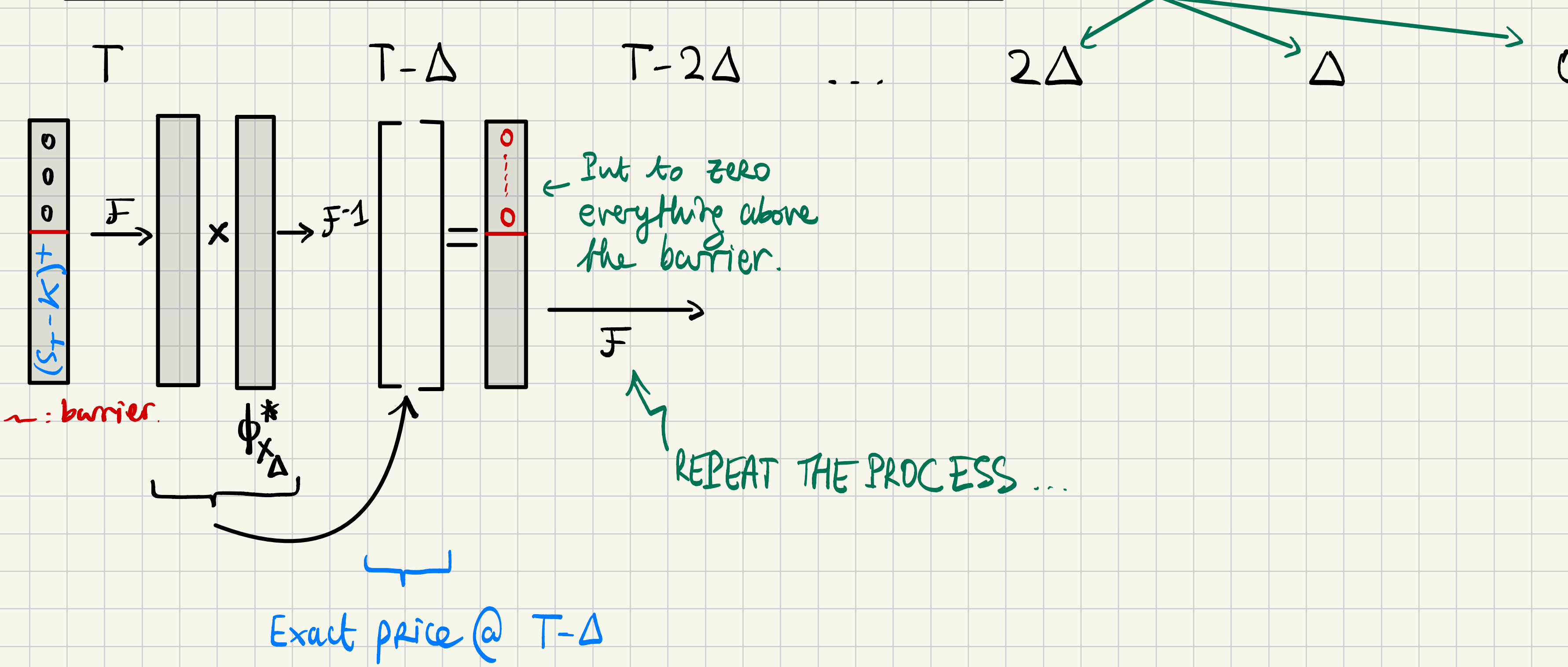
↑ Price @ time 0.

However, use CM if you can (more accurate).

We can exploit this method for <u>BARRIER OPTIONS w/ DISCRETE MONITORING.</u>

CONV METHOD FOR BARRIER OPTION:

Monitoring dates.



↳ The idea is that between two monitoring dates, the evaluation is the same as European option. So we repeat the process till $t=0$, taking each time into account the barrier.

cf CONV_Method.pdf → fully explained method.

Now, let's look @ the code :

- we need a function to compute characteristic fct : charfunction.m
  - with a flag if we want the complex conjugate.
  - there is a correction to take into account the risk free rate & the potential dividends, and also to have $\phi_X(-i) = 1$ (i.e $\psi_X(-i) = 0$ under $\mathbb{Q}$).

- main.m :
  - Ndate = 12 → Monthly monitoring.
  - $N = 2^{12}$: grid of the log returns : $\log(S_T/S_0)$.

- kernel.m : we need two grids : one for log price and one for Fourier space.

  for Fourier space → dw, w (with 2π).

  log price → N, dx, x.

  - we invert the characteristic function using "fft" (⚠ not ifft) see why in the CM video.
  - we need to correct the shift induced by MATLAB :

h = real(fftshift(fft(ifftshift(H)))) (shifts of the GRID)

This corrects the fact that x is −N → N−1. corrects the w grid which doesn't go from 0 to ... but from −N to N−1.

h is the density of $(-X)$ (since it was $\Phi_X^* = H$).

$\longrightarrow$ see the comments that explain the shifts in grids.

- CONV.m : new algorithm that uses kernel & goes backward in times.

  $\hookrightarrow$ once again we have to transform H to be in MATLAB style : H = ifftshift(H).

  $\hookrightarrow$ if we remove the line " V(S<=Barriera) = 0 " we are basically pricing a EU option.

As a conclusion, this code can be used to price a barrier option with discrete monitoring under any Lévy, exactly like we previously did with MC. And it's a general method: the only thing we have to change if we want to use another kind of Lévy is the computation of the characteristic function. Nothing else.

⚠ Important to understand why we shift: MATLAB wants $0 \to 2N$, we want $-N \to N$.

⚠ We don't use char. fct for continuous monitoring path-dependent options: it is a Nightmare because it is not sufficient to know ϕ...

↳ We have PDE or we just take discrete monitoring.