Matlab for Carr-Madan : see Lecture 7.

Carr-Madan Method

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Goal: [To price EU options exploiting Fourier Transform.]

11/10/2022 recording @37'30''

- **Framework** :
cf Notes, page 71.

$S_t = S_0 \exp(rt + X_t)$ where $(X_t)_t$ is Lévy s.t :

$$\underbrace{\int_{|x|>1} e^x \nu(dx) < +\infty}_{}, \qquad \underbrace{\Psi_X(-i) = 0}_{}.$$

Requirement on a bound for the exp. of large jumps.

(we know that in Lévy, large jumps are limited, but we also need smthg about the exp. of the large jumps).

We are risk-neutral (under $\mathbb{Q}$) iff $\Psi_X(-i) = 0$.

$\downarrow$

Then the discounted stock price is a Martingale.

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} S(T) \,\middle|\, \mathcal{F}_t\right] = S_t.$$

- We know that : $\quad price_{CALL}(t) = price(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)^+ \,\middle|\, \mathcal{F}_t\right]$

for $t < T$.

- In 1998, Carr & Madan Formula :

DEF:

- F.T of a smooth enough function f : $\mathcal{F}(f)(v) = \int_{-\infty}^{+\infty} e^{ixv} f(x)\, dx$.
- Inverse Fourier Transform (I.F.T) : $\mathcal{F}^{-1}(f)(v) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ixv} f(x)\, dx$.
- ϕ_T is the F.T of X at T : $\phi_T(u) = \mathbb{E}\left[e^{iuX_T}\right] = \int_{-\infty}^{+\infty} e^{iux} \rho_T(x)\, dx$

where ρ_T is the density of X_T. So : $\phi_T(u) = \mathcal{F}(\rho_T)(u)$.

⚠ In Lévy, we don't know ρ, but ϕ is known, i.e the FT of ρ.
So the idea of Carr & Madan is to look in the Fourier Space.

- $\phi_T(-i) = 1$ because $\Psi_X(-i) = 0$ [ASSUMPTION] & we know that : $\phi_T(u) = e^{\Psi_X(u)T}$ so $\phi_T(-i) = e^0 = 1$.

- $\phi_T(0) = 1$ because $\phi_T(0) = \int_{-\infty}^{+\infty} e^0 \rho_T(x)\, dx = \int_{-\infty}^{+\infty} \rho_T(x)\, dx \overset{\rho_T \text{ is a density}}{=} 1$

So, using all the information of the previous slide, we want to compute the price, given by: $\mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}(S_T-K)^+ \mid \mathcal{F}_t]$ (cf slide 2).

- Let's define : $k = \log(K/S_0)$ so that, at $t=0$:

$$c(k) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\left(S_0 e^{rT+X(T)} - S_0 e^{k}\right)^+\right]$$

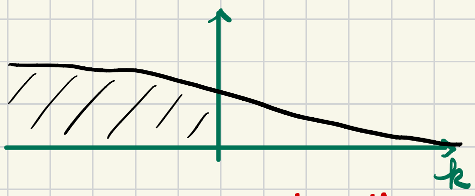
$$= S_0\, \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\left(e^{rT+X(T)} - e^{k}\right)^+\right]$$

($K = \exp(k)\, S_0$)

(we can forget S_0, take $S_0 = 1$ & then $\times S_0$ if it is $\neq 1$).

The goal is to compute that $\mathbb{E}[\ ]$.

- Let's forget S_0 $(S_0 = 1)$: $c(k) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\left(e^{rT+X(T)} - e^{k}\right)^+\right]$.
  - $c(k) \xrightarrow[k \to +\infty]{} 0$ and $c(k) \xrightarrow[k \to -\infty]{} \mathbb{E}^{\mathbb{Q}}[e^{X(T)}] = \phi_T(-i) = 1$ (cf slide 3).
  - So this is a non-integrable function since:



The idea now is to define a new function, $c(k)$-something, such that it is integrable (in F.E we probably use $\exp(---) \times c(k)$ to do idem).

- Let's define: $z(k) = c(k) - \underbrace{\left(1 - e^{k-rT}\right)^+}$.
  - $z(k) \xrightarrow[k \to +\infty]{} 0$.
  - $z(k) \xrightarrow[k \to +\infty]{} 0$.

$\Big[$ Nothing stochastic here: we can compute the price $c(k)$ once we know z: $c(k) = z(k) + \left(1 - e^{k-rT}\right)^+$.

Now we will apply the F.T of z.

- Let: $g(v) = \mathcal{F}(z)(v) = \displaystyle\int_{-\infty}^{+\infty} e^{ivk} z(k)\, dk$.

If it's not $\mathcal{L}$-$\int$, we should write: $\underbrace{\rho_T(dx)}$.

Let's proove: $\Big\}$

$$z(k) = e^{-rT} \int_{-\infty}^{+\infty} \left(e^{rT+x} - e^{k}\right)\left(\mathbb{1}_{\{k \leq x + rT\}} - \mathbb{1}_{\{k \leq rT\}}\right) \rho_T(x)\, dx$$

$$= e^{-rT} \int_{-\infty}^{+\infty} \underbrace{\left(e^{rT+x} - e^{k}\right)\mathbb{1}_{\{k \leq x+rT\}}}_{= 1 \text{ if } e^{rT+x} - e^{k} \geq 0,\ 0 \text{ otherwise}} \rho_T(dx) - \left(\int_{-\infty}^{+\infty} e^{x} - e^{k-rT} \rho_T(dx)\right)\mathbb{1}_{\{k \leq rT\}}$$

$$= \underbrace{e^{-rT} \int_{-\infty}^{+\infty} \left(e^{rT+x} - e^{k}\right)^+ \rho_T(dx)}_{= c(k)} - \Bigg(\underbrace{\int_{-\infty}^{+\infty} e^{x} \rho_T(dx)}_{\substack{= 1 \text{ since we are under } \mathbb{Q}. \\ \left(\mathbb{E}[e^{X_T}] = \phi_T(-i) = 1\right)}} - e^{k-rT} \underbrace{\int_{-\infty}^{+\infty} \rho_T(dx)}_{= 1.}\Bigg)\mathbb{1}_{\{k \leq rT\}}$$

$$= c(k) - \left(1 - e^{k-rT}\right)\mathbb{1}_{\{k - rT \leq 0\}} = \underbrace{c(k) - \left(1 - e^{k-rT}\right)^+}_{= z(k)\ \checkmark} \quad \blacksquare$$

• Let's compute the Fourier Transform using the previous formula:

$$g_T(v) = e^{-rT}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{ivk}\left(e^{rT+x}-e^{k}\right)\left(\mathbb{1}_{\{k\le x+rT\}}-\mathbb{1}_{\{k\le rT\}}\right)\rho_T(dx)\,dk$$

We switch $\int$

$$= e^{-rT}\int_{-\infty}^{+\infty}\int_{rT}^{rT+x} e^{ivk}\left(e^{rT+x}-e^{k}\right)dk\,\rho_T(dx)$$

$$= e^{-rT}\int_{-\infty}^{+\infty}\int_{rT}^{rT+x}\left(e^{ivk+rT+x}-e^{(iv+1)k}\right)dk\,\rho_T(dx)$$

$$= \left[\frac{e^{ivk+rT+x}}{iv}-\frac{e^{(iv+1)k}}{iv+1}\right]_{rT}^{rT+x}$$

$$= \frac{e^{iv(rT+x)+rT+x}}{iv}-\frac{e^{(iv+1)(rT+x)}}{iv+1}-\frac{e^{ivrT+rT+x}}{iv}+\frac{e^{(iv+1)rT}}{iv+1}$$

$$= \frac{e^{(iv+1)(rT+x)}}{iv}-\frac{e^{(iv+1)(rT+x)}}{iv+1}-\frac{e^{(iv+1)rT+x}}{iv}+\frac{e^{(iv+1)rT}}{iv+1}$$

$$= e^{(iv+1)(rT+x)}\underbrace{\left(\frac{1}{iv}-\frac{1}{iv+1}\right)}_{\frac{1}{iv(iv+1)}} + \dots$$

So let's come back to $g_T(v)$:

$$g_T(v) = \int_{-\infty}^{+\infty} e^{-rT}\left(\frac{e^{(iv+1)(rT+x)}}{iv(iv+1)} - \frac{e^{(iv+1)rT+x}}{iv} + \frac{e^{(iv+1)rT}}{iv+1} \right) \rho_T(dx)$$

$$= \int_{-\infty}^{+\infty} \left(\frac{e^{ivrT + (iv+1)x}}{iv(iv+1)} - \frac{e^{ivrT+x}}{iv} + \frac{e^{ivrT}}{iv+1} \right) \rho_T(dx)$$

$$= \frac{e^{ivrT}}{iv(iv+1)} \int_{-\infty}^{+\infty} e^{(iv+1)x} \rho_T(dx) - \frac{e^{ivrT}}{iv} \underbrace{\int_{-\infty}^{+\infty} e^{x} \rho_T(dx)}_{=1} + \frac{e^{ivrT}}{iv+1} \underbrace{\int_{-\infty}^{+\infty} \rho_T \, dx}_{=1}$$

$$= \frac{e^{ivrT}}{iv(iv+1)} \left(\int_{-\infty}^{+\infty} e^{i(v-i)x} \rho_T(dx) \right) + \frac{e^{ivrT}}{iv(iv+1)} \underbrace{\left(-iv - 1 + iv \right)}_{=-1}$$

$$\boxed{g_T(v) = \frac{e^{ivrT}}{iv(iv+1)} \left[\phi_{X_T}(v-i) - 1 \right]}$$

<u>Carr Madan Formula</u>.

Then we can collect $z(k)$ by inverting the formula using the IFT:

$$z(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikv} g(v) \, dv$$

and finally get the price:

$$\downarrow$$

$$c(k) = z(k) + \left(1 - e^{k-rT}\right)^+ .$$

RMK: it is a very general framework that can be applied to other models: 1) you compute $g(v)$ analytically;

we cannot compute this $\int$ analytically, we need to approximate it! ← 2) you invert the formula by IFT, numerically;

3) you get the price of the EU Call option $C(k)$.

End of 11/10/2022 recording

21/10/2024

12/10/2022 RECORDING
04/10/2021 RECORDING

Characteristic Function

(Recap on what we did last time)

Let us consider a process X_t such that the underlying asset S_t is given by

$$S_t = S_0\, e^{rt+X_t}$$

and for which the characteristic function

$$\phi_{X_t}(v) = E\left[e^{ivX_t}\right] = \int_{\mathbb{R}} e^{ivx} f_{X_t}(x)dx = \mathcal{F} f_{X_t}$$

FT of the pdf of X_t.

is well-known. Here f_{X_t} is the pdf, and $\mathcal{F}$ is the Fourier transform operator.

we just assume that there exists a characteristic fct.

In the following we will use ϕ_t as a short for ϕ_{X_t}.
Moreover, in the risk-neutral measure, it must be $\phi_t(-i) = 1$.

Proof: under $\mathbb{Q}$, $\mathbb{E}[S_t | \mathcal{F}_0] = \mathbb{E}[S_t] = S_0$. But $\mathbb{E}[S_t] = \mathbb{E}[S_0 e^{X_t}] = S_0 \underbrace{\mathbb{E}[e^{i(-i)X_t}]}_{\phi_t(-i)}$. ■

For example, considering the risk-neutral measure, for the GBM we have

$$\phi_{X_t}(v) = e^{-\frac{\sigma^2}{2}ivT-\frac{\sigma^2}{2}v^2T}$$

Even for the Heston model, the characteristic function is known analytically

Another example of a more complex model.

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^1$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^2$$

$$\phi_{X_T}(\omega) = e^{\,B(\omega)+C(\omega)}$$

$$B(\omega) = \frac{2\zeta(\omega)(1-e^{-\psi(\omega)T})V_0}{2\psi(\omega)-(\psi(\omega)-\gamma(\omega))(1-e^{-\psi(\omega)T})}$$

$$C(\omega) = -\frac{2\kappa\theta}{\sigma^2}\left[2\log\left(\frac{2\psi(\omega)-(\psi(\omega)-\gamma(\omega))(1-e^{-\psi(\omega)T}}{2\psi(\omega)}\right)+(\psi(\omega)-\gamma(\omega))T\right]$$

$$\zeta(\omega) = -\frac{1}{2}(\omega^2+i\omega)$$

$$\psi(\omega) = \sqrt{\gamma(\omega)^2-2\sigma^2\zeta(\omega)}$$

$$\gamma(\omega) = \kappa - \rho\sigma\omega i$$

Carr-Madan

- Peter Carr and Dilip B. Madan, *Option Valuation Using the Fast Fourier Transform*, 1999 ⟶ This is the initial article.
- Rama Cont and Peter Tankov, *Financial Modelling with Jump Processes,* Chapman & Hall / CRC Press, 2003 ⟶ This is the formulation that we are going to use.

Idea: compute the option price of european options (in our case, call options) moving to the Fourier space, where an analitycal formula is provided.

The following notes are based on Cont-Tankov (2003), Section 11.1.3

DEFINITIONS:

Recall the definition of the Fourier transform of a function f:

$$\mathbf{F}f(v) = \int_{-\infty}^{\infty} e^{ixv} f(x)dx$$

Usually v is real but it can also be taken to be a complex number. The inverse Fourier transform is given by:

$$\mathbf{F}^{-1}f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixv} f(v)dx$$

For $f \in L^2(\mathbb{R})$, $\mathbf{F}^{-1}\mathbf{F}f = f$, but this inversion formula holds in other cases as well. In what follows we denote by $k = \ln K$ the log strike and assume without loss of generality that $t = 0$.

In order to compute the price of a call option

$$C(k) = e^{-rT} E[(e^{rT+X_T} - e^k)^+]$$

IDEA:
we would like to express its Fourier transform in strike in terms of the characteristic function $\Phi_T(v)$ of X_T and then find the prices for a range of strikes by Fourier inversion. However we cannot do this directly because $C(k)$ is not integrable (it tends to a positive constant as $k \to -\infty$). The key idea of the method is to instead compute the Fourier transform of the (modified) time value of the option, that is, the function

$$z_T(k) = \underbrace{e^{-rT} E[(e^{rT+X_T} - e^k)^+]}_{C(k)} - (1 - e^{k-rT})^+. \qquad (11.17)$$

Let $\zeta_T(v)$ denote the Fourier transform of the time value:

$$\zeta_T(v) = \mathbf{F} z_T(v) = \int_{-\infty}^{+\infty} e^{ivk} z_T(k) dk. \tag{11.18}$$

It can be expressed in terms of characteristic function of X_T in the following way. First, we note that since the discounted price process is a martingale, we can write

$$z_T(k) = e^{-rT} \int_{-\infty}^{\infty} \rho_T(x) dx (e^{rT+x} - e^k)(1_{k \le x+rT} - 1_{k \le rT}).$$

Now we can compute $\zeta_T(v) = \mathcal{F} z_T(v)$.

$$
\begin{aligned}
\zeta_T(v) &= e^{-rT}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx e^{ivk}\rho_T(x)(e^{rT+x}-e^k)(1_{k\le x+rT}-1_{k\le rT}) \\
&= e^{-rT}\int_{-\infty}^{\infty}\rho_T(x)dx\int_{x+rT}^{rT} e^{ivk}(e^k-e^{rT+x})dk \\
&= \int_{-\infty}^{\infty}\rho_T(x)dx\left\{\frac{e^{ivrT}(1-e^x)}{iv+1}-\frac{e^{x+ivrT}}{iv(iv+1)}+\frac{e^{(iv+1)x+ivrT}}{iv(iv+1)}\right\}
\end{aligned}
$$

The first term in braces disappears due to martingale condition and, after computing the other two, we conclude that

$$
\boxed{\zeta_T(v) = e^{ivrT}\frac{\Phi_T(v-i)-1}{iv(1+iv)}} \tag{11.19}
$$

The martingale condition guarantees that the numerator is equal to zero for $v = 0$ and the fraction has a finite limit for $v \to 0$.

So the above formula is well defined, even for $v = 0$.

Option prices can now be found by inverting the Fourier transform:

$$z_T(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ivk}\zeta_T(v)dv \tag{11.20}$$

$$\Big\Downarrow$$

once $z_T(k)$ is obtained, one can get $C(k)$, call price.

$$C(k) = z_T(k) + (1 - e^{k-rT})^+.$$

Now the missing point is how to compute 11.20 in an accurate & fast way: FFT.

Using the FFT

fft Discrete Fourier transform.

fft(X) is the discrete Fourier transform (DFT) of vector X.

For length N input vector x, the DFT is a length N vector X, with elements

$$X(k) = \sum_{n=1}^{N} x(n)*\exp(-1i*2*pi*(k-1)*(n-1)/N),\ 1 <= k <= N.$$

The inverse DFT (computed by IFFT) is given by

$$x(n) = (1/N) \sum_{k=1}^{N} X(k)*\exp(\ 1i*2*pi*(k-1)*(n-1)/N),\ 1 <= n <= N.$$

We have $Z_T(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ivk} g_T(v)\,dv$. And we want to compute:

DFT $\quad F_n = \sum_{k=0}^{N-1} F_k\, e^{-2\pi i n \frac{k}{N}} \quad (n = k+1,\ k = n-1)$

1st step: to troncate the integral:

$$Z_T(k) \simeq \frac{1}{2\pi}\int_{-A/2}^{A/2} e^{-ivk} g_T(v)\,dv$$

À checker ça...

2nd step: quadrature:

$$Z_T(k) \simeq \frac{1}{2\pi}\frac{A}{N-1}\sum_{j=0}^{N-1} w_j\, e^{-iv_j k} g_T(v_j)$$

where: $v_j = -\frac{A}{2} + j \times \underbrace{\frac{A}{N-1}}_{=A}$ and w_j: weights.

Pourquoi ?? d'un coup c'est $g_T(v_j)$??

$$g_T(v_j) = g_j = \frac{1}{2\pi}\frac{A}{N-1}\sum_{j=0}^{N-1} w_j\, g_j\, e^{-ik\left(-\frac{A}{2} + j\frac{A}{N-1}\right)}$$

$$= \frac{1}{2\pi} \frac{A}{N-1} e^{ik A/2} \sum_{j=0}^{N-1} w_j g_j e^{-ikj \frac{A}{N-1}}.$$

Let's define : $k = \dfrac{2\pi n}{N \frac{A}{N-1}}$ so that :

$$F_n = \frac{1}{2\pi} \Delta e^{i \frac{\pi n (N-1)}{N}} \sum_{j=0}^{N-1} w_j g_j e^{-i \frac{2\pi n}{N} j}.$$

So we find the same expression as DFT :

$$F_n = \underbrace{\frac{1}{2\pi} \Delta e^{i \frac{\pi n (N-1)}{N}}}_{\text{Known quantity}} \cdot \mathrm{DFT}(w_j g_j)$$

We have to compute

$$z_T(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ivk} \zeta_T(v) dv.$$

Since $z_T(k)$ must be real, then it holds

Trick of Functional Analysis.

$$z_T(k) = \frac{1}{\pi} \int_{0}^{+\infty} e^{-ivk} \zeta_T(v) dv$$

To compute this integral, we can use a quadrature formula, i.e.,

$$\begin{aligned} z_T(k) &\approx \frac{1}{\pi} \int_{0}^{A(N-1)/N} e^{-ivk} \zeta_T(v) dv \\ &\approx \frac{1}{\pi} \sum_{j=0}^{N-1} w_j \eta e^{-i\eta jk} \zeta_T(\eta j). \end{aligned}$$

If we set $w_0 = w_{N-1} = 0.5$, 1 otherwise, we are using a trapezoidal formula with nodes $j\eta$, with $\eta = A/N$.

If we now consider the following grid for the log-strike $k_l = -\lambda N/2 + \lambda l$, with $\lambda = 2\pi/(N\eta)$ and $l = 0, ..., N-1$, we obtain

$$
\begin{aligned}
z_T(k_l) &\approx \frac{1}{\pi}\sum_{j=0}^{N-1} w_j \eta e^{-i\eta j(-\lambda N/2+\lambda l)}\zeta_T(\eta j) \\
&= \frac{1}{\pi}\sum_{j=0}^{N-1} w_j \eta e^{i\eta j\lambda N/2} e^{-i\eta j\lambda l}\zeta_T(\eta j) \\
&= \frac{1}{\pi}\sum_{j=0}^{N-1} w_j \eta e^{ij\pi} e^{-ijl2\pi/N}\zeta_T(\eta j) \\
&= \frac{1}{\pi} FFT(\{w_j \eta e^{ij\pi}\zeta_T(\eta j)\}_{j=0}^{N-1})
\end{aligned}
$$

Not exactly the same formula, but in the end it is the same result.

Matlab code

```
function fii = trasf_fourier_BS (r,sig,T,v)
fii = (exp(1i*r*v*T)).*…
   ((characteristic_func_BS(sig,T,v-1i)…
     -1)./(1i*v.*(1+1i*v))  );

function f = characteristic_func_BS(sig,T,y)
f = exp(1i* (-sig^(2)/2*T).*y …
    -(T*sig.^(2)*y.^(2))/2) ;
```

```
function [P_i] = FFT_BS(K_i)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% European Call - BS model
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% [P_i] = FFT_BS(K_i) - could be a vector
% INPUT: K_i = strikes - could be a vector
% OUTPUT: P_i = prices
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%--- model parameters
S = 100; % spot price
T = 1; % maturity
r = 0.0367; % risk-free interest rate
sig = 0.17801; % volatility
Npow = 15;
N = 2^(Npow); % grid point
A = 600; % upper bound
eta = A/N;
lambda = 2*pi/(N*eta);
k = -lambda * N/2  + lambda *(0:N-1); % log-strike grid
K = S * exp(k); % strike
v = eta*(0:N-1);
v(1)=1e-22; %correction term: could not be equal to zero
            %   (otherwise NaN)
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%PRICING
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
tic
% Fourier transform of z_k
tr_fou = trasf_fourier_BS(r,sig,T,v);
% Trapezoidal rule
w = [0.5  ones(1,N-2)  0.5];
h = exp(1i*(0:N-1)*pi).*tr_fou.*w*eta;
P = S * real( fft(h)/pi + max(1-exp(k-r*T),0)); % prices
time=toc


% delete too small and too big strikes
index=find( (K>0.1*S & K<3*S) );
K=K(index); P=P(index);
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% PLOT
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
figure
plot(K,P, 'r') ;
hold on
axis([0  2*S  0  S]) ;
xlabel('strike') ;
ylabel('option price') ;


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% INTERPOLATION
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
P_i = interp1(K,P,K_i,'spline');
```

22/10/2024 Topic of today : FFT & other derivatives. 14/10/2022 Recording

We want to see if we can use FFT technique when we deal with other kinds of derivatives, like path-dependent ones.

↳ Answer: we can do something, but not as much as Carr Madan. We are not able to get closed formulas.

Let's consider for example a Barrier Option:

$$\begin{cases} M_t = \max\limits_{0\le s\le t} X_s \\ m_t = \min\limits_{0\le s\le t} X_s \end{cases} ; \quad \begin{cases} S_t = S_0 e^{X_t} \\ k = \log\left(\frac{K}{S_0}\right) \end{cases} ; \quad b = \log(B/S_0) \text{ where } B \text{ is a barrier.}$$

→ Let's consider an up & out call:

$$\underbrace{C(T; k, b) = S_0\left(e^{X_T} - \underbrace{e^{k}}_{K/S_0}\right)^+}_{\text{Payoff of EU Call style}} \underbrace{\mathbb{1}_{\underbrace{\{M_T \le b\}}_{\text{Up \& Out}}}}_{\text{Barrier}}$$

] Payoff of our up&out Call, i.e., "Price @ Maturity T".

So as we can see from the expression above: now, we are not only interested in the distribution of X_t anymore. We are now interested in the **joint distribution of X_t and M_t**.

For simplicity, we assume:
- $r = 0 \longrightarrow$ NO DISCOUNT;
- $\psi_X(-i) = 0 \longrightarrow$ DISCOUNTED STOCK PRICE IS A MG.

Let's move to $t < T$:

- $C(t; k, b) = \mathbb{E}^{\mathbb{Q}}\left[S_0 (e^{X_t} - e^k)^+ \mathbb{1}_{\{M_t \leq b\}} \,\middle|\, \mathcal{F}_t \right]$;

WE ARE UNDER $\mathbb{Q}$ (cf assumptions above)

ρ_T: joint density of (X_T, M_T).

- $C(0; k, b) = \int_{\mathbb{R}} \int_{\mathbb{R}} S_0 (e^x - e^k)^+ \mathbb{1}_{\{y < b\}} \rho_T(dx, dy)$.

So if we want to make something close to CM, we need to compute this integral, but now we have joint density.

↳ Can we define a similar "bidimensional F.T"? The answer is NO.

Hint (out of our scope): apply a "double FT" in k and in b:

$\int_{\mathbb{R}} \int_{\mathbb{R}} e^{iuk} e^{ivb} C(0; k, b)\, dk\, db$, not surprisingly we arrive to a similar result as Carr Madan :

$$\int_{\mathbb{R}} \int_{\mathbb{R}} e^{iuk} e^{ivb} \, C(0; k, b) \, dk \, db = \frac{F(u-i, v)}{uv(1+\dots u)}$$, with F the

characteristic function of (X_T, M_T).

MAJOR PROBLEM : we don't know F. So even if we are able to get a formula, it is not useful since we don't know F.

However, researchers made computations and discovered :

$$q \int_0^{+\infty} \iint_{\mathbb{R}\,\mathbb{R}} e^{iuk + ivb} e^{-qt} C(t; k, b) \, dt \qquad \text{(Laplace transform)}$$

$$\parallel$$

$$\frac{\phi_q^+(v+u-i)\,\phi_q^-(u-i)}{uv(1+iu)}$$ with $\phi_q^{\pm}$: "Wiener Hopf Factors".

No more the joint density!

$$\left\{ \phi_q^+(u)\,\phi_q^-(u) = \frac{q}{\underbrace{q - \psi_X(u)}_{\text{This is KNOWN.}}} \right.$$ (ϕ_q^+ has support in $\mathbb{R}^+$, ϕ_q^- in $\mathbb{R}^-$)

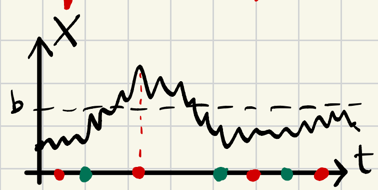
So the idea is : $\boxed{1}$ compute $\frac{q}{q-\psi_X(u)}$ then $\boxed{2}$ take the inverse Laplace & two times the inverse Fourier transforms. SIMILAR IDEA AS CM.

What is the problem? It is very hard, numerically, to compute $\frac{q}{q-\psi_X(u)}$.

→ That's why we won't go further: it's complex & not used.

Clearly, what we have shown in ~~ was not the correct way to perform: so we have to change our POV.

So, our idea will be: apply FOURIER TRANSFORM TO BARRIER OPTIONS BUT IN THE DISCRETE MONITORING CASE.
(we check for the barrier only on some dates)

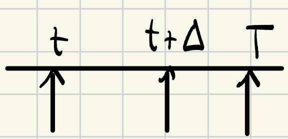


~ : OK

~ : NOT OK

→ cf "conv" / "convolution" on WEBEEP. What we'll see now is a convolution method:

Let's consider first a EU option: $T > t$, $t + \Delta < T$.



we know that: $C(x,t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\left[C(X_T, T) \mid \mathcal{F}_t \right]$, $X_t = x$

we always need to be under $\mathbb{Q}$.

And this is TRUE for any $t < T$, so it means that:

$$C(x,t) = e^{-r\Delta}\,\mathbb{E}^{\mathbb{Q}}\Big[C(X_{t+\Delta}, \underbrace{t+\Delta}_{<T}) \,\Big|\, X_t = x\Big]$$

$$= e^{-r\Delta}\int_{\mathbb{R}} C(y, t+\Delta)\, p_{x,t,\Delta}(y)\, dy$$

$$p_{x,t,\Delta}(y) = \mathbb{P}(X_{t+\Delta} = y \mid X_t = x)$$

conditional!

Indeed, this is "conditional expectation".

Now, in the Lévy framework: $\mathbb{P}(X_{t+\Delta} = y \mid X_t = x) = \mathbb{P}(X_\Delta = y - x)$ using the stationary and identically distributed increments.

So: $$C(x,t) = e^{-r\Delta}\int_{\mathbb{R}} C(y, t+\Delta)\, f_\Delta(y-x)\, dy \quad (E) \quad [\text{Lévy framework}]$$

f_Δ ↔ density of the increments of X.

↔ density of X_Δ.

DEF: [CONVOLUTION] $$g_1(x) = \int_{\mathbb{R}} g_2(x-y)\, g_3(y)\, dy. \qquad (g_1 = g_2 * g_3).$$

PROP: $$\mathcal{F}(g_1) = \mathcal{F}(g_2) \times \mathcal{F}(g_3) \qquad \left(* \xrightarrow{\mathcal{F}} \times \right).$$

Let's define : $\boxed{f_\Delta^b(x) = f_\Delta(-x)}$ such that (E)

becomes : $$C(x,t) = e^{-r\Delta} \int_{\mathbb{R}} f_\Delta^b(x-y)\, C(y, t+\Delta)\, dy.$$

This is written as a convolution. Now we use the above property :

$$\mathcal{F}(C(\cdot, t)) = e^{-r\Delta}\, \mathcal{F}(f_\Delta^b) \times \mathcal{F}(C(\cdot, t+\Delta))$$

i.e : $$C(\cdot, t) = \mathcal{F}^{-1}\left[e^{-r\Delta}\, \mathcal{F}(C(\cdot, t+\Delta))\, \underbrace{\mathcal{F}(f_\Delta^b)}_{\text{? Key point: what is it?}} \right]$$

$\mathcal{F}(f_\Delta) = \phi_{X_\Delta}$ ⟵ "$\mathcal{F}$(density) $= \phi$".

$\mathcal{F}(f_\Delta^b) = \mathcal{F}(f_\Delta(-\cdot)) = \underbrace{\phi_{X_\Delta}^*}_{\text{we know it so we can use it.}}$ ⟵ conjugate (by $\mathbb{C}$ analysis).

THIS IS THE SO CALLED "CONV METHOD"

take payoff $\to \mathcal{F}$(payoff) $\to \times \phi^* \to \mathcal{F}^{-1}(\ )$.

CONV METHOD FOR EUROPEAN OPTION:

$c(x, T) = S_0 (e^x - e^k)^+$. Compute:

$\mathcal{F}(S_0(e^x - e^k)^+) \times \phi_{X_T}^*(u)$ and then take $\mathcal{F}^{-1}$:

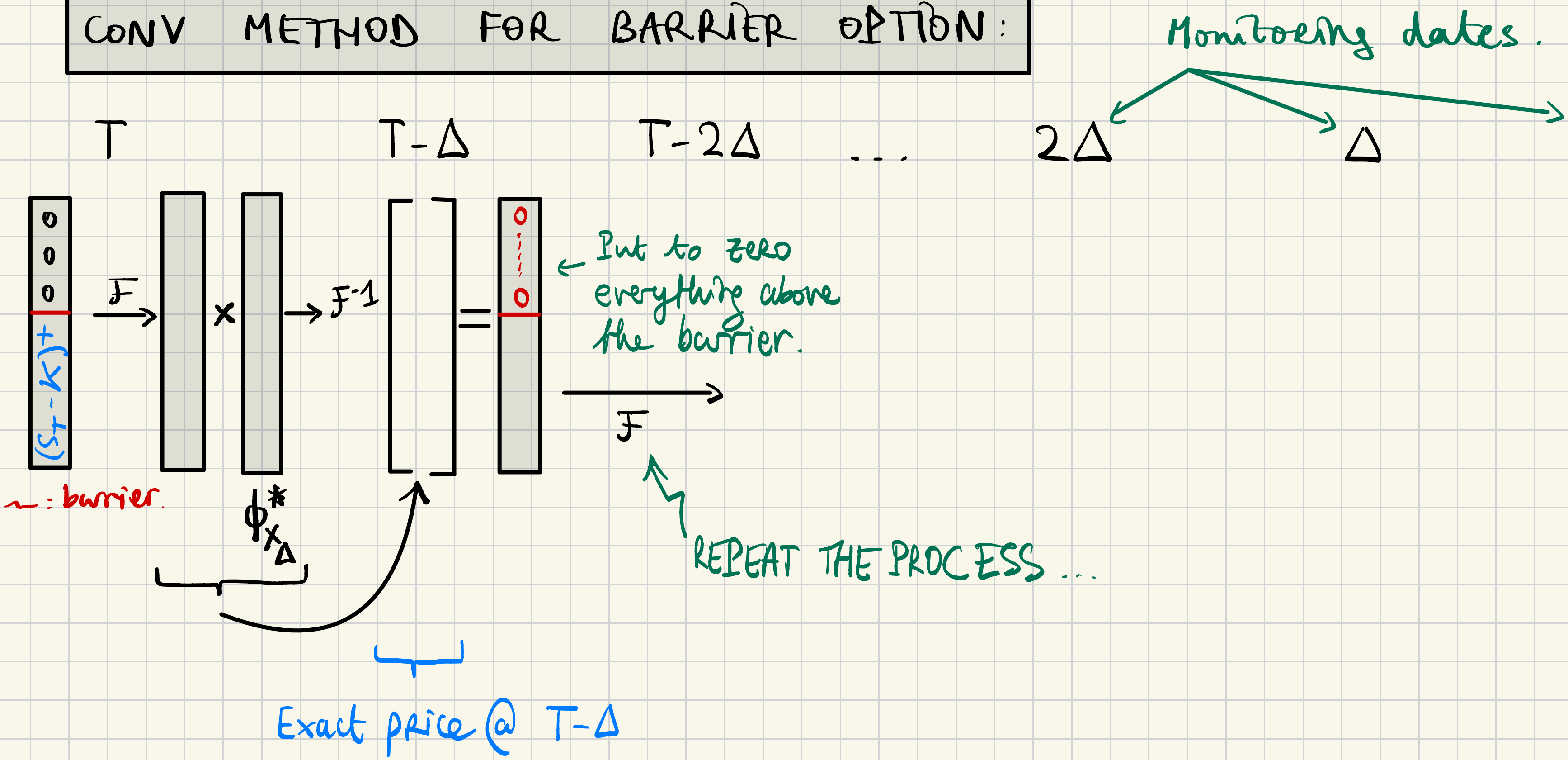
$$C(x, 0) = \mathcal{F}^{-1}_{u \to x}\left[\mathcal{F}_{x \to u}\left(S_0(e^x - e^k)^+\right) \times \phi_{X_T}^*(u)\right].$$

Price @ time 0.

However, use CM if you can (more accurate).

We can exploit this method for <u>BARRIER OPTIONS w/ DISCRETE MONITORING.</u>

CONV METHOD FOR BARRIER OPTION:



↳ The idea is that between two monitoring dates, the evaluation is the same as European option. So we repeat the process till $t=0$, taking each time into account the barrier.

cf CONV_Method.pdf → fully explained method.

Now, let's look @ the code:

- we need a function to compute characteristic fct : <u>charfunction.m</u>
  - ↳ with a flag if we want the complex conjugate.
  - ↳ there is a correction to take into account the risk free rate & the potential dividends, and also to have $\phi_X(-i) = 1$ (i.e $\psi_X(-i) = 0$ under $\mathbb{Q}$).

- <u>main.m</u> :
  - ↳ Ndate = 12 → Monthly monitoring.
  - ↳ N = 2112 : grid of the log returns : $\log(S_T/S_0)$.

- <u>kernel.m</u> : we need two grids : one for log price and one for Fourier space.

  log price ↓ N, dx, x

  Fourier space ↓ dw, w (with 2π).

  see why in the CM video.

  - ↳ we invert the characteristic function using "fft" (⚠ not ifft)
  - ↳ we need to correct the shift induced by MATLAB:

$$h = \text{real}(\underbrace{\text{fftshift}}_{\text{This corrects the fact that } x \text{ is } -N \to N-1.}(\text{fft}(\underbrace{\text{ifftshift}}_{\text{corrects the } w \text{ grid which doesn't go from } 0 \text{ to} \dots \text{ but from } -N \text{ to } N-1.}(H))))$$

(shifts of the GRID)

h is the density of $(-X)$ (since it was $\Phi_X^* = H$).

→ see the comments that explain the shifts in grids.

- **CONV.m**: new algorithm that uses kernel & goes backward in times.
  - ↳ once again we have to transform H to be in MATLAB style: `H = ifftshift(H)`.
  - ↳ if we remove the line "V(S<=Barriera) = 0" we are basically pricing a EU option.

As a conclusion, this code can be used to price a barrier option with discrete monitoring under any Lévy, exactly like we previously did with MC. And it's a general method: the only thing we have to change if we want to use another kind of Lévy is the computation of the characteristic function. Nothing else.

⚠ Important to understand why we shift: MATLAB wants $0 \to 2N$, we want $-N \to N$.

⚠ We don't use char. fct for continuous monitoring path-dependent options: it is a nightmare because it is not sufficient to know ϕ...

↳ We have PDE or we just take discrete monitoring.