Lévy Processes

Noé Debrois

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1 Stochastic Jump Processes

We want to define "stochastic jump processes". We need two things which are the following. When does the jump occur ? And what is the size of the jump ? But of course, the jump cannot go $+\infty$. So, basically, in this course, we will work with right-continuous-left-limited processes, i.e, "CADLAG" (continu à droite, (admet une) limite à gauche). Let's define the tools we need to work in this framework.

2 Basic tools

Definition: [*Characteristic Function*] Let X be a random variable in $\mathbb{R}^d$. Its characteristic function $\Phi_X : \mathbb{R} \to \mathbb{R}$ is defined as :

$$z \to \Phi_X(z) = \mathbb{E}[e^{iz\cdot X}] = \int_{\mathbb{R}^d} e^{iz\cdot X} d\mu_X(x)$$

where μ_X is the measure associated to the distribution of X, i.e,

$$\forall A \in B(\mathbb{R}), \mu_X(A) = \mathbb{P}(X \in A)$$

Remark: If μ_X has a density p_X, we can write $d\mu_X(x) = p_X(x)\,dx$, where dx is the Lebesgue measure. This is equivalent to the **absolute continuity** of the distribution of X with respect to the Lebesgue measure.

Definition: [*Moments*] Let $n \in \mathbb{N}$,

$$\text{Moment: } m_n(X) = \mathbb{E}[X^n] \;;$$

$$\text{Centered moment: } \mu_n(X) = \mathbb{E}[(X - \mathbb{E}(X))^n].$$

Property:

- If $\mathbb{E}[|X|^n] < +\infty$, then $\Phi_X \in C^n(I)$ where I is an open set contianing 0, and :

$$m_k = \frac{1}{i^k}\frac{\partial^k}{\partial z^k}\Phi_X(0), k \in \{1, ..., n\}.$$

- If Φ_X has n continuous derivatives in 0, then :

$$m_k = \frac{1}{i^k}\frac{\partial^k}{\partial z^k}\Phi_X(0), k \in \{1, ..., n\}.$$

Definition: [*Moment Generating Function*]

$$M_X(u) = \mathbb{E}[e^{u\cdot X}] \;;$$

$$m_n = \frac{\partial^n}{\partial u^n} M_X(0).$$

Remark: It is easy to go from M_X to Φ_X and vice versa since : $M_X(u) = \Phi_X(-iu)$.

Definition: [*Characteristic Exponent*] When it exists, Ψ_X such that :

$$\Phi_X(u) = e^{\Psi_X(u)}.$$

Remark: $\Psi_X(0) = 0$.

Definition: [*Exponential Random Variable*]

Theorem: [*Absence of memory*] Let $T \geq 0$ a random variable such that :

$$\forall t, s > 0, \mathbb{P}(T > t + s | T > t) = \mathbb{P}(T > s).$$

This is equivalent to : $T \sim Exp$.

Definition: [*Poisson Distribution*] Let N random variable with values in $\mathbb{N}$. $N \sim Poiss(\lambda)$ if and only if :

$$\mathbb{P}(N = n) = e^{-\lambda}\frac{\lambda^n}{n!}.$$

Then we have the following moment generating function : $M(u) = e^{\lambda(e^u - 1)}$.

Property: Let $(\tau_i)_{i\geq 1}$ **i.i.d** random variables following an $Exp(\lambda)$ distribution. Let :

$$\forall t > 0, N_t = inf\{n \geq 0 : \sum_{i=1}^{n+1} \tau_i > t\}.$$

We then have that $N_t \sim Poisson(\lambda t)$.

Property:

- Let $Y_1, Y_2 \sim Poisson(\lambda_1), Poisson(\lambda_2)$ and $Y_1 \perp Y_2$. Then :

$$Y_1 + Y_2 \sim Poisson(\lambda_1 + \lambda_2).$$

- *[Infinite Divisibility]* Let $Y \sim Poisson(\lambda)$. Then :

$$\forall n, Y = \sum_{i=1}^{n} Y_i$$

  where $Y_1, ..., Y_n$ **i.i.d** $\sim Poisson(\lambda/n)$.

3 Poisson Process

Definition: [*Poisson Process*] Let $(\tau_i)_{i\leq 1}$ be a sequence of **i.i.d** random variables following an exponential distribution of parameter λ. Let $T_n = \sum_{i=1}^{n} \tau_i$. Then the following is a Poisson Process with intensity λ :

$$N_t = \sum_{n=1}^{+\infty} \mathbf{1}_{t\geq T_n}.$$

Remark: This definition is equivalent to the one above with the inf. If we fix time, $N_t \sim Poisson(\lambda t)$.
Remark: Fortunately, with Poisson Process we don't have mass probability. It means that : $\forall t \geq 0, N_{t^-} = N_t$ with probability 1.
Remark: $(N_t)_{t\geq 0}$ is a CADLAG process.

Property: [*About Poisson Process*]

- $\forall t \geq 0, \forall n \geq 0, \mathbb{P}(N_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$;
- $\Phi_{N_t}(u) = \mathbb{E}[e^{iuN_t}] = e^{\lambda t(e^{iu}-1)}$;
- $(N_t)_{t\geq 0}$ has **independent** increments :

$$\forall t_1 < ... < t_n, N_{t_n} - N_{t_{n-1}} \perp N_{t_{n-1}} - N_{t_{n-2}}, ..., N_{t_2} - N_{t_1}, N_{t_1};$$

- $\forall t \geq 0, \mathbb{E}[N_t] = \lambda t.$

Remark: We don't have Martingale Property with $(N_t)_t$, so we need to define the "*Compensated Poisson Process*".

Definition: [*Compensated Poisson Process*] Let $\forall t \geq 0, \hat{N}_t = N_t - \lambda t$, and $\hat{N}_0 = 0$. Clearly, $\hat{N}$ is **not** a Poisson Process, since it doesn't even take only integer values.

Property: $\Phi_{\hat{N}_t}(z) = e^{\lambda t(e^{iz}-1-iz)}$.

Property: $\hat{N}$ is a Martingale.

Theorem: Let $(X_t)_{t\geq 0}$ be a counting process with *independent* and *stationary* increments. Then $(X_t)_{t\geq 0}$ is a Poisson Process.
Remark: Stationary increments means that :

$$\forall t > s, h > 0, X_{t+h} - X_{s+h} \sim X_t - X_s.$$

Remark: The previous theorem tells us that the Poisson Process is our only "counting process" choice **if** we want to work with **independent and stationary** increments : this will be the **framework of this course.**

Definition: Let's introduce a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, $\omega \in \Omega$ a realisation, and $T_i(\omega)$ the i-th jump time occurring in the realisation ω. Then, let's define the following integer-valued **random measure** :

$$\forall A, M(\omega, A) = \#\{i \geq 1 : T_i(\omega) \in A\}.$$

We can write :

$$N_t(\omega) = M(\omega, [0,t]) = \int_0^t M(\omega, ds).$$

Definition: [*Compensated Random Measure*] $\hat{M}(\omega, A) = M(\omega, A) - \int_A \lambda dt = M(\omega, A) - \lambda|A|$.

Now we are ready to introduce **Lévy** processes.

4 Lévy Process

Definition: Let $(\Omega, \mathbb{F}, \mathbb{P})$ a probability space, $\Omega \subseteq \mathbb{R}^d$. A **CADLAG** process $(X_t)_{t\geq 0}$ such that $X_0 = 0$ is Lévy if :

- Increments are **independent** :

$$0 \leq t_0 \leq t_1 \leq ... \leq t_n, X_{t_0} \perp X_{t_1} - X_{t_0} \perp ... \perp X_{t_n} - X_{t_{n-1}};$$

- Increments are **stationary** :

$$\forall t, h > 0, X_{t+h} - X_t \sim X_h;$$

- **Stochastic Continuity :** [*defined starting from the limit in probability*]

$$\forall t > 0, \forall \varepsilon > 0, \lim_{h\to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0.$$

Example: $(N_t)_{t\geq 0}$; $(W_t)_{t\geq 0}$ is the only Lévy process which is continuous.

Property: We have **infinite divisibility**, i.e

$$\forall t > 0, n \geq 1, \Delta = \frac{t}{n}, X_t = X_{n\Delta} = (X_{n\Delta} - X_{(n-1)\Delta}) + (X_{(n-1)\Delta} - X_{(n-2)\Delta}) + ... + (X_{2\Delta} - X_{\Delta}) + X_{\Delta};$$

i.e $X_t = \sum_{i=1}^n Y_i$ where $Y_i = X_{i\Delta} - X_{(i-1)\Delta}$ with $(Y_i)_i$ that are **i.i.d** (Y_i are **independent** and $\forall i, Y_i \sim X_\Delta$).

Remark: The infinite divisibility is precisely the ability to write X_t as a sum of **i.i.d** terms.

Definition: The characteristic function of a Lévy process is defined as:

$$\Phi_{X_t}(u) = \mathbb{E}[e^{iuX_t}].$$

- It is multiplicative:

$$\Phi_{X_{t+s}}(u) = \mathbb{E}[e^{iu(X_{t+s} - X_s + X_s)}] = \mathbb{E}[e^{iu(X_{t+s} - X_s)} e^{X_s}].$$

  Therefore, by independence :

$$\Phi_{X_{t+s}}(u) = \mathbb{E}[e^{iu(X_{t+s} - X_s)}] \cdot \mathbb{E}[e^{iuX_s}].$$

  And since $X_{t+s} - X_s \sim X_t$ by definition, hence :

$$\Phi_{X_{t+s}}(u) = \Phi_{X_t}(u) \times \Phi_{X_s}(u).$$

- Lévy processes admit a **characteristic exponent** :

$$\exists \Psi_X : \mathbb{R}^d \to \mathbb{R} \text{ such that : } \Phi_{X_t}(u) = e^{t\Psi_X(u)}.$$

Remark: t is "outside", which is really useful. That's the reason why we will **never** write the characteristic function of a Lévy Process. Instead, we will use the characteristic exponent, from which we cna easily get Φ_{X_t}.

Example: For a Poisson Process,

$$\Phi_{X_t}(u) = e^{\lambda t(e^{iu}-1)} \text{ so that : } \Psi_X(u) = \lambda(e^{iu} - 1).$$

Now that we are able to count the jumps when they occur, we would like to capture the size of the jumps.

5 Compounded Poisson Process

Definition: A Compounded Poisson Process (CPP) $(X_t)_{t\geq 0}$ with **intensity** λ and **jumpsize distribution** f is defined as:

$$X_t = \sum_{i=1}^{N_t} Y_i, \text{ where :}$$

- $(N_t)_{t\geq 0}$ is a **counting process** with intensity λ ;
- Y_i are **i.i.d** random variables with distribution f.

Properties: Let $(X_t)_{t\geq 0}$ be a CPP as above. Then :

- $(X_t)_{t\geq 0}$ is for sure **CADLAG** ;
- A Poisson Process is a special case of a CPP, where f is such that $\forall i, P(Y_i = 1) = 1$;
- $(X_t)_{t\geq 0}$ is a CPP **if and only if** $(X_t)_{t\geq 0}$ is a **Lévy Process with piecewise constant paths**.

Property: [*Computation of the characteristic function of a CPP*] For all $t \geq 0$, $\Phi_{X_t}(u)$ of a CPP is given by:

$$\Phi_{X_t}(u) = \exp\left(t\lambda \int_{\mathbb{R}^d} \left(e^{iux} - 1\right) f(dx)\right) = \exp(t\Psi_X(u)) \text{ where : } \Psi_X(u) = \lambda \int_{\mathbb{R}^d} \left(e^{iux} - 1\right) f(dx).$$

Remark: As previously said, we notice that using $f(dx) = \delta_1(dx)$, we find back the formula for a Poisson Process.

Proof: By the **Tower Property** :

$$\Phi_{X_t}(u) = \mathbb{E}\left[e^{iuX_t}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{iuX_t}|N_t\right]\right]$$

Using the expression of $(X_t)_{t\geq 0}$; and the fact that the sum in the exponential is **measurable w.r.t** N_t with all the Y_i being **independent** (therefore all the e^{iuY_i} too) we get :

$$\Phi_{X_t}(u) = \mathbb{E}\left[\mathbb{E}\left[e^{iu\sum_{i=1}^{N_t} Y_i}|N_t\right]\right] = \mathbb{E}\left[\prod_{i=1}^{N_t} \mathbb{E}\left[e^{iuY_i}|N_t\right]\right]$$

Then we use the fact that $\forall i, Y_i \perp N_t$; and then that all the Y_i are identically distributed :

$$\Phi_{X_t}(u) = \mathbb{E}\left[\prod_{i=1}^{N_t} \mathbb{E}\left[e^{iuY_i}\right]\right] = \mathbb{E}\left[\mathbb{E}\left[e^{iuY_1}\right]^{N_t}\right] = \mathbb{E}\left[\hat{f}(u)^{N_t}\right]$$

where $\hat{f}(u)$ will be computed afterwards. Then by the **Transfer Theorem** for $\hat{f}(u)^{N_t}$ with $N_t \sim \mathit{Poisson}\,(\lambda t)$:

$$\Phi_{X_t}(u) = \sum_{n=0}^{+\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \left(\hat{f}(u)\right)^n = e^{\lambda t(\hat{f}(u)-1)}.$$

Now we just need to compute $\hat{f}(u)$, which, by the **Transfer Theorem** is :

$$\hat{f}(u) = \mathbb{E}\left(e^{iuY_1}\right) = \int_{\mathbb{R}^d} e^{ius} f(ds).$$

Hence, putting everything together we get :

$$\Phi_{X_t}(u) = e^{\lambda t\left(\int_{\mathbb{R}^d} e^{ius} f(ds) - 1\right)} = e^{\lambda t\left(\int_{\mathbb{R}^d}\left(e^{ius}-1\right) f(ds)\right)} \square$$

Definition: [*Lévy Measure*] The **Lévy Measure** of a CPP is :

$$\forall A, \nu(A) = \lambda f(A) \ \textbf{WARNING : this is not a probability measure !}$$

so that the **Characteristic Exponent** is :

$$\Psi_X(u) = \int_{\mathbb{R}^d} \left(e^{iux} - 1\right) \nu(dx)$$

Remark: As previously mentioned in section 2, in most of the cases : $\exists K$ s.t $f(dx) = K(x)dx$ which makes everything easier.

Properties: [*About the Lévy Measure of a CPP*]

- $\forall A, \nu(A) = \mathbb{E}\left[\#\left\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\right\}\right]$ the **Lévy Measure of** A, where :
  - $\Delta X_t = X_t - X_{t^-} = X_t - lim_{h \to 0^+} X_{t-h}$;
  - It means that the Lévy Measure of A is $\nu(A) = \mathbb{E}\left[\text{nombre de sauts} \neq 0, \in A, \text{for } t \in [0,1]\right]$.
- $\forall B \in [0,+\infty) \times \mathbb{R}^d, J_X(B) = \#\left\{(t, \Delta X_t) \in B\right\}$, i.e,
  it is a **Muldi-dimensional Random Measure** (1 realization gives 1 measure) where :
  - $J_X : \Omega \to \mathbb{M}\left([0,+\infty) \times \mathbb{R}^d\right)$ (space of measures) ;
  - $[0,+\infty)$ stands for the **time** ;
  - $\mathbb{R}^d$ stands for the **jump**.
- Then we can write :

$$X_t = \sum_{i=1}^{N_t} Y_i = \sum_{s \in [0,t]} \Delta X_s = \int_{[0,t] \times \mathbb{R}^d} x J_X(ds \times dx).$$

Now we would like to write Lévy Processes. Here follows the first possibility, which is about **Finite Activity Lévy**. The other form of Lévy is **Infinite Activity Lévy**.

Definition: [*Finite Activity Lévy*] The following two forms are two equivalent ways to write **Finite Activity (FA) Lévy** processes :

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$$

$$X_t = \mu t + \sigma W_t + \sum_{s \in [0,t]} \Delta X_s$$

Because it is a **FA Lévy**, the term $\sum_{s \in [0,t]} \Delta X_s \in \mathbb{R}$. We can also write it as a **CPP** term : $\sum_{i=1}^{N_t} Y_i$ (cf above). There is a third way of writing such a process (**FA Lévy**), exploiting the **random measure** we have seen before :

$$X_t = \mu t + \sigma W_t + \int_{[0,t] \times \mathbb{R}^d} x J_X(ds \times dx)$$

where J_X is the **Poisson Random Measure** with intensity $\nu(dx)dt$ ($\nu(dx)$ being the Lévy Measure (cf above)). Notice that all the Lévy processes can be written using this third form. It is not the case with the first two forms which are **only valid for FA Lévy** processes.

Remark: Remember that $X_t = log(S_t/S_0)$.

Remark: Now, let's consider **General Lévy processes**. We can always define the Lévy Measure $\nu(.)$:

$$\forall \text{ compact set } A \in \mathbb{R}^d, \text{ such that } 0 \notin A, \nu(A) < +\infty.$$

Remember previously :

$$\nu(A) = \mathbb{E}\left[\#\left\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\right\}\right].$$

So for example : $\nu([1,2]) = +\infty$ is **not possible**.

Now let's define **Infinite Activity (IA) Lévy**.

Definition: [*Infinite Activity Lévy*] A Lévy process with the following characteristics :

- Finite number of "*large jumps*" : for the moment we choose the jumps of size ≥ 1 ;
- Infinite number of "*infinitesimal jumps*".

Property: [*Lévy-Ito Decomposition of IA Lévy*] Let $(X_t)_t$ be a Lévy process in $\mathbb{R}^d$, ν be its Lévy measure. Then :

- ν is a Radon measure on $\mathbb{R}^d - \{0\}$;
- $\int_{|x|\geq 1} \nu(dx) < +\infty$ [*large jumps*];
- $\int_{|x|<1} |x^2|\nu(dx) < +\infty$ [*small jumps*];
- $\exists \gamma \in \mathbb{R}^d, A \in \mathbb{R}^{d\times d}$ (A is a **variance-covariance** matrix) and let $(B_t)_t$ be a Brownian Motion with A as variance-covariance matrix, such that : $X_t = \gamma t + B_t + X_t^l + lim_{\varepsilon\to 0}\tilde{X}_t^\varepsilon$ where :

  $$X_t^l = \int_{|x|\geq 1, s\in[0,t]} xJ_X(dx \times ds) = \sum_{|\Delta X_s|\geq 1, s\in[0,t]} \Delta X_s \text{ [\textit{Compound Poisson}]};$$

  and :

  $$X_t^\varepsilon = \int_{\varepsilon<|x|<1, s\in[0,t]} xJ_X(dx \times ds) \text{ [\textit{Compound Poisson}]};$$

  and :

  $$\tilde{X}_t^\varepsilon = \int_{\varepsilon<|x|<1, s\in[0,t]} x\left(J_X(dx \times ds) - \nu(dx)ds\right) \text{ [We \textbf{cannot split} in two with \textbf{IA Lévy}].}$$

Definition: In this case, we define the **Lévy triplet** (γ, A, ν).

Remarks:

- $\gamma t + B_t$ is the "continuous part" of the process ;
- X_t^ε, X_t^l are **Compound Poisson** processes ;
- $\tilde{X}_t^\varepsilon$ is a **Compensated Compound Poisson** process, therefore it is a martingale : $\mathbb{E}_0\left[\tilde{X}_t^\varepsilon\right] = \tilde{X}_0^\varepsilon = 0$ a.s.

Remarks: What happens when $\varepsilon \to 0$?

- $|lim_{\varepsilon\to 0}X_t^\varepsilon| = +\infty$ for **some** Lévy processes (IA Lévy) ;
- $|lim_{\varepsilon\to 0}\tilde{X}_t^\varepsilon| < +\infty$ for **all** Lévy processes (MG prop + Central Limit Theorem) : that's why we need $\tilde{X}_t^\varepsilon$;

So the **Ito-Lévy Decomposition**, tells us that we can only consider the "full integral" :

$$\tilde{X}_t^\varepsilon = \int_{\varepsilon<|x|<1, s\in[0,t]} x\left(J_X(dx \times ds) - \nu(dx)ds\right)$$

but we cannot "split the integral" in two parts :

$$X_t^\varepsilon = \int_{\varepsilon<|x|<1, s\in[0,t]} x J_X(dx \times ds) = \sum_{s\in[0,t], 0<|\Delta X_s|<1} \Delta X_s$$

and :

$$X_t^{\varepsilon,b} = \int_{\varepsilon<|x|<1, s\in[0,t]} x\nu(dx)ds$$

because even though we know that for all Lévy, $lim_{\varepsilon\to 0}\tilde{X}_t^\varepsilon \in \mathbb{R}$, we have for some Lévy (IA Lévy) that $lim_{\varepsilon\to 0}X_t^\varepsilon, lim_{\varepsilon\to 0}X_t^{\varepsilon,b} = +\infty$. In fact, it is a "$+\infty - (+\infty) = c \in \mathbb{R}$".

So we were able to write a **GENERAL LEVY PROCESS** (γ, A, ν) the following way :

$$X_t = \gamma t + B_t + X_t^l + \lim_{\varepsilon\to 0^+} \tilde{X}_t^\varepsilon$$

An interesting question is : can we derive a FA ("jump diffusion") expression starting from the general one above ? Yes :

$$X_t = \gamma t + B_t + \sum_{s\in[0,t], |\Delta X_s|\geq 1} \Delta X_s + lim_{\varepsilon\to 0} \int_{[0,t]\times\mathbb{R}^d, \varepsilon<|x|<1} x(J_X(dx \times ds) - \nu(dx)ds)$$

$$X_t = \left(\gamma - \int_{0<|x|<1} x\nu(dx)\right) t + B_t + \sum_{s\in[0,t]} \Delta X_s$$

by cutting in two the integral (we can since it's FA Lévy) and since :

$$\int_{0<|x|<1, s\in[0,t]} x J_X(dx \times ds) = \sum_{s\in[0,t], 0<|\Delta X_s|<1} \Delta X_s.$$

So we obtained the expression of a FA Lévy, with :

$$\mu = \left(\gamma - \int_{0<|x|<1} x\nu(dx)\right).$$

Therefore, we write a FA Lévy the following way :

$$X_t = \mu t + B_t + \sum_{s\in[0,t]} \Delta X_s$$

with $\mu = \gamma - \int_{0<|x|<1} x\nu(dx)$ for $\int_{|x|<1} \nu(dx) < +\infty$. And :

$$X_t = \mu t + B_t + \int_{[0,t]\times\mathbb{R}^d} x J_X(dx, ds)$$

for $\int_{|x|<1} |x|\nu(dx) < +\infty$ (**Finite Variation Lévy**, cf just below).

Definition: [*Total Variation / Finite Variation*]

- Total variation : for $f : [a,b] \to \mathbb{R}^d$,

$$TV = sup_{a=t_0<t_1<\ldots<t_n=b} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|.$$

- Finite variation : it is when $TV < +\infty$, i.e,

$$A = 0 \text{ (because CM has infinite TV) and } \int_{|x|<1} |x|\nu(dx) < +\infty.$$

Remark: FV Lévy is really an intermediary situation between FA Lévy and IA Lévy.

Theorem: [*Lévy-Khincin Formula*] Let $(X_t)_{t\geq 0}$ be a Lévy Process (γ, A, ν).

$$\mathbb{E}\left[e^{izX_t}\right] = e^{t\Psi(z)}$$

where :

$$\Psi(z) = i\gamma z - \frac{1}{2}z^T Az + \int_{\mathbb{R}^d} \left(e^{izx} - 1 - izx1_{|x|\leq 1}\right)\nu(dx).$$

Proof: cf my written notes.□

Summing up: we start from a general Lévy process (γ, A, ν) : it can be FA or IA. And depending on the situation we can write it using the FA formula (if it's FA!). So what we did is :

- From the writing of Jump Diffusion (FA) that we saw first we went to Lévy Ito Decomposition (for IA Lévy Processes) and ended up with the triplet (γ, A, ν) ;
- From General Lévy (γ, A, ν) we wrote Jump Diffusion (FA) with the formula with the summation, defining $\mu = \gamma - \int_{0<|x|<1} x\nu(dx) = \gamma - \lambda \int_{0<|x|<1} xf(x)dx$.

6 Subordinator

6.1 Idea

We will use subordinator for *variance* and *construction of Lévy processes.*
How will we construct Lévy Processes ? By **time change**. Assume we have a process $(X_t)_t$ (ex: BM) and another one $(S_t)_t$. We can build $(X_{S_t})_t$ but we need $(S_t)_t$ to be a *positive and non-decreasing* time process.

Theorem: Let $(X_t)_t$ be a Lévy process (γ, A, ν). We have 4 **equivalent** conditions :

- $\forall t > 0, X_t \geq 0$ as ;
- $\exists t > 0, X_t \geq 0$ as ;
- $(X_t)_t$ non-decreasing ;
- $(X_t)_t$ finite variation process with $\nu\left((-\infty, 0]\right) = 0$ **(non-negative jumps)** and $\mu = \left(\gamma - \int_{|x|<1} x\nu(dx)\right) > 0$.

Proof: cf my written notes.□

6.2 Constructing a subordinator

Theorem: Let $(X_t)_t$ be a Lévy process in $\mathbb{R}^d$. Let $f : \mathbb{R}^d \to [0, +\infty)$ be a positive function such that $f(x) = O(|x|^2)$ in a neighborhood of 0. Then a subordinator is the following :

$$S_t = \sum_{s\leq t, \Delta X_s \neq 0} f(\Delta X_s).$$

Remark: In the Lévy Khincin Representation $\Psi(z) = i\gamma z - \frac{1}{2}z^T Az + \int_{\mathbb{R}^d} \left(e^{izx} - 1 - izx1_{|x|\leq 1}\right)\nu(dx)$, we chose that the "small jumps" were the one with $|x| < 1$. Let then be $g : x \to 1_{|x|<1}$. We could do the same with $|x| < 1/2$; in fact we could choose any g of the form :

$$g : \mathbb{R}^d \to \mathbb{R} \text{ s.t } g(x) =_{x\to 0} 1 + o(|x|) \;\&\; g(x) =_{x\to +\infty} O\left(\frac{1}{|x|}\right).$$

See my written notes to have a glimpse of how we cut the integral and define a new $\tilde{\gamma}$ to obtain an analogous formula to the one of Lévy Khincin but with a separation between large and small jumps which is given by g. The message is that all the theory that we developed using a separation of 1 can be redeveloped if we change the separation : it's not a problem at all ! Only γ changes, but neither A nor ν. But **why do we need this remark for the previous theorem** ? Because f can change a jump of let's say 3/4 to a jump of let's say 1.5. But according to the present remark, it is not a problem.

Proof: See my written notes.□

Let's now "meet" some Lévy processes and implement on MATLAB. Keep in mind that for us : $X_t = log\left(\frac{S_t}{S_0}\right)$. (46/89 written notes)

Now what we still have to do is the following :

- To introduce other kinds of Lévy processes that are not Jump Diffusion processes ;
- To see in which case we are under the **risk-neutral measure** because to price derivatives we need to be able to have a risk-neutral measure. In B&S it's very simple : $\mu = r$, but here it will be different (not difficult though) ;
- To perform some MC simulations to price derivatives under the risk-neutral measure for Lévy models.

7 Other kinds of Lévy processes : no Jump Diffusion anymore

There are 3 main ways to construct a Lévy process starting from a Lévy Process :

- Linear Transformation ;
- Subordination ;
- Exponential Tilting.

7.1 Linear Transformation

Theorem: Let $(X_t)_t$ be a Lévy process in $\mathbb{R}^d$ with (γ, A, ν). Let M be a $n \times d$ matrix. Then : $(Y_t)_t = (MX_t)_t$ (with $\forall t \geq 0, Y_t = MX_t \in \mathbb{R}^n$) is a Lévy process $(\tilde{\gamma}, \tilde{A}, \tilde{\nu})$ where :

$$\tilde{A} = MAM^T; \tilde{\gamma} = M\gamma + \int_{\mathbb{R}^n} y \left(1_{|y| \leq 1} - 1_{S_1}\right) \tilde{\nu}(dy) \text{ where : } S_1 = \{Mx : |x| \leq 1\}; \forall B, \tilde{\nu}(B) = \nu\left(\{x : Mx \in B\}\right).$$

Remark: The extra term in the integral comes from the separation ebtween small and large jumps. Maybe a small jump for X is transformed into a big one for Y by M. It's a "correction integral" for the small jumps. See the simple example in the written notes if needed.

Property: If $(X_t)_t$ is Lévy (γ_1, A_1, ν_1) and $(Y_t)_t$ is Lévy (γ_2, A_2, ν_2), then : $(X_t + Y_t)_t$ is Lévy (γ, A, ν) where :

$$A = A_1 + A_2; \nu(B) = \nu_1(B) + \nu_2(B); \gamma = \gamma_1 + \gamma_2 - \int_{[-\sqrt{2}, -1] \cup [1, \sqrt{2}]} y\nu(dy).$$

Proof: $X = (X_1\ X_2)^T$ and $M = (1\ 1)$ and use the previous theorem. $\square$

7.2 Subordination

Theorem: Let $(S_t)_t$ a subordinator. The moment-generating function of the subordinator is : $\mathbb{E}\left[e^{uS_t}\right] = e^{tl(u)}$ where : $l(u) = bu + \int_0^{+\infty} (e^{ux} - 1)\rho(dx)$ with $b \geq 0, \rho((-\infty, 0]) = 0, (S_t)_t$ finite variations. Let $(X_t)_t$ be a Lévy process (γ, A, ν) in $\mathbb{R}^d$, let Ψ_X be its Lévy exponent. Then the process defined by :

$$\forall t \geq 0, Y_t = X_{S_t}$$

is a Lévy process (γ^Y, A^y, ν^y) with $\Psi_Y(u) = l(\Psi_X(u))$, and :

$$A^y = bA; \nu^y(B) = b\nu(B) + \int_0^{+\infty} p_s^X(B)\rho(ds); \gamma^Y = bY + \int_0^{+\infty} \int_{|x| \leq 1} xp_s^X(dx)\rho(ds)$$

where p_s^X is the PDF of X_s, which is sometimes **unknown**.

Example: we can speak about *Stable Processes*, that are used in finance. The definition is "horrible" (cf written notes). Fortunately we will always work in 1D where it is way simpler.

Definition: In 1D, $(X_t)_t$ is a (real-)"α-stable process", $0 < \alpha < 2$ if it is Lévy $(\gamma, 0, \nu)$ (**no Wiener**) and $\exists A, B > 0, \nu(x) = \frac{A}{x^{1+\alpha}} 1_{x>0} + \frac{B}{|x|^{1+\alpha}} 1_{x<0}$.

Definition: $(X_t)_t$ is an α-stable subordinator if :

- First :
$$(X_t)_t \text{ is a real Lévy process } (\gamma, 0, \rho);$$
- Then :
$$\rho(x) = \frac{A}{x^{1+\alpha}} 1_{x>0};$$
- Furthermore :
$$\alpha \in (0,1)\,;$$
- And finally :
$$b = \gamma - \int_{|x|<1} x\rho(dx) > 0.$$

And then : $l(u) = C_1 \int_0^{+\infty} \frac{e^{-ux}-1}{x^{1+\alpha}} dx, C_1(A, \alpha, \gamma)$.

Remark: In finance, we will take a Wiener and define $(X_{W_t})_t$. We will talk about it after the third way to construct a Lévy starting from a Lévy.

7.3 Exponential Tilting

Theorem: $(X_t)_t$ Lévy (γ, A, ν) such that :
$$\int_{|x|\leq 1} x^2 \nu(dx) < +\infty \text{ (small jumps)};$$
And :
$$\int_{|x|>1} \nu(dx) < +\infty \text{ (large jumps)}.$$
Here is the *exponential tilting* : $\tilde{\nu}(dx) = e^{\theta x}\nu(dx)$. If $\exists \theta \in \mathbb{R}^d$ such that :
$$\int_{|x|\leq 1} x^2 \tilde{\nu}(dx) < +\infty$$
And :
$$\int_{|x|>1} \tilde{\nu}(dx) < +\infty$$
Then $(\gamma, A, \tilde{\nu})$ defines a new Lévy process.

Definition: 1D tempering is the following :
$$\tilde{\nu}(dx) = \left(1_{x>0}e^{-\lambda^+ x} + 1_{x<0}e^{-\lambda^-|x|}\right)\nu(dx), \lambda^-, \lambda^+ > 0.$$

Remark: Tempering & Subordination give rise to our infinite activity Lévy process that we need in finance !

8 Brownian Subordination

Let $(S_t)_t$ be a subordinator with Laplace exponent $l(u)$, $(W_t)_t$ a Wiener process, **independent** of $(S_t)_t$. We define the following :
$$X_t = \mu S_t + \sigma W_{S_t}.$$
This Lévy process has the following characteristic exponent :
$$\Psi_X(u) = l(i\mu u - u^2\frac{\sigma^2}{2}).$$
Now we have to choose $(S_t)_t$! We will take an α-stable subordinator. We construct the class of *Normal Tempered Stable Processes* :

- "normal" : we start by a BM $\mu t + \sigma W_t$;
- "tempered" : we use tempered **to model tails** ;

- "stable" : $(S_t)_t$ is used as an α-stable subordinator.

All this together is called "*Brownian Subordination*". This is the class of processes that we are going to study.

Definition: [*Normal Tempered Stable Process*]

- **α-stable subordinator :** (cf above)

$$\rho(x) = \frac{A}{x^{1+\alpha}} 1_{x>0}, \alpha \in [0,1) \text{ "}S_t\text{";}$$

- Tempering :

$$\tilde{\rho}(x) = \frac{Ae^{-\lambda^+ x}}{x^{1+\alpha}} 1_{x>0}, A, \lambda^+ > 0 \text{ "}\tilde{S}_t\text{".}$$

Putting things together, we obtain the process defined by : $\mu\tilde{S}_t + \sigma W_{\tilde{S}_t}$, a "N.T.S Process", which is **again Lévy**. Here is the meaning of the different parameters :

- A : number impacting on the intensity of **all jumps** ;
- λ^+ : impacts on **large jumps** ;
- α : impacts on **small jumps**.

What is the Laplace exponent of $(\tilde{S}_t)_t$?

$$\tilde{l}(u) = A\Gamma(-\alpha)\left[(\lambda - u)^\alpha - 1\right]$$

where : $\Gamma(x) = \int_0^{+\infty} t^{z-1}e^{-t}dt$ (**Gamma Function**).

Now let's take $\mu\tilde{S}_t + W_{\tilde{S}_t}$. **What if we want to perform a MC simulation ?**

- To simulate $(\tilde{S}_t)_t$: we need the PDF of $\tilde{S}_t$ but it is not always known. It is known only for $\alpha = 0$ and $\alpha = \frac{1}{2}$;
- To simulate $(W_{\tilde{S}_t})_t$

The PDF is known is two case :

- $\alpha = 0$: fix $t > 0, \tilde{S}_t \sim GAMMA$ and its PDF is :

$$p_t^{\tilde{S}}(x) = \frac{\lambda_+^{At}}{\Gamma(At)} x^{At-1} e^{-\lambda_+ x}$$

- $\alpha = \frac{1}{2}$: fix $t > 0, \tilde{S}_t \sim INVERSE\ GAUSSIAN$ and its PDF is :

$$p_t^{\tilde{S}}(x) = \frac{At}{x^{3/2}} exp\left(2At\sqrt{\pi\lambda_+} - \lambda_+ x - \pi\frac{A^2t^2}{x}\right)$$

Definition: The following processes are the most famous examples of **Infinite Activity Lévy** processes :

- Variance gamma : $\alpha = 0$, i.e $\left(\mu\tilde{S}_t + \sigma W_{\tilde{S}_t}\right)_t$ with $\forall t \geq 0, \tilde{S}_t \sim Gamma$;
- Normal Inverse Gaussian : $\alpha = \frac{1}{2}$, i.e $\left(\mu\tilde{S}_t + \sigma W_{\tilde{S}_t}\right)_t$ with $\forall t \geq 0, \tilde{S}_t \sim IG$.

Remark: From above $(\gamma, 0, \nu)$ (no BM), we can add a BM : $X_t = \mu\tilde{S}_t + \sigma W_{\tilde{S}_t} + \tilde{\sigma}\tilde{W}_t$ where $dW_t d\tilde{W}_t = 0dt$, to obtain $(\gamma, \tilde{\sigma}, \nu)$. These are called *Extended VG* and *Extended NIG*. But the thing is that we don't really need a BM since the infinitesimal small jumps "mimic" a BM (cf pages 62-65 of my written notes !).
$\left(\mu\tilde{S}_t + \sigma W_{\tilde{S}_t}\right)_t$ is rich enough to model the evolution of the log-price.

Remark: cf page 59 of my written notes to see the algorithm for when you want to perform a MC simulation with this type of processes (VG and NIG).

RECAP OF WHERE WE ARE :

So, in our framework we have:

$$\left.\begin{array}{l} S_t = S_0 e^{X_t} \\ (X_t)_t \text{ LEVY} \end{array}\right\}$$ And we have seen, for $(X_t)_t$:

LEVY Proc. that we usually consider to model log. prices of a stock

- $\longrightarrow \mu t + \sigma W_t$ (LEVY) $\longrightarrow$ B&S Model.
- $\longrightarrow \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$: JUMP DIFFUSION
  - Merton Model.
  - Kou Model.
- $\longrightarrow \mu \tilde{S}_t + \sigma W_{\tilde{S}_t}$: INFINITE ACTIVITY
  - NIG Model.
  - VG Model.
- $\longrightarrow \mu \tilde{S}_t + \sigma W_{\tilde{S}_t} + \tilde{\sigma} \tilde{W}_t$: INFINITE ACTIVITY
  - Extended NIG.
  - Extended VG.

  $(dW_t \, d\tilde{W}_t = 0dt)$
- $\longrightarrow$ α-stable process (generalized, tempering).

Now let's take a 1D Lévy process which comes from an α-stable and tempering process :

$$\nu(x) = \frac{C_-}{|x|^{1+\alpha}} e^{-\lambda_-|x|} 1_{x<0} + \frac{C_+}{x^{1+\alpha}} e^{-\lambda_+ x} 1_{x>0}$$

with $C_-, C_+, \lambda_-, \lambda_+ > 0$. Remember that α impacts both positive and negative small jumps. We can generalize easily !

Definition: Let $\alpha_-, \alpha_+ \in [0, 1)$. Then a **Generalized Tempered α-stable process** is such that :

$$\nu(x) = \frac{C_-}{|x|^{1+\alpha_-}} e^{-\lambda_-|x|} 1_{x<0} + \frac{C_+}{x^{1+\alpha_+}} e^{-\lambda_+ x} 1_{x>0}$$

Now we can construct a Lévy (γ, σ, ν) process and use it with that ν Lévy Measure in order to work in a Lévy framework.

Theorem: *(It is used in pages 62-65 to show how infinitesimal small jumps mimic BM, read it!)* Let $(X_t)_t$ be a (γ, A, ν) Lévy. Then, $\forall t \geq 0$:

$$\mathbb{E}[X_t] = t\left(\gamma + \int_{|x|\geq 1} x\nu(dx)\right); Var(X_t) = t\left(A + \int_{\mathbb{R}} x^2 \nu(dx)\right)$$

Read pages 62-65 of my written notes to see how the infinitesimal small jumps can "mimic" the BM, and to see the error we make when pricing with no infinitesimal small jumps.

So now what we still need is the following :

- Itö Lemma for processes *with jumps* ;
- What does it mean to be **risk-neutral** ?
  - First, we need to define the "stock price" : $S_t = S_0 exp(X_t)$;
  - Next, what is the risk-neutral measure $\mathbb{Q}$? We know how to do it for GMB (cf page 66), but how to do it for Lévy processes ?

Theorem: [*Lévy Itö Formula*] Let $(X_t)_t$ be a (γ, σ^2, ν) Lévy process, and $f : \mathbb{R} \to \mathbb{R} \in C^2(\mathbb{R})$. Then the following Itö formula holds :

$$f(X_T) = f(X_0) + \int_0^T f''(X_s)ds + \int_0^T f'(X_s)dX_s + \sum_{s \leq T, \Delta X_s \neq 0} \left[f(X_s^- + \Delta X_s) - f(X_s^-) \right] - \Delta X_s f'(X_s^-)$$

Proof:

- Let $(X_t)_t$ be a **Jump Diffusion** process : $dX_t = \mu dt + \sigma dW_t + d\left(\sum_{i=1}^{N(t)} Y_i\right) = dX_t^C + dJ_t$ where : $dX_t^C = \mu dt + \sigma dW_t$ and $dJ_t = d\left(\sum_{i=1}^{N(t)} Y_i\right)$;
- Let $f \in C^2(\mathbb{R})$ and $\forall t \geq 0, Y_t = f(X_t)$;
- Let $T > 0$, $(T_i)_{i \in \{1,\dots,N(T)\}}$ the jumptimes ;
- Let $t \in]T_i, T_{i+1}[$, then :

$$\int_{T_i}^t dX_s = \int_{T_i}^t dX_s^C + \int_{T_i}^t dJ_s = \int_{T_i}^t (\mu ds + \sigma dW_s)$$

  since $\int_{T_i}^t dJ_s = 0$. So : $X_t - X_{T_i} = \mu(t - T_i) + \sigma(W_t - W_{T_i})$. Let's do it for $Y_t = f(X_t)$, which has **no jump** in $]T_i, T_{i+1}[$, hence the use of the classical Itö formula :

$$\int_{T_i}^t dY_s = \int_{T_i}^t df(X_s) = \int_{T_i}^t f'(X_s)dX_s + \frac{1}{2}\int_{T_i}^t f''(X_s)(dX_s)^2 = \int_{T_i}^t f'(X_s)\mu ds + \int_{T_i}^t f'(X_s)\sigma dW_s + \frac{\sigma^2}{2}\int_{T_i}^t f''(X_s)ds.$$

It means :

$$\forall t \in]T_i, T_{i+1}[, f(X_t) = f(X_{T_i}) + \frac{\sigma^2}{2}\int_{T_i}^t f''(X_s)ds + \mu\int_{T_i}^t f'(X_s)ds + \sigma\int_{T_i}^t f'(X_s)dW_s$$

Let then $t \to T_{i+1}$:

$$f(X_{T_{i+1}^-}) = f(X_{T_i}) + \frac{\sigma^2}{2}\int_{T_i}^{T_{i+1}} f''(X_s)ds + \int_{T_i}^{T_{i+1}} f'(X_s)dX_s^C$$

since $\int_{T_i}^t f'(X_s)dX_s^C = \mu\int_{T_i}^t f'(X_s)ds + \sigma\int_{T_i}^t f'(X_s)dW_s$.

By right-continuity :

$$f(X_{T_{i+1}}) = f(X_{T_{i+1}}^+) = f(X_{T_{i+1}}^- + \Delta X_{i+1}) + f(X_{T_{i+1}^-}) - f(X_{T_{i+1}^-}) = f(X_{T_{i+1}^-}) + \left(f(X_{T_{i+1}}^- + \Delta X_{i+1}) - f(X_{T_{i+1}^-})\right)$$

Which means (by replacing $f(X_{T_{i+1}}^-)$ in the previous expression by its expression above) :

$$f(X_{T_{i+1}}) = f(X_{T_i}) + \frac{\sigma^2}{2}\int_{T_i}^{T_{i+1}} f''(X_s)ds + \int_{T_i}^{T_{i+1}} f'(X_s)dX_s^C + \left(f(X_{T_{i+1}}^- + \Delta X_{i+1}) - f(X_{Ti+1}{}^-)\right)$$

Let $t > 0$, by summing everything :

$$f(X_T) = f(X_0) + \frac{\sigma^2}{2}\int_0^T f''(X_s)ds + \int_0^T f'(X_s)dX_s^C + \sum_{s \leq T, \Delta X_s \neq 0} \left[f(X_s) - f(X_s^-)\right]$$

And since $dX_s = dX_s^C + \Delta X_s$, we have $dX_s^C = dX_s - \Delta X_s$, so we can replace :

$$f(X_T) = f(X_0) + \frac{\sigma^2}{2}\int_0^T f''(X_s)ds + \int_0^T f'(X_s)dX_s + \sum_{s \le T, \Delta X_s \neq 0} [f(X_s^- + \Delta X_s) - f(X_s^-) - \Delta X_s f'(X_s^-)]$$

And **this formula can be used for any Lévy, since the sum converges if you have infinite activity Lévy**.

Remark: The key point here is to understand why the $\sum$ is finite. For **FA** Lévy it's ok, but the question is when we work with infinitesimal jumps : because they can be in infinite number. But we can use *Taylor formula* : cf page 69 of my written notes. So the formula holds true for **general Lévy processes**.

Property: Let $(X_t)_t$ be a (γ, σ^2, ν) Lévy process, $f \in C^2(\mathbb{R})$. Then :

$$Y_t = f(X_t) = M_t + V_t$$

where :

- $M_t = f(X_0) + \int_0^t f'(X_s)\sigma dW_s + \int_{[0,t)\times\mathbb{R}} \hat{J}_X(ds, dy)\left(f(X_{S^-} + y) - f(X_{S^-})\right)$ is a **martingale** ;
- $V_t = \int_0^t \frac{\sigma^2}{2} f''(X_s)ds + \int_0^t \gamma f'(X_s)ds + \int_{[0,t)\times\mathbb{R}} \left(f(X_{S^-} + y) - f(X_{S^-}) - yf'(X_{S^-})1_{|y|\le 1}\right) ds\nu(dy)$ is a **finite variation (FV) process**.

Example: Let $S_t = S_0 exp(X_t), \hat{S}_t = e^{-rt}S_t$. We have :

$$Y_t = f(X_t) = S_0 exp(X_t) = f'(X_t) = f''(X_t)$$

Therefore :

$$Y_t = Y_0 + \int_0^t Y_s dX_s + \frac{\sigma^2}{2}\int_0^t Y_s ds + \sum_{s \le t, \Delta X_s \neq 0} \left(f(X_{S^-} + \Delta X_s) - f(X_{S^-}) - \Delta X_s f'(X_{s^-})\right)$$

i.e :

$$Y_t = S_0 + \int_0^t Y_s dX_s + \frac{\sigma^2}{2}\int_0^t Y_s ds + \sum_{s \le t, \Delta X_s \neq 0} S_0 e^{X_{s^-}}\left(e^{\Delta X_s} - 1 - \Delta X_s\right)$$

i.e :

$$Y_t = S_0 + \int_0^t Y_s dX_s + \frac{\sigma^2}{2}\int_0^t Y_s ds + \int_{[0,t]\times\mathbb{R}} Y_{s^-}(e^z - 1 - z)J_X(ds, dz)$$

What is $(V_t)_t$ here ?

$$V_t = \int_0^t \frac{\sigma^2}{2} Y_s ds + \int_0^t \gamma Y_s ds + \int_{[0,t)\times\mathbb{R}} Y_{s^-}(e^y - 1 - y1_{|y|\le 1} ds\nu(dy) = \int_0^t Y_{s^-}\left(\gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} e^y - 1 - y1_{|y|\le 1}\nu(dy)\right) ds;$$

So the **condition in order to be under $\mathbb{Q}$ (risk-neutral measure)** is :

$$\gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left(e^y - 1 - y1_{|y|\le 1}\right)\nu(dy) = 0$$

i.e :

$$\textbf{For } (S_t)_t \textbf{ : } \gamma = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} \left(e^y - 1 - y1_{|y|\le 1}\right)\nu(dy)$$

and :

$$\textbf{For } (\hat{S}_t)_t \textbf{ : } \gamma = r - \frac{\sigma^2}{2} - \int_{\mathbb{R}} \left(e^y - 1 - y1_{|y|\le 1}\right)\nu(dy)$$

So γ **is fully defined by the risk-free IR r, σ and the Lévy measure**. Another interesting result is that since $\gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left(e^y - 1 - y1_{|y|\le 1}\right)\nu(dy) = 0$ under $\mathbb{Q}$, and we know that $\Psi_X(-i) = 0$, then :

$$\Psi_X(-i) = \gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left(e^y - 1 - y1_{|y|\le 1}\right)\nu(dy) = 0$$

So the most useful idea is the following : let $S_t = S_o e^{rt+X_t}$, then the discounted stock price is : $e^{-rt}S_t = S_0 e^{X_t}$. And we are under $\mathbb{Q}$ iff $\Psi_X(-i) = 0$. And usually, the best thing **(*)** is to start working with $\hat{X} : (0, \sigma^2, \nu)$ (no drift). Then we take $X : (-\Psi_{\hat{X}}(-i), \sigma^2, \nu)$, and by construction $\Psi_X(-i) = 0$. Hence we are under risk-neutral measure $\mathbb{Q}$. Usually, we calibrate the process without drift ($(0, \sigma^2, \nu) : \hat{X}$) and then we add the exact drift ($(-\Psi_{\hat{X}}(-i), \sigma^2, \nu) : X$).

(*) Take $\hat{X} : (0, \sigma^2, \nu)$. Then, by the formula for the characteristic exponent on top of page 9 we have :

$$\Psi_{\hat{X}}(-i) = -\frac{1}{2}\sigma^2(-i)^2 + \int_{\mathbb{R}} \left(e^{-i^2 y} - 1 + i^2 y 1_{|y|\leq 1}\right) \nu(dy) = \frac{1}{2}\sigma^2 + \int_{\mathbb{R}} \left(e^y - 1 - y 1_{|y|\leq 1}\right) \nu(dy)$$

which is exactly $-\gamma$ (cf the above expression for γ of S_t). Therefore, by defining a new process $X : (-\Psi_{\hat{X}}(-i), \sigma^2, \nu)$ we have exactly : $\gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left(e^y - 1 - y 1_{|y|\leq 1}\right) \nu(dy) = -\Psi_{\hat{X}}(-i) + \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left(e^y - 1 - y 1_{|y|\leq 1}\right) \nu(dy) = 0$ which is exactly the condition to be under the **risk-neutral measure** $\mathbb{Q}$.

Now that we know how to simulate Lévy processes and we know the condition to be under the risk-neutral measure, we can price derivatives (for example by MC) !