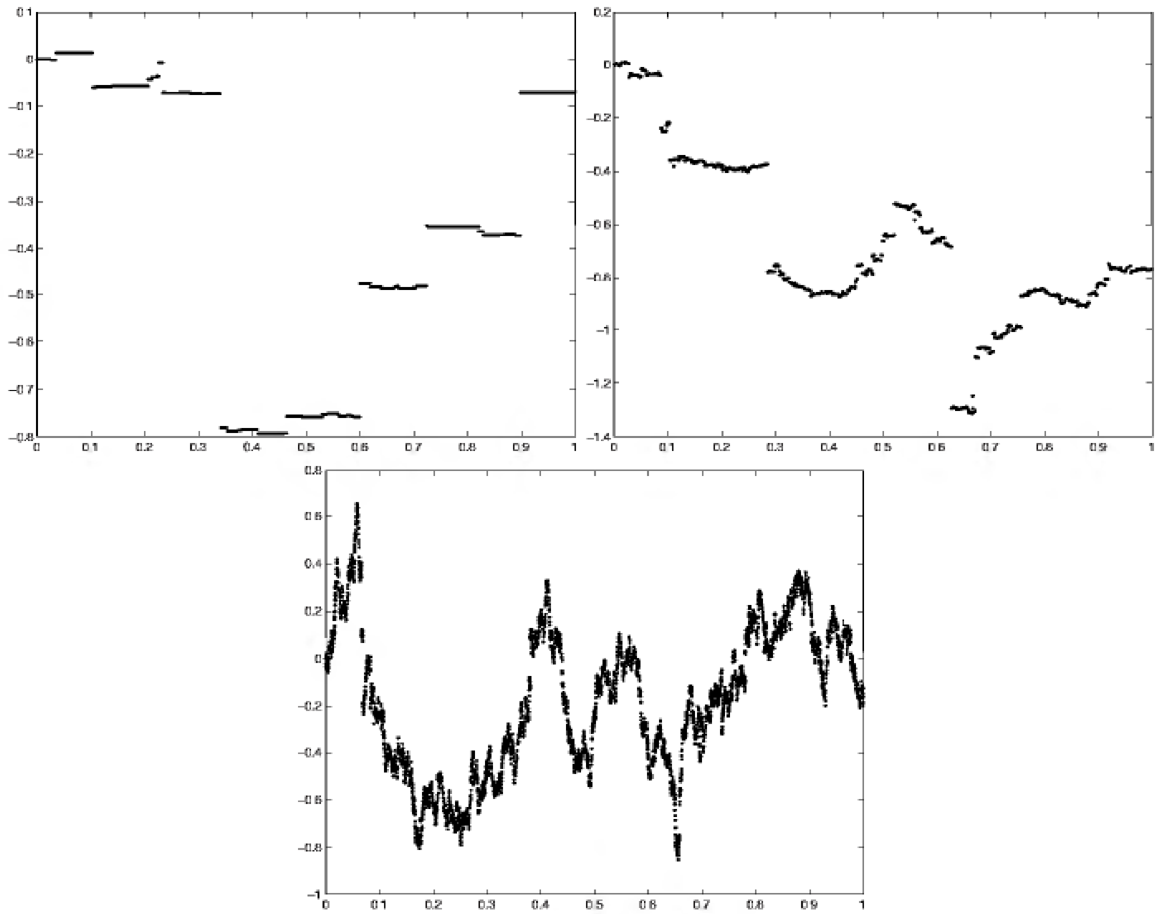

FIGURE 6.3: Simulated trajectories of α-stable processes with $\alpha = 0.5$ (left), $\alpha = 1$ (right) and $\alpha = 1.9$ (bottom).

have a significant effect. Comparing the three graphs with Figure 6.2 makes one think that stable processes may be approximated by a combination of compound Poisson process and Brownian motion. We will see shortly that this is indeed the case for stable processes and many other Lévy processes.

Example 6.4 Gamma process

The gamma subordinator was defined in Table 4.4. At a fixed time t this process has the well-studied gamma distribution with density

$$p_t(x) = \frac{\lambda^{ct}}{\Gamma(ct)} x^{ct-1} e^{-\lambda x}.$$

Gamma process has the following scaling property: if S_t is a gamma process with parameters c and λ then λS_t is a gamma process with parameters c and 1. Therefore it is sufficient to be able to simulate gamma random variables with density of the form

$$p(x) = \frac{x^{a-1}}{\Gamma(a)} e^{-x}.$$

There exist many algorithms for generating such random variables. A survey of available methods can be found in [113]. Below we reproduce two algorithms from this book. The first one should be used if $a \leq 1$ (which is most often the case in applications) and the second one if $a > 1$. Typical trajectories of gamma process with different values of c are shown in Figure 6.4. □

ALGORITHM 6.7 Johnk's generator of gamma variables, $a \leq 1$

REPEAT

  Generate i.i.d. uniform $[0, 1]$ *random variables* U, V

  Set $X = U^{1/a}$, $Y = V^{1/(1-a)}$

UNTIL $X + Y \leq 1$

Generate an exponential random variable E

RETURN $\frac{XE}{X+Y}$

ALGORITHM 6.8 Best's generator of gamma variables, $a \geq 1$

Set $b = a - 1$, $c = 3a - \frac{3}{4}$

REPEAT

  Generate i.i.d. uniform $[0, 1]$ *random variables* U, V

  Set $W = U(1-U)$, $Y = \sqrt{\frac{c}{W}}(U - \frac{1}{2})$, $X = b + Y$

  If $X < 0$ *go to REPEAT*

  Set $Z = 64W^3V^3$

UNTIL $\log(Z) \leq 2(b\log(\frac{X}{b}) - Y)$

RETURN X

Example 6.5 Inverse Gaussian process

The inverse Gaussian Lévy process gives another example of a subordinator for which both Lévy measure and probability density are known in explicit form (see Table 4.4). The inverse Gaussian density has the form

$$p(x) = \sqrt{\frac{\lambda}{2\pi x^3}} e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}} 1_{x>0}. \tag{6.5}$$

Below we reproduce the algorithm of Michael, Schucany and Haas for simulating inverse Gaussian variables (see [113]). □

ALGORITHM 6.9 Generating inverse Gaussian variables

Generate a normal random variable N

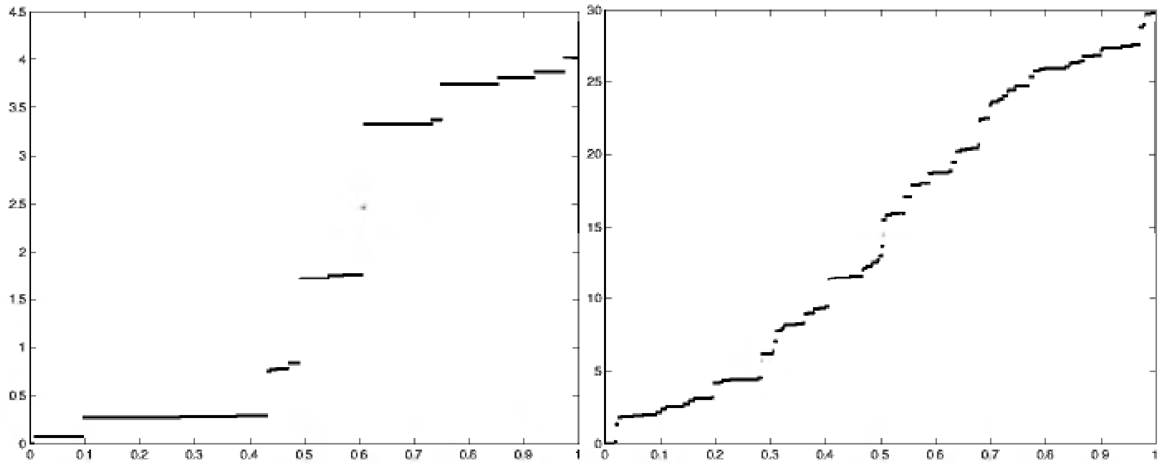

FIGURE 6.4: Two trajectories of gamma process with $c = 3$ (left), $c = 30$ (right) and $\lambda = 1$ for both graphs. These trajectories were simulated using series representation (6.16).

Set $Y = N^2$
Set $X_1 = \mu + \frac{\mu^2 Y}{2\lambda} - \frac{\mu}{2\lambda}\sqrt{4\mu\lambda Y + \mu^2 Y^2}$
Generate a uniform $[0, 1]$ *random variable* U
IF $U \leq \frac{\mu}{X_1+\mu}$ *RETURN* X_1 *ELSE RETURN* $\frac{\mu^2}{X_1}$

Example 6.6 Subordinated Brownian motion
A popular class of processes for stock price modelling is obtained by subordinating the standard Brownian motion or a Brownian motion with drift with an independent positive Lévy process. If the subordinator is denoted by V_t then the resulting process will be

$$X_t = W_{V_t} + bV_t, \tag{6.6}$$

where W is standard Brownian motion. When V_t is the gamma process or the inverse Gaussian process, we obtain, respectively, the variance gamma process and the normal inverse Gaussian process. Processes of type (6.6) possess a number of useful properties because they are conditionally Gaussian. In particular, if one knows how to simulate the increments of the subordinator, the increments of X_t can be simulated using the following algorithm. □

ALGORITHM 6.10 Generating the subordinated Brownian motion on a fixed time grid
Simulation of $(X(t_1), \dots, X(t_n))$ *for* n *fixed times* $t_1, \dots, t_n$ *where* $X(t) = B(S(t))$ *is Brownian motion with volatility* σ *and drift* b*, time changed with subordinator* (S_t).

- *Simulate increments of the subordinator:* $\Delta S_i = S_{t_i} - S_{t_{i-1}}$ *where* $S_0 = 0$.

- *Simulate n independent standard normal random variables $N_1, \dots, N_n$. Set $\Delta X_i = \sigma N_i \sqrt{\Delta X_i} + b \Delta S_i$.*

The discretized trajectory is given by $X(t_i) = \sum_{k=1}^{i} \Delta X_k$.

The following two algorithms show how the above method can be used to simulate variance gamma processes and normal inverse Gaussian processes on a fixed time grid:

ALGORITHM 6.11 Simulating a variance gamma process on a fixed time grid
Simulation of $(X(t_1), \dots, X(t_n))$ for fixed times $t_1, \dots, t_n$: a discretized trajectory of the variance gamma process with parameters σ, θ, κ.

- *Simulate, using Algorithms 6.7 and 6.8, n independent gamma variables $\Delta S_1, \dots, \Delta S_n$ with parameters $\frac{t_1}{\kappa}, \frac{t_2 - t_1}{\kappa}, \dots, \frac{t_n - t_{n-1}}{\kappa}$. Set $\Delta S_i = \kappa \Delta S_i$ for all i.*
- *Simulate n i.i.d. $N(0,1)$ random variables $N_1, \dots, N_n$. Set $\Delta X_i = \sigma N_i \sqrt{\Delta S_i} + \theta \Delta S_i$ for all i.*

The discretized trajectory is $X(t_i) = \sum_{k=1}^{i} \Delta X_i$.

ALGORITHM 6.12 Simulating normal inverse Gaussian process on a fixed time grid
Simulation of $(X(t_1), \dots, X(t_n))$ for fixed times $t_1, \dots, t_n$: a discretized trajectory of the normal inverse Gaussian process with parameters σ, θ, κ.

- *Simulate, using Algorithm 6.9 n independent inverse Gaussian variables $\Delta S_1, \dots, \Delta S_n$ with parameters $\lambda_i = \frac{(t_i - t_{i-1})^2}{\kappa}$ and $\mu_i = t_i - t_{i-1}$ where we take $t_0 = 0$.*
- *Simulate n i.i.d. $N(0,1)$ random variables $N_1, \dots, N_n$. Set $\Delta X_i = \sigma N_i \sqrt{\Delta S_i} + \theta \Delta S_i$ for all i.*

The discretized trajectory is $X(t_i) = \sum_{k=1}^{i} \Delta X_i$.

6.3 Approximation of an infinite activity Lévy process by a compound Poisson process

Let $(X_t)_{t \geq 0}$ be an infinite activity Lévy process with characteristic triplet $(0, \nu, \gamma)$. The goal of this and the following section is to find a process $(X_t^\varepsilon)_{t \geq 0}$