Exercise I : $T > 0$; $\sigma, \beta, \alpha : [0,T] \longrightarrow \mathbb{R}$: deterministic measurable and bounded ; B is a Real standard $\mathcal{F}^0$ BM ;

$$\begin{cases} dX_{(t)} = \alpha(t)dt + \beta(t)X_{(t)}\,dt + \sigma(t)\,dB_t & 0 \leq t \leq T ; \\ X_{(0)} = x_0 \in \mathbb{R} . \end{cases}$$

1) In our case : $\begin{cases} b(t,x) = \alpha(t) + \beta(t)x \\ \sigma(t,x) = \sigma(t) \\ x_0 \in \mathbb{R} \end{cases}$

* $b(t,x)$: • Measurable since it's the sum of 2 meas. fcts.

• sublinear growth : let $x \in \mathbb{R}$, $t \in [0,T]$.

$$|b(t,x)| = |\alpha(t) + \beta(t)x| \leq |\alpha(t)| + |\beta(t)|\,|x| \leq M(1+|x|)$$

where $M = \max\left(\sup_{t\in[0,T]} |\beta(t)| , \sup_{t\in[0,T]} |\alpha(t)| \right)$ (it exists since α, β are bdd).

$t \in [0,T]$ $t \in [0,T]$

- local Lipschitz continuity: it is indeed Lipschitz continuous since $\forall x,y \in \mathbb{R}$, $\forall t \in [0,T]$, $|b(t,x)-b(t,y)| = |\beta(t)(x-y)| \leq L|x-y|$ where $L = \sup_{t \in [0,T]} |\beta(t)|$.

* $\sigma(t,x)$: • Measurable by assumption.

- sublinear growth: $\forall x \in \mathbb{R}$, $\forall t \in [0,T]$, $|\sigma(t,x)| = |\sigma(t)| \leq M_\sigma$ where $M_\sigma = \sup_{t \in [0,T]} |\sigma(t)|$ (it exists since σ is bdd by assumption).

- Local Lipschitz Continuity: $\forall x,y \in \mathbb{R}$, $\forall t \in [0,T]$, $|\sigma(t,x) - \sigma(t,y)| = 0 \leq L|x-y|$ with $L = 1$ (for example).

$\longrightarrow$ We can apply the thm of existence and uniqueness under assumptions (A') and: there exists a strong solution X, pathwise unique. $\square$

2) $\begin{cases} dX_{(t)} = (\alpha(t) + \beta(t) X_{(t)}) dt + \sigma(t) dB_t & 0 \leq t \leq T \\ X(0) = x_0 \in \mathbb{R} \end{cases}$

- We know that $\exists!$ strong solution: if we find a solution, we are done.
- We want to use a "variation of constant" method.

a) Solve the homogeneous equation associated to SDE:

$\begin{cases} dY_{(t)} = \beta(t) Y_{(t)} dt \\ Y_{(0)} = x_0 \in \mathbb{R}. \end{cases}$ | The solution in this case is: $Y(t) = x_0 e^{\int_0^t \beta(s) ds} \quad \forall t \in [0,T].$

b) Inspired by a), look for particular solution:

Inspired by a) we look for solution of the form:

$$\begin{cases} X(t) = \tilde{X}_{(t)} e^{\int_0^t \beta(s) ds} \\ X(0) = \tilde{X}(0) = x_0 \in \mathbb{R} \end{cases}$$

for a certain $\tilde{X}$ we have to determine.

We notice that : $\int \tilde{X}_{(t)} = X(t) e^{-\int_0^t \beta(s) ds} = X(t) e^{-\Lambda(t)}$;

$\tilde{X}(0) = X(0) = x_0$.

we compute the stochastic differential of $\tilde{X}$:

$$d\tilde{X}_{(t)} = dX_{(t)}e^{-\Lambda(t)} - X_{(t)}\beta(t)e^{-\Lambda(t)}dt$$

$$= e^{-\Lambda(t)}\left(\alpha(t) + \cancel{\beta(t)X(t)dt} - \cancel{X_{(t)}\beta(t)dt}\right) + \sigma(t)e^{-\Lambda(t)}dB_t$$

$$= e^{-\Lambda(t)}\alpha(t)dt + \sigma(t)e^{-\Lambda(t)}dB_t$$

Magically, it doesn't depend on X.

So: $\tilde{X}(t) = x_0 + \int_0^t e^{-\Lambda(s)}\alpha(s)ds + \int_0^t \sigma(s)e^{-\Lambda(s)}dB_s$

$\Rightarrow \boxed{X(t) = e^{\Lambda(t)}\tilde{X}(t)}$. So:

$$\boxed{X(t) = e^{\Lambda(t)}\left(x_0 + \int_0^t e^{-\Lambda(s)}\alpha(s)ds + \int_0^t \sigma(s)e^{-\Lambda(s)}dB_s\right)}$$

$$d\left(e^{-\Lambda(t)}X(t)\right) = \cancel{e^{-\Lambda(t)}}\alpha(t)dt + \sigma(t)\cancel{e^{-\Lambda(t)}}dB_t$$

$$\parallel \text{ Itô } : f(t,x) := e^{-\Lambda(t)}x$$

$$\left[-\beta(t)\cancel{e^{-\Lambda(t)}}X(t) + (\alpha(t)+\beta(t)X_t)\cdot \cancel{e^{-\Lambda(t)}}\right]dt + \sigma(t)\cancel{e^{-\Lambda(t)}}dB_t$$

$$-\cancel{\beta(t)X(t)}dt + \cancel{\alpha(t)dt} + \cancel{\beta(t)X(t)}dt + \sigma(t)dB_t =$$

$$\cancel{\alpha(t)dt} + \sigma(t)dB_t \qquad \underline{OK}.$$

CHECK THAT $dX(t)$ COINCIDES WITH OUR SDE

Our unique strong solution (pathwise unique) is :

$$\boxed{X(t) = e^{-\Lambda(t)}\left(x_0 + \int_0^t e^{-\Lambda(s)}\alpha(s)\,ds + \int_0^t e^{-\Lambda(s)}\sigma(s)\,dB_s\right)}$$ $\square$

3) $\begin{cases} F_1(t) := e^{-\Lambda(t)}\alpha(t) \in M^1_{loc}[0,T] \text{ (since it is continuous)} \\ G_1(t) := e^{-\Lambda(t)}\sigma(t) \in M^2_{loc}[0,T] \text{ (since it is continuous)} \end{cases}$ So :

$dX_t = F_1(t)dt + G_1(t)dB_t$ and X is then an Ito Process.

- The theorem also tells us that :

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^2\right] < C(M,T)\left(1 + |x_0|^2\right) < +\infty .$$

so $\forall t \in [0,T]$, $\mathbb{E}\left[|X_t|^2\right] < +\infty$ so $\forall t \in [0,T]$, $\mathbb{E}\left[|X_t|\right] < +\infty$

so $\mathbb{E}[X_t]$ and $\mathrm{Var}(X_t)$ do exist.

- In general, $X(t)$ is not a martingale since the drift is not zero ($F_1(t) \not\equiv 0$). (If $\alpha \equiv \beta \equiv 0$ then the process is a mg).
- It's gaussian: indeed, $e^{-\Lambda(s)}\sigma(s)$ is deterministic so $\int_0^t e^{-\Lambda(s)}\sigma(s)\,dB_s$ is gaussian. $e^{\Lambda(t)}x_0$ and $e^{-\Lambda(t)}\int_0^t e^{-\Lambda(s)}\alpha(s)\,ds$ are deterministic $\Rightarrow$ they don't change the law of X.

As a conclusion: X is a gaussian process. $\square$

4) $\langle X \rangle_t$? Directly from the SDE, since X is Itô, we get the diffusion coefficient and:

$$\langle X \rangle_t = \int_0^t (\sigma(s))^2\,ds \quad \square$$

5) $\mu_t = \mathbb{E}[X_t]$?

$$\mu_t = e^{\Lambda(t)} x_0 + e^{\Lambda(t)} \mathbb{E}\left[\int_0^t e^{-\Lambda(s)} \alpha(s)\, ds\right]$$ since $G_1(t) \in M^2[0,T]$

$$\mu_t = e^{\Lambda(t)} x_0 + e^{\Lambda(t)} \int_0^t e^{-\Lambda(s)} \alpha(s)\, ds$$ since $\int_0^t e^{-\Lambda(s)} \alpha(s)\, ds$ & $e^{\Lambda(t)} x_0$ are deterministic.

$$\boxed{\forall t \in [0,T], \quad \mu_t = e^{\Lambda(t)}\left(x_0 + \int_0^t e^{-\Lambda(s)} \alpha(s)\, ds\right)} \quad \square$$

6) YES,

$$\begin{cases} \dfrac{d\mu_t}{dt} = \beta(t)\mu_t + \alpha(t) \\ \mu_0 = x_0 \end{cases}$$

WE CAN SHOW THIS BY 2 METHODS.

• 1st METHOD: compute $\frac{d\mu_t}{dt}$ in the previous expression in 5):

$$\frac{d\mu_t}{dt} = \beta(t) e^{\Lambda(t)} \left(x_0 + \int_0^t e^{-\Lambda(s)} (\alpha) ds \right) +$$

$$e^{\cancel{\Lambda(t)}} \times e^{-\cancel{\Lambda(t)}} \alpha(t)$$

$$\boxed{\frac{d\mu_t}{dt} = \beta(t)\mu_t + \alpha(t)} \quad \text{and} \quad \boxed{\mu_0 = x_0} \; .$$

• 2nd method: consider the SDE in integral form: $\forall t \in [0,T]$,

$$X(t) = x_0 + \int_0^t \alpha(s)ds + \int_0^t \beta(s)X(s)ds + \int_0^t \underbrace{\sigma(s)dB_s}_{\in M^2[0,T]} \quad \text{so:}$$

mg.

$$\mathbb{E}[X(t)] = \mu_t = x_0 + \mathbb{E}\left[\int_0^t \alpha(s)ds\right] + \mathbb{E}\left[\int_0^t \beta(s)X(s)ds\right] + 0$$

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$$\mu_t = x_0 + \int_0^t \alpha(s)ds + \int_0^t \beta(s)\mathbb{E}[X_s]ds$$

$$\mu_t = x_0 + \int_0^t \alpha(s)\,ds + \int_0^t \beta(s)\,\mathbb{E}[X(s)]\,ds$$

$$\mu_t = x_0 + \int_0^t \alpha(s)\,ds + \int_0^t \beta(s)\mu_s\,ds \quad \text{so:}$$

$$d\mu_t = \alpha(t)\,dt + \beta(t)\mu_t\,dt \quad \text{so:}$$

$$\boxed{\frac{d\mu_t}{dt} = \alpha(t) + \beta(t)\mu_t} \quad \text{and} \quad \boxed{\mu_0 = x_0}. \ \square$$

7)

$$\mathrm{Var}(X_t) = \mathrm{Var}\left(e^{\Lambda(t)}\left(x_0 + \int_0^t e^{-\Lambda(s)}\alpha(s)\,ds + \int_0^t e^{-\Lambda(s)}\sigma(s)\,dB_s\right)\right)$$

$$= \mathrm{Var}\left(\int_0^t e^{-\Lambda(s)}\sigma(s)\,dB_s\right) \times e^{2\Lambda(t)}$$

$$= e^{2\Lambda(t)} \times \mathbb{E}\left[\left(\int_0^t e^{-\Lambda(s)} \sigma(s) \, dB_s\right)^2\right]$$

$$= e^{2\Lambda(t)} \times \mathbb{E}\left[\int_0^t e^{-2\Lambda(s)} \sigma^2(s) \, ds\right]$$

$$\mathrm{Var}(X_t) = e^{2\Lambda(t)} \int_0^t e^{-2\Lambda(s)} \sigma^2(s) \, ds \qquad (\forall t \in [0, T]). \quad \square$$

8) $\mathrm{Cov}(X_s, X_t)$?

$$\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}\left(e^{\Lambda(t)}\int_0^s e^{-\Lambda(r)}\sigma(r)dB_r, e^{\Lambda(t)}\int_0^t e^{-\Lambda(r)}\sigma(r)dB_r\right)$$

Covariance doesn't see constants...

$$= e^{\Lambda(t)+\Lambda(s)}\,\mathrm{Cov}\left(\int_0^s e^{-\Lambda(r)}\sigma(r)dB_r, \int_0^t e^{-\Lambda(r)}\sigma(r)dB_r\right)$$

$$\boxed{\mathrm{Cov}(X_s, X_t) = e^{\Lambda(t)+\Lambda(s)} \times \int_0^{t\wedge s} e^{-2\Lambda(r)}\sigma^2(r)dr} \quad \square$$

9) Let's suppose that $\beta \in \mathcal{C}^0$ and $\sigma \in \mathcal{C}^1$. Let's rewrite the solution $X(t)$ without using stochastic integral:

- From TD. 10. 1.c), we know that if $f \in \mathcal{C}^1$,

$$\int_0^t f(s)dB(s) = f(t)B(t) - \int_0^t f'(s)B(s)ds .$$

→ In our case, we have that if $\beta \in \mathcal{C}^0$ and $\sigma \in \mathcal{C}^1$, then $e^{-\Lambda(t)}\sigma(t) \in \mathcal{C}^1$. Indeed, since $\Lambda(t) = \int_0^t \beta(s)ds$, we have

$$\frac{d}{dt}\left(e^{-\Lambda(t)}\sigma(t)\right) = e^{-\Lambda(t)}\frac{d}{dt}(\sigma(t)) - \sigma(t)e^{-\Lambda(t)}\frac{d}{dt}(\Lambda(t))$$

$$= \underbrace{e^{-\Lambda(t)}}_{\in\mathcal{C}^\infty}\left(\underbrace{\frac{d}{dt}[\sigma(t)] - \sigma(t)\beta(t)}_{\in\mathcal{C}^0}\right) \in \mathcal{C}^0.$$

$$\rightarrow e^{-\Lambda(t)}\sigma(t) \in \mathcal{C}^1$$

"stochastic IPP" → $$\int_0^t e^{-\Lambda(s)}\sigma(s)dB(s) = e^{-\Lambda(t)}\sigma(t)B(t) - \int_0^t \frac{d}{ds}\left[e^{-\Lambda(s)}\sigma(s)\right]B(s)ds$$

So we can write $X(t)$ without S.I ! (<u>Remark</u>: generalization of Ornstein-Uhlenbeck process, which can be obtained with b, σ cst and $a = 0$ (i.e $dX_t = bX_t\,dt + \sigma dB_t$)).