I SPREADING, INVASION, SPEED OF PROPAGATION

Basic example: "Generalized Logistic Reaction"

$$\begin{cases} u_t - \Delta u = u - u^2 & \Omega \times (0,+\infty) \\ u(x,0) = g(x) \geq 0 & \bar{\Omega} \\ \partial_\nu u = 0 & \partial\Omega \times (0,+\infty) \end{cases}$$

w/ $f(x,u) := \mu(x)u - u^2$ (it satisfies all the assumptions).

Here $\boxed{\mu \equiv 1}$

stationary solution

$$U(x) := 1$$

So the eigenvalue pb is:

$$\begin{cases} -\Delta u - u = \lambda u & \Omega \\ \partial_\nu u = 0 & \partial\Omega \end{cases}$$

1 is eigenfunction so $\lambda = -1 < 0$

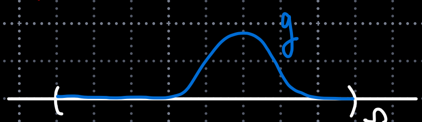
PERSISTENCE.

$\forall g,\ u_g(x,t) \xrightarrow[t\to+\infty]{} U(x) = 1$ ← By previous theory

(2) Persistence vs Extinction

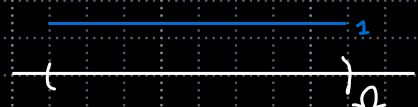
⚠ Remember that the behavior (PERSISTENCE vs EXT.) only depends on $\lambda_1^{NEU}(\mu, D)$ achieved by $\varphi_1^{(>0)}$ solving

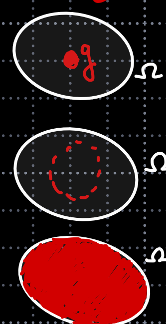
$t=0$



¿ TRANSIENT REGIME ?

$t \to +\infty$





To avoid considerations about the effects of the boundary, we will deal with the "Global Cauchy Problem" & homogeneous reaction $f(u)$.

Global Cauchy PB for RDEs:

$$\begin{cases} u_t - \Delta u = f(\cancel{x}, u) & x \in \mathbb{R}^N,\ t > 0 \\ u(x,0) = g(x) & x \in \mathbb{R}^N \end{cases}$$

f C^1 ⊕ f & $\partial_u f$ bdd on $\mathbb{R}^N \times [-A, A]$, $\forall A$
($x \in \mathbb{R}^N$, $u \in [-A,A]$)

g continuous & bdd.

THEN: $\boxed{\text{NO PROOF}}$

Max. time of existence.

1) $\exists$ classical solution $u_g := u_g(x,t)$, defined on $\mathbb{R}^N \times [0, T)$, $0 < T \leq +\infty$;
2) It is unique among solutions that are bdd on $\mathbb{R}^N \times [0, T']$, $\forall T' < T$;
3) If u_g is bounded on $\mathbb{R}^N \times [0, T)$, then $T = +\infty$.

$\longrightarrow$ We have WMP, Comp. Thm, & S.M.P for this equation also.

$$\begin{cases} u_t - \Delta u \geq f(x,u), & x \in \mathbb{R}^N,\ 0 < t < T \\ V_t - \Delta V \leq h(x,V), & x \in \mathbb{R}^N,\ 0 < t < T \\ V(x,0) \leq u(x,0), & x \in \mathbb{R}^N \end{cases}$$

With $f, h, u(\cdot,0), V(\cdot,0)$ as above & assume $h \leq f$. Then:

(WMP) $\Rightarrow$ $V(x,t) \leq u(x,t)$ in $\mathbb{R}^N \times (0,T)$;

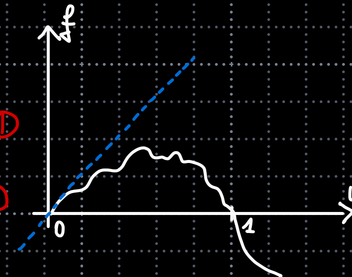
(SMP) $\Rightarrow$ if $V(x,0) \not\equiv u(x,0)$ then $V < u$ for $t > 0$.

SPREADING FOR GLOBAL CAUCHY PROBLEM:

(GCP) $\begin{cases} u_t - \Delta u = f(u) & x \in \mathbb{R}^N,\ t > 0\ ; \\ u(x,0) = u_0(x) & x \in \mathbb{R}^N. \end{cases}$

ASSUMPTIONS: (e.g. $f(u) = u - u^2$)

- $f \in \mathcal{C}^1$; • $f(0) = f(1) = 0$;
- $0 < f(u) < f'(0)u$ for $0 < u < 1$;
- $f(u) < 0$ for $u > 1$;
- $u_0 \in \mathcal{C}^0(\mathbb{R}^N)$, $u_0 \geq 0$, $u_0 \not\equiv 0$, u_0 bdd.



UNDER THE ABOVE ASSUMPTIONS $\exists!\, u$ BDD$^{\text{V}}$ POSITIVE SOLUTION, DEFINED IN $\mathbb{R}^N \times (0, +\infty)$ $(T = +\infty)$, AND COMPARISON HOLDS.

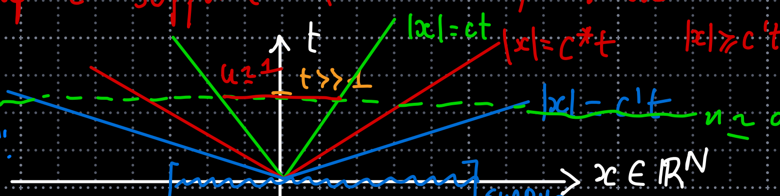
QUESTION: what is the speed of spreading ?? $\longrightarrow$ $c^* = 2\sqrt{f'(0)}$.

Thm [Aronson - Weinberger, 1978] Let $c^* = 2\sqrt{f'(0)}$.

1) $0 \leq c < c^*$. Then $\max\limits_{|x| \leq ct} |u(x,t) - 1| \xrightarrow[t \to +\infty]{} 0$.

2) If u_0 is compact support & $c' > c^*$, then: $\max\limits_{|x| \geq c't} |u(x,t)| \xrightarrow[t \to +\infty]{} 0$.

In particular: $\forall x$, $u(x,t) \xrightarrow[t \to +\infty]{} 1$ "Hair Trigger Effect".



PROOF: 1) VERY LONG : see the notebook.

2) By assumption, $f(u) < f'(0)u \quad \forall u > 0$. Take v such that:

$$\begin{cases} v_t - \Delta v = f'(0)v & \mathbb{R}^N \times (0, +\infty) \\ v(x,0) = \underset{=}{u_0}(x) & \mathbb{R}^N \end{cases}$$

→ if 0 instead of $f'(0)v$, solution is the convolution of $u_0(x)$ w/ the fundamental sol. of the Heat equation. Trick: multiply by e^{at} to include $f'(0)v$ into v_t.

cf the original pb: $f(u) < f'(0)v$.

Then, by $\boxed{\text{comparison}}$ we have $0 \leq u(x,t) \leq v(x,t)$, $\forall x, \forall t > 0$. So, I'm left to prove 2) for v, since v is explicit: $v(x,t) = e^{at}z(x,t)$, $a \in \mathbb{R}$ to be fixed. Then:

$$\begin{cases} ae^{at}z + e^{at}z_t - e^{at}\Delta z = e^{at}f'(0)z. \\ z(x,0) = u_0(x) \end{cases}$$

By taking $a = f'(0)$, we get:

$$\begin{cases} z_t - \Delta z = 0 & \mathbb{R}^N \times (0, +\infty). \\ z(x,0) = u_0(x) & \mathbb{R}^N \end{cases}$$

cf above.

We have, in this case, a fundamental solution:

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}. \quad \text{Then: } z(x,t) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} u_0(y) e^{-\frac{|x-y|^2}{4t}} dy \;\&$$

convolution w/ $u_0(x)$

finally:
$$V(x,t) = \frac{e^{f'(0)t}}{(4\pi t)^{N/2}} \int_{B_R(0)} u_0(y) \exp\left(-\frac{|x-y|^2}{4t}\right) dy \qquad (\operatorname{supp} u_0 \subset B_R(0)).$$

Fix $c' > c^* = 2\sqrt{f'(0)}$ & $x \notin B_{c't}(0)$, $y \in B_R(0)$.

Claim: $\forall \gamma$ s.t $c' > \gamma > c^*$,

$\exists t_0 > 0$ s.t $\begin{cases} t \geq t_0 \\ |x| \geq c't \\ |y| \leq R \end{cases} \Rightarrow |x-y| \geq \gamma t.$

Indeed: $|x-y| \geq |x| - |y| \geq c't - R \geq \gamma t$ as long as $t \geq t_0 = \dfrac{R}{c'-\gamma}$.

Then: $t \geq t_0$,
$$V(x,t) \leq \frac{\|u_0\|_\infty e^{f'(0)t}}{(4\pi t)^{N/2}} \int_{B_R(0)} e^{-\gamma^2 t^2/4t}\, dy \leq C \exp\left(\left[f'(0) - \frac{\gamma^2}{4}\right]t\right) \xrightarrow[t\to+\infty]{} 0$$

since $\gamma > 2\sqrt{f'(0)}$. $\square$

- Weakened assumptions: same kind of result but invasion happens @ $c^* \geq 2\sqrt{f'(0)}$...
- Other reaction (like bistable, with $\theta \in (0,1)$ unstable): no more "Hair Trigger Effect" (cf above)...

TRAVELLING FRONTS: (waves)

FRAMEWORK: $u(x,t)$, $x \in \mathbb{R}^N$, $t \cancel{>0}$ "eternal solutions": No Cauchy PB anymore.

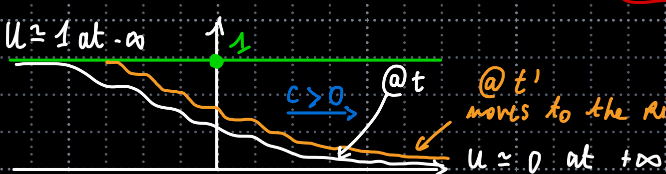
DEF: A travelling wave is a solution of the following type:

$k \in \mathbb{R}^N$, $|k| = 1$, $c \in \mathbb{R}$, s.t $u(x,t) = U(k \underset{\substack{\uparrow \\ \text{scalar product}}}{\cdot} x - ct)$ where $U := U(z)$, $z \in \mathbb{R}$.

EXAMPLE: Fisher-KPP Equation: travelling fronts (or waves):

(1) $u_t - \Delta u = u - u^2$, $u = u(x,t)$, $x \in \mathbb{R}^N$. For a travelling wave solution in $\mathbb{R}^N$, we look for solutions of the form $u(t,x) = U(k\cdot x - ct)$, where: U: the wave profile; k: unit vector indicating the direction of the wave propagation; c is the constant speed of wave. So the equation (1) becomes: (2)

$$\underbrace{-cU'(k\cdot x - ct)}_{u_t} - \underbrace{U''(k\cdot x - ct)}_{\Delta u}\underbrace{\cancel{k\cdot k}}_{k \text{ unit vector}} = \underbrace{U - U^2}_{f(u)}.$$



@t' moves to the right if $c>0$) Maintaining the shape.

$c>0 \to$ shift to the right.
$c<0 \to$ shift to the left.

Typically, for a biological or chemical scenario, we look for a solution satisfying $\lim_{z\to-\infty} U(z) = 1$, $\lim_{z\to+\infty} U(z) = 0$. INTERPRETATION: Far behind the wave FRONT, $(z\to-\infty)$, the population is fully established (density $=1$), & far ahead of the wave front $(z\to+\infty)$, the population is absent: it has not yet spread.

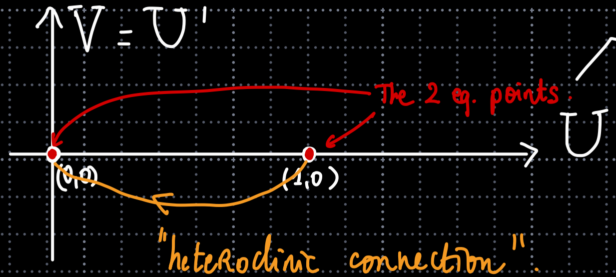
Regions of high population density invades the region of low population density. I.e: stable state 1 invades unstable state 0.

Biological interpretation: example of the spread of an advantageous gene in a population. Initially, only a small region may have individuals with the advantageous gene (high u). Over time, the gene spreads through the population at a constant speed c, w/ the density of individuals carrying the gene transitioning from high behind the wave front to low ahead of the wave front.

By a trick: (2) becomes

$$\begin{cases} U' = V \\ V' = U^2 - U - cV \end{cases} \quad \text{w/} \quad \begin{cases} U(-\infty) = 1, V(-\infty) = 0; \\ U(+\infty) = 0, V(+\infty) = 0. \end{cases}$$

Phase portrait :



$\begin{cases} U' = 0 \\ V' = 0 \end{cases} \iff \begin{cases} V = 0 \\ U = 1 \text{ or } 0 \end{cases}$

In Dipierro & Valdinoci's lecture notes we can see that :

- for $c^* < 2 \to$ no "biologically" interesting solutions because <u>not non-negative</u>.
- they exist only $\forall c^* \geq 2$: $\exists$ TRAVELLING WAVES $\forall c \geq 2\sqrt{f'(0)}$.

Presentation 10' RDE PART 2 :

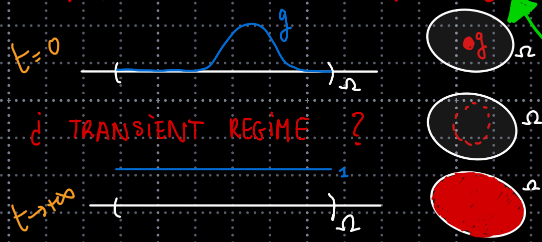
1 INTRO :

1 Generalized log. Reaction (or Fisher KPP)

$$\begin{cases} u_t - \Delta u = u - u^2 & \Omega \times (0,+\infty) \\ u(x,0) = g(x) \geq 0 & \bar{\Omega} \\ \partial_\nu u = 0 & \partial\Omega \times (0,+\infty) \end{cases}$$

2 To know more about extinction vs persistence :

3 ⚠ Remember that the behavior (PERSISTENCE vs EXT.) only depends on $\lambda_1^{NEU}(\mu, D)$ achieved by $\varphi_1^{(>0)}$ solving the EIG-PB.



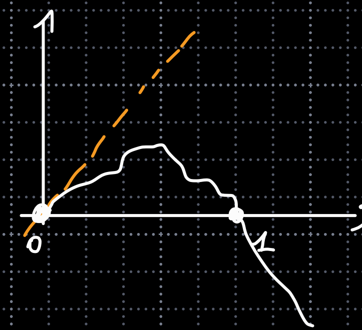
4 Here : we can show eigenvalue < 0 : PERSISTENCE.

$\forall g,\ u_g(x,t) \xrightarrow[t\to+\infty]{} U(x) = 1$ By previous theory

5 what is of interest is then what is happening between $t=0$ & $t \to +\infty$, i.e "transient regime". More specifically, I am going to focus on the SPEED OF PROPAGATION which is 1 of the topics of the 2nd part of the course.

2 Spreading for global Cauchy Problem:

1. GCP : $\begin{cases} u_t - \Delta u = f(u) & x \in \mathbb{R}^N, \ t > 0 \\ u(x,0) = u_0(x) & x \in \mathbb{R}^N \end{cases}$



2. Additional assumptions:
   - $f \in \mathcal{C}^1$
   - $f(0) = f(1) = 0$
   - $f(u) \in (0, f'(0)u) \quad \forall u \in (0,1)$
   - $f(u) < 0 \quad \forall u > 1$
   - $u_0 \ \mathcal{C}^0, \ \geq 0, \ \not\equiv 0, \ $ bdd

   best example: Fisher KPP logistic Reaction $f(u) = u - u^2$

3. we know $\exists !$ u_g positive bdd solution defined on $\mathbb{R}^N \times (0, +\infty)$ & Comp Princ. holds.
4. What is the speed of spreading ? $c^* = 2\sqrt{f'(0)}$.
5. Thm Aronson Weinberger 1978 Enoncé (+) Preuve du pt. 2) (+) schéma.

   → (+) "Hair Trigger Effect" : $\forall x, \ u(x,t) \xrightarrow[t \to +\infty]{} 1$

③ Travelling waves / Fronts : No $\nexists$ → "eternal solution" → No Cauchy pb anymore.

① Def. of travelling ⊕ trick to solve : $\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} U \\ U' \end{pmatrix}$.

② Focus on Fisher-KPP case → what equation we obtain ⊕ schéma avec direction selon signe de c ⊕ $\lim_{z \to -\infty} U(z)$ / $\lim_{z \to +\infty} U(z)$.

③ Biological Interpretation : advantageous gene spreading in a population

④ Phase portrait ("heteroclinic connection") ⊕ Dipierro & Valdinoci 's notes :

↓

$\exists$ Travelling waves for $c \geq 2\sqrt{f'(0)}$

⑤ : sols. become negative ...