$\boxed{\text{EXERCISE 3 :}}$ Let $(X_t)_t$ a stochastic process, solution of:

$$dX_t = \lambda X_t dt + \sigma dB_t \quad , \lambda < 0 \, , \sigma \neq 0 .$$

a) We are in the setting of EX I with $\begin{cases} \alpha(t) \equiv 0 \\ \beta(t) = \lambda \\ \sigma(t) = \sigma \end{cases}$, $\forall t$.

- So :

$$X(t) = e^{\lambda t}\left(x_0 + \int_0^t e^{-\lambda r} \times 0 \, dr + \int_0^t e^{-\lambda r} \sigma \, dB_r \right)$$

$$\boxed{X(t) = e^{\lambda t}\left(x_0 + \int_0^t e^{-\lambda r} \sigma \, dB_r\right)}$$

So $X(t) \sim \mathcal{N}\left(x_0 e^{\lambda t} , \sigma^2 \int_0^t e^{2\lambda (t-r)} dr \right)$

$$\boxed{\forall t, \; X(t) \sim \mathcal{N}\left(x_0 e^{\lambda t} , \sigma^2 \frac{1-e^{2\lambda t}}{2|\lambda|} \right)} .$$

- $\forall t$, X_t is gaussian with $\mathbb{E}[X_t] = m_t$, $Var(X_t) = \sigma_t^2$.

If the 2 limits $\lim_{t \to +\infty} m_t = m$ and $\lim_{t\to+\infty} \sigma_t^2 = \sigma^2$ exist and are finite , then $\exists X$: $X_t \xrightarrow[t\to+\infty]{\mathcal{L}} X \sim \mathcal{N}(m, \sigma^2)$ (see ex. 1.14, Baldi).

<u>In our case</u> :

$$\left.\begin{aligned} &\lim_{t\to+\infty} x_0 e^{\lambda t} = 0 \text{ since } \lambda < 0 . \\ &\lim_{t\to+\infty} \sigma^2 \frac{1-e^{2\lambda t}}{2|\lambda|} = \frac{\sigma^2}{2|\lambda|} \text{ since } \lambda < 0 . \end{aligned}\right\} X_t \xrightarrow{\mathcal{L}} X \sim \mathcal{N}\left(0, \frac{\sigma^2}{2|\lambda|}\right) = \mu .$$

b) If $X_0 \sim \mu$ we have that $\exists!$ solution to : $\begin{cases} dX_t = \lambda X_t dt + \sigma dB_t . \\ X(0) = X_0 \; , X_0 \perp\!\!\!\perp B_t \end{cases}$

The solution is given by:

$$X_t = e^{\lambda t}\left(X_0 + \int_0^t \sigma e^{-\lambda s} dB_s \right) = e^{\lambda t} X_0 + \sigma \int_0^t e^{\lambda(t-s)} dB_s .$$

First, we notice that, since $x_0 \perp\!\!\!\perp B_t$ and $e^{\lambda(t-s)}$, $e^{\lambda t}$ are deterministic, we have : $e^{\lambda t} X_0 \perp\!\!\!\perp \sigma \int_0^t e^{\lambda(t-s)} dB_s$.

We compute the laws of $e^{\lambda t} X_0$ and $\sigma \int_0^t e^{\lambda(t-s)} dB_s$ separately. They are both gaussian. Then we can sum the result:

- $e^{\lambda t} X_0 \sim \mathcal{N}(0, e^{2\lambda t} \cdot \frac{\sigma^2}{2|\lambda|})$

- $\int_0^t \underbrace{\sigma e^{\lambda(t-s)}}_{\in M^2[0,t]\ \forall t.} dB_s \sim \mathcal{N}(0, \sigma^2 \frac{1-e^{2\lambda t}}{2|\lambda|})$

$$\rightarrow \boxed{X_t \sim \mathcal{N}\left(0, \frac{e^{2\lambda t}\sigma^2}{2|\lambda|} + \sigma^2 \frac{1-e^{2\lambda t}}{2|\lambda|} = \frac{\sigma^2}{2|\lambda|}\right) = \mu, \forall t.}$$

(The Σ is gaussian with Var equal to the Σ of the variances due to the <u>independence</u>).

□

⇒ $X_0 \in L^2(\mathcal{F}_0)$ and $Y_t = X_t^2 + \frac{\sigma^2}{2\lambda}$, $\forall t$.

c) $dY_t = d(X_t^2) + \underbrace{d\left(\frac{\sigma^2}{2\lambda}\right)}_{=0} = d(X_t^2)$.

$d(X_t^2) = d(f(X_t))$ with $f: x \mapsto x^2$ $\in \mathcal{C}^{1,2}$. By applying Itô:

$$d(X_t^2) = \left[0 + \lambda X_t \times 2X_t + \cancel{2} \times \frac{\sigma^2}{\cancel{2}}\right] dt + \sigma \cdot 2X_t \, dB_t$$

$$d(X_t^2) = \left[2\lambda X_t^2 + \cancel{2}\frac{\sigma^2}{\cancel{2}}\right] dt + 2\sigma X_t \, dB_t$$

So: $\boxed{dY_t = d(X_t^2) = 2\left[\lambda X_t^2 + \frac{\sigma^2}{2}\right] dt + 2\sigma X_t \, dB_t}$. □

d) $t \mapsto e^{-2\lambda t}$ is deterministic so :

$$d(Z_t) = d(e^{-2\lambda t} Y_t) = e^{-2\lambda t} d(Y_t) + Y_t \, d(e^{-2\lambda t})$$

$$= e^{-2\lambda t}\left[2\left(\lambda X_t^2 + \frac{\sigma^2}{2}\right)dt + 2\sigma X_t dB_t \right] + Y_t \cdot (-2)\lambda \cdot e^{-2\lambda t} dt$$

$$= 2e^{-2\lambda t}\left(\lambda X_t^2 + \frac{\sigma^2}{2} - \lambda Y_t\right) dt + 2e^{-2\lambda t}\sigma X_t dB_t$$

$$= 2e^{-2\lambda t}\left(\cancel{\lambda X_t^2} + \cancel{\frac{\sigma^2}{2}} - \cancel{\lambda X_t^2} - \cancel{\frac{\sigma^2}{2}}\right) dt + 2e^{-2\lambda t}\sigma X_t dB_t$$

$$= 2e^{-2\lambda t}\sigma X_t dB_t .$$

We are in presence of an Itô process with $F_t \equiv 0$: it's a local martingale. Furthermore, $G_t = 2e^{-2\lambda t}\sigma X_t$ is a $\Pi^2[0,t]$ process : $Z_t = e^{-2\lambda t} Y_t$ is then a martingale. $\square$

e) $\mathbb{E}[Y_t] = ?$

$$\left.\begin{array}{l} \mathbb{E}[Z_t] = \mathbb{E}[e^{-2\lambda t} Y_t] = e^{-2\lambda t} \cdot \mathbb{E}[Y_t] \\ \quad \| \text{mg} \\ \mathbb{E}[Z_0] = \mathbb{E}[Y_0] = \mathbb{E}[X_0^2] + \frac{\sigma^2}{2\lambda} \end{array}\right\} \Rightarrow \begin{array}{l} \forall t, \\ \mathbb{E}[Y_t] = e^{2\lambda t}\left(\mathbb{E}[X_0^2] + \frac{\sigma^2}{2\lambda}\right) \end{array}$$

i.e : $\forall t, \quad \mathbb{E}[Y_t] = e^{2\lambda t}\Big(\underbrace{Var(X_0)}_{\substack{\text{bc. } \mathbb{E}[X_0]=0 . \\ \text{(cf. b)).}}} + \frac{\sigma^2}{2\lambda} \Big) = e^{2\lambda t} \underbrace{\Big(\frac{\sigma^2}{2|\lambda|} + \frac{\sigma^2}{2\lambda} \Big)}_{=0 \text{ since } \lambda = -|\lambda|}$

$= e^{2\lambda t} . 0 = 0 .$

So : $\boxed{\forall t, \quad \mathbb{E}[Y_t] = e^{2\lambda t}\, \mathbb{E}[Y_0] = e^{2\lambda t}\Big(\mathbb{E}[X_0^2] + \frac{\sigma^2}{2\lambda} \Big) = 0 \xrightarrow[t \to +\infty]{} 0 \quad \text{(since } \lambda < 0)}$. $\square$