Stochastic Dynamical Models A.A. 2024-2025

Exercise Session 3

October 15, 2024

Keywords: Convergence to equilibrium; hitting probability; mean hitting time.

Exercise 1. Let $(X_n)_{n\geq 0}$ be an irreducible, fast recurrent and aperiodic Markov chain. Prove that X_n converges in distribution as $n \to +\infty$. cf my written notes.

Exercise 2. Consider a Markov Chain $(X_n)_{n\geq 0}$ with state space $I = \mathbb{N}$ and transition matrix

$$P = \begin{pmatrix} 1-p & p & 0 & 0 & 0 & \cdots \\ 1-p & 0 & p & 0 & 0 & \cdots \\ 1-p & 0 & 0 & p & 0 & \cdots \\ 1-p & 0 & 0 & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$P^n \xrightarrow[n\to+\infty]{} \begin{bmatrix} \pi_0 & \pi_1 & \pi_2 & \cdots \\ \pi_0 & \pi_1 & \pi_2 & \cdots \\ \pi_0 & \pi_1 & \pi_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Easy to show $\exists!$ invariant density by writing $\pi P = \pi$.

where $0 < p < 1$.

$\forall i \in I,\ p_{i,i+1} > 0,\ p_{i,0} > 0.$

1. Show that the Markov chain is irreducible, aperiodic, and fast recurrent.

   $\begin{cases} 0 \to 1 \to 0 \\ 0 \to 0 \end{cases}$ with >0 proba.

2. Compute $\lim_{n\to+\infty} P^n$. corollary pages 37/38 written notes (*)
3. Suppose that chain starts from state 0, compute the mean time of returning back to state 0. Let $T_0 = \min\{n \geq 1 : X_n = 0\}$. $\mathbb{E}_0[T_0]$? By the thm page 37: $\lim_{n\to+\infty} p_{00}^{(n)} = \frac{1}{\mathbb{E}_0[T_0]}$ i.e: $\mathbb{E}_0[T_0] = \frac{1}{1-p}$.

Exercise 3. Consider a random walk on the vertices of a graph $G = (V, E)$, where V represents the set of vertices and E the set of edges. The valence (or degree) v_i of a vertex $i \in V$ is the number of edges at i. We assume that every vertex has a finite valence and the random walks on G picks edges with equal probability. It is easy to see, if G is a connected graph, i.e. it is possible to get from every vertex in the graph to every other vertex through a series of edges, then the Markov chain is irreducible.

1. Let G be a connected graph and define $\sigma := \sum_{i\in V} v_i < \infty$. Show that $\pi_i := v_i/\sigma$ is reversible (and hence invariant). $\pi_i p_{ij} = \pi_i \times \frac{1}{v_i} = \frac{1}{\sigma} = \pi_j p_{ji} = v_j/\sigma \times \frac{1}{v_j}$.
2. Specifically, let us imagine the random walk of a knight on a chess board: The vertices of the random graph are the squares and the edges are the moves that the knight can take. Find the valences of the 64 vertices of the graph, and if the knight starts in a corner, how long on average will it take to return? Thm @ page 37: $\mathbb{E}_i[T_i] = \frac{1}{\pi_i} = \frac{\sigma}{v_i}$.

Exercise 4. A circus acrobat is doing his balancing act on the rope. There are $2N+1$ positions on the rope which are indicated by integers $\{0, 1, 2, \cdots, 2N-1, 2N\}$. The acrobat successfully completes his performance when he manages to reach any of two ends 0 or $2N$ of the rope. Today, however, our acrobat has lifted his elbow a little too much before the show, and so at this moment he is dangerously oscillating on the rope. More precisely,

- With each step he takes there is a probability equal to $1/5$ he falls below and therefore fails his performance.
- If instead he manages to stay in balance, it is equally probable that he takes a step forward or backward.

(*) Corollary: Let $(X_n)_{n\geq 0}$ irreducible MC with fast recurrent states such that: $\forall i \in I,\ \pi_i = \lim_{n\to+\infty} p_{ii}^{(n)} > 0$. Then $(\pi_i)_{i\in I}$ is the unique inv. density and $\forall j \in I,\ \lim_{n\to+\infty} p_{ji}^{(n)} = \pi_i \quad (\forall i \in I)$.

Suppose that at the initial time the acrobat is exactly in the middle of the string (i.e. in position N), then

1. Indicate with -1 the state "the acrobat fell down" and model the state of the acrobat with a suitable discrete Markov chain.

2. What is the probability that the acrobat can successfully finish his performance?

3. How long does the number of acrobatics lasts on average (independently of its success or not)?

1)

$$
P = \begin{array}{c}
\begin{array}{ccccccc} -1 & 0 & 1 & 2 & 3 & \cdots & 2N \end{array} \\
\left[\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & \cdots & \cdots \\
0 & 1 & 0 & 0 & 0 & \cdots & \cdots \\
1/5 & 2/5 & 0 & 2/5 & 0 & \cdots & \cdots \\
1/5 & 0 & 2/5 & 0 & 2/5 & \cdots & \cdots \\
 & & & \ddots & \ddots & \ddots & 2/5 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & 1
\end{array}\right]
\end{array}
\begin{array}{c} -1 \\ 0 \\ 1 \\ 2 \\ \vdots \\ 2N \end{array}
$$

2) Use "hitting probability" part of the course: Thm page 55.

↳ in fact, we need a + general result (cf written exercise sheet).

3) Idem: we need a + general result (cf written ex. sheet).