Mathematical Engineering - A.Y. 2022-23

Real and Functional Analysis - Written exam - February 15, 2023

Exercise 1. Consider the functions $f, g : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}, \qquad g(x) = \begin{cases} 0 & 0 < x \leq 1 \\ 1, & x = 0 \end{cases}.$$

(1) Is f absolutely continuous in $[0,1]$? Is f of bounded variation in $[0,1]$? Justify the answers.
(2) Is g absolutely continuous in $[0,1]$? Is g of bounded variation in $[0,1]$? Justify the answers.
(3) Consider the function $h : [0,1] \to \mathbb{R}$, $h(x) := f(x) + g(x)$. Determine whether h is absolutely continuous in $[0,1]$. Justify the answer.

Solution.
(1) We show that f is absolutely continuous by showing that f satisfies
- f is differentiable almost everywhere with $f' \in L^1([0,1])$,
- $f(x) = f(0) + \int_0^x f'(t)\,\mathrm{d}t$, for every $x \in [0,1]$.

Since f is written as product and composition of elementary functions, it is differentiable in $(0,1]$, hence f is differentiable a.e. in $[0,1]$ ($\lambda(\{0\}) = 0$). For $x \in (0,1]$, we have

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

In particular, $f' \in L^1([0,1])$, indeed f' is bounded on a bounded interval. Hence, it remains to verify the calculus formula

$$f(x) - f(0) = \int_0^x f'(t)\ \mathrm{d}t, \qquad \forall x \in [0,1].$$

Let then $0 < c < 1$ be arbitrarily fixed. Since $f \in C^1([c,1])$, by the standard fundamental theorem of calculus in $[c,1]$ we have

$$f(x) - f(c) = \int_c^x f'(t)\ \mathrm{d}t, \qquad \forall x \in [c,1]$$

We now aim to pass to the limit as $c \to 0^+$ on both sides of the equality above. To this end we first observe that, since f is continuous at 0, we have

$$f(c) \to f(0) \text{ as } c \to 0^+.$$

Moreover, since $f' \in L^1([0,1])$ then the map

$$c \mapsto \int_c^x f'(t)\ \mathrm{d}t$$

is absolutely continuous (hence continuous) in $[0,1]$, so that

$$\lim_{c\to 0^+} \int_c^x f'(t)\ \mathrm{d}t = \int_0^x f'(t)\ \mathrm{d}t.$$

Gathering these facts, we obtain

$$f(x) - f(0) = f(x) - \lim_{c\to 0^+} f(c) = \lim_{c\to 0^+} \int_c^x f'(t)\ \mathrm{d}t = \int_0^x f'(t)\ \mathrm{d}t,$$

holding for all $x \in [0,1]$. This proves that $f \in \mathrm{AC}([0,1])$. Being $\mathrm{AC}([0,1]) \subset \mathrm{BV}([0,1])$, f is also of bounded variation.

(2) Since g is not continuous, it cannot be absolutely continuous. Moreover g is monotone, therefore it has bounded variation. In particular we have $V_0^1(g) = |g(1) - g(0)| = 1$.

(3) We have

$$h(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & 0 < x \leq 1 \\ 1, & x = 0. \end{cases}$$

Hence the function h is not continuous at $x = 0$, thus it cannot be absolutely continuous on $[0,1]$.

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(2) Is g absolutely continuous in $[0,1]$? Is g of bounded variation in $[0,1]$? Justify the answers.
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(1) $f : x \longmapsto \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$

strategy for the f

- We would like to show that $f(x) = f(0) + \int_0^x f'(t)\,dt$ with $f' \in L^1((0,1])$. Then, by the regularity theorem of the integral function, we would have $f \in AC[0,1]$.

- f is differentiable on $(0,1]$ as the product of two $(0,1]$-differentiable fcts:

$$\forall x \in (0,1],\ f'(x) = 2x \sin\left(\tfrac{1}{x}\right) + x^2 \times \frac{-1}{x^2} \times \cos\left(\tfrac{1}{x}\right)$$
$$f'(x) = 2x \sin\left(\tfrac{1}{x}\right) - \cos(1/x).$$

- f' is a bounded function on a bounded interval $[0,1]$, so $f' \in L^1([0,1])$. So it remains to verify the FFC: $f(x) - f(0) = \int_0^x f'(t)\,dt$, $\forall x \in [0,1]$.

- Since $\forall c \in (0,1]$, $f \in \mathcal{C}^1((c,1])$ so:

$$f(x) - f(c) = \int_c^x f'(t)\,dt, \quad \forall x \in [c,1]$$ (by the classical fundamental theorem of calculus).

We have how to extend this to $[0,1]$.

- Since f is continuous at 0, $\lim_{c\to 0^+} f(c) = f(0) = 0$.

- We can also notice that $f' \in L^1([0,1])$ since it is bounded on a bounded interval. So that

$$c \longmapsto \int_c^x f'(t)\,dt \quad \text{is absolutely continuous}$$

on $[0,1]$. Hence continuous. So: $\lim_{c\to 0^+} \int_c^x f'(t)\,dt = \int_0^x f'(t)\,dt$.

Gathering everything, we have that: $f(x) - \lim_{c\to 0^+} f(c) = \lim_{c\to 0^+} \int_c^x f'(t)\,dt$, i.e:

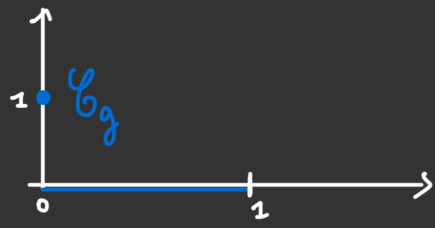
$$f(x) - \underbrace{\cancel{f(0)}}_{=0} = \int_0^x f'(t)\,dt, \quad \forall x \in [0,1].$$

Since $f' \in L^1([0,1])$, we have:

$$\boxed{f \in AC([0,1])}. \square$$

$AC([0,1]) \subset BV([0,1])$ so $\boxed{f \in BV([0,1])}. \square$

(2) $g(x) = \begin{cases} 0 & 0<x\leq 1 \\ 1 & x=0 \end{cases}$:



- g is not continuous on $[0,1]$, since it is not continuous at $x=0$. Hence $\boxed{g \text{ is not A.C on } [0,1].}$ $\square$
- g constant $\neq 0$ on $(0,1]$ so it's easy to compute its total variation on $[0;1]$:

$$V_0^1(g) = 1 < +\infty .$$

So $\boxed{g \in BV([0,1])}$. $\square$

(3) $h(x) := f(x) + g(x)$

$$:= \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & 0<x\leq 1 \\ 1 & x=0 \end{cases}$$

$\forall x \in (0,1), \left| x^2 \sin\left(\frac{1}{x}\right) \right| \leq x^2 \xrightarrow[x\to 0]{} 0$ so $\lim\limits_{x\to 0^+} h(x) = h(0) = 1$

$\neq \lim\limits_{x\to 0^+} x^2 \sin(1/x) = 0.$

h is thus not continuous at 0. Hence $\boxed{h \notin AC([0,1])}$. $\square$

Exercise 2. Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset L^1(0,1)$ defined by

$$f_n(x) := ne^{-nx}, \quad x \in (0,1).$$

(1) Discuss the pointwise a.e. convergence of the sequence $\{f_n\}_{n\in\mathbb{N}}$.
(2) Compute

$$\lim_{n\to+\infty} \int_0^1 f_n(x)\,\mathrm{d}x.$$

(3) Does there exist a function $g \in L^1(0,1)$ such that $|f_n(x)| \le g(x)$ for a.e. $x \in (0,1)$ and every $n \in \mathbb{N}$? In the affirmative case, determine such a function g. If not, justify your answer.
(4) Discuss the weak convergence of $\{f_n\}_{n\in\mathbb{N}}$ in $L^1(0,1)$.

Solution. (1) For all $x \in (0,1)$, we have

$$\lim_{n\to+\infty} f_n(x) = 0.$$

Therefore, the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges pointwisely a.e. to $f \equiv 0$.

(2) For every $n \in \mathbb{N}$, we have

$$\int_0^1 f_n(x)\,\mathrm{d}x = 1 - e^{-n}.$$

Letting $n \to +\infty$ we conclude

$$\lim_{n\to+\infty} \int_0^1 f_n(x)\,\mathrm{d}x = 1.$$

(3) Notice that if such a function g exists, by dominated convergence theorem, we would have

$$\lim_{n\to+\infty} \int_0^1 f_n(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x = 0.$$

This is in contradiction with the previous point. Hence, such a function g does not exist.

(4) By (1) the candidate limit is $f \equiv 0$. By the Riesz theorem, the dual space $(L^1(0,1))^*$ can be identified with the space $L^\infty(0,1)$; as a consequence, $f_n \rightharpoonup 0$ in $L^1(0,1)$ *if and only if*

$$\int_0^1 f_n\varphi\ \mathrm{d}x \to 0, \quad \text{as } n \to +\infty \qquad \forall\varphi\ \in L^\infty(0,1).$$

On the other hand, choosing $\varphi \equiv 1$, we get

$$\int_0^1 f_n\varphi\ \mathrm{d}x = 1 - e^{-n} \to 1,$$

and this proves that $\{f_n\}_{n\in\mathbb{N}}$ does not converges weakly to 0 in $L^1(0,1)$.

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(1) Discuss the pointwise a.e. convergence of the sequence $\{f_n\}_{n\in\mathbb{N}}$.
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$$\lim_{n\to+\infty} \int_0^1 f_n(x)\,dx.$$

(3) Does there exist a function $g \in L^1(0,1)$ such that $|f_n(x)| \le g(x)$ for a.e. $x \in (0,1)$ and every $n \in \mathbb{N}$? In the affirmative case, determine such a function g. If not, justify your answer.
(4) Discuss the weak convergence of $\{f_n\}_{n\in\mathbb{N}}$ in $L^1(0,1)$.

(1) $\forall x \in (0,1)$, $\quad f_n(x) = ne^{-nx} = \underbrace{\frac{nx}{e^{nx}}}_{\xrightarrow[n\to+\infty]{} 0} \times \underbrace{\frac{1}{x}}_{<+\infty} \xrightarrow[n\to+\infty]{} 0$.

by comparison of convergence speed.

So: $\{f_n\}_{n\in\mathbb{N}}$ converges to $f \equiv 0$.

Hence $\boxed{\{f_n\}_n \ \text{cv}^{\text{pointwisely}}_{\text{a.e.}} \text{ to } f \equiv 0}$.

(2) $\displaystyle\lim_{n\to+\infty} \int_0^1 f_n(x)\,dx$?

$f_n \in \mathcal{C}^0((0,1))$. Then: $\quad \forall n \in \mathbb{N}$,

$$\int_0^1 f_n(x)\,dx = n\int_0^1 e^{-nx}\,dx = n \cdot \left[\frac{-1}{n} e^{-nx}\right]_0^1 = -\left[e^{-nx}\right]_0^1 = 1 - e^{-n}.$$

Letting $n \to +\infty$ we obtain:

$$\boxed{\lim_{n\to+\infty} \int_0^1 f_n(x)\,dx = 1}. \ \square$$

(3) No it does not exist such a function. Indeed,

($f_n : x \mapsto ne^{-nx}$ ~~so~~ f is ~~continuous~~ on ~~[0,1]~~ and ~~unfortunately~~
$\cancel{(f_n(0))_n} = \cancel{(n)_n}$ ~~is~~ not ~~bounded~~.)

if such a function exists, then (since f_n are all meas. bc of continuity) by using the DCT we would get:

$$\lim_{n\to+\infty} \int_0^1 f_n(t)\,dt = \int_0^1 \lim_{n\to+\infty} f_n(t)\,dt = \int_0^1 0\,dt = 0 \neq 1 \ ⚡. \ \square$$

(4) [Revisit Weak Convergence :]

Exercise 3. Let $X = \ell^2$ and consider the linear operator $T : X \to X$ defined by

$$[T(x)]^{(k)} := \begin{cases} \dfrac{x^{(1)}}{2^k}, & \text{if } k \text{ odd} \\ \\ x^{(k)}, & \text{if } k \text{ even} \end{cases} \qquad \forall x = (x^{(k)})_{k\in\mathbb{N}} \in X.$$

(1) Prove that T is a bounded operator.
(2) Is T injective? Justify the answer.
(3) Let $\{\mathrm{e}_n\}_{n\in\mathbb{N}} \subset X$ be the sequence whose elements e_n are given by

$$\mathrm{e}_n = (\mathrm{e}_n^{(k)})_{k\in\mathbb{N}}, \quad \mathrm{e}_n^{(k)} := \begin{cases} 1, & \text{if } k = 2n \\ 0, & \text{otherwise} \end{cases}.$$

Show that $\{\mathrm{e}_n\}_{n\in\mathbb{N}}$ is bounded. Considering the behaviour of $\{T(\mathrm{e}_n)\}_{n\in\mathbb{N}}$, what can one say about the compactness of T?

Solution. (1) Let us show that the operator T is bounded, indeed

$$\|T(x)\|_2^2 = |x^{(1)}|^2 \sum_{k=1}^{\infty} \frac{1}{2^{2(2k-1)}} + \sum_{k=1}^{+\infty} |x^{(2k)}|^2 \le |x^{(1)}|^2 + \sum_{k=1}^{+\infty} |x^{(2k)}|^2 \le \|x\|_2^2,$$

holds for every $x \in \ell_2$.

(2) No, consider for instance $x = (0, 0, 1, 0, \dots) \in \ell_2$. Since $T(x) = \mathbf{0} = (0, 0, \dots)$, we conclude that $\ker T \neq \{\mathbf{0}\}$, being T linear this implies that T is not injective.

(3) Notice that $\|\mathrm{e}_n\|_2 = 1$, for every $n \in \mathbb{N}$. Hence the sequence $\{\mathrm{e}_n\}$ is bounded in ℓ_2. Moreover, $T(\mathrm{e}_n) = \mathrm{e}_n$, for every $n \in \mathbb{N}$. Observing that

$$\|T\mathrm{e}_n - T\mathrm{e}_m\|_2 = \sqrt{2}$$

for $n \neq m$, the sequence $\{T\mathrm{e}_n\}$ admits no convergent subsequences. We conclude that T is not a compact operator.

Exercise 3: $X = \ell^2$ and $T : X \to X$ linear operator:

$$\forall x = (x^{(k)})_{k\in\mathbb{N}} \in X,\ (Tx)^{(k)} := \begin{cases} x^{(1)}/2^k & \text{if } k \text{ odd} \\ x^{(k)} & \text{if } k \text{ even} \end{cases};$$

1) T bdd? Let $x \in X$. we want to show:

$$\|Tx\|_Y \le M\|x\|_X \quad \text{i.e} \quad \|Tx\|_{\ell^2} \le M\|x\|_{\ell^2}.$$

$$\|Tx\|_{\ell^2}^2 = \sum_{k=1}^{+\infty} |(Tx)^{(k)}|^2 = \sum_{k=1}^{+\infty}\left|\frac{x^{(1)}}{2^{2k-1}}\right|^2 + \sum_{k=1}^{+\infty}|x^{(2k)}|^2$$

$$\le |x^{(1)}|^2 \underbrace{\sum_{k=1}^{+\infty}\frac{1}{2^{2(2k-1)}}}_{\le 1 \text{ since } \sum_{k=1}^{+\infty}\frac{1}{2^k} = 1} + \sum_{k=1}^{+\infty}|x^{(2k)}|^2$$

$$\le |x^{(1)}|^2 + \sum_{k=1}^{+\infty}|x^{(2k)}|^2 \le \|x\|_{\ell^2}^2 \quad (\forall x \in \ell^2).$$

So $\boxed{T \text{ is bounded with } M = 1}$. □

2) Intuition: I don't think so bc in $(Tx)^{(k)}$ the $x^{(k)}$, with k odd, don't appear.

Let $x \in \ell^2$ and $y^{(k)} := \begin{cases} x^{(k)} & \text{for all } k \in \mathbb{N}^* \setminus \{2\} \\ x^{(3)} - 1 & \text{for } k = 3 \end{cases}$.

So: $\begin{cases} y = (x^{(1)}, x^{(2)}, x^{(3)} - 1, \dots) \\ x = (x^{(1)}, x^{(2)}, x^{(3)}, \dots) \end{cases}$. Clearly, since $x \in \ell^2$, $y \in \ell^2$ also.

We have: $Tx = Ty$.

Indeed $(Tx)^{(k)} = \begin{cases} x^{(1)}/2^k & \text{if } k \text{ odd} \\ x^{(k)} & \text{if } k \text{ even} \end{cases}$ whereas $(Ty)^{(k)} = \begin{cases} \frac{y^{(1)}}{2^k} = \frac{x^{(1)}}{2^k} & \text{if } k \text{ odd} \\ y^{(k)} = x^{(k)} & \text{if } k \text{ even} \end{cases}$

But: $x \ne y$. As a conclusion: $\boxed{T \text{ is not injective}}$ □

Correction: using the $\ker(T)$: $(0,0,1,0\dots) \in \operatorname{Ker}(T)$. □

3) $\{e_n\}_{n\in\mathbb{N}} \subset X = \ell^2$

$e_n = (e_n^{(k)})_{k\in\mathbb{N}}$ with $e_n^{(k)} := \begin{cases} 1 & \text{if } k = 2n \\ 0 & \text{otherwise} \end{cases}$.

- <u>$\{e_n\}_n$ bounded?</u> $\|e_n\|_2^2 = \sum_{k=0}^{+\infty} |e_n^{(k)}|^2 = |e_n^{(2n)}|^2 = 1^2 = 1$

So : $\forall n\in\mathbb{N}$, $\|e_n\|_2 = 1$. Hence $\{e_n\}_{n\in\mathbb{N}}$ is bounded.

- $Te_n = e_n \; \forall n\in\mathbb{N}$.

$$\|Te_n - Te_m\|_2 = \sqrt{2} \text{ for } n \neq m.$$

So the sequence $\{Te_n\}_n$ admits no (strongly) converging subsequences. We conclude that T is not a compact operator. $\square$

Theory

Question 1. *(4 points)* State and prove the properties of regularity of the Lebesgue measure (it is sufficient to present the proof regarding the outer approximation of measurable sets).

Solution. See Lecture 4. $\longrightarrow$ À remaniller !!

√ **Question 2.** *(4 points)* Let $(X, \mathcal{A}, \mu)$ be a complete measure space, and let $\{f_n\}_{n\in\mathbb{N}} \subset L^1(X)$, $f \in L^1(X)$. Give the following definitions:

(a) $f_n \to f$ in $L^1(X)$;
(b) $f_n \to f$ in measure in X.

Discuss the validity of the following implications: (i) (a) $\Rightarrow$ (b); (ii) (b) $\Rightarrow$ (a).
If yes, give a proof; if not, provide a counterexample.

Solution. See Lecture 10.

Question 3 *(4 points)* (i) Give the definition of bidual space. Introduce the canonical (or evaluation) map $\tau : X \to X^{**}$, proving in particular that it maps $x \in X$ to an element $\tau(x) \in X^{**}$. Now, τ is linear and continuous. Prove that τ is an isometry. Give the definition of reflexive space.

(ii) Exhibit an example of infinite dimensional Banach space which is reflexive and another one which is not reflexive.

Solution. See Lecture 20.

Question 4 *(4 points)* State and prove the theorem of the projections in Hilbert spaces.

Solution. See Lecture 23.

(2) (a) $f_n \xrightarrow[n\to+\infty]{L^1} f$ means that: $\underbrace{\int_X |f_n - f|\, d\mu}_{\|f_n - f\|_1} \xrightarrow[n\to+\infty]{} 0$;

(b) $f_n \xrightarrow[n\to+\infty]{} f$ in measure : $\forall \alpha > 0$, $\mu(\{x \in X : |f_n(x) - f(x)| \geq \alpha\}) \xrightarrow[n\to+\infty]{} 0$.

(i) • (a) $\Rightarrow$ (b) : YES.

Let's suppose that $\int_X |f_n - f|\, d\mu \xrightarrow[n\to+\infty]{} 0$.

Suppose by contradiction that $f_n \not\xrightarrow[n\to+\infty]{} f$ in measure, i.e that

$\left.\begin{array}{l}\exists \bar{\alpha}, \bar{\varepsilon} > 0 \\ \exists \{f_{n_k}\}_{k\in\mathbb{N}}\end{array}\right\}$ s.t $\mu(\underbrace{\{|f_{n_k} - f| \geq \bar{\alpha}\}}_{E_k}) \geq \bar{\varepsilon} > 0$.

$\int_X |f_{n_k} - f|\, d\mu \underset{\substack{\uparrow \\ E_k \subset X}}{\geq} \int_{E_k} |f_{n_k} - f|\, d\mu \geq \int_{E_k} \bar{\alpha}\, d\mu \underset{\substack{\uparrow \\ \mu(E_k) \geq \bar{\varepsilon}}}{\geq} \bar{\alpha}\bar{\varepsilon} > 0$ ↯

So (a) $\Rightarrow$ (b) i.e $\boxed{L^1\text{-CVnce} \Rightarrow \text{CVnce in measure}}$. □

• (b) $\Rightarrow$ (a) : Not in general. Contrexample: $f_n(x) := n\, \mathbb{1}_{[0,1/n]}(x)$.

$\{f_n\}_n$ cv. a.e to $f = 0$
$\Downarrow$
$\{f_n\}_n$ cv in measure to $f = 0$.

but $\int_{[0,1]} |f_n(x) - f(x)|\, dx = n \times \frac{1}{n} = 1 \underset{n\to+\infty}{\not\to} 0$