MIDA I - EXERCISE CLASS 2 -

$$y(t) = -a\underbrace{y(t-1)}_{AR(1)} + e(t) + \underbrace{c\,e(t-1)}_{MA(1)} \quad , \quad e(\cdot) \sim WN(0,\lambda^2)$$

$\longrightarrow$ ARMA(1,1)

TF: $y(t)(1+az^{-1}) = e(t)(1+cz^{-1})$

$$\boxed{y(t) = \underbrace{\frac{1+cz^{-1}}{1+az^{-1}}}_{W(z)} e(t) = \frac{z+c}{z+a}\, e(t)}$$

$y(\cdot)$ stationary (thm) $\begin{cases} \cdot\ e(\cdot) \text{ is stationary} \\ \cdot\ z_{pole} = -a \ , \ |a|<1 \longrightarrow \text{asympt. stable.} \end{cases}$

Expected value: $\mathbb{E}[y(t)] = -a\underbrace{\mathbb{E}[y(t-1)]}_{=\mathbb{E}[y(t)]} + \cancel{\mathbb{E}[e(t)]} + \cancel{c\,\mathbb{E}[e(t-1)]}$

so $\boxed{\mathbb{E}[y(t)] = 0}$

Covariance function:

- $\gamma_y(0) = \mathbb{E}[y(t)^2] = \mathbb{E}[(-ay(t-1) + e(t) + ce(t-1))^2]$

$$= a^2\,\mathbb{E}[y(t-1)^2] + \mathbb{E}[e^2(t)] + c^2\,\mathbb{E}[e(t)^2]$$

$y(t-1) = fct(e(t-1), e(t-2), \dots)$ $\quad - 2a\,\cancel{\mathbb{E}[y(t-1)e(t)]} - 2ac\,\mathbb{E}[y(t-1)e(t-1)]$

$$+ 2c\,\cancel{\mathbb{E}[e(t)e(t-1)]}$$

$$= a^2\gamma_y(0) + \lambda^2 + c^2\lambda^2 - 2ac\,\mathbb{E}[y(t-1)e(t-1)]$$

$$\mathbb{E}[y(t-1)e(t-1)] = \mathbb{E}\left[\left(-ay(t-2) + e(t-1) + ce(t-2)\right)e(t-1)\right]$$

$$= \mathbb{E}\left[e(t-1)^2\right] = \lambda^2$$

So : $\gamma_y(0) = a^2\gamma_y(0) + \lambda^2 + c^2\lambda^2 - 2ac\lambda^2$

i.e $(1-a^2)\,\gamma_y(0) = \lambda^2(1+c^2-2ac)$ i.e $\boxed{\gamma_y(0) = \dfrac{\lambda^2(1+c^2-2ac)}{1-a^2}}$.

- $\gamma_y(1) = \mathbb{E}[y(t)y(t-1)] = \mathbb{E}\left[\left(-ay(t-1) + e(t) + ce(t-1)\right)y(t-1)\right]$

$$= -a\underbrace{\mathbb{E}\left[y(t-1)^2\right]}_{=\gamma_y(0)} + \underbrace{\mathbb{E}\left[\cancel{e(t)y(t-1)}\right]}_{y(t-1) = fct(e(t-1), e(t-2), \dots)} + c\underbrace{\mathbb{E}\left[e(t-1)y(t-1)\right]}_{=\lambda^2 \text{ (cf above)}}$$

$$= -a\gamma_y(0) + 0 + c\lambda^2 = -a\frac{\lambda^2(1+c^2-2ac)}{1-a^2} + c\lambda^2$$

$$\boxed{\gamma_y(1) = -a\gamma_y(0) + c\lambda^2}.$$

- $\boxed{\gamma_y(\tau) = -a\,\gamma_y(\tau-1) \qquad \forall \tau \geq 2}$.

Now let $c = 1/a$, $|a| < 1$, $a \neq 0$. [SPECIFIC CASE]

$\boxed{\gamma_y(0) = \dfrac{1}{a^2}\lambda^2 = c^2\lambda^2}$ $\boxed{\gamma_y(1) = 0}$ $\boxed{\gamma_y(\tau) = 0 \quad \forall \tau \geq 1}$.

What does it mean?

$$\underset{\substack{\lambda^2 \\ WN}}{e(t)} \longrightarrow \boxed{\frac{1+cz^{-1}}{1+az^{-1}}} \xrightarrow[\substack{c^2\lambda^2 \\ WN}]{} y(t)$$

$c = \frac{1}{a}$

"ALL PASS FILTER"

cf Prediction chapter.

$$\eta_1(t) \longrightarrow \boxed{\frac{1}{1-az^{-1}}} \xrightarrow{y_1(t)} \overset{+}{\bigcirc} \longrightarrow y(t), \qquad \eta_2(t) \uparrow \bigcirc$$

- $(\eta_1(t))_t \sim WN(0, \lambda_1^2)$
- $(\eta_2(t))_t \sim WN(0, \lambda_2^2)$
- $\eta_1 \perp \eta_2$

Let's suppose $a = 1/2$, $\lambda_1^2 = 3/4$, $\lambda_2^2 = 1$.

Expected value:

$$y(t) = y_1(t) + \eta_2(t)$$

$$\mathbb{E}[y(t)] = \mathbb{E}[y_1(t)] + \cancel{\mathbb{E}[\eta_2(t)]}$$

$\mathbb{E}[y_1(t)]$? $\quad y_1(t) = \dfrac{1}{1-az^{-1}}\eta_1(t)$ so :

AR(1) $\quad y_1(t) = a\,y(t-1) + \eta_1(t)$

y_1 stationary
- $|a| < 1$;
- η_1 stat.

$$\mathbb{E}[y_1(t)] = a\,\mathbb{E}[y_1(t)] + 0 \text{ i.e } \boxed{\mathbb{E}[y_1(t)] = 0}.$$

So : $\boxed{\mathbb{E}[y(t)] = 0}$.

<u>Covariance function</u> :

$$\gamma_y(\tau) = \mathbb{E}[y(t)y(t-\tau)] = \mathbb{E}\left[(y_1(t) + \eta_2(t))(y_1(t-\tau) + \eta_2(t-\tau))\right]$$

$$= \gamma_{y_1}(\tau) + \cancel{\mathbb{E}[y_1(t)\eta_2(t-\tau)]} + \cancel{\mathbb{E}[\eta_2(t)y_1(t-\tau)]} + \gamma_{\eta_2}(\tau)$$

$$\boxed{\gamma_y(\tau) = \gamma_{y_1}(\tau) + \gamma_{\eta_2}(\tau)}$$

(1) (2)

(2) very easy : $\boxed{\gamma_{\eta_2}(\tau) = \begin{cases} \lambda_2^2 & \text{if } \tau = 0 \\ 0 & \text{if } \tau \neq 0 \end{cases}}$

(1) <u>COV. FUN. OF y_1</u> :

- $\gamma_{y_1}(0) = \mathbb{E}[y_1^2(t)] = \mathbb{E}[(a y_1(t-1) + \eta_1(t))^2]$

$$= a^2 \gamma_{y_1}(0) + \gamma_{\eta_1}(0) + \cancel{2a\,\mathbb{E}[y_1(t-1)\eta_1(t)]}$$

$y_1(t-1) = fct(\eta_1(t-1), \dots)$

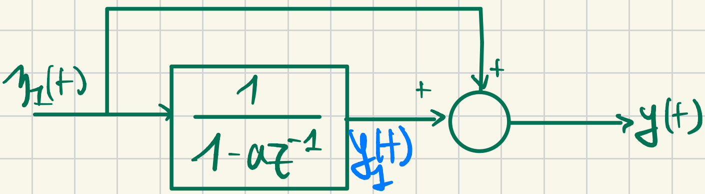
$$\boxed{\gamma_{y_1}(0) = \frac{\lambda_1^2}{1-a^2}} \quad (=1)$$

- $\gamma_{y_1}(1) = \mathbb{E}[y_1(t)y_1(t-1)] = a\gamma_{y_1}(0)$

And $\boxed{\gamma_{y_1}(\tau) = a\gamma_{y_1}(\tau-1),\ \forall |\tau| \geq 1}$

So:

$$\boxed{\begin{aligned}
\gamma_y(0) &= 1 + \underbrace{1}_{\lambda_2^2} = 2 \; ; \\
\gamma_y(1) &= \frac{1}{2} + 0 = \frac{1}{2} \; ; \\
\gamma_y(\tau) &= \frac{1}{2}\,\gamma_y(\tau-1) \quad \forall |\tau| \geq 1 .
\end{aligned}}$$



- $\mathbb{E}[y(t)] = 0$

- <u>COVARIANCE FUNCTION:</u>

1st METHOD: Very long

$$\gamma_y(\tau) = \mathbb{E}[y(t)y(t-\tau)] = \mathbb{E}\left[\left(y_1(t)+\eta_1(t)\right)\left(y_1(t-\tau)+\eta_1(t-\tau)\right)\right]$$

$$= \gamma_{y_1}(\tau) + \gamma_{\eta_1}(\tau) + \underbrace{\mathbb{E}[y_1(t)\eta_1(t-\tau)]}_{\neq 0} + \underbrace{\mathbb{E}[\eta_1(t)y_1(t-\tau)]}_{\neq 0}$$

LONG

$$\mathbb{E}[y_1(t)\eta_1(t)] = \lambda_1^2$$

$$\mathbb{E}[y_1(t)\eta_1(t-1)] = a\,\mathbb{E}[y_1(t-1)\eta_1(t-1)] + \mathbb{E}[\cancel{\eta_1(t)\eta_1(t-1)}]$$

$$y_1(t) = a y_1(t-1) + \eta_1(t) \qquad = a\,\gamma_{\eta_1}(0) = a\lambda_1^2 .$$

LONG

$$\mathbb{E}[y_1(t)\eta_1(t-2)] = a\,\mathbb{E}[\cancel{y_1(t-1)\eta_1(t-2)}] + 0$$

$$\mathbb{E}[\eta_1(t) y_1(t-\tau)] = \begin{cases} \lambda_1^2 & \text{if } \tau = 0 \\ a\lambda_1^2 & \text{if } \tau = 1 \\ 0 & \text{if } \tau > 1 \end{cases}$$

2nd METHOD : WAY SHORTER

We can write y :

$$y(t) = \eta_1(t) + \frac{1}{1-az^{-1}}\eta_1(t) = \frac{2-az^{-1}}{1-az^{-1}}\eta_1(t)$$

$$= 2\eta_1(t) \times \frac{1-\frac{a}{2}z^{-1}}{1-az^{-1}}$$

$2\eta_1(t) \sim WN(0, 4\lambda_1^2)$

$\equiv \eta(t)$

So $y(t) = \dfrac{1-a/2\, z^{-1}}{1-az^{-1}}\eta(t)$

$\eta(\cdot) \sim WN(0, 4\lambda_1^2)$.

$$\boxed{y(t) = \frac{1-\frac{1}{4}z^{-1}}{1-\frac{1}{2}z^{-1}}\eta(t) \quad , \eta(t) \sim WN(0,3)}$$

Use the formula from ARMA(1,1)

We know that for $y(t) = \dfrac{1+cz^{-1}}{1+az^{-1}}e(t)$, $e \sim WN(0,\lambda^2)$

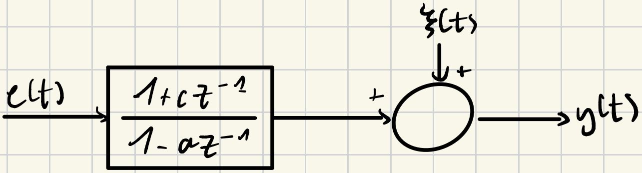
with $c = -1/4$, $a = -1/2$, $\lambda^2 = 3$, we have :

$$\gamma_y(0) = \frac{1+c^2-2ac}{1-a^2}\lambda^2 = \frac{13}{4}$$

$$\gamma_y(1) = -a\gamma_y(0) + c\lambda^2 = 7/8$$

$$\gamma_y(\tau) = -\left(-\tfrac{1}{2}\right)\gamma_y(\tau-1) = \tfrac{1}{2}\gamma_y(\tau-1).$$

EXERCISE SESSION 2 - PART 2



- $e(t), \xi(t) \sim WN(0,1)$,
- $e(t) \perp\!\!\!\perp \xi(t)$

- $(y(t))_t$ stationary if $|a|<1$. $\left(W(z) = \dfrac{z+c}{z-a}\right)$

- $\gamma_y(0) = 6$, $\gamma_y(1) = -2$, $\gamma_y(\tau) = 0$ if $\tau \geq 2$. a & c ?

MA(1)

$\boxed{a = 0}$

$\rightarrow y(t) = e(t) + ce(t-1) + \xi(t)$

$\mathbb{E}[y(t)^2] = 1 + c^2 + 1 = 2 + c^2$

$\parallel$

$6 \iff \boxed{c = \pm 2}$

$\rightarrow \mathbb{E}[y(t)y(t-1)] = \mathbb{E}\Big[\big(e(t) + ce(t-1) + \xi(t)\big)\big(e(t-1) + ce(t-2) + \xi(t-1)\big)\Big]$

$= c \times \gamma_e(0) = c \times 1 = c$

$\mathbb{E}[y(t)y(t-1)] = -2 \iff \boxed{c = -2}$.

v_1 s.t. $\mathbb{E}[v_1] = 0$, $\mathbb{E}[v_1^2] = 4$.

$v_2 = 10\, v_1 + 10$

$\rightarrow \mathbb{E}[v_2] = 10\, \mathbb{E}[v_1] + 10 = 10$.

$\rightarrow \text{Var}(v_2) = \mathbb{E}\left[\left(v_2 - \mathbb{E}[v_2]\right)^2 \right] = \mathbb{E}\left[(10\, v_1 + 10 - 10)^2 \right]$

$= \mathbb{E}[100\, v_1^2] = 100 \times 4 = 400$.

$y(t) = a\, y(t-1) + e(t)$, $e(\cdot) \sim WN(0, 9)$, $a \in \mathbb{R}$.

- $a = 20$. $y(\cdot)$ stat.? NO, because the pole $z = a$ is outside the unit disk.
- $a = 0.5 \rightarrow$ YES, $y(\cdot)$ stat.
- $a = 0.5$. $\mathbb{E}[y^2(t)] = a^2 \underbrace{\mathbb{E}[y^2(t-1)]}_{} + 9$

  i.e. $\mathbb{E}[y^2(t)] = \dfrac{9}{1 - 0.25} = 12$. | $= \mathbb{E}[y(t)^2]$ (STAT.)

- If $|a| < 1$, is $v(t) = y(2t)$ a WN?

  NO: because the cov. function of $v(\cdot)$ is not zero for $\tau = 1$:

  $\mathbb{E}[v(t)\, v(t-1)] = \mathbb{E}[y(2t)\, y(2t-2)]$

$$= \mathbb{E}\Big[\big(ay(2t-1)+e(2t)\big)\big(ay(2t-2)+e(2t-2)\big)\Big]$$

$$= \mathbb{E}\Big[\big(a^2 y(2t-2) + e(2t-1)\big)\big(ay(2t-2)+ e(2t-2)\big)\Big]$$

$y(t) = e(t) + e(t-1), \quad e \sim WN(1,1).$

⚠ BE CAREFUL.

- $\mathbb{E}[y(t)] = 1 + 1 = 2.$
- $\mathrm{Var}(y(t)) = \mathbb{E}\big[(y(t)-2)^2\big] = \mathbb{E}\big[(e(t)+e(t-1)-2)^2\big]$

$$= \mathbb{E}[e^2] + \mathbb{E}[e^2] + 4 + 2\,\mathbb{E}[e(t)e(t-1)] - 4\,\mathbb{E}[e(t)] - 4\,\mathbb{E}[e(t-1)]$$

⚠ $\neq 0$

$$= \cancel{2} + \cancel{2} + \cancel{4} + 2 - \cancel{4} - \cancel{4} = 2.$$

⚠ $\mathbb{E}[e(t)^2] \neq 1$

but:

$\mathbb{E}\big[(e(t)-1)^2\big] = 1$ i.e.

$\mathbb{E}[e(t)^2] = +\cancel{1} - \cancel{1} + 2\,\mathbb{E}[e(t)] = 2.$

$\mathbb{E}\Big[\big(e(t)-1\big)\big(e(t-1)-1\big)\Big] = 0 \iff$

$\mathbb{E}[e(t)e(t-1)] = \cancel{-1} + \underbrace{\mathbb{E}[e(t)]}_{1} + \underbrace{\mathbb{E}[e(t-1)]}_{1} = 1.$

- $\gamma_y(2)$? ($\rightarrow$ $y \sim MA(1)$ so $\gamma_y(2) = 0$) Let's do the computations:

$$\mathbb{E}\Big[\big(y(t)-2\big)\big(y(t-2)-2\big)\Big] = \cancel{4} + \mathbb{E}[y(t)y(t-2)] - \cancel{2}\,\overbrace{\mathbb{E}[y(t)]}^{2} - 2\,\overbrace{\mathbb{E}[y(t-2)]}^{2}$$

$$= -4 + \mathbb{E}[y(t)y(t-2)]$$

$$\mathbb{E}[y(t)y(t-2)] = \mathbb{E}\left[\left(e(t) + e(t-1)\right)\left(e(t-2) + e(t-3)\right)\right]$$

$$= \mathbb{E}[e(t)e(t-2)] + \mathbb{E}[e(t-1)e(t-3)] +$$

$$\mathbb{E}[e(t)e(t-3)] + \mathbb{E}[e(t-1)e(t-3)]$$

$$\mathbb{E}\left[\left(e(t)-1\right)\left(e(t-2)-1\right)\right] = 0 \iff \underbrace{\mathbb{E}[e(t)e(t-2)]}_{=1} = -\cancel{1} + \overbrace{\mathbb{E}[e(t)]}^{=1} + \underbrace{\mathbb{E}[e(t-2)]}_{=1}$$

same for the 3 other terms.

So: $\boxed{\mathbb{E}[(y(t)-2)(y(t-2)-2)] = \gamma_y(2) = -4 + 4 = 0}$.

- Is $v(t) = y(2t)$ a white noise? YES

$$v(t) = y(2t) = e(2t) + e(2t-1)$$

t	$y(t)$	$v(t)$
2	$e(2) + e(1) = y(2)$	$= v(1)$
3	$e(3) + e(2) = y(3)$	
4	$e(4) + e(3) = y(4)$	$= v(2)$

uncorrelated