Mathematical Engineering - A.Y. 2022-23

Real and Functional Analysis - Exam with Solutions- January 25, 2023

Answers and solutions can be written in English or in Italian.

Theory

Question 1. *(4 points)* (i) State and prove the property of continuity of measure along monotone decreasing sequences $\{E_n\}$ of measurable subsets.

(ii) Does the property hold, if E_1 (or some of the sets E_n) has infinite measure? If not, provide a counterexample.

Solution. See Lecture 2.

Question 2. *(4 points)* State and prove the Fatou's Lemma.

Solution. See Lecture 7.

Question 3 *(4 points)* (i) Write the definition of open mapping. State the Open Mapping theorem.

(ii) State and prove the Inverse Bounded Mapping Theorem.

Solution. See Lecture 18.

Question 4 *(4 points)* Let X be a Banach space. (I) Write the definitions of weak convergence and of (strong) convergence for a sequence $\{x_n\} \subset X$.

(II) Consider now the following properties:

(a) $x_n \rightharpoonup x$ (weakly) in X; (a') $\{x_n\}$ possesses a weakly convergent subsequence;

(b) $\{x_n\}$ is bounded;

(c) $x_n \to x$ (strongly) in X.

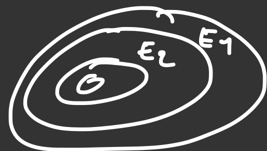
(i) Does (a) imply (b)? (ii) Does (b) imply (a') (if necessary, under additional assumptions)?

(iii) Does (a) imply (c)? (iv) Does (c) imply (a)?

For questions 4.(II)(i)-(iv), justify the answers, only quoting some theorems or briefly discussing a counterexample. No proofs are required.

Solution. See Lectures 14, 21, 22.

<u>Theory:</u>



1) (i) $\{E_n\}_n \searrow$ $\qquad G_n := E_1 \setminus E_n$.

- $\{G_n\}_n$ is an increasing sequence: $\lim\limits_{n\to+\infty} G_n = \bigcup\limits_{n\in\mathbb{N}} G_n = \bigcup\limits_{n\in\mathbb{N}} (E_1 \setminus E_n) = E_1 \setminus \overbrace{\bigcap\limits_{n\in\mathbb{N}} E_n}^{\lim E_n} = E_1 - \lim\limits_{n\to+\infty} E_n$
- We apply the result of continuity of μ on a sequence:

$$\mu(\lim_{n\to+\infty} G_n) = \lim_{n\to+\infty} \mu(G_n) \quad \text{i.e:}$$

$$\mu(E_1) - \mu(\lim_{n\to+\infty}(E_n)) = \lim_{n\to+\infty} \mu(G_n) = \mu(E_1) - \lim_{n\to+\infty} \mu(E_n)$$

i.e: $\lim\limits_{n\to+\infty} \mu(E_n) = \mu(\lim\limits_{n\to+\infty} E_n)$. $\square$

(ii) E_1 is the biggest set (in $\subset$ sense) of $\{E_n\}_n$.

<u>PROP:</u> Given $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M} \searrow$. $E := \lim\limits_{n\to+\infty} E_n = \bigcap\limits_{n\in\mathbb{N}} E_n$.
If $\mu(E_1) < +\infty$ then $\mu(E) = \lim\limits_{n\to+\infty} \mu(E_n)$.

⚠ Essential to have $\mu(E_1) < +\infty$.
Counterexample: $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_\#)$ and $E_n := \{k\in\mathbb{N} : k \geq n\}$ $\forall n \in \mathbb{N}$. $\forall n\in\mathbb{N}$, $E_{n+1} \subsetneq E_n$.
But: $E = \lim\limits_{n\to+\infty} E_n = \bigcap\limits_{n\in\mathbb{N}} E_n = \emptyset$
and: $\lim\limits_{n\to+\infty} \mu_\#(E_n) = +\infty$.
So: $\mu_\#(E) \neq \lim\limits_{n\to+\infty} \mu_\#(E_n)$

2) <u>Fatou's Lemma:</u> $(X, \mathfrak{M}, \mu)$ complete measure space. Let $f_n : X \to [0,+\infty]$ measurable $\forall n\in\mathbb{N}$.

$$\int_X \left(\liminf_{n\to+\infty} f_n\right) d\mu \leq \liminf_{n\to+\infty} \int_X f_n \, d\mu .$$

<u>Proof:</u> $(\liminf\limits_{n\to+\infty} f_n): x \mapsto \liminf\limits_{n\to+\infty} f_n(x) = \sup\limits_{n\in\mathbb{N}} \underbrace{\inf\limits_{k\geq n} f_k(x)}_{g_n(x)}$

- (g_n) is an increasing sequence:

$$\forall x\in X, \quad g_n(x) \leq g_{n+1}(x).$$

- By MCT: $\displaystyle\int_X \liminf f_n \, d\mu = \int_X \lim g_n \, d\mu \overset{MCT}{=} \lim \int_X g_n \, d\mu$

$$= \liminf_{n\to+\infty} \int_X g_n \, d\mu \leq \liminf_{n\to+\infty} \int_X f_n \, d\mu . \ \square$$

$g_n \leq f_n$.

3) (i) Def [open map]: let $T: X \to Y$. T is an open map if
$\forall A \subset X$ open, $T(A) \subset Y$ is open.

Thm [Open Map Theorem]: Let $T \in \mathcal{L}(X,Y)$ with X and Y Banach Spaces.
T surjective $\Rightarrow$ T open.

(ii) Corollary I: "Inverse Bounded Mapping Theorem" The inverse of a linear & bdd operator between two Banach Spaces X and Y is bounded.

"$T \in \mathcal{L}(X,Y)$, $T^{-1} \in \mathcal{L}(Y,X)$"

Proof:
- T^{-1} linear: $T \circ T^{-1} = Id_Y$ and $T^{-1} \circ T = Id_X$; and T linear.
- T^{-1} continuous: let $A \subset X$ open,
  $(T^{-1})^{-1}(A) = T(A) \subset Y$ open.
  Indeed T is bijective, hence surjective, so we use the OMT.
  As a conclusion: $\forall A \subset X$ open, $(T^{-1})^{-1}(A) \subset Y$ open.
  T^{-1} is continuous. $\square$

4)

Exercises

Exercise 1. Consider the measure space $([0,+\infty), \mathcal{L}([0,+\infty)))$ with the Lebesgue measure. Define the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ by

$$f_n(x) = \frac{\sin^2(x)}{1+x}\,\chi_{[0,n]}(x), \quad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

(1) Prove that $f_n \in L^p([0,+\infty))$ for any $n \in \mathbb{N}$ and any $p \in (1,+\infty)$.
(2) Study the convergence a.e. of the sequence $\{f_n\}_{n\in\mathbb{N}}$.
(3) Study the convergence in $L^p([0,+\infty))$ of the sequence $\{f_n\}_{n\in\mathbb{N}}$ for $p \in (1,+\infty)$.

Solution.
(1) Let $n \in \mathbb{N}$ and $p \in (1,+\infty)$, we have

$$\int_0^{+\infty} |f_n(x)|^p \,\mathrm{d}x = \int_0^n \left|\frac{\sin^2 x}{1+x}\right|^p \mathrm{d}x < +\infty,$$

indeed, the function $h(x) := \left|\frac{\sin^2 x}{1+x}\right|^p$ is continuous in the bounded interval $[0,n]$, it is integrable. Therefore $f_n \in L^p([0,+\infty))$.
(2) Since for any $x \in [0,+\infty)$, we have

$$\chi_{[0,n]}(x) \to 1, \text{ as } n \to +\infty,$$

then f_n converges pointwisely everywhere (thus a.e.) to the function f, with

$$f(x) := \frac{\sin^2 x}{1+x}.$$

(3) Let $p \in (1,+\infty)$ be fixed. On account of point (2), we already know that $(f_n)_n$ converges pointwisely a.e. to the function f; hence, to study the convergence of $(f_n)_n$ in $L^p([0,+\infty))$ we need to check whether $\|f_n - f\|_{L^p([0,+\infty))} \to 0$ as $n \to +\infty$ with f as in item (2). We have

$$\|f_n - f\|_{L^p}^p = \int_0^{+\infty} |f_n(x) - f(x)|^p \,\mathrm{d}x = \int_0^{+\infty} \left|\frac{\sin^2 x}{1+x}\right|^p \chi_{(n,+\infty)} \,\mathrm{d}x.$$

Denote by $h_n(x) := \left|\frac{\sin^2 x}{1+x}\right|^p \chi_{(n,+\infty)}$, we have that h_n is measurable in $[0,+\infty)$ (indeed it is the product of two measurable functions) and

$$h_n(x) \le \left|\frac{1}{1+x}\right|^p =: g(x), \qquad \text{for any } n \in \mathbb{N} \text{ and any } x \in [0,+\infty).$$

Since $g \in L^1([0,+\infty))$ and $h_n = |f_n - f|^p$ converges pointwisely a.e. to 0, we can apply Dominated Convergence Theorem and have

$$\lim_{n\to+\infty} \|f_n - f\|_{L^p}^p = \lim_{n\to+\infty} \int_0^{+\infty} h_n(x)\,\mathrm{d}x = \int_0^{+\infty} \lim_{n\to+\infty} h_n(x)\,\mathrm{d}x = 0.$$

Hence $\{f_n\}_{n\in\mathbb{N}}$ converges to f in $L^p([0,+\infty))$, for any $p \in (1,+\infty)$.

Exercise 2. Let $p \in (1, +\infty)$ and consider the sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in \ell^p$ for any $n \in \mathbb{N}$, defined by

$$x_n = (x_n^{(k)})_{k\in\mathbb{N}}, \quad \text{with} \quad x_n^{(k)} := \begin{cases} e^{1/n}, & \text{if } k = n \\ 0, & \text{otherwise} \end{cases}.$$

Discuss weak and strong convergence of $\{x_n\}_{n\in\mathbb{N}}$.

Solution. We start by studying the pointwise convergence of $\{x_n\}_{n\in\mathbb{N}}$. Observe that for any $n \in \mathbb{N}$, $x_n \in \ell^p$ is a sequence of real numbers;

$$\begin{aligned} &x_1 := (e,\, 0,\, 0, \dots), \\ &x_2 := (0,\, e^{1/2},\, 0, \dots), \\ &\dots \\ &x_n := (0,\, \dots,\, 0,\, e^{1/n},\, 0, \dots), \\ &\dots \end{aligned}$$

The study of the pointwise convergence of $\{x_n\}_{n\in\mathbb{N}}$ consists in determining the limit of $x_n^{(k)}$ as $n \to +\infty$, for any fixed $k \in \mathbb{N}$ (i.e. of each component of x_n). Fix $k \in \mathbb{N}$, by definition we have

$$x_n^{(k)} = 0, \quad \text{for any } n > k,$$

hence $x_n^{(k)} \to 0$ in $\mathbb{R}$ as $n \to +\infty$. Therefore

$$x_n \to \mathbf{0} = (0, 0, \dots), \text{ as } n \to +\infty.$$

Thus $\{x_n\}_{n\in\mathbb{N}}$ converges pointwisely to $\mathbf{0}$.

Since weak convergence in ℓ^p implies pointwise convergence, then if $\{x_n\}_{n\in\mathbb{N}}$ converges weakly in ℓ^p to x, then $x = \mathbf{0}$. Since $p \in (1, +\infty)$, by Riesz representation theorem we can identify $(\ell^p)^*$ with ℓ^q, with $q \in (1, +\infty)$ conjugate index of p. Thus, $x_n \rightharpoonup \mathbf{0}$ in ℓ^p if and only if

$$\sum_{k=1}^{+\infty} x_n^{(k)} y^{(k)} \to 0, \quad \text{for any } y = (y^{(k)}) \in \ell^q.$$

Let $y \in \ell^q$ be arbitrarily fixed, we have

$$\sum_{k=1}^{+\infty} x_n^{(k)} y^{(k)} = e^{1/n} y^{(n)} \to 0,$$

indeed $e^{1/n} \to 1$ as $n \to +\infty$, and $y^{(n)} \to 0$ by necessary condition of convergence of series (remind that $y \in \ell^q$, so the series $\sum_{k=1}^{+\infty} |y^{(k)}|^q$ converges).

Concerning strong convergence, recall that strong convergence implies weak convergence, so that if $\{x_n\}_{n\in\mathbb{N}}$ converges strongly in ℓ^p to x, then $x = \mathbf{0}$. Therefore we study whether $\|x_n - \mathbf{0}\|_p$ tends to 0 as $n \to +\infty$. We have

$$\|x_n - \mathbf{0}\|_p = \|x_n\|_p = e^{1/n} \not\to 0,$$

hence $\{x_n\}_{n\in\mathbb{N}}$ does not converge strongly in ℓ^p.

Exercise 3. Let $X = C([0,1])$ endowed with the norm $\|\cdot\|_\infty$. Consider the linear operator T defined by

$$(Tu)(t) = \int_0^t e^{\sin(s)}\, u(s)\, \mathrm{d}s, \quad \forall u \in X.$$

(1) Show that $T : X \to X$ is well-defined and bounded.
(2) Is T surjective? Justify the answer.
(3) Prove that T is a compact operator.

Solution. Set $g(s) := e^{\sin(s)}$ and notice that g is continuous on $[0,1]$ and $\lambda([0,1]) < +\infty$, hence $g \in L^1([0,1])$.

(1) We have

-*T is well defined.* Taken $u \in X$ we have $gu \in L^1([0,1])$, since ug is continuous on $[0,1]$; thus, by the First and Second Fundamental Theorems of Calculus we get

$$(\star) \qquad T(u) \in AC([0,1]).$$

From $(\star)$ we have that, in particular, $T(u)$ is continuous in $[0,1]$, hence it belongs to X and T is well-posed.

-*Boundedness.* For every $u \in X$, we have

$$|T(u)(t)| \le \int_0^t |g(s)|\,|u(s)|\, \mathrm{d}s \le \|g\|_{L^1}\|u\|_\infty \qquad \forall\, t \in [0,1].$$

As a consequence, we obtain

$$(\star\star) \qquad \|T(u)\|_\infty = \sup_{t\in[0,1]} |T(u)(t)| \le \|g\|_{L^1}\, \|u\|_\infty.$$

This, together with the arbitrariness of $u \in X$, ensures that $T : X \to X$ is bounded.

(2) The operator $T : X \to X$ is not surjective, indeed $v(t) \equiv 1$ belongs to X, but since $v(0) \neq 0$ there are no $u \in X$ such that $v = Tu$.

(3) We consider a bounded set B of $X = C([0,1])$ and we prove that its image under T, say $E = T(B)$, is *equi-bounded and equi-continuous.* In view of the Ascoli-Arzelà Theorem, this will ensure that

$$\overline{E} \text{ is compact in } C([0,1]),$$

and thus that T is a compact operator.

-*Equi-boundedness.* By item (1) and in particular from $(\star\star)$, we get that $T \in \mathcal{L}(X)$ and $\|T\|_{\mathcal{L}(X)} \le \|g\|_{L^1} = M$, with $M < +\infty$. We remind that, since B is bounded in X, there exists $R > 0$ such that $\|u\|_\infty \le R$ for any $u \in B$. Hence, for every $u \in B$ we have

$$\max_{t\in[0,1]} |T(u)(t)| = \|T(u)\|_\infty \le M\, \|u\|_\infty \le M \cdot R.$$

Thus, $E = T(B)$ is equi-bounded.

-*Equi-continuity.* Let $u \in B$ be arbitrarily fixed. Taking into account the very definition of T, for every $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we have

$$\begin{aligned} |T(u)(t_1) - T(u)(t_2)| &= \left| \int_{t_1}^{t_2} g(s)\, u(s)\ \mathrm{d}s \right| \leq \int_{t_1}^{t_2} g(s)\, |u(s)|\ \mathrm{d}s \\ &\leq \max_{[0,1]} |u| \cdot \int_{t_1}^{t_2} g(s)\, ds = \|u\|_\infty \cdot \int_{t_1}^{t_2} g(s)\, ds \\ &\qquad (\text{since } \|u\|_\infty \leq R, \text{ as } u \in B) \\ &\leq R \left| \int_{t_1}^{t_2} e^{\sin(s)}\, ds \right| \leq M \cdot R(t_2 - t_1), \end{aligned}$$

where we have used the fact that g is bounded on $[0,1]$. Thanks to the above estimate, we easily infer that $E = T(B)$ is *equi-continuous* on $[0,1]$ (actually, $T(B)$ is *equi-Lipschitz* on $[0,1]$).

Exercise 1. Consider the measure space $([0,+\infty), \mathcal{L}([0,+\infty)))$ with the Lebesgue measure. Define the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ by

$$f_n(x) = \frac{\sin^2(x)}{1+x}\chi_{[0,n]}(x), \quad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

(1) Prove that $f_n \in L^p([0,+\infty))$ for any $n \in \mathbb{N}$ and any $p \in (1,+\infty)$.
(2) Study the convergence a.e. of the sequence $\{f_n\}_{n\in\mathbb{N}}$.
(3) Study the convergence in $L^p([0,+\infty))$ of the sequence $\{f_n\}_{n\in\mathbb{N}}$ for $p \in (1,+\infty)$.

$\to \forall p\in(1,+\infty)\ \forall n\in\mathbb{N},\ \int_{[0;+\infty)} |f_n(x)|^p dx = \int_0^n \underbrace{\left|\frac{\sin^2(x)}{1+x}\right|^p}_{=: h_n(x)} dx$

(1) Let $n\in\mathbb{N}$. Let $p\in(1,+\infty)$. $\forall x\in\mathbb{R}$,

$$|f_n(x)|^p = \left|\frac{\sin^2(x)}{1+x}\mathbb{1}_{[0,n]}(x)\right|^p \leqslant \left|\frac{\sin^2(x)}{1+x}\right|^p \leqslant \frac{1}{|1+x|^p} \leqslant \frac{1}{x^p}$$

h_n $\mathcal{C}^0$ and bdd on $[0;n] \to$ integrable.

We know that $x \mapsto \frac{1}{x^p}$ is integrable for $p>1$ $(p<+\infty)$ in the sense of Riemann. When a fct is integrable in the sense of Riemann, $\mathcal{R}$ and $\mathcal{L}$ integrals coincide : so $\boxed{f_n \in L^p([0;+\infty))\ \forall n\in\mathbb{N},\ \forall p\in(1,+\infty)}$ □

(2) Let $x\in\mathbb{R}^+$, let $n\in\mathbb{N}$,

$$f_n(x) = \frac{\sin^2(x)}{1+x}\mathbb{1}_{[0;n]}(x) = \begin{cases} \frac{\sin^2(x)}{1+x} & \text{if } x\leqslant n \\ 0 & \text{otherwise} \end{cases} \xrightarrow[n\to+\infty]{} \frac{\sin^2(x)}{1+x} =: f(x).$$

since $\forall x\in\mathbb{R}$, $\mathbb{1}_{[0;n]}(x) \xrightarrow[n\to+\infty]{} 1$.

So : $\boxed{\{f_n\}_{n\in\mathbb{N}} \text{ converges everywhere \& thus a.e to } f \text{ on } [0;+\infty).}$ □

(3) If $\{f_n\}_n$ converges in $L^p([0,+\infty))$, it is to f.
Let $p\in(1,+\infty)$. We want to show if :

$$\|f_n - f\|_p \xrightarrow[n\to+\infty]{} 0.$$

$$\|f_n - f\|_p^p = \int_{[0;+\infty)} \left|\frac{\sin^2(x)}{1+x}\left(\mathbb{1}_{[0;n]}(x) - 1\right)\right|^p dx = \int_{[0;+\infty)} \left|\frac{\sin^2(x)}{1+x}\right|^p \mathbb{1}_{[n;+\infty)}(x)\, dx$$

Let's define $h_n(x) := \left|\frac{\sin^2(x)}{1+x}\right|^p \mathbb{1}_{[n;+\infty)}(x)$, $\forall x\in\mathbb{R}, \forall n\in\mathbb{N}, \forall p\in(1,+\infty)$.

h_n is measurable in $[0,+\infty)$ as the product of two measurable functions.

- By previous question, we know that $f_n(x) \xrightarrow[n\to+\infty]{} f(x)$ a.e in $\mathbb{R}^+$.

So $h_n(x) \xrightarrow[n\to+\infty]{} 0$ a.e in $\mathbb{R}^+$.

- Furthermore, $\forall x \in \mathbb{R}$, $(\forall n \in \mathbb{N})$

$$\left| h_n(x) \right| \leq \frac{1}{|1+x|^p}$$ which is in $\mathcal{L}^1([0,+\infty))$ since $p>1$.

We can then apply the DCT to h_n:

$$\int_{[0,+\infty)} \lim_{n\to+\infty} h_n(x)\,dx = \lim_{n\to+\infty} \int_{[0,+\infty)} h_n(x)\,dx \quad \text{i.e}$$

$$\lim_{n\to+\infty} \int_{[0,+\infty)} h_n(x)\,dx = \int_{[0,+\infty)} 0\,dx = 0$$

$0 \cdot \infty = 0$ in order to insure that the null fct will have $\int$ null even on infinite sets.

So:

$$\lim_{n\to+\infty} \int_{[0,+\infty)} h_n(x)\,dx = \lim_{n\to+\infty} \int_{[0,+\infty)} |f_n(x)-f(x)|^p\,dx = \lim_{n\to+\infty} \|f_n-f\|_p^p = 0$$

As a conclusion: $\boxed{\|f_n - f\|_p \xrightarrow[n\to+\infty]{} 0 \quad \text{i.e} \quad f_n \xrightarrow[n\to+\infty]{L^p([0,+\infty))} f}$ $\square$

Exercise 2 : $p \in (1,+\infty)$, $\{x_n\}_n$ w/ $\forall n \in \mathbb{N},\ x_n \in \ell^p = \mathcal{L}^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_\#)$.

$\forall n \in \mathbb{N}^*,\quad x_n = (x_n^{(k)})_{k \in \mathbb{N}^*}$ with $x_n^{(k)} := \begin{cases} e^{1/n} & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$.

- We start by studying the pointwise CVence of $\{x_n\}_n$:

Let $k \in \mathbb{N}^*$, we want to find the limit when $n \to +\infty$ of $x_n^{(k)}$.

$$x_1 = \begin{pmatrix} e^1 & 0 & 0 & \dots \end{pmatrix}$$
$$x_2 = \begin{pmatrix} 0 & e^{1/2} & 0 & \dots \end{pmatrix}$$
$$x_3 = \begin{pmatrix} 0 & 0 & e^{1/3} & \dots \end{pmatrix}$$

Let $k \in \mathbb{N}^*$, $x_n^{(k)} := \begin{cases} e^{1/n} & \text{if } k = n \\ 0 & \text{otherwise} \end{cases} \xrightarrow[n \to +\infty]{} 0$ since, for any fixed $k \in \mathbb{N}^*$, $x_n^{(k)} = 0$ for $n > k$.

So : $\boxed{(x_n)_{n \in \mathbb{N}^*} \text{ converges pointwisely to } x = (x^{(k)})_{k \in \mathbb{N}^*} := (0)_{k \in \mathbb{N}}}$.

- If $(x_n)_{n \in \mathbb{N}}$ weakly converges, it should be to x.

We are in ℓ^p so we can use Riesz Representation Thm $(1 < p < +\infty)$:

So we can identify $(\ell^p)^*$ with ℓ^q

$$x_n \xrightarrow[n \to +\infty]{} x \text{ weakly in } \ell^p$$
$$\Updownarrow \text{ R.R.Thm}$$
$$\forall y \in \ell^q(X) \ \left(\tfrac{1}{p} + \tfrac{1}{q} = 1\right),\ \sum_{k=1}^{+\infty} x_n^{(k)} y^{(k)} \xrightarrow[n \to +\infty]{} \sum_{k=1}^{+\infty} x^{(k)} y^{(k)}.$$

$\rightarrow \sum_{k=1}^{+\infty} x^{(k)} y^{(k)} = 0$ since $x = (0)_{k \in \mathbb{N}^*}$.

$\rightarrow \sum_{k=1}^{+\infty} x_n^{(k)} y^{(k)} = x_n^{(n)} y^{(n)} = e^{1/n} y^{(n)} \xrightarrow[n \to +\infty]{} 1 \times 0 = 0$

indeed, $y^{(n)} \xrightarrow[n \to +\infty]{} 0$ since $\sum_{k=1}^{+\infty} |y^{(k)}|^q < +\infty$

So : $\forall y \in \ell^q,\ \sum_{k=1}^{+\infty} x_n^{(k)} y^{(k)} \xrightarrow[n \to +\infty]{} 0 = \sum_{k=1}^{+\infty} x^{(k)} y^{(k)}$.

As a conclusion: $\boxed{\{x_n\}_{n\in\mathbb{N}^*} \text{ converges weakly to } x=(0)_{k\in\mathbb{N}^*} \text{ in } \ell^p}$. $\square$

3) If we have strong convergence of $\{x_n\}_{n\in\mathbb{N}}$ in ℓ^p, it should be to $x=(0)_{k\in\mathbb{N}}$. Strong cvgce in ℓ^p means that:

$$\|x_n - x\|_{\ell^p} \xrightarrow[n\to+\infty]{} 0$$

. Let's compute, then, $\|x_n - x\|_{\ell^p}$.

$$\|x_n - x\|_{\ell^p}^p = \sum_{k=1}^{+\infty} |x_n^{(k)} - x^{(k)}|^p \underset{\uparrow}{=} \sum_{k=1}^{+\infty} |x_n^{(k)}| = e^{1/n} \xrightarrow[n\to+\infty]{} 1 \neq 0$$

$\forall k\in\mathbb{N}^*, x^{(k)} = 0$.

As a conclusion: $\boxed{\{x_n\}_{n\in\mathbb{N}^*} \text{ doesn't converge strongly in } \ell^p}$. $\square$

Exercice 3: $X = (\mathcal{C}^0[0,1], \|.\|_\infty)$.

$$T : X \longrightarrow X, \quad u \longmapsto Tu \quad \text{with } (Tu)(t) = \int_0^t e^{\sin(s)} u(s)\,ds$$
$$(\forall u \in X).(\forall t \in [0,1])$$

1) • T is clearly linear by linearity of $\int$.

• T is well defined: Let $u \in X$,

$Tu : t \longmapsto \int_0^t e^{\sin(s)} u(s)\,ds$. We have $s \mapsto e^{\sin(s)} u(s)$ which is continuous on $[0,t]$ (since $\sin$ is $\mathcal{C}^0$ as well as $u \in X$). So it means that Tu is the integral function of a continuous function on an interval, so it is continuous: $(Tu) \in X$.
(Classical Fundamental Theorem of Calculus)

• ~~T is bounded since it is continuous (cf just before).~~

Bof

Correction: Set $g(s) := e^{\sin(s)}$. Notice that g is continuous on $[0,1]$ and $\lambda([0,1]) < +\infty$. Hence $g \in L^1([0,1])$.

- T is well defined: let $u \in X$. $ug \in L^1([0,1])$ since $ug \in \mathcal{C}^0([0,1])$.
Thus, by the 1st and 2nd Fundamental Theorems of Calculus we get:

$$(*) \quad T(u) \in AC([0,1])$$

From $(*)$ we have that, in particular, $T(u)$ is continuous on $[0,1]$. Hence $T(u) \in X$ and T is well defined.

Commentaire : juste la Regularity
Thm of Integral Function is OK:
$ug \in L^1([0,1]) \Rightarrow T(u) \in AC([0,1])$.

- Boundedness: For every $u \in X$, we have:

$$|T(u)(t)| \leq \int_0^t |g(s)||u(s)|\,ds \leq \|g\|_{L^1} \times \|u\|_\infty \quad \forall t \in [0,1].$$

As a consequence, we obtain:

$$(**) \quad \|Tu\|_\infty = \sup_{t \in [0,1]} |T(u)(t)| \leq \|g\|_1 \cdot \|u\|_\infty$$

This, together with the arbitrariness of $u \in X$, ensures that $T : X \to X$ is bounded. $\square$

2) T surjective ? let $v \in X$. Can we find $u \in X$ s.t : $T(u) = v$?

$v \equiv 1 \in X$ but $v(0) \neq 0$ so there is no $u \in X$ s.t $T(u) = v$.

As a conclusion, $\boxed{T \text{ isn't surjective}}$. □

3) We consider a bounded set B of $X = \mathcal{C}^0([0,1])$ and we prove that its image under T, $T(B) =: E$, is equibounded & equicontinuous.

Using Ascoli-Arzelà Theorem, this will ensure that:

$$\bar{E} = \overline{T(B)} \text{ is compact in } \mathcal{C}^0([0,1]).$$

And thus that T is a compact operator.

(1) Equi-boundedness: By (**) from (1) we have: $T \in \mathcal{L}(X)$ and $\|T\|_{\mathcal{L}(X)} \leq M < +\infty$. Since B is bdd in X, $\exists R > 0$ s.t $\|u\|_\infty \leq R$ for any $u \in B$. Hence, $\forall u \in B$,

$$\max_{t \in [0,1]} |Tu(t)| = \|Tu\|_\infty \leq M \|u\|_\infty \leq MR$$

Thus, $E = T(B)$ is equibdd.

(2) Equi-continuity: let $u \in B$. We have:

$$|T(u)(t_1) - (Tu)(t_2)| \leq MR\,(t_2 - t_1)$$

So $T(B)$ is equi-lipschitz. Hence Equi-continuous on $[0,1]$.

$\rightarrow$ we apply Ascoli Arzelà then:

$T(B) = E$ is precompact, i.e,

$\overline{T(B)} = \bar{E}$ is compact in $\mathcal{C}^0([0,1])$. $\boxed{T \text{ compact}}$ □