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Stochastic Dynamical Models A.A. 2024-2025

Exercise Session 1

September 26, 2024

Keywords: Markov property; classification of states; period.

Exercise 1. Let X_0 be a random variable with values in a countable set I. Let $(Z_n)_{n\geq 0}$ be a sequence of independent and identically distributed (i.i.d.) random variables with values in a set E. Assume that X_0 and the sequence $(Z_n)_{n\geq 0}$ are also independent. Given a measurable function $f : I \times E \to I$ and define inductively $\hookrightarrow \forall n \in \mathbb{N},\ X_0 \perp\!\!\!\perp Z_n$.

"Lemme de regroupement" {Préservation de l'$\perp\!\!\!\perp$ par passage dans 1 fct mesurable

$$X_{n+1} := f(X_n, Z_n), \quad n \geq 0.$$

Show that $(X_n)_{n\geq 0}$ is a Markov chain. $\longrightarrow$ First, show that $Z_n \perp\!\!\!\perp (X_0, \dots, X_n)$.
(since $(X_k)_{1\leq k\leq n}$ are built by $(Z_k)_{1\leq k\leq n-1}$ & X_0)

Exercise 2. Let $(Z_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with Bernoulli distribution $\mathcal{B}(1, q)$, $0 < q < 1$, and $(Y_n)_{n\geq 0}$ be a sequence of random variables defined by

$$Y_0 := 1, \quad Y_{n+1} := Y_n \left(2Z_{n+1} - 1\right).$$

$\longrightarrow$ Notice : $2Z_{n+1} - 1 \in \{-1, 1\}$.
So $Y_n \in \{-1, 1\}$.

Prove that $(Y_n)_{n\geq 0}$ is a Markov chain and find its transition matrix.
EX 1

Exercise 3. Let $(X_n)_{n\geq 0}$ be a Markov chain with state space $I = \{1, 2, 3\}$ and transition matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \end{pmatrix} \end{array}.$$

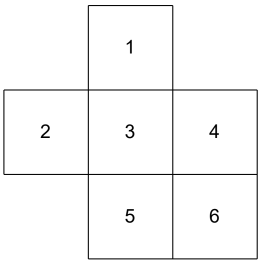
OR : $\mathbb{P}(X_3 = 2) = (\pi P^3)_2$

Suppose that the initial distribution λ of X_0 is uniform on I, i.e. $\lambda = (1/3, 1/3, 1/3)$.

1. Compute $\mathbb{P}\{X_3 = 2\}$. $\mathbb{P}(X_3 = 2) = \sum_{i=1}^{3} \mu(i)\, \mathbb{P}_i(X_3 = 2) = \sum_{i=1}^{3} \mu(i)\, P^3_{i,2} = \frac{1}{3}\left(P^3_{1,2} + P^3_{2,2} + P^3_{3,2}\right)$.
2. Compute $\mathbb{P}\{X_0 = 1, X_2 = 3\}$ and $\mathbb{P}\{X_0 + X_2 = 4\}$. (*) $\mathbb{P}(X_0 = 1, X_2 = 3) = \mathbb{P}(X_2 = 3 \mid X_0 = 1)\,\mathbb{P}(X_0 = 1)$.
$= (P^2)_{1,3} \cdot \pi_0 = \frac{4}{9} \cdot \frac{1}{3} = \frac{4}{27}$.
3. Compute $\mathbb{P}\{X_0 = 1, X_2 = 3 \mid X_0 + X_2 = 4\}$. $= \dfrac{\mathbb{P}(X_0 = 1, X_2 = 3, X_0 + X_2 = 4)}{\mathbb{P}(X_0 + X_2 = 4)} = \dfrac{\mathbb{P}(X_0 = 1, X_2 = 3)}{\mathbb{P}(X_0 + X_2 = 4)}$
4. Determine the joint distribution of (X_3, X_4) and compute $\mathbb{E}[X_3 X_4]$. (**)

$$\mathbb{E}[X_3 X_4] = \sum_{\substack{i,j \\ \in \{1,2,3\}^2}} i\,j\ \mathbb{P}((X_3, X_4) = (i,j)) = \sum_{i,j \in \{1,2,3\}^2} i\,j\ \frac{p_{ij}}{3}$$

Exercise 4. A mouse runs through the maze shown below. At each step it leaves the room it is in by choosing randomly and with equal probability one of the doors our of the room. Suppose there is always a door between two neighboring rooms.



Let X_n represent the position of the mouse at time n. Then,

(*) $\mathbb{P}(X_0 + X_2 = 4) = \sum_{k=1}^{3} \mathbb{P}(X_0 + X_2 = 4 \mid X_0 = k)\,\mathbb{P}(X_0 = k) = \frac{1}{3}\sum_{k=1}^{3} \mathbb{P}(X_2 = 4 - k \mid X_0 = k)$.

(**) $\forall i,j \in \{1,2,3\}^2 \quad \mathbb{P}(X_3 = i, X_4 = j) = \mathbb{P}(X_4 = j \mid X_3 = i)\,\mathbb{P}(X_3 = i) = p_{ij}/3$.

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

or we can notice that there are only 2 possible routes:
$1 \to 3 \to 4 \to 6$ & $1 \to 3 \to 5 \to 6$.

1. Give the transition matrix P for the Markov chain $(X_n)_{n\geq 0}$.

2. Suppose that the mouse starts in room 1, i.e. $X_0 = 1$, compute the probability that it arrives room 6 after 3 steps. $\mathbb{P}(X_3 = 6 \mid X_0 = 1) = (P^3)_{1,6} = 1/4$

Exercise 5. Let $(X_n)_{n\geq 0}$ be a Markov chain with state space $I = \{1,2,3,4,5\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1/6 & 1/3 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/10 & 4/5 & 0 & 1/10 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

1. Identify the communicating classes. $\to$ $1 \to 2 \to 4 \to 3 \to 1$ so $\mathcal{C}_1 = \{1,2,3,4\}$.
   5 is absorbing so $\mathcal{C}_2 = \{5\}$.

2. Find the period of each state.

All states from a same class have same period:

- <u>state 1:</u>
  - $1 \to 2 \to 1$ / $1 \to 3 \to 1$: $p_{11}^{(2)} > 0$.
  - $1 \to 2 \to 4 \to 3 \to 1$: $p_{11}^{(4)} > 0$
    $1 \to 2 \to 4 \to 2 \to 1$
    $1 \to 3 \to 4 \to 2 \to 1$
    $1 \to 3 \to 4 \to 3 \to 1$

  $\to$ $GCD(2,4) = 2$: period of states in $\mathcal{C}_1$ is 2.

- <u>state 5:</u> it's absorbing therefore its period is 1.