Stochastic Processes

Comparison of 2 proc. 1

Indistinguishable

Same trajectories w/ proba 1

⇓

Modification

t is fixed

⇓

Equivalent

$\forall t_1, \dots, t_m$, $(X_{t_1}, \dots, X_{t_m})$ and $(X'_{t_1}, \dots, X'_{t_m})$ same law

Construction of a stoch. proc. by only introducing a fdd system 2

[Consistency condition]

Kolmogorov Existence Theorem

Exit & Entrance Times 7

τ_B exit time from B.

1) B open

2) B closed ⊕ $(\mathcal{F}_t)_t$ right co

$\Rightarrow \tau_B$: stopping time

Properties of a stochastic process

TRAJECTORIES: 3

Continuity not preserved when modification of a continuous process

Kolmogorov continuity Theorem

MEASURABILITY: 4

Progressively Measurable

$(t, \omega) \longmapsto X_t(\omega)$

$(T \times \Omega, \mathcal{B}(T) \otimes \mathcal{F}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$

⇓

Measurable

$\forall u \in T$,

$(t, \omega) \longmapsto X_t(\omega)$

$([0;u] \times \Omega, \mathcal{B}([0;u]) \otimes \mathcal{F}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$

⇓

Adapted

Continuity / Right-Continuity

Continuity a.s / R-Continuity a.s

FILTRATION: 5

Standard process ⊕ A.S continuous

⇓

It has a continuous very

Stopping times 6

σ, τ 2 S.T

Stopped σ-algebra : $\mathcal{F}_\infty = \sigma\left(\bigcup_t \mathcal{F}_t\right)$

Given τ a stopping time,

$\sigma(\tau) - \sigma\{\{\tau < t\} \mid t \in T\} \subseteq \mathcal{F}_\tau = \{A \in \mathcal{F}_\infty \mid A \cap \{\tau \le t\} \in \mathcal{F}_t\}$

1) $\{\tau > t\}, \{\tau < t\}, \{\tau \geq t\} \in \mathcal{F}_t$.

2) τ is $\mathcal{F}_\tau$ - measurable.

3) $\sigma \wedge \tau$, $\sigma \vee \tau$ are S.T.

4) $\sigma \le \tau, \forall \omega \in \Omega \Rightarrow \mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.

5) $\mathcal{F}_{\sigma \wedge \tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$.

6) $X_t = \mathbb{1}_{\{t < \tau\}}$, $Y_t = \mathbb{1}_{\{\tau \le t\}}$

are PROGR. MEASURABLE PROCESSES.

Stopped A.V : $X_\sigma : \omega \in \Omega \mapsto X_{\sigma(\omega)}(\omega)$,

with X a meas. sto. proc. and

$\sigma : (\Omega, \mathcal{F}) \longrightarrow ([0;+\infty), \mathcal{B}([0;+\infty)))$ a R.V.

Let X be a prog. measurable process and τ an a.s finite stopping time. Then X_τ is measurable wrt the stopped σ-algebra $\mathcal{F}_\tau$.