NOÉ DEBROIS
MSc, 2024.

Politecnico di Milano.

LIFE INSURANCE:

LIFE INSURANCE MATHEMATICS: SURVIVAL MODELS

ASSUMPTIONS:

(2.1) $\forall y \geq 0, \forall t \geq 0$, we assume that: $\mathbb{P}[T_y \leq t] = \mathbb{P}[T_0 \leq y+t \mid T_0 > y]$.

Interpretation: starting from the cdf of T_0, the only additional information used to determine survival probabilites at age x & beyond is: SURVIVAL OR NOT.

(2.2) (Corollary) $\forall t, u \geq 0$, we have that: $\mathbb{P}[T_{x+t} \leq u] = \mathbb{P}[T_x \leq t+u \mid T_x > t]$.

---

1) Lifetime distributions F_x & F_0: $F_x(t) = \dfrac{F_0(x+t) - F_0(x)}{S_0(x)}$ (2.2).

PROOF: By assumption (2.1) we know: $\mathbb{P}[T_y \leq t] = \mathbb{P}[T_0 \leq y+t \mid T_0 > y]$.

So:
$$\mathbb{P}[T_y \leq t] = \frac{\mathbb{P}[T_0 \leq y+t \cap T_0 > y]}{\mathbb{P}[T_0 > y]} = \frac{\mathbb{P}[y < T_0 \leq y+t]}{\mathbb{P}[T_0 > y]}$$
$$= \frac{F_0(y+t) - F_0(y)}{S_0(y)} \quad \square$$

2) μ_x in terms of S_0: $\mu_x = -\dfrac{1}{S_0(x)} \dfrac{d}{dx} S_0(x)$ (2.9).

PROOF:
$$\mu_x = \lim_{h \to 0^+} \frac{\mathbb{P}[T_x \leq h]}{h} = \frac{1}{S_0(x)} \lim_{h \to 0^+} \frac{S_0(x)(1 - S_x(h))}{h}$$
$$= \frac{1}{S_0(x)} \lim_{h \to 0^+} \frac{S_0(x) - \overbrace{S_0(x) S_x(h)}^{= S_0(x+h)}}{h}$$
$$= \frac{1}{S_0(x)} \lim_{h \to 0^+} \frac{S_0(x) - S_0(x+h)}{x+h-x}$$
$$= \frac{-1}{S_0(x)} \frac{d}{dx} S_0(x) \quad \square$$

3) An expression of $S_x(t)$ in terms of force of mortality:

$$S_x(t) = \exp\left(-\int_0^t \mu_{x+s}\, ds\right) \quad (2.11).$$

PROOF: Remember that $\frac{d}{dx}\ln(h(x)) = \frac{1}{h(x)}\frac{d}{dx}h(x)$. So:

$$\mu_x = \frac{-1}{S_0(x)}\frac{d}{dx}S_0(x) = \frac{d}{dx}\ln(S_0(x)) \quad \text{so:}$$

$$\int_0^y \mu_x\, dx = -\int_{\ln(S_0(0))}^{\ln(S_0(y))} d\ln(S_0(x)) = -\left[\ln(S_0(y)) - \underbrace{\cancel{\ln(S_0(0))}}_{=0 \text{ since } S_0(0)=1}\right]$$

$$= -\ln(S_0(y)) \quad \text{i.e} \quad S_0(y) = \exp\left(-\int_0^y \mu_x\, dx\right).$$

We can do exactly the same using

$$\mu_{x+t} = \frac{f_x(t)}{S_x(t)} \quad \& \quad f_x(t) = -\frac{d}{dt}S_x(t). \quad \square$$

So, using the force of mortality expression, we can get the survival function. Example: Gombertz & Makeham's laws of mortality (cf notebook).

$$\mu_x \overset{def}{=} \lim_{h\to 0^+} \frac{\mathbb{P}(T_x \le h)}{h} = \lim_{h\to 0^+} \frac{\mathbb{P}(T_0 \le x+h \mid T_0 > x)}{h} \overset{(2.9)}{=} \frac{-1}{S_0(x)}\frac{d}{dx}S_0(x) = \frac{f_0(x)}{S_0(x)}$$

Actuarial Notations:

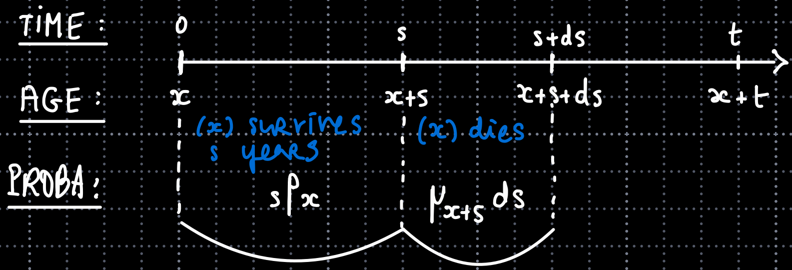
$$\mu_x = \lim_{h\to 0^+} \frac{{}_hq_x}{h} \overset{(2.9)}{=} \frac{-1}{{}_xp_0}\frac{d}{dx}\,{}_xp_0 = \frac{f_0(x)}{{}_xp_0}$$

$$\mu_{x+t} = \frac{-1}{{}_tp_x}\frac{d}{dt}\,{}_tp_x = \frac{f_x(t)}{{}_tp_x}$$

4) Death Rates & forces-of-mortality:

$$ {}_tq_x = \int_0^t {}_sp_x \, \mu_{x+s} \, ds \quad (2.21) $$

"PROOF" (with a drawing): (Fig 2.3) time-line diagram for ${}_tq_x$:



$\square$

LIFE INSURANCE MATHEMATICS: LIFE TABLES & SELECTION

Def: a survival model is a set of forces-of-mortality $\{\mu_y \mid y \geq x_0\}$ used to determine the survival probabilities of a specified group of people.

Consider the following "survival model": $\{\mu_y \mid y \geq x_0\}$.

Survival probabilities for (x) whose survival probabilities follow from this model:

$${}_t p_{x+u} = \exp\left(-\int_0^t \mu_{x+u+s}\, ds\right), \ \forall x \geq x_0, \ \forall u \geq 0.$$

Life table constructed from this survival model:

- l_{x_0} = arbitrary positive number, called the "radix".
- $\forall t \geq 0, \quad l_{x_0+t} \underset{\text{def.}}{=} l_{x_0} \times {}_t p_{x_0}$.

Life table: $\{l_y \mid y \geq x_0\}$.

HOW TO GET INFO FROM THE LIFE TABLE ?

5) Survival probabilities for (x):

$$\forall t \geq 0, \quad {}_t p_x = \frac{l_{x+t}}{l_x} \quad (3.1).$$

PROOF: $l_{x+t} = l_{x_0+x+t-x_0} \underset{\text{def.}}{=} l_{x_0} \times {}_{x+t-x_0} p_{x_0}$ *

$= \underbrace{l_{x_0} \times \overbrace{{}_{x-x_0} p_{x_0}}^{} \times {}_t p_x}$

$\underset{\text{def.}}{=} l_x \times {}_t p_x$. $\square$

* ${}_{x+t-x_0} p_{x_0} = \mathbb{P}[T_{x_0} \geq x+t-x_0]$

$= {}_{x-x_0} p_{x_0} \times {}_t p_x$

to live up to $x_0 + (x-x_0) = x$ & from x to $x+t$.

$d_x \overset{\text{not.}}{=} \ell_x - \ell_{x-1}$ (3.4)

One year mortality rates: $q_x = {}_1q_x = \dfrac{d_x}{\ell_x}$

Deferred mortality rates: ${}_{t|u}q_x = \dfrac{\ell_{x+t} - \ell_{x+t+u}}{\ell_x}$

About fractional age assumptions...

PROBLEM: In practice, often $\{\ell_0, \ell_1, \ldots\}$ is the only available info. about the survival model. While we want to compute ${}_tp_x$ for non-$\mathbb{N}$ x &/or t.

SOLUTION: We make a fractional age assumption.

Hereafter, we consider the following fractional age assumptions:
- → Uniform Distribution of Deaths;
- → Constant Force of Mortality.

(1) Uniform Distribution of Deaths:

6) $R_x \overset{\text{def}}{=} T_x - K_x$: fraction lived by (x) in his year-of-death.

UDD 1: (useful for computations) for any integer x and any $0 \leq s < 1$, assume that:

$${}_sq_x = s \times q_x \quad (3.6)$$

$\boxed{\text{UDD2:}}$ (uniform distribution of deaths) for any integer age x, assume that:

$$R_x \sim \mathcal{U}(0,1)$$

& moreover: R_x is $\perp\!\!\!\perp$ of K_x.

$\boxed{\text{THM:}}$ UDD 1 $\iff$ UDD 2.

PROOF:

HARDEST

$\boxed{\text{UDD 1} \Rightarrow \text{UDD2}:}$ Idea = Compute the cdf of R_x & show that: $\mathbb{P}[R_x \leq s] = s$, $\forall s \in [0,1]$.

$$\text{(CDF)}: \mathbb{P}[R_x \leq s] = \sum_{k=0}^{+\infty} \mathbb{P}\Big[\underbrace{(R_x \leq s) \cap (K_x = k)}\Big]$$

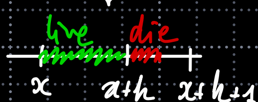
$\begin{cases} K_x = k \\ 0 \leq R_x \leq s \iff \end{cases}$ $k \leq T_x \leq k+s$

disjoint events since K_x cannot be equal to both k & k'.

$$= \sum_{k=0}^{+\infty} \mathbb{P}[k \leq T_x \leq k+s] = \sum_{k=0}^{+\infty} {}_kp_x \cdot {}_sq_{x+k}$$

$$= \sum_{k=0}^{+\infty} {}_kp_x \times (s \times q_{x+k}) = s \underbrace{\sum_{k=0}^{+\infty} {}_kp_x \times q_{x+k}}$$

Deferred Mortality Rate



$$= s \times 1 = s.$$

UDD1

cf just before

$$(\perp\!\!\!\perp): \mathbb{P}[(R_x \leq s) \cap (K_x = k)] = \mathbb{P}[k \leq T_x \leq k+s] = s \cdot {}_kp_x \cdot q_{x+k}$$

$$= \underbrace{\mathbb{P}(R_x \leq s)}_{s} \times \underbrace{\mathbb{P}(K_x = k)}_{{}_kp_x \times q_{x+k}} \qquad \square$$

$\boxed{\text{UDD2} \Rightarrow \text{UDD1}:}$

$$
\begin{aligned}
{}_s q_x &= \mathbb{P}[T_x \leq s] \\
&= \mathbb{P}[(K_x = 0) \cap (R_x \leq s)] \\
&= \mathbb{P}[K_x = 0] \times \mathbb{P}[R_x \leq s] \\
&= s \times \mathbb{P}[K_x = 0] = s \times ({}_0 p_x \times q_x) \\
&= s \times q_x \quad \square .
\end{aligned}
$$

UDD2

Curtate Future Lifetime. (2.6)

(2) Constant Force of Mortality:

(7) • CFM assumption: for any integer age x, and any $0 \leq s < 1$, assume that: $\mu_{s+x} = \mu_x^*$ (piecewise deterministic fct).

• Value of μ_x^*: $\mu_x^* = -\ln(p_x)$

PROOF: $p_x = {}_1p_x = \exp\left(-\int_0^1 \mu_{x+s}\, ds\right) = \exp(-\mu_x^*)$. $\square$

• For any integer age x, and any $0 \leq r < 1$, we have: ${}_r p_x = (p_x)^r$.

PROOF: ${}_r p_x = \exp\left(-\int_0^r \mu_{x+s}\, ds\right) = \exp(-\mu_x^* \times r) = (p_x)^r$. $\square$

• For any integer age x, any $r, t > 0$, such that $r + t < 1$, we obtain:

$$ {}_r p_{x+t} = (p_x)^r \quad (3.12). $$

• Sometimes, UDD $\sim$ CFM:

$$\begin{cases} q_x = 1 - p_x \underset{CFM}{=} 1-\exp(-\mu_x^*) \underset{\mu_x^*\to 0}{\sim} \mu_x^* \\ {}_tq_x = 1 - {}_tp_x \underset{CFM}{=} 1-\exp(-\mu_x^* \times t) \underset{\mu_x^* t\to 0}{\sim} \mu_x^* t \quad (t\in(0,1)) \end{cases}$$

So CFM $\Rightarrow$ $\underbrace{{}_tq_x \sim t \times q_x}_{\text{UDD assumption}}$

⚠ Remember that μ_x^* IS INDEED close to zero, so we do not make a huge mistake here.

NOW:

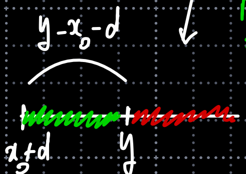
HOW TO BUILD A SELECT LIFE TABLE ?

(1) <u>After the selection period:</u>

- $l_{x_0} > 0$: arbitrary
- d : select period
- $y \geq x_0 + d$: why? Because, after d years, there is no difference; the only criteria = your age.

<u>DEF:</u> $l_y = \underbrace{{}_{y-x_0-d}p_{x_0+d}}_{} \cdot \underbrace{l_{x_0+d}}_{}$

"proba to survive till $x_0+d + [y-x_0-d] = y$"   "those who survived till x_0+d"



$$= \left({}_{x-x_0-d}p_{x_0+d} \times {}_{y-x}p_x\right) \times l_{x_0+d}$$

$$= {}_{y-x}p_x \times \underbrace{{}_{x-x_0-d}p_{x_0+d} \times l_{x_0+d}}_{l_x}$$

$$\longrightarrow \boxed{l_y = l_x \; {}_{y-x}p_x}.$$

(2) <u>Before the end of the selection period:</u>

→ when it is less than d years after the health check.

Let $x \geq x_0$, $t \in [0,d]$, $\underbrace{\ell_{[x]+d}}_{\text{You are } x+d \text{ y.o \& bought the policy when } x \text{ y.o.}} = \dfrac{\ell_{x+d}}{{}_{d-t}p_{[x]+t}} \quad (\geq \ell_{x+d})$.

EXAMPLE 3.13

LIFE INSURANCE MATHEMATICS : LIFE INSURANCE BENEFITS

ASSUMPTIONS: Technical basis = a set of assumptions used for performing life insurance or pension calculations. In this chapter:

- We use the standard ultimate model:

$$\mu_x = 0{,}00022 + 2{,}7 \cdot 10^{-6} \times 1{,}124^x .$$

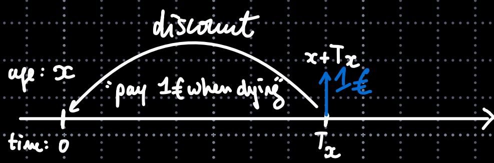
- constant interest.

<u>CONVENTION:</u> time = 0 now; time unit = 1 year.

NOTATIONS:

- i = <u>annual</u> rate of interest
  - ↳ $i^{(p)}$: nominal rate of interest, compounded p times per year : $1+i = \left(1+\frac{i^{(p)}}{p}\right)^p$;
- v : yearly discount factor : $v = \frac{1}{1+i}$ $(= e^{-\underbrace{\delta}_{\text{force of interest.}}})$;
- d : discount rate per year : $d = 1 - v = iv$ $(= 1 - e^{-\delta})$
  - ↳ $d^{(p)}$ = nominal discount rate compounded p times per year : $d^{(p)} = p\left(1 - v^{1/p}\right) = i^{(p)} v^{1/p}$.
- Cash flow notations:
  - cash flow w/ payment c @ time t : (c, t);
  - $(\alpha c, t)$ is often denoted $\alpha(c, t)$;
  - c & t may be deterministic or random.

7 contracts :



1- WHOLE LIFE INSURANCE ($\mathscr{C}^0$ CASE)

2- ———————————— (ANNUAL CASE)

3- TERM INSURANCE ($\mathscr{C}^0$ CASE)

4- ———————————— (ANNUAL CASE)

5- PURE ENDOWMENT

6- ENDOWMENT INSURANCE ($\mathscr{C}^0$ CASE)

7- ———————————— (ANNUAL CASE)

"Fair price" of the contract (no arbitrage)

OR "EPV" : "Expected Present Value".

METHOD (ALWAYS THE SAME) : Benefit Cash Flows $\rightsquigarrow$ Present Value $\rightsquigarrow$ Actuarial value

1- $(1, T_x) \rightsquigarrow Z = v^{T_x} = e^{-\delta T_x} \rightsquigarrow \mathbb{E}[e^{-\delta T_x}] = \int_0^{+\infty} e^{-\delta t} \, {}_tp_x \, \mu_{x+t} \, dt$

2- $(1, K_x+1) \rightsquigarrow Z = v^{K_x+1} \rightsquigarrow A_x \overset{not}{=} \mathbb{E}[v^{K_x+1}] = \sum_{k=0}^{+\infty} v^{k+1} \, {}_{k|1}q_x$

3- $(\mathbb{1}_{\{T_x \le n\}}, T_x) \rightsquigarrow Z = e^{-\delta T_x} \mathbb{1}_{\{T_x \le n\}} \rightsquigarrow \mathbb{E}[Z] = \int_0^n e^{-\delta t} \, {}_tp_x \, \mu_{x+t} \, dt$

(discount only on $\{\omega / T_x(\omega) \le n\}$)

4- $(\mathbb{1}_{\{K_x+1 \le n\}}, K_x+1) \rightsquigarrow Z = v^{K_x+1} \mathbb{1}_{\{K_x+1 \le n\}} \rightsquigarrow \mathbb{E}[Z] = \sum_{k=0}^{n-1} v^{k+1} \, {}_{k|1}q_x$

5- $(\mathbb{1}_{\{T_x > n\}}, n) \rightsquigarrow Z = v^n \mathbb{1}_{\{T_x > n\}} \rightsquigarrow \mathbb{E}[Z] = v^n \, {}_np_x$

6- $(1, \min(T_x, n)) = \underbrace{(\mathbb{1}_{\{T_x \le n\}}, T_x)}_{\text{TERM INSURANCE}} + \underbrace{(\mathbb{1}_{\{T_x > n\}}, n)}_{\text{PURE ENDOWMENT}} \rightsquigarrow Z = v^{\min(T_x, n)} \rightsquigarrow \mathbb{E}[Z] = \int_0^n e^{-\delta t} \, {}_tp_x \, \mu_{x+t} \, dt + v^n \, {}_np_x$

7- $(1, \min(K_x+1, n)) = (\mathbb{1}_{\{K_x+1 \le n\}}, K_x+1) + (\mathbb{1}_{\{K_x+1 > n\}}, n) \rightsquigarrow Z = v^{\min(K_x+1, n)} \rightsquigarrow \sum_{k=0}^{n-1} v^{k+1} \, {}_{k|1}q_x + v^n \, {}_np_x$