$\boxed{\text{Ex I – TD 8 1/3 – PART II :}}$

Let's consider $\beta \in \mathcal{M}(m \times m)$ and 2 deterministic fcts bdd & measurable

$$\alpha : [0,T] \to \mathbb{R}^m \quad , \quad \sigma : [0,T] \to \mathcal{M}(m \times d) .$$

We consider the following SDE for m-dim. process $X(t)$ driven by a $\mathcal{F}^0$ & standard d-dimensional BM $B(t)$:

$$\begin{cases} dX(t) = \alpha(t)dt + \beta X(t)\,dt + \sigma(t)dB(t) \;,\; 0 \leq t \leq T \\ X(0) = x_0 \in \mathbb{R}^m \end{cases}$$

1) **VERIFICATION OF ASSUMPTIONS (A) :** In our case : $\begin{cases} b(t,x) = \alpha(t) + \beta X(t) \\ \sigma(t,x) = \sigma(t) \\ x_0 \in \mathbb{R}^m \end{cases}$

- **MEASURABILITY:**
  - b : ok since it's the Σ of 2 meas. fcts.
  - σ : it is measurable by assumption.

- **SUB_LINEAR GROWTH :** Let $x \in \mathbb{R}^m$, $t \in [0,T]$,
  - $|b(t,x)| = |\alpha(t) + \beta x| \leq |\alpha(t)| + |\beta|\,|x| \leq M_b(1 + |x|)$ with :

$$M_b = \max\left(\sup_{t\in[0,T]} |\alpha(t)|, \|\beta\|_\infty\right)$$

  - $|\sigma(t,x)| = |\sigma(t)| \leq M_\sigma \left(\underbrace{1 + |x|}_{\geq 1}\right)$ with :

$$M_\sigma = \sup_{t\in[0,T]} |\sigma(t)| .$$

Notation :

$A \in \mathcal{M}(n \times d)$,

$$|A|^2 = \sum_{i=1}^{n}\sum_{j=1}^{d} |A_{i,j}|^2$$

$$\|A\|_\infty = \max_{i,j} |A_{i,j}|$$

- **LIPSCHITZIANITY :** Let $x, y \in \mathbb{R}^m$, $t \in [0,T]$,
  - $|b(t,x) - b(t,y)| = |\beta(x-y)| \leq L_b |x-y|$ with $L_b = \|\beta\|_\infty$.
  - $|\sigma(t,x) - \sigma(t,y)| = |\sigma(t) - \sigma(t)| = 0 \leq L_\sigma |x-y|$ with $L_\sigma \geq 0$ (any positive nb.).

$\longrightarrow$ $\boxed{M = M_b \vee M_\sigma \quad \text{and} \quad L = L_b}$ $\square$

11) SOLVE THE SDE FOR $0 \le t \le T$ IN TERMS OF THE INITIAL DATUM $x_0 \in \mathbb{R}^m$.

Use variation of constant like in 2):

(1) Homogeneous equation associated: $\begin{cases} dY_{(t)} = \beta Y_{(t)} dt \\ Y(0) = x_0 \end{cases} \Rightarrow Y_{(t)} = x_0 e^{\beta t}$ where $e^{\beta t} = \sum_{n=0}^{+\infty} \frac{(\beta t)^n}{n!}$ (exp. matrix).

(2) We look for solutions of the form: $\begin{cases} X(t) = e^{\beta t} \tilde{X}(t) \\ X(0) = \tilde{X}(0) = x_0 \end{cases}$ (i.e. $\begin{cases} \tilde{X}(t) = e^{-\beta t} X(t) \\ \tilde{X}(0) = x_0 \end{cases}$).

Let's compute the stochastic differential of $\tilde{X}$:

$$d\tilde{X}_{(t)} = -\beta e^{-\beta t} X(t)\, dt + e^{-\beta t} dX_t$$

$$= \cancel{-\beta e^{-\beta t} X(t)\, dt} + e^{-\beta t}\left(\alpha(t) dt + \cancel{\beta X(t) dt} + \sigma(t)\, dB_{(t)}\right)$$

$$= e^{-\beta t}\left(\alpha(t)\, dt + \sigma(t)\, dB_{(t)}\right)$$

$$\rightarrow \tilde{X}(t) = x_0 + \int_0^t e^{-\beta r} \alpha(r) dr + \int_0^t e^{-\beta r} \sigma(r) dB_r$$

so that: $$\boxed{X(t) = e^{\beta t}\left(x_0 + \int_0^t e^{-\beta r}\alpha(r) dr + \int_0^t e^{-\beta r} \sigma(r) dB_r\right)}.$$

our pathwise unique strong solution □

12) $r \mapsto e^{-\beta r}\alpha(r) \in M^1_{loc}[0,T]$ since it is bdd on the closed interval $[0,T]$.

$r \mapsto e^{-\beta r}\sigma(r) \in M^2_{loc}[0,T]$ ——————————————.

So X is an Itô Process.

- X is Itô since it is the solution of a SDE.
- X has finite variance & mean (in particular, $X(t) \in L^2(\Omega)$, $\forall t \in [0,T]$).
- Moreover, X is gaussian since $r \mapsto e^{-\beta r}\sigma(r)$ is deterministic. □

13) $\mu_t = \mathbb{E}[X_t] = e^{\beta t}\left(x_0 + \int_0^t e^{-\beta r}\alpha(r) dr + 0\right)$ since $r \mapsto e^{-\beta r}\sigma(r) \in M^2[0,T]$ so $\left(\int_0^t e^{-\beta r}\sigma(r) dB_r\right)_t$ is a mg w/ null expectation. □

14) Does μ_t satisfy a differential equation? which one?

1st METHOD

$$\frac{d\mu_t}{dt} = \beta e^{\beta t}\underbrace{\left(x_0 + \int_0^t e^{-\beta r}\alpha(r)\,dr\right)}_{\mu_t} + e^{\beta t}\left(e^{-\beta t}\alpha(t)\right) = \beta\mu_t + \alpha(t)$$

$$\boxed{\frac{d\mu_t}{dt} = \beta\mu_t + \alpha(t)}\ . \ \square$$

OR:

2nd METHOD

Consider the SDE: $X(t) = x_0 + \int_0^t \alpha(r)\,dr + \int_0^t \beta X(r)\,dr + \int_0^t \sigma(r)\,dB_r$

Then take the expectation: $\mu_t = x_0 + \int_0^t \alpha(r)\,dr + \mathbb{E}\left[\int_0^t \beta X(r)\,dr\right] + 0$

$$\rightarrow \mu_t = x_0 + \int_0^t \alpha(r)\,dr + \int_0^t \beta\, \underbrace{\mathbb{E}[X_r]}_{\mu_r}\,dr$$

$$\rightarrow \boxed{\mu_t = x_0 + \int_0^t \alpha(r)\,dr + \int_0^t \beta\mu_r\,dr}$$

i.e: $d\mu_t = \alpha(t)\,dt + \beta\mu_t\,dt$ i.e

$$\boxed{\frac{d\mu_t}{dt} = \alpha(t) + \beta\mu_t}\ .$$

$\square$