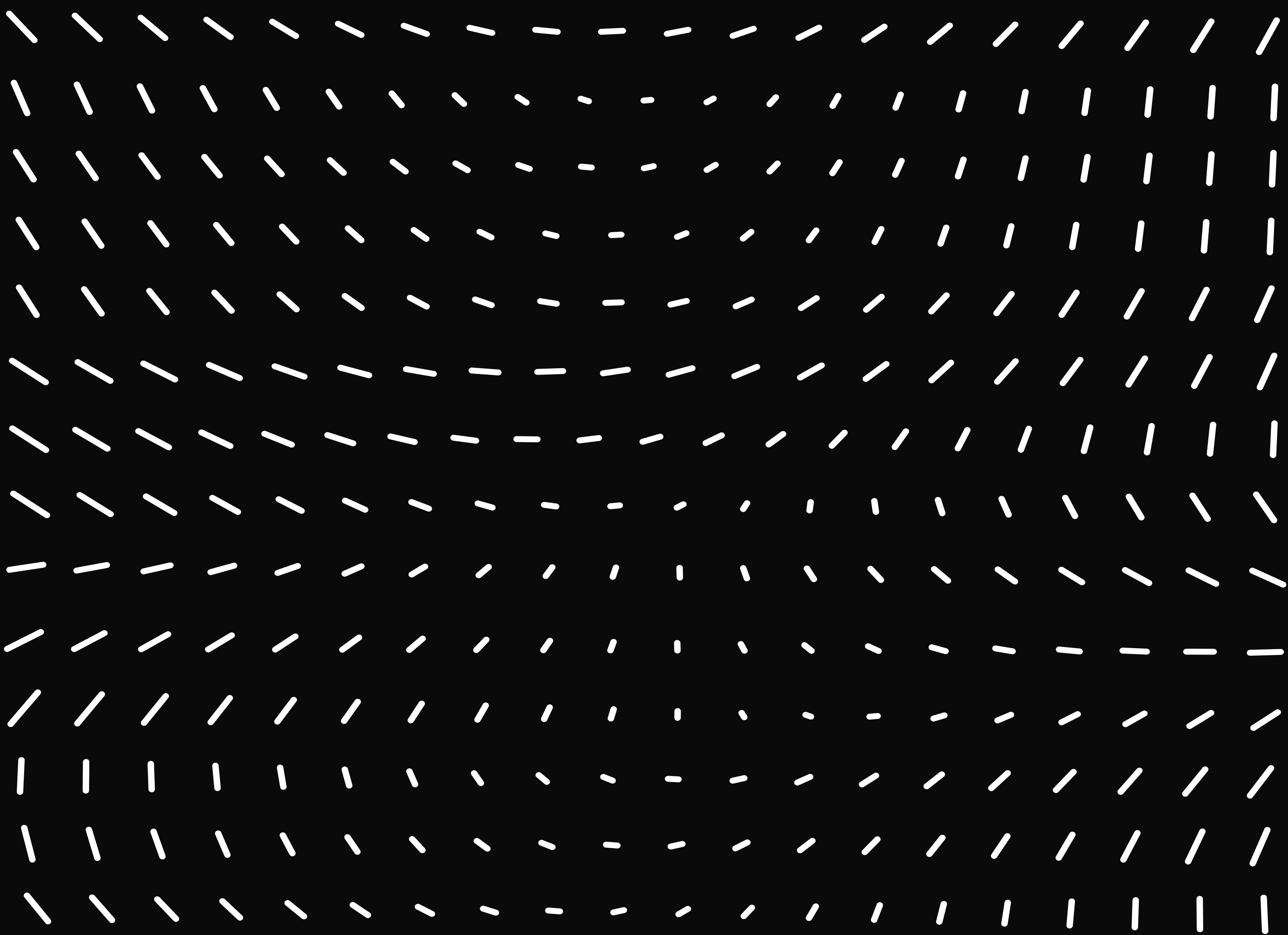
MiDA 1

Exercise sessions



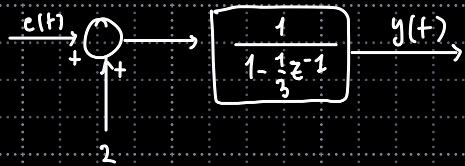
Exercise session 3 :

$y(t) = \frac{1}{3} y(t-1) + e(t) + 2$, $e(t) \sim WN(1,1)$.

Operator delay : $y(t) = \frac{1}{3} z^{-1} y(t) + e(t) + 2$

i.e $y(t)\left(1 - \frac{1}{3} z^{-1}\right) = e(t) + 2$

i.e $\boxed{y(t) = \frac{1}{1-\frac{1}{3}z^{-1}}\left(e(t)+2\right)}$



$W(z) = \frac{1}{1-\frac{1}{3}z^{-1}}$

$z_{pole} = \frac{1}{3} \in$ unit circle

$\Downarrow$

$y(t)$ is stationary

(since $W(z)$: asymptotically stable
$e(t)$: stationary since WN)

y stationary
$\downarrow$

- $\mathbb{E}[y(t)] = \frac{1}{3}\mathbb{E}[y(t)] + \mathbb{E}[e(t)] + 2$

$= \frac{1}{3}\mathbb{E}[y(t)] + 1 + 2$

$= \frac{1}{3}\mathbb{E}[y(t)] + 3$ i.e $\mathbb{E}[y(t)] \times \frac{2}{3} = 3$

so : $\boxed{\mathbb{E}[y(t)] = \frac{9}{2}}$.

- **Cov. function:** $\mathbb{E}[e(t)] = 1$

$\mathbb{E}[e(t)e(t-\tau)] \neq 0$ in general

⚠ It is $\mathbb{E}\big[\underbrace{(e(t)-1)}_{\tilde{e}(t)}\underbrace{(e(t-\tau)-1)}_{\tilde{e}(t-\tau)}\big] = 0$

Let's define: $\boxed{\forall t,\ \tilde{e}(t) = e(t) - \mathbb{E}[e(t)] = e(t) - 1}$

And define: $\boxed{\forall t,\ \tilde{y}(t) = y(t) - \mathbb{E}[y(t)] = y(t) - \frac{9}{2}}$

These are called "debias" or "detrend" proc.

- $r_y(\tau) = \mathbb{E}\Big[\big(y(t) - \mathbb{E}[y(t)]\big)\big(y(t-\tau) - \mathbb{E}[y(t-\tau)]\big)\Big] = \mathbb{E}[\tilde{y}(t)\tilde{y}(t-\tau)]$

- $\tilde{e}(t) = e(t) - 1.$ $\begin{cases} \mathbb{E}[\tilde{e}(t)] = 0 \\ \mathbb{E}[\tilde{e}(t)^2] = \mathbb{E}[(e(t)-1)^2] = \mathrm{Var}(e(t)) = 1 \end{cases}$

- $y(t) = \tilde{y}(t) + \cancel{\frac{9}{2}} = \frac{1}{3}y(t-1) + e(t) + 2$

$$= \frac{1}{3}\left(\tilde{y}(t-1) + \frac{9}{2}\right) + \left(\tilde{e}(t) + 1\right) + 2$$

$$= \frac{1}{3}\tilde{y}(t-1) + \frac{3}{2} + 3 + \tilde{e}(t)$$

$$= \frac{1}{3}\tilde{y}(t-1) + \tilde{e}(t) + \cancel{\frac{9}{2}}$$

So: $\boxed{\tilde{y}(t) = \frac{1}{3}\tilde{y}(t-1) + \tilde{e}(t)}$ This AR(1) process.

So we use these results of the Course on AR(1) model:
(cf page 12 my notes of MIDA 1)

Yule-Walker Equations:
$$\begin{cases}\gamma_{\tilde y}(0) = \dfrac{1}{1-\left(\frac{1}{3}\right)^2}\cdot 1 = \dfrac{1}{8/9} = 9/8 \\[2mm] \gamma_{\tilde y}(\tau) = \dfrac{1}{3}\gamma_{\tilde y}(\tau-1) \quad , \forall|\tau|\geq 1\end{cases}$$

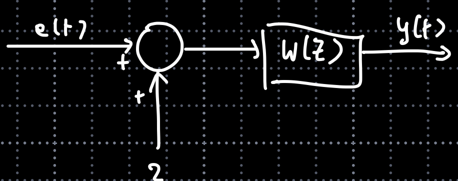
Let's show that $\gamma_{\tilde y}(\tau) = \gamma_y(\tau) \quad , \forall\tau$:

$$\begin{aligned}\gamma_{\tilde y}(\tau) &= \mathbb{E}\Big[\big(\tilde y(t) - \mathbb{E}[\tilde y(t)]\big)\big(\tilde y(t-\tau) - \mathbb{E}[\tilde y(t-\tau)]\big)\Big] \\ &= \mathbb{E}\Big[\big(\tilde y(t)\big)\big(\tilde y(t-\tau)\big)\Big] = \mathbb{E}\Big[\big(y(t)-\tfrac{9}{2}\big)\big(y(t-\tau)-9/2\big)\Big] \\ &= \mathbb{E}\Big[\big(y(t) - \mathbb{E}[y(t)]\big)\big(y(t-\tau) - \mathbb{E}[y(t-\tau)]\big)\Big] \\ &= \gamma_y(\tau)\end{aligned}$$

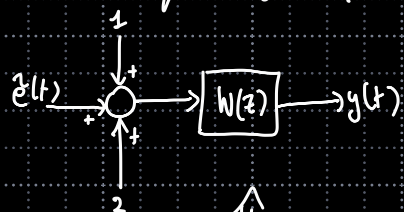
Conclusion:

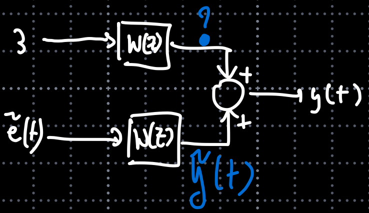
$$\boxed{\gamma_y(\tau) = \begin{cases} 9/8 & \text{if } \tau = 0 \\ \frac{1}{3}\gamma_y(\tau-1) & \text{otherwise}\end{cases}}$$

$$\begin{cases} y(t) = \frac{1}{3}y(t-1) + e(t) + 2 & , e(t) \sim WN(1,1) \\ W(z) = \dfrac{1}{1-\frac{1}{3}z^{-1}} \end{cases}$$

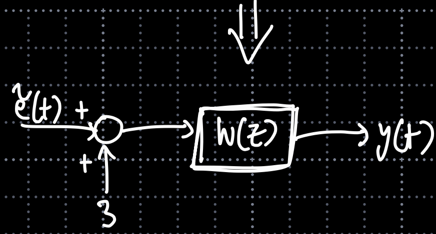


we have defined $\tilde e(t) = e(t) - 1 \sim WN(0,1)$.





$\Longleftrightarrow$



$$\boxed{\tilde{y}(t) = W(z)\tilde{e}(t)}$$

TO FIND WHERE IN THE LECTURES

Page 18 of: LECTURE 4.

<u>Final value theorem:</u>

$$y(\infty) = \lim_{z \to 1} \cancel{\frac{z-1}{z}} \cdot W(z)\, U \cdot \cancel{\frac{z}{z-1}} = \underline{\underline{W(1) \cdot U}}$$

Theorem of Gain

Where does it come from??

$$\mathbb{E}[y(t)] = W(1) \cdot \mathbb{E}[u(t)] = \frac{1}{1-\frac{1}{3}} \cdot 3 = \frac{9}{2}.$$

$$u(t) = \tilde{e}(t) + 3$$

$$y(t) = 0.5\, e(t-1) + 0.5\, e(t-2) + e(t), \quad e(t) \sim WN(2,1).$$

- <u>OPERATOR DELAY:</u>

$$y(t) = \left(0.5 z^{-1} + 0.5 z^{-2} + 1\right) e(t)$$

$$\boxed{W(z) = 0.5 z^{-1} + 0.5 z^{-2} + 1 = \frac{z^2 + 0.5z + 0.5}{z^2}}$$

- <u>UNBIASED / DETREND PROCESSES:</u>

$$\mathbb{E}[e(t)] = 2 \implies \boxed{\tilde{e}(t) = e(t) - 2 \sim WN(0,1)}$$

$$\mathbb{E}[y(t)] = 0.5 \times 2 + 0.5 \times 2 + 2 = 1 + 1 + 2 = 4$$

$$\Rightarrow \boxed{\tilde{y}(t) = y(t) - 4} \qquad \left(\mathbb{E}[\tilde{y}(t)] = 0\right)$$

so we will be able to use the formula for ARMA(m,n) (page 13 of my notes)

cf. page 14 of my notes.

- WE KNOW $\gamma_y(\tau) = \gamma_{\tilde{y}}(\tau)$ $(\forall \tau)$ SO WE WILL COMPUTE $\gamma_{\tilde{y}}(\tau)$:

$$y(t) = \tilde{y}(t) + \cancel{4} = 0.5\, e(t-1) + 0.5\, e(t-2) + e(t)$$

$$= 0.5\left(\tilde{e}(t-1) + 2\right) + 0.5\left(\tilde{e}(t-2) + 2\right) + \tilde{e}(t) + 2$$

$$= 0.5\,\tilde{e}(t-1) + 0.5\,\tilde{e}(t-2) + \tilde{e}(t) + \cancel{4}$$

THE EQUATION ON y vs. e IS THE SAME AS THE ONE ON $\tilde{y}$ vs. $\tilde{e}$.

$$\rightarrow \boxed{\tilde{y}(t) = 0.5\,\tilde{e}(t-1) + 0.5\,\tilde{e}(t-2) + \tilde{e}(t)}$$

$$\downarrow$$

$$\boxed{\text{ARMA}(2,0) \text{ PROC.}}$$

$$\downarrow$$

"YULE WALKER FOR ARMA(m,n)" (13/30 MIDA1)

$$\begin{cases} \gamma_{\tilde{y}}(0) = (1^2 + 0.5^2 + 0.5^2) \cdot 1 = 1.5 \\ \gamma_{\tilde{y}}(1) = (c_0 c_1 + c_1 c_2)\, \lambda^2 = 0.75 \\ \gamma_{\tilde{y}}(2) = 0.5 \; ; \quad \gamma_{\tilde{y}}(\tau) = 0 \quad \forall |\tau| \geq 3. \end{cases}$$