JUMP PROCESSES :

DEF: Poisson Process ($T_n - T_{n-1} \sim Exp(\lambda)$)

1. $N_0 = 0$ a.s.
2. Independent Increments: $\forall 0 \leq s \leq t < +\infty$, $N_t - N_s \perp\!\!\!\perp N_s - N_0$.
3. $\forall 0 \leq s \leq t < +\infty$, $N_t - N_s \sim \mathcal{P}(\lambda(t-s))$.

1) Thm: $M = \{M_t\}_{t \geq 0}$ defined by $\forall t \geq 0$, $M_t = N_t - \lambda t$ (where N is a Poisson Process) is a martingale called "compensated Poisson Process".

PROOF: $0 \leq s \leq t$, $\mathbb{E}[M_t | \mathcal{F}_s] = \mathbb{E}[M_t - M_s + M_s | \mathcal{F}_s] = \mathbb{E}[M_t - M_s | \mathcal{F}_s] + \mathbb{E}[M_s | \mathcal{F}_s]$

M_s is $\mathcal{F}_s$-meas.

$$= \mathbb{E}[N_t - N_s - \lambda(t-s) | \mathcal{F}_s] + M_s$$

$$= M_s - \lambda(t-s) + \mathbb{E}[N_t - N_s | \mathcal{F}_s]$$

$N_t - N_s \perp\!\!\!\perp \mathcal{F}_s$

$$= M_s - \lambda(t-s) + \lambda(t-s) = M_s . \square$$

Rmk: $\{N_t\}_{t \geq 0}$ could not be a mg since, by def, it is "expected to ↑".

2) MG characterisation of the intensity: Let N be a jump process. Suppose $\lambda = \{\lambda_t\}_{t \geq 0}$ such that:

$N_{t \wedge T_n} - \int_0^{t \wedge T_n} \lambda_s ds$ is a MG. Then: λ is the intensity of N.

PROOF: Let $t \geq 0$, by the cst expectation of MG property:

$$\mathbb{E}\left[N_{t \wedge T_n} - \int_0^{t \wedge T_n} \lambda_s ds\right] = 0 \quad \text{i.e}$$

$$\mathbb{E}\left[N_{t \wedge T_n}\right] = \mathbb{E}\left[\int_0^{t \wedge T_n} \lambda_s ds\right] \quad \text{i.e}$$

$$\mathbb{E}\left[\int_0^{t \wedge T_n} dN_s\right] = \mathbb{E}\left[\int_0^{t \wedge T_n} \lambda_s ds\right]$$

According to the "MONOTONE CLASS THM" (cf. txtbook Brémaud), if we can prove a result for such a process : $C_s = \mathbb{1}_A \mathbb{1}_{(r,t]}(s)$ ($A \in \mathcal{F}_r$, $s \in (r,t]$), then the result is true for any predictable process.

So, using the fact that $N_{t\wedge\tau_n} - \int_0^{t\wedge\tau_n} \lambda_s ds$ is a MG, we can write: $\forall A \in \mathcal{F}_r$, $\mathbb{E}\left[\mathbb{1}_A\left(N_{t\wedge\tau_n} - \int_0^{t\wedge\tau_n} \lambda_s ds\right)\right] = \mathbb{E}\left[\mathbb{1}_A\left(N_{r\wedge\tau_n} - \int_0^{r\wedge\tau_n}\lambda_s ds\right)\right]$ (eq. def of MG). So: $\forall A \in \mathcal{F}_r$,

$$\mathbb{E}\left[\mathbb{1}_A\left(N_{t\wedge\tau_n} - N_{r\wedge\tau_n}\right)\right] = \mathbb{E}\left[\mathbb{1}_A\left(\int_0^{t\wedge\tau_n}\lambda_s ds - \int_0^{r\wedge\tau_n}\lambda_s ds\right)\right]$$

i.e $\mathbb{E}\left[\mathbb{1}_A \int_{r\wedge\tau_n}^{t\wedge\tau_n} dN_s\right] = \mathbb{E}\left[\mathbb{1}_A \int_{r\wedge\tau_n}^{t\wedge\tau_n} \lambda_s ds\right]$

i.e $\mathbb{E}\left[\int_0^{t\wedge\tau_n} \mathbb{1}_A \mathbb{1}_{(r,t]}(s)\, dN_s\right] = \mathbb{E}\left[\int_0^{t\wedge\tau_n} \mathbb{1}_A \mathbb{1}_{(r,t]}(s)\, \lambda_s ds\right]$

i.e (Monotone Class Thm from Brémaud's textbook) $\mathbb{E}\left[\int_0^{t\wedge\tau_n} C_s dN_s\right] = \mathbb{E}\left[\int_0^{t\wedge\tau_n} C_s \lambda_s ds\right]$ for any predictable (non-neg.) process C. $\square$

3) <u>Integration thm:</u> Assume that N has intensity λ.

Then: $\left(N_{t\wedge\tau_n} - \int_0^{t\wedge\tau_n}\lambda_s ds\right)_t$ is a MG, $\forall n \geq 1$.

(i.e $\left(N_t - \int_0^t \lambda_s ds\right)_t$ is a local MG).

<u>PROOF:</u> [USUALLY ASKED]

since $\begin{cases} \text{if } t < \tau_n : N_{t\wedge\tau_n} = N_t < n ; \\ \text{if } t \geq \tau_n : N_{t\wedge\tau_n} = N_{\tau_n} = n . \end{cases}$

Let $n \geq 1$, $N_{t\wedge\tau_n} \leq n$.

We have to prove that $\mathbb{E}\left[\int_0^{t\wedge\tau_n}\lambda_s ds\right] < +\infty$.

- $\mathbb{E}\left[\int_0^{t\wedge\tau_n} \lambda_s ds\right] = \mathbb{E}\left[\int_0^{t\wedge\tau_n} 1\, dN_s\right] = \mathbb{E}\left[\int_0^t \mathbb{1}_{\{s\le\tau_n\}}\, dN_s\right]$

λ intensity of N

by def of $\int dN_s$ $= \mathbb{E}\left[\sum_{k\geq 1} \mathbb{1}_{\{\tau_k\le\tau_n\}} \cdot \mathbb{1}_{\{\tau_k\le t\}}\right]$

$\{\tau_i\}_i$ is $\nearrow$ $= \mathbb{E}\left[\sum_{k=1}^{n} \mathbb{1}_{\{\tau_k\le\tau_n\}} \cdot \mathbb{1}_{\{\tau_k\le t\}}\right]$

$= \sum_{k=1}^{n} \mathbb{E}\left[\mathbb{1}_{\{\tau_k\le\tau_n\}} \mathbb{1}_{\{\tau_k\le t\}}\right] < +\infty$

$\longrightarrow$ so it is in L^1.

- Now let's prove the property of the MG:

let $M_t = N_t - \int_0^t \lambda_s ds$. Let $r\le t$,

$\mathbb{E}[M_{t\wedge\tau_n} \mid \mathcal{F}_r] = \mathbb{E}[M_{t\wedge\tau_n} - M_{r\wedge\tau_n} + M_{r\wedge\tau_n} \mid \mathcal{F}_r]$

$= M_{r\wedge\tau_n} + \underbrace{\mathbb{E}\left[\int_{r\wedge\tau_n}^{t\wedge\tau_n} dN_s - \int_{r\wedge\tau_n}^{t\wedge\tau_n} \lambda_s ds \,\Big|\, \mathcal{F}_r\right]}_{\text{WE WANT TO PROVE IT'S } = 0.}$

$$\mathbb{E}\left[\int_{r\wedge\tau_n}^{t\wedge\tau_n} dN_s \Big| \mathcal{F}_r\right] = \mathbb{E}\left[\int_{r\wedge\tau_n}^{t\wedge\tau_n} \lambda_s ds \,\Big|\, \mathcal{F}_r\right] \iff$$

$$\forall A\in\mathcal{F}_r,\quad \int_A \left(\int_{r\wedge\tau_n}^{t\wedge\tau_n} dN_s\right) d\mathbb{P} = \int_A \left(\int_{r\wedge\tau_n}^{t\wedge\tau_n} \lambda_s ds\right) d\mathbb{P} \iff$$

$$\forall A\in\mathcal{F}_r,\ \int_\Omega \mathbb{1}_A(\dots)\, d\mathbb{P} = \int_\Omega \mathbb{1}_A(\dots)\, d\mathbb{P} \iff$$

$$\forall A \in \mathcal{F}_r, \quad \mathbb{E}\left[\int_{r\wedge T_n}^{t\wedge T_n} \mathbb{1}_A \, dN_s\right] = \mathbb{E}\left[\int_{r\wedge T_n}^{t\wedge T_n} \mathbb{1}_A \lambda_s \, ds\right] \iff$$

$$\forall A \in \mathcal{F}_r, \quad \mathbb{E}\left[\int_0^t \mathbb{1}_A \mathbb{1}_{(r\wedge T_n, t\wedge T_n)}(s) \, dN_s\right] = \mathbb{E}\left[\int_0^t \mathbb{1}_A \mathbb{1}_{(r\wedge T_n, t\wedge T_n)}(s) \lambda_s \, ds\right]$$

$$\iff \forall A \in \mathcal{F}_r, \quad \mathbb{E}\left[\int_0^t \mathbb{1}_A \mathbb{1}_{(r,t]}(s) \mathbb{1}_{\{s \leq T_n\}} dN_s\right] = \mathbb{E}\left[\int_0^t \mathbb{1}_A \mathbb{1}_{(r,t]}(s) \mathbb{1}_{\{s\leq T_n\}} \lambda_s ds\right]$$

But the last equality is TRUE since the involved processes are <u>positive</u> & <u>predictable</u>; and λ is an intensity (so we can use the def of the stoch. $\int$). $\square$

4) <u>Uniqueness theorem:</u> Let N a jump process with intensities $\lambda, \tilde{\lambda}$. Then $\lambda_t(\omega) = \tilde{\lambda}_t(\omega)$ $d\mathbb{P}\, dN_t(\omega)$-a.e.

<u>PROOF:</u> [NEVER HAS BEEN ASKED, AS OF TODAY]

$\forall C$ non-negative & predictable:

$$\mathbb{E}\left[\int_0^t C_s dN_s\right] = \mathbb{E}\left[\int_0^t C_s \lambda_s ds\right] = \mathbb{E}\left[\int_0^t C_s \tilde{\lambda}_s ds\right] \quad \forall t \geq 0.$$

Let's fix t, we can define $C_s = \mathbb{1}_{\{\lambda_s > \tilde{\lambda}_s\}} \mathbb{1}_{\{s\leq t\}}$. It is non-neg. & predictable. So:

$$\mathbb{E}\left[\int_0^{+\infty} \underbrace{\mathbb{1}_{\{\lambda_s > \tilde{\lambda}_s\}} \mathbb{1}_{\{s\leq t\}}}_{C_s} \tilde{\lambda}_s \, ds\right] = \mathbb{E}\left[\int_0^{+\infty} \underbrace{\mathbb{1}_{\{\lambda_s > \tilde{\lambda}_s\}} \mathbb{1}_{\{s\leq t\}}}_{C_s} \lambda_s ds\right]$$

i.e:

$\mathbb{E}\left[\int_0^{+\infty} \mathbb{1}_{\{\lambda_s > \tilde{\lambda}_s\}} \mathbb{1}_{\{s \leq t\}} (\tilde{\lambda}_s - \lambda_s)\, ds\right] = 0$ so :

$\mathbb{1}_{\{\lambda_s > \tilde{\lambda}_s\}} = 0$ $\left| \begin{array}{l} d\mathbb{P}\, \lambda_s\, ds \\ d\mathbb{P}\, \tilde{\lambda}_s\, ds \end{array}\right.$. And we can do the same

with $\mathbb{1}_{\{\lambda_s < \tilde{\lambda}_s\}}$. And from $\mathbb{E}\left[\int_0^t C_s\, dN_s\right] = \mathbb{E}\left[\int_0^t C_s \lambda_s\, ds\right]$

we can say that $d\mathbb{P}\, dN_t = d\mathbb{P}\, \lambda_t\, dt$ on non-neg.-predictable processes. $\blacksquare$

<u>RMK</u>: λ depends on $\mathbb{P}$ & on $\mathbb{F}$: λ is a $(\mathbb{P}, \mathbb{F})$ - intensity of N.

5) <u>PROP</u>: Let $\{\tau_n\}_{n \geq 1}$ jump times, λ an intensity.
Then : $\forall n \geq 1$, $\lambda_{\tau_n}(\omega) > 0 \ \langle \mathbb{P} = 1 \rangle$.

<u>PROOF</u>: By definition of the intensity,

$$\mathbb{E}\left[\int_0^{+\infty} C_t\, dN_t\right] = \mathbb{E}\left[\int_0^{+\infty} C_t \lambda_t\, dt\right] \quad \forall C \text{ non-neg \& pred.}$$

Take $n \geq 1$, and $C_t = \mathbb{1}_{\{\lambda_t = 0\}} \mathbb{1}_{\{(\tau_{n-1} ; \tau_n]\}}(t)$.
Then :

- the left hand side is : $\mathbb{E}\left[\sum_{k \geq 1} C_{\tau_k} \mathbb{1}_{\{\tau_k \leq +\infty\}}\right] =$

$$\mathbb{E}\left[\sum_{k \geq 1} \mathbb{1}_{\{\lambda_{\tau_k} = 0\}} \mathbb{1}_{\{\tau_{n-1} < \tau_k \leq \tau_n\}}\right] = \mathbb{E}\left[\sum_{k \geq 1} \mathbb{1}_{\{\lambda_{\tau_k} = 0\}} \mathbb{1}_{\{\tau_k = \tau_n\}}\right]$$

$$= \mathbb{E}\left[\mathbb{1}_{\{\lambda_{T_n} = 0\}} \right] = \mathbb{P}(\lambda_{T_n} = 0).$$

- the right hand side is :

$$\mathbb{E}\left[\int_0^{+\infty} \mathbb{1}_{\{\lambda_t = 0\}} \mathbb{1}_{\{(T_{n-1}; T_n]\}}(t) \lambda_t \, dt \right] = \mathbb{E}\left[\int_{T_{n-1}}^{T_n} \mathbb{1}_{\{\lambda_t = 0\}} \lambda_t \, dt \right]$$

$$= 0 .$$

So $\mathbb{P}(\lambda_{T_n} = 0) = 0$. $\square$

A NEW KIND OF JUMP PROCESSES: MARKED POINT PROCESSES

See the def + def of indexed process + def of intensity kernel.

Notation: $\int_0^t \int_E H(s,z)\, \mu(ds,dz) = \sum_{n\geq 1} H(\tau_n, z_n)\, \mathbb{1}_{\{\tau_n \leq t\}}$, $\forall t \geq 0$.

"Random measure"

6) Projection thm: Given $(\tau_n, z_n)_{n\geq 1}$ with $\lambda_t(dz)$; for any non-negative predictable & indexed $(H_t)_{t\geq 0}$:

$$\mathbb{E}\left[\int_0^{+\infty}\int_E H(s,z)\,\mu(ds,dz)\right] = \mathbb{E}\left[\int_0^{+\infty}\int_E H(s,z)\,\lambda_s(dz)\,ds\right].$$

PROOF: It's enough to prove the result for $H_t(z) = C_t \mathbb{1}_A(z)$, $\forall A \in \mathcal{E}$. So:

$$\begin{aligned}
\mathbb{E}\left[\int_0^{+\infty}\int_E H(s,z)\,\mu(ds,dz)\right] &= \mathbb{E}\left[\int_0^{+\infty}\int_E C_t \mathbb{1}_A(z)\,\mu(dt,dz)\right] \\
&= \mathbb{E}\left[\int_0^{+\infty}\int_A C_t\,\mu(dt,dz)\right] \\
\text{by the notation} \quad &= \mathbb{E}\left[\sum_{n\geq 1} C_{\tau_n} \mathbb{1}_A(z_n)\, \mathbb{1}_{\{\tau_n \leq +\infty\}}\right] \\
\text{by def of the stoch. int.} \quad &= \mathbb{E}\left[\int_0^{+\infty} C_t\, dN_t(A)\right] \\
\text{by def of } \lambda_t(A) \text{ being the intensity of } N_t(A). \quad &= \mathbb{E}\left[\int_0^{+\infty} C_t \lambda_t(A)\, dt\right] \\
&= \mathbb{E}\left[\int_0^{+\infty}\int_A C_t \lambda_t(dz)\, dt\right] \\
&= \mathbb{E}\left[\int_0^{+\infty}\int_E \underbrace{C_t \mathbb{1}_A(z)}_{H_t(z)} \lambda_t(dz)\,dt\right] \quad \square.
\end{aligned}$$

IDEA OF THE PROOF: the thm is true for H of the form $H_t(z) = C_t \mathbb{1}_A(z)$, & then use Monotone Class Thm.

CRAMER - LUNDBERG MODEL:

1) Expected surplus: $\forall t \geq 0, \quad \mathbb{E}[R_t] = R_0 + ct - \frac{\lambda t}{\xi}$.

$\left((N_t)_t \sim \text{Poisson Process} \ \& \ X \sim \exp(\xi) \right)$.

PROOF: $\mathbb{E}[R_t] = R_0 + ct - \mathbb{E}\left[\sum_{n=1}^{N_t} X_n\right] = R_0 + ct - \mathbb{E}\left[\sum_{n \geq 1} \sum_{i=1}^{n} X_i \mathbb{1}_{\{N_t = n\}}\right]$

$$= R_0 + ct - \sum_{n \geq 1} \sum_{i=1}^{n} \mathbb{E}\left[X_i \mathbb{1}_{\{N_t = n\}}\right]$$

$N_t \perp\!\!\!\perp X_n$

$$= R_0 + ct - \mathbb{E}[X] \sum_{n \geq 1} n \, \mathbb{P}[N_t = n]$$

$\mathbb{E}[N_t] = \lambda t$

$$= R_0 + ct - \mathbb{E}[X] \lambda t$$

$\mathbb{E}[X] = \frac{1}{\xi} \quad (X \sim \mathrm{Exp}(\xi))$.

$$= R_0 + ct - \frac{\lambda t}{\xi} \quad \square$$

2) $\mathrm{Var}(L_t) = \mathrm{Var}\left(\sum_{n=1}^{N_t} X_n\right) = \mu_2 \cdot \lambda t$ where $\mu_2 = \mathbb{E}[X^2]$.

PROOF: $\mathrm{Var}\left(\sum_{n=1}^{N_t} X_n\right) = \mathbb{E}\left[\left(\sum_{n=1}^{N_t} X_n\right)^2\right] - \mathbb{E}\left[\sum_{n=1}^{N_t} X_n\right]^2$

$$= \mathbb{E}\left[\left(\sum_{n=1}^{N_t} X_n\right)^2\right] - \left(\frac{\lambda t}{\xi}\right)^2$$

$$\mathbb{E}\left[\left(\sum_{n=1}^{N_t} X_n\right)^2\right] = \sum_{k \geq 1} \mathbb{E}\left[\left(\sum_{n=1}^{k} X_n\right)^2 \mathbb{1}_{\{N_t = k\}}\right]$$

$N_t \perp\!\!\!\perp X_n$

$$= \sum_{k \geq 1} \mathbb{E}\left[\left(\sum_{n=1}^{k} X_n\right)^2\right] \mathbb{P}[N_t = k]$$

$$= \sum_{k \geq 1} \mathbb{P}(N_t = k) \left[\mathrm{Var}\left(\sum_{n=1}^{k} X_n\right) + \mathbb{E}\left[\sum_{n=1}^{k} X_n\right]^2\right]$$

finite sum & $(X_n)_n$ iid

$$= \sum_{k \geq 1} \mathbb{P}(N_t = k) \left[\sum_{n=1}^{k} \mathrm{Var}(X_n) + \mathbb{E}\left[\sum_{n=1}^{k} X_n\right]^2\right]$$

$\mu = \mathbb{E}[X]$

$\mu_2 = \mathbb{E}[X^2]$

$$= \sum_{k \geq 1} \mathbb{P}(N_t = k) \left[k \cdot (\mu_2 - \mu^2) + k^2 \mu^2\right]$$

So : $\mathrm{Var}\left(\sum_{n=1}^{N_t} X_n\right) = \sum_{k\geq 1} \mathbb{P}(N_t = k)\left[k(\mu_2 - \mu^2) + k^2\mu^2\right] - \mu^2(\lambda t)^2$

$\mathbb{E}[N_t] = \sum_k k\,\mathbb{P}(N_t = k)$

$\mathbb{E}[N_t^2] = \sum_k k^2\,\mathbb{P}(N_t = k)$

$= \mu_2\,\mathbb{E}[N_t] - \mu^2\,\mathbb{E}[N_t] + \mu^2\,\mathbb{E}[N_t^2] - \mu^2(\lambda t)^2$

$= \lambda t + \lambda^2 t^2$

$k^2 = k(k-1) + k$

$= \mu_2\,\lambda t - \cancel{\mu^2 \lambda t} + \cancel{\mu^2(\lambda t} + \cancel{\lambda^2 t^2}) - \cancel{\mu^2(\lambda t)^2}$

$= \mu_2 \lambda t$.

So $\mathrm{Var}\left(\sum_{n=1}^{N_t} X_n\right) = \mu_2 \lambda t$. $\square$

3) $\mathbb{E}\left[e^{r\sum_{i=1}^{N_t} X_i}\right] = e^{\lambda t(M_X(r) - 1)}$ with $r \in \mathbb{R}$ and

$M_X(r) = \mathbb{E}\left[e^{rX}\right]$ (moment generating function).

PROOF: see the slide 36/49 .

4) <u>Lemma 5.3:</u> Let $r \in \mathbb{R}$, $M_X(r) < +\infty$ and $\theta(r) = \lambda(M_X(r) - 1) - cr$. Then: $\left(e^{-rC_t - \theta(r)t}\right)_{t\geq 0}$ is a martingale, with $C_t = u + ct - \sum_{n=1}^{N_t} X_n$.

PROOF: Fix $0 \leq s \leq t$.

$$\mathbb{E}\left[e^{-rC_t - \theta(r)t} \,\middle|\, \mathcal{F}_s\right] = \mathbb{E}\left[e^{-rC_t - \theta(r)t + rC_s + \theta(r)s} \times e^{-rC_s - \theta(r)s} \,\middle|\, \mathcal{F}_s\right]$$

$$= \mathbb{E}\left[e^{-r(C_t - C_s)} \middle| \mathcal{F}_s\right] \cdot \underbrace{e^{-rC_s}}_{\mathcal{F}_s\text{-meas}} \cdot \underbrace{e^{-\lambda(M_X(r)-1)t + crt}}_{\text{deterministic term.}}$$

...

$\square$

Let's study $r \mapsto \theta(r)$: