Introduction to PDE

Important Example: Motivation to Study Eigenvalue Problems

STATIONARY PDEs

N=3, $\begin{cases} c = c(x) \in L^{\infty}(\Omega) \\ c \geq 0 \text{ a.e. on } \Omega \end{cases}$, we want to solve, given $f \in L^2(\Omega)$ (*) $\begin{cases} -\Delta u + c(x)u = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ (Dirichlet Pbm)

GOAL: show that $\forall f \in L^2(\Omega)$, $\exists! u \in H_0^2(\Omega)$ solving (*) in weak sense.

IDEA: We rewrite (*) $\Leftrightarrow$ $u = T(f)$, $T: L^2(\Omega) \to H_0^2(\Omega)$. Then we can study the properties of T ...

HOW TO PROCEED?

SOLVING

I Use Lax Milgram Theorem.

Lax Milgram Thm: $\begin{cases} \cdot\ H: \text{Hilbert space} \\ \cdot\ a: H \times H \to \mathbb{R} \text{ bilinear, } \mathcal{C}^0, \text{ coercive} \\ \cdot\ F \in H^* \end{cases}$ such that $\forall v \in H$, $a(u,v) = F(v)$

Then $\exists! u \in H$ such that $\forall v \in H$, $a(u,v) = F(v)$ (Consider)

Furthermore: $\|u\|_H \leq C_1 \|F\|_{H^*}$

Spectral Thm: H separable Hilbert space, $T \in K(H)$ (compact linear operator, self-adjoint) Then:

$\exists (\lambda_i)$, enumerated list of eigenvalues of T with eigenvalues $(e_i)_i \subset H$

- $e_i(x) \cdot e_j(x) = \delta_{ij}(x) \cdot e_i(x)$
- $\|x\|_H^2 = \sum_i |\langle x, e_i \rangle|^2$

$\begin{cases} \cdot\ \text{Radius of ess. spectrum of } T = 0 \\ \cdot\ \sigma(T): \text{spectrum of } T = \{0\} \cup \{\lambda_i\} \\ \cdot\ \dim(\lambda_i): \text{mult. of eig. value } \lambda_i \end{cases}$

II Use Spectral Theorem.

⚠ In the thm, T has to be $H \to H$.

NOT A PB BECAUSE: $H_0^2(\Omega) \xhookrightarrow{\text{compact}} L^2(\Omega)$

We introduce a new operator K: $L^2(\Omega) \xrightarrow[\text{linear, } \mathcal{C}^0]{T} H_0^2(\Omega) \xhookrightarrow[\text{linear, } \mathcal{C}^0, \text{ compact}]{i} L^2(\Omega)$

$\boxed{K = i \circ T}$

↳ K is linear, $\mathcal{C}^0$, compact, $L^2(\Omega) \to L^2(\Omega)$.

Prop. of the operator

III Study the relations between eigenvalues of K & the elliptic equation.

↳ Let $\varphi \in L^2(\Omega)$ s.t. $K\varphi = \mu\varphi$. ($\mu \in \mathbb{R}$: eigenvalue of K).

$\mu = 0$: $\varphi = 0$ ⇓ φ not eigenfunction.

$\mu \neq 0$: $-\Delta(\mu\varphi) + c(x)(\mu\varphi) = \varphi$ ie $-\Delta\varphi + c(x)\varphi = \frac{1}{\mu}\varphi = \lambda\varphi$, $\lambda = \frac{1}{\mu}$

Conclusion: $\left\{ \begin{array}{l} \mu \text{ eigenvalue of } K \\ \text{in } L^2(\Omega) \text{ with} \\ \text{eigenfunction } \varphi \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \frac{1}{\mu} \text{ eigenvalue of} \\ -\Delta \cdot + c(x) \cdot \text{ in } H_0^2(\Omega) \\ \text{with eigenfunction } \varphi \end{array} \right\}$

NOW: LET'S MAKE THINGS PRECISE & STUDY "EIGENVALUE PROBLEMS"

Plan: About linear equations,
- Eigenvalues, eigenfunctions, solvability (WEAK)
- Elliptic regularity (WEAK → Strong, Classical)
- Non-linear analysis (calculus in Banach Space)
- Maximum principles (weak, strong boundary (Hopf) ...)

They extend to non-linear equations

Introduction to PDE

Pb DC

⚠ non-linearities in the PDE ⚠

STATIONARY PDE: $\begin{cases} -\Delta u = f(x,u) & x \in \Omega \\ u = g & x \in \partial\Omega \end{cases}$

$\exists! u \in V$, $\forall v \in V$, $\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f(x,u) v \, dx$

e.g. Wk: $V = H_0^1(\Omega)$ for Dirichlet BC, $V = H^1(\Omega)$ for Neumann BC

RELATIONS BETWEEN NORMS: For Ω bounded, Hölder Inequality $\int_\Omega u^2 \leq C \int_\Omega |\nabla u|^2$, $\forall u \in H_0^1(\Omega)$

RELATIONS BETWEEN SPACES: Rellich Thm: $H_0^1(\Omega) \xhookrightarrow{\text{compact}} L^2(\Omega)$

Sobolev Embedding: $H_0^1(\Omega) \xhookrightarrow{\text{continuously}} L^{2^*}(\Omega)$, $\|u\|_{2^*} \leq C\|u\|_{H_0^1}$, $2^* = \frac{2n}{n-2}$

- N=1: $H_0^1(\Omega) \hookrightarrow L^\infty(\Omega)$
- N=2: $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, $2^* \in [2, +\infty)$
- N=3: $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, $2^* = \frac{2n}{n-2}$

Compact Embeddings: $\forall 2 < 2^*$, $H_0^1(\Omega) \xhookrightarrow{\text{compact}} L^2(\Omega)$

Heat equation: $\begin{cases} \partial_t u - \Delta u = f(u,x,t) = f(u) & x \in \Omega, t > 0 \\ u(x,0) = g(x) & x \in \Omega \\ \partial_\nu u(x,t) = 0 & x \in \partial\Omega, t > 0 \end{cases}$ — not for $f(u)$! — Neumann!

Then $u = u(x,t)$ is $\begin{cases} v(x,0) = f(v) & t > 0 \\ v(x,0) = g(x) & \end{cases}$ is solution of $\begin{cases} u(x,t) = v(x) \\ u(x,0) = v(x) \end{cases}$

1D, linear (or bilinear), autonomous

$\Omega = (0,L) \subset \mathbb{R}$, $u = u(x,t)$, $\{x, t\} \subset \mathbb{R}$.

$\begin{cases} \partial_t u - D\partial_{xx} u = \gamma u & x \in (0,L), t > 0 \\ u(0,t) = g(t) \geq 0 & \text{for } 0 \leq x \leq L \\ u(L,t) = u(L,t) \geq 0 & t > 0 \end{cases}$

→ eigenvalue of the Laplacian: $w(x,t) = v(x)u(t)$

↳ eigenvalues, etc

In the N-dimensional case, we saw: $\begin{cases} \cdot\ \Omega \subset \mathbb{R}^n \text{ bdd, smooth} \\ \cdot\ \text{Dirichlet BC} \\ \cdot\ c \in L^\infty(\Omega), c \geq 0 \text{ a.e.} \end{cases}$ $\Rightarrow$ $\exists$ eigenvalue of $(EP) \Leftrightarrow$ $\exists$ non trivial $\Leftrightarrow$ $\ker\{-\Delta + c - \lambda I\} \neq \{0\}$

$\ker\{-\Delta + c - \lambda I\} = \{u \in H_0^2(\Omega) : u \text{ solves } (EP)\}$

EIGENVALUE PROBLEM:

↳ Dirichlet BC; $c \in L^\infty(\Omega)$

$(EP) \begin{cases} -\Delta u + c(x)u = \lambda u \\ u \in H_0^2(\Omega) \end{cases}$ which means: $\forall v \in H_0^2(\Omega)$, $\int_\Omega \nabla u \cdot \nabla v + c(x)uv = \int_\Omega \lambda u v$ (weak formulation)

We will use spectral theorem in separable Hilbert spaces to show:

Thm 1: Assume $c \in L^\infty(\Omega)$, $c \geq 0$. Then: $\exists (\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{R}$, $\exists (\varphi_n)_{n \in \mathbb{N}} \subset H_0^2(\Omega)$, such that:

1) If $\forall n \in \mathbb{N}$, $\lambda_n \neq \lambda$ then $\begin{cases} \ker(-\Delta + c - \lambda I) = \{0\} \\ \operatorname{Im}(-\Delta + c - \lambda I) = L^2(\Omega) \end{cases}$ → $\frac{1}{\lambda} \in \operatorname{EV}(K) \Rightarrow$ injectivity → surjectivity

2) λ eigenvalue of $(EP) \Leftrightarrow \ker(-\Delta + c - \lambda I) < +\infty$ & $\lambda = \lambda_n$ for some n (and with the associated eigenfct)

[Remark: $\dim(\ker(-\Delta + c - \lambda I)) < +\infty$ ie $\ker(-\Delta + c - \lambda I) \oplus \operatorname{Im}(-\Delta + c - \lambda I) = L^2(\Omega)$]

3) $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \xrightarrow{n \to +\infty} +\infty$

4) $(\varphi_n)_{n \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$. → need a lot of work ... → Spectral Thm (...details).

Proof of 3): Claim: $\forall n \in \mathbb{N}$, $\lambda_n \geq 0$.

$-\Delta\varphi_n + c\varphi_n = \lambda_n \varphi_n$. So: $\underbrace{\int_\Omega [\nabla\varphi_n \cdot \nabla\varphi_n + c\varphi_n^2]}_{\geq 0} = \lambda_n \underbrace{\int_\Omega \varphi_n^2}_{=1}$ (since $\|\varphi_n\|_2 = 1$ because φ_n eigenfct of $\|\varphi_n\| = 1$). So: $\lambda_n \geq 0$ $\forall n \in \mathbb{N}$.

So: $\begin{cases} \forall n \in \mathbb{N}, \lambda_n \geq 0 \\ \forall n \in \mathbb{N}, \lambda_n \in [-\|c\|_\infty, \|c\|_\infty] \\ \lambda_n \xrightarrow[n \to +\infty]{} 0 \end{cases}$ (Since $\|c\|_\infty$ is bounded by hypothesis)

This means: $\begin{cases} \frac{1}{\lambda_n} \in [-\|c\|_\infty, \|c\|_\infty] \\ \lambda_n \geq 0 \end{cases}$ and $\lambda_n = \frac{1}{\lambda_{n \to +\infty}}$

which implies that: $\lambda_n \geq \frac{1}{\|c\|_\infty} > 0$, $\lambda_n \xrightarrow[n \to +\infty]{} \infty$

so finally: $\lambda_n \xrightarrow[n \to +\infty]{} \infty$ ■

NOTE: I haven't written the video of the proof.

Range $R(\lambda I - c) = \{f \in L^2(\Omega) : \exists u \in H_0^2(\Omega), -\Delta u + cu = \lambda f\}$

Equivalent Scalar Product (⊕) orthogonal basis:

- OK to have $c \geq \|c\|_\infty$
- OK to have $c \in L^\infty(\Omega)$
- Ω can be open
- $-\Delta u + c(x)u$ ($u \in H_0^2$)

C>0 not necessary:

If Ω enough to have $c \geq -a$, $a \geq 0$.

Indeed: $-\Delta u + c(x)u = \lambda u$

$-\Delta u + \underbrace{(c(x)+a)}_{\tilde{c} \geq 0}u = (\lambda + a)u$

Define $\tilde{c} = c + a$, $\tilde{\lambda} = \lambda + a$. We see now solving: $-\Delta u + \tilde{c}(x)u = \tilde{\lambda}u$ with $\tilde{c} \geq 0$ a.e. in Ω.

Finally we will also use the following theorem:

Thm 2: Consider (EP): $-\Delta u + cu = \lambda u$ in $H_0^2(\Omega)$ (Dirichlet) with $H^2(\Omega)$ (Neumann)

- $\Omega \subset \mathbb{R}^n$ bdd & smooth
- $c \in L^\infty(\Omega)$ (⚠ NO SIGN CONDITION)

Then: $\forall \lambda \in \mathbb{R}$, the "Fredholm Alternative" holds & $\begin{cases} \cdot\ \text{The eigenvalues sequence } (\lambda_n)_{n \in \mathbb{N}} \text{ satisfies } -\|c\|_\infty \leq \lambda_1 \leq \lambda_2 \leq \dots \xrightarrow{n \to +\infty} +\infty \\ \cdot\ \text{The eigenfunctions sequence } (\varphi_n)_{n \in \mathbb{N}} \text{ can be chosen to be an orthonormal basis of } L^2(\Omega) \text{, and} \end{cases}$

EXAMPLE:

"Grown-up example"

Continuation of the example with separation of variable (reminded version):

$u = u(x,t)$

For the example, $\begin{cases} u_t - \Delta u = -c(x)u & \text{in } \Omega \times \mathbb{R}_*^+ \\ u(x,0) = g(x) \geq 0 \ (\not\equiv 0) & (g \in L^2(\Omega)) \\ u(x,t) = 0 & \text{in } \partial\Omega \times \mathbb{R}_*^+ \end{cases}$

We want to solve this pb using Thm 2.

Role of Malthus term w/growth term depending on x.

$\begin{cases} \mathbb{R} \longrightarrow H_0^2(\Omega) \\ t \longmapsto u(\cdot,t) \end{cases}$: $\begin{cases} \Omega \longrightarrow \mathbb{R} \\ x \longmapsto u(x,t) \end{cases}$

SEPARATION OF VARIABLES:

Again we look for solutions in the form: $u(x,t) = \sum_{k=1}^{+\infty} c_k(t)\varphi_k(x)$ where: $-\Delta\varphi_k + c\varphi_k = \lambda_k\varphi_k$ in $H_0^2(\Omega)$. Substitute in the equation and derive the series (if c is enough regular):

$0 = \sum_k c_k'(t)\varphi_k(x) + \sum_k c_k(t)(-\Delta\varphi_k + c(x)\varphi_k)$

ie $0 = \sum_k c_k'(t)\varphi_k(x) + \sum_k c_k(t)\lambda_k\varphi_k(x)$

ie $0 = \sum_k \underbrace{(c_k'(t) + \lambda_k c_k(t))}_{=0}\varphi_k(x)$ i.e. $\boxed{\forall k \in \mathbb{N}, c_k(t) = he^{-\lambda_k t}}$

$\Rightarrow \forall k \in \mathbb{N}$, $c_k'(t) + \lambda_k c_k(t) = 0$

And we can write g using the basis: coeff = projection on the basis (scalar prod.)

$g(x) = \sum_k g_k\varphi_k(x)$ and $g_k = \int_\Omega g(x)\varphi_k(x)dx$ ie $g_k = \langle g, \varphi_k \rangle_{L^2(\Omega)}$

↳ We will see that $g \geq 0 \Rightarrow g_1 > 0$.

Finally: $u(x,t) = \sum_{k=1}^{+\infty} g_k e^{-\lambda_k t}\varphi_k(x) = g_1 e^{-\lambda_1 t}\varphi_1(x)\left[1 + \sum_{k=2}^{+\infty} \frac{g_k}{g_1} e^{\overbrace{(\lambda_1 - \lambda_k)}^{<0}t} \frac{\varphi_k(x)}{\varphi_1(x)}\right]$

for the moment we have to believe that $\varphi_1 \neq 0$

Initial condition: $\begin{cases} u(x,0) = g(x) \text{ i.e.} \\ \sum_k c_k(0)\varphi_k(x) = g(x) = \sum_k g_k\varphi_k(x) \end{cases}$ identification with

Conclusion: $\boxed{\text{PERSISTENCE} \Leftrightarrow \lambda_1(-\Delta + c) \leq 0}$