Mathematical Engineering - A.Y. 2022-23

Real and Functional Analysis - Written exam - June 20, 2023

Exercise 1. Consider the functions $f, f_n : [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$, defined respectively by

$$f(x) = \begin{cases} x\cos\left(\frac{1}{x}\right), & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}, \qquad f_n(x) = \begin{cases} x\cos\left(\frac{1}{x}\right) & \frac{1}{2\pi n} < x \leq 1 \\ \frac{1}{2\pi n}, & 0 \leq x \leq \frac{1}{2\pi n} \end{cases}.$$

(1) Is f of bounded variation in $[0,1]$? Is f absolutely continuous in $[0,1]$? Justify the answers.
(2) Prove that f_n converges pointwisely a.e. to f as $n \to +\infty$. Does f_n converges to f in L^∞ as $n \to +\infty$? Justify the answers.
(3) Considering that $f_n \in \mathrm{AC}([0,1])$ for any $n \in \mathbb{N}$ (not to be proven), use items (1),(2) to answer to the following question: is the normed space $(\mathrm{AC}([0,1]), \|\cdot\|_\infty)$ complete? Justify the answer.

Solution.
(1) We show that f is not of bounded variation in $[0,1]$, hence it is not absolutely continuous in $[0,1]$ as well (recall that $\mathrm{AC}([0,1]) \subset \mathrm{BV}([0,1])$). Indeed, for $n \in \mathbb{N}$, $n \geq 2$, consider for instance the following partition P_n of $[0,1]$:

$$P_n = \{x_k\}_{k=0}^n, \qquad x_k = \begin{cases} 1, & k = 0 \\ \frac{1}{k\pi}, & k = 1, \dots, n-1 \\ 0, & k = n \end{cases}.$$

Then

$$f(x_k) = \begin{cases} \cos 1, & k = 0 \\ \frac{(-1)^k}{\pi k}, & k = 1, \dots, n-1 \\ 0, & k = n \end{cases},$$

so that we get

$$\begin{aligned} \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| &\geq \sum_{k=2}^{n-1} |f(x_k) - f(x_{k-1})| \\ &= \sum_{k=2}^{n-1} \left|\frac{(-1)^k}{\pi k} - \frac{(-1)^{k-1}}{\pi(k-1)}\right| = \sum_{k=2}^{n-1} \frac{2k-1}{\pi k(k-1)} \end{aligned}$$

In conclusion,

$$V_0^1(f) \geq \sup_n \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \geq \sum_{k=2}^{\infty} \frac{2k-1}{\pi k(k-1)} = +\infty.$$

Hence, f does not belong to $\mathrm{BV}([0,1])$.

(2) The pointwise convergence of f_n to f follows immediately by noting that for any $x \in [0,1]$, we have $\frac{1}{2\pi n} \to 0$ as $n \to +\infty$. Concerning the convergence in $L^\infty([0,1])$, we notice that both f_n and f are continuous in $[0,1]$ and they coincide in $\left[\frac{1}{2\pi n}, 1\right]$. So that $f_n - f$ is continuous

and, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} \|f_n - f\|_\infty &= \mathrm{esssup}_{x\in[0,1]}|f_n(x) - f(x)| \\ &= \sup_{x\in[0,1]} |f_n(x) - f(x)| \\ &= \sup_{x\in\left[0,\frac{1}{2\pi n}\right]} \left|\frac{1}{2\pi n} - x\cos\left(\frac{1}{x}\right)\right| \\ &\le \frac{1}{2\pi n} + \sup_{x\in\left[0,\frac{1}{2\pi n}\right]} \left|x\cos\left(\frac{1}{x}\right)\right| \\ &= \frac{1}{\pi n} \to 0, \quad \text{as } n \to +\infty. \end{aligned}$$

In particular, f_n converges to f in $L^\infty([0,1])$, as $n \to +\infty$.

(3) The normed space $(\mathrm{AC}([0,1]), \|\cdot\|_\infty)$ cannot be complete, since we found a sequence $\{f_n\}$ of absolutely continuous function that is convergent in L^∞, hence it is a Cauchy sequence in L^∞, however the limit f is not absolutely continuous by item (1).

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(1) Is f of bounded variation in $[0,1]$? Is f absolutely continuous in $[0,1]$? Justify the answers.
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(3) Considering that $f_n \in \mathrm{AC}([0,1])$ for any $n \in \mathbb{N}$ (not to be proven), use items (1),(2) to answer to the following question: is the normed space $(\mathrm{AC}([0,1]), \|\cdot\|_\infty)$ complete? Justify the answer.

(1) • Consider the partition: $P = \{x_k\}_{k\in[[0;n]]}$ with

$$x_k := \begin{cases} 0 & \text{if } k=0 \\ 1/\pi k & \text{if } k\in[[1;n-1]] \\ 1 & \text{if } k=n. \end{cases};$$

Then: $V_0^1(f,P) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$

$$V_0^1(f,P) = \sum_{k=1}^{n} \left| \frac{1}{\pi k}\cos(\pi k) - \frac{1}{\pi(k-1)}\cos(\pi(k-1)) \right|$$

$$= \sum_{k=1}^{n} \left| \frac{(-1)^k}{\pi k} - \frac{(-1)^{k-1}}{\pi(k-1)} \right| = \sum_{k=1}^{n} \left| (-1)^k \left(\frac{1}{\pi k} + \frac{1}{\pi(k-1)} \right) \right|$$

$$= \sum_{k=1}^{n} \frac{1}{\pi k} + \sum_{k=1}^{n} \frac{1}{\pi(k-1)} > \frac{1}{\pi} \sum_{k=1}^{n} \frac{1}{k} \xrightarrow[n\to+\infty]{} +\infty.$$

So $V_0^1(f) = +\infty$ and $\boxed{f \notin BV([0,1])}$. $\square$ Hence $\boxed{f \notin AC([0,1])}$. $\square$

(2)

- $f_n(x) := \begin{cases} x\cos\left(\frac{1}{x}\right) & \frac{1}{\pi x n} \leq x < 1 \\ \frac{1}{2\pi n} & 0 \leq x \leq \frac{1}{\pi x n} \end{cases}$

When $n \to +\infty$, $\frac{1}{\pi x n} \xrightarrow[n \to +\infty]{} 0$ so $f_n(x) \longrightarrow \underbrace{\begin{cases} x\cos\left(\frac{1}{x}\right) & 0 < x < 1 \\ 0 & x = 0 \end{cases}}_{= f(x).}$

$\boxed{\{f_n\} \text{ cv pointwisely a.e to } f \text{ as } n \to +\infty.}$ □

- $\|f_n - f\|_\infty = \sup_{x \in [0,1]} |f_n(x) - f(x)| = \sup_{x \in [0,1]} \left| \frac{1}{2\pi n} - x\cos\left(\frac{1}{x}\right) \right| \leq \frac{1}{2\pi n} + \sup_{x \in [0,1]} \left|x\cos\left(\frac{1}{x}\right)\right|$

$\|f_n - f\|_\infty \leq \frac{1}{2\pi n} + \frac{1}{2\pi n} = \frac{1}{\pi n} \xrightarrow[n \to +\infty]{} 0.$

$\boxed{f_n \xrightarrow[n \to +\infty]{L^\infty} f}$. □

(3) It is given that: $\forall n \in \mathbb{N}$, $f_n \in AC([0,1])$.

Well, $\{f_n\}_n \subset AC([0,1])$ which converges in L^∞ to $f \notin AC([0,1])$.

As $f_n \xrightarrow[n \to +\infty]{L^\infty} f$ (item 2), $\{f_n\}_n$ is a Cauchy sequence. So we found a Cauchy sequence $\{f_n\}_n \subset AC([0,1])$ in L^∞, but converging to a limit which is NOT in $AC([0,1])$. (item 1)

So $\boxed{(AC([0,1]), \|\cdot\|_\infty) \text{ is not a complete n.v.s}}$. □

Exercise 2. Consider the sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in \ell^2$ for any $n \in \mathbb{N}$, defined by

$$x_n = (x_n^{(k)})_{k\in\mathbb{N}}, \quad \text{with} \quad x_n^{(k)} := \begin{cases} \dfrac{k}{n+1}, & \text{if } n \le k \le n+1 \\ 0, & \text{otherwise} \end{cases}.$$

(1) Study the pointwise convergence of $\{x_n\}_{n\in\mathbb{N}}$.
(2) Denoted by $x = (x^{(k)})_{k\in\mathbb{N}}$ the pointwise limit of the sequence $\{x_n\}_{n\in\mathbb{N}}$ of item (1), prove that $\{x_n\}_{n\in\mathbb{N}}$ converges weakly in ℓ^2 to x as $n \to +\infty$.

Solution.
(1) We first observe that

$$\begin{aligned} x_1 &= (1/2, 1, 0, \dots) \\ x_2 &= (0, 2/3, 1, 0, \dots) \\ x_3 &= (0, 0, 3/4, 1, 0, \dots). \end{aligned}$$

Therefore for each fixed $k \in \mathbb{N}$, $x_n^{(k)} = 0$ for every $n > k$. Thus, the sequence $\{x_n\}_n$ converges pointwisely to $x = \mathbf{0} = (0, 0, \dots)$, as $n \to +\infty$.

(2) Since weak convergence in ℓ^2 implies pointwise convergence, then if $\{x_n\}_{n\in\mathbb{N}}$ converges weakly in ℓ^2 to x, then $x = \mathbf{0}$, by item (1). Let $\varphi \in (\ell^2)^*$, by the Riesz representation theorem, there exists $z = (z^{(k)})_{k\in\mathbb{N}} \in \ell_2$ such that

$$\varphi(y) = \sum_{k=1}^{\infty} z^{(k)} y^{(k)},$$

for each $y = (y^{(k)})_{k\in\mathbb{N}} \in \ell_2$. Hence, x_n converges weakly to x in ℓ^2 if and only if

$$\left| \sum_{k=1}^{\infty} z^{(k)} x^{(k)} - \sum_{k=1}^{\infty} z^{(k)} x_n^{(k)} \right| = \left| \sum_{k=1}^{\infty} z^{(k)} x_n^{(k)} \right| \to 0, \quad \text{for any } z = (z^{(k)})_{k\in\mathbb{N}} \in \ell^2.$$

Let $z \in \ell^2$ be arbitrarily fixed, we have

$$\left| \sum_{k=1}^{\infty} z^{(k)} x^{(k)} - \sum_{k=1}^{\infty} z^{(k)} x_n^{(k)} \right| = \left| \sum_{k=1}^{\infty} z^{(k)} x_n^{(k)} \right| = \frac{n}{n+1} z^{(n)} + z^{(n+1)} \to 0$$

indeed $\frac{n}{n+1} \to 1$ as $n \to +\infty$, and $z^{(n)} \to 0$ by necessary condition of convergence of series (remind that $z \in \ell^2$, so the series $\sum_{k=1}^{+\infty} |z^{(k)}|^2$ converges).

By the above observations we get the weak convergence of the sequence $\{x_n\}_{n\in\mathbb{N}}$ to $x \in \ell^2$.

Exercise 2: $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in \ell^2 \ \forall n \in \mathbb{N}$.

$$x_n = (x_n^{(k)})_{k\in\mathbb{N}} \quad , \text{ with } \quad x_n^{(k)} := \begin{cases} \frac{k}{n+1} & \text{if } n \leq k \leq n+1 \,; \\ 0 & \text{otherwise}. \end{cases}$$

1) Pointwise convergence: Let $k \in \mathbb{N}$ arbitrarily fixed.

We observe that, $\forall k < n$, $x_n^{(k)} = 0$.

Adding to this the fact that: $\frac{k}{n+1} \xrightarrow[n\to+\infty]{} 0$, we have that

$$x_n^{(k)} \xrightarrow[n\to+\infty]{} 0 \ , \quad \forall k \in \mathbb{N} \ . \quad \text{So :} \quad \boxed{\{x_n\}_{n\in\mathbb{N}} \text{ converges pointwisely to } (0)_k = x} . \square$$

2) If $\{x_n\}_{n\in\mathbb{N}}$ converges weakly in ℓ^2, this is to $x = (0)_{k\in\mathbb{N}}$.

Using Riesz Representation Theorem for ℓ^p $(1<p<+\infty)$, we can identify $(\ell^2)^*$ to ℓ^2 and:

$$x_n \xrightarrow[n\to+\infty]{} x \text{ in } \ell^2 \iff \sum_{k=1}^{+\infty} x_n^{(k)} y^{(k)} \xrightarrow[n\to+\infty]{} \sum_{k=1}^{+\infty} x^{(k)} y^{(k)} \ ,$$

$$\forall y \in \ell^2 .$$

Let $y \in \ell^2$ be arbitrarily fixed.

- $$\sum_{k=1}^{+\infty} x_n^{(k)} y^{(k)} = \frac{n}{n+1} y^{(n)} + 1 \times y^{(n+1)} = \underbrace{\left(\frac{1}{1+1/n}\right)}_{\xrightarrow[n\to+\infty]{} 1} . \underbrace{y^{(n)}}_{\xrightarrow[n\to+\infty]{} 0} + \underbrace{y^{(n+1)}}_{\xrightarrow[n\to+\infty]{} 0} \xrightarrow[n\to+\infty]{} 0$$

- $$\sum_{k=1}^{+\infty} x^{(k)} y^{(k)} = 0 \quad \text{since } x = (0)_{k\in\mathbb{N}}$$

[since $y \in \ell^2$: $\sum_{k=1}^{+\infty} |y^{(k)}|^2 < +\infty$

so that $y^{(n)} \xrightarrow[n\to+\infty]{} 0$]

As a conclusion:

$$\boxed{\{x_n\}_{n\in\mathbb{N}} \text{ converges weakly to } x = \{0\}_{k\in\mathbb{N}} \text{ in } \ell^2} . \square$$

Exercise 3. Let $X_1 = L^\infty([-1,1])$ and $X_2 = C([-1,1])$ both endowed with the norm $\|\cdot\|_\infty$. Consider the linear operators $T_i : X_i \to X_i$, $i = 1,2$, defined by

$$T_i\, g = \int_{-1}^{1} \frac{x}{1+x^2}\, g(x)\ \mathrm{d}x, \quad \forall g \in X_i.$$

(1) Show that T_1 is a continuous operator.
(2) Compute the operator norm of T_1.
(3) Compute the operator norm of T_2.
(4) Let $g(x) = x^2$. Does g belong to the kernel of T_1? Is T_1 injective? Justify the answer.

Solution.
(1) Let $g \in L^\infty[-1,1]$,

$$\|T_1 g\|_\infty = \left\| \int_{-1}^{1} \frac{x}{1+x^2}\, g(x)\ \mathrm{d}x \right\|_\infty .$$

Since Tg is a constant function, we obtain

$$\|T_1 g\|_\infty = \left| \int_{-1}^{1} \frac{x}{1+x^2}\, g(x)\ \mathrm{d}x \right| \le \int_{-1}^{1} \frac{|x|}{1+x^2}\, |g(x)|\ \mathrm{d}x \le \|g\|_\infty \int_{-1}^{1} \frac{|x|}{1+x^2}\ \mathrm{d}x.$$

The function $f(x) = \frac{|x|}{1+x^2}$ is even in the symmetric interval $[-1,1]$, therefore we obtain

$$\|g\|_\infty \int_{-1}^{1} \frac{|x|}{1+x^2}\ \mathrm{d}x = 2\,\|g\|_\infty \int_{0}^{1} \frac{x}{1+x^2}\ \mathrm{d}x = \log(2)\,\|g\|_\infty.$$

Thus we obtain

$$\|T_1 g\|_\infty \le \log(2)\,\|g\|_\infty.$$

This proves the boundedness of the operator T_1; since T_1 is linear, it is a continuous operator. The same computation holds for the operator T_2.

(2) By item (1), we have

$$\|T_1 g\|_\infty \le \log(2)\|g\|_\infty, \quad \text{for any } g \in L^\infty[-1,1].$$

Consider the function $g \in L^\infty[-1,1]$ defined by

$$g(x) = \begin{cases} 1, & x \in [0,1], \\ -1, & x \in [-1,0), \end{cases}$$

and notice that $\|g\|_\infty = 1$. We obtain

$$\|T_1 g\|_\infty = \int_{-1}^{1} \frac{|x|}{1+x^2}\ \mathrm{d}x = \log(2).$$

Therefore, recalling the definition of operator norm, we deduce that the norm of T_1 is equal to $\log(2)$.

(3) The function g defined in the above item is not continuous, therefore we define, for each $n \in \mathbb{N}$, $g_n \in X_2$ by

$$g_n(x) = \begin{cases} 1, & x \in [1/n, 1], \\ nx, & x \in (-1/n, 1/n) \\ -1, & x \in [-1, -1/n]. \end{cases}$$

Notice that, for any $n \in \mathbb{N}$, we have $\|g_n\|_\infty = 1$ and

$$\|T_2 g_n\|_\infty \ge 2 \int_{1/n}^{1} \frac{x}{1+x^2}\ \mathrm{d}x = \log(2) - \log(1 + 1/n^2).$$

Thus, since the sequence $(\log(1+1/n^2))_{n\in\mathbb{N}}$ converges to zero, the operator norm of T_2 is equal to $\log(2)$ as well, by using the definition of operator norm.

(4) Let $g(x) = x^2$. We compute

$$T_1 g = \int_{-1}^{1} \frac{x^3}{1+x^2}\, \mathrm{d}x = 0.$$

Then g belongs to the kernel of T_1. Therefore, since $\ker T_1 \neq \{0\}$, T_1 is not injective.

Exercise 3: $X_1 = L^\infty([-1;1])$ and $X_2 = \mathcal{C}([-1;1])$ both endowed by the $\|\cdot\|_\infty$ norm. Consider $\forall i \in \{1,2\}$,

$$T_i : X_i \to X_i \quad w/ \quad \forall g \in X_i,\ T_i g = \int_{-1}^{1} \frac{x}{1+x^2} g(x)\,dx .$$

1) Let $g \in L^\infty([-1;1])$,

$\|T_1 g\|_\infty = \left\| \int_{-1}^{1} \frac{x}{1+x^2} g(x)\,dx \right\|_\infty$. Since Tg is a constant function, we obtain:

$$\|T_1 g\|_\infty = \left| \int_{-1}^{+1} \frac{x}{1+x^2} g(x)\,dx \right| \leq \int_{-1}^{1} \frac{|x|}{1+x^2} \times |g(x)|\,dx \leq \underbrace{\|g\|_\infty}_{<+\infty \text{ since } g \in L^\infty([-1;1])} \cdot \int_{-1}^{1} \frac{|x|}{1+x^2}\,dx$$

$$\leq \|g\|_\infty \times \frac{2}{2} \int_0^1 \frac{2x}{1+x^2}\,dx = \|g\|_\infty \times \ln(2)$$

$\forall g \in L^\infty([-1;1])$, $\|T_1 g\|_\infty \leq M \|g\|_\infty$ with $M = \ln(2)$. Hence T_1 is bounded. Since T_1 is clearly linear (using the linearity of the integral) we conclude that:

$$\boxed{T_1 \text{ is a continuous operator}} . \square$$

2) Operator norm of T_1?

- Using Thm 1, we have : $\|T_1\|_{\mathcal{L}(X_1,X_1)} \leq \ln(2)$.

(Indeed: $\|T_1\|_{\mathcal{L}(X_1,X_1)} = \inf \{M \geq 0 : \forall x \in X_1,\ \|T_1 x\|_{X_1} \leq M \cdot \|x\|_{X_1}\}$).

- $\|T_1\|_{\mathcal{L}(X_1)} = \sup\limits_{\|g\|_\infty = 1} \|T_1 g\|_\infty \geq \|T_1 g_0\|_\infty$ where $g_0 : x \in [-1;1] \mapsto \mathbb{1}_{[0;1]}(x) - \mathbb{1}_{[-1;0]}(x)$.

($g_0 \in X_1$ and $\|g_0\|_{X_1} = 1$) ($\|g_0\|_{X_1} = \|g_0\|_\infty$)

$$\|T_1 g_0\|_\infty = \left| \int_0^1 \frac{x}{1+x^2}\,dx - \int_{-1}^{0} \frac{x}{1+x^2}\,dx \right| = 2\left| \int_0^1 \frac{x}{1+x^2}\,dx \right| = \ln(2).$$

So $\|T_1\|_{\mathcal{L}(X_1)} \geq \ln(2)$.

As a conclusion: $\boxed{\|T_1\|_{\mathcal{L}(X_1)} = \ln(2)} . \square$

3) • using (1), we have : $\|T_2\|_{\mathcal{L}(X_2)} \leq \ln(2)$.

• For $\geqslant$ we cannot use g_0 since g_0 isn't continuous on $[-1;1]$.

Therefore, we define $\forall n \in \mathbb{N}$, $g_n \in X_2$ by

$$g_n(x) = \begin{cases} 1, & x \in [1/n; 1] \\ nx, & x \in (-1/n; 1/n) \\ -1, & x \in [-1; -1/n] \end{cases}.$$

Notice that, for any $n \in \mathbb{N}$, we have $\|g_n\|_\infty = 1$. And:

$$\|T_2 g_n\|_\infty \geqslant 2 \int_{1/n}^{1} \frac{x}{1+x^2} dx = \ln(2) - \ln\left(1 + \frac{1}{n^2}\right) \xrightarrow[n \to +\infty]{} \ln(2).$$

So the operator norm of T_2 is equal to $\ln(2)$ as well.

$$\boxed{\|T_2\|_{\mathcal{L}(X_2)} = \ln(2)}. \square$$

4) $g(x) = x^2$. $g \in X_1$.

$T_1 g = \int_{-1}^{1} \frac{x}{1+x^2} \times x^2 dx = \int_{-1}^{1} \frac{x^3}{1+x^2} dx = 0$ since we integrate an "odd" function $\left(\frac{x^3}{1+x^2} = - \frac{(-x)^3}{1+(-x)^2} \right)$ on a symmetric interval.

So : $\boxed{g \in \operatorname{Ker}(T_1)}$. Since $g \neq [x \mapsto 0]$, $\operatorname{Ker}(T_1) \neq \{0\}$.

As a conclusion: $\boxed{T_1 \text{ is not injective}}. \square$

Theory

Question 1. *(4 points)* (i) Let $(X_1, \mathcal{A}_1, \mu_1), (X_2, \mathcal{A}_2, \mu_2)$ be measure spaces. Define the product $\sigma-$algebra $\mathcal{A}_1 \times \mathcal{A}_2$, and the product measure $\mu_1 \times \mu_2$.

(ii) Let $(\mathbb{R}^k, \mathcal{L}(\mathbb{R}^k), \lambda_k)$ be the standard Lebesgue measure space on $\mathbb{R}^k$. What is the relation between $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^m) \times \mathcal{L}(\mathbb{R}^n), \lambda_m \times \lambda_n)$ and $(\mathbb{R}^{m+n}, \mathcal{L}(\mathbb{R}^{m+n}), \lambda_{m+n})$?

Solution. See Lecture 10.

Question 2. *(4 points)* State and prove the (Lebesgue) dominated convergence theorem. State a sufficient condition in order to apply the theorem, when the sequence of functions is defined on a set with finite measure.

Solution. See Lectures 8 and 9.

Question 3. *(4 points)* Let $(X, \|\cdot\|)$ be a Banach space. Is it true that $C \subset X$ is compact $\iff$ C is closed and bounded? Is it true under some assumption on X?

State and prove the Riesz theorem (about the compactness of the unit ball; if you use a lemma, state it, but it is not necessary to write its proof).

Solution. See Lecture 15.

Question 4 *(4 points)* Let $(X, \mathcal{A}, \mu)$ be a measure space. Show that the normed space $L^p(X, \mathcal{A}, \mu)$, endowed with the usual norm $\|\cdot\|_p$, is a Banach space.

Solution. See Lecture 16.