Lecture Notes

SDE

YEAR 2023/2024

Semester Review

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Course 1

Information

		Major / Elective
		☐ Major
		☐ Elective
Credit	Time	Room
Professor	Phone	E-mail
Notes		

Score

Midterm	Final	Grade

Attendance

Week 1	Week 2	Week 3	Week 4	Week 5	Week 6	Week 7	Week 8
Week 9	Week 10	Week 11	Week 12	Week 13	Week 14	Week 15	Week 16

Notice

Assignment

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Course 1 › # Lecture 1

Title

Chapter : Stochastic Differential Equations (END OF THE COURSE)

Summary

Assignment

Deadline.

Notes

Course. Stochastic Differential Equation | Title. END OF THE COURSE : Study of SDEs. | Date. 05/02/2024
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Definitions :
- solution of a SDE : a) b) & c) conditions.
- strong solution . (+ eq. def w/ $(\mathcal{G}_t)_t$)
- Weak solution .

$\Downarrow$

- Uniqueness in law .
- Pathwise uniqueness .

$\Uparrow$

Def [Solution of SDE(b,σ)]: The couple of processes

$(X,B) = (\Omega, \mathcal{F}, (\mathcal{F}_t)_t, (X_t)_t, (B_t)_t, \mathbb{P})$ is a solution of

$$(SDE(b,\sigma)) : \begin{cases} dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t \\ X_u = \eta \sim \mathcal{L}(\eta) \end{cases}$$ if :

a) $B = (\Omega, \mathcal{F}, (\mathcal{F}_t)_t, (B_t)_t, \mathbb{P})$ is a d-dim $\mathcal{C}^0$ & standard BM ;

b) $\forall t \in [0,T]$, $X_t = \eta + \int_u^t b(s,X_s)ds + \int_u^t \sigma(s,X_s)dB_s$ $\mathbb{P}$-a.s . (∗)

c) η is a R.V defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t)$ such that $\eta \in \mathcal{F}$ (meas) & $\eta \sim \mathcal{L}(\eta)$.

(∗) i.e : $\mathbb{P}\left(\left\{\forall t \in [0,T],\ X_t = \eta + \int_u^t b(s,X_s)ds + \int_u^t \sigma(s,X_s)dB_s\right\}\right) = 1$.

N.B: Note that when we speak about a solution of SDE(b,σ), we do not fix a priori the filtered proba. space and the brownian motion B. When we fix these objects, we will say so explicitely.
∃ several notions of existence and uniqueness of SDEs.

Def [Strong solution]: A SDE has a strong solution, if for every continuous & standard BM $B = (\Omega, \mathcal{F}, (\mathcal{F}_t)_t, (B_t)_t, \mathbb{P})$ and for every RV η defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$, $\mathcal{F}_u$- measurable and following $\mathcal{L}(\eta)$, there exists a process X on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ such that (X,B) is a solution of the SDE (cf. previous DEF).
I.e : (cos DEF) (X,B) is a strong solution of the SDE if X is adapted to the augmented natural filtration of B, i.e X is $\mathcal{G}_t$-adapted.

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Def [Weak Solution]: We say that SDE has a weak solution if there exists a BM $B = (\Omega, \mathcal{F}, (\mathcal{F}_t)_t, (B_t)_t, \mathbb{P})$, a RV η defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ such that η is $\mathcal{F}_u$-measurable, $\eta \sim \mathcal{L}(\eta)$ and X defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ such that (X, B) is a solution of the SDE (cf 1^{st} DEF).

Example: $\begin{cases} dX_t = \alpha X_t dB_t \\ X_0 = 1 \end{cases}$. For every BM $B = (\Omega, \mathcal{F}, (\mathcal{F}_t)_t, (B_t)_t, \mathbb{P})$ we can find the process such that (X, B) satisfies : $X_t = 1 + \int_0^t \alpha X_s dB_s$. It is: $X_t = e^{\alpha B_t - \frac{1}{2}\alpha^2 t}$. (X, B) is a STRONG SOLUTION.

Def [Weak uniqueness / Uniqueness in law]: There is uniqueness in law (or weak uniqueness) if, given two solutions $(\Omega^i, \mathcal{F}^i, (\mathcal{F}_t^i)_t, (X_t^i)_t, (B_t^i)_t, \mathbb{P}^i)$ $(i \in \{1,2\})$ possibly defined on two different probability spaces and wrt to two different BMs, (X^1, B^1) and (X^2, B^2) are equivalent, i.e have the same law, i.e: X^1 and X^2 have the same law.
In other (simpler) words : if all the solutions have the same law.

Def [Pathwise Uniqueness]: There is pathwise uniqueness if, given two solutions $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, (X_t^i)_t, (B_t), \mathbb{P})$ $(i \in \{1,2\})$ defined on the same probability space and wrt the same BM, X^1 and X^2 are indistinguishable, i.e: $\mathbb{P}(\{\forall t \in [0,T], X_t^1 = X_t^2\}) = 1$.
In other words : if, whenever the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ and the $(\mathcal{F}_t)_t$-BM are fixed, two solutions X^1 and X^2 such that $X_0^1 = X_0^2$ $\mathbb{P}$-a.s are indistinguishable.

N.B: Strong solution ⇒ Weak Solution ; Pathwise Unique ⇒ Unique in law.

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$\boxed{dX_t = \mathrm{sign}(X_t)\,dB_t \;,\; X_0 = 0}$

Example : Weak Solution with uniqueness in law BUT NO pathwise uniqueness : TANAKA'S EQUATION. $dX_t = \mathrm{sgn}(X_t)\,dB_t \;,\; X_0 = 0.$

⟶ To show this : Lévy's characterisation of a BM and its corollary : "The real BM is the only local mg with quadratic variation equal to t".

Examples : SDEs with explicit strong and pathwise unique solution :

- Ornstein Uhlenbeck process :
  $\left[\begin{array}{l} A \in \mathcal{M}_{n\times n}(\mathbb{R}),\ X : n\text{dim process}, \\ \sigma \in \mathcal{M}_{n\times d}(\mathbb{R}),\ B : d\text{dim } \mathcal{F}^0 \text{ std BM} \end{array}\right]$ $\begin{cases} dX_t = -AX_t\,dt + \sigma\,dB_t \\ X_0 = x \end{cases}$ $\rightarrow \boxed{X_t = e^{-At}x + \int_0^t e^{-A(t-s)}\sigma\,dB_s}$

- Geometric Brownian Motion : $\begin{cases} dX_t = X_t\,\mu\,dt + X_t\,\sigma\,dB_t \\ \boxed{X_t = e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}} \end{cases}$.

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EXISTENCE AND UNIQUENESS THMS :

$$(SDE): \begin{cases} dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t \\ X_u = \eta \end{cases}$$

ASSUMPTIONS A : $b : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$
$\sigma : [0,T] \times \mathbb{R}^n \longrightarrow \mathcal{M}_{n \times d}(\mathbb{R})$

1) b and σ measurable in (t,x) ;
2) Sublinear growth: $\exists M>0$ s.t $\forall x \in \mathbb{R}^n, \forall t \in [0,T]$, $\begin{cases} |b(t,x)| \leq M(1+|x|) \\ |\sigma(t,x)| \leq M(1+|x|) \end{cases}$;
3) Lipschitz continuity: $\exists L>0$ s.t $\forall x,y \in \mathbb{R}^n, \forall t \in [0,T]$, $\begin{cases} |b(t,x)-b(t,y)| \leq L|x-y| \\ |\sigma(t,x)-\sigma(t,y)| \leq L|x-y| \end{cases}$.

N.B: We can define $S^2 = \left\{ \text{continuous processes } X \text{ s.t } \mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^2\right] < +\infty \right\}$
$= L^2(\Omega, \mathcal{C}^0([0,T];\mathbb{R}^n))$

$X : \Omega \to \mathcal{C}^0([0,T];\mathbb{R}^n)$, $\omega \mapsto X_\cdot(\omega)$; $\int |\ |^2 d\mathbb{P} < +\infty$ i.e $\mathbb{E}[\ldots] < +\infty$; $[0,T] \to \mathbb{R}^n$, $t \mapsto X_t(\omega)$ continuous.

S^2 is Banach with the norm $\|X\|_{S^2} = \left(\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^2\right]\right)^{1/2}$.

We can observe that: $S^2[0,T] \subset M^2[0,T]$.

Thm [∃! in S^2]: Assume b, σ satisfy A. Let $u \geq 0$, $\eta \in L^2(\Omega, \mathcal{F}_u, \mathbb{P}; \mathbb{R}^n)$.
Then: $\exists$ X strong solution, pathwise unique, s.t :
$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^2\right] < +\infty \quad (\text{i.e in } S^2).$$

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$$(SDE): \begin{cases} dX_t = b(t,X_t)\,dt + \sigma(t,X_t)\,dB_t \\ X_u = \eta \end{cases}$$ with Assumptions A

Thm: $\exists!$ in S^2

ONLY (A)1) & (A)2): Prop: Exit time from open ball of radius R.

ONLY (A)1) & (A)3): Localisation Thm

Same BM!

Indistinguishability of X_1 & X_2 up to the exit time of the open set $D \subset \mathbb{R}^n$ where $b_1 = b_2$ & $\sigma_1 = \sigma_2$

ONLY (A)1) & 2): Thm: a priori estimate

$$\|X\|_{S^2}^2 \leq C(\underset{\uparrow\,(A)2)}{T,M})\left(1 + \|\eta\|_{L^2(\Omega)}^2\right)$$

WE WOULD LIKE TO WEAKEN ASSUMPTIONS A

(SDE) with Assumptions A'

Thm: $\exists!$ in S^2

EXISTENCE & UNIQUENESS IN LAW: EVEN WEAKER ASSUMPTIONS

$$(SDE): \begin{cases} dX_t = b(t,X_t)\,dt + \sigma(t,X_t)\,dB_t \\ X_u = x \quad : \text{DETERMINISTIC!} \end{cases}$$

- x deterministic
- b meas. & bdd
- σ satisfies A'; $\sigma = \sigma^T$; $\forall t,x$, the smallest eigenvalue of $\sigma(t,x)$ is bounded below by $\lambda > 0$.

($\Leftrightarrow \sigma(t,x)^{-1}$ is well defined & bdd).

Thm: $\exists$ a weak solution; Uniqueness in law.

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1 JAN 2 FEB 3 MAR 4 APR 5 MAY 6 JUN

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