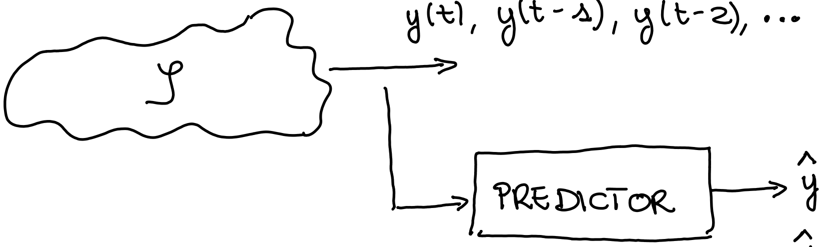
MIDA 1 - EXERCISE CLASS 5

PREDICTION – THEORY OF KOLMOGOROV - WIENER

the data we have.

$y(t), y(t-1), y(t-2), \dots$



what we want to compute.

$\hat{y}(t+1|t)$

$\hat{y}(t+2|t)$

$\hat{y}(t+k|t)$

K : prediction horizon

STEP 0 : ANALYSIS OF THE SYSTEM

$y(t) = \frac{1}{2} y(t-2) + \eta(t) + 4\eta(t-1)$, $\eta(t) \sim WN(0,1)$

$$y(t) = \frac{1 + 4z^{-1}}{1 - \frac{1}{2}z^{-2}}\eta(t) = \underbrace{\frac{z^2 + 4z}{z^2 - \frac{1}{2}}}_{\text{positive coefficient}}\eta(t)$$

zeros : $z_1 = 0$, $z_2 = -4$

poles : $z_{1,2} = \frac{1}{\sqrt{2}}$ Because poles are inside the unit.

$y(t)$ is stationary

STEP 1: EVALUATION OF THE CANONICAL REPRESENTATION

VERY IMPORTANT

$y(t) = \frac{C(z)}{A(z)} \cdot \mu(t)$

Always check that $W(z)$ is written in canonical representation when we want to do prediction.

what we have to check:

- NUM and DEN must be monic (coef. of highest power of z equal 1) ✓
- NUM and DEN must be coprime (no common factors to simplify) ✓
- NUM and DEN must be with the same degree ✓
- POLES and ZEROS inside the unit disk ✗ → $z_2 = -4$ from the zeros. So we have to "remove" it.

WE USE ALL-PASS FILTER !!

$$y(t) = \frac{1 + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-2}} \cdot \frac{1 + 4z^{-1}}{1 + \frac{1}{4}z^{-1}} \cdot \eta(t)$$

This is still a white noise.

$e(t) = \frac{1 + 4z^{-1}}{1 + \frac{1}{4}z^{-1}}\eta(t)$

$(z=1)$ $\frac{1+4}{5/4} = \frac{5}{5/4} = 4$

so: $4\eta(t) \sim WN(0,16)$.

$e(t) \sim WN(0, 16)$

$$\boxed{y(t) = \frac{1 + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-2}}e(t)}$$

CANONICAL REP.

Now that we have the CANONICAL REPRESENTATION, we can compute the predictor.

STEP 2: COMPUTATION OF THE PREDICTOR

$$\begin{array}{l|l} \underbrace{1 + \tfrac{1}{4} z^{-1}}_{C(z)} & \underbrace{1 + 0 z^{-1} - \tfrac{1}{2} z^{-2}}_{A(z)} \\ \hline 1 + 0 z^{-1} - \tfrac{1}{2} z^{-2} & \underbrace{1}_{E(z)} \\ \hline / \ \underbrace{\tfrac{1}{4} z^{-1} + \tfrac{1}{2} z^{-2}}_{F(z)} & \end{array}$$

$$y(t) = \frac{C(z)}{A(z)} e(t) = E(z) e(t) + \frac{F(z)}{A(z)} e(t)$$

$$y(t) = e(t) + \frac{\frac{1}{4} z^{-1} + \frac{1}{2} z^{-2}}{1 - \frac{1}{2} z^{-2}} e(t) = e(t) + \frac{\frac{1}{4} + \frac{1}{2} z^{-1}}{1 - \frac{1}{2} z^{-2}} e(t-1)$$

$e(t)$ → not predictable
1. white noise at time t
2. $E[e(t)] = 0$

$\frac{\frac{1}{4} + \frac{1}{2} z^{-1}}{1 - \frac{1}{2} z^{-2}} e(t-1)$ → is predictable

QUESTION: what is predictable given data at time $t-1$?

$$\hat{y}(t|t-1) = \frac{\frac{1}{4} + \frac{1}{2} z^{-1}}{1 - \frac{1}{2} z^{-2}} e(t-1)$$

PREDICTOR FROM NOISE

$$y(t) = \frac{C(z)}{A(z)} e(t), \quad e(t) = \frac{A(z)}{C(z)} y(t)$$

$$\hat{y}(t|t-1) = \frac{\frac{1}{4} + \frac{1}{2} z^{-1}}{\cancel{1 - \frac{1}{2} z^{-2}}} \cdot \frac{\cancel{1 - \frac{1}{2} z^{-2}}}{1 + \frac{1}{4} z^{-1}} \cdot y(t-1) = \frac{\frac{1}{4} + \frac{1}{2} z^{-1}}{1 + \frac{1}{4} z^{-1}} y(t-1)$$

PREDICTOR FROM DATA

$$\hat{y}(t+1|t) = \frac{\frac{1}{4} + \frac{1}{2} z^{-1}}{1 + \frac{1}{4} z^{-1}} y(t)$$

$$\hat{y}(t+1|t) = -\frac{1}{4} \hat{y}(t|t-1) + y(t) + \frac{1}{2} y(t-1)$$

} Expression of the predictor "in the time domain".

↓

useful if we have real data.

for example...

t	$y(t)$	$\hat{y}(t+1\|t)$
≤ 0	0	$\hat{y}(1\|0) = 0$
1	1	$\hat{y}(2\|1)$
2	0	$\hat{y}(3\|2)$
3	$-\frac{1}{2}$	$\hat{y}(4\|3)$

STANDARD INITIALIZATION

$$\hat{y}(2|1) = -\frac{1}{4}\hat{y}(1|0) + \frac{1}{4}y(1) + \frac{1}{2}y(0) = -\frac{1}{4}\cdot 0 + \frac{1}{4}\cdot 1 + \frac{1}{2}\cdot 0 = \frac{1}{4}$$

$$\hat{y}(3|2) = -\frac{1}{4}\hat{y}(2|1) + \frac{1}{4}y(2) + \frac{1}{2}y(1) = -\frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot 0 + \frac{1}{2}\cdot 1 = -\frac{1}{16} + \frac{8}{16} = +\frac{7}{16}$$

$$\hat{y}(4|3) = -\frac{1}{4}\hat{y}(3|2) + \frac{1}{4}y(3) + \frac{1}{2}y(2) = -\frac{1}{4}\cdot\frac{7}{16} + \frac{1}{4}\cdot\left(-\frac{1}{2}\right) + \frac{1}{2}\cdot 0 = -\frac{15}{64}$$

Remark: when we have to compute the one step prediction

$$\hat{y}(t|t-1) = \frac{C(z)-A(z)}{A(z)}e(t) \qquad \text{PRED. from NOISE}$$

$$\hat{y}(t|t-1) = \frac{C(z)-A(z)}{C(z)}y(t) \qquad \text{PRED. from DATA}$$

with $y(t) = \frac{C(z)}{A(z)}e(t)$ in $\boxed{\text{canonical representation}}$ (!)

These formula are only for 1-step predictions... If we want >1-steps predictions, we need to use the long-division method.

Example:

$$C(z) = 1 + \frac{1}{4}z^{-1} \qquad A(z) = 1 - \frac{1}{2}z^{2}$$

$$\hat{y}(t|t-1) = \frac{\cancel{1} + \frac{1}{4}z^{-1} - \cancel{1} + \frac{1}{2}z^{2}}{1+\frac{1}{4}z^{-1}}\cdot y(t) = \frac{\frac{1}{4}+\frac{1}{2}z^{-1}}{1+\frac{1}{4}z^{-1}}\,y(t-1) \qquad \text{PR. from DATA}$$

STEP 3: EVALUATE THE PREDICTION ERROR

LAST THING WE USUALLY HAVE TO DO.

$$\varepsilon(t|t-1) = y(t) - \hat{y}(t|t-1) = E(z)e(t) + \cancel{\frac{F(z)}{A(z)}e(t-1)} - \cancel{\frac{F(z)}{A(z)}e(t-1)} = E(z)e(t)$$

$$y(t) = E(z)e(t) + \frac{F(z)}{A(z)}e(t-1)$$

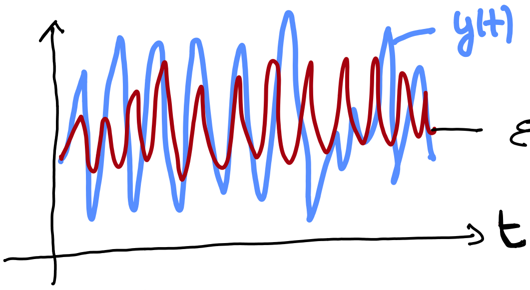
$$\boxed{\varepsilon(t|t-k) = E(z)e(t)} \rightarrow \text{GENERAL FORMULA}$$

$$E\left[\varepsilon(t|t-k)^2\right] = E\left[\left(E(z)e(t)\right)^2\right] = E\left[e(t)^2\right] = 16$$

- The best we can do is to obtain an error that is a white noise (no information about the process)
- If the error is white the variance of the prediction error is minimum.

$\gamma_y(0) = \frac{68}{3} \approx 23$

$E[\varepsilon(t|t-1)^2] = 16$



The error has around the same variance as the process, so the prediction will not be very good.

But the theory says that one prediction is the best, bc the error is WN.

→ Now we want to compute $\hat{y}(t|t-2) = \hat{y}(t+2|t)$ GIVEN THE SAME PROCESS.

$$C(z) = 1 + \frac{1}{4}z^{-1}$$

$$A(z) = 1 + 0z^{-1} - \frac{1}{2}z^{-2}$$

$$E(z) = 1 + \frac{1}{4}z^{-1}$$

$$\begin{array}{l} 1 + \frac{1}{4}z^{-1} \\ 1 + 0z^{-1} - \frac{1}{2}z^{-2} \\ \hline / \quad \frac{1}{4}z^{-1} + \frac{1}{2}z^{-2} \\ \quad \frac{1}{4}z^{-1} + 0z^{-2} - \frac{1}{8}z^{-3} \\ \hline / \quad \frac{1}{2}z^{-2} + \frac{1}{8}z^{-3} \end{array}$$

$$F(z) = \frac{1}{2}z^{-2} + \frac{1}{8}z^{-3}$$

2 steps ahead

long division w/ 2 steps.

$$y(t) = \underbrace{e(t) + \frac{1}{4}e(t-1)}_{\text{NOT PRED.}} + \underbrace{\frac{\frac{1}{2} + \frac{1}{8}z^{-1}}{1 - \frac{1}{2}z^{-2}}e(t-2)}_{\text{PREDICTABLE}}$$

$$\left(y(t) = \frac{C(z)}{A(z)}e(t) = \frac{A(z)E(z) + F(z)}{A(z)}e(t) = E(z)e(t) + \frac{F(z)}{A(z)}e(t) \right).$$

⚠ what is predictable given data at time t-2? ⚠

$$\hat{y}(t|t-2) = \frac{\frac{1}{2} + \frac{1}{8}z^{-1}}{1 - \frac{1}{2}z^{-2}}e(t-2)$$

PR. from NOISE

$y(t) = \frac{C(z)}{A(z)}e(t)$ so $e(t) = \frac{A(z)}{C(z)}y(t)$.

$$\hat{y}(t|t-2) = \frac{\frac{1}{2} + \frac{1}{8}z^{-1}}{\cancel{1 - \frac{1}{2}z^{-2}}} \cdot \frac{\cancel{1 - \frac{1}{2}z^{-2}}}{1 + \frac{1}{4}z^{-1}} \cdot y(t-2) = \frac{\left(\frac{1}{2} + \frac{1}{8}z^{-1}\right)}{\left(1 + \frac{1}{4}z^{-1}\right)}y(t-2) = \frac{1}{2} \cdot \frac{\cancel{1 + \frac{1}{4}z^{-1}}}{\cancel{1 + \frac{1}{4}z^{-1}}}y(t-2)$$

$$= \frac{1}{2}y(t-2)$$

very simple predictor in that case.

$$\hat{y}(t|t-2)\left(1 + \frac{1}{4}z^{-1}\right) = \left(\frac{1}{2} + \frac{1}{8}z^{-1}\right)y(t-2)$$

$$\hat{y}(t|t-2) + \frac{1}{4}\hat{y}(t-1|t-3) = \frac{1}{2}y(t-2) + \frac{1}{8}y(t-3)$$

$$\hat{y}(t|t-2) = -\frac{1}{4}\hat{y}(t-1|t-3) + \frac{1}{2}y(t-2) + \frac{1}{8}y(t-3)$$

$$\hat{y}(t+2|t) = -\frac{1}{4}\hat{y}(t+1|t-1) + \frac{1}{2}y(t) + \frac{1}{8}y(t-1)$$

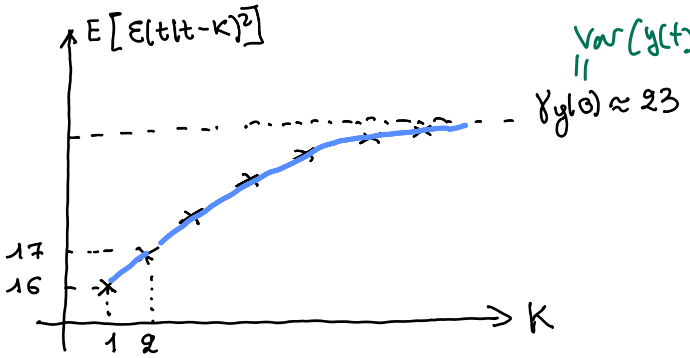
$$\varepsilon(t|t-2) = \overset{\text{general formula}}{E(z)}\, e(t) = \left(1+\frac{1}{4}z^{-1}\right)e(t) = e(t)+\frac{1}{4}e(t-1)$$

$$E\left[\varepsilon(t|t-2)^2\right] = \left(1+\frac{1}{16}\right)16 = 17 \longrightarrow$$ this variance is greater than the variance of the prediction with one step ⚠

what is $\underbrace{\varepsilon(t|t-k) = E(z)e(t)}_{MA(\infty) \text{ of } y(t)}$ if $k\to\infty$?

$$E\left[\varepsilon(t|t-k)^2\right] = E\left[\left(y(t)-E[y(t)]\right)^2\right] = E\left[\left(y(t)-\hat{y}(t|t-k)\right)^2\right]$$

The prediction of $y(t)$ for $k\to\infty$ is only its expected value, because all the information about $y(t)$ is in the prediction error, that is the $MA(\infty)$ representation of $y(t)$.



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ex. $e(t) \sim WN(0,1)$

$$\frac{1}{2}y(t) = -\frac{1}{3}y(t-1) - \frac{1}{18}y(t-2) + 3e(t-2) - 8e(t-3) - 3e(t-4)$$

Q: $\hat{y}(t|t-1)$?

$$\frac{1}{2}y(t) + \frac{1}{3}z^{-1}y(t) + \frac{1}{18}z^{-2}y(t) = \left(3z^{-2} - 8z^{-3} - 3z^{-4}\right)e(t)$$

$$y(t) = \frac{3z^{-2} - 8z^{-3} - 3z^{-4}}{\frac{1}{2} + \frac{1}{3}z^{-1} + \frac{1}{18}z^{-2}}\, e(t)$$

→ But we have to check if the TF is canonical, or not.

1. NUM and DEN are not monic

$$y(t) = \frac{3z^{-2}\left(1 - \frac{8z^{-3}}{3z^{-2}} - \frac{3z^{-4}}{3z^{-2}}\right)}{\frac{1}{2}\left(1 + \frac{1/3}{1/2}z^{-1} + \frac{1/18}{1/2}z^{-2}\right)}\, e(t) = \frac{\left(1 - \frac{8}{3}z^{-1} - z^{-2}\right)}{\left(1 + \frac{2}{3}z^{-1} + \frac{1}{9}z^{-2}\right)} \cdot \frac{3z^{-2}}{\frac{1}{2}} \cdot e(t)$$

again WN.

$\eta(t) = 6\cdot e(t-2)$, $\eta(t) \sim WN(0, 6^2\cdot 1)$

$$y(t) = \frac{z^2 - \frac{8}{3}z - 1}{z^2 + \frac{2}{3}z + \frac{1}{9}} \eta(t)$$

Positive power notation.

zeros:

$z^2 - \frac{8}{3}z - 1 = 0 \quad , \quad z_{1,2} = \begin{cases} 3 \\ -\frac{1}{3} \end{cases}$

poles:

$z_{1,2} = -\frac{1}{3}$ (root of multiplicity 2).

$$z_{1,2} = \frac{\frac{8}{3} \pm \sqrt{\frac{64}{9} + 4}}{2} = \frac{\frac{8}{3} \pm \sqrt{\frac{100}{9}}}{2} = \frac{\left(\frac{8}{3} \pm \frac{10}{3}\right)}{2} = \begin{cases} \frac{6}{2} = 3 \\ -\frac{2}{3} \cdot \frac{1}{2} = -\frac{1}{3} \end{cases}$$

$$y(t) = \frac{(1 - 3z^{-1})\cancel{(1 + \frac{1}{3}z^{-1})}}{(1 + \frac{1}{3}z^{-1})\cancel{(1 + \frac{1}{3}z^{-1})}} \cdot \eta(t) = \frac{1 - 3z^{-1}}{1 + \frac{1}{3}z^{-1}} \eta(t)$$

$1 - 3z^{-1} = 0 \Leftrightarrow z = 3$

Not in the unit disk.

We have to remove it!

$$y(t) = \frac{1 - \boxed{\frac{1}{3}}z^{-1}}{1 + \frac{1}{3}z^{-1}} \cdot \frac{1 - 3z^{-1}}{1 - \boxed{\frac{1}{3}}z^{-1}} \cdot \eta(t)$$

still a WN!

$$\xi(t) = \frac{1 - 3z^{-1}}{1 - \frac{1}{3}z^{-1}} \eta(t)$$

$\xi(t) \sim WN(0, 3^2 \cdot 36)$

The canonical representation:

$$\boxed{y(t) = \frac{1 - \frac{1}{3}z^{-1}}{1 + \frac{1}{3}z^{-1}} \xi(t) \qquad \xi(t) \sim WN(0, 18^2)}$$

$$\hat{y}(t|t-1) = \frac{C(z) - A(z)}{C(z)} y(t) = \frac{(1 - \frac{1}{3}z^{-1}) - (1 + \frac{1}{3}z^{-1})}{(1 - \frac{1}{3}z^{-1})} = \frac{-\frac{2}{3}z^{-1}}{1 - \frac{1}{3}z^{-1}} y(t)$$

$$\boxed{\hat{y}(t|t-1) = \frac{-\frac{2}{3}}{1 - \frac{1}{3}z^{-1}} y(t-1)}$$

So, in the time domain:

$$\hat{y}(t|t-1)\left(1 - \frac{1}{3}z^{-1}\right) = -\frac{2}{3}y(t-1) \quad , \quad \hat{y}(t|t-1) = \frac{1}{3}\hat{y}(t-1|t-2) - \frac{2}{3}y(t-1)$$

$$\boxed{\hat{y}(t+1|t) = \frac{1}{3}\hat{y}(t|t-1) - \frac{2}{3}y(t)}$$

Predictor in the time domain.

$$E[\varepsilon(t|t-1)^2] = E[\xi(t)^2] = 18^2 = 324$$

1 step predictor so $Var(\varepsilon(t|t-1)) = Var(\xi(t))$