Exercise 2 - Brownian Bridge (V/IV - Bonus):

Real process X, SDE :
(Real and $\mathcal{C}^0$ standard BM)

$$\begin{cases} dX_t = -\frac{1}{1-t} X_t \, dt + dB_t \quad , \quad 0 \le t \le 1 \, ; \\ X_0 = 0 \, . \end{cases}$$

1) $\begin{cases} dX_t = b(t,X_t)\, dt + \sigma(t,X_t)\, dB_t \\ X_0 = 0 \in \mathbb{R} \end{cases}$ w/ : $\begin{cases} b(t,x) := -\frac{x}{1-t} \\ \sigma(t,x) := 1 \end{cases}$.

. We have that X solves SDE on $[0,1)$ if it solves SDE on $[0,T]$, $\forall T \in [0,1)$.

. SDE is of the form from ex I : $dX_t = \alpha(t)dt + \beta(t) X_t dt + \sigma(t) dB_t$

with : $\alpha(t) = 0$, $\beta(t) = -\frac{1}{1-t}$, $\sigma(t) = 1$. And $\sup_{[0,T]} |\beta(t)| = \frac{1}{1-T} < +\infty$.

$\longrightarrow$ For the same argument we used in Ex I , we have that :

$\boxed{\exists \text{ pathwise unique strong solution to the SDE, } \forall\, 0 \le T < 1}$. □

2) <u>Find the solution X ?</u>

Using the ex I we have :

$$\Lambda(t) = \int_0^t \beta(s)\, ds = -\int_0^t \frac{1}{1-s} ds = \log(1-t)$$

$$\Rightarrow \quad X_t = e^{\Lambda(t)} \left[0 + \int_0^t e^{-\Lambda(s)} \sigma(s) dB_s \right]$$

$$\boxed{X_t = (1-t) \int_0^t \frac{1}{1-s} \, dB_s \quad : \text{ THE UNIQUE SOLUTION OF SDE.}}$$

... It seems familiar ... (Brownian Bridge) ...

3) $X_t = (1-t) \int_0^t \frac{1}{1-s} dB_s$ is a continuous gaussian process since $f(s) = \frac{1}{1-s}$ is deterministic and in $L^2([0,T])$ $\forall\, 0 \le T \le 1$.

Let's now precise its parameters.

$\mathbb{E}[X_t] = 0$ ($f \in L^2[0,T]$ so its S.I is a mg.).

$$\mathrm{Cov}(X_t, X_s) = \mathrm{Cov}\left((1-t)\int_0^t \frac{1}{1-r} dB_r ,\ (1-s)\int_0^s \frac{1}{1-r} dB_r \right)$$

$$= (1-t)(1-s) \cdot \mathrm{Cov}\left(\int_0^t \frac{1}{1-r} dB_r ,\ \int_0^s \frac{1}{1-r} dB_r \right)$$

$$= (1-t)(1-s) \cdot \left[\mathbb{E}\left[\int_0^t \frac{1}{1-r} dB_r \int_0^s \frac{1}{1-r} dB_r \right] - \underbrace{\mathbb{E}[\ldots]\mathbb{E}[\ldots]}_{=0} \right]$$

$$= (1-t)(1-s)\, \mathbb{E}\left[\int_0^t \frac{1}{1-r} dB_r \int_0^s \frac{1}{1-r} dB_r \right]$$

$$= (1-t)(1-s)\, \mathbb{E}\left[\int_0^T \frac{\mathbb{1}_{[0,t]}(r)}{1-r} dB_r \int_0^T \frac{\mathbb{1}_{[0,s]}(r)}{1-r} dB_r \right]$$

$$= (1-t)(1-s)\, \mathbb{E}\left[\int_0^T \left(\frac{1}{1-r}\right)^2 \mathbb{1}_{[0,t\wedge s]}(r)\, dr \right]$$

$$= (1-t)(1-s) \int_0^{t\wedge s} \left(\frac{1}{1-r}\right)^2 dr = (1-t)(1-s) \left[\frac{1}{1-r} \right]_0^{t\wedge s}$$

$$= (1-t)(1-s) \left[\frac{1}{1-t\wedge s} - \frac{1-t\wedge s}{1-t\wedge s} \right] = (1-t)(1-s) \frac{t\wedge s}{1-t\wedge s}$$

$$= \cancel{(1-t\wedge s)}(1-t\vee s)\frac{t\wedge s}{\cancel{1-t\wedge s}} = t\wedge s\,(1-t\vee s)$$

$$\boxed{\mathrm{Cov}(X_t, X_s) = t\wedge s\,(1-t\vee s) \quad \forall s,t \in [0,T],\ \forall T \in (0,1)} \ .\square$$

So :

$$\boxed{X_t \sim \mathcal{N}(0,\ t(1-t))} \ .\square$$

Let's introduce the "Brownian Bridge":

$$\tilde{X}_t := B_t - tB_1 \quad \forall t \in [0,1].$$

4) $\cdot$ $\tilde{X}_t$ follows a gaussian law as a Σ of two gaussian-distributed RV, $\forall t \in [0,1]$.

$\cdot$ $\mathbb{E}[\tilde{X}_t] = \mathbb{E}[B_t - tB_1] = \mathbb{E}[B_t] - t\,\mathbb{E}[B_1] = 0$.

$\cdot$ $Cov(\tilde{X}_t, \tilde{X}_s) = \mathbb{E}[\tilde{X}_t \tilde{X}_s] = \mathbb{E}[(B_t - tB_1)(B_s - sB_1)]$

$$= \mathbb{E}[B_t B_s] - s\,\mathbb{E}[B_t B_1] - t\,\mathbb{E}[B_1 B_s] + ts\,\mathbb{E}[B_1^2]$$

$$= t\wedge s - s\cdot t - \cancel{t\cdot s} + \cancel{ts} = t\wedge s - s\vee t\cdot s\wedge t$$

$$= t\wedge s\,(1 - t\vee s)$$

So $\boxed{\tilde{X}_t \sim \mathcal{N}(0, t(1-t)) \quad \forall t \in [0,1]}$.

In other words, $\boxed{X_t \text{ and } \tilde{X}_t,\ t\in[0,1),\ \text{are equivalent}}$. $\square$

5) Indistinguishable would mean that: $\mathbb{P}(\forall t \in [0,1),\ X_t = \tilde{X}_t) = 1$, i.e,

$\mathbb{P}(\{\omega \in \Omega \mid \forall t \in [0,1),\ X_t(\omega) = \tilde{X}_t(\omega)\}) = 1$.

$\boxed{\begin{array}{l}\rightarrow \text{They are not indistinguishable since } (X_t)_t \text{ is } \mathcal{F}_t\text{-adapted} \\ (\mathcal{F}_t)\text{: filtration of the B.M) but } (\tilde{X}_t)_t \text{ is not (clearly, } \tilde{X}_t \text{ depends of } \underline{\underline{B_1}}).\end{array}}$ $\square$

Eventually, we introduce :

$$\tilde{B}_t = B_t - \int_0^t \frac{B_1 - B_u}{(1-u)}\, du\ , \quad 0 \leq t < 1$$

$$\tilde{\mathcal{F}}_t = \sigma(B_s \mid 0 \leq s \leq t) \vee \sigma(B_1) \vee \mathcal{N}\ .$$

6) $(\tilde{B}_t)_t$ is :

- $\tilde{B}_0 = 0$ a.s ;
- $\tilde{B}_t$ is gaussian (standard argument : Riemann-Stieltjes sum) ;
- $\mathbb{E}[\tilde{B}_t] = 0 \quad \forall\, 0 \leq t < 1$;
- $\mathrm{Cov}(\tilde{B}_t, \tilde{B}_s) = \underset{\substack{\uparrow \\ \text{very long !}}}{\cdots} = s \wedge t$.

As a conclusion: $\boxed{(\tilde{B}_t)_t \text{ is a continuous natural BM}}$. □

7) Suppose that $\tilde{B}_t$ is $\mathcal{F}_t$ - measurable :

Then :
$$B_1 = \Bigg(\underbrace{\tilde{B}_t}_{\substack{\mathcal{F}_t\text{-meas} \\ \text{by assumption}}} \underset{\substack{\uparrow \\ \mathcal{F}_t\text{ meas}}}{-\, B_t} + \underbrace{\int_0^t B_u \cdot \frac{1}{1-u}\, du}_{\substack{\uparrow \\ \tilde{\mathcal{F}}_t\text{-meas}}} \Bigg) \cdot \underbrace{\frac{1}{\left(\int_0^t \frac{1}{1-u}\, du\right)}}_{\text{deterministic}} .$$

So B_1 would be $\mathcal{F}_t$ - measurable $\forall\, 0 \leq t < 1$. ↯

Contradiction. So : $\boxed{\tilde{B}_t \text{ is not } \mathcal{F}_t \text{ - measurable}}$. □

8) • $\tilde{B}_t$ is $\tilde{\mathcal{F}}_t$-adapted by def of $\tilde{\mathcal{F}}_t$.

• $\tilde{\mathcal{F}}_t$ is standard since $\mathcal{N} \subset \tilde{\mathcal{F}}_t$, it is right continuous (since the augmented Brownian filtration $\sigma(B_s \mid 0 \le s \le t) \vee \mathcal{N}$ is Right Continuous).

• $\tilde{B}_t$ is continuous by construction.

• $\tilde{B}_t - \tilde{B}_s \perp\!\!\!\perp \tilde{\mathcal{F}}_s$ $\forall\, 0 \le s \le t \le T < 1$. Indeed: $\forall\, 0 \le r \le s \le t$

$$\mathbb{E}\left[(\tilde{B}_t - \tilde{B}_s) B_r\right] = 0 .$$

$$\text{Moreover: } \underbrace{\mathbb{E}\left[(\tilde{B}_t - \tilde{B}_s) B_1\right] = 0 .}$$

So they are uncorrelated. And by the usual "Gaussian family" argument, we conclude that $\tilde{B}_t - \tilde{B}_s \perp\!\!\!\perp \sigma(B_1) \vee \sigma(B_r \mid 0 \le r \le s)$ and so:

$$\tilde{B}_t - \tilde{B}_s \perp\!\!\!\perp \tilde{\mathcal{F}}_s .$$

• The gaussianity of the increments follows from 6.

9) Show that $\tilde{X}_t$ is the solution to:

$$(\tilde{BB}) \begin{cases} dX_t = -\frac{1}{1-t} X_t\, dt + d\tilde{B}_t & , 0 \le t < 1 . \\ X_0 = 0 \end{cases}$$

⚠ Here we are studying the behaviour of an Itô Process under a change of filtration, with $\mathbb{P}$ fixed. So this situation is $\neq$ from the usual change of probability "Girsanov's style".

We showed that $\tilde{B}_t$ was a $\mathcal{C}^0$ std. BM wrt $(\tilde{\mathcal{F}}_t)_t$.

What we can do is to try to write $\tilde{X}_t$ as a fct. of $\tilde{B}_t$ and show that it is Itô with a stochastic differential given by $(\widetilde{BB})$.

- $\tilde{X}_t = B_t - tB_1$ so $\tilde{X}_t$ is $\tilde{\mathcal{F}}_t$ measurable.

  So: $\tilde{X}_t = \tilde{B}_t + \int_0^t \frac{B_1 - B_u}{1-u}\, du - tB_1$.

  → It's an Itô process, wrt $\tilde{\mathcal{F}}_t$.

- Let's compute $d\tilde{X}_t$:

$$d\tilde{X}_t = d\tilde{B}_t + \frac{B_1 - B_t}{1-t}\, dt - B_1\, dt$$

$$d\tilde{X}_t = d\tilde{B}_t - \frac{1}{1-t}\tilde{X}_t\, dt \; : \; \boxed{\tilde{X}_t \text{ solves } (\widetilde{BB})}. \qquad \square$$

⚠ X_t and $\tilde{X}_t$ solve the same equation, with different B.M and filtration.