Mathematical Engineering - A.Y. 2022-23

Real and Functional Analysis - Written exam - July 12, 2023

Exercise 1. Consider the measure space $([0,+\infty), \mathcal{L}([0,+\infty)))$ with the Lebesgue measure. Define the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ by

$$f_n(x) = \frac{e^{-nx} + n^2 e^{-x}}{n + n^2 + (1+n^2)x^2}, \quad x \in [0,+\infty), \quad n \in \mathbb{N}.$$

(1) Study the convergence a.e. of the sequence $\{f_n\}_{n\in\mathbb{N}}$.
(2) Study the convergence in $L^1([0,+\infty))$ of the sequence $\{f_n\}_{n\in\mathbb{N}}$.
(3) Consider the sequence $g_n(x) = f_n(x) + \chi_{[n,n+1]}$. Study the convergence in $L^1([0,+\infty))$ of the sequence $\{g_n\}_{n\in\mathbb{N}}$.

Solution.
(1) Let $x \in (0,+\infty)$

$$f_n(x) = \frac{e^{-nx} + n^2 e^{-x}}{n + n^2 + (1+n^2)x^2} = \frac{\frac{e^{-nx}}{n^2} + e^{-x}}{1 + \frac{1}{n} + \frac{(1+n^2)}{n^2}x^2} \longrightarrow \frac{e^{-x}}{1+x^2} \qquad \text{for } n \to +\infty$$

Therefore the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to the function $f(x) = \frac{e^{-x}}{1+x^2}$ almost everywhere in $[0,\infty)$.

(2) In order to show that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges in $L^1([0,+\infty))$ to the function f, we are going to use the dominated convergence theorem. We first observe that for each $n \in \mathbb{N}$, we have

$$n + n^2 + (1+n^2)x^2 \ge n^2 + (1+n^2)x^2 \ge n^2(1+x^2)$$

and

$$e^{-nx} + n^2 e^{-x} = n^2 \left(\frac{e^{-nx}}{n^2} + e^{-x} \right) \le n^2(1+e^{-x})$$

Therefore we have

$$|f_n(x)| = \frac{e^{-nx} + n^2 e^{-x}}{n + n^2 + (1+n^2)x^2} \le \frac{n^2(1+e^{-x})}{n^2(1+x^2)} = \frac{1+e^{-x}}{1+x^2} \le \frac{2}{1+x^2}$$

Letting $g(x) = \frac{2}{1+x^2}$ and observing that $g \in L^1([0,+\infty))$, we get that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges in $L^1([0,+\infty))$ to f, thanks to the dominated convergence theorem.

(3) Let $h_n(x) = \chi_{[n,n+1]}(x)$. The sequence $\{h_n\}_{n\in\mathbb{N}}$ converges almost everywhere to $h(x) \equiv 0$. Indeed, for every $x \in [0,+\infty)$, there exists $n_0 \in \mathbb{N}$ such that $n > x$ for every $n \ge n_0$, which implies $h_n(x) = 0$ for every $n \ge n_0$. Notice that the sequence $\{h_n\}_{n\in N}$ does not converge in $L^1([0,+\infty))$. Indeed, the candidate limit is h, which has norm zero, and

$$\|h_n\|_1 = \int_n^{n+1} 1 dx = 1$$

for every $n \in \mathbb{N}$. Hence the sequence $\{h_n\}_{n\in\mathbb{N}}$ does not converge in $L^1([0,+\infty))$. Finally, suppose by contradiction that the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges in $L^1([0,+\infty))$. Observing that $h_n = g_n - f_n$, and that $\{f_n\}_{n\in\mathbb{N}}$ converges in $L^1([0,+\infty))$ by item (1), we would obtain the convergence in $L^1([0,+\infty))$ of the sequence $\{h_n\}_{n\in\mathbb{N}}$, which is a contradiction.

We conclude that $\{g_n\}_{n\in\mathbb{N}}$ does not converge in $L^1([0,+\infty))$.

Exercise 1 :

$([0;+\infty), \mathcal{B}([0;+\infty)))$ with λ : Lebesgue measure.

$\{f_n\}_{n\in\mathbb{N}}$: $\quad f_n(x) = \dfrac{e^{-nx}+n^2e^{-x}}{n+n^2+(1+n^2)x^2}$, $x \in [0;+\infty)$, $n \in \mathbb{N}$.

1) Let $x \in [0;+\infty)$. $\quad f_n(x) = \dfrac{e^{-nx}+n^2e^{-x}}{n+n^2+(1+n^2)x^2} = \dfrac{e^{-nx}}{n+n^2+(1+n^2)x^2} + \dfrac{n^2e^{-x}}{n+n^2+(1+n^2)x^2}$

$$f_n(x) = \frac{e^{-nx}}{n+x^2+n^2(1+x^2)} + \frac{n^2e^{-x}}{n+x^2+n^2(1+x^2)} = \underbrace{\frac{e^{-nx}}{\dots}}_{\xrightarrow[n\to+\infty]{} 0} + \underbrace{\frac{e^{-x}}{\frac{1}{n}+\frac{x^2}{n^2}+(1+x^2)}}_{\xrightarrow[n\to+\infty]{} \frac{e^{-x}}{1+x^2}}$$

So $\{f_n\}_n$ converges to f ($f(x) := \frac{e^{-x}}{1+x^2}$) pointwisely, so a.e in $[0;+\infty)$.

$$\boxed{f_n \xrightarrow[n\to+\infty]{} f \text{ a.e in } [0;+\infty)} . \square$$

2) If $\{f_n\}_n$ CV in $L^1([0;+\infty))$, it is to the same limit f.

$$\|f_n - f\|_{L^1} = \int_0^{+\infty} |f_n(x) - f(x)|\,dx = \int_0^{+\infty} \left| \frac{e^{-nx}+n^2e^{-x}}{n+n^2+(1+n^2)x^2} - \frac{e^{-x}}{1+x^2} \right| dx$$

- $e^{-nx} + n^2e^{-x} = n^2\left(\dfrac{e^{-nx}}{n^2} + e^{-x}\right) \leq n^2(1+e^{-x})$
- $n+n^2+(1+n^2)x^2 = n^2(1+x^2) + n + x^2 \geq n^2(1+x^2)$.

So : $\forall n \in \mathbb{N}$, $\quad |f_n(x)| \leq \dfrac{\cancel{n^2}(1+e^{-x})}{\cancel{n^2}(1+x^2)} \leq \dfrac{2}{1+x^2} =: g(x)$.

We have : f_n are all measure in $\mathcal{B}([0;+\infty))$; $f_n \xrightarrow[n\to+\infty]{} f$ a.e in $[0;+\infty)$;
and $\forall n \in \mathbb{N}$, $|f_n(x)| \leq g(x)$ $\forall x \in [0;+\infty)$.

We then can apply the DCT :

$$\lim_{n\to+\infty}\int_0^{+\infty}\left|f_n(x)-f(x)\right|dx = \int_0^{+\infty}\underbrace{\lim_{n\to+\infty}\left|f_n(x)-f(x)\right|}_{=0 \text{ a.e in } [0;+\infty)}dx$$

$$= 0$$

So $\|f_n - f\|_{L^1} \xrightarrow[n\to+\infty]{} 0$. Hence $\boxed{\{f_n\}_n \text{ CV in } L^1 \text{ to } f}$. $\square$

3) $g_n(x) = f_n(x) + \mathbb{1}_{[n;n+1]}$.

- Clearly : $g_n(x) \xrightarrow[n\to+\infty]{} f(x)$ a.e in $[0;+\infty)$.
- If $\{g_n\}_{n\in\mathbb{N}}$ CV in L^1, it is to f also.

$$\int_0^{+\infty}\left|g_n(x)-f(x)\right|dx = \int_0^{+\infty}\left|f_n(x)-f(x)+\mathbb{1}_{[n;n+1]}(x)\right|dx$$

$$\geq \left|\int_0^{+\infty} f_n(x)-f(x)+\mathbb{1}_{[n;n+1]}(x)\,dx\right|$$

$$\geq \left|\int_0^{+\infty}(f_n(x)-f(x))dx + \int_0^{+\infty}\mathbb{1}_{[n;n+1]}(x)dx\right|$$

$$\geq \left|\underbrace{\int_0^{+\infty}[f_n(x)-f(x)]dx}_{\xrightarrow[n\to+\infty]{} 0 \text{ because } f_n \xrightarrow[n\to+\infty]{L^1} f} + 1\right| \xrightarrow[n\to+\infty]{} 1$$

Correction: [proposition, I think my technic works].

let $h_n(x) = \mathbb{1}_{[n;n+1]}(x)$.

→ show $\{h_n\}_n$ cv a.e to 0.

→ show $\{h_n\}_n$ doesn't converge in L^1:

$$\int_0^{+\infty}|h_n(x)-0|dx = \int_0^{+\infty}|h_n(x)|dx = 1.$$

→ $h_n = g_n - f_n$. $\{f_n\}_n$ cv in L^1 to f and suppose by contradiction that $\{g_n\}$ CV in L^1 as well. We would obtain L^1-CVce for h_n which is a contradiction.

So $g_n \overset{L^1}{\underset{n\to+\infty}{\not\longrightarrow}} f$ so $\boxed{\{g_n\}_n \text{ doesn't CV in } L^1([0;+\infty))}$. $\square$

$$\underbrace{\int_0^{+\infty} |f_n(x) - f(x)|\, dx}_{\xrightarrow[n \to +\infty]{} 0} \geqslant \int_0^{+\infty} f_n(x) - f(x)\, dx$$

So, a.e : $|f_n(x) - f(x)| = 0$

So, a.e : $f_n(x) = f(x)$

So : $\int_A f_n(x)\,dx = \int_A f(x)\,dx$

So : $\int_A f_n(x) - f(x)\, dx = 0$ ok.

Exercise 2. Consider the sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in \ell^\infty$ for any $n \in \mathbb{N}$, defined by

$$x_n = (x_n^{(k)})_{k\in\mathbb{N}}, \quad \text{with} \quad x_n^{(k)} := \begin{cases} (-1)^k, & \text{if } k \le n \\ 0, & \text{otherwise} \end{cases}.$$

(1) Is the sequence $\{x_n\}_{n\in\mathbb{N}}$ bounded in ℓ^∞? Justify the answer.
(2) Compute the pointwise limit of the sequence $\{x_n\}_{n\in\mathbb{N}}$.
(3) Denote by x the pointwise limit of the sequence $\{x_n\}_{n\in\mathbb{N}}$. Does $\{x_n\}_{n\in\mathbb{N}}$ converge weak* to x in ℓ^∞? Justify the answer.

Solution.
(1) For any $n \in \mathbb{N}$, we have

$$\|x_n\|_\infty = \sup_k |x_n^{(k)}| = 1,$$

hence

$$\sup_n \|x_n\|_\infty = 1$$

so that we get the boundedness of $\{x_n\}_{n\in\mathbb{N}}$ in ℓ^∞.
(2) We first observe that

$$\begin{aligned} x_1 &= (-1, 0, 0, \dots) \\ x_2 &= (-1, 1, 0, 0, \dots) \\ x_3 &= (-1, 1, -1, 0, 0, \dots). \end{aligned}$$

Therefore for each fixed $k \in \mathbb{N}$, $x_n^{(k)} = (-1)^k$ for every $n \ge k$. Thus, the sequence $\{x_n\}_n$ converges pointwisely to $x = (x^{(k)})_{k\in\mathbb{N}} = \big((-1)^k\big)_{k\in\mathbb{N}}$, as $n \to +\infty$.

(2) Since $\ell^\infty \simeq (\ell^1)^*$, x_n converges weakly* to x in ℓ^∞, as $n \to +\infty$, if and only if

$$\left| \sum_{k=1}^{\infty} x_n^{(k)}\, y^{(k)} - \sum_{k=1}^{\infty} x^{(k)}\, y^{(k)} \right| \to 0, \quad \text{for any } y = (y^{(k)})_{k\in\mathbb{N}} \in \ell^1.$$

Let $y \in \ell^1$ be arbitrarily fixed, we have

$$\left| \sum_{k=1}^{\infty} x_n^{(k)}\, y^{(k)} - \sum_{k=1}^{\infty} x^{(k)}\, y^{(k)} \right| = \left| \sum_{k=n+1}^{\infty} (-1)^k y^{(k)} \right| \le \sum_{k=n+1}^{\infty} \left| y^{(k)} \right| \to 0$$

indeed this is the remainder of a convergent series ($y \in \ell^1$).

By the above observations, we get the weak* convergence of the sequence $\{x_n\}_{n\in\mathbb{N}}$ to $x \in \ell^\infty$.

Exercise 2 :

$\{x_n\}_{n\in\mathbb{N}} \subset \ell^\infty$. $\forall n\in\mathbb{N}$, $x_n = (x_n^{(k)})_{k\in\mathbb{N}}$, with $x_n^{(k)} := \begin{cases} (-1)^k & \text{if } k \leq n \\ 0 & \text{otherwise} \end{cases}$.

1) For any $n\in\mathbb{N}$, we have $\|x_n\|_\infty = \sup_{k\in\mathbb{N}} |x_n^{(k)}| = 1$.

So $\sup_{n\in\mathbb{N}} \|x_n\|_\infty = 1$. So $\boxed{\{x_n\}_{n\in\mathbb{N}} \text{ is bounded in } \ell^\infty}$. $\square$

2) We can see that $\forall k > n$, $x_n^{(k)} = 0$. And $(-1)^k \xrightarrow[n\to+\infty]{} (-1)^k$. $\Big\}$ So : $\forall k\in\mathbb{N}$, $x_n^{(k)} \xrightarrow[n\to+\infty]{} (-1)^k$.

$\boxed{\{x_n\}_{n\in\mathbb{N}} \text{ pointwisely converges to } x = ((-1)^k)_{k\in\mathbb{N}}}$. $\square$

3) "We can identify $(\ell^1)^*$ and ℓ^∞".

- Since $\ell^\infty \simeq (\ell^1)^*$, x_n converges weakly$*$ to x in ℓ^∞, as $n\to+\infty$, if and only if : $\left| \sum_{k=1}^{+\infty} x_n^{(k)} y^{(k)} - \sum_{k=1}^{+\infty} x^{(k)} y^{(k)} \right| \xrightarrow[n\to+\infty]{} 0$, $\forall y = (y^{(k)})_{k\in\mathbb{N}} \in \ell^1$.

→ Let $y \in \ell^1$ be arbitrarily fixed, we have :

$$\left| \sum_{k=1}^{+\infty} x_n^{(k)} y^{(k)} - \sum_{k=1}^{+\infty} x^{(k)} y^{(k)} \right| = \left| \cancel{\sum_{k=1}^{n} (-1)^k y^{(k)} - \sum_{k=1}^{n} (-1)^k y^{(k)}} - \sum_{k=n+1}^{+\infty} (-1)^k y^{(k)} \right|$$

$$= \left| \sum_{k=n+1}^{+\infty} (-1)^k y^{(k)} \right| \leq \sum_{k=n+1}^{+\infty} |y^{(k)}| \xrightarrow[n\to+\infty]{} 0.$$

Indeed, $y \in \ell^1$ so $\sum_{k=1}^{+\infty} |y^{(k)}| < +\infty$ so the remainder of this convergent series has to vanish.

Then $\boxed{\{x_n\}_{n\in\mathbb{N}} \text{ weakly}* \text{ converges to } x \text{ in } \ell^\infty}$. $\square$

Exercise 3. Let $X = \ell^2$ and consider the linear operator $T : X \to X$ defined by

$$[T(x)]^{(k)} := \begin{cases} x^{(1)} - x^{(2)}, & \text{if } k = 1 \\ \\ \dfrac{x^{(k+1)}}{2}, & \text{if } k \geq 2 \end{cases} \qquad \forall x = (x^{(k)})_{k\in\mathbb{N}} \in X.$$

(1) Prove that T is continuous.
(2) Is T surjective? Is T injective? Justify the answers.
(3) Is T compact? Justify the answer.

Solution.
(1) Let $x \in X$, we have

$$\begin{aligned} \|T(x)\|_2^2 &= \sum_{k=1}^{+\infty} \left|[T(x)]^{(k)}\right|^2 = |x^{(1)} - x^{(2)}|^2 + \sum_{k=3}^{+\infty} \frac{|x^{(k)}|^2}{4} \\ &\leq 2|x^{(1)}|^2 + 2|x^{(2)}|^2 + \sum_{k=3}^{+\infty} |x^{(k)}|^2 \\ &\leq 2\|x\|_2^2, \end{aligned}$$

where the inequality at the second row comes by Young's inequality. Thus we obtain

$$\|Tx\|_2 \leq \sqrt{2}\|x\|_2.$$

This proves the boundedness of the operator T; since T is linear, it is a continuous operator.

(2) *Surjectivity:* T is surjective, indeed let y be an arbitrary element of ℓ^2 and let us prove that there exists $x \in \ell^2$ such that $y = T(x)$. Given $y = (y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, \dots)$, it is sufficient to take

$$x = (2y^{(1)}, y^{(1)}, 2y^{(2)}, 2y^{(3)}, \dots),$$

so that

$$T(x) = \left(2y^{(1)} - y^{(1)}, \frac{2y^{(2)}}{2}, \frac{2y^{(3)}}{2}, \dots\right) = (y^{(1)}, y^{(2)}, y^{(3)}, \dots) = y.$$

Injectivity: T is not injective, indeed if we take $x = (1, 1, 0, \dots)$ then $T(x) = (0, 0, 0, \dots)$. Hence, $x \in \ker(T)$, so that $\ker T \neq \{\mathbf{0}\}$.

(3) Since X is an infinite dimensional Banach space and T is surjective, T cannot be compact (cf. open mapping theorem).

Exercise 3 :

$X = \ell^2$ and consider the linear operator $T : X \to X$ defined by :

$$[T(x)]^{(k)} := \begin{cases} x^{(1)} - x^{(2)} & \text{if } k = 1 \\ \dfrac{x^{(k+1)}}{2} & \text{if } k \geq 2 \end{cases}, \quad \forall x = (x^{(k)})_{k \in \mathbb{N}^*} \in X.$$

1) • As mentionned, T is linear.

• T continuous: we can show that T is bounded.

Let $x \in \ell^2$.
$$\|Tx\|_2^2 = \sum_{k=1}^{+\infty} |(Tx)^{(k)}|^2 = \sum_{k=2}^{+\infty} \left|\frac{x^{(k+1)}}{2}\right|^2 + |(x^{(1)} - x^{(2)})|^2$$
$$\leq \frac{1}{4} \sum_{k=3}^{+\infty} |x^{(k)}|^2 + |x^{(1)} - x^{(2)}|^2$$
$$\leq \sum_{k=3}^{+\infty} |x^{(k)}|^2 + 2|x^{(1)}|^2 + 2|x^{(2)}|^2 \leq 2 \|x\|_{\ell^2}^2$$

Thus : $\|Tx\|_2 \leq \sqrt{2}\, \|x\|_2$.

T is bounded. Therefore : $\boxed{T \text{ is continuous}}$. □

2) • Injectivity: Let $x = (1, 1, 0, 0 \ldots) \neq 0_{\ell^2}$

$x \in \ell^2$ clearly since $\sum_{k=1}^{+\infty} |x^{(k)}|^2 = 2 = \|x\|_2^2$.

We have : $Tx = (0, 0, 0, \ldots) = 0_{\ell^2}$ whereas

$x \neq 0$. So $\ker(T) \neq \{0_{\ell^2}\}$. By this observation :

$$\boxed{T \text{ is not injective}}.$$

• Surjectivity: let $y \in \ell^2$. Can we find $x \in \ell^2$ such that

$y = Tx$? $\begin{cases} y = (y^{(1)}, y^{(2)}, y^{(3)}, \ldots) \\ x = (x^{(1)}, x^{(2)}, x^{(3)}, \ldots) \end{cases}$. If $y = Tx$, then :

$\begin{cases} y^{(1)} = x^{(1)} - x^{(2)} \\ \forall k \geq 2,\ y^{(k)} = \frac{1}{2} x^{(k+1)} \end{cases}$ $\longrightarrow$ It is sufficient to take $x = (2y^{(1)}, y^{(1)}, 2y^{(2)}, \ldots)$. It seems OK but we have to check that

$x \in \ell^2$: $\quad x = \left(2y^{(1)}, y^{(1)}, 2y^{(2)}, 2y^{(3)}, \dots\right)$

So $$\|x\|_{\ell^2}^2 = \sum_{k=1}^{+\infty} |x^{(k)}|^2 = 4\underbrace{\sum_{k=1}^{+\infty} |y^{(k)}|^2}_{< \sum_{k=1}^{+\infty}|y^{(k)}|^2 < +\infty} - 3|y^{(1)}|^2 < +\infty .$$

So $x \in \ell^2$. As a conclusion:

$$\boxed{T \text{ is surjective}} . \quad \square$$

3) $e_n^{(k)} := \begin{cases} 1 & \text{if } k = n \\ 0 & \text{otherwise} \end{cases} \rightarrow$ bounded in ℓ^2 ? $\qquad (Te_n)^{(k)} = \left(0, \dots, 0, \frac{1}{2}, \dots 0\right)$

$$\|Te_n - Te_m\|_2^2 \underset{m \neq n}{=} 1/2 \Rightarrow \|Te_n - Te_m\|_2 = \frac{\sqrt{2}}{2} .$$

So the sequence $\{Te_n\}_{n\in\mathbb{N}}$ admits no (strongly) converging subsequence.

We conclude that T is not a compact operator. $\square$

<u>Correction:</u> Since X is an infinite dimensional Banach space and T is surjective $(T \in \mathcal{L}(X,X))$, we can apply the Open Map Theorem to get that T is open.

Hence T cannot be compact.

Theory

Question 1. *(4 points)* State and prove the theorem of absolute continuity of the Lebesgue integral.

Solution. See Lecture 12.

Question 2. *(4 points)* Let $(X, \mathcal{A})$ be a measurable space. Prove or disprove the following statements:

(1) $A \in \mathcal{A} \Leftrightarrow \chi_A \in \mathcal{M}(X, \mathcal{A})$;
(2) $f \in \mathcal{M}(X, \mathcal{A}) \Leftrightarrow f_\pm \in \mathcal{M}(X, \mathcal{A})$;
(3) $f \in \mathcal{M}(X, \mathcal{A}) \Leftrightarrow |f| \in \mathcal{M}(X, \mathcal{A})$.

Solution. See Lectures 5 and 6.

Question 3. *(4 points)* State the Banach-Alaoglu theorem. State and prove the variant of the Banach-Alaoglu theorem in reflexive spaces.

Solution. See Lecture 22.

Question 4. *(4 points)* Let $T : H \to H$ be a linear and bounded operator on the Hilbert space H.

(i) Write the definitions of:

(1) T is symmetric;
(2) T is compact;
(3) resolvet set and spectrum of T.

(ii) State the spectral theorem for T.

Solution. See Lectures 22 and 24.