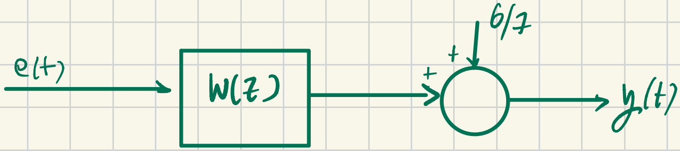


$e(\cdot) \sim WN\left(1, \frac{63}{64}\right)$

$W(z) = \dfrac{1}{1 - \frac{1}{8} z^{-3}}$.

• $\mathbb{E}$? covfct? Spectrum?

■ Analyze $W(z)$ to see if stationary ?

$W(z) = \dfrac{z^3}{z^3 - 1/8}$ ZEROS: $z = 0$.

poles: $z_1 = \sqrt[3]{1/8} = 1/2$

so $z^3 - 1/8 = (z - 1/2)(z^2 + \frac{1}{2} z + 1/4)$

and $z^2 + 1/2\, z + 1/4 = 0 \iff z_{2,3} = -\frac{1}{4} \pm \frac{\sqrt{3}}{4} i$

so $|z_{2,3}| = 1/2 < 1$. $\boxed{z_1, z_2, z_3 \in \text{Unit disk}}$

$\Rightarrow$ $W(z)$ asympt. stable

$e(\cdot)$ is a WN, so stationary. So $y(\cdot)$ is stationary.

■ So now, we can compute the expected value:

$y(t) = 6/7 + \dfrac{1}{1 - \frac{1}{8} z^{-3}} e(t)$ so $\tilde{y}(t) = y(t) - 6/7 = \dfrac{1}{1 - \frac{1}{8} z^{-3}} e(t)$

i.e $\tilde{y}(t) = \frac{1}{8} \tilde{y}(t-3) + e(t)$ so: $m_{\tilde{y}} = \frac{1}{8} m_{\tilde{y}} + m_e$ so:

$7/8\, m_{\tilde{y}} = 1$ i.e $m_{\tilde{y}} = m_y - 6/7 = 8/7$ i.e: $\boxed{m_y = \mathbb{E}[y(t)] = 2}$.

OR, remember the "theorem of the GAIN":

$y(t) = 6/7 + W(z) e(t) \longrightarrow \mathbb{E}[y(t)] = 6/7 + W(1) \cdot \mathbb{E}[e(t)] = \frac{6}{7} + \frac{1}{1 - 1/8} \cdot 1 = \frac{6}{7} + \frac{8}{7} = 2$

- We want to compute the covariance function:

Remember: we have to have null expectations. So let's define:

$$\left.\begin{aligned}\tilde{y} &= y - m_y = y - 2 \\ \tilde{e} &= e - m_e = e - 1\end{aligned}\right\} \Rightarrow \quad \tilde{y}(t) = W(z)\tilde{e}(t) \quad (\forall t)$$

$$\tilde{y}(t) = \frac{1}{1 - \frac{1}{8}z^{-3}}\tilde{e}(t) \quad \text{so} \quad \tilde{y}(t) = \frac{1}{8}\tilde{y}(t-3) + \tilde{e}(t) \quad AR(3)$$

So: • $\gamma_{\tilde{y}}(0) = \mathbb{E}[\tilde{y}(t)^2] = \mathbb{E}\left[\left(\frac{1}{8}\tilde{y}(t-3) + \tilde{e}(t)\right)^2\right] = \frac{1}{64}\gamma_{\tilde{y}}(0) + \gamma_{\tilde{e}}(0) + \frac{1}{4}\cancel{\mathbb{E}[\tilde{y}(t-3)\tilde{e}(t)]}$

$$= \frac{1}{64}\gamma_{\tilde{y}}(0) + \frac{63}{64} \quad \text{so:}$$

$$\boxed{\gamma_{\tilde{y}}(0) = \frac{63/64}{1 - 1/64} = 1}$$

• $\gamma_{\tilde{y}}(1) = \mathbb{E}[\tilde{y}(t)\tilde{y}(t-1)] = \mathbb{E}\left[\left(\frac{1}{8}\tilde{y}(t-3) + \tilde{e}(t)\right)\tilde{y}(t-1)\right]$

$$= \frac{1}{8}\gamma_{\tilde{y}}(2)$$

so we need an equation for $\gamma_{\tilde{y}}(2)$.

• $\gamma_{\tilde{y}}(2) = \mathbb{E}[\tilde{y}(t)\tilde{y}(t-2)] = \mathbb{E}\left[\left(\frac{1}{8}\tilde{y}(t-3) + \tilde{e}(t)\right)\tilde{y}(t-2)\right]$

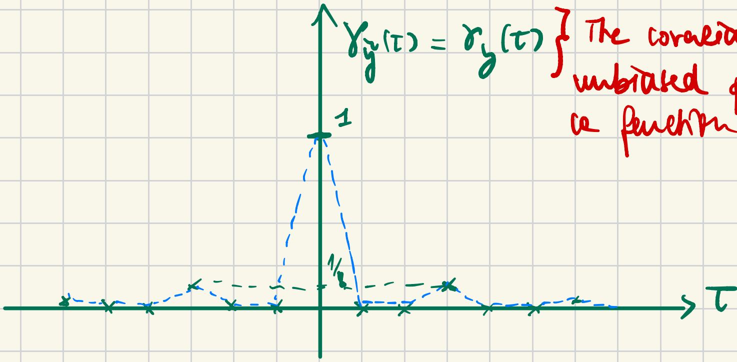
$$= \frac{1}{8}\gamma_{\tilde{y}}(1).$$

It means that $\boxed{\gamma_{\tilde{y}}(1) = 0 = \gamma_{\tilde{y}}(2) = 0}$

• $\boxed{\gamma_{\tilde{y}}(3) = \mathbb{E}\left[\left(\frac{1}{8}\tilde{y}(t-3) + \tilde{e}(t)\right)\tilde{y}(t-3)\right] = 1/8\,\gamma_{\tilde{y}}(0)}$

So: $\boxed{\gamma_{\tilde{y}}(\tau) = 1/_8\, \gamma_{\tilde{y}}(\tau-3), \quad \forall |\tau| \geq 3}$.

Let's draw $\gamma_{\tilde{y}}(\tau)$:



} The covariance function of the unbiased process = the covariance function of the original process.

Now we want to compute the spectrum:

Remember the Fundamental theorem of Spectral Analysis:

$$\Gamma_y(\omega) = \left|W(z=e^{j\omega})\right|^2 \cdot \Gamma_e(\omega).$$

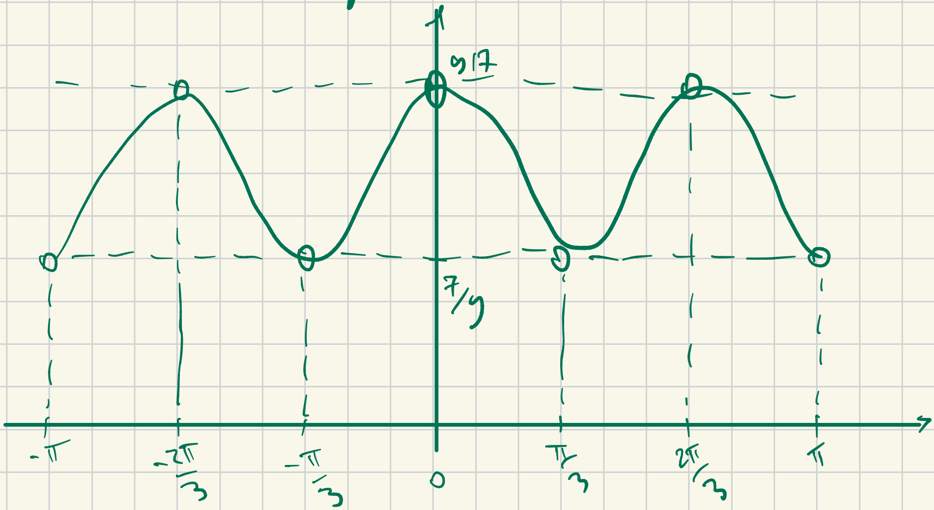
$$\Gamma_y(\omega) = \left|\frac{1}{1-\frac{1}{8}e^{-3j\omega}}\right|^2 \cdot \Gamma_e(\omega) = \frac{1}{\left(1-\frac{1}{8}e^{-3j\omega}\right)\left(1-\frac{1}{8}e^{3j\omega}\right)} \cdot \frac{63}{64}$$

$$= \frac{63/64}{\left(1+\frac{1}{64}\right) - \frac{1}{8}\left(e^{-3j\omega}+e^{3j\omega}\right)} = \frac{63/64}{\frac{65}{64} - \frac{1}{4}\times\cos(3\omega)} = \frac{63/16}{65/16 - \cos(3\omega)}$$

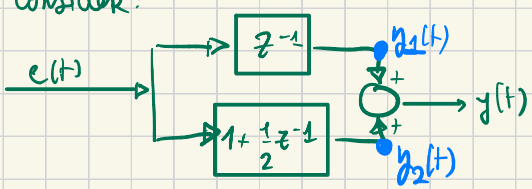
$$\boxed{\Gamma_y(\omega) = \frac{63}{65-16\cos(3\omega)}}.$$

- if $\cos(3\omega) = 1$, $\Gamma_y(\omega) = 9/7$ → max point → $\omega = 0 \pm \frac{2k\pi}{3}$.
- if $\cos(3\omega) = -1$, $\Gamma_y(\omega) = 7/9$ → min point → $\omega = \frac{\pi}{3} \pm \frac{2k\pi}{3}$

So we can draw $\Gamma_y(\omega)$:



Consider:



$$\begin{cases} e(\cdot) \sim WN(0,1) \\ \hat{y}(t|t-1) = ? \end{cases}$$

CHECK CANONICAL REPRESENTATION!

$$y(t) = e(t-1) + \frac{1}{2}e(t-1) + e(t) = \left(1+\frac{3}{2}z^{-1}\right)e(t)$$

It is not canonical since a zero is $\left(-\frac{3}{2}\right) \notin$ Unit Circle

→ we use ALL-PASS filter:

$$y(t) = \frac{1+\frac{3}{2}z^{-1}}{1+\frac{2}{3}z^{-1}} \times \left(1+\frac{2}{3}z^{-1}\right) \times e(t) = \left(1+\frac{2}{3}z^{-1}\right) \times \frac{1+\frac{3}{2}z^{-1}}{1+\frac{2}{3}z^{-1}} e(t)$$

STILL A WN: $\eta(t) \sim WN\left(0, \frac{9}{4}\right)$

$$\boxed{y(t) = \frac{C(z)}{A(z)}\eta(t) = \frac{1+\frac{2}{3}z^{-1}}{1}\eta(t)}$$ (In canonical form)

$$\hat{y}(t|t-1) = \frac{C(z)-A(z)}{C(z)}\, y(t) = \frac{\frac{2}{3}z^{-1}}{1+\frac{2}{3}z^{-1}}\, y(t) = \frac{1}{1+\frac{2}{3}z^{-1}} \times \frac{2}{3} y(t-1)$$

so: $\boxed{\hat{y}(t|t-1) = \frac{2}{3} y(t-1) - \frac{2}{3}\hat{y}(t-1|t-2)}$.

And by the lectures we know that:

$$\mathbb{E}\left[\varepsilon(t|t-1)^2\right] = \mathbb{E}\left[\left(y(t)-\hat{y}(t|t-1)\right)^2\right] \overset{!}{=} \mathbb{E}\left[\eta^2(t)\right] = 9/4 .$$

since we deal with the 1 step ahead predictor.

Compute the prediction of $y(t)$ starting from the measurement of $y_2(t)$?

"compute a prediction from a signal inside the scheme"

$$y(t) = e(t) + \frac{3}{2}e(t-1) = \left(1+\frac{3}{2}z^{-1}\right)e(t)$$

& $y_2(t) = \left(1+\frac{1}{2}z^{-1}\right)e(t)$.

What is predictable at $t-1$?

$$\rightarrow \hat{y}_2(t|t-1) = \frac{3}{2}e(t-1) = \frac{3}{2}z^{-1}\, e(t) = \frac{3}{2}z^{-1}\frac{1}{1+\frac{1}{2}z^{-1}}\, y_2(t)$$

$$\hat{y}_2(t|t-1) = \frac{\frac{3}{2}y_2(t-1)}{1+\frac{1}{2}z^{-1}} = \frac{3}{2}y_2(t-1) - \frac{1}{2}\hat{y}_2(t-1|t-2)$$

$\hat{y}_2(t|t-1) = y(t) - e(t)$

$$\mathbb{E}\left[\left(y(t)-\hat{y}_2(t|t-1)\right)^2\right] = \mathbb{E}\left[e(t)^2\right] = 1$$

So the better predictor is the 2nd one!

NOT MONIC!!

$$y(t) = \frac{2+5z^{-1}+2z^{-2}}{\left(1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}\right)\left(1+\frac{1}{2}z^{-1}\right)} e(t) , \quad e(t) \sim WN(0,1)$$

We want to compute $\begin{cases} \hat{y}(t|t-1) \\ \hat{y}(t|t-2) \end{cases}$.

$(\cdot 4z^2)$ $1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2} = 0 \iff 4z^2+2z+1=0$

$\Delta = 4-16 = -12 = -3 \times 4$

$z_{1,2} = \frac{-2 \pm i\sqrt{12}}{8} = \frac{-1}{4} \pm \frac{i}{8}2\sqrt{3} = \frac{-1}{4} \pm i\frac{\sqrt{3}}{4}$

$|z_{1,2}| = \sqrt{\frac{1}{16}+\frac{3}{16}} = \sqrt{\frac{4}{16}} = \frac{1}{2} < 1$

POLES OK

$2+5z^{-1}+2z^{-2} = 0 \iff 2z^2+5z+2=0$

$\Delta = 25-16 = 9 > 0 \rightarrow z_{3,4} = \frac{-5 \pm \sqrt{9}}{4} = \frac{-5}{4} \pm \frac{3}{4} = \begin{cases} -2 \;! \\ -1/2 \end{cases}$

↳ 1 ZERO IS NOT OK. Let's use ALL PASS FILTER:

$$y(t) = \frac{\left(1+2z^{-1}\right)\cancel{\left(1+\frac{1}{2}z^{-1}\right)}\times 2}{\left(1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}\right)\cancel{\left(1+\frac{1}{2}z^{-1}\right)}} e(t)$$

$$= \frac{\left(1+\frac{1}{2}z^{-1}\right)}{1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}} \times \frac{2\times\left(1+2z^{-1}\right)}{1+\frac{1}{2}z^{-1}} e(t) \rightarrow \eta(t) \sim WN(0,16)$$

CANONICAL FORM $\rightarrow$ $= \frac{C(z)}{A(z)}\eta(t)$

$\frac{2(1+2)}{1+1/2} = \frac{6}{3/2} = \frac{6\times 2}{3} = 4$

THEN:

- $$\hat{y}(t|t-1) = \frac{C(z)-A(z)}{C(z)}\,y(t) = \frac{\cancel{1}+\cancel{\tfrac{1}{2}z^{-1}}-(\cancel{1}+\cancel{\tfrac{1}{2}z^{-1}}+\tfrac{1}{4}z^{-2})}{1+\frac{1}{2}z^{-1}}\,y(t)$$

$$= \frac{-\frac{1}{4}z^{-2}}{1+\frac{1}{2}z^{-1}}\,y(t)$$

so $$\boxed{\hat{y}(t|t-1) = -\frac{1}{4}y(t-2) - \frac{1}{2}\hat{y}(t-1|t-2)}\ .$$

- $\hat{y}(t|t-2)$?

LONG DIVISION:

$$\begin{array}{l|l}
\overbrace{\cancel{1}+\cancel{\tfrac{1}{2}z^{-1}}}^{C(z)} & \overbrace{1+\tfrac{1}{2}z^{-1}+\tfrac{1}{4}z^{-2}}^{A(z)} \\ \cline{2-2}
-(\cancel{1}+\cancel{\tfrac{1}{2}z^{-1}}+\tfrac{1}{4}z^{-2}) & \underbrace{1}_{E(z)} \\ \cline{1-1}
\underbrace{-\tfrac{1}{4}z^{-2}}_{F(z)} &
\end{array}$$

$$\frac{C(z)}{A(z)} = E(z) + \frac{F(z)}{A(z)} \qquad \text{so} \qquad y(t) = \left(E(z)+\frac{F(z)}{A(z)}\right)\cdot\eta(t)$$

$$y(t) = \underbrace{E(z)\eta(t)}_{=1\cdot\eta(t)\text{: NOT PREDICTABLE}} + \underbrace{\frac{F(z)}{\cancel{A(z)}}\times\frac{\cancel{A(z)}}{C(z)}\,y(t)}_{=\frac{-1/4\,z^{-2}}{1+\frac{1}{2}z^{-1}}y(t)\text{ : PREDICTABLE}}$$

$$\longrightarrow \boxed{\hat{y}(t|t-2) = \frac{-1/4\,y(t-2)}{1+\frac{1}{2}z^{-1}} = -\frac{1}{4}y(t-2) - \frac{1}{2}\hat{y}(t-1|t-3)}$$

IDENTIFICATION:

$\mathcal{S}: \quad y(t) = \frac{1}{2} y(t-2) + e(t) \quad , \quad e(\cdot) \sim WN(0,1)$

$\mathcal{M}_k: \quad y(t) = a_1 y(t-1) + \dots + a_k y(t-k) + \xi(t) \, , \; \xi(\cdot) \sim WN(0, \lambda^2)$

↖ ALWAYS CANONICAL.

→ Identify $\forall k$ our system with the model $\mathcal{M}_k$.

- If $k=2$, $a_1^* = 0$, $a_2^* = \frac{1}{2}$, $\lambda^{*2} = 1$: $\mathcal{S}$ ∈ family Model.
- If $k \geq 3$, $a_1^* = 0$, $a_2^* = 1/2$, $a_3^* = 0, \dots, a_k^* = 0$, $\lambda^{*2} = 1$
- What if $k=1$?

$\varepsilon = y(t) - \hat{y}(t|t-1)$

$= \frac{A(z)}{C(z)} y(t)$: Inject here $\mathcal{S}$ info if $\mathcal{S}$ has MA part.

1. $\hat{y}(t|t-1) = a_1 y(t-1)$

2. $$\underbrace{\mathbb{E}[\varepsilon(t|t-1)^2]}_{= J(a_1)} = \mathbb{E}\left[\left(y(t) - \hat{y}(t|t-1)\right)^2\right] = \mathbb{E}\left[\left(y(t) - a_1 y(t-1)\right)^2\right]$$

$$J(a_1) = \mathbb{E}[y(t)^2] + a_1^2 \mathbb{E}[y(t-1)^2] - 2a_1 \mathbb{E}[y(t) y(t-1)]$$

$$= \gamma_y(0)(1 + a_1^2) - 2a_1 \gamma_y(1)$$

3. $$\frac{dJ(a_1)}{da_1} = 0 \iff 2a_1^* \gamma_y(0) - 2\gamma_y(1) = 0$$

$$\iff a_1^* = \frac{\gamma_y(1)}{\gamma_y(0)}$$ THAT'S WHERE WE NEED $\mathcal{S}$.

Using $\mathcal{S}$ we compute $\gamma_y(0), \gamma_y(1)$:

$y(t-2) = f(H[e(t-2)], \dots)$

- $$\gamma_y(0) = \mathbb{E}[y(t)^2] = \mathbb{E}\left[\left(\frac{1}{2} y(t-2) + e(t)\right)^2\right] = \frac{1}{4}\gamma_y(0) + \gamma_e(0) + \cancel{\mathbb{E}[y(t-2) e(t)]}$$

So $\boxed{\gamma_y(0)\left(1-\frac{1}{4}\right)=1} \iff \gamma_y(0)=\frac{4}{3}$

- $\gamma_y(1)=\mathbb{E}[y(t)y(t-1)]=\mathbb{E}\left[\left(\frac{1}{2}y(t-2)+e(t)\right)(y(t-1))\right]$

$$=\frac{1}{2}\mathbb{E}[y(t-2)y(t-1)]+\underbrace{\mathbb{E}[\cancel{y(t-1)}e(t)]}_{y(t-1)=f(t)(e(t-1),\dots)}$$

$$=\frac{1}{2}\gamma_y(1)$$

So $\boxed{\gamma_y(1)=0}$ So $\boxed{a_1^*=0}$

4. $N^{*2}=\bar{J}(a_1^*)=\gamma_y(0)=4/3$