MIDA I - Exercise session 7 - Identification

$\mathcal{S}: \quad y(t) = 3e(t) + 9e(t-1) \quad , \; e(t) \sim WN(0,1)$

$\mathcal{M}: \quad y(t) = \eta(t) + b\,\eta(t-1) \quad , \; \eta(t) \sim WN(0,\lambda^2)$

$\rightarrow$ Find b^* & λ^{*2}.

1) Computation of the 1 step predictor of the model:

$$\mathcal{M}: \; y(t) = \eta(t) + b\eta(t-1) \longrightarrow y(t) = \underbrace{(1 + bz^{-1})}_{C_m(z)/A_m(z)}\eta(t)$$

$$\rightarrow \quad \hat{y}(t|t-1) = \frac{C_m(z) - A_m(z)}{C_m(z)}\, y(t)$$

$$= \frac{\cancel{1} + bz^{-1} - \cancel{1}}{1 + bz^{-1}}\, y(t) = \frac{bz^{-1}}{1+bz^{-1}} \times y(t)$$

2) Computation of the prediction error:

$$\varepsilon(t|t-1) = y(t) - \hat{y}(t|t-1) = y(t) - \frac{bz^{-1}}{1+bz^{-1}}\, y(t)$$

$$= \frac{1}{1+bz^{-1}}\, \underbrace{y(t)}_{y(t)\ (\text{from } \mathcal{S})} = \frac{1}{1+bz^{-1}} \times (3 + 9z^{-1})\, e(t)$$

$$= \frac{3 + 9z^{-1}}{1+bz^{-1}}\, e(t) = \frac{3(1 + 3z^{-1})}{1+bz^{-1}} \times \frac{(1 + \frac{1}{3}z^{-1})}{1 + \frac{1}{3}z^{-1}}\, e(t)$$

$$= \frac{1 + 1/3\, z^{-1}}{1 + bz^{-1}} \times \frac{3(1 + 3z^{-1})}{1 + \frac{1}{3}z^{-1}}\, e(t) = \xi(t) \sim WN(0, 81)$$

$$= \frac{1+\frac{1}{3}z^{-1}}{1+bz^{-1}}\xi(t)$$

N.B: Here we can see that in order to have $\varepsilon(t) \sim WN$ equal to $\xi(t) \rightarrow b^* = 1/3$.

3. Computation of the variance of the prediction error:

$$y(t) = \frac{1+cz^{-1}}{1+az^{-1}}e(t) \qquad y(t) = -ay(t-1) + e(t) + ce(t-1) \quad ARMA(1,1)$$

$$\rightarrow Var(y(t)) = a^2 Var(y(t)) + (1+c^2)Var(e(t)) + -2ac\,Var(e(t))$$

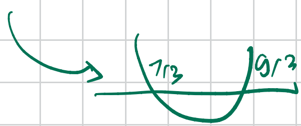
$$So: \quad Var(\varepsilon(t)) = \frac{1+c^2-2ac}{1-a^2}Var(\xi(t)) = \frac{1+1/9-2b/3}{1-b^2}81 = \frac{90-54b}{1-b^2} = \bar{J}$$

4. Minimization of $\bar{J}$ to find θ^*:

$$\frac{d\bar{J}}{db} = \frac{-54(1-b^2)+2b(90-54b)}{(1-b^2)^2} = \frac{-54b^2+180b-54}{(\quad)^2}$$

$$\frac{d\bar{J}}{db} = 0 \iff -54b^2+180b-54=0$$

$$\iff b^2 - \underbrace{\frac{180}{54}}_{=\frac{90}{27}=\frac{10}{3}}b + 1 = 0 \iff b^2 - \frac{10}{3}b + 1 = 0$$



$$\Delta = \frac{100}{9} - 4 = \frac{100-36}{9} = \frac{64}{9} = \frac{8^2}{3^2} = \left(\frac{8}{3}\right)^2$$

$$b_{1,2} = \frac{10/3 \pm 8/3}{2} = \begin{cases} \frac{2/3}{2} = 1/3 \quad \text{OR:} \\ \frac{18/3}{2} = 9/3 \end{cases}$$

x	$-$	$1/3$	$+$	$9/3$	$-$
sign of $\frac{d\bar{J}}{db}$	$-$		$+$		$-$

Because we divide by (-54). $\left[-(\text{sign}(\dots))\right] \equiv$

$b^* = 1/3$

5. Computation of the Var of the WN:

$$\lambda^{*2} = \bar{J}(b^*) = \frac{90 - 54 \times 1/3}{1 - 1/9} = \frac{90 - 18}{8/9} = \frac{72}{8/9}$$

$$= 9 \times \frac{72}{8} = 9 \times 9 = 81 = \lambda^{*2}$$

6. Computation of the prediction error:

$$\varepsilon(t|t-1) = \xi(t)$$

7. Analysis of the results:

1) The pred. error is a WN:

YES $\rightarrow \mathcal{S} \in \mathcal{M}$

MIDA I - Exercise session 8 - Identification

$\mathcal{S}: y(t) = e(t) + \frac{1}{3} e(t-1) \quad e(t) \sim WN(0,1)$] GENERATE DATA

$\mathcal{M}: y(t) = -a y(t-1) + \eta(t) \quad \eta(t) \sim WN(0,\lambda^2)$] MODEL WE WANT TO USE

$y(t) = \frac{C_m(z)}{A_m(z)} \eta(t)$

AR(1) MA(1)

We want to compute the value of a & λ^2 that minimize the variance of the prediction error.

1. $\hat{\mathcal{M}}: \quad \hat{y}(t|t-1) = -a y(t-1)$

2. $\varepsilon: \quad \hat{\varepsilon}(t|t-1) = y(t) - \hat{y}(t|t-1) = \eta(t) = \frac{A_m(z)}{C_m(z)} y(t)$

$$= \frac{1+az^{-1}}{1} y(t)$$

⚠ usually we substitute $y(t)$ from the system $\mathcal{S}$

3. $\mathbb{E}[\hat{\varepsilon}^2(t|t-1)] = \mathbb{E}[(y(t) + a y(t-1))^2] = \mathbb{E}[y(t)^2] + a^2 \mathbb{E}[y(t)^2] + 2a\, \mathbb{E}[y(t) y(t-1)]$

$$= \gamma_y(0) + a^2 \gamma_y(0) + 2a \gamma_y(1) = \bar{J}$$

Performance Index $\bar{J}$ that we want to minimize.

4. $\frac{d\bar{J}}{da} = 2a \gamma_y(0) + 2\gamma_y(1)$

$\frac{d\bar{J}}{da} = 0 \iff a^* = -\gamma_y(1) / \gamma_y(0)$

⚠ ↑ ONLY NOW WE USE $\mathcal{S}$: to compute $\gamma_y(0), \gamma_y(1)$.

$\mathcal{S}$: $y(t)$ is MA(1) so: $\gamma_y(0) = 1 + \frac{1}{9} = \frac{10}{9}$, $\gamma_y(1) = -1/3$

so: $\boxed{a^* = -\frac{1/3}{10/9} = -3/10}$.

5. $\lambda^{*2} = \bar{J}(a^*) = (1 + (3/10)^2)\frac{10}{9} - \frac{6}{10}\frac{1}{3} = \frac{109}{100} \times \frac{10}{9} - \frac{1}{5}$

$\boxed{\lambda^{*2} = \frac{109}{90} - \frac{18}{90} = \frac{91}{90}}$

THE VAR OF THE PRED. ERROR IS NOT WAY BIGGER THAN $\gamma_e(0)$, SO OUR IDENTIFICATION IS GOOD.

<u>REMARK:</u> use this procedure "Yule Walker" (i.e don't replace in the expr. of ε the value $y(t)$ of $\mathcal{S}$ but only in the final expression of a^*) If you have an AR model to identify. → COMPUTATION SIMPLER.

$\mathcal{S}$: $y(t) = \frac{1}{3}y(t-1) + u(t-1) + \eta(t) - \frac{1}{2}\eta(t-1)$,

$\eta(t) \sim WN(0,1)$, $u(t) \sim WN(0,1)$, $u(t) \perp \eta(t)$

ARMAX(1,1,1)

$\mathcal{M}$: $y(t) = -a\,y(t-1) + b\,u(t-1) + e(t)$ $\quad e(t) \sim WN(0,\lambda^2)$

ARX(1,1).

1. $\hat{\mathcal{M}}: \quad \hat{y}(t|t-1) = -a y(t-1) + b u(t-1)$

(immediately, since $e(t)$ not pred. @ $t-1$)

2. $\varepsilon: \quad \hat{\varepsilon}(t|t-1) = y(t) - \hat{y}(t|t-1)$

$$= y(t) + a y(t-1) - b u(t-1)$$

3. $\bar{J} = \mathbb{E}\left[\hat{\varepsilon}(t|t-1)^2\right] = \mathbb{E}[y(t)^2] + a^2 \mathbb{E}[y^2(t-1)] +$

$b^2 \mathbb{E}[u^2(t-1)] + 2a \mathbb{E}[y(t)y(t-1)]$

$-2b\, \mathbb{E}[y(t)u(t-1)] \cancel{-2ab\, \mathbb{E}[y(t-1)u(t-1)]}$

$\bar{J} = \gamma_y(0)(1+a^2) + b^2 \gamma_u(0) + 2a\gamma_y(1) - 2b\, \mathbb{E}[y(t)u(t-1)]$

$y(t-1) = f(u(t-2), \ldots)$

4. $\frac{d\bar{J}}{da} = 2a\gamma_y(0) + 2\gamma_y(1)$

$\frac{d\bar{J}}{db} = 2b\gamma_u(0) - 2\mathbb{E}[y(t)u(t-1)]$

$a^* = -\gamma_y(1)/\gamma_y(0)$

$b^* = \mathbb{E}[y(t)u(t-1)]/\gamma_u(0)$

only @ this point we use info of $\mathcal{S}$:

- $\gamma_y(0) = 69/32 \quad \gamma_y(1) = 7/32 \longrightarrow \boxed{a^* = -7/69}$

- $\mathbb{E}[y(t)u(t-1)] = \mathbb{E}\left[\left(\frac{1}{3}y(t-1) + u(t-1) + \eta(t) - \frac{1}{2}\eta(t-1)\right)u(t-1)\right]$

$$= \mathbb{E}\left[v^2(t-1)\right] = 1 \rightarrow \boxed{b^* = 1}$$

$\gamma_u(0) = 1$

5. $\boxed{\bar{J}(a^*, b^*) \overset{\sim}{=} 1.134 \cong \lambda^{*2}}$

Remark: $\Rightarrow$ the identified model written in canonical form ? $|a| < 1 \rightarrow$ TRUE FOR $a^* = -7/69$.

MIDA I - EXERCISE SESSION 1 - IDENTIFICATION STARTING FROM DATA:

Let $y(\cdot)$ stationary & w/ null expected value.

We know: $y(1) = 1,\ y(2) = 0,\ y(3) = -1$.

$\mathcal{M}$: $y(t) = a y(t-1) + \xi(t) + a\xi(t-1)$, $\xi(\cdot) \sim WN(0, \lambda^2)$.

We want to fit this model to the data.

- $y(t) = \dfrac{1+az^{-1}}{1-az^{-1}}\xi(t) = \dfrac{z+a}{z-a}\xi(t)$: $|a| < 1$ in order to have the canonical rep. of the model.

- $$\hat{a} = \underset{a}{\text{argmin}}\ \underbrace{\frac{1}{3}\sum_{i=1}^{3}\left(y(i) - \hat{y}(i|i-1)\right)^2}_{\text{Sample variance of the pred. error.}}$$

$$\hat{\mathcal{M}} : \quad \hat{y}(t|t-1) = \frac{C_m(z)-A_m(z)}{C_m(z)}\, y(t) = \frac{2az^{-1}}{1+az^{-1}}\, y(t)$$

i.e.:

$$\hat{y}(t|t-1) = -a\hat{y}(t-1|t-2) + 2a\,y(t-1)$$

so:

t	$y(t)$	$\hat{y}(t\|t-1)$
0	0	0
1	1	$\hat{y}(1\|0) = -a\hat{y}(0\|-1) + 2a\,y(0) = 0$
2	0	$\hat{y}(2\|1) = -a\hat{y}(1\|0) + 2a\,y(1) = 2a$
3	-1	$\hat{y}(3\|2) = -a\hat{y}(2\|1) + 2a\,y(2) = -2a^2$

We write the sample variance with the values:

$$\hat{J}_3 = \frac{1}{3}\sum_{i=1}^{3}\left(y(i) - \hat{y}(i|i-1)\right)^3$$

$$= \frac{1}{3}\left((1-0)^2 + (0-2a)^2 + (-1+2a^2)^2\right)$$

$$= \frac{2}{3}\left[1 + 2a^4\right]$$

So: $\boxed{\hat{a} = 0}$. And: $\boxed{\hat{\lambda}^2 = \hat{J}_3(\hat{a}) = 2/3}$.

To apply this method, you need to check.

- the process is stationary;

- Expected value is equal to 0.

Consider I/O system:

$u(t) = -1$

$y(t) = \begin{cases} 1 & \text{if } t \text{ odd} \\ -1 & \text{if } t \text{ even} \end{cases}$ from $t=0$ to $t=15$.

we have these data

we want to identify these data with the following model:

$\mathcal{M}: \quad y(t) = a\,y(t-1) + b\,u(t-1) + \xi(t), \quad \xi(\cdot) \sim WN(0,\lambda^2).$

- $\theta = \begin{bmatrix} a \\ b \end{bmatrix}$, $\hat{J}_{15} = \frac{1}{15}\sum_{i=1}^{15}\left(y(i) - \hat{y}(i|i-1)\right)^2$

- We know that: $\hat{\mathcal{M}}: \hat{y}(t|t-1) = a\,y(t-1) + b\,u(t-1)$.

Since the $\mathcal{M}$ is AR, we can use the least square formula:

$\mathcal{M}: \quad y(t) = \theta^T \varphi(t) + \xi(t), \quad \theta = \begin{bmatrix} a \\ b \end{bmatrix} \quad \varphi(t) = \begin{bmatrix} y(t-1) \\ u(t-1) \end{bmatrix}.$

$\hat{\mathcal{M}}: \quad \hat{y}(t|t-1) = \theta^T \varphi(t)$

$$\hat{J}_{15} = \frac{1}{15} \sum_{i=1}^{15} \left(y(i) - \theta^T \varphi(i)\right)^2$$

i.e $$\hat{\theta}_{15} = \left[\sum_{i=1}^{15} \varphi(i)\varphi(i)^T\right]^{-1} \sum_{i=1}^{15} \varphi(i) y(i) .$$

So : $\hat{\theta}_{15} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.   because it was a really simple example ...

$\hat{\lambda}^2 = \dots = 0$ the model perfectly fit our data.

$$\boxed{M: \; y(t) = a\,y(t-1) + b\,u(t-1) + \xi(t) \quad \hat{\lambda}^2 = 0 \qquad \boxed{y(t) = -y(t-1)}}$$

$M: \; y(t) = \frac{1}{4} y(t-1) + \frac{1}{a} y(t-2) + e(t) \quad e(\cdot) \sim WN(0, \lambda^2)$

$y(0) = 2 \; ; \; y(1) = 0 \; ; \; y(2) = -1 \; ; \; y(t) = 0 \; \forall t < 0 .$

$$J(a) = \frac{1}{3} \sum_{i=0}^{2} \left(y(t) - \hat{y}(t|t-1)\right)^2$$

1. $\hat{M} : \quad \hat{y}(t|t-1) = \frac{1}{4} y(t-1) + \frac{1}{a} y(t-2) \, , \; a \neq 0 .$

2.

t	$y(t)$	$\hat{y}(t\|t-1)$
0	2	$\hat{y}(0\|-1) = 0$

1	0	$\hat{y}(1\|0) = \frac{1}{4}y(0) + \frac{1}{a}y(-1) = \frac{1}{4}\cdot 2 = \frac{1}{2}$
2	-1	$\hat{y}(2\|1) = \frac{1}{4}y(1) + \frac{1}{a}y(0) = 0 + \frac{2}{a}$

3. $$\hat{J}(a) = \frac{1}{3}\left((2-0)^2 + \left(0-\frac{1}{2}\right)^2 + \left(-1-\frac{2}{a}\right)^2\right)$$

$$= \frac{1}{3}\left(4 + \frac{1}{4} + 1 + \frac{4}{a} + \frac{4}{a^2}\right)$$

$$= \frac{1}{3}\left(\frac{21}{4} + \frac{4a+4}{a^2}\right) = \frac{7}{4} + \frac{4}{3}\,\frac{a+1}{a^2}$$

4. $$\frac{d\hat{J}(a)}{da} = \frac{4}{3} \times \left(\frac{a^2 - 2a(a+1)}{a^4}\right) = \frac{4}{3}\left(\frac{-2a-a^2}{a^4}\right)$$

$$\frac{d\hat{J}(a)}{da} = -\frac{4}{3}\left(\frac{2a+a^2}{a^4}\right)$$

sgn: • NUM $\rightarrow$ $-(2a+a^2) \geq 0 \iff 2a+a^2 \leq 0$

$$\iff a(2+a) \leq 0 \iff \begin{cases} a \leq 0 \\ 2+a \geq 0 \end{cases} \text{ OR } \begin{cases} a \geq 0 \\ 2+a \leq 0 \end{cases}$$

$$\iff \begin{cases} a \leq 0 \\ a \geq -2 \end{cases} \text{ OR } \cancel{\begin{cases} a \geq 0 \\ a \leq -2 \end{cases}} \iff -2 < a < 0$$

IMP

• DEN $\rightarrow$ always ≥ 0

So :

a	$-\infty$		-2		0		$+\infty$
$\frac{d\hat{J}}{da}$		$-$		$+$		$-$	
$\hat{J}(a)$		$\searrow$		$\nearrow$		$\searrow$	

So $\boxed{a^* = -2}$

⚠ We have to check the canonical representation of the model : ⚠

$$y(t) = \frac{1}{4} y(t-1) + \frac{1}{a^*} y(t-2) + e(t)$$

$$= \frac{1}{4} z^{-1} y(t) - \frac{1}{2} z^{-2} y(t-2) + e(t)$$

i.e $$y(t) = \frac{1}{1 - \frac{1}{4} z^{-1} + \frac{1}{2} z^{-2}} e(t) = \frac{z^2}{z^2 - \frac{1}{4} z + \frac{1}{2}} e(t)$$

We have to solve $z^2 - \frac{1}{4} z + \frac{1}{2} = 0$ i.e

$$\left(z - \frac{1}{8}\right)^2 + \frac{1}{2} - \frac{1}{64} = 0 \quad \text{i.e} \quad \left(z - \frac{1}{8}\right)^2 + \frac{31}{64} = 0$$

i.e $$\left(z - 1/8\right)^2 + \left(\frac{\sqrt{31}}{8}\right)^2 = 0 \quad \text{i.e} \quad \boxed{z = \frac{1}{8} \pm j \frac{\sqrt{31}}{8}}$$

$|z| = \sqrt{\frac{1}{64} + \frac{31}{64}} = \sqrt{\frac{32}{64}} < 1 \longrightarrow$ poles of $W_m(z)$ are inside the unit disk $\rightarrow$ OK: the obtained $W_m(z)$ is in CANONICAL FORM.

ALWAYS CHECK.

So $\hat{a} = -2$ is the (valid) solution.

5. $\hat{\lambda}^2 = \bar{J}(\hat{a}) = \frac{7}{4} + \cancel{\frac{4}{3}} \frac{-2+1}{\cancel{(-2)^2}} = \frac{7}{4} - \frac{1}{3} = \frac{21}{12} - \frac{4}{12} = \frac{17}{12}$.

$$\boxed{\hat{\lambda}^2 = 17/12 \cong 1.417}$$

Real system $\mathcal{S}$ with: $\left[\gamma_y(0) = 3,\ \gamma_y(1) = 1,\ \gamma_y(\tau) = 0\ \forall |\tau| \geq 2\right]$

Information about the covariance!

$\mathcal{M}$: $y(t) = a_1 y(t-1) + a_2 y(t-2) + e(t)$, $e(\cdot) \sim WN(0, \lambda^2)$.

1. $\hat{\mathcal{M}}$: $\hat{y}(t|t-1) = a_1 y(t-1) + a_2 y(t-2)$ (immediate)

2. $\varepsilon(t|t-1) = y(t) - \hat{y}(t-1) = y(t) - a_1 y(t-1) - a_2 y(t-2)$

3. $$\bar{J}(a_1, a_2) = \mathbb{E}[\varepsilon(t|t-1)^2] = \mathbb{E}[y^2(t)] + a_1^2 \mathbb{E}[y(t-1)^2] + a_2^2 \mathbb{E}[y(t-2)^2] - 2a_1 \mathbb{E}[y(t)y(t-1)] - 2a_2 \mathbb{E}[y(t)y(t-2)] + 2a_1a_2 \mathbb{E}[y(t-1)y(t-2)].$$

$$\bar{J}(a_1,a_2) = (1+a_1^2+a_2^2)\gamma_y(0) + 2a_1(-1+a_2)\gamma_y(1) - 2a_2\gamma_y(2)$$

4. $$\frac{d\bar{J}(a_1,a_2)}{da_1} = \cancel{2}a_1\gamma_y(0) + \cancel{2}(a_2-1)\gamma_y(1) = 0$$

$$\frac{d\bar{J}(a_1,a_2)}{da_2} = \cancel{2}a_2\gamma_y(0) + \cancel{2}a_1\gamma_y(1) - \cancel{2}\gamma_y(2) = 0$$

$$\Leftrightarrow \begin{bmatrix} \gamma_y(0) & \gamma_y(1) \\ \gamma_y(1) & \gamma_y(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \gamma_y(1) \\ \gamma_y(2) \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} a_1^* \\ a_2^* \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Info.

$$= \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} \times \frac{1}{8} = \begin{bmatrix} 3/8 \\ -1/8 \end{bmatrix}$$

So $\boxed{a_1^* = 3/8 \quad \& \quad a_2^* = -1/8}$.

So our model $\hat{\mathcal{M}}$ is:

$$y(t) = \frac{3}{8} y(t-1) - \frac{1}{8} y(t-2) + e(t)$$

i.e
$$y(t) = \frac{1}{1 - \frac{3}{8} z^{-1} + \frac{1}{8} z^{-2}} e(t) = \frac{z^2}{z^2 - \frac{3}{8} z + \frac{1}{8}} e(t)$$

⚠ We have to check the poles:

$$z^2 - 3/8 + 1/8 = 0 \iff \left(z - \frac{3}{16}\right)^2 - \frac{9}{16^2} + \frac{1}{\underbrace{8}_{2^3}} = 0$$

$$\iff \left(z - 3/16\right)^2 - \frac{9}{2^8} + \frac{25}{2^8} = 0$$

$$\iff \left(z - 3/16\right)^2 + \frac{16}{2^8} = 0$$

$$\iff \left(z - 3/16\right)^2 + \frac{1}{2^4} = 0$$

$$\iff z = \frac{3}{16} \pm j \frac{1}{4} \longrightarrow |z| < 1.$$

So our model's TF is in CANONICAL FORM.

6. $\hat{\lambda}^2 = \bar{J}(\hat{a}_1, \hat{a}_2) = 21/8 \simeq 2.625$.