Exercise session 4 - part 1:

25/08/2024

THEORY RECAP

Stochastic process:

- Expected value
- Covariance function:
  - → time domain (τ)
  - → $\gamma(\tau)$

→ Spectrum:
- → frequency domain (ω)
- → $\Gamma(\omega)$.

SAME INFORMATION BUT IN THE FREQUENCY DOMAIN.

FOURIER TRANSFORM (DISCRETE)

__DEF:__ SPECTRUM: $$\Gamma(\omega) = \sum_{\tau=-\infty}^{\tau=+\infty} \gamma(\tau) e^{-j\omega\tau}$$

__PROP:__ $(y(t))_t$ stationary process. Then:

1- $\Gamma_y(\omega)$ is real;

2- $\Gamma_y(\omega)$ is positive;

3- $\Gamma_y(\omega)$ is an even function;

4- $\Gamma_y(\omega)$ is periodic with $T=2\pi$.

5- **Fundamental THM of spectral analysis:**

If $y(t) = W(z)u(t)$ with $W(z)$ - stable, $u(\cdot)$ stationary.

Then: $$\boxed{\Gamma_y(\omega) = \left|W(e^{j\omega})\right|^2 \cdot \Gamma_u(\omega)}.$$

__PROP:__ 6- $$\gamma_y(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_y(\omega)\,d\omega.$$

Example:

That's why we call it "white".

- $u(\cdot) \sim WN(0, \lambda^2) \longrightarrow \Gamma_u(\omega) = \lambda^2 e^{-j\omega \times 0} = \lambda^2 \quad \forall \omega$

- $y(t) = \dfrac{1 + cz^{-1}}{1 + az^{-1}} e(t), \quad e(\cdot) \sim WN(0, \lambda^2), \quad |a| < 1$

Classical ARMA(1,1) process

Let's compute $\Gamma_y(\omega)$:

$$y(t) = \frac{c + z}{a + z} e(t), \quad \Gamma_y(\omega) = \frac{|e^{j\omega} + c|^2}{|e^{j\omega} + a|^2} \Gamma_e(\omega)$$

$$\rightarrow \Gamma_y(\omega) = \frac{(e^{j\omega} + c)(e^{-j\omega} + c)}{(e^{j\omega} + a)(e^{-j\omega} + a)} \lambda^2$$

$$= \frac{1 + ce^{j\omega} + ce^{-j\omega} + c^2}{1 + ae^{j\omega} + ae^{-j\omega} + a^2} \lambda^2$$

$$= \frac{1 + c^2 + c \times 2\cos(\omega)}{1 + a^2 + a \times 2\cos(\omega)} \lambda^2$$

If $c = 1/a$:

$$\Gamma_y(\omega) = \ldots = \frac{1}{a^2}\lambda^2 = \underbrace{c^2\lambda^2}$$

constant $\forall \omega \rightarrow$ so y is still a WN.

ALL PASS FILTER

$$\underset{WN(0,\lambda^2)}{e(t)} \longrightarrow \boxed{\frac{1 + \frac{1}{a}z^{-1}}{1 + az^{-1}}} \longrightarrow \underset{WN(0, c^2\lambda^2)}{y(t)}$$

$|a| < 1$

Application:

$$y(t) = \frac{1}{2} y(t-1) + e(t) - e(t-1)$$

$e(t) \sim WN(0, 9)$, ARMA(1,1)

so : $c = -1$, $a = -1/2$ & we can apply the formula:

$$y(t) = \frac{1 - z^{-1}}{1 - \frac{1}{2} z^{-1}} e(t) = \frac{z - 1}{z - 1/2} e(t) .$$

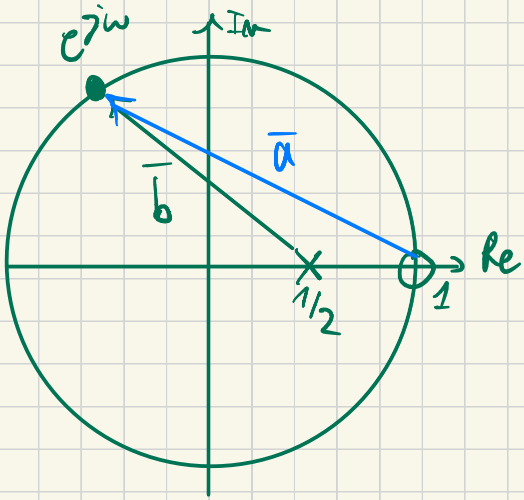
$$\Gamma_y(\omega) = \frac{1 + (-1)^2 + (-1) \times 2 \times \cos(\omega)}{1 + (-1/2)^2 + \left(\frac{-1}{2}\right) \times 2 \times \cos(\omega)} \lambda^2$$

$$= \frac{2 - 2\cos(\omega)}{\frac{5}{4} - \cos(\omega)} \cdot 9 = \frac{18 - 18\cos(\omega)}{\frac{5}{4} - \cos(\omega)}$$

ANOTHER METHOD : *graphical representation.*

Here, we start again from the thm: $\Gamma_y(\omega) = \frac{|e^{j\omega} - 1|^2}{|e^{j\omega} - \frac{1}{2}|^2} \cdot 9$

$\begin{cases} \text{Zero} : z = 1 \\ \text{Pole} : z = 1/2 \end{cases}$

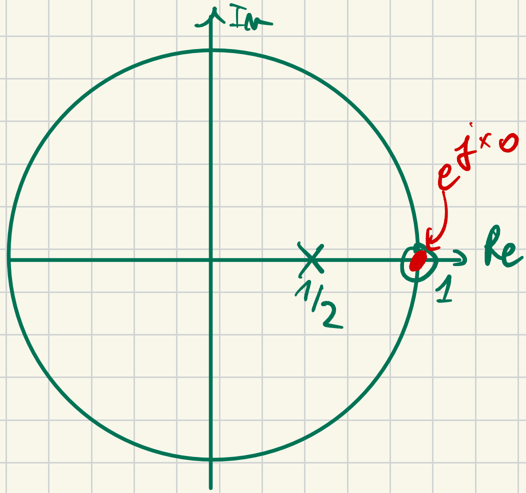


so $\Gamma_y(\omega) = 9 \times \frac{|\bar{a}|^2}{|\bar{b}|^2}$

Usually we only look at particular values for ω: $\omega = 0$, $\omega = \pi/2$, $\omega = \pi$.

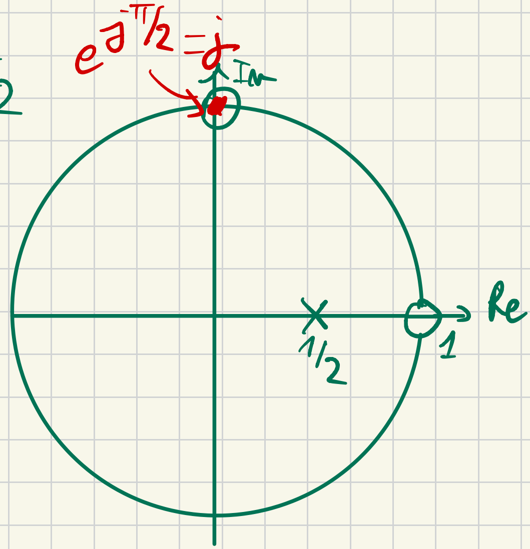
And we extrapolate...

$\omega = 0$:



$$\left.\begin{aligned} |\bar{a}|^2 &= 0 \\ |\bar{b}| &= 1/2 \end{aligned}\right\} \boxed{\Gamma_y(0) = 0}$$

$\omega = \pi/2$



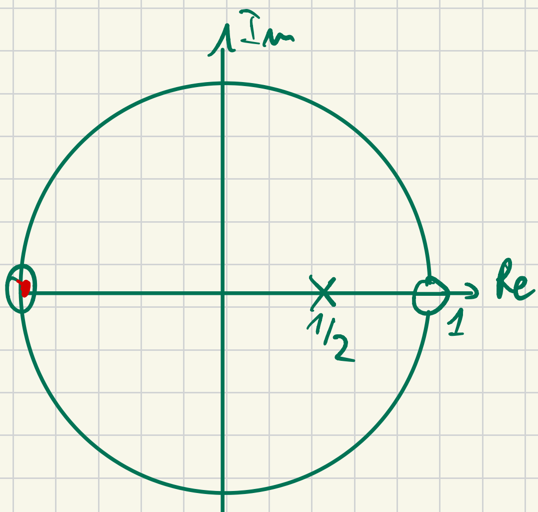
Pythagorean Thm

$$|\bar{a}|^2 = 1^2 + 1^2 = 2$$

$$|\bar{b}|^2 = \left(\tfrac{1}{2}\right)^2 + 1^2 = 5/4$$

$$\boxed{\Gamma_y(\omega = \pi/2)} = 9 \cdot 2 \cdot \frac{4}{5} = \frac{72}{5} \boxed{= 14.4}$$

$\omega = \pi$

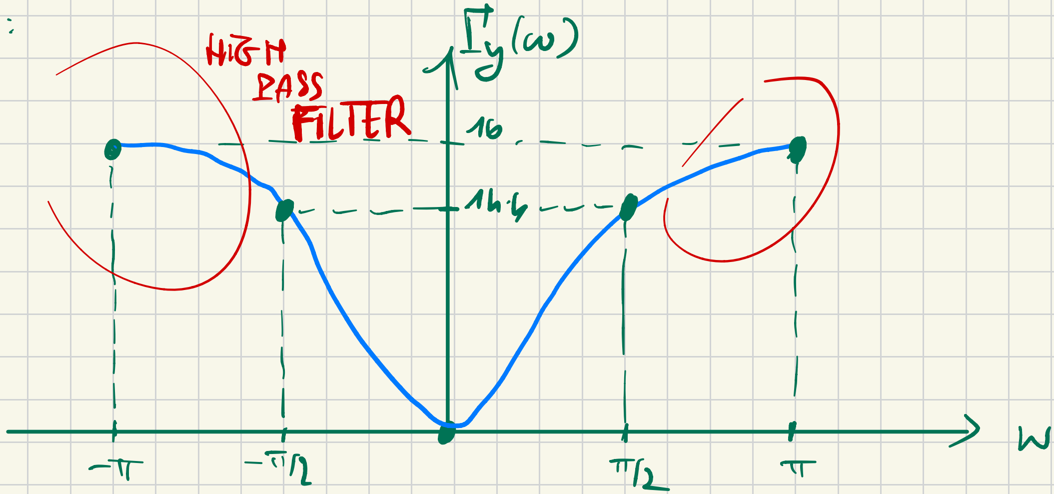


$$|\bar{a}|^2 = 2^2 = 4$$

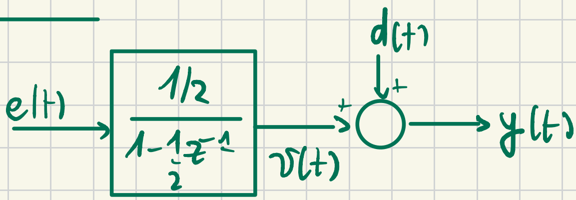
$$|\bar{b}|^2 = \left(3/2\right)^2 = 9/4$$

$$\boxed{\Gamma_y(\omega = \pi)} = 9 \times \frac{4}{9/4} = \boxed{16}.$$

So:



Consider:



$$\left.\begin{array}{l} e(\cdot) \sim WN(0,1) \\ d(\cdot) \sim WN(0,1) \end{array}\right\} e(\cdot) \perp d(\cdot).$$

- $\Gamma_v(\omega) = 0$, $\omega = 0$? **FALSE**

$$v(t) = \frac{1}{2} \times \frac{1}{1-\frac{1}{2}z^{-1}} e(t) = \frac{1}{2} \cdot \frac{z}{z-\frac{1}{2}} e(t)$$

$$\Gamma_v(\omega) = \frac{1}{4} \times \frac{|e^{j\omega}|^2}{|e^{j\omega}-\frac{1}{2}|^2} \times \overbrace{1}^{\Gamma_e(\omega)} = \frac{1}{4} \times \frac{1}{(e^{j\omega}-\frac{1}{2})(e^{-j\omega}-\frac{1}{2})}$$

$$= \frac{1/4}{1+\frac{1}{4}-\frac{1}{2}(e^{j\omega}+e^{-j\omega})} = \frac{1/4}{\frac{5}{4}-\cos(\omega)}$$

so $\Gamma_v(0) = \frac{1/4}{5/4-1} = \frac{1/4}{1/4} = 1.$

- $\Gamma_y(\omega = \pi) = 10/9$? TRUE

$y(t) = d(t) + v(t)$, $\gamma_y(\tau) = \gamma_d(\tau) + \gamma_v(\tau)$

d & e uncorrelated

$\hookrightarrow v = \text{fct}(e(t), e(t-1), \dots)$

So: $\Gamma_y(\omega) = \sum_{\tau=-\infty}^{+\infty} \gamma_y(\tau) e^{-j\omega\tau} = \Gamma_d(\omega) + \Gamma_v(\omega)$.

$\Gamma_d(\omega) = 1 \quad \forall \omega$. So:

$$\Gamma_y(\pi) = \frac{1/4}{9/4} + 1 = \frac{1}{9} + 1 = \frac{10}{9}.$$

- $\Gamma_y(\pi/3) = -1/9$? FALSE : spectrum cannot be negative!

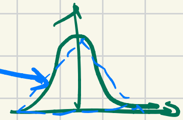
Exercise session 4 - part 2 :

Another type of question : we have a plot of the spectrum of $y(t) = W(z)e(t)$, $e(\cdot) \sim WN(0,1)$.

- Is $W(z) = \frac{z-1}{z}$ possible ? $z = 1 \rightarrow W(z=1) = 0$.

$\Gamma_y(\omega) = 0$ for $\omega = \pm\pi$ (cf plot) $\longrightarrow$ i.e for $z = -1$ FALSE.

- Is $\gamma_y(0) = \frac{2}{\pi}$?



$$\gamma_y(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_y(\omega)\,d\omega \simeq \frac{1}{\cancel{2\pi}} \times \frac{\cancel{2\pi}\times 4}{2} = 2$$ FALSE.

- Is $W(z) = \frac{z+1}{z}$ possible? $z=-1$: the zero is correct.

$$\Gamma_y(\pi/2) = \dots = 2 \rightsquigarrow$$ COHERENT WITH THE PLOT.

$$\Gamma_y(0) = \dots = 4 \rightsquigarrow$$ " ".