

Same traj-
ectories w/ probas

Indistinguishable

t is fixed

Modification

stochastic
(X_{t_1}, \dots, X_{t_n}) and
($X'_{t_1}, \dots, X'_{t_n}$) same law

Equivalent

τ_B exit time from B.
1) B open
2) B closed $\oplus (F_t)_t$ right c.o. $\Rightarrow \tau_B$: stopping time

Exit & Entrance Times

Comparison
of 2 proc. 1

Stochastic
Processes

Construction of a
stoch. proc. by
only introducing
a fdd system 2

Kolmogorov Existence
Theorem

Stopped σ -algebra: $\mathcal{F}_\infty = \sigma(\cup_t \mathcal{F}_t)$
Given τ a stopping time,
 $\sigma(\tau) = \sigma\{t \leq \tau \mid t \in T\} \subseteq \mathcal{F}_\tau = \{\mathcal{F}_t \mid t \leq \tau\}$

Stopping times 6

(τ, T 2 S.T.)

Properties of a
stochastic process

FILTRATION: 5

TRAJECTORIES: 3

Continuity not preserved
when modification
of a continuous process

Kolmogorov continuity
Theorem

MEASURABILITY: 4

Standard process
 \oplus A.S. continuous
 \Downarrow
It has a continuous
very

- 1) $\{t > \tau\}, \{t \leq \tau\}, \{\tau \geq t\} \in \mathcal{F}_t$.
- 2) τ is \mathcal{F}_t -measurable.
- 3) $\sigma \wedge \tau, \sigma \vee \tau$ are S.T.
- 4) $\sigma \leq \tau, \forall \omega \in \Omega \Rightarrow \mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.
- 5) $\mathcal{F}_{\sigma \wedge \tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$.
- 6) $X_t = \mathbb{1}_{\{t < \tau\}} + Y_t = \mathbb{1}_{\{t \leq \tau\}}$
one PROG. MEASURABLE PROCESSES.

Continuity/
Right-Continuity

Continuity a.s/
R-Continuity a.s

$(t, \omega) \mapsto X_t(\omega)$
 $(T \times \Omega, \mathcal{B}(T) \otimes \mathcal{F}) \rightarrow (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$

Progressively Measurable

Measurable

Adapted

$\forall u \in T,$
 $(t, \omega) \mapsto X_t(\omega)$
 $([0, u] \times \Omega, \mathcal{B}([0, u]) \otimes \mathcal{F}) \rightarrow (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$

Stopped A.V: $X_\sigma: \omega \in \Omega \mapsto X_{\sigma(\omega)}(\omega),$
with X a meas. stoch. proc. and
 $\sigma: (\Omega, \mathcal{F}) \rightarrow ([0, +\infty), \mathcal{B}([0, +\infty)))$ a R.V.

Let X be a prog. measurable process
and τ an a.s. finite stopping time.
Then X_τ is measurable wrt the stopped
 σ -algebra \mathcal{F}_τ .