⊛ Linear operators : $(X,\|\cdot\|_X)$, $(Y,\|\cdot\|_Y)$ n.v.s

- $T: X\to Y$ linear, bdd $\Leftrightarrow$ Lip $\Leftrightarrow$ continuous $\forall x_0\in X$ $\Leftrightarrow$ continuous at $0\in X$ $\Big\}$: $(1)\underset{(I)}{\Rightarrow}(2)\underset{(II)}{\Rightarrow}(3)\underset{(III)}{\Rightarrow}(4)\underset{(IV)}{\Rightarrow}1$

(I) : Take $x=u-v$ in the def of bdd.

(II) : Take $x_n\xrightarrow[n\to+\infty]{}x_0$. Then: $0\le \|Tx_n-Tx_0\|_Y \overset{\text{Lip}}{\le} L\|x_n-x_0\|_X\xrightarrow[n\to+\infty]{}0$: $\mathcal{C}^0$.

(III) : Trivial

(IV) : Suppose by contradiction that T $\mathcal{C}^0$ at 0_X but not bdd :

$\forall n\in\mathbb{N}, \exists x_n\in X$ s.t $\|Tx_n\|_Y\ge n\|x_n\|_X$ $\to$ $z_n=\frac{x_n}{n\|x_n\|_X}$, $\|z_n\|_X=1/n\xrightarrow[+\infty]{}0$

but $\|Tz_n\|_h\ge 1$ ↯

- Operator norm : $\|T\|_{\mathcal{L}(X,Y)}=\sup\limits_{\|x\|_X\le 1}\|Tx\|_Y=\sup\limits_{\|x\|_X=1}\|Tx\|_Y=\sup\limits_{x\ne 0_X}\frac{\|Tx\|_Y}{\|x\|_X}=\inf\{M>0: \|Tx\|_Y\le M\|x\|_X, \forall x\in X\}$.

- X n.v.s and Y Banach space $\Rightarrow$ $(\mathcal{L}(X,Y), \|\cdot\|_{\mathcal{L}(X,Y)})$ Banach space.

- __Def:__ • Let $X\subset Y$ with X a vector subspace of Y. Take : $J: X\to Y$, $x\mapsto x$ be the __immersion__ ($\triangle$ $J\ne id_X$). J is __linear__ (but not necessarily continuous).

  • If J is also __continuous__ ($J\in\mathcal{L}(X,Y)$), i.e, $\|x\|_Y\le M\|x\|_X$ $\forall x\in X$, then J is an __embedding__ : $X\hookrightarrow Y$.

  __Rmk:__ • Immersion preserves linearity ; • Embedding preserves ——— & continuity.

  __Example:__ $\mu(X)<+\infty$, $p<q$, $L^q(X)\hookrightarrow L^p(X)$.

__In the following : 3 "big" theorems :__

- __Banach Steinhaus thm (uniform bddness principle)__ : X,Y Banach spaces, $\mathcal{F}\subset\mathcal{L}(X,Y)$.
  • Suppose $\mathcal{F}$ is pointwise bdd (PB: $\forall x\in X, \exists M_x: \|Tx\|_Y\le M_x$ $\forall T\in\mathcal{F}$).
  • Then $\mathcal{F}$ is uniformly bdd (UB: $\exists M$, $\|T\|_{\mathcal{L}(X,Y)}\le M$ $\forall T\in\mathcal{F}$).

  ↳ __Proof:__ • Based on __Baire's lemma__

  ↳ X complete measure space: $\{C_n\}_{n\in\mathbb{N}}$, $C\subset X$, closed & $\bigcup\limits_{n\in\mathbb{N}} C_n=X$. Then $\exists n_0\in\mathbb{N}$, s.t C_{n_0} has non-empty interior (i.e $\exists r>0, x_0\in C_{n_0}$ s.t $\overline{B_r(x_0)}\subset C_{n_0}$).

  • $\forall n\in\mathbb{N}$, $C_n:=\{x\in X: \|Tx\|_Y\le n, \forall T\in\mathcal{F}\}$. C_n is closed.

  • Moreover $\bigcup\limits_{n\in\mathbb{N}}C_n=X$: __by PB__ : $\forall x\in X, \exists M_x, \forall T\in\mathcal{F}, \|Tx\|_Y\le M_x$. So $\forall n\ge M_x$, $x\in C_n$.

  • Use Baire's Lemma : $\exists n_0\in\mathbb{N}$, $x_0\in X$, $r>0$: $\overline{B_r(x_0)}\subset C_{n_0}$.
  $y\in\overline{B_r(x_0)}\Leftrightarrow y=x_0+rz$ ($\|z\|_X\le 1$). $y\in C_{n_0}\Leftrightarrow \|Ty\|_Y\le n_0$ $\forall T\in\mathcal{F}$.
  $\|T(x_0+rz)\|_Y\le n_0$ $\forall T\in\mathcal{F}, \forall \|z\|_X\le 1$

$$\|Tx_0 + Trz\|_y \geq r\,\|Tz\|_y - \|Tx_0\|_y \underset{PB}{\geq} r\|Tz\|_y - M_{x_0}$$

i.e : $n_0 \geq r\|Tz\|_y - M_{x_0} \quad \forall \|z\|_x \leq 1, \forall T \in \mathcal{F}$.

i.e : $\|Tz\|_y \underset{(r>0)}{\leq} \frac{n_0 + M_{x_0}}{r} \quad \forall \|z\|_x \leq 1, \forall T \in \mathcal{F}$.

Finally : $\|T\|_{\mathcal{L}(X,Y)} = \sup_{\|z\|_x \leq 1} \|Tz\|_y \leq \frac{n_0 + M_{x_0}}{r} = M$. $\square$

· **Corollary:** X, Y Banach. $\{T_n\}_n \subset \mathcal{L}(X,Y)$. Assume : $\forall x \in X$, $T_n x \xrightarrow[n\to+\infty]{} Tx$. Then $T \in \mathcal{L}(X,Y)$.

→ Cool car Pointwise CVence facile à prouver !

Proof: · T linear : $T_n(\alpha u + \beta v) = \alpha T_n u + \beta T_n v$, $n \to +\infty$:

$$T(\alpha u + \beta v) = \alpha T u + \beta T v$$

· T bdd : $\forall x$, $T_n x$ CV then $T_n x$ is bdd on Y: $\|T_n x\|_y \leq M_x$.

Apply Banach-Steinhaus : $\|T_n\|_{\mathcal{L}(X,Y)} \leq M$, $\forall n$, i.e :

$\|T_n z\|_y \leq M \quad \forall \|z\|_x \leq 1$ and $\|T_n z\|_y \xrightarrow[n\to+\infty]{} \|Tz\|_y \leq M$, $\forall \|z\|_x \leq 1$. $\square$

OPEN MAP THEOREM

T open : $\forall A \subset X$ open, $T(A) \subset Y$ open.

T continuous : $\forall A \subset Y$ open, $T^{-1}(A) \subset X$ open.

Open Map Theorem : X, Y Banach spaces, $T \in \mathcal{L}(X,Y)$. T surjective $\Rightarrow$ T open.

COROLLARY I : The inverse of a linear & bdd operator between 2 Banach spaces is bounded.

Proof: · T^{-1} is linear :

$T^{-1} \circ T = id_X$ and $T \circ T^{-1} = id_Y$,

and T linear.

· T^{-1} continuous : $\forall A \subset X$ open, $\underbrace{(T^{-1})^{-1}(A)}_{= T(A)} \subset Y$ open.

T bijective $\Rightarrow$ T surjective $\underset{(OMT)}{\Rightarrow}$ T open $\square$

Corollary II : on the equivalence of norms. $(X, \|\cdot\|_a)$, $(X, \|\cdot\|_b)$ Banach.

Assume $\exists C_1 > 0$, $\|x\|_b \leq C_1 \|x\|_a$ $\forall x \in X$.

Then: $\|\cdot\|_b$ and $\|\cdot\|_a$ are equivalent

$\exists C_2 > 0$, $\|x\|_a \leq C_2 \|x\|_b$.

Proof: $J : (X, \|\cdot\|_a) \to (X, \|\cdot\|_b)$ and apply Corollary I. $\square$

· **Def :** $T : X \to Y$ closed if $\left.\begin{array}{l} x_n \to x \in X \\ T(x_n) \to y \in Y \end{array}\right\} \Rightarrow y = T(x)$.

· **Closed Graph Theorem :** X, Y Banach Spaces, $T : X \to Y$ linear & closed.

Then $T \in \mathcal{L}(X,Y)$. Consequence of corollary II before :

Define the "graph norm" on X: $\forall x \in X$, $\|x\|_g = \|x\|_X + \|Tx\|_Y$.

To apply corollary II:

① $\|.\|_g$ has to be a norm: it is! $\longrightarrow$ ② $(X, \|.\|_g)$ has to be Banach: it is (use Cauchy seq). ④ T is closed ($y = Tx$) $\longrightarrow$ ③ By choosing $C_1 = 1$: $\|x\|_X \leq \|x\|_g \times \underbrace{C_1}_{=1}$. $\searrow$ Apply Corollary II: $\exists C_2$, $\|x\|_g \leq C_2 \|x\|_X$ i.e T is $\mathcal{C}$. $\square$

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* DUAL SPACES : Duality/Reflexivity:

$$T : L^q(X) \longrightarrow (L^p(X))^*$$

$$u \longmapsto Tu = L_u \; : \; \begin{array}{ccc} L^p(X) & \longrightarrow & \mathbb{R} \\ v & \longmapsto & L_u v = \int_{\mathbb{R}} uv \, d\mu \end{array}$$

$L^p(X) = (L^q(X))^*$ so $(L^p(X))^* = (L^q(X))^{**}$

$$L_u \in (L^q(X))^{**} : \begin{array}{ccc} (L^q(X))^* = L^p(X) & \longrightarrow & \mathbb{R} \\ v & \longmapsto & L_u v \end{array}$$

HAHN BANACH CONTINUOUS EXTENSION THEOREM: Let X a normed space, $Y \subset X$ subspace, $L_0 \in Y^*$. Then: $\exists \tilde{L}_0 \in X^*$ s.t. $\begin{cases} \tilde{L}_0 y = L_0 y \ \forall y \in Y \\ \|\tilde{L}_0\|_{X^*} = \|L_0\|_{Y^*} \end{cases}$

"Coïncident sur Y, normes duales égales"

NO PROOF $\square$

Corollary I: X normed space, $x_0 \in X \setminus \{0\}$. Then: $\exists L \in X^*$ s.t $\begin{cases} Lx_0 = \|x_0\|_X ; \\ \|L\|_{X^*} = 1 . \end{cases}$

Proof: • Take $Y = \text{span}\{x_0\} = \{tx_0 : t \in \mathbb{R}\}$ and take $L_0(tx_0) = t\|x_0\|_X$ $(L_0 \in Y^*)$.

Apply Hahn Banach (L_0 is a functional on Y: $L_0 : Y \to \mathbb{R}$; L_0 linear & continuous; $\|L_0\|_{Y^*} = 1$). $\square$

Corollary II: "Bounded linear functionals separate points".

$\forall x, y \in X$, $x \neq y$, $\exists L \in X^*$ s.t : $Lx \neq Ly$.

Proof: • Take $x \neq y$ and apply corollary I to $x_0 = x - y$. $\square$

Corollary III: "Bounded linear functionals separate closed subspace and points".

Let X be a normed space, $Y \subsetneq X$ a closed subspace.

$x_0 \in X \setminus Y$. Then $\exists L \in X^*$ s.t. $\begin{cases} Lx_0 \neq 0 ; \\ Ly = 0 \ \forall y \in Y . \end{cases}$

DUAL OF L^p : L^q CAN BE IDENTIFIED BY AN ISOMETRY AS A SUBSPACE OF $(L^p)^*$:

$X = L^p(X, m, \mu)$; $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p \leq +\infty$.

Take $u \in L^q(X, m, \mu)$. Define : $L_u \in (L^p(x))^*$ by :

$$\forall v \in L^p(X), \quad L_u v = \int_X u v \, d\mu .$$

We have :

0) L_u is well defined. <u>Hölder</u> : $|L_u v| \leq \int_X |uv| d\mu \leq \|u\|_q \cdot \|v\|_p$.

1) L_u is linear . Linearity of the integral.

2) L_u is $\mathcal{C}^0$. $|L_u v| \leq \|u\|_q \cdot \|v\|_p = M \cdot \|v\|_p$. $(M = \|u\|_q)$.

3) Evaluate $\|L_u\|_*$. $\|L_u\|_* = \sup_{v \neq 0} \frac{|L_u v|}{\|v\|_p}$

Hölder $\leq \|u\|_q$

$\forall \bar{v} \neq 0 \geq \underbrace{\frac{|L_u \bar{v}|}{\|\bar{v}\|_p}}$ (w/ $\bar{v} = |u|^{q/p} \mathrm{sgn}(u)$).

$= \frac{|\int u \bar{v}|}{(\int |\bar{v}|^p)^{1/p}} = \|u\|_q$.

$$\boxed{\|L_u\|_{(L^p(x))^*} = \|u\|_{L^q(x)}} \quad \square$$

→ In a more abstract way :

$$\underbrace{\begin{aligned} T : \quad & L^q(X) \longrightarrow (L^p(X))^* \\ & u \longmapsto T(u) = L_u \end{aligned}}$$

$: \; L^p(x) \longrightarrow \mathbb{R}$

$\quad v \longmapsto L_u v$.

If $1 < p < +\infty$: T is an isometric embedding : "$L^q \subset (L^p)^*$"

Q: $\boxed{\text{ARE ALL THE ELEMENTS OF } (L^p)^* \text{ OF TYPE } L_u ,\ u \in L^q \text{ ?}}$

↳ <u>ANSWER</u>: $\boxed{(L^p)^* \simeq L^q \text{ for } \underbrace{1 < p < +\infty}_{⚠}}$.

(<u>SPECIFIC CASES</u>: • $p = +\infty$ $(q = 1)$: $\|L_u\|_{(L^\infty)^*} = \|u\|_{L^1}$;

• $p = 1$: IF X IS σ-FINITE : $\|L_u\|_{(L^1)^*} = \|u\|_{L^\infty}$.)

Reflexive spaces: X Banach, X^* Banach too.

Notation: $\left.\begin{array}{l} L \in X^* \\ x \in X \end{array}\right\}$ $L(x) = Lx \underset{\text{notation}}{=} \langle L, x\rangle = \langle L, x\rangle_{X^*,X}$ "duality pairing" (bilinear)

Q: $\boxed{\text{Relation between } X \text{ and } X^{**} \text{ ?}}$

Canonical Map: Given $x \in X$, we can construct $\Lambda_x \in X^{**}$:

T: "canonical/evaluation map"

$$T : X \longrightarrow X^{**}$$
$$x \longmapsto T(x) = \Lambda_x : X^* \longrightarrow \mathbb{R}$$
$$L \longmapsto \Lambda_x(L) = Lx = \langle L, x\rangle$$

i.e: $\boxed{\forall L \in X^*, \ {}_{X^{**}}\langle \Lambda_x, L\rangle_{X^*} = {}_{X^*}\langle L, x\rangle_X}$.

Properties of the canonical map:

- **PROP:** $\forall x \in X$, $\Lambda_x \in X^{**}$ and $\|\Lambda_x\|_{**} = \|x\|_X$.

Proof:
- Linear ok.
- $\|\Lambda_x\|_{**} = \sup_{L \neq 0} \frac{|\Lambda_x(L)|}{\|L\|_*} = \sup_{L \neq 0} \frac{|Lx|}{\|L\|_*}$ $\geq \|x\|_X$: by corollary (I) Hahn Banach. $\leq \|x\|_X$ ok. □

- **THM:** $T : X \longrightarrow X^{**}$, $x \longmapsto T(x) = \Lambda_x$ is
  - linear & continuous;
  - an isometry;
  - injective.

Proof:
- Linear ok.
- $\|x\|_X = \|\Lambda_x\|_{X^{**}} = \|T(x)\|_{**}$ => Isometry (+) continuity.
- Injectivity: assume $x, y \in X$, $x \neq y$, by the 2nd H-Banach theorem: $\exists L \in X^*$ s.t. $Lx \neq Ly$. Then: ${}_{**}\langle T(x), L\rangle_* = {}_*\langle L, x\rangle_X \neq {}_*\langle L, y\rangle_X = {}_{**}\langle T(y), L\rangle_*$
  $T(x) \neq T(y)$. □

DEF: X is reflexive if T is surjective.

Rmk: X reflexive $\Leftrightarrow$ $\forall \Phi \in X^{**}$, $\exists ! x \in X$, $\forall L \in X^*$, ${}_{**}\langle \Phi, L\rangle_* = {}_*\langle L, x\rangle_X$

(i.e: $\forall L \in X^*$, $\Phi L = Lx$).

Rmk bis: $\left.\begin{array}{l}\text{By a corollary of} \\ \text{the Open Map Thm}\end{array}\right\}$: X reflexive $\Leftrightarrow$ T is an isometric isomorphism between X & X^{**}.

DUAL SPACE OF L^p:

Riesz Representation Theorem for $(L^p(X))^*$: $(X,\mathcal{M},\mu)$ complete measure space; $1<p<+\infty$.
q is the conjugate exponent of p: $\frac{1}{p}+\frac{1}{q}=1$. Then:

$\forall L\in (L^p(X))^*$, $\exists!\, u\in L^q(X)$ such that:

$\forall v\in L^p(X)$, $Lv=\int_X u\cdot v\,d\mu$. Moreover: $\|L_u\|_{(L^p)^*}=\|u\|_{L^q}$. $\square$

Rmk: i.e.: $\underbrace{\begin{array}{rccc} T: & L^q(X) & \longrightarrow & (L^p(X))^* \\ & u & \longmapsto & L_u \end{array}}_{\text{is an ISOMETRIC ISOMORPHISM.}}$: $\begin{array}{rcl} L^p(X) & \longrightarrow & \mathbb{R} \\ v & \longmapsto & L_u v=\int_X uv\,d\mu \end{array}$

Thm: The same holds for $p=1$ $(q=+\infty)$ AS LONG AS μ IS σ-FINITE. $\square$

Rmk: Also true for $(\Omega,\mathcal{L}(\Omega),\lambda)$ w/ $\Omega\in\mathcal{L}(\mathbb{R}^N)$;
but also for $(\mathbb{N},\mathcal{P}(\mathbb{N}),\mu_\#)$: ℓ^p spaces.

Dual spaces for L^∞: $(p=+\infty)$

$u\in L^1$: $\begin{array}{rcl} L_u: L^\infty & \longrightarrow & \mathbb{R} \\ v & \longmapsto & L_u v=\int_X uv\,d\mu \end{array}$ $\Bigg\}$ $\leadsto$ "$L^1\subsetneq(L^\infty)^*$".
$\exists$ elements of $(L^\infty)^*$ of different types ⚠

WEAK CONVERGENCE :

DEF: X Banach, $\{x_n\}_n \subset X$, $x \in X$. $x_n \underset{n\to+\infty}{\rightharpoonup} x$ if $\forall L \in X^*$, $Lx_n \underset{n\to+\infty}{\longrightarrow} Lx$.

PROP: strong convergence to x $\Rightarrow$ weak convergence to x. **Proof:** If $x_n \underset{n\to+\infty}{\longrightarrow} x$, then, by continuity, $\forall L \in X^*$, $Lx_n \longrightarrow Lx$.
continuous linear!

Rmk: $1 \leq p < +\infty$, $u_n \underset{n\to+\infty}{\rightharpoonup} u$ in $L^p(X) \Leftrightarrow \forall L \in (L^p(X))^*$, $Lu_n \underset{n\to+\infty}{\longrightarrow} Lu$

$$\Leftrightarrow \forall v \in L^q, \int_X u_n v \, d\mu \underset{n\to+\infty}{\longrightarrow} \int_X uv \, d\mu .$$

PROP: "the weak limit is unique" **Proof:** $\begin{cases} x_n \rightharpoonup y \\ x_n \rightharpoonup z \end{cases}$. Then $\forall L \in X^*$, $\begin{cases} Lx_n \to Ly \\ Lx_n \to Lz \end{cases}$

Then by corollary of HB: $\forall L \in X^*$, $Ly = Lz \Rightarrow y = z$. $\square$

PROP: If $x_n \underset{n\to+\infty}{\rightharpoonup} x$ in X, then $\{x_n\}_n$ is bounded.

Proof: Use Banach-Steinhaus on X^*.

$T(x_n) \in X^{**}$: $\underbrace{{}_{**}\langle T(x_n), L \rangle_*}_{= Lx_n} \longrightarrow \underbrace{{}_{**}\langle T(x), L \rangle_*}_{= Lx}$ $\forall L \in X^*$

$\Rightarrow {}_{**}\langle T(x_n), L \rangle_*$ is bounded (as cv. seq in $\mathbb{R}$).

$\Rightarrow \forall L \in X^*$, $\exists M_L$: $|T(x_n)L| \leq M_L$.

$\overset{B.S}{\Rightarrow} \exists M > 0$, $\|T(x_n)\|_{**} \leq M$ $\forall n$. So : $\forall n \in \mathbb{N}$, $\|x_n\|_X \leq M$. $\square$

PROP: $x_n \underset{n\to+\infty}{\rightharpoonup} x$ weakly in X. Then $\|x\| \leq \liminf_n \|x_n\|$.

PROOF: By a corollary of Hahn-Banach : $\exists L \in X^*$, $\begin{cases} \|L\|_* = 1 \\ Lx = \|x\|_X \end{cases}$. We have :

$Lx_n \leq |Lx_n| \leq \|L\|_* \|x_n\|_X$

$\|x\| = Lx = \lim_{n\to+\infty} Lx_n = \liminf_{n\to+\infty} Lx_n \leq \liminf_{n\to+\infty} \|L\|_* \|x_n\|_X$ $\square$

WEAK * CONVERGENCE :

DEF: X Banach, X^* Banach. $\{L_n\}_{n\in\mathbb{N}} \subset X^*$, $L \in X^*$.

$L_n \xrightarrow[n\to+\infty]{*} L$ in X^* iff $\forall x \in X$, $L_n x \xrightarrow[n\to+\infty]{} Lx$.

Rmk: $L_n \xrightarrow{*} L \iff \forall x \in X$, $L_n x \xrightarrow[n\to+\infty]{} Lx \iff \forall x \in X$, $\tau(x) L_n \xrightarrow[n\to+\infty]{} \tau(x) L$.

WHEREAS: $L_n \xrightarrow[n\to+\infty]{} L \iff \Phi L_n \longrightarrow \Phi L \quad \forall \Phi \in X^{**}$.

PROP: If X reflexive then $L_n \xrightarrow[n\to+\infty]{} L$ weakly in $X^* \iff L_n \xrightarrow[n\to+\infty]{*} L$ weakly* in X^*.

($\Longrightarrow$: general X)

PROOF: If X reflexive: $\forall \Phi \in X^{**}$, $\exists! x \in X$, $\Phi = \tau(x)$. □

BANACH-ALAOGLU THM:

$\boxed{\text{VARIANT I}}$: X Banach, reflexive. Then: every bounded sequence $\{x_n\}_n \subset X$ admits a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ which weakly converges.

$\boxed{\text{VARIANT II}}$: X Banach, separable. Then: every bounded sequence $\{L_n\}_{n\in\mathbb{N}} \subset X^*$ admits a subsequence $\{L_{n_k}\}_{k\in\mathbb{N}}$ which weakly* converges.

COMPACT OPERATORS:

DEF: · X, Y Banach, $K : X \to Y$, linear. K is compact if:

$$\forall E \subset X \text{ bounded}, \quad K(E) \subset Y \text{ is precompact.}$$

· Equivalently: K is compact $\Leftrightarrow$ $\forall \{x_n\}_n \subset X$, the sequence $\{Kx_n\}_n \subset Y$ admits a (strongly) converging subsequence.

PROP: Linear & compact operators are bounded.

PROOF: $B_1 \subset X$, $B_1 := \{x \in X : \|x\| < 1\}$. B_1 is bdd.

So $\overline{K(B_1)}$ is compact in Y. $\overline{K(B_1)}$ is then bdd:

$\exists M > 0$, $\forall x \in X$, $\|x\| \leq 1$, $\|Kx\| \leq M$. $\square$

PROP: $T \in \mathcal{K}(X, Y)$, $\dim(Y) = +\infty$ $\Rightarrow$ T cannot be surjective.

"Compact operators (in ∞ dim) cannot be surjective".

PROOF: Assume T surjective. Then T is open (OMT).

$B_1 \subset X$: so $T(B_1)$ is open (T open) and $T(B_1)$ is precompact (T compact).

↳ $B_1 := \{x \in X : \|x\| \leq 1\}$

$T(B_1)$ open $\Rightarrow$ $T(B_1) \supset B_\varepsilon(y)$. So $\underbrace{\overline{T(B_1)}}_{\text{compact}} \supset \overline{B_\varepsilon(y)}$ is compact.

$\overline{B_\varepsilon(y)}$ compact $\Rightarrow$ $\dim Y < +\infty$. $\square$

HILBERT SPACES :

- Cauchy-Schwarz ; Parallelogram identity : $\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+2\|y\|^2 \quad \forall x,y \in H$.
- **DEF:** $(H, \|.\|)$ is a Hilbert space if $(H, \langle . | . \rangle)$ is an innerproduct-space which is complete wrt the induced norm (i.e $(H, \|.\|)$ is Banach).

ORTHOGONAL PROJECTIONS :

PROJECTION THEOREM (on closed convex set): H Hilbert, $x \in H$, $S \subset H$ convex set and closed (non empty).

$\exists ! h \in S$ s.t. $\|x-h\| = dist(x,S) = \inf_{v \in S} \|x-v\|$. (1)

Moreover, h is caracterized by the "variational inequality" :

$$\forall v \in S, \ \langle x-h \,|\, v-h \rangle \leq 0 \ . \quad (2)$$

I.e: h satisfies (1) $\Leftrightarrow$ h satisfies (2).

PROJECTION THEOREM FOR CLOSED SUBSPACES: H Hilbert, $x \in H$, $V \subset H$ closed subspace.

$\exists h \in V$, $\|x-h\| = \min_{v \in V} \|x-v\|$. h is caracterized by: $\langle x-h, v \rangle = 0 \ \forall v \in V$.

DUAL OF A HILBERT SPACE:

Riesz Representation Theorem: (For Hilbert Space) H Hilbert, $\forall L \in H^*$, $\exists ! u \in H$ s.t.

$$\forall v \in H, \quad Lv = \langle u, v \rangle \qquad (L = L_u).$$

Moreover, $\|u\|_H = \|L_u\|_*$.

Corollary: H Hilbert $\Rightarrow$ H reflexive.

CONSEQUENCE OF RIESZ THM IN HILBERT SPACE: H Hilbert. $x_n \xrightarrow[n \to +\infty]{} x$ weakly in H

$$\Leftrightarrow \ \langle u, x_n \rangle \xrightarrow[n \to +\infty]{} \langle u, x \rangle, \ \forall u \in H.$$

PROP: "Weak CVence + CVence of norms" H Hilbert, $\{x_n\}_n \subset H$.

$$\begin{cases} x_n \to x \text{ weakly in } H \\ \|x_n\| \to \|x\| \end{cases} \Rightarrow \ x_n \to x \text{ strongly in } H \ (\|x_n - x\| \to 0).$$

PROOF: $0 \leq \|x_n - x\|^2 = \langle x_n - x, x_n - x \rangle = \|x_n\|^2 + \|x\|^2 - 2\langle x_n, x \rangle \longrightarrow 0$ $\square$

PROP: H separable, $\{e_n\}_{n \in \mathbb{N}}$ orthonormal basis.

$\Rightarrow$ $e_n \xrightarrow[n \to +\infty]{} 0$ weakly in H. But not strongly.

PROOF: $\forall n$, $\|e_n\| = 1 \not\to 0$ so $e_n \not\to 0$ strongly.

On the other hand: $e_n \rightharpoonup 0 \Leftrightarrow \langle x, e_n \rangle \longrightarrow 0 \quad \forall x \in H$. This is the case using Parseval: $\forall x \in H$, $\|x\|^2 = \sum_{n \in \mathbb{N}} \langle x, e_n \rangle^2 < +\infty$ so necessarily: $\langle x, e_n \rangle \xrightarrow[n \to +\infty]{} 0$. $\square$