# Lévy Processes

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# 1 Stochastic Jump Processes

We want to define "stochastic jump processes". We need two things which are the following. When does the jump occur ? And what is the size of the jump ? But of course, the jump cannot go $+\infty$. So, basically, in this course, we will work with right-continuous-left-limited processes, i.e, "CADLAG" (continu à droite, (admet une) limite à gauche). Let's define the tools we need to work in this framework.

# 2 Basic tools

**Definition:** [*Characteristic Function*] Let $X$ be a random variable in $\mathbb{R}^d$. Its characteristic function $\Phi_X : \mathbb{R} \to \mathbb{R}$ is defined as :

$$z \to \Phi_X(z) = \mathbb{E}[e^{iz\cdot X}] = \int_{\mathbb{R}^d} e^{iz\cdot X} d\mu_X(x)$$

where $\mu_X$ is the measure associated to the distribution of $X$, i.e,

$$\forall A \in B(\mathbb{R}), \mu_X(A) = \mathbb{P}(X \in A)$$

**Remark:** If $\mu_X$ has a density $p_X$, we can write $d\mu_X(x) = p_X(x)\,dx$, where $dx$ is the Lebesgue measure. This is equivalent to the **absolute continuity** of the distribution of $X$ with respect to the Lebesgue measure.

**Definition:** [*Moments*] Let $n \in \mathbb{N}$,

$$\text{Moment: } m_n(X) = \mathbb{E}[X^n] \; ;$$

$$\text{Centered moment: } \mu_n(X) = \mathbb{E}[(X - \mathbb{E}(X))^n].$$

**Property:**

- If $\mathbb{E}[|X|^n] < +\infty$, then $\Phi_X \in C^n(I)$ where $I$ is an open set contianing 0, and :

$$m_k = \frac{1}{i^k}\frac{\partial^k}{\partial z^k}\Phi_X(0), k \in \{1, ..., n\}.$$

- If $\Phi_X$ has $n$ continuous derivatives in 0, then :

$$m_k = \frac{1}{i^k}\frac{\partial^k}{\partial z^k}\Phi_X(0), k \in \{1, ..., n\}.$$

**Definition:** [*Moment Generating Function*]

$$M_X(u) = \mathbb{E}[e^{u\cdot X}] \; ;$$

$$m_n = \frac{\partial^n}{\partial u^n} M_X(0).$$

**Remark:** It is easy to go from $M_X$ to $\Phi_X$ and vice versa since : $M_X(u) = \Phi_X(-iu)$.

**Definition:** [*Characteristic Exponent*] When it exists, $\Psi_X$ such that :

$$\Phi_X(u) = e^{\Psi_X(u)}.$$

**Remark:** $\Psi_X(0) = 0$.

**Definition:** [*Exponential Random Variable*]

**Theorem:** [*Absence of memory*] Let $T \geq 0$ a random variable such that :

$$\forall t, s > 0, \mathbb{P}(T > t + s | T > t) = \mathbb{P}(T > s).$$

This is equivalent to : $T \sim Exp$.

**Definition:** [*Poisson Distribution*] Let $N$ random variable with values in $\mathbb{N}$. $N \sim Poiss(\lambda)$ if and only if :

$$\mathbb{P}(N = n) = e^{-\lambda} \frac{\lambda^n}{n!}.$$

Then we have the following moment generating function : $M(u) = e^{\lambda(e^u - 1)}$.

**Property:** Let $(\tau_i)_{i \geq 1}$ **i.i.d** random variables following an $Exp(\lambda)$ distribution. Let :

$$\forall t > 0, N_t = inf\{n \geq 0 : \sum_{i=1}^{n+1} \tau_i > t\}.$$

We then have that $N_t \sim Poisson(\lambda t)$.

**Property:**

- Let $Y_1, Y_2 \sim Poisson(\lambda_1), Poisson(\lambda_2)$ and $Y_1 \perp Y_2$. Then :

$$Y_1 + Y_2 \sim Poisson(\lambda_1 + \lambda_2).$$

- *[Infinite Divisibility]* Let $Y \sim Poisson(\lambda)$. Then :

$$\forall n, Y = \sum_{i=1}^{n} Y_i$$

  where $Y_1, ..., Y_n$ **i.i.d** $\sim Poisson(\lambda/n)$.

# 3 Poisson Process

**Definition:** [*Poisson Process*] Let $(\tau_i)_{i \leq 1}$ be a sequence of **i.i.d** random variables following an exponential distribution of parameter $\lambda$. Let $T_n = \sum_{i=1}^{n} \tau_i$. Then the following is a Poisson Process with intensity $\lambda$ :

$$N_t = \sum_{n=1}^{+\infty} \mathbf{1}_{t \geq T_n}.$$

**Remark:** This definition is equivalent to the one above with the $inf$. If we fix time, $N_t \sim Poisson(\lambda t)$.
**Remark:** Fortunately, with Poisson Process we don't have mass probability. It means that : $\forall t \geq 0, N_{t^-} = N_t$ with probability 1.
**Remark:** $(N_t)_{t \geq 0}$ is a CADLAG process.

**Property:** [*About Poisson Process*]

- $\forall t \geq 0, \forall n \geq 0, \mathbb{P}(N_t = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$ ;
- $\Phi_{N_t}(u) = \mathbb{E}[e^{iuN_t}] = e^{\lambda t(e^{iu} - 1)}$ ;
- $(N_t)_{t \geq 0}$ has **independent** increments :

$$\forall t_1 < ... < t_n, N_{t_n} - N_{t_{n-1}} \perp N_{t_{n-1}} - N_{t_{n-2}}, ..., N_{t_2} - N_{t_1}, N_{t_1};$$

- $\forall t \geq 0, \mathbb{E}[N_t] = \lambda t.$

**Remark:** We don't have Martingale Property with $(N_t)_t$, so we need to define the "*Compensated Poisson Process*".

**Definition:** [*Compensated Poisson Process*] Let $\forall t \geq 0, \hat{N}_t = N_t - \lambda t$, and $\hat{N}_0 = 0$. Clearly, $\hat{N}$ is **not** a Poisson Process, since it doesn't even take only integer values.

**Property:** $\Phi_{\hat{N}_t}(z) = e^{\lambda t(e^{iz} - 1 - iz)}$.

**Property:** $\hat{N}$ is a Martingale.

**Theorem:** Let $(X_t)_{t\geq 0}$ be a counting process with *independent* and *stationary* increments. Then $(X_t)_{t\geq 0}$ is a Poisson Process.
**Remark:** Stationary increments means that :

$$\forall t > s, h > 0, X_{t+h} - X_{s+h} \sim X_t - X_s.$$

**Remark:** The previous theorem tells us that the Poisson Process is our only "counting process" choice **if** we want to work with **independent and stationary** increments : this will be the **framework of this course.**

**Definition:** Let's introduce a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, $\omega \in \Omega$ a realisation, and $T_i(\omega)$ the i-th jump time occurring in the realisation $\omega$. Then, let's define the following integer-valued **random measure** :

$$\forall A, M(\omega, A) = \#\{i \geq 1 : T_i(\omega) \in A\}.$$

We can write :

$$N_t(\omega) = M(\omega, [0,t]) = \int_0^t M(\omega, ds).$$

**Definition:** [*Compensated Random Measure*] $\hat{M}(\omega, A) = M(\omega, A) - \int_A \lambda dt = M(\omega, A) - \lambda|A|$.

Now we are ready to introduce **Lévy** processes.

# 4 Lévy Process

**Definition:** Let $(\Omega, \mathbb{F}, \mathbb{P})$ a probability space, $\Omega \subseteq \mathbb{R}^d$. A **CADLAG** process $(X_t)_{t\geq 0}$ such that $X_0 = 0$ is Lévy if :

- Increments are **independent** :

$$0 \leq t_0 \leq t_1 \leq ... \leq t_n, X_{t_0} \perp X_{t_1} - X_{t_0} \perp ... \perp X_{t_n} - X_{t_{n-1}};$$

- Increments are **stationary** :

$$\forall t, h > 0, X_{t+h} - X_t \sim X_h;$$

- **Stochastic Continuity :** [*defined starting from the limit in probability*]

$$\forall t > 0, \forall \varepsilon > 0, \lim_{h\to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0.$$

**Example:** $(N_t)_{t\geq 0}$ ; $(W_t)_{t\geq 0}$ is the only Lévy process which is continuous.

**Property:** We have **infinite divisibility**, i.e

$$\forall t > 0, n \geq 1, \Delta = \frac{t}{n}, X_t = X_{n\Delta} = (X_{n\Delta} - X_{(n-1)\Delta}) + (X_{(n-1)\Delta} - X_{(n-2)\Delta}) + ... + (X_{2\Delta} - X_{\Delta}) + X_{\Delta};$$

i.e $X_t = \sum_{i=1}^n Y_i$ where $Y_i = X_{i\Delta} - X_{(i-1)\Delta}$ with $(Y_i)_i$ that are **i.i.d** ($Y_i$ are **independent** and $\forall i, Y_i \sim X_\Delta$).

**Remark:** The infinite divisibility is precisely the ability to write $X_t$ as a sum of **i.i.d** terms.

**Definition:** The characteristic function of a Lévy process is defined as:

$$\Phi_{X_t}(u) = \mathbb{E}[e^{iuX_t}].$$

- It is multiplicative:

$$\Phi_{X_{t+s}}(u) = \mathbb{E}[e^{iu(X_{t+s} - X_s + X_s)}] = \mathbb{E}[e^{iu(X_{t+s} - X_s)}e^{X_s}].$$

  Therefore, by independence :

$$\Phi_{X_{t+s}}(u) = \mathbb{E}[e^{iu(X_{t+s} - X_s)}] \cdot \mathbb{E}[e^{iuX_s}].$$

  And since $X_{t+s} - X_s \sim X_t$ by definition, hence :

$$\Phi_{X_{t+s}}(u) = \Phi_{X_t}(u) \times \Phi_{X_s}(u).$$

- Lévy processes admit a **characteristic exponent** :

$$\exists \Psi_X : \mathbb{R}^d \to \mathbb{R} \text{ such that : } \Phi_{X_t}(u) = e^{t\Psi_X(u)}.$$

**Remark:** $t$ is "outside", which is really useful. That's the reason why we will **never** write the characteristic function of a Lévy Process. Instead, we will use the characteristic exponent, from which we cna easily get $\Phi_{X_t}$.

**Example:** For a Poisson Process,

$$\Phi_{X_t}(u) = e^{\lambda t(e^{iu}-1)} \text{ so that : } \Psi_X(u) = \lambda(e^{iu} - 1).$$

Now that we are able to count the jumps when they occur, we would like to capture the size of the jumps.

# 5 Compounded Poisson Process

**Definition:** A Compounded Poisson Process (CPP) $(X_t)_{t\geq 0}$ with **intensity** $\lambda$ and **jumpsize distribution** $f$ is defined as:

$$X_t = \sum_{i=1}^{N_t} Y_i, \text{ where :}$$

- $(N_t)_{t\geq 0}$ is a **counting process** with intensity $\lambda$ ;
- $Y_i$ are **i.i.d** random variables with distribution $f$.

**Properties:** Let $(X_t)_{t\geq 0}$ be a CPP as above. Then :

- $(X_t)_{t\geq 0}$ is for sure **CADLAG** ;
- A Poisson Process is a special case of a CPP, where $f$ is such that $\forall i, P(Y_i = 1) = 1$ ;
- $(X_t)_{t\geq 0}$ is a CPP **if and only if** $(X_t)_{t\geq 0}$ is a **Lévy Process with piecewise constant paths**.

**Property:** [*Computation of the characteristic function of a CPP*] For all $t \geq 0$, $\Phi_{X_t}(u)$ of a CPP is given by:

$$\Phi_{X_t}(u) = \exp\left(t\lambda \int_{\mathbb{R}^d} \left(e^{iux} - 1\right) f(dx)\right) = \exp(t\Psi_X(u)) \text{ where : } \Psi_X(u) = \lambda \int_{\mathbb{R}^d} \left(e^{iux} - 1\right) f(dx).$$

**Remark:** As previously said, we notice that using $f(dx) = \delta_1(dx)$, we find back the formula for a Poisson Process.

**Proof:** By the **Tower Property** :

$$\Phi_{X_t}(u) = \mathbb{E}\left[e^{iuX_t}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{iuX_t}|N_t\right]\right]$$

Using the expression of $(X_t)_{t\geq 0}$ ; and the fact that the sum in the exponential is **measurable w.r.t** $N_t$ with all the $Y_i$ being **independent** (therefore all the $e^{iuY_i}$ too) we get :

$$\Phi_{X_t}(u) = \mathbb{E}\left[\mathbb{E}\left[e^{iu\sum_{i=1}^{N_t} Y_i}|N_t\right]\right] = \mathbb{E}\left[\prod_{i=1}^{N_t} \mathbb{E}\left[e^{iuY_i}|N_t\right]\right]$$

Then we use the fact that $\forall i, Y_i \perp N_t$ ; and then that all the $Y_i$ are identically distributed :

$$\Phi_{X_t}(u) = \mathbb{E}\left[\prod_{i=1}^{N_t} \mathbb{E}\left[e^{iuY_i}\right]\right] = \mathbb{E}\left[\mathbb{E}\left[e^{iuY_1}\right]^{N_t}\right] = \mathbb{E}\left[\hat{f}(u)^{N_t}\right]$$

where $\hat{f}(u)$ will be computed afterwards. Then by the **Transfer Theorem** for $\hat{f}(u)^{N_t}$ with $N_t \sim Poisson\,(\lambda t)$ :

$$\Phi_{X_t}(u) = \sum_{n=0}^{+\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \left(\hat{f}(u)\right)^n = e^{\lambda t(\hat{f}(u)-1)}.$$

Now we just need to compute $\hat{f}(u)$, which, by the **Transfer Theorem** is :

$$\hat{f}(u) = \mathbb{E}\left(e^{iuY_1}\right) = \int_{\mathbb{R}^d} e^{ius} f(ds).$$

Hence, putting everything together we get :

$$\Phi_{X_t}(u) = e^{\lambda t\left(\int_{\mathbb{R}^d} e^{ius} f(ds) - 1\right)} = e^{\lambda t\left(\int_{\mathbb{R}^d} \left(e^{ius} - 1\right) f(ds)\right)} \square$$

**Definition:** [*Lévy Measure*] The **Lévy Measure** of a CPP is :

$$\forall A, \nu(A) = \lambda f(A) \text{ \textbf{WARNING : this is not a probability measure !}}$$

so that the **Characteristic Exponent** is :

$$\Psi_X(u) = \int_{\mathbb{R}^d} \left(e^{iux} - 1\right) \nu(dx)$$

**Remark:** As previously mentioned in section 2, in most of the cases : $\exists K$ s.t $f(dx) = K(x)dx$ which makes everything easier.

**Properties:** [*About the Lévy Measure of a CPP*]

- $\forall A, \nu(A) = \mathbb{E}\left[\#\left\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\right\}\right]$ the **Lévy Measure of** $A$, where :
  - $\Delta X_t = X_t - X_{t^-} = X_t - lim_{h \to 0^+} X_{t-h}$ ;
  - It means that the Lévy Measure of $A$ is $\nu(A) = \mathbb{E}\left[\text{nombre de sauts} \neq 0, \in A, \text{for } t \in [0,1]\right]$.
- $\forall B \in [0, +\infty) \times \mathbb{R}^d, J_X(B) = \#\left\{(t, \Delta X_t) \in B\right\}$, i.e,
  it is a **Muldi-dimensional Random Measure** (1 realization gives 1 measure) where :
  - $J_X : \Omega \to \mathbb{M}\left([0, +\infty) \times \mathbb{R}^d\right)$ (space of measures) ;
  - $[0, +\infty)$ stands for the **time** ;
  - $\mathbb{R}^d$ stands for the **jump**.
- Then we can write :

$$X_t = \sum_{i=1}^{N_t} Y_i = \sum_{s \in [0,t]} \Delta X_s = \int_{[0,t] \times \mathbb{R}^d} x J_X(ds \times dx).$$

Now we would like to write Lévy Processes. Here follows the first possibility, which is about **Finite Activity Lévy**. The other form of Lévy is **Infinite Activity Lévy**.

**Definition:** [*Finite Activity Lévy*] The following two forms are two equivalent ways to write **Finite Activity (FA) Lévy** processes :

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$$

$$X_t = \mu t + \sigma W_t + \sum_{s \in [0,t]} \Delta X_s$$

Because it is a **FA Lévy**, the term $\sum_{s \in [0,t]} \Delta X_s \in \mathbb{R}$. We can also write it as a **CPP** term : $\sum_{i=1}^{N_t} Y_i$ (cf above). There is a third way of writing such a process (**FA Lévy**), exploiting the **random measure** we have seen before :

$$X_t = \mu t + \sigma W_t + \int_{[0,t] \times \mathbb{R}^d} x J_X(ds \times dx)$$

where $J_X$ is the **Poisson Random Measure** with intensity $\nu(dx)dt$ ($\nu(dx)$ being the Lévy Measure (cf above)). Notice that all the Lévy processes can be written using this third form. It is not the case with the first two forms which are **only valid for FA Lévy** processes.

**Remark:** Remember that $X_t = log(S_t/S_0)$.

**Remark:** Now, let's consider **General Lévy processes**. We can always define the Lévy Measure $\nu(.)$ :

$$\forall \text{ compact set } A \in \mathbb{R}^d, \text{ such that } 0 \notin A, \nu(A) < +\infty.$$

Remember previously :

$$\nu(A) = \mathbb{E}\left[\#\left\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\right\}\right].$$

So for example : $\nu([1,2]) = +\infty$ is **not possible**.

Now let's define **Infinite Activity (IA) Lévy**.

**Definition:** [*Infinite Activity Lévy*] A Lévy process with the following characteristics :

- Finite number of "*large jumps*" : for the moment we choose the jumps of size $\geq 1$ ;
- Infinite number of "*infinitesimal jumps*".

**Property:** [*Lévy-Ito Decomposition of IA Lévy*] Let $(X_t)_t$ be a Lévy process in $\mathbb{R}^d$, $\nu$ be its Lévy measure. Then :

- $\nu$ is a Radon measure on $\mathbb{R}^d - \{0\}$ ;
- $\int_{|x|\geq 1} \nu(dx) < +\infty$ [*large jumps*];
- $\int_{|x|<1} |x^2|\nu(dx) < +\infty$ [*small jumps*];
- $\exists \gamma \in \mathbb{R}^d, A \in \mathbb{R}^{d\times d}$ ($A$ is a **variance-covariance** matrix) and let $(B_t)_t$ be a Brownian Motion with $A$ as variance-covariance matrix, such that : $X_t = \gamma t + B_t + X_t^l + lim_{\varepsilon\to 0}\tilde{X}_t^\varepsilon$ where :

$$X_t^l = \int_{|x|\geq 1, s\in[0,t]} xJ_X(dx \times ds) = \sum_{|\Delta X_s|\geq 1, s\in[0,t]} \Delta X_s \text{ [\textit{Compound Poisson}]};$$

  and :

$$X_t^\varepsilon = \int_{\varepsilon<|x|<1, s\in[0,t]} xJ_X(dx \times ds) \text{ [\textit{Compound Poisson}]};$$

  and :

$$\tilde{X}_t^\varepsilon = \int_{\varepsilon<|x|<1, s\in[0,t]} x\left(J_X(dx \times ds) - \nu(dx)ds\right) \text{ [We \textbf{cannot split} in two with \textbf{IA Lévy}]}.$$

**Definition:** In this case, we define the **Lévy triplet** $(\gamma, A, \nu)$.

**Remarks:**

- $\gamma t + B_t$ is the "continuous part" of the process ;
- $X_t^\varepsilon, X_t^l$ are **Compound Poisson** processes ;
- $\tilde{X}_t^\varepsilon$ is a **Compensated Compound Poisson** process, therefore it is a martingale : $\mathbb{E}_0\left[\tilde{X}_t^\varepsilon\right] = \tilde{X}_0^\varepsilon = 0$ a.s.

**Remarks:** What happens when $\varepsilon \to 0$ ?

- $|lim_{\varepsilon\to 0}X_t^\varepsilon| = +\infty$ for **some** Lévy processes (IA Lévy) ;
- $|lim_{\varepsilon\to 0}\tilde{X}_t^\varepsilon| < +\infty$ for **all** Lévy processes (MG prop + Central Limit Theorem) : that's why we need $\tilde{X}_t^\varepsilon$ ;

So the **Ito-Lévy Decomposition**, tells us that we can only consider the "full integral" :

$$\tilde{X}_t^\varepsilon = \int_{\varepsilon<|x|<1, s\in[0,t]} x\left(J_X(dx \times ds) - \nu(dx)ds\right)$$

but we cannot "split the integral" in two parts :

$$X_t^\varepsilon = \int_{\varepsilon<|x|<1, s\in[0,t]} xJ_X(dx\times ds) = \sum_{s\in[0,t], 0<|\Delta X_s|<1} \Delta X_s$$

and :

$$X_t^{\varepsilon,b} = \int_{\varepsilon<|x|<1, s\in[0,t]} x\nu(dx)ds$$

because even though we know that for all Lévy, $lim_{\varepsilon\to 0}\tilde{X}_t^\varepsilon \in \mathbb{R}$, we have for some Lévy (IA Lévy) that $lim_{\varepsilon\to 0}X_t^\varepsilon, lim_{\varepsilon\to 0}X_t^{\varepsilon,b} = +\infty$. In fact, it is a "$+\infty - (+\infty) = c \in \mathbb{R}$".

So we were able to write a **GENERAL LEVY PROCESS** $(\gamma, A, \nu)$ the following way :

$$X_t = \gamma t + B_t + X_t^l + \lim_{\varepsilon\to 0^+} \tilde{X}_t^\varepsilon$$

An interesting question is : can we derive a FA ("jump diffusion") expression starting from the general one above ? Yes :

$$X_t = \gamma t + B_t + \sum_{s\in[0,t], |\Delta X_s|\geq 1} \Delta X_s + lim_{\varepsilon\to 0} \int_{[0,t]\times\mathbb{R}^d, \varepsilon<|x|<1} x(J_X(dx\times ds) - \nu(dx)ds)$$

$$X_t = \left(\gamma - \int_{0<|x|<1} x\nu(dx)\right) t + B_t + \sum_{s\in[0,t]} \Delta X_s$$

by cutting in two the integral (we can since it's FA Lévy) and since :

$$\int_{0<|x|<1, s\in[0,t]} xJ_X(dx\times ds) = \sum_{s\in[0,t], 0<|\Delta X_s|<1} \Delta X_s.$$

So we obtained the expression of a FA Lévy, with :

$$\mu = \left(\gamma - \int_{0<|x|<1} x\nu(dx)\right).$$

Therefore, we write a FA Lévy the following way :

$$X_t = \mu t + B_t + \sum_{s\in[0,t]} \Delta X_s$$

with $\mu = \gamma - \int_{0<|x|<1} x\nu(dx)$ for $\int_{|x|<1} \nu(dx) < +\infty$. And :

$$X_t = \mu t + B_t + \int_{[0,t]\times\mathbb{R}^d} xJ_X(dx, ds)$$

for $\int_{|x|<1} |x|\nu(dx) < +\infty$ (**Finite Variation Lévy**, cf just below).

**Definition:** [*Total Variation / Finite Variation*]

- Total variation : for $f : [a,b] \to \mathbb{R}^d$,

$$TV = sup_{a=t_0<t_1<...<t_n=b} \sum_{i=1}^n |f(t_i) - f(t_{i-1})|.$$

- Finite variation : it is when $TV < +\infty$, i.e,

$$A = 0 \text{ (because CM has infinite TV) and } \int_{|x|<1} |x|\nu(dx) < +\infty.$$

**Remark:** FV Lévy is really an intermediary situation between FA Lévy and IA Lévy.

**Theorem:** [*Lévy-Khincin Formula*] Let $(X_t)_{t\geq 0}$ be a Lévy Process $(\gamma, A, \nu)$.

$$\mathbb{E}\left[e^{izX_t}\right] = e^{t\Psi(z)}$$

where :

$$\Psi(z) = i\gamma z - \frac{1}{2}z^T Az + \int_{\mathbb{R}^d} \left(e^{izx} - 1 - izx1_{|x|\leq 1}\right)\nu(dx).$$

**Proof:** cf my written notes.□

**Summing up:** we start from a general Lévy process $(\gamma, A, \nu)$ : it can be FA or IA. And depending on the situation we can write it using the FA formula (if it's FA!). So what we did is :

- From the writing of Jump Diffusion (FA) that we saw first we went to Lévy Ito Decomposition (for IA Lévy Processes) and ended up with the triplet $(\gamma, A, \nu)$ ;
- From General Lévy $(\gamma, A, \nu)$ we wrote Jump Diffusion (FA) with the formula with the summation, defining $\mu = \gamma - \int_{0<|x|<1} x\nu(dx) = \gamma - \lambda \int_{0<|x|<1} xf(x)dx$.

# 6 Subordinator

## 6.1 Idea

We will use subordinator for *variance* and *construction of Lévy processes.*
How will we construct Lévy Processes ? By **time change**. Assume we have a process $(X_t)_t$ (ex: BM) and another one $(S_t)_t$. We can build $(X_{S_t})_t$ but we need $(S_t)_t$ to be a *positive and non-decreasing* time process.

**Theorem:** Let $(X_t)_t$ be a Lévy process $(\gamma, A, \nu)$. We have 4 **equivalent** conditions :

- $\forall t > 0, X_t \geq 0$ as ;
- $\exists t > 0, X_t \geq 0$ as ;
- $(X_t)_t$ non-decreasing ;
- $(X_t)_t$ finite variation process with $\nu((-\infty, 0]) = 0$ **(non-negative jumps)** and $\mu = \left(\gamma - \int_{|x|<1} x\nu(dx)\right) > 0$.

**Proof:** cf my written notes.□

## 6.2 Constructing a subordinator

**Theorem:** Let $(X_t)_t$ be a Lévy process in $\mathbb{R}^d$. Let $f : \mathbb{R}^d \to [0, +\infty)$ be a positive function such that $f(x) = O(|x|^2)$ in a neighborhood of 0. Then a subordinator is the following :

$$S_t = \sum_{s\leq t, \Delta X_s \neq 0} f(\Delta X_s).$$

**Remark:** In the Lévy Khincin Representation $\Psi(z) = i\gamma z - \frac{1}{2}z^T Az + \int_{\mathbb{R}^d} \left(e^{izx} - 1 - izx1_{|x|\leq 1}\right)\nu(dx)$, we chose that the "small jumps" were the one with $|x| < 1$. Let then be $g : x \to 1_{|x|<1}$. We could do the same with $|x| < 1/2$ ; in fact we could choose any $g$ of the form :

$$g : \mathbb{R}^d \to \mathbb{R} \text{ s.t } g(x) =_{x\to 0} 1 + o(|x|) \ \& \ g(x) =_{x\to +\infty} O\left(\frac{1}{|x|}\right).$$

See my written notes to have a glimpse of how we cut the integral and define a new $\tilde{\gamma}$ to obtain an analogous formula to the one of Lévy Khincin but with a separation between large and small jumps which is given by $g$. The message is that all the theory that we developed using a separation of 1 can be redeveloped if we change the separation : it's not a problem at all ! Only $\gamma$ changes, but neither $A$ nor $\nu$. But **why do we need this remark for the previous theorem** ? Because $f$ can change a jump of let's say 3/4 to a jump of let's say 1.5. But according to the present remark, it is not a problem.

**Proof:** See my written notes.□
Let's now "meet" some Lévy processes and implement on MATLAB. Keep in mind that for us : $X_t = log\left(\frac{S_t}{S_0}\right)$.
(46/89 written notes)