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Stochastic Dynamical Models A.A. 2024-2025

Exercise Session 2

October 1, 2024

Keywords: Classification of states, Period, Invariant distributions, Recursion relations.

Exercise 1. Let $(X_n)_{n\geq 0}$ be a Markov chain with state space $I = \{1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 0 & 2/3 & 1/3 \end{pmatrix}.$$

1. Classify the states and determine the period of each state. Irreducible. Recurrent? it would mean: $\mathbb{P}\left(\bigcup_{n=1}^{+\infty} X_n = 3 \mid X_0 = 3\right) = 1$.
(*)2. Compute $\mathbb{P}\{X_2 = 1 \mid X_0 = 3\}$. $= 1 - \mathbb{P}\left(\bigcap_{n=1}^{+\infty} X_n \neq 3 \mid X_0 = 3\right) = 1 - \mathbb{P}(X_1 = 2, X_2 = 1, X_3 = 2, \dots \mid X_0 = 3)$
(**)3. Study invariant distributions (existence, uniqueness and explicit computation). $\underbrace{= 1 - \frac{2}{3}\cdot\frac{1}{3}\cdot\frac{2}{3}\cdots = 1}_{\text{all states are recurrent.}}$

Exercise 2. Consider a Markov Chain $(X_n)_{n\geq 0}$ with state space $I = \mathbb{N}$ and transition matrix $P = (p_{i,j})_{i,j\in I}$. The non-zero entries of P are

$$p_{i,i+1} = c_i, \quad p_{i,0} = 1 - c_i, \quad 0 < c_i < 1, \quad i \in I.$$

Assume that $(c_i)_{i\in I}$ satisfies $\prod_{i\in I} c_i > 0$. Prove that there is no invariant distribution for this Markov chain. (***)

Exercise 3. Consider a Markov chain $(X_n)_{n\geq 0}$ with state space $I = \mathbb{N}$ and transition matrix

$$P = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1/3 & 1/3 & 0 & 1/3 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1/3 & 1/3 & 0 & 1/3 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1/3 & 1/3 & 0 & 1/3 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1/3 & 1/3 & 0 & 1/3 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 1/3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

$$\begin{cases} \pi_0 = \frac{1}{3}\pi_2 & (1) \\ \pi_1 = \pi_0 + \frac{1}{3}\pi_2 + \frac{1}{3}\pi_3 & (2) \\ \pi_2 = \pi_1 + \frac{1}{3}\pi_3 + \frac{1}{3}\pi_4 & (3) \\ \pi_i = \frac{1}{3}\pi_{i-1} + \frac{1}{3}\pi_{i+1} + \frac{1}{3}\pi_{i+2} \quad (i \geq 3) \end{cases}$$

Compute its invariant distributions if they exist.

$\longrightarrow$ to be solved. (****)

(*) $\mathbb{P}(X_2 = 1 \mid X_0 = 3) = (P^2)_{3,1} = \frac{2}{9}$ (or you can notice the only way: $3 \to 2 \to 1$).

(**) $\pi P = \pi$ + normalisation condition $\longrightarrow$ $\pi = \left(\frac{2}{15}, \frac{6}{15}, \frac{7}{15}\right)$ (only one π).

(***) we have $P = \begin{bmatrix} 1-c_0 & c_0 & 0 & \cdots \\ 1-c_1 & 0 & c_1 & \cdots \\ 1-c_2 & 0 & 0 & c_2 \cdots \\ \vdots & & \ddots & \vdots \end{bmatrix}$ so $\pi P = \pi$ gives: $\pi_0 = \sum_{i\in I} \pi_i (1 - c_i) > 0$

$\forall i \in I \setminus \{0\},\ \pi_i = c_{i-1}\pi_{i-1}$

$= \dots = \left(\prod_{k=0}^{i-1} c_k\right)\pi_0$

since $\forall k \in I$, $0 < c_k < 1$, $\prod_{k=0}^{i-1} c_k > \prod_{k=0}^{\infty} c_k$ so that:

$\sum_{i\in I} \pi_i = \sum_{i\in I}\left(\prod_{k=0}^{i-1} c_k\right)\pi_0 > \sum_{i\in I}\left(\prod_{k\in I} c_k\right)\pi_0 = +\infty$ since by hyp, $\begin{cases} \pi_0 > 0 \\ \prod_{k\in I} c_k > 0 \end{cases}$. So no inv. density. $\square$

(****) characteristic equation: $\lambda^3 + \lambda^2 - 3\lambda + 1 = 0$

$$\lambda^3 + \lambda^2 - 3\lambda + 1 = (\lambda - 1)(\lambda^2 + 2\lambda - 1)$$

multiplicity 1.

- with simple real roots $\lambda_1 = 1$, $\lambda_2 = -(\sqrt{2}+1)$, $\lambda_3 = \sqrt{2} - 1$.

Therefore:

$$T_k = \cancel{A}\, 1^k + \cancel{B}\left(-(\sqrt{2}+1)\right)^k + C\left(\sqrt{2}-1\right)^k$$

$\forall k \geq 2$.

We need to determine A, B, C.

- B has to be 0: otherwise $\exists k_0 \in I$ s.t

$$B\left(-(\sqrt{2}+1)\right)^{k_0} < -A\, 1^{k_0} - C\,(\sqrt{2}-1)^{k_0}$$

$\forall A, C \in \mathbb{R}$.

- A has to be zero otherwise it would not be summable.

- The boundary conditions (1), (2), (3) give:

$$C = \left(\frac{10 - 5\sqrt{2}}{6}\right)^{-1} = \frac{3(2+\sqrt{2})}{5} \quad \left(\sum_{k \in \mathbb{N}} \pi_k = 1\right). \quad \square$$