Modelling Volatility

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June 16, 2024

Introduction

This chapter deals with the limitations of linear models in explaining features of financial data and introduces non-linear models, specifically focusing on volatility modelling.

Motivation for Non-linear Models

- Linear models fail to explain:
  - Leptokurtosis
  - Volatility clustering or pooling
  - Leverage effects
- Traditional structural model: $y_t = \beta_1 + \beta_2 x_{2t} + ... + \beta_k x_{kt} + u_t$ where $u_t \sim N(0, \sigma^2)$ or more compactly : $y = X\beta + u$.

Non-linear Models

- Defined as $y_t = f(u_t, u_{t-1}, u_{t-2}, \ldots)$ where u_t is an iid error term and f is a non-linear function.
- Specific definition: $y_t = g(u_{t-1}, u_{t-2}, \ldots) + u_t \sigma^2(u_{t-1}, u_{t-2}, \ldots)$
- Types of non-linear models include:
  - ARCH / GARCH
  - Switching models
  - Bilinear models
  - Neural networks

Many relationships in finance are intrinsically non-linear.

Testing for Non-linearity

- Traditional tools (acf's, spectral analysis) may not detect the need for non-linear models.
- Ramsey's RESET test: Tests for non-linearity by regressing residuals on the squares, cubes, etc., of the fitted values: $\hat{u}_t = \beta_0 + \beta_1 \hat{y}_t^2 + \beta_2 \hat{y}_t^3 + \dots + \beta_{p-1} \hat{y}_t^p + v_t$.

Models for Volatility

- Volatility is crucial in finance for risk measurement, VaR models, and option pricing (Black-Scholes formula).
- Historical volatility: Calculated as the variance or standard deviation of returns over a historical period, useful as a benchmark.

Heteroscedasticity Revisited

$Var(u_t) = \sigma_u^2$ for example in: $y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + u_t$.

- Traditional assumption: Constant variance of errors (homoscedasticity).
- Financial data often exhibit heteroscedasticity (non-constant variance).

To estimate these models: not linear so ~~OLS~~ → MLE

ARCH Models

Model which does not assume that variance is cst.

- Autoregressive Conditional Heteroscedasticity (ARCH) models: Variance of the error term depends on previous squared error terms.
- Example: ARCH(1) model where $\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2$
- General ARCH(q) model: $\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \ldots + \alpha_q u_{t-q}^2$

↳ full model: $y_t = \beta_1 + \beta_2 x_{2t} + \dots + \beta_k x_{kt} + u_t$, $u_t \sim \mathcal{N}(0, \sigma_t^2)$, w/ $\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \dots + \alpha_q u_{t-q}^2$.
or $u_t = v_t \sigma_t$ where $\sigma_t = \sqrt{\dots}$ and $v_t \sim \mathcal{N}(0,1)$.

GARCH Models

- Generalized ARCH (GARCH) models: Include lagged values of the variance itself.
- Example: GARCH(1,1) model where $\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta \sigma_{t-1}^2$: like ARMA(1,1) model for the variance.

  $\beta < 1$ for non-exploding volatility.
- Extensions: EGARCH, GJR, GARCH-M models to address issues like non-negativity constraints and leverage effects.

↳ GARCH(p,q) : $\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \dots + \alpha_q u_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \dots + \beta_p \sigma_{t-p}^2$
$= \alpha_0 + \sum_{i=1}^{q} \alpha_i u_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2$.

In general, GARCH(1,1) model is sufficient to capture the volatility clustering in the data.

Garch > Arch : - more parsimonious → avoids overfitting

Stochastic Volatility Models

- Differ from GARCH in that the conditional variance equation includes an error term.
- Example: $\log(\sigma_t^2) = \omega + \beta \log(\sigma_{t-1}^2) + \gamma u_{t-1} + \eta_t$ where η_t is another error term.

Forecasting Volatility

- Variance forecasts are additive over time.
- GARCH models can forecast conditional variance similarly to ARMA models for mean forecasts.
- Example: GARCH(1,1) model produces one-step ahead forecast σ_{T+1}^2 as $\alpha_0 + \alpha_1 u_T^2 + \beta \sigma_T^2$.

Practical Applications

- Option pricing
- Variance swap contracts
- Risk measurement
- Optimal hedge ratio calculation

References

- Day, T. E., and Lewis, C. M. (1992). "Stock Market Volatility and the Information Content of Stock Index Options." *Journal of Econometrics.*

METHOD:

1) Perform time-series analysis neglecting arch effects (ARMA...).
2) Check whether the model residuals present an ARCH effect (heteroskedasticity).
3) If so, find the GARCH specification w/ the best I.C : stick if possible to (1,1).
4) Hypothesis testing & volatility forecasting.