Control Systems 2

FS24 ETHZ

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This PDF, the source code as well as the disclaimer can be found in the GitHub repository `https://github.com/MeierTobias/eth-control-systems-2`.

1 Discrete-Time Systems

1.1 Digital Control Systems

All continuous-time (CT) system properties hold for discrete-time (DT) in the same way as DT values form a subset of CT values.

1.1.1 Holding, D → A

Hold: Linear system that takes DT signal as input and outputs CT signal.
Holding results in a CT signal with average delay $T/2$.

Zero-Order Hold (ZOH)

$$y(t) = u[k], \qquad \forall\, kT \le t < (k+1)T,\ k \in \mathbb{Z}$$
$$y(t) = u\left[\left\lfloor \frac{t}{T} \right\rfloor\right], \qquad \forall t \in \mathbb{R}$$

1.1.2 Sampling, A → D

Sampler: A linear system that takes CT signal as input and outputs DT sequence.

$$y[k] = u(kT), \quad \forall k \in \mathbb{Z}$$

1.1.3 Aliasing

Nyquist-Shannon Sampling Theorem
A signal can be reconstructed without aliasing from samples at rate $1/T$ if it does not contain frequencies higher than the Nyquist (folding) frequency:

$$\omega_f := \frac{\pi}{T}[rad/s] = \frac{1}{2T}[Hz]$$

A signal
$$y[k] = \cos[\omega \cdot kT]$$
has the same samples as a sinusoid with frequency
$$\omega' = \left|\omega \pm n\frac{2\pi}{T}\right|, \ n \in \mathbb{N}$$

Anti-Aliasing Filter
Low-pass filter before the A → D converter with
$$\omega_{c,LP} < \omega_f$$
where $\omega_{c,LP}$ must be high enough to capture system dynamics.

1.2 DT State Space Models

1.2.1 Difference Equations

	Time-Invariant	Time-Variant ($t = kT$)
Nonlinear	$\mathbf{x}[k+1] = f(\mathbf{x}[k], \mathbf{u}[k])$	$\mathbf{x}[k+1] = f(t, \mathbf{x}[k], \mathbf{u}[k])$
	$\mathbf{y}[k] = h(\mathbf{x}[k], \mathbf{u}[k])$	$\mathbf{y}[k] = h(t, \mathbf{x}[k], \mathbf{u}[k])$
	$\mathbf{x}[0] = \mathbf{x}_0$	$\mathbf{x}[0] = \mathbf{x}_0$
Linear	$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k]$	$\mathbf{x}[k+1] = \mathbf{A}[k]\mathbf{x}[k] + \mathbf{B}[k]\mathbf{u}[k]$
	$\mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k]$	$\mathbf{y}[k] = \mathbf{C}[k]\mathbf{x}[k] + \mathbf{D}[k]\mathbf{u}[k]$
	$\mathbf{x}[0] = \mathbf{x}_0$	$\mathbf{x}[0] = \mathbf{x}_0$

As in CT, the system is **strictly causal** iff $\mathbf{D} = 0$.

1.2.2 Time Response

$$y[k] = \underbrace{\mathbf{C}\mathbf{A}^k\mathbf{x}_0}_{\text{homogeneous r.}} + \underbrace{\mathbf{C}\sum_{i=0}^{k-1}\mathbf{A}^{k-i-1}\mathbf{B}\mathbf{u}[i] + \mathbf{D}\mathbf{u}[k]}_{\text{forced response}}$$

1.2.3 Internal Stability for DT Systems

If A is Diagonalizable
For diagonalizable matrices A:

$$\mathbf{A}^k = \left(\mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}\right)^k = \mathbf{T}\mathbf{\Lambda}^k\mathbf{T}^{-1} = \mathbf{T}\begin{bmatrix}\lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k\end{bmatrix}\mathbf{T}^{-1}$$

$\lim_{k\to+\infty}\mathbf{A}^k = 0$ is fulfilled if
$$|\lambda_i| < 1, \quad \forall i = 1, \dots, n$$
which means that all eigenvalues must lie within the unit circle for the system to be stable.
Reminder:
$$\exp\left(\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}t\right) = \begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}$$

If A is Defective
Similarly, each block in the Jordan form will converge to zero if and only if
$$|\lambda_i| < 1, \quad \forall i = 1, \dots, n$$

Reminder:
$$\exp\left(\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}t\right) = \begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}e^{\lambda t}$$
$$\exp\left(\begin{bmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{bmatrix}t\right) = \begin{bmatrix}1 & t & \frac{1}{2!}t^2\\ 0 & 1 & t\\ 0 & 0 & 1\end{bmatrix}e^{\lambda t}$$

1.3 Transfer Functions (TF) of DT LTI Systems

Time Response to Elementary Inputs
Similar to the elementary inputs in the CT case, a *geometric sequence* z^k corresponds to an input sequence
$$u[k] = u_0 z^k = u_0 e^{ksT} = u(kT), \ z = e^{sT}$$
The corresponding time response is
$$y[k] = \underbrace{\mathbf{C}\mathbf{A}^k\left(\mathbf{x}_0 - \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}u_0\right) + \dots}_{\text{transient}}$$
$$+ \underbrace{\mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}u_0 z^k + \mathbf{D}u_0 z^k}_{\text{steady-state}}$$

Pulse/Discrete TF
For asymptotically stable DT LTI systems, for large k and $u[k] = u_0 z^k$:
$$y[k] \approx G(z)u[k]$$
$$G(z) := \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Remarks
- z^k can be interpreted as samples from an exponential

1.3.1 Special TF and Realizations

DC Gain
If $G(z)$ is asymptotically stable and $\mathbf{A}$ is invertible, i.e. has no eigenvalue at 1, then
$$\lim_{k\to+\infty} y[k] = G(1)u_0 = \left(\mathbf{C}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right)u_0$$

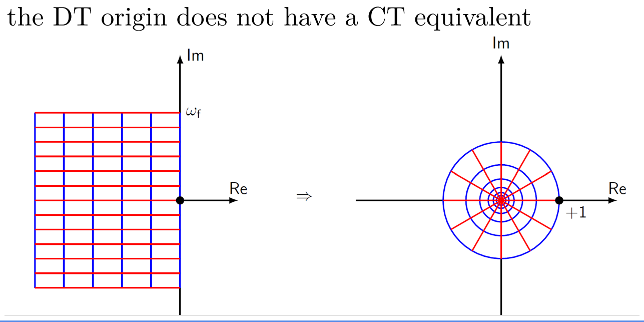
Time Delay
$$y[k] = u[k-1] = u_0 z^{k-1} = \frac{1}{z}u[k]$$

Realizations of DT TF
All methods to create (state space) realizations of CT TF as LTI models can be adapted to DT TF.

1.4 System Discretization

Similar to CT, the natural dynamics of a DT LTI system are given by the eigenvalues of $\mathbf{A}_d$.
For given eigenvalues λ_i of $\mathbf{A}$ the eigenvalues of $\mathbf{A}_d$ are
$$z_i = e^{\lambda_1 T}, e^{\lambda_2 T}, \dots, e^{\lambda_n T}$$

Mapping
1. CT left half plane → DT unit disk
2. CT neighborhood of origin → DT neighborhood of $+1$ point
3. CT poles at $j\omega_f$ → DT -1 point (because $e^{j\omega_f T} = -1$)
4. CT finite poles cannot be mapped to the DT origin and the DT origin does not have a CT equivalent



1.4.1 Discretization: Hold and Sample

To discretize the $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ matrices of a time continuous system one has to apply the following transformations:

$$\mathbf{A}_d := e^{\mathbf{A}T} \qquad \mathbf{B}_d := \left(\int_0^T e^{\mathbf{A}\theta}d\theta\right)\mathbf{B} \overset{\mathbf{A}\text{ inv.}}{=} \mathbf{A}^{-1}\left(\mathbf{A}_d - \mathbf{I}\right)\mathbf{B}$$
$$\mathbf{C}_d := \mathbf{C} \qquad \mathbf{D}_d := \mathbf{D}$$

Calculations
Assuming sampling time T
- Check invertibility by looking at the determinant.
- For diagonal matrices $\mathbf{\Lambda}$, one can apply exponentiation elementwise on the main diagonal.
- For non-diagonal matrices use
  - either diagonalization via eigenvectors ($\mathbf{A} = \mathbf{V}^{-1}\cdot\mathbf{\Lambda}\cdot\mathbf{V}$) and
    $$e^{\mathbf{A}T} = \mathbf{V}^{-1}\cdot e^{\mathbf{\Lambda}T}\cdot\mathbf{V}$$
  - or power series expansion
    $$e^{\mathbf{A}T} = \sum_{k=0}^{\infty}\frac{1}{k!}(\mathbf{A}T)^k$$

1.4.2 Discretization: Frequency Domain

Euler Forward
$$s = \frac{z-1}{T}$$

Euler Backward
$$s = \frac{z-1}{zT}$$

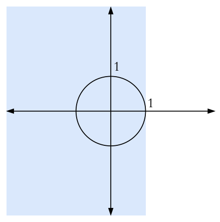
Tustin's Method
$$G_d(z) = G\left(s = \frac{2}{T}\frac{z-1}{z+1}\right)$$
which corresponds to a rearranged Padé approximation for z.

1.4.3 Discretization: Time Domain

1.4.3.1 Forward Difference

$$\mathbf{x}(t) - \mathbf{x}(t-T) \approx T\cdot\dot{\mathbf{x}}(t-T) = T\cdot\left(\mathbf{A}\mathbf{x}(t-T) + \mathbf{B}\mathbf{u}(t-T)\right)$$
yields
$$\mathbf{x}(t) \approx (\mathbf{I} + \mathbf{A}T)\mathbf{x}(t-T) + T\mathbf{B}\mathbf{u}(t-T)$$

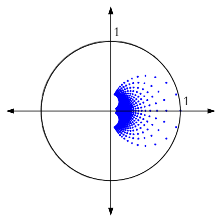
which is simple but can result in an unstable compensator as
$$s = \lambda \to z = 1 + \lambda T$$
i.e. for large $|\lambda|$ there is no mapping into the unit disk.



1.4.3.2 Backward Difference

$$\mathbf{x}(t) - \mathbf{x}(t-T) \approx T\cdot\dot{\mathbf{x}}(T) = T\cdot\left(\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\right)$$
yields
$$\mathbf{x}(t) \approx (\mathbf{I} - \mathbf{A}T)^{-1}\mathbf{x}(t-T) + (\mathbf{I} - \mathbf{A}T)^{-1}T\mathbf{B}\mathbf{u}(t)$$

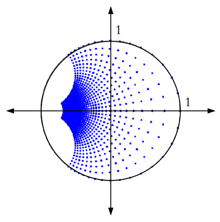
which avoids instability as
$$s = \lambda \to z = (1 - \lambda T)^{-1}$$
This transforms poles into a disk smaller than the unit disk. DT poles cannot get negative real part.



1.4.3.3 Trapezoidal Rule

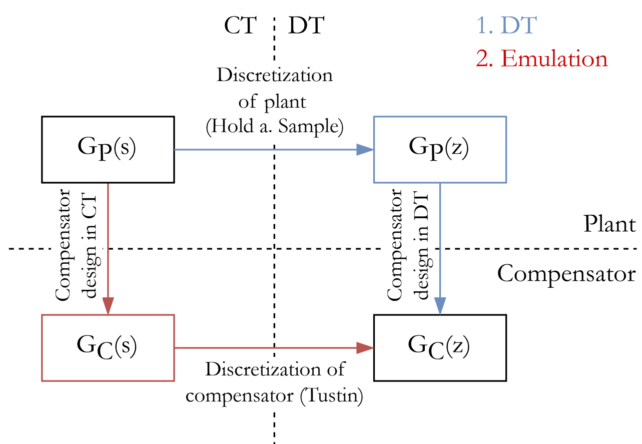
$$\mathbf{x}(t) - \mathbf{x}(t-T) \approx \frac{T}{2}\left(\dot{\mathbf{x}}(t) + \dot{\mathbf{x}}(t-T)\right)$$
yields
$$\mathbf{x}(t) \approx \left(\mathbf{I} - \frac{\mathbf{A}T}{2}\right)^{-1}\left(\mathbf{I} + \frac{\mathbf{A}T}{2}\right)\mathbf{x}(t-T)$$
$$+ \left(\mathbf{I} - \frac{\mathbf{A}T}{2}\right)^{-1}\mathbf{B}T\frac{u(t) + u(t-T)}{2}$$

which maps the left half plane to the unit disk. This discretization method corresponds to Tustin's method in frequency domain.



1.5 Design and Implementation of DT Control Systems

There are two main design paradigms for DT control systems:
1. DT: The CT system is converted to a DT system and the design process happens in DT
2. Emulation: the design process happens in CT and the result is transferred to DT



Remarks
- A common method is to
  - Use DT design if T is known
  - Use CT design with emulation if T is unknown

1.5.1 DT

1. First the CT model of the plant is transformed into a DT model (use the "Hold and Sample" discretization method as shown in 1.4.1).
2. Then the discrete compensator can be designed.

1.5.2 Emulation

1. First the compensator is designed in the CT domain.
2. This compensator is then transformed into a discrete compensator with one of the following methods:
   - Frequency domain → Tustin's method (1.4.2)
   - Time domain → Trapezoid method (1.4.3.3)

2 Modal Decomposition

2.1 Recall: Eigendecomposition

Let λ_i and $\mathbf{v}_i$ be an eigenvalue and an eigenvector of $\mathbf{A}$.

$$\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

$$\mathbf{A}\begin{bmatrix} \vdots & & \vdots \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & & \vdots \\ \lambda_1\mathbf{v}_1 & \cdots & \lambda_n\mathbf{v}_n \\ \vdots & & \vdots \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \vdots & & \vdots \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \vdots & & \vdots \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}}_{\Lambda}$$

$$\mathbf{A}\mathbf{V} = \mathbf{V}\mathbf{\Lambda}$$

$$\mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$$

As shown $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues of $\mathbf{A}$ and $\mathbf{V}$ is a column stack of the eigenvectors.

2.2 Similarity Transformation

The state space representation is not unique. This can be used to transform the model into a more «useful/readable» representation.

$$\tilde{\mathbf{x}} = \mathbf{T}\mathbf{x}$$
$$\tilde{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$$
$$\tilde{\mathbf{B}} = \mathbf{T}\mathbf{B}$$
$$\tilde{\mathbf{C}} = \mathbf{C}\mathbf{T}^{-1}$$
$$\tilde{\mathbf{D}} = \mathbf{D}$$

2.2.1 Modal Transformation

If $\mathbf{T}$ is set to the inverse of the eigenvector matrix of $\mathbf{A}$ we get:

$$\mathbf{T} = \mathbf{V}^{-1}$$
$$\tilde{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{x}$$
$$\tilde{\mathbf{A}} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{\Lambda}$$
$$\tilde{\mathbf{B}} = \mathbf{V}^{-1}\mathbf{B}$$
$$\tilde{\mathbf{C}} = \mathbf{C}\mathbf{V}$$
$$\tilde{\mathbf{D}} = \mathbf{D}$$

Where $\tilde{\mathbf{A}}$ is a diagonal matrix with the eigenvalues of $\mathbf{A}$ on its main diagonal.

2.3 Modal Time Response

2.3.1 Modal Coordinates

- The eigenvector $\mathbf{v}_i$ defines the shape of the i-th mode.
- The modal coordinate $\tilde{\mathbf{x}}_i = \mathbf{V}^{-1}\mathbf{x}$ scales the mode (e.g., at the initial condition)
- The eigenvalue λ_i defines how the amplitude of the mode evolves over time.
  - Real λ_i $\quad \rightarrow e^{\lambda_i t}x_i(0)$
  - Complex $\lambda_i = \sigma_i + j\omega_i$ $\quad \rightarrow e^{\sigma_i t}\sin(\omega_i t + \phi_0)x_i(0)$
  - Repeated λ_i $\quad \rightarrow t^p e^{\lambda_i t}$

2.3.1.1 Example: Interpretation of Modal Decomposition

The modal decomposition gives valuable insight regarding the behavior of a system. The modes (eigenvalues and corresponding eigenvectors) are analyzed one at a time.

1. The magnitude of the eigenvalue describes the influence of the corresponding mode.
2. The property (real, complex or repeated) of the eigenvalue describes the behavior over time of that mode.
3. The corresponding eigenvector describes the states/dimensions the mode is acting on. For example x, y, z or $\varphi, \dot{\varphi}$

2.3.2 Homogeneous Response

For a given initial condition $x(0) = x_0$ the homogeneous response can be computed as follows:

In Modal Coordinates

$$\tilde{\mathbf{x}}(t) = e^{\tilde{\mathbf{A}}t}\tilde{\mathbf{x}}(0)$$
$$\tilde{x}_i(t) = e^{\lambda_i t}\tilde{x}_i(0)$$

which means that

- each mode evolves independently (of other modes) over time.
- $\tilde{\mathbf{x}}(t)$ is obtained elementwise from $\tilde{x}_i(t)$.

In Standard Coordinates

The modes in standard coordinates are given by

$$\mathbf{m}_i(t) = e^{\lambda_i t}\tilde{x}_i(0)\mathbf{v}_i$$

so that

$$\mathbf{x}(t) = \sum_{i=1}^{n}\mathbf{m}_i(t) = \sum_{i=1}^{n} e^{\lambda_i t}\tilde{x}_i(0)\mathbf{v}_i$$

because for an initial condition $x_0 = v_i$ the time response is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{v}_i = e^{\lambda_i t}\mathbf{v}_i$$

and $\mathbf{v}_i$ form a basis of the corresponding state space.

Remarks

- Note that the standard coordinates time response is obtained by simply multiplying the modal components with their eigenvectors and summing over all components.
- Attention, the initial conditions have to be transformed as well.

2.3.2.1 Example: Simple Pendulum

For a simple pendulum with no damping and no input the state model is given by:

$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}}_{\mathbf{A}}\mathbf{x}$$

where

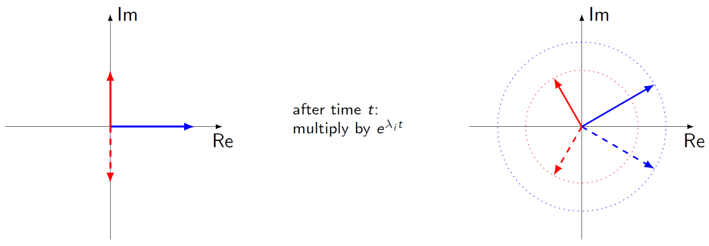
$$\mathbf{x} = \begin{bmatrix} \varphi \\ \dot{\varphi} \end{bmatrix}$$

The eigenvalues and eigenvectors of $\mathbf{A}$ are

$$\lambda_{1,2} = \pm j\sqrt{\frac{g}{l}} \qquad \mathbf{v}_{1,2} = \begin{bmatrix} 1 \\ \pm j\sqrt{\frac{g}{l}} \end{bmatrix}$$

With the transformed initial condition

$$\mathbf{V}^{-1}\mathbf{x}(0) = \mathbf{V}^{-1}[1, 0]^{\mathsf{T}} = \left[\frac{1}{2}, \frac{1}{2}\right]^{\mathsf{T}} = \tilde{\mathbf{x}}(0)$$



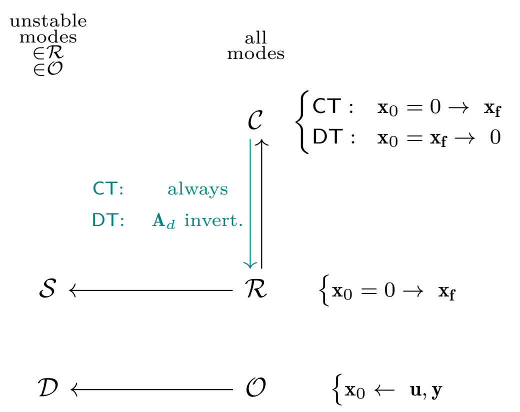
Due to the purely complex eigenvalues, the eigenvectors only rotate when evolving over time.
As we transformed the initial conditions into modal coordinates, the time response in standard coordinates can be recovered by multiplication with the eigenvectors

$$\mathbf{x}_{ICR}(t) = e^{j\sqrt{\frac{g}{l}}t}\frac{1}{2}\begin{bmatrix} 1 \\ +j\sqrt{\frac{g}{l}} \end{bmatrix} + e^{-j\sqrt{\frac{g}{l}}t}\frac{1}{2}\begin{bmatrix} 1 \\ -j\sqrt{\frac{g}{l}} \end{bmatrix}$$

$$\mathbf{x}_{ICR}(t) = \begin{bmatrix} \varphi \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} \cos(\omega t) \\ -\omega\sin(\omega t) \end{bmatrix}$$

3 Reachability and Observability

$$\mathcal{R} \subseteq \mathcal{C}, \quad \mathcal{R} \subset \mathcal{S}, \quad \mathcal{O} \subset \mathcal{D}$$



- If a transfer function does not contain any pole-zero-cancellations, it is observable, controllable and reachable.
- Duality: $\mathbf{A}, \mathbf{B}$ are reachable if and only if $\mathbf{A}^{\mathbf{T}}, \mathbf{B}^{\mathbf{T}}$ are observable.
- Structural properties of a system (**not** affected by similarity transforms or even feedback control!)

3.1 Reachability/Controllability

An LTI system is reachable if there exists a control signal that takes the system from an initial condition $\mathbf{x_0} = \mathbf{0}$ to any desired final state (vector) $\mathbf{x_f}$, in finite time. This means that each state can be influenced by the input.

$$\mathbf{R} := \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

$$\mathbf{x}[n] = \mathbf{R}\mathbf{U}, \quad \mathbf{U} = \begin{bmatrix} \mathbf{u}[0], \dots, \mathbf{u}[n] \end{bmatrix}^{\mathsf{T}}$$

Mathematically expressed, a system is reachable if and only if the reachability matrix $\mathbf{R}$ has **full row rank** n.

$$\operatorname{rank}(\mathbf{R}) = n$$
$$\mathbf{x} \in \operatorname{Range}(\mathbf{R})$$

Remarks

- If the system is reachable, then $\mathbf{x}[n] = \mathbf{R}\mathbf{U}$ can be solved for $\mathbf{U}$ which represents the desired sequence of input signals.
- Note that the condition arises from the Cayley Hamilton Theorem.
- For **continuous-time systems** controllability and reachability are the same.

3.1.1 SISO diagonal systems

The i-th mode of a system in diagonal form is reachable only if $\tilde{b}_i \neq 0$ because

$$\dot{\tilde{x}}_i = \lambda_i\tilde{x}_i + \tilde{b}_i u$$

This is a necessary but not sufficient condition for reachability of the whole system.

3.1.2 DT Systems: Controllability

- A DT LTI system is controllable if, for any initial condition $\mathbf{x}_0$, there exists a control input that brings the state x to 0 in finite time (Note: For Reachability one has the "opposite" condition).
- Reachability always implies controllability and uncontrollable systems are never reachable
- Controllability only implies reachability iff $\mathbf{A}_d$ is invertible
- An unreachable DT system with non-invertible $\mathbf{A}_d$ could be controllable (Eigenvalues at 0). E.g.:
  - $\mathbf{x}[k+1] = 0\mathbf{x}[k] + 0\mathbf{u}[k]$ is controllable (state goes to 0) but unreachable as $\det(\mathbf{R}) = 0$.

3.2 Observability

An LTI system is observable if, for any initial condition $\mathbf{x}_0$, this initial condition can be reconstructed uniquely based on the knowledge of the input and output signal over a finite time interval. This means that each state must influence the output.

$$\mathbf{O} := \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}$$

$$\mathbf{Y} = \mathbf{O}\mathbf{x}[n], \quad \mathbf{Y} = [\mathbf{y}[0], \dots, \mathbf{y}[n]]^{\mathsf{T}}$$

Mathematically expressed, a system is observable if and only if the observability matrix $\mathbf{O}$ has **full column rank** n.

$$\operatorname{rank}(\mathbf{O}) = n$$
$$\mathbf{x} \notin \ker(\mathbf{O})$$

Remarks

- Note that $\mathbf{x}$ must not be in the null space of $\mathbf{O}$ so $\mathbf{O}\mathbf{x} = \mathbf{0}$ iff $\mathbf{x} = \mathbf{0}$ (trivial null space).

SISO Diagonal Systems

The i-th mode of a system in diagonal form is observable only if $\tilde{c}_i \neq 0$ because

$$y = \tilde{c}_1\tilde{x}_1 + \ldots + \tilde{c}_n\tilde{x}_n + \mathbf{D}\mathbf{u}$$

This is a necessary but not sufficient condition for observability of the whole system.

3.3 Modal View: Stabilizability and Detectability

In a system that is stabilizable and detectable all the modes that cannot be controlled and/or observed must behave "nicely" on their own.

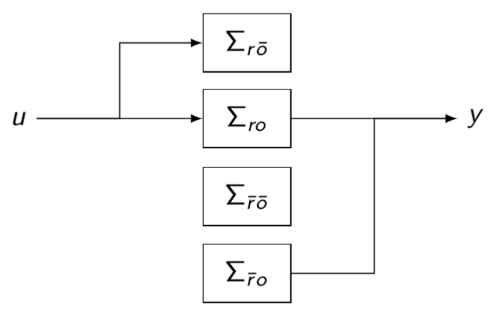
Stabilizability

- A system is stabilizable if all **unstable modes are reachable**.
- Reachability always implies stabilizability

Detectability

- A system is detectable if all **unstable modes are observable**.
- Observability always implies detectability

3.4 Kalman Decomposition



A specific method to group basis vectors.

Diagonalizable system

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{A}_{r\bar{o}} & 0 & 0 & 0 \\ 0 & \mathbf{A}_{ro} & 0 & 0 \\ 0 & 0 & \mathbf{A}_{\bar{r}\bar{o}} & 0 \\ 0 & 0 & 0 & \mathbf{A}_{\bar{r}o} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \mathbf{B}_{r\bar{o}} \\ \mathbf{B}_{ro} \\ 0 \\ 0 \end{bmatrix}\mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} 0 & \mathbf{C}_{ro} & 0 & \mathbf{C}_{\bar{r}o} \end{bmatrix}\mathbf{x} + \mathbf{D}\mathbf{u}$$

General case

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{A}_{r\bar{o}} & \mathbf{A}_{12} & \mathbf{A}_{13} & \mathbf{A}_{14} \\ 0 & \mathbf{A}_{ro} & 0 & \mathbf{A}_{24} \\ 0 & 0 & \mathbf{A}_{\bar{r}\bar{o}} & \mathbf{A}_{34} \\ 0 & 0 & 0 & \mathbf{A}_{\bar{r}o} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \mathbf{B}_{r\bar{o}} \\ \mathbf{B}_{ro} \\ 0 \\ 0 \end{bmatrix}\mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} 0 & \mathbf{C}_{ro} & 0 & \mathbf{C}_{\bar{r}o} \end{bmatrix}\mathbf{x} + \mathbf{D}\mathbf{u}$$

r reachable
$\bar{r}$ not reachable
o observable
$\bar{o}$ not observable

Remarks

- In the transfer function $u \rightarrow y$ only the modes corresponding to the reachable and observable modes will appear (others will be cancelled by a zero).

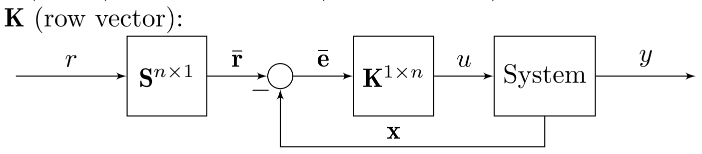
- A realization is minimal **iff** it is reachable and observable.

4 SISO Pole Placement

This section primarily holds for SISO systems but with some restrictions also for the MIMO case (see end of this section).

4.1 Full State Feedback

Assuming the internal state of a system can be accessed completely, the system can be controlled with a *reference signal* r (scalar) *scaling vector* $\mathbf{S}$ (column vector) and a static gain $\mathbf{K}$ (row vector):



The corresponding closed-loop system is

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\bar{N}r, \qquad \bar{N} = \mathbf{K}\mathbf{S}$$
$$y = (\mathbf{C} - \mathbf{D}\mathbf{K})\mathbf{x} + \mathbf{D}\bar{N}r$$

Remarks:
- $(\mathbf{A} - \mathbf{B}\mathbf{K})$ determines the closed-loop dynamics
- zeros are not affected by feedback

4.2 Pole Placement

Pole placement can be used to tune $\mathbf{K}$ and $\mathbf{S}$ to achieve the desired closed loop poles λ_i of a controllable system. The desired characteristic equation is

$$\begin{aligned}\varphi_{cl,des} &= \det(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}) = (s-\lambda_1)(s-\lambda_2)\dots(s-\lambda_n)\\ &= s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_0\end{aligned}$$

4.2.1 Reachable Canonical Form

If the system is in **reachable canonical form** and $\mathbf{K} = [k_0, k_1, \ldots, k_{n-1}]$, then

$$\mathbf{A}_{\text{cl}} = \mathbf{A} - \mathbf{B}\mathbf{K} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 1 \\ -a_0 - k_0 & -a_1 - k_1 & \dots & -a_{n-1} - k_{n-1} \end{bmatrix}$$

and the closed-loop characteristic polynomial can be used to choose the parameters k_i based on the desired parameters α_i:

$$\varphi(s) = s^n + (a_{n-1} + k_{n-1})s^{n-1} + \ldots + (a_0 + k_0) \overset{!}{=} \varphi_{cl,des}$$
$$k_i = \alpha_i - a_i, i = 0, \ldots, n-1$$

4.2.2 General Case

If the system is **controllable** but not in reachable canonical form, the following steps have to be applied to transform the system into reachable canonical form $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \tilde{\mathbf{D}}$.

1. Calculate $\tilde{\mathbf{A}}$ by comparing the characteristic polynomial of $\mathbf{A}$ with the one of the parametric reachable canonical form $\tilde{\mathbf{A}}$.
2. Find transformation matrix $\mathbf{T}$ (using $\tilde{\mathbf{A}}$ from 1. and the known form for $\tilde{\mathbf{B}}$ to calculate $\tilde{\mathbf{R}}$):
$$\tilde{\mathbf{R}} = \begin{bmatrix}\tilde{\mathbf{B}} & \tilde{\mathbf{A}}\tilde{\mathbf{B}} & \dots & (\tilde{\mathbf{A}})^{n-1}\tilde{\mathbf{B}}\end{bmatrix}$$
$$= \mathbf{T}\mathbf{R} = \begin{bmatrix} 0 & 0 & \dots & 1 \\ \vdots & & \ddots & \\ 0 & 1 & -a_{n-1} & \dots \\ 1 & -a_{n-1} & a_{n-1}^2 - a_{n-2} & \dots \end{bmatrix}$$
$$\mathbf{T} = \tilde{\mathbf{R}}\mathbf{R}^{-1}$$
with $\tilde{\mathbf{R}}$ the reachability matrix of the transformed system and a_i are the coefficients of $\tilde{\mathbf{A}}$
3. Calculate the $\tilde{\mathbf{C}}$ matrix.
4. Apply method for reachable canonical form
$$\mathbf{K}' = [\alpha_0 - a_0, \quad \alpha_1 - a_1, \quad \ldots, \quad \alpha_{n-1} - a_{n-1}]$$
5. Transform $\tilde{\mathbf{K}}$ back to the original system:
$$\mathbf{K} = \tilde{\mathbf{K}}\tilde{\mathbf{R}}\mathbf{R}^{-1}$$
which is possible if $\mathbf{R}$ is invertible (corresponds to controllability).

Reminder

The reachable canonical form is given by

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} + d$$

and

$$\mathbf{A}' = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \\ & & & & & 1 \\ -a_0 & -a_1 & & \cdots & & -a_{n-1} \end{bmatrix}, \quad \mathbf{B}' = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$
$$\mathbf{C}' = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix}, \qquad \mathbf{D}' = [d];$$

4.2.2.1 Unreachable/Uncontrollable Modes

Controllability (or reachability) is a **necessary and sufficient condition** for **arbitrary** pole placement. Conversely, in an uncontrollable system in modal coordinates there will be at least one state

$$\dot{\tilde{x}}_i = \lambda_i\tilde{x}_i + \tilde{b}_i u$$

for which $b_i = 0$.

- Hence, no matter how we choose $\mathbf{K}$, the pole λ_i will remain in it's original location.
- Note however, that this doesn't make a statement on stability:
  - If the system is not stabilizable (see definition) this is an issue.
  - If the system is stabilizable, the unreachable modes will remain in their locations and e.g. slow down the system dynamics but the overall system will still be stable.

4.2.3 Ackermann's Formula

Assuming that the system is **controllable**, Ackermann's formula can be used to calculate $\mathbf{K}$ for **both** CT and DT systems:

$$\mathbf{K} = \begin{bmatrix}0, & \ldots, & 0, & 1\end{bmatrix}\mathbf{R}^{-1}\varphi_{cl}(\mathbf{A})$$
$$\varphi_{cl,des}(s) = s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_0 = (s-\lambda_1)\ldots(s-\lambda_n)$$
$$\begin{aligned}\varphi_{cl,des}(\mathbf{A}) &= \mathbf{A}^n + \alpha_{n-1}\mathbf{A}^{n-1} + \ldots + \alpha_0\mathbf{I}\\ &= (\mathbf{A} - \lambda_1\mathbf{I})\ldots(\mathbf{A} - \lambda_n\mathbf{I})\end{aligned}$$

Remarks
- The formulas are the same for **DT** as for CT but of course the poles must be placed in the unit disk instead of the LHP.
- **MATLAB**: `place` or `acker` (place uses more stable algorithm)

4.2.4 Reference Scaling

To ensure that the closed-loop systems follows unit steps with zero steady state error, the scaling vector $\mathbf{S}$ has to be chosen accordingly

$$G_{yr}^{cl}(s) = G_{y\leftarrow r}^{cl}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K})^{-1}\mathbf{B}\bar{N}r, \quad \bar{N} = \mathbf{K}\mathbf{S}$$
$$G_{y\leftarrow r}^{cl}(0) \overset{!}{=} 1$$
$$\bar{N} = -\left[\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{B}\right]^{-1}$$
$$\bar{N} = \underbrace{\left[\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right]}_{\text{DC ol.}} \cdot \underbrace{\left[\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{B}\right]^{-1}}_{\text{DC cl.}}$$

In other words, the scaling vector $\mathbf{S}$ has to be chosen such that $\mathbf{K}\mathbf{S} = \bar{N}$.

4.2.5 Pole Placement for MIMO Systems

- If the system is controllable from any one of the inputs, pole placement works even though we might not use the full potential of all the inputs
- In addition to the poles, the closed-loop eigenvectors (modal shapes) can be placed

5 LQR

More details on the derivation of LQR can be found under 10.4 $\mathcal{H}_2$ Synthesis.

5.1 Cost Functional

LQR can be used to systematically optimize the **static gain vector** $\mathbf{K}$ based on *tracking errors* and *control effort*.

For the reference input $r(t) = 0$ the cost functional J is defined as

$$J(\mathbf{x}, \mathbf{t}) = \int_0^{+\infty} \left[\underbrace{\mathbf{x}(t)^\mathsf{T}\mathbf{Q}\mathbf{x}(t)}_{\text{tracking error}} + \underbrace{\mathbf{u}(t)^\mathsf{T}\mathbf{R}\mathbf{u}(t)}_{\text{control effort}}\right]dt$$

Remarks:
- $\mathbf{Q}$
  - is symmetric i.e. $\mathbf{Q} = \mathbf{Q}^\mathsf{T}$
  - is positive semidefinite i.e. $\mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} \geq 0$ or $\text{eig}(\mathbf{Q}) \geq 0$
  - must give positive weights to unstable modes to penalize them in the cost function (the pair $\mathbf{A}, \sqrt{\mathbf{Q}}$ is detectable)
- $\mathbf{R}$
  - must be positive definite i.e. $\mathbf{u}^\mathsf{T}\mathbf{R}\mathbf{u} > 0 \ \forall \mathbf{u} \neq \mathbf{0}$ or $\text{eig}(\mathbf{R}) > 0$ (otherwise not every control effort is penalized).
  - is symmetric (see derivation in H2-Synthesis Section 10.4)
- The pair $\mathbf{A}, \mathbf{B}$ must be stabilizable
- In general, there is an additional cross-coupling term $\mathbf{x}^\mathsf{T}\mathbf{N}\mathbf{u}$ in J which is often neglected i.e. equal to 0

5.1.1 Bryson's Rule

Bryson's rule can be used as a *baseline guess* for $\mathbf{Q}, \mathbf{R}$:

$$\mathbf{Q} = \begin{bmatrix} \frac{q_1^2}{|x_{1,\max}|^2} & & & \\ & \frac{q_2^2}{|x_{2,\max}|^2} & & \\ & & \ddots & \\ & & & \frac{q_n^2}{|x_{n,\max}|^2} \end{bmatrix}$$
$$\mathbf{R} = \rho\begin{bmatrix} \frac{r_1^2}{|u_{1,\max}|^2} & & & \\ & \frac{r_2^2}{|u_{2,\max}|^2} & & \\ & & \ddots & \\ & & & \frac{r_n^2}{|u_{n,\max}|^2} \end{bmatrix}$$

where

$$\sum_{i=1}^{n} q_i^2 = 1 \text{ and } \sum_{i=1}^{m} r_i^2 = 1$$

and $x_{i,\max}$, $u_{j,\max}$ are the **maximum deviations** that we are willing to tolerate for the i-th state and j-th input, respectively.

5.2 Continuous Time

First, check $\mathbf{A}, \mathbf{B}$ for reachability!

In continuous time, first the **symmetric, positive definite** solution

$$\mathbf{P} = \begin{bmatrix} p_{1,1} & \dots & p_{1,n} \\ & \ddots & \\ p_{1,n} & \dots & p_{n,n} \end{bmatrix}$$

with

$$\begin{aligned} p_{1,1} &> 0 \\ p_{1,1}p_{2,2} - p_{1,2}^2 &> 0 \\ &\vdots \end{aligned}$$

of the *continuous algebraic Riccati equation* (***CARE***) has to be found

$$\mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{P} = 0$$

then the *static gain matrix* $\mathbf{K}$ can be obtained by setting $\mathbf{u} = -\mathbf{K}\mathbf{x}$:

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{P}$$

Remarks:
- LQR applies directly to MIMO-systems

Matlab: `[K,P]=lqr(A,B,Q,R)`

5.3 Discrete Time

In discrete time, the cost functional is

$$J(\mathbf{x}, \mathbf{u}) = \sum_{k=0}^{+\infty}\left(\mathbf{x}[k]^\mathsf{T}\mathbf{Q}\mathbf{x}[k] + \mathbf{u}[k]^\mathsf{T}\mathbf{R}\mathbf{u}[k]\right),$$

the **symmetric, positive definite** solution $\mathbf{P}$ to the *discrete algebraic Riccati equation* (***DARE***)

$$\mathbf{P} = \mathbf{A}^\mathsf{T}\mathbf{P}\mathbf{A} - (\mathbf{A}^\mathsf{T}\mathbf{P}\mathbf{B})(\mathbf{B}'\mathbf{P}\mathbf{B} + \mathbf{R})^{-1}(\mathbf{B}^\mathsf{T}\mathbf{P}\mathbf{A}) + \mathbf{Q}$$

and the *static gain matrix* obtained with $\mathbf{u} = -\mathbf{K}\mathbf{x}$ is

$$\mathbf{K} = (\mathbf{R} + \mathbf{B}^\mathsf{T}\mathbf{P}\mathbf{B})^{-1}\mathbf{B}^\mathsf{T}\mathbf{P}\mathbf{A}$$

Matlab: `[K_d,P_d]=dlqr(A,B,Q,R)`

5.4 LQR Servo: Integrator

To compensate model errors or disturbances an integrator can be added. This is simply done by adding the integral of the error to the state space model

$$\dot{x}_I = -y + r = -\mathbf{C}\mathbf{x} - \mathbf{D}\mathbf{u} + r$$
$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}\mathbf{x}\\x_I\end{bmatrix} &= \begin{bmatrix}\mathbf{A} & 0\\-\mathbf{C} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}\\x_I\end{bmatrix} + \begin{bmatrix}\mathbf{B}\\\mathbf{D}\end{bmatrix}\mathbf{u} + \begin{bmatrix}0\\1\end{bmatrix}r\\ &= \begin{bmatrix}\mathbf{A} & 0\\-\mathbf{C} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}\\x_I\end{bmatrix} + \begin{bmatrix}\mathbf{B} & 0\\\mathbf{D} & 1\end{bmatrix}\begin{bmatrix}\mathbf{u}\\r\end{bmatrix},\end{aligned}$$

then apply LQR to the modified state space model which yields a controller

$$\mathbf{K}' = \begin{bmatrix}\mathbf{K} & K_I\end{bmatrix}$$
$$\mathbf{u} = -\mathbf{K}'\begin{bmatrix}\mathbf{x}\\x_I\end{bmatrix} + \mathbf{K}'\mathbf{S}r$$

5.5 Symmetric Root Locus

The symmetric root locus can be used to understand the influence of ρ (penalizes control effort) on the poles of a closed loop LQR feedback system. By using $\mathbf{Q} = \mathbf{C}^\mathsf{T}\mathbf{C}$ and $\mathbf{R} = \rho\mathbf{I}$, the symmetric root locus condition simplifies to

$$\rho\mathbf{I} + \mathbf{G}(s)\mathbf{G}(-s) = \mathbf{0} \qquad \Leftrightarrow$$
$$\rho\mathbf{D}(s)\mathbf{D}(-s) + \mathbf{N}(s)\mathbf{N}(-s) = \mathbf{0}$$

where $\mathbf{G}(-s)$ corresponds to flipping signs in all the odd coefficients.

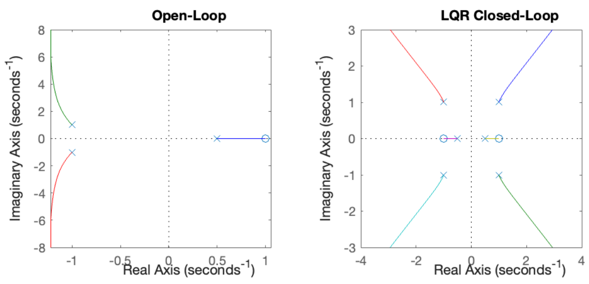
Symmetric Root Locus Rules
- $2n$ branches, where n is the size of $\mathbf{A}$
  - like in standard RL, every branch starts in an OL pole
  - but asymptotic behavior different from standard RL
- symmetric to the real and imaginary axis
- LQR closed-loop poles are all in the LHP
- $\rho \to \infty$ (expensive control):
  CL poles approach stable OL poles and the mirror-images of the unstable OL poles (all on LHP)
- $\rho \to 0$ (cheap control):
  CL poles approach MP OL zeros and the mirror-images of the NMP OL zeros or go to infinity along the LHP asymptotes (all on LHP)
- note that for $\rho \to \infty$ there are no asymptotes to ∞ but only convergence towards (mirror) OL poles
  - for $\rho \to 0$ there is also convergence to ∞

5.5.1 Example: Symmetric Root Locus

Consider the LTI system

$$G = \frac{s-1}{(s-0.5)(s^2+2s+s)}$$

the corresponding root-loci are



Here the open-loop system features both a non-minimum-phase zero and an unstable pole. When looking at the left half of the *symmetric root locus*, the mirrored NMP zero and mirrored unstable pole stabilize the system such that for any ρ the closed loop poles will stay on the LHP.

5.6 LQR Margins: Kalman Frequency Domain Equality

Considering a SISO LTI system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x}$$
$$\mathbf{Q} = \mathbf{C}^\mathsf{T}\mathbf{C}, \quad \mathbf{R} = \rho$$

and

$$G(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} \qquad r \to y$$
$$L(s) = \mathbf{K}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} \qquad r \to u$$

the *Kalman Frequency Domain Equality*

$$(1+L(-s))(1+L(s)) = 1 + \frac{1}{\rho}G(-s)G(s)$$

holds.

5.6.1 Margins

In the frequency domain ($s = j\omega$)

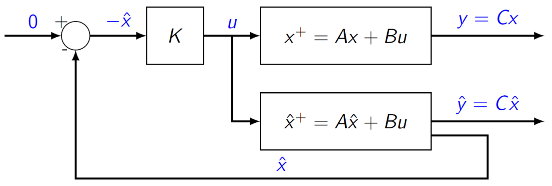
$$|1+L(j\omega)|^2 = 1 + \frac{1}{\rho}|G(j\omega)|^2 \geq 1$$

In the Nyquist plot, the transfer function using the LQR gain is always outside of a unit circle centered at -1 i.e.

- The phase margin is at least 60° because the Nyquist curve cannot enter a unit circe centered at -1
- Be aware that LQR synthesizes an optimal controller for the **given model**. However the model is never perfect and the margins to the real system can be smaller due to uncertainties.
- The gain margin is $(\frac{1}{2}, \infty)$ because the loop transfer function (Nyquist curve) will never cross the real axis between $(-2, 0)$ (unit circle argument again) $\to -\frac{1}{k} = (\frac{1}{2}, \infty)$

6 Observers

In case we desire full-state feedback but there are no sensors available to measure each state, we could try to simulate the system with its estimated states $\hat{\mathbf{x}}$.



- we choose $\hat{\mathbf{x}}_0 = \mathbf{0}$
- the control input u is computed from the simulated state
- both the simulated model and the physical plant receive the same control input

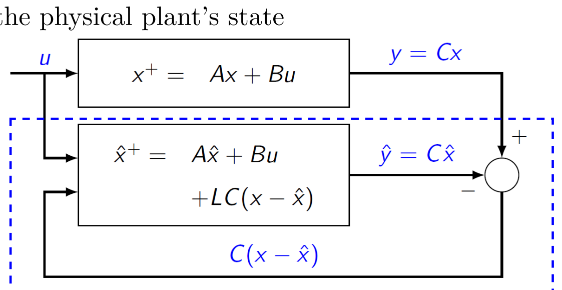
Estimation Error Dynamics

- the estimation error $\boldsymbol{\eta} := \mathbf{x} - \hat{\mathbf{x}}$ shares the dynamics of the OL system $\dot{\boldsymbol{\eta}} = \mathbf{A}\boldsymbol{\eta}$.
- this can lead to undesirable estimation error dynamics (slow convergence, oscillations, divergence in case of unstable OL system)

6.1 The Luenberger Observer

We don't have to fully rely on our state estimation but can instead, use the available sensory information to improve it.

- the signal $\mathbf{y} - \hat{\mathbf{y}} = \mathbf{C}(\mathbf{x} - \hat{\mathbf{x}}) = \mathbf{C}\boldsymbol{\eta}$ is called **innovation** and is the input to the Luenberger **observer gain L**
- the Luenberger observer corrects the estimated state by a linear feedback on the innovation
- in other words we design a controller to get a good estimate of the physical plant's state



Luenberger Observer

Estimation Error Dynamics

The states evolve as follows:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}\mathbf{C}(\mathbf{x} - \hat{\mathbf{x}})$$
$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}y$$

Hence,

$$\dot{\eta} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{A}(\mathbf{x} - \hat{\mathbf{x}}) - \mathbf{L}\mathbf{C}(\mathbf{x} - \hat{\mathbf{x}}) = (\mathbf{A} - \mathbf{L}\mathbf{C})\boldsymbol{\eta}$$

Remarks:

- The system has to be **observable**.
- The observer dynamics are given by $\mathbf{A} - \mathbf{L}\mathbf{C}$.

6.1.1 Observer Pole Placement

For state feedback we obtained the CL eigenvalues $\mathbf{A} - \mathbf{B}\mathbf{K}$. For the Luenberger observer we can state the **dual** problem, namely placing the poles of

$$\mathbf{A} - \mathbf{L}\mathbf{C}$$

respectively

$$\mathbf{A}^\mathsf{T} - \mathbf{C}^\mathsf{T}\mathbf{L}^\mathsf{T} \text{ (same Eigenvalues)}$$

using the methods mentioned in Section 4.2 (Pole Placement) or LQE (see 6.3).

Observer Pole Placement

$\mathbf{L} = [\ell_0, \cdots, \ell_{n-1}]^\mathsf{T}$ can be obtained by comparing the eigenvalues of the observer with the desired ones:

$$\det\big(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{L}\mathbf{C})\big) \overset{!}{=} \varphi_{\text{cl,des}}(\lambda)$$

Ackermann Observer Design

Similarly to state feedback we get

$$\mathbf{L} = \varphi_{\text{cl,des}}(\mathbf{A})\mathbf{O}^{-1}\begin{bmatrix}0, & \ldots, & 0, & 1\end{bmatrix}^\mathsf{T}$$
$$\varphi_{\text{cl,des}}(s) = s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_0 = (s-\lambda_1)\ldots(s-\lambda_n)$$
$$\varphi_{\text{cl,des}}(\mathbf{A}) = \mathbf{A}^n + \alpha_{n-1}\mathbf{A}^{n-1} + \ldots + \alpha_0\mathbf{I}$$
$$= (\mathbf{A} - \lambda_1\mathbf{I})\ldots(\mathbf{A} - \lambda_n\mathbf{I})$$

Remarks:

- **O** has to be invertible. This is the case for observable SISO LTI systems.
- **observability** is a **necessary and sufficient** condition for arbitrary pole placement
- **detectability** is **necessary** for stable observer dynamics (as we need to observe the unstable poles)
- If the system is
  - not observable (so at least one $c_k = 0$)
  - diagonalizable and given in modal coordinates

  $$\dot{\boldsymbol{\eta}}_k = \lambda_k\boldsymbol{\eta}_k - \mathbf{l}_k\sum_{j=1}^{k}(\tilde{c}_j\boldsymbol{\eta}_j)$$

  then the unobservable poles cannot be moved (cannot assign these poles)

Noise Sensitivity

- Large values for **L** make our measurements more **sensitive to noise**.
- Hence, we need to find a trade-off between estimation dynamics and robustness to noise.

6.2 Noise Models

We model noise as a stochastic process with a **noise signal**

$$\mathbf{w} : t \mapsto \mathbf{w}(t) \in \mathbb{R}^n$$

where each $\mathbf{w}(t)$ is the realization of a random variable.

6.2.1 White Noise

"Perfect" but unphysical (values change instantaneously) noise can be modeled as follows:

1. $\mathbb{E}[\mathbf{w}(t)] = 0$:
   the signal has zero mean.
2. $\mathbb{E}[\mathbf{w}(t)\mathbf{w}(t)^\mathsf{T}] = \mathbf{W}$:
   the signal value at each time instance has covariance matrix $\mathbf{W}$, for some positive definite matrix $\mathbf{W}$.
3. $\mathbb{E}[\mathbf{w}(t)\mathbf{w}(\tau)^\mathsf{T}] = 0$:
   the signal values at two different times are not correlated. In fact, we will assume that signal values at two different times are **independent** i.e. the matrix **W** is diagonal

If such a signal additionally is Gaussian-distributed it is called "zero mean Gaussian white noise".

6.2.2 Colored Noise

To simulate physical noise we let a white noise input w propagate through an LTI system model (filter) and use the system's output as noise signal:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{w}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

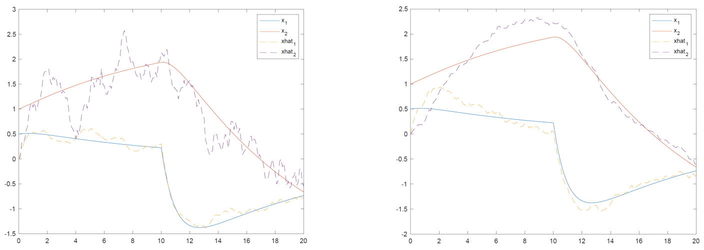
The LTI system usually acts as low-pass filter in this case.

6.3 Linear Quadratic Estimator (LQE) / Kalman Filter

Noise affects our state estimation adversely. The LQE / Kalman filter aims at an optimal trade-off between

- trusting the sensory information and hence, having higher observer gains in **L** (increases noise-sensitivity, left image)
- trusting the state-space model and hence, having lower observer gains (slower convergence) in **L** (decreases noise-sensitivity, right image)

and corrects state estimation based on available measurements and assumptions on noise.



Remark: LQE can be applied only to **LTI systems**.

6.3.1 Optimal LQE Design

For the LQE problem, we get the following LTI Model

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{w}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{n}(t)$$

with the **Gaussian white noise** signals

- $\mathbf{w}(t)$: process noise
- $\mathbf{n}(t)$: sensor noise

We model the errors with the covariance matrices

$$\mathbf{Q} = \mathbb{E}[\mathbf{w}(t)\mathbf{w}(t)^\mathsf{T}] \qquad \text{(process)}$$
$$\mathbf{R} = \mathbb{E}[\mathbf{n}(t)\mathbf{n}(t)^\mathsf{T}], \qquad \forall t \geq 0 \text{ (sensor)}$$

The (Luenberger) observer is given by

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}\mathbf{y}(t)$$
$$\hat{\mathbf{y}}(t) = \mathbf{C}\hat{\mathbf{x}}(t)$$

Minimization Problem

The steady-state covariance of the state estimation error

$$\lim_{t\to+\infty}\mathbb{E}\left[(\mathbf{x}(t) - \hat{\mathbf{x}}(t))(\mathbf{x}(t) - \hat{\mathbf{x}}(t))^\mathsf{T}\right]$$

is minimized by solving the Riccati equation (*ARE*)

$$\mathbf{A}\mathbf{Y} + \mathbf{Y}\mathbf{A}^\mathsf{T} - \mathbf{Y}\mathbf{C}^\mathsf{T}\mathbf{R}^{-1}\mathbf{C}\mathbf{Y} + \mathbf{Q} = \mathbf{0}$$

for the **positive definite** matrix **Y** and choosing the *optimal estimation gain* **L** as

$$\mathbf{L}^\mathsf{T} = \mathbf{R}^{-1}\mathbf{C}\mathbf{Y}$$

Remarks:

- the system has to be detectable (check for observability) and $(\mathbf{A}, \mathbf{Q})$ stabilizable (check for reachability, treat **Q** like **B**)
  (e.g. $n = 2 : \mathcal{R} = [\mathbf{Q} \quad \mathbf{A}\mathbf{Q}]$)
- **Y** is *real*, *symmetric*, *positive definite* ($\mathrm{Re}(\lambda_i) > 0, \mathrm{Im}(\lambda_i) = 0$) and has to fulfil the *Sylvester criterion* e.g. for $n = 2$:

  $$\mathbf{Y} = \begin{bmatrix} a & b \\ b & d \end{bmatrix}; \quad a > 0, \quad ad - b^2 > 0$$

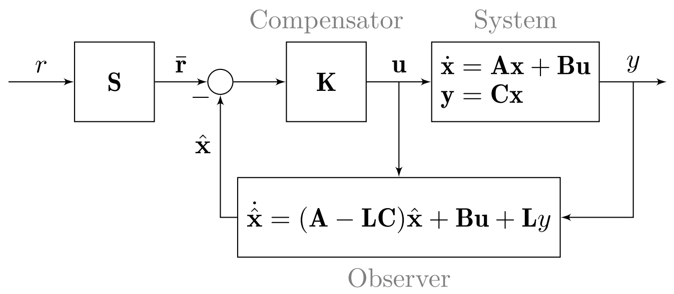
- for symmetric **Y** one has that $\mathbf{Y}\mathbf{A}^\mathsf{T} = (\mathbf{A}\mathbf{Y})^\mathsf{T}$
- as expected from duality, **L** is the transpose of **K**, obtained for the pair $(\mathbf{A}^\mathsf{T}, \mathbf{C}^\mathsf{T})$, and for weight matrices **Q** and **R**.
- in practice, we can measure the noise to get a first estimate of **Q**, **R**
- as a guideline, one should make the innovations $\mathbf{C}\boldsymbol{\eta}$ as white as possible

7 LQG / DOFC

Linear-Quadratic Gaussian Compensator (LQG)
Dynamic output Feedback Compensator (DOFC)

7.1 LQG/DOFC State Space Description

The combination of the LQR and the LQE is referred to as a LQG/DOFC.



The closed-loop system is given by

$$\begin{bmatrix}\dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}}\end{bmatrix} = \begin{bmatrix}\mathbf{A} & -\mathbf{B}\mathbf{K} \\ \mathbf{L}\mathbf{C} & (\mathbf{A} - \mathbf{L}\mathbf{C} - \mathbf{B}\mathbf{K})\end{bmatrix}\begin{bmatrix}\mathbf{x} \\ \hat{\mathbf{x}}\end{bmatrix} + \begin{bmatrix}\mathbf{B}\mathbf{K}\mathbf{S} \\ \mathbf{B}\mathbf{K}\mathbf{S}\end{bmatrix}r$$

$$y = \begin{bmatrix}\mathbf{C} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{x} \\ \hat{\mathbf{x}}\end{bmatrix}$$

If we substitute the estimation error $\boldsymbol{\eta} = \mathbf{x} - \hat{\mathbf{x}}$ using the similarity transformation

$$\mathbf{T} = \mathbf{T}^{-1} = \begin{bmatrix}\mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I}\end{bmatrix}$$

we get

$$\begin{bmatrix}\dot{\mathbf{x}} \\ \dot{\boldsymbol{\eta}}\end{bmatrix} = \underbrace{\begin{bmatrix}\mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ 0 & (\mathbf{A} - \mathbf{L}\mathbf{C})\end{bmatrix}}_{\mathbf{A}_{\text{cl}}}\begin{bmatrix}\mathbf{x} \\ \boldsymbol{\eta}\end{bmatrix} + \begin{bmatrix}\mathbf{B}\mathbf{K}\mathbf{S} \\ \mathbf{0}\end{bmatrix}r$$

$$y = \begin{bmatrix}\mathbf{C} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{x} \\ \boldsymbol{\eta}\end{bmatrix}$$

The determinant of the closed-loop dynamic output feedback system $\mathbf{A_{cl}}$ is

$$\det(\mathbf{A}_{cl}) = \det(\mathbf{A}-\mathbf{BK})\det(\mathbf{A}-\mathbf{LC})$$

Therefore the eigenvalues of the closed-loop system are given by the **union of the eigenvalues** of $\mathbf{A}-\mathbf{BK}$ and of $\mathbf{A}-\mathbf{LC}$.

7.1.1 Separation Principle

Based on the independence of the Eigenvalues of the observer and controller and the assumption of a **LTI System with Gaussian noise**, the *separation principle* states that both the observer and the controller can be designed individually i.e.

- choose $\mathbf{K}$ as the solution of the LQR problem
- choose $\mathbf{L}$ as the solution of the LQE problem (approx. $10\times$ faster than poles of $\mathbf{K}$)

Recall:
The determinant of a 2×2 block matrix if $\mathbf{M}$ is invertible

$$\det\begin{bmatrix}\mathbf{M} & \mathbf{N}\\ \mathbf{P} & \mathbf{Q}\end{bmatrix} = \det(\mathbf{M})\det(\mathbf{Q}-\mathbf{P}\mathbf{M}^{-1}\mathbf{N})$$

if $\mathbf{Q}$ is invertible

$$\det\begin{bmatrix}\mathbf{M} & \mathbf{N}\\ \mathbf{P} & \mathbf{Q}\end{bmatrix} = \det(\mathbf{Q})\det(\mathbf{M}-\mathbf{N}\mathbf{Q}^{-1}\mathbf{P})$$

In particular

$$\det\begin{bmatrix}\mathbf{M} & \mathbf{N}\\ \mathbf{0} & \mathbf{Q}\end{bmatrix} = \det(\mathbf{M})\det(\mathbf{Q}) = \det\begin{bmatrix}\mathbf{M} & \mathbf{0}\\ \mathbf{P} & \mathbf{Q}\end{bmatrix}$$

7.1.2 Properties of LQG/DOFC

- The estimation error is uncontrollable from the reference input (The estimation error is not influenced by the reference).
- The LQG is often referred to as dynamic output feedback compensator (**DOFC**) as the control is not just based on a static gain but on a dynamic state estimation.

7.2 Transfer Function

The **closed-loop** transfer function of the LQG/DOFC from r to y is

$$G_{yr}^{cl} = \mathbf{C}(s\mathbf{I}-\mathbf{A}+\mathbf{BK})^{-1}\mathbf{BKS}$$

and from u to y (**open-loop**)

$$G_{uy} = -\mathbf{K}(s\mathbf{I}-\mathbf{A}+\mathbf{LC}+\mathbf{BK})^{-1}\mathbf{L}$$

Remarks

- In G_{yr}^{cl}, only the closed-loop poles of the controller (observer and system are cancelled) and open-loop zeros are retained.
- G_{uy}, has the same order as the plant and the number of closed-loop poles is doubled (all stable)

Transmission Zeros

$$\left|\det\begin{bmatrix}s\mathbf{I}-\mathbf{A}+\mathbf{BK} & -\mathbf{BK} & -\mathbf{BKS}\\ \mathbf{0} & s\mathbf{I}-\mathbf{A}+\mathbf{LC} & \mathbf{0}\\ \mathbf{C} & \mathbf{0} & 0\end{bmatrix}\right|$$
$$= |\det(s\mathbf{I}-\mathbf{A}+\mathbf{LC})|\cdot\left|\det\begin{bmatrix}s\mathbf{I}-\mathbf{A}+\mathbf{BK} & -\mathbf{BKS}\\ \mathbf{C} & 0\end{bmatrix}\right|$$

- The eigenvalues of $(\mathbf{A}-\mathbf{LC})$ are also zeros
- In the closed loop, only observer poles are eliminated due to pole zero cancellation

Reference Scaling
To ensure a DC-Gain of 1 we need the reference scaling matrix S to satisfy
in the **continuous-time** case

$$\lim_{s\to 0} G_{yr}^{cl}(s) = \mathbf{I}$$
$$-\mathbf{C}(\mathbf{A}-\mathbf{BK})^{-1}\mathbf{BKS} = \mathbf{I}$$

for a SISO system

$$\bar{N} = \mathbf{KS} = -\frac{1}{\mathbf{C}(\mathbf{A}-\mathbf{BK})^{-1}\mathbf{B}}$$

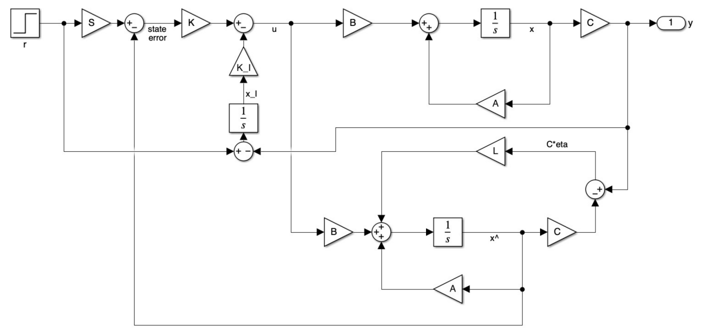
in the **discrete-time** case

$$\lim_{z\to 1} G_{yr}^{cl}(z) = \mathbf{I}$$
$$\mathbf{C}(\mathbf{I}-\mathbf{A}+\mathbf{BK})^{-1}\mathbf{BKS} = \mathbf{I}$$

for a SISO system

$$\bar{N} = \mathbf{KS} = \frac{1}{\mathbf{C}(\mathbf{I}-\mathbf{A}+\mathbf{BK})^{-1}\mathbf{B}}$$

7.3 LQG/DOFC with Integrator



Like with the LQR compensator an integrator can be added by augmenting the state vector with new states x_I and calculating the new $\mathbf{K}'$ matrix.

$$\dot{x}_I = r - y$$
$$\mathbf{K}' = \begin{bmatrix}\mathbf{K} & K_I\end{bmatrix}$$

The closed-loop system is then given by

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{x}_I\\ \dot{\hat{\mathbf{x}}}\end{bmatrix} = \begin{bmatrix}\mathbf{A} & -\mathbf{BK}_I & -\mathbf{BK}\\ -\mathbf{C} & 0 & \mathbf{0}\\ \mathbf{LC} & -\mathbf{BK}_I & \mathbf{A}-\mathbf{BK}-\mathbf{LC}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ x_I\\ \hat{\mathbf{x}}\end{bmatrix} + \begin{bmatrix}\mathbf{BKS}\\ \mathbf{I}\\ \mathbf{BKS}\end{bmatrix} r$$
$$y = \begin{bmatrix}\mathbf{C} & 0 & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ x_I\\ \hat{\mathbf{x}}\end{bmatrix}$$

If we substitute the estimation error $\boldsymbol{\eta} = \mathbf{x}-\hat{\mathbf{x}}$ we get

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{x}_I\\ \dot{\boldsymbol{\eta}}\end{bmatrix} = \begin{bmatrix}\mathbf{A}-\mathbf{BK} & -\mathbf{BK}_I & \mathbf{BK}\\ -\mathbf{C} & 0 & \mathbf{0}\\ \mathbf{0} & 0 & \mathbf{A}-\mathbf{LC}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ x_I\\ \boldsymbol{\eta}\end{bmatrix} + \begin{bmatrix}\mathbf{BKS}\\ \mathbf{I}\\ \mathbf{0}\end{bmatrix} r$$
$$y = \begin{bmatrix}\mathbf{C} & 0 & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ x_I\\ \boldsymbol{\eta}\end{bmatrix}$$

Remarks:

- The integrator states don't have to be estimated as they are calculated separately and are not part of the observer or the physical plant.
- The error dynamics are uncontrollable form the reference (good)
- The closed-loop eigenvalues are a union of the eigenvalues of $\mathbf{A}-\mathbf{LC}$ (the observer's) and of the eigenvalues of the augmented controller

$$\begin{bmatrix}\mathbf{A}-\mathbf{BK} & -\mathbf{BK}_I\\ -\mathbf{C} & 0\end{bmatrix} = \mathbf{A}'-\mathbf{B}'\mathbf{K}'$$

- The controller output with integrator becomes $\mathbf{u} = -\mathbf{Kx}-\mathbf{K_I}x_I+\mathbf{KS}r$. Hence, the term $\mathbf{u_0} = \mathbf{KS}r = \bar{\mathbf{N}}r$ (scaled for the reference) can be considered a feedforward control input.

7.3.1 Transfer Function

The transfer function from y to u is

$$G_{uy} = -\begin{bmatrix}\mathbf{K}_I & \mathbf{K}\end{bmatrix}\left(s\mathbf{I}-\begin{bmatrix}0 & 0\\ -\mathbf{BK}_I & \mathbf{A}-\mathbf{BK}-\mathbf{LC}\end{bmatrix}\right)^{-1}\begin{bmatrix}-\mathbf{I}\\ \mathbf{L}\end{bmatrix}$$

7.4 Margins

While LQG/DOFC ensures the closed-loop stability of the nominal system, the stability margins can be **arbitrarily small**.

8 MIMO Systems

8.1 State Space Representation

$$\underbrace{\dot{\mathbf{x}}}_{n\times 1} = \underbrace{\mathbf{A}}_{n\times n}\underbrace{\mathbf{x}}_{n\times 1} + \underbrace{\mathbf{B}}_{n\times m}\underbrace{\mathbf{u}}_{m\times 1}$$
$$\underbrace{\mathbf{y}}_{\ell\times 1} = \underbrace{\mathbf{C}}_{\ell\times n}\underbrace{\mathbf{x}}_{n\times 1} + \underbrace{\mathbf{D}}_{\ell\times m}\underbrace{\mathbf{u}}_{m\times 1}$$

with

- $\mathbf{x}\in\mathbb{R}^n$, where n is the order of the system.
- $\mathbf{u}\in\mathbb{R}^m$, where m is the number of inputs.
- $\mathbf{y}\in\mathbb{R}^\ell$, where ℓ is the number of outputs.

8.2 TF Matrix

Continuous Time
For an elementary input vector $\mathbf{u}(t)$ we get the steady state output vector

$$\mathbf{y}_{ss}(t) = \mathbf{G}(s)\mathbf{u}(t) = \begin{bmatrix}G_{11}(s) & \dots & G_{1m}(s)\\ \vdots & \ddots & \vdots\\ G_{\ell 1}(s) & \dots & G_{\ell m}(s)\end{bmatrix}\underbrace{\begin{bmatrix}u_1\\ \vdots\\ u_m\end{bmatrix}e^{st}}_{\mathbf{u}(t)}$$

where $\mathbf{G}(s)\in\mathbb{C}^{\ell\times m}$ is a complex-valued matrix for $s\in\mathbb{C}$. The transfer function is then given by

$$\mathbf{G}(s) = \mathbf{C}\cdot(s\cdot\mathbf{I}-\mathbf{A})^{-1}\cdot\mathbf{B}+\mathbf{D}$$

Discrete Time
The same holds for the discrete time case where e^{st} is replaced by z^k and s by z. For further details see 1.3.

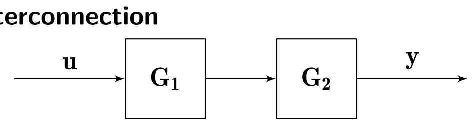
Remarks

- Diagonal TF matrices represent **decoupled** systems
- non-diagonal entries describe the relation of a certain input and a certain output

8.2.1 Interconnections

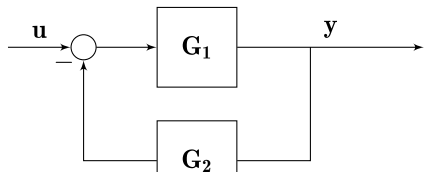
To find the interconnection start at output $\mathbf{y}$ and proceed against signal flow towards the input $\mathbf{u}$.

Series Interconnection



$$\mathbf{y} = \mathbf{G_2 G_1 u}$$

feedback interconnection



$$\mathbf{y} = (\mathbf{I}+\mathbf{G_1G_2})^{-1}\mathbf{G_1 u}$$

8.3 Poles

Reminder: Even if the excitation signal does not contain frequency p the output signal will contain this frequency as the initial condition response $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x_0}$ in modal form contains oscillations with frequency p.

8.3.1 State-Space Representation

The poles are given by the eigenvalues of $\mathbf{A}$.

8.3.2 Transfer Function Representation

- The poles are given by the **roots** of the **characteristic polynomial** of $\mathbf{G}(s)$.
- The **characteristic polynomial** $d(s)$ of a minimal realization of $\mathbf{G}(s)$ is the least common multiple (LCM) of the denominators of all possible minors G_{ij}.
- The **minors** are the determinants of all single entries of the $\mathbf{G}(s)$ matrix as well as the determinants of all possible square submatrices (of all sizes!) in $\mathbf{G}(s)$.

8.3.2.1 Example: Poles of a MIMO-TF

Given the transfer function $\mathbf{G}(s)$ of a MIMO system

$$\mathbf{G}(s) = \begin{bmatrix}\frac{s+3}{s^2} & \frac{3-S}{s^2(s-1)}\\ \frac{1}{s} & \frac{3s+1}{s(s-1)}\end{bmatrix}$$

The first order minors are

$$M_{1,1} = \det(G_{2,2}) = \frac{3s+1}{s(s-1)}$$
$$M_{1,2} = \det(G_{2,1}) = \frac{1}{s}$$
$$M_{2,1} = \det(G_{1,2}) = \frac{3-S}{s^2(s-1)}$$
$$M_{2,2} = \det(G_{1,1}) = \frac{s+3}{s^2}$$

And the second order minor is

$$M = \det(\mathbf{G}) = \frac{3s+11}{s^2(s-1)}$$

The characteristic polynomial is the LCM of the denominators of all minors

$$s^2(s-1)$$

The poles are the roots of the characteristic polynomial and therefore are

$$s_{1,2} = 0$$
$$s_3 = 1$$

8.3.3 Stability

Same condition as in SISO systems:

- **Stable** if $\mathrm{Re}(p_i\le 0)$ and the algebraic multiplicity of poles with zero real part is equal to their geometric multiplicity.
  Reminder:
  - Algebraic multiplicity: multiplicity of eigenvalue λ_i ($\lambda_i = 0$ for in case)
  - Geometric multiplicity:
    - Number of independent eigenvectors to eigenvalue λ_i
    - Dimension of eigenspace $\ker(\lambda\not{\mathbb{1}}-\mathbf{A})$ (number of free parameters).
    - E.g. if λ_i has algebraic multiplicity 2, then $\ker(\lambda\not{\mathbb{1}}-\mathbf{A})$ must be a plane.
- **Asymptotically stable** if $\mathrm{Re}(p_i<0)$

8.4 Zeros

- One can find MIMO zeros either from $\mathbf{G}(s)$ (transmission zeros) or from state space representation (invariant zeros).
- For minimal realizations they are the same.

8.4.1 Transmission Zeros

z_0 is a transmission zero (frequency) if there exists a non-zero, rational function $\mathbf{u}_0(s)$ such that

$$\lim_{s\to z_0}\left[\mathbf{G}(s)\mathbf{u}_0(s)\right]e^{st} = 0, \quad \forall t\ge 0$$

which means that $\mathbf{G}(s)$ **loses rank** for $s\to z_0$.

Method to Determine Transmission Zeros

1. Calculate all minors (of all orders) of $\mathbf{G}(s)$
2. Find the least common (multiple) denominator $d_{\min}(s)$ of the minors
3. Normalize highest order minors s.t. they have $d_{\min}(s)$ as denominator
4. Find largest common numerator $n_{\max}(s)$ (usually product of $(s-z_i)$ terms) of the normalized highest order minors.
5. The roots of $n_{\max}(s)$ are the MIMO zeros with their multiplicities

6. To find the directions $\mathbf{u}_{0,i}$ one has to solve the equation
$$\lim_{s\to z_i}\left[\mathbf{G}(s)\mathbf{u}_{0,i}(s)\right]=\mathbf{0}$$
for each found zero z_i.

Remarks
- A MIMO transmission zero is a pair of frequency z_0 and direction $\mathbf{u}_0(z_0)$.
- It can happen that a zero of $\mathbf{G}$ is not visible in any of it's entries $G_{ij}(s)$.
- The direction of the zero decides whether a MIMO pole and zero at the same frequency **cancel** (i.e. there can be a pole and a zero at same frequency without cancellation!).
- A zero means not just one entry of $\mathbf{y}$ is zero but that given an input vector $\mathbf{u}\neq\mathbf{0}$, $\mathbf{y}$ becomes the zero vector.

8.4.1.1 Example (From Script of Gioele Zardini)

Given TF matrix:
$$\mathbf{G}(s)=\begin{pmatrix}\frac{1}{s+1} & \frac{1}{s+2} & \frac{2\cdot(s+1)}{(s+2)\cdot(s+3)}\\ 0 & \frac{s+3}{(s+1)^2} & \frac{s+4}{s+1}\end{pmatrix}$$

1. Minors (already simplified determinants):
$$\frac{1}{s+1},\frac{1}{s+2},\frac{2\cdot(s+1)}{(s+2)\cdot(s+3)},0,\frac{s+3}{(s+1)^2},\frac{s+4}{s+1}\ \text{(1st order)}$$
$$\frac{s+3}{(s+1)^3},\frac{1}{s+1},-\frac{s+4}{(s+1)^2}\ \text{(2nd order)}$$
2. Find $d_{\min}(s)$ (LCM of all minor denominators):
$$(s+1)^3\cdot(s+2)\cdot(s+3)$$
3. Normalize highest order (here 2nd order) minors:
$$\frac{(s+3)^2\cdot(s+2)}{(s+1)^3\cdot(s+2)\cdot(s+3)},$$
$$\frac{(s+1)^2\cdot(s+2)\cdot(s+3)}{(s+1)^3\cdot(s+2)\cdot(s+3)},$$
$$-\frac{(s+4)\cdot(s+1)\cdot(s+2)\cdot(s+3)}{(s+1)^3\cdot(s+2)\cdot(s+3)}$$
4. Find $n_{\max}(s)$:
$$n_{\max}(s)=(s+3)(s+2)$$
5. Identify zeros: $z_1=-3,\ z_2=-2$

8.4.2 Invariant Zeros

Invariant zeros are the values of s (and vectors) for which the following matrix becomes singular and therefore the determinant of the matrix is zero:
$$\det\left(\begin{bmatrix}s\mathbf{I}-\mathbf{A} & -\mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\right)\stackrel{!}{=}0$$
The corresponding input vectors can then be determined by plugging the invariant zeros into the equation below:
$$\begin{bmatrix}s\mathbf{I}-\mathbf{A} & -\mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{x}_i\\ \mathbf{u}_i\end{bmatrix}=\mathbf{0}$$
and then either solving the linear system.

Remarks
- An invariant zero z_i is associated with a vector
$$\begin{bmatrix}\mathbf{x_i}\\ \mathbf{u_i}\end{bmatrix}$$
the same way a pole is associated with an eigenvector.
- Given zero initial condition and excitation $\mathbf{u}(t)=\mathbf{u}_i e^{s_i t}$ (zero direction and zero frequency) we get
$$\mathbf{y}(t)=-\mathbf{C}e^{\mathbf{A}t}\mathbf{x_i}$$
which contains no components of the input and becomes $\mathbf{0}$ for a corresponding non-zero initial condition.
- For non-minimal realizations invariant zeros can give additional zeros (correspond to uncontrollable/unobservable modes).

8.5 Realizations of MIMO Systems

8.5.1 Naive Realization

Compute realization of each component of $\mathbf{G}(s)$ and assemble them to a new state space model.
$$\mathbf{A}=\begin{bmatrix}\mathbf{A}_{11} & & & \\ & \mathbf{A}_{12} & & \\ & & \ddots & \\ & & & \mathbf{A}_{lm}\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}\mathbf{b}_{11} & & & & \\ & \mathbf{b}_{12} & & & \\ & & \ddots & & \\ \mathbf{b}_{21} & & & & \\ & & \ddots & & \\ & & & & \mathbf{b}_{lm}\end{bmatrix}$$
$$\mathbf{C}=\begin{bmatrix}\mathbf{c}_{11} & \mathbf{c}_{12} & \cdots & & & & \\ & & & \mathbf{c}_{21} & \cdots & & \\ & & & & & \ddots & \\ & & & & \mathbf{c}_{l1} & \dots & \mathbf{c}_{lm}\end{bmatrix}$$
$$\mathbf{D}=\begin{bmatrix}\mathbf{d}_{11} & \dots & \mathbf{d}_{1m}\\ \vdots & \ddots & \vdots\\ \mathbf{d}_{l1} & \dots & \mathbf{d}_{lm}\end{bmatrix}$$

Problems
- non-minimal
- redundant poles (sum of entrie's poles)

8.5.2 Gilbert's Realization

Given
$$\mathbf{G}(s)=\frac{\mathbf{H}(s)}{d(s)}+\mathbf{D},\quad \mathbf{G}\in\mathbb{C}^{l\times m}$$
where
- $d(s)$ is the least common denominator of all entries in $\mathbf{G}(s)$
- $\mathbf{D}=\lim_{s\to\infty}\mathbf{G}(s)$ is the feed-through term.

1. Calculate $\mathbf{D}=\lim_{s\to\infty}\mathbf{G}(s)$
2. Calculate the **poles** p_i of the system by determining the roots of $d(s)$.
   - If $d(s)$ has **no repeated roots**, one can use Gilbert's method (otherwise the generalized Gilbert's method has to be used).
3. Perform a (matrix) partial fraction expansion of $\mathbf{G}(s)$
$$\mathbf{G}(s)=\frac{\mathbf{R_1}}{s-p_1}+\frac{\mathbf{R_2}}{s-p_2}+\ldots+\frac{\mathbf{R_{n_d}}}{s-p_{n_d}}+\mathbf{D}$$
with the residues
$$\mathbf{R_i}=\lim_{s\to p_i}(s-p_i)\mathbf{G}(s)$$
4. Calculate the ranks $r_i=\operatorname{rank}(\mathbf{R_i})$ which indicate the number of poles at location p_i that are needed.
   - The order of the resulting state space model will be $n=\sum_{i=1}^{n_d}r_i\geq n_d$
   - Reminder: The row rank and column rank of a matrix are **always equal**!
5. Now the $\mathbf{A}$ matrix can be assembled
$$\mathbf{A}=\begin{bmatrix}p_1\mathbf{I}_{r_1\times r_1} & & & \\ & p_2\mathbf{I}_{r_2\times r_2} & & \\ & & \ddots & \\ & & & p_{n_d}\mathbf{I}_{r_{n_d}\times r_{n_d}}\end{bmatrix}$$
6. The $\mathbf{B}_i$ and $\mathbf{C}_i$ components can be constructed from the following relation
$$\overbrace{\mathbf{R_i}}^{l\times m}=\overbrace{\mathbf{C}_i}^{l\times r_i}\overbrace{\mathbf{B}_i}^{r_i\times m}$$
so that
   - $\mathbf{B_i}$ has m (# inputs) columns
   - $\mathbf{C_i}$ has l (# outputs) rows
   - $\mathbf{B_i}$ and $\mathbf{C_i}$ have r_i independent rows/columns
7. Assemble the remaining state space matrices
$$\mathbf{B}=\begin{bmatrix}\mathbf{B}_1\\ \mathbf{B}_2\\ \vdots\\ \mathbf{B}_{n_d}\end{bmatrix},\quad \mathbf{C}=\begin{bmatrix}\mathbf{C_1} & \mathbf{C_2} & \dots & \mathbf{C}_{n_d}\end{bmatrix},\quad \mathbf{D}=\mathbf{D}$$

Remarks
- Figures out the minimum number of "copies" of each pole that we need to construct a realization of a MIMO transfer function.
- $\mathbf{A}$ simply contains diagonal matrices of dimension $r_i\times r_i$ with p_i on their diagonals.
- Transmission zeros and invariant zeros are equal in this (minimal) realization.

8.6 Singular Value Decomposition (SVD)

Any(!) matrix $\mathbf{A}\in\mathbb{C}^{m\times n}$ can be decomposed as
$$\mathbf{A}=\mathbf{U}\Sigma\mathbf{V^H}\quad\begin{cases}\mathbf{U}\in\mathbb{C}^{m\times m} & \text{left singular vectors}\\ \Sigma\in\mathbb{C}^{m\times n} & \text{singular values }\sigma_i\text{, desc. order}\\ \mathbf{V}\in\mathbb{C}^{n\times n} & \text{right singular vectors}\end{cases}$$
where $\mathbf{U},\mathbf{V}$ are unitary, $\mathbf{\Sigma}$ is diagonal.

2 × 3 Example
$$\mathbf{A}=\underbrace{\begin{bmatrix}| & |\\ \mathbf{u}_1 & \mathbf{u}_2\\ | & |\end{bmatrix}}_{\text{Rotation}}\underbrace{\begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\end{bmatrix}}_{\text{Scaling \& Dimensions}}\underbrace{\begin{bmatrix}— & \mathbf{v_1}^H & —\\ — & \mathbf{v_2}^H & —\\ — & \mathbf{v_3}^H & —\end{bmatrix}}_{\text{Rotation}}$$
with eigenvalues
$$\lambda_{u_1}=\lambda_{v_1}=\sigma_1{}^2$$
$$\lambda_{u_2}=\lambda_{v_2}=\sigma_2{}^2$$
$$\lambda_{v_3}=0$$

Remarks
- See Appendix: 13.1.1 for detailed understanding.
- $\mathbf{U},\mathbf{V}$ are unitary i.e. $\mathbf{U}^H\mathbf{U}=\mathbf{U}\mathbf{U}^H=\mathbf{I}$.
- For real $\mathbf{A}$, $\mathbf{A}^H\mathbf{A}$ and $\mathbf{A}\mathbf{A}^H$ are **symmetric**.
- Unitary implies positive semi-definite and real, non-negative eigenvalues.
- (Upper- and lower-) diagonal matrices have their singular values (or eigenvalues) on the main diagonal.

8.6.1 Procedure

Assuming $\mathbf{A}$ has rank $=\min(m,n)$:
1. calculate the eigenvalues
(use smaller matrix, pad with 0 up to $\max(m,n)$)
$$\lambda_{\mathbf{V}}\in\mathbb{R}^n:\qquad \det\left(\lambda\mathbf{I}-\mathbf{A^HA}\right)\stackrel{!}{=}0$$
or
$$\lambda_{\mathbf{U}}\in\mathbb{R}^m:\qquad \det\left(\lambda\mathbf{I}-\mathbf{AA^H}\right)\stackrel{!}{=}0$$
2. calculate the singular values σ and compose $\Sigma\in\mathbb{R}^{m\times n}$. Order the singular values σ in descending order.
$$\sigma_i=\sqrt{\lambda_{u_i}}=\sqrt{\lambda_{v_i}}\quad i=0,1,\cdots,\min(m,n)$$
$$\Sigma=\begin{bmatrix}\sigma_1 & \cdots & 0 & \mathbf{0}\\ \vdots & \ddots & \vdots & \mathbf{0}\\ 0 & \cdots & \sigma_m & \mathbf{0}\end{bmatrix}\text{ or }\begin{bmatrix}\sigma_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \sigma_n\\ \mathbf{0} & \cdots & \mathbf{0}\end{bmatrix}$$
3. calculate the eigenvectors of $\mathbf{A^HA}$ (or start with 4. if $m<n$). Use the same order as before and normalize the vectors if necessary.
$$\mathbf{A^HA}\mathbf{v}_i=\lambda_i\mathbf{v}_i$$
$$\left(\lambda_i\mathbf{I}-\mathbf{A^HA}\right)\mathbf{v}_i\stackrel{!}{=}\mathbf{0}$$
or if $m<n$ use for the $\sigma_i\neq 0$
$$\mathbf{v}_i=\sigma_i\mathbf{A}^H\mathbf{u}_i$$
then assemble $\mathbf{V}$
$$\mathbf{V}=\begin{bmatrix}| & | & \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots\\ | & | & \end{bmatrix}$$
4. calculate eigenvectors of $\mathbf{AA^H}$ either with
$$\mathbf{AA^H}\mathbf{u}_i=\lambda_i\mathbf{u}_i$$
$$\left(\lambda_i\mathbf{I}-\mathbf{AA^H}\right)\mathbf{u}_i\stackrel{!}{=}\mathbf{0}$$
or with the formula below (this can only be used for the singular values $\sigma_i\neq 0$)
$$\mathbf{u}_i=\frac{1}{\sigma_i}\mathbf{A}\mathbf{v}_i$$
or for $\sigma_i=0$ solve the linear system
$$\mathbf{AA^H}\mathbf{u}_i=\mathbf{0}$$
then assemble $\mathbf{U}$
$$\mathbf{U}=\begin{bmatrix}| & | & \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots\\ | & | & \end{bmatrix}$$
5.
$$\mathbf{A}=\mathbf{U}\Sigma\mathbf{V^H}$$

Remarks
- Write out the dimensions of $\mathbf{U}$, Σ and $\mathbf{V}$ directly
- Each column of $\mathbf{U}$ and $\mathbf{V}$ must be normalized (unitary!).
- Hence, one can normalize the $\mathbf{v}_i$ separately (same for $\mathbf{u}_i$).

8.6.2 Interpetation

When looking at the columns of $\mathbf{A}$ separately
$$\mathbf{A}\mathbf{v}_i=\sigma_i\mathbf{u}_i$$
shows that
- a unit input in direction $\mathbf{v}_i$ results in an output in direction $\mathbf{u}_i$ with amplification σ_i.
- in contrast to multiplying $\mathbf{A}$ by a general vector, in case of a singular vector one can exactly predict the output direction and magnitude.
- SVD can be seen as generalized eigenvalues/-vectors.

8.6.3 Zeros and Poles via SVD

For a MIMO transfer function $\mathbf{A}(s)$, the directions of poles and zeros can be found with SVD:
1. Set $s=z$ or $s=p+0.0001$
2. Calculate SVD of $\mathbf{A}(s)$
3. Evaluate singular values:
   (a) The **input zero direction** is the column $\mathbf{v_i}$ with corresponding $\sigma_i=0$.
   (b) The output zero direction is the column $\mathbf{u_i}$ with corresponding $\sigma_i=0$.
   (c) The **input pole direction** is the column $\mathbf{v_i}$ with largest σ_i.
   (d) The output pole direction is the column $\mathbf{u_i}$ with largest σ_i.

Condition Number
$$\kappa(\mathbf{A})=\frac{\sigma_{\max}(\mathbf{A})}{\sigma_{\min}(\mathbf{A})}$$

Remarks:
- A high condition number suggests that the system is *strongly directional* i.e. *ill-conditioned* and thus difficult to control.
- Often, considering only the *strong* singular values is sufficient.

8.7 Norms

- Input vectors with same but permuted entries can generate wildly different outputs due to directionality.
- Directionality depends on frequency as well.
- Norms are a way to measure "sizes" in order to compare effects in different directions.

Properties
$$\|a\mathbf{x}\|=|a|\,\|\mathbf{x}\|,\quad\forall a\in\mathbb{R},\mathbf{x}\in V\qquad\text{(homogeneity)}$$

$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality)

$\|\mathbf{x}\| > 0$ for $\mathbf{x} \neq 0$, $\|\mathbf{x}\| = 0 \leftrightarrow \mathbf{x} = 0$ (positivity)

Example Norms

- Euclidean norm: $\|\mathbf{x}\| = \sqrt{\mathbf{x}^\mathsf{H}\mathbf{x}}$
- For Hermitian, pos. definite matrices: $\|\mathbf{x}\| = \sqrt{\mathbf{x}^\mathsf{H}\mathbf{Q}\mathbf{x}}$ is a norm
- p-norm: $\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ with the important cases

$$\|\mathbf{x}\|_1 = \sum_1^n |x_i|\,,$$

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2} = \sqrt{\mathbf{x}^\mathsf{H}\mathbf{x}}$$

$$\|\mathbf{x}\|_\infty = \max_i |x_i|\,.$$

8.7.1 Matrix Norms

A $m \times n$ complex matrix can be seen as an operator between the vector spaces $\mathbb{C}^n$ and $\mathbb{C}^m$ i.e.

$$\mathbf{A}^{m\times n} : \mathbb{C}^n \to \mathbb{C}^m, \mathbf{x} \mapsto \mathbf{A}x$$

If we then provide the vector spaces with a norm, a **matrix norm is induced** by the norms of the vector spaces it operates on.

$$\|\mathbf{A}\|_p := \sup_{\mathbf{x}\neq 0} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p} = \max_{\|\mathbf{x}\|_p=1} \|\mathbf{A}\mathbf{x}\|_p$$

which means that the induced p-norm of $\mathbf{A}$ measures how much multiplication by $\mathbf{A}$ amplifies the p-norm of a vector. In other words, $\|\mathbf{A}\|_p$ is the **gain** of the operator $\mathbf{A}$. The p-norm finds the maximum possible amplification of a vector with length 1.

Properties of Induced Norms

As for vectors, homogeneity, triangle inequality and positivity hold for matrix norms too. Additionally we have:

$$\|\mathbf{A}\mathbf{x}\|_p \leq \|a\|_p \|\mathbf{x}\|_p$$

and the submultiplicative property:

$$\|\mathbf{A}\mathbf{B}\|_p \leq \|\mathbf{A}\|_p\|\mathbf{B}\|_p$$
$$\|\mathbf{A}\mathbf{B}\mathbf{x}\|_p \leq \|\mathbf{A}\|_p\|\mathbf{B}\mathbf{x}\|_p \leq \|\mathbf{A}\|_p\|\mathbf{B}\|_p\|\mathbf{x}\|_p$$
$$\frac{\|\mathbf{A}\mathbf{B}\mathbf{x}\|_p}{\|\mathbf{x}\|_p} \leq \|\mathbf{A}\|_p\|\mathbf{B}\|_p$$

8.7.1.1 Specific Norms

Induced p-norms

For the $p=1$ norm we get the **max column-sum**:

$$\|\mathbf{A}\|_1 = \max_j \sum_{i=1}^m |a_{ij}|, \qquad x_k = \begin{cases} \pm 1 & k=j \\ 0 & \text{else}\end{cases}$$

For the $p=2$ norm we get the **maximal amplification**:

$$\|\mathbf{A}\|_{2,\text{ind}} := \sup_{x\neq 0} \frac{\|\mathbf{A}\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \sigma_{\max}(\mathbf{A}), \qquad \mathbf{x} = \mathbf{v}_1$$

and **minimal amplification** (assuming rank($\mathbf{A}$)=n):

$$\inf_{x\neq 0} \frac{\|\mathbf{A}\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \sigma_n(\mathbf{A}), \qquad \mathbf{x} = \mathbf{v}_n$$

For the $p=\infty$ norm we get the **max row-sum**:

$$\|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^n |a_{ij}| \qquad x_k = \pm 1\ \forall k$$

Frobenius Norm

$$\|\mathbf{A}\|_F := \left(\sum_{i=1}^m\sum_{j=1}^n |a_{ij}|^2\right)^{1/2} = \text{Trace}(\mathbf{A}^\mathsf{H}\mathbf{A})^{1/2}$$
$$= \left(\sum_{i=1}^r \sigma_i(\mathbf{A})^2\right)^{1/2}$$

- non-induced
- has submultiplicative property even though not induced

Remarks: x_k describes the elements of the vector $\mathbf{x}$ for which the norm-specific maximum amplification happens.

8.7.2 Signal Norms

Signals are functions and functions are vectors.

$$w : \mathbb{T} \to \mathbb{R}^n, \quad w(t) = (w_1(t), w_2(t), \dots w_n(t))$$

p-norms

1-norm: Action

$$\|w\|_1 = \begin{cases} \sum_{k\in\mathbb{Z}} \|w[k]\|_1 & \text{(DT)} \\ \int_{-\infty}^{\infty} \|w(t)\|_1 dt & \text{(CT)}\end{cases}$$

2-norm: Square of the Energy

$$\|w\|_2^2 = \begin{cases} \sum_{k\in\mathbb{Z}} w[k]^\mathsf{H} w[k] = \sum_{k\in\mathbb{Z}} \|w[k]\|_2^2 & \text{(DT)} \\ \int_{-\infty}^{\infty} w(t)^\mathsf{H} w(t) dt = \int_{-\infty}^{\infty} \|w(t)\|_2^2 dt & \text{(CT)}\end{cases}$$

∞-norm: Peak Magnitude

$$\|w\|_\infty = \sup_{t\in\mathbb{T}} \|w(t)\|_\infty = \sup_{t\in\mathbb{T}} \max_{i=1\dots n} |w_i(t)|$$

where the absolute value of the signal must be taken because the maximum could be negative.

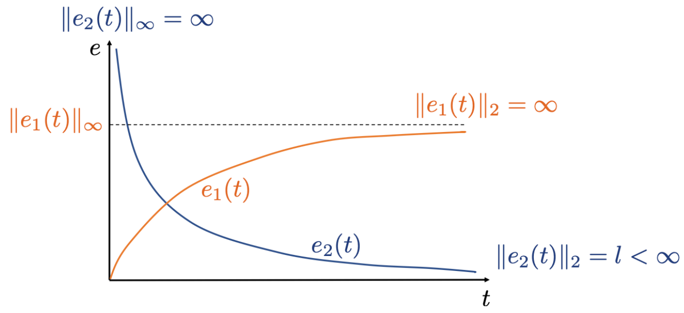
Power (not a norm)

$$\text{pow}(w)^2 = \begin{cases} \lim_{N\to+\infty} \frac{1}{2N} \sum_{k=-N}^{N} \|w[k]\|_2^2 & \text{(DT)} \\ \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \|w(t)\|_2^2 dt & \text{(CT)}\end{cases}$$

8.7.2.1 Relationships

$$\|w\|_2 < \infty \Rightarrow \text{pow}(w) = 0$$
$$\text{DT}: \quad \|w\|_2 < \infty \Rightarrow \|w\|_\infty < \infty$$
$$\text{CT}: \quad \|w\|_2 < \infty \not\Rightarrow \|w\|_\infty < \infty$$
$$(\text{pow}(w) < \infty) \wedge (\|w\|_\infty < \infty) \Rightarrow \text{pow}(w) < \|w\|_\infty^2$$
$$(\|w\|_1 < \infty) \wedge (\|w\|_\infty < \infty) \Rightarrow \|w\|_2 < \sqrt{\|w\|_1\|w\|_\infty} < \infty$$

Example



which illustrates property 3 from above (in both directions).

8.7.3 System Norms

Similar to a matrix, a system maps an input u to an output y:

$$y = Su$$

The induced norm is therefore defined as

$$\|S\|_{p,\text{ind}} := \sup_{u\neq 0} \frac{\|Su\|_p}{\|u\|_p} = \sup_{u\neq 0} \frac{\|y\|_p}{\|u\|_p}$$

8.7.3.1 Terminology

The $\mathcal{L}_2$ norm

- is the *2-norm of a signal* in continuous time along the real (time) axis.

The l_2 norm

- is the discrete analogon of the $\mathcal{L}_2$ norm.

The $\mathcal{H}_\infty$ norm

- is the *2-norm of a complex function* along the imaginary axis.

8.7.3.2 Parseval's Identity

$$\underbrace{\|y\|_{\mathcal{L}_2}^2}_{\text{Time}} = \int_0^\infty \|y(t)\|_2^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \|Y(j\omega)\|_2^2 d\omega = \underbrace{\|Y(s)\|_{\mathcal{H}_2}^2}_{\text{Freq.}(s=j\omega)}$$

i.e. energy is conserved from time to frequency domain.

8.7.3.3 $\mathcal{H}_2$ norm

Properties

- Measure of system's energy (calculated in frequency domain)
- Average direction and frequency
- Not induced: Hence, the submultiplicative property does not hold.

Computation

The $\mathcal{H}_2$ norm of a system is given by

$$\|\mathbf{G}\|_{\mathcal{H}_2}^2 = \|g\|_{\mathcal{L}_2}^2 = \text{Tr}\left[\mathbf{C}\mathcal{R}\mathbf{C}^T\right] = \text{Tr}\left[\mathbf{B}^\mathbf{T}\mathcal{O}\mathbf{B}\right]$$

where the **symmetric** matrices $\mathcal{R}$ or $\mathcal{O}$ are computed by solving the Lyapunov (matrix) equation

$$\mathbf{A}\mathcal{R} + \mathcal{R}\mathbf{A}^T = -\mathbf{B}\mathbf{B}^T$$

or

$$\mathbf{A}^\mathbf{T}\mathcal{O} + \mathcal{O}\mathbf{A} = -\mathbf{C}^\mathbf{T}\mathbf{C}$$

respectively. Note that there are $m\times n$ equations to solve. $\mathcal{R}$ and $\mathcal{O}$ are called Reachability and Observability **Gramians** and can also be computed by:

$$\mathcal{R} := \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t}\, dt$$

$$\mathcal{O} := \int_0^\infty e^{\mathbf{A}^T t}\mathbf{C}^\mathbf{T}\mathbf{C}e^{\mathbf{A}t}\, dt$$

Significance

The $\mathcal{H}_2$-norm measures

1. The energy of the impulse response:

$$\|g\|_{\mathcal{L}_2}^2 := \|\mathbf{G}\|_{\mathcal{H}_2}^2$$

2. The energy of the response to initial conditions of the form $x(0) = \mathbf{B}u_0$ for $u_0 = (1, 1, \dots, 1)^\mathsf{T}$
3. The (expected) power of the response to white noise:

$$\mathbb{E}\left[\lim_{T\to+\infty}\frac{1}{T}\text{Tr}\left(\int_0^T y(t)y(t)^\mathsf{H} dt\right)\right] = \|g\|_{\mathcal{L}_2}^2 := \|\mathbf{G}\|_{\mathcal{H}_2}^2$$

Remarks:

- The LQR/LQE problem can be though as a $\mathcal{H}_2$ minimization problem.
- In order for $\|g\|_{\mathcal{L}_2}^2 := \|\mathbf{G}\|_{\mathcal{H}_2}^2 < \infty$, the system needs to be strictly causal, i.e. $\lim_{\omega\to\infty} \mathbf{G}(j\omega) = 0$
- Lyapunov equations are linear in the unknown matrices whereas Riccati equations have order 2 in the unknown matrices. MATLAB: `lyap`.
- Note the duality in the gramians as we had it for reachability and observability with A, B and A^H, C^H.
- The $\mathcal{H}_2$ norm is a quantity in frequency domain. Note that we found a method to compute it **in state space** by solving linear equations.
- The gramians are (if they are not singular) equivalent conditions to the reachability and observability matrices.

8.7.3.4 $\mathcal{H}_\infty$ norm

The $\mathcal{H}_\infty$ norm is induced by the $\mathcal{L}_2$ norm.

$$\sup_{u\neq 0} \frac{\|y\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}} = \sup_{\omega\in\mathbb{R}} \sigma_{\max}[G(j\omega)] =: \|G\|_{\mathcal{H}_\infty}$$

Remarks

- In words, the maximum amplification of the *signal* energy in time domain corresponds to the maximum singular value of the *system* it propagates through.
- In SISO systems, the $\mathcal{H}_\infty$ norm corresponds to the peak value of the Bode plot.
- Submultiplicative property for $\mathcal{H}_\infty$ holds (because induced).
- Note that the singular values are frequency-dependent i.e. can be shown in a Bode plot.

Computation

Again one wants to compute the norm in state space (instead of frequency domain), now by using **bisection**:

1. Check technical conditions: The system must be **stable** and **strictly causal** to use this criterium.
2. Define a lower and an upper bound for $\|\mathbf{G}\|_{\mathcal{H}_\infty}$ (0 is a good lower bound as it is the minimum for any norm)
3. Compute the **Hamiltonian matrix**

$$\mathbf{H}_\gamma = \begin{bmatrix} \mathbf{A} & \frac{1}{\gamma}\mathbf{B}\mathbf{B}^T \\ -\frac{1}{\gamma}\mathbf{C}^T\mathbf{C} & -\mathbf{A}^T \end{bmatrix}$$

   where γ is an estimate of $\|\mathbf{G}\|_{\mathcal{H}_\infty}$.
4. Does H_γ have eigenvalues on the imaginary axis?
   Yes: $\|\mathbf{G}\|_{\mathcal{H}_\infty} > \gamma$, choose larger gamma.
   No: $\|\mathbf{G}\|_{\mathcal{H}_\infty} < \gamma$, choose lower gamma.
5. Go back to 2. until enough precision is reached. The final γ is $\|\mathbf{G}\|_{\mathcal{H}_\infty}$

Remarks to Computation

- There is no easy analytical way to compute $\|\mathbf{G}\|_{\mathcal{H}_\infty}$

9 Stability and Performance Robustness

One main achievement of computing the $\mathcal{H}_\infty$ norm is to access stability and performance robustness and achieve robust control.

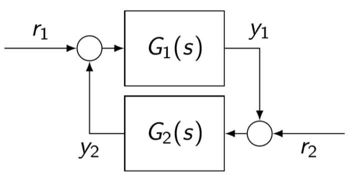
9.1 The Small Gain Theorem (SGT)

The (unstructured) SGT attempts to access interconnection stability. In contrast to the structured SGT, the SGT does not make any assumptions on the structure of the uncertainty matrix.

9.1.1 Sufficient Condition

The interconnection of two **stable** (MIMO) systems, with transfer functions $\mathbf{G_1}(s)$ and $\mathbf{G_2}(s)$, is stable if

$$\|\mathbf{G_1}\|_{\mathcal{H}_\infty} \cdot \|\mathbf{G_2}\|_{\mathcal{H}_\infty} < 1$$



Remarks

- There could be some systems that violate this condition but still lead to a stable feedback interconnection.
- **Any** norm fulfilling the submultiplicative property can be used for the SGT!
- The small-gain theorem also applies to non-linear, time-varying systems.
- To have sufficiency, $(\mathbf{I} - \mathbf{G_1}\mathbf{G_2})$ must not have zeros in the RHP.

9.1.2 Necessary and Sufficient Condition

Given a feedback interconnection of $\mathbf{G}$ and an unknown system Δ where

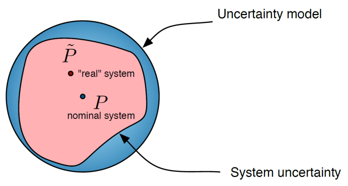
- Δ is stable
- and $\|\Delta\|_{\mathcal{H}_\infty} < 1$

the FB interconnection of $\mathbf{G}$ and Δ is **guaranteed** to be **stable iff**

$$\|\mathbf{G}\|_{\mathcal{H}_\infty} < 1$$

- Otherwise, one can find a Δ so that $(\mathbf{I}-\mathbf{G}\Delta)$ singular.
- Note that one only makes the conservative assumptions: Δ stable and $\|\Delta\|_{\mathcal{H}_\infty} < 1$
  - In fact, one can sometimes make more assumptions (see SSV).

9.2 Modelling Uncertainty



To take uncertainty into account, one first defines an **uncertainty model**, consisting of

1. A nominal model $\mathbf{P}$
2. A set of models that is guaranteed to contain the system uncertainty, and is easier to handle

and then designs a control system that meets the stability and performance specifications for all possible models in the *uncertainty model*.

Assumptions on Uncertainty

In the following uncertainty models one assumes that Δ

- is minimum-phase
- does not cancel unstable poles of the nominal system
- $|\Delta(j\omega)| < 1, \quad \forall\omega$

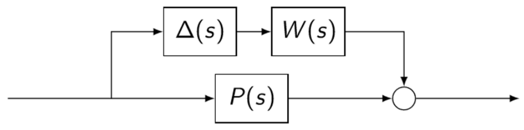
and that there is a high-pass transfer function W called **frequency weight** so that the total uncertainty is given by

$$\mathbf{W}(s)\Delta(s)$$

9.2.1 Common Uncertainty Models

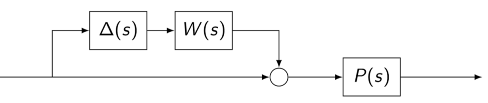
Additive Uncertainty

$$\tilde{\mathbf{P}}(s) = \mathbf{P}(s) + \mathbf{W}(s)\Delta(s)$$



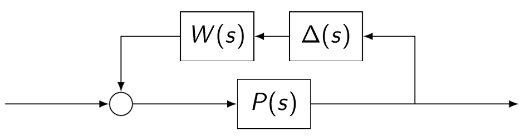
Multiplicative Uncertainty

$$\tilde{\mathbf{P}}(s) = \mathbf{P}(s)(1 + \mathbf{W}(s)\Delta(s))$$



Feedback Uncertainty

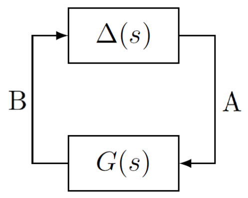
$$\tilde{\mathbf{P}}(s) = (\mathbf{I} - \mathbf{P}(s)\mathbf{W}(s)\Delta(s))^{-1}\mathbf{P}(s)$$



9.3 Robust Stability

- Equivalent to SISO bode plot obstacles (now obstacles are given by singular values and in time domain)
- Now we impose constraints on the $\mathcal{H}_\infty$ norm
- **Caution**: depending on the convention the block diagrams vary and some matrices in a specific $\mathbf{G}$ will be permuted to the forms shown below.

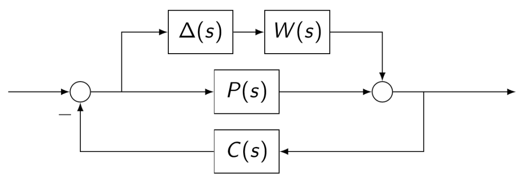
General Procedure



Given an uncertainty and a controller designed for the nominal plant $\mathbf{P}$ one computes the transfer function without the system $\mathbf{\Delta}$ and then applies the SGT. The transfer function $\mathbf{G}$ to be found is from the output of $\mathbf{\Delta}$ to it's input i.e. find:

$$\mathbf{G}_{BA} = \mathbf{B}\mathbf{A}^{-1}$$

Additive Uncertainty

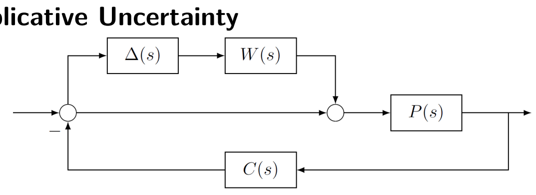


The CL system is robustly stable iff

$$\|\mathbf{G}(s)\|_{\mathcal{H}_\infty} = \|-(\mathbf{I}+\mathbf{CP})^{-1}\mathbf{CW}\|_{\mathcal{H}_\infty} < 1$$

Remember: this is a necessary and sufficient condition.

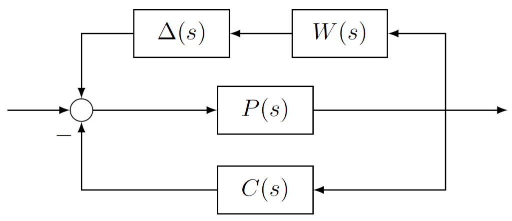
Multiplicative Uncertainty



The CL system is robustly stable iff

$$\|\mathbf{G}(s)\|_{\mathcal{H}_\infty} = \|-(\mathbf{I}+\mathbf{CP})^{-1}\mathbf{CPW}\|_{\mathcal{H}_\infty} < 1$$

Feedback Uncertainty



The CL system is robustly stable iff

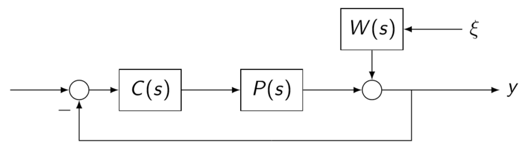
$$\|\mathbf{G}(s)\|_{\mathcal{H}_\infty} = \|\mathbf{W}(\mathbf{I}+\mathbf{PC})^{-1}\mathbf{P}\|_{\mathcal{H}_\infty} < 1$$

9.4 Robust Performance

The "robust performance" problem can be formulated in a way that matches the "robust stability" problem.

- Isolate the two signals between which amplification is unwanted.
- Apply the SGT to the rest of the system as done for robust control.

Disturbance at Output

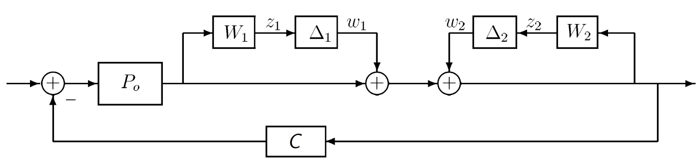


- We don't want the energy (induced 2-norm) of disturbance ξ to be amplified at the output.
- Therefore we want to limit the gain from ξ to y
  - Instead of unwanted amplification in the FB path we now look at $\xi \to y$.

Applying the SGT (assuming $\mathbf{\Delta} = \mathbf{G}_{y\xi}$) yields

$$\|\mathbf{G}_{y\xi}\|_{\mathcal{H}_\infty} = \|(\mathbf{I}+\mathbf{PC})^{-1}\mathbf{W}\|_{\mathcal{H}_\infty} < 1$$

9.5 Robust Disturbance Rejection



We want to consider robust stability and robust performance **simultaneously**. To do so we need

- An **uncertainty block**
- A **transfer function matrix** describing all uncertainty input-output relations

We model an uncertainty block by

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}}_{\mathbf{\Delta}} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad \begin{matrix} \text{stability} \\ \text{robustness} \end{matrix}$$

And the transfer function matrix for the SGT by

$$\begin{aligned} \mathbf{M} &= \begin{bmatrix} G_{z_1 w_1} & G_{z_1 w_2} \\ G_{z_2 w_1} & G_{z_2 w_2} \end{bmatrix} \\ &= \begin{bmatrix} -\mathbf{W}_1\mathbf{P}_0\mathbf{K}(\mathbf{I}+\mathbf{P}_0\mathbf{K})^{-1} & -\mathbf{W}_1\mathbf{P}_0\mathbf{K}(\mathbf{I}+\mathbf{P}_0\mathbf{K})^{-1} \\ \mathbf{W}_2(\mathbf{I}+\mathbf{P}_0\mathbf{K})^{-1} & \mathbf{W}_2(\mathbf{I}+\mathbf{P}_0\mathbf{K})^{-1} \end{bmatrix} \end{aligned}$$

9.5.1 Robustness Assessment

Robustness can now be accessed by applying the SGT to $\mathbf{M}$

$$\|\mathbf{M}\|_{\mathcal{H}_\infty} < 1$$

to achieve robust **disturbance rejection**.

Remarks

- Note that we ignored the diagonal structure of $\mathbf{\Delta}$ which makes our assumptions too conservative (see SSV for less conservatism).
- Also note that $\mathbf{M}$ has rank 1. In this case one can calculate the SSV exactly (see below).

9.6 Structured Singular Value (SSV)

The condition from the *unstructured SGT* is a conservative assumption as the SGT could be applied to an arbitrary $\mathbf{\Delta}$. For example, as the $\mathbf{\Delta}$ from 9.5 has block-diagonal structure, less conservative robustness conditions could be applied.

Definition of SSV

The SSV is defined with respect to a **class of perturbations** $\mathbb{D}$ as the inverse value of the smallest $\sigma_{\max}$ making $\mathbf{M}$ singular:

$$\mu(\mathbf{M}) := \frac{1}{\inf\{\sigma_{\max}(\mathbf{\Delta}) : \det(1 - M\mathbf{\Delta}) = 0\}}, \qquad \mathbf{\Delta} \in \mathbb{D}$$

- Convention: If $\det(I - M\mathbf{\Delta}) \neq 0, \forall \mathbf{\Delta} \in \mathbb{D}$, then $\mu(M) = 0$.
- If $\mathbf{\Delta}$ is diagonal and $\mathbf{M}$ has rank one the SSV can be computed analytically.

Properties of the SSV

- $\mu(\mathbf{M}) \geq 0$
- If $\mathbb{D}$ is arbitrary: $\mu(\mathbf{M}) = \|\mathbf{M}\|_{\mathcal{H}_\infty}$ (unstructured case)
- If $\mathbb{D} = \{\lambda I : \lambda \in \mathbb{C}\}$: $\mu(\mathbf{M}) = \rho(\mathbf{M})$ (spectral radius = largest eigenvalue, easy to compute)
- If $\mathbb{D}$ is the set of diagonal (complex) matrices then

  $$\mu(\mathbf{M}) = \mu(\mathbf{D}^{-1}\mathbf{M}\mathbf{D})$$

  for any invertible diagonal (complex) scaling matrix $\mathbf{D}$ and

  $$\rho(\mathbf{M}) \leq \mu(\mathbf{M}) \leq \inf_{\mathbf{D}} \sigma_{\max}(\mathbf{D}^{-1}\mathbf{M}\mathbf{D}) \leq \sigma_{\max}(\mathbf{M})$$

9.6.1 Accessing Stability Using SSV

Stability Condition

The $\mathbf{M}-\mathbf{\Delta}$ FB system is stable for all $\mathbf{\Delta} \in \mathbb{D}$, $\|\mathbf{\Delta}\|_{\mathcal{H}_\infty} < 1$ iff

$$\mu(\mathbf{M}(j\omega)) \leq 1, \quad \forall\omega \in \mathbb{R}$$

which is less conservative than the SGT.

9.6.1.1 Rank-One Problems

In general, there is no closed-form method for computing the SSV (must resort to, e.g., estimating upper bounds (e.g., by scaling matrices)).

However, when

- $\mathbf{\Delta}$ is a diagonal (complex) matrix,
- and $\mathbf{M}$ is a rank 1 matrix, i.e.,

  $$\mathbf{M} = \mathbf{a}\mathbf{b}^{\mathbf{H}}$$

  for some complex vectors $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$

then one can compute the structured singular value exactly. One has

$$\det(\mathbf{I} - \mathbf{M}\mathbf{\Delta}) = 1 - \sum_i \mathbf{a}_i \mathbf{b}_i^H \delta_i$$

and since $\sigma_{\max}(\mathbf{\Delta}) = \max_i |\delta_i|$ one has

$$\mu(\mathbf{M}) = \sum_i |\mathbf{a}_i \mathbf{b}_i^H|$$

Remarks

- Given the mentioned conditions one can express $\mathbf{M}$ as the product of two vectors.
- One can then easily calculate the SSV from these vectors (instead of from $\mathbf{M}$).

9.6.1.2 Example: Scalar Combined Robustness

The aforementioned conditions are fulfilled as $\mathbf{\Delta}$ diagonal and $\mathbf{M}$ has rank 1. We have

$$\mathbf{M} = \mathbf{a}\mathbf{b}^{\mathbf{H}} = \begin{bmatrix} -\frac{\mathbf{W}_1\mathbf{P}_0\mathbf{K}}{1+\mathbf{P}_0\mathbf{K}} \\ \frac{\mathbf{W}_2}{1+\mathbf{P}_0\mathbf{K}} \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$$

which yields

$$\begin{aligned} \mu(\mathbf{M}(j\omega)) &= \left|\frac{\mathbf{W}_1\mathbf{P}_0\mathbf{K}}{1+\mathbf{P}_0\mathbf{K}}(j\omega)\right| + \left|\frac{\mathbf{W}_2}{1+\mathbf{P}_0\mathbf{K}}(j\omega)\right| \leq 1 \\ &= |\mathbf{W}_1\mathbf{L}(j\omega)| + |\mathbf{W}_2(j\omega)| \leq |1 + \mathbf{L}(j\omega)| \quad \forall\omega \in R \end{aligned}$$

where typically

- $\mathbf{L} = \mathbf{P}_0\mathbf{K}$
- $\mathbf{W}_1$ is a high-pass
- $\mathbf{W}_2$ is a low-pass
- $|\mathbf{L}|$ is a low-pass (low-pass assumption for physical systems)

Therefore, one has for low frequencies

$$|\mathbf{W}_1(j\omega)| + \frac{|\mathbf{W}_2(j\omega)|}{|\mathbf{L}(j\omega)|} \leq 1$$

$$|\mathbf{L}(j\omega)| \geq \frac{|\mathbf{W}_2(j\omega)|}{1 - |\mathbf{W}_1(j\omega)|} \approx |\mathbf{W}_2(j\omega)|$$

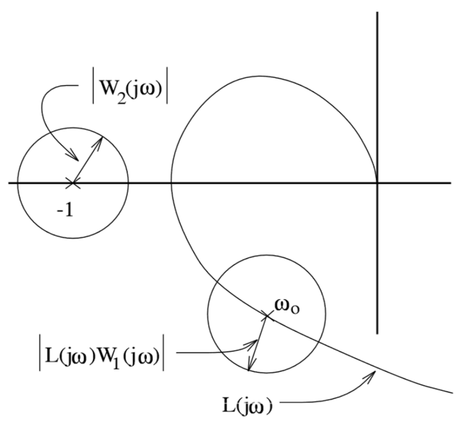
and for high frequencies

$$|\mathbf{W}_1(j\omega)||\mathbf{L}(j\omega)| + |\mathbf{W}_2(j\omega)| \leq 1$$

$$|\mathbf{L}(j\omega)| \leq \frac{1 - |\mathbf{W}_2(j\omega)|}{|\mathbf{W}_1(j\omega)|} \approx \frac{1}{|\mathbf{W}_1(j\omega)|}$$

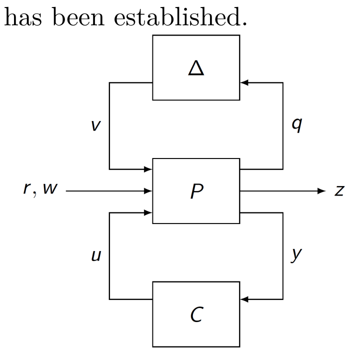
which corresponds to the CS1 Bode obstacles.

One can visualize the condition by an advanced Nyquist criterion.



10 Modern Controller Synthesis

To unify the controller synthesis the following standard form has been established.

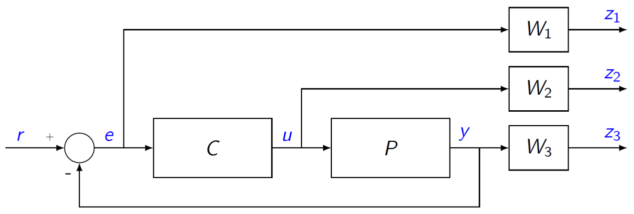


u: control input
y: measured output
r: reference signal
w: disturbance
z: performance output
q: uncertainty block input
v: uncertainty block output

The objective is to stabilize the system in presence of $\mathbf{\Delta}$ and minimize the performance outputs z, given the exogenous reference and disturbances.

10.1 Frequency Weights

To guide the optimization process the performance outputs z_i are weighted with frequency dependent weighting functions W_i.
The most common performance outputs are



with the known transfer functions

$$\begin{aligned} L(s) &= P(s)C(s) && \text{loop TF}\\ S(s) &= G_{er}(s) = (I + L(s))^{-1} && \text{sensitivity}\\ T(s) &= G_{yr}(s)\\ &= (I + L(s))^{-1}L(s) = I - S(s) && \text{comp. sensitivity} \end{aligned}$$

Tracking Error
Is given by

$$z_1 = W_1 G_{er} = W_1 S(s)$$

- If $W_1(s)$ is chosen large at low frequencies $S(s)$ must be small in order to minimize z_1. In other words, the tracking error must be small at low frequencies.
- This is the same as requiring $\sigma_{\min}[L(s)] \gg |W_1(s)|$, which corresponds to the low-frequency "Bode obstacle" in the SISO case.

Control Effort
Is given by

$$z_2 = W_2 G_{ur}(s) = W_2 C(s) S(s)$$

- Useful to limit the maximum control effort (large W_2 yields small u)

Noise Rejection
Is given by

$$z_3 = W_3 G_{yn} = -W_3 T(s)$$

- If W_3 is chosen very large for high frequencies, the complementary sensitivity must be very small at high frequencies.
- This is the same as requiring $\sigma_{\max}[L(s)] \ll |W_3(s)|$, which corresponds to the high-frequency "Bode obstacle" in the SISO case.

Stability Robustness
For stability robustness in the presence of uncertainty Δ the same weighting method can be applied.

$$z_\Delta = \Delta(s) W_\Delta(s)$$

with

$$\begin{aligned} \|\Delta(s)\|_{\mathcal{H}_\infty} &< 1\\ \|M(s)\|_{\mathcal{H}_\infty} &< 1 \end{aligned}$$

where $M(s)$ is the TF from the output of Δ to its input.

10.1.1 First Order Weights

Weights with magnitude m can be achieved by:

Lowpass

$$W(s) = \frac{m}{\frac{ms}{\omega} + 1}$$

Highpass

$$W(s) = \frac{ms}{s + m\omega}$$

MATLAB: `makeweight`

10.2 State Space Representation

To synthesize a controller an assembled state space model of the generalized system (G) can be constructed. This yields a similar approach as the transfer function-based one before.

Plant

$$P : \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right]$$

$$P(s) = C_p(sI - A_p)^{-1}B_p + D_p$$

MATLAB: `P=ss(Ap, Bp, Cp, Dp)`

Frequency Weights
For each weight we get

$$W : \left[\begin{array}{c|c} -p & 1 \\ \hline r - qp & q \end{array}\right]$$

$$W(s) = \frac{qs + r}{s + p}$$

Generalized System
The generalized system is obtained by stacking all the state-space models:

$$G : \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{array}\right]$$

$$G(s) = \begin{bmatrix} G_{zw}(s) & G_{zu}(s) \\ G_{yw}(s) & G_{yu}(s) \end{bmatrix}$$

The matrix A contains both the dynamics of the plant and the frequency weights.

MATLAB: `augw(P, W1, W2, W3)`

Controller

$$K : \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right]$$

$$K(s) = C_c(sI - A_c)^{-1}B_c + D_c$$

As $u(s) = K(s)y(s)$ the closed loop TF from the exogenous inputs w to the performance outputs z is given by the *Linear Fractional Transformation*:

$$F(s) = G_{zw}(s) + G_{zu}(s)K(s)(I - G_{yu}(s)K(s))^{-1}G_{yw}(s)$$

MATLAB: `F = lft(G, K)`

10.2.1 Controller Synthesis

Given F one can use various control synthesis techniques, e.g.

$\mathcal{H}_2$ Design
Minimize

$$\|F\|_2 := \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{Tr}[F(j\omega)^* F(j\omega)]\, \mathrm{d}\omega\right)^{1/2}$$

i.e.

$$\int_0^{+\infty} \|f(t)\|_2^2 dt$$

$\mathcal{H}_\infty$ Design
Minimize

$$\|F\|_\infty := \sup_\omega \sigma_{\max}[F(j\omega)]$$

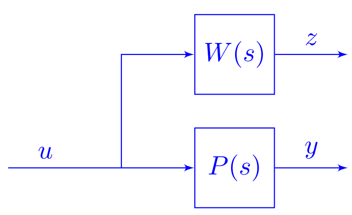
i.e.

$$\int_0^{+\infty} \|z(t)\|_2^2 dt$$

for unit-energy inputs i.e. $\int_0^{+\infty} \|w(t)\|_2^2 dt = 1$

10.2.1.1 Example Assembly

For the system



the generalized system $G(s)$ has to be calculated

$$\begin{bmatrix} y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} G_{yu}(s) \\ G_{zu}(s) \end{bmatrix}}_{G(s)} u$$

The combined states evolve with

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_w \end{bmatrix} = \underbrace{\begin{bmatrix} A_p & 0 \\ 0 & A_w \end{bmatrix}}_{A_G} \begin{bmatrix} x_p \\ x_w \end{bmatrix} + \underbrace{\begin{bmatrix} B_p \\ B_w \end{bmatrix}}_{B_G} u$$

and the combined output is given by

$$\begin{bmatrix} y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} C_p & 0 \\ 0 & C_w \end{bmatrix}}_{C_G} \begin{bmatrix} x_p \\ x_w \end{bmatrix} + \underbrace{\begin{bmatrix} D_p \\ D_w \end{bmatrix}}_{D_G} u$$

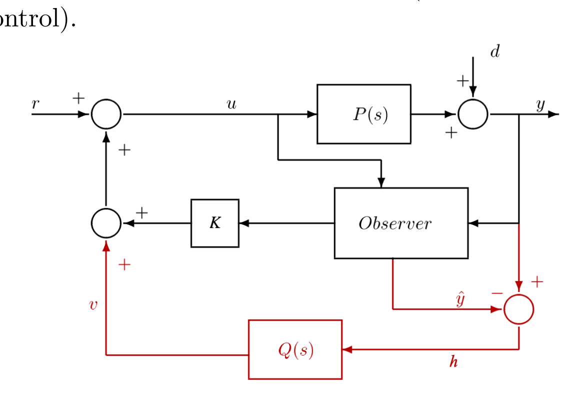
The generalized system representation is then given by

$$G = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right] = \left[\begin{array}{cc|c} A_p & 0 & B_p \\ 0 & A_w & B_w \\ \hline C_p & 0 & D_p \\ 0 & C_w & D_w \end{array}\right]$$

10.3 Youla's Q Parameterization

One can get all stabilizing controllers for a given plant as a function of a single stable transfer function Q(s). This is called the Youla parameterization (Q-parameterization).

- Extension of a full state feedback controller (including observer) with a system Q that takes the innovation as an input.
- Valid in a very general setting, including MIMO
- Q needs to be **stable** and has to consist of a **proper rational** transfer function. Reminder:
  - Proper: $\deg(Den) \leq \deg(Num)$
  - Rational: fraction of two polynomials
- Note that positive feedback is used (common in modern control).



Assuming a controllable and observable plant, the closed-loop dynamics are given by

$$\begin{bmatrix} \dot{x} \\ \dot{x}_Q \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A + BK & BC_Q & -BK \\ 0 & A_Q & B_Q C \\ 0 & 0 & A + LC \end{bmatrix} \begin{bmatrix} x \\ x_Q \\ \eta \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & C \\ 0 & C \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix}$$

The closed-loop poles are the union of the full-state feedback poles $A + BK$, the observer poles $A + LC$ and the poles of $Q(s)$.

10.3.1 SISO Controller Design

The TF from y to u hence the TF of the controller $C(s)$ can be written as

$$C(s) = -(X_0(s) + Q(s)N(s))^{-1}(Y_0(s) - Q(s)D(s))$$

where

$$C_0(s) = -X_0(s)^{-1}Y_0(s)$$

is the TF when $Q(s) = 0$ with controller gain K_0 and observer gain L_0

$$\begin{aligned} X_0(s) &= -K_0(sI - A - L_0C)^{-1}B\\ Y_0(s) &= -K_0(sI - A - L_0C)^{-1}L_0 \end{aligned}$$

and N, D are the nominator and denominator of the plant

$$\begin{aligned} N(s) &= C(sI - A - L_0C)^{-1}B\\ D(s) &= -C(sI - A - L_0 - C^{-1}L_0) \end{aligned}$$

where L_0 is an observer gain that would stabilize the error dynamics.

10.3.1.1 Finding a Stabilizing Controller

It can be shown that the interconnection of the system is stable if the **Bezout identity** is fulfilled

$$D(s)X(s) - N(s)Y(s) = 1$$

and that all feedback stabilizing controllers for a **SISO**(!) system P are given by

$$C(s) = \frac{Y_0(s) - D(s)Q(s)}{X_0(s) - N(s)Q(s)}$$

One can then

1. Find a default "dummy" controller $C_0(s)$.
2. Plug it into $C(s)$.
3. Use optimization algorithms to find a better controller $C(s)$ by systematically iterating through many **stable** Q functions.

10.3.2 (Complementary) Sensitivity

The sensitivity function is given by

$$S(s) = D(s)(X_0(s) - N(S)Q(s))$$

and the complementary sensitivity function by

$$T(s) = -N(s)(Y_0(s) + D(s)Q(s))$$

10.4 $\mathcal{H}_2$ Synthesis

The dynamics of the standard setup for modern control synthesis are given by

$$\begin{aligned} \mathbf{x}(0) &= \mathbf{x}_0\\ \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}_w\mathbf{w}(t) + \mathbf{B}_u\mathbf{u}(t)\\ \mathbf{z}(t) &= \mathbf{C}_z\mathbf{x}(t) + \mathbf{D}_{zw}\mathbf{w}(t) + \mathbf{D}_{zu}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}_y\mathbf{x}(t) + \mathbf{D}_{yw}\mathbf{w}(t) + \mathbf{D}_{yu}\mathbf{u}(t) \end{aligned}$$

LQR, LQE, LQG are applied to special cases of the standard setup.

10.4.1 LQR

Problem Statement
In the LQR problem we assume

- Full state feedback: $\mathbf{C_y} = \mathbb{I}$
- No disturbance input: $w = 0$ ($\mathbf{B}_w = \mathbf{D}_{zw} = \mathbf{D}_{yw} = \mathbf{0}$)
- no cross-coupling term N: ($\mathbf{C}_z^\mathsf{T}\mathbf{D}_{zu} = 0$)
- i.e.

$$\begin{bmatrix} \mathbf{A} & 0 & \mathbf{B}_u \\ \begin{bmatrix}\sqrt{\mathbf{Q}} & 0\end{bmatrix}^\mathsf{T} & 0 & \begin{bmatrix}0 & \sqrt{\mathbf{R}}\end{bmatrix}^\mathsf{T} \\ \mathbb{I} & 0 & \mathbf{D}_{yu} \end{bmatrix}$$

and try to find a control signal $u(t, x)$ that minimizes

$$\|\mathbf{z}\|_2^2 = \int_0^{+\infty} \|\mathbf{C}_z\mathbf{x} + \mathbf{D}_{zu}\mathbf{u}\|_2^2 dt$$

where

$$\begin{aligned} \mathbf{C}_z &= \begin{bmatrix}\sqrt{\mathbf{Q}} & 0\end{bmatrix}^\mathsf{T} && \Leftrightarrow \mathbf{Q} = \mathbf{C}_z^\mathsf{T}\mathbf{C}_z\\ \mathbf{D}_{zu} &= \begin{bmatrix}0 & \sqrt{\mathbf{R}}\end{bmatrix}^\mathsf{T} && \Leftrightarrow \mathbf{R} = \mathbf{D}_{zu}^\mathsf{T}\mathbf{D}_{zu} \end{aligned}$$

which yields the known LQR problem

$$\|\mathbf{z}\|_2^2 = \int_0^{+\infty} (\mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} + \mathbf{u}^\mathsf{T}\mathbf{R}\mathbf{u})\, dt$$

Solving for the Controller Matrix
One can show that minimizing $\|\mathbf{z}\|_2^2$ imposes the following ARE on $\mathbf{X}_F$:

$$\mathbf{A}^\mathsf{T}\mathbf{X}_F + \mathbf{X}_F\mathbf{A} - \mathbf{X}_F\mathbf{B}_u\big(\underbrace{\mathbf{D}_{zu}^\mathsf{T}\mathbf{D}_{zu}}_{\mathbf{R}}\big)^{-1}\mathbf{B}_u^\mathsf{T}\mathbf{X}_F + \underbrace{\mathbf{C}_z^\mathsf{T}\mathbf{C}_z}_{Q} = \mathbf{0}$$

Solving this ARE yields the controller

$$\mathbf{F} = -\left(\mathbf{D}_{zu}^{\mathsf{T}}\mathbf{D}_{zu}\right)^{-1}\mathbf{B}_u^{\mathsf{T}}\mathbf{X}_F$$

where $\mathbf{X}_F$ is symmetric and positive semidefinite.

Technical Considerations

1. $(\mathbf{A},\mathbf{B}_u)$ stabilizable.
2. $(\mathbf{C}_z,\mathbf{A})$ detectable.
   This ensures that any unstable mode of $\mathbf{A}$ is detected by the performance output.
3. $\begin{bmatrix}\mathbf{A}-j\omega\mathbf{I} & \mathbf{B}_u\\ \mathbf{C}_z & \mathbf{D}_{zu}\end{bmatrix}$ has full column rank for all $\omega\in\mathbb{R}$
   Ensures that control effort is penalized at all ω so that the Hamiltonian does not have purely imaginary eigenvalues.
4. $\mathbf{D}_{zu}^{\mathsf{T}}\mathbf{D}_{zu}=\mathbf{R}$, invertible, i.e., $\mathbf{D}_{zu}$ has full column rank
   Is for convenience.

Remarks

- One can show that this control law is optimal to achieve the stated minimization.
- Technical considerations 1.-3. ensure that the Riccati equation admits a solution $\mathbf{Y}$ that is positive semi-definite.

10.4.2 LQE / Kalman Filter

Problem Statement

In the LQE problem we assume

- That $\mathbf{u}$ takes the role uf the observer update
- Full state updates: $\mathbf{B}_u=\mathbb{I}$ (u updates the states)
- Zero initial conditions: $\tilde{\mathbf{x}}(0)=\mathbf{x}(0)-\hat{\mathbf{x}}(0)=0$
- $\mathbf{B}_w\mathbf{D}_{yw}^{\mathsf{T}}=0$ i.e., process noise and sensor noise are uncorrelated.
- i.e.

$$\begin{bmatrix}\mathbf{A} & \begin{bmatrix}\sqrt{\mathbf{Q}} & \mathbf{0}\end{bmatrix}^{\mathsf{T}} & \mathbb{I}\\ \mathbf{C}_z & \mathbf{D}_{zw} & \mathbf{D}_{zu}\\ \mathbf{C}_y & \begin{bmatrix}\mathbf{0} & \sqrt{\mathbf{R}}\end{bmatrix}^{\mathsf{T}} & \mathbf{D}_{yu}\end{bmatrix}$$

and try to find an update signal $\mathbf{u}(t,\tilde{x})\in\mathcal{L}_2$ that minimizes power in the error signal due to white noise disturbance w.

Noise

Process noise and sensor noise affect the system as

$$\begin{aligned}\frac{d\tilde{\mathbf{x}}}{dt} &= \mathbf{A}\tilde{\mathbf{x}}+\mathbf{B}_w\mathbf{w} && \text{affects real, physical state}\\ \mathbf{y} &= \mathbf{C}_y\tilde{\mathbf{x}}+\mathbf{D}_{yw}\mathbf{w} && \text{affects measurements}\end{aligned}$$

and are modelled by

$$\begin{aligned}\mathbf{B}_w &= \begin{bmatrix}\sqrt{\mathbf{Q}} & \mathbf{0}\end{bmatrix}^{\mathsf{T}} && \Leftrightarrow \mathbf{Q}=\mathbf{B}_w\mathbf{B}_w^{\mathsf{T}}\\ \mathbf{D}_{yw} &= \begin{bmatrix}\mathbf{0} & \sqrt{\mathbf{R}}\end{bmatrix} && \Leftrightarrow \mathbf{R}=\mathbf{D}_{yw}\mathbf{D}_{yw}^{\mathsf{T}}\end{aligned}$$

where process noise and sensor noise are not correlated i.e.

$$\mathbb{E}\left[\mathbf{w}^{\mathsf{T}}\mathbf{B}_w^{\mathsf{T}}\mathbf{D}_{yw}\mathbf{w}^{\mathsf{T}}\right]=0$$

Solving for the Observer Matrix

One can show that the given problem statement on $\mathbf{u}$ imposes the following ARE on $\mathbf{Y}_L$:

$$\mathbf{A}\mathbf{Y}_L+\mathbf{Y}_L\mathbf{A}^{\mathsf{T}}-\mathbf{Y}_L\mathbf{C}_y^{\mathsf{T}}\left(\mathbf{D}_{yw}\mathbf{D}_{yw}^{\mathsf{T}}\right)^{-1}\mathbf{C}_y\mathbf{Y}_L+\mathbf{B}_w\mathbf{B}_w^{\mathsf{T}}=\mathbf{0}$$

The optimal observer matrix is then given by

$$\mathbf{L}=-\mathbf{Y}_L\mathbf{C}_y^{\mathsf{T}}\underbrace{\left(\mathbf{D}_{yw}\mathbf{D}_{yw}^{\mathsf{T}}\right)}_{=\mathbf{R}}^{-1}$$

where $\mathbf{Y}_L$ is symmetric and positive semidefinite.

Technical Considerations

1. $(\mathbf{C}_y,\mathbf{A})$ detectable.
   To observe unstable modes.
2. $(\mathbf{A},\mathbf{B}_w)$ stabilizable
   That unstable modes are affected by disturbance to actually track it.
3. $\begin{bmatrix}\mathbf{A}-j\omega I & \mathbf{B}_w\\ \mathbf{C}_y & \mathbf{D}_{yw}\end{bmatrix}$ has full row rank for all $\omega\in\mathbb{R}$
   That errors are penalized at all frequencies and condition on Hamiltonian.
4. $\mathbf{D}_{yw}\mathbf{D}_{yw}^{\mathsf{T}}=\mathbf{R}_{ww}$ invertible, i.e., $\mathbf{D}_{yw}$ has full row rank.
   For convenience.

Remarks

- One can show that this control law is optimal to achieve the stated minimization.
- Technical considerations 1.-3. ensure that the Riccati equation admits a solution $\mathbf{Y}$ that is positive semi-definite.

10.4.3 LQG

Due to the **separation principle** we can design the LQR and LQE controller separately (assuming all technical considerations are fulfilled). This process yields an **optimal controller** of the form

$$\mathbf{K}=\left[\begin{array}{c|cc}\mathbf{A}+\mathbf{B}_u\mathbf{F}+\mathbf{L}\mathbf{C}_y & -\mathbf{L} & \mathbf{B}_u\\ \hline \mathbf{F} & 0 & \mathbf{I}\\ -\mathbf{C}_y & \mathbf{I} & 0\end{array}\right]$$

with

$$\mathbf{F}=-\mathbf{R}_{uu}^{-1}(\mathbf{B}_u^{\mathsf{T}}X+\mathbf{D}_{zu}^{\mathsf{T}}\mathbf{C}_z)$$

where $\mathbf{X}$ is the stabilizing solution to the corresponding ARE and an **optimal observer**

$$\mathbf{L}=-(\mathbf{Y}\mathbf{C}_y^{\mathsf{T}}+\mathbf{B}_w\mathbf{D}_{yw}^{\mathsf{T}})\mathbf{R}_{ww}^{-1}$$

where $\mathbf{Y}$ is the stabilizing solution to the corresponding ARE.

10.5 $\mathcal{H}_\infty$ Synthesis

$\mathcal{H}_\infty$ minimizes the worst-case input-output gain (singular values).

Differentiation from $\mathcal{H}_2$ Synthesis

- $\mathcal{H}_2$ is optimal given
  1. The cost trade-offs between the state error and the control effort
  2. The expected covariances of the disturbance and sensor noise

  i.e. **optimal** w.r.t. time-domain specifications.
- $\mathcal{H}_2$ specifications are given in **time domain**: difficult to handle frequency-domain specifications.
- For $\mathcal{H}_2$ design one could use a "mixed sensitivity" approach, and further augment the plant P with frequency-dependent weighting functions.
- $\mathcal{H}_\infty$ provides a more direct way to handle **frequency-domain specifications.** i.e. it allows to achieve a desired level of robustness to disturbances and noise ("Bode obstacles").

Remarks

- $\mathcal{H}_\infty$ synthesis requires (as $\mathcal{H}_2$) a detectable and stabilizable plant.

10.5.1 Optimal $\mathcal{H}_\infty$ Synthesis

In principle, we would like to find a controller K that minimizes the energy ($\mathcal{L}_2$) gain of the CL system i.e.

$$\|\mathbf{T}_{zw}\|_{\mathcal{H}_\infty}=\sup_{w\neq0}\frac{\|\mathbf{z}\|_{\mathcal{L}_2}}{\|\mathbf{w}\|_{\mathcal{L}_2}}$$

However, the optimal controller(s) are such that

1. $\sigma_{\max}(\mathbf{T}_{zw}(j\omega))$ is a constant over all frequencies, the response does not roll off at high frequencies (*Bode's integral law*).
2. The controller is not strictly proper. (The optimal controller is not unique)
3. Computing an optimal controller is numerically challenging.

10.5.2 Suboptimal $\mathcal{H}_\infty$ Synthesis

Problem Statement

In practice one pursues a sub-optimal design, i.e., given $\gamma>0$, find a controller K such that

$$\|\mathbf{T}_{zw}\|_{\mathcal{H}_\infty}<\gamma$$

if one exists. This can be formulated mathematically as a cost function:

$$\|\mathbf{z}\|_{\mathcal{L}_2}^2-\gamma^2\|\mathbf{w}\|_{\mathcal{L}_2}^2<0$$

where we search for the smallest γ such that the controller can achieve negative cost. We **need $\gamma\leq1$** for a stabilizing controller as otherwise for larger gamma one could easily achieve negative cost even though the energy of the disturbance gets not damped at all! The formula can be understood as

- $\mathbf{z}$ is the performance output given some disturbance $\mathbf{w}$
- We want to damp $\mathbf{z}$ given $\mathbf{w}$ (remember the definition of the induced $\mathcal{L}_2$ norm above)
- The aforementioned formula becomes negative if the disturbance energy term is larger than the (hopefully damped) performance output $\mathbf{z}$. A small γ gives an even faster controller as it is more difficult to achieve negative cost then.
- Faster is **not** necessarily better as control effort rises.

Conditions

For feasibility one must have

1. $(\mathbf{A}_{ext},\mathbf{B}_u)$ must be stabilizable
2. $(\mathbf{C}_{ext,y},\mathbf{A}_{ext})$ must be detectable
3. $\begin{bmatrix}\mathbf{A}-j\omega\mathbf{I} & \mathbf{B}_w\end{bmatrix},\begin{bmatrix}\mathbf{A}^{\mathsf{H}}-j\omega\mathbf{I} & \mathbf{C}_z^{\mathsf{H}}\end{bmatrix}$ must have full row rank

A controller $\mathbf{K}$ fulfilling the cost inequality exists, only if

1. The following ARE has a solution for $\mathbf{X}_\infty$
$$\mathbf{A}^{\mathsf{H}}\mathbf{X}_\infty+\mathbf{X}_\infty\mathbf{A}+\mathbf{C}_z^{\mathsf{H}}\mathbf{C}_z=\mathbf{X}_\infty(\mathbf{B}_u\mathbf{B}_u^{\mathsf{H}}-\gamma^{-2}\mathbf{B}_w\mathbf{B}_w^{\mathsf{H}})\mathbf{X}_\infty$$
2. The following ARE has a solution for $\mathbf{Y}_\infty$
$$\mathbf{A}\mathbf{Y}_\infty+\mathbf{Y}_\infty\mathbf{A}^{\mathsf{H}}+\mathbf{B}_w^{\mathsf{H}}\mathbf{B}_w=\mathbf{Y}_\infty(\mathbf{C}_y\mathbf{C}_y^{\mathsf{H}}-\gamma^{-2}\mathbf{C}_z\mathbf{C}_z^{\mathsf{H}})\mathbf{Y}_\infty$$
3. The matrix $\gamma^2\mathbf{I}-\mathbf{Y}_\infty\mathbf{X}_\infty$ is positive definite

Bisection Algorithm

1. Initialize to, e.g, $\gamma_-=0,\gamma_+=$ the $\mathcal{H}_\infty$ norm of the $\mathcal{H}_2$ **optimal design** (approximation of a reasonable bound). Let $\mathbf{K}_+$ be the optimal $\mathcal{H}_2$ controller.
2. Let $\gamma\leftarrow\frac{\gamma_-+\gamma_+}{2}$. Check whether a controller exists such that $\|\mathbf{T}_{zw}\|_{\mathcal{H}_\infty}<\gamma$.
3. If yes, set $\gamma_+\leftarrow\gamma$, and set $\mathbf{K}_+$ to the controller just designed. Otherwise, set $\gamma_-\leftarrow\gamma$.
4. Repeat from step 2 until $\gamma_+-\gamma_-<\epsilon$.
5. Return $\mathbf{K}_+$.

Remarks

- The conditions are only fulfilled if $\gamma\leq1$
- For $\gamma>1$ use relaxed weights
- The final controller gains are

$$\mathbf{F}_u=-\mathbf{B}_u^{\mathsf{H}}\mathbf{X}_\infty,\quad \mathbf{F}_w=\frac{1}{\gamma^2}\mathbf{B}_w^{\mathsf{H}}\mathbf{X}_\infty$$

- The final observer gain is

$$\mathbf{L}=-\left(\mathbf{I}-\gamma^{-2}\mathbf{Y}_\infty\mathbf{X}_\infty\right)^{-1}\mathbf{Y}_\infty\mathbf{C}_y^{\mathsf{H}}$$

Simplified Setup

Solving the problem is simplified if

- $\mathbf{C}_z^{\mathsf{T}}\mathbf{D}_{zu}=0$, i.e.,the cost if of the form $\int_0^{+\infty}\mathbf{x}^{\mathsf{T}}\mathbf{Q}\mathbf{x}+\mathbf{u}^{\mathsf{T}}\mathbf{R}\mathbf{u}\,dt$.
- $\mathbf{B}_w\mathbf{D}_{yw}^{\mathsf{T}}=0$, i.e., process noise and sensor noise are uncorrelated.
- $\mathbf{D}_{zu}^{\mathsf{T}}\mathbf{D}_{zu}=1,\mathbf{D}_{yw}\mathbf{D}_{yw}^{\mathsf{T}}=1$.

10.5.3 Applying Frequency Weights

One can apply frequency weights to different performance outputs, similar to the "Bode obstacles" for SISO systems.

- In order for $\mathcal{H}_\infty$ synthesis to be possible, frequency weights must be **stable and proper**.
- Integrators (not asymptotically stable) can be approximated by very slow poles.
- MATLAB:
  - `Pinf=augw(G,W1,0,W3)`
    creates the augmented (weighted) system
  - `[Kinf, CL, GAM, INFO]=hinfsyn(Pinf,1,1)`
    applies optimal $\mathcal{H}_\infty$ synthesis
  - `[K09, CL, GAM, I]=hinfsyn(Pinf,1,1,0.9)`
    applies suboptimal $\mathcal{H}_\infty$ synthesis targeting $\gamma=0.9$. This can yield a more feasible controller (less control effort)

11 Nonlinear Systems

11.1 Concepts of Stability

A nonlinear system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= f(\mathbf{x}(t),\mathbf{u}(t)) && \mathbf{x}(0)=\mathbf{x}_0\\ \mathbf{y}(t) &= h(\mathbf{x}(t),\mathbf{u}(t))\end{aligned}$$

does not satisfy the superposition property and therefore

- Effects from inputs and from initial conditions cannot be separated.
- The input cannot be separated into elementary inputs and the output will not be a composition of elementary outputs.

Properties

- A nonlinear system can have zero, one or multiple **equilibrium points** s.t. $f(\mathbf{x}_e,0)=0$.
- In contrast to linear systems that require infinite time to go to infinity, nonlinear systems can **go to infinity in finite time**.
- Nonlinear systems may generate permanent oscillations of fixed amplitude **- limit cycles -** independent of initial conditions.
- Nonlinear systems can generate outputs at frequencies that are submultiples or multiples of the input frequency (**Subharmonic oscillations**).
- Deterministic nonlinear systems can generate **chaos**.

11.2 Lyapunov Stability Theory

Lyapunov's theorem only gives a sufficient condition for stability

- If we can find a Lyapunov function, then we know the equilibrium is stable.
- However, if a candidate Lyapunov function does not satisfy the conditions in the theorem, this does not prove that the equilibrium is unstable.

General Lyapunov functions are

- Energy
- Quadratic Lyapunov functions are commonly used; these can be derived from linearization of the system near equilibrium points.
- "Sum-of-Squares" (SoS) methods can be used to construct polynomial Lyapunov functions

11.2.1 Types of Stability

Assuming $\mathbf{u}=0,\mathbf{x}_e=0$, then in the equilibrium $\mathbf{x}_e$ of the system

$$\dot{\mathbf{x}}=f(\mathbf{x})$$

one has

$$f(\mathbf{x}_e)=0$$

The equilibrium is said to be

- **Stable in the sense of Lyapunov**
$$\|\mathbf{x}(0)\|<\delta\Rightarrow\|\mathbf{x}(t)\|\leq\varepsilon,\qquad\forall t\geq0,\ \delta>0,\ \varepsilon\geq0$$
- **Asymptotically stable**
$$\|\mathbf{x}(0)\|<\delta\Rightarrow\lim_{t\to+\infty}\mathbf{x}(t)=0,\qquad\delta>0$$
- **Exponentially stable**
$$\|\mathbf{x}(0)\|<\delta\Rightarrow\|\mathbf{x}(t)\|<\beta e^{-\alpha t},\qquad\forall t\geq0,\ \alpha,\beta,\delta>0$$

11.2.2 Lyapunov Functions

Lyapunov functions are, in a sense, a notion of energy: non-negative, minimized (0) only at the equilibrium point and non-increasing along all trajectories.

For a given compact subset D of the state space containing

$\mathbf{x}_e$, a Lyapunov function is defined as

$$V : D \to \mathbb{R} \mapsto V(\mathbf{x})$$

If this Lyapunov function satisfies

$$\begin{aligned} V(\mathbf{x}) &\geq 0 && \forall \mathbf{x} \in D \\ V(\mathbf{x}) = 0 &\Leftrightarrow \mathbf{x}_e = \mathbf{x} && \text{(if and only if)} \\ \dot{V}(\mathbf{x}(t)) = \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\frac{\partial \mathbf{x}(t)}{\partial t} &= \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} f(\mathbf{x}) \leq 0 && \forall \mathbf{x}(t) \in D \end{aligned}$$

the equilibrium point $\mathbf{x_e}$ is **stable in the sense of Lyapunov**. Remember that V is defined on a set containing $\mathbf{x}_e$.

Futhermore, $\mathbf{x}_e$ is **asymptotically stable** if

$$\dot{V}(\mathbf{x}(t)) = 0 \Leftarrow \mathbf{x}(t) = \mathbf{x}_e \quad \text{(only if)}$$

Finally, $\mathbf{x}_e$ is **exponentially stable** if

$$\dot{V}(\mathbf{x}(t)) < -\alpha V(\mathbf{x}(t)), \qquad \alpha > 0$$

Global Stability

According to the Barbashin-Krasovski theorem, to ensure global stability (not necessarily asymptotical or exponential), i.e. $D = \mathbb{R}^n$, a Lyapunov function V satisfying the aforementioned three properties **additionally** needs to be **radially unbounded**:

$$\|\mathbf{x}\| \to +\infty \Rightarrow V(\mathbf{x}) \to +\infty$$

Remarks

- If there could be multiple equilibrium points one needs to use LaSalle.
- $V(\mathbf{x}) = 0$ must hold in any equilibrium point, even if there are more then one.
- If one has more than one state vector, then
$$\dot{V} = \sum_i \frac{\partial V(\mathbf{x}_i)}{\partial \mathbf{x}_i}\frac{\partial \mathbf{x}_i(t)}{\partial t}$$
- For asymptotical stability, given $\dot{V}(\mathbf{x}(t)) = 0$ we know that the system is in the **only** equilibrium point.

11.2.2.1 Indirect Method

For a linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

A common choice is a *quadratic Lyapunov function*

$$V(x) = \mathbf{x}^T\mathbf{P}\mathbf{x}$$

where $\mathbf{P}$ is symmetric and positive definite matrix.
The Derivation along trajectories is given by

$$\dot{V}(\mathbf{x}) = \mathbf{x}^T\left(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A}\right)\mathbf{x} = -\mathbf{x}^T\mathbf{Q}\mathbf{x}$$

An appropriate matrix P defining the desired Lyapunov function can be found by solving

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$$

This is called the **Lyapunov Equation**.

Remark

The solution $\mathbf{P}$ of the LQR Riccati equation fulfills the criteria of a Lyapunov function $V(x) = \mathbf{x}^T\mathbf{P}\mathbf{x}$.

Hartman-Grobman Theorem

According to the **Hartman-Grobman Theorem**, a linearized system

$$A = \left.\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=0}$$

with no unstable pole behaves qualitatively the same as the nonlinear around the equilibrium point.

As a result, the Lyapunov function of the linear system also describes the nonlinear system for sufficiently small deviations from the equilibrium point.

11.2.3 LaSalle's Invariance Principle

LaSalle's invariance theorem is useful if Lyapunov stability is given by the existence of a Lyapunov function but $\dot{V} = 0$ not only at $\mathbf{x}_e$:

- Let $\Omega \subset D$ be a compact positively invariant set w.r.t. $\dot{\mathbf{x}}$ ($\mathbf{x}$ stays in the set) i.e., $\mathbf{x}(t_0) \in \Omega \Rightarrow \mathbf{x}(t) \in \Omega\ \forall t \geq t_0$.
- Let $V : D \to \mathbb{R}^n$, such that $\dot{V}(\mathbf{x}) \leq 0\ \forall \mathbf{x} \in \Omega$.
- Let $E = \left\{\mathbf{x} : \dot{V}(\mathbf{x}) = 0\right\} \subset \Omega$ (possibly multiple pts.).
- ▷ Then every solution starting in Ω approaches $M \subset E$ (largest invariant set in E) as $t \to \infty$.
- ▷ This proves asymptotic stability.

$$M \subset E \subset \Omega \subset D \subset \mathbb{R}^n$$

Procedure

1. Find the largest invariant set $E = \left\{\mathbf{x} : \dot{V}(\mathbf{x}) = 0\right\}$
2. Plug the found set E into the dynamics of the system and check if the system can "escape" this set.

Remarks

- For the pendulum with Lyapunov function $\dot{V} = -c\dot{\theta}^2$, one has multiple states (turning points) where $\dot{V} = 0$ namely whenever $\dot{\theta} = 0$. However the only invariant $\dot{\theta} = 0$ is the one in the origin (when the pendulum stopped).

11.3 Control Lyapunov Functions

If a Control Lyapunov Function (CLF) exists, it provides a way to construct a stabilizing feedback with an appropriate control input $\tilde{\mathbf{u}}$ (*Artstein's Theorem*).

A CLF satisfies

$$\begin{aligned} V(\mathbf{x}) &\geq 0 && \text{positive definite} \\ V(\mathbf{x}) = 0 &\Leftrightarrow \mathbf{x} = 0 && \text{radially unbounded} \\ \frac{d}{dt}V(\mathbf{x}, \tilde{\mathbf{u}}(\mathbf{x})) = \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} & f(\mathbf{x}, \tilde{\mathbf{u}}(\mathbf{x})) \leq 0 && \forall \mathbf{x} \neq 0 \end{aligned}$$

i.e. using the control input u, V becomes a Lyapunov function. In the following methods, one aims to find a control Lyapunov function to prove stabilizing behavior with a specific control input.

11.3.1 Gain Scheduling

In gain scheduling, the state space is partitioned into disjoint subspaces and for a number of design equilibria $(\mathbf{x}_i, \mathbf{u}_i)$ nominal control laws $\mathbf{K}_i$ are designed using the aforementioned techniques.

The dynamics of the system around a equilibrium point $\mathbf{x}_i$ can be approximated by

$$\dot{\mathbf{x}} = \mathbf{A}_i(\mathbf{x} - \mathbf{x}_i) + \mathbf{B}_i(\mathbf{u} - \mathbf{u}_i)$$

11.3.1.1 Common Lyapunov Function

The stability of

$$\mathbf{u} = \mathbf{u}_i + \mathbf{K}_i(\mathbf{x} - \mathbf{x}_i)$$

in **all** regions i can be proven by finding a common Lyapunov function

$$V(\mathbf{x}) = \min_i\,(\mathbf{x} - \mathbf{x}_i)^\mathsf{T}\mathbf{P}(\mathbf{x} - \mathbf{x}_i)$$

- Note however, that this is **not** always possible.
- This means that, even if one designs a specific controller for each region, the CL could still be unstable.

11.3.2 Multiple Lyapunov Functions

Another option is to consider multiple Lyapunov-like functions V_i with corresponding sets X_i that satisfy

- positive definite in X_i
- $\dot{V}_i(\mathbf{x}) \leq 0$ when $\mathbf{x} \in X_i$

Then, define $V_i[k]$ as infimum taken by V_i in the given time interval and check if the system satisfies the **sequence non-increasing condition**:

$$V_i[k+1] < V_i[k] \qquad \forall k \in \mathbb{N}$$

Rationale

Assuming the system can move from region to region, we look at all time intervals where the system is in a certain region X_i. We want to have the Lyapunov function decreased every time the system reenters a certain region again. Defining V_i as infimum (largest lower bound), we have stable behavior if the infimum decreases at every re-entry of region X_i.

11.3.3 Linear Parameter-Varying (LPV) Systems

A third option is to rewrite the nonlinear system in a "linear" fashion with slow varying parameter $\sigma(\mathbf{x})$

$$\dot{\mathbf{x}}_\delta = \mathbf{A}(\sigma(\mathbf{x}))\mathbf{x}_\delta + \mathbf{B}(\sigma(\mathbf{x}))\mathbf{u}_\delta \qquad \begin{cases} \mathbf{x}_\delta = \mathbf{x} - \mathbf{x}_e(\sigma(\mathbf{x})) \\ \mathbf{u}_\delta = \mathbf{u} - \mathbf{u}_e(\sigma(\mathbf{x})) \end{cases}$$

Lyapunov Condition for LPV

This system is **exponentially stable** if

$$\bar{\mathbf{A}}(\sigma)^\mathsf{T}\mathbf{P} + \mathbf{P}\bar{\mathbf{A}}(\sigma) + \sum \rho_i \frac{\partial \mathbf{P}(\sigma)}{\partial \sigma_i} < 0, \qquad \begin{cases} \sigma(t) \in S \\ \dot{\sigma}(t) \in R \\ \forall \sigma \in S, \rho \in R \end{cases}$$

In other words, the existence of a family of positive matrices $\mathbf{P}(\sigma)$ (common Lyapunov functions) prove the systems exponential stability.

LPV Controllers

- One assumes σ is measurable
- Then, one will design a stabilizing controller for every σ
- LPV can be imagined as continuous gain scheduling: one has controllers depending on a (possibly infinite) number of σs instead of a number of equilibrium points

11.3.4 Backstepping Control

Backstepping is a form of *cascaded control* where a control input $\mathbf{u}$ is chosen s.t. the output $\mathbf{z}$ of the outer loop stabilizes the inner loop:

$$\begin{aligned} \dot{\mathbf{x}} &= f_0(\mathbf{x}) + g_0(\mathbf{x})\mathbf{z} && \text{inner loop} \\ \dot{\mathbf{z}} &= f_1(\mathbf{x}, \mathbf{z}) + g_1(\mathbf{x}, \mathbf{z})\mathbf{u} && \text{outer loop} \end{aligned}$$

where the inner system has an equilibrium point in $\mathbf{x} = 0, \mathbf{z} = 0$.

Assuming the inner system is stable for a certain $\mathbf{z} = \mathbf{u}_0(\mathbf{x})$ then

$$\frac{d}{dt}V_0(\mathbf{x}) = -W(\mathbf{x}) < 0$$

Then, in order to ensure stability, Lyapunov stability of the error $\mathbf{e} = \mathbf{z} - \mathbf{u}_0$ needs to be established. The error evolves as

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\mathbf{u}_0(x) + \mathbf{g}_0(\mathbf{x})\mathbf{e},$$

$$\dot{\mathbf{e}} = \underbrace{\mathbf{f}_1(\mathbf{x}, \mathbf{z}) + \mathbf{g}_1(\mathbf{x}, \mathbf{z})\mathbf{u}}_{\dot{\mathbf{z}}} - \underbrace{\frac{\partial \mathbf{u}_0(\mathbf{x})}{\partial \mathbf{x}}\overbrace{(\mathbf{f}_0(\mathbf{x}) + \mathbf{g}_0(\mathbf{x})\mathbf{z})}^{\dot{\mathbf{x}}}}_{\dot{\mathbf{u}}_0(\mathbf{x})} = \mathbf{v}$$

Then, the Lyapunov candidate

$$V_1(\mathbf{x}, \mathbf{e}) = V_0(\mathbf{x}) + \frac{1}{2}\mathbf{e}^2$$

satisfies

$$\frac{d}{dt}V_1(\mathbf{x}, \mathbf{e}) = -W(\mathbf{x}) - k_1\mathbf{e}^2 < 0, \qquad \forall (\mathbf{x}, \mathbf{e}) \neq 0$$

if one cleverly picks

$$\mathbf{v} = -\frac{\partial V_0(\mathbf{x})}{\partial \mathbf{x}}\mathbf{g}_0(\mathbf{x}) - k_1\mathbf{e}, \quad k_1 > 0$$

and thus proves the stability of the "error" system.
Using $\dot{\mathbf{u}}_0 = \dot{\mathbf{z}} - \dot{\mathbf{e}}$ one finally finds the stabilizing control law as

$$\begin{aligned} u_1(\mathbf{x}, \mathbf{z}) = \frac{1}{g_1(\mathbf{x}, \mathbf{z})}\Bigg(&\frac{\partial \mathbf{u}_0(\mathbf{x})}{\partial \mathbf{x}}(f_0(\mathbf{x}) + g_0(\mathbf{x})\mathbf{z}) - f_1(\mathbf{x}, \mathbf{z}) \\ &- \frac{\partial V_0(\mathbf{x})}{\partial \mathbf{x}}g_0(\mathbf{x}) - k_1(\mathbf{z} - \mathbf{u}_0(\mathbf{x}))\Bigg) \end{aligned}$$

Remarks

- The control input $\mathbf{u}$ affects $\mathbf{z}$, which in turn affects $\mathbf{x}$
- One assumes that the inner system is easily controllable by $\mathbf{u}_0$ e.g. because
  - it is linear
  - one can find a control Lyapunov function for it
- The idea can be extended to multiple nested systems

11.3.4.1 Recursive Backstepping

The same approach can be used recursively as

$$\begin{aligned} \dot{\mathbf{x}} &= f_0(\mathbf{x}) + g_0(\mathbf{x})\mathbf{z}_1, \\ \dot{\mathbf{z}}_1 &= f_1(\mathbf{x}, \mathbf{z}_1) + g_1(\mathbf{x}, \mathbf{z}_1)\mathbf{z}_2, \\ &\vdots \\ \dot{\mathbf{z}}_m &= f_m(\mathbf{x}, \mathbf{z}_1, \ldots, \mathbf{z}_m) + g_1(\mathbf{x}, \mathbf{z}_m)\mathbf{u} \end{aligned}$$

11.4 Feedback Linearization

Feedback linearization is an approach based on a change of variables, transforming the nonlinear system into a linear one (seen from the controller perspective). In other words the nonlinearity gets "hidden" in a transformed input signal.
In order to apply this technique the system needs to be **differentially flat** with a **flat output** (see Section 11.4.2).

To find the input transformation that linearizes a system of the form

$$\begin{aligned} \dot{\mathbf{x}} &= f(\mathbf{x}) + g(\mathbf{x})\mathbf{u} \\ \mathbf{y} &= h(\mathbf{x}) \end{aligned}$$

the output y has to be differentiated γ times, where γ (the relative degree) is the smallest number (if it exists) such that

$$L_g L_f^{\gamma-1}h(\mathbf{x}) \neq 0$$

with

$$\begin{aligned} L_f h(\mathbf{x}) &:= \frac{\partial h}{\partial \mathbf{x}} f(\mathbf{x}) \\ L_g h(\mathbf{x}) &:= \frac{\partial h}{\partial \mathbf{x}} g(\mathbf{x}) \\ L_f^2 h(\mathbf{x}) &:= L_f(L_f h(\mathbf{x})) = \frac{\partial L_f h(\mathbf{x})}{\partial \mathbf{x}} f(\mathbf{x}) \\ L_g L_f h(\mathbf{x}) &:= L_g(L_f h(\mathbf{x})) = \frac{\partial L_f h(\mathbf{x})}{\partial \mathbf{x}} g(\mathbf{x}) \end{aligned}$$

where $L_f h(\mathbf{x})$ and $L_g h(\mathbf{x})$ are called the *Lie derivatives* of $h(\mathbf{x})$ in direction f or g respectively.

The system then can be transformed into a (virtually) linear system

$$\frac{d^\gamma \mathbf{y}}{dt^\gamma} = \mathbf{v}$$

by choosing a physical input

$$\mathbf{u} = \frac{1}{L_g L_f^{\gamma-1}h(\mathbf{x})}(-L_f^\gamma h(\mathbf{x}) + \mathbf{v})$$

A (linear, "virtual") realization of this system is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\mathbf{v}$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}\mathbf{x}$$

Explicit examples for $\gamma = 1$

$$\begin{aligned} \dot{\mathbf{y}} &= L_f h(\mathbf{x}) + L_g h(\mathbf{x})\mathbf{u} \\ \dot{\mathbf{y}} &= \mathbf{v} \\ \mathbf{u} &= \frac{1}{L_g h(\mathbf{x})}(-L_f h(\mathbf{x}) + \mathbf{v}) \end{aligned}$$

and for $\gamma = 2$

$$\ddot{\mathbf{y}} = L_f^2 h(\mathbf{x}) + L_g L_f h(\mathbf{x})\mathbf{u}$$

$$\ddot{\mathbf{y}} = \mathbf{v}$$

$$\mathbf{u} = \frac{1}{L_g L_f h(\mathbf{x})}(-L_f^2 h(\mathbf{x}) + \mathbf{v})$$

Remarks:

- Only works for **controllable** systems
- The actuator basically cancels out the behavior of the original system to replace it with the designed linear one.
- This has the disadvantage that potential good characteristics of the nonlinear system (like damping) get actively canceled out.
- One can imagine the FB linearization as controlling a virtual linear system with control input v but under the hood applying a cleverly chosen physical control input u.
- Feedback linearization provides an exact linear model of a nonlinear system. Not like the Jacobian-Linearization which linearizes the system around an equilibrium point.

11.4.1 Pendulum

In the case of the nonlinear pendulum with a torque input u and an output $y = \theta$

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{x}_2\\ \dot{\mathbf{x}}_2 &= -a\sin(\mathbf{x}_1) - c\mathbf{x}_2 + \mathbf{u}\\ \mathbf{y} &= \mathbf{x}_1\end{aligned}$$

The Lie derivatives are

$$L_f h(\mathbf{x}) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}_2\\ -a\sin(\mathbf{x}_1) - c\mathbf{x}_2\end{bmatrix} = \mathbf{x}_2$$

$$L_g h(\mathbf{x}) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 0$$

Since $L_g h(\mathbf{x}) = 0$ we need to differentiate again

$$L_f^2 h(\mathbf{x}) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{x}_2\\ -a\sin(\mathbf{x}_1) - c\mathbf{x}_2\end{bmatrix} = -a\sin(\mathbf{x}_1) - c\mathbf{x}_2$$

$$L_g L_f h(\mathbf{x}) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 1$$

This results in

$$\mathbf{u} = \frac{1}{L_g L_f h(\mathbf{x})}(-L_f^2 h(\mathbf{x}) + \mathbf{v}) = a\sin(\mathbf{x}_1) + c\mathbf{x}_2 + \mathbf{v}$$

and the linear system

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{x}_2\\ \dot{\mathbf{x}}_2 &= \mathbf{v}\\ \mathbf{y} &= \mathbf{x}_1\end{aligned}$$

with

$$\mathbf{v} = -a\sin(\mathbf{x}_1) - c\mathbf{x}_2 + \mathbf{u}$$

11.4.2 Differental Flatness

A dynamic system

$$\begin{aligned}\dot{\mathbf{x}} &= f(\mathbf{x}, \mathbf{u})\\ \mathbf{y} &= h(\mathbf{x}, \mathbf{u})\end{aligned}$$

is differentially flat, with flat output $\mathbf{y}$, if one can compute the state and input trajectories as a function (map Ξ) of the flat outputs and a finite number of derivatives.

$$(\mathbf{x}, \mathbf{u}) = \Xi(\mathbf{y}, \dot{\mathbf{y}}, \dots, \mathbf{y}^{(m)})$$

11.4.2.1 Pendulum

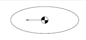
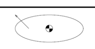
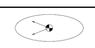
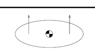
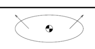
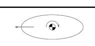
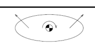
The nonlinear pendulum with a torque input $\mathbf{u}$ and an output $\mathbf{y} = \theta$

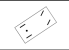
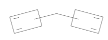
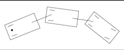
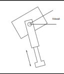
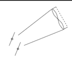
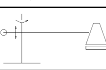
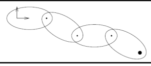
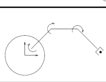
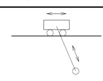
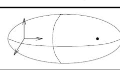
$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{x}_2\\ \dot{\mathbf{x}}_2 &= -a\sin(\mathbf{x}_1) - c\mathbf{x}_2 + \mathbf{u}\\ \mathbf{y} &= \mathbf{x}_1\end{aligned}$$

Is differentially flat with flat output y. As per definition x and u can be reconstructed from y, $\dot{y}$ and $\ddot{y}$.

$$\begin{aligned}\mathbf{x}_1 &= \mathbf{y}\\ \mathbf{x}_2 &= \dot{\mathbf{y}}\\ \mathbf{u} &= \ddot{\mathbf{y}} + c\dot{\mathbf{y}} + a\sin(\mathbf{y})\end{aligned}$$

11.4.2.2 Differental flat systems

Inputs	Picture	Controllable?	Flat Output
Single torque		no	not flat
Single force, at center of mass		no	not flat
Single force, off center		yes	not flat
Two forces, at center of mass		no	not flat
Two colinear forces, off center		yes	center of mass
Two noncolinear forces, off center		yes	center of oscillation
One force, one torque		yes	center of mass
Two noncolinear forces, one torque		yes	center of mass

System	Picture	Flat Output	Comments
Mobile robot, car		rear wheel position	perfect rolling
Car with 1 trailer		function of hitch angle/position	perfect rolling
Car with N trailers, special hitching		last trailer wheel position	perfect rolling
Hopping robot		position of end of leg	
Ducted fan/PVTOL		center of oscillation	no drag
Ducted fan with ideal stand		quasi-center of oscillation	$m_x \neq m_y$, no drag
Planar rigid body chain		center of oscillation, last rigid body	attachment points at centers of oscillation
Planar satellite with actuated robotic arm		body fixed point plus linear functions of joint angles	
Simplified planar crane		position of the load	
Rigid body with S^1 symmetry and body fixed forces in $\mathbb{R}^3$		axisymmemtric center of oscillation	
Towed cable system		position of end of cable	Drag doesn't destroy flatness
Fully actuated mechanical system	$M(q)\ddot{q}+N(q,\dot{q}) = \tau$	configuration coordinates	covers robot manipulators
(Dynamic) feedback linearizable system	$\dot{x} = f(x, u)$	end of integrator chain	Equivalent to flatness on open dense set
Nonholonomic system with 2–4 states, 2 controls	$\dot{x} = g(x)u$	system dependent	Assume controllable
Nonholonomic system with 5 states, 3 controls	$\dot{x} = g(x)u$	system dependent	Assume controllable
Nonholonomic system in chained form with single generator	$\dot{x} = g(x)u$	first and last states	

11.4.3 Feedback control of differentially-flat systems

The geometric property of differential flatness allows to compute the input control signal and the internal state of a system given a desired, sufficient differentiable output trajectory. A problem arises if the actual initial state does not match the computed initial state. This deviation can be controlled by introducing a feedback loop into the virtual signal $\mathbf{v}$ depending on the desired output $\mathbf{y}_d$

$$\mathbf{v} = \mathbf{y}_d^{(\gamma)} + K\begin{bmatrix}\mathbf{y}_d - \mathbf{y}\\ \dot{\mathbf{y}}_d - \dot{\mathbf{y}}\\ \vdots\\ \mathbf{y}_d^{(\gamma-1)} - \mathbf{y}^{(\gamma-1)}\end{bmatrix}$$

or the desired internal state $\mathbf{x}_d$

$$\mathbf{v} = \mathbf{y}_d^{(\gamma)} + K\begin{bmatrix}\mathbf{x}_d - \mathbf{x}\\ \dot{\mathbf{x}}_d - \dot{\mathbf{x}}\\ \vdots\\ \mathbf{x}_d^{(\gamma-1)} - \mathbf{x}^{(\gamma-1)}\end{bmatrix}$$

Remarks:

- Due to the coordinate transformation it becomes hard to check the control input e.g. for saturation bounds.
- The feedback is non-proper, i.e., it contains a number of differentiators that is equal to the relative degree
- The closed-loop dynamics are (internally) stable iff the zero dynamics are stable, i.e., if the system is minimum phase.

11.4.4 Zero dynamics

Checking for Asymptotic Minimum Phase

If it is possible to find an input $\mathbf{u}_0(\mathbf{x})$ that satisfies

$$\mathbf{y} = \dot{\mathbf{y}} = \cdots = \mathbf{y}^{(\gamma)} = 0$$

the **zero dynamics** (if they exist) are given by

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}_0(\mathbf{x})$$

Given an equilibrium point $f(\mathbf{x}_0) = 0$ with zero output $h(\mathbf{x}_0) = 0$, if the zero dynamics for the equilibrium point $\mathbf{x}_0$, i.e.

$$\dot{\mathbf{x}} = f(\mathbf{x}_0) + g(\mathbf{x}_0)\mathbf{u}_0(\mathbf{x}_0)$$

are **asymptotically** (exponentially) **stable**, the system is said to be **asymptotically** (exponentially) **minimum-phase**.

Breaking Condition

If the system is not minimum-phase, the feedback linearization approach does not ensure internal stability of the closed-loop system, i.e., there are "unobservable modes" that are unstable.

12 Model Predictive Control (MPC)

For a discrete-time nonlinear control system of the form

$$\begin{aligned}\mathbf{x}[k+1] &= f(\mathbf{x}[k], \mathbf{u}[k])\\ \mathbf{y}[k] &= h(\mathbf{x}[k], \mathbf{u}[k])\end{aligned}$$

a general description of the cost function and the constraints with the horizon length H can be formulated as following

$$\begin{aligned}\min_{u[0],\dots,u[H-1]} J_H(\mathbf{x}, \mathbf{u}) &:= \sum_{k=0}^{H-1} g(\mathbf{x}[k], \mathbf{u}[k])\\ \mathbf{x}[k+1] &= f(\mathbf{x}[k], \mathbf{u}[k])\\ \mathbf{x}[0] &= \mathbf{x}_0\\ \mathbf{x}[k] &\in X\\ \mathbf{u}[k] &\in U\\ k &= 0, \dots, H-1\end{aligned}$$

By solving these equations one can obtain the best control input u for the next step. This is then done over and over again.

The remaining costs from $k = H, \dots, \infty$ are called the tail. If the prediction is too short-sighted (hence the tail is too large) the system can get unstable. To fix this, one can

- shorten the tail (enlarge the horizon). This introduces more computational effort each step.
- make the tail zero (or sufficiently small). See Section 12.1.
- replace the tail with an estimate. See Section 12.2.

The formulation becomes

$$\begin{aligned}\min_{u[0],\dots,u[H-1]} J_H(\mathbf{x}, \mathbf{u}) &:= \sum_{k=0}^{H-1} g(\mathbf{x}[k], \mathbf{u}[k]) + V(\mathbf{x}[H])\\ \mathbf{x}[k+1] &= f(\mathbf{x}[k], \mathbf{u}[k])\\ \mathbf{x}[0] &= \mathbf{x}_0\\ \mathbf{x}[k] &\in X\\ \mathbf{u}[k] &\in U\\ \mathbf{x}[H] &\in X_H\\ k &= 0, \dots, H-1\end{aligned}$$

12.1 Terminal Constraints

If the state is forced to an equilibrium at the end of the horizon it will remain there and the tail will have no further effect. On the downside this may put excessive pressure on the actuators/control effort.

12.2 Terminal Cost

The best terminal cost estimate would be $V = J_\infty^*$ the cost from the end of the horizon up to infinity.

Because this is not determinable we are looking for a **global Control Lyapunov Function (CLF)** such that

$$V(\mathbf{x}) \geq \mathbf{c}_v \|\mathbf{x}\|^2$$

and

$$\min_u (V(f(\mathbf{x}, \mathbf{u})) + \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}) \leq 0 \quad \forall\, \mathbf{x}$$

which is an upper bound for the optimal cost

$$V(\mathbf{x}) \geq J_\infty^*(\mathbf{x})$$

If we ensure that the terminal cost of the next step $V(f(\mathbf{x}, \mathbf{u}))$ is less or equal to the current terminal cost $V(\mathbf{x})$ we get a stabilizing control law.

However, the MPC control law will have better performance – if not optimal, i.e.

$$J_\infty^*(\mathbf{x}) \leq J_H^*(\mathbf{x}) \leq J_{CLF}^*(\mathbf{x})$$

An example of a CLF candidate may be given by the solution $\mathbf{P}$ of the Algebraic Riccati Equation of the LQR problem, i.e., given $V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$.

12.3 Solving the Finite Horizon Optimization

For an unconstrained, differentiable and convex optimization problem one can

- set the gradient to zero to find the stationary points.
- use a gradient descent method.

In most cases one deals with limiting constraints. To incorporate these into the cost function a so called **barrier function** is added.

$$V(\mathbf{x}, \mathbf{u}) - \log(-c(\mathbf{x}, \mathbf{u}))$$

Note that the function c is defined in a way such that

$$c(\mathbf{x}, \mathbf{u}) \leq 0$$

Hence the barrier function is not defined for (x, u) not satisfying the constraints, and it becomes very large when approaching the boundaries of the constraint set.

To solve this kind of optimization problem the **barrier interior-point method** can be used.

12.3.1 Barrier Interior-Point Method

Interior point methods solve **convex problems** of the form

$$\begin{aligned}\min\quad & f_0(\mathbf{x})\\ \text{s.t.:}\quad & f_i(\mathbf{x}) \leq 0, \quad 0 = 1, \dots, m\\ & \mathbf{A}\mathbf{x} = \mathbf{b}\end{aligned}$$

where the scalar functions $f_0, f_1, \dots, f_m$ are convex and twice differentiable.

The problem is converted into a minimization with affine equality constraints by adding **barrier functions** in the following way

$$\begin{aligned} \min \quad & f_0(\mathbf{x}) - \frac{1}{t}\sum_{i=1}^{m} \log(-f_i(\mathbf{x})) \\ \text{s.t.:} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$

As $t \to +\infty$, the objective function gets closer to f_0, but diverges very steeply at the boundary of the feasible set.

The problem is then solved iteratively using gradient descent or Newton method for $t = \mu^k t_0$ with $t_0 > 0, \mu > 1$ where k is the iteration variable. The solution of the pervious step is used as a starting point for the next step.

Remarks:

- The sequence of intermediate optimal solutions forms what is known as the **central path**, which is always contained in the interior of the feasible set.
- This method terminates with $\mathcal{O}(\sqrt{m})$ time.

13 Appendix

13.1 Matrices

13.1.1 Matrix Properties

Orthogonal Matrix

$$\mathbf{U}^\mathsf{T}\mathbf{U} = \mathbf{U}\mathbf{U}^\mathsf{T} = \mathbf{I} \quad \mathbf{U} \in \mathbb{R}^{n\times n}$$

Unitary Matrix

$$\mathbf{U}^\mathsf{H}\mathbf{U} = \mathbf{U}\mathbf{U}^\mathsf{H} = \mathbf{I} \quad \mathbf{U} \in \mathbb{R}^{n\times n}$$

- preserves the euclidean norm:

$$\|\mathbf{U}\mathbf{x}\|_2 = \|\mathbf{x}\|_2 \quad \forall \mathbf{x} \in \mathbb{C}^n$$

Hermitian Matrix

$$\mathbf{S} = \mathbf{S}^\mathsf{H}$$

For any

$$\mathbf{A} \in \mathbb{R}^{m\times n}$$

both

$$\mathbf{A}^\mathbf{H}\mathbf{A} \in \mathbb{R}^{n\times n}$$
$$\mathbf{A}\mathbf{A}^\mathbf{H} \in \mathbb{R}^{m\times m}$$

are hermitian, positive semi-definite and their eigenvalues are real and non-negative.

- For every hermitian matrix $\mathbf{S}$ exists a unitary matrix $\mathbf{U}$ s.t. $\mathbf{U}^\mathsf{H}\mathbf{S}\mathbf{U}$ is a diagonal matrix.
  - In other words, unitary matrices can diagonalize hermitian matrices which is what one uses for SVD.

A is symmetric

$$\mathbf{A}^\mathsf{T} = \mathbf{A}$$
$$\mathbf{A}^\mathsf{H} = \mathbf{A}^*$$

A is invertible

$$\left(\mathbf{A}^\mathsf{T}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^\mathsf{T}$$
$$\left(\mathbf{A}^\mathsf{H}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^\mathsf{H}$$

A is square

$$\text{eigvals}(\mathbf{A}^\mathsf{T}) = \text{eigvals}(\mathbf{A})$$
$$\text{eigvals}(\mathbf{A}^\mathsf{H}) = \text{eigvals}(\mathbf{A})^*$$
$$\text{eigvals}(\mathbf{A}^{-1}) = \text{eigvals}(\mathbf{A})^{-1}$$

Blocks

$$\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{pmatrix}^\mathsf{H} = \begin{pmatrix} A_{11}^\mathsf{H} & \cdots & A_{m1}^\mathsf{H} \\ \vdots & & \vdots \\ A_{1n}^\mathsf{H} & \cdots & A_{nm}^\mathsf{H} \end{pmatrix}$$

13.1.2 Matrix Algebra

Taken from *Petersen & Pedersen, The Matrix Cookbook*

Basic Rules

$$\mathbf{A}^\mathsf{H} = \left(\mathbf{A}^\mathsf{T}\right)^*$$

$$\left(\mathbf{A}^\mathsf{T}\right)^\mathsf{T} = \mathbf{A} \qquad \left(\mathbf{A}^\mathsf{H}\right)^\mathsf{H} = \mathbf{A} \qquad \left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$

$$(\lambda\mathbf{A})^\mathsf{T} = \lambda\mathbf{A}^\mathsf{T} \qquad (\lambda\mathbf{A})^\mathsf{H} = \lambda\mathbf{A}^\mathsf{H} \qquad (\lambda\mathbf{A})^{-1} = \lambda^{-1}\mathbf{A}^{-1}$$

$$(\mathbf{A}+\mathbf{B})^\mathsf{T} = \mathbf{A}^\mathsf{T} + \mathbf{B}^\mathsf{T} \qquad (\mathbf{A}+\mathbf{B})^\mathsf{H} = \mathbf{A}^\mathsf{H} + \mathbf{B}^\mathsf{H}$$

$$(\mathbf{A}\mathbf{B})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T} \qquad (\mathbf{A}\mathbf{B})^\mathsf{H} = \mathbf{B}^\mathsf{H}\mathbf{A}^\mathsf{H} \qquad (\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$\det(\mathbf{A}^\mathsf{T}) = \det(\mathbf{A}) \qquad \det(\mathbf{A}^\mathsf{H}) = \det(\mathbf{A})^* \qquad \det(\mathbf{A}^{-1}) = \det(\mathbf{A})^{-1}$$

$$\text{rank}(\mathbf{A}^\mathsf{H}) = \text{rank}(\mathbf{A}) \qquad \text{rank}(\mathbf{A}^{-1}) = \text{rank}(\mathbf{A})$$

Derivatives

For any matrix $\mathbf{Y}(x)$

$$\frac{\partial \mathbf{Y}^\mathsf{T}}{\partial x} = \left(\frac{\partial \mathbf{Y}}{\partial x}\right)^\mathsf{T}$$
$$\frac{\partial \mathbf{Y}^\mathsf{H}}{\partial x} = \left(\frac{\partial \mathbf{Y}}{\partial x}\right)^\mathsf{H}$$
$$\frac{\partial \mathbf{Y}^{-1}}{\partial x} = -\mathbf{Y}^{-1}\frac{\partial \mathbf{Y}}{\partial x}\mathbf{Y}^{-1}$$

Disclaimer The following section uses *Hessian notation* for derivatives. When using *Jacobian notation*, the following formulas need be transposed.

For $\mathbf{w} \in \mathbb{R}^d, \mathbf{A} \in \mathbb{R}^{d\times d}$ and $f : \mathbb{R}^d \to \mathbb{R}$

$$\frac{\partial f}{\partial \mathbf{w}} = \begin{pmatrix} \frac{\partial f}{\partial \mathbf{w}_1} \\ \vdots \\ \frac{\partial f}{\partial \mathbf{w}_d} \end{pmatrix} \qquad \in \mathbb{R}^d$$

$$\frac{\partial \mathbf{w}^\mathsf{T}\mathbf{w}}{\partial \mathbf{w}} = \frac{\partial \|\mathbf{w}\|^2}{\partial \mathbf{w}} = 2\mathbf{w} \qquad \in \mathbb{R}^d$$
$$\frac{\partial \mathbf{A}\mathbf{w}}{\partial \mathbf{w}} = \mathbf{A}^\mathsf{T} \qquad \mathbf{A} \in \mathbb{R}^{d\times d}$$
$$\frac{\partial \mathbf{w}^\mathsf{T}\mathbf{A}}{\partial \mathbf{w}} = \mathbf{A} \qquad \mathbf{A} \in \mathbb{R}^{d\times d}$$
$$\frac{\partial \mathbf{w}^\mathsf{T}\mathbf{A}\mathbf{w}}{\partial \mathbf{w}} = \left(\mathbf{A} + \mathbf{A}^\mathsf{T}\right)\mathbf{w} \overset{\mathbf{A}\text{ sym.}}{=} 2\mathbf{A}\mathbf{w} \qquad \in \mathbb{R}^{d\times d}$$

Matrix Exponential Function

$$e^\mathbf{X} = \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{X}^\mathbf{k}$$

Determinant

$$\det(A) = \sum_{j=1}^{n} \underbrace{(-1)^{i+j}}_{\text{checkboard pattern}} \cdot a_{i,j} \cdot \underbrace{m_{i,j}}_{\text{minor: } |\text{submatrix of } \mathbf{A}|}$$

Remark: Any row or column can be chosen (choose wisely: 0)

Inversion

$$\mathbf{A}^{-1} = \frac{\text{adj}(\mathbf{A})}{\det(\mathbf{A})}$$

13.1.3 Specific Solutions

Adjoint of a 3×3 matrix

$$\text{adj}(\mathbf{A}) = \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$
$$= \begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ a_{23}a_{31} - a_{21}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{13}a_{21} - a_{11}a_{23} \\ a_{21}a_{32} - a_{22}a_{31} & a_{12}a_{31} - a_{11}a_{32} & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}$$

Inversion of a 2×2 matrix

$$\mathbf{A}^{-1} = \frac{\text{adj}(\mathbf{A})}{\det(\mathbf{A})} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$