SF1686/1626 Flervariabelanalys
Exam (08:00-11:00)
June 4, 2021

No books/notes/calculators etc. allowed
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This exam consists of three parts: A, B and C; each worth 12 points. The bonus points from the seminars will be automatically added to the total score of part A, which however cannot exceed 12 points.

The thresholds for the respective grades are as follows:

Grade	A	B	C	D	E	Fx
Total sum	27	24	21	18	16	15
of which in part C	6	3	–	–	–	–

For full score you should define your notation; clearly explain the logical structure of your argument in words or symbols; and motivate and explain your argument.

Part A.

Question A1. Consider the vector field $\mathbf{F}(x,y,z) = (2xy, x^2+2yz, y^2)$ in $\mathbb{R}^3$.

a) Find the divergence div($\mathbf{F}$). **1 p.**

b) Find the rotation curl($\mathbf{F}$). **1 p.**

c) Find a potential function to $\mathbf{F}$, if it exists. **2 p.**

Solution to Question 1a) By definiton

$$\mathrm{div}(\mathbf{F})(x,y,z) = \frac{\partial(2xy)}{\partial x} + \frac{\partial(x^2+2yz)}{\partial y} + \frac{\partial(y^2)}{\partial z} = 2y+2z$$

Answer to Question 1a) $\mathrm{div}(\mathbf{F})(x,y,z) = 2y+2z$

b) By definition

$$\mathrm{curl}(\mathbf{F})(x,y,z) = \left(\frac{\partial(y^2)}{\partial y} - \frac{\partial(x^2+2yz)}{\partial z}, \frac{\partial(2xy)}{\partial z} - \frac{\partial(y^2)}{\partial x}, \frac{\partial(x^2+2yz)}{\partial x} - \frac{\partial(2xy)}{\partial y}\right) =$$
$$(2y-2y, 0-0, 2x-2x) = (0,0,0).$$

Answer to Question 1b) $\mathrm{curl}(\mathbf{F}(x,y,z) = (0,0,0)$.

c) Since the curl of $\mathbf{F}$ is identically zero and since $\mathbb{R}^3$ is simply connected there must exist a potential function $\phi(x,y,x)$ to $\mathbf{F}(x,y,z)$. I.e. a function ϕ such that $\nabla\phi(x,y,z) = \mathbf{F}(x,y,z)$. We write $\nabla\phi(x,y,z) = \mathbf{F}(x,y,z)$ componentwise

$$\begin{aligned} \tfrac{\partial\phi}{\partial x}(x,y,z) &= 2xy \\ \tfrac{\partial\phi}{\partial y}(x,y,z) &= x^2+2yz \\ \tfrac{\partial\phi}{\partial z}(x,y,z) &= y^2. \end{aligned} \tag{1}$$

Integrating the first equation we obtain, for some h independent of x,

$$\phi(x,y,z) = x^2y + h(y,z).$$

Differentiating ϕ w.r.t. y and inserting in the second equation (1) we have

$$x^2 + \frac{\partial h}{\partial y}(y,z) = x^2+2yz \Rightarrow \frac{\partial h}{\partial y}(y,z) = 2yz \Rightarrow h(y,z) = y^2z + g(z)$$

where $g(z)$ depends on z only. Inserting this in the last eq. (1) we obtain

$$g'(z) = 0,$$

i.e. g is constant. Hence

$$\phi(x, y, z) = x^2y + y^2z + C.$$

We may obviously choose $C = 0$, or any other constant.

Answer to Question 1c) $\phi(x, y, z) = x^2y + y^2z$ is a potential function to $\mathbf{F}(x, y, z)$.

Question A2. The bull Ferdinand has escaped from the shadow of his beloved tree and is on a hill described by the graph of the function

$$f(x, y) = \frac{x - 2y}{1 + x^2 + y^2}.$$

The exact location of Ferdinand is given by the coordinates $(x, y, z) = (2, 1, 0)$.

a) In which direction in (x, y)-coordinates should Ferdinand go in order to go down the steepest? **2 p.**

b) Determine a trajectory (a curve) γ in (x, y) coordinates so that Ferdinand can walk without changing the height $z = 0$. vandra utan att ändra höjden $z = 0$. **2 p.**

2 p.

Solution to Question A2a) Since the hill is a level curve to $f(x, y)$, the direction in which the height of the hill decreases fastest at the point $(x, y) = (2, 1)$ is given by the opposite direction of the gradient of the function:

$$-\nabla f(2, 1) = -\left(\frac{1}{1 + x^2 + y^2} - \frac{(x - 2y)(2x)}{(1 + x^2 + y^2)^2}, \frac{-2}{1 + x^2 + y^2} - \frac{(x - 2y)(2y)}{(1 + x^2 + y^2)^2}\right)\Bigg\lfloor_{(x,y)=(2,1)} =$$

$$\left(-\frac{1}{6}, \frac{1}{3}\right).$$

Answer to Question A2a: The direction in which the height decreases fastest is $\left(-\frac{1}{6}, \frac{1}{3}\right)$.

b) We need to find $\gamma(t) = (x(t), y(t))$ such that

$$0 = f(x(t), y(t)) = \frac{x(t) - 2y(t)}{1 + x(t)^2 + y(t)^2} \Rightarrow x(t) - 2y(t) = 0.$$

We parametrise $\gamma(t) = (2t, t)$ where $-\infty < t < \infty$.

Answer to Question A2b: A possible parametrisation is given by $\gamma(t) = (2t, t)$ where $-\infty < t < \infty$.

Question A3. Consider the function

$$g(x, y, z) = x^2 + 2y^2 + 3z^2.$$

a) Find all points on the level surface $g(x, y, z) = 5$, whose tangent plane is parallel with the plane that is given by the equation $x - 2y + 3z = 0$. **2 p.**

b) Let K be the body which is given by $K = \{(x, y, z) : \; g(x, y, z) < 5\}$. Find the mass of the body K, when the density of K is given by the density function $\rho(x, y, z) = z^2$. **2 p.**

Solution to Question A3a) The tangent plane of the level curve $g(x, y, z) = 5$ is parallel to the plane $x - 2y + 3z = 0$ if the normal to the level curve points in the same direction as the normal to the plane: i.e. if there is a $\lambda \neq 0$ such that $\nabla g(x, t, z) = \lambda(1, -2, 3)$. If $\lambda = 0$, this method does not provide any unambiguous information.

The gradient of g is calculated

$$\nabla g(x,y,z) = \left(\frac{\partial(x^2+2y^2+3z^3)}{\partial x}, \frac{\partial(x^2+2y^2+3z^3)}{\partial y}, \frac{\partial(x^2+2y^2+3z^3)}{\partial z}\right) = (2x, 4y, 6z)\,.$$

The tangent plane at (x,y,z) on the level surface

(2) $$g(x,y,z) = x^2 + 2y^2 + 3z^2 = 5$$

is parallel with the plane $x - 2y + 3z = 0$ if there exists $\lambda \neq 0$ such that

(3) $$(2x, 4y, 6z) = \lambda(1, -2, 3).$$

The solution to (3) is given by

(4) $$x = \frac{\lambda}{2}, \quad y = -\frac{\lambda}{2} \text{ och } z = \frac{\lambda}{2}.$$

Inserting in (2) we arrive at

$$\lambda^2 = \frac{10}{3} \Rightarrow \lambda = \pm\sqrt{\frac{10}{3}}.$$

Inserting these λ in (4) we have two solutions $(x,y,z) = \pm\sqrt{\frac{5}{6}}(1,-1,1)$.

Answer to Question A3a: At points $(x,y,z) = \pm\sqrt{\frac{5}{6}}(1,-1,1)$ the tangent plane to $g(x,y,z) = 5$ is parallel to $x - 2y + 3z = 0$.

A3b) We use the coordinates $u = x$, $v = \sqrt{2}y$ and $w = \sqrt{3}z$, which implies that $g(x,y,z) = x^2 + 2y^2 + 3z^2 < 5$ is given by $u^2 + v^2 + w^2 < 5$, $z^2 = \frac{w^2}{3}$ and $dxdydz = \frac{1}{\sqrt{6}}dudvdw$.
Hence

$$\iiint_K \rho(x,y,z)dV = \iiint_D \frac{w^2}{3\sqrt{6}}dudvdw,$$

where D is the domain $u^2 + v^2 + w^2 < 5$. In spherical coordinates we have

$$\begin{aligned} u &= R\cos(\theta)\sin(\phi) \\ v &= R\sin(\theta)\sin(\phi) \\ w &= R\cos(\phi), \end{aligned}$$

where D av $0 \leq R < \sqrt{5}$, $0 \leq \theta < 2\pi$ and $0 \leq \phi \leq \pi$.
Hence

$$\iiint_D \frac{w^2}{3\sqrt{6}}dudvdw = \frac{1}{3\sqrt{6}}\int_0^{2\pi}\int_0^{\pi}\int_0^{\sqrt{5}} R^4\cos^2(\phi)\sin(\phi)dRd\phi d\theta =$$

$$\frac{1}{3\sqrt{6}}\int_0^{2\pi}\int_0^{\pi}\left[\frac{R^5}{5}\cos^2(\phi)\sin(\phi)\right]_0^{\sqrt{5}} d\phi d\theta = \frac{5^{5/2}}{15\sqrt{6}}\int_0^{2\pi}\int_0^{\pi}\cos^2(\phi)\sin(\phi)d\phi d\theta =$$

$$\frac{5^{3/2}}{3\sqrt{6}}\int_0^{2\pi}\left[-\frac{\cos^3(\phi)}{3}\right]_0^{\pi} d\theta = \frac{2\cdot 5^{3/2}}{3^2\sqrt{6}}\int_0^{2\pi} d\theta = \frac{4\cdot 5^{3/2}\pi}{3^2\sqrt{6}}.$$

Answer to Question A3b: $\iiint_K \rho(x,y,z)dV = \frac{4\cdot 5^{3/2}\pi}{3^2\sqrt{6}}$.

Del B.

Question B1. Let D be the domain in $\mathbb{R}^2$ that is defined by the inequalities $x^2 + 4y^2 \leq 4$,

$$f(x,y) = \frac{x+y}{x^2+4y^2+1}.$$

a) Find a parametrisation to the boundary of D: i.e. for the curve $x^2 + 4y^2 = 4$. **2 p.**

b) Compute the maximum value of the function f on the domain D. **4 p.**

Solution to Question B1:

a) This is standard ellipse and can be parametrised as $x(t) = 2\cos(t)$ and $y(t) = \sin(t)$, where $0 \leq t < 2\pi$.

Answer to Question B1a: A parametraization is given by $x(t) = 2\cos(t)$ and $y(t) = \sin(t)$, where $0 \leq t < 2\pi$.

b) In the interior of the domain the maximum value of a differentiable function can only occur at critical points: i.e. $\nabla f(x,y) = (0,0)$.

We have

$$\nabla f(x,y) = \left(\frac{-x^2+4y^2+1-2xy}{(x^2+4y^2+1)^2}, \frac{x^2-4y^2+1-8xy}{(x^2+4y^2+1)^2} \right).$$

Hence inner critical points are given by

$$\begin{aligned} -x^2+4y^2+1-2xy &= 0 \\ x^2-4y^2+1-8xy &= 0, \end{aligned}$$

which gives us the solution

$$(x,y) = \left(\frac{2}{\sqrt{5}}, \frac{1}{2\sqrt{5}} \right) \text{ och } (x,y) = \left(-\frac{2}{\sqrt{5}}, -\frac{1}{2\sqrt{5}} \right).$$

Both solutions satisfy

$$\left(\frac{2}{\sqrt{5}} \right)^2 + 4\left(\frac{1}{2\sqrt{5}} \right)^2 = \left(-\frac{2}{\sqrt{5}} \right)^2 + 4\left(-\frac{1}{2\sqrt{5}} \right)^2 = 1 < 4,$$

so both of these points are inner points. We can calculate that

$$f\left(\frac{2}{\sqrt{5}}, \frac{1}{2\sqrt{5}} \right) = \frac{1}{2\sqrt{5}} \frac{5}{\frac{4}{5}+\frac{1}{5}+1} = \frac{\sqrt{5}}{4}$$

and similarly $f\left(\frac{2}{\sqrt{5}}, \frac{1}{2\sqrt{5}} \right) = -\frac{\sqrt{5}}{4}$. A possible maximum is $\frac{\sqrt{5}}{4}$ at $\left(\frac{2}{\sqrt{5}}, \frac{1}{2\sqrt{5}} \right)$.

To find find possible maximum values on the boundary, we use the parameterization from a)

$$f(2\cos(t), \sin(t)) = \frac{2\cos(t)+\sin(t)}{5}. \tag{5}$$

We find critical points as usual by deriving (5) and set equal to zero:

$$\frac{-2\sin(t)+\cos(t)}{5} = 0. \tag{6}$$

We get from (6) that possible critical points on the boundary are $(2\cos(t), \sin(t))$ when

$$t = \arctan(1/2) \quad \text{eller} \quad t = \arctan(1/2) + \pi.$$

inserting in f we have

$$f(2\cos(\arctan(1/2)), \sin(\arctan(1/2))) = \frac{1}{\sqrt{5}}$$

and

$$f(2\cos(\arctan(1/2)+\pi), \sin(\arctan(1/2)+\pi)) = -\frac{1}{\sqrt{5}}.$$

Since

$$f(2\cos(\arctan(1/2)), \sin(\arctan(1/2))) = \frac{1}{\sqrt{5}} < \frac{\sqrt{5}}{4} = f\left(\frac{2}{\sqrt{5}}, \frac{1}{2\sqrt{5}}\right)$$

and the max. is obtained at

$$\left(\frac{2}{\sqrt{5}}, \frac{1}{2\sqrt{5}}\right).$$

Answer to Question B1b: Maximum is $\frac{\sqrt{5}}{4}$ and attained at

$$(x, y) = \left(\frac{2}{\sqrt{5}}, \frac{1}{2\sqrt{5}}\right).$$

Question B2. Compute the curve integral $\int_\gamma (2xy - x^2)dx + (3x+y)dy$ where γ is the positively
6 p. oriented edge of the bounded area in the first quadrant bounded by the curves $y = x^2$, and $x = y^2$.

Solution to Question B2: We can either calculate the line integral directly or we can use Green's Theorem. Let us use Green's Theorem:

$$\oint_\gamma F_{(1)}(x,y)dx + F_{(2)}(x,y)dy = \iint_D \left(\frac{\partial F_{(2)}(x,y)}{\partial x} - \frac{\partial F_{(1)}(x,y)}{\partial y}\right) dA,$$

if D is the area inside the closed curve γ.
The region inside γ will be given by the inequalities $0 \le x \le 1$ and $x^2 \le y \le \sqrt{x}$ so the line integral becomes

$$\int_\gamma (2xy - x^2)dx + (3x+y)dy = \iint_D \left(\frac{\partial(3x+y)}{\partial x} - \frac{\partial(2xy - x^2)}{\partial y}\right) dA =$$

$$\int_0^1 \int_{x^2}^{\sqrt{x}} (3 - 2x)\, dydx = \int_0^1 \left(3\sqrt{x} - 3x^2 - 2x^{3/2} + 2x^3\right) dx = \frac{7}{10}.$$

Answer to Question B2. $\int_\gamma (2xy - x^2)dx + (3x+y)dy = \frac{7}{10}$.

Del C.

Question C1. Compute the surface integral $\iint_\mathcal{S} f(x,y,z)dS$ where $f(x,y,z) = \frac{z}{\sqrt{x^2+z^2}}$ and the surface $\mathcal{S}$ is given by

$$\mathcal{S} = \{(x,y,z):\ x^2+y^2+z^2 = 1, \quad z \ge 0, \quad -1/\sqrt{2} < y < 1/\sqrt{2}\,\}.$$

6 p.

Solution to Question C1: We parametrize the surface

$$\mathbf{r}(\theta, v) = (\sqrt{1-v^2}\cos(\theta), v, \sqrt{1-v^2}\sin(\theta))$$

where $0 \le \theta \le \pi$ and $-\frac{1}{\sqrt{2}} \le v \le \frac{1}{\sqrt{2}}$.
The surface element dS can be calculated

$$dS = \left|\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial v}\right| d\theta dv =$$

$$\left|\left(-\sqrt{1-v^2}\cos(\theta), -v(\cos^2(\theta)+\sin^2(\theta)), -\sqrt{1-v^2}\sin(\theta)\right)\right| d\theta dv =$$

$$\left((1-v^2)(\cos^2(\theta)+\sin^2(\theta)) + v^2\right)^{1/2} d\theta dv = d\theta dv.$$

We have

$$\iint_{\mathcal{S}} f(x,y,z)dS = \int_0^{\pi}\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{\sqrt{1-v^2}\sin(\theta)}{\sqrt{1-v^2}} dv d\theta = \sqrt{2}\int_0^{\pi} \sin(\theta)d\theta = 2\sqrt{2}.$$

Alternative geometric method: In the (x,z) plane we have semicircles (we call them γ_y) of radius $\sqrt{1-y^2}$, where $-1/\sqrt{2} < y < 1/\sqrt{2}$, ie the integral can be written as we integrate over these semicircles and then we integrate along the y-axis. This give

$$\iint_{\mathcal{S}} f(x,y,z)dS = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\int_{\gamma_y} \frac{z}{\sqrt{1-y^2}} d\sigma dy = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-y^2}}\left(\int_{\gamma_y} z d\sigma\right) dy.$$

Now we can use polar coordinates in the (x,z) plane to parameterize γ_y which gives

$$x = \sqrt{1-y^2}\cos\theta, \qquad z = \sqrt{1-y^2}\sin\theta, \qquad 0 \le \theta \le \pi.$$

$$\int_{\gamma_y} z d\sigma = \int_0^{\pi} \sqrt{1-y^2}\sin\theta d\theta = 2\sqrt{1-y^2}.$$

If we insert it in the integral above, we get

$$\iint_{\mathcal{S}} f(x,y,z)dS = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-y^2}}\left(\int_{\gamma_y} z d\sigma\right) dy = 2\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} dy = 2\sqrt{2}.$$

Answer to Question C1: $\iint_{\mathcal{S}} f(x,y,z)dS = 2\sqrt{2}$.

Question C2. Let

$$p(a_0, a_1, a_2, x) = x^3 + a_2x^2 + a_1x + a_0$$

be a third degree polynomial with coefficients $a_0, a_1, a_2 \in \mathbb{R}$. We can write the polynomial as

$$(x - x_1)(x - x_2)(x - x_3)$$

where x_1, x_2 and x_3 are the roots of the polynomial. We may consider the roots to be functions of the coefficients a_0, a_1, a_2: $x_1(a_0, a_1, a_2)$, $x_2(a_0, a_1, a_2)$ and $x_3(a_0, a_1, a_2)$. We will assume that the roots are real functions of the coefficients.

a) Use the implicit function theorem to find conditions for the roots $x_1(a_0, a_1, a_2)$, $x_2(a_0, a_1, a_2)$ and $x_3(a_0, a_1, a_2)$ so that they are differentiable w.r.t. a_0, a_1 and a_2. **3 p.**

b) Let $\mathbf{a}(t) = (a_0(t), a_1(t), a_2(t))$ be differentiable curve in $\mathbb{R}^3$ such that $\mathbf{a}(0) = (-6, 11, -6)$. Then

$$p(a_0(0), a_1(0), a_2(0), x) = x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3).$$

Find $\mathbf{a}'(0)$ if $\frac{dx_1(\mathbf{a}(t))}{dt}\lfloor_{t=0} = 0$, $\frac{dx_2(\mathbf{a}(t))}{dt}\lfloor_{t=0} = 0$ och $\frac{dx_3(\mathbf{a}(t))}{dt}\lfloor_{t=0} = 1$. Here x_1, x_2 and x_3 are the roots such that $x_1(-6, 11, -6) = 1$, $x_2(-6, 11, -6) = 2$ and $x_3(-6, 11, -6) = 3$. **3 p.**

Solution to Question C2a: We will use implicit the function theorem. Therefore, we define the function

$$\mathbf{P}(a_0, a_1, a_2, x_1, x_2, x_3) = \begin{bmatrix} p(a_0, a_1, a_2, x_1) \\ p(a_0, a_1, a_2, x_2) \\ p(a_0, a_1, a_2, x_3) \end{bmatrix}.$$

We observe that $\mathbf{P}$ is differentiable (because each component is a polynomial in the variables). If x_1, x_2 och x_3 are the roots to $p(a_1, a_1, a_2, x) = 0$ we will have

$$\mathbf{P}(a_0, a_1, a_2, x_1, x_2, x_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The implicit function statement then says that we can write (x_1, x_2, x_3) as a differentiable function of $\mathbf{a} = (a_0, a_1, a_2)$ so that

$$\mathbf{P}(a_0, a_1, a_2, x_1(\mathbf{a}), x_2(\mathbf{a}), x_3(\mathbf{a})) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

in a small neighborhood of a point where

$$\frac{\partial(\mathbf{P})}{\partial(a_0, a_1, a_2)} \neq 0.$$

We compute

$$\frac{\partial(\mathbf{P})}{\partial(a_0, a_1, a_2)} = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} =$$

$$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 \\ 0 & x_3 - x_1 & x_3^2 - x_1^2 \end{vmatrix} =$$

$$(x_2 - x_1)(x_3 - x_1) \begin{vmatrix} 1 & x_1 & x_1^2 \\ 0 & 1 & x_2 + x_1 \\ 0 & 1 & x_3 + x_1 \end{vmatrix} =$$

$$(x_2 - x_1)(x_3 - x_1) \begin{vmatrix} 1 & x_1 & x_1^2 \\ 0 & 1 & x_2 + x_1 \\ 0 & 0 & x_3 - x_2 \end{vmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2),$$

where we used that the determinant is not affected by row operations so we could subtract the first row from the second and third without affecting the determinant, we also broke $(x_2 - x_1)$ from the second and $(x_3 - x_1)$ from the third row and subtracted the second row from the last in the penultimate step.

We see that the determinant is non-zero if and only if all roots are different.

Answer to Question C2a: *We can write the roots (x_1, x_2, x_3) as differentiable functions of (a_0, a_1, a_2) if all roots are different: $x_1 \neq x_2$, $x_1 \neq x_3$ and $x_2 \neq x_3$.*

C2b We know from sub-question a) that x_1, x_2 and x_3 can be written as differentiable functions of $\mathbf{a}(t)$. We can therefore calculate, for $i = 0, 1, 2$ and $j = 1, 2, 3$,

$$0 = \frac{dp(a_0, a_1, a_2, x_j(a_0, a_1, a_2))}{da_i} = x_j^i + \frac{\partial p(a_0, a_1, a_2, x_j)}{\partial x_j} \frac{\partial x_j}{\partial a_i}.$$

We insert that the derivative of the polynomial with the values $(a_0, a_1, a_2) = (-6, 11, -6)$ so we get that

$$\frac{\partial p(-6, 11, -6, x_j)}{\partial x_j} = 3x_j^2 - 12x_j + 11.$$

Since $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$ we get

$$\begin{bmatrix} \frac{\partial x_1}{\partial a_0} & \frac{\partial x_1}{\partial a_1} & \frac{\partial x_1}{\partial a_2} \\ \frac{\partial x_2}{\partial a_0} & \frac{\partial x_2}{\partial a_1} & \frac{\partial x_2}{\partial a_2} \\ \frac{\partial x_3}{\partial a_0} & \frac{\partial x_3}{\partial a_1} & \frac{\partial x_3}{\partial a_2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -1 & -2 \\ -\frac{1}{2} & -\frac{3}{2} & -\frac{9}{2} \end{bmatrix}.$$

By the chain rule

$$\begin{bmatrix} \frac{dx_1(\mathbf{a}(t))}{dt} \\ \frac{dx_2(\mathbf{a}(t))}{dt} \\ \frac{dx_3(\mathbf{a}(t))}{dt} \end{bmatrix} \Bigg|_{t=0} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -1 & -2 \\ -\frac{1}{2} & -\frac{3}{2} & -\frac{9}{2} \end{bmatrix} \begin{bmatrix} a_0'(0) \\ a_1'(0) \\ a_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where we used the given values of the derivatives.
We can easily solve the last system of equations which gives

$$\begin{bmatrix} a_0'(0) \\ a_1'(0) \\ a_2'(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}.$$

Answer to Question C2b: *The derivatives area* $a_0'(0) = -2$, $a_1'(0) = 3$ *and* $a_2'(0) = -1$.