How can vector calculus be used to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle, and what mathematical principles underpin the observed fluid behaviour?

Mathematics AA HL

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Contents

1 Introduction

Fluid dynamics is today a cornerstone to several fields of study, including ærospace engineering and meteorology. Real world fluid behaviour is intricate and complex. Therefore, to gain insights into the governing principles of fluid flow, simplified and idealized models are used. This essay investigates the application of vector calculus to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle. These idealizations allow for the derivation of some of fluid dynamic's key mathematical formulæ and provides a foundation for understanding less idealized fluids.

This essay will address the question: "How can vector calculus be used to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle, and what mathematical principles underpin the observed fluid behaviour?" Through the derivation of the velocity potential and vector field, this essay aims to demonstrate how fundamental laws of fluid motion can be expressed and used through vector calculus.

1.1 Aim & scope

The scope of this essay will be limited to the theoretical modelling of fluid flow in a two-dimensional space as a vector field under idealized conditions forming steady, inviscid and incompressible fluid flow through the derivation of the velocity-potential. The analysis will be centred on the application of vector calculus to derive fundamental formulæ and describe fluid behaviour around a stationary circular obstacle. Consequently, this essay will not touch on viscous effects, turbulent flow or three-dimensional analysis, nor will it involve any experimental validation. The focus is on the mathematical derivation and analysis of the idealized model.

1.2 Background

1.2.1 Definitions

Definition 1.1. STEADY FLOW refers to flow in which the velocity at every point does not change over time [CRACIUNOIU and CIOCIRLAN, 2001]. Mathematically this can be expressed as: $\frac{\partial \vec{V}}{\partial t} = 0$.

Definition 1.2. INVISCID FLOW refers to the flow of a fluid with 0 viscosity [Anderson, 2003].

1.2.2 Notation

In this paper, the gradient, divergence and curl operators will be denoted using their explicit ∇ forms as follows:

$$\begin{aligned} \text{grad } \mathbf{F} &\equiv \nabla \mathbf{F} \\ \text{div } \mathbf{F} &\equiv \nabla \cdot \mathbf{F} \\ \text{curl } \mathbf{F} &\equiv \nabla \times \mathbf{F} \end{aligned}$$

The directional vector will also be denoted using ∇ as $\nabla_{\vec{v}}\mathbf{F}$.

For the purposes of clarity, vectors in cartesian systems will be denoted $\begin{pmatrix} x \\ y \end{pmatrix}$ whilst vectors in polar systems will be denoted as $\langle r, \vartheta \rangle$

To ensure point-uniqueness within polar systems, all polar coordinates are restricted such as $r \geq 0$, $\vartheta \in [0, 2\pi)$.

6 References

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