How can vector calculus be used to model and analyze incompressible fluid flow in two-dimensional spaces, and what insights can this provide about the vector fields of real-world fluid systems with circular obstacles?

Mathematics AA HL

Word Count: 90

Contents

1 Introduction

Vector calculus provides the foundation and tools for the analysis and modeling of several real-world phenomona, and is integral to understanding several important fields such as aero- & hydrodynamics, as well as the modeling of weather & climates.

Through the use of pure mathematics, this essay will investigate the flow of fluids in 2 dimensional spaces around circular obstacles.

1.1 Aim & scope

This essay will for simplicity's sake only cover fluid flow around circular obstacles in $\mathbb{R}^2$ spaces; an alaysis of fluid flow in $\mathbb{R}^3$ spaces would be much more complex. Furthermore, only incompressible fluids sans sinks and sources ($\mathbf{F} \ni \nabla \cdot \mathbf{F} = 0$), will be analyzed.

Most of the analysis will take place using Green's theorem[see 1.4].

1.2 Background

Vector calculus is the mathematical study of applying multi-variable calculus to vector valued functions, often for the spaces $\mathbb{R}^2$ and $\mathbb{R}^3$.

$$f : \mathbb{R}^n \to \mathbb{R}^n \ni n > 1, n \in \mathbb{Z}$$

$$\rightsquigarrow \nabla = \left.\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ \vdots \end{bmatrix}\right\} n \text{ times} \tag{1}$$

$$\frac{\mathrm{D}f}{\mathrm{D}t} \triangleq \frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f \tag{2}$$

$$\vec{v_1} \otimes \vec{v_2}$$

1.3 Fluid dynamics

An incompressible fluid is any fluid such that $\nabla \cdot \mathbf{F} = 0$, which is to say that the divergence of the fluid is 0.

1.4 Green's theorem

Green's theorem 1 *The double integral over some reigon R of the curl of a vector field* $\mathbf{F}$ *is equal to the line integral over some curve C of* $\mathbf{F}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\iint_R \nabla \times \mathbf{F} \mathrm{d}A = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

$$\frac{\mathrm{D}f}{\mathrm{D}\mathbf{t}} = \iiint_V \left(\frac{\mathrm{D}\rho}{\mathrm{D}\mathbf{t}} + \rho(\nabla \cdot u)\right) dV \tag{3}$$

Lorem ipsum dolor sit amet [Peyret and Taylor, 2012]

2 References

[Peyret and Taylor, 2012] Peyret, R. and Taylor, T. D. (2012). *Computational methods for fluid flow*. Springer Science & Business Media.

3 List of Figures

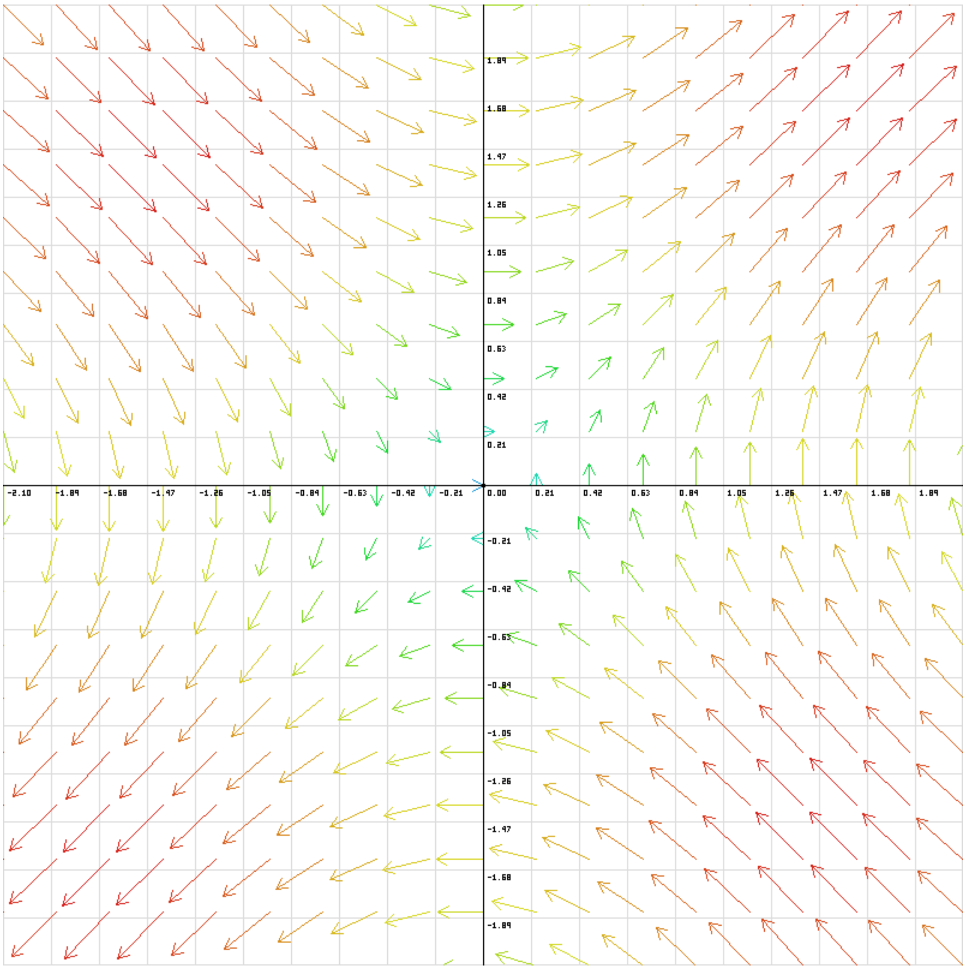

Figure 1: Vector field for $f(x, y) = \begin{pmatrix} \sin y \\ \sin x \end{pmatrix}$

$$\mathbf{F}(t) : \mathbb{R} \to \mathbb{R}^2$$

$$\rightsquigarrow \mathbf{P}(t, \vec{p}) = \vec{p} + \hat{\imath} \iint_0^t \mathbf{F}_x \ \mathrm{d}t + \hat{\jmath} \iint_0^t \mathbf{F}_y \ \mathrm{d}t$$

$$\mathbf{P}(t, \vec{p}) = \vec{p} + \iint_0^t \hat{\imath}\mathbf{F}_x + \hat{\jmath}\mathbf{F}_y \ \mathrm{d}t$$

$$\hat{\jmath} + \hat{\imath}$$