

How can vector calculus be used to model
and analyze incompressible fluid flow in
two-dimensional spaces, and what insights
can this provide about the vector fields of
real-world fluid systems with circular
obstacles?

Mathematics AA HL

Word Count: 398

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1 Introduction

Vector calculus provides the foundation and tools for the analysis and modeling of several real-world phenomena, and is integral to understanding several important fields such as aero- & hydrodynamics, as well as the modeling of weather & climates.

Through the use of pure mathematics, this essay will investigate the flow of fluids in 2 dimensional spaces around circular obstacles. Visual representations through mediums such as vector field plots (plotted through a custom program

1.1 Aim & scope

This essay will for simplicity's sake only cover fluid flow around circular obstacles in \mathbb{R}^2 spaces; an analysis of fluid flow in \mathbb{R}^3 spaces would be much more complex. Furthermore, only incompressible fluids sans sinks and sources ($\mathbf{F} \ni \nabla \cdot \mathbf{F} = 0$), will be analyzed.

Most of the analysis will take place using Green's theorem^[see 1.4].

1.2 Background

Vector calculus is the mathematical study of applying multi-variable calculus to vector valued functions, often for the spaces \mathbb{R}^2 and \mathbb{R}^3 .

1.2.1 The nabla operator

In one dimensional calculus, the gradient (slope) of a graph can be calculated at any point by differentiating the function and evaluating it at said point.

$$f(x) = x^2 \implies \frac{df}{dx} = 2x$$

Defining this as the equation of greatest ascent at point x , even though there in this case is only one valid direction of ascent for the function since we are working in the space \mathbb{R}^1 , will help us when extending this concept of greatest ascent to \mathbb{R}^n spaces.

In multi-variable calculus a new tool called the partial derivative helps us differentiate multi-variable functions. It works identically to a regular derivative but treats all variables other than the one being differentiated by as constants.

$$f(x, y) = x^2 + y^2 \implies \frac{\partial f}{\partial x} = 2x$$

This also finds the equation of greatest ascent for f in the direction differentiated by, so if we were to combine the partial derivatives in all directions f takes as parameters into a vector, then we end up with a vector pointing in the direction of steepest ascent.

The nabla (alt. gradient) operator, denoted by the symbol ∇ (pronounced

nabla or grad), is thereby defined for some function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ as:

$$\nabla = \left\{ \begin{array}{c} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ \vdots \end{array} \right\} n \text{ times} \quad (1)$$

Applying this to our previous example $f(x, y) = x^2 + y^2$, we get:

$$\nabla f = \left[\begin{array}{c} \frac{\partial}{\partial x} x^2 + y^2 \\ \frac{\partial}{\partial y} x^2 + y^2 \end{array} \right] = \left[\begin{array}{c} 2x \\ 2y \end{array} \right]$$

Meaning that at some point (x, y) , the direction of steepest incline will be $2(x, y)$. The nabla operator proves foundational to vector calculus and is the backbone of several important concepts.

1.2.2 Other

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^n \ni n > 1, n \in \mathbb{Z}$$

$$\frac{Df}{Dt} \triangleq \frac{\partial f}{\partial t} + \underbrace{\vec{v} \cdot \nabla f}_{\text{Directional derivative } \nabla_{\vec{v}} f} \quad (2)$$

$$\vec{v}_1 \otimes \vec{v}_2$$

1.3 Fluid dynamics

An incompressible fluid is any fluid such that $\nabla \cdot \mathbf{F} = 0$, which is to say that the divergence of the fluid is 0.

1.4 Green's theorem

Green's theorem 1. *The double integral over some region R of the curl of a vector field \mathbf{F} is equal to the line integral over some curve C of \mathbf{F}*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\iint_R \nabla \times \mathbf{F} dA = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

$$\frac{Df}{Dt} = \iiint_V \left(\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{u}) \right) dV \quad (3)$$

Lorem ipsum dolor sit amet [Peyret and Taylor, 2012]

2 References

[Peyret and Taylor, 2012] Peyret, R. and Taylor, T. D. (2012). *Computational methods for fluid flow*. Springer Science & Business Media.

3 List of Figures

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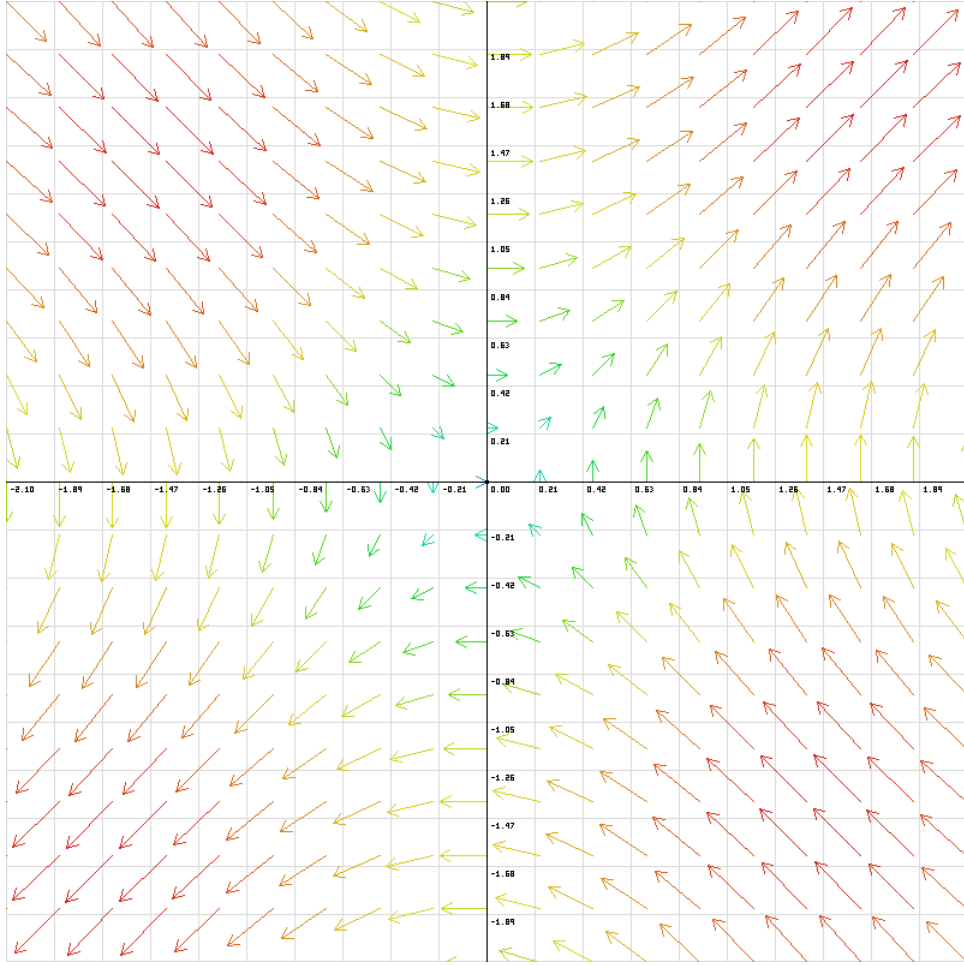


Figure 1: Vector field for $f(x, y) = \begin{pmatrix} \sin y \\ \sin x \end{pmatrix}$

$$\mathbf{F}(t) : \mathbb{R} \rightarrow \mathbb{R}^2$$

$$\rightsquigarrow \mathbf{P}(t, \vec{p}) = \vec{p} + \hat{i} \iint_0^t \mathbf{F}_x \, dt + \hat{j} \iint_0^t \mathbf{F}_y \, dt$$

$$\mathbf{P}(t, \vec{p}) = \vec{p} + \iint_0^t \hat{i} \mathbf{F}_x + \hat{j} \mathbf{F}_y \, dt$$

$$\hat{j} + \hat{i}$$

$$A = B$$

$$= C$$

substitution

Proof. Green's Theorem

$$B \triangleq C$$

$$A = B$$

$$= C$$

□

$$a \underbrace{\quad a \quad}_{\text{banana}}$$