1 Introduction

Fluid dynamics is today a cornerstone to several fields of study, including ærospace engineering and meteorology. Real world fluid behaviour is intricate and complex. Therefore, to gain insights into the governing principles of fluid flow, simplified and idealised models are used. This essay investigates the application of vector calculus to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle. These idealisations allow for the derivation of some of fluid dynamic's key mathematical formulæ and provides a foundation for understanding less idealised fluids.

This essay will address the question: "How can vector calculus be used to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle, and what mathematical principles underpin the observed fluid behaviour?" Through the derivation of the velocity potential and vector field, this essay aims to demonstrate how fundamental laws of fluid motion can be expressed and used through vector calculus.

1.1 Aim & scope

The scope of this essay will be limited to the theoretical modelling of fluid flow in a two-dimensional space as a vector field under idealised conditions forming steady, inviscid and incompressible fluid flow through the derivation of the velocity-potential. The analysis will be centred on the application of vector calculus to derive fundamental formulæ and describe fluid behaviour around a stationary circular obstacle. Consequently, this essay will not touch on viscous effects, turbulent flow or three-dimensional analysis, nor will it involve any experimental validation. The focus is on the mathematical derivation and analysis of the idealised model.

1.2 Background

1.2.1 Glossary

Definition 1.1. ***Steady flow*** refers to flow in which the velocity at every point does not change over time [CRACIUNOIU and CIOCIRLAN, 2001].

Definition 1.2. ***Inviscid flow*** is the flow of a fluid with 0 viscosity [Anderson, 2003].

Definition 1.3. An ***incompressible fluid*** is a fluid whose density at every point does not change over time [Ahmed, 2019].

Definition 1.4. A ***scalar field*** is a function mapping points in space to scalar quantities such as temperatures.

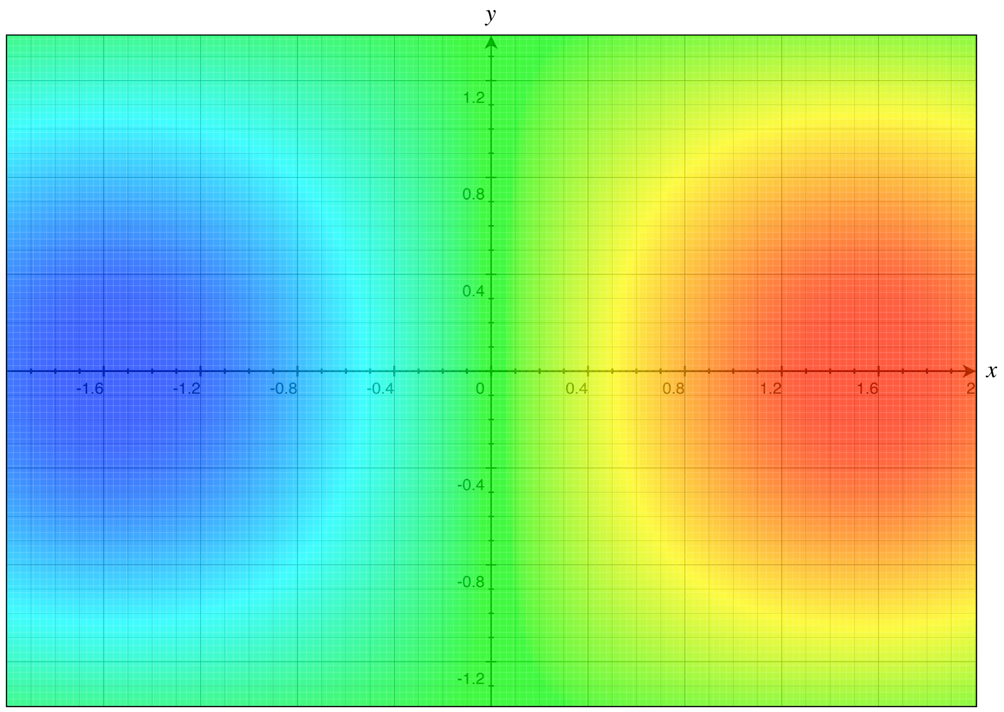


Figure 1: Scalar field plotted for the function $f(x, y) = \sin(x)\cos y$

Definition 1.5. A ***vector field*** is a function mapping points in space to vector quantities [Brezinski, 2006]. In the case of fluid dynamics, vector fields often model quantities like fluid velocity.

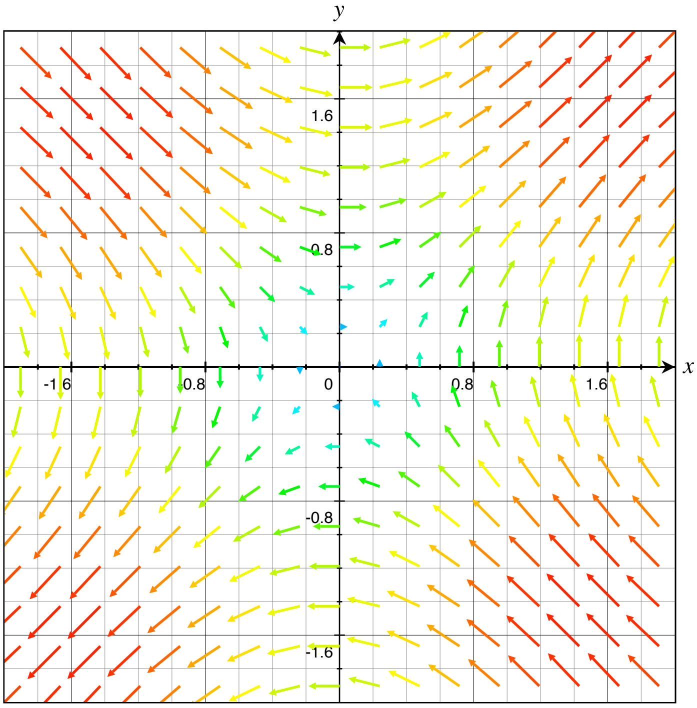


Figure 2: Vector field plotted for the function $f(x, y) = \begin{pmatrix} \sin y \\ \sin x \end{pmatrix}$

Definition 1.6. The ***velocity potential*** ϕ is a scalar field whose gradient is the velocity vector field of some fluid, mathematically $\mathbf{V} = \nabla\phi$. The quantity is defined for irrotational flow which is a resulting property of the idealisations made in this essay[see ??].

1.2.2 Notation

Vector calculus, like one-variable calculus, has no standardized notation. This essay will employ the following notation:

- ∇:
  - ∇F: The gradient of some scalar field F.
  - $\nabla \cdot \mathbf{F}$: The divergence of some vector field $\mathbf{F}$.
  - $\nabla \times \mathbf{F}$: The curl of some vector field $\mathbf{F}$.
  - $\nabla_{\mathbf{v}} f$: The directional derivative of f in the direction of some vector $\mathbf{v}$
- Δ: The Laplacian operator
- $\mathbb{D}_\delta(\langle x,\, y\rangle)$: The set of the points in an open disk centred at $(x,\, y)$ with radius δ

- $\hat{\imath}$, $\hat{\jmath}$ & $\hat{k}$: Unit vectors in the positive x, y and z directions respectively.
- $\hat{r}$ & $\hat{\vartheta}$: Unit vectors in the positive r and ϑ directions respectively.

1.2.3 The mean value theorem

To support the derivations made later in this essay, particularly in the proof of Clairaut's theorem[see 2.1], fundamental concepts and theorems from single-variable calculus are introduced here, including the Mean Value Theorem and the lemmas it builds upon.

Lemma 1.1 (The extreme value theorem)**.** If a function f is continuous on the finite interval $[a, b]$, then there exists $A, B \in [a, b]$ such that $f(A) \leq f(x) \leq f(B)\,\forall x \in [a, b]$. Thus, at the points A and B, f has an absolute minimum $m = f(A)$ and an absolute maximum $M = f(B)$.

Lemma 1.2 (Rolle's theorem)**.** If a function f is continuous on the interval $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then $\exists\, c \in (a, b)$ such that $f'(c) = 0$.

Proof. Consider two cases:

Case 1: f remains constant over $[a, b]$

If $f(x) = f(a) = f(b)\,\forall x \in (a, b)$, then $f'(x) = 0$, and the theorem holds trivially.

Case 2: f is not constant over $[a, b]$

If f is not constant over $[a, b]$ and $f(a) = f(b)$, then Lemma 1.1 asserts that there must exist an absolute maximum or minimum that occur at some point $\eta \in (a, b)$. Since f is differentiable over (a, b), then any point η where an absolute extremum occurs must also be a local extremum. Consider the case where η is a local maxima (the proof for the case of local minima is analogous). Then let the interval $I = (\eta - \delta, \eta + \delta)$ for some $\delta > 0$ such that $\forall X \in I, f(X) \leq f(\eta)$.

Let $h < 0$ be a number sufficiently small such that $\eta + h \in I$. $f(\eta + h) \leq f(\eta) \implies f(\eta + h) - f(\eta) \leq 0$. Thus,

$$\frac{f(\eta + h) - f(\eta)}{h} \geq 0 \because \begin{cases} f(\eta + h) - f(\eta) & \leq 0 \\ h & \leq 0 \end{cases}$$

Taking the left-hand limit as $h \to 0$,

$$\lim_{h \to 0^-} \frac{f(\eta + h) - f(\eta)}{h} = f'(\eta)$$

Now let $H > 0$ be a number sufficiently small such that $\eta - H \in I$.

$$\begin{aligned} \frac{f(\eta + H) - f(\eta)}{H} &\leq 0 \because \begin{cases} f(\eta + H) - f(\eta) & \leq 0 \\ H & \geq 0 \end{cases} \\ \lim_{H \to 0^+} \frac{f(\eta + H) - f(\eta)}{H} &= f'(\eta) \end{aligned}$$

Thus,

$$0 \geq \lim_{H \to 0^+} \frac{f(\eta + H) - f(\eta)}{H} = f'(\eta) = \lim_{h \to 0^-} \frac{f(\eta + h) - f(\eta)}{h} \geq 0$$

$$\therefore f'(\eta) = 0$$

Since the same would apply for local minima, then for any local extrema $\eta \in (a, b)$, of which Lemma 1.1 asserts there must exist at least one, $f'(\eta) = 0$. ∎

Lemma 1.3 (The mean value theorem)**.** For any function f continuous on the interval $[a, b]$ and differentiable on the interval (a, b), $\exists\, c \in (a, b)$ such that

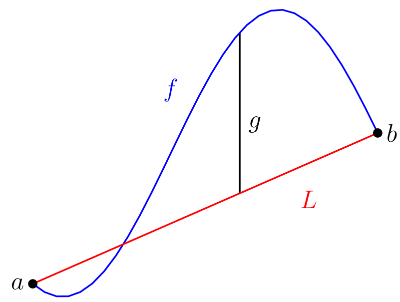
$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1}$$

Proof. Consider the region of some function f on the finite interval $[a, b]$ over which f is continuous and differentiable over (a, b). Let the function L represent the straight line between the points $(a,\ f(a))$ and $(b,\ f(b))$, which is given by the expression:

$$L(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

Now consider the function g, defined as the difference between f and L:

$$g(x) = L(x) - f(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) - f(x)$$



Computing the derivative of g with respect to x gives:

$$g'(x) = \frac{f(b) - f(a)}{b - a} - f'(x)$$

Since $g(a) = g(b) = 0$, Lemma 1.2 asserts that there is at least one point $c \in (a, b)$ such that $g'(c) = 0$. Thus, at c,

$$\begin{aligned} 0 = g'(c) &= \frac{f(b) - f(a)}{b - a} - f'(c) \\ \implies f'(c) &= \frac{f(b) - f(a)}{b - a} \end{aligned}$$

■

2 Vector calculus

2.1 The fundamentals of vector calculus

Definition 2.1. ***Partial derivatives*** are an extension of single-variable derivatives in which all variables save the one being differentiated by are treated as constants [Mortimer, 2013]. A formal definition of the partial derivative of some function f with respect to a parameter x_n can be expressed as:

$$\frac{\partial f}{\partial x_n} = \lim_{\delta \to 0} \frac{f(x_1, x_2, \cdots, x_n + \delta, \cdots) - f(x_1, x_2, \cdots, x_n, \cdots)}{\delta} \tag{2}$$

Partial derivatives allow for the analysis of how multi-variable functions such as scalar- or vector fields change with respect to just one spatial dimension. For example, consider the function $f(x, y) = x^2 y + \sin(x)\sin y$:

$$\frac{\partial f}{\partial x} = 2xy + \cos(x)\sin y \qquad\qquad \frac{\partial f}{\partial y} = x^2 + \sin(x)\cos y$$

n-th order partial derivatives are denoted, similarly to normal calculus, as

$$\frac{\partial^n f}{\partial x^n} = \underbrace{\frac{\partial}{\partial x}\cdots\frac{\partial}{\partial x}\frac{\partial f}{\partial x}}_{n \text{ times}}$$

Definition 2.2. ***Mixed partial derivatives*** are partial derivatives of a function taken with respect to multiple variables [Garrett, 2015]. This is denoted as

$$\frac{\partial^2 f}{\partial\alpha\partial\beta} \equiv \frac{\partial}{\partial\beta}\frac{\partial f}{\partial\alpha}$$

where both α and β are parameters of f.

Lemma 2.1 (Clairaut's theorem)**.** Let $f(\alpha, \beta)$ be a function of two parameters α and β. If the mixed partial derivatives $\frac{\partial^2 f}{\partial\alpha\partial\beta}$ and $\frac{\partial^2 f}{\partial\beta\partial\alpha}$ exist and are continuous in the open disk $\mathbb{D}_\delta(\langle\alpha_0,\, \beta_0\rangle)$ centred at $(\alpha_0,\, \beta_0)$ with radius $\delta > 0$, then

$$\left.\frac{\partial^2 f}{\partial\alpha\partial\beta}\right|_{(\alpha_0,\, \beta_0)} = \left.\frac{\partial^2 f}{\partial\beta\partial\alpha}\right|_{(\alpha_0,\, \beta_0)}$$

[Garrett, 2015]

Proof. Let $(\alpha_0,\, \beta_0)$ and $(\alpha_1,\, \beta_1)$ be points in the domain of f. Consider a rectangular region bound by the points $W\,(\alpha_0,\, \beta_0)\,, X\,(\alpha_1,\, \beta_0)\,, Y\,(\alpha_1,\, \beta_1)$ and $Z\,(\alpha_0,\, \beta_1)$. $\frac{\partial f}{\partial\alpha}$ and $\frac{\partial f}{\partial\beta}$ exist in a neighbourhood of this rectangle, and the mixed partial derivatives $\frac{\partial^2 f}{\partial\beta\partial\alpha}$ and $\frac{\partial^2 f}{\partial\alpha\partial\beta}$ exist and are continuous in this neighbourhood. Let Q be such that

$$Q = [f(\alpha_1, \beta_1) - f(\alpha_0, \beta_1)] - [f(\alpha_1, \beta_0) - f(\alpha_0, \beta_0)]$$

According to the mean value theorem (MVT) $\exists\, \xi_0, \xi_1 \in [\alpha_0, \alpha_1]$ such that

$$
\begin{aligned}
\left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_0,\, \beta_0)} &= \frac{f(\alpha_1, \beta_0) - f(\alpha_0, \beta_0)}{\alpha_1 - \alpha_0} \\
\left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_1,\, \beta_1)} &= \frac{f(\alpha_1, \beta_1) - f(\alpha_0, \beta_1)}{\alpha_1 - \alpha_0}
\end{aligned}
$$

Thus Q can be expressed as

$$
\begin{aligned}
Q &= \left(\left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_0,\, \beta_0)} (\alpha_1 - \alpha_0) \right) - \left(\left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_1,\, \beta_1)} (\alpha_1 - \alpha_0) \right) \\
&= \left(\left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_0,\, \beta_0)} - \left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_1,\, \beta_1)} \right) (\alpha_1 - \alpha_0)
\end{aligned}
$$

Now let R be the equivalent of Q in the direction of β,

$$
R = [f(\alpha_1, \beta_1) - f(\alpha_1, \beta_0)] - [f(\alpha_0, \beta_1) - f(\alpha_0, \beta_0)]
$$

By the MVT $\exists\, \zeta_0, \zeta_1 \in [\beta_0, \beta_1]$ such that

$$
\begin{aligned}
\left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_0,\, \zeta_0)} &= \frac{f(\alpha_0, \beta_1) - f(\alpha_0, \beta_0)}{\beta_1 - \beta_0} \\
\left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_1,\, \zeta_1)} &= \frac{f(\alpha_1, \beta_1) - f(\alpha_1, \beta_0)}{\beta_1 - \beta_0}
\end{aligned}
$$

Thus R can be expressed as

$$
\begin{aligned}
R &= \left(\left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_0,\, \zeta_0)} (\beta_1 - \beta_0) \right) - \left(\left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_1,\, \zeta_1)} (\beta_1 - \beta_0) \right) \\
&= \left(\left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_0,\, \zeta_0)} - \left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_1,\, \zeta_1)} \right) (\beta_1 - \beta_0)
\end{aligned}
$$

Rearranging Q and R,

$$
\begin{aligned}
Q &= [f(\alpha_1,\beta_1) - f(\alpha_0,\beta_1)] - [f(\alpha_1,\beta_0) - f(\alpha_0,\beta_0)] \\
&= f(\alpha_1,\beta_1) - f(\alpha_1,\beta_0) - f(\alpha_0,\beta_1) + f(\alpha_0,\beta_0) \\
&= [f(\alpha_1,\beta_1) - f(\alpha_1,\beta_0)] - [f(\alpha_0,\beta_1) - f(\alpha_0,\beta_0)] = R \\
\therefore Q &= R
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\left(\left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_0,\,\beta_0)} - \left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_1,\,\beta_1)} \right)(\alpha_1-\alpha_0) = \left(\left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_0,\,\zeta_0)} - \left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_1,\,\zeta_1)} \right)(\beta_1-\beta_0) \\
\rightsquigarrow\ &\frac{\partial f/\partial\alpha\,\big|_{(\xi_0,\,\beta_0)} - \partial f/\partial\alpha\,\big|_{(\xi_1,\,\beta_1)}}{\beta_1-\beta_0} = \frac{\partial f/\partial\beta\,\big|_{(\alpha_0,\,\zeta_0)} - \partial f/\partial\beta\,\big|_{(\alpha_1,\,\zeta_1)}}{\alpha_1-\alpha_0} \qquad (3)
\end{aligned}
$$

Applying the MVT again $\exists\, \xi^* \in (\xi_0, \xi_1), \beta^* \in (\beta_0, \beta_1)$ such that

$$
\begin{aligned}
\left.\frac{\partial^2 f}{\partial\alpha\partial\beta}\right|_{(\xi^*,\,\beta^*)} &= \frac{\partial f/\partial\alpha\,\big|_{(\xi_1,\,\beta_1)} - \partial f/\partial\alpha\,\big|_{(\xi_0,\,\beta_0)}}{\beta_1-\beta_0} \\
\implies -\left.\frac{\partial^2 f}{\partial\alpha\partial\beta}\right|_{(\xi^*,\,\beta^*)} &= \frac{\partial f/\partial\alpha\,\big|_{(\xi_0,\,\beta_0)} - \partial f/\partial\alpha\,\big|_{(\xi_1,\,\beta_1)}}{\beta_1-\beta_0}
\end{aligned}
$$

Similarly, $\exists\, \alpha^* \in (\alpha_0, \alpha_1), \zeta^* \in (\zeta_0, \zeta_1)$ such that

$$
\begin{aligned}
\left.\frac{\partial^2 f}{\partial\beta\partial\alpha}\right|_{(\alpha^*,\,\zeta^*)} &= \frac{\partial f/\partial\beta\,\big|_{(\alpha_1,\,\zeta_1)} - \partial f/\partial\beta\,\big|_{(\alpha_0,\,\zeta_0)}}{\alpha_1-\alpha_0} \\
\implies -\left.\frac{\partial^2 f}{\partial\beta\partial\alpha}\right|_{(\alpha^*,\,\zeta^*)} &= \frac{\partial f/\partial\beta\,\big|_{(\alpha_0,\,\zeta_0)} - \partial f/\partial\beta\,\big|_{(\alpha_1,\,\zeta_1)}}{\alpha_1-\alpha_0}
\end{aligned}
$$

Substituting back into (3),

$$
\begin{aligned}
-\left.\frac{\partial^2 f}{\partial\alpha\partial\beta}\right|_{(\xi^*,\,\beta^*)} &= -\left.\frac{\partial^2 f}{\partial\beta\partial\alpha}\right|_{(\alpha^*,\,\zeta^*)} \\
\implies \left.\frac{\partial^2 f}{\partial\alpha\partial\beta}\right|_{(\xi^*,\,\beta^*)} &= \left.\frac{\partial^2 f}{\partial\beta\partial\alpha}\right|_{(\alpha^*,\,\zeta^*)}
\end{aligned}
$$

Consequently, as $\alpha_1 \to \alpha_0$ and $\beta_1 \to \beta_0$, $\xi^* \to \alpha_0, \beta^* \to \beta_0, \alpha^* \to \alpha_0$ and $\zeta^* \to \beta_0$. Since the derivatives are continuous,

$$\left.\frac{\partial^2 f}{\partial\beta\partial\alpha}\right|_{(\alpha_0,\,\beta_0)} = \left.\frac{\partial^2 f}{\partial\alpha\partial\beta}\right|_{(\alpha_0,\,\beta_0)}$$

Because $(\alpha_0,\, \beta_0)$ is an arbitrary point in the domain, $\frac{\partial^2 f}{\partial\beta\partial\alpha} = \frac{\partial^2 f}{\partial\alpha\partial\beta}$ at all points in the domain where the mixed partial derivatives are continuous. ■

Definition 2.3. The ***nabla*** operator ∇ is a vector containing one partial derivative for each parameter of the scalar valued function applied to [Rapp, 2017]. Thus applying the operator is taking the scalar multiple of the vector ∇ and some function f. For some function $f : \mathbb{R}^n \to \mathbb{R}$, ∇ and ∇f would be given by:

$$\nabla = \begin{bmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \vdots \\ \partial/\partial x_n \end{bmatrix}, \quad \nabla f = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \vdots \\ \partial f/\partial x_n \end{bmatrix}$$

Definition 2.4. The directional derivative in the direction of some vector $\mathbf{v}$ of the function f which is differentiable in the open disk $\mathbb{D}_\delta(\langle x_0,\, y_0\rangle)$ centred at $(x_0,\, y_0)$ with radius $\delta > 0$ is defined as

$$\left.\nabla_{\mathbf{v}} f\right|_{(x_0,\,y_0)} = \frac{\left.\nabla f\right|_{(x_0,\,y_0)} \cdot \mathbf{v}}{\|\mathbf{v}\|}$$

[Giannakidis and Petrou, 2010]

2.2 Divergence & curl

Definition 2.5. The ***divergence*** of a vector field $\mathbf{F}$ is a scalar field denoted as $\nabla \cdot \mathbf{F}$, defined as the dot product of the nabla operator (∇) and the vector field. For the function $\mathbf{F} : x, y \mapsto X(x,y)\hat{\imath} + Y(x,y)\hat{\jmath}$, $\nabla \cdot \mathbf{F}$ would be given by:

$$\nabla \cdot \mathbf{F} = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}$$

Definition 2.6. The ***curl*** of a vector field $\mathbf{F}$ is a vector field denoted as $\nabla \times \mathbf{F}$, defined as the cross product of the nabla operator and the vector field. Typically, since the cross product is only defined for 3 dimensional spaces, 2 dimensional curl is defined as a scalar field which for some function $\mathbf{F} : x, y \mapsto X(x,y)\hat{\imath} + Y(x,y)\hat{\jmath}$ is derived from the coefficient of $\hat{k}$ if the cross product was done as if the z term of both vectors was set to 0. Thus, for such a function $\mathbf{F}$:

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & 0 \\ X & Y & 0 \end{vmatrix} \\
&= \begin{vmatrix} \partial/\partial y & 0 \\ Y & 0 \end{vmatrix} \hat{\imath} - \begin{vmatrix} \partial/\partial x & 0 \\ X & 0 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ X & Y \end{vmatrix} \hat{k} \\
&= 0\hat{\imath} - 0\hat{\jmath} + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) \hat{k} = \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) \hat{k} \\
\leadsto \nabla \times \mathbf{F} &= \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}
\end{aligned}$$

4 References

[Ahmed, 2019] Ahmed, T. (2019). Chapter 6 - fundamentals of reservoir fluid flow. In Ahmed, T., editor, *Reservoir Engineering Handbook (Fifth Edition)*, pages 331–456. Gulf Professional Publishing, fifth edition edition.

[Anderson, 2003] Anderson, J. D. (2003). Flight (aerodynamics). In Meyers, R. A., editor, *Encyclopedia of Physical Science and Technology (Third Edition)*, pages 1–21. Academic Press, New York, third edition edition.

[Brezinski, 2006] Brezinski, M. E. (2006). 1 - the basics (math, waves, etc.) for the non-physical scientist. In Brezinski, M. E., editor, *Optical Coherence Tomography*, pages 3–29. Academic Press, Burlington.

[CRACIUNOIU and CIOCIRLAN, 2001] CRACIUNOIU, N. and CIOCIRLAN, B. O. (2001). 8 - fluid dynamics. In Marghitu, D. B., editor, *Mechanical Engineer's Handbook*, Academic Press Series in Engineering, pages 559–610. Academic Press, San Diego.

[Garrett, 2015] Garrett, S. (2015). Chapter 12 - multivariate calculus. In Garrett, S., editor, *Introduction to Actuarial and Financial Mathematical Methods*, pages 377–410. Academic Press, San Diego.

[Giannakidis and Petrou, 2010] Giannakidis, A. and Petrou, M. (2010). Chapter 4 - conductivity imaging and generalized radon transform: A review. In Hawkes, P. W., editor, *Advances in Imaging and electron Physics*, volume 162 of *Advances in Imaging and Electron Physics*, pages 129–172. Elsevier.

[Mortimer, 2013] Mortimer, R. G. (2013). Chapter 8 - differential calculus with several independent variables. In Mortimer, R. G., editor, *Mathematics for Physical Chemistry (Fourth Edition)*, pages 89–106. Elsevier, Boston, fourth edition edition.

[Rapp, 2017] Rapp, B. E. (2017). Chapter 7 - vector calculus. In Rapp, B. E., editor, *Microfluidics: Modelling, Mechanics and Mathematics*, Micro and Nano Technologies, pages 137–188. Elsevier, Oxford.

5 List of Figures