

6 References

- [Peyret and Taylor, 2012] Peyret, R. and Taylor, T. D. (2012). *Computational methods for fluid flow*. Springer Science & Business Media.
- [Stony Brook University, 2021] Stony Brook University (2021). Mat132 episode 25: Second-order differential equations.

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8 Research

8.1 Potential flow around a circular cylinder

A cylinder of radius L is placed in two-dimensional, incompressible, inviscid flow which flows in the direction of \hat{i} . Far away from the cylinder the velocity field \mathbf{V} can be described as:

$$\mathbf{V} = U\hat{i} \tag{6}$$

Where U is some constant. Since the cylinder is impermissible, at the boundary $\mathbf{V} \cdot \hat{n} = 0$ where the vector \hat{n} is the unit vector normal to the surface.

Since in this model the viscosity $\nu = 0$, the flow can be modeled using the Euler equations. If the Euler equations, apply, so does Kelvin's theorem:

Theorem 8.1 (Kelvin's circulation theorem). *The circulation around a closed material loop moving with an inviscid, barotropic fluid in the presence of conservative body forces remains constant over time.*^[Citation needed]

If Γ denotes the circulation around a material loop $C(t)$ moving with the fluid, then:

$$\frac{D\Gamma}{Dt} = 0$$

Id est, if the vorticity of \mathbf{V} is 0 initially, it must remain 0 everywhere, thus $\nabla \times \mathbf{V} = 0$. Since the flow is irrotational, \mathbf{V} can be expressed as $\mathbf{V} = \nabla\phi$, where ϕ is the velocity potential.

Furthermore, if \mathbf{V} is incompressible, that being that $\nabla \cdot \mathbf{V} = 0$, then ϕ must satisfy Laplace's equation: $\nabla^2\phi = 0$.

8.2 Polar coordinate boundary conditions

8.2.1 $\mathbf{V} = U\hat{i}$

In polar coordinates, the base vectors \hat{r} and $\hat{\vartheta}$ are defined as:

$$\hat{r} \triangleq \hat{i} \cos \vartheta + \hat{j} \sin \vartheta$$

$$\hat{\vartheta} \triangleq -\hat{i} \sin \vartheta + \hat{j} \cos \vartheta$$

Solving for \hat{i} and \hat{j} gives:

$$\hat{i} = \frac{\hat{r} - \hat{j} \sin \vartheta}{\cos \vartheta} \quad (7)$$

$$\hat{j} = \frac{\hat{\vartheta} + \hat{i} \sin \vartheta}{\cos \vartheta} \quad (8)$$

Substituting 8 into 7 and isolating \hat{i} shows that

$$\begin{aligned} \hat{i} &= \frac{\hat{r} - \frac{\hat{\vartheta} + \hat{i} \sin \vartheta}{\cos \vartheta} \sin \vartheta}{\cos \vartheta} \\ &= \frac{\hat{r}}{\cos \vartheta} - \frac{\hat{\vartheta} \sin \vartheta + \hat{i} \sin^2 \vartheta}{\cos^2 \vartheta} \\ &= \frac{\hat{r}}{\cos \vartheta} - \frac{\hat{\vartheta} \sin \vartheta}{\cos^2 \vartheta} - \frac{\hat{i} \sin^2 \vartheta}{\cos^2 \vartheta} \\ \implies \hat{i} + \frac{\sin^2 \vartheta}{\cos^2 \vartheta} \hat{i} &= \frac{\hat{r}}{\cos \vartheta} - \frac{\hat{\vartheta} \sin \vartheta}{\cos^2 \vartheta} \\ \hat{i} \left(1 + \frac{\sin^2 \vartheta}{\cos^2 \vartheta} \right) &= \frac{\hat{r}}{\cos \vartheta} - \frac{\hat{\vartheta} \sin \vartheta}{\cos^2 \vartheta} \\ \hat{i} \left(\frac{\sin^2 \vartheta + \cos^2 \vartheta}{\cos^2 \vartheta} \right) &= \frac{\hat{r}}{\cos \vartheta} - \frac{\hat{\vartheta} \sin \vartheta}{\cos^2 \vartheta} \\ \frac{\hat{i}}{\cos^2 \vartheta} &= \frac{\hat{r}}{\cos \vartheta} - \frac{\hat{\vartheta} \sin \vartheta}{\cos^2 \vartheta} \\ \hat{i} &= \hat{r} \cos \vartheta - \hat{\vartheta} \sin \vartheta \quad \blacksquare \end{aligned} \quad (9)$$

The condition stated in 6 was that *in infinitum*, $\mathbf{V} = U\hat{i}$. By substituting in 9, the statement becomes in terms of \hat{r} and $\hat{\vartheta}$:

$$\text{as } r \rightarrow \infty, \quad \mathbf{V} = U(\hat{r} \cos \vartheta - \hat{\vartheta} \sin \vartheta)$$

8.2.2 $\mathbf{V} \cdot \hat{n} = 0$

In polar coordinates, the base vector \hat{r} points in the direction of positive change of r , that being outwards from the center. If the cylinder is assumed to be the center of the coordinate system, then \hat{r} will always point normal to the surface of the cylinder. Therefore, at the boundary of the cylinder when $r = L$,

$$\mathbf{V} \cdot \hat{r} = 0$$

8.2.3 $\nabla^2 \phi = 0$

Lemma 8.2 (Multivariable chain rule). *Let $X(t, u)$ and $Y(t, u)$ be functions where $X, Y : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $X, Y \in C^1(\mathbb{R}^2)$. Then define $Z(x, y)$ to be a function where $Z : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $Z \in C^1(\mathbb{R}^2)$. Then the partial derivatives of the composite function $z(t, u) = Z(X(t, u), Y(t, u))$ are given by:*

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial Z}{\partial x} \frac{\partial X}{\partial t} + \frac{\partial Z}{\partial y} \frac{\partial Y}{\partial t} \\ \frac{\partial z}{\partial u} &= \frac{\partial Z}{\partial x} \frac{\partial X}{\partial u} + \frac{\partial Z}{\partial y} \frac{\partial Y}{\partial u} \end{aligned}$$

Lemma 8.3 (Polar-Form Laplacian). *For some scalar field $\phi(x, y)$ defined in a Cartesian system, the Laplacian of ϕ in polar coordinates $\langle r, \vartheta \rangle$ is given by:*

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \vartheta^2}$$

Proof. In Cartesian coordinates, the Laplacian operator ∇^2 is defined as $\nabla \cdot \nabla$, which for the scalar field ϕ becomes:

$$\begin{aligned} \nabla^2 \phi &= \nabla \cdot \nabla \phi \\ &= \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} \cdot \begin{pmatrix} \partial \phi / \partial x \\ \partial \phi / \partial y \end{pmatrix} \\ &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \end{aligned}$$

Translating x and y to polar coordinates and calculating their derivatives with respect to r and ϑ gives:

$$\begin{aligned} x &= r \cos \vartheta, & y &= r \sin \vartheta \\ \frac{\partial x}{\partial r} &= \cos \vartheta, & \frac{\partial y}{\partial r} &= \sin \vartheta \end{aligned} \quad (10)$$

$$\frac{\partial x}{\partial \vartheta} = -r \sin \vartheta, \quad \frac{\partial y}{\partial \vartheta} = r \cos \vartheta \quad (11)$$

Consequently, by the chain rule and substitution from 10:

$$\begin{aligned} \frac{\partial \phi}{\partial r} &= \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial r} \\ &= \frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial \phi}{\partial y} \sin \vartheta \end{aligned} \quad (12)$$

Taking the derivative of 12 with respect to r again gives:

$$\begin{aligned} \frac{\partial^2 \phi}{\partial r^2} &= \frac{\partial}{\partial r} \frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial}{\partial r} \frac{\partial \phi}{\partial y} \sin \vartheta \\ &= \frac{\partial}{\partial x} \frac{\partial \phi}{\partial r} \cos \vartheta + \frac{\partial}{\partial y} \frac{\partial \phi}{\partial r} \sin \vartheta \end{aligned} \quad (13)$$

Substituting 12 into 13 gives:

$$\begin{aligned} \frac{\partial^2 \phi}{\partial r^2} &= \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial \phi}{\partial y} \sin \vartheta \right) \cos \vartheta + \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial \phi}{\partial y} \sin \vartheta \right) \sin \vartheta \\ &= \frac{\partial^2 \phi}{\partial x^2} \cos^2 \vartheta + \frac{\partial^2 \phi}{\partial x \partial y} \sin \vartheta \cos \vartheta + \frac{\partial^2 \phi}{\partial y \partial x} \cos \vartheta \sin \vartheta + \frac{\partial^2 \phi}{\partial y^2} \sin^2 \vartheta \\ &= \frac{\partial^2 \phi}{\partial x^2} \cos^2 \vartheta + 2 \frac{\partial^2 \phi}{\partial x \partial y} \sin \vartheta \cos \vartheta + \frac{\partial^2 \phi}{\partial y^2} \sin^2 \vartheta \end{aligned} \quad (14)$$

Applying the same process for $\frac{\partial \phi}{\partial \vartheta}$ with substitution from 11 yields:

$$\begin{aligned} \frac{\partial \phi}{\partial \vartheta} &= \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial \vartheta} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial \vartheta} \\ &= -\frac{\partial \phi}{\partial x} r \sin \vartheta + \frac{\partial \phi}{\partial y} r \cos \vartheta \end{aligned} \quad (15)$$

Taking the derivative of 15 with respect to ϑ again gives:

$$\frac{\partial^2 \phi}{\partial \vartheta^2} = -\frac{\partial}{\partial \vartheta} \frac{\partial \phi}{\partial x} r \sin \vartheta + \frac{\partial}{\partial \vartheta} \frac{\partial \phi}{\partial y} r \cos \vartheta$$

Since both terms contain a product of two functions dependent on ϑ the product rule needs to be applied. This gives:

$$\begin{aligned} \frac{\partial^2 \phi}{\partial \vartheta^2} &= -\frac{\partial^2 \phi}{\partial \vartheta \partial x} r \sin \vartheta - \frac{\partial \phi}{\partial x} r \cos \vartheta + \frac{\partial^2 \phi}{\partial \vartheta \partial y} r \cos \vartheta - \frac{\partial \phi}{\partial y} r \sin \vartheta \\ &= -r \left(\frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial \phi}{\partial y} \sin \vartheta \right) + r \frac{\partial \phi}{\partial \vartheta} \left(-\frac{\partial}{\partial x} \sin \vartheta + \frac{\partial}{\partial y} \cos \vartheta \right) \end{aligned} \quad (16)$$

Substituting 15 into 16 gives:

$$\frac{\partial^2 \phi}{\partial \vartheta^2} = -r \left(\frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial \phi}{\partial y} \sin \vartheta \right) + r \underbrace{\left(-\frac{\partial \phi}{\partial x} r \sin \vartheta + \frac{\partial \phi}{\partial y} r \cos \vartheta \right) \left(-\frac{\partial}{\partial x} \sin \vartheta + \frac{\partial}{\partial y} \cos \vartheta \right)}_{\Phi} \quad (17)$$

Expanding Φ :

$$\begin{aligned} \Phi &= \left(-\frac{\partial \phi}{\partial x} r \sin \vartheta + \frac{\partial \phi}{\partial y} r \cos \vartheta \right) \left(-\frac{\partial}{\partial x} \sin \vartheta + \frac{\partial}{\partial y} \cos \vartheta \right) \\ &= \left(-\frac{\partial \phi}{\partial x} r \sin \vartheta \right) \left(-\frac{\partial}{\partial x} \sin \vartheta \right) + \left(-\frac{\partial \phi}{\partial x} r \sin \vartheta \right) \left(\frac{\partial}{\partial y} \cos \vartheta \right) \\ &\quad + \left(\frac{\partial \phi}{\partial y} r \cos \vartheta \right) \left(-\frac{\partial}{\partial x} \sin \vartheta \right) + \left(\frac{\partial \phi}{\partial y} r \cos \vartheta \right) \left(\frac{\partial}{\partial y} \cos \vartheta \right) \\ &= \frac{\partial^2 \phi}{\partial x^2} r \sin^2 \vartheta - 2 \frac{\partial^2 \phi}{\partial x \partial y} r \cos \vartheta \sin \vartheta + \frac{\partial^2 \phi}{\partial y^2} r \cos^2 \vartheta \end{aligned}$$

Substituting Φ back into 17 gives:

$$\begin{aligned} \frac{\partial^2 \phi}{\partial \vartheta^2} &= -r \left(\frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial \phi}{\partial y} \sin \vartheta \right) + r \left(\frac{\partial^2 \phi}{\partial x^2} r \sin^2 \vartheta - 2 \frac{\partial^2 \phi}{\partial x \partial y} r \cos \vartheta \sin \vartheta + \frac{\partial^2 \phi}{\partial y^2} r \cos^2 \vartheta \right) \\ &= r^2 \left(\frac{\partial^2 \phi}{\partial x^2} \sin^2 \vartheta - 2 \frac{\partial^2 \phi}{\partial x \partial y} \cos \vartheta \sin \vartheta + \frac{\partial^2 \phi}{\partial y^2} \cos^2 \vartheta \right) - r \left(\frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial \phi}{\partial y} \sin \vartheta \right) \\ &= r^2 \left(\frac{\partial^2 \phi}{\partial x^2} \sin^2 \vartheta - 2 \frac{\partial^2 \phi}{\partial x \partial y} \cos \vartheta \sin \vartheta + \frac{\partial^2 \phi}{\partial y^2} \cos^2 \vartheta \right) - r \frac{\partial \phi}{\partial r} \end{aligned} \quad (18)$$

Combining 14 and 18 yields:

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial \vartheta^2} &= \frac{\partial^2 \phi}{\partial x^2} \cos^2 \vartheta + 2 \frac{\partial^2 \phi}{\partial x \partial y} \sin \vartheta \cos \vartheta + \frac{\partial^2 \phi}{\partial y^2} \sin^2 \vartheta \\
&\quad + r^2 \left(\frac{\partial^2 \phi}{\partial x^2} \sin^2 \vartheta - 2 \frac{\partial^2 \phi}{\partial x \partial y} \cos \vartheta \sin \vartheta + \frac{\partial^2 \phi}{\partial y^2} \cos^2 \vartheta \right) - r \frac{\partial \phi}{\partial r} \\
\Rightarrow \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \vartheta^2} &= \frac{\partial^2 \phi}{\partial x^2} \cos^2 \vartheta + \frac{\partial^2 \phi}{\partial x^2} \sin^2 \vartheta + \frac{\partial^2 \phi}{\partial y^2} \cos^2 \vartheta + \frac{\partial^2 \phi}{\partial y^2} \sin^2 \vartheta - \frac{1}{r} \frac{\partial \phi}{\partial r} \\
&= \frac{\partial^2 \phi}{\partial x^2} (\cos^2 \vartheta + \sin^2 \vartheta) + \frac{\partial^2 \phi}{\partial y^2} (\cos^2 \vartheta + \sin^2 \vartheta) - \frac{1}{r} \frac{\partial \phi}{\partial r} \\
&= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} \\
\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} &= \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \vartheta^2} \\
\therefore \nabla^2 \phi &= \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \vartheta^2} \tag{19}
\end{aligned}$$

■

8.3 Ad confluōrem

Summarized, the conditions translated to polar form in sections 8.2.1, 8.2.2 and 8.2.3 are:

$$\begin{aligned}
\mathbf{V} &= U(\hat{r} \cos \vartheta - \hat{\vartheta} \sin \vartheta) & \text{as} & \quad r \rightarrow \infty \\
\mathbf{V} \cdot \hat{r} &= 0 & \text{when} & \quad r = L \\
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \vartheta^2} &= 0
\end{aligned}$$

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