How can vector calculus be used to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle, and what mathematical principles underpin the observed fluid behaviour?

Mathematics AA HL

Contents

1 Introduction

Fluid dynamics is today a cornerstone to several fields of study, including ærospace engineering and meteorology. Real world fluid behaviour is intricate and complex. Therefore, to gain insights into the governing principles of fluid flow, simplified and idealised models are used. This essay investigates the application of vector calculus to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle. These idealisations allow for the derivation of some of fluid dynamic's key mathematical formulæ and provides a foundation for understanding less idealised fluids.

This essay will address the question: "How can vector calculus be used to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle, and what mathematical principles underpin the observed fluid behaviour?" Through the derivation of the velocity potential and vector field, this essay aims to demonstrate how fundamental laws of fluid motion can be expressed and used through vector calculus.

1.1 Aim & scope

The scope of this essay will be limited to the theoretical modelling of fluid flow in a two-dimensional space as a vector field under idealised conditions forming steady, inviscid and incompressible fluid flow through the derivation of the velocity-potential. The analysis will be centred on the application of vector calculus to derive fundamental formulæ and describe fluid behaviour around a stationary circular obstacle. Consequently, this essay will not touch on viscous effects, turbulent flow or three-dimensional analysis, nor will it involve any experimental validation. The focus is on the mathematical derivation and analysis of the idealised model.

1.2 Background

1.2.1 Glossary

Definition 1.1. ***Steady flow*** refers to flow in which the velocity at every point does not change over time [CRACIUNOIU and CIOCIRLAN, 2001].

Definition 1.2. ***Inviscid flow*** is the flow of a fluid with 0 viscosity [Anderson, 2003].

Definition 1.3. An ***incompressible fluid*** is a fluid whose density at every point does not change over time [Ahmed, 2019].

Definition 1.4. A ***scalar field*** is a function mapping points in space to scalar quantities such as temperatures.

Figure 1: Scalar field plotted for the function $f(x, y) = \sin(x) \cos y$

Definition 1.5. A ***vector field*** is a function mapping points in space to vector quantities [Brezinski, 2006]. In the case of fluid dynamics, vector fields often model quantities like fluid velocity.
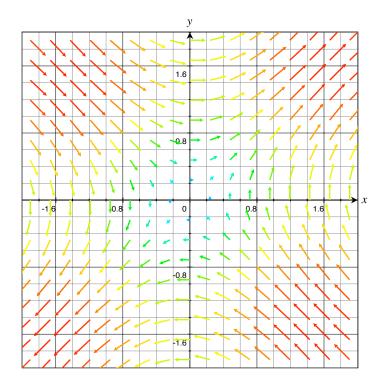
Figure 2: Vector field plotted for the function $f(x, y) = \begin{pmatrix} \sin y \\ \sin x \end{pmatrix}$

Definition 1.6. The ***velocity potential*** ϕ is a scalar field whose gradient is the velocity vector field of some fluid, mathematically $\mathbf{V} = \nabla \phi$. The quantity is defined for irrotational flow which is a resulting property of the idealisations made in this essay[see 8.1].

1.2.2 Notation

Vector calculus, like one-variable calculus, has no standardized notation. This essay will employ the following notation:

- ∇:

    - ∇F: The gradient of some scalar field F.

    - $\nabla \cdot \mathbf{F}$: The divergence of some vector field $\mathbf{F}$.

    - $\nabla \times \mathbf{F}$: The curl of some vector field $\mathbf{F}$.

- Δ: The Laplacian operator

- $\hat{\imath} \,\&\, \hat{\jmath}$: Unit vectors in the positive x and y directions respectively.

- $\hat{r} \,\&\, \hat{\vartheta}$: Unit vectors in the positive r and ϑ directions respectively.

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2 Vector calculus

2.1 The fundamentals of vector calculus

Definition 2.1. ***Partial derivatives*** are an extension of single-variable derivatives in which all variables save the one being differentiated by are treated as constants [Mortimer, 2013]. A formal definition of the partial derivative of some function f with respect to a parameter x_n can be expressed as:

$$\frac{\partial f}{\partial x_n} = \lim_{\delta \to 0} \frac{f(x_1, x_2, \cdots, x_n + \delta, \cdots) - f(x_1, x_2, \cdots, x_n, \cdots)}{\delta} \tag{1}$$

Partial derivatives allow for the analysis of how multi-variable functions such as scalar- or vector fields change with respect to just one spatial dimension. For example, consider the function $f(x, y) = x^2 y + \sin(x) \sin y$:

$$\frac{\partial f}{\partial x} = 2xy + \cos(x) \sin y \qquad\qquad \frac{\partial f}{\partial y} = x^2 + \sin(x) \cos y$$

n-th order partial derivatives are denoted, similarly to normal calculus, as

$$\frac{\partial^n f}{\partial x^n} = \lim_{\delta_1 \to 0} \lim_{\delta_2 \to 0} \ldots \lim_{\delta_n \to 0} \frac{f(x_1, x_2, \cdots, x_n + \delta, \cdots) - f(x_1, x_2, \cdots, x_n, \cdots)}{\delta}$$

Definition 2.2. ***Mixed partial derivatives*** are partial derivatives of a function taken with respect to multiple variables [Garrett, 2015]. This is denoted as

$$\frac{\partial^2 f}{\partial \alpha \partial \beta} \equiv \frac{\partial}{\partial \alpha} \frac{\partial f}{\partial \beta}$$

where both α and β are parameters of f.

Lemma 2.1 (Clairaut's theorem)**.** The order of mixed partial derivatives has no effect on the outcome of the derivative, id est

$$\frac{\partial^2 f}{\partial \alpha \partial \beta} = \frac{\partial^2 f}{\partial \beta \partial \alpha}$$

Proof. Let a rectangular region be bound by the points $W\langle\alpha_0, \beta_0\rangle, X\langle\alpha_1, \beta_0\rangle, Y\langle\alpha_1, \beta_1\rangle$ and $Z\langle\alpha_0, \beta_1\rangle$. The function f is continuous on the intervals $[\alpha_0, \alpha_1]$ and $[\beta_0, \beta_1]$ and differentiable on the intervals (α_0, α_1) and (β_0, β_1). Let Q represent the difference in the change of values of f at the regions of the function given at $f(\alpha, \beta_1)$ and $f(\alpha, \beta_2)$ on the interval $[\alpha_0, \alpha_1]$.

$$Q = [f(\alpha_1, \beta_1) - f(\alpha_0, \beta_1)] - [f(\alpha_1, \beta_0) - f(\alpha_0, \beta_0)]$$

The mean value theorem (MVT) states that on both regions $\vec{WX}$ and $\vec{ZY} \exists \xi_0$ and $\xi_1 \ni$

$$\left.\frac{\partial f}{\partial \alpha}\right|_{\langle\xi_0, \beta_0\rangle} = \frac{f(\alpha_1, \beta_0) - f(\alpha_0, \beta_0)}{\|\vec{XY}\|} = \frac{f(\alpha_1, \beta_0) - f(\alpha_0, \beta_0)}{\alpha_1 - \alpha_0}$$

$$\left.\frac{\partial f}{\partial \alpha}\right|_{\langle\xi_1, \beta_1\rangle} = \frac{f(\alpha_1, \beta_1) - f(\alpha_0, \beta_1)}{\|\vec{ZY}\|} = \frac{f(\alpha_1, \beta_1) - f(\alpha_0, \beta_1)}{\alpha_1 - \alpha_0}$$

Thus Q can be expressed as

$$Q = \left(\left.\frac{\partial f}{\partial \alpha}\right|_{\langle\xi_0, \beta_0\rangle}(\alpha_1 - \alpha_0)\right) - \left(\left.\frac{\partial f}{\partial \alpha}\right|_{\langle\xi_1, \beta_1\rangle}(\alpha_1 - \alpha_0)\right)$$

$$= \left(\left.\frac{\partial f}{\partial \alpha}\right|_{\langle\xi_0, \beta_0\rangle} - \left.\frac{\partial f}{\partial \alpha}\right|_{\langle\xi_1, \beta_1\rangle}\right)(\alpha_1 - \alpha_0)$$

Now let R be the equivalent of Q in the direction of β,

$$R = [f(\alpha_1, \beta_1) - f(\alpha_1, \beta_0)] - [f(\alpha_0, \beta_1) - f(\alpha_0, \beta_0)]$$

The MVT asserts that on the regions $\vec{ZW}$ and $\vec{ZY} \exists \zeta_0$ and $\zeta_1 \ni$

$$\left.\frac{\partial f}{\partial \beta}\right|_{\langle\alpha_0, \zeta_0\rangle} = \frac{f(\alpha_0, \beta_1) - f(\alpha_0, \beta_0)}{\|\vec{ZW}\|} = \frac{f(\alpha_0, \beta_1) - f(\alpha_0, \beta_0)}{\beta_1 - \beta_0}$$

$$\left.\frac{\partial f}{\partial \beta}\right|_{\langle\alpha_1, \zeta_1\rangle} = \frac{f(\alpha_1, \beta_1) - f(\alpha_1, \beta_0)}{\|\vec{ZY}\|} = \frac{f(\alpha_1, \beta_1) - f(\alpha_1, \beta_0)}{\beta_1 - \beta_0}$$

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Thus R can be expressed as

$$R = \left(\left. \frac{\partial f}{\partial \beta} \right|_{\langle \alpha_0, \zeta_0 \rangle} (\beta_1 - \beta_0) \right) - \left(\left. \frac{\partial f}{\partial \beta} \right|_{\langle \alpha_1, \zeta_1 \rangle} (\beta_1 - \beta_0) \right)$$

$$= \left(\left. \frac{\partial f}{\partial \beta} \right|_{\langle \alpha_0, \zeta_0 \rangle} - \left. \frac{\partial f}{\partial \beta} \right|_{\langle \alpha_1, \zeta_1 \rangle} \right) (\beta_1 - \beta_0)$$

Rearranging Q and R,

$$Q = [f(\alpha_1, \beta_1) - f(\alpha_0, \beta_1)] - [f(\alpha_1, \beta_0) - f(\alpha_0, \beta_0)]$$

$$= f(\alpha_1, \beta_1) - f(\alpha_1, \beta_0) - f(\alpha_0, \beta_1) + f(\alpha_0, \beta_0)$$

$$= [f(\alpha_1, \beta_1) - f(\alpha_1, \beta_0)] - [f(\alpha_0, \beta_1) - f(\alpha_0, \beta_0)] = R$$

$$\therefore Q = R$$

Thus

$$\left(\left. \frac{\partial f}{\partial \alpha} \right|_{\langle \xi_0, \beta_0 \rangle} - \left. \frac{\partial f}{\partial \alpha} \right|_{\langle \xi_1, \beta_1 \rangle} \right) (\alpha_1 - \alpha_0) = \left(\left. \frac{\partial f}{\partial \beta} \right|_{\langle \alpha_0, \zeta_0 \rangle} - \left. \frac{\partial f}{\partial \beta} \right|_{\langle \alpha_1, \zeta_1 \rangle} \right) (\beta_1 - \beta_0)$$

$$\rightsquigarrow \frac{\partial f / \partial \alpha \left.\right|_{\langle \xi_0, \beta_0 \rangle} - \partial f / \partial \alpha \left.\right|_{\langle \xi_1, \beta_1 \rangle}}{\beta_1 - \beta_0} = \frac{\partial f / \partial \beta \left.\right|_{\langle \alpha_0, \zeta_0 \rangle} - \partial f / \partial \beta \left.\right|_{\langle \alpha_0, \zeta_0 \rangle}}{\alpha_1 - \alpha_0} \qquad (2)$$

Applying the MVT again, there exists some $\xi^* \in (\xi_0, \xi_1), \beta^* \in (\beta_0, \beta_1) \ni$

$$- \left. \frac{\partial^2 f}{\partial \beta \partial \alpha} \right|_{\langle \xi^*, \beta^* \rangle} = \frac{\partial f / \partial \alpha \left.\right|_{\langle \xi_1, \beta_1 \rangle} - \partial f / \partial \alpha \left.\right|_{\langle \xi_0, \beta_0 \rangle}}{\beta_1 - \beta_0}$$

$$\implies \left. \frac{\partial^2 f}{\partial \beta \partial \alpha} \right|_{\langle \xi^*, \beta^* \rangle} = \frac{\partial f / \partial \alpha \left.\right|_{\langle \xi_0, \beta_0 \rangle} - \partial f / \partial \alpha \left.\right|_{\langle \xi_1, \beta_1 \rangle}}{\beta_1 - \beta_0}$$

Similarly, there exists some $\alpha^* \in (\alpha_0, \alpha_1), \zeta^* \in (\zeta_0, \zeta_1) \ni$

$$- \left. \frac{\partial^2 f}{\partial \alpha \partial \beta} \right|_{\langle \alpha^*, \zeta^* \rangle} = \frac{\partial f / \partial \beta \left.\right|_{\langle \alpha_1, \zeta_1 \rangle} - \partial f / \partial \beta \left.\right|_{\langle \alpha_0, \zeta_0 \rangle}}{\alpha_1 - \alpha_0}$$

$$\implies \left. \frac{\partial^2 f}{\partial \alpha \partial \beta} \right|_{\langle \alpha^*, \zeta^* \rangle} = \frac{\partial f / \partial \beta \left.\right|_{\langle \alpha_0, \zeta_0 \rangle} - \partial f / \partial \beta \left.\right|_{\langle \alpha_1, \zeta_1 \rangle}}{\alpha_1 - \alpha_0}$$

Substituting back into (2),

$$\frac{\partial^2 f}{\partial \beta \partial \alpha}\bigg|_{\langle \xi^*, \beta^* \rangle} = \frac{\partial^2 f}{\partial \alpha \partial \beta}\bigg|_{\langle \alpha^*, \zeta^* \rangle}$$

Consequently, as $\alpha_1 \to \alpha_0$ and $\beta_1 \to \beta_0$, $\xi^* \to \alpha_0, \beta^* \to \beta_0, \alpha^* \to \alpha_0$ and $\zeta^* \to \beta_0$. Since the derivatives are continuos,

$$\frac{\partial^2 f}{\partial \beta \partial \alpha}\bigg|_{\langle \alpha_0, \beta_0 \rangle} = \frac{\partial \alpha}{\partial \beta}\bigg|_{\langle \alpha_0, \beta_0 \rangle}$$

Because $\langle \alpha_0, \beta_0 \rangle$ is an arbitrary point in the domain $\frac{\partial^2 f}{\partial \beta \partial \alpha} = \frac{\partial^2 f}{\partial \alpha \partial \beta}$ at all points in the domain where the mixed partial derivatives are continuous. ∎

Definition 2.3. The **nabla** operator ∇ is a vector containing one partial derivative for each parameter of the function applied to [Rapp, 2017]. For some function $f : \mathbb{R}^n \to \mathbb{R}$, ∇f would be given by:

$$\nabla f = \begin{bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \\ \vdots \\ \partial f / \partial x_n \end{bmatrix}$$

The nabla of a function will be the vector which points in the direction of the greatest change for the function at the point evaluated at.

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7 List of Figures

8 Research

8.1 Potential flow around a circular cylinder

A cylinder of radius L is placed in two-dimensional, incompressible, inviscid flow which flows in the direction of $\hat{\imath}$. Far away from the cylinder the velocity field $\mathbf{V}$ can be described as:

$$\mathbf{V} = U\hat{\imath} \tag{6}$$

Where U is some constant. Since the cylinder is impermissible, at the boundary $\mathbf{V} \cdot \hat{n} = 0$ where the vector $\hat{n}$ is the unit vector normal to the surface.

Since in this model the viscosity $\nu = 0$, the flow can be modelled using the Euler equations. If the Euler equations, apply, so does Kelvin's theorem:

Theorem 8.1 (Kelvin's circulation theorem)**.** The circulation around a closed material loop moving with an inviscid, barotropic fluid in the presence of conservative body forces remains constant over time.[Citation needed]

If Γ denotes the circulation around a material loop $C(t)$ moving with the fluid, then:

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = 0$$

Id est, if the vorticity of $\mathbf{V}$ is 0 initially, it must remain 0 everywhere, thus $\nabla \times \mathbf{V} = 0$. Since the flow is irrotational, $\mathbf{V}$ can be expressed as $\mathbf{V} = \nabla\phi$, where ϕ is the velocity potential.

Furthermore, if $\mathbf{V}$ is incompressible, that being that $\nabla \cdot \mathbf{V} = 0$, then ϕ must satisfy Laplace's equation: $\Delta\phi = 0$.

8.2 Polar coordinate boundary conditions

8.2.1 $\mathbf{V} = U\hat{\imath}$

In polar coordinates, the base vectors $\hat{r}$ and $\hat{\vartheta}$ are defined as:

$$\hat{r} \overset{\Delta}{=} \hat{\imath}\cos\vartheta + \hat{\jmath}\sin\vartheta$$

$$\hat{\vartheta} \overset{\Delta}{=} -\hat{\imath}\sin\vartheta + \hat{\jmath}\cos\vartheta$$

Solving for $\hat{\imath}$ and $\hat{\jmath}$ gives:

$$\hat{\imath} = \frac{\hat{r} - \hat{\jmath}\sin\vartheta}{\cos\vartheta} \tag{7}$$

$$\hat{\jmath} = \frac{\hat{\vartheta} + \hat{\imath}\sin\vartheta}{\cos\vartheta} \tag{8}$$

Substituting 8 into 7 and isolating $\hat{\imath}$ shows that

$$\hat{\imath} = \frac{\hat{r} - \frac{\hat{\vartheta} + \hat{\imath}\sin\vartheta}{\cos\vartheta}\sin\vartheta}{\cos\vartheta}$$

$$= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta + \hat{\imath}\sin^2\vartheta}{\cos^2\vartheta}$$

$$= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta} - \frac{\hat{\imath}\sin^2\vartheta}{\cos^2\vartheta}$$

$$\implies \hat{\imath} + \frac{\sin^2\vartheta}{\cos^2\vartheta}\hat{\imath} = \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta}$$

$$\hat{\imath}\left(1 + \frac{\sin^2\vartheta}{\cos^2\vartheta}\right) = \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta}$$

$$\hat{\imath}\left(\frac{\sin^2\vartheta + \cos^2\vartheta}{\cos^2\vartheta}\right) = \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta}$$

$$\frac{\hat{\imath}}{\cos^2\vartheta} = \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta}$$

$$\hat{\imath} = \hat{r}\cos\vartheta - \hat{\vartheta}\sin\vartheta \quad \blacksquare \tag{9}$$

The condition stated in 6 was that *in infinitum*, $\mathbf{V} = U\hat{\imath}$. By substituting in 9, the statement becomes in terms of $\hat{r}$ and $\hat{\vartheta}$:

$$\mathbf{V} = U(\hat{r}\cos\vartheta - \hat{\vartheta}\sin\vartheta) \quad \text{as} \quad r \to \infty$$

8.2.2 $\mathbf{V} \cdot \hat{n} = 0$

In polar coordinates, the base vector $\hat{r}$ points in the direction of positive change of r, that being outwards from the center. If the cylinder is assumed to be the center of the coordinate system, then $\hat{r}$ will always point normal to the surface of the cylinder. Therefore, at the boundary of the cylinder when $r = L$,

$$\mathbf{V} \cdot \hat{r} = 0$$

8.2.3 $\Delta\phi = 0$

Lemma 8.2 (Jacobian Shmaycobian). The derivative of composite functions corresponds to the product Jacobian of Jacobian matrices:

$$J_{f \circ g} = (J_f \circ g) J_g$$

Proof. I finna fix it later frfr. ∎

Lemma 8.3 (Multivariable chain rule). Let $X(t, u)$ and $Y(t, u)$ be functions where $X, Y : \mathbb{R}^2 \to \mathbb{R}$ such that $X, Y \in C^1(\mathbb{R}^2)$. Then define $Z(x, y)$ to be a function where $Z : \mathbb{R}^2 \to \mathbb{R}$ and $Z \in C^1(\mathbb{R}^2)$. Then the partial derivatives of the composite function $z(t, u) = Z(X(t, u), Y(t, u))$ are given by:

$$\frac{\partial z}{\partial t} = \frac{\partial Z}{\partial x}\frac{\partial X}{\partial t} + \frac{\partial Z}{\partial y}\frac{\partial Y}{\partial t}$$
$$\frac{\partial z}{\partial u} = \frac{\partial Z}{\partial x}\frac{\partial X}{\partial u} + \frac{\partial Z}{\partial y}\frac{\partial Y}{\partial u}$$

Proof. Let $g : \mathbb{R}^n \to \mathbb{R}^p$ and $f : \mathbb{R}^p \to \mathbb{R}^m$, the dimensions of the Jacobian matrices must then be given as:

$$J_g \in \mathbb{R}^{m \times p}, \ (J_f \circ g) \in \mathbb{R}^{p \times n}$$
$$\therefore (J_f \circ g) J_g \in \mathbb{R}^{n \times m}$$

Let the parameters of f be called x_1, x_2, $\ldots$, x_n and the parameters of g be called y_1, y_2, $\ldots$, y_n. The Jacobian of the the composite function $f \circ g$ is defined as:

$$J_{f \circ g} = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} & \cdots & \frac{\partial f}{\partial y_n} \end{bmatrix}.$$

Because $f \circ g : \mathbb{R}^n \to \mathbb{R}^m$, $J_{f \circ g} \in \mathbb{R}^{n \times m}$. The element at position (i, j) of some Jacobian J_F is given by:

$$(J_F)_{ij} = \frac{\partial (f \circ g)_j}{\partial x_i} \tag{10}$$

By matrix multiplication, $((J_f \circ g) J_g)_{ij}$ can be computed as:

$$((J_f \circ g) J_g)_{ij} = \sum_{k=1}^{p} (J_f \circ g)_{ik} (J_g)_{kj}$$

Applying the form given in 10 gives:

$$((J_f \circ g) J_g)_{ij} = \sum_{k=1}^{p} \left. \frac{\partial f_k}{\partial x_i} \right|_{x=g} \frac{\partial g_j}{\partial y_k}$$

$$\rightsquigarrow ((J_f \circ g) J_g)_{ij} = \sum_{k=1}^{p} \frac{\partial f}{\partial g} \frac{\partial g}{\partial y_k} \qquad \blacksquare$$

For the case given above with the composite function z,

$$J_z =$$

Lemma 8.4 (Polar-Form Laplacian). For some scalar field $\phi(x, y)$ defined in a Cartesian system, the Laplacian of ϕ in polar coordinates $\langle r, \vartheta \rangle$ is given by:

$$\Delta \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \vartheta^2}$$

Proof. In Cartesian coordinates, the Laplacian operator Δ is defined as $\nabla \cdot \nabla$, which for the

scalar field ϕ becomes:

$$\Delta\phi = \nabla \cdot \nabla\phi$$
$$= \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} \cdot \begin{pmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \end{pmatrix}$$
$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$$

Translating x and y to polar coordinates and calculating their derivatives with respect to r and ϑ gives:

$$x = r\cos\vartheta, \quad y = r\sin\vartheta$$
$$\frac{\partial x}{\partial r} = \cos\vartheta, \quad \frac{\partial y}{\partial r} = \sin\vartheta \tag{11}$$
$$\frac{\partial x}{\partial\vartheta} = -r\sin\vartheta, \quad \frac{\partial y}{\partial\vartheta} = r\cos\vartheta \tag{12}$$

Consequently, by the chain rule and substitution from 11:

$$\frac{\partial\phi}{\partial r} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial r}$$
$$= \frac{\partial\phi}{\partial x}\cos\vartheta + \frac{\partial\phi}{\partial y}\sin\vartheta \tag{13}$$

Taking the derivative of 13 with respect to r again gives:

$$\frac{\partial^2\phi}{\partial r^2} = \frac{\partial}{\partial r}\frac{\partial\phi}{\partial x}\cos\vartheta + \frac{\partial}{\partial r}\frac{\partial\phi}{\partial y}\sin\vartheta$$
$$= \frac{\partial}{\partial x}\frac{\partial\phi}{\partial r}\cos\vartheta + \frac{\partial}{\partial y}\frac{\partial\phi}{\partial r}\sin\vartheta \tag{14}$$

Substituting 13 into 14 gives:

$$\frac{\partial^2\phi}{\partial r^2} = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\cos\vartheta + \frac{\partial\phi}{\partial y}\sin\vartheta\right)\cos\vartheta + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\cos\vartheta + \frac{\partial\phi}{\partial y}\sin\vartheta\right)\sin\vartheta$$
$$= \frac{\partial^2\phi}{\partial x^2}\cos^2\vartheta + \frac{\partial^2\phi}{\partial x\partial y}\sin\vartheta\cos\vartheta + \frac{\partial^2\phi}{\partial y\partial x}\cos\vartheta\sin\vartheta + \frac{\partial^2\phi}{\partial y^2}\sin^2\vartheta$$
$$= \frac{\partial^2\phi}{\partial x^2}\cos^2\vartheta + 2\frac{\partial^2\phi}{\partial x\partial y}\sin\vartheta\cos\vartheta + \frac{\partial^2\phi}{\partial y^2}\sin^2\vartheta \tag{15}$$

Applying the same process for $\frac{\partial \phi}{\partial \vartheta}$ with substitution from 12 yields:

$$\frac{\partial \phi}{\partial \vartheta} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \vartheta} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \vartheta}$$

$$= -\frac{\partial \phi}{\partial x}r\sin\vartheta + \frac{\partial \phi}{\partial y}r\cos\vartheta \qquad (16)$$

Taking the derivative of 16 with respect to ϑ again gives:

$$\frac{\partial^2 \phi}{\partial \vartheta^2} = -\frac{\partial}{\partial \vartheta}\frac{\partial \phi}{\partial x}r\sin\vartheta + \frac{\partial}{\partial \vartheta}\frac{\partial \phi}{\partial y}r\cos\vartheta$$

Since both terms contain a product of two functions dependent on ϑ the product rule needs to be applied. This gives:

$$\frac{\partial^2 \phi}{\partial \vartheta^2} = -\frac{\partial^2 \phi}{\partial \vartheta \partial x}r\sin\vartheta - \frac{\partial \phi}{\partial x}r\cos\vartheta + \frac{\partial^2 \phi}{\partial \vartheta \partial y}r\cos\vartheta - \frac{\partial \phi}{\partial y}r\sin\vartheta$$

$$= -r\left(\frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial \phi}{\partial y}\sin\vartheta\right) + r\frac{\partial \phi}{\partial \vartheta}\left(-\frac{\partial}{\partial x}\sin\vartheta + \frac{\partial}{\partial y}\cos\vartheta\right) \qquad (17)$$

Substituting 16 into 17 gives:

$$\frac{\partial^2 \phi}{\partial \vartheta^2} = -r\left(\frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial \phi}{\partial y}\sin\vartheta\right) + r\underbrace{\left(-\frac{\partial \phi}{\partial x}r\sin\vartheta + \frac{\partial \phi}{\partial y}r\cos\vartheta\right)\left(-\frac{\partial}{\partial x}\sin\vartheta + \frac{\partial}{\partial y}\cos\vartheta\right)}_{\Phi}$$

$$(18)$$

Expanding Φ:

$$\Phi = \left(-\frac{\partial \phi}{\partial x}r\sin\vartheta + \frac{\partial \phi}{\partial y}r\cos\vartheta\right)\left(-\frac{\partial}{\partial x}\sin\vartheta + \frac{\partial}{\partial y}\cos\vartheta\right)$$

$$= \left(-\frac{\partial \phi}{\partial x}r\sin\vartheta\right)\left(-\frac{\partial}{\partial x}\sin\vartheta\right) + \left(-\frac{\partial \phi}{\partial x}r\sin\vartheta\right)\left(\frac{\partial}{\partial y}\cos\vartheta\right)$$

$$+ \left(\frac{\partial \phi}{\partial y}r\cos\vartheta\right)\left(-\frac{\partial}{\partial x}\sin\vartheta\right) + \left(\frac{\partial \phi}{\partial y}r\cos\vartheta\right)\left(\frac{\partial}{\partial y}\cos\vartheta\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2}r\sin^2\vartheta - 2\frac{\partial^2 \phi}{\partial x \partial y}r\cos\vartheta\sin\vartheta + \frac{\partial^2 \phi}{\partial y^2}r\cos^2\vartheta$$

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Substituting Φ back into 18 gives:

$$\frac{\partial^2 \phi}{\partial \vartheta^2} = -r\left(\frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial \phi}{\partial y}\sin\vartheta\right) + r\left(\frac{\partial^2 \phi}{\partial x^2}r\sin^2\vartheta - 2\frac{\partial^2 \phi}{\partial x \partial y}r\cos\vartheta\sin\vartheta + \frac{\partial^2 \phi}{\partial y^2}r\cos^2\vartheta\right)$$

$$= r^2\left(\frac{\partial^2 \phi}{\partial x^2}\sin^2\vartheta - 2\frac{\partial^2 \phi}{\partial x \partial y}\cos\vartheta\sin\vartheta + \frac{\partial^2 \phi}{\partial y^2}\cos^2\vartheta\right) - r\left(\frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial \phi}{\partial y}\sin\vartheta\right)$$

$$= r^2\left(\frac{\partial^2 \phi}{\partial x^2}\sin^2\vartheta - 2\frac{\partial^2 \phi}{\partial x \partial y}\cos\vartheta\sin\vartheta + \frac{\partial^2 \phi}{\partial y^2}\cos^2\vartheta\right) - r\frac{\partial \phi}{\partial r} \tag{19}$$

Combining 15 and 19 yields:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial \vartheta^2} = \frac{\partial^2 \phi}{\partial x^2}\cos^2\vartheta + 2\frac{\partial^2 \phi}{\partial x \partial y}\sin\vartheta\cos\vartheta + \frac{\partial^2 \phi}{\partial y^2}\sin^2\vartheta$$

$$+ r^2\left(\frac{\partial^2 \phi}{\partial x^2}\sin^2\vartheta - 2\frac{\partial^2 \phi}{\partial x \partial y}\cos\vartheta\sin\vartheta + \frac{\partial^2 \phi}{\partial y^2}\cos^2\vartheta\right) - r\frac{\partial \phi}{\partial r}$$

$$\implies \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \vartheta^2} = \frac{\partial^2 \phi}{\partial x^2}\cos^2\vartheta + \frac{\partial^2 \phi}{\partial x^2}\sin^2\vartheta + \frac{\partial^2 \phi}{\partial y^2}\cos^2\vartheta + \frac{\partial^2 \phi}{\partial y^2}\sin^2\vartheta - \frac{1}{r}\frac{\partial \phi}{\partial r}$$

$$= \frac{\partial^2 \phi}{\partial x^2}\left(\cos^2\vartheta + \sin^2\vartheta\right) + \frac{\partial^2 \phi}{\partial y^2}\left(\cos^2\vartheta + \sin^2\vartheta\right) - \frac{1}{r}\frac{\partial \phi}{\partial r}$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - \frac{1}{r}\frac{\partial \phi}{\partial r}$$

$$\implies \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \vartheta^2}$$

$$\therefore \Delta\phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \vartheta^2} \tag{20}$$

$\blacksquare$

8.3   Ad confluōrem

Summarized, the conditions translated to polar form in sections 8.2.1, 8.2.2 and 8.2.3 are:

$$\mathbf{V} = U(\hat{r}\cos\vartheta - \hat{\vartheta}\sin\vartheta) \qquad \text{as} \qquad r \to \infty$$

$$\mathbf{V} \cdot \hat{r} = 0 \qquad \text{when} \qquad r = L$$

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \vartheta^2} = 0$$

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