How can vector calculus be used to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle, and what mathematical principles underpin the observed fluid behaviour?

Mathematics AA HL

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1 Introduction

Fluid dynamics is today a cornerstone to several fields of study, including ærospace engineering and meteorology. Real world fluid behaviour is intricate and complex. Therefore, to gain insights into the governing principles of fluid flow, simplified and idealized models are used. This essay investigates the application of vector calculus to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle. These idealizations allow for the derivation of some of fluid dynamic's key mathematical formulæ and provides a foundation for understanding less idealized fluids. The essay will address the question: "How can vector calculus be used to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle, and what mathematical principles underpin the observed fluid behaviour?".

1.1 Aim & scope

This essay will for simplicity's sake only cover fluid flow around circular obstacles in $\mathbb{R}^2$ spaces; an alaysis of fluid flow in $\mathbb{R}^3$ spaces would be much more complex. Furthermore, only incompressible fluids sans sinks and sources ($\mathbf{F} \ni \nabla \cdot \mathbf{F} = 0$), will be analyzed.

Most of the analysis will take place using Green's theorem[see 2.3].

1.2 Background

1.2.1 Notation

In this paper, the gradient, divergence and curl operators will be denoted using their explicit ∇ forms as follows:

$$\begin{aligned} \text{grad } \mathbf{F} &\equiv \nabla \mathbf{F} \\ \text{div } \mathbf{F} &\equiv \nabla \cdot \mathbf{F} \\ \text{curl } \mathbf{F} &\equiv \nabla \times \mathbf{F} \end{aligned}$$

The directional vector will also be denoted using ∇ as $\nabla_{\vec{v}}\mathbf{F}$.

For the purposes of clarity, vectors in cartesian systems will be denoted $\begin{pmatrix} x \\ y \end{pmatrix}$ whilst vectors in polar systems will be denoted as $\langle r, \vartheta \rangle$

To ensure point-uniqueness within polar systems, all polar coordinates are restricted such as $r \geq 0$, $\vartheta \in [0, 2\pi)$.

6 References

[Peyret and Taylor, 2012] Peyret, R. and Taylor, T. D. (2012). *Computational methods for fluid flow*. Springer Science & Business Media.

[Stony Brook University, 2021] Stony Brook University (2021). Mat132 episode 25: Second-order differential equations.

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1 Vector field for $f(x, y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ 5