6 References

[Peyret and Taylor, 2012] Peyret, R. and Taylor, T. D. (2012). *Computational methods for fluid flow*. Springer Science & Business Media.

[Stony Brook University, 2021] Stony Brook University (2021). Mat132 episode 25: Second-order differential equations.

7 List of Figures

8 Research

8.1 Potential flow around a circular cylinder

A cylinder of radius L is placed in two-dimensional, incompressible, inviscid flow which flows in the direction of $\hat{\imath}$. Far away from the cylinder the velocity field $\mathbf{V}$ can be described as:

$$\mathbf{V} = U\hat{\imath} \tag{6}$$

Where U is some constant. Since the cylinder is impermissible, at the boundary $\mathbf{V} \cdot \hat{n} = 0$ where the vector $\hat{n}$ is the unit vector normal to the surface.

Since in this model the viscocity $\nu = 0$, the flow can be modeled using the Euler equations. If the Euler equations, apply, so does Kelvin's theorem:

Theorem 8.1 (Kelvin's circulation theorem)**.** *The circulation around a closed material loop moving with an inviscid, barotropic fluid in the presence of conservative body forces remains constant over time.*[Citation needed]

If Γ denotes the circulation around a material loop $C(t)$ moving with the fluid, then:

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = 0$$

Id est, if the vorticity of $\mathbf{V}$ is 0 initialy, it must remain 0 everywhere, thus $\nabla \times \mathbf{V} = 0$. Since the flow is irrotational, $\mathbf{V}$ can be expressed as $\mathbf{V} = \nabla\phi$, where ϕ is the velocity potential.

Furthermore, if $\mathbf{V}$ is incompressible, that bieng that $\nabla \cdot \mathbf{V} = 0$, then ϕ must satisfy Laplace's equation: $\nabla^2\phi = 0$.

8.2 Polar coordinate boundary conditions

8.2.1 $\mathbf{V} = U\hat{\imath}$

In polar coordinates, the base vectors $\hat{r}$ and $\hat{\vartheta}$ are defined as:

$$
\begin{aligned}
\hat{r} &\triangleq \hat{\imath}\cos\vartheta + \hat{\jmath}\sin\vartheta \\
\hat{\vartheta} &\triangleq -\hat{\imath}\sin\vartheta + \hat{\jmath}\cos\vartheta
\end{aligned}
$$

Solving for $\hat{\imath}$ and $\hat{\jmath}$ gives:

$$
\hat{\imath} = \frac{\hat{r} - \hat{\jmath}\sin\vartheta}{\cos\vartheta} \tag{7}
$$

$$
\hat{\jmath} = \frac{\hat{\vartheta} + \hat{\imath}\sin\vartheta}{\cos\vartheta} \tag{8}
$$

Substituting 8 into 7 and isolating $\hat{\imath}$ shows that

$$
\begin{aligned}
\hat{\imath} &= \frac{\hat{r} - \frac{\hat{\vartheta}+\hat{\imath}\sin\vartheta}{\cos\vartheta}\sin\vartheta}{\cos\vartheta} \\
&= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta + \hat{\imath}\sin^2\vartheta}{\cos^2\vartheta} \\
&= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta} - \frac{\hat{\imath}\sin^2\vartheta}{\cos^2\vartheta} \\
\implies \hat{\imath} + \frac{\sin^2\vartheta}{\cos^2\vartheta}\hat{\imath} &= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta} \\
\hat{\imath}\left(1 + \frac{\sin^2\vartheta}{\cos^2\vartheta}\right) &= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta} \\
\hat{\imath}\left(\frac{\sin^2\vartheta + \cos^2\vartheta}{\cos^2\vartheta}\right) &= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta} \\
\frac{\hat{\imath}}{\cos^2\vartheta} &= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta} \\
\hat{\imath} &= \hat{r}\cos\vartheta - \hat{\vartheta}\sin\vartheta \quad \blacksquare
\end{aligned} \tag{9}
$$

The condition stated in 6 was that *in infinitum*, $\mathbf{V} = U\hat{\imath}$. By substituting in 9, the statement becomes in terms of $\hat{r}$ and $\hat{\vartheta}$:

$$
\mathbf{V} = U(\hat{r}\cos\vartheta - \hat{\vartheta}\sin\vartheta) \quad \text{as} \quad r \to \infty
$$

8.2.2 $\mathbf{V} \cdot \hat{n} = 0$

In polar coordinates, the base vector $\hat{r}$ points in the direction of positive change of r, that being outwards from the center. If the cylinder is assumed to be the center of the coordinate system, then $\hat{r}$ will always point normal to the surface of the cylinder. Therefore, at the boundary of the cylinder when $r = L$,

$$\mathbf{V} \cdot \hat{r} = 0$$

8.2.3 $\nabla^2 \phi = 0$

Lemma 8.2 (Jacobian Shmaycobian)**.** *The derivative of composite functions corresponds to the product Jacobian of Jacobian matrices:*

$$J_{f \circ g} = (J_f \circ g) J_g$$

Proof. The Jacobian of $J_{f \circ g}$ is defined as

$$J_{f \circ g} = \begin{bmatrix} \nabla^\top (f \circ g)_1 \\ \nabla^\top (f \circ g)_2 \\ \vdots \\ \nabla^\top (f \circ g)_m \end{bmatrix}$$

for the composite function $(f \circ g) : \mathbb{R}^n \to \mathbb{R}^m$, where

$$\operatorname{row}_i J_{f \circ g} = \nabla^\top (f \circ g)_i = \begin{bmatrix} \frac{\partial (f \circ g)_i}{\partial x_1} & \frac{\partial (f \circ g)_i}{\partial x_2} & \dots & \frac{\partial (f \circ g)_i}{\partial x_n} \end{bmatrix}$$

$$\operatorname{col}_i J_{f \circ g} = \begin{bmatrix} \partial (f \circ g)_1 / \partial x_i \\ \partial (f \circ g)_2 / \partial x_i \\ \vdots \\ \partial (f \circ g)_m / \partial x_i \end{bmatrix}$$

■

Lemma 8.3 (Multivariable chain rule)**.** *Let $X(t, u)$ and $Y(t, u)$ be functions where $X, Y : \mathbb{R}^2 \to \mathbb{R}$ such that $X, Y \in C^1(\mathbb{R}^2)$. Then define $Z(x, y)$ to be a function where $Z : \mathbb{R}^2 \to \mathbb{R}$ and $Z \in C^1(\mathbb{R}^2)$. Then the partial derivatives of the composite function $z(t, u) =$*

$Z(X(t,u),Y(t,u))$ are given by:

$$\frac{\partial z}{\partial t} = \frac{\partial Z}{\partial x}\frac{\partial X}{\partial t} + \frac{\partial Z}{\partial y}\frac{\partial Y}{\partial t}$$
$$\frac{\partial z}{\partial u} = \frac{\partial Z}{\partial x}\frac{\partial X}{\partial u} + \frac{\partial Z}{\partial y}\frac{\partial Y}{\partial u}$$

Proof. To be proven... ■

Lemma 8.4 (Polar-Form Laplacian). *For some scalar field $\phi(x,y)$ defined in a Cartesian system, the Laplacian of ϕ in polar coordinates $\langle r, \vartheta\rangle$ is given by:*

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2}$$

Proof. In Cartesian coordinates, the Laplacian operator ∇^2 is defined as $\nabla\cdot\nabla$, which for the scalar field ϕ becomes:

$$\begin{aligned}\nabla^2\phi &= \nabla\cdot\nabla\phi\\ &= \begin{pmatrix}\partial/\partial x\\ \partial/\partial y\end{pmatrix}\cdot\begin{pmatrix}\partial\phi/\partial x\\ \partial\phi/\partial y\end{pmatrix}\\ &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\end{aligned}$$

Translating x and y to polar coordinates and calculating their derivatives with respect to r and ϑ gives:

$$x = r\cos\vartheta, \quad y = r\sin\vartheta$$
$$\frac{\partial x}{\partial r} = \cos\vartheta, \quad \frac{\partial y}{\partial r} = \sin\vartheta \tag{10}$$
$$\frac{\partial x}{\partial\vartheta} = -r\sin\vartheta, \quad \frac{\partial y}{\partial\vartheta} = r\cos\vartheta \tag{11}$$

Consequently, by the chain rule and substitution from 10:

$$\begin{aligned}\frac{\partial\phi}{\partial r} &= \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial r}\\ &= \frac{\partial\phi}{\partial x}\cos\vartheta + \frac{\partial\phi}{\partial y}\sin\vartheta \qquad (12)\end{aligned}$$

Taking the derivative of 12 with respect to r again gives:

$$\begin{aligned}\frac{\partial^2\phi}{\partial r^2} &= \frac{\partial}{\partial r}\frac{\partial\phi}{\partial x}\cos\vartheta + \frac{\partial}{\partial r}\frac{\partial\phi}{\partial y}\sin\vartheta \\ &= \frac{\partial}{\partial x}\frac{\partial\phi}{\partial r}\cos\vartheta + \frac{\partial}{\partial y}\frac{\partial\phi}{\partial r}\sin\vartheta \end{aligned}\tag{13}$$

Substituting 12 into 13 gives:

$$\begin{aligned}\frac{\partial^2\phi}{\partial r^2} &= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\cos\vartheta + \frac{\partial\phi}{\partial y}\sin\vartheta\right)\cos\vartheta + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\cos\vartheta + \frac{\partial\phi}{\partial y}\sin\vartheta\right)\sin\vartheta \\ &= \frac{\partial^2\phi}{\partial x^2}\cos^2\vartheta + \frac{\partial^2\phi}{\partial x\partial y}\sin\vartheta\cos\vartheta + \frac{\partial^2\phi}{\partial y\partial x}\cos\vartheta\sin\vartheta + \frac{\partial^2\phi}{\partial y^2}\sin^2\vartheta \\ &= \frac{\partial^2\phi}{\partial x^2}\cos^2\vartheta + 2\frac{\partial^2\phi}{\partial x\partial y}\sin\vartheta\cos\vartheta + \frac{\partial^2\phi}{\partial y^2}\sin^2\vartheta \end{aligned}\tag{14}$$

Applying the same process for $\frac{\partial\phi}{\partial\vartheta}$ with substitution from 11 yields:

$$\begin{aligned}\frac{\partial\phi}{\partial\vartheta} &= \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial\vartheta} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial\vartheta} \\ &= -\frac{\partial\phi}{\partial x}r\sin\vartheta + \frac{\partial\phi}{\partial y}r\cos\vartheta \end{aligned}\tag{15}$$

Taking the derivative of 15 with respect to ϑ again gives:

$$\frac{\partial^2\phi}{\partial\vartheta^2} = -\frac{\partial}{\partial\vartheta}\frac{\partial\phi}{\partial x}r\sin\vartheta + \frac{\partial}{\partial\vartheta}\frac{\partial\phi}{\partial y}r\cos\vartheta$$

Since both terms contain a product of two functions dependent on ϑ the product rule needs to be applied. This gives:

$$\begin{aligned}\frac{\partial^2\phi}{\partial\vartheta^2} &= -\frac{\partial^2\phi}{\partial\vartheta\partial x}r\sin\vartheta - \frac{\partial\phi}{\partial x}r\cos\vartheta + \frac{\partial^2\phi}{\partial\vartheta\partial y}r\cos\vartheta - \frac{\partial\phi}{\partial y}r\sin\vartheta \\ &= -r\left(\frac{\partial\phi}{\partial x}\cos\vartheta + \frac{\partial\phi}{\partial y}\sin\vartheta\right) + r\frac{\partial\phi}{\partial\vartheta}\left(-\frac{\partial}{\partial x}\sin\vartheta + \frac{\partial}{\partial y}\cos\vartheta\right) \end{aligned}\tag{16}$$

Substituting 15 into 16 gives:

$$\frac{\partial^2\phi}{\partial\vartheta^2} = -r\left(\frac{\partial\phi}{\partial x}\cos\vartheta + \frac{\partial\phi}{\partial y}\sin\vartheta\right) + r\underbrace{\left(-\frac{\partial\phi}{\partial x}r\sin\vartheta + \frac{\partial\phi}{\partial y}r\cos\vartheta\right)\left(-\frac{\partial}{\partial x}\sin\vartheta + \frac{\partial}{\partial y}\cos\vartheta\right)}_{\Phi} \tag{17}$$

Expanding Φ:

$$\begin{aligned}
\Phi &= \left(-\frac{\partial\phi}{\partial x}r\sin\vartheta + \frac{\partial\phi}{\partial y}r\cos\vartheta\right)\left(-\frac{\partial}{\partial x}\sin\vartheta + \frac{\partial}{\partial y}\cos\vartheta\right)\\
&= \left(-\frac{\partial\phi}{\partial x}r\sin\vartheta\right)\left(-\frac{\partial}{\partial x}\sin\vartheta\right) + \left(-\frac{\partial\phi}{\partial x}r\sin\vartheta\right)\left(\frac{\partial}{\partial y}\cos\vartheta\right)\\
&+ \left(\frac{\partial\phi}{\partial y}r\cos\vartheta\right)\left(-\frac{\partial}{\partial x}\sin\vartheta\right) + \left(\frac{\partial\phi}{\partial y}r\cos\vartheta\right)\left(\frac{\partial}{\partial y}\cos\vartheta\right)\\
&= \frac{\partial^2\phi}{\partial x^2}r\sin^2\vartheta - 2\frac{\partial^2\phi}{\partial x\partial y}r\cos\vartheta\sin\vartheta + \frac{\partial^2\phi}{\partial y^2}r\cos^2\vartheta
\end{aligned}$$

Substituting Φ back into 17 gives:

$$\begin{aligned}
\frac{\partial^2\phi}{\partial\vartheta^2} &= -r\left(\frac{\partial\phi}{\partial x}\cos\vartheta + \frac{\partial\phi}{\partial y}\sin\vartheta\right) + r\left(\frac{\partial^2\phi}{\partial x^2}r\sin^2\vartheta - 2\frac{\partial^2\phi}{\partial x\partial y}r\cos\vartheta\sin\vartheta + \frac{\partial^2\phi}{\partial y^2}r\cos^2\vartheta\right)\\
&= r^2\left(\frac{\partial^2\phi}{\partial x^2}\sin^2\vartheta - 2\frac{\partial^2\phi}{\partial x\partial y}\cos\vartheta\sin\vartheta + \frac{\partial^2\phi}{\partial y^2}\cos^2\vartheta\right) - r\left(\frac{\partial\phi}{\partial x}\cos\vartheta + \frac{\partial\phi}{\partial y}\sin\vartheta\right)\\
&= r^2\left(\frac{\partial^2\phi}{\partial x^2}\sin^2\vartheta - 2\frac{\partial^2\phi}{\partial x\partial y}\cos\vartheta\sin\vartheta + \frac{\partial^2\phi}{\partial y^2}\cos^2\vartheta\right) - r\frac{\partial\phi}{\partial r} \qquad (18)
\end{aligned}$$

Combining 14 and 18 yields:

$$
\begin{aligned}
\frac{\partial^2\phi}{\partial r^2}+\frac{\partial^2\phi}{\partial\vartheta^2} &= \frac{\partial^2\phi}{\partial x^2}\cos^2\vartheta+2\frac{\partial^2\phi}{\partial x\partial y}\sin\vartheta\cos\vartheta+\frac{\partial^2\phi}{\partial y^2}\sin^2\vartheta \\
&\quad + r^2\left(\frac{\partial^2\phi}{\partial x^2}\sin^2\vartheta-2\frac{\partial^2\phi}{\partial x\partial y}\cos\vartheta\sin\vartheta+\frac{\partial^2\phi}{\partial y^2}\cos^2\vartheta\right)-r\frac{\partial\phi}{\partial r} \\
\implies \frac{\partial^2\phi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2} &= \frac{\partial^2\phi}{\partial x^2}\cos^2\vartheta+\frac{\partial^2\phi}{\partial x^2}\sin^2\vartheta+\frac{\partial^2\phi}{\partial y^2}\cos^2\vartheta+\frac{\partial^2\phi}{\partial y^2}\sin^2\vartheta-\frac{1}{r}\frac{\partial\phi}{\partial r} \\
&= \frac{\partial^2\phi}{\partial x^2}\left(\cos^2\vartheta+\sin^2\vartheta\right)+\frac{\partial^2\phi}{\partial y^2}\left(\cos^2\vartheta+\sin^2\vartheta\right)-\frac{1}{r}\frac{\partial\phi}{\partial r} \\
&= \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}-\frac{1}{r}\frac{\partial\phi}{\partial r} \\
\implies \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} &= \frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2} \\
\therefore \nabla^2\phi &= \frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2} \qquad (19)
\end{aligned}
$$

■

8.3 Ad confluōrem

Summarized, the conditions translated to polar form in sections 8.2.1, 8.2.2 and 8.2.3 are:

$$
\begin{aligned}
\mathbf{V} = U(\hat{r}\cos\vartheta-\hat{\vartheta}\sin\vartheta) &\qquad \text{as} \qquad r\to\infty \\
\mathbf{V}\cdot\hat{r}=0 &\qquad \text{when} \qquad r=L \\
\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2}=0 &
\end{aligned}
$$

testing hello hello! [Stony Brook University, 2021]