6 References

[Peyret and Taylor, 2012] Peyret, R. and Taylor, T. D. (2012). *Computational methods for fluid flow.* Springer Science & Business Media.

[Stony Brook University, 2021] Stony Brook University (2021). Mat132 episode 25: Second-order differential equations.

7 List of Figures

8 Research

8.1 Potential flow around a circular cylinder

A cylinder of radius L is placed in two-dimensional, incompressible, inviscid flow which flows in the direction of $\hat{\imath}$. Far away from the cylinder the velocity field $\mathbf{V}$ can be described as:

$$\mathbf{V} = U\hat{\imath} \tag{6}$$

Where U is some constant. Since the cylinder is impermissible, at the boundary $\mathbf{V} \cdot \hat{n} = 0$ where the vector $\hat{n}$ is the unit vector normal to the surface.

Since in this model the viscocity $\nu = 0$, the flow can be modeled using the Euler equations. If the Euler equations, apply, so does Kelvin's theorem:

Theorem 8.1 (Kelvin's circulation theorem). *The circulation around a closed material loop moving with an inviscid, barotropic fluid in the presence of conservative body forces remains constant over time.*[Citation needed]

If Γ denotes the circulation around a material loop $C(t)$ moving with the fluid, then:

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = 0$$

Id est, if the vorticity of $\mathbf{V}$ is 0 initialy, it must remain 0 everywhere, thus $\nabla \times \mathbf{V} = 0$. Since the flow is irrotational, $\mathbf{V}$ can be expressed as $\mathbf{V} = \nabla\phi$, where ϕ is the velocity potential.

Furthermore, if $\mathbf{V}$ is incompressible, that bieng that $\nabla \cdot \mathbf{V} = 0$, then ϕ must satisfy Laplace's equation: $\nabla^2\phi = 0$.

8.2 Polar coordinate boundary conditions

8.2.1 $\mathbf{V} = U\hat{\imath}$

In polar coordinates, the base vectors $\hat{r}$ and $\hat{\vartheta}$ are defined as:

$$\hat{r} \triangleq \hat{\imath}\cos\vartheta + \hat{\jmath}\sin\vartheta$$

$$\hat{\vartheta} \triangleq -\hat{\imath}\sin\vartheta + \hat{\jmath}\cos\vartheta$$

Solving for $\hat{\imath}$ and $\hat{\jmath}$ gives:

$$\hat{\imath} = \frac{\hat{r} - \hat{\jmath}\sin\vartheta}{\cos\vartheta} \tag{7}$$

$$\hat{\jmath} = \frac{\hat{\vartheta} + \hat{\imath}\sin\vartheta}{\cos\vartheta} \tag{8}$$

Substituting 8 into 7 and isolating $\hat{\imath}$ shows that

$$\hat{\imath} = \frac{\hat{r} - \frac{\hat{\vartheta}+\hat{\imath}\sin\vartheta}{\cos\vartheta}\sin\vartheta}{\cos\vartheta}$$

$$= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta + \hat{\imath}\sin^2\vartheta}{\cos^2\vartheta}$$

$$= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta} - \frac{\hat{\imath}\sin^2\vartheta}{\cos^2\vartheta}$$

$$\implies \hat{\imath} + \frac{\sin^2\vartheta}{\cos^2\vartheta}\hat{\imath} = \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta}$$

$$\hat{\imath}\left(1 + \frac{\sin^2\vartheta}{\cos^2\vartheta}\right) = \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta}$$

$$\hat{\imath}\left(\frac{\sin^2\vartheta + \cos^2\vartheta}{\cos^2\vartheta}\right) = \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta}$$

$$\frac{\hat{\imath}}{\cos^2\vartheta} = \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta}$$

$$\hat{\imath} = \hat{r}\cos\vartheta - \hat{\vartheta}\sin\vartheta \quad \blacksquare \tag{9}$$

The condition stated in 6 was that *in infinitum*, $\mathbf{V} = U\hat{\imath}$. By substituting in 9, the statement becomes in terms of $\hat{r}$ and $\hat{\vartheta}$:

$$\mathbf{V} = U(\hat{r}\cos\vartheta - \hat{\vartheta}\sin\vartheta) \quad \text{as} \quad r \to \infty$$

8.2.2 $\mathbf{V} \cdot \hat{n} = 0$

In polar coordinates, the base vector $\hat{r}$ points in the direction of positive change of r, that being outwards from the center. If the cylinder is assumed to be the center of the coordinate system, then $\hat{r}$ will always point normal to the surface of the cylinder. Therefore, at the boundary of the cylinder when $r = L$,

$$\mathbf{V} \cdot \hat{r} = 0$$

8.2.3 $\nabla^2 \phi = 0$

Lemma 8.2 (Jacobian Shmaycobian). *The derivative of composite functions corresponds to the product Jacobian of Jacobian matrices:*

$$J_{f \circ g} = (J_f \circ g) J_g$$

Proof. Let $g : \mathbb{R}^n \to \mathbb{R}^p$ and $f : \mathbb{R}^p \to \mathbb{R}^m$, the Jacobian matrices must then be given by

$$J_g \in \mathbb{R}^{m \times p}, (J_f \circ g) \in \mathbb{R}^{p \times n}$$

$$\therefore (J_f \circ g) J_g \in \mathbb{R}^{n \times m}$$

The Jacobian of the composite $(f \circ g)$ must also have the dimensions $n \times m$.

$$(J_{f \circ g})_{ij} = \frac{\partial (f \circ g)_j}{\partial x_i}$$

$$(\mathbf{AB})_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

$$\implies ((J_f \circ g) J_g)_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

$$= \sum_{k=1}^{p} \frac{\partial f_k}{\partial x_i} \bigg|_{x=g} \frac{\partial g_j}{\partial x_k} \rightsquigarrow \sum_{k=1}^{p} \frac{\partial f}{\partial g} \frac{\partial g}{\partial x}$$

∎

Lemma 8.3 (Multivariable chain rule). *Let $X(t, u)$ and $Y(t, u)$ be functions where X, Y :*

$\mathbb{R}^2 \rightarrow \mathbb{R}$ *such that* $X, Y \in C^1(\mathbb{R}^2)$. *Then define* $Z(x, y)$ *to be a function where* $Z :$ $\mathbb{R}^2 \rightarrow \mathbb{R}$ *and* $Z \in C^1(\mathbb{R}^2)$. *Then the partial derivatives of the composite function* $z(t, u) =$ $Z(X(t, u), Y(t, u))$ *are given by:*

$$\frac{\partial z}{\partial t} = \frac{\partial Z}{\partial x}\frac{\partial X}{\partial t} + \frac{\partial Z}{\partial y}\frac{\partial Y}{\partial t}$$
$$\frac{\partial z}{\partial u} = \frac{\partial Z}{\partial x}\frac{\partial X}{\partial u} + \frac{\partial Z}{\partial y}\frac{\partial Y}{\partial u}$$

Proof. To be proven... ∎

Lemma 8.4 (Polar-Form Laplacian)**.** *For some scalar field* $\phi(x, y)$ *defined in a Cartesian system, the Laplacian of* ϕ *in polar coordinates* $\langle r, \vartheta \rangle$ *is given by:*

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \vartheta^2}$$

Proof. In Cartesian coordinates, the Laplacian operator ∇^2 is defined as $\nabla \cdot \nabla$, which for the scalar field ϕ becomes:

$$\nabla^2 \phi = \nabla \cdot \nabla \phi$$
$$= \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} \cdot \begin{pmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \end{pmatrix}$$
$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

Translating x and y to polar coordinates and calculating their derivatives with respect to r and ϑ gives:

$$x = r\cos\vartheta, \quad y = r\sin\vartheta$$
$$\frac{\partial x}{\partial r} = \cos\vartheta, \quad \frac{\partial y}{\partial r} = \sin\vartheta \tag{10}$$
$$\frac{\partial x}{\partial \vartheta} = -r\sin\vartheta, \quad \frac{\partial y}{\partial \vartheta} = r\cos\vartheta \tag{11}$$

Consequently, by the chain rule and substitution from 10:

$$\begin{aligned}
\frac{\partial \phi}{\partial r} &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial r} \\
&= \frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial \phi}{\partial y}\sin\vartheta
\end{aligned} \tag{12}$$

Taking the derivative of 12 with respect to r again gives:

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial r^2} &= \frac{\partial}{\partial r}\frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial}{\partial r}\frac{\partial \phi}{\partial y}\sin\vartheta \\
&= \frac{\partial}{\partial x}\frac{\partial \phi}{\partial r}\cos\vartheta + \frac{\partial}{\partial y}\frac{\partial \phi}{\partial r}\sin\vartheta
\end{aligned} \tag{13}$$

Substituting 12 into 13 gives:

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial r^2} &= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial \phi}{\partial y}\sin\vartheta\right)\cos\vartheta + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial \phi}{\partial y}\sin\vartheta\right)\sin\vartheta \\
&= \frac{\partial^2 \phi}{\partial x^2}\cos^2\vartheta + \frac{\partial^2 \phi}{\partial x \partial y}\sin\vartheta\cos\vartheta + \frac{\partial^2 \phi}{\partial y \partial x}\cos\vartheta\sin\vartheta + \frac{\partial^2 \phi}{\partial y^2}\sin^2\vartheta \\
&= \frac{\partial^2 \phi}{\partial x^2}\cos^2\vartheta + 2\frac{\partial^2 \phi}{\partial x \partial y}\sin\vartheta\cos\vartheta + \frac{\partial^2 \phi}{\partial y^2}\sin^2\vartheta
\end{aligned} \tag{14}$$

Applying the same process for $\frac{\partial \phi}{\partial \vartheta}$ with substitution from 11 yields:

$$\begin{aligned}
\frac{\partial \phi}{\partial \vartheta} &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \vartheta} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \vartheta} \\
&= -\frac{\partial \phi}{\partial x}r\sin\vartheta + \frac{\partial \phi}{\partial y}r\cos\vartheta
\end{aligned} \tag{15}$$

Taking the derivative of 15 with respect to ϑ again gives:

$$\frac{\partial^2 \phi}{\partial \vartheta^2} = -\frac{\partial}{\partial \vartheta}\frac{\partial \phi}{\partial x}r\sin\vartheta + \frac{\partial}{\partial \vartheta}\frac{\partial \phi}{\partial y}r\cos\vartheta$$

Since both terms contain a product of two functions dependent on ϑ the product rule needs

to be applied. This gives:

$$\frac{\partial^2 \phi}{\partial \vartheta^2} = -\frac{\partial^2 \phi}{\partial \vartheta \partial x} r \sin \vartheta - \frac{\partial \phi}{\partial x} r \cos \vartheta + \frac{\partial^2 \phi}{\partial \vartheta \partial y} r \cos \vartheta - \frac{\partial \phi}{\partial y} r \sin \vartheta$$

$$= -r \left(\frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial \phi}{\partial y} \sin \vartheta \right) + r \frac{\partial \phi}{\partial \vartheta} \left(-\frac{\partial}{\partial x} \sin \vartheta + \frac{\partial}{\partial y} \cos \vartheta \right) \quad (16)$$

Substituting 15 into 16 gives:

$$\frac{\partial^2 \phi}{\partial \vartheta^2} = -r \left(\frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial \phi}{\partial y} \sin \vartheta \right) + r \underbrace{\left(-\frac{\partial \phi}{\partial x} r \sin \vartheta + \frac{\partial \phi}{\partial y} r \cos \vartheta \right) \left(-\frac{\partial}{\partial x} \sin \vartheta + \frac{\partial}{\partial y} \cos \vartheta \right)}_{\Phi}$$

$$(17)$$

Expanding Φ:

$$\Phi = \left(-\frac{\partial \phi}{\partial x} r \sin \vartheta + \frac{\partial \phi}{\partial y} r \cos \vartheta \right) \left(-\frac{\partial}{\partial x} \sin \vartheta + \frac{\partial}{\partial y} \cos \vartheta \right)$$

$$= \left(-\frac{\partial \phi}{\partial x} r \sin \vartheta \right) \left(-\frac{\partial}{\partial x} \sin \vartheta \right) + \left(-\frac{\partial \phi}{\partial x} r \sin \vartheta \right) \left(\frac{\partial}{\partial y} \cos \vartheta \right)$$

$$+ \left(\frac{\partial \phi}{\partial y} r \cos \vartheta \right) \left(-\frac{\partial}{\partial x} \sin \vartheta \right) + \left(\frac{\partial \phi}{\partial y} r \cos \vartheta \right) \left(\frac{\partial}{\partial y} \cos \vartheta \right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} r \sin^2 \vartheta - 2 \frac{\partial^2 \phi}{\partial x \partial y} r \cos \vartheta \sin \vartheta + \frac{\partial^2 \phi}{\partial y^2} r \cos^2 \vartheta$$

Substituting Φ back into 17 gives:

$$\frac{\partial^2 \phi}{\partial \vartheta^2} = -r \left(\frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial \phi}{\partial y} \sin \vartheta \right) + r \left(\frac{\partial^2 \phi}{\partial x^2} r \sin^2 \vartheta - 2 \frac{\partial^2 \phi}{\partial x \partial y} r \cos \vartheta \sin \vartheta + \frac{\partial^2 \phi}{\partial y^2} r \cos^2 \vartheta \right)$$

$$= r^2 \left(\frac{\partial^2 \phi}{\partial x^2} \sin^2 \vartheta - 2 \frac{\partial^2 \phi}{\partial x \partial y} \cos \vartheta \sin \vartheta + \frac{\partial^2 \phi}{\partial y^2} \cos^2 \vartheta \right) - r \left(\frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial \phi}{\partial y} \sin \vartheta \right)$$

$$= r^2 \left(\frac{\partial^2 \phi}{\partial x^2} \sin^2 \vartheta - 2 \frac{\partial^2 \phi}{\partial x \partial y} \cos \vartheta \sin \vartheta + \frac{\partial^2 \phi}{\partial y^2} \cos^2 \vartheta \right) - r \frac{\partial \phi}{\partial r} \quad (18)$$

Combining 14 and 18 yields:

$$
\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial \vartheta^2} = \frac{\partial^2 \phi}{\partial x^2} \cos^2 \vartheta + 2 \frac{\partial^2 \phi}{\partial x \partial y} \sin \vartheta \cos \vartheta + \frac{\partial^2 \phi}{\partial y^2} \sin^2 \vartheta
$$

$$
+ r^2 \left(\frac{\partial^2 \phi}{\partial x^2} \sin^2 \vartheta - 2 \frac{\partial^2 \phi}{\partial x \partial y} \cos \vartheta \sin \vartheta + \frac{\partial^2 \phi}{\partial y^2} \cos^2 \vartheta \right) - r \frac{\partial \phi}{\partial r}
$$

$$
\implies \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \vartheta^2} = \frac{\partial^2 \phi}{\partial x^2} \cos^2 \vartheta + \frac{\partial^2 \phi}{\partial x^2} \sin^2 \vartheta + \frac{\partial^2 \phi}{\partial y^2} \cos^2 \vartheta + \frac{\partial^2 \phi}{\partial y^2} \sin^2 \vartheta - \frac{1}{r} \frac{\partial \phi}{\partial r}
$$

$$
= \frac{\partial^2 \phi}{\partial x^2} \left(\cos^2 \vartheta + \sin^2 \vartheta \right) + \frac{\partial^2 \phi}{\partial y^2} \left(\cos^2 \vartheta + \sin^2 \vartheta \right) - \frac{1}{r} \frac{\partial \phi}{\partial r}
$$

$$
= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - \frac{1}{r} \frac{\partial \phi}{\partial r}
$$

$$
\implies \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \vartheta^2}
$$

$$
\therefore \nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \vartheta^2} \tag{19}
$$

∎

8.3 Ad confluōrem

Summarized, the conditions translated to polar form in sections 8.2.1, 8.2.2 and 8.2.3 are:

$$
\mathbf{V} = U(\hat{r} \cos \vartheta - \hat{\vartheta} \sin \vartheta) \qquad \text{as} \qquad r \to \infty
$$

$$
\mathbf{V} \cdot \hat{r} = 0 \qquad \text{when} \qquad r = L
$$

$$
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \vartheta^2} = 0
$$

testing hello hello! [Stony Brook University, 2021]