6 References

[Peyret and Taylor, 2012] Peyret, R. and Taylor, T. D. (2012). *Computational methods for fluid flow*. Springer Science & Business Media.

[Stony Brook University, 2021] Stony Brook University (2021). Mat132 episode 25: Second-order differential equations.

7 List of Figures

8 Research

8.1 Potential flow around a circular cylinder

A cylinder of radius L is placed in two-dimensional, incompressible, inviscid flow which flows in the direction of $\hat{\imath}$. Far away from the cylinder the velocity field $\mathbf{V}$ can be described as:

$$\mathbf{V} = U\hat{\imath} \tag{6}$$

Where U is some constant. Since the cylinder is impermissible, at the boundary $\mathbf{V} \cdot \hat{n} = 0$ where the vector $\hat{n}$ is the unit vector normal to the surface.

Since in this model the viscocity $\nu = 0$, the flow can be modeled using the Euler equations. If the Euler equations, apply, so does Kelvin's theorem:

Theorem 8.1 (Kelvin's circulation theorem)**.** *The circulation around a closed material loop moving with an inviscid, barotropic fluid in the presence of conservative body forces remains constant over time.*[Citation needed]

If Γ denotes the circulation around a material loop $C(t)$ moving with the fluid, then:

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = 0$$

Id est, if the vorticity of $\mathbf{V}$ is 0 initialy, it must remain 0 everywhere, thus $\nabla \times \mathbf{V} = 0$. Since the flow is irrotational, $\mathbf{V}$ can be expressed as $\mathbf{V} = \nabla\phi$, where ϕ is the velocity potential.

Furthermore, if $\mathbf{V}$ is incompressible, that bieng that $\nabla \cdot \mathbf{V} = 0$, then ϕ must satisfy Laplace's equation: $\nabla^2\phi = 0$.

8.2 Polar coordinate boundary conditions

8.2.1 $\mathbf{V} = U\hat{\imath}$

In polar coordinates, the base vectors $\hat{r}$ and $\hat{\vartheta}$ are defined as:

$$\hat{r} \triangleq \hat{\imath}\cos\vartheta + \hat{\jmath}\sin\vartheta$$
$$\hat{\vartheta} \triangleq -\hat{\imath}\sin\vartheta + \hat{\jmath}\cos\vartheta$$

Solving for $\hat{\imath}$ and $\hat{\jmath}$ gives:

$$\hat{\imath} = \frac{\hat{r} - \hat{\jmath}\sin\vartheta}{\cos\vartheta} \tag{7}$$
$$\hat{\jmath} = \frac{\hat{\vartheta} + \hat{\imath}\sin\vartheta}{\cos\vartheta} \tag{8}$$

Substituting 8 into 7 and isolating $\hat{\imath}$ shows that

$$\begin{aligned}
\hat{\imath} &= \frac{\hat{r} - \frac{\hat{\vartheta}+\hat{\imath}\sin\vartheta}{\cos\vartheta}\sin\vartheta}{\cos\vartheta} \\
&= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta + \hat{\imath}\sin^2\vartheta}{\cos^2\vartheta} \\
&= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta} - \frac{\hat{\imath}\sin^2\vartheta}{\cos^2\vartheta} \\
\implies \hat{\imath} + \frac{\sin^2\vartheta}{\cos^2\vartheta}\hat{\imath} &= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta} \\
\hat{\imath}\left(1 + \frac{\sin^2\vartheta}{\cos^2\vartheta}\right) &= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta} \\
\hat{\imath}\left(\frac{\sin^2\vartheta + \cos^2\vartheta}{\cos^2\vartheta}\right) &= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta} \\
\frac{\hat{\imath}}{\cos^2\vartheta} &= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta} \\
\hat{\imath} &= \hat{r}\cos\vartheta - \hat{\vartheta}\sin\vartheta \quad \blacksquare
\end{aligned} \tag{9}$$

The condition stated in 6 was that *in infinitum*, $\mathbf{V} = U\hat{\imath}$. By substituting in 9, the statement becomes in terms of $\hat{r}$ and $\hat{\vartheta}$:

$$\text{as} \quad r \to \infty\,, \quad \mathbf{V} = U(\hat{r}\cos\vartheta - \hat{\vartheta}\sin\vartheta)$$

8.2.2 $\mathbf{V} \cdot \hat{n} = 0$

In polar coordinates, the base vector $\hat{r}$ points in the direction of positive change of r, that being outwards from the center. If the cylinder is assumed to be the center of the coordinate system, then $\hat{r}$ will always point normal to the surface of the cylinder. Therefore, at the boundary of the cylinder when $r = L$,

$$\mathbf{V} \cdot \hat{r} = 0$$

8.2.3 $\nabla^2 \phi = 0$

Lemma 8.2 (Multivariable chain rule)**.** *Let $X(t,u)$ and $Y(t,u)$ be functions where $X, Y : \mathbb{R}^2 \to \mathbb{R}$ such that $X, Y \in C^1(\mathbb{R}^2)$. Then define $Z(x,y)$ to be a function where $Z : \mathbb{R}^2 \to \mathbb{R}$ and $Z \in C^1(\mathbb{R}^2)$. Then the partial derivatives of the composite function $z(t,u) = Z(X(t,u), Y(t,u))$ are given by:*

$$\frac{\partial z}{\partial t} = \frac{\partial Z}{\partial x}\frac{\partial X}{\partial t} + \frac{\partial Z}{\partial y}\frac{\partial Y}{\partial t},$$

$$\frac{\partial z}{\partial u} = \frac{\partial Z}{\partial x}\frac{\partial X}{\partial u} + \frac{\partial Z}{\partial y}\frac{\partial Y}{\partial u}$$

Lemma 8.3 (Polar-Form Laplacian)**.** *For some scalar field $\phi(x,y)$ defined in a Cartesian system, the Laplacian of ϕ in polar coordinates $\langle r, \vartheta \rangle$ is given by:*

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \vartheta^2}$$

Proof. In Cartesian coordinates, the Laplacian operator ∇^2 is defined as $\nabla \cdot \nabla$, which for the scalar field ϕ becomes:

$$\begin{aligned}
\nabla^2 \phi &= \nabla \cdot \nabla \phi \\
&= \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} \cdot \begin{pmatrix} \partial \phi/\partial x \\ \partial \phi/\partial y \end{pmatrix} \\
&= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}
\end{aligned}$$

Translating x and y to polar coordinates and calculating their derivatives with respect to r and ϑ gives:

$$
\begin{aligned}
x = r\cos\vartheta, &\quad y = r\sin\vartheta \\
\frac{\partial x}{\partial r} = \cos\vartheta, &\quad \frac{\partial y}{\partial r} = \sin\vartheta \qquad (10)\\
\frac{\partial x}{\partial \vartheta} = -r\sin\vartheta, &\quad \frac{\partial y}{\partial \vartheta} = r\cos\vartheta \qquad (11)
\end{aligned}
$$

Consequently, by the chain rule and substitution from 10:

$$
\begin{aligned}
\frac{\partial \phi}{\partial r} &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial r} \\
&= \frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial \phi}{\partial y}\sin\vartheta \qquad (12)
\end{aligned}
$$

Taking the derivative of 12 with respect to r again gives:

$$
\begin{aligned}
\frac{\partial^2 \phi}{\partial r^2} &= \frac{\partial}{\partial r}\frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial}{\partial r}\frac{\partial \phi}{\partial y}\sin\vartheta \\
&= \frac{\partial}{\partial x}\frac{\partial \phi}{\partial r}\cos\vartheta + \frac{\partial}{\partial y}\frac{\partial \phi}{\partial r}\sin\vartheta \qquad (13)
\end{aligned}
$$

Substituting 12 into 13 gives:

$$
\begin{aligned}
\frac{\partial^2 \phi}{\partial r^2} &= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial \phi}{\partial y}\sin\vartheta\right)\cos\vartheta + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial \phi}{\partial y}\sin\vartheta\right)\sin\vartheta \\
&= \frac{\partial^2 \phi}{\partial x^2}\cos^2\vartheta + \frac{\partial^2 \phi}{\partial x \partial y}\sin\vartheta\cos\vartheta + \frac{\partial^2 \phi}{\partial y \partial x}\cos\vartheta\sin\vartheta + \frac{\partial^2 \phi}{\partial y^2}\sin^2\vartheta \\
&= \frac{\partial^2 \phi}{\partial x^2}\cos^2\vartheta + 2\frac{\partial^2 \phi}{\partial x \partial y}\sin\vartheta\cos\vartheta + \frac{\partial^2 \phi}{\partial y^2}\sin^2\vartheta \qquad (14)
\end{aligned}
$$

Applying the same process for $\frac{\partial \phi}{\partial \vartheta}$ with substitution from 11 yields:

$$
\begin{aligned}
\frac{\partial \phi}{\partial \vartheta} &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \vartheta} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \vartheta} \\
&= -\frac{\partial \phi}{\partial x}r\sin\vartheta + \frac{\partial \phi}{\partial y}r\cos\vartheta \qquad (15)
\end{aligned}
$$

Taking the derivative of 15 with respect to ϑ again gives:

$$\frac{\partial^2 \phi}{\partial \vartheta^2} = -\frac{\partial}{\partial \vartheta}\frac{\partial \phi}{\partial x} r \sin\vartheta + \frac{\partial}{\partial \vartheta}\frac{\partial \phi}{\partial y} r \cos\vartheta$$

Since both terms contain a product of two functions dependent on ϑ the product rule needs to be applied. This gives:

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial \vartheta^2} &= -\frac{\partial^2 \phi}{\partial \vartheta \partial x} r \sin\vartheta - \frac{\partial \phi}{\partial x} r \cos\vartheta + \frac{\partial^2 \phi}{\partial \vartheta \partial y} r \cos\vartheta - \frac{\partial \phi}{\partial y} r \sin\vartheta \\
&= -r\left(\frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial \phi}{\partial y}\sin\vartheta\right) + r\frac{\partial \phi}{\partial \vartheta}\left(-\frac{\partial}{\partial x}\sin\vartheta + \frac{\partial}{\partial y}\cos\vartheta\right)
\end{aligned} \tag{16}$$

Substituting 15 into 16 gives:

$$\frac{\partial^2 \phi}{\partial \vartheta^2} = -r\left(\frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial \phi}{\partial y}\sin\vartheta\right) + r\underbrace{\left(-\frac{\partial \phi}{\partial x} r \sin\vartheta + \frac{\partial \phi}{\partial y} r \cos\vartheta\right)\left(-\frac{\partial}{\partial x}\sin\vartheta + \frac{\partial}{\partial y}\cos\vartheta\right)}_{\Phi} \tag{17}$$

Expanding Φ:

$$\begin{aligned}
\Phi &= \left(-\frac{\partial \phi}{\partial x} r \sin\vartheta + \frac{\partial \phi}{\partial y} r \cos\vartheta\right)\left(-\frac{\partial}{\partial x}\sin\vartheta + \frac{\partial}{\partial y}\cos\vartheta\right) \\
&= \left(-\frac{\partial \phi}{\partial x} r \sin\vartheta\right)\left(-\frac{\partial}{\partial x}\sin\vartheta\right) + \left(-\frac{\partial \phi}{\partial x} r \sin\vartheta\right)\left(\frac{\partial}{\partial y}\cos\vartheta\right) \\
&+ \left(\frac{\partial \phi}{\partial y} r \cos\vartheta\right)\left(-\frac{\partial}{\partial x}\sin\vartheta\right) + \left(\frac{\partial \phi}{\partial y} r \cos\vartheta\right)\left(\frac{\partial}{\partial y}\cos\vartheta\right) \\
&= \frac{\partial^2 \phi}{\partial x^2} r \sin^2\vartheta - 2\frac{\partial^2 \phi}{\partial x \partial y} r \cos\vartheta \sin\vartheta + \frac{\partial^2 \phi}{\partial y^2} r \cos^2\vartheta
\end{aligned}$$

Substituting Φ back into 17 gives:

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial \vartheta^2} &= -r\left(\frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial \phi}{\partial y}\sin\vartheta\right) + r\left(\frac{\partial^2 \phi}{\partial x^2} r \sin^2\vartheta - 2\frac{\partial^2 \phi}{\partial x \partial y} r \cos\vartheta \sin\vartheta + \frac{\partial^2 \phi}{\partial y^2} r \cos^2\vartheta\right) \\
&= r^2\left(\frac{\partial^2 \phi}{\partial x^2}\sin^2\vartheta - 2\frac{\partial^2 \phi}{\partial x \partial y}\cos\vartheta \sin\vartheta + \frac{\partial^2 \phi}{\partial y^2}\cos^2\vartheta\right) - r\left(\frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial \phi}{\partial y}\sin\vartheta\right) \\
&= r^2\left(\frac{\partial^2 \phi}{\partial x^2}\sin^2\vartheta - 2\frac{\partial^2 \phi}{\partial x \partial y}\cos\vartheta \sin\vartheta + \frac{\partial^2 \phi}{\partial y^2}\cos^2\vartheta\right) - r\frac{\partial \phi}{\partial r}
\end{aligned} \tag{18}$$

Combining 14 and 18 yields:

$$
\begin{aligned}
\frac{\partial^2\phi}{\partial r^2} + \frac{\partial^2\phi}{\partial\vartheta^2} &= \frac{\partial^2\phi}{\partial x^2}\cos^2\vartheta + 2\frac{\partial^2\phi}{\partial x\partial y}\sin\vartheta\cos\vartheta + \frac{\partial^2\phi}{\partial y^2}\sin^2\vartheta \\
&\quad + r^2\left(\frac{\partial^2\phi}{\partial x^2}\sin^2\vartheta - 2\frac{\partial^2\phi}{\partial x\partial y}\cos\vartheta\sin\vartheta + \frac{\partial^2\phi}{\partial y^2}\cos^2\vartheta\right) - r\frac{\partial\phi}{\partial r} \\
\implies \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2} &= \frac{\partial^2\phi}{\partial x^2}\cos^2\vartheta + \frac{\partial^2\phi}{\partial x^2}\sin^2\vartheta + \frac{\partial^2\phi}{\partial y^2}\cos^2\vartheta + \frac{\partial^2\phi}{\partial y^2}\sin^2\vartheta - \frac{1}{r}\frac{\partial\phi}{\partial r} \\
&= \frac{\partial^2\phi}{\partial x^2}\left(\cos^2\vartheta + \sin^2\vartheta\right) + \frac{\partial^2\phi}{\partial y^2}\left(\cos^2\vartheta + \sin^2\vartheta\right) - \frac{1}{r}\frac{\partial\phi}{\partial r} \\
&= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - \frac{1}{r}\frac{\partial\phi}{\partial r} \\
\implies \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} &= \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2} \\
\therefore \nabla^2\phi &= \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2} \qquad (19)
\end{aligned}
$$

■

8.3 Ad confluōrem

Summarized, the conditions translated to polar form in sections 8.2.1, 8.2.2 and 8.2.3 are:

$$
\begin{array}{ccc}
\mathbf{V} = U(\hat{r}\cos\vartheta - \hat{\vartheta}\sin\vartheta) & \text{as} & r \to \infty \\
\mathbf{V}\cdot\hat{r} = 0 & \text{when} & r = L \\
\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2} = 0 & &
\end{array}
$$

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