How can vector calculus be used to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle, and what mathematical principles underpin the observed fluid behaviour?

Mathematics AA HL

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Contents

1 Introduction

Fluid dynamics is today a cornerstone to several fields of study, including ærospace engineering and meteorology. Real world fluid behaviour is intricate and complex. Therefore, to gain insights into the governing principles of fluid flow, simplified and idealised models are used. This essay investigates the application of vector calculus to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle. These idealisations allow for the derivation of some of fluid dynamic's key mathematical formulæ and provides a foundation for understanding less idealised fluids.

This essay will address the question: "How can vector calculus be used to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle, and what mathematical principles underpin the observed fluid behaviour?" Through the derivation of the velocity potential and vector field, this essay aims to demonstrate how fundamental laws of fluid motion can be expressed and used through vector calculus.

1.1 Aim & scope

The scope of this essay will be limited to the theoretical modelling of fluid flow in a two-dimensional space as a vector field under idealised conditions forming steady, inviscid and incompressible fluid flow through the derivation of the velocity-potential. The analysis will be centred on the application of vector calculus to derive fundamental formulæ and describe fluid behaviour around a stationary circular obstacle. Consequently, this essay will not touch on viscous effects, turbulent flow or three-dimensional analysis, nor will it involve any experimental validation. The focus is on the mathematical derivation and analysis of the idealised model.

1.2 Background

1.2.1 Glossary

Definition 1.1. ***Steady flow*** refers to flow in which the velocity at every point does not change over time [CRACIUNOIU and CIOCIRLAN, 2001].

Definition 1.2. ***Inviscid flow*** is the flow of a fluid with 0 viscosity [Anderson, 2003].

Definition 1.3. An ***incompressible fluid*** is a fluid whose density at every point does not change over time [Ahmed, 2019].

Definition 1.4. A ***scalar field*** is a function mapping points in space to scalar quantities such as temperatures.

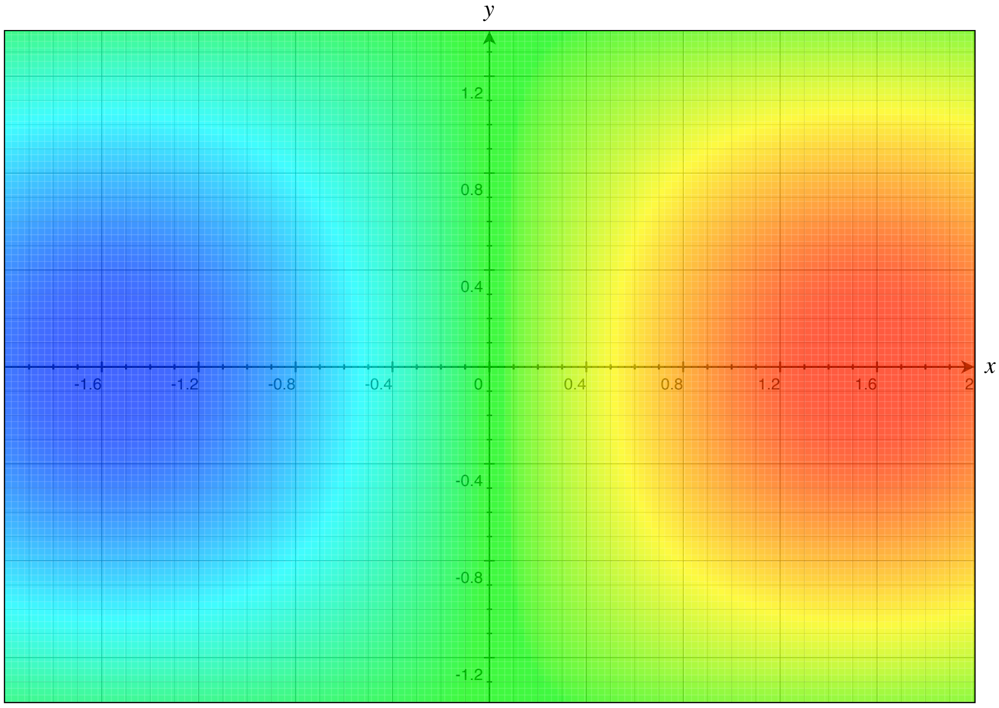

Figure 1: Scalar field plotted for the function $f(x, y) = \sin(x)\cos y$

Definition 1.5. A ***vector field*** is a function mapping points in space to vector quantities [Brezinski, 2006]. In the case of fluid dynamics, vector fields often model quantities like fluid velocity.

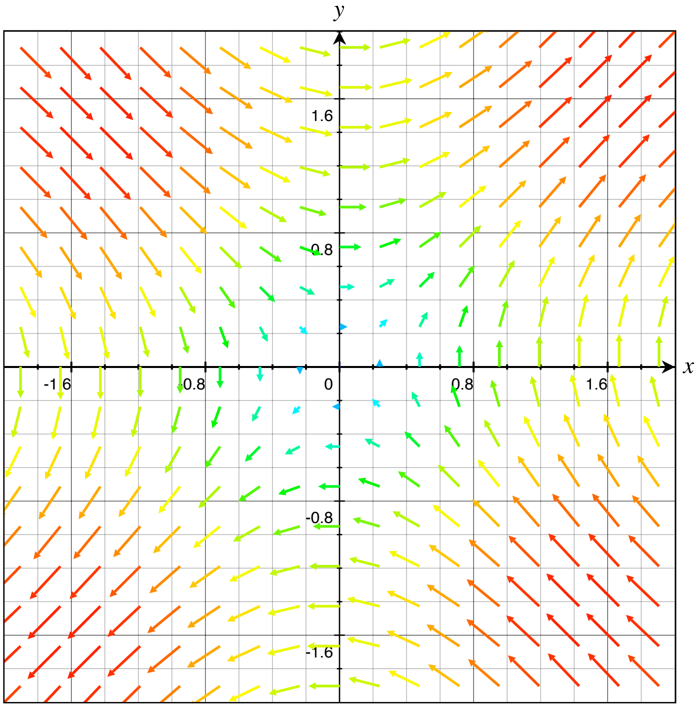

Figure 2: Vector field plotted for the function $f(x, y) = \begin{pmatrix} \sin y \\ \sin x \end{pmatrix}$

Definition 1.6. The ***velocity potential*** ϕ is a scalar field whose gradient is the velocity vector field of some fluid, mathematically $\mathbf{V} = \nabla\phi$. The quantity is defined for irrotational flow which is a resulting property of the idealisations made in this essay[see 8.1].

1.2.2 Notation

Vector calculus, like one-variable calculus, has no standardized notation. This essay will employ the following notation:

- ∇:
  - ∇F: The gradient of some scalar field F.
  - $\nabla \cdot \mathbf{F}$: The divergence of some vector field $\mathbf{F}$.
  - $\nabla \times \mathbf{F}$: The curl of some vector field $\mathbf{F}$.
- Δ: The Laplacian operator
- $\hat{\imath}$ & $\hat{\jmath}$: Unit vectors in the positive x and y directions respectively.
- $\hat{r}$ & $\hat{\vartheta}$: Unit vectors in the positive r and ϑ directions respectively.

2 Vector calculus

2.1 The fundamentals of vector calculus

Definition 2.1. ***Partial derivatives*** are an extension of single-variable derivatives in which all variables save the one being differentiated by are treated as constants [Mortimer, 2013]. A formal definition of the partial derivative of some function f with respect to a parameter x_n can be expressed as:

$$\frac{\partial f}{\partial x_n} = \lim_{\delta \to 0} \frac{f(x_1, x_2, \cdots, x_n + \delta, \cdots) - f(x_1, x_2, \cdots, x_n, \cdots)}{\delta} \tag{1}$$

Partial derivatives allow for the analysis of how multi-variable functions such as scalar- or vector fields change with respect to just one spatial dimension. For example, consider the function $f(x, y) = x^2 y + \sin(x) \sin y$.

$$\frac{\partial f}{\partial x} = 2xy + \cos(x) \sin y \qquad \frac{\partial f}{\partial y} = x^2 + \sin(x) \cos y$$

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7 List of Figures