On the modelling of steady, inviscid and incompressible fluid flow around a two-dimensional cylinder

Research question: "**How can vector calculus be used to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle, and what mathematical principles underpin the observed fluid behaviour?**"

Mathematics AA HL

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Contents

1 Introduction

Fluid dynamics is today a cornerstone to several fields of study, including ærospace engineering and meteorology. Real world fluid behaviour is intricate and complex. Therefore, to gain insights into the governing principles of fluid flow, simplified and idealised models are used. This essay investigates the application of vector calculus to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle. These idealisations allow for the derivation of some of fluid dynamic's key mathematical formulæ and provides a foundation for understanding less idealised fluids.

This essay will address the question: "How can vector calculus be used to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle, and what mathematical principles underpin the observed fluid behaviour?" Through the derivation of the velocity potential and vector field, this essay aims to demonstrate how fundamental laws of fluid motion can be expressed and used through vector calculus.

1.1 Aim & scope

The scope of this essay will be limited to the theoretical modelling of fluid flow in a two-dimensional space as a vector field under idealised conditions forming steady, inviscid and incompressible fluid flow through the derivation of the velocity-potential. The analysis will be centred on the application of vector calculus to derive key equations and describe the behaviour of idealised fluid around a stationary circular obstacle. Consequently, this essay will not touch on viscous effects, turbulent flow or three-dimensional analysis, nor will it involve any experimental validation. The focus is on the mathematical derivation and analysis of the idealised model.

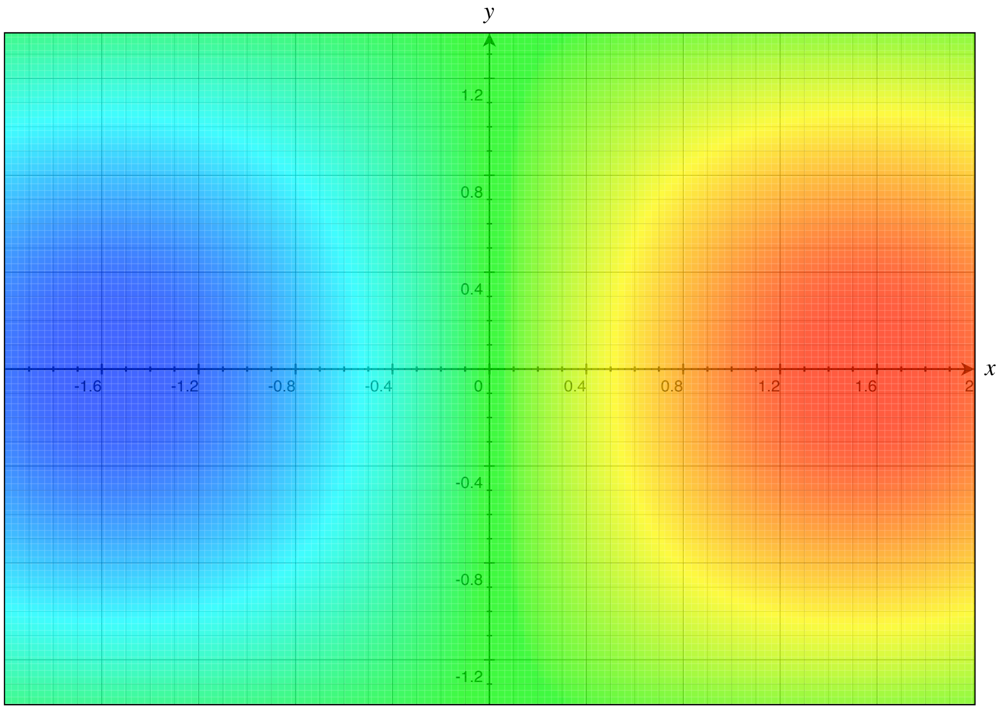


Figure 1: Scalar field plotted for the function $f : x, y \mapsto \sin(x)\cos y$

1.2 Background

1.2.1 Glossary

Definition 1.1. ***Steady flow*** refers to flow in which the velocity at every point does not change over time [CRACIUNOIU and CIOCIRLAN, 2001].

Definition 1.2. ***Inviscid flow*** is the flow of a fluid with 0 viscosity [Anderson, 2003].

Definition 1.3. An ***incompressible fluid*** is a fluid whose density at every point does not change over time [Ahmed, 2019].

Definition 1.4. A ***scalar field*** is a function mapping points in space to scalar quantities such as temperatures[see Figure 1].

Definition 1.5. A ***vector field*** is a function mapping points in space to vector quantities [Brezinski, 2006]. In the case of fluid dynamics, vector fields often model quantities like fluid velocity[see Figure 2].

Definition 1.6. The ***velocity potential*** ϕ is a scalar field whose gradient is the velocity vector field of some fluid, mathematically $\mathbf{V} = \nabla\phi$. The quantity is defined for irrotational

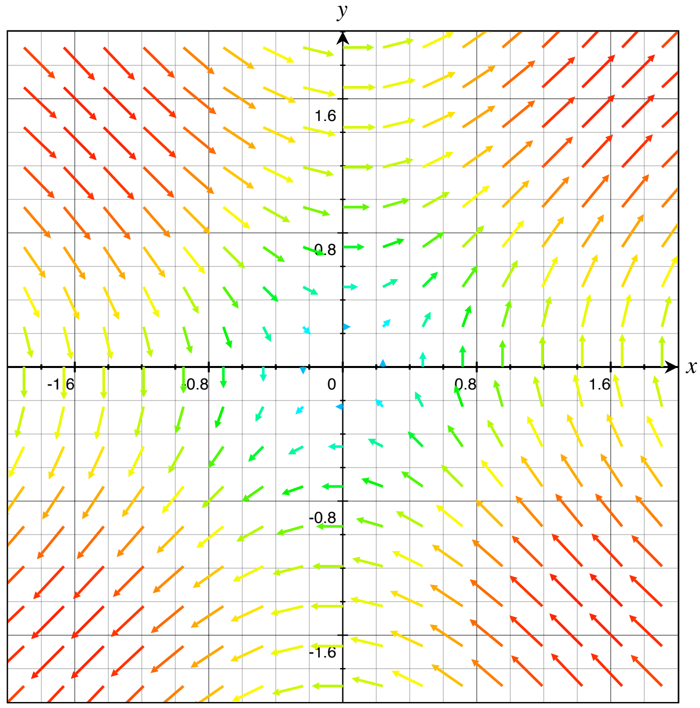


Figure 2: Vector field plotted for the function $\mathbf{F} : x, y \mapsto \binom{\sin y}{\sin x}$

flow (which is a resulting property of the idealisations made in this essay[see Section 3.2.1]) and is essential to its analysis.

1.2.2 Notation

Vector calculus, like one-variable calculus, has no standardized notation. This essay will employ the following notation:

- ∇:
  - ∇F: The gradient of some scalar field F.
  - $\nabla \cdot \mathbf{F}$: The divergence of some vector field $\mathbf{F}$.
  - $\nabla \times \mathbf{F}$: The curl of some vector field $\mathbf{F}$.
  - $\nabla_{\mathbf{v}} f$: The directional derivative of f in the direction of some vector $\mathbf{v}$
- Δ: The Laplacian operator
- δ_x: A small change in some variable x, used in place of Δx to avoid confusion with the Laplacian operator.

- $\mathbf{J_F}$: The Jacobian matrix of the function $\mathbf{F}$

- $\mathbf{V}^\top$: The transpose of some matrix or vector $\mathbf{V}$

- $\Im(z)$: The imaginary part of some complex number z

- $0 \notin \mathbb{N}$: 0 is not a member of the set of natural numbers

- $\mathbb{D}_\delta(\langle x,\, y\rangle)$: The set of the points in an open disk centred at $(x,\, y)$ with radius δ

- $\hat{\imath},\, \hat{\jmath}\,\&\,\hat{k}$: Unit vectors in the positive x, y and z directions respectively.

- $\hat{r}\,\&\,\hat{\vartheta}$: Unit vectors in the positive r and ϑ directions respectively.

1.2.3 The mean value theorem

To support the derivations made later in this essay, particularly in the proof of Clairaut's theorem[see lemma 2.1], fundamental concepts and theorems from single-variable calculus are introduced here, including the Mean Value Theorem and the lemmas it builds upon.

Theorem 1.1 (The extreme value theorem)**.** If a function f is continuous on the finite interval $[a, b]$, then there exists $A, B \in [a, b]$ such that $f(A) \leq f(x) \leq f(B)\,\forall x \in [a, b]$. Thus, at the points A and B, f has an absolute minimum $m = f(A)$ and an absolute maximum $M = f(B)$.

Theorem 1.2 (Rolle's theorem)**.** If a function f is continuous on the interval $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then $\exists\, c \in (a, b)$ such that $f'(c) = 0$.

Proof. Consider two cases:

Case 1: f remains constant over $[a, b]$

If $f(x) = f(a) = f(b)\,\forall x \in (a, b)$, then $f'(x) = 0$, and the theorem holds trivially.

Case 2: f is not constant over $[a, b]$

If f is not constant over $[a, b]$ and $f(a) = f(b)$, then Theorem 1.1 asserts that there must exist an absolute maximum or minimum that occur at some point $\eta \in (a, b)$. Since f is differentiable over (a, b), then any point η where an absolute extremum occurs must also be a local extremum. Consider the case where η is a local maxima (the proof for the case of local minima is analogous). Then let the interval $I = (\eta - \delta, \eta + \delta)$ for some $\delta > 0$ such that $\forall X \in I, f(X) \leq f(\eta)$.

Let $h < 0$ be a number sufficiently small such that $\eta + h \in I$. $f(\eta + h) \leq f(\eta) \implies f(\eta + h) - f(\eta) \leq 0$. Thus,

$$\frac{f(\eta+h) - f(\eta)}{h} \geq 0 \because \begin{cases} f(\eta+h) - f(\eta) & \leq 0 \\ h & \leq 0 \end{cases}$$

Taking the left-hand limit as $h \to 0$,

$$\lim_{h \to 0^-} \frac{f(\eta+h) - f(\eta)}{h} = f'(\eta)$$

Now let $H > 0$ be a number sufficiently small such that $\eta - H \in I$.

$$\begin{aligned} \frac{f(\eta+H) - f(\eta)}{H} &\leq 0 \because \begin{cases} f(\eta+H) - f(\eta) & \leq 0 \\ H & \geq 0 \end{cases} \\ \lim_{H \to 0^+} \frac{f(\eta+H) - f(\eta)}{H} &= f'(\eta) \end{aligned}$$

Thus,

$$0 \geq \lim_{H \to 0^+} \frac{f(\eta+H) - f(\eta)}{H} = f'(\eta) = \lim_{h \to 0^-} \frac{f(\eta+h) - f(\eta)}{h} \geq 0$$
$$\therefore f'(\eta) = 0$$

Since the same would apply for local minima, then for any local extrema $\eta \in (a, b)$, of which Theorem 1.1 asserts there must exist at least one, $f'(\eta) = 0$. ■

Theorem 1.3 (The mean value theorem)**.** For any function f continuous on the interval $[a, b]$ and differentiable on the interval (a, b), $\exists\, c \in (a, b)$ such that

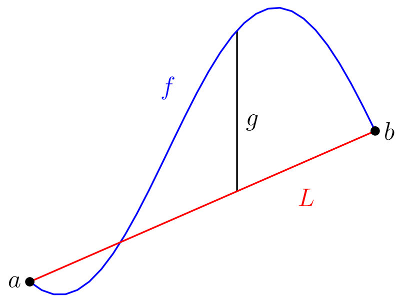
$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1}$$

Proof. Consider the region of some function f on the finite interval $[a, b]$ over which f is continuous and differentiable over (a, b). Let the function L represent the straight line between the points $(a,\ f(a))$ and $(b,\ f(b))$, which is given by the expression:

$$L(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

Now consider the function g, defined as the difference between f and L:

$$g(x) = L(x) - f(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) - f(x)$$



Computing the derivative of g with respect to x gives:

$$g'(x) = \frac{f(b) - f(a)}{b - a} - f'(x)$$

Since $g(a) = g(b) = 0$, Theorem 1.2 asserts that there is at least one point $c \in (a, b)$ such that $g'(c) = 0$. Thus, at c,

$$\begin{aligned} 0 = g'(c) &= \frac{f(b) - f(a)}{b - a} - f'(c) \\ \implies f'(c) &= \frac{f(b) - f(a)}{b - a} \end{aligned}$$

■

2 Vector calculus

2.1 The fundamentals of vector calculus

Definition 2.1. ***Partial derivatives*** are a multivariable extension of single-variable derivatives in which all variables save the one being differentiated by are treated as constants [Mortimer, 2013]. A formal definition of the partial derivative of some function f with respect to a parameter x_n can be expressed as:

$$\frac{\partial f}{\partial x_n} = \lim_{\delta_x \to 0} \frac{f(x_1, x_2, \cdots, x_n + \delta_x, \cdots) - f(x_1, x_2, \cdots, x_n, \cdots)}{\delta_x} \tag{2}$$

Partial derivatives allow for the analysis of how multi-variable functions such as scalar- or vector fields change with respect to just one spatial dimension. For example, consider the function $f(x, y) = x^2 y + \sin(x) \sin y$:

$$\frac{\partial f}{\partial x} = 2xy + \cos(x) \sin y \qquad \qquad \frac{\partial f}{\partial y} = x^2 + \sin(x) \cos y$$

n-th order partial derivatives are denoted, similarly to normal calculus, as

$$\frac{\partial^n f}{\partial x^n} \equiv \underbrace{\frac{\partial}{\partial x} \cdots \frac{\partial}{\partial x} \frac{\partial f}{\partial x}}_{n \text{ times}}$$

Definition 2.2. ***Mixed partial derivatives*** are partial derivatives of a function taken with respect to multiple variables [Garrett, 2015]. This is denoted as

$$\frac{\partial^2 f}{\partial \alpha \partial \beta} \equiv \frac{\partial}{\partial \beta} \frac{\partial f}{\partial \alpha}$$

where both α and β are parameters of f. The order of mixed partial derivatives, provided that both of the first order partial derivatives are continuous, does not matter, as guaranteed by Theorem 2.1 (Clairaut's theorem).

Theorem 2.1 (Clairaut's theorem)**.** Let $f(\alpha, \beta)$ be a function of two parameters α and β. If the mixed partial derivatives $\frac{\partial^2 f}{\partial \alpha \partial \beta}$ and $\frac{\partial^2 f}{\partial \beta \partial \alpha}$ exist and are continuous in the open disk $\mathbb{D}_\delta(\langle \alpha_0,\, \beta_0 \rangle)$ centred at $(\alpha_0,\, \beta_0)$ with radius $\delta > 0$, then

$$\left. \frac{\partial^2 f}{\partial \alpha \partial \beta} \right|_{(\alpha_0,\, \beta_0)} = \left. \frac{\partial^2 f}{\partial \beta \partial \alpha} \right|_{(\alpha_0,\, \beta_0)}$$

[Garrett, 2015]

Proof. Let $(\alpha_0,\ \beta_0)$ and $(\alpha_1,\ \beta_1)$ be points in the domain of f. Consider a rectangular region bound by the points $W\left(\alpha_0,\ \beta_0\right), X\left(\alpha_1,\ \beta_0\right), Y\left(\alpha_1,\ \beta_1\right)$ and $Z\left(\alpha_0,\ \beta_1\right)$. $\frac{\partial f}{\partial \alpha}$ and $\frac{\partial f}{\partial \beta}$ exist in a neighbourhood of this rectangle, and the mixed partial derivatives $\frac{\partial^2 f}{\partial \beta \partial \alpha}$ and $\frac{\partial^2 f}{\partial \alpha \partial \beta}$ exist and are continuous in this neighbourhood. Let Q be such that

$$Q = [f(\alpha_1, \beta_1) - f(\alpha_0, \beta_1)] - [f(\alpha_1, \beta_0) - f(\alpha_0, \beta_0)]$$

According to Theorem 1.3, the mean value theorem (MVT), $\exists\, \xi_0, \xi_1 \in [\alpha_0, \alpha_1]$ such that

$$\begin{aligned}
\left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_0,\, \beta_0)} &= \frac{f(\alpha_1, \beta_0) - f(\alpha_0, \beta_0)}{\alpha_1 - \alpha_0} \\
\left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_1,\, \beta_1)} &= \frac{f(\alpha_1, \beta_1) - f(\alpha_0, \beta_1)}{\alpha_1 - \alpha_0}
\end{aligned}$$

Thus Q can be expressed as

$$\begin{aligned}
Q &= \left(\left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_0,\, \beta_0)} (\alpha_1 - \alpha_0)\right) - \left(\left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_1,\, \beta_1)} (\alpha_1 - \alpha_0)\right) \\
&= \left(\left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_0,\, \beta_0)} - \left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_1,\, \beta_1)}\right) (\alpha_1 - \alpha_0)
\end{aligned}$$

Now let R be the equivalent of Q in the direction of β,

$$R = [f(\alpha_1, \beta_1) - f(\alpha_1, \beta_0)] - [f(\alpha_0, \beta_1) - f(\alpha_0, \beta_0)]$$

By the MVT $\exists\, \zeta_0, \zeta_1 \in [\beta_0, \beta_1]$ such that

$$\begin{aligned}
\left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_0,\, \zeta_0)} &= \frac{f(\alpha_0, \beta_1) - f(\alpha_0, \beta_0)}{\beta_1 - \beta_0} \\
\left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_1,\, \zeta_1)} &= \frac{f(\alpha_1, \beta_1) - f(\alpha_1, \beta_0)}{\beta_1 - \beta_0}
\end{aligned}$$

Thus R can be expressed as

$$\begin{aligned} R &= \left(\left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_0,\,\zeta_0)} (\beta_1 - \beta_0) \right) - \left(\left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_1,\,\zeta_1)} (\beta_1 - \beta_0) \right) \\ &= \left(\left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_0,\,\zeta_0)} - \left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_1,\,\zeta_1)} \right) (\beta_1 - \beta_0) \end{aligned}$$

Rearranging Q and R,

$$\begin{aligned} Q &= [f(\alpha_1,\beta_1) - f(\alpha_0,\beta_1)] - [f(\alpha_1,\beta_0) - f(\alpha_0,\beta_0)] \\ &= f(\alpha_1,\beta_1) - f(\alpha_1,\beta_0) - f(\alpha_0,\beta_1) + f(\alpha_0,\beta_0) \\ &= [f(\alpha_1,\beta_1) - f(\alpha_1,\beta_0)] - [f(\alpha_0,\beta_1) - f(\alpha_0,\beta_0)] = R \\ \therefore Q &= R \end{aligned}$$

Thus

$$\begin{aligned} &\left(\left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_0,\,\beta_0)} - \left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_1,\,\beta_1)} \right) (\alpha_1 - \alpha_0) = \left(\left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_0,\,\zeta_0)} - \left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_1,\,\zeta_1)} \right) (\beta_1 - \beta_0) \\ &\rightsquigarrow \frac{\partial f/\partial \alpha\,\big|_{(\xi_0,\,\beta_0)} - \partial f/\partial \alpha\,\big|_{(\xi_1,\,\beta_1)}}{\beta_1 - \beta_0} = \frac{\partial f/\partial \beta\,\big|_{(\alpha_0,\,\zeta_0)} - \partial f/\partial \beta\,\big|_{(\alpha_1,\,\zeta_1)}}{\alpha_1 - \alpha_0} \end{aligned} \tag{3}$$

Applying the MVT again $\exists\, \xi^* \in (\xi_0, \xi_1), \beta^* \in (\beta_0, \beta_1)$ such that

$$\begin{aligned} \left.\frac{\partial^2 f}{\partial \alpha \partial \beta}\right|_{(\xi^*,\,\beta^*)} &= \frac{\partial f/\partial \alpha\,\big|_{(\xi_1,\,\beta_1)} - \partial f/\partial \alpha\,\big|_{(\xi_0,\,\beta_0)}}{\beta_1 - \beta_0} \\ \implies -\left.\frac{\partial^2 f}{\partial \alpha \partial \beta}\right|_{(\xi^*,\,\beta^*)} &= \frac{\partial f/\partial \alpha\,\big|_{(\xi_0,\,\beta_0)} - \partial f/\partial \alpha\,\big|_{(\xi_1,\,\beta_1)}}{\beta_1 - \beta_0} \end{aligned}$$

Similarly, $\exists\, \alpha^* \in (\alpha_0, \alpha_1), \zeta^* \in (\zeta_0, \zeta_1)$ such that

$$\begin{aligned} \left.\frac{\partial^2 f}{\partial \beta \partial \alpha}\right|_{(\alpha^*,\,\zeta^*)} &= \frac{\partial f/\partial \beta\,\big|_{(\alpha_1,\,\zeta_1)} - \partial f/\partial \beta\,\big|_{(\alpha_0,\,\zeta_0)}}{\alpha_1 - \alpha_0} \\ \implies -\left.\frac{\partial^2 f}{\partial \beta \partial \alpha}\right|_{(\alpha^*,\,\zeta^*)} &= \frac{\partial f/\partial \beta\,\big|_{(\alpha_0,\,\zeta_0)} - \partial f/\partial \beta\,\big|_{(\alpha_1,\,\zeta_1)}}{\alpha_1 - \alpha_0} \end{aligned}$$

Substituting back into (3),

$$
\begin{aligned}
-\left.\frac{\partial^2 f}{\partial\alpha\partial\beta}\right|_{(\xi^*,\,\beta^*)} &= -\left.\frac{\partial^2 f}{\partial\beta\partial\alpha}\right|_{(\alpha^*,\,\zeta^*)} \\
\implies \left.\frac{\partial^2 f}{\partial\alpha\partial\beta}\right|_{(\xi^*,\,\beta^*)} &= \left.\frac{\partial^2 f}{\partial\beta\partial\alpha}\right|_{(\alpha^*,\,\zeta^*)}
\end{aligned}
$$

Consequently, as $\alpha_1 \to \alpha_0$ and $\beta_1 \to \beta_0$, $\xi^* \to \alpha_0, \beta^* \to \beta_0, \alpha^* \to \alpha_0$ and $\zeta^* \to \beta_0$. Since the derivatives are continuous,

$$
\left.\frac{\partial^2 f}{\partial\beta\partial\alpha}\right|_{(\alpha_0,\,\beta_0)} = \left.\frac{\partial^2 f}{\partial\alpha\partial\beta}\right|_{(\alpha_0,\,\beta_0)}
$$

Because $(\alpha_0,\,\beta_0)$ is an arbitrary point in the domain, $\frac{\partial^2 f}{\partial\beta\partial\alpha} = \frac{\partial^2 f}{\partial\alpha\partial\beta}$ at all points in the domain where the mixed partial derivatives are continuous. ■

Definition 2.3. The ***nabla*** or ***gradient*** operator ∇ is a vector containing the partial derivatives of a scalar valued function with respect to each of its parameters [Rapp, 2017]. Thus applying the operator is taking the product of the vector ∇ and a scalar valued function f. For some function $f : \mathbb{R}^n \to \mathbb{R}$, ∇ and ∇f would be given by:

$$
\nabla = \begin{bmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \vdots \\ \partial/\partial x_n \end{bmatrix}, \quad \nabla f = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \vdots \\ \partial f/\partial x_n \end{bmatrix}
$$

2.2 Divergence & curl

Definition 2.4. The ***divergence*** of a vector field $\mathbf{F}$ is a scalar field denoted as $\nabla \cdot \mathbf{F}$, defined as the dot product of the nabla operator (∇) and the vector field. For the function $\mathbf{F} : x, y \mapsto X(x,y)\hat{\imath} + Y(x,y)\hat{\jmath}$, $\nabla \cdot \mathbf{F}$ would be given by:

$$
\nabla \cdot \mathbf{F} = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}
$$

Definition 2.5. The ***curl*** of a vector field $\mathbf{F}$ is a vector field denoted as $\nabla \times \mathbf{F}$, defined as the cross product of the nabla operator and the vector field. Typically, since the cross product is only defined for 3 dimensional spaces, 2 dimensional curl is defined as a scalar field which for some function $\mathbf{F} : x, y \mapsto X(x,y)\hat{\imath} + Y(x,y)\hat{\jmath}$ is derived from the coefficient of $\hat{k}$ if the cross product was done as if the z term of both vectors was set to 0. Thus, for such a function $\mathbf{F}$:

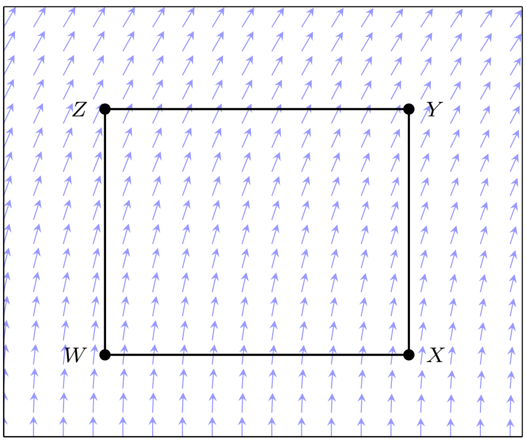
$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & 0 \\ X & Y & 0 \end{vmatrix} \\
&= \begin{vmatrix} \partial/\partial y & 0 \\ Y & 0 \end{vmatrix} \hat{\imath} - \begin{vmatrix} \partial/\partial x & 0 \\ X & 0 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ X & Y \end{vmatrix} \hat{k} \\
&= 0\hat{\imath} - 0\hat{\jmath} + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) \hat{k} = \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) \hat{k} \\
\rightsquigarrow \nabla \times \mathbf{F} &= \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}
\end{aligned}$$

3 Governing equations for ideal fluid flow

3.1 The continuity equation for incompressible flow

The idealised fluid in this essay is incompressible. This means that if ρ represents the density of the fluid, then as per definition 1.3, ρ must remain constant over time at every point in the domain of the vector field representing the fluid flow. This means that for any arbitrary closed volume within the fluid, the net mass flow rate across its boundaries must be zero.

Let the velocity vector field of the fluid be $\mathbf{V} : x, y \mapsto X(x,y)\hat{\imath} + Y(x,y)\hat{\jmath}$. To quantify the mass flow, consider an arbitrary infinitesimal rectangular volume within the fluid, with vertices $W(\alpha_0, \beta_0)$, $X(\alpha_1, \beta_0)$, $Y(\alpha_1, \beta_1)$, and $Z(\alpha_0, \beta_1)$, as depicted below. Let $\bar{\alpha} = \frac{\alpha_0 + \alpha_1}{2}$ and $\bar{\beta} = \frac{\beta_0 + \beta_1}{2}$. Assume $\mathbf{V}$ is continuous and differentiable over this region.



The mass flow rate ($\dot{m}$, mass per unit time) across a given surface is defined as the flux of mass, which is computed as the product of density, the velocity component normal to the surface, and the area of the surface. Thus, along the x axis (in the direction of $\hat{\imath}$) the mass flow rate into WZ is given as,

$$\dot{m}_{\to WZ} = \rho X(\alpha_0, \bar{\beta})(\beta_1 - \beta_0)$$

and similarly the mass flow rate out of the opposite side XY is given as:

$$\dot{m}_{XY\to} = \rho X(\alpha_1, \bar{\beta})(\beta_1 - \beta_0)$$

Thus, the net mass flow rate out of the rectangular region along the x axis is:

$$\begin{aligned}\dot{m}_{\hat{\imath}} &= \dot{m}_{XY\to} - \dot{m}_{\to WZ} \\ &= \rho X(\alpha_1, \bar{\beta})(\beta_1 - \beta_0) - \rho X(\alpha_0, \bar{\beta})(\beta_1 - \beta_0)\end{aligned}$$

Which, factoring out $\rho(\beta_1 - \beta_0)$, leads to:

$$\dot{m}_{\hat{\imath}} = \rho(\beta_1 - \beta_0)\left[X(\alpha_1, \bar{\beta}) - X(\alpha_0, \bar{\beta})\right]$$

Analogously, across the y axis, the net mass flow rate out of the rectangular region between sides WX and ZY is given by the expression:

$$\dot{m}_{\hat{\jmath}} = \rho(\alpha_1 - \alpha_0)\left[Y(\bar{\alpha}, \beta_1) - Y(\bar{\alpha}, \beta_0)\right]$$

Therefore, the net mass flow rate out of the rectangular region, which must be equal to 0 for the fluid to incompressible, is given by:

$$\begin{aligned}\dot{m} &= \dot{m}_{\hat{\imath}} + \dot{m}_{\hat{\jmath}} \\ &= \rho(\beta_1 - \beta_0)\left[X(\alpha_1, \bar{\beta}) - X(\alpha_0, \bar{\beta})\right] + \rho(\alpha_1 - \alpha_0)\left[Y(\bar{\alpha}, \beta_1) - Y(\bar{\alpha}, \beta_0)\right] = 0\end{aligned}$$

Dividing through by ρ, $(\alpha_1 - \alpha_0)$ and $(\beta_1 - \beta_0)$:

$$\frac{X(\alpha_1, \bar{\beta}) - X(\alpha_0, \bar{\beta})}{\alpha_1 - \alpha_0} + \frac{Y(\bar{\alpha}, \beta_1) - Y(\bar{\alpha}, \beta_0)}{\beta_1 - \beta_0} = 0$$

Now consider the limit as $\alpha_1 \to \alpha_0$ and $\beta_1 \to \beta_0$, the difference $\delta_\alpha = \alpha_1 - \alpha_0 \to 0$ and $\delta_\beta = \beta_1 - \beta_0 \to 0$.

$$\lim_{\delta_\alpha \to 0} \frac{X(\alpha_0 + \delta_\alpha, \bar{\beta}) - X(\alpha_0, \bar{\beta})}{\delta_\alpha} = \left.\frac{\partial X}{\partial x}\right|_{(\alpha_0, \bar{\beta})}$$
$$\lim_{\delta_\beta \to 0} \frac{Y(\bar{\alpha}, \beta_0 + \delta_\beta) - Y(\bar{\alpha}, \beta_0)}{\delta_\beta} = \left.\frac{\partial Y}{\partial y}\right|_{(\bar{\alpha}, \beta_0)}$$

Furthermore, as $\alpha_1 \to \alpha_0$ and $\beta_1 \to \beta_0$, $\bar{\alpha} \to \alpha_0$ and $\bar{\beta} \to \beta_0$, consequently

$$\dot{m} = \left.\frac{\partial X}{\partial x}\right|_{(\alpha_0, \beta_0)} + \left.\frac{\partial Y}{\partial y}\right|_{(\alpha_0, \beta_0)} = \nabla \cdot \mathbf{V}\Big|_{(\alpha_0, \beta_0)} = 0$$

Because $(\alpha_0,\, \beta_0)$ is any point in the domain of $\mathbf{V}$ where the function is differentiable, the expression can be generalised as

$$\nabla \cdot \mathbf{V} = 0$$

This equation is known as the continuity equation for incompressible fluids [Pham et al., 2014] and will underpin following derivations made in this essay.

3.2 Irrotational flow

As mentioned in definition 1.6, one resulting property of the idealisations (steady, inviscid and incompressible flow) made in this essay is irrotational flow. If flow is rotational, then there exists points at which $\nabla \times \mathbf{F} \neq 0$. In other words, if one were to imagine a water wheel at some point in the fluid, and it spins, then the flow is rotational, and vice versa for irrotational flow. However, flow being irrotational does not imply that it cannot curve, for example $\nabla \times \mathbf{F} = 0$ in cases such as:

$$\mathbf{F} : x, y \mapsto X(x, y)\hat{\imath} + Y(x, y)\hat{\jmath} \quad \forall\, (x,\, y) \neq (0,\, 0)$$
$$X : x, y \mapsto -\frac{y}{x^2 + y^2}\,, \quad Y : x, y \mapsto \frac{x}{x^2 + y^2}$$

Applying the quotient rule to compute the derivatives for both X and Y gives:

$$\begin{aligned}
\frac{\partial X}{\partial y} &= -\frac{\left(\frac{\partial}{\partial y} y\right)(x^2 + y^2) - y\frac{\partial}{\partial y}(x^2 + y^2)}{(x^2 + y^2)^2} \\
&= -\frac{x^2 + y^2 - y(2y)}{(x^2 + y^2)^2} \\
&= -\frac{x^2 - y^2}{(x^2 + y^2)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial Y}{\partial x} &= \frac{\left(\frac{\partial}{\partial x} x\right)(x^2 + y^2) - x\frac{\partial}{\partial x}(x^2 + y^2)}{(x^2 + y^2)^2} \\
&= \frac{x^2 + y^2 - x(2x)}{(x^2 + y^2)^2} \\
&= \frac{-x^2 + y^2}{(x^2 + y^2)^2} = -\frac{x^2 - y^2}{(x^2 + y^2)^2}
\end{aligned}$$

Thus,

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x} \implies \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} = 0$$
$$\therefore \nabla \times \mathbf{F} = 0$$

Plotting the vector field for $\mathbf{F}$ reveals circulation around the origin, suggesting rotational flow, but which, with a curl of 0 (everywhere except for the origin, where $\mathbf{F}$ is undefined), is irrotational[see figure 3].

3.2.1 Kelvin's circulation theorem

Kelvin's circulation theorem is crucial in the modelling of ideal fluids. While a full proof of the theorem is outside the scope of this essay for the purposes of keeping focused, it provides

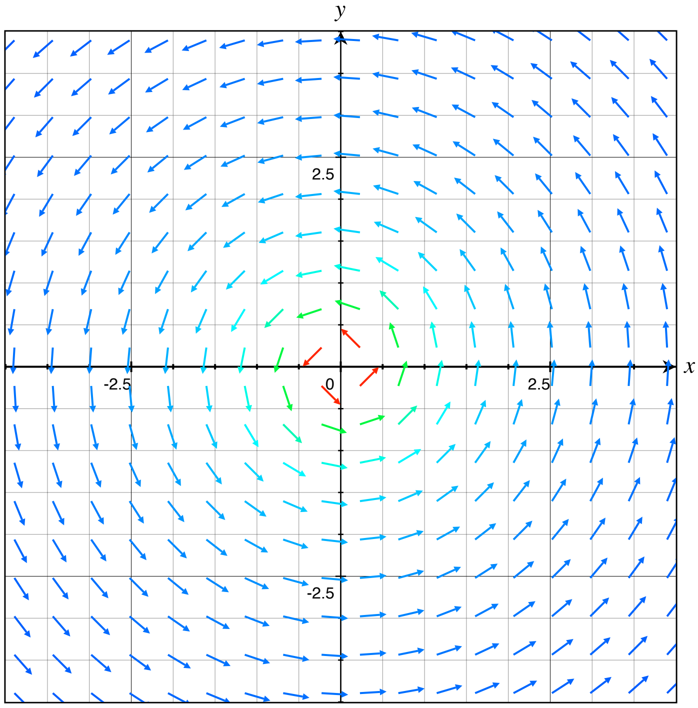


Figure 3: The function $\mathbf{F} : x, y \mapsto \begin{pmatrix} -y\left(x^2+y^2\right)^{-1} \\ x\left(x^2+y^2\right)^{-1} \end{pmatrix}$ is irrotational despite curving

the guarantee of irrotational flow which underpins the rest of this essay.

The theorem states that for an ideal (crucially inviscid) fluid, the circulation around a closed loop moving with the fluid remains constant over time. Therefore, implying that if a fluid begins its flow irrotational, it remains irrotational. Irrotational flow enables the use of a velocity potential[see Section 3.3], greatly simplifying the further analysis of the fluid flow.

Theorem 3.1 (Kelvin's circulation theorem)**.** The rate of change of the circulation (Γ) of an ideal and inviscid fluid with respect to time around a closed contour moving with the flow of the fluid (in a velocity field $\mathbf{V}$) is zero. Mathematically,

$$\overbrace{\frac{\partial \Gamma}{\partial t} + \mathbf{V} \cdot \nabla \Gamma}^{\text{Commonly denoted as } \frac{\mathrm{D}\Gamma}{\mathrm{D}t}} = 0$$

3.3 The velocity potential

As established in Section 3.2.1 through Theorem 3.1 (Kelvin's circulation theorem), for an incompressible and inviscid fluid initially irrotational, the flow remains irrotational, meaning the velocity field $\mathbf{V}$ modelling the flow satisfies $\nabla \times \mathbf{V} = 0$. This condition leads to the existence of a scalar-valued function, known as the velocity potential, whose gradient is equal to the vector field. To demonstrate this, consider the identity $\nabla \times \nabla\phi$, where ϕ is a function of x and y and is scalar valued. $\nabla\phi$ is defined as:

$$\nabla\phi = \begin{bmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \end{bmatrix}$$

Taking the curl of this expression and applying Theorem 2.1 (Clairaut's theorem) gives

$$\begin{aligned} \nabla \times \nabla\phi &= \frac{\partial^2\phi}{\partial y\partial x} - \frac{\partial^2\phi}{\partial x\partial y} \\ &= \frac{\partial^2\phi}{\partial y\partial x} - \frac{\partial^2\phi}{\partial y\partial x} = 0 \end{aligned}$$

Thus, by the definition of irrotational flow,

$$\nabla \times \mathbf{V} = 0 = \nabla \times \nabla\phi$$
$$\therefore \mathbf{V} = \nabla\phi$$

Here, ϕ is known as the velocity potential of $\mathbf{V}$. The velocity potential makes derivations made in this essay easier due its scalar valued nature.

3.4 The multivariable chain rule

3.4.1 The Jacobian matrix

Definition 3.1. The ***Jacobian matrix*** of some function $\mathbf{F}$, denoted $\mathbf{J_F}$, is defined as the matrix containing all the first order partial derivatives of the function. For the function

$$\mathbf{F} : x_1, x_2, \cdots, x_n \mapsto \begin{bmatrix} f_1(x_1, x_2, \cdots, x_n) \\ f_2(x_1, x_2, \cdots, x_n) \\ \vdots \\ f_m(x_1, x_2, \cdots, x_n) \end{bmatrix}$$

the Jacobian of $\mathbf{F}$ would be given as

$$\mathbf{J_F} = \begin{bmatrix} \nabla^\top f_1 \\ \nabla^\top f_2 \\ \vdots \\ \nabla^\top f_m \end{bmatrix} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots & \partial f_2/\partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial f_m/\partial x_1 & \partial f_m/\partial x_2 & \cdots & \partial f_m/\partial x_n \end{bmatrix} \in \mathbb{R}^{m \times n}$$

The Jacobian matrix, and its determinant, has many uses within vector calculus, but for the purposes of this essay, the Jacobian chain rule is a fundamental result which allows for the calculation of the derivatives of a composite functions. The following result and proof underpins the proof for the more generalised version in Section 3.4.2.

Lemma 3.2 (The Jacobian chain rule)**.** Let the function $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$ and the function $\mathbf{G} : \mathbb{R}^l \to \mathbb{R}^m$. The Jacobian of the composite $\mathbf{F} \circ \mathbf{G}$ evaluated at some point $\mathbf{p}$ can be expressed as

$$\mathbf{J_{F \circ G}}(\mathbf{p}) = [\mathbf{J_F} \circ \mathbf{G}(\mathbf{p})] \, \mathbf{J_G}(\mathbf{p})$$

Proof. A function $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$ is considered differentiable at some point $\mathbf{a}$ if there exists a linear map such (represented by its Jacobian matrix $\mathbf{J_F}(\mathbf{a})$)that for an infinitesimal vector $\mathbf{h}$

$$\mathbf{F}(\mathbf{a} + \mathbf{h}) - \mathbf{F}(\mathbf{a}) = \mathbf{J_F}(\mathbf{a})\mathbf{h} + \epsilon(\mathbf{h})$$

where $\epsilon(\mathbf{h})$ is an error term such that it vanishes faster than the magnitude of $\mathbf{h}$:

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{\|\epsilon(\mathbf{h})\|}{\|\mathbf{h}\|} = \mathbf{0} \tag{4}$$

Consider the function $\mathbf{Z} : \mathbf{t} \mapsto (\mathbf{F} \circ \mathbf{G})(\mathbf{t})$, where $\mathbf{G} : \mathbb{R}^l \to \mathbb{R}^m$ and $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$. For some small change in $\mathbf{t}$, called $\delta_\mathbf{t}$, the change in the inner function $\mathbf{G}$ can be expressed using its differentiability at $\mathbf{t}$

$$\delta_\mathbf{G} = \mathbf{G}(\mathbf{t} + \delta_\mathbf{t}) - \mathbf{G}(t) = \mathbf{J}_\mathbf{G}(\mathbf{t})\delta_\mathbf{t} + \epsilon_\mathbf{G}(\delta_\mathbf{t}) \tag{5}$$

The error term $\epsilon_\mathbf{G}(\delta_t)$ satisfies (4). Then consider the change of the outer function $\mathbf{F}$, also expressed using its differentiability at the point $\mathbf{u} = \mathbf{G}(\mathbf{t})$,

$$\delta_\mathbf{F} = \mathbf{F}(\mathbf{u} + \delta_\mathbf{G}) - \mathbf{F}(\mathbf{u}) = \mathbf{J}_\mathbf{F}(\mathbf{u})\delta_\mathbf{G} + \epsilon_\mathbf{F}(\delta_\mathbf{G})$$

Which, since $\mathbf{Z} = (\mathbf{F} \circ \mathbf{G})(\mathbf{t})$, means that

$$\delta_\mathbf{Z} = \delta_\mathbf{F} = \mathbf{F}(\mathbf{u} + \delta_\mathbf{G}) - \mathbf{F}(\mathbf{u}) = \mathbf{J}_\mathbf{F}(\mathbf{u})\delta_\mathbf{G} + \epsilon_\mathbf{F}(\delta_\mathbf{G}) \tag{6}$$

Let the error term $\epsilon_\mathbf{F}(\delta_\mathbf{G})$ also satisfy (4). Then, substituting (5) into (6),

$$\begin{aligned}\delta_\mathbf{Z} &= \mathbf{J}_\mathbf{F}(\mathbf{u})\left[\mathbf{J}_\mathbf{G}(\mathbf{t})\delta_\mathbf{t} + \epsilon_\mathbf{G}(\delta_\mathbf{t})\right] + \epsilon_\mathbf{F}(\mathbf{J}_\mathbf{G}(\mathbf{t})\delta_\mathbf{t} + \epsilon_\mathbf{G}(\delta_\mathbf{t})) \\ &= \mathbf{J}_\mathbf{F}(\mathbf{u})\mathbf{J}_\mathbf{G}(\mathbf{t})\delta_\mathbf{t} + \mathbf{J}_\mathbf{F}(\mathbf{u})\epsilon_\mathbf{G}(\delta_\mathbf{t}) + \epsilon_\mathbf{F}(\mathbf{J}_\mathbf{G}(\mathbf{t})\delta_\mathbf{t} + \epsilon_\mathbf{G}(\delta_\mathbf{t}))\end{aligned}$$

Dividing through by $\|\delta_\mathbf{t}\|$,

$$\frac{\delta_\mathbf{Z}}{\|\delta_\mathbf{t}\|} = \mathbf{J}_\mathbf{F}(\mathbf{u})\mathbf{J}_\mathbf{G}(\mathbf{t})\overbrace{\frac{\delta_\mathbf{t}}{\|\delta_\mathbf{t}\|}}^{\text{Henceforth } \hat{\delta_\mathbf{t}}} + \mathbf{J}_\mathbf{F}(\mathbf{u})\frac{\epsilon_\mathbf{G}(\delta_\mathbf{t})}{\|\delta_\mathbf{t}\|} + \frac{\epsilon_\mathbf{F}(\mathbf{J}_\mathbf{G}(\mathbf{t})\delta_\mathbf{t} + \epsilon_\mathbf{G}(\delta_\mathbf{t}))}{\|\delta_\mathbf{t}\|}$$

Considering the value of the error terms as $\delta_\mathbf{t} \to \mathbf{0}$,

$$\lim_{\delta_\mathbf{t} \to \mathbf{0}} \mathbf{J}_\mathbf{F}(\mathbf{u})\frac{\epsilon_\mathbf{G}(\delta_\mathbf{t})}{\|\delta_\mathbf{t}\|} + \frac{\epsilon_\mathbf{F}(\mathbf{J}_\mathbf{G}(\mathbf{t})\delta_\mathbf{t} + \epsilon_\mathbf{G}(\delta_\mathbf{t}))}{\|\delta_\mathbf{t}\|}$$

For any matrix $\mathbf{M}$ and vector $\mathbf{v}$, the magnitude of their product $\|\mathbf{M}\mathbf{v}\| \leq \|\mathbf{M}\|\|\mathbf{v}\|$, thus for the first term,

$$\begin{aligned}
\lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \|\mathbf{J_F}(\mathbf{u})\frac{\epsilon_{\mathbf{G}}(\boldsymbol{\delta_t})}{\|\boldsymbol{\delta_t}\|}\| &\leq \lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \|\mathbf{J_F}(\mathbf{u})\|\|\frac{\epsilon_{\mathbf{G}}(\boldsymbol{\delta_t})}{\|\boldsymbol{\delta_t}\|}\| \\
&= \lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \|\mathbf{J_F}(\mathbf{u})\|\frac{\|\epsilon_{\mathbf{G}}(\boldsymbol{\delta_t})\|}{\|\boldsymbol{\delta_t}\|}
\end{aligned}$$

By the definition of the definition of the error term $\lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \frac{\|\epsilon_{\mathbf{G}}(\boldsymbol{\delta_t})\|}{\|\boldsymbol{\delta_t}\|} = 0$, therefore

$$\begin{aligned}
&\lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \|\mathbf{J_F}(\mathbf{u})\frac{\epsilon_{\mathbf{G}}(\boldsymbol{\delta_t})}{\|\boldsymbol{\delta_t}\|}\| \leq \lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \|\mathbf{J_F}(\mathbf{u})\|\frac{\|\epsilon_{\mathbf{G}}(\boldsymbol{\delta_t})\|}{\|\boldsymbol{\delta_t}\|} \\
\implies &\lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \|\mathbf{J_F}(\mathbf{u})\frac{\epsilon_{\mathbf{G}}(\boldsymbol{\delta_t})}{\|\boldsymbol{\delta_t}\|}\| \leq 0 \\
\therefore &\lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \|\mathbf{J_F}(\mathbf{u})\frac{\epsilon_{\mathbf{G}}(\boldsymbol{\delta_t})}{\|\boldsymbol{\delta_t}\|}\| = 0 \implies \lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \mathbf{J_F}(\mathbf{u})\frac{\epsilon_{\mathbf{G}}(\boldsymbol{\delta_t})}{\|\boldsymbol{\delta_t}\|} = \mathbf{0}
\end{aligned}$$

The second error term can be shown to also be equal to $\mathbf{0}$ through rearrangment,

$$\begin{aligned}
\lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \frac{\epsilon_{\mathbf{F}}(\mathbf{J_G}(\mathbf{t})\boldsymbol{\delta_t} + \epsilon_{\mathbf{G}}(\boldsymbol{\delta_t}))}{\|\boldsymbol{\delta_t}\|} &= \lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \frac{\epsilon_{\mathbf{F}}(\mathbf{J_G}(\mathbf{t})\boldsymbol{\delta_t} + \epsilon_{\mathbf{G}}(\boldsymbol{\delta_t}))}{\|\mathbf{J_G}(\mathbf{t})\boldsymbol{\delta_t} + \epsilon_{\mathbf{G}}(\boldsymbol{\delta_t})\|}\frac{\|\mathbf{J_G}(\mathbf{t})\boldsymbol{\delta_t} + \epsilon_{\mathbf{G}}(\boldsymbol{\delta_t})\|}{\|\boldsymbol{\delta_t}\|} \\
&= \lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \frac{\epsilon_{\mathbf{F}}(\mathbf{J_G}(\mathbf{t})\boldsymbol{\delta_t} + \epsilon_{\mathbf{G}}(\boldsymbol{\delta_t}))}{\|\mathbf{J_G}(\mathbf{t})\boldsymbol{\delta_t} + \epsilon_{\mathbf{G}}(\boldsymbol{\delta_t})\|} \lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \frac{\|\mathbf{J_G}(\mathbf{t})\boldsymbol{\delta_t} + \epsilon_{\mathbf{G}}(\boldsymbol{\delta_t})\|}{\|\boldsymbol{\delta_t}\|} \\
&= \mathbf{0} \lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \frac{\|\mathbf{J_G}(\mathbf{t})\boldsymbol{\delta_t} + \epsilon_{\mathbf{G}}(\boldsymbol{\delta_t})\|}{\|\boldsymbol{\delta_t}\|} \\
\lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \frac{\|\mathbf{J_G}(\mathbf{t})\boldsymbol{\delta_t} + \epsilon_{\mathbf{G}}(\boldsymbol{\delta_t})\|}{\|\boldsymbol{\delta_t}\|} &\leq \lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \frac{\|\mathbf{J_G}(\mathbf{t})\boldsymbol{\delta_t}\|}{\|\boldsymbol{\delta_t}\|} + \frac{\|\epsilon_{\mathbf{G}}(\boldsymbol{\delta_t})\|}{\|\boldsymbol{\delta_t}\|} \\
&= \lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \frac{\|\mathbf{J_G}(\mathbf{t})\boldsymbol{\delta_t}\|}{\|\boldsymbol{\delta_t}\|} + \mathbf{0} \leq \lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \frac{\|\mathbf{J_G}(\mathbf{t})\|\|\boldsymbol{\delta_t}\|}{\|\boldsymbol{\delta_t}\|} + \mathbf{0} \\
&= \|\mathbf{J_G}(\mathbf{t})\| \\
\therefore \lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \frac{\epsilon_{\mathbf{F}}(\mathbf{J_G}(\mathbf{t})\boldsymbol{\delta_t} + \epsilon_{\mathbf{G}}(\boldsymbol{\delta_t}))}{\|\boldsymbol{\delta_t}\|} &\leq \mathbf{0}\|\mathbf{J_G}(\mathbf{t})\| = \mathbf{0}\,\forall\|\mathbf{J_G}(\mathbf{t})\| \\
\therefore \lim_{\boldsymbol{\delta_t}\to\mathbf{0}} \frac{\epsilon_{\mathbf{F}}(\mathbf{J_G}(\mathbf{t})\boldsymbol{\delta_t} + \epsilon_{\mathbf{G}}(\boldsymbol{\delta_t}))}{\|\boldsymbol{\delta_t}\|} &= \mathbf{0}
\end{aligned}$$

Thus, both of the error terms approach $\mathbf{0}$, meaning that

$$\begin{aligned}\lim_{\delta_{\mathbf{t}}\to\mathbf{0}}\frac{\delta_{\mathbf{Z}}}{\|\delta_{\mathbf{t}}\|} &= \lim_{\delta_{\mathbf{t}}\to 0}\mathbf{J_F}(\mathbf{u})\mathbf{J_G}(\mathbf{t})\hat{\delta_{\mathbf{t}}} + \mathbf{J_F}(\mathbf{u})\frac{\epsilon_{\mathbf{G}}(\delta_{\mathbf{t}})}{\|\delta_{\mathbf{t}}\|} + \frac{\epsilon_{\mathbf{F}}(\mathbf{J_G}(\mathbf{t})\delta_{\mathbf{t}} + \epsilon_{\mathbf{G}}(\delta_{\mathbf{t}}))}{\|\delta_{\mathbf{t}}\|} \\ &= \lim_{\delta_t\to 0}\mathbf{J_F}(\mathbf{u})\mathbf{J_G}(\mathbf{t})\hat{\delta_{\mathbf{t}}} + \mathbf{0} + \mathbf{0} \\ &= \mathbf{J_F}(\mathbf{G}(\mathbf{t}))\mathbf{J_G}(\mathbf{t})\hat{\delta_{\mathbf{t}}} = [\mathbf{J_F}\circ\mathbf{G}(\mathbf{t})]\,\mathbf{J_G}(\mathbf{t})\hat{\delta_{\mathbf{t}}} \end{aligned} \tag{7}$$

Using the definition of differentiability again for $\mathbf{Z}$ at $\mathbf{t}$,

$$\begin{aligned} \delta_{\mathbf{Z}} &= \mathbf{Z}(\mathbf{t}+\delta_{\mathbf{t}}) - \mathbf{Z}(\mathbf{t}) = \mathbf{J_Z}(\mathbf{t})\delta_{\mathbf{t}} + \epsilon_{\mathbf{Z}}(\delta_{\mathbf{t}}) \\ \implies \lim_{\delta_{\mathbf{t}}\to\mathbf{0}}\frac{\delta_{\mathbf{Z}}}{\|\delta_{\mathbf{t}}\|} &= \mathbf{J_Z}(\mathbf{t})\hat{\delta_{\mathbf{t}}} \end{aligned} \tag{8}$$

Therefore, substituting (8) into (7),

$$\begin{aligned} \mathbf{J_Z}(\mathbf{t})\hat{\delta_{\mathbf{t}}} &= [\mathbf{J_F}\circ\mathbf{G}(\mathbf{t})]\,\mathbf{J_G}(\mathbf{t})\hat{\delta_{\mathbf{t}}} \\ \therefore \mathbf{J_{F\circ G}}(\mathbf{t})\hat{\delta_{\mathbf{t}}} &= [\mathbf{J_F}\circ\mathbf{G}(\mathbf{t})]\,\mathbf{J_G}(\mathbf{t})\hat{\delta_{\mathbf{t}}} \end{aligned}$$

Now let $\mathbf{A} = \mathbf{J_{F\circ G}}(\mathbf{t})$ and $\mathbf{B} = [\mathbf{J_F}\circ\mathbf{G}(\mathbf{t})]\,\mathbf{J_G}(\mathbf{t})$,

$$\begin{aligned} \mathbf{A}\hat{\delta_{\mathbf{t}}} = \mathbf{B}\hat{\delta_{\mathbf{t}}} \implies \mathbf{0} &= \mathbf{A}\hat{\delta_{\mathbf{t}}} - \mathbf{B}\hat{\delta_{\mathbf{t}}} \\ &= (\mathbf{A}-\mathbf{B})\,\hat{\delta_{\mathbf{t}}} \end{aligned}$$

Since this equality holds for all non-zero vectors $\hat{\delta_{\mathbf{t}}}$, the matrix mapping the unit vector $\hat{\delta_{\mathbf{t}}}$ must be the zero matrix, consequently,

$$\begin{aligned} &\mathbf{A}-\mathbf{B} = \mathbf{0} \\ &\implies \mathbf{A} = \mathbf{B} \\ &\therefore \mathbf{J_{F\circ G}}(\mathbf{t}) = [\mathbf{J_F}\circ\mathbf{G}(\mathbf{t})]\,\mathbf{J_G}(\mathbf{t}) \end{aligned}$$

$\blacksquare$

3.4.2 The multivariable chain rule

The more generalised form of the multivariable chain rule can now be proven using the Jacobian matrix version of the chain rule proved in Lemma 3.2.

Lemma 3.3 (The multivariable chain rule)**.** Let $X : \mathbb{R}^2 \to \mathbb{R}$ and $Y : \mathbb{R}^2 \to \mathbb{R}$ be functions of some parameters α, β. Then let $Z : \mathbb{R}^2 \to \mathbb{R}$ be a function of the parameters x and y. The partial derivatives of the composite function $z : \alpha, \beta \mapsto Z(X(\alpha, \beta), Y(\alpha, \beta))$ are given by:

$$\frac{\partial z}{\partial \alpha} = \frac{\partial Z}{\partial x}\frac{\partial X}{\partial \alpha} + \frac{\partial Z}{\partial y}\frac{\partial Y}{\partial \alpha}$$
$$\frac{\partial z}{\partial \beta} = \frac{\partial Z}{\partial x}\frac{\partial X}{\partial \beta} + \frac{\partial Z}{\partial y}\frac{\partial Y}{\partial \beta}$$

Proof. Let $\mathbf{G} : \mathbb{R}^l \to \mathbb{R}^m$ and $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$. Suppose the output functions and parameters of $\mathbf{F}$ are called $f_1, f_2, \cdots, f_n$ and $x_1, x_2, \cdots, x_m$ respectively, and the output functions and parameters of $\mathbf{G}$ are called $g_1, g_2, \cdots, g_m$ and $y_1, y_2, \cdots, y_l$ respectively. The Jacobian matrix of the composite function $\mathbf{F} \circ \mathbf{G}$ is defined as:

$$\mathbf{J}_{\mathbf{F}\circ\mathbf{G}} = \begin{bmatrix} \partial(f_1 \circ \mathbf{G})/\partial y_1 & \partial(f_1 \circ \mathbf{G})/\partial y_2 & \cdots & \partial(f_1 \circ \mathbf{G})/\partial y_l \\ \partial(f_2 \circ \mathbf{G})/\partial y_1 & \partial(f_2 \circ \mathbf{G})/\partial y_2 & \cdots & \partial(f_2 \circ \mathbf{G})/\partial y_l \\ \vdots & \vdots & \ddots & \vdots \\ \partial(f_n \circ \mathbf{G})/\partial y_1 & \partial(f_n \circ \mathbf{G})/\partial y_2 & \cdots & \partial(f_n \circ \mathbf{G})/\partial y_l \end{bmatrix}$$

Thus the element at indices i, j of the Jacobian matrix $\mathbf{J}_{\mathbf{F}\circ\mathbf{G}}$ is given by:

$$(\mathbf{J}_{\mathbf{F}\circ\mathbf{G}})_{ij} = \frac{\partial(f_i \circ \mathbf{G})}{\partial y_j}$$

For the product of two matrices $\mathbf{C} = \mathbf{AB}$, where $\mathbf{A} \in \mathbb{R}^{n\times m}$ and $\mathbf{B} \in \mathbb{R}^{m\times l}$, the element at indices i, j is computed as:

$$\mathbf{C}_{ij} = \sum_{k=1}^{m} \mathbf{A}_{ik}\mathbf{B}_{kj}$$

Thus, for the product of the two Jacobian matrices $\mathbf{J_F} \circ \mathbf{G}$ and $\mathbf{J_G}$,

$$
\begin{aligned}
((\mathbf{J_F} \circ \mathbf{G})\mathbf{J_G})_{ij} &= \sum_{k=1}^{m} (\mathbf{J_F} \circ \mathbf{G})_{ik} (\mathbf{J_G})_{kj} \\
&= \sum_{k=1}^{m} \left.\frac{\partial f_i}{\partial x_k}\right|_{\mathbf{G}} \frac{\partial g_k}{\partial y_j}
\end{aligned}
$$

As shown in Lemma 3.2, $\mathbf{J_{F\circ G}} = (\mathbf{J_F} \circ \mathbf{G})\mathbf{J_G}$, therefore,

$$
\begin{aligned}
(\mathbf{J_{F\circ G}})_{ij} &= [(\mathbf{J_F} \circ \mathbf{G})\mathbf{J_G}]_{ij} \\
\implies \frac{\partial (f_i \circ \mathbf{G})}{\partial y_j} &= \sum_{k=1}^{m} \left.\frac{\partial f_i}{\partial x_k}\right|_{\mathbf{G}} \frac{\partial g_k}{\partial y_j} \qquad (9)
\end{aligned}
$$

Considering again the functions $X : \mathbb{R}^2 \to \mathbb{R}$ and $Y : \mathbb{R}^2 \to \mathbb{R}$ of parameters α, β and $Z : \mathbb{R}^2 \to \mathbb{R}$ of parameters x and y. Applying the form derived in (9) to the composite function $z : \alpha, \beta \mapsto Z(X(\alpha, \beta), Y(\alpha, \beta))$ leads to

$$
\begin{aligned}
\frac{\partial z}{\partial \alpha} &= \sum_{k=1}^{2} \frac{\partial Z}{\partial x_k} \frac{\partial g_k}{\partial \alpha} = \frac{\partial Z}{\partial x} \frac{\partial X}{\partial \alpha} + \frac{\partial Z}{\partial y} \frac{\partial Y}{\partial \alpha} \\
\frac{\partial z}{\partial \beta} &= \sum_{k=1}^{2} \frac{\partial Z}{\partial x_k} \frac{\partial g_k}{\partial \beta} = \frac{\partial Z}{\partial x} \frac{\partial X}{\partial \beta} + \frac{\partial Z}{\partial y} \frac{\partial Y}{\partial \beta}
\end{aligned}
$$

■

3.5 Laplace's equation

3.5.1 Definition of the Laplacian

Definition 3.2. The ***laplacian*** operator, denoted Δ, is defined for some scalar field ϕ as $\Delta\phi = \nabla \cdot \nabla\phi$, which when expanded gives:

$$
\begin{aligned}
\Delta\phi &= \nabla \cdot \nabla\phi \\
&= \nabla \cdot \begin{bmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \end{bmatrix} \\
&= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}
\end{aligned}
$$

As discussed in Section 3.3, the vector field representing this essay's idealised fluid can be expressed through the gradient of a velocity potential ϕ. The fluid must also satisfy the continuity equation, as established in Section 3.1, thus

$$\begin{aligned} \nabla \cdot \nabla\phi &= 0 \\ \implies \Delta\phi &= 0 \end{aligned}$$

This equation is known as Laplace's equation [Lewis et al., 2022]. As will become evident in Section 4.2, it is helpful to define this equation in polar coordinates.

3.5.2 The polar form of the Laplacian

Lemma 3.4 (The polar form of the Laplacian)**.** For the scalar field ϕ, the Laplacian is defined in polar coordinates as

$$\Delta\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2}$$

Proof. The Cartesian coordinates x and y are defined in terms of the polar coordinates r and ϑ as

$$\begin{aligned} x &= r\cos\vartheta \\ y &= r\sin\vartheta \end{aligned}$$

Consequently the partial derivatives for x and y in terms of r and ϑ become

$$\begin{aligned} \frac{\partial x}{\partial r} &= \cos\vartheta \qquad & \frac{\partial x}{\partial\vartheta} &= -r\sin\vartheta \\ \frac{\partial y}{\partial r} &= \sin\vartheta \qquad & \frac{\partial y}{\partial\vartheta} &= r\cos\vartheta \end{aligned}$$

Therefore, using the chain rule derived in Lemma 3.3, the partial derivative of ϕ in terms of

r becomes:

$$\begin{aligned}\frac{\partial \phi}{\partial r} &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial \phi}{\partial y}\sin\vartheta \end{aligned} \tag{10}$$

And in terms of ϑ (applying the chain rule again),

$$\begin{aligned}\frac{\partial \phi}{\partial \vartheta} &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \vartheta} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \vartheta} \\ &= -\frac{\partial \phi}{\partial x} r\sin\vartheta + \frac{\partial \phi}{\partial y} r\cos\vartheta \\ &= r\left(\frac{\partial \phi}{\partial y}\cos\vartheta - \frac{\partial \phi}{\partial x}\sin\vartheta\right) \end{aligned} \tag{11}$$

Taking the second order partial derivatives with respect to r and using Theorem 2.1 (Clairaut's theorem), which will henceforth be taken for granted, gives

$$\begin{aligned}\frac{\partial^2 \phi}{\partial r^2} &= \frac{\partial}{\partial r}\frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial}{\partial r}\frac{\partial \phi}{\partial y}\sin\vartheta \\ &= \frac{\partial}{\partial x}\frac{\partial \phi}{\partial r}\cos\vartheta + \frac{\partial}{\partial y}\frac{\partial \phi}{\partial r}\sin\vartheta \end{aligned}$$

Substituting back from (10),

$$\begin{aligned}\frac{\partial^2 \phi}{\partial r^2} &= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial \phi}{\partial y}\sin\vartheta\right)\cos\vartheta + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial \phi}{\partial y}\sin\vartheta\right)\sin\vartheta \\ &= \frac{\partial^2 \phi}{\partial x^2}\cos^2\vartheta + \frac{\partial^2 \phi}{\partial y \partial x}(\sin\vartheta)\cos\vartheta + \frac{\partial^2 \phi}{\partial x \partial y}(\cos\vartheta)\sin\vartheta + \frac{\partial^2 \phi}{\partial y^2}\sin^2\vartheta \\ &= \frac{\partial^2 \phi}{\partial x^2}\cos^2\vartheta + \frac{\partial^2 \phi}{\partial y^2}\sin^2\vartheta + \left(\frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \phi}{\partial x \partial y}\right)\cos\vartheta\sin\vartheta \\ &= \frac{\partial^2 \phi}{\partial x^2}\cos^2\vartheta + \frac{\partial^2 \phi}{\partial y^2}\sin^2\vartheta + 2\frac{\partial^2 \phi}{\partial y \partial x}\cos\vartheta\sin\vartheta \end{aligned}$$

Applying the same process for the second partial derivative with respect to ϑ leads to

$$\begin{aligned}\frac{\partial^2 \phi}{\partial \vartheta^2} &= \frac{\partial}{\partial \vartheta}\left(\frac{\partial \phi}{\partial y}\cos\vartheta - \frac{\partial \phi}{\partial x}\sin\vartheta\right) r \\ &= \frac{\partial}{\partial \vartheta}\left(\frac{\partial \phi}{\partial y} r\cos\vartheta\right) - \frac{\partial}{\partial \vartheta}\left(\frac{\partial \phi}{\partial x} r\sin\vartheta\right) \end{aligned}$$

$\partial\phi/\partial x$, $\partial\phi/\partial y$, $\cos\vartheta$ and $\sin\vartheta$ are functions of ϑ, so the product rule is applied leading to:

$$\begin{aligned}
\frac{\partial^2\phi}{\partial\vartheta^2} &= \left(\left[\frac{\partial}{\partial\vartheta}\frac{\partial\phi}{\partial y}\right] r\cos\vartheta + \frac{\partial\phi}{\partial y}\left[\frac{\partial}{\partial\vartheta} r\cos\vartheta\right]\right) - \left(\left[\frac{\partial}{\partial\vartheta}\frac{\partial\phi}{\partial x}\right] r\sin\vartheta + \frac{\partial\phi}{\partial x}\left[\frac{\partial}{\partial\vartheta} r\sin\vartheta\right]\right) \\
&= \left(\frac{\partial^2\phi}{\partial y\partial\vartheta} r\cos\vartheta - \frac{\partial\phi}{\partial y} r\sin\vartheta\right) - \left(\frac{\partial^2\phi}{\partial x\partial\vartheta} r\sin\vartheta + \frac{\partial\phi}{\partial x} r\cos\vartheta\right) \\
&= r\left(\frac{\partial}{\partial y}\frac{\partial\phi}{\partial\vartheta} r\cos\vartheta - \frac{\partial}{\partial x}\frac{\partial\phi}{\partial\vartheta} r\sin\vartheta\right) - r\left(\frac{\partial\phi}{\partial x}\cos\vartheta + \frac{\partial\phi}{\partial y}\sin\vartheta\right) \\
&= \underbrace{r^2\left(\frac{\partial}{\partial y}\frac{\partial\phi}{\partial\vartheta}\cos\vartheta - \frac{\partial}{\partial x}\frac{\partial\phi}{\partial\vartheta}\sin\vartheta\right)}_{\Phi} - r\left(\frac{\partial\phi}{\partial x}\cos\vartheta + \frac{\partial\phi}{\partial y}\sin\vartheta\right) \qquad (12)
\end{aligned}$$

Substituting (11) into Φ,

$$\begin{aligned}
\Phi &= r^2\left(\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\cos\vartheta - \frac{\partial\phi}{\partial x}\sin\vartheta\right)\cos\vartheta - \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\cos\vartheta - \frac{\partial\phi}{\partial x}\sin\vartheta\right)\sin\vartheta\right) \\
&= r^2\left(\left(\frac{\partial^2\phi}{\partial y^2}\cos\vartheta - \frac{\partial^2\phi}{\partial x\partial y}\sin\vartheta\right)\cos\vartheta - \left(\frac{\partial^2\phi}{\partial y\partial x}\cos\vartheta - \frac{\partial^2\phi}{\partial x^2}\sin\vartheta\right)\sin\vartheta\right) \\
&= r^2\left(\frac{\partial^2\phi}{\partial y^2}\cos^2\vartheta - \frac{\partial^2\phi}{\partial x\partial y}\cos\vartheta\sin\vartheta - \frac{\partial^2\phi}{\partial x\partial y}\cos\vartheta\sin\vartheta + \frac{\partial^2\phi}{\partial x^2}\sin^2\vartheta\right) \\
&= r^2\left(\frac{\partial^2\phi}{\partial y^2}\cos^2\vartheta + \frac{\partial^2\phi}{\partial x^2}\sin^2\vartheta - 2\frac{\partial^2\phi}{\partial y\partial x}\cos\vartheta\sin\vartheta\right)
\end{aligned}$$

Substituting Φ back into (12),

$$\frac{\partial^2\phi}{\partial\vartheta^2} = r^2\left(\frac{\partial^2\phi}{\partial y^2}\cos^2\vartheta + \frac{\partial^2\phi}{\partial x^2}\sin^2\vartheta - 2\frac{\partial^2\phi}{\partial y\partial x}\cos\vartheta\sin\vartheta\right) - r\left(\frac{\partial\phi}{\partial x}\cos\vartheta + \frac{\partial\phi}{\partial y}\sin\vartheta\right)$$

Substituting in (10) leads to

$$\frac{\partial^2\phi}{\partial\vartheta^2} = r^2\left(\frac{\partial^2\phi}{\partial y^2}\cos^2\vartheta + \frac{\partial^2\phi}{\partial x^2}\sin^2\vartheta - 2\frac{\partial^2\phi}{\partial y\partial x}\cos\vartheta\sin\vartheta\right) - r\frac{\partial\phi}{\partial r}$$

Taking the sum of $\partial^2\phi/\partial r^2$ and $(r^{-2})\,\partial^2\phi/\partial\vartheta^2$ gives,

$$\begin{aligned}
\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2} &= \left(\frac{\partial^2\phi}{\partial x^2}\cos^2\vartheta + \frac{\partial^2\phi}{\partial y^2}\sin^2\vartheta + 2\frac{\partial^2\phi}{\partial y\partial x}\cos\vartheta\sin\vartheta\right) \\
&\quad + \left(\frac{\partial^2\phi}{\partial y^2}\cos^2\vartheta + \frac{\partial^2\phi}{\partial x^2}\sin^2\vartheta - 2\frac{\partial^2\phi}{\partial y\partial x}\cos\vartheta\sin\vartheta - \frac{1}{r}\frac{\partial\phi}{\partial r}\right) \\
&= \frac{\partial^2\phi}{\partial x^2}\cos^2\vartheta + \frac{\partial^2\phi}{\partial x^2}\sin^2\vartheta + \frac{\partial^2\phi}{\partial y^2}\cos^2\vartheta + \frac{\partial^2\phi}{\partial y^2}\sin^2\vartheta - \frac{1}{r}\frac{\partial\phi}{\partial r} \\
&= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - \frac{1}{r}\frac{\partial\phi}{\partial r} \\
\therefore \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2} &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \\
\therefore \Delta\phi &= \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2}
\end{aligned}$$

■

4 Formulation of the problem

Let the velocity field of the fluid be denoted as $\mathbf{V}$. The 2 dimensional solid cylinder is placed at the origin (0, 0) and has a radius L. Due to the circular nature of the cylinder, it is helpful to define the problem in polar coordinates.

4.1 Polar form of the gradient

Since the velocity field is defined in terms of the velocity potential (such that $\mathbf{V} = \nabla\phi$), the polar coordinate form of the gradient, similarly to the Laplacian[see Section 3.5.2], must be derived.

Lemma 4.1. In Polar coordinates, the gradient of some scalar valued function ϕ is given by

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\vartheta}\hat{\vartheta}$$

Proof. In Cartesian coordinates, the gradient is defined as

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath}$$

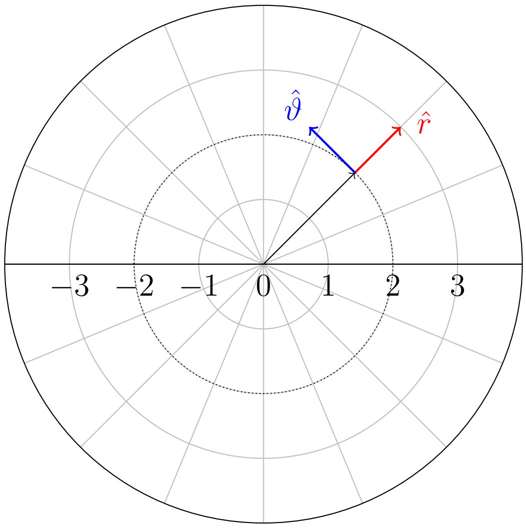


Figure 4: The polar coordinate unit vectors $\hat{r}$ and $\hat{\vartheta}$

Let the unit vectors in the direction of positive change for r and ϑ be called $\hat{r}$ and $\hat{\vartheta}$ respectively[see Figure 4]. In terms of Cartesian coordinates, the polar coordinate unit vectors would be defined as

$$\hat{r} = \hat{\imath}\cos\vartheta + \hat{\jmath}\sin\vartheta$$
$$\hat{\vartheta} = -\hat{\imath}\sin\vartheta + \hat{\jmath}\cos\vartheta$$

Thus, stating the relationship using a transformation matrix,

$$\begin{bmatrix} \hat{r} \\ \hat{\vartheta} \end{bmatrix} = \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix} \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \end{bmatrix}$$

To invert the matrix (and thereby the relationship), use the formula

$$\begin{aligned} \mathbf{M}^{-1} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det \mathbf{M}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ \implies \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix}^{-1} &= \frac{1}{\cos^2\vartheta + \sin^2\vartheta} \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} \end{aligned}$$

Consequently,

$$\begin{bmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{bmatrix}\begin{bmatrix}\hat{r}\\ \hat{\vartheta}\end{bmatrix} = \overbrace{\begin{bmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{bmatrix}\begin{bmatrix}\cos\vartheta & \sin\vartheta\\ -\sin\vartheta & \cos\vartheta\end{bmatrix}}^{\mathbf{MM}^{-1}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}\begin{bmatrix}\hat{\imath}\\ \hat{\jmath}\end{bmatrix}$$
$$\implies \begin{bmatrix}\hat{\imath}\\ \hat{\jmath}\end{bmatrix} = \begin{bmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{bmatrix}\begin{bmatrix}\hat{r}\\ \hat{\vartheta}\end{bmatrix}$$

Thus, the Cartesian coordinate unit vectors are defined in terms of the polar coordinate vectors as

$$\hat{\imath} = \hat{r}\cos\vartheta - \hat{\vartheta}\sin\vartheta$$
$$\hat{\jmath} = \hat{\vartheta}\cos\vartheta + \hat{r}\sin\vartheta$$

Substituting these into the definition of the gradient leads to

$$\begin{aligned}\nabla\phi &= \frac{\partial\phi}{\partial x}\left(\hat{r}\cos\vartheta - \hat{\vartheta}\sin\vartheta\right) + \frac{\partial\phi}{\partial y}\left(\hat{\vartheta}\cos\vartheta + \hat{r}\sin\vartheta\right)\\ &= \left(\frac{\partial\phi}{\partial x}\cos\vartheta + \frac{\partial\phi}{\partial y}\sin\vartheta\right)\hat{r} + \left(\frac{\partial\phi}{\partial y}\cos\vartheta - \frac{\partial\phi}{\partial x}\sin\vartheta\right)\hat{\vartheta}\end{aligned}$$

Substituting in the equations of $\partial\phi/\partial r$ from (10) and $\frac{1}{r}\,\partial\phi/\partial\vartheta$ from (11) derived in the proof of the polar coordinate form of the Laplacian[see Section 3.5.2],

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\vartheta}\hat{\vartheta}$$

■

4.2 Boundary equations

4.2.1 Fluid behaviour infinitely far away from the cylinder

As a consequence of the flow being steady, infinitely far away from the cylinder, the fluid particles are moving at constant velocity along the x axis[see Figure 5]. Let the speed of the undisturbed fluid be denoted U.

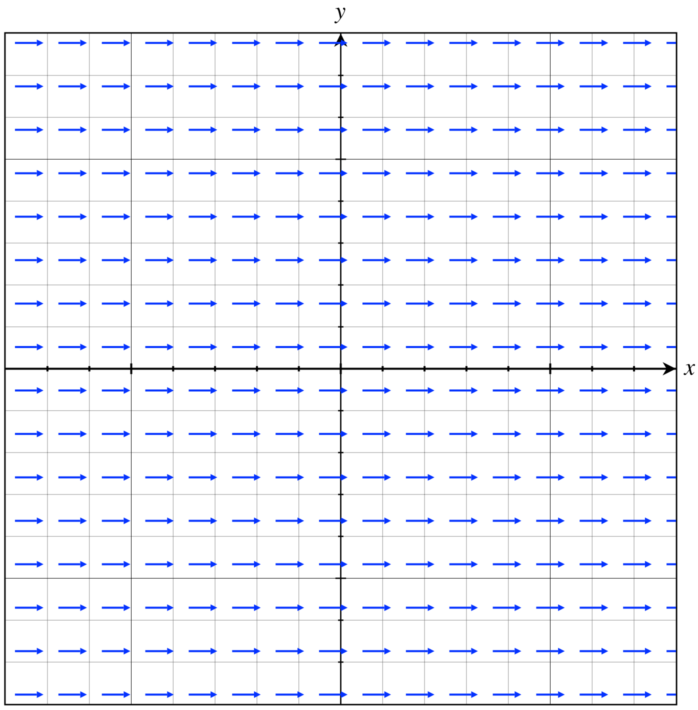


Figure 5: Uniform steady flow in the direction of $\hat{\imath}$

$$\mathbf{V} : x, y \mapsto U\hat{\imath} \qquad \text{as } x, y \to \infty$$

Or, in terms of the velocity potential ϕ:

$$\nabla\phi = U\hat{\imath} \qquad \text{as } x, y \to \infty$$

Thus, in polar coordinates the equation of the vector field in terms of the velocity potential infinitely far away from the cylinder is given by the equation

$$\nabla\phi = U(\hat{r}\cos\vartheta - \hat{\vartheta}\sin\vartheta) \qquad \text{as } r \to \infty$$

4.2.2 Fluid behaviour when $r = L$

The fluid may not penetrate the cylinder, thus, at the points $r = L$ the normal component of the velocity must be zero. Since $\hat{r}$ always points in the direction of positive radial change, which is normal to the surface of the cylinder,

$$\mathbf{V} \cdot \hat{r} = 0 \qquad \text{when } r = L$$

In terms of the velocity potential,

$$\begin{aligned} \nabla\phi \cdot \hat{r} &= 0 \\ \implies \left(\frac{\partial \phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \phi}{\partial \vartheta}\hat{\vartheta} \right) \cdot \hat{r} &= 0 \\ \therefore \frac{\partial \phi}{\partial r} &= 0 \qquad \text{when } r = L \end{aligned}$$

5 Solving Laplace's equation

To solve the equation $\Delta\phi = 0$, assume the function ϕ can be separated into two functions R and Θ dependent on r and ϑ respectively such that

$$\phi : r, \vartheta \mapsto R(r)\Theta(\vartheta)$$

Thus, the partial derivatives of ϕ would be given as

$$\begin{aligned} \frac{\partial \phi}{\partial r} &= R'(r)\Theta(\vartheta) & \frac{\partial \phi}{\partial \vartheta} &= R(r)\Theta'(\vartheta) \\ \frac{\partial^2 \phi}{\partial r^2} &= R''(r)\Theta(\vartheta) & \frac{\partial^2 \phi}{\partial \vartheta^2} &= R(r)\Theta''(\vartheta) \end{aligned}$$

Substituting these expressions into the polar form of the Laplacian derived in Section 3.5.2 gives

$$\begin{aligned} \Delta\phi &= \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \vartheta^2} \\ &= R''(r)\Theta(\vartheta) + \frac{1}{r}R'(r)\Theta(\vartheta) + \frac{1}{r^2}R(r)\Theta''(\vartheta) = 0 \end{aligned}$$

Consequently the expressions can be separated as

$$
\begin{aligned}
r^2R''(r)\Theta(\vartheta) + rR'(r)\Theta(\vartheta) + R(r)\Theta''(\vartheta) &= 0 \\
r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \frac{\Theta''(\vartheta)}{\Theta(\vartheta)} &= 0 \\
\implies r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} &= -\frac{\Theta''(\vartheta)}{\Theta(\vartheta)}
\end{aligned}
$$

Then, let $\lambda \in \mathbb{R}$ be such that

$$\lambda = r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} \tag{13}$$

$$-\lambda = \frac{\Theta''(\vartheta)}{\Theta(\vartheta)} \tag{14}$$

5.1 Solving for Θ

Through (14) it can be shown that

$$\frac{\Theta''(\vartheta)}{\Theta(\vartheta)} = -\lambda$$

$$\implies \Theta''(\vartheta) + \lambda\Theta(\vartheta) = 0 \tag{15}$$

Since Θ must be periodic such that $\Theta(\vartheta) = \Theta(\vartheta + 2\pi)$, one function fitting this differential equation is the complex exponential equation

$$\Theta : \vartheta \mapsto Ce^{\mu\vartheta} \tag{16}$$

Where C is some complex constant and $\mu \in \mathbb{C}$ such that $\Im(\mu) \neq 0$. Taking the second derivative of (16) with respect to ϑ using the chain rule,

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}\vartheta^2} = C\mu^2e^{\mu\vartheta}$$

Thus, for the equation to satisfy (15), $\mu = \pm\sqrt{-\lambda} = \pm i\sqrt{\lambda}$. To ensure $\Im(\mu) \neq 0$, thereby keeping Θ periodic, $\lambda > 0$.

$$\Theta : \vartheta \mapsto Ce^{\pm i\sqrt{\lambda}\vartheta}$$

Since the two solutions are distinct, the general solution is the linear combination of them,

$$\Theta : \vartheta \mapsto C_1 e^{i\sqrt{\lambda}\vartheta} + C_2 e^{-i\sqrt{\lambda}\vartheta}, \quad C_1, C_2 \in \mathbb{C}$$

By applying Euler's formula ($e^{i\alpha} = \cos\alpha + i\sin\alpha$) and some trigonometric identities it can be shown that

$$\begin{aligned}
C_1 e^{i\sqrt{\lambda}\vartheta} + C_2 e^{-i\sqrt{\lambda}\vartheta} &= C_1\left(\cos\left(\sqrt{\lambda}\vartheta\right) + i\sin\left(\sqrt{\lambda}\vartheta\right)\right) + C_2\left(\cos\left(-\sqrt{\lambda}\vartheta\right) + i\sin\left(-\sqrt{\lambda}\vartheta\right)\right)\\
&= C_1\cos\left(\sqrt{\lambda}\vartheta\right) + C_1 i\sin\left(\sqrt{\lambda}\vartheta\right) + C_2\cos\left(\sqrt{\lambda}\vartheta\right) - C_2 i\sin\left(\sqrt{\lambda}\vartheta\right)\\
&= (C_1 + C_2)\cos\left(\sqrt{\lambda}\vartheta\right) + (C_1 - C_2)\, i\sin\left(\sqrt{\lambda}\vartheta\right)
\end{aligned}$$

Defining the constants $A = C_1 + C_2$ and $B = (C_1 - C_2)i$ leads to the definition of Θ as

$$\Theta : \vartheta \mapsto A\cos\left(\sqrt{\lambda}\vartheta\right) + B\sin\left(\sqrt{\lambda}\vartheta\right)$$

Finally, to ensure the periodicity of Θ, $\sqrt{\lambda}$ must be an integer, which will be called $n \in \mathbb{N}$, leading to the discrete solutions of Θ_n as

$$\Theta_n : \vartheta \mapsto A\cos(n\vartheta) + B\sin(n\vartheta) \tag{17}$$

5.2 Solving for R

Rearranging (15) shows that

$$\begin{aligned}
r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} &= \lambda\\
\implies r^2 R''(r) + rR'(r) - \lambda R(r) &= 0
\end{aligned} \tag{18}$$

This type of differential equation, known as a Cauchy–Euler equation, can be solved using Frobenius method, which has the goal of finding a power series such that

$$R : r \mapsto \sum_{k=0}^{\infty} C_k r^{k+s}, \quad C_0 \neq 0 \tag{19}$$

The first and second order derivatives of (19) are given by

$$\begin{aligned}
\frac{\mathrm{d}R}{\mathrm{d}r} &= \sum_{k=0}^{\infty} (k+s) C_k r^{k+s-1} \\
\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} &= \sum_{k=0}^{\infty} (k+s)(k+s-1) C_k r^{k+s-2}
\end{aligned}$$

Consequently,

$$rR'(r) = \sum_{k=0}^{\infty} (k+s) C_k r^{k+s} \tag{20}$$

$$r^2 R''(r) = \sum_{k=0}^{\infty} (k+s)(k+s-1) C_k r^{k+s} \tag{21}$$

Let $C_k r^{k+s}$ be called σ temporarily. Substituting the equations of (20) and (21) back into (22),

$$\begin{aligned}
\sum_{k=0}^{\infty} (k+s)(k+s-1)\sigma + \sum_{k=0}^{\infty} (k+s)\sigma - \lambda \sum_{k=0}^{\infty} \sigma = 0 \\
\implies \sum_{k=0}^{\infty} (k+s)(k+s-1)\sigma + (k+s)\sigma - \lambda\sigma = 0
\end{aligned}$$

Let $\mu = k + s$,

$$\begin{aligned}
0 &= \sum_{k=0}^{\infty} \mu(\mu-1)\sigma + \mu\sigma - \lambda\sigma \\
&= \sum_{k=0}^{\infty} \sigma \left(\mu(\mu-1) + \mu - \lambda\right) \\
&= \sum_{k=0}^{\infty} \sigma \left(\mu^2 - \lambda\right)
\end{aligned}$$

Therefore,

$$\sum_{k=0}^{\infty} C_k r^{k+s}(\mu^2 - \lambda) = 0, \quad C_0 \neq 0$$

Thus, for the equation to hold when $k = 0$,

$$\begin{aligned} \mu^2 - \lambda &= 0 \\ \implies \pm\sqrt{\lambda} &= \mu \\ &= k + s \\ &= s \end{aligned}$$

Using the definition of $n = \sqrt{\lambda}$ from Section 5.1,

$$\begin{aligned} 0 + s &= \pm n \\ \implies s &= \pm n \end{aligned}$$

For the cases $k > 0$, to ensure the equation holds, because the term $(k+s)^2 - \lambda$ will no longer be 0, $C_k r^{k+s} = 0$. Therefore, all terms but $k = 0$ must vanish, meaning that

$$R : r \mapsto C_0 r^{\pm n}$$

Since $n \in \mathbb{N}^{[\text{see Section 5.1}]}$, these solutions are distinct, and therefore the general solution is the linear combination of them,

$$R_n : r \mapsto ar^n + br^{-n} \tag{22}$$

Where a and b are some real constants.

5.3 The equation of the velocity potential

Finally, the equation of the velocity potential ϕ can be derived by combining the equation of Θ_n from (17) and R_n from (22),

$$\begin{aligned} &\phi_n : r, \vartheta \mapsto R_n(r)\Theta_n(\vartheta) \\ \implies &\phi_n : r, \vartheta \mapsto \left(ar^n + \frac{b}{r^n}\right)(A\cos n\vartheta + B\sin n\vartheta) \end{aligned}$$

The values of the constants a, b, A, B and n must now be chosen such that they obey the boundary equations, which were

$$\begin{aligned} &\nabla\phi = U(\hat{r}\cos\vartheta - \hat{\vartheta}\sin\vartheta) && \text{as } r \to \infty \\ &\frac{\partial\phi}{\partial r} = 0 && \text{when } r = L \end{aligned}$$

Computing the partial derivatives of ϕ,

$$\begin{aligned} \frac{\partial\phi_n}{\partial r} &= \left(anr^{n-1} - \frac{nb}{r^{n+1}}\right)(A\cos n\vartheta + B\sin n\vartheta) \\ &= n\left(ar^{n-1} - \frac{b}{r^{n+1}}\right)(A\cos n\vartheta + B\sin n\vartheta) \\ \frac{\partial\phi_n}{\partial\vartheta} &= \left(ar^n + \frac{b}{r^n}\right)(-An\sin n\vartheta + Bn\cos n\vartheta) \\ &= n\left(ar^n + \frac{b}{r^n}\right)(B\cos n\vartheta - A\sin n\vartheta) \end{aligned}$$

Therefore, using the polar coordinate form of the gradient derived in Section 4.1,

$$\begin{aligned} \nabla\phi_n &= \frac{\partial\phi_n}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi_n}{\partial\vartheta}\hat{\vartheta} \\ &= n\left(ar^{n-1} - \frac{b}{r^{n+1}}\right)(A\cos n\vartheta + B\sin n\vartheta)\,\hat{r} + \frac{n}{r}\left(ar^n + \frac{b}{r^n}\right)(B\cos n\vartheta - A\sin n\vartheta)\,\hat{\vartheta} \\ &= n\left[\left(ar^{n-1} - \frac{b}{r^{n+1}}\right)(A\cos n\vartheta + B\sin n\vartheta)\,\hat{r} + \left(ar^{n-1} + \frac{b}{r^{n+1}}\right)(B\cos n\vartheta - A\sin n\vartheta)\,\hat{\vartheta}\right] \end{aligned}$$

Considering the case where $n = 1$,

$$\begin{aligned}
\nabla\phi_1 &= \left(ar^{1-1} - \frac{b}{r^{1+1}}\right)(A\cos\vartheta + B\sin\vartheta)\,\hat{r} + \left(ar^{1-1} + \frac{b}{r^{1+1}}\right)(B\cos\vartheta - A\sin\vartheta)\,\hat{\vartheta} \\
&= \left(a - \frac{b}{r^2}\right)(A\cos\vartheta + B\sin\vartheta)\,\hat{r} + \left(a + \frac{b}{r^2}\right)(B\cos\vartheta - A\sin\vartheta)\,\hat{\vartheta}
\end{aligned}$$

As $r \to \infty$,

$$\begin{aligned}
\lim_{r\to\infty}\nabla\phi_1 &= \left(a - \frac{b}{\infty^2}\right)(A\cos\vartheta + B\sin\vartheta)\,\hat{r} + \left(a + \frac{b}{\infty^2}\right)(B\cos\vartheta - A\sin\vartheta)\,\hat{\vartheta} \\
&= (a-0)(A\cos\vartheta + B\sin\vartheta)\,\hat{r} + (a+0)(B\cos\vartheta - A\sin\vartheta)\,\hat{\vartheta} \\
&= a\Big((A\cos\vartheta + B\sin\vartheta)\,\hat{r} + (B\cos\vartheta - A\sin\vartheta)\,\hat{\vartheta}\Big)
\end{aligned}$$

To match the first boundary equation which was $U(\hat{r}\cos\vartheta - \hat{\vartheta}\sin\vartheta)$ as $r \to \infty$, the value of the variables must be

$$\begin{aligned}
\nabla\phi_1 &\begin{cases} a = U \\ b \in \mathbb{R} \\ A = 1 \\ B = 0 \end{cases} \\
\rightsquigarrow \lim_{r\to\infty}\nabla\phi_1 &= U\Big((\cos\vartheta + 0\sin\vartheta)\,\hat{r} + (0\cos\vartheta - 1\sin\vartheta)\,\hat{\vartheta}\Big) \\
&= U\Big(\hat{r}\cos\vartheta - \hat{\vartheta}\sin\vartheta\Big)
\end{aligned}$$

Now consider the value of $\partial\phi_1/\partial r$ when $r = L$,

$$\begin{aligned}
\left.\frac{\partial\phi_1}{\partial r}\right|_{r=L} &= \left(Ur^{1-1} - \frac{b}{r^{1+1}}\right)(\cos\vartheta + 0\sin\vartheta) \\
&= \left(U - \frac{b}{r^2}\right)\cos\vartheta
\end{aligned}$$

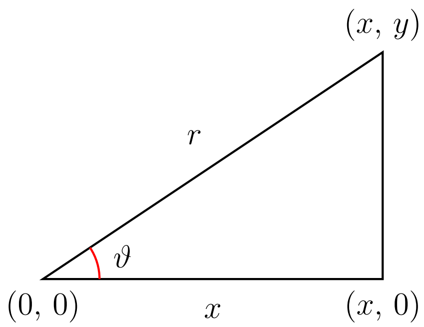
The second boundary equation stated that $\partial\phi_1/\partial r = 0$ when $r = L$, therefore $b = r^2 = L^2$. Thus, for the case $n = 1$, it has been demonstrated that

$$\begin{aligned}
\nabla\phi_1 &\begin{cases} a = U \\ b = L^2 \\ A = 1 \\ B = 0 \end{cases} \\
\implies \phi_1 : r, \vartheta &\mapsto \left(Ur + \frac{L^2}{r}\right)\cos\vartheta \\
\implies \mathbf{V}_1 : r, \vartheta \mapsto \nabla\phi_1 &= \hat{r}\left(U - \frac{L^2}{r^2}\right)\cos\vartheta - \hat{\vartheta}\left(U + \frac{L^2}{r^2}\right)\sin\vartheta
\end{aligned}$$

Since this equation matches the boundary equations perfectly, the case of $n = 1$ is accepted as the principal solution. Lastly, to avoid undefined behaviour inside the cylinder, let the velocity of the fluid be defined as $\mathbf{0}$ when $r < L$.

$$\mathbf{V} : r, \vartheta \mapsto \begin{cases} \hat{r}\left(U - \frac{L^2}{r^2}\right)\cos\vartheta - \hat{\vartheta}\left(U + \frac{L^2}{r^2}\right)\sin\vartheta & r \geq L \\ \mathbf{0} & r < L \end{cases}$$

Finally, to convert back to Cartesian coordinates, r is equivalent to the magnitude of the vector pointing from the origin to a point on the graph, $\sqrt{x^2 + y^2}$. For the angular part, since the velocity potential is only dependent on $\cos\vartheta$, consider the geometric meaning of the expression.



Cosine is defined as the ratio between the adjacent side (x) and hypotenuse (r) in a triangle, thus, $\cos\vartheta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$. Substituting this into the equation of the velocity potential gives

$$\begin{aligned}\phi : x, y \mapsto & \left(Ur + \frac{L^2}{r}\right)\cos\vartheta \\ &= \left(U\sqrt{x^2+y^2} + \frac{L^2}{\sqrt{x^2+y^2}}\right)\frac{x}{\sqrt{x^2+y^2}} \\ &= Ux + \frac{L^2 x}{x^2+y^2}\end{aligned}$$

The partial derivative of ϕ with respect to x can be computed using the quotient rule:

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= U + L^2\frac{x^2+y^2-x(2x)}{\left(x^2+y^2\right)^2} \\ &= U + L^2\frac{y^2-x^2}{\left(x^2+y^2\right)^2}\end{aligned}$$

And for the partial derivative of ϕ with respect to y, the chain rule can be applied, giving:

$$\begin{aligned}\frac{\partial\phi}{\partial y} &= 0 + L^2 x\frac{\partial}{\partial y}\left(x^2+y^2\right)^{-1} \\ &= -L^2 x(2y)\left(x^2+y^2\right)^{-2} \\ &= -L^2\frac{2xy}{(x^2+y^2)^2}\end{aligned}$$

Therefore, the velocity field, $\mathbf{V} = \nabla\phi = \hat{\imath}(\partial\phi/\partial x) + \hat{\jmath}(\partial\phi/\partial y)$, in a Cartesian coordinate system must be given by the equation

$$\mathbf{V} : x, y \mapsto \begin{cases}\left(U + L^2\frac{y^2-x^2}{(x^2+y^2)^2}\right)\hat{\imath} - L^2\frac{2xy}{(x^2+y^2)^2}\hat{\jmath} & x^2+y^2 \geq L^2 \\ \mathbf{0} & x^2+y^2 < L^2\end{cases}$$

6 Conclusion

7 References

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