How can vector calculus be used to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle, and what mathematical principles underpin the observed fluid behaviour?

Mathematics AA HL

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Contents

1 Introduction

Fluid dynamics is today a cornerstone to several fields of study, including ærospace engineering and meteorology. Real world fluid behaviour is intricate and complex. Therefore, to gain insights into the governing principles of fluid flow, simplified and idealised models are used. This essay investigates the application of vector calculus to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle. These idealisations allow for the derivation of some of fluid dynamic's key mathematical formulæ and provides a foundation for understanding less idealised fluids.

This essay will address the question: "How can vector calculus be used to model and analyse steady, inviscid, and incompressible fluid flow in two-dimensional spaces around a circular obstacle, and what mathematical principles underpin the observed fluid behaviour?" Through the derivation of the velocity potential and vector field, this essay aims to demonstrate how fundamental laws of fluid motion can be expressed and used through vector calculus.

1.1 Aim & scope

The scope of this essay will be limited to the theoretical modelling of fluid flow in a two-dimensional space as a vector field under idealised conditions forming steady, inviscid and incompressible fluid flow through the derivation of the velocity-potential. The analysis will be centred on the application of vector calculus to derive fundamental formulæ and describe fluid behaviour around a stationary circular obstacle. Consequently, this essay will not touch on viscous effects, turbulent flow or three-dimensional analysis, nor will it involve any experimental validation. The focus is on the mathematical derivation and analysis of the idealised model.

1.2 Background

1.2.1 Glossary

Definition 1.1. ***Steady flow*** refers to flow in which the velocity at every point does not change over time [CRACIUNOIU and CIOCIRLAN, 2001].

Definition 1.2. ***Inviscid flow*** is the flow of a fluid with 0 viscosity [Anderson, 2003].

Definition 1.3. An ***incompressible fluid*** is a fluid whose density at every point does not change over time [Ahmed, 2019].

Definition 1.4. A ***scalar field*** is a function mapping points in space to scalar quantities such as temperatures.

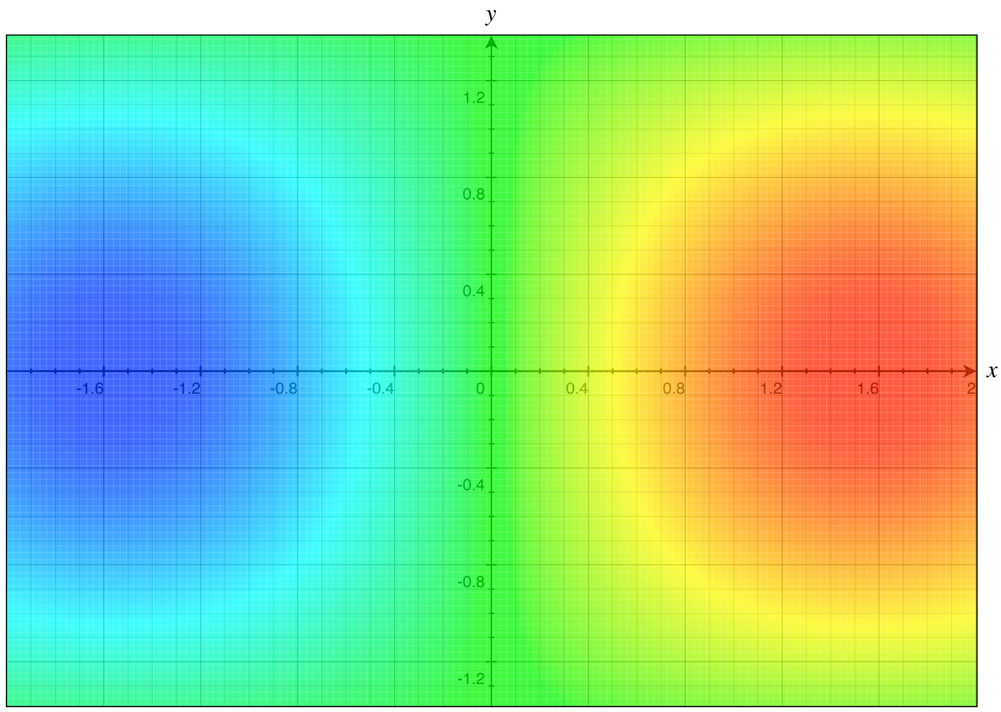


Figure 1: Scalar field plotted for the function $f(x, y) = \sin(x)\cos y$

Definition 1.5. A ***vector field*** is a function mapping points in space to vector quantities [Brezinski, 2006]. In the case of fluid dynamics, vector fields often model quantities like fluid velocity.

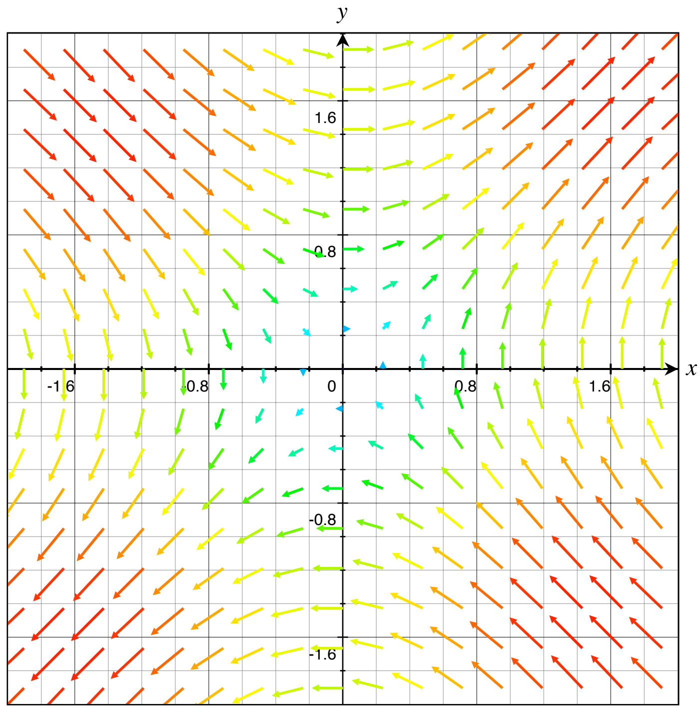


Figure 2: Vector field plotted for the function $f(x, y) = \binom{\sin y}{\sin x}$

Definition 1.6. The ***velocity potential*** ϕ is a scalar field whose gradient is the velocity vector field of some fluid, mathematically $\mathbf{V} = \nabla\phi$. The quantity is defined for irrotational flow which is a resulting property of the idealisations made in this essay[see 6.1].

1.2.2 Notation

Vector calculus, like one-variable calculus, has no standardized notation. This essay will employ the following notation:

- ∇:
  - ∇F: The gradient of some scalar field F.
  - $\nabla \cdot \mathbf{F}$: The divergence of some vector field $\mathbf{F}$.
  - $\nabla \times \mathbf{F}$: The curl of some vector field $\mathbf{F}$.
  - $\nabla_{\mathbf{v}} f$: The directional derivative of f in the direction of some vector $\mathbf{v}$
- Δ: The Laplacian operator
- $\mathbb{D}_\delta(\langle x,\, y\rangle)$: The set of the points in an open disk centred at $(x,\, y)$ with radius δ

- such that : "Such that"
- $\hat{\imath}\,\&\,\hat{\jmath}$: Unit vectors in the positive x and y directions respectively.
- $\hat{r}\,\&\,\hat{\vartheta}$: Unit vectors in the positive r and ϑ directions respectively.

2 The mean value theorem

Lemma 2.1 (The extreme value theorem)**.** If a function f is continuous on the finite interval $[a, b]$, then there exists $A, B \in [a, b]$ such that $f(A) \leq f(x) \leq f(B)\,\forall x \in [a, b]$. Thus, at the points A and B, f has an absolute minimum $m = f(A)$ and an absolute maximum $M = f(B)$.

Lemma 2.2 (Rolle's theorem)**.** If a function f is continuous on the interval $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then $\exists\, c \in (a, b)$ such that $f'(c) = 0$.

Proof. Consider two cases:

Case 1: f remains constant over $[a, b]$

If $f(x) = f(a) = f(b)\,\forall x \in (a, b)$, then $f'(x) = 0$, and the theorem holds trivially.

Case 2: f is not constant over $[a, b]$

If f is not constant over $[a, b]$ and $f(a) = f(b)$, then Lemma 2.1 asserts that there must exist an absolute maximum or minimum that occur at some point $\eta \in (a, b)$. Since f is differentiable over (a, b), then any point η where an absolute extremum occurs must also be a local extremum. Consider the case where η is a local maxima (the proof for the case of local minima is analogous). Then let the interval $I = (\eta - \delta, \eta + \delta)$ for some $\delta > 0$ such that $\forall X \in I, f(X) \leq f(\eta)$.

Let $h < 0$ be a number sufficiently small such that $\eta + h \in I$. $f(\eta + h) \leq f(\eta) \implies f(\eta + h) - f(\eta) \leq 0$. Thus,

$$\frac{f(\eta + h) - f(\eta)}{h} \geq 0 \because \begin{cases} f(\eta + h) - f(\eta) & \leq 0 \\ h & \leq 0 \end{cases}$$

Taking the left-hand limit as $h \to 0$,

$$\lim_{h\to 0^-} \frac{f(\eta+h)-f(\eta)}{h} = f'(\eta)$$

Now let $H > 0$ be a number sufficiently small such that $\eta - H \in I$.

$$\begin{aligned} \frac{f(\eta+H)-f(\eta)}{H} &\leq 0 \because \begin{cases} f(\eta+H)-f(\eta) & \leq 0 \\ H & \geq 0 \end{cases} \\ \lim_{H\to 0^+} \frac{f(\eta+H)-f(\eta)}{H} &= f'(\eta) \end{aligned}$$

Thus,

$$0 \geq \lim_{H\to 0^+} \frac{f(\eta+H)-f(\eta)}{H} = f'(\eta) = \lim_{h\to 0^-} \frac{f(\eta+h)-f(\eta)}{h} \geq 0$$
$$\therefore f'(\eta) = 0$$

Since the same would apply for local minima, then for any local extrema $\eta \in (a, b)$, of which Lemma 2.1 asserts there must exist at least one, $f'(\eta) = 0$. ■

Lemma 2.3 (The mean value theorem)**.** For any function f continuous on the interval $[a, b]$ and differentiable on the interval (a, b), $\exists\, c \in (a, b)$ such that

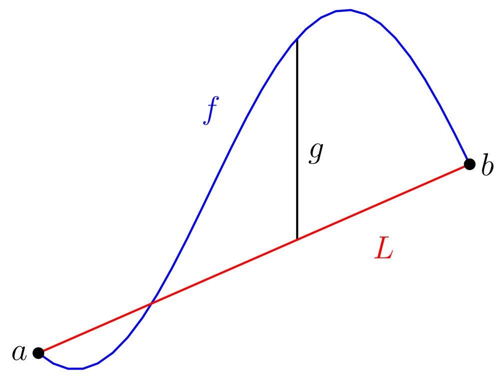
$$f'(c) = \frac{f(b)-f(a)}{b-a} \tag{1}$$

Proof. Consider the region of some function f on the finite interval $[a, b]$ over which f is continuous and differentiable over (a, b). Let the function L represent the straight line between the points $(a,\ f(a))$ and $(a,\ f(b))$, given by the expression:

$$L(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

Now consider the function g, defined as the difference between f and L:

$$g(x) = L(x) - f(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a) - f(x)$$



Computing the derivative of g with respect to x gives:

$$g'(x) = \frac{f(b) - f(a)}{b - a} - f'(x)$$

Since $g(a) = g(b) = 0$, Lemma 2.2 asserts that there is some point $c \in (a, b)$ such that $g'(c) = 0$. Thus, at c,

$$\begin{aligned} 0 = g'(c) &= \frac{f(b) - f(a)}{b - a} - f'(c) \\ \implies f'(c) &= \frac{f(b) - f(a)}{b - a} \end{aligned}$$

■

3 Vector calculus

3.1 The fundamentals of vector calculus

Definition 3.1. ***Partial derivatives*** are an extension of single-variable derivatives in which all variables save the one being differentiated by are treated as constants [Mortimer, 2013]. A formal definition of the partial derivative of some function f with respect to a parameter x_n can be expressed as:

$$\frac{\partial f}{\partial x_n} = \lim_{\delta \to 0} \frac{f(x_1, x_2, \cdots, x_n + \delta, \cdots) - f(x_1, x_2, \cdots, x_n, \cdots)}{\delta} \tag{2}$$

Partial derivatives allow for the analysis of how multi-variable functions such as scalar- or vector fields change with respect to just one spatial dimension. For example, consider the function $f(x, y) = x^2y + \sin(x)\sin y$:

$$\frac{\partial f}{\partial x} = 2xy + \cos(x)\sin y \qquad\qquad \frac{\partial f}{\partial y} = x^2 + \sin(x)\cos y$$

n-th order partial derivatives are denoted, similarly to normal calculus, as

$$\frac{\partial^n f}{\partial x^n} = \underbrace{\frac{\partial}{\partial x} \cdots \frac{\partial}{\partial x}\frac{\partial f}{\partial x}}_{n \text{ times}}$$

Definition 3.2. ***Mixed partial derivatives*** are partial derivatives of a function taken with respect to multiple variables [Garrett, 2015]. This is denoted as

$$\frac{\partial^2 f}{\partial\alpha\partial\beta} \equiv \frac{\partial}{\partial\beta}\frac{\partial f}{\partial\alpha}$$

where both α and β are parameters of f.

Lemma 3.1 (Clairaut's theorem)**.** Let $f(\alpha, \beta)$ be a function of two parameters α and β. If the mixed partial derivatives $\frac{\partial^2 f}{\partial\alpha\partial\beta}$ and $\frac{\partial^2 f}{\partial\beta\partial\alpha}$ exist and are continuous in the open disk $\mathbb{D}_\delta(\langle\alpha_0,\, \beta_0\rangle)$ centred at $(\alpha_0,\, \beta_0)$ with radius $\delta > 0$, then

$$\left.\frac{\partial^2 f}{\partial\alpha\partial\beta}\right|_{(\alpha_0,\, \beta_0)} = \left.\frac{\partial^2 f}{\partial\beta\partial\alpha}\right|_{(\alpha_0,\, \beta_0)}$$

[Garrett, 2015]

Proof. Let $(\alpha_0,\, \beta_0)$ and $(\alpha_1,\, \beta_1)$ be points in the domain of f. Consider a rectangular region bound by the points $W\,(\alpha_0,\, \beta_0)\,, X\,(\alpha_1,\, \beta_0)\,, Y\,(\alpha_1,\, \beta_1)$ and $Z\,(\alpha_0,\, \beta_1)$. $\frac{\partial f}{\partial\alpha}$ and $\frac{\partial f}{\partial\beta}$ exist in a neighbourhood of this rectangle, and the mixed partial derivatives $\frac{\partial^2 f}{\partial\beta\partial\alpha}$ and $\frac{\partial^2 f}{\partial\alpha\partial\beta}$ exist and are continuous in this neighbourhood. Let Q be such that

$$Q = [f(\alpha_1, \beta_1) - f(\alpha_0, \beta_1)] - [f(\alpha_1, \beta_0) - f(\alpha_0, \beta_0)]$$

According to the mean value theorem (MVT) $\exists\, \xi_0, \xi_1 \in [\alpha_0, \alpha_1]$ such that

$$\left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_0,\beta_0)} = \frac{f(\alpha_1,\beta_0) - f(\alpha_0,\beta_0)}{\alpha_1 - \alpha_0}$$
$$\left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_1,\beta_1)} = \frac{f(\alpha_1,\beta_1) - f(\alpha_0,\beta_1)}{\alpha_1 - \alpha_0}$$

Thus Q can be expressed as

$$\begin{aligned} Q &= \left(\left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_0,\beta_0)} (\alpha_1 - \alpha_0)\right) - \left(\left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_1,\beta_1)} (\alpha_1 - \alpha_0)\right) \\ &= \left(\left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_0,\beta_0)} - \left.\frac{\partial f}{\partial \alpha}\right|_{(\xi_1,\beta_1)}\right) (\alpha_1 - \alpha_0) \end{aligned}$$

Now let R be the equivalent of Q in the direction of β,

$$R = [f(\alpha_1,\beta_1) - f(\alpha_1,\beta_0)] - [f(\alpha_0,\beta_1) - f(\alpha_0,\beta_0)]$$

By the MVT $\exists\, \zeta_0, \zeta_1 \in [\beta_0, \beta_1]$ such that

$$\left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_0,\zeta_0)} = \frac{f(\alpha_0,\beta_1) - f(\alpha_0,\beta_0)}{\beta_1 - \beta_0}$$
$$\left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_1,\zeta_1)} = \frac{f(\alpha_1,\beta_1) - f(\alpha_1,\beta_0)}{\beta_1 - \beta_0}$$

Thus R can be expressed as

$$\begin{aligned} R &= \left(\left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_0,\zeta_0)} (\beta_1 - \beta_0)\right) - \left(\left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_1,\zeta_1)} (\beta_1 - \beta_0)\right) \\ &= \left(\left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_0,\zeta_0)} - \left.\frac{\partial f}{\partial \beta}\right|_{(\alpha_1,\zeta_1)}\right) (\beta_1 - \beta_0) \end{aligned}$$

Rearranging Q and R,

$$\begin{aligned} Q &= [f(\alpha_1, \beta_1) - f(\alpha_0, \beta_1)] - [f(\alpha_1, \beta_0) - f(\alpha_0, \beta_0)] \\ &= f(\alpha_1, \beta_1) - f(\alpha_1, \beta_0) - f(\alpha_0, \beta_1) + f(\alpha_0, \beta_0) \\ &= [f(\alpha_1, \beta_1) - f(\alpha_1, \beta_0)] - [f(\alpha_0, \beta_1) - f(\alpha_0, \beta_0)] = R \\ \therefore Q &= R \end{aligned}$$

Thus

$$\left(\frac{\partial f}{\partial \alpha}\bigg|_{(\xi_0,\,\beta_0)} - \frac{\partial f}{\partial \alpha}\bigg|_{(\xi_1,\,\beta_1)} \right)(\alpha_1 - \alpha_0) = \left(\frac{\partial f}{\partial \beta}\bigg|_{(\alpha_0,\,\zeta_0)} - \frac{\partial f}{\partial \beta}\bigg|_{(\alpha_1,\,\zeta_1)} \right)(\beta_1 - \beta_0)$$
$$\rightsquigarrow \frac{\partial f/\partial \alpha\,\big|_{(\xi_0,\,\beta_0)} - \partial f/\partial \alpha\,\big|_{(\xi_1,\,\beta_1)}}{\beta_1 - \beta_0} = \frac{\partial f/\partial \beta\,\big|_{(\alpha_0,\,\zeta_0)} - \partial f/\partial \beta\,\big|_{(\alpha_1,\,\zeta_1)}}{\alpha_1 - \alpha_0} \tag{3}$$

Applying the MVT again $\exists\, \xi^* \in (\xi_0, \xi_1), \beta^* \in (\beta_0, \beta_1)$ such that

$$\begin{aligned} \frac{\partial^2 f}{\partial \alpha \partial \beta}\bigg|_{(\xi^*,\,\beta^*)} &= \frac{\partial f/\partial \alpha\,\big|_{(\xi_1,\,\beta_1)} - \partial f/\partial \alpha\,\big|_{(\xi_0,\,\beta_0)}}{\beta_1 - \beta_0} \\ \implies -\frac{\partial^2 f}{\partial \alpha \partial \beta}\bigg|_{(\xi^*,\,\beta^*)} &= \frac{\partial f/\partial \alpha\,\big|_{(\xi_0,\,\beta_0)} - \partial f/\partial \alpha\,\big|_{(\xi_1,\,\beta_1)}}{\beta_1 - \beta_0} \end{aligned}$$

Similarly, $\exists\, \alpha^* \in (\alpha_0, \alpha_1), \zeta^* \in (\zeta_0, \zeta_1)$ such that

$$\begin{aligned} \frac{\partial^2 f}{\partial \beta \partial \alpha}\bigg|_{(\alpha^*,\,\zeta^*)} &= \frac{\partial f/\partial \beta\,\big|_{(\alpha_1,\,\zeta_1)} - \partial f/\partial \beta\,\big|_{(\alpha_0,\,\zeta_0)}}{\alpha_1 - \alpha_0} \\ \implies -\frac{\partial^2 f}{\partial \beta \partial \alpha}\bigg|_{(\alpha^*,\,\zeta^*)} &= \frac{\partial f/\partial \beta\,\big|_{(\alpha_0,\,\zeta_0)} - \partial f/\partial \beta\,\big|_{(\alpha_1,\,\zeta_1)}}{\alpha_1 - \alpha_0} \end{aligned}$$

Substituting back into (3),

$$\begin{aligned} -\frac{\partial^2 f}{\partial \alpha \partial \beta}\bigg|_{(\xi^*,\,\beta^*)} &= -\frac{\partial^2 f}{\partial \beta \partial \alpha}\bigg|_{(\alpha^*,\,\zeta^*)} \\ \implies \frac{\partial^2 f}{\partial \alpha \partial \beta}\bigg|_{(\xi^*,\,\beta^*)} &= \frac{\partial^2 f}{\partial \beta \partial \alpha}\bigg|_{(\alpha^*,\,\zeta^*)} \end{aligned}$$

Consequently, as $\alpha_1 \to \alpha_0$ and $\beta_1 \to \beta_0$, $\xi^* \to \alpha_0, \beta^* \to \beta_0, \alpha^* \to \alpha_0$ and $\zeta^* \to \beta_0$. Since the derivatives are continuous,

$$\left.\frac{\partial^2 f}{\partial\beta\partial\alpha}\right|_{(\alpha_0,\,\beta_0)} = \left.\frac{\partial^2 f}{\partial\alpha\partial\beta}\right|_{(\alpha_0,\,\beta_0)}$$

Because $(\alpha_0,\ \beta_0)$ is an arbitrary point in the domain, $\frac{\partial^2 f}{\partial\beta\partial\alpha} = \frac{\partial^2 f}{\partial\alpha\partial\beta}$ at all points in the domain where the mixed partial derivatives are continuous. ■

Definition 3.3. The ***nabla*** operator ∇ is a vector containing one partial derivative for each parameter of the function applied to [Rapp, 2017]. For some function $f : \mathbb{R}^n \to \mathbb{R}$, ∇f would be given by:

$$\nabla f = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \vdots \\ \partial f/\partial x_n \end{bmatrix}$$

Definition 3.4. The directional derivative in the direction of some vector $\mathbf{v}$ of the function f which is differentiable in the open disk $\mathbb{D}_\delta(\langle x_0,\ y_0\rangle)$ centred at $(x_0,\ y_0)$ with radius $\delta > 0$ is defined as

$$\left.\nabla_{\mathbf{v}} f\right|_{(x_0,\,y_0)} = \frac{\left.\nabla f\right|_{(x_0,\,y_0)} \cdot \mathbf{v}}{\|\mathbf{v}\|}$$

[Giannakidis and Petrou, 2010]

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5 List of Figures

6 Research

6.1 Potential flow around a circular cylinder

A cylinder of radius L is placed in two-dimensional, incompressible, inviscid flow which flows in the direction of $\hat{\imath}$. Far away from the cylinder the velocity field $\mathbf{V}$ can be described as:

$$\mathbf{V} = U\hat{\imath} \tag{3}$$

Where U is some constant. Since the cylinder is impermissible, at the boundary $\mathbf{V} \cdot \hat{n} = 0$ where the vector $\hat{n}$ is the unit vector normal to the surface.

Since in this model the viscosity $\nu = 0$, the flow can be modelled using the Euler equations. If the Euler equations, apply, so does Kelvin's theorem:

Theorem 6.1 (Kelvin's circulation theorem)**.** The circulation around a closed material loop moving with an inviscid, barotropic fluid in the presence of conservative body forces remains constant over time.[Citation needed]

If Γ denotes the circulation around a material loop $C(t)$ moving with the fluid, then:

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = 0$$

Id est, if the vorticity of $\mathbf{V}$ is 0 initially, it must remain 0 everywhere, thus $\nabla \times \mathbf{V} = 0$. Since the flow is irrotational, $\mathbf{V}$ can be expressed as $\mathbf{V} = \nabla\phi$, where ϕ is the velocity potential.

Furthermore, if $\mathbf{V}$ is incompressible, that being that $\nabla \cdot \mathbf{V} = 0$, then ϕ must satisfy Laplace's equation: $\Delta\phi = 0$.

6.2 Polar coordinate boundary conditions

6.2.1 $\mathbf{V} = U\hat{\imath}$

In polar coordinates, the base vectors $\hat{r}$ and $\hat{\vartheta}$ are defined as:

$$\begin{aligned} \hat{r} &\triangleq \hat{\imath}\cos\vartheta + \hat{\jmath}\sin\vartheta \\ \hat{\vartheta} &\triangleq -\hat{\imath}\sin\vartheta + \hat{\jmath}\cos\vartheta \end{aligned}$$

Solving for $\hat{\imath}$ and $\hat{\jmath}$ gives:

$$\hat{\imath} = \frac{\hat{r} - \hat{\jmath}\sin\vartheta}{\cos\vartheta} \tag{4}$$

$$\hat{\jmath} = \frac{\hat{\vartheta} + \hat{\imath}\sin\vartheta}{\cos\vartheta} \tag{5}$$

Substituting 5 into 4 and isolating $\hat{\imath}$ shows that

$$\begin{aligned} \hat{\imath} &= \frac{\hat{r} - \frac{\hat{\vartheta}+\hat{\imath}\sin\vartheta}{\cos\vartheta}\sin\vartheta}{\cos\vartheta} \\ &= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta + \hat{\imath}\sin^2\vartheta}{\cos^2\vartheta} \\ &= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta} - \frac{\hat{\imath}\sin^2\vartheta}{\cos^2\vartheta} \\ \implies \hat{\imath} + \frac{\sin^2\vartheta}{\cos^2\vartheta}\hat{\imath} &= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta} \\ \hat{\imath}\left(1 + \frac{\sin^2\vartheta}{\cos^2\vartheta}\right) &= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta} \\ \hat{\imath}\left(\frac{\sin^2\vartheta + \cos^2\vartheta}{\cos^2\vartheta}\right) &= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta} \\ \frac{\hat{\imath}}{\cos^2\vartheta} &= \frac{\hat{r}}{\cos\vartheta} - \frac{\hat{\vartheta}\sin\vartheta}{\cos^2\vartheta} \\ \hat{\imath} &= \hat{r}\cos\vartheta - \hat{\vartheta}\sin\vartheta \quad \blacksquare \end{aligned} \tag{6}$$

The condition stated in 3 was that *in infinitum*, $\mathbf{V} = U\hat{\imath}$. By substituting in 6, the statement becomes in terms of $\hat{r}$ and $\hat{\vartheta}$:

$$\mathbf{V} = U(\hat{r}\cos\vartheta - \hat{\vartheta}\sin\vartheta) \quad \text{as} \quad r \to \infty$$

6.2.2 $\mathbf{V} \cdot \hat{n} = 0$

In polar coordinates, the base vector $\hat{r}$ points in the direction of positive change of r, that being outwards from the center. If the cylinder is assumed to be the center of the coordinate system, then $\hat{r}$ will always point normal to the surface of the cylinder. Therefore, at the boundary of the cylinder when $r = L$,

$$\mathbf{V} \cdot \hat{r} = 0$$

6.2.3 $\Delta\phi = 0$

Lemma 6.2 (Jacobian Shmaycobian)**.** The derivative of composite functions corresponds to the product Jacobian of Jacobian matrices:

$$J_{f \circ g} = (J_f \circ g) J_g$$

Proof. I finna fix it later frfr. ∎

Lemma 6.3 (Multivariable chain rule)**.** Let $X(t,u)$ and $Y(t,u)$ be functions where $X, Y : \mathbb{R}^2 \to \mathbb{R}$ such that $X, Y \in C^1(\mathbb{R}^2)$. Then define $Z(x,y)$ to be a function where $Z : \mathbb{R}^2 \to \mathbb{R}$ and $Z \in C^1(\mathbb{R}^2)$. Then the partial derivatives of the composite function $z(t,u) = Z(X(t,u), Y(t,u))$ are given by:

$$\frac{\partial z}{\partial t} = \frac{\partial Z}{\partial x}\frac{\partial X}{\partial t} + \frac{\partial Z}{\partial y}\frac{\partial Y}{\partial t}$$
$$\frac{\partial z}{\partial u} = \frac{\partial Z}{\partial x}\frac{\partial X}{\partial u} + \frac{\partial Z}{\partial y}\frac{\partial Y}{\partial u}$$

Proof. Let $g : \mathbb{R}^n \to \mathbb{R}^p$ and $f : \mathbb{R}^p \to \mathbb{R}^m$, the dimensions of the Jacobian matrices must then be given as:

$$J_g \in \mathbb{R}^{m \times p},\ (J_f \circ g) \in \mathbb{R}^{p \times n}$$
$$\therefore (J_f \circ g) J_g \in \mathbb{R}^{n \times m}$$

Let the parameters of f be called $x_1,\ x_2,\ \ldots,\ x_n$ and the parameters of g be called $y_1,\ y_2,\ \ldots,\ y_n$. The Jacobian of the the composite function $f \circ g$ is defined as:

$$J_{f \circ g} = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} & \cdots & \frac{\partial f}{\partial y_n} \end{bmatrix}.$$

Because $f \circ g : \mathbb{R}^n \to \mathbb{R}^m$, $J_{f \circ g} \in \mathbb{R}^{n \times m}$. The element at position (i, j) of some Jacobian J_F is given by:

$$(J_F)_{ij} = \frac{\partial (f \circ g)_j}{\partial x_i} \tag{7}$$

By matrix multiplication, $((J_f \circ g)J_g)_{ij}$ can be computed as:

$$((J_f \circ g)J_g)_{ij} = \sum_{k=1}^{p} (J_f \circ g)_{ik} (J_g)_{kj}$$

Applying the form given in 7 gives:

$$\begin{aligned} ((J_f \circ g)J_g)_{ij} &= \sum_{k=1}^{p} \left.\frac{\partial f_k}{\partial x_i}\right|_{x=g} \frac{\partial g_j}{\partial y_k} \\ \rightsquigarrow ((J_f \circ g)J_g)_{ij} &= \sum_{k=1}^{p} \frac{\partial f}{\partial g} \frac{\partial g}{\partial y_k} \end{aligned} \qquad \blacksquare$$

For the case given above with the composite function z,

$$J_z =$$

Lemma 6.4 (Polar-Form Laplacian)**.** For some scalar field $\phi(x, y)$ defined in a Cartesian system, the Laplacian of ϕ in polar coordinates $\langle r, \vartheta \rangle$ is given by:

$$\Delta \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \vartheta^2}$$

Proof. In Cartesian coordinates, the Laplacian operator Δ is defined as $\nabla \cdot \nabla$, which for the scalar field ϕ becomes:

$$\begin{aligned}\Delta\phi &= \nabla\cdot\nabla\phi \\ &= \begin{pmatrix}\partial/\partial x\\ \partial/\partial y\end{pmatrix}\cdot\begin{pmatrix}\partial\phi/\partial x\\ \partial\phi/\partial y\end{pmatrix} \\ &= \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\end{aligned}$$

Translating x and y to polar coordinates and calculating their derivatives with respect to r and ϑ gives:

$$x = r\cos\vartheta, \quad y = r\sin\vartheta$$

$$\frac{\partial x}{\partial r} = \cos\vartheta, \quad \frac{\partial y}{\partial r} = \sin\vartheta \tag{8}$$

$$\frac{\partial x}{\partial\vartheta} = -r\sin\vartheta, \quad \frac{\partial y}{\partial\vartheta} = r\cos\vartheta \tag{9}$$

Consequently, by the chain rule and substitution from 8:

$$\begin{aligned}\frac{\partial\phi}{\partial r} &= \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial\phi}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial\phi}{\partial x}\cos\vartheta+\frac{\partial\phi}{\partial y}\sin\vartheta\end{aligned} \tag{10}$$

Taking the derivative of 10 with respect to r again gives:

$$\begin{aligned}\frac{\partial^2\phi}{\partial r^2} &= \frac{\partial}{\partial r}\frac{\partial\phi}{\partial x}\cos\vartheta+\frac{\partial}{\partial r}\frac{\partial\phi}{\partial y}\sin\vartheta \\ &= \frac{\partial}{\partial x}\frac{\partial\phi}{\partial r}\cos\vartheta+\frac{\partial}{\partial y}\frac{\partial\phi}{\partial r}\sin\vartheta\end{aligned} \tag{11}$$

Substituting 10 into 11 gives:

$$\begin{aligned}\frac{\partial^2\phi}{\partial r^2} &= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\cos\vartheta+\frac{\partial\phi}{\partial y}\sin\vartheta\right)\cos\vartheta+\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\cos\vartheta+\frac{\partial\phi}{\partial y}\sin\vartheta\right)\sin\vartheta \\ &= \frac{\partial^2\phi}{\partial x^2}\cos^2\vartheta+\frac{\partial^2\phi}{\partial x\partial y}\sin\vartheta\cos\vartheta+\frac{\partial^2\phi}{\partial y\partial x}\cos\vartheta\sin\vartheta+\frac{\partial^2\phi}{\partial y^2}\sin^2\vartheta \\ &= \frac{\partial^2\phi}{\partial x^2}\cos^2\vartheta+2\frac{\partial^2\phi}{\partial x\partial y}\sin\vartheta\cos\vartheta+\frac{\partial^2\phi}{\partial y^2}\sin^2\vartheta\end{aligned} \tag{12}$$

Applying the same process for $\frac{\partial \phi}{\partial \vartheta}$ with substitution from 9 yields:

$$\begin{aligned}\frac{\partial \phi}{\partial \vartheta} &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \vartheta} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \vartheta} \\ &= -\frac{\partial \phi}{\partial x} r \sin\vartheta + \frac{\partial \phi}{\partial y} r \cos\vartheta \end{aligned} \tag{13}$$

Taking the derivative of 13 with respect to ϑ again gives:

$$\frac{\partial^2 \phi}{\partial \vartheta^2} = -\frac{\partial}{\partial \vartheta}\frac{\partial \phi}{\partial x} r \sin\vartheta + \frac{\partial}{\partial \vartheta}\frac{\partial \phi}{\partial y} r \cos\vartheta$$

Since both terms contain a product of two functions dependent on ϑ the product rule needs to be applied. This gives:

$$\begin{aligned}\frac{\partial^2 \phi}{\partial \vartheta^2} &= -\frac{\partial^2 \phi}{\partial \vartheta \partial x} r \sin\vartheta - \frac{\partial \phi}{\partial x} r \cos\vartheta + \frac{\partial^2 \phi}{\partial \vartheta \partial y} r \cos\vartheta - \frac{\partial \phi}{\partial y} r \sin\vartheta \\ &= -r\left(\frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial \phi}{\partial y}\sin\vartheta\right) + r\frac{\partial \phi}{\partial \vartheta}\left(-\frac{\partial}{\partial x}\sin\vartheta + \frac{\partial}{\partial y}\cos\vartheta\right)\end{aligned} \tag{14}$$

Substituting 13 into 14 gives:

$$\frac{\partial^2 \phi}{\partial \vartheta^2} = -r\left(\frac{\partial \phi}{\partial x}\cos\vartheta + \frac{\partial \phi}{\partial y}\sin\vartheta\right) + r\underbrace{\left(-\frac{\partial \phi}{\partial x} r \sin\vartheta + \frac{\partial \phi}{\partial y} r \cos\vartheta\right)\left(-\frac{\partial}{\partial x}\sin\vartheta + \frac{\partial}{\partial y}\cos\vartheta\right)}_{\Phi} \tag{15}$$

Expanding Φ:

$$\begin{aligned}\Phi &= \left(-\frac{\partial \phi}{\partial x} r \sin\vartheta + \frac{\partial \phi}{\partial y} r \cos\vartheta\right)\left(-\frac{\partial}{\partial x}\sin\vartheta + \frac{\partial}{\partial y}\cos\vartheta\right) \\ &= \left(-\frac{\partial \phi}{\partial x} r \sin\vartheta\right)\left(-\frac{\partial}{\partial x}\sin\vartheta\right) + \left(-\frac{\partial \phi}{\partial x} r \sin\vartheta\right)\left(\frac{\partial}{\partial y}\cos\vartheta\right) \\ &+ \left(\frac{\partial \phi}{\partial y} r \cos\vartheta\right)\left(-\frac{\partial}{\partial x}\sin\vartheta\right) + \left(\frac{\partial \phi}{\partial y} r \cos\vartheta\right)\left(\frac{\partial}{\partial y}\cos\vartheta\right) \\ &= \frac{\partial^2 \phi}{\partial x^2} r \sin^2\vartheta - 2\frac{\partial^2 \phi}{\partial x \partial y} r \cos\vartheta \sin\vartheta + \frac{\partial^2 \phi}{\partial y^2} r \cos^2\vartheta\end{aligned}$$

Substituting Φ back into 15 gives:

$$\begin{aligned}
\frac{\partial^2\phi}{\partial\vartheta^2} &= -r\left(\frac{\partial\phi}{\partial x}\cos\vartheta + \frac{\partial\phi}{\partial y}\sin\vartheta\right) + r\left(\frac{\partial^2\phi}{\partial x^2} r\sin^2\vartheta - 2\frac{\partial^2\phi}{\partial x\partial y} r\cos\vartheta\sin\vartheta + \frac{\partial^2\phi}{\partial y^2} r\cos^2\vartheta\right) \\
&= r^2\left(\frac{\partial^2\phi}{\partial x^2}\sin^2\vartheta - 2\frac{\partial^2\phi}{\partial x\partial y}\cos\vartheta\sin\vartheta + \frac{\partial^2\phi}{\partial y^2}\cos^2\vartheta\right) - r\left(\frac{\partial\phi}{\partial x}\cos\vartheta + \frac{\partial\phi}{\partial y}\sin\vartheta\right) \\
&= r^2\left(\frac{\partial^2\phi}{\partial x^2}\sin^2\vartheta - 2\frac{\partial^2\phi}{\partial x\partial y}\cos\vartheta\sin\vartheta + \frac{\partial^2\phi}{\partial y^2}\cos^2\vartheta\right) - r\frac{\partial\phi}{\partial r} \qquad (16)
\end{aligned}$$

Combining 12 and 16 yields:

$$\begin{aligned}
\frac{\partial^2\phi}{\partial r^2} + \frac{\partial^2\phi}{\partial\vartheta^2} &= \frac{\partial^2\phi}{\partial x^2}\cos^2\vartheta + 2\frac{\partial^2\phi}{\partial x\partial y}\sin\vartheta\cos\vartheta + \frac{\partial^2\phi}{\partial y^2}\sin^2\vartheta \\
&\quad + r^2\left(\frac{\partial^2\phi}{\partial x^2}\sin^2\vartheta - 2\frac{\partial^2\phi}{\partial x\partial y}\cos\vartheta\sin\vartheta + \frac{\partial^2\phi}{\partial y^2}\cos^2\vartheta\right) - r\frac{\partial\phi}{\partial r} \\
\implies \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2} &= \frac{\partial^2\phi}{\partial x^2}\cos^2\vartheta + \frac{\partial^2\phi}{\partial x^2}\sin^2\vartheta + \frac{\partial^2\phi}{\partial y^2}\cos^2\vartheta + \frac{\partial^2\phi}{\partial y^2}\sin^2\vartheta - \frac{1}{r}\frac{\partial\phi}{\partial r} \\
&= \frac{\partial^2\phi}{\partial x^2}\left(\cos^2\vartheta + \sin^2\vartheta\right) + \frac{\partial^2\phi}{\partial y^2}\left(\cos^2\vartheta + \sin^2\vartheta\right) - \frac{1}{r}\frac{\partial\phi}{\partial r} \\
&= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - \frac{1}{r}\frac{\partial\phi}{\partial r} \\
\implies \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} &= \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2} \\
\therefore \Delta\phi &= \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2} \qquad (17)
\end{aligned}$$

■

6.3 Ad confluōrem

Summarized, the conditions translated to polar form in sections 6.2.1, 6.2.2 and 6.2.3 are:

$$\begin{aligned}
\mathbf{V} = U(\hat{r}\cos\vartheta - \hat{\vartheta}\sin\vartheta) &\qquad \text{as} \qquad r \to \infty \\
\mathbf{V}\cdot\hat{r} = 0 &\qquad \text{when} \qquad r = L \\
\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\vartheta^2} = 0 &
\end{aligned}$$

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