

## 6 References

- [Peyret and Taylor, 2012] Peyret, R. and Taylor, T. D. (2012). *Computational methods for fluid flow*. Springer Science & Business Media.
- [Stony Brook University, 2021] Stony Brook University (2021). Mat132 episode 25: Second-order differential equations.

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## 8 Research

### 8.1 Potential flow around a circular cylinder

A cylinder of radius  $L$  is placed in two-dimensional, incompressible, inviscid flow which flows in the direction of  $\hat{i}$ . Far away from the cylinder the velocity field  $\mathbf{V}$  can be described as:

$$\mathbf{V} = U\hat{i} \tag{6}$$

Where  $U$  is some constant. Since the cylinder is impermissible, at the boundary  $\mathbf{V} \cdot \hat{n} = 0$  where the vector  $\hat{n}$  is the unit vector normal to the surface.

Since in this model the viscosity  $\nu = 0$ , the flow can be modeled using the Euler equations. If the Euler equations, apply, so does Kelvin's theorem:

**Theorem 8.1** (Kelvin's circulation theorem). *The circulation around a closed material loop moving with an inviscid, barotropic fluid in the presence of conservative body forces remains constant over time.*<sup>[Citation needed]</sup>

*If  $\Gamma$  denotes the circulation around a material loop  $C(t)$  moving with the fluid, then:*

$$\frac{D\Gamma}{Dt} = 0$$

*Id est*, if the vorticity of  $\mathbf{V}$  is 0 initially, it must remain 0 everywhere, thus  $\nabla \times \mathbf{V} = 0$ . Since the flow is irrotational,  $\mathbf{V}$  can be expressed as  $\mathbf{V} = \nabla\phi$ , where  $\phi$  is the velocity potential.

Furthermore, if  $\mathbf{V}$  is incompressible, that being that  $\nabla \cdot \mathbf{V} = 0$ , then  $\phi$  must satisfy Laplace's equation:  $\nabla^2\phi = 0$ .

## 8.2 Polar coordinate boundary conditions

### 8.2.1 $\mathbf{V} = U\hat{i}$

In polar coordinates, the base vectors  $\hat{r}$  and  $\hat{\vartheta}$  are defined as:

$$\hat{r} \triangleq \hat{i} \cos \vartheta + \hat{j} \sin \vartheta$$

$$\hat{\vartheta} \triangleq -\hat{i} \sin \vartheta + \hat{j} \cos \vartheta$$

Solving for  $\hat{i}$  and  $\hat{j}$  gives:

$$\hat{i} = \frac{\hat{r} - \hat{j} \sin \vartheta}{\cos \vartheta} \quad (7)$$

$$\hat{j} = \frac{\hat{\vartheta} + \hat{i} \sin \vartheta}{\cos \vartheta} \quad (8)$$

Substituting 8 into 7 and isolating  $\hat{i}$  shows that

$$\begin{aligned} \hat{i} &= \frac{\hat{r} - \frac{\hat{\vartheta} + \hat{i} \sin \vartheta}{\cos \vartheta} \sin \vartheta}{\cos \vartheta} \\ &= \frac{\hat{r}}{\cos \vartheta} - \frac{\hat{\vartheta} \sin \vartheta + \hat{i} \sin^2 \vartheta}{\cos^2 \vartheta} \\ &= \frac{\hat{r}}{\cos \vartheta} - \frac{\hat{\vartheta} \sin \vartheta}{\cos^2 \vartheta} - \frac{\hat{i} \sin^2 \vartheta}{\cos^2 \vartheta} \\ \implies \hat{i} + \frac{\sin^2 \vartheta}{\cos^2 \vartheta} \hat{i} &= \frac{\hat{r}}{\cos \vartheta} - \frac{\hat{\vartheta} \sin \vartheta}{\cos^2 \vartheta} \\ \hat{i} \left( 1 + \frac{\sin^2 \vartheta}{\cos^2 \vartheta} \right) &= \frac{\hat{r}}{\cos \vartheta} - \frac{\hat{\vartheta} \sin \vartheta}{\cos^2 \vartheta} \\ \hat{i} \left( \frac{\sin^2 \vartheta + \cos^2 \vartheta}{\cos^2 \vartheta} \right) &= \frac{\hat{r}}{\cos \vartheta} - \frac{\hat{\vartheta} \sin \vartheta}{\cos^2 \vartheta} \\ \frac{\hat{i}}{\cos^2 \vartheta} &= \frac{\hat{r}}{\cos \vartheta} - \frac{\hat{\vartheta} \sin \vartheta}{\cos^2 \vartheta} \\ \hat{i} &= \hat{r} \cos \vartheta - \hat{\vartheta} \sin \vartheta \quad \blacksquare \end{aligned} \quad (9)$$

The condition stated in 6 was that *in infinitum*,  $\mathbf{V} = U\hat{i}$ . By substituting in 9, the statement becomes in terms of  $\hat{r}$  and  $\hat{\vartheta}$ :

$$\mathbf{V} = U(\hat{r} \cos \vartheta - \hat{\vartheta} \sin \vartheta) \quad \text{as } r \rightarrow \infty$$

### 8.2.2 $\mathbf{V} \cdot \hat{n} = 0$

In polar coordinates, the base vector  $\hat{r}$  points in the direction of positive change of  $r$ , that being outwards from the center. If the cylinder is assumed to be the center of the coordinate system, then  $\hat{r}$  will always point normal to the surface of the cylinder. Therefore, at the boundary of the cylinder when  $r = L$ ,

$$\mathbf{V} \cdot \hat{r} = 0$$

### 8.2.3 $\nabla^2 \phi = 0$

**Lemma 8.2** (Jacobian Shmaycobian). *The derivative of composite functions corresponds to the product Jacobian of Jacobian matrices:*

$$J_{f \circ g} = (J_f \circ g) J_g$$

*Proof.* The Jacobian of  $J_{f \circ g}$  is defined as

$$J_{f \circ g} = \begin{bmatrix} \nabla^\top (f \circ g)_1 \\ \nabla^\top (f \circ g)_2 \\ \vdots \\ \nabla^\top (f \circ g)_m \end{bmatrix}$$

for some function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , where

$$\text{row}_i J_{f \circ g} = \nabla^\top (f \circ g)_i = \left[ \frac{\partial (f \circ g)_i}{\partial x_1} \quad \frac{\partial (f \circ g)_i}{\partial x_2} \quad \dots \quad \frac{\partial (f \circ g)_i}{\partial x_n} \right]$$

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**Lemma 8.3** (Multivariable chain rule). *Let  $X(t, u)$  and  $Y(t, u)$  be functions where  $X, Y : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that  $X, Y \in C^1(\mathbb{R}^2)$ . Then define  $Z(x, y)$  to be a function where  $Z : \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $Z \in C^1(\mathbb{R}^2)$ . Then the partial derivatives of the composite function  $z(t, u) =$*

$Z(X(t, u), Y(t, u))$  are given by:

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial Z}{\partial x} \frac{\partial X}{\partial t} + \frac{\partial Z}{\partial y} \frac{\partial Y}{\partial t} \\ \frac{\partial z}{\partial u} &= \frac{\partial Z}{\partial x} \frac{\partial X}{\partial u} + \frac{\partial Z}{\partial y} \frac{\partial Y}{\partial u}\end{aligned}$$

*Proof.* To be proven... ■

**Lemma 8.4** (Polar-Form Laplacian). *For some scalar field  $\phi(x, y)$  defined in a Cartesian system, the Laplacian of  $\phi$  in polar coordinates  $\langle r, \vartheta \rangle$  is given by:*

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \vartheta^2}$$

*Proof.* In Cartesian coordinates, the Laplacian operator  $\nabla^2$  is defined as  $\nabla \cdot \nabla$ , which for the scalar field  $\phi$  becomes:

$$\begin{aligned}\nabla^2 \phi &= \nabla \cdot \nabla \phi \\ &= \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} \cdot \begin{pmatrix} \partial \phi / \partial x \\ \partial \phi / \partial y \end{pmatrix} \\ &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\end{aligned}$$

Translating  $x$  and  $y$  to polar coordinates and calculating their derivatives with respect to  $r$  and  $\vartheta$  gives:

$$\begin{aligned}x &= r \cos \vartheta, & y &= r \sin \vartheta \\ \frac{\partial x}{\partial r} &= \cos \vartheta, & \frac{\partial y}{\partial r} &= \sin \vartheta\end{aligned}\tag{10}$$

$$\frac{\partial x}{\partial \vartheta} = -r \sin \vartheta, \quad \frac{\partial y}{\partial \vartheta} = r \cos \vartheta\tag{11}$$

Consequently, by the chain rule and substitution from 10:

$$\begin{aligned}\frac{\partial \phi}{\partial r} &= \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial r} \\ &= \frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial \phi}{\partial y} \sin \vartheta\end{aligned}\tag{12}$$

Taking the derivative of 12 with respect to  $r$  again gives:

$$\begin{aligned}\frac{\partial^2 \phi}{\partial r^2} &= \frac{\partial}{\partial r} \frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial}{\partial r} \frac{\partial \phi}{\partial y} \sin \vartheta \\ &= \frac{\partial}{\partial x} \frac{\partial \phi}{\partial r} \cos \vartheta + \frac{\partial}{\partial y} \frac{\partial \phi}{\partial r} \sin \vartheta\end{aligned}\quad (13)$$

Substituting 12 into 13 gives:

$$\begin{aligned}\frac{\partial^2 \phi}{\partial r^2} &= \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial \phi}{\partial y} \sin \vartheta \right) \cos \vartheta + \frac{\partial}{\partial y} \left( \frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial \phi}{\partial y} \sin \vartheta \right) \sin \vartheta \\ &= \frac{\partial^2 \phi}{\partial x^2} \cos^2 \vartheta + \frac{\partial^2 \phi}{\partial x \partial y} \sin \vartheta \cos \vartheta + \frac{\partial^2 \phi}{\partial y \partial x} \cos \vartheta \sin \vartheta + \frac{\partial^2 \phi}{\partial y^2} \sin^2 \vartheta \\ &= \frac{\partial^2 \phi}{\partial x^2} \cos^2 \vartheta + 2 \frac{\partial^2 \phi}{\partial x \partial y} \sin \vartheta \cos \vartheta + \frac{\partial^2 \phi}{\partial y^2} \sin^2 \vartheta\end{aligned}\quad (14)$$

Applying the same process for  $\frac{\partial \phi}{\partial \vartheta}$  with substitution from 11 yields:

$$\begin{aligned}\frac{\partial \phi}{\partial \vartheta} &= \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial \vartheta} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial \vartheta} \\ &= -\frac{\partial \phi}{\partial x} r \sin \vartheta + \frac{\partial \phi}{\partial y} r \cos \vartheta\end{aligned}\quad (15)$$

Taking the derivative of 15 with respect to  $\vartheta$  again gives:

$$\frac{\partial^2 \phi}{\partial \vartheta^2} = -\frac{\partial}{\partial \vartheta} \frac{\partial \phi}{\partial x} r \sin \vartheta + \frac{\partial}{\partial \vartheta} \frac{\partial \phi}{\partial y} r \cos \vartheta$$

Since both terms contain a product of two functions dependent on  $\vartheta$  the product rule needs to be applied. This gives:

$$\begin{aligned}\frac{\partial^2 \phi}{\partial \vartheta^2} &= -\frac{\partial^2 \phi}{\partial \vartheta \partial x} r \sin \vartheta - \frac{\partial \phi}{\partial x} r \cos \vartheta + \frac{\partial^2 \phi}{\partial \vartheta \partial y} r \cos \vartheta - \frac{\partial \phi}{\partial y} r \sin \vartheta \\ &= -r \left( \frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial \phi}{\partial y} \sin \vartheta \right) + r \frac{\partial \phi}{\partial \vartheta} \left( -\frac{\partial}{\partial x} \sin \vartheta + \frac{\partial}{\partial y} \cos \vartheta \right)\end{aligned}\quad (16)$$

Substituting 15 into 16 gives:

$$\frac{\partial^2 \phi}{\partial \vartheta^2} = -r \left( \frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial \phi}{\partial y} \sin \vartheta \right) + r \underbrace{\left( -\frac{\partial \phi}{\partial x} r \sin \vartheta + \frac{\partial \phi}{\partial y} r \cos \vartheta \right) \left( -\frac{\partial}{\partial x} \sin \vartheta + \frac{\partial}{\partial y} \cos \vartheta \right)}_{\Phi} \quad (17)$$

Expanding  $\Phi$ :

$$\begin{aligned} \Phi &= \left( -\frac{\partial \phi}{\partial x} r \sin \vartheta + \frac{\partial \phi}{\partial y} r \cos \vartheta \right) \left( -\frac{\partial}{\partial x} \sin \vartheta + \frac{\partial}{\partial y} \cos \vartheta \right) \\ &= \left( -\frac{\partial \phi}{\partial x} r \sin \vartheta \right) \left( -\frac{\partial}{\partial x} \sin \vartheta \right) + \left( -\frac{\partial \phi}{\partial x} r \sin \vartheta \right) \left( \frac{\partial}{\partial y} \cos \vartheta \right) \\ &\quad + \left( \frac{\partial \phi}{\partial y} r \cos \vartheta \right) \left( -\frac{\partial}{\partial x} \sin \vartheta \right) + \left( \frac{\partial \phi}{\partial y} r \cos \vartheta \right) \left( \frac{\partial}{\partial y} \cos \vartheta \right) \\ &= \frac{\partial^2 \phi}{\partial x^2} r \sin^2 \vartheta - 2 \frac{\partial^2 \phi}{\partial x \partial y} r \cos \vartheta \sin \vartheta + \frac{\partial^2 \phi}{\partial y^2} r \cos^2 \vartheta \end{aligned}$$

Substituting  $\Phi$  back into 17 gives:

$$\begin{aligned} \frac{\partial^2 \phi}{\partial \vartheta^2} &= -r \left( \frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial \phi}{\partial y} \sin \vartheta \right) + r \left( \frac{\partial^2 \phi}{\partial x^2} r \sin^2 \vartheta - 2 \frac{\partial^2 \phi}{\partial x \partial y} r \cos \vartheta \sin \vartheta + \frac{\partial^2 \phi}{\partial y^2} r \cos^2 \vartheta \right) \\ &= r^2 \left( \frac{\partial^2 \phi}{\partial x^2} \sin^2 \vartheta - 2 \frac{\partial^2 \phi}{\partial x \partial y} \cos \vartheta \sin \vartheta + \frac{\partial^2 \phi}{\partial y^2} \cos^2 \vartheta \right) - r \left( \frac{\partial \phi}{\partial x} \cos \vartheta + \frac{\partial \phi}{\partial y} \sin \vartheta \right) \\ &= r^2 \left( \frac{\partial^2 \phi}{\partial x^2} \sin^2 \vartheta - 2 \frac{\partial^2 \phi}{\partial x \partial y} \cos \vartheta \sin \vartheta + \frac{\partial^2 \phi}{\partial y^2} \cos^2 \vartheta \right) - r \frac{\partial \phi}{\partial r} \quad (18) \end{aligned}$$

Combining 14 and 18 yields:

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial \vartheta^2} &= \frac{\partial^2 \phi}{\partial x^2} \cos^2 \vartheta + 2 \frac{\partial^2 \phi}{\partial x \partial y} \sin \vartheta \cos \vartheta + \frac{\partial^2 \phi}{\partial y^2} \sin^2 \vartheta \\
&\quad + r^2 \left( \frac{\partial^2 \phi}{\partial x^2} \sin^2 \vartheta - 2 \frac{\partial^2 \phi}{\partial x \partial y} \cos \vartheta \sin \vartheta + \frac{\partial^2 \phi}{\partial y^2} \cos^2 \vartheta \right) - r \frac{\partial \phi}{\partial r} \\
\Rightarrow \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \vartheta^2} &= \frac{\partial^2 \phi}{\partial x^2} \cos^2 \vartheta + \frac{\partial^2 \phi}{\partial x^2} \sin^2 \vartheta + \frac{\partial^2 \phi}{\partial y^2} \cos^2 \vartheta + \frac{\partial^2 \phi}{\partial y^2} \sin^2 \vartheta - \frac{1}{r} \frac{\partial \phi}{\partial r} \\
&= \frac{\partial^2 \phi}{\partial x^2} (\cos^2 \vartheta + \sin^2 \vartheta) + \frac{\partial^2 \phi}{\partial y^2} (\cos^2 \vartheta + \sin^2 \vartheta) - \frac{1}{r} \frac{\partial \phi}{\partial r} \\
&= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} \\
\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} &= \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \vartheta^2} \\
\therefore \nabla^2 \phi &= \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \vartheta^2} \tag{19}
\end{aligned}$$

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### 8.3 Ad confluōrem

Summarized, the conditions translated to polar form in sections 8.2.1, 8.2.2 and 8.2.3 are:

$$\begin{aligned}
\mathbf{V} &= U(\hat{r} \cos \vartheta - \hat{\vartheta} \sin \vartheta) & \text{as } r &\rightarrow \infty \\
\mathbf{V} \cdot \hat{r} &= 0 & \text{when } r &= L \\
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \vartheta^2} &= 0
\end{aligned}$$

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