How can vector calculus be used to model and analyze incompressible fluid flow in two-dimensional spaces, and what insights can this provide about the vector fields of real-world fluid systems with circular obstacles?

Mathematics AA HL

Word Count: 399

Contents

1 Introduction

Vector calculus provides the foundation and tools for the analysis and modeling of several real-world phenomona, and is integral to understanding several important fields such as aero- & hydrodynamics, as well as the modeling of weather & climates.

Through the use of pure mathematics, this essay will investigate the flow of fluids in 2 dimensional spaces around circular obstacles. Visual representations through mediums such as vector field plots (plotted through a custom program

1.1 Aim & scope

This essay will for simplicity's sake only cover fluid flow around circular obstacles in $\mathbb{R}^2$ spaces; an alaysis of fluid flow in $\mathbb{R}^3$ spaces would be much more complex. Furthermore, only incompressible fluids sans sinks and sources ($\mathbf{F} \ni \nabla \cdot \mathbf{F} = 0$), will be analyzed.

Most of the analysis will take place using Green's theorem[see 1.4].

1.2 Background

Vector calculus is the mathematical study of applying multi-variable calculus to vector valued functions, often for the spaces $\mathbb{R}^2$ and $\mathbb{R}^3$.

1.2.1 The gradient operator

In one dimensional calculus, the gradient (slope) of a graph can be calculated at any point by differentiating the function and evaluating it at said point.

$$f(x) = x^2 \implies \frac{\mathrm{d}f}{\mathrm{d}x} = 2x$$

Defining this as the equation of greatest incline at point x, even though there in this case is only one true rate of the function since we are working in the space $\mathbb{R}^1$, will help us when extending this concept of greatest incline to $\mathbb{R}^n$ spaces.

In multi-variable calculus a new tool called the partial derivative helps us differentiate multi-variable functions. It works identically to a regular derivative just that we treat all variables other than the one we are differentiating with respect to as constants.

$$f(x, y) = x^2 + y^2 \implies \frac{\partial f}{\partial x} = 2x$$

This finds the greatest change the equation of greatest change for f in the direction differentiated by, so if we combine the partial derivatives in all directions f takes as parameters into a vector, then we end up with a vector pointing in the direction of steepest ascent.

The gradient operator, denoted by the symbol ∇ (pronounced nabla or grad), is thereby defined for some function $f : \mathbb{R}^n \to \mathbb{R}^m$ as:

$$\nabla = \left.\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ \vdots \end{bmatrix}\right\} n \text{ times} \tag{1}$$

Aplying this to our previous example $f(x, y) = x^2 + y^2$, we get:

$$\nabla f = \begin{bmatrix} \frac{\partial}{\partial x} x^2 + y^2 \\ \frac{\partial}{\partial y} x^2 + y^2 \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

Meaning that at some point (x, y), the direction of steepest incline will be $2(x, y)$. The nabla operator proves foundational to vector calculus and is the backbone of several important concepts.

1.2.2 Other

$$f : \mathbb{R}^n \to \mathbb{R}^n \ni n > 1, n \in \mathbb{Z}$$

$$\frac{\mathrm{D}f}{\mathrm{D}t} \triangleq \frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f \tag{2}$$

$$\vec{v_1} \otimes \vec{v_2}$$

1.3 Fluid dynamics

An incompressible fluid is any fluid such that $\nabla \cdot \mathbf{F} = 0$, which is to say that the divergence of the fluid is 0.

1.4 Green's theorem

Green's theorem 1. *The double integral over some reigon R of the curl of a vector field $\mathbf{F}$ is equal to the line integral over some curve C of $\mathbf{F}$*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\iint_R \nabla \times \mathbf{F} \mathrm{d}A = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

$$\frac{\mathrm{D}f}{\mathrm{D}\mathbf{t}} = \iiint\limits_V (\frac{\mathrm{D}\rho}{\mathrm{D}\mathbf{t}} + \rho(\nabla \cdot u))dV \tag{3}$$

Lorem ipsum dolor sit amet [Peyret and Taylor, 2012]

2 References

[Peyret and Taylor, 2012] Peyret, R. and Taylor, T. D. (2012). *Computational methods for fluid flow*. Springer Science & Business Media.

3 List of Figures

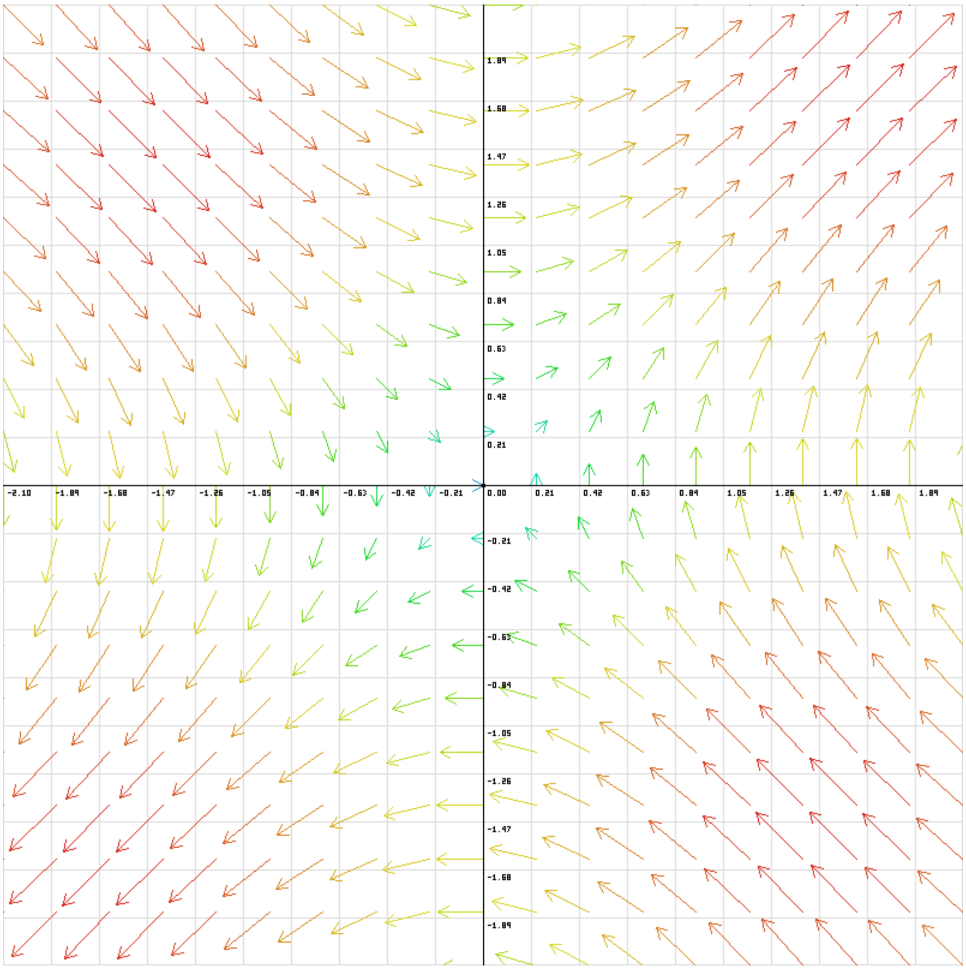

Figure 1: Vector field for $f(x,y) = \begin{pmatrix} \sin y \\ \sin x \end{pmatrix}$

$$\mathbf{F}(t) : \mathbb{R} \to \mathbb{R}^2$$

$$\rightsquigarrow \mathbf{P}(t, \vec{p}) = \vec{p} + \hat{\imath} \iint_0^t \mathbf{F}_x \ \mathrm{d}t + \hat{\jmath} \iint_0^t \mathbf{F}_y \ \mathrm{d}t$$

$$\mathbf{P}(t, \vec{p}) = \vec{p} + \iint_0^t \hat{\imath}\mathbf{F}_x + \hat{\jmath}\mathbf{F}_y \ \mathrm{d}t$$

$$\hat{\jmath} + \hat{\imath}$$

$$
\begin{aligned}
A &= B \\
&= C && \text{substitution}
\end{aligned}
$$

Proof. Green's Theorem

$$B \triangleq C$$

$$
\begin{aligned}
A &= B \\
&= C
\end{aligned}
$$

□

$$a \underbrace{a}_{\text{banana}}$$