Improved Waddle

For the IB Math AA HL course

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1 Vectors

1.1 Planes

Fundementally, a point $\mathbf{P}$ lies on a plane Π if it satisfies the equation:

$$\overrightarrow{AP} = \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}, \quad \lambda, \mu \in \mathbb{R}$$

where $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are three other points on the plane. The normal vector $\mathbf{n}$ of a plane is found by taking the cross product of two vectors on the plane:

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$$

There are three main formulaeic ways of describe a plane, these are:

- The vector equation:

  $$\mathbf{r} = \mathbf{a} + \lambda\mathbf{u} + \mu\mathbf{v}, \quad \lambda, \mu \in \mathbb{R}$$

  where $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three position vectors in space. This form is used to generate positions $\mathbf{r}$ through the setting of λ and μ. This form is equivalently stated as:

  $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \mu \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}$$

- The parametric form is obtained by making the vector equation in to a system of equations.

- The Cartesian form:

  $$ax + by + cz = d, \quad a, b, c, d \in \mathbb{R}$$

  This form is obtained by removing λ and μ from the parametric form's system of equations. This form is homogenous to the equation of a straight line ($ax + by = c$) in 2 dimensional Cartesian x-y space, extended for 3 variables. If we know the normal vector $\mathbf{n}$ of the plane and one point $\mathbf{a}$, this form is also equivalently stated as (with $\mathbf{p}$ being any arbritrary point):

  $$\mathbf{n} \cdot \mathbf{p} = \mathbf{n} \cdot \mathbf{a}$$

  The normal vector of a plane stated in Cartesian form $ax + by + cz = d$ is:

  $$\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Example

The point $A\langle 0,2,1\rangle$, $B\langle 3,0,-1\rangle$ and $C\langle -2,1,1\rangle$ lie on the plane Π. The vector equation of the plane is given by first finding $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{AC}$:

$$\mathbf{u} = \overrightarrow{AB} = \langle 3,0,-1\rangle - \langle 0,2,1\rangle = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix}$$
$$\mathbf{v} = \overrightarrow{AC} = \langle -2,1,1\rangle - \langle 0,2,1\rangle = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

Thus the vector equation is given as:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}$$

Analagously, the parametric form is given by the system of equations:

$$\begin{cases} x = 3\lambda - 2\mu \\ y = 2 - 2\lambda - 1\mu \\ z = 1 - 2\lambda \end{cases}$$

To convert to Cartesian form, we need to eliminate μ and λ from the system. We see:

$$x - 2y = 3\lambda - 2\mu - 4 + 4\lambda + 2\mu = -4 + 7\lambda$$

Thus:

$$\begin{aligned} x - 2y + 3.5z &= -4 + 7\lambda + 3.5 - 7\lambda = -0.5 \\ \implies 2x - 4y + 7z &= -1 \end{aligned}$$

The normal vector is therefore also given as:

$$\mathbf{n} = \begin{pmatrix} 2 \\ -4 \\ 7 \end{pmatrix}$$

If the normal vector of two planes $\mathbf{n}_1$ and $\mathbf{n}_2$ are collinear ($\mathbf{n}_1 = a\mathbf{n}_2$, $a \in \mathbb{R}$), then the planes are parallel. Otherwise, the planes intersect at a line given by two different methods. The first is the solution to the system of equations of the planes' Cartesian forms letting one of the coordinates equal some variable λ. The other is taking the cross product of the two normal vector $\mathbf{n}_1 \times \mathbf{n}_2$, which gives the direction vector of the line.

Example

Let $\Pi_1 : 2x - 4y + 7z = 1$ and $\Pi_2 : -x + y + 2z = 0$. The normal vectors of the planes are:

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ -4 \\ 7 \end{pmatrix}$$
$$\mathbf{n}_2 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$\because \nexists a \in \mathbb{R}$ s.t. $\mathbf{n}_1 = a\mathbf{n}_2$, $\neg(\Pi_1 \parallel \Pi_2)$.

Method 1 The planes' intersection is given at the line which is the solution to the system, letting $z = \lambda$:

$$\begin{aligned}
&\begin{pmatrix} 2x - 4y = 1 - 7\lambda \\ -x + y = -2\lambda \end{pmatrix} \\
\sim &\begin{pmatrix} 2x - 4y = 1 - 7\lambda \\ -2x + 2y = -4\lambda \end{pmatrix}_{2R_2 \to R_2} \\
\sim &\begin{pmatrix} -2y = 1 - 11\lambda \\ -2x = 1 - 15\lambda \end{pmatrix}_{R_1 + R_2 \to R_1,\, 2R_2 + R_1 \to R_2} \\
\sim &\begin{pmatrix} y = \frac{11\lambda - 1}{2} \\ x = \frac{15\lambda - 1}{2} \end{pmatrix}_{-\frac{R_1}{2} \to R_1,\, -\frac{R_2}{2} \to R_2}
\end{aligned}$$

The intersection is thus the line:

$$L : \lambda \mapsto \left\langle \frac{15\lambda - 1}{2}, \frac{11\lambda - 1}{2}, \lambda \right\rangle$$

Method 2 The direction vector $\mathbf{d}$ is given by:

$$\begin{aligned}
\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2 &= \begin{pmatrix} 2 \\ -4 \\ 7 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -4 & 7 \\ -1 & 1 & 2 \end{vmatrix} = \hat{\imath} \begin{vmatrix} -4 & 7 \\ 1 & 2 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 2 & 7 \\ -1 & 2 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & -4 \\ -1 & 1 \end{vmatrix} \\
&= -15\hat{\imath} - 11\hat{\jmath} - 2\hat{k}
\end{aligned}$$

We then find a point common on both planes by choosing $z = 0$ and their equating equations (using $\Pi_1 - 1$):

$$2x - 4y - 1 = -x + y$$

$$3x - 5y = 1$$

We can then choose the point $\left\langle \frac{1}{3}\,,0\,,0 \right\rangle$. The equation can then finally be written as:

$$L : \lambda \mapsto \left\langle \frac{1}{3}\,,0\,,0 \right\rangle + \langle 15\,,11\,,2 \rangle\, \lambda$$