Summation of the Leibniz Series

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Abstract

In this study the Leibniz series was used to calculate the value of Pi to an arbitrary degree of accuracy. The fractional error was estimated as a function of the number of terms in the series, and convergence to the known value was demonstrated using numerical and graphical techniques.

1 Background

In many areas of physics, engineering, and mathematics, numerical precision plays an important role in the efficacy of models and simulations. Transcendental numbers such as Pi must therefore be approximated using various techniques to acquire the degree of precision required by the model.

One of the earliest known techniques for calculating the value of Pi is the Madhava-Gregory-Leibniz series, named after the three independent discoverers of the technique (Madhava of Sangamagrama in the 14th century and James Gregory & Gottfried Leibniz in the 16th century). The series is as follows:

$$\frac{\pi}{4} = arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \sum_{k=1}^{\infty} (-1)^{k+1}/(2k-1)$$

This is a harmonic series which converges slowly. It is therefore of limited utility in actual numerical applications, but represents one of the simplest series to execute programmatically, and so is of pedagogical interest as an exercise in computational programming.

2 Procedure

The series was coded in C using the Cloud9 IDE (integrated development environment). The program takes in the number of terms in the series, and outputs the associated partial sum and its fractional error as follows:

```
#include <stdio.h>
#include <math.h>
void main() {
        int user_input;
        double s=1.0;
        printf("Enter the number of terms in the Leibniz Series: ");
        scanf("%i", &user_input);
        int a;
        while (a<=user_input)    {
        if (a % 2 == 0){
                double k = 2*a-1;
                s+=1/k;
                a++;} else {
                double k = 2*a-1;
                s-=1/k;
                a++;}                           }
        double pi=4*fabs(s);
        printf("Value of Pi: %f \n", pi);
        double c=pi;
        double fracerror=(acos(-1.0)-c)/(acos(-1.0));
        printf("Fractional error: %f \n", fracerror);}
```

3 Results

The first 10 partial sums given by the series are displayed below in Figure 1:

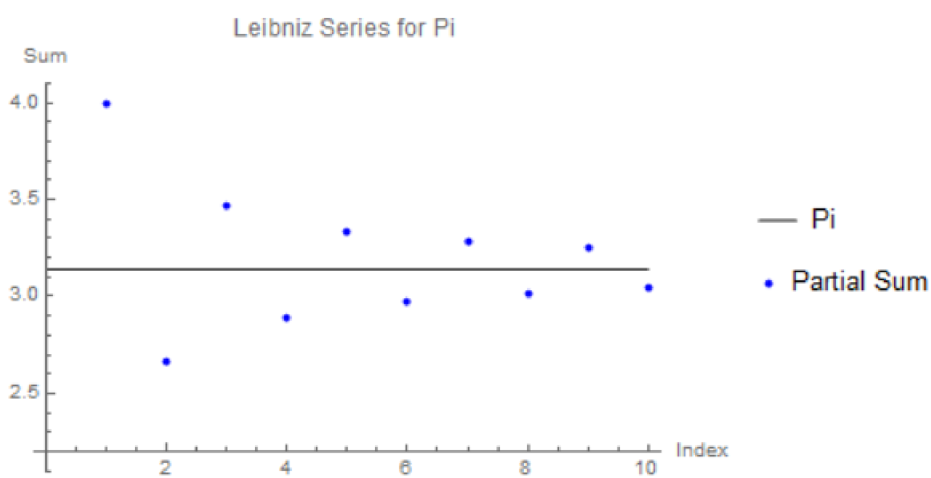

Figure 1: Partial sums of the first 10 terms in the Leibniz series (blue dots) shown alongside Pi (black line).

Note the oscillating convergence of the partial sums to the true value. This behavior can be further verified by taking higher terms, such as the first 1000, shown below in Figure 2:

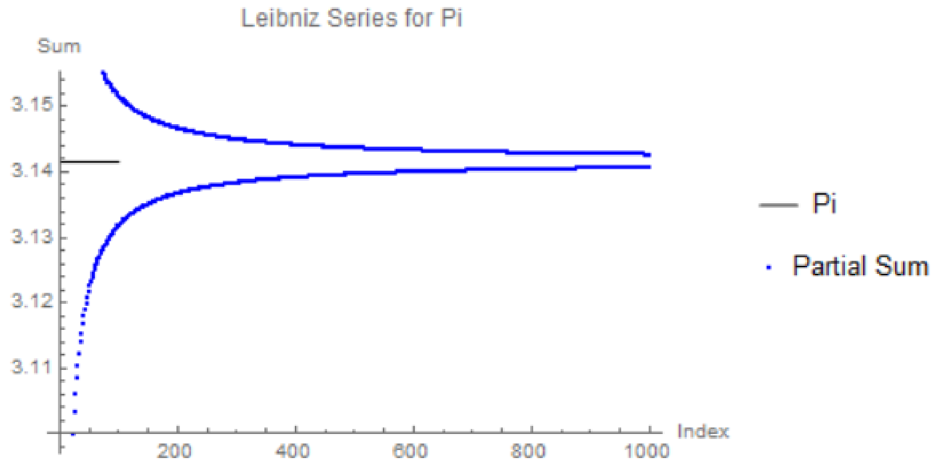

Figure 2: Partial sums of the first 1000 terms in the Leibniz series (blue dots) shown alongside Pi (black line).

As shown above the series converges very slowly, consistently oscillating about Pi. In fact the error can be given as a remainder $R = S - s_n$ where R is the remainder, S is the convergent value of the series, and s_n is the partial sum of the nth term in the sequence. The remainder for the first 50 terms is given below in Figure 3:

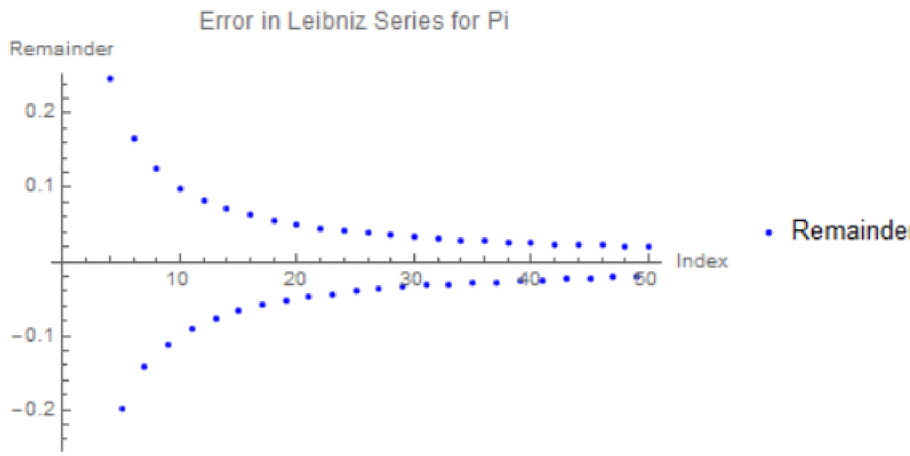

Figure 3: Error in Leibniz series given as a remainder for the first 50 partial sums.

Unsurprisingly the error coverges to 0 in the same oscillatory manner that the series converges to Pi.

4 Discussion

The oscillatory behavior of the Leibniz series' makes it difficult to precisely approximate Pi without using a very large number of terms. In general the convergent behavior of a series determines how useful it is in numerical applications. Other series for Pi, which converge much more quickly (such as Ramanujan's series), are of greater utility for situations in which optimizing computational resources is an important factor.

5 Conclusion

In summary the Leibniz series was used to calculate the value of Pi using the C programming language. The behavior of the series and its error were explored graphically using Mathematica, confirming the nature of its convergence.

References

[1] "The Madhava-Gregory series," Math. Education 7 (1973), B67-B70 [James A. Landau].