Autoregressive Moving Average (ARMA) Models

Arif Nurwahid

Seminar on 21 Feb 2023

We consider a mix of Autoregressive and Moving Average Models. Firstly, ARMA(1,1) model will be reviewed. Next part is revieweing the general MA(p,q) model.

ARMA(1,1)

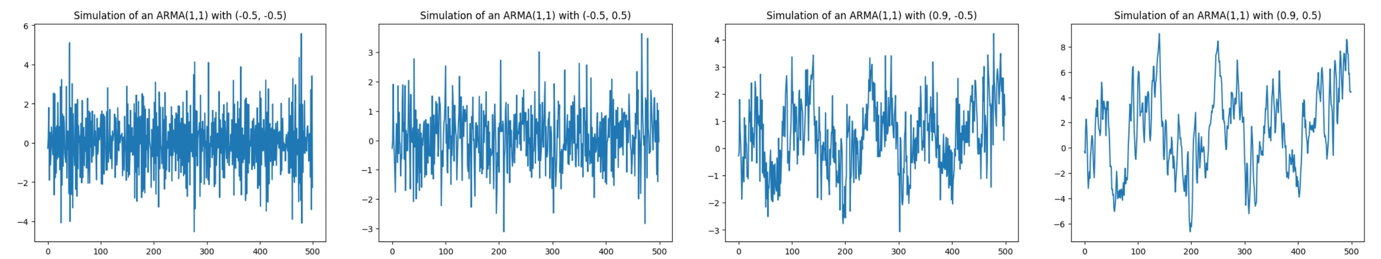
Definition 1

A Mixed autoregressive moving average model of order 1, an ARMA(1,1) process, is a stationary process $\{X_t\}$ which satisfies the following:

$$\begin{aligned} X_t &= \mu + \varphi X_{t-1} + \epsilon_t + \theta \epsilon_{t-1} \\ \varphi(B) X_t &= \mu + \theta(B) \epsilon_t \end{aligned}$$

where:

- $\{\epsilon_t\} \sim WN(0, \sigma_\epsilon^2)$
- μ is a constant term
- $\varphi \neq 0$
- $\theta \neq 0.$
- $\varphi(B) = 1 - \varphi B$
- $\theta(B) = 1 + \theta B$



The property of an ARMA(1,1) process are a mixture of those an AR(1) and MA(1) processes:

- The stationary condition is the one of an AR(1) process: $|\varphi| < 1$
- The invertibility condition is the one of an MA(1) process: $|\theta| < 1$
- The representation of an ARMA(1,1) process is causal if: $|\varphi| < 1$ and $|\theta| < 1.$

If $\{X_t\}$ is stationary, then it has an infinite moving average process:

$$\begin{aligned} X_t &= \frac{\mu}{1-\varphi} + \left(\frac{1+\theta B}{1-\varphi B}\right)\epsilon_t \\ &= \frac{\mu}{1-\varphi} + \sum_{k=0}^{\infty} a_k \epsilon_{t-k} \end{aligned}$$

where:

- $a_0 = 1$
- $a_k = \varphi^k + \theta\varphi^{k-1}$

If $\{X_t\}$ is invertible, then it has an infinite autoregressive process:

$$\begin{aligned} \left(\frac{1-\varphi B}{1+\theta B}\right) X_t &= \frac{\mu}{1+\theta} + \epsilon_t \\ \epsilon_t &= -\frac{\mu}{1+\theta} + X_t + \sum_{k=1}^{\infty} b_k X_{t-k} \end{aligned}$$

where:

- $b_k = (-\theta)^k - \varphi(-\theta)^{k-1}$

Moments

By $\{X_t\}$ defined above, we have the following:

1. $E[X_t] = \dfrac{\mu}{1-\varphi}$
2. $\gamma_0 = \dfrac{1+2\varphi\theta+\theta^2}{1-\varphi^2}\sigma_\epsilon^2$
3. $\gamma_1 = \dfrac{(\varphi+\theta)(1+\varphi\theta)}{1-\varphi^2}\sigma_\epsilon^2$
4. $\gamma_h = \varphi\gamma_{h-1}$ for $|h| > 1$

Autocorrelation Function of an ARMA(1,1)

The autocorrelation Function of an ARMA(1,1) process satisfies:

$$\rho_h = \begin{cases} 1 & , h = 0 \\ \dfrac{(\varphi+\theta)(1+\varphi\theta)}{1+2\varphi\theta+\theta^2} & , |h| = 1 \\ \varphi\rho_{h-1} & , |h| > 1 \end{cases}$$

ARMA(p,q)

Main idea of MA(p,q) models

- Approximate Wold (https://en.wikipedia.org/wiki/Wold%27s_theorem) form of stationary time series by parsimonious paramteric models
  - AR and MA models can be cumbersome because we may need a high-order model with many parameters to adequately describe the data.
  - By mixing both models into a more compact form, the number of parameters is kept small.

Definition 2

A Mixed autoregressive moving average model of order (p,q), an ARMA(p,q) process, is a stationary process $\{X_t\}$ which satisfies the relation:

$$X_t = \mu + \sum_{i=1}^{p} \varphi_i X_{t-i} + \epsilon_t + \sum_{j=1}^{q} \theta_j \epsilon_{t-j}$$
$$\varphi(B) X_t = \mu + \theta(B)\epsilon_t$$

where:

- $\{\epsilon_t\} \sim WN(0, \sigma_\epsilon^2)$
- $E[X_s \epsilon_t] = 0$ if $s < t$
- μ is a constant term
- $\varphi_p \neq 0$
- $\theta_q \neq 0.$
- $\varphi(z) = 1 - \varphi_1 z - \varphi_2 z^2 - \cdots - \varphi_p z^p$
- $\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$

The properties of ARMA(p,q) process are a mixture of those an AR(p) and MA(q) processes:

- The stationary conditions are those of an AR(p) process (i.e. **ARMA(p,0)**):
  - all roots of $\varphi(z)$ do not lie on the unit circle.
- The invertibility conditions are those of an MA(q) process (i.e. **ARMA(0,q)**):
  - all roots of $\theta(z)$ lie outside the unit circle.
- The representation of an ARMA(p,q) process is causal if it is stationary and invertible.

Moment of ARMA(p,q)

The mean is the same as the one of an AR(p) model: $E[X_t] = \dfrac{\mu}{1 - \sum_{j=1}^{p} \varphi_k}$