The Invertibility of MA Model

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Backshift Operator *B* Notation

Using **B** before either a value of the series X_t or an error term ϵ_t means to move that element back one time. For instance,

$$BX_t = X_{t-1}$$

A **power** of **B** means to repeatedly apply the backshift in order to move back a number of time periods that equals the **power**. As an example,

$$B^4 X_t = X_{t-4}$$

MA Polynomial

Using the backshift operator *B*, the MA(q) model can be rewritten as follows:

$$X_t = \theta(B)\epsilon_t$$

where $\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$ is the MA Polynomial.

Invertibility

Let $\{\epsilon_t\} \sim WN(0, \sigma_\epsilon^2)$.

Consider MA(1) Model $X_t = \epsilon_t - \theta\epsilon_{t-1}$

We may write above model as:

$$\epsilon_t = X_t + \theta\epsilon_{t-1} = X_t + \theta(X_{t-1} + \theta\epsilon_{t-2}) = X_t + \theta X_{t-1} + \theta^2\epsilon_{t-2}$$

By repeating the substitution process, we will have

$$\epsilon_t = \sum_{j=0}^{\infty} \theta^j X_{t-j}, \text{ with } |\theta| < 1$$

Theorem 1

The MA(1) Model given by $X_t = \epsilon_t - \theta\epsilon_{t-1}$ is invertible if $|\theta| < 1$

We can justify that $|\theta| < 1$ is necessary by calculating the variance of previous equation.

$$\sigma_\epsilon^2 = Var(\epsilon_t) = Var\left(\sum_{j=0}^{\infty}\theta^j X_{t-j}\right) = \sum_{j=0}^{\infty}\theta^{2j}Var(X_{t-j}) = \gamma_0\sum_{j=0}^{\infty}\theta^{2j} = \frac{\gamma_0}{1-\theta^2}$$

Furthermore, the condition $|\theta| < 1$ guarantees the summability of $\sum_{j=0}^{\infty}\theta^j < \infty$ and

$$\sum_{j=0}^{\infty}(\theta^j)^2 < \infty$$

Corollary 2

The MA(1) process given by $X_t = \epsilon_t - \theta\epsilon_{t-1} = (1-\theta B)\epsilon_t$ is invertible if the root of MA polynomial $1-\theta z = 0$ is outside the unit circle. That is, the root satisfies $|z| > 1$.

Proof

The solution $1-\theta z = 0$ implies the solution is $z = 1/\theta$. Therefore, the invertibility condition $|\theta| < 1$ is equivalent to $|z| > 1$.

Example

Suppose that an MA(1) model is $X_t = \epsilon_t + .7\epsilon_{t-1}$, where $\epsilon_t \sim WN(0,1)$.

This is invertible since the coefficient $|\theta| = 0.7 < 1$. Therefore, we have the following invertible solution

$$\epsilon_t = \sum_{j=0}^{\infty}0.7^j X_{t-j}.$$

Invertibility of MA(q) Model

Theorem 2

Consider an MA(q) model given by $X_t = \theta(B)\epsilon_t$; $\{\epsilon_t\} \sim WN(0, \sigma_\epsilon^2)$.

If all the roots of the associated algebraic equation $\theta(z) = 0$ lie outside the unit circle (i.e. the roots $|\delta_i| > 1; i = 1, 2, \cdots, q$), then there exists a valid invertible solution.

Proof

The fundamental theorem in algebra states that any polynomial of degree q can be factorized into q linear factors such that

$$\theta(z) = (1-\beta_1 z)(1-\beta_2 z)\cdots(1-\beta_q z)$$

Here, we have $\epsilon_t = (1-\beta_1 B)^{-1}(1-\beta_2 B)^{-1}\cdots(1-\beta_q B)^{-1}X_t$.

From the previous argument, it is clear that each linear factor has a valid inversion if $|\beta_i| < 1$ for all $i = 1, 2, \cdots, q$. This is equivalent to the argument that the roots of $\theta(z) = 0$ satisfy $|z| > 1$.

This is clear since the roots of $\theta(z) = 0$ are $\frac{1}{\beta_1}, \frac{1}{\beta_2}, \cdots, \frac{1}{\beta_q}$, then $|\beta_i| < 1$ (as convergence requirement) implies the result.

Note

This invertibility of an MA model is very important in practice. The main reason is that the unobservable ϵ_t can be expressed as a linear combination of the observable X_t.

Example

Show that the MA(2) model given by $X_t = \epsilon_t + 0.5\epsilon_{t-1} + 0.06\epsilon_{t-2}$ is invertible!

Solution

Let $\theta(B) = 1 + 0.5B + 0.06B^2$. It is clear that $\theta(B) = (1 + 0.2B)(1 + 0.3B)$.

Then, the roots of $\theta(B)$ are -5 and $-10/3$, which are outside the unit circle. Hence the above MA(2) model is invertible.

Furthermore, the invertible solution is:

$$\begin{aligned}
\epsilon_t &= (1 + 0.5B + 0.06B^2)^{-1} X_t && (1) \\
&= ((1 + 0.2B)(1 + 0.3B))^{-1} X_t && (2) \\
&= \left(\frac{3}{1 + 0.3B} - \frac{2}{1 + 0.2B} \right) X_t && (3) \\
&= \sum_{j=0}^{\infty} (3 \cdot 0.3^j - 2 \cdot 0.2^j) X_t && (4)
\end{aligned}$$

Note

In the case of MA(2) with polynomial $\theta(z) = 1 + \theta_1 z + \theta_2 z^2$, it can bee seen that the polynomial has roots outside the unit circle if it satisfies all of the following :

- $|\theta_2| < 1$
- $\theta_2 + \theta_1 > -1$
- $\theta_2 - \theta_1 > -1$

Harder Example

Show whether the following MA model is invertible or not:

$X_t = \epsilon_t - 0.2\epsilon_{t-1} + 0.6\epsilon_{t-2} - 0.1\epsilon_{t-3} + 0.1\epsilon_{t-4} + 0.2\epsilon_{t-5}$

In [2]:
```python
# check the absolute value of the roots
import numpy as ny
poly = [0.2, 0.1, -0.1, 0.6, -0.2, 1] #Plynomial of MA Model above
roots = ny.roots(poly)
abs(roots)
```

Out[2]:
```
array([1.97706234, 1.36630391, 1.36630391, 1.16393173, 1.16393173])
```

Since all of roots are outside of the unit circle, therefore the MA model above is invertible.