# Electrodynamics of Pyramidal Horn

Lumos

This document is a compilation of various sections from the book *Antenna Theory: Analysis and Design* by Constantine A. Balanis. We strongly encourage readers to review the introductory chapters, as well as the chapters on aperture antennas and horn antennas, in order to gain an understanding of concepts such as radiation pattern, directivity, E-sectoral horns, and H-sectoral horns. In this document, we have derived the electric and magnetic fields inside an E-sectoral horn to provide insights into the field distributions of horn antennas. Additionally, we have consolidated the methods and approximations used to derive the results, which are scattered throughout the book.

To accurately characterize the radiation pattern and enhance gain and directivity, it is crucial to analyze aperture distributions to obtain the far-field patterns. Initially, we examine the aperture distributions of the E-sectoral horn to gain insights into the aperture distributions of the pyramidal horn.

### E- Sectoral Horn Aperture Distributions

In the study of electromagnetic waves inside the horn, we begin by examining the aperture distribution of fields. This involves analyzing the spatial variation of electric and magnetic fields at the aperture in cylindrical coordinates $\rho, \phi, x$.
Since the region within the horn is source-free, Maxwell's equations must be satisfied, leading to the following general equations *(time-harmonic fields)*:

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \tag{1}$$

$$\nabla \times \vec{H} = j\omega\epsilon\vec{E} \tag{2}$$

Maxwell's equations can be expressed in cylindrical coordinates $\rho, \psi, x$. The electric and magnetic field components transform as follows (cylindrical coordinates):

$$j\omega\epsilon E_\rho = \frac{1}{\rho}\frac{\partial H_x}{\partial \psi} - \frac{\partial H_\psi}{\partial x} \tag{3}$$

$$j\omega\epsilon E_\psi = \frac{\partial H_\rho}{\partial x} - \frac{\partial H_x}{\partial \rho} \tag{4}$$

$$j\omega\epsilon E_x = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\psi) - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \psi} \tag{5}$$

$$-j\omega\mu H_\rho = \frac{1}{\rho}\frac{\partial E_x}{\partial \psi} - \frac{\partial E_\psi}{\partial x} \tag{6}$$

$$-j\omega\mu H_\psi = \frac{\partial E_\rho}{\partial x} - \frac{\partial E_x}{\partial \rho} \tag{7}$$

$$-j\omega\mu H_x = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\psi) - \frac{1}{\rho}\frac{\partial E_\rho}{\partial \psi} \tag{8}$$

If we assume that the waveguide feeding the horn only supports the dominant $TE_{10}$ mode, the lowest order mode within the sectoral guide (horn) is that which is analogous to the $TE_{10}$ mode of the rectangular guide, with all the other modes attenuated in the transition region (throat) between the waveguide and the horn. Thus the dominant mode within the horn is one whose only non-vanishing components are $E_\psi, H_\rho, H_x$. *(The assumption comes from assuming the flaring to be similar to a radial waveguide - This is mentioned in appendix)* Thus,

$$E_\psi = E_x = H_\psi = 0 \tag{9}$$

Using (9), we can write (6) and (8) as:

$$j\omega\mu H_\rho = \frac{\partial E_\psi}{\partial x} \tag{11}$$

$$-j\omega\mu H_x = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\psi) \tag{12}$$

Substituting (11) and (12) into (4), we can write,

$$-\omega^2\mu\epsilon E_\psi = \frac{\partial^2 E_\psi}{\partial x^2} + \frac{\partial}{\partial \rho}\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\psi)\right] \tag{13}$$

which when expanded can be written as:

$$\frac{\partial^2 E_\psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_\psi}{\partial \rho} + \frac{\partial^2 E_\psi}{\partial x^2} + \left(k^2 - \frac{1}{\rho^2}\right)E_\psi = 0 \tag{14}$$

where,

$$k^2 = \omega^2\mu\epsilon \tag{14a}$$

To solve (14), we use the method of separation of variables. We assume that:

$$E_\psi(\rho, x) = R(\rho)X(x) \tag{15}$$

Substituting (15) into (14) leads to,

$$X\left(\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho}\right) + R\frac{d^2X}{dx^2} + \left(k^2 - \frac{1}{\rho^2}\right)RX = 0 \tag{16}$$

Dividing by $RX$ and changing the partials to total derivatives, (16) reduces to,

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2, \quad k_x^2 = \text{constant} \tag{17}$$

$$\frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{R\rho}\frac{dR}{d\rho} + \left(k^2 - k_x^2 - \frac{1}{\rho^2}\right) = 0 \tag{18}$$

Multiplying (18) by $\rho^2 R$ reduces to,

$$\rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + \left[(k_\rho^2\rho^2) - 1\right]R = 0 \tag{19}$$

where,

$$k_\rho^2 = k^2 - k_x^2 \tag{19a}$$

Equation (19) is recognized as a special form ($n = 1$) of Bessel's differential equation with a solution of the form:

$$R(\rho) = AH_1^{(2)}(k_\rho\rho) + BH_1^{(1)}(k_\rho\rho) \tag{20}$$

where $A$ and $B$ are constants. The Hankel functions of the first and second kind of order one ($n = 1$) were chosen as solutions because they represent traveling waves in the inward and outward, respectively, radial directions. The solution of (17) is of the form:

$$X(x) = C\cos(k_x x) + D\sin(k_x x) \tag{21}$$

where $C$ and $D$ are constants. Using (20) and (21), we can write (15) as,

$$E_\psi(\rho, x) = [AH_1^{(2)}(k_\rho\rho) + BH_1^{(1)}(k_\rho\rho)][C\cos(k_x x) + D\sin(k_x x)] \tag{22}$$

Applying the boundary conditions,

$$E_\psi(\rho, x = a/2) = E_\psi(\rho, x = -a/2) = 0 \tag{23}$$

leads to,

$$C\cos\left(\frac{k_x a}{2}\right) + D\sin\left(\frac{k_x a}{2}\right) = 0 \tag{24}$$

$$C\cos\left(\frac{k_x a}{2}\right) - D\sin\left(\frac{k_x a}{2}\right) = 0 \tag{25}$$

$$\Rightarrow D = 0, \quad k_x = \frac{m\pi}{a}, \quad m = 1, 2, 3, \ldots$$

Equation (22) can be rewritten as,

$$E_\psi(\rho, x) = A_m\cos(k_x x)\left[H_1^{(2)}(k_\rho\rho) + \alpha_m H_1^{(1)}(k_\rho\rho)\right] \tag{26}$$

where $A_m$ and $\alpha_m$ are constants for the modes $m = 1, 3, 5, \ldots$
The non-vanishing magnetic field components can be obtained from (11) and (12). That is,

$$H_\rho(\rho, x) = \frac{1}{j\omega\mu}\frac{\partial E_\psi}{\partial x} = \frac{k_x}{j\omega\mu} A_m \sin(k_x x)\left[H_1^{(2)}(k_\rho\rho) + \alpha_m H_1^{(1)}(k_\rho\rho)\right] \tag{27}$$

$$H_x(\rho, x) = \frac{1}{-j\omega\mu}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\psi) = j\frac{k_\rho}{\omega\mu} A_m \cos(k_x x)\left[H_0^{(2)}(k_\rho\rho) + \alpha_m H_0^{(1)}(k_\rho\rho)\right] \tag{28}$$

If we consider only the lowest order mode ($m = 1$, $k_x = \pi/a$) and no reflected component $[\alpha_m H_1^{(1)}(k_\rho\rho) = \alpha_m H_0^{(1)}(k_\rho\rho) = 0]$, the fields within the horn can be written as:

$$E_\rho = E_x = H_\psi = 0 \tag{29a}$$

$$E_\psi(\rho, x) = A_1 \cos\left(\frac{\pi x}{a}\right) H_1^{(2)}(k_\rho\rho) \tag{29b}$$

$$H_\rho(\rho, x) = \frac{1}{j\omega\mu} A_1 \sin\left(\frac{\pi x}{a}\right) H_1^{(2)}(k_\rho\rho) \tag{29c}$$

$$H_x(\rho, x) = j\frac{k_\rho}{\omega\mu} A_1 \cos\left(\frac{\pi x}{a}\right) H_0^{(2)}(k_\rho\rho) \tag{29d}$$

$$k_\rho = \left[k^2 - \left(\frac{\pi}{a}\right)^2\right]^{1/2} = k\left[1 - \left(\frac{f_c}{f}\right)^2\right]^{1/2} \tag{29e}$$

$$f_c = \frac{c}{2a}, \quad k = \frac{\omega\mu}{\eta} \tag{29f}$$

The cylindrical components $E_\psi(\rho, x)$ and $H_\rho(\rho, x)$ can be resolved to any point within the horn to their rectangular counterparts. Thus,

$$E_z = -E_\psi \sin\psi, \quad H_z = H_\rho \cos\psi \tag{30}$$

$$E_y = E_\psi \cos\psi, \quad H_y = H_\rho \sin\psi \tag{31}$$

which for horns with small flares ($\psi$ small) reduce to,

$$E_z = H_y = 0 \tag{32}$$

$$E_y \simeq E_\psi, \quad H_z \simeq H_\rho \tag{33}$$

If the length of the horn is large, the Hankel functions can be approximated by their asymptotic expansions, or:

$$H_1^{(2)}(k_\rho\rho) \simeq \sqrt{\frac{2}{\pi k_\rho\rho}} e^{-j(k_\rho\rho - \pi/4)}, \quad H_0^{(2)}(k_\rho\rho) \simeq \sqrt{\frac{2}{\pi k_\rho\rho}} e^{-j(k_\rho\rho - \pi/4)} \tag{34}$$

where,

$$\rho = (y^2 + z^2)^{1/2} \tag{34a}$$

Choosing a new coordinate system $(x'; y'; z')$ *(at aperture of horn)*, such that

$$x' = x, \quad y' = y, \quad z' = z - \rho_1, \quad \rho_1 = \rho \cos \psi_1 \tag{35}$$

we can write $\rho$ of (34a) as,

$$\rho = (z^2 + y'^2)^{1/2} = [\rho_1^2 + y'^2]^{1/2} \textit{(for z' = 0)} = \rho_1 \left[ 1 + \frac{1}{2} \left( \frac{y'}{\rho_1} \right)^2 - \frac{1}{8} \left( \frac{y'}{\rho_1} \right)^4 + \dots \right] \tag{36}$$

For narrow horns ($y' \ll \rho_1$):

$$\rho \simeq \rho_1 \left[ 1 + \frac{1}{2} \left( \frac{y'}{\rho_1} \right)^2 \right] \text{ for phase terms} \tag{37a}$$

$$\rho \simeq \rho_1 \text{ for amplitude terms} \tag{37b}$$

Using (30)-(37), we can write (29a)-(29b) as:

$$E'_z = F'_x = H'_y = 0$$

$$E_1 = jA_1 \sqrt{\frac{2}{\pi k_\rho \rho_1}} e^{-jk\rho_1} \tag{38}$$

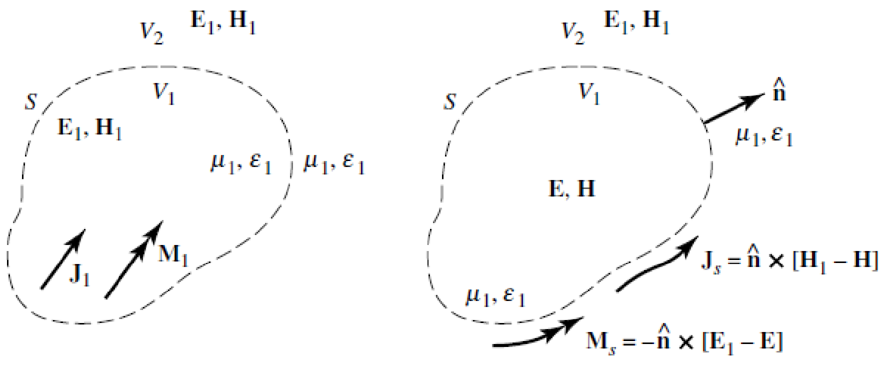
$$E_y(x', y') = E_1 \cos \left( \frac{\pi x'}{a} \right) e^{-j\frac{k}{2}\left(\frac{y'^2}{\rho_1^2}\right)} \tag{39}$$

$$H'_x(x', y') \simeq jE_1 \left( \frac{\pi \eta}{ka} \right) \sin \left( \frac{\pi x'}{a} \right) e^{-j\frac{k}{2}\left(\frac{y'^2}{\rho_1^2}\right)} \tag{40}$$

$$H'_z(x', y') \simeq \frac{E_1}{\eta} \cos \left( \frac{\pi x'}{a} \right) e^{-j\frac{k}{2}\left(\frac{y'^2}{\rho_1^2}\right)} \tag{41}$$

## Equivalent Densities J & M

The radiation characteristics of wire antennas can be determined once the current distribution on the wire is known. For many configurations, however, the current distribution is not known exactly, and only physical intuition or experimental measurements can provide a reasonable approximation to it. This is even more evident in aperture antennas (slits, slots, waveguides, horns, reflectors, lenses). It is therefore expedient to have alternate methods to compute the radiation characteristics of antennas. Emphasis will be placed on techniques that for their solution rely primarily not on the current distribution but on reasonable approximations of the fields on or in the vicinity of the antenna structure. One such technique is the Field Equivalence Principle.




(a) Actual problem (b) Equivalent problem

## Vector Potentials & Far-Field Approximation

The Vector potential *(A & F)* are related to the current densities as follows:

$$\mathbf{A} = \frac{\mu}{4\pi} \iint_S \mathbf{J}_s(x', y', z') \frac{e^{-jkR}}{R} \, ds' \tag{1}$$

$$\mathbf{F} = \frac{\epsilon}{4\pi} \iint_S \mathbf{M}_s(x', y', z') \frac{e^{-jkR}}{R} \, ds' \tag{2}$$

where $R$ is the distance from any point in the source to the observation point and the primed coordinates denote the frame attached to the aperture of the antenna.
The total fields are then determined by: *(under Lorentz gauge)*

$$\boxed{\mathbf{E} = \mathbf{E}_A + \mathbf{E}_F = -j\omega\mathbf{A} - \frac{j}{\omega\mu\epsilon}\nabla(\nabla \cdot \mathbf{A}) - \frac{1}{\epsilon}\nabla \times \mathbf{F}} \tag{3}$$

or

$$\boxed{\mathbf{E} = \mathbf{E}_A + \mathbf{E}_F = \frac{1}{j\omega\epsilon}\nabla \times \mathbf{H}_A - \frac{1}{\epsilon}\nabla \times \mathbf{F}} \tag{4}$$

and

$$\boxed{\mathbf{H} = \mathbf{H}_A + \mathbf{H}_F = \frac{1}{\mu}\nabla \times \mathbf{A} - j\omega\mathbf{F} - \frac{j}{\omega\mu\epsilon}\nabla(\nabla \cdot \mathbf{F})} \tag{5}$$

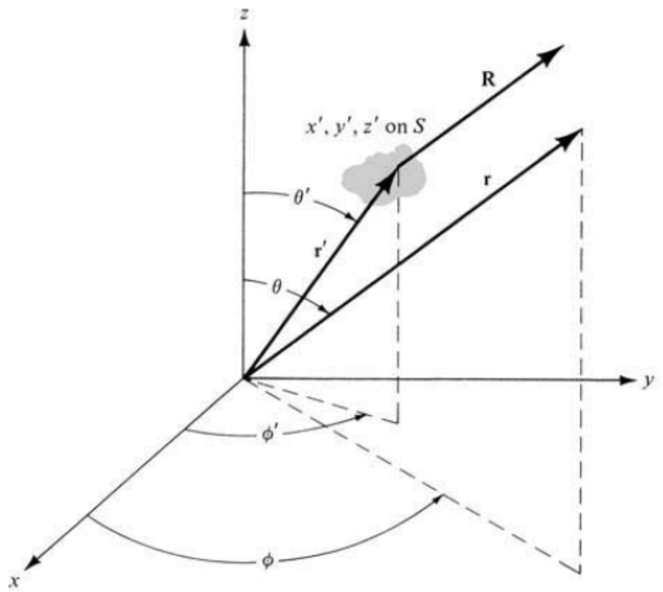
or

$$\boxed{\mathbf{H} = \mathbf{H}_A + \mathbf{H}_F = \frac{1}{\mu}\nabla \times \mathbf{A} - \frac{1}{j\omega\mu}\nabla \times \mathbf{E}_F} \tag{6}$$

Since the HI clouds are present in the interstellar medium we can reduce the complexity of the formulation using far-field approximation which is:

$$R \simeq r - r'\cos\psi \quad \text{for phase variations} \tag{3}$$

$$R \simeq r \quad \text{for amplitude variations} \tag{4}$$




where $\psi$ is the angle between the vectors $\mathbf{r}$ and $\mathbf{r}'$, as shown in the figure . The primed coordinates $(x', y', z', r', \theta', \phi')$ indicate the space occupied by the sources $\mathbf{J}_s$ and $\mathbf{M}_s$, over which integration must be performed. The unprimed coordinates $(x, y, z, r, \theta, \phi)$ represent the observation point. Geometrically, the approximation of (1) & (2) assumes that the vectors $\mathbf{R}$ and $\mathbf{r}$ are parallel, as shown in the figure. The vector potentials, using the approximation can be written as:

$$\mathbf{A} = \frac{\mu}{4\pi}\iint_S \mathbf{J}_s \frac{e^{-jkR}}{R}\, ds' \simeq \frac{\mu e^{-jkr}}{4\pi r}\mathbf{N} \tag{5}$$

$$\mathbf{N} = \iint_S \mathbf{J}_s e^{jkr'\cos\psi}\, ds' \tag{6}$$

$$\mathbf{F} = \frac{\epsilon}{4\pi}\iint_S \mathbf{M}_s \frac{e^{-jkR}}{R}\, ds' \simeq \frac{\epsilon e^{-jkr}}{4\pi r}\mathbf{L} \tag{7}$$

$$\mathbf{L} = \iint_S \mathbf{M}_s e^{jkr'\cos\psi}\, ds' \tag{8}$$

For far-field the $\theta$ and $\phi$ components of the $\mathbf{E}$ and $\mathbf{H}$ fields are dominant. Although the radial components are not necessarily zero, they are negligible compared to the $\theta$ and $\phi$ components. Therefore the components E and H fields are:

$$\begin{aligned}(E_A)_\theta &\simeq -j\omega A_\theta\\ (E_A)_\phi &\simeq -j\omega A_\phi\\ (H_F)_\theta &\simeq -j\omega F_\theta\\ (H_F)_\phi &\simeq -j\omega F_\phi\end{aligned}$$

$$\begin{aligned}(E_F)_\theta &\simeq +\eta(H_F)_\phi = -j\omega\eta F_\phi\\ (E_F)_\phi &\simeq -\eta(H_F)_\theta = +j\omega\eta F_\theta\\ (H_A)_\theta &\simeq \frac{(E_A)_\phi}{\eta} = +\frac{j\omega}{\eta}A_\phi\\ (H_A)_\phi &\simeq -\frac{(E_A)_\theta}{\eta} = -\frac{j\omega}{\eta}A_\theta\end{aligned}$$

Therefore, the total **E**- and **H**-fields can be written as:

$$E_r \simeq 0 \qquad\qquad H_r \simeq 0$$
$$E_\theta \simeq -\frac{jke^{-jkr}}{4\pi r}(L_\phi + \eta N_\phi) \qquad\qquad H_\theta \simeq +\frac{jke^{-jkr}}{4\pi r}\left(\frac{N_\phi \eta - L_\phi}{\eta}\right)$$
$$E_\phi \simeq +\frac{jke^{-jkr}}{4\pi r}(L_\theta - \eta N_\theta) \qquad\qquad H_\phi \simeq -\frac{jke^{-jkr}}{4\pi r}\left(\frac{N_\theta \eta + L_\theta}{\eta}\right)$$

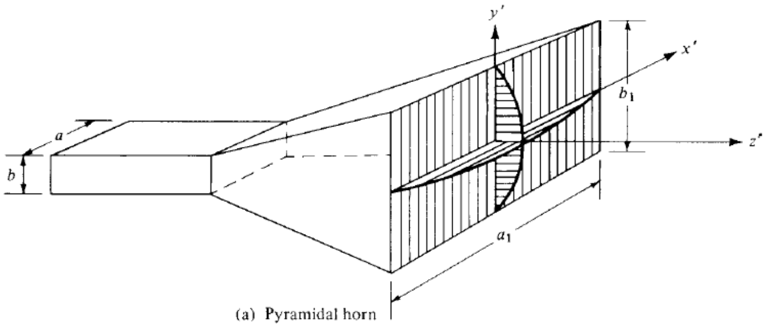
The $N_\theta$, $N_\phi$, $L_\theta$, and $L_\phi$ can be obtained from (5) and (6). That is,

$$\mathbf{N} = \iint_S \mathbf{J}_s e^{jkr' \cos\psi}\, ds' = \iint_S (\hat{a}_x J_x + \hat{a}_y J_y + \hat{a}_z J_z)\, e^{jkr' \cos\psi}\, ds' \tag{1}$$
$$\mathbf{L} = \iint_S \mathbf{M}_s e^{jkr' \cos\psi}\, ds' = \iint_S (\hat{a}_x M_x + \hat{a}_y M_y + \hat{a}_z M_z)\, e^{jkr' \cos\psi}\, ds' \tag{2}$$

Using the rectangular-to-spherical component transformation, $N$ and $L$ are reduced as follows:

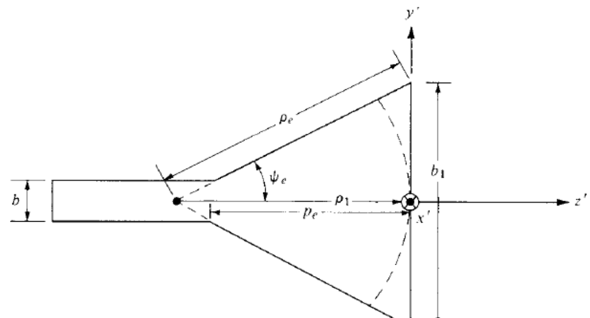
$$N_\theta = \iint_S [J_x \cos\theta \cos\phi + J_y \cos\theta \sin\phi - J_z \sin\theta]\, e^{jkr' \cos\psi}\, ds'$$
$$N_\phi = \iint_S [-J_x \sin\phi + J_y \cos\phi]\, e^{jkr' \cos\psi}\, ds'$$
$$L_\theta = \iint_S [M_x \cos\theta \cos\phi + M_y \cos\theta \sin\phi - M_z \sin\theta]\, e^{jkr' \cos\psi}\, ds'$$
$$L_\phi = \iint_S [-M_x \sin\phi + M_y \cos\phi]\, e^{jkr' \cos\psi}\, ds'$$

## Aperture Distribution for Pyramidal Horn Antenna and Radiated Fields

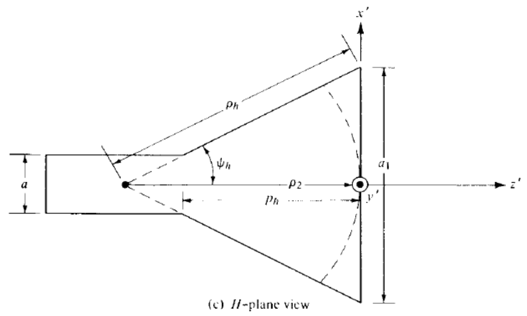


(a) Pyramidal horn

To simplify the analysis and to maintain modeling that leads to computations that have been shown to correlate well with experimental data, the tangential components of the $E$- and $H$-fields over the aperture of the horn are approximated by:



(b) $E$-plane view



(c) $H$-plane view

$$E'_y(x', y') = E_0 \cos\left(\frac{\pi}{a_1} x'\right) e^{-jk\left(\frac{x'^2}{\rho_2} + \frac{y'^2}{\rho_1}\right)/2} \tag{1}$$

$$H'_x(x', y') = -\frac{E_0}{\eta} \cos\left(\frac{\pi}{a_1} x'\right) e^{-jk\left(\frac{x'^2}{\rho_2} + \frac{y'^2}{\rho_1}\right)/2} \tag{2}$$

and the equivalent current densities by:

$$J'_y(x', y') = -\frac{E_0}{\eta} \cos\left(\frac{\pi}{a_1} x'\right) e^{-jk\left(\frac{x'^2}{\rho_2} + \frac{y'^2}{\rho_1}\right)/2} \tag{3}$$

$$M'_x(x', y') = E_0 \cos\left(\frac{\pi}{a_1} x'\right) e^{-jk\left(\frac{x'^2}{\rho_2} + \frac{y'^2}{\rho_1}\right)/2} \tag{4}$$

where $\eta = \sqrt{\frac{\mu}{\epsilon}}$ is the intrinsic impedance of the medium. The above expressions contain a cosinusoidal amplitude distribution in the $x'$ direction and quadratic phase variations in both the $x'$ and $y'$ directions, similar to those of the sectoral $E$- and $H$-plane horns. The $N_\theta$, $N_\phi$, $L_\theta$, and $L_\phi$ can now be formulated as derived, and they are given by:

$$N_\theta = -\frac{E_0}{\eta} \cos\theta \sin\phi \, I_1 I_2 \tag{5}$$

$$N_\phi = -\frac{E_0}{\eta} \cos\phi \, I_1 I_2 \tag{6}$$

$$L_\theta = E_0 \cos\theta \cos\phi \, I_2 \tag{7}$$

$$L_\phi = -E_0 \sin\phi \, I_1 I_2 \tag{8}$$

where,

$$I_1 = \int_{-a/2}^{+a/2} \cos\left(\frac{\pi}{a}x'\right) e^{-jk\left(x'^2/2\rho_1 - x' \sin\cos\phi\right)} dx' \tag{9}$$

$$I_2 = \int_{-b/2}^{+b/2} e^{-jk\left(y'^2/2\rho_1 - y' \sin\sin\phi\right)} dy' \tag{10}$$

Using Fresnel Integrations,

$$\begin{aligned} I_1 = \frac{1}{2}\sqrt{\frac{\pi\rho_2}{k}}\Big[ & e^{j({k'_x}^2\rho_2/2k)} \left(C(t'_2) - C(t'_1)\right) - j\left(S(t'_2) - S(t'_1)\right) \\ & + e^{j(k"^2_x\rho_2/2k)} \left(C(t''_2) - C(t''_1)\right) - j\left(S(t''_2) - S(t''_1)\right)\Big] \end{aligned} \tag{11}$$

where the fresnel integrals are as follows:

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt \tag{12a}$$

$$S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt \tag{12b}$$

where $t'_1$, $t'_2$, $k'_1$, $t''_1$, $t''_2$, $k''_1$ are given by:

$$t'_1 = \sqrt{\frac{1}{\pi k\rho_2}}\left(-\frac{ka_1}{2} - k'_x\rho_2\right) \tag{3}$$

$$t'_2 = \sqrt{\frac{1}{\pi k\rho_2}}\left(+\frac{ka_1}{2} - k'_x\rho_2\right) \tag{4}$$

$$k'_x = k\sin\theta\cos\phi + \frac{\pi}{a_1} \tag{5}$$

$$t''_1 = \sqrt{\frac{1}{\pi k\rho_2}}\left(-\frac{ka_1}{2} - k''_x\rho_2\right) \tag{6}$$

$$t''_2 = \sqrt{\frac{1}{\pi k\rho_2}}\left(+\frac{ka_1}{2} - k''_x\rho_2\right) \tag{7}$$

$$k''_x = k\sin\theta\cos\phi - \frac{\pi}{a_1} \tag{8}$$

$$I_2 = \sqrt{\frac{\pi\rho_1}{k}} e^{j(k_y^2\rho_1/2k)} \left[\left(C(t_2) - C(t_1)\right) - j(S(t_2) - S(t_1))\right] \tag{19}$$

where $k_y$, $t_1$, $t_2$ are given by:

$$k_y = k \sin\theta \sin\phi$$

$$t_1 = \sqrt{\frac{1}{\pi k \rho_1}} \left( -\frac{k b_1}{2} - k_y \rho_1 \right)$$

$$t_2 = \sqrt{\frac{1}{\pi k \rho_1}} \left( \frac{k b_1}{2} - k_y \rho_1 \right)$$

Therefore the $E$ and $H$ field components at the aperture are:

$$E_r = 0 \tag{20}$$

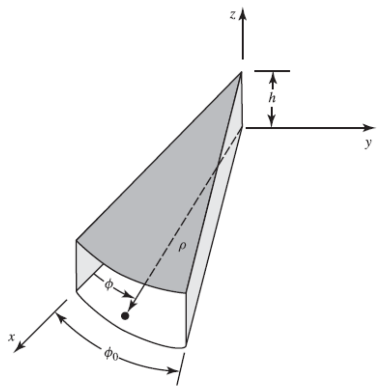
$$\begin{aligned} E_\theta &= -\frac{jke^{jkr}}{4\pi r} \left[ L_\theta + \eta N_\theta \right] \\ &= \frac{jkE_0 e^{-jkr}}{4\pi r} \left[ \sin\phi (1 + \cos\theta) I_1 I_2 \right] \end{aligned} \tag{21}$$

$$\begin{aligned} E_\phi &= \frac{jke^{-jkr}}{4\pi r} \left[ L_\phi - \eta N_\phi \right] \\ &= \frac{jkE_0 e^{-jkr}}{4\pi r} \left[ \cos\phi (\cos\theta + 1) I_1 I_2 \right] \end{aligned} \tag{22}$$

## Appendix: Radial Waveguide Approximation

A radial type of waveguide structure is the wedged-plate geometry of the figure below with plates along $z = 0, h$, and $\phi = 0, \phi_0$. This type of configuration resembles and can be used to represent the structures of $E$- and $H$-plane sectoral horns. In fact, the fields within the horns are found using the procedure outlined here. In general, both $\mathrm{TE}^z$ and $\mathrm{TM}^z$ modes can exist in the space between the plates. The independent sets of boundary conditions of this structure that can be used to solve for the $\mathrm{TE}^z_{\rho n}$ and $\mathrm{TM}^z_{\rho n}$ modes are:

$$\begin{aligned} E_\rho(0 \le \rho \le \infty,\ 0 \le \phi \le \phi_0,\ z = 0) &= E_\rho(0 \le \rho \le \infty,\ 0 \le \phi \le \phi_0,\ z = h) = 0 \\ E_\rho(0 \le \rho \le \infty,\ \phi = 0,\ 0 \le z \le h) &= E_\rho(0 \le \rho \le \infty,\ \phi = \phi_0,\ 0 \le z \le h) = 0 \\ E_\phi(0 \le \rho \le \infty,\ 0 \le \phi \le \phi_0,\ z = 0) &= E_\phi(0 \le \rho \le \infty,\ 0 \le \phi \le \phi_0,\ z = h) = 0 \\ E_z(0 \le \rho \le \infty,\ \phi = 0,\ 0 \le z \le h) &= E_z(0 \le \rho \le \infty,\ \phi = \phi_0,\ 0 \le z \le h) = 0 \end{aligned}$$



Transverse Electric (TE$^z$) Modes can be derived using Maxwell's Equations stated above, the results of this set of TE$^z_{pn}$ modes will be summarized here. Only the outward radial ($+\rho$) parts will be included here.

$$
\begin{aligned}
F_z^+(\rho,\phi,z) &= A_{pn}H_m^{(2)}(\beta_\rho\rho)\cos(m\phi)\sin(\beta_z z)\\
\beta_\rho^2+\beta_z^2 &= \beta^2\\
\beta_z &= \frac{n\pi}{h},\quad n=0,1,2,\dots\\
m &= \frac{p\pi}{\phi_0},\quad p=1,2,3,\dots
\end{aligned}
$$

$$
\begin{gathered}
E^+\rho = -\frac{1}{\varepsilon}\frac{1}{\rho}\frac{\partial F_z^+}{\partial\phi} = A_{pn}\frac{p\pi}{\phi_0\varepsilon\rho}H_m^{(2)}(\beta_\rho\rho)\sin\left(\frac{p\pi}{\phi_0}\phi\right)\sin\left(\frac{n\pi}{h}z\right)\\
E^+\phi = \frac{1}{\varepsilon}\frac{\partial F_z^+}{\partial\rho} = \beta\rho\frac{A_{pn}}{\varepsilon}H_m^{(2)\prime}(\beta_\rho\rho)\cos\left(\frac{p\pi}{\phi_0}\phi\right)\sin\left(\frac{n\pi}{h}z\right)\\
E_z^+ = 0\\
H^+\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2F_z^+}{\partial\rho\partial z} = -jA_{pn}\frac{\beta\rho\beta_z}{\omega\mu\varepsilon}H_m^{(2)\prime}(\beta_\rho\rho)\cos\left(\frac{p\pi}{\phi_0}\phi\right)\cos\left(\frac{n\pi}{h}z\right)\\
H^+\phi = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2F_z^+}{\partial\phi\partial z} = jA_{pn}\frac{\beta_z p\pi/\phi_0}{\omega\mu\varepsilon}\frac{1}{\rho}H_m^{(2)}(\beta\rho\rho)\sin\left(\frac{p\pi}{\phi_0}\phi\right)\cos\left(\frac{n\pi}{h}z\right)\\
H^+z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)F_z^+ = -jA_{pn}\frac{\beta_\rho^2}{\omega\mu\varepsilon}H_m^{(2)}(\beta_\rho\rho)\cos\left(\frac{p\pi}{\phi_0}\phi\right)\sin\left(\frac{n\pi}{h}z\right)\\
Z_w^{+\rho}(TE^z_{pn}) = \frac{E_\phi^+}{H_z^+} = j\frac{\omega\mu H_m^{(2)\prime}(\beta_\rho\rho)}{\beta_\rho H_m^{(2)}(\beta_\rho\rho)},\quad {}^\prime\equiv\frac{\partial}{\partial(\beta_\rho\rho)}
\end{gathered}
$$

Transverse Magnetic ($TM^z$) Modes can also be derived from the Maxwell's Equations. Therefore, the results will be summarized here. Only the outward radial ($+\rho$) parts will be included here.

$$
\begin{gathered}
A_z^+(\rho,\phi,z) = B_{pn}H_m^{(2)}(\beta_\rho\rho)\sin(m\phi)\cos(\beta_z z)\\
\beta_\rho^2+\beta_z^2=\beta^2\\
\beta_z=\frac{n\pi}{h},\quad n=0,1,2,\dots\\
m=\frac{p\pi}{\phi_0},\quad p=1,2,3,\dots
\end{gathered}
$$

$$E_\rho^+ = -\frac{j}{\omega\mu\epsilon}\frac{\partial^2 A_z^+}{\partial\rho\partial z} = j\beta_\rho\beta_z\frac{B_{pn}}{\omega\mu\epsilon}H_m^{(2)'}(\beta_\rho\rho)\sin\left(\frac{p\pi}{\phi_0}\phi\right)\sin\left(\frac{n\pi}{h}z\right)$$

$$E_\phi^+ = -\frac{j}{\omega\mu\epsilon\rho}\frac{\partial^2 A_z^+}{\partial\phi\partial z} = j\frac{\beta_z p\pi}{\phi_0\rho}\frac{B_{pn}}{\omega\mu\epsilon}H_m^{(2)}(\beta_\rho\rho)\cos\left(\frac{p\pi}{\phi_0}\phi\right)\sin\left(\frac{n\pi}{h}z\right)$$

$$E_z^+ = -\frac{j}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)A_z^+ = -j\frac{\beta_\rho^2}{\omega\mu\epsilon}B_{pn}H_m^{(2)}(\beta_\rho\rho)\sin\left(\frac{p\pi}{\phi_0}\phi\right)\cos\left(\frac{n\pi}{h}z\right)$$

$$H_\rho^+ = \frac{1}{\mu}\frac{1}{\rho}\frac{\partial A_z^+}{\partial\phi} = \frac{B_{pn}p\pi/\phi_0}{\mu\rho}H_m^{(2)}(\beta_\rho\rho)\cos\left(\frac{p\pi}{\phi_0}\phi\right)\cos\left(\frac{n\pi}{h}z\right)$$

$$H_\phi^+ = -\frac{1}{\mu}\frac{\partial A_z^+}{\partial\rho} = -\frac{\beta_\rho B_{pn}}{\mu}H_m^{(2)'}(\beta_\rho\rho)\sin\left(\frac{p\pi}{\phi_0}\phi\right)\cos\left(\frac{n\pi}{h}z\right)$$

$$H_z^+ = 0$$

$$Z_w^{+\rho}(\mathrm{TM}_{pn}^z) = \frac{E_z^+}{-H_\phi^+} = -j\frac{\beta_\rho}{\omega\epsilon}\frac{H_m^{(2)}(\beta_\rho\rho)}{H_m^{(2)'}(\beta_\rho\rho)}, \quad {}' \equiv \frac{\partial}{\partial(\beta_\rho\rho)}.$$