Origin of 21cm HI Line

Lumos

The 21cm emission arises due to the spin-spin coupling between the electron and proton. The electron is in orbit around the nucleus. The proton sets up a magnetic field **B** due to it's spin magnetic moment, which interacts with the spin magnetic moment of the electron and thus exerts a torque on the spinning electron, tending to align its spin magnetic moment (**m**) along the direction of the field resulting into the splitting of the ground state energy level because of this coupling. The magnetic field of an ideal dipole can thus be written as:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right] + \frac{2}{3}\mu_0 \mathbf{m}\delta^3(\mathbf{r})$$

The Perturbing Hamiltonian is:

$$H = -\mathbf{m}\cdot\mathbf{B} \tag{1}$$

In particular, the energy of one magnetic dipole ($\mathbf{m}_1$) in the field of another magnetic dipole ($\mathbf{m}_2$) is:

$$= -\mathbf{m}_1 \cdot \left(\frac{\mu_0}{4\pi r^3}\left[3(\mathbf{m}_2\cdot\hat{r})\hat{r} - \mathbf{m}_2\right] - \frac{2}{3}\mu_0\mathbf{m}_2\delta^3(r)\right) \tag{2}$$

$$= -\frac{\mu_0}{4\pi r^3}\left[3(\mathbf{m}_1\cdot\hat{r})(\mathbf{m}_2\cdot\hat{r}) - \mathbf{m}_1\cdot\mathbf{m}_2\right] - \frac{2}{3}\mu_0(\mathbf{m}_1\cdot\mathbf{m}_2)\delta^3(r) \tag{3}$$

$$\begin{aligned} H &= -\frac{\mu_0}{4\pi r^3}\left[3(m_1\cos\theta)(m_2\cos\theta) - \mathbf{m}_1\cdot\mathbf{m}_2\right] - \frac{2}{3}\mu_0(\mathbf{m}_1\cdot\mathbf{m}_2)\delta^3(r) \\ &= -\frac{\mu_0}{4\pi r^3}\left[\mathbf{m}_1\cdot\mathbf{m}_2\left(3\cos^2\theta - 1\right)\right] - \frac{2}{3}\mu_0(\mathbf{m}_1\cdot\mathbf{m}_2)\delta^3(r) \end{aligned} \tag{4}$$

where r is their separation. The formula is symmetric in treating $\mathbf{m}_1$ and $\mathbf{m}_2$, as it should be; it represents the energy of interaction of the two dipoles. In most applications, $\mathbf{m}_1$ and $\mathbf{m}_2$ are physically separated, and the delta-function term can be ignored; however, it is precisely this part which accounts for hyperfine splitting in the ground state of hydrogen. (m_1 and m_2 represent the magnetic moment of electron and proton respectively.)

In first-order perturbation theory, a quantum state's energy change is given by the expectation value of the perturbing Hamiltonian ($H = -m.B$). The ground-state wave function for atomic hydrogen is:

$$\psi_0 = (\pi a^3)^{-\frac{1}{2}} e^{-r/a} \left|s\right\rangle \tag{5}$$

where $a = 0.5291770$ Å is the Bohr radius, and $|s\rangle$ denotes the spin of the electron.
Treating the dipole-dipole interaction as a perturbation, the energy of the ground state

is shifted by an amount.

$$
\begin{aligned}
E &= \int \psi_0^* H \psi_0 d\tau \\
&= \int \psi_0^* \left[-\frac{\mu_0}{4\pi r^3} (\mathbf{m}_1 \cdot \mathbf{m}_2) \left(3\cos^2\theta - 1 \right) - \frac{2}{3}\mu_0 (\mathbf{m}_1 \cdot \mathbf{m}_2) \delta^3(r) \right] \psi_0 d\tau \\
&= -\frac{\mu_0}{4\pi} (\mathbf{m}_1 \cdot \mathbf{m}_2) \int \frac{1}{r^3} \left(3\cos^2\theta - 1 \right) |\psi_0|^2 \, d\tau - \frac{2}{3}\mu_0 (\mathbf{m}_1 \cdot \mathbf{m}_2) \int \delta^3(r) \, |\psi_0|^2 \, d\tau. \qquad (6)
\end{aligned}
$$

Because ψ_0 (and indeed any $l = 0$ state) is spherically symmetric, the θ integral gives zero. Accordingly,

$$
E = -\frac{\mu_0}{4\pi} (\mathbf{m}_1 \cdot \mathbf{m}_2) \int |\psi_0|^2 \, \frac{1}{r^3} \left(3\cos^2\theta - 1 \right) dr = 0 \qquad (7)
$$

Thus,

$$
\begin{aligned}
E &= -\frac{2}{3}\mu_0 (\mathbf{m}_1 \cdot \mathbf{m}_2) \int |\psi_0|^2 \, \delta^3(r) d\tau \qquad (8) \\
&= -\frac{2}{3}\mu_0 (\mathbf{m}_1 \cdot \mathbf{m}_2) \frac{1}{\pi a^3} \int \delta^3(r) dr \qquad (9) \\
&= -\frac{2}{3}\mu_0 (\mathbf{m}_1 \cdot \mathbf{m}_2) \frac{1}{\pi a^3}. \qquad (10) \\
E &= -\frac{2}{3}\frac{\mu_0}{\pi a^3} (\mathbf{m}_1 \cdot \mathbf{m}_2) \qquad (11)
\end{aligned}
$$

Here $\mathbf{m}_1$ is the magnetic dipole moment of the proton and $\mathbf{m}_2$ is that of the electron; they are proportional to the respective spins:

$$
\mathbf{m}_1 = \gamma_p \mathbf{S}_p, \quad \mathbf{m}_2 = -\gamma_e \mathbf{S}_e \qquad (12)
$$

where γ are the two gyromagnetic ratios. Thus,

$$
E = \left(\frac{2}{3}\frac{\mu_0}{\pi a^3} \right) \gamma_p \gamma_e (\mathbf{S}_e \cdot \mathbf{S}_p) \qquad (13)
$$

In the presence of such "spin-spin coupling," the z components of $\mathbf{S}_e$ and $\mathbf{S}_p$ are no longer separately conserved; the quantum numbers for the system are rather the eigenvalues of the *total angular momentum*:

$$
\begin{aligned}
\mathbf{J} &= \mathbf{S}_e + \mathbf{S}_p \qquad (14) \\
J^2 &= (\mathbf{S}_e + \mathbf{S}_p)^2 \qquad (15) \\
J^2 &= S_e^2 + S_p^2 + 2\mathbf{S}_e \cdot \mathbf{S}_p \qquad (16)
\end{aligned}
$$

so that,

$$
\mathbf{S}_e \cdot \mathbf{S}_p = \frac{1}{2} \left(J^2 - S_e^2 - S_p^2 \right) \qquad (17)
$$

The electron and proton carry spin $\frac{1}{2}$, so the eigenvalues of S_e^2 and S_p^2 are $\frac{3}{4}\hbar^2$. The two spins combine to form a spin-1, so-called a triplet state ($J^2 = 2\hbar^2$), and a spin-0, so-called a singlet state ($J^2 = 0$). Thus,

$$\mathbf{S}_e \cdot \mathbf{S}_p = \frac{1}{4}\hbar^2 \quad \text{(triplet)} \tag{18}$$
$$\mathbf{S}_e \cdot \mathbf{S}_p = -\frac{3}{4}\hbar^2 \quad \text{(singlet)} \tag{19}$$

and hence,

$$E = \left(\frac{2}{3}\frac{\mu_0}{\pi a^3}\right)\gamma_p\gamma_e\left(\frac{1}{4}\hbar^2\right) \quad \text{(triplet)} \tag{20}$$
$$E = \left(\frac{2}{3}\frac{\mu_0}{\pi a^3}\right)\gamma_p\gamma_e\left(-\frac{3}{4}\hbar^2\right) \quad \text{(singlet)} \tag{21}$$

Evidently, the singlet state, in which the spins are antiparallel, carries somewhat lower energy than the triplet combination. The energy gap is:

$$\Delta E = E_{\text{trip}} - E_{\text{sing}}. \tag{22}$$

$$\Delta E = \left(\frac{2\mu_0}{3\pi a^3}\right)\gamma_e\gamma_p\left(\frac{1}{4} - \left(-\frac{3}{4}\right)\right)$$

thus,

$$\Delta E = \left(\frac{2\mu_0}{3\pi a^3}\right)\gamma_e\gamma_p \tag{23}$$

Now, the gyromagnetic ratios are given by:

$$\gamma = \left(\frac{e}{2m}\right)g \tag{24}$$

where e is the proton charge, m is the mass of the particle, and g is its ”g factor” (2.0023 for the electron, 5.5857 for the proton). So, finally:

$$\Delta E_{\text{hyd}} = \left(\frac{\mu_0 h^2 e^2}{6\pi a^3}\right)\frac{g_e g_p}{m_e m_p} \tag{25}$$

$$= 9.427539063 \times 10^{-25}\,\text{m}^5 \cdot \text{kg} \cdot \text{m}^{-3} \cdot \text{s}^2$$

$$= 5.88420775 \times 10^{-6}\,\text{eV} \tag{26}$$

The frequency of the photon emitted in a transition from the triplet to the singlet state is then:

$$\nu = \frac{\Delta E}{h} = 1422.8\,\text{MHz} \tag{27}$$

and its wavelength is then:

$$\lambda = \frac{c}{\nu} = 21.07\,\text{cm} \tag{28}$$