

Nof owerzzk

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§ 1.1

1.1.1 (Peano) ()

1.1.2() $+$: $\mathbb{N} \rightarrow \mathbb{N} / \mathbb{N}$,

:

- $a + b = b + a$
- $a + 1 = S(a)$
- $a + S(b) = S(a + b)$

$;$ $\mathbb{N} \rightarrow \mathbb{N} / \mathbb{N}$, :

- $a - b = b - a$
- $a - 1 = a$
- $a - S(b) = a + (a - b)$

1.1.3() $a < b$ $c \in \mathbb{N}; b = a + c.$

§ 1.2

()

§ 1.3

131 : \mathbb{R} .

. \mathbb{R} , $\mathbb{R} = f(x_1; x_2; \dots; x_n; \dots) g$.

(1) $[a_1; b_1] \quad x_1 \notin [a_1; b_1]$.

(2) $[a_1; b_1]$, x_2 . $[a_2; b_2] :$

$f[a_n; b_n]_{g_{n=1}^1} \cdot \bigvee_{n=1}^{\infty} , g \notin \mathbb{R}; g_n \notin \mathbb{N}; \notin [a_n; b_n]$.
 $g_k \notin \mathbb{N}; x_k \notin [a_k; b_k], \quad x_k \notin \bigvee_{n=1}^{\infty} [a_n; b_n]. \quad \mathbb{R} \setminus \bigvee_{n=1}^{\infty} [a_n; b_n] = ? , \quad !$

§ 21

211 : $\lim_{n \rightarrow \infty} \frac{P_n}{n} = 1$.

$$\frac{P_n}{n} = 1 + y_n,$$

$$n = (1 + y_n)^n > 1 + \frac{n(n-1)}{2} y_n.$$

$$\frac{P_n}{n} - 1 = |y_n| < \frac{2}{n}; \quad g_n \in \mathbb{N}:$$

$$|y_n| > 0, \quad N = \frac{2}{|y_n|} + 1, \quad n < N \quad \frac{P_n}{n} - 1 < |y_n|$$

212 . , ; , :

$$a_n \quad p, \quad \lim_{n \rightarrow \infty} j a_n \quad a_{n+p} j = 0.$$

$$: \quad a_n = \frac{P_n}{n}, \quad g_p > 0$$

$$j a_{n+p} \quad a_n j = \frac{P_{n+p}}{n+p} - \frac{P_n}{n} \neq 0;$$

.

213 $\lim_{n \rightarrow \infty} \sin \sqrt[n]{4n^2 + n} = ?$

$$\begin{aligned} \sin \sqrt[n]{4n^2 + n} &= \sin \sqrt[n]{4n^2 + n + 2n} \\ &= \sin \sqrt[n]{\frac{n}{4n^2 + n + 2n}} \\ &= \sin \sqrt[n]{\frac{1}{4 + \frac{1}{n} + 2}} \end{aligned}$$

$$\lim_{n \rightarrow \infty} \sin \sqrt[n]{4n^2 + n} = \sin \frac{1}{4} = \frac{\sqrt{2}}{2}$$

§ 31

311

$f(x)$ is continuous at x_0 if and only if $\lim_{x \rightarrow x_0} f(x) = f(x_0)$.
 If $f(x)$ is continuous at x_0 and $f'(x_0) > 0$, then $f(x) > f(x_0)$ for $x > x_0$ and $f(x) < f(x_0)$ for $x < x_0$.
 If $f(x)$ is continuous at x_0 and $f'(x_0) < 0$, then $f(x) < f(x_0)$ for $x > x_0$ and $f(x) > f(x_0)$ for $x < x_0$.
 If $f(x)$ is continuous at x_0 and $f'(x_0) = 0$, then $f(x) = f(x_0)$ for $x = x_0$.

§ 32

321 ()

311

321

$$\lim_{x \nearrow x_0} f(x) < f(x_0) < \lim_{x \searrow x_0} f(x).$$

$$\lim_{x \nearrow x_0} f(x); \lim_{x \searrow x_0} f(x) \text{ ,}$$

.

.

.

$$\mathbf{321} \quad f: \mathbb{R} \rightarrow \mathbb{R} \quad . \quad L(f) = f(x) \geq \inf_{x \in \mathbb{R}} f(x) = 0g.$$

 $L(f)$ $L(f)$

.

§ 41

$$\mathbf{41.1} \quad f(x) \in C[a; b] \setminus D(a; b), \quad a > 0 \quad f(a) = 0. \quad \exists \in (a; b); f(\cdot) = \frac{b}{a} f'(\cdot).$$

$$F(x) = (b - x)^a f(x), \quad F(x) \in C[a; b] \setminus D(a; b); F(a) = F(b) = 0,$$

Rolle $\exists \in (a; b); F'(\cdot) = 0 \Rightarrow f(\cdot) = \frac{b}{a} f'(\cdot)$

Rolle

- $g(\cdot) = 0$

$$Z$$

- $G(x) = \int_a^x g(x) dx$

- $G(x)$ **Rolle**

$$\frac{f'(x)}{f(x)} - \frac{a}{b-x} = 0, \quad G(x) = (b-x)^a f(x)$$

$$\mathbf{412} \quad f(x) \in [0; 1] \quad f(0) = f(1); f'(1) = 1, \quad 2 \\ (0; 1); f''(x) = 2.$$

$$f''(x) - 2 = 0 \quad f'(x) - 2x = C_1 \quad f'(1) = 1 \quad C_1 = -1 \\ f'(x) - 2x + 1 = 0 \quad F(x) = f(x) = x^2 + x.$$

$$\mathbf{413} \quad 2^x - x^2 = 1$$

.

$$\mathbf{414} \quad f(x) \in (-1; +1) \quad f(0) = 0, |f'(x)| \leq p|f(x)|, \quad f(x) \\ 0; x \in \mathbb{R}$$

$$x \in [0; \frac{1}{2p}] \quad |f(x)| \leq \frac{1}{2p} \quad |f(x_0)| = M > 0$$

$$M = |f(x) - f(0)| = |f'(x_0)| \cdot x_0 \leq \frac{1}{2p} \cdot p|f(x_0)| \leq \frac{1}{2}M$$

$$M = 0.$$

.

$$\mathbf{411} \quad (\text{Darboux} \quad (\quad)) \quad f(x) \quad f'(a) \notin f'(b), \\ f'(a); f'(b) \quad r, \quad \in (a; b), \quad r = f'(\xi)$$

$$f'(a) < f'(b), \quad r \in (f'(a); f'(b)), \quad F(x) = f(x) - rx, \quad F(a) < \\ 0; F(b) > 0. \quad > 0; \exists x \in (a; a + \delta); \frac{F(x) - F(a)}{x - a} < 0,$$

$$F(x) < F(a), F(a) \quad .$$

$$F(b) \quad .$$

$$F'(\xi) \in (a; b)$$

$$F'(\xi) = 0 \quad f'(\xi) = r.$$

$$F(x) = \begin{cases} \frac{f(x) - f(a)}{x - a}; & x \neq a \\ f'(a); & x = a \end{cases}; \quad G(x) = \begin{cases} \frac{f(x) - f(b)}{x - b}; & x \neq b \\ f'(b); & x = b \end{cases}$$

$$r \quad F(a); F(b) \quad G(a); G(b) \quad .$$

$$\frac{f(x_0) - f(a)}{x_0 - a} \quad F(x) \quad \text{Lagrange} \quad . \quad r \quad G(a); G(b) \quad .$$

§ 42 Taylor

$$\mathbf{421} \quad r(x_0) = r'(x_0) = r''(x_0) = \dots = r^{(n)}(x_0) = 0 \quad r(x) = o((x - x_0)^n) \quad (x \rightarrow x_0).$$

$$n \quad .$$

$$n = 1 \quad r(x) = r(x_0) + r'(x_0)(x - x_0) + o(x - x_0) = o(x - x_0)$$

$$n > 1 \quad r^{(n)}(x) = o((x - x_0)^n) \quad .$$

$$n + 1 \quad \text{Lagrange}$$

$$r(x) = r(x_0) + r'(x_0 + \theta(x - x_0))(x - x_0) = r'(x_0 + \theta(x - x_0))(x - x_0)$$

$$x \rightarrow x_0$$

$$r'(x_0 + \theta(x - x_0))(x - x_0) = o((x - x_0)^n) \quad (x \rightarrow x_0) = o((x - x_0)^{n+1})$$

.

$$\mathbf{421} \quad \text{Lagrange} \quad) \quad f(x) \quad [a; b] \quad n \quad (a; b)$$

$$n+1 \quad . \quad x_0 \quad [a; b] \quad x \in [a; b] \quad \in (0; 1),$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \dots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n + \frac{1}{(n+1)!}f^{(n+1)}(x_0 + \theta(x - x_0))(x - x_0)^{(n+1)}$$

$$r(x) = f(x) - f(x_0) - f'(x_0)(x - x_0) - \frac{1}{2}f''(x_0)(x - x_0)^2 - \dots - \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n, \quad r(x_0) = r'(x_0) = r''(x_0) = \dots = r^{(n)}(x_0) = 0.$$

Cauchy

$$\begin{aligned} \frac{r(x)}{(x - x_0)^{n+1}} &= \frac{r(x) - r(x_0)}{(x - x_0)^{n+1} - (x_0 - x_0)^{n+1}} \\ &= \frac{r'(x_1)}{(n+1)((x_1 - x_0)^n - (x_0 - x_0)^n)} \quad x_1 \in (x_0; x) \quad (x; x_0) \\ &= \frac{r''(x_2)}{(n+1)n((x_2 - x_0)^{n-1} - (x_0 - x_0)^{n-1})} \quad x_2 \in (x_0; x_1) \quad (x; x_1) \\ &= \dots \\ &= \frac{1}{(n+1)!} \frac{r^{(n)}(x_n)}{x_n - x_0} \quad x_n \in (x_0; x_{n-1}) \quad (x_{n-1}; x_0) \\ &= \frac{1}{(n+1)!} r^{(n+1)}(\xi) \quad \xi \in (x_0; x_n) \quad (x_n; x_0) \\ &= \frac{1}{(n+1)!} f^{(n+1)}(\xi) \end{aligned}$$

$$r(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x - x_0)^{n+1}$$

421 (Taylor) $f(x) \in [a; b] \quad f(a) = f(b) = 0,$

$$\max_{a \leq x \leq b} |f(x)| \leq \frac{1}{8}(b-a)^2 \max_{a \leq x \leq b} |f''(x)|$$

$$x_0 \in [a; b]; |f(x_0)| = \max_{a \leq x \leq b} |f(x)|, \quad f'(x_0) = 0$$

$$f(x) \quad x = x_0$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(\xi)(x - x_0)^2; \quad \xi \in (x; x_0)$$

$$x = a; x = b : \quad$$

$$\begin{aligned} \frac{1}{2}f''(x_1)(a - x_0)^2 &= f(a) - f(x_0) - f'(x_0)(a - x_0) & a < x_1 < x_0 \\ \frac{1}{2}f''(x_2)(x_0 - b)^2 &= f(b) - f(x_0) - f'(x_0)(b - x_0) & x_0 < x_2 < b \end{aligned}$$

$$\begin{aligned} |f''(x_1)|j + |f''(x_2)|j &= 2|f'(x_0)|j \left(\frac{1}{(a - x_0)^2} + \frac{1}{(b - x_0)^2} \right) \\ &> 2|f'(x_0)|j \frac{(1 + 1)^3}{(a - x_0 + x_0 - b)^2} \\ &= \frac{16}{(a - b)^2} |f'(x_0)|j \end{aligned}$$

$$\begin{aligned} \max_{a \leq x \leq b} |f''(x)|j &> \frac{1}{2} (|f''(x_1)|j + |f''(x_2)|j) \\ &= \frac{8}{(a - b)^2} |f'(x_0)|j \\ &= \frac{8}{(a - b)^2} \max_{a \leq x \leq b} |f'(x)|j \end{aligned}$$

$$\max_{a \leq x \leq b} |f'(x)|j \leq \frac{1}{8} (a - b)^2 \max_{a \leq x \leq b} |f''(x)|j$$

□

$$422 \quad f(x) \in C^1; f'(x) \in C^0; |f'(x)| \leq k_0; |f''(x)| \leq k_1,$$

$$|f''(x)| \leq \frac{2k_0k_1}{h^2}$$

Taylor

$$\begin{aligned} f(x + h) &= f(x) + f'(x)h + \frac{1}{2}f''(x_1)h^2 \\ f(x - h) &= f(x) - f'(x)h + \frac{1}{2}f''(x_2)h^2 \end{aligned}$$

$$f''(x) = \frac{f(x + h) - f(x - h)}{2h} + \frac{1}{4} (f''(x_2) - f''(x_1))h$$

$$|f'(x)| \leq \frac{|f(x+h) - f(x-h)|}{2h} + \frac{1}{4} (|f''(\xi_1)| + |f''(\xi_2)|) h \leq \frac{k_0}{h} + \frac{h}{2} k_1 \quad \forall h > 0$$

$$|f'(x)| \leq \min_{h \in (0, +\infty)} \left(\frac{k_0}{h} + \frac{h}{2} k_1 \right) = \sqrt{2k_0 k_1}$$

□

§ 43 Lipschitz

431 () $f(x)$ $[a; b]$ $\in [0; 1]$,

$$f(\lambda x_1 + (1 - \lambda)x_2) = \lambda f(x_1) + (1 - \lambda)f(x_2)$$

$f(x)$

431 $f(x)$ I $\forall x \in I; f'(x) > 0$ $f(x)$ I

.

$x_1, x_2 \in I$ $x_1 < x_2$ $\in (0; 1)$ **Lagrange**

$$\begin{aligned} & f(x_1) + (1 - \lambda)f(x_2) - f(\lambda x_1 + (1 - \lambda)x_2) \\ = & (f(\lambda x_1 + (1 - \lambda)x_2) - f(x_1)) + (1 - \lambda)(f(x_2) - f(\lambda x_1 + (1 - \lambda)x_2)) \\ = & f'(\xi_1)(1 - \lambda)(x_2 - x_1) + (1 - \lambda)f'(\xi_2)(x_2 - x_1) \\ & (x_1 < \xi_1 < \lambda x_1 + (1 - \lambda)x_2 < \xi_2 < x_2) \\ = & (1 - \lambda)f'(\xi_1)(x_2 - x_1) + (1 - \lambda)f'(\xi_2)(x_2 - x_1) \\ > & 0 \end{aligned}$$

$f(x)$ I $x_0 \in I$ $\forall x > 0$

$$\frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2} = 0 \quad x_0 = \frac{1}{2}(x_0 - h) + \frac{1}{2}(x_0 + h)$$

Peano Taylor ,

$$\begin{aligned} & \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2} \\ &= \frac{1}{h^2} [f(x_0) + f^{(j)}(x_0)h + \frac{1}{2}f^{(j+2)}(x_0)h^2 + o(h^2)] \\ & \quad + f(x_0) + f^{(j)}(x_0)(-h) + \frac{1}{2}f^{(j+2)}(x_0)(-h)^2 + o(h^2) - 2f(x_0) \\ &= f^{(j+2)}(x_0) + \frac{o(h^2)}{h^2} \quad (h \neq 0) \\ &= f^{(j+2)}(x_0) \\ &> 0 \end{aligned}$$

Lipschitz

432(**Lipschitz**) $x_0 \in D_f$, $(x_0 - \delta; x_0 + \delta) \subset D_f$,
 $C \in C^j(x_0)$,

$$\forall x, x' \in (x_0 - \delta; x_0 + \delta) \setminus D_f: |f(x) - f(x')| \leq C|x - x'|^j$$

$f(x)$ **Lipschitz** .

433(**Lipschitz**) C ,

$$\forall x, x' \in D_f: |f(x) - f(x')| \leq C|x - x'|^j$$

$f(x)$ **Lipschitz** .

Lipschitz **Lipschitz** .

432 Lipschitz

431 $f(x)$ $(a; b)$ $(x_1; x_2)$ $(a; b)$, $x \in (x_1; x_2)$,

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}$$

$$x = \frac{x_2 - x}{x_2 - x_1}x_1 + \frac{x - x_1}{x_2 - x_1}x_2,$$

$$f(x) \leq \frac{x_2 - x}{x_2 - x_1}f(x_1) + \frac{x - x_1}{x_2 - x_1}f(x_2) \quad (8a < x_1 < x < x_2 < b)$$

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

432

$[a; b]$, $x \in [a; b]$, $\theta \in (0; 1)$; $x = \theta a + (1 - \theta)b$; $f(x) \leq \theta f(a) + (1 - \theta)f(b) \leq \max\{f(a), f(b)\}$.

$f(x)$ $[c; d]$ $[a; b]$ $[c; d]$ $f(x)$ $[c; d]$.

$f(a) \leq \theta f(x) + (1 - \theta)f(c)$, $f(x) > \frac{1}{\theta}f(a) - \frac{1}{1 - \theta}f(c)$.

$f(x) > \frac{1}{\theta}f(b) - \frac{1}{1 - \theta}f(d)$. $f(x)$.

433

$\theta > 0$, $x_0 \in (a; b)$, $\theta > 0$; $[x_0 - \theta; x_0 + \theta] \subset (a; b)$.

$x > y \in (x_0 - \theta; x_0 + \theta)$, $x = (x_0 + \theta) + (1 - \theta)y$; $\theta \in (0; 1)$.

$f(x) \leq f(x_0 + \theta) + (1 - \theta)f(y)$, $f(x) - f(y) \leq [f(x_0 + \theta) - f(y)]$.

332 $|f(x)| < M$; $\theta M > 0$. $|f(y) - f(x)| \leq [f(x_0 + \theta) - f(y)] \leq 2M$.

$$x = (x_0 + 2) + (1 - 2)y, \quad |x - y| = |(x_0 + 2) - y| > 2M < \frac{|x - y|}{2M}.$$

$$|f(x) - f(y)| < \frac{|x - y|}{2M}, \quad \text{Lipschitz} \quad f(x).$$

434

Lipschitz .

431 $f(x)$ $(a; b)$:

$$(1) \quad x \in (a; b) \quad f'(x) = f'_+(x) = f'_-(x).$$

$$(2) \quad a < x_1 < x_2 < b \quad f'_+(x_1) \leq f'_-(x_2).$$

$$(3) \quad f(x) \quad .$$

$$(1) \quad x_0 \in (a; b), \quad \text{431,} \quad \frac{f(x_0) - f(x_0 - \Delta x)}{\Delta x} > \frac{f(x_0) - f(x_0 - 2\Delta x)}{2\Delta x},$$

$$F(\Delta x) = \frac{f(x_0) - f(x_0 - \Delta x)}{\Delta x}.$$

$$x' > x_0; F(\Delta x) \leq \frac{f(x') - f(x_0)}{x' - x_0}, \quad F(\Delta x) \rightarrow 0^+.$$

$$f'(x_0) = \lim_{x' \rightarrow 0} F(\Delta x) = f'_+(x_0).$$

$$f'(x_0) = \lim_{x' \rightarrow x_0} \frac{f(x_0) - f(x)}{x_0 - x} \leq \lim_{x' \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = f'_-(x_0) \quad (\text{431})$$

$$(2) \quad x \in (x_1; x_2),$$

$$f'_+(x_1) = \lim_{x' \rightarrow x_1^+} \frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x} \leq$$

$$\lim_{x' \rightarrow x_2^-} \frac{f(x_2) - f(x)}{x_2 - x} = f'_-(x_2)$$

$$(3) \quad (2) \quad x_0 \in (a; b), \quad f'(x_0) < f'_+(x_0),$$

$$(f'(x_0); f'_+(x_0)) \quad f'(x); f'_+(x); \delta x \in (a; b) \quad .$$

$$x_1 < x_2, \quad f'(x_i) < f'_+(x_i); \quad i = 1, 2. \quad f'(x_1) < f'_+(x_1) \leq$$

$$f'(x_2) < f'_+(x_2).$$

321,

 x_0 .

435 (Young) $p, q > 0, \frac{1}{p} + \frac{1}{q} = 1, a, b,$

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q \quad p > 1$$

$$ab > \frac{1}{p}a^p + \frac{1}{q}b^q \quad p < 1$$

.

436 (Holder) $p, q > 0, \frac{1}{p} + \frac{1}{q} = 1,$

$f, a_k g_k^n; f, b_k g_k^n,$

$$\sum_{i=1}^n a_i b_i \leq \left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q \right)^{\frac{1}{q}} \quad p > 1$$

$$\sum_{i=1}^n a_i b_i > \left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q \right)^{\frac{1}{q}} \quad p < 1$$

$p > 1$ $A = \sum_{i=1}^n a_i^p; B = \sum_{i=1}^n b_i^q$ **Young**

$$\frac{a_i b_i}{A^{\frac{1}{p}} B^{\frac{1}{q}}} \leq \frac{1}{p} \frac{a_i^p}{A} + \frac{1}{q} \frac{b_i^q}{B}$$

$$\frac{1}{A^{\frac{1}{p}} B^{\frac{1}{q}}} \sum_{i=1}^n a_i b_i \leq \frac{1}{p} + \frac{1}{q} = 1$$

$$\sum_{i=1}^n a_i b_i \leq A^{\frac{1}{p}} B^{\frac{1}{q}} = \left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q \right)^{\frac{1}{q}}$$

(Jensen) $p > 1$ $f(x) = x^p$.
 $q = \frac{p}{p-1}, \quad x_i y_i$

$$x_i y_i = y_i^q \prod_{i=1}^n y_i^{q-1} x_i y_i^{1-q} \prod_{i=1}^n y_i^q$$

$$\prod_{i=1}^n @ y_i^q \prod_{i=1}^n y_i^{q-1} A = 1$$

Jensen

$$\begin{aligned} \prod_{i=1}^n x_i y_i &= \frac{8}{4} \prod_{i=1}^n y_i^q \prod_{i=1}^n y_i^{q-1} x_i y_i^{1-q} \prod_{i=1}^n y_i^q \\ &= \frac{6}{4} \prod_{i=1}^n y_i^q \prod_{i=1}^n y_i^{q-1} x_i^p y_i^{p-pq} \prod_{i=1}^n y_i^q \\ &= \prod_{i=1}^n x_i^p \prod_{i=1}^n y_i^{q-p-1} \end{aligned}$$

$$\prod_{i=1}^n x_i y_i = \prod_{i=1}^n a_i^p \prod_{i=1}^n b_i^q$$

§ 51

()

511()

- $\int \frac{x dx}{1+x^2} = \frac{1}{2} \ln(1+x^2)$
- $\int x^{2k+1} \sqrt{a+bx^2} dx = \frac{1}{2b} \int \sqrt{a+bt} dt$; ($bx dx = t dt$)

512()

- $\int f(x) \ln g(x) dx = \int \ln g(x) dF(x)$
- $\int f(x) \arctan g(x) dx = \int \arctan g(x) dF(x) - \int f(x) \arctan g(x) dF(x)$.
 $t = \arctan g(x)$

511()

$$\int_a^x f(t) dt = F(x) - F(a)$$

512(!)

- $\int \frac{x^2}{a^2+x^2} dx = \frac{x^2}{2} - \frac{a^2}{2} \ln|x| + \frac{a^2}{2} \ln|x| + \frac{a^2}{2} \ln|x| + C$
- $I_n = \int \frac{dx}{(x^2+a^2)^n}$

$$I_n = \frac{1}{2a^2(n-1)} I_{n-1} + \frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}} \quad n > 2$$

$$I_1 = \frac{1}{a} \arctan \frac{x}{a} + C \quad n = 1$$

$$513(\quad) \quad \int \frac{7 \sin x + \cos x}{3 \sin x + 4 \cos x} dx.$$

$$7 \sin x + \cos x = 3 \sin x + 4 \cos x - (3 \sin x + 4 \cos x)^0,$$

$$\int \frac{7 \sin x + \cos x}{3 \sin x + 4 \cos x} dx = \int dx - \int \frac{d(3 \sin x + 4 \cos x)}{3 \sin x + 4 \cos x} = x - \ln|3 \sin x + 4 \cos x| + C$$

§ 52

Darboux

521 (Riemann) ()

522 (Darboux) ()

— Darboux

521 (Darboux) [a, b] f(x)

$$\lim_{(P) \rightarrow 0} \bar{S}(P) = \int_a^b f(x) dx; \quad \lim_{(P) \rightarrow 0} \underline{S}(P) = \int_a^b f(x) dx;$$

$$" > 0 \quad > 0 \quad P \quad 0 < (P) <$$

$$\bar{S}(P) - \int_a^b f(x) dx < "; \quad \underline{S}(P) - \int_a^b f(x) dx < "$$

521 (Darboux) f(x) [a; b]

1. f(x) [a; b] Riemann

$$\int_a^b f(x) dx = \int_a^b f(x) dx;$$

$$2 \int_a^b f(x) dx = \int_a^b f(x) dx;$$

$$3 \quad \lim_{(P) \downarrow 0} \overline{S}(P) - \underline{S}(P) = 0;$$

$$4 \quad \lim_{l \rightarrow 1} \overline{S}(P_l) - \underline{S}(P_l) = 0.$$

$$4. \quad \epsilon > 0 \quad P \quad \overline{S}(P) - \underline{S}(P) < \epsilon$$

$$f(x)$$

$$\int_a^b f(x) dx = \int_a^b f(x) dx = \int_a^b f(x) dx:$$

— Lebesgue

522

$$523(\quad) \quad I = (a; b) \quad |I| = b - a$$

$$S \subset \mathbb{R}$$

$$\epsilon > 0$$

$$\lim_{n \rightarrow \infty} \int_{I_n} f$$

$$S \quad \bigcup_{n=1}^{\infty} I_n \quad \sum_{n=1}^{\infty} |I_n| < \epsilon;$$

$$S$$

$$521 \quad \mathbb{R}$$

$$S \subset \mathbb{R}$$

$$S = \{f(x_1; x_2; x_3; \dots; x_n; \dots; g)\}$$

$$\epsilon > 0 \quad n$$

$$I_n = \left[x_n - \frac{\epsilon}{2^{n+2}}, x_n + \frac{\epsilon}{2^{n+2}} \right] :$$

$$S = \sum_{n=1}^{\infty} I_n;$$

$$\sum_{n=1}^{\infty} |I_n| = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = 1;$$

S

.

522

$$f(x), f(x) \quad \epsilon; \quad \epsilon > 0,$$

$P,$

$$\sum_{i=1}^{\infty} |x_i| < \epsilon$$

$f(x)$

$$\lim_{(P) \rightarrow 0} \sum_{i=1}^{\infty} w_i |x_i| = 0. \quad \max_{x \in [a; b]} f(x) = M; \min_{x \in [a; b]} f(x) = m$$

•

$$\epsilon > 0, \quad \epsilon = \frac{\epsilon}{2(p-1)}; \quad \epsilon = \frac{\epsilon}{2(p-1)(M-m)}; \quad \epsilon_0 = \frac{\epsilon}{2(p-1)}$$

$P \quad (P) < \epsilon,$

$$\sum_{i=1}^{\infty} w_i |x_i| = \sum_{w_i > \epsilon_0} w_i |x_i| + \sum_{w_i < \epsilon_0} w_i |x_i| < (p-1)(M-m) \epsilon + (p-1) \epsilon_0 = \epsilon$$

$$\lim_{(P) \rightarrow 0} \sum_{i=1}^{\infty} w_i |x_i| = 0, f(x) \quad .$$

•

$$\epsilon_0 > 0, \quad \epsilon = \inf_{a, b} \{ \epsilon; \epsilon = \frac{\epsilon_0}{P} \}.$$

P

$$\sum_{i=1}^{\infty} x_i > \epsilon, \\ w_i > \epsilon$$

$$\sum_{i=1}^{\infty} w_i x_i = \sum_{w_i > \epsilon_0} w_i x_i + \sum_{w_i < \epsilon_0} w_i x_i > \epsilon + 0 = \epsilon_0 \\ f(x) \quad \epsilon > 0; \quad \epsilon > 0, \quad P, \sum_{w_i > \epsilon} x_i <$$

$$523 \text{ (Lebesgue)} \quad f(x) \quad [a; b] \quad f(x) \quad [a; b] \\ f(x) \quad (f(x))$$

$$x \in D_f, \\ !f(x; \epsilon) := \sup_{x_1, x_2 \in (x - \epsilon; x + \epsilon)} |f(x_1) - f(x_2)| \\ !f(x; \epsilon) \quad \epsilon \quad \epsilon \rightarrow 0^+ \\ !f(x) := \lim_{\epsilon \rightarrow 0^+} !f(x; \epsilon)$$

$$522 \quad f(x) \quad x = x_0 \quad !f(x_0) = 0.$$

$$f(x) \quad [a; b] \quad |f(x)| \in M. \\ K \quad [a; b] \quad f(x) \quad K \\ \epsilon > 0. \quad f|_{I_k} \in \mathcal{G}_{k=1}^1$$

$$K \quad \bigcup_{k=1}^l I_k; \quad \sum_{k=1}^l |I_k| < \frac{\epsilon}{4M}. \\ x \in [a; b] \cap K \quad x \\ !f(x) = \lim_{\epsilon \rightarrow 0^+} !f(x; \epsilon) = 0:$$

$$x > 0 \\ !f(x; 2^{-x}) := \sup_{x^0, x^0 \in (x - 2^{-x}; x + 2^{-x}) \setminus [a; b]} |f(x^0) - f(x^0)| < \frac{\epsilon}{2(b-a)}$$

$$f(x_{n-1}; x_n) g_{x \in [a; b] n K} [f I_k g_{k=1}^l [a; b] \\ [a; b] n K K .$$

$$[a; b] \prod_{j=1}^N (y_j - y_j; y_j + y_j) \prod_{l=1}^M I_{k_l} : \\ [a; b] P : a = x_0 < x_1 < x_2 < \\ < x_n = b ([a; b] [x_{i-1}; x_i]).$$

$$\text{I. } (x_{i-1}; x_i) I_{k_l} . \quad [x_{i-1}; x_i] \\ I_{k_l} \frac{1}{4M} . \quad i \quad 1$$

$$\text{II. } (x_{i-1}; x_i) I_{k_l} . \quad x_{i-1} \\ [x_{i-1}; x_i]$$

$$- x_{i-1} I_{k_l} . \quad x_i \quad x_{i-1} \quad x_{i-1} \\ I_{k_l} . \quad (x_{i-1}; x_i) I_{k_l} . \\ - x_{i-1} I_{k_l} (y_j - y_j; y_j + y_j) .$$

$$(x_{i-1}; x_i) (y_j - y_j; y_j + y_j) :$$

$$[x_{i-1}; x_i] (y_j - 2 y_j; y_j + 2 y_j) : \\ \frac{1}{2(b-a)} . \quad i \quad 2$$

$$S(P) = S(P) = \prod_{i \in \mathbb{Z}_1}^P !_i x_i + \prod_{i \in \mathbb{Z}_2} !_i x_i < 2M \frac{1}{4M} + \frac{1}{2(b-a)} (b-a) = 1 :$$

$$523 \quad f(x) \in [a; b] \quad > 0,$$

$$f(x) \in [a; b] \quad f(x) > g$$

.

522 .

$$x \in [a; b] \text{ и } f(x) > 0, \quad \exists n \in \mathbb{N}; \quad \int_a^b f(x) dx > \frac{1}{n}$$

$$, \quad \forall n \in \mathbb{N} \quad \exists x \in [a; b] \text{ и } f(x) > \frac{1}{n}$$

$$\int_a^b f(x) dx > 0 \Leftrightarrow \exists n \in \mathbb{N} \quad \exists x \in [a; b] \text{ и } f(x) > \frac{1}{n}$$

$$\int_a^b f(x) dx > 0 \Leftrightarrow \exists x \in [a; b] \text{ и } f(x) > 0$$

523

321 .

524 () $f(x); g(x) \in [a; b]$ $g(x) \in [m; M]$.

$$M = \sup_{x \in [a; b]} f(x); m = \inf_{x \in [a; b]} f(x), \quad \int_a^b f(x) g(x) dx = \int_a^b f(x) dx \int_a^b g(x) dx$$

$$\int_a^b f(x) g(x) dx = \int_a^b f(x) dx \int_a^b g(x) dx$$

$$\int_a^b f(x) g(x) dx = f(\xi) \int_a^b g(x) dx$$

524 $f(x) \in [a; b]$ $x \in [a; b], F(x) = \int_a^x f(x) dx$

$$F(x)$$

525

• $F(x)$ Lipschitz .

• $f(x) = F'(x) \implies F'(x) = f(x)$

• $|f(x)| \leq M$,

$$\begin{aligned} |f(x) - f(y)| &= \left| \int_y^x f(t) dt \right| \\ &\leq \int_y^x |f(t)| dt \\ &\leq \int_y^x M dt = M|x - y|. \end{aligned}$$

• $x_0 \in (a; b)$. $\epsilon > 0$. $f(x) = x_0$ $\epsilon > 0$

$$\delta \in (x_0 - \epsilon; x_0 + \epsilon) : |f(x) - f(x_0)| < \epsilon:$$

x

$$\frac{F(x_0 + \epsilon) - F(x_0)}{\epsilon} = \frac{1}{\epsilon} \int_{x_0}^{x_0 + \epsilon} f(t) dt = f(\xi);$$

$$\delta \in [x_0 - \epsilon; x_0 + \epsilon] \cap [x_0, x_0 + \epsilon]. \quad 0 < \epsilon - \delta < \epsilon$$

$$\frac{F(x_0 + \epsilon) - F(x_0)}{\epsilon} = f(\xi) = f(x_0) + |f(\xi) - f(x_0)| < \epsilon:$$

$$F'(x_0) = \lim_{\epsilon \rightarrow 0} \frac{F(x_0 + \epsilon) - F(x_0)}{\epsilon} = f(x_0):$$

$$526 \quad \left(\int_a^x f(x) dx \right)' = f(x) \quad [a; b]$$

$$F(x) = \int_a^x f(x) dx \quad f(x) \quad .$$

$$527 \quad \left(\int_a^x f(x) dx \right)' = f(x) \quad [a; b]$$

$$F(x) = \int_a^x f(x) dx \quad f(x) \quad .$$

$$\int_a^b f(x) dx = F(b) - F(a) = F(x) \Big|_a^b$$

527

$$528 \quad F(x) \quad [a; b] \quad (a; b) \quad . \quad f(x) \quad [a; b]$$

$$F'(x) = f(x), \quad x \in (a; b)$$

$$\int_a^b f(x) dx = F(b) - F(a)$$

$$529 \quad \left(\int_a^x f(x) dx \right)' = f(x) \quad [a; b] \quad g(x) \quad [a; b] \quad .$$

$$\int_a^b f(x) g(x) dx = g(a) \int_a^b f(x) dx + g(b) \int_a^b f(x) dx$$

$$F(x) = \int_a^x f(x) dx. \quad F(x) \quad , \quad F(x) \quad M; m.$$

$$\int_a^b f(x) g(x) dx = g(a) \int_a^b f(x) dx + g(b) \int_a^b f(x) dx$$

$$= F(b)g(b) - F(a)g(a) - F(x)(g(b) - g(a))$$

$$\int_a^b f(x) g(x) dx + F(x)g(x) \Big|_a^b = F(x)g(x) \Big|_a^b$$

$$F(x) \quad g(x)$$

$$mg(x) \Big|_a^b - \int_a^b f(x)g(x)dx + F(x)g(x) \Big|_a^b - Mg(x) \Big|_a^b$$

$$mg(x) \Big|_a^b + \int_a^b f(x)g(x)dx - F(x)g(x) \Big|_a^b - Mg(x) \Big|_a^b + \int_a^b f(x)g(x)dx \quad (1)$$

$$[a; b] \quad P : a = x_0 < x_1 < \dots < x_n = b,$$

$$\begin{aligned} F(x)g(x) \Big|_a^b &= \sum_{i=1}^n (F(x_i)g(x_i) - F(x_{i-1})g(x_{i-1})) \\ &= \sum_{i=1}^n F(x_i)(g(x_i) - g(x_{i-1})) + \sum_{i=1}^n g(x_{i-1})(F(x_i) - F(x_{i-1})) \end{aligned}$$

$$mg(x) \Big|_a^b - \sum_{i=1}^n F(x_i)(g(x_i) - g(x_{i-1})) - Mg(x) \Big|_a^b$$

$$\begin{aligned} & \sum_{i=1}^n g(x_{i-1})(F(x_i) - F(x_{i-1})) - \int_a^b f(x)g(x)dx \\ &= \sum_{i=1}^n g(x_{i-1}) \int_{x_{i-1}}^{x_i} f(t)dt - \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(t)g(t)dt \\ &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (g(x_{i-1}) - g(t))f(t)dt \\ &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (-g'(t))f(t)dt \\ &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(t)g'(t)dt \\ &= \int_a^b f(t)g'(t)dt \end{aligned}$$

$$(P) \neq 0 \quad 0. \quad (1) \quad .$$

525 (p) $[a; b]$ $f(x)$, $f(x)$ p -

$$\|f(x)\|_p = \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}$$

5210 (Holder) $f(x); g(x)$ $[a; b]$ $p; q > 1$ $\frac{1}{p} + \frac{1}{q} = 1$,

$$\left| \int_a^b f(x)g(x) dx \right| \leq \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx \right)^{\frac{1}{q}}$$

$$\left| \int_a^b f(x)g(x) dx \right| \leq \|f\|_p \|g\|_q$$

$p = q = 2$ **Cauchy-Schwarz**

$$\left| \int_a^b f(x)g(x) dx \right| \leq \left(\int_a^b |f(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_a^b |g(x)|^2 dx \right)^{\frac{1}{2}}$$

Young

Holder

.

524 $f(x)$ $[a; b]$ $f(a) = f(b) = 0$ $\int_a^b f(x)^2 dx =$

1.

$$\int_a^b x^2 (f'(x))^2 dx = \frac{1}{4}$$

$$\begin{aligned}
1 &= \int_a^b f(x)^2 dx \\
&= \int_a^b x f(x)^2 \int_a^b x (f(x)^2)^\theta dx \\
&= \int_a^b 2x f^\theta(x) f(x) dx \\
&= 2 \int_a^b (x f^\theta(x)) f(x) dx \\
&= 2 \int_a^b (x f^\theta(x))^{\frac{1}{2}} \int_a^b f(x)^2 dx \\
&= 2 \int_a^b x^2 (f^\theta(x))^2 dx
\end{aligned}$$

□

526() • $[a; b] \subset \mathbb{R}^2$

•

527() $[a; b]$ $P : a = t_0 < t_1 < \dots < t_n = b,$
 $\lim_{(P)! 0} \sum_{i=1}^n \frac{\mathbb{X}^n}{(t_{i-1}) - (t_i)}$.

$$\lim_{(P)! 0} \sum_{i=1}^n \frac{\mathbb{X}^n}{(t_{i-1}) - (t_i)} = \sup_P \sum_{i=1}^n \frac{\mathbb{X}^n}{(t_{i-1}) - (t_i)}$$

528() $(t) = (x(t); y(t)), \quad x(t); y(t)$

$$(x^\theta(t))^2 + (y^\theta(t))^2 \notin 0$$

5211

$$\int_a^b \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} dt$$

()

Lagrange

$$\begin{aligned} \frac{1}{\sqrt{(t_{i-1} - t_i)^2}} &= \frac{1}{\sqrt{(x(t_{i-1}) - x(t_i))^2 + (y(t_{i-1}) - y(t_i))^2}} \\ &= \frac{1}{\sqrt{(x'(t_i))^2 + (y'(t_i))^2}} \frac{1}{t_i} \end{aligned}$$

$$\begin{aligned} &\frac{1}{\sqrt{(t_{i-1} - t_i)^2}} \int_{t_{i-1}}^{t_i} \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} dt \\ &= \int_{t_{i-1}}^{t_i} \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} \frac{1}{t_i} dt \\ &\leq \int_{t_{i-1}}^{t_i} \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} \frac{1}{t_{i-1}} dt \\ &\leq \int_{t_{i-1}}^{t_i} \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} dt \\ &\leq \int_{t_{i-1}}^{t_i} \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} dt \\ &\leq \int_{t_{i-1}}^{t_i} \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} dt \end{aligned}$$

$$\int_a^b \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} dt = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} dt,$$

$$\begin{aligned} &\sum_{i=1}^n \frac{1}{\sqrt{(t_{i-1} - t_i)^2}} \int_{t_{i-1}}^{t_i} \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} dt \\ &\leq \sum_{i=1}^n \frac{1}{\sqrt{(t_{i-1} - t_i)^2}} \int_{t_{i-1}}^{t_i} \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} dt \\ &\leq \sum_{i=1}^n \frac{1}{\sqrt{(t_{i-1} - t_i)^2}} \int_{t_{i-1}}^{t_i} \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} dt \end{aligned}$$

525 $f(x) \in C[a; b], f(a) = 0, \sup_{x \in [a; b]} f^2(x) \leq (b-a) \int_a^b (f'(x))^2 dx$

$$(b-a) \int_a^x (f'(t))^2 dt \geq (x-a) \int_a^x (f'(t))^2 dt \geq \int_a^x f^2(t) dt = f^2(x)$$

$$x_0 \in [a; b], f^2(x_0) = \sup_{x \in [a; b]} f^2(x)$$

$$(b-a) \int_a^b (f'(x))^2 dx \geq (b-a) \int_a^{x_0} (f'(x))^2 dx \geq \sup_{x \in [a; b]} f^2(x)$$

□

526 $f(x); g(x) \in C[a; b]$

$$\int_a^b f(x) dx \int_a^b g(x) dx \leq (b-a) \int_a^b f(x)g(x) dx$$

Chebyshev .

□

§ 53

()

531 $I_n = \int_0^{+1} x^n e^{-x} dx$

$$I_n = \int_0^{+1} x^n e^{-x} dx + n \int_0^{+1} x^{n-1} e^{-x} dx = n I_{n-1}$$

$$I_0 = \int_0^{+1} e^{-x} dx = 1$$

$$I_n = n!$$

$$(x) = \int_0^{x+1} t^x e^{-t} dt$$

532
$$I = \int_0^{\pi/2} \ln \sin x dx.$$

$$\begin{aligned} I &\stackrel{x=2t}{=} \int_0^{\pi/4} 2 \ln 2 \sin t \cos t dt \\ &= \frac{1}{2} \ln 2 + 2 \int_0^{\pi/4} \ln \sin x dx + \int_0^{\pi/4} \ln \cos x dx \\ &= \frac{1}{2} \ln 2 + 2 \int_0^{\pi/2} \ln \sin x dx = \frac{1}{2} \ln 2 + 2I \end{aligned}$$

$$I = -\frac{1}{2} \ln 2$$

533
$$I = \int_0^{x+1} \frac{1}{1+x^2} \frac{1}{1+x} dx.$$

$$\begin{aligned} I &= \int_0^1 \frac{1}{1+x} dx + \int_1^{x+1} \frac{1}{1+\frac{1}{t^2}} \frac{1}{1+\frac{1}{t}} d\frac{1}{t} \\ &= \int_0^1 \frac{1}{1+x} dx + \int_1^0 \frac{t dt}{(1+t^2)(1+t)} \\ &= \int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4} \end{aligned}$$

531 () $f(x); f'(x) > 0$

Riemann

$$\lim_{x \rightarrow +1} \frac{f'(x)}{f(x)} = k > 0 \quad +1$$

• $k \in [0; +1)$,
$$\int_a^{x+1} f'(x) dx \quad) \quad \int_a^{x+1} f(x) dx \quad .$$

$$\bullet k \in (0; +\infty], \quad \int_a^{+\infty} f(x) dx = \int_a^{+\infty} f'(x) dx.$$

$$f'(x) = \frac{1}{x^p}, \quad \text{Cauchy}.$$

532 (Abel-Dirichlet) $\int_a^{+\infty} f(x) dx$ converges if $f(x)$ is monotonic and $f'(x) > 0$. Riemann

$$\int_a^{+\infty} f(x) g(x) dx$$

$$\bullet \text{ (Abel) } \int_a^{+\infty} f(x) dx \quad g(x)$$

$$\bullet \text{ (Dirichlet) } F(x) = \int_a^x f(t) dt \quad g(x) \quad 0$$

$$\int_{A^0}^{A^\infty} f(x) g(x) dx = g(A^0) \int_{A^0}^{A^\infty} f(x) dx + g(A^\infty) \int_{A^0}^{A^\infty} f(x) dx$$

$$\int_{A^0}^{A^\infty} f(x) dx \quad g(A) \quad \text{Cauchy}.$$

§ 61

611 $S_n = \sum_{i=1}^n a_i \quad \sum_{n=1}^\infty a_n$.

611 $\sum_{n=1}^\infty a_n \quad \lim_{n \rightarrow \infty} a_n = 0.$

$$\mathbb{X} \quad a_n$$

$$n=1$$

§ 62

$$f x_n g \quad 2 \mathbb{R} \quad f x_n g \quad f x_{n_k} g$$

$$\lim_{k \rightarrow \infty} x_{n_k} = \quad ;$$

$$f x_n g \quad f x_n g \quad E. \sup E$$

$$f x_n g \quad \overline{\lim}_{n \rightarrow \infty} x_n \quad \inf E \quad f x_n g \quad \lim_{n \rightarrow \infty} x_n.$$

$$621 \quad x_n$$

$$\bullet \quad f x_{n_i} g_{i=1}^1 \quad \lim_{i \rightarrow \infty} x_{n_i} = \overline{\lim}_{n \rightarrow \infty} x_n;$$

$$\bullet \quad f x_{n_k} g_{k=1}^1 \quad \lim_{k \rightarrow \infty} x_{n_k} = \underline{\lim}_{n \rightarrow \infty} x_n.$$

$$622 \quad x_n$$

$$\bullet \quad H = \overline{\lim}_{n \rightarrow \infty} x_n \quad " > 0$$

$$- \quad N \in \mathbb{N} \quad \delta n \quad N : x_n < H + " ;$$

$$- \quad f x_n g \quad H - " .$$

$$\bullet \quad h = \underline{\lim}_{n \rightarrow \infty} x_n \quad " > 0$$

$$- \quad N \in \mathbb{N} \quad \delta n \quad N : x_n > h - " ;$$

$$- \quad f x_n g \quad h + " .$$

$$623 \quad f x_n g \quad \overline{\lim}_{n \rightarrow \infty} x_n = \underline{\lim}_{n \rightarrow \infty} x_n.$$

624 $f x_n g; f y_n g$

$$\bullet \quad \overline{\lim}_{n \rightarrow \infty} (x_n + y_n) = \overline{\lim}_{n \rightarrow \infty} x_n + \overline{\lim}_{n \rightarrow \infty} y_n, \quad \liminf_{n \rightarrow \infty} (x_n + y_n) = \liminf_{n \rightarrow \infty} x_n + \liminf_{n \rightarrow \infty} y_n;$$

$$\bullet \quad \lim_{n \rightarrow \infty} x_n$$

$$\overline{\lim}_{n \rightarrow \infty} (x_n + y_n) = \lim_{n \rightarrow \infty} x_n + \overline{\lim}_{n \rightarrow \infty} y_n; \quad \lim_{n \rightarrow \infty} (x_n + y_n) = \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} y_n;$$

625 $\overline{\lim}_{n \rightarrow \infty} x_k = \lim_{n \rightarrow \infty} \sup_{k > n} x_k; \quad \lim_{n \rightarrow \infty} x_k = \lim_{n \rightarrow \infty} \inf_{k > n} x_k:$

626 $\sum_{n=1}^{\infty} a_n$.

627 () $\sum_{n=1}^{\infty} x_n, \sum_{n=1}^{\infty} y_n$ $A > 0,$
 $x_n \leq A y_n; \quad n \in \mathbb{N}:$

$$\bullet \quad \sum_{n=1}^{\infty} y_n \leq \sum_{n=1}^{\infty} x_n$$

$$\bullet \quad \sum_{n=1}^{\infty} x_n \leq \sum_{n=1}^{\infty} y_n$$

628 () $\sum_{n=1}^{\infty} x_n, \sum_{n=1}^{\infty} y_n$

$$\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = l \in [0; +\infty):$$

$$\bullet \quad 0 \leq l < +\infty \quad \sum_{n=1}^{\infty} y_n < \sum_{n=1}^{\infty} x_n$$

$$\bullet \quad 0 < l \leq 1 \quad \sum_{n=1}^{\infty} y_n \quad \sum_{n=1}^{\infty} x_n$$

$$\textbf{629(Cauchy)} \quad \sum_{n=1}^{\infty} x_n \quad \lim_{n \rightarrow \infty} \sqrt[n]{x_n} = r$$

$$\bullet \quad r < 1 \quad \sum_{n=1}^{\infty} x_n$$

$$\bullet \quad r > 1 \quad \sum_{n=1}^{\infty} x_n$$

$$\bullet \quad r = 1$$

$$\textbf{6210(d'Alembert)} \quad \sum_{n=1}^{\infty} x_n \quad x_n \neq 0$$

$$\bullet \quad \overline{\lim}_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = r < 1 \quad \sum_{n=1}^{\infty} x_n$$

$$\bullet \quad \underline{\lim}_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = \underline{r} > 1 \quad \sum_{n=1}^{\infty} x_n$$

$$\bullet \quad r = 1 \quad \underline{r} = 1$$

$$\textbf{6211} \quad (\quad) \quad f(x) \quad [a; +\infty) \\ [a; A] \quad [a; +\infty) \quad \textbf{Riemann} \quad f a_n g$$

$$a = a_1 < a_2 < a_3 < \dots < a_n < \dots \rightarrow +\infty$$

$$u_n := \int_{a_n}^{a_{n+1}} f(x) dx \\ \int_a^{+\infty} f(x) dx = \sum_{n=1}^{\infty} u_n \quad +\infty.$$

$$f(x) \quad a_n = n \quad \sum_{n=1}^{\infty} \int_a^{a_{n+1}} f(x) dx = \sum_{n=[a]+1}^{\infty} f(n) \quad .$$

621

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad 0 < p \leq 1 \qquad p > 1 \quad .$$

§ 63

631(

$$\sum_{n=1}^{\infty} a_n \qquad \sum_{n=1}^{\infty} a_n \qquad \sum_{n=1}^{\infty} ja_n^j$$

$\sum_{n=1}^{\infty} a_n$

.

631

$$\sum_{n=1}^{\infty} a_n \quad .$$

Leibniz

632(Leibniz

$$f u_n g_1^l \qquad \sum_{n=1}^{\infty} (-1)^{n+1} u_n \quad .$$

u_n

0,

$$\sum_{n=1}^{\infty} (-1)^{n+1}$$

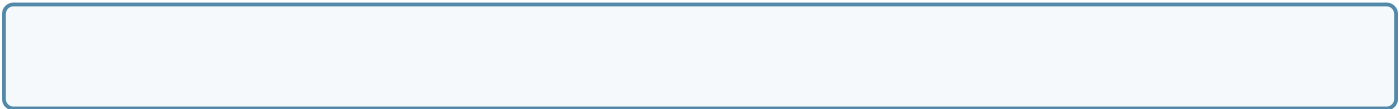
Leibniz

.

632 Leibniz

.

Abel Dirichlet



§ 7.1

7.1.1 () X $X \times X$ $d: X \times X \rightarrow \mathbb{R}$

- $d(x; y) \geq 0$ $d(x; y) = 0$ $x = y$
- $d(x; y) = d(y; x)$
- $d(x; z) \leq d(x; y) + d(y; z)$.

d $X \times X$ $(X; d)$.

7.1.2 () X $X \times X$ $h; i: X \times X \rightarrow \mathbb{R}$

- $h(x; x) \geq 0$ $h(x; x) = 0$ $x = 0$
- $h(x; y) = h(y; x)$
- $h(x + y; z) = h(x; z) + h(y; z)$ $h(x; y) = h(x; y)$.

$h; i$ $X \times X$ $(X; h; i)$.

7.1.3 () X X $k: X \rightarrow \mathbb{R}$

- $k(x) \geq 0$ $k(x) = 0$ $x = 0$
- $k(\alpha x) = |\alpha| k(x)$
- $k(x + y) \leq k(x) + k(y)$.

k X .

Euclid

7.1.4(Euclid) \mathbb{R}^n

$$(x, y) = x_1 y_1 + x_2 y_2 + \dots + x_n y_n;$$

$(\mathbb{R}^n; \langle \cdot, \cdot \rangle)$ n Euclid .

Euclid

- 1- $\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$
- 2- $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$
- 1 - $\|x\|_1 = \max\{|x_1|, |x_2|, \dots, |x_n|\}$.
- p - $\|x\|_p = (|x_1|^p + |x_2|^p + \dots + |x_n|^p)^{\frac{1}{p}}$.

§ 7.2

7.21() $\delta > 0$, $A \setminus B(x; \delta) \neq \emptyset$, $x \in A$.
 $\{x_n\} \subset A$ $x_n \notin x$, $\lim_{n \rightarrow \infty} x_n = x$.

7.22() $\{x_n\}_{n=1}^{\infty} \subset X$ $\delta > 0$,
 $N > 0$, $n > N \implies d(x_n, x) < \delta$, $x_n \in X$.

7.23() $(X; d)$, $A \subset X$ $x \in X$.

(1) () $\delta > 0$, $B(x; \delta) \cap A \neq \emptyset$. A

(2) () $\delta > 0$, $B(x; \delta) \cap A = \emptyset$ $x \in A$. A
 (A^c) .

(3) () $\delta > 0$, $B(x; \delta) \cap A \neq \emptyset$ $B(x; \delta) \cap A^c \neq \emptyset$ $x \in X \setminus A$

$$\cdot A \quad @A.$$

$$A \quad X$$

$$A \setminus (A^c) \setminus @A = X:$$

$$\mathbb{R}^n$$

•

•

$$\mathbf{7.24}(\quad)(X; d) \quad x \in X \quad \epsilon > 0 \quad B(x; \epsilon)$$

$$\mathbf{7.25}(\quad)(X; d) \quad U \subseteq X \quad x \in U$$

$$\mathbf{7.26}(\quad)(X; d) \quad F \subseteq X \quad X \cap F$$

$$\overline{A} = A \cup A^0 \quad A$$

$$\mathbf{7.21} \quad (X; d) \quad .$$

$$(1) A \subseteq X$$

$$(2) A = \overline{A}$$

$$(3) \quad f_{X_n} g_{n=1}^1 \quad A \quad A$$

$$(4) A^c \quad .$$

7.22 () . ()

. . .

7.23 $(X; d)$ $A \subseteq X$

(1) $\overline{A} \subseteq A$ $\overline{A} \subseteq K \subseteq A \subseteq K$
 $\overline{A} \subseteq K$

(2) $A \subseteq A$ $A^o \subseteq U \subseteq U \subseteq A$
 $U \subseteq A^o$

§ 7.3

7.31 () $(X; d)$ $A \subseteq X$
 $fU \subseteq g \subseteq 2I \subseteq A \subseteq U \subseteq I \subseteq fU \subseteq g \subseteq 2I \subseteq A$.

7.32 () $(X; d)$ $A \subseteq X$ A
 $fU \subseteq g \subseteq 2I \subseteq fU \subseteq g_{i=1}^n \subseteq A \subseteq \bigcup_{i=1}^n U_i \subseteq A$.

7.31 .

$x \in A, fB(x; n) \subseteq g_{n=1}^1 \subseteq A$. $fB(x; n) \subseteq g_{i=1}^k$.
 $A \subseteq \bigcup_{i=1}^k B(x; n_i) = B(x; n_k), \quad A \subseteq A$.

7.33 () A A A

.

7.32 .

A . $f_{X_n g_{n=1}^1}$ A
 A .

7.31 $x \in A$ $x > 0$ $B(x; x)$ $f_{X_n g_{n=1}^1}$

.

$x \in A$ > 0 $B(x;)$ $f_{X_n g_{n=1}^1}$
 $= 1; 1/2; 1/3; \dots$ $x_{k_1} \in B(x; 1) \setminus A$ $x_{k_2} \in B(x; 1/2) \setminus A$
 $(k_2 > k_1)$ $x_{k_3} \in B(x; 1/3) \setminus A$ $(k_3 > k_2)$ \therefore $f_{X_{k_i} g_{i=1}^1}$ $x \in A$.
 $f_{X_n g_{n=1}^1}$ A . .

$fB(x; x) g_{x \in A}$ A . A
 $fB(x_{k_i}; x_{k_i}) g_{i=1}^N$.

$f_{X_n g_{n=1}^1}$ A $\bigcap_{i=1}^N B(x_{k_i}; x_{k_i})$

.

7.33 . .

§ 7.4

7.41 (Cauchy)

7.42 () (X; d) Cauchy

7.41

-
-

7.42 () $f A_n g_{n=1}^1 A_n$
 $n \neq 0. \quad n = \sup_{x,y \in A} d(x,y)$
 $\exists A_n; \delta n \in \mathbb{N}.$

7.43 () (X; d) $A \subset X, \quad \delta > 0; \exists x_1, \dots, x_n ($),

$$A \subseteq \bigcup_{i=1}^n B(x_i; \delta)$$

$A = \bigcup_{k=1}^n B(x_k; \delta_k) \quad A \subset X.$

7.41 A

- A
- \bar{A}

7.43 () (X; d) $A \subset X,$

(1) A

(2) A

(3) A

(1)) (2) 7.32

(2)) (3) . + .

(3)) (1) A
A
"1 = 1 A x1^1;x1^2;:::;x1^n1

$$A \bigcup_{i=1}^{[n_1]} B(x_1^i;1):$$

$$A \setminus B(x_1^i;1) \qquad \qquad \qquad A_1 \qquad A_1$$
$$"2 = \frac{1}{2} \qquad \qquad \qquad x_2^1;x_2^2;:::;x_2^{n_2}$$

$$A_1 \bigcup_{i=1}^{[n_2]} B \left(x_2^i;\frac{1}{2} \right) :$$

$$A_1 \setminus B \left(x_2^i;\frac{1}{2} \right) \qquad \qquad \qquad A_2$$
$$fA_n g_{n=1}^l$$

(1) A_{n+1} A_n 8n 2 N

(2) A_n 2/n

(3) A_n

$$\bigcap_{n=1}^{\infty} A_n \qquad \qquad \qquad 2 A \qquad A$$
$$U \qquad \qquad \qquad 2 U \qquad \qquad \qquad > 0 \qquad \qquad \qquad B(;) \qquad U$$
$$n \qquad \qquad \qquad \frac{2}{n} <$$

$$A_n \quad B \quad ; \frac{2}{n} \quad U$$

$$A_n \qquad \qquad \qquad A$$

(3) (1)

§ 7.5 Euclid