Discrete Math Homework 16

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1

a), b), d), f)

2

Proof.

- If the simple path passes through w, due to the path is simple, it's obvious that $d(u,v) = d(u,w) + d(w,v)$.
- If $d(u,v) = d(u,w) + d(w,v)$, let the path from u to w is $u = x_0, x_1, \cdots, x_n = w$, $d(u,v) = n$, let the path from v to w is $v = y_0, y_1, \cdots, y_m = w$, $d(v,w) = m$, then connect the two path $u = x_0, x_1, \cdots, x_n = w = y_m, y_{m-1}, \cdots, y_0 = v$, then length of the path is $m + n = d(u,v)$, so the path is the simple path between u, v, i.e. is passes through w.

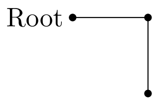
□

3

Let the root denoted by r.

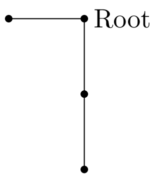
(a) *Proof.* For any $(u,v) \in R_1$, there is a unique simple path from r to v which passes through u and $u \neq v$, i.e. u's level is strictly lower than v, so $(u,v) \in R_2$. So $R_1 \subseteq R_2$. □

(b) *Proof.*



□

(c) *Proof.*



□

4

Proof. According to the definition of ancestor, there is a unique path from root r to w which passes through u, v, i.e. $r = x_0, x_1, \cdots, x_n = w$, $\exists i, j \in \{1, 2, \cdots, n\}$, $u = x_i, v = x_j$.
If $i = j$, then $u = v$.
If $i < j$, then $r = x_0, x_1, \cdots, x_i = u, \cdots, x_j = v$, so u is the ancestor of v.
If $i > j$, then similarly v is the ancestor of u. □

5

Let $P(G)$ denote that the rooted tree G is a tree such that every internal vertex has at least two children, $Q(G)$ denote that G's leaves is more than its internal vertexes, i.e. $n(\text{internal vertexes}) \leqslant n(\text{leaves}) - 1$.

Proof.

- Let T_1 is a root tree with height of 1, then the number of internal vertex is 1 and the number of leaves is more than 2. So $Q(T_1)$.
- Assume that all the root tree T_k with $P(T_k)$ and height less than n satifies $Q(T_k)$, then for any rooted tree T with height n and $P(T)$, let T's root be r, then it has t $(t \geqslant 2)$ children.
  All the subtree with root of r's children $(S_1, S_2, \cdots, S_p)$ satifies property P and their height less than n, so according to the assumption, their leaves are more than their internal vertexes. so

$$
\begin{aligned}
n(\text{internal vertexes of T}) &= 1 + \sum_{i=1}^{p} n(\text{internal vertices of } S_i) \qquad (p \geq 2) \\
&\leqslant 1 - p + \sum_{i=1}^{p} n(\text{leaves of } S_i) \\
&< n(\text{leaves of } T)
\end{aligned}
$$

According to inducing principle, all the rooted trees such that every internal vertex has at least two children have more leaves than internal vertexes. ☐