Dscrete Math Homework 13

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1

Proof.

- Noiting that $0 = \varnothing$, so $\forall x(x \subseteq A \wedge \neg x = \varnothing \to \exists y(y \in x \wedge \forall z(z \in x \to y \in z \vee y = z)))$ $\quad (*)$ is always true. So 0 is $\in$-well-ordered.
- If $x \subseteq n$, then according to the condition, x satisfies the expression $(*)$.
  If $x = x' \cup \{n\}$, where $x' \subseteq n$, let y be the $\in$-least element of x', so $y \in x' \subseteq n \Rightarrow y \in \{n\}$. And for any $z \in x'$, it holds that $y \in z \vee y = z$.
  In summary, $n \cup \{n\}$ is also $\in$-well-ordered.

$\square$

2

Proof. Obviously Inducive$(u) \to u \subseteq v$, Inducive$(u) \to v \subseteq u$.
So $\forall a(a \in v \to a \in u), \forall a(a \in v \to a \in u)$, i.e. $u = v$. $\square$

3

Proof.

- Noting that $\varnothing \in u, \varnothing \in v$, so $varnothing \in u \cap v$.
- $\forall a \in u, a \cup \{a\} \in u, \forall a \in v, a \cup \{a\} \in u$, so $\forall a \in u \cap v$, we have $a \cup \{a\} \in u, a \cup \{a\} \in v$, i.e. $a \cup \{a\} \in u \cap v$.

So $u \cap v$ is alse inducive. $\square$

4

Let V denote $\{x \in u \mid \forall v(v \subseteq u \wedge \text{Inducive}(v) \to x \in v)\}$

- *Proof.* Given $v \subseteq u \wedge \text{Inducive}(v)$, we know that $\forall x \in V, x \in v$. Noting that v is inducive, $x \cup \{x\} \in v$. Due to the arbitrariness of v, $x \cup \{x\} \in V$. And obviously $\varnothing \in V$. So $V = \{x \in u \mid \forall v(v \subseteq u \wedge \text{Inducive}(v) \to x \in v)\}$ is also inducive. $\square$
- *Proof.* If there exists an inducive subset v_0 of u that $v_0 \subseteq V$, we prove that $v_0 = V$.
  According to the definition of V, for any $x \in V$, due to v_0 is inducive, $x \in v_0$, so $V \subseteq v_0$. So $V = v_0$, V is the smallest inducive subset of u. $\square$

5

Proof. Let K denote the set of all inducive sets, T_u denote the smallest inducive subseteq of the inducive set u.
Then we prove that $\bigcap\limits_{u \in K} T_u$ is the smallest inducive set.
According to **3**, $\bigcap\limits_{u \in K} T_u$ is an inducive set.
For any inducive set $u' \subseteq \bigcap\limits_{u \in K} T_u, \text{Inducive}(u')$. Noting that $u' \in K$, so $T_{u'} \subseteq u'$, so $\bigcap\limits_{u \in K} T_u \subseteq u'$, i.e. $u' = \bigcap\limits_{u \in K} T_u$.
So $\bigcap\limits_{u \in K} T_u$ is the smallest inducive set. □

6

Proof.

a) Noting that $0 = \varnothing \in X$, so $\varnothing = 0 \in \mathbb{N} \cap X$.
Also, for any $n \in \mathbb{N} \cap X$, $n \cup \{n\} \in X$, so $\mathbb{N} \cap X$ is an inducive set.

b) Noting that $\mathbb{N} \cap X \subseteq \mathbb{N}$, $\mathbb{N} \cap X$ is an inducive set and $\mathbb{N}$ is the smallest inducive set, we have $\mathbb{N} = \mathbb{N} \cap X$, i.e. $\forall n \in \mathbb{N}$, $n \in X$ always holds.

□