# Discrete Math Homework 11

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## 1

Let $A = [0,1], B = (0,1)$, then $C_0 = A \setminus \{G(b)|b \in B\} = \{0,1\}$, $D_0 = \{F(a)|a \in C_0\} = \left\{\frac{1}{3}, \frac{2}{3}\right\}$.
According to the define,

$$C_{n+1} = \{G(b)|b \in D_n\} = \{G(F(a))|a \in C_n\} = \left\{\frac{1}{3}x + \frac{1}{3}|x \in C_n\right\}$$

Then we have $C_n = \left\{\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{3^n}, \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3^n}\right\}$, $D_n = \left\{\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{3^{n+1}}, \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3^{n+1}}\right\} = C_{n+1}$
Let $H : A \to B$,

$$H(a) = \begin{cases} \frac{1}{3}a + \frac{1}{3}, & a \in \bigcup\limits_{n=0}^{\infty} C_n \\ a, & a \notin \bigcup\limits_{n=0}^{\infty} C_n \end{cases}$$

obviously $H$ is a function.
$\forall b \in D_n$, $\exists a \in C_n, F(a) = b$. $\forall b \in B \setminus \bigcup\limits_{n=0}^{\infty} D_n, H(b) = b$, because if $a \in C_i$, then $b = H(a) \in D_i$, contradicts with $b \in B \setminus \bigcup\limits_{n=0}^{\infty} D_n$. So $H$ is surjective.
If $a, a' \in \bigcup\limits_{n=0}^{\infty} C_n$, then $H(a), H(a') \in \bigcup\limits_{n=0}^{\infty} D_n$. If $H(a) = H(a') \in D_n$, then $\frac{1}{3}a + \frac{1}{3} = \frac{1}{3}a' + \frac{1}{3} \Rightarrow a = a'$.
If $a, a' \notin \bigcup\limits_{n=0}^{\infty} C_n$, if $H(a) = H(a')$, then $a = a'$. So $H$ is injective.
So $H$ is a bijection from $[0,1]$ to $(0,1)$.

## 2

a) *Proof.* If $(a, [a]_R), (a, [a']_R)$, then $a = a = a'$, so $F$ is a function. $\square$

b) *Proof.* $\forall b = [a]_R \in B$, $\exists a \in A, F(a) = [a]_R$, so $F$ is a surjection from $A$ to $B$. $\square$

c) *Proof.* For any $(a,b) \in \{(a,b)|F(a) = F(b)\}$, we have $[a]_R = [b]_R$, i.e. $(a,b) \in R$, $\{(a,b)|F(a) = F(b)\} \subseteq R$.
For any $(a,b) \in R$, we have $[a]_R = [b]_R$, so $F(a) = F(b)$, $(a,b) \in \{(a,b)|F(a) = F(b)\}$, $R \subseteq \{(a,b)|F(a) = F(b)\}$.
In summary, $R = \{(a,b)|F(a) = F(b)\}$ $\square$

## 3

- *Proof.* According to $aR_1a, bR_2b$, $((a,b),(a,b)) \in R$, $R$ is reflexive.
According to $aR_1b, bR_1c \Rightarrow aR_1c$, $a'R_2b', b'R_2c' \Rightarrow a'R_2c'$, then $((a,a'),(b,b')) \in R$, $((b,b'),(c,c')) \in R$, then $((a,a'),(c,c')) \in R$, $R$ is transitive.

According to $aR_1a' \Leftrightarrow a'R_1a$, $bR_2b' \Leftrightarrow b'R_2b$, we have $((a,b),(a',b')) \in R \Leftrightarrow ((a',b'),(a,b)) \in R$, $R$ is symmetric.
In summary, $R$ is a equivalence relations on $A_1 \times A_2$. □

- *Proof.* Let $F \subseteq (P_1 \times P_2) \times P$, $\forall a_1 \in A_1, a_2 \in A_2$, $(([a_1]_{R_1}, [a_2]_{R_2}), [(a_1,a_2)]_R) \in F$. Now we prove that $F$ is a bijection from $P_1 \times P_2$ to $P$.
  According to $F$'s definition, $F$ is a function.
  $\forall [(a_1,a_2)]_R \in P$, there exists $([a_1]_{R_1}, [a_2]_{R_2}) \in P_1 \times P_2$, $F(([a_1]_{R_1}, [a_2]_{R_2})) = [(a_1,a_2)]_R$, $F$ is a injection.
  If $F(([a]_{R_1}, [b]_{R_2})) = F(([a']_{R_1}, [b']_{R_2})) = [(a_0,b_0)]_R$, then $[(a,b)]_R = [(a',b')]_R = [(a_0,b_0)]_R$, so $((a,b),(a',b')) \in R$, i,e, $aR_1a', bR_2b'$, so $[a]_{R_1} = [a']_{R_1}, [b]_{R_2} = [b']_{R_2}$, $([a]_{R_1}, [b]_{R_2}) = ([a']_{R_1}, [b']_{R_2})$, $F$ is a surjection.
  So $F$ is a bijection, so $P_1 \times P_2 \approx P$. □