Discrete Math Homework 18

noflowerzzk

2024.12.26

1

Let $A_k = \left\{ f \;\middle|\; k \notin \bigcup \{ f(x) \mid x \in \mathbb{N} \text{ and } 1 \leqslant x \leqslant n \} \right\}$, we have

$$
\begin{aligned}
|A_k| = \left(2^{m-1}\right)^n = 2^{(m-1)n} \\
\left| \bigcap_{j=1}^{l} A_{k_j} \right| = 2^{(m-l)n}
\end{aligned}
$$

So

$$
\begin{aligned}
\left| \bigcup_{i=1}^{m} A_i \right| &= \sum_{l=1}^{m} (-1)^{l+1} \sum_{1 \leqslant k_1 < k_2 < \cdots < k_l \leqslant m} \left| \bigcap_{j=1}^{l} A_{k_j} \right| \\
&= \sum_{l=1}^{m} (-1)^{l+1} \binom{m}{l} 2^{(m-l)n} \\
&= -2^{mn} \sum_{l=1}^{m} \binom{m}{l} \left(-\frac{1}{2^n} \right)^l
\end{aligned}
$$

So the number of such function that is

$$
\begin{aligned}
& 2^{mn} - \left| \bigcup_{i=1}^{m} A_i \right| \\
=& 2^{mn} \sum_{l=0}^{m} \binom{m}{l} \left(-\frac{1}{2^n} \right)^l \\
=& 2^{mn} \left(1 - \frac{1}{2^n} \right)^m = (2^n - 1)^m
\end{aligned}
$$

2

Let $A_k = \{ f \mid f(k) = 2k - 1 \}$. We have

$$
|A_k| = \begin{cases} 0, & k = 0 \text{ or } k \geqslant \left\lceil \dfrac{n}{2} \right\rceil + 1 \\ (n)!, & 1 \leqslant k \leqslant \left\lceil \dfrac{n}{2} \right\rceil \end{cases}
$$

$$
\left| \bigcap_{j=1}^{l} A_{k_j} \right| = \begin{cases} 0, & \exists j, k_j = 0 \text{ or } k_j \geqslant \left\lceil \dfrac{n}{2} \right\rceil + 1 \\ (n+1-l)!, & \text{other} \end{cases}
$$

So the number of such functions is

$$\begin{aligned}
&(n+1)! - \sum_{l=1}^{n} (-1)^{l+1} \sum_{1\leqslant k_1<k_2<\cdots<k_l\leqslant n} \left|\bigcap_{j=1}^{l} A_{k_j}\right| \\
=&(n+1)! - \sum_{l=1}^{n} (-1)^{l+1} \binom{\lceil \frac{n}{2} \rceil}{l} (n+1-l)! \\
=&\sum_{l=0}^{n} (-1)^{l} \binom{\lceil \frac{n}{2} \rceil}{l} (n+1-l)!
\end{aligned}$$