作业七

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P135 T1

$$
\begin{aligned}
&(11)\quad \frac{\partial u}{\partial x}=(3x^2+y^2+z^2)e^{x(x^2+y^2+z^2)},\quad \frac{\partial u}{\partial y}=2xye^{x(x^2+y^2+z^2)},\quad \frac{\partial u}{\partial z}=2xze^{x(x^2+y^2+z^2)}.\\
&(12)\quad \frac{\partial u}{\partial x}=\frac{y}{z}x^{z-1},\quad \frac{\partial u}{\partial y}=\frac{\ln x}{z}x^{\frac{y}{z}},\quad \frac{\partial u}{\partial z}=-\frac{y\ln x}{z^2}x^{\frac{y}{z}}.\\
&(13)\quad \frac{\partial u}{\partial x}=-\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}},\quad \frac{\partial u}{\partial y}=-\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}},\quad \frac{\partial u}{\partial z}=-\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}.\\
&(14)\quad \frac{\partial u}{\partial x}=y^zx^{z-1},\quad \frac{\partial u}{\partial y}=zy^{z-1}xy^z\ln x,\quad \frac{\partial u}{\partial z}=y^zx^z\ln x\ln y.\\
&(15)\quad \frac{\partial u}{\partial x_i}=a_i,\quad i=1,2,\cdots,n.\\
&(16)\quad \frac{\partial u}{\partial x_i}=\sum_{j=1}^{n}a_{ij}y_j,\quad i=1,2,\cdots,n,\quad \frac{\partial u}{\partial y_j}=\sum_{i=1}^{n}a_{ij}x_i,\quad j=1,2,\cdots,n.
\end{aligned}
$$

P135 T3

证明. 由于

$$\frac{\partial z}{\partial x}=\frac{1}{y^2}e^{\frac{x}{y^2}},\quad \frac{\partial z}{\partial y}=-\frac{2x}{y^3}e^{\frac{x}{y^2}},$$

所以

$$2x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0.$$

□

P135 T12

证明.

$$\lim_{(x,y)\to(0,0)}f(x,y)=\lim_{(x,y)\to(0,0)}\sqrt[3]{xy}=0=f(0,0),$$

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{\sqrt[3]{\Delta x\cdot 0}-0}{\Delta x}=0,\quad f_y(0,0)=\lim_{\Delta y\to 0}\frac{\sqrt[3]{0\cdot\Delta y}-0}{\Delta y}=0,$$

所以函数在原点 $(0,0)$ 连续且可偏导。取 $v=(\cos\alpha,\sin\alpha)$ ，则在该方向，

$$\frac{\mathrm{d}f}{\mathrm{d}v}=\lim_{t\to 0^+}\frac{f(0+t\cos\alpha,0+t\sin\alpha)-f(0,0)}{t}$$

$$= \lim_{t\to 0^+} \frac{\sqrt[3]{t\cos\alpha \cdot t\sin\alpha}}{t} = \lim_{t\to 0^+} \frac{\sqrt[3]{\sin 2\alpha}}{\sqrt[3]{2t}},$$

当 $\sin 2\alpha = 0$ ，即 $\alpha = \dfrac{k\pi}{2}$ 时，极限存在且为零；当 $\sin 2\alpha \neq 0$ ，即 $\alpha \neq \dfrac{k\pi}{2}$ 时，极限不存在。所以除方向 $\boldsymbol{e_i}$，$-\boldsymbol{e_i}(i=1,2)$ 外，在原点的沿其他方向的方向导数都不存在。

□

P135 T13

证明.

$$\frac{|xy|}{\sqrt{x^2+y^2}} \leq \sqrt{x^2+y^2} \to 0$$

故

$$\lim_{(x,y)\to(0,0)} \frac{|xy|}{\sqrt{x^2+y^2}} = 0$$

. 由此可得函数在原点连续。又 $f_x(0,0) = f_y(0,0) = 0$，但 $f(\Delta x, \Delta y) - f(0,0) - (f_x(0,0)\Delta x + f_y(0,0)\Delta y) \neq o(\sqrt{\Delta x^2 + \Delta y^2})$. 故 $f(x)$ 在 $(0,0)$ 不可微。 □

P135 T16

(4)

$$\frac{\partial^4 u}{\partial x^4} = -\frac{3\cdot 2a^3}{(ax+by+cz)^4}\frac{\partial(ax+by+cz)}{\partial x} = -\frac{6a^4}{(ax+by+cz)^4},$$
$$\frac{\partial^4 u}{\partial x^2\partial y^2} = \frac{\partial^4 u}{\partial y^2\partial x^2} = -\frac{3\cdot 2a^2 b}{(ax+by+cz)^4}\frac{\partial(ax+by+cz)}{\partial y} = -\frac{6a^2b^2}{(ax+by+cz)^4}.$$

(5)

$$\begin{aligned}\frac{\partial^{p+q} z}{\partial x^p \partial y^q} &= \frac{\partial^p}{\partial x^p}\left(\frac{\partial^q z}{\partial y^q}\right) = \frac{\partial^p}{\partial x^p}\left((x-a)^p \frac{\partial^q (y-b)^q}{\partial y^q}\right) \\ &= \frac{\mathrm{d}^p (x-a)^p}{\mathrm{d}x^p}\frac{\mathrm{d}^q (y-b)^q}{\mathrm{d}y^q} = p!\,q!\,.\end{aligned}$$

(6) 由 Leibniz 公式可得

$$\begin{aligned}\frac{\partial^{p+q+r} u}{\partial x^p \partial y^q \partial z^r} &= \frac{\partial^p (xe^x)}{\partial x^p}\frac{\partial^q (ye^y)}{\partial y^q}\frac{\partial^r (ze^z)}{\partial z^r} \\ &= \frac{\mathrm{d}^p (xe^x)}{\mathrm{d}x^p}\frac{\mathrm{d}^q (ye^y)}{\mathrm{d}y^q}\frac{\mathrm{d}^r (ze^z)}{\mathrm{d}z^r} \\ &= (x+p)e^x \cdot (y+q)e^y \cdot (z+r)e^z \\ &= (x+p)(y+q)(z+r)e^{x+y+z}.\end{aligned}$$

由 x 的偏导数可得 $f(x,y) = -x\sin y - \dfrac{1}{y}\ln(1-xy) + g(y)$. 令 $x=0$ 有 $g(y) = 2\sin y + y^3$. 因此 $f(x,y) = (2-x)\sin y - \dfrac{1}{y}\ln(1-xy) + y^3$

P135 T20

(1)

$$f_1'(x,y,z) = (1,0,0), \quad f_2'(x,y,z) = (0,1,0), \quad f_3'(x,y,z) = (0,0,1),$$

因此 $\boldsymbol{f}$ 的导数为单位阵.

(2) $f_x'(x,y,z) = (1,0,0)$, 则 $f_x(x,y,z) = x + C_1$. 同理 $f_y(x,y,z) = y + C_2, f_z(x,y,z) = z + C_3$.

(3) 类似 (2) 有 $f_x(x,y,z) = \displaystyle\int p(x)\mathrm{d}x, f_y(x,y,z) = \int q(y)\mathrm{d}y, f_z(x,y,z) = \int r(z)\mathrm{d}z$.