Mutual Implication of Theorems in the Real Number System

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1 Supremum Axiom & Monotone Convergence Theorem

1.1 Supremum Axiom⇒Monotone Convergence Theorem

Proof. Take an in increasing sequence $\{x_n\}$ s.t. $\forall n \in \mathbb{N}, x_n < M$
By Supremum Axiom, we know that $\{x_n\}$ has a supremum $L = \sup\{x_n\}$
For $\forall \varepsilon > 0, \exists n_0$ s.t.

$$x_{n_0} + \varepsilon > L$$

Because $\{x_n\}$ is monotonically increasing, for all $n > n_0$, we have

$$L - x_n < \varepsilon$$

i.e.

$$|L - x_n| < \varepsilon$$

That means

$$\lim_{n\to\infty} x_n = L = \sup\{x_n\}$$

□

1.2 Monotone Convergence Theorem⇒Supremum Axiom

Proof. Referring to **The Nested Closed Interval Theorem** □

2 Monotone Convergence Theorem & The Nested Closed Interval Theorem

2.1 Monotone Convergence Theorem ⇒ The Nested Closed Interval Theorem

Theorem 1. *For a sequence of losed intervals $[a_n, b_n]$ with the following properties:*

(1) $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$

(2) $\lim_{n\to\infty} a_n - b_n = 0$

Then there exist a unique ξ

$$\bigcap_{i=1}^{\infty}[a_n, b_n] = \{\xi\}$$

Proof. For nested closed intervals $\{[a_n, b_n]\}$, obviously

$$\bigcap_{i=1}^{\infty}[a_n, b_n]$$

is not empty. Noting that

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \leqslant \cdots \leqslant b_n \leqslant \cdots \leqslant b_2 \leqslant b_1$$

So we have

$$\lim_{n\to\infty} a_n = A, \lim_{n\to\infty} b_n = B$$

If there exist

$$\xi, \xi' \in \bigcap_{i=1}^{\infty} [a_n, b_n]$$

Because $\lim_{n\to\infty}(a_n - b_n) = 0$, we have $A = B$. And $A \leqslant \xi, \xi' \leqslant B$, So $\xi = \xi'$. □

2.2 The Nested Closed Interval Theorem⇒Monotone Convergence Theorem

3 Supremum Axiom & The Nested Closed Interval Theorem

3.1 The Nested Closed Interval Theorem ⇒ Supremum Axiom

Proof. Assume a number set A. When A is finit, obviously its supremum is $\max A$.
If A is infinit, without loss of generality, let A has upper bounds. Let B be the set of upper bounds of A.
We choose $a_1 \in A$, and let $C = \{x | x > a_1, x \notin B\}$.
Then choose $c_1 \in C, b_1 \in B$, we have $c_1 < b_1$.
If $\dfrac{c_1 + b_1}{2} \in C$, let $c_2 = \dfrac{c_1 + b_1}{2}, b_2 = b_1$. Otherwise let $c_2 = c_1, b_2 = \dfrac{c_1 + b_1}{2}$. By analogy, we have construct a sequence of closed intervals

$$\{[c_n, b_n]\}$$

which satisfies all the condition of The Nested Closed Interval Theorem.
So we have

$$\bigcap_{i=1}^{\infty} [c_n, b_n] = \{\xi\}, \lim_{n\to\infty} c_n = \lim_{n\to\infty} b_n = \xi$$

We will now prove that $\xi = \sup A$.
According to the definition of limit, $\forall \varepsilon > 0, \exists N, \forall n > N, \xi - c_n < \varepsilon$.
Noting that c_n is not the upper bound of A,
so $\exists \varphi \in A, \varphi > \xi - \varepsilon$. That means ξ is the supremum of A. □

3.2 Supremum Axiom ⇒ The Nested Closed Interval Theorem

Proof. Referring to **Monotone Convergence Theorem** □

4 Cauchy's convergence test & Supremum Axiom

4.1 Cauchy's convergence test ⇒ Supremum Axiom

Proof. Assume that S is a set with upper bounds.
According to Archimedean property, for any $a > 0$, there exist k, $ka = \lambda_a$ is the upper bound of S while $\lambda_a - a = (k-1)a$ is not.
Now let $a_n = \frac{1}{n}$, then we have a sequence of λ_n.
Noting that exist $a' > \lambda_n - \frac{1}{n}$.
Also, $\lambda_m \geqslant a'$, so we have

$$\lambda_n - \lambda_m < \frac{1}{n}$$

Simiarly,

$$\lambda_m - \lambda_n < \frac{1}{m}$$

So

$$|\lambda_m - \lambda_n| < \max\left\{\frac{1}{n}, \frac{1}{m}\right\}$$

So according to **Cauchy's convergence test**, λ_n have a limit.
Now we will prove that the limit is the supremum of S.
Let $\lambda = \lim\limits_{n\to\infty} \lambda_n$. Obviously λ is the upper bound of S. Also $\exists a' > \lambda - \delta$.
So λ is the supremum of S. □