作业八

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P184 T7

(1) 证明. $\ln(x+1)-(x-\dfrac{x^2}{2})=\dfrac{x^3}{3(1+\xi)^3}>0\quad 0<\xi<x,$

$\ln(x+1)-(x-\dfrac{x^2}{2}+\dfrac{x^3}{3})=-\dfrac{x^4}{4(1+\xi)^4}<0\quad 0<\xi<x.$ □

(2) 证明.

$$
\begin{aligned}
&(1+x)^\alpha-(1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2)\\
=&\frac{\alpha(\alpha-1)(\alpha-2)}{6}x^3\xi^{\alpha-2} \qquad \xi\in(0,x)\\
<&0
\end{aligned}
$$

□

P184 T10

证明. $\left|\dfrac{f'(x_1)-f'(x_2)}{x_1-x_2}\right|=|f''(\xi)|\leqslant 1,\ \forall x_1,x_2\in[0,1],\xi\in(x_1,x_2).$
故 $|f'(x_1)-f'(x_2)|\leqslant|x_1-x_2|\leqslant 1.$
又 $f(x)$ 在 $(0,1)$ 能取到最大值，$\exists x_0\in(0,1),f(x_0)=\dfrac{1}{4},\ f'(x_0)=0.$
因此 $f(x)=f(x_0)+\dfrac{1}{2}f''(\xi)(x-x_0)^2,\xi$ 在 x,x_0 之间.

$$
\begin{aligned}
&|f(1)|+|f(0)|\\
=&\left|\frac{1}{4}+\frac{1}{2}f''(\xi_1)(0-x_0)^2\right|\\
&\left|+\frac{1}{4}+\frac{1}{2}f''(\xi_2)(1-x_0)^2\right| \qquad 0<\xi_1<x_0<\xi_2<1\\
\leqslant&\frac{1}{2}+\left|\frac{1}{2}f'(\xi_1)(0-x_0)^2\right|+\left|\frac{1}{2}f'(\xi_2)(1-x_0)^2\right|\\
\leqslant&1
\end{aligned}
$$

□

P184 T11

证明. 对 $x, x_0 \in [0,1], \exists \xi$ 在 x, x_0 之间，

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(\xi)(x - x_0)^2$$

取 $x = 0$ 有

$$\begin{aligned}|f'(x_0)| &= \left|f(0) - f(x_0) - \frac{1}{2}f''(\xi)x_0^2\right| \\ &\leqslant 1 + 1 + \frac{1}{2}\cdot 2 \cdot 1^2 \\ &= 3\end{aligned}$$

由 x_0 任意性即证. □

P184 T12

证明. 由于 $f(0) = f(1) \neq \min\limits_{0\leqslant x\leqslant 1} f(x) = -1$, 因此 $\min\limits_{0<x<1} f(x) = -1$, 即存在 $0 < x_0 < 1, f(x_0) = -1, f'(x_0) = 0$.
把 $f(x)$ 在 $x = x_0$ 处展开，有

$$f(x) = -1 + f'(x_0)(x - x_0) + \frac{1}{2}f''(\xi)(x - x_0)^2, \quad \xi在x, x_0之间$$

有

$$\begin{aligned}0 &= -1 + 0 + \frac{1}{2}f''(\xi_1)x_0^2 && 0 < \xi_1 < x_0 \\ 0 &= -1 + 0 + \frac{1}{2}f''(\xi_2)(1 - x_0)^2 && x_0 < \xi_2 < 1\end{aligned}$$

因此

$$\begin{aligned}f''(\xi_1) + f''(\xi_2) &= 2\left(\frac{1}{x_0^2} + \frac{1}{(1 - x_0)^2}\right) \\ &\geqslant 2\left(\frac{(1+1)^3}{(1 - x_0 + x_0)^2}\right) \\ &= 16\end{aligned}$$

故 $f''(\xi_1), f''(\xi_2)$ 中至少有一个大于等于 8, 即 $\max\limits_{0\leqslant x\leqslant 1} f(x) \geqslant 8$. □

P184 T13

证明. 由最值存在定理，$\exists x_0 \in [a,b], |f(x_0)| = \max\limits_{a\leqslant x\leqslant b}|f(x)|$, 且 $f'(x_0) = 0$
把 $f(x)$ 在 $x = x_0$ 处展开，有

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(\xi)(x - x_0)^2, \quad \xi在x, x_0之间$$

代入 $x = a, x = b$ 有:

$$\begin{aligned}\frac{1}{2}f''(\xi_1)(a - x_0)^2 &= f(a) - f(x_0) - f'(x_0)(a - x_0) && a < \xi_1 < x_0 \\ \frac{1}{2}f''(\xi_2)(x_0 - b)^2 &= f(b) - f(x_0) - f'(x_0)(b - x_0) && x_0 < \xi_2 < b\end{aligned}$$

相加有

$$\begin{aligned}
|f''(\xi_1)| + |f''(\xi_2)| &= 2\,|f(x_0)| \left(\frac{1}{(a-x_0)^2} + \frac{1}{(b-x_0)^2} \right) \\
&\geqslant 2\,|f(x_0)| \left(\frac{(1+1)^3}{(a-x_0+x_0-b)^2} \right) \\
&= \frac{16}{(a-b)^2}\,|f(x_0)|
\end{aligned}$$

而

$$\begin{aligned}
\max_{a\leqslant x\leqslant b} |f''(x)| &\geqslant \frac{1}{2}\left(|f''(\xi_1)| + |f''(\xi_2)|\right) \\
&= \frac{8}{(a-b)^2}\,|f(x_0)| \\
&= \frac{8}{(a-b)^2} \max_{a\leqslant x\leqslant b} |f(x)|
\end{aligned}$$

即

$$\max_{a\leqslant x\leqslant b} |f(x)| \leqslant \frac{1}{8}(a-b)^2 \max_{a\leqslant x\leqslant b} |f''(x)|$$

□

补充题

证明. 设 $\lim\limits_{x\to+\infty} \dfrac{f'(x)}{g'(x)} = A$
注意到对任意 $a \leqslant x, y$,

$$\begin{aligned}
\frac{f(x)}{g(x)} &= \left(1 - \frac{g(y)}{g(x)}\right) \frac{f(x)-f(y)}{g(x)-g(y)} + \frac{f(y)}{g(x)} \\
&= \left(1 - \frac{g(y)}{g(x)}\right) \frac{f'(\xi)}{g'(\xi)} + \frac{f(y)}{g(x)} \qquad \xi\text{在}x,y\text{之间}
\end{aligned}$$

(1) 当 $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} g(x) = 0$ 时，对给定的 $x > a, 1 > \varepsilon > 0$, 有

$$\exists N' > x, \forall y > N', \left|\frac{g(y)}{g(x)}\right| < \varepsilon, \left|\frac{f(y)}{g(x)}\right| < \varepsilon$$

当 A 是有限量时，
对 $1 > \varepsilon > 0$, 有

$$\exists N > a, \forall x > N, \left|\frac{f'(x)}{g'(x)} - A\right| < \varepsilon$$

因此 $x > \max\{N, N'\}$ 时

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - A\right| &= \left|\left(1 - \frac{g(y)}{g(x)}\right)\left(\frac{f'(\xi)}{g'(\xi)} - A\right) + \frac{f(y)}{g(x)} + A\left(1 - \frac{g(y)}{g(x)}\right)\right| \\
&\leqslant (1+\varepsilon)\varepsilon + \varepsilon + |A|\,\varepsilon \\
&< (3 + |A|)\varepsilon
\end{aligned}$$

当 $A = +\infty$ 时，
对 $M > 0$, 有

$$\exists N > a, \forall x > N, \frac{f'(x)}{g'(x)} > \frac{M+\varepsilon}{1-\varepsilon}$$

因此 $x > \max\{N, N'\}$ 时

$$\begin{aligned}\frac{f(x)}{g(x)} &= \left(1-\frac{g(y)}{g(x)}\right)\frac{f'(\xi)}{g'(\xi)}+\frac{f(y)}{g(x)}\\&> (1-\varepsilon)\frac{M+\varepsilon}{1-\varepsilon}-\varepsilon\\&= M\end{aligned}$$

$A=-\infty$ 同理.

(2) $\lim\limits_{x\to+\infty} g(x)=+\infty$ 时，对给定的 $y>a, 1>\varepsilon>0$, 有

$$\exists N' > y, \forall x > N', \frac{g(y)}{g(x)} < \varepsilon, \frac{f(y)}{g(x)} < \varepsilon$$

当 A 是有限量时，
对 $1>\varepsilon>0$, 有

$$\exists N > a, \forall x > N, \left|\frac{f'(x)}{g'(x)}-A\right| < \varepsilon$$

因此 $x > \max\{N, N'\}$ 时

$$\begin{aligned}\left|\frac{f(x)}{g(x)}-A\right| &= \left|\left(1-\frac{g(y)}{g(x)}\right)\left(\frac{f'(\xi)}{g'(\xi)}-A\right)+\frac{f(y)}{g(x)}+A\left(1-\frac{g(y)}{g(x)}\right)\right|\\&\leqslant (1+\varepsilon)\varepsilon+\varepsilon+|A|\,\varepsilon\\&< (3+|A|)\varepsilon\end{aligned}$$

当 $A=+\infty$ 时，
对 $M>0$, 有

$$\exists N > a, \forall x > N, \frac{f'(x)}{g'(x)} > \frac{M+\varepsilon}{1-\varepsilon}$$

因此 $x > \max\{N, N'\}$ 时

$$\begin{aligned}\frac{f(x)}{g(x)} &= \left(1-\frac{g(y)}{g(x)}\right)\frac{f'(\xi)}{g'(\xi)}+\frac{f(y)}{g(x)}\\&> (1-\varepsilon)\frac{M+\varepsilon}{1-\varepsilon}-\varepsilon\\&= M\end{aligned}$$

$A=-\infty$ 同理.

$\square$