作业九

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P221 T1

(16)

$$\int \frac{\mathrm{d}x}{(\arcsin x)^2 \sqrt{1-x^2}} = \int \frac{\mathrm{d}\left(\arcsin x\right)}{(\arcsin x)^2} = -\frac{1}{\arcsin x} + C$$

(17)

$$\int \frac{\mathrm{d}x}{x^2-2x+2} = \int \frac{\mathrm{d}(x-1)}{(x-1)^2+1} = \arctan(x-1) + C$$

(18)

$$\int \frac{(1-x)\mathrm{d}x}{\sqrt{9-4x^2}} = \frac{1}{2}\int \frac{\mathrm{d}(\frac{2x}{3})}{\sqrt{1-\left(\frac{2}{3}x\right)^2}} + \frac{3}{8}\int \frac{\mathrm{d}\left(1-\frac{4}{9}x^2\right)}{\sqrt{1-\frac{4}{9}x^2}} = \frac{1}{2}\arcsin\left(\frac{2}{3}x\right) + \frac{1}{4}\sqrt{9-4x^2} + C$$

(19)

$$\int \tan\sqrt{1+x^2}\frac{x}{\sqrt{1+x^2}}\mathrm{d}x = \int \tan\sqrt{1+x^2}\mathrm{d}(\sqrt{1+x^2}) = -\ln\left|\cos\sqrt{1+x^2}\right| + C$$

(20)

$$\int \frac{\sin x\cos x}{1+\sin^4 x}\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}(\sin^2 x)}{1+\sin^4 x} = \frac{1}{2}\arctan\left(\sin^2 x\right) + C$$

P222 T3

(16)

$$\begin{aligned}\int \cos\left(\ln x\right)\mathrm{d}x &= x\cos\left(\ln x\right) - \int x\mathrm{d}\left(\cos\left(\ln x\right)\right)\\ &= x\cos\left(\ln x\right) + \int \sin\left(\ln x\right)\mathrm{d}x\\ &= x\cos\left(\ln x\right) + x\sin\left(\ln x\right) - \int x\mathrm{d}\left(\sin\left(\ln x\right)\right)\\ &= x\cos\left(\ln x\right) + x\sin\left(\ln x\right) - \int \cos\left(\ln x\right)\mathrm{d}x\end{aligned}$$

解得

$$\int \cos\left(\ln x\right)\mathrm{d}x = \frac{x}{2}\left(\cos\left(\ln x\right) + \sin\left(\ln x\right)\right)$$

(17)

$$\begin{aligned}\int(\arcsin x)^2\,\mathrm{d}x &= x(\arcsin x)^2-2\int\frac{x\arcsin x}{\sqrt{1-x^2}}\mathrm{d}x\\ &= x(\arcsin x)^2+2\int\arcsin x\,\mathrm{d}\left(\sqrt{1-x^2}\right)\\ &= x(\arcsin x)^2+2\sqrt{1-x^2}\arcsin x-2\int\sqrt{1-x^2}\mathrm{d}(\arcsin x)\\ &= x(\arcsin x)^2+2\sqrt{1-x^2}\arcsin x-2x+C\end{aligned}$$

(18)

$$\begin{aligned}\int\sqrt{x}\mathrm{e}^{\sqrt{x}}\mathrm{d}x \xlongequal{t=\sqrt{x}} &\int 2t^2\mathrm{d}\mathrm{e}^t\\ &= 2t^2\mathrm{e}^t-\int 4t\mathrm{d}\mathrm{e}^t\\ &= 2t^2\mathrm{e}^t-4t\mathrm{e}^t+4\int\mathrm{e}^t\mathrm{d}t\\ &= 2t^2\mathrm{e}^t-4t\mathrm{e}^t+4\mathrm{e}^t+C\\ &= 2x\mathrm{e}^{\sqrt{x}}-4\sqrt{x}\mathrm{e}^{\sqrt{x}}+4\mathrm{e}^{\sqrt{x}}+C\end{aligned}$$

(19)

$$\int\mathrm{d}x \xlongequal{t=\sqrt{x+1}} \int 2t\mathrm{d}\mathrm{e}^t=2t\mathrm{e}^t-2\int\mathrm{e}^t\mathrm{d}t=2t\mathrm{e}^t-2\mathrm{e}^t+C=2\sqrt{x+1}\mathrm{e}^{\sqrt{x+1}}-2\mathrm{e}^{\sqrt{x+1}}+C$$

(20)

$$\begin{aligned}\int\ln\left(x+\sqrt{1+x^2}\right)\mathrm{d}x &= x\ln\left(x+\sqrt{1+x^2}\right)-\int x\mathrm{d}\left(\ln\left(x+\sqrt{1+x^2}\right)\right)\\ &= x\ln\left(x+\sqrt{1+x^2}\right)-\int\frac{x}{\sqrt{1+x^2}}\mathrm{d}x\\ &= x\ln\left(x+\sqrt{1+x^2}\right)-\sqrt{1+x^2}+C\end{aligned}$$

P222 T8

(7)

$$\begin{aligned}I_n &= \int -x^{n-1}\mathrm{d}\left(\sqrt{1-x^2}\right)\\ &= -x^{n-1}\sqrt{1-x^2}+(n-1)\int\sqrt{1-x^2}x^{n-2}\mathrm{d}x\\ &= -x^{n-1}\sqrt{1-x^2}+(n-1)\int\left(\frac{x^{n-2}}{\sqrt{1-x^2}}-\frac{x^n}{\sqrt{1-x^n}}\right)\mathrm{d}x\\ &= -x^{n-1}\sqrt{1-x^2}+(n-1)I_{n-2}-(n-1)I_n\end{aligned}$$

故

$$I_n=-\frac{x^{n-1}\sqrt{1-x^2}}{n}+\frac{n-1}{n}I_{n-2}$$

其中

$$I_0=\arcsin x+C,\quad I_1=-\sqrt{1-x^2}+C$$

(8)

$$\begin{aligned} I_n &= -\frac{1}{n-1}\int\frac{\mathrm{d}x^{-(n-1)}}{\sqrt{1+x}} \\ &= -\frac{1}{n-1}\cdot\frac{1}{x^{n-1}\sqrt{1+x}}+\frac{1}{n-1}\int\frac{\mathrm{d}x}{2\sqrt{1+x}x^{n-1}} \\ &= -\frac{1}{n-1}\cdot\frac{1}{x^{n-1}\sqrt{1+x}}+\frac{1}{2(n-1)}I_{n-1} \end{aligned}$$

其中

$$I_0 = 2\sqrt{1+x}+C$$

P243 T2

证明. 假设 $f(X)$ 无界，不妨 $f(x)$ 无上界. 则对任意的一个划分 P, 有一个区间 $[x_k, x_{k+1}]$ 上 $f(x)$ 无上界. 即 $\forall M>0, \exists \xi_k \in [x_k, x_{k+1}], f(\xi_k) > M$.
现对任意的 $M>0$, $p-1$ 等分 $P: a = x_1 < x_2 < \cdots < x_p = b$，取 $\delta = \dfrac{b-a}{p-1}$, 在有上界区间取代表元 $\xi_i = \sup\limits_{x\in[x_{i-1},x_i]} f(x)$, 记所有有上界区间的上确界的最小值为 m_0; 无上界区间取 ξ_i 使得 $f(\xi_i) > \max\left\{\dfrac{M}{\delta}-(p-2)m_0, \dfrac{M}{\delta}\right\}$, 则此时

$$\sum_{i=1}^{p} f(\xi_i)\Delta x_i \geqslant (p-1)m_0\delta + \left(\frac{M}{\delta}-(p-2)m_0\right)\delta = M \qquad m_0 < 0$$

$$\sum_{i=1}^{p} f(\xi_i)\Delta x_i \geqslant \frac{M}{\delta}\delta = M \qquad m_0 \geqslant 0$$

综上，存在一个特定的划分与代表元的选取方式使得 $\sum\limits_{i=1}^{p} f(\xi_i)\Delta x_i \quad (\lambda(P)\to 0)$ 无界，因此其极限是否存在依赖代表元与划分的选取，该极限不存在，矛盾！
因此，$f(x)$ 有界. □

P243 T8

证明. $f(x)$ 可积等价于 $\lim\limits_{\lambda(P)\to 0}\sum\limits_{i=1}^{p} w_i\Delta x_i = 0$. 令 $\max\limits_{x\in[a,b]} f(x) = M$, $\min\limits_{x\in[a,b]} f(x) = m$

- 充分性：
  对任意 $\varepsilon > 0$, 存在 $\delta = \sqrt{\dfrac{\varepsilon}{2(p-1)}}, \sigma = \dfrac{\varepsilon}{2(p-1)(M-m)}, \varepsilon_0 = \sqrt{\dfrac{\varepsilon}{2(p-1)}}$ 取分划 P 满足 $\lambda(P) < \delta$, 有

$$\sum_{i=1}^{p} w_i\Delta x_i = \sum_{w_i\geqslant\varepsilon_0} w_i\Delta x_i + \sum_{w_i<\varepsilon_0} w_i\Delta x_i < (p-1)(M-m)\sigma + (p-1)\varepsilon_0\delta = \varepsilon$$

  即 $\lim\limits_{\lambda(P)\to 0}\sum\limits_{i=1}^{p} w_i\Delta x_i = 0$, $f(x)$ 可积.

- 必要性：
取定 $\varepsilon_0 > 0$, 存在 $\sigma = \min\{b-a, \sqrt{\varepsilon_0}\}, \varepsilon = \dfrac{\varepsilon_0}{\sigma}$, 对任意分划 P 满足 $\sum\limits_{w_i \geqslant \varepsilon} \Delta x_i \geqslant \sigma$, 有

$$\sum_{i=1}^{p} w_i \Delta x_i = \sum_{w_i \geqslant \varepsilon_0} w_i \Delta x_i + \sum_{w_i < \varepsilon_0} w_i \Delta x_i \geqslant \varepsilon\sigma + 0 = \varepsilon_0$$

表明 $f(x)$ 不可积，矛盾！因此对任意 $\varepsilon > 0, \sigma > 0$, 存在划分 P, $\sum\limits_{w_i \geqslant \varepsilon} \Delta x_i < \sigma$

$\square$