下作业八

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P145 T1

(8)

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= yf_1\left(xy, \frac{x}{y}\right) + \frac{1}{y}f_2\left(xy, \frac{x}{y}\right) \\
\frac{\partial u}{\partial y} &= xf_1\left(xy, \frac{x}{y}\right) - \frac{x}{y^2}f_2\left(xy, \frac{x}{y}\right) \\
\frac{\partial^2 u}{\partial x \partial y} &= f_1\left(xy, \frac{x}{y}\right) - \frac{1}{y^2}f_2\left(xy, \frac{x}{y}\right) + xyf_{11}\left(xy, \frac{x}{y}\right) - \frac{x}{y^2}f_{22}\left(xy, \frac{x}{y^3}\right) \\
\frac{\partial^2 u}{\partial y^2} &= \frac{2x}{y^3}f_2\left(xy, \frac{x}{y}\right) + x^2 f_{11}\left(xy + \frac{x}{y}\right) - \frac{2x^2}{y}f_{12}\left(xy + \frac{x}{y}\right) + \frac{x^2}{y^4}f_{22}\left(xy + \frac{x}{y}\right)
\end{aligned}
$$

(9)

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= 2xf'(x^2+y^2+z^2) \\
\frac{\partial u}{\partial y} &= 2yf'(x^2+y^2+z^2) \\
\frac{\partial u}{\partial z} &= 2zf'(x^2+y^2+z^2) \\
\frac{\partial^2 u}{\partial x^2} &= 2f'(x^2+y^2+z^2) + 4x^2f''(x^2+y^2+z^2) \\
\frac{\partial^2 u}{\partial x \partial y} &= 4xyf''(x^2+y^2+z^2)
\end{aligned}
$$

(10)

$$
\begin{aligned}
\frac{\partial w}{\partial u} &= f_x + f_y + vf_z \\
\frac{\partial w}{\partial v} &= f_x - f_y + uf_z
\end{aligned}
$$

$$
\frac{\partial^2 w}{\partial u \partial v} = f_{xx} + (u+v)f_{xz} - f_{yy} + (u-v)f_{yz} + f_z + uvf_{zz}
$$

P144 T9

证明. (1) 由于 $\dfrac{\partial f(tx,ty)}{\partial t} = nt^{n-1}f(x,y) = xf_1(tx,ty) + yf_2(tx,ty)$. 代入 $t=1$ 有 $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf$

(2) 易得 $n=1$. $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z = \sqrt{x^2+y^2}$

□

P144 T11

$f'(u,v)=2uv$

$$\begin{pmatrix}\cos\theta & -r\sin\theta\\ r\cos\theta & \sin\theta\end{pmatrix}$$

所以

$$(fg)'(r,0)=f'(g(r,0))g'(r,0)=\begin{pmatrix}2r\cos\theta & -2r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}r\cos\theta\\ r\sin\theta\end{pmatrix}=\begin{pmatrix}2r^2\cos^2\theta\\ 2r^2\cos\theta\sin\theta\end{pmatrix}$$

P144 T17

取 $x=r\cos\theta, y=r\sin\theta$, 有

$$\frac{\partial}{\partial r}f(x,y)=\frac{1}{r}(f_x(x,y)+f_y(x,y))=0$$

故 $f(x,y)$ 只与 θ 有关. 又 $\lim\limits_{(x,y)\to(0,0)}=f(0,0)$ 是常数，故对任意 θ, 有 $f(x,y)$ 是常数.

P151 T2

$f(x,y)=-14-13(x-1)-6(y-2)+5(x-1)^2-12(x-1)(y-2)+4(y-2)^2+3(x-1)^3-2(x-1)^2(y-2)-2(x-1)(y-2)^2+(y-2)^3$

P151 T3

$f(x,y)=xy-\frac{1}{2}xy^2+0\left(\left(\sqrt{x^2+y^2}\right)^3\right)$

P151 T4

$f(x,y)=1+(x+y)+\frac{1}{2}(x+y)^2+\cdots+\frac{1}{n!}(x+y)^n+R_n$, $R_n=\frac{1}{(n+1)!}(x+y)^{n+1}\mathrm{e}^{\theta(x+y)},\theta\in(0,1)$.

P189 T1

(2) $\begin{cases}f_x=0\\ f_y=0\end{cases}$ 得驻点 $(x,y)=(0,0),(1,1),(-1,-1)$. 由于 $f_{xx}=2(6x^2-1),f_{xy}=-2,f_{yy}=2(6y^2-1)$, 有 $H=4(6x^2-1)(6y^2-1)-4$, 得 $(1,1),(-1,-1)$ 是极值点. 又 $f(x,x)$ 在 $(0,0)$ 附近小于 0，$f(x,-x)$ 附近大于 0，故 $f(x,y)$ 在 $(0,0)$ 变号，不是极值点.

(4) 求得驻点 $(0,0),(1,1),(-1,1),\left(\frac{\sqrt{2}}{2},\frac{3}{8}\right),\left(\frac{-\sqrt{2}}{2},\frac{3}{8}\right)$. 又 $H=2(30x^4-12x^2y-2y)-(4x^3+2x)^2$ 得 $\left(\frac{\sqrt{2}}{2},\frac{3}{8}\right),\left(\frac{-\sqrt{2}}{2},\frac{3}{8}\right)$ 上取极小值 $-\frac{1}{64}$.

(6) 求得驻点 $\left(2^{\frac{1}{4}},2^{\frac{1}{2}},2^{\frac{3}{4}}\right)$, 又 Hesse 矩阵正定，故取得极小值 $4\times2^{\frac{1}{4}}$

P189 T2

$$f(x,y,z) = \frac{1}{2}(x-2y)^2 + \frac{1}{2}(x+2z) + y^2$$

最小值为 0, 当 $x = y = z = 0$ 是取到.

P189 T6

$$S = \frac{R^2}{2}(\sin\alpha_1 + \sin\alpha_2 - \sin(\alpha_1+\alpha_2))$$

求导得当 $\alpha_1 = \alpha_2 = \dfrac{2\pi}{3}$ 有 $S_{\max} = \dfrac{3\sqrt{3}}{4}R^2$.

P189 T11

设圆为单位圆，两个顶角为 $2\alpha, 2\beta$.

$$S = \cot\alpha + \cot\beta + \tan(\alpha+\beta)$$

求偏导计算得 $\alpha = \beta = \dfrac{\pi}{2} - \alpha - \beta$ 时取到极值. 即 $\alpha = \beta = \dfrac{\pi}{6}$.

P189 T12

同理设单位圆，各边圆心角为 α_i.

$$S = \frac{1}{2}(\sin\alpha_1 + \cdots + \sin\alpha_n)$$

求偏导计算得

$$\frac{\partial S}{\partial \alpha_k} = \frac{1}{2}(\cos\alpha_k - \cos(\alpha_1 + \cdots + \alpha_{n-1})) = 0$$

解得 $\alpha_k = \dfrac{2\pi}{n}$, 即正 n 边形时面积最大.