Discrete Math Homework 9

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1

b) not reflexive; symmetric; not antisymmetric; not transitive.

f) reflexive; symmetric; not antisymmetric; transitive.

2

a) $R_1 \circ R_1 = \{(a, b) \in \mathbb{R}^2 | a > b\} = R_1$

b) $R_1 \circ R_2 = \{(a, b) \in \mathbb{R}^2 | a > b\} = R_1$

c) $R_1 \circ R_3 = \mathbb{R}^2$

e) $R_1 \circ R_5 = \{(a, b) \in \mathbb{R}^2 | a > b\} = R_1$

f) $R_1 \circ R_6 = \mathbb{R}^2$

g) $R_2 \circ R_3 = \mathbb{R}^2$

h) $R_3 \circ R_3 = \{(a, b) \in \mathbb{R}^2 | a > b\} = R_1$

3

Let $R_1 = \{(1, 2), (2, 1)\} \cup I_A$, $R_2 = \{(2, 3), (3, 2)\} \cup I_A$, $A = \{1, 2, 3\}$.
Then $(1, 2), (2, 3) \in R_1 \cup R_2$, but $(1, 3) \notin R_1 \cup R_2$, $R_1 \cup R_2$ is not transitive.
So it is not a an equivalence relation on A.

4

Proof. Reflexive: Noting that $I_A \subseteq R_1, I_A \subseteq R_2$, so $I_A \subseteq R_1 \cap R_2$.
Symmetric: $\forall a, b \in A$, if $a(R_1 \cap R_2)b$, then aR_1b and aR_2b. Then $bR_1a \Rightarrow b(R_1 \cap R_2)a$, i.e. $a(R_1 \cap R_2)b$ iff. $b(R_1 \cap R_2)a$.
Transitive: $\forall a, b, c \in A, a(R_1 \cap R_2)b, b(R_1 \cap R_2)c$, then aR_1b, bR_1c, so aR_1c, similarly, aR_2c. So $a(R_1 \cap R_2)c$. □

5

a) $[1]_R = \mathbb{Z}$.

b) $\left[\frac{1}{2}\right]_R = \{x | x = a + \frac{1}{2}, a \in \mathbb{Z}\}$.

6

a) *Proof.* Let the set of all equivalence classes of R is R'.
Reflexive: $I_{R'} \subseteq S_2$.
Symmetric: If $[a]_R S_2 [b]_R$ ($[a]_R \neq [b]_R$), then $a, b \in \mathbb{R}, a - b = \frac{1}{2}$. Noting that $b + 1 - a = \frac{1}{2}, b + 1 \in [b]_R \Rightarrow [b]_R = [b+1]_R$, so $[b]_R S_2 [a]_R$.
Transitive: Assume $[a]_R S_2 [b]_R, [b]_R S_2 [c]_R$. If $[a] = [b]$ or $[b] = [c]$, it is obvious that S_2 is transitive. If $[a] \neq [b], [b] \neq [c]$, then $a - b = \frac{1}{2}, b - c = \frac{1}{2}$, so $a - c = 1$, $[a]_R = [c]_R$, $[a]_R S_2 [c]_R$, S_2 is transitive. □

b) *Proof.* Let the set of all equivalence classes of R is R'.
Reflexive: $I_{R'} \subseteq S_3$.
Symmetric: If $[a]_R S_3 [b]_R$ ($[a]_R \neq [b]_R$), then $a, b \in \mathbb{R}, |a - b| = \frac{1}{3} \Leftrightarrow |b - a| = \frac{1}{3}$, i.e. $[b]_R S_3 [a]_R$.
Transitive: Assume $[a]_R S_3 [b]_R, [b]_R S_3 [c]_R$. If $[a] = [b]$ or $[b] = [c]$, it is obvious that S_3 is transitive. If $[a] \neq [b], [b] \neq [c]$, then $|a - b| = \frac{1}{3}, |b - c| = \frac{1}{3}$, so either $a = c$ or $c = a + \frac{2}{3} = (a+1) - \frac{1}{3}$ or $c = a - \frac{2}{3} = (a-1) + \frac{1}{3}$, i.e. $[a]_R S_3 [c]_R$, S_3 is transitive.
□

c) Let $a = 0, b = \frac{1}{4}, c = \frac{1}{2}$, $|a - b| = |b - c| = \frac{1}{4} \Rightarrow [a]_R S_3 [b]_R, [b]_R S_3 [c]_R$, but $a - c \notin \mathbb{Z}, |a - c| \neq \frac{1}{4}$, $[a]_R \not S_4 [c]_R$, S_4 is not transitive.