Discrete Math Homework 8

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a) *Proof.* For all interpretation $\mathcal{J}$, if $[\![\phi]\!]_{\mathcal{J}} = \mathbf{T}$, then $[\![\psi]\!]_{\mathcal{J}} = \mathbf{T}$.
Assuming a $\mathcal{J}_0$, $[\![\forall x\phi]\!]_{\mathcal{J}_0} = \mathbf{T}$, i.e.$\forall a \in \mathcal{J}$'s domain, $[\![\phi]\!]_{\mathcal{J}_0[x\mapsto a]} = \mathbf{T}$, so $[\![\psi]\!]_{\mathcal{J}_0[x\mapsto a]} = \mathbf{T}$, which means $[\![\forall x\psi]\!]_{\mathcal{J}_0} = \mathbf{T}$.
So $\forall x\phi \models \forall x\psi$. □

b) *Proof.* Suppose a interpretation $\mathcal{J}$, $\forall \phi_0 \in \Phi, [\![\phi_0]\!]_{\mathcal{J}} = \mathbf{T}$ and $[\![\forall x\phi]\!]_{\mathcal{J}} = \mathbf{T}$.
So for all $a \in \mathcal{J}$'s domain, , $[\![\phi]\!]_{\mathcal{J}_0[x\mapsto a]} = \mathbf{T}$.
And noting that x does not freely occur in Φ, so $\forall \phi_0 \in \Phi, [\![\phi_0]\!]_{\mathcal{J}[x\mapsto a]} = [\![\phi_0]\!]_{\mathcal{J}} = \mathbf{T}$.
So according to the condition, $[\![\psi]\!]_{\mathcal{J}[x\mapsto a]} = \mathbf{T}$, i.e. $[\![\forall x\psi]\!]_{\mathcal{J}} = \mathbf{T}$.
Thus $\Phi, \forall x\phi \models \forall x\psi$. □

c)
Let $Phi = \{\chi\}$, and a interpretation $\mathcal{J}$, where $[\![\chi]\!]_{\mathcal{J}} = \mathbf{T}$ iff. $\mathcal{J}(x) = a(a \in \mathcal{J}$'s domain), and $[\![\forall x\phi]\!]_{\mathcal{J}} = \mathbf{T}, [\![\forall x\psi]\!]_{\mathcal{J}} = \mathbf{T}$
Then let $b \in \mathcal{J}$'s domain , $[\![\phi]\!]_{\mathcal{J}[x\mapsto b]} = \mathbf{T}, [\![\psi]\!]_{\mathcal{J}[x\mapsto b]} = \mathbf{T}$, but $[\![\chi]\!]_{\mathcal{J}[x\mapsto b]} = \mathbf{F}$.
That indicates that $\Phi, \phi \not\models \psi$.

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a) WRONG

b) CORRECT

c) WRONG

d) CORRECT

3

$A = \{1\}, B = \{1, \{1\}\}$.

4

$A = \{1\}, B = \{1, \{1\}\}, C = \{1, \{1\}, \{1, \{1\}\}\}$

5

Proof.

$$\begin{aligned}\forall X, \quad & X \in \mathcal{P}(A) \cap \mathcal{P}(B) \\ & \leftrightarrow X \in \mathcal{P}(A) \wedge X \in \mathcal{P}(B) \\ & \leftrightarrow X \subseteq A \wedge X \subseteq B \\ & \leftrightarrow \forall x \in X, x \in A \wedge x \in B \\ & \leftrightarrow \forall x \in X, x \in A \cap B \\ & \leftrightarrow X \subseteq A \cap B \\ & \leftrightarrow X \in \mathcal{P}(A \cap B).\end{aligned}$$

So $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$ □

5.1

Proof.

$$\begin{aligned}\forall (x, y), \quad & (x, y) \in A \times \bigcup B \\ & \leftrightarrow x \in A \wedge \exists Y (Y \in B \wedge y \in Y) \\ & \leftrightarrow \exists Y (Y \in B \wedge (x, y) \in A \times Y) \\ & \leftrightarrow \exists T (T \in \{A \times X | X \in B\} \wedge (x, y) \in T) \\ & \leftrightarrow (x, y) \in \bigcup \{A \times X | X \in B\}\end{aligned}$$

So $A \times \bigcup B = \bigcup \{A \times X | X \in B\}$ □