Discrete Math Homework 10

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2024.11.14

1

Proof.

$$\begin{aligned}
R_1 \subseteq R_2 &\Leftrightarrow \forall(x,y)((x,y) \in R_1 \to (x,y) \in R_2) \\
&\Leftrightarrow \forall(x,y)(\exists X(X \in P_1 \wedge x \in X \wedge y \in X) \to \exists Y(Y \in P_2 \wedge x \in Y \wedge y \in Y)) \\
&\Leftrightarrow \forall X(X \in P_1 \to \forall x(x \in X \to \forall y(y \in X \to \exists Y(Y \in P_2 \wedge x \in Y \wedge y \in Y)))) \\
&\Leftrightarrow \forall X(X \in P_1 \to \forall x(x \in X \to \exists Y(Y \in P_2 \wedge \forall y(y \in X \to y \in Y \wedge x \in Y)))) \\
&\Leftrightarrow \forall X(X \in P_1 \to \exists Y(Y \in P_2 \wedge \forall x(x \in X \to x \in Y))) \\
&\Leftrightarrow \forall X(X \in P_1 \to \exists Y(Y \in P_2 \wedge X \subseteq Y)) \\
&\Leftrightarrow P_1 \text{ is the refinement of } P_2
\end{aligned}$$

□

2

Proof. Noting that R is a symmetric relation, we have $R = R^{-1}$.
Then $(R^n)^{-1} = \underbrace{R^{-1} \circ R^{-1} \circ \cdots \circ R^{-1}}_{n \text{ times}} = \left(R^{-1}\right)^n = R^n$, i.e. R^n is a symmetric relation. □

3

Proof.

$\Leftarrow$:
Noting that $R \subseteq \bigcup_{n \in \mathbb{Z}^+} R^n$, $S \subseteq S$. Then $R \circ S \subseteq \bigcup_{n \in \mathbb{Z}^+} R^n \circ S \subseteq S$.

$\Rightarrow$:
Let's proof $\forall n \in \mathbb{Z}^+$, $R^n \circ S \subseteq S$.
Induction on n:
$n = 1$, obviously the conclusion holds.
Assume the conclusion holds at $n \geqslant 1$. Then we have for any a, b, c, $(a, b) \in S, (b, c) \in R^n$, we have $(a, c) \in R^n \circ S$, i,e, $(a, c) \in S$
Then if there exists d, $(c, d) \in R$, then $(a, d) \in S$, i.e. $R \circ (R^n \circ S) = R^{n+1} \circ S \subseteq S$. So by the principle of induction, $\forall n \in \mathbb{Z}^+$, $R^n \circ S \subseteq S$.
So $\bigcup_{n \in \mathbb{Z}^+} R^n \circ S = \bigcup_{n \in \mathbb{Z}^+} (R^n \circ S) \subseteq S$.

□

4

a) *Proof.*
For any $a \in \mathbb{R}$, obviously there is no integer n that satisfies $a < n \leqslant a$, so $(a, a) \in R$, R is reflexive.

Noting that $(a,b) \in R \Leftrightarrow \exists n \in \mathbb{Z}, n \leqslant a < b < n+1 \vee b \leqslant a$. So $\forall a,b,c \in \mathbb{R}, (a,b) \in R, (b,c) \in R$, then $\exists n \in \mathbb{Z}, n \leqslant a < b < n+1, n \leqslant b < c < n+1$ or $n \leqslant a < b < n+1, c \leqslant b$ or $b \leqslant a, n \leqslant b < c < n+1$ or $b \leqslant a, c \leqslant b$, in any case, $(a,c) \in R$, R is transitive.
But $(0, \dfrac{1}{2}) \in R$ and $(\dfrac{1}{2}, 0) \in R$, but $0 \neq \dfrac{1}{2}$, R is not antisymmetric. □

b) *Proof.*
Given R's transitive, so $R \circ R = R^2 \subseteq R$, according to the principle of induction, $\forall n \in \mathbb{Z}^+, R^n \subseteq R$.
So $R^+ = \bigcup\limits_{n \in \mathbb{Z}^+} R^n \subseteq R$.
And it's obvious that $R \subseteq \bigcup\limits_{n \in \mathbb{Z}^+} R^n = R^+$.
So $R = \bigcup\limits_{n \in \mathbb{Z}^+} R^n = R^+$. □

c) *Proof.*
$I_A \subseteq R \Rightarrow I_A \subseteq R \cap R^{-1}$, $R \cap R^{-1}$ is reflexive.
$\forall (x,y) \in R \cap R^{-1}$, obviously that $(x,y) \in R \Rightarrow (y,x) \in R^{-1}, (x,y) \in R^{-1} \Rightarrow (y,x) \in R$, i.e. $(y,x) \in R \cap R^{-1}$, $R \cap R^{-1}$ is symmetric.
$\forall x,y,z \in A, (x,y) \in R \cap R^{-1}, (y,z) \in R \cap R^{-1}$. Given R is transitive, so $R \circ R \subseteq R \Rightarrow (R \circ R)^{-1} = R^{-1} \circ R^{-1} \subseteq R^{-1}$, R^{-1} is transitive. So $(x,z) \in R$ and $(x,z) \in R^{-1} \Leftrightarrow (x,z) \in R \cap R^{-1}$, i.e. $R \cap R^{-1}$ is transitive.
In summary, $R \cap R^{-1}$ is an equivalence relation on A. □

d) *Proof.*
$\forall a,b \in A, [a]_{R \cap R^{-1}} = [b]_{R \cap R^{-1}} \Leftrightarrow a(R \cap R^{-1})b \Rightarrow aRb$. So $\forall [a]_{R \cap R^{-1}} = [b]_{R \cap R^{-1}}, ([a]_{R \cap R^{-1}}, [b]_{R \cap R^{-1}}) \in S$, S is reflexive.
$\forall [a]_{R \cap R^{-1}}, [b]_{R \cap R^{-1}} \in B$, if $([a]_{R \cap R^{-1}}, [b]_{R \cap R^{-1}}) \in S$ and $([b]_{R \cap R^{-1}}, [a]_{R \cap R^{-1}}) \in S$, then aRb and $bRa \Leftrightarrow aR^{-1}b$, i.e. $a(R \cap R^{-1})b$, so $[b]_{R \cap R^{-1}} = [a]_{R \cap R^{-1}}$, S is antisymmetric.
$\forall [a]_{R \cap R^{-1}}, [b]_{R \cap R^{-1}}, [c]_{R \cap R^{-1}} \in B$, if $([a]_{R \cap R^{-1}}, [b]_{R \cap R^{-1}}) \in S, ([b]_{R \cap R^{-1}}, [c]_{R \cap R^{-1}}) \in S$, so aRb, bRc.
Noting that R is transitive, $aRc \Leftrightarrow ([a]_{R \cap R^{-1}}, [c]_{R \cap R^{-1}}) \in S$. S is transitive.
In summary, S is a partial order on B. □