作业十二

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P314 T3

(7)

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(x^2+1)^{3/2}} \xlongequal{x=\tan t} 2\int_0^{\pi/2} \cos t\mathrm{d}t = 2\sin x\bigg|_0^{\pi/2} = 2$$

(8)

$$\int_0^{+\infty} \frac{\mathrm{d}x}{(\mathrm{e}^x+\mathrm{e}^{-x})^2} \xlongequal{x=\frac{\ln t}{2}} \frac{1}{2}\int_1^{+\infty} \frac{\mathrm{d}t}{(t+1)^2} = \frac{1}{2}\left(-\frac{1}{t+1}\bigg|_1^{+\infty}\right) = \frac{1}{4}$$

(9)

$$\begin{aligned}\int_0^{+\infty} \frac{\mathrm{d}x}{x^4+1} &= \int_0^1 \frac{\mathrm{d}x}{x^4+1} + \int_1^{+\infty} \frac{\mathrm{d}x}{x^4+1} \\ &= \int_0^1 \frac{\mathrm{d}x}{x^4+1} + \int_0^1 \frac{t^2\mathrm{d}t}{t^4+1} = \int_0^1 \frac{1+x^2}{1+x^4}\mathrm{d}x \\ &= \int_0^1 \frac{\mathrm{d}\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2} = \frac{1}{\sqrt{2}}\arctan\frac{x-x^{-1}}{\sqrt{2}}\bigg|_0^1 \\ &= \frac{\sqrt{2}\pi}{4}\end{aligned}$$

(10)

$$\int_0^{+\infty} \frac{\ln x\mathrm{d}x}{x^2+1} = \int_0^1 \frac{\ln x\mathrm{d}x}{x^2+1} + \int_1^0 \frac{-\ln t\mathrm{d}\left(\frac{1}{t}\right)}{1+\frac{1}{t^2}} = \int_0^1 \frac{\ln x\mathrm{d}x}{x^2+1} + \int_1^0 \frac{\ln t\mathrm{d}t}{1+t^2} = 0$$

P314 T4

(4)

$$\int_0^1 \frac{\mathrm{d}x}{(2-x)\sqrt{1-x}} \xlongequal{x=1-t^2} -2\int_1^0 \frac{\mathrm{d}t}{1+t^2} = 2\arctan x\bigg|_0^1 = \frac{\pi}{2}$$

(5) 由于令 $\dfrac{1}{x^2}=t$, 有

$$\int_{-1}^1 \frac{1}{x^3}\sin\frac{1}{x^2}\mathrm{d}x = 2\int_1^{+\infty} t^{3/2}\sin t\mathrm{d}\left(t^{-1/2}\right) = -\int_1^{+\infty} \sin t\mathrm{d}t = \cos x\bigg|_1^{+\infty}$$

不收敛！因此积分不收敛.

(6)

$$\int_0^{\pi/2}\frac{\mathrm{d}x}{\sqrt{\tan x}}\xlongequal{\sqrt{\tan x}=t}2\int_0^{+\infty}\frac{\mathrm{d}t}{t^4+1}=\frac{\sqrt{2}\pi}{2}\quad\text{(由前面的作业)}$$

P314 T5

由于

$$\ln\frac{\sqrt[n]{n!}}{n}=\sum_{i=1}^{n}\frac{1}{n}\ln\frac{i}{n}$$

因此

$$\ln\left(\lim_{n\to+\infty}\frac{\sqrt[n]{n!}}{n}\right)=\lim_{n\to+\infty}\ln\frac{\sqrt[n]{n!}}{n}=\int_0^1\ln x\mathrm{d}x=(x\ln x-x)\Big|_0^1=-1$$

故

$$\lim_{n\to+\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{\mathrm{e}}$$

P314 T6

(1)

$$\int_0^{\pi/2}\ln\cos x\mathrm{d}x\xlongequal{t=x-\pi/2}\int_{\pi/2}^{0}\ln\sin t\mathrm{d}t=-\int_0^{\pi/2}\sin t\mathrm{d}t=\frac{\pi}{2}\ln 2$$

(2)

$$\int_0^{\pi}x\ln\sin x\mathrm{d}x\xlongequal{x=2t}4\int_0^{\pi/2}\left(t\ln 2+\ln\cos t+\ln\sin t\right)\mathrm{d}t=2t^2\Big|_0^{\pi/2}=\frac{\pi^2}{2}$$

(3)

$$\int_0^{\pi/2}x\cot x\mathrm{d}x=\int_0^{\pi/2}x\mathrm{d}\left(\ln\sin x\right)=x\ln\sin x\Big|_0^{\pi/2}-\int_0^{\pi/2}\ln\sin x\mathrm{d}x=\frac{\pi}{2}\ln 2$$

(4)

$$\begin{aligned}\int_0^1\frac{\arcsin x}{x}\mathrm{d}x&\xlongequal{\sin t=x}\int_0^{\pi/2}\frac{x}{\sin x}\mathrm{d}\left(\sin x\right)=\int_0^{\pi/2}x\mathrm{d}(\ln\sin x)\\&=x\ln\sin x\Big|_0^{\pi/2}-\int_0^{\pi/2}\ln\sin x\mathrm{d}x=\frac{\pi}{2}\ln 2\end{aligned}$$

(5)

$$\int_0^1\frac{\ln x}{\sqrt{1-x^2}}\xlongequal{x=\sin t}\int_0^{\pi/2}\ln\sin t\mathrm{d}t=-\frac{\pi}{2}\ln 2$$

P315 T10

证明.

$$\begin{aligned}\int_0^{+\infty} f\left(\frac{a}{x}+\frac{x}{a}\right)\frac{\ln x}{x}\mathrm{d}x &= \int_0^{a} f\left(\frac{a}{x}+\frac{x}{a}\right)\frac{\ln x}{x}\mathrm{d}x + \int_a^{+\infty} f\left(\frac{a}{x}+\frac{x}{a}\right)\frac{\ln x}{x}\mathrm{d}x \\ &\xlongequal{t=\frac{a^2}{x}} \int_0^{a} f\left(\frac{a}{x}+\frac{x}{a}\right)\frac{\ln x}{x}\mathrm{d}x + \frac{a}{0} f\left(\frac{a}{t}+\frac{t}{a}\right)\frac{2\ln a-\ln t}{-t}\mathrm{d}t \\ &= 2\ln a\int_0^{+\infty} f\left(\frac{a}{x}+\frac{x}{a}\right)\frac{1}{x}\mathrm{d}x \\ &= \ln a\int_0^{+\infty} f\left(\frac{a}{x}+\frac{x}{a}\right)\frac{1}{x}\mathrm{d}x + \ln a\int_0^{a} f\left(\frac{a}{x}+\frac{x}{a}\right)\mathrm{d}\ln x \\ &\xlongequal{t=\frac{a^2}{x}} \ln a\int_0^{+\infty} f\left(\frac{a}{x}+\frac{x}{a}\right)\frac{1}{x}\mathrm{d}x + \ln a\int_a^{+\infty} f\left(\frac{a}{t}+\frac{t}{a}\right)\mathrm{d}\ln t \\ &= \ln a\int_0^{+\infty} f\left(\frac{a}{x}+\frac{x}{a}\right)\frac{1}{x}\mathrm{d}x\end{aligned}$$

□

P315 T11

证明. 假设 $A \neq 0$, 不妨 $A > 0$, 则存在 $M_1 > 0, \forall x > M_1, f(x) > \dfrac{A}{2} \quad (1)$.
则对任意 $M > 0$, 存在 $A_2 > A_1 > \max\{M_1, M, a\}, A_2 - A_1 = \dfrac{2}{A}$, 使得存在 $\eta > \dfrac{A}{2}$,

$$\int_{A_1}^{A_2} f(x)\mathrm{d}x = \eta\int_{A_1}^{A_2}\mathrm{d}x > \frac{A}{2}\cdot\frac{2}{A} = 1$$

由 Cauchy 收敛准则，$\displaystyle\int_a^{\infty} f(x)\mathrm{d}x$ 不收敛，矛盾！
因此 $A = 0$, $\lim\limits_{n\to+\infty} f(x) = 0$. □

P315 T12

证明. 由微积分第二基本定理，$f(x)$ 在 $[a, +\infty)$ 上连续且可导，则有 $\displaystyle\int_a^x f'(t)\mathrm{d}t = f(x) - f(a)$. 因此 $\lim\limits_{n\to+\infty} f(x)$ 存在. 由上题结论知，$\lim\limits_{n\to+\infty} f(x) = 0$. □

补充题

证明. 设 $\displaystyle\int_\alpha^\beta f(\varphi(t))\cdot\varphi(t)\mathrm{d}t = I$. 由无穷积分收敛的定义，任意 $\varepsilon > 0$, 存在 $\delta > 0$, 任意 $\eta > \delta$ 有 $\left|\displaystyle\int_\alpha^{\beta-\eta} f(\varphi(t))\cdot\varphi(t)\mathrm{d}t - I\right| < \varepsilon$.
因此欲证 $\displaystyle\int_a^{+\infty} f(x)\mathrm{d}x$ 收敛于 I, 只需证存在 $M > 0, M > \varphi(\beta-\eta)$, 任意 $x_0 > M$ 有 $\left|\displaystyle\int_a^{x_0} f(x)\mathrm{d}x - I\right| < \varepsilon$.
由于 $\lim\limits_{x\to\beta}\varphi(x) = +\infty$, 因此存在 $\delta' > 0$，任意 $t_0 \in (\beta-\delta', \beta)$, $\varphi(t_0) > x_0$. 又由于 $\varphi(t)$ 在

$[\beta-\eta,t_0]$ 上连续且 $x_0\in[\varphi(\beta-\eta),\varphi(t_0)]$, 则存在 $t_1>\beta-\eta,\varphi(t_1)=x_0$, 且 $\left|\int_a^{x_0}f(x)\mathrm{d}x-I\right|=\left|\int_\alpha^{t_1}f(\varphi(t))\varphi'(t)\mathrm{d}t-I\right|<\varepsilon$, 故积分 $\int_a^{x_0}f(x)\mathrm{d}x$ 存在. $\square$