作业十一

Noflowerzzk

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P264 T1(3)

$$
\begin{aligned}
F(x) &= \arctan x \Big|_a^{\int_0^x \sin^2 t\mathrm{d}t} \\
&= \arctan \int_0^x \sin^2 t\mathrm{d}t - \arctan a \\
\text{且} F'(x) &= \frac{1}{1 + \left(\int_0^x \sin^2 t\mathrm{d}t\right)^2} \left(\int_0^x \sin^2 t\mathrm{d}t\right)'
\end{aligned}
$$

又

$$
\begin{aligned}
\int_0^x \sin^2 t\mathrm{d}t &= -\sin t\cos t\Big|_0^x + \int_0^x \cos^2 \mathrm{d}t \\
&= -\sin x\cos x + \int_0^x \mathrm{d}t - \int_0^x \sin^2 t\mathrm{d}t \\
&= -\sin x\cos x + x - \int_0^x \sin^2 t\mathrm{d}t \\
\text{且} \left(\int_0^x \sin^2 t\mathrm{d}t\right)' &= \sin^2 x
\end{aligned}
$$

因此 $\displaystyle\int_0^x \sin^2 t\mathrm{d}t = \frac{1}{2}\left(-\sin x\cos x + x\right)$, 代入原式即有

$$
F'(x) = \frac{1}{1 + \left(\int_0^x \sin^2 t\mathrm{d}t\right)^2} \left(\int_0^x \sin^2 t\mathrm{d}t\right)' = \frac{4\sin^2 x}{4 + \left(-\sin x\cos x + x\right)^2}
$$

P264 T2

(1) 由洛必达法则，

$$
\lim_{x\to 0} \frac{\int_0^x \cos^t \mathrm{d}t}{x} = \lim_{x\to 0} \frac{\cos^2 x}{1} = 1 \tag{1}
$$

(2) 由于 $\displaystyle\left(\int_{\cos x}^1 \mathrm{e}^{-w^2}\mathrm{d}w\right)' = -\mathrm{e}^{-\cos^2 x}\left(\cos x\right)' = +\sin x\mathrm{e}^{-\cos^2 x}$,

$$
\begin{aligned}
\lim_{x\to 0} \frac{x^2}{\int_{\cos x}^1 \mathrm{e}^{-w^2}\mathrm{d}w} &= \lim_{x\to 0} \frac{2x}{\sin x\mathrm{e}^{-\cos^2 x}} \\
&= \lim_{x\to 0} \frac{2}{\mathrm{e}^{-\cos^2 x}} = 2\mathrm{e}
\end{aligned}
$$

(3) 由洛必达法则，

$$\lim_{x\to+\infty}\frac{\int_0^x\arctan^2 t\mathrm{d}t}{\sqrt{1+x^2}}=\lim_{x\to+\infty}\frac{\arctan^2 x}{\frac{x}{\sqrt{1+x^2}}}=\frac{\pi}{2}$$

(4) 由洛必达法则，

$$\begin{aligned}\lim_{x\to+\infty}\frac{\left(\int_0^x\mathrm{e}^{u^2}\mathrm{d}u\right)^2}{\int_0^x\mathrm{e}^{2u^2}\mathrm{d}u}&=\lim_{x\to+\infty}\frac{2\left(\int_0^x\mathrm{e}^{u^2}\mathrm{d}u\right)\mathrm{e}^{x^2}}{\mathrm{e}^{2x^2}}\\&=\lim_{x\to+\infty}2\frac{\mathrm{e}^{x^2}}{2x\mathrm{e}^{x^2}}=0\end{aligned}$$

P264 T3

证明. 由积分第一中值定理，存在 $\xi\in(0,x)$, $\displaystyle\int_0^x tf(t)\mathrm{d}t=\xi\int_0^x f(t)\mathrm{d}t$. 因此

$$\begin{aligned}g'(x)&=\frac{xf(x)\int_0^x f(t)\mathrm{d}t-f(x)\int_0^x tf(t)\mathrm{d}t}{\left(x\int_0^x f(t)\mathrm{d}t\right)^2}\\&=\frac{(x-\xi)f(x)\int_0^x f(t)\mathrm{d}t}{\left(x\int_0^x f(t)\mathrm{d}t\right)^2}>0\end{aligned}$$

因此 $g(x)$ 是 $[0,+\infty)$ 上的单调增加函数. □

P265 T5

(1) 由积分第一中值定理,存在 $\xi\in[0,1]$, $\displaystyle\int_0^1\frac{x^n}{1+x}\mathrm{d}x=\xi^n\int_0^1\frac{\mathrm{d}x}{x+1}=\xi^n\ln 2$. 而 $\displaystyle\lim_{n\to\infty}\xi^n\ln 2=0$, 因此原极限为 0.

(2) 由积分第一中值定理，存在 $\xi\in[n,n+p]$, $\displaystyle\int_n^{n+p}\frac{\sin x}{x}\mathrm{d}x=\frac{1}{\xi}\int_n^{n+p}\sin x\mathrm{d}x$.
由于 $\displaystyle\left|\int_n^{n+p}\sin x\mathrm{d}x\right|\leqslant\left|\int_n^{n+p}\mathrm{d}x\right|=p$, 有界，因此 $\displaystyle\lim_{n\to\infty}\frac{1}{\xi}\int_n^{n+p}\sin x\mathrm{d}x=0$, 即原极限为 0.

P265 T6

(16) 由于函数是偶函数，有

$$\begin{aligned}\int_{-\frac12}^{\frac12}\frac{\mathrm{d}x}{\sqrt{(1-x^2)^3}}&\xlongequal{x=\sin t}2\int_0^{\frac{\pi}{6}}\frac{\mathrm{d}(\sin t)}{\cos^3 t}\\&=2\int_0^{\frac{\pi}{6}}\frac{\mathrm{d}x}{\cos^2 x}\\&=2\tan x\Big|_0^{\frac{\pi}{6}}=\frac{2\sqrt{3}}{3}\end{aligned}$$

(17) 令 $t=\frac{x-1}{x+1}$, 有 $x=\frac{1+t}{1-t},\mathrm{d}x=\frac{2\mathrm{d}t}{(1-t)^2}$.

$$\begin{aligned}\int_0^1\left(\frac{x-1}{x+1}\mathrm{d}x\right)&=\int_{-1}^0\frac{2t^4\mathrm{d}t}{(1-t)^2}\\&=2\int_{-1}^0\left(t^2+2t+3-\frac{1}{1-t}+\frac{4}{(1-t)^2}\right)\mathrm{d}t\\&=2\left(\frac{1}{3}-2\frac{1}{2}+3-4\ln 2+1-\frac{1}{2}\right)=\frac{17}{3}-8\ln 2\end{aligned}$$

(18)

$$\begin{aligned}\int_0^1\frac{x^2+1}{x^4+1}\mathrm{d}x&=\int_0^1\frac{\mathrm{d}(x-x^{-1})}{(x-x^{-1})^2+2}\\&=\frac{1}{\sqrt{2}}\arctan\frac{x-x^{-1}}{\sqrt{2}}\Big|_0^1\\&=\frac{\sqrt{2}\pi}{4}\end{aligned}$$

(19)

$$\begin{aligned}\int_1^{\sqrt{2}}\frac{\mathrm{d}x}{x\sqrt{1+x^2}}&\xlongequal{t=\sqrt{1+x^2}}\int_{\sqrt{2}}^{\sqrt{3}}\frac{\mathrm{d}t}{t^2-1}\\&=\frac{1}{2}\left(\ln|x-1|\Big|_{\sqrt{2}}^{\sqrt{3}}-\ln|x+1|\Big|_{\sqrt{2}}^{\sqrt{3}}\right)\\&=\frac{1}{2}\ln\frac{\sqrt{3}-1}{\sqrt{2}-1}\frac{\sqrt{2}+1}{\sqrt{3}+1}\end{aligned}$$

(20) 令 $x=1+\sin t$.

$$\begin{aligned}\int_0^1 x\sqrt{\frac{x}{2-x}}\mathrm{d}x&=\int_{-\frac{\pi}{2}}^0\left|\frac{\sin\frac{t}{2}+\cos\frac{t}{2}}{\sin\frac{t}{2}-\cos\frac{t}{2}}\right|(1+\sin t)\cos t\mathrm{d}t\\&=\int_{-\frac{\pi}{2}}^0\left|\frac{\sin\frac{t}{2}+\cos\frac{t}{2}}{\sin\frac{t}{2}-\cos\frac{t}{2}}\right|(1+\sin t)\left(\cos^2\frac{t}{2}-\sin^2\frac{t}{2}\right)\mathrm{d}t\\&=\int_{-\frac{\pi}{2}}^0(1+\sin t)^2\mathrm{d}t\\&=\int_{-\frac{\pi}{2}}^0\left(1+2\sin t+\frac{1-\cos 2t}{2}\right)\mathrm{d}t\\&=\frac{3}{2}\frac{\pi}{2}-2\cos t\Big|_{-\frac{\pi}{2}}^0-\frac{1}{4}\sin 4t\Big|_{-\frac{\pi}{2}}^0\\&=\frac{3\pi}{4}-2\end{aligned}$$

P265 T7

(1)

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n-1}{n^2}\right)&=\lim_{n\to\infty}\sum_{i=1}^n\frac{i}{n}\frac{1}{n}\\&=\int_0^1\mathrm{d}x=\frac{1}{2}\end{aligned}$$

(2)

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{1^p}{n^{p+1}}+\frac{2^p}{n^{p+1}}+\cdots+\frac{(n-1)^p}{n^{p+1}}\right)&=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^p\frac{1}{n}\\&=\int_0^1 x^p\mathrm{d}x=\frac{1}{p+1}\end{aligned}$$

(3)

$$\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\cdots+\sin\frac{(n-1)\pi}{n}\right)=\int_0^1\sin\pi x\mathrm{d}x=\frac{2}{\pi}$$

P265 T8

(4) 令 $x=\dfrac{\cos t}{2}$，

$$\begin{aligned}\int_0^{\frac{1}{2}}x^2(1-4x^2)^10\mathrm{d}x&=-\frac{1}{8}\int_{\frac{\pi}{2}}^0\cos t\sin^2 1t\mathrm{d}t\\&=-\frac{1}{8}\left(\int_{\frac{\pi}{2}}^0\sin^2 1t\mathrm{d}t-\int_{\frac{\pi}{2}}^0\sin^2 3t\mathrm{d}t\right)\\&=\frac{1}{8}\left(\frac{20!!}{21!!}-\frac{22!!}{23!!}\right)=\frac{1}{184}\frac{20!!}{21!!}\end{aligned}$$

(5) 令 $I(n,m)=\displaystyle\int_0^1 x^n\ln^m x\mathrm{d}x$. 有

$$I(n,m)=\frac{1}{n+1}\left(x^{n+1}\ln^m x\Big|_0^1-m\int_0^1 x^n\ln^{m-1}\mathrm{d}x\right)=-\frac{m}{n+1}I(n,m-1)$$

因此 $I(n,m)=\left(-\dfrac{1}{n+1}\right)^{m-1}m!I(n,1)$. 又 $I(n,1)=-\dfrac{1}{(n+1)^2}$. 故

$$I(n,m)=\frac{(-1)^m}{(n+1)^{m+1}}m!$$

(6) 令 $I_n=\displaystyle\int_1^{\mathrm{e}}x\ln^n x\mathrm{d}x$. $I_0=\displaystyle\int_1^{\mathrm{e}}x\mathrm{d}x=\frac{1}{2}(\mathrm{e}^2-1)$.

$$I_n=\frac{1}{2}\left(x^2\ln^n x\Big|_1^{\mathrm{e}}-n\int_1^{e}x\ln^{n-1}x\mathrm{d}x\right)=\frac{1}{2}\mathrm{e}^2-\frac{n}{2}I_{n-1}$$

通项……？

P266 T14

证明.

$$f(x)=\frac{1}{2}\left(x^2\int_0^x g(t)\mathrm{d}t-2x\int_0^x tg(t)\mathrm{d}t+\int_0^x t^2g(t)\mathrm{d}t\right)$$

$$
\begin{aligned}
f'(x) &= \frac{1}{2}\left(2x\int_0^x g(t)\mathrm{d}t + x^2 g(x) - 2\int_0^x tg(t)\mathrm{d}t - 2x^2 g(x) + x^2 g(x)\right) \\
&= x\int_0^x g(t)\mathrm{d}t - \int_0^x tg(t)\mathrm{d}t
\end{aligned}
$$

因此

$$
f''(x) = \int_0^x g(t)\mathrm{d}t + xg(x) - g(x)
$$

$f''(1) = 2 + 1 \times 5 - 5 = 2.$

$$
f'''(x) = g(x) + g(x) + xg'(x) - g'(x)
$$

$f'''(1) = 5 + 5 + g'(1) - g'(1) = 10$ □

P266 T15

设 $f(x)$ 的一个原函数是 $F(x)$. $F(x)$ 可导，且 $F'(x) = f(x)$. 有 $F(x) = \int f(x)\mathrm{d}x = x\ln x - x\int_1^{\mathrm{e}} f(t)\mathrm{d}t + C$.

故 $\int_1^{\mathrm{e}} f(x)\mathrm{d}x = F(x)\Big|_1^{\mathrm{e}} = \mathrm{e} - (\mathrm{e}-1)\int_1^{\mathrm{e}} f(x)\mathrm{d}x$, 解得 $\int_1^{\mathrm{e}} f(x)\mathrm{d}x = 1$.

P266 T16

令 $m = 2x - t$. 有

$$
\begin{aligned}
\frac{1}{2}\arctan x^2 &= \int_{2x-1}^{2x} (2x - m)f(m)\mathrm{d}m \\
&= 2x\int_{2x-1}^{2x} f(t)\mathrm{d}t - \int_{2x-1}^{2x} tf(t)\mathrm{d}t
\end{aligned}
$$

两边求导，得

$$
\begin{aligned}
\frac{x}{1+x^4} &= 2\int_{2x-1}^{2x} f(t)\mathrm{d}t + 4x\left(f(2x) - f(2x-1)\right) - 4xf(2x) + (2x-1)f(2x-1) \\
&= 2\int_{2x-1}^{2x} f(t)\mathrm{d}t - f(2x-1)
\end{aligned}
$$

令 $x = 1$ 得

$$
\int_1^2 f(x)dx = \frac{5}{4}
$$

P266 T19

证明. $g(x)$ 两边求导有：

$$
af(ax) - f(x) = 0, \quad \forall x \in (0, +\infty)
$$

取定 $x = 1$, 有 $af(a) - f(1) = 0, \quad \forall a > 0$. 因此 $f(x) = \dfrac{f(1)}{x}$, $\forall x \in (0, +\infty)$. □

补充题

证明. 取 φ 的一个划分 $P: a = x_0 < x_1 < \cdots < x_n = b$. 有

$$\begin{aligned}
\int_a^b f(g(x))\varphi(x)\mathrm{d}x &= \lim_{\lambda(P)\to 0}\sum_{i=1}^n f(g(\xi_i))\varphi(\xi_i)\Delta x_i \\
&= \lim_{\lambda(P)\to 0}\sum_{i=1}^n \varphi(\xi_i)\Delta x_i \sum_{i=1}^n f(g(\xi_i))\frac{\varphi(x_i)\Delta x_i}{\sum_{i=1}^n \varphi(\xi_i)\Delta x_i} \\
&\geqslant \lim_{\lambda(P)\to 0}\left(\sum_{i=1}^n \varphi(\xi_i)\Delta x_i\right) f\left(\sum_{i=1}^n g(\xi_i)\varphi(\xi_i)\Delta x_i\right) \\
&= 1\cdot f\left(\lim_{\lambda(P)\to 0}\sum_{i=1}^n g(\xi_i)\varphi(\xi_i)\Delta x_i\right) \\
&= f\left(\int_a^b g(x)\varphi(x)\mathrm{d}x\right)
\end{aligned}$$

□