作业十四

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P289 T1

(6) 设 $L_1 : y = 0, x : 0 \to 2a$, 有

$$\int_{L+L_1} \mathrm{e}^x \in y - b(x+y)\mathrm{d}x + \mathrm{e}^2 \cos y - ax\mathrm{d}y = \iint_D (b-a)\mathrm{d}x\mathrm{d}y = \frac{\pi}{2}a^2(b-a)$$

故原式为 $\left(2 + \frac{\pi}{2}\right) a^2 b - \frac{\pi}{a^3}$

(7) 原式为 π

(8) 原式为 $\int_0^{2\pi} \frac{1}{2}\mathrm{d}x = \pi$

(9) 原式为 $\int_0^{2\pi} \mathrm{e}^{r\cos t} \cos(r \sin t)\mathrm{d}t$. 令 $r \to 0$ 有原式为 2π

P289 T3

(1) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = -1$, 故其与路径无关. 故 $\int_{(0,0)}^{(1,1)} (x-y)(\mathrm{d}x - \mathrm{d}y) = 0$

(2) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 0$, 故取 $L : (2,1) \to (1,1) \to (1,2)$, 有原式为 $\int_1^2 (\psi(t) - \phi(t))\mathrm{d}t$

(3) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{-xy}{\sqrt{(x^2+y^2)^3}}$, 故取 $L : (1,0) \to (6,0) \to (6,8)$, 有原式为 $\int_1^6 \mathrm{d}x + \frac{0}{8}\frac{y\mathrm{d}y}{\sqrt{y^2+36}} = 9$

P289 T5

证明. 易得 $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = -\frac{2xy}{(x^2+y^2)^2}$. 故其是某个全微分. $u(x,y) = \frac{1}{2}\ln\left(x^2+y^2\right) + C$. □

P289 T9

(6) 原式为 $-\frac{\pi}{2}$

(7) 原式为 $-\frac{1}{2}\pi a^4$

(8) 原式为 4π

1 P289 T11

(1) 设 $\Sigma_1 : z = 0(x^2+y^2 \le a^2)$，方向取下侧；$\Sigma_2 : z = h(x^2+y^2 \le a^2)$，方向取上侧

$$\iint_{\Sigma_1} v\,dS = -\iint_D xy\,dx\,dy = 0, \quad \iint_{\Sigma_2} v\,dS = \iint_D xy\,dx\,dy = 0,$$

由 Gauss 公式，

$$\iint_{\Sigma+\Sigma_1+\Sigma_2} v\,dS = \iiint_\Omega 0\,dx\,dy\,dz = 0,$$

故

$$\iint_\Sigma v\,dS = 0.$$

(2) 由 (1) 可知，流过该圆柱的全表面的流量 $\iint_\Sigma v\,dS = 0$.

P289 T13

证明. 取 $D = \{(x,y) \mid (x-a)^2 + (y-a)^2 \le 1\}$。由 Green 公式，

$$\int_L xf(y)\,dy - \frac{y}{f(x)}\,dx = \iint_D \left[f(y) + \frac{1}{f(x)}\right]dx\,dy$$

$$= \iint_D \left[f(x) + \frac{1}{f(x)}\right]dx\,dy \ge \iint_D 2\,dx\,dy = 2\pi,$$

□

P289 T18

$$\int_L \begin{vmatrix} dx & dy & dz \\ \cos\alpha & \cos\beta & \cos\gamma \\ x & y & z \end{vmatrix}$$

$$= \int_L (z\cos\beta - y\cos\gamma)dx + (x\cos\gamma - z\cos\alpha)dy + (y\cos\alpha - x\cos\beta)dz$$

$$= \iint_D \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z\cos\beta - y\cos\gamma & x\cos\gamma - z\cos\alpha & y\cos\alpha - x\cos\beta \end{vmatrix} dS$$

$$= 2\iint_D (\cos^2\alpha + \cos^2\beta + \cos^2\gamma)dS = 2\iint_D dS = 2S,$$

所以

$$S = \frac{1}{2}\int_L \begin{vmatrix} dx & dy & dz \\ \cos\alpha & \cos\beta & \cos\gamma \\ x & y & z \end{vmatrix}$$

P310 T3

(1) $f(r) = \dfrac{c}{r^3}$

(2) $f(r) = \dfrac{c_1}{r} + c_2$

P310 T4

$$\text{原式} = \boldsymbol{c} + \frac{\boldsymbol{1}}{\boldsymbol{2}} \frac{\boldsymbol{c}}{\boldsymbol{c} \cdot \boldsymbol{r}}$$

P310 T9

- (1) $\nabla \cdot (a \times r) = \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \\ x & y & z \end{vmatrix} = \dfrac{\partial(a_x - a_y)}{\partial x} + \dfrac{\partial(a_x - a_z)}{\partial y} + \dfrac{\partial(a_y - a_x)}{\partial z} = 0.$
- (2) $\nabla \times (a \times r) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_y z - a_z y & a_z x - a_x z & a_x y - a_y x \end{vmatrix} = 2(a_x i + a_y j + a_z k) = 2a.$
- (3) $\nabla \cdot ((r \cdot r)a) = \dfrac{\partial(a_x x^2)}{\partial x} + \dfrac{\partial(a_y y^2)}{\partial y} + \dfrac{\partial(a_z z^2)}{\partial z} = 2r \cdot a.$

P310 T18

(1) 原式为 $\displaystyle\iiint_\Omega \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0.$