Simplified Harsanyi's Theorem

Assumptions: Assume that a finite social state space X equipped with sigma-algebra $\mathcal{X}$, a finite set of probability measures on X: $\mathcal{M}(\mathcal{X})$. Suppose n agent have welfare ordering $\succeq_i$ on $\mathcal{M}(\mathcal{X})$ that satisfy the expected utility assumption with expected utility $\mathbb{E}_\mu v_i(x)$. Suppose that there's a social preference $\succeq$ on $\mathcal{M}(\mathcal{X})$ that satisfy the expected utility assumption with expected utility $\mathbb{E}_\mu w(x)$ as well. Pareto indifference is satisfied. If $\forall i, \mathbb{E}_\mu v_i = \mathbb{E}_\nu v_i$ then we have $\mathbb{E}_\mu w = \mathbb{E}_\nu w$.

Statement: there exists a weight α_i and a constant α such that $w(x) = \alpha + \sum_i v_i(x)$.

Proof: Denote the elements of X as $x_1, x_2, ..., x_k$. Denote $v_i = (v_i(x_1), v_i(x_2), ..., v_i(x_k))$ and $v_i \in \mathbb{R}^k$. Similarly, $w = (w(x_1), w(x_2), ..., w(x_k))$ and $w \in \mathbb{R}^k$. Hence it's enough to show $w = \sum_i \alpha_i v_i + \alpha$. Denote $v_0 = \mathbf{1}$. Let $S = \text{span}(v_0, v_1, ..., v_n)$, if $w \in S$, the theorem is obvious. Assume $w \notin S$, by separating hyperplane theorem, we have that $\exists r \in \mathbb{R}^k, \gamma \neq \beta \in \mathbb{R} \quad \forall v \in S$ s.t. $rw = \gamma, rv = \beta$. Since $\mathbf{0} \in S$, $\beta = 0$. Because $\mathbf{1} \in S$ we have $r\mathbf{1} = 0 \Rightarrow r^+\mathbf{1} = r^-\mathbf{1} = K$, hence r^+/K and r^-/K are in $\mathcal{M}(\mathcal{X})$. Because $\forall v_i, rv_i = 0$, we have that $r^+/Kv_i = r^-/Kv_i \Rightarrow \mathbb{E}_{r^+/K} v_i = \mathbb{E}_{r^-/K} v_i$, according to Pareto indifference, we have that $\mathbb{E}_{r^+/K} w_i = \mathbb{E}_{r^-/K} w_i$ which means $rw = 0$ and a contradiction.