Preuves en logique du premier ordre

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Logique du premier ordre

Définitions préliminaires

- $\mathcal{V} \equiv$ ensemble de variables d'individu x, y, etc. ;
- $\mathcal{S}_\mathcal{F} \equiv$ ensemble de symboles de fonctions f, g, etc. ;
- $\mathcal{S}_\mathcal{P} \equiv$ ensemble de symboles de prédicats P, Q, etc. ;
- $\mathcal{S}_\mathcal{F} \cap \mathcal{S}_\mathcal{P} = \emptyset$;
- Arité $m : \mathcal{S}_\mathcal{F} \cup \mathcal{S}_\mathcal{P} \to \mathbb{N}$.

Termes du premier ordre

- Plus petit ensemble $\mathcal{T}$ t.q. :
  - Si $x \in \mathcal{V}$ alors $x \in \mathcal{T}$;
  - Si $f \in \mathcal{S}_\mathcal{F}$ d'arité n et $t_1, \ldots, t_n \in \mathcal{T}$, alors $f(t_1, \ldots, t_n) \in \mathcal{T}$.

Logique du premier ordre

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Formules du premier ordre

- Plus petit ensemble $\mathcal{F}$ t.q. :
  - Si $P \in \mathcal{S}_\mathcal{P}$ d'arité n et $t_1, \ldots, t_n \in \mathcal{T}$, alors $P(t_1, \ldots, t_n) \in \mathcal{F}$;
  - $\bot, \top \in \mathcal{F}$;
  - Si $\Phi \in \mathcal{F}$ alors $\neg\Phi \in \mathcal{F}$;
  - Si $\Phi, \Phi' \in \mathcal{F}$ alors $\Phi \wedge \Phi', \Phi \vee \Phi', \Phi \Rightarrow \Phi', \Phi \Leftrightarrow \Phi' \in \mathcal{F}$;
  - Si $x \in \mathcal{V}$ et $\Phi \in \mathcal{F}$, alors $\forall x.\Phi, \exists x.\Phi \in \mathcal{F}$.

Logique du premier ordre

Associativité des connecteurs

- $\wedge$, $\vee$, et $\Leftrightarrow$ associent à gauche :
  - $A \wedge B \wedge C \equiv (A \wedge B) \wedge C$.
- $\Rightarrow$ associe à droite :
  - $A \Rightarrow B \Rightarrow C \equiv A \Rightarrow (B \Rightarrow C)$.

Précédence des connecteurs

- On a la précédence suivante : $\neg \succ \wedge \succ \vee \succ \Rightarrow \succ \Leftrightarrow$;
- Exemples :
  - $A \wedge B \Rightarrow C \equiv (A \wedge B) \Rightarrow C$;
  - $A \wedge \neg B \vee C \Rightarrow D \equiv ((A \wedge \neg B) \vee C) \Rightarrow D$;
  - $A \Rightarrow B \Leftrightarrow C \wedge D \equiv (A \Rightarrow B) \Leftrightarrow (C \wedge D)$.

Logique du premier ordre

Notation pointée pour les quantificateurs

- La portée d'un quantificateur va jusqu'à la parenthèse fermante de la formule du quantificateur ;
- Si la formule du quantificateur n'est pas parenthésée, la portée du quantificateur va jusqu'à la fin de la formule ;
- Donc, si on veut arrêter la portée d'un quantificateur, il suffit d'utiliser des parenthèses pour limiter explicitement la portée du quantificateur ;
- Exemple :
  - $\exists x.P(x) \Rightarrow P(a) \wedge P(b) \equiv \exists x.(P(x) \Rightarrow P(a) \wedge P(b))$;
  - Si on veut que le $\exists$ ne porte que sur $P(x)$, on doit écrire : $(\exists x.P(x)) \Rightarrow P(a) \wedge P(b)$.
- Notation : $\forall x, y.\Phi \equiv \forall x.\forall y.\Phi$ (idem pour $\exists$).

Sémantiques

Logique classique

- Une formule est toujours vraie ou fausse ;
- Que je puisse en démontrer la validité ou non ;
- Logique bi-valuée (vrai, faux) ;
- Logique du « tiers exclu » : $A \vee \neg A$.

Logique intuitionniste ou constructive

- Une formule est vraie, fausse, ou « on ne sait pas » ;
- Si on ne sait en démontrer la validité, alors « on ne sait pas » ;
- Logique tri-valuée d'une certaine manière ;
- Le « tiers exclu » n'est pas admis dans cette logique.

Systèmes de preuves

Plusieurs systèmes

- Systèmes à la Frege-Hilbert ;
- Systèmes à la Gentzen :
  - Déduction naturelle ;
  - Calcul des séquents.

Adéquation vis-à-vis de la sémantique

- Correction et complétude par rapport à la sémantique ;
- Correction : si je trouve une preuve de P alors P est vraie ;
- Complétude : si P est vraie alors il existe une preuve de P ;
- Preuve $\equiv$ moyen syntaxique de vérifier la validité d'une formule.

Calcul des séquents intuitionniste (LJ)

Règles

$$\frac{}{\Gamma, A \vdash A}\,\text{ax} \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\text{cont}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \Rightarrow B \vdash C}\,\Rightarrow_{\text{left}} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\,\Rightarrow_{\text{right}}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \Leftrightarrow B \vdash C}\,\Leftrightarrow_{\text{left1}}$$

$$\frac{\Gamma \vdash B \qquad \Gamma, A \vdash C}{\Gamma, A \Leftrightarrow B \vdash C}\,\Leftrightarrow_{\text{left2}} \qquad \frac{\Gamma, A \vdash B \qquad \Gamma, B \vdash A}{\Gamma \vdash A \Leftrightarrow B}\,\Leftrightarrow_{\text{right}}$$

Calcul des séquents intuitionniste (LJ)

Règles

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge_{\text{left}} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge_{\text{right}}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee_{\text{right1}}$$

$$\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\ \vee_{\text{left}} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee_{\text{right2}}$$

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash B}\ \neg_{\text{left}} \qquad \frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}\ \neg_{\text{right}}$$

$$\frac{}{\Gamma, \bot \vdash A}\ \bot_{\text{left}} \qquad \frac{}{\Gamma \vdash \top}\ \top_{\text{right}}$$

Calcul des séquents intuitionniste (LJ)

Règles

$$\frac{\Gamma, A(t) \vdash B}{\Gamma, \forall x.A(x) \vdash B}\ \forall_{\text{left}} \qquad \frac{\Gamma \vdash A(x)}{\Gamma \vdash \forall x.A(x)}\ \forall_{\text{right}},\ \ x \notin \Gamma$$

$$\frac{\Gamma, A(x) \vdash B}{\Gamma, \exists x.A(x) \vdash B}\ \exists_{\text{left}},\ \ x \notin \Gamma, B \qquad \frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x.A(x)}\ \exists_{\text{right}}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, A \vdash B}{\Gamma \vdash B}\ \text{cut}$$

Calcul des séquents classique (LJ_{em})

Règles

$$\frac{\Gamma, A(t) \vdash B}{\Gamma, \forall x.A(x) \vdash B}\ \forall_{\text{left}} \qquad \frac{\Gamma \vdash A(x)}{\Gamma \vdash \forall x.A(x)}\ \forall_{\text{right}},\ x \notin \Gamma$$

$$\frac{\Gamma, A(x) \vdash B}{\Gamma, \exists x.A(x) \vdash B}\ \exists_{\text{left}},\ x \notin \Gamma, B \qquad \frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x.A(x)}\ \exists_{\text{right}}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, A \vdash B}{\Gamma \vdash B}\ \text{cut} \qquad \frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}\ \text{em}$$

Calcul des séquents classique (LK)

Règles

$$\frac{}{\Gamma, A \vdash \Delta, A}\,\text{ax} \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, B}\,\text{cut}$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\,\text{cont}_{\text{left}} \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\,\text{cont}_{\text{right}}$$

Calcul des séquents classique (LK)

Règles

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta} \Rightarrow_{\text{left}} \qquad \frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \Rightarrow B} \Rightarrow_{\text{right}}$$

$$\frac{\Gamma \vdash \Delta, A, B \qquad \Gamma, A, B \vdash \Delta}{\Gamma, A \Leftrightarrow B \vdash \Delta} \Leftrightarrow_{\text{left}}$$

$$\frac{\Gamma, A \vdash \Delta, B \qquad \Gamma, B \vdash \Delta, A}{\Gamma \vdash \Delta, A \Leftrightarrow B} \Leftrightarrow_{\text{right}}$$

Calcul des séquents classique (LK)

Règles

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge_{\mathrm{left}} \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\ \wedge_{\mathrm{right}}$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \vee_{\mathrm{left}} \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}\ \vee_{\mathrm{right}}$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, \neg A \vdash \Delta}\ \neg_{\mathrm{left}} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ \neg_{\mathrm{right}}$$

$$\frac{}{\Gamma, \bot \vdash \Delta}\ \bot_{\mathrm{left}} \qquad \frac{}{\Gamma \vdash \Delta, \top}\ \top_{\mathrm{right}}$$

Calcul des séquents classique (LK)

Règles

$$\frac{\Gamma, A(t) \vdash \Delta}{\Gamma, \forall x.A(x) \vdash \Delta}\ \forall_{\text{left}} \qquad \frac{\Gamma \vdash \Delta, A(x)}{\Gamma \vdash \Delta, \forall x.A(x)}\ \forall_{\text{right}},\ x \notin \Gamma, \Delta$$

$$\frac{\Gamma, A(x) \vdash \Delta}{\Gamma, \exists x.A(x) \vdash \Delta}\ \exists_{\text{left}},\ x \notin \Gamma, \Delta \qquad \frac{\Gamma \vdash \Delta, A(t)}{\Gamma \vdash \Delta, \exists x.A(x)}\ \exists_{\text{right}}$$

Exemple de preuve propositionnelle dans LJ/LK

Une preuve simple

▸ Règles LJ

$$\dfrac{\dfrac{\dfrac{A, B \vdash A \qquad A, B \vdash B}{A, B \vdash A \wedge B}}{A \vdash B \Rightarrow A \wedge B}}{\vdash A \Rightarrow B \Rightarrow A \wedge B}$$

Exemple de preuve propositionnelle dans LJ/LK

Une preuve simple

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$$\dfrac{\dfrac{\dfrac{A, B \vdash A \qquad A, B \vdash B}{A, B \vdash A \wedge B}}{A \vdash B \Rightarrow A \wedge B}}{\vdash A \Rightarrow B \Rightarrow A \wedge B} \Rightarrow_{\text{right}}$$

Exemple de preuve propositionnelle dans LJ/LK

Une preuve simple

Règles LJ

$$\dfrac{\dfrac{\dfrac{A, B \vdash A \qquad A, B \vdash B}{A, B \vdash A \wedge B}}{A \vdash B \Rightarrow A \wedge B} \Rightarrow_{\text{right}}}{\vdash A \Rightarrow B \Rightarrow A \wedge B} \Rightarrow_{\text{right}}$$

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Une preuve simple

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\dfrac{\dfrac{\dfrac{A, B \vdash A \qquad A, B \vdash B}{A, B \vdash A \wedge B}\ \wedge_{\text{right}}}{A \vdash B \Rightarrow A \wedge B}\ \Rightarrow_{\text{right}}}{\vdash A \Rightarrow B \Rightarrow A \wedge B}\ \Rightarrow_{\text{right}}
$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{A, B \vdash A}\,\text{ax} \qquad A, B \vdash B}{A, B \vdash A \wedge B}\,\wedge_{\text{right}}}{A \vdash B \Rightarrow A \wedge B}\,\Rightarrow_{\text{right}}}{\vdash A \Rightarrow B \Rightarrow A \wedge B}\,\Rightarrow_{\text{right}}$$

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Exemple de preuve au premier ordre dans LJ/LK

Négation et quantificateurs

Règles LJ

$$P(x) \vdash P(x)$$

$$P(x) \vdash \exists x.P(x)$$

$$\neg\exists x.P(x), P(x) \vdash \bot$$

$$\neg\exists x.P(x) \vdash \neg P(x)$$

$$\neg\exists x.P(x) \vdash \forall x.\neg P(x)$$

$$\vdash \neg(\exists x.P(x)) \Rightarrow \forall x.\neg P(x)$$

Exemple de preuve au premier ordre dans LJ/LK

Négation et quantificateurs

Règles LJ

$$P(x) \vdash P(x)$$

$$P(x) \vdash \exists x.P(x)$$

$$\neg\exists x.P(x), P(x) \vdash \bot$$

$$\neg\exists x.P(x) \vdash \neg P(x)$$

$$\dfrac{\neg\exists x.P(x) \vdash \forall x.\neg P(x)}{\vdash \neg(\exists x.P(x)) \Rightarrow \forall x.\neg P(x)} \Rightarrow_{\text{right}}$$

Exemple de preuve au premier ordre dans LJ/LK

Négation et quantificateurs

▸ Règles LJ

$$P(x) \vdash P(x)$$

$$P(x) \vdash \exists x.P(x)$$

$$\neg\exists x.P(x), P(x) \vdash \bot$$

$$\dfrac{\dfrac{\neg\exists x.P(x) \vdash \neg P(x)}{\neg\exists x.P(x) \vdash \forall x.\neg P(x)}\ \forall_{\text{right}}}{\vdash \neg(\exists x.P(x)) \Rightarrow \forall x.\neg P(x)}\ \Rightarrow_{\text{right}}$$

Exemple de preuve au premier ordre dans LJ/LK

Négation et quantificateurs

▸ Règles LJ

$$P(x) \vdash P(x)$$

$$P(x) \vdash \exists x.P(x)$$

$$\cfrac{\cfrac{\cfrac{\neg\exists x.P(x), P(x) \vdash \bot}{\neg\exists x.P(x) \vdash \neg P(x)}\;\neg_{right}}{\neg\exists x.P(x) \vdash \forall x.\neg P(x)}\;\forall_{right}}{\vdash \neg(\exists x.P(x)) \Rightarrow \forall x.\neg P(x)}\;\Rightarrow_{right}$$

Exemple de preuve au premier ordre dans LJ/LK

Négation et quantificateurs

‣ Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{P(x) \vdash \exists x.P(x)}}{\neg\exists x.P(x), P(x) \vdash \bot}\neg_{\text{left}}}{\neg\exists x.P(x) \vdash \neg P(x)}\neg_{\text{right}}}{\neg\exists x.P(x) \vdash \forall x.\neg P(x)}\forall_{\text{right}}}{\vdash \neg(\exists x.P(x)) \Rightarrow \forall x.\neg P(x)}\Rightarrow_{\text{right}}$$

Exemple de preuve au premier ordre dans LJ/LK

Négation et quantificateurs

▸ Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{P(x) \vdash \exists x.P(x)}\ \exists_{\text{right}}}{\neg\exists x.P(x), P(x) \vdash \bot}\ \neg_{\text{left}}}{\neg\exists x.P(x) \vdash \neg P(x)}\ \neg_{\text{right}}}{\neg\exists x.P(x) \vdash \forall x.\neg P(x)}\ \forall_{\text{right}}}{\vdash \neg(\exists x.P(x)) \Rightarrow \forall x.\neg P(x)}\ \Rightarrow_{\text{right}}
$$

Exemple de preuve au premier ordre dans LJ/LK

Négation et quantificateurs

▸ Règles LJ

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(x) \vdash P(x)$}
\RightLabel{$\exists_{\text{right}}$}
\UnaryInfC{$P(x) \vdash \exists x.P(x)$}
\RightLabel{$\neg_{\text{left}}$}
\UnaryInfC{$\neg\exists x.P(x), P(x) \vdash \bot$}
\RightLabel{$\neg_{\text{right}}$}
\UnaryInfC{$\neg\exists x.P(x) \vdash \neg P(x)$}
\RightLabel{$\forall_{\text{right}}$}
\UnaryInfC{$\neg\exists x.P(x) \vdash \forall x.\neg P(x)$}
\RightLabel{$\Rightarrow_{\text{right}}$}
\UnaryInfC{$\vdash \neg(\exists x.P(x)) \Rightarrow \forall x.\neg P(x)$}
\end{prooftree}
$$

Logiques classique/intuitionniste

Sémantique du « il existe »

- En logique classique : $\exists x.P(x) \equiv$ il existe n termes $t_1, t_2, \ldots, t_n$ tels que $P(t_1) \vee P(t_2) \vee \ldots \vee P(t_n)$ est vraie (théorème de Herbrand) ;
- En logique intuitionniste : $\exists x.P(x) \equiv$ il existe un terme t tel que $P(t)$ est vraie.
  On doit construire un témoin t qui vérifie P et en avoir l'intuition.
  D'où le nom de logique « intuitionniste » ou « constructive ».

Logique classique

- La logique classique est une logique assez « exotique » ;
- On peut démontrer une formule $\exists x.P(x)$ sans jamais montrer un seul témoin qui fonctionne (c'est-à-dire qui vérifie P) !
- De ce fait, c'est plus facile de faire des preuves en logique classique qu'en logique intuitionniste.

Exemple de preuve en logique classique

Petit théorème mathématique

- Il existe a et b irrationnels tels que a^b est rationnel ;
- Preuve :
  - Utilisation du tiers exclu : $\sqrt{2}^{\sqrt{2}}$ est rationnel ou non ; deux cas :
    - Si $\sqrt{2}^{\sqrt{2}}$ est rationnel, alors le théorème est vrai ;
    - Si $\sqrt{2}^{\sqrt{2}}$ est irrationnel, alors $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$, qui est rationnel.

En logique intuitionniste

- Le théorème est vrai en logique intuitionniste ;
- Mais on doit montrer un a et b qui fonctionnent ;
- Plusieurs pages de théorie des nombres non triviales !

Un autre exemple de preuve en logique classique

Preuve dans LK

Démontrer : $\exists x.P(x) \Rightarrow P(a) \wedge P(b)$.

Cette formule est-elle valide ?

Un autre exemple de preuve en logique classique

Preuve dans LK

Règles LK

$$\Gamma \vdash P(a), P(a) \wedge P(b) \qquad \Gamma \vdash P(b), P(a) \wedge P(b)$$

$$\Gamma = P(a), P(b) \vdash P(a) \wedge P(b), P(a) \wedge P(b)$$

$$P(a) \vdash P(a) \wedge P(b), P(b) \Rightarrow P(a) \wedge P(b)$$

$$P(a) \vdash P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)$$

$$\vdash P(a) \Rightarrow P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)$$

$$\vdash \exists x.P(x) \Rightarrow P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)$$

$$\vdash \exists x.P(x) \Rightarrow P(a) \wedge P(b)$$

Un autre exemple de preuve en logique classique

Preuve dans LK

Règles LK

$$\Gamma \vdash P(a), P(a) \wedge P(b) \qquad \Gamma \vdash P(b), P(a) \wedge P(b)$$

$$\Gamma = P(a), P(b) \vdash P(a) \wedge P(b), P(a) \wedge P(b)$$

$$P(a) \vdash P(a) \wedge P(b), P(b) \Rightarrow P(a) \wedge P(b)$$

$$P(a) \vdash P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)$$

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$$\dfrac{\vdash \exists x.P(x) \Rightarrow P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)}{\vdash \exists x.P(x) \Rightarrow P(a) \wedge P(b)}\ \mathrm{cont}_{\mathrm{right}}$$

Un autre exemple de preuve en logique classique

Preuve dans LK

▸ Règles LK

$$\Gamma \vdash P(a), P(a) \wedge P(b) \qquad \Gamma \vdash P(b), P(a) \wedge P(b)$$

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Un autre exemple de preuve en logique classique

Preuve dans LK

Règles LK

$$\Gamma \vdash P(a), P(a) \wedge P(b) \qquad \Gamma \vdash P(b), P(a) \wedge P(b)$$

$$\Gamma = P(a), P(b) \vdash P(a) \wedge P(b), P(a) \wedge P(b)$$

$$P(a) \vdash P(a) \wedge P(b), P(b) \Rightarrow P(a) \wedge P(b)$$

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Un autre exemple de preuve en logique classique

Preuve dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash P(a), P(a) \wedge P(b) \qquad \Gamma \vdash P(b), P(a) \wedge P(b)}{\Gamma = P(a), P(b) \vdash P(a) \wedge P(b), P(a) \wedge P(b)}}{P(a) \vdash P(a) \wedge P(b), P(b) \Rightarrow P(a) \wedge P(b)}}{P(a) \vdash P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)}\exists_{right}}{\vdash P(a) \Rightarrow P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)}\Rightarrow_{right}}{\vdash \exists x.P(x) \Rightarrow P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)}\exists_{right}}{\vdash \exists x.P(x) \Rightarrow P(a) \wedge P(b)}\text{cont}_{right}$$

Un autre exemple de preuve en logique classique

Preuve dans LK

▸ Règles LK

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Preuve dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash P(a), P(a) \wedge P(b) \qquad \Gamma \vdash P(b), P(a) \wedge P(b)}{\Gamma = P(a), P(b) \vdash P(a) \wedge P(b), P(a) \wedge P(b)} \wedge_{\text{right}}}{P(a) \vdash P(a) \wedge P(b), P(b) \Rightarrow P(a) \wedge P(b)} \Rightarrow_{\text{right}}}{P(a) \vdash P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)} \exists_{\text{right}}}{\vdash P(a) \Rightarrow P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)} \Rightarrow_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)} \exists_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow P(a) \wedge P(b)} \text{cont}_{\text{right}}$$

Un autre exemple de preuve en logique classique

Preuve dans LK

Règles LK

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash P(a), P(a) \wedge P(b)$}
\AxiomC{$\Gamma \vdash P(b), P(a) \wedge P(b)$}
\RightLabel{$\wedge_{\text{right}}$}
\BinaryInfC{$\Gamma = P(a), P(b) \vdash P(a) \wedge P(b), P(a) \wedge P(b)$}
\RightLabel{$\Rightarrow_{\text{right}}$}
\UnaryInfC{$P(a) \vdash P(a) \wedge P(b), P(b) \Rightarrow P(a) \wedge P(b)$}
\RightLabel{$\exists_{\text{right}}$}
\UnaryInfC{$P(a) \vdash P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)$}
\RightLabel{$\Rightarrow_{\text{right}}$}
\UnaryInfC{$\vdash P(a) \Rightarrow P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)$}
\RightLabel{$\exists_{\text{right}}$}
\UnaryInfC{$\vdash \exists x.P(x) \Rightarrow P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)$}
\RightLabel{$\text{cont}_{\text{right}}$}
\UnaryInfC{$\vdash \exists x.P(x) \Rightarrow P(a) \wedge P(b)$}
\end{prooftree}
$$

Un autre exemple de preuve en logique classique

Preuve dans LK

Règles LK

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P(a), P(a) \wedge P(b)}\,\text{ax} \qquad \dfrac{}{\Gamma \vdash P(b), P(a) \wedge P(b)}\,\text{ax}}{\Gamma = P(a), P(b) \vdash P(a) \wedge P(b), P(a) \wedge P(b)}\,\wedge_{\text{right}}}{P(a) \vdash P(a) \wedge P(b), P(b) \Rightarrow P(a) \wedge P(b)}\,\Rightarrow_{\text{right}}}{P(a) \vdash P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)}\,\exists_{\text{right}}}{\vdash P(a) \Rightarrow P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)}\,\Rightarrow_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)}\,\exists_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow P(a) \wedge P(b)}\,\text{cont}_{\text{right}}
$$

Une preuve classique un peu plus complexe

Paradoxe (pas si paradoxal) des buveurs

Énoncé : « Il y a quelqu'un dans un bar tel que, s'il boit alors tout le monde dans le bar boit ».

$$P(x), P(y) \vdash P(y), \forall y.P(y)$$
$$P(x) \vdash P(y), P(y) \Rightarrow \forall y.P(y)$$
$$P(x) \vdash P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$
$$P(x) \vdash \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$
$$\vdash P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$
$$\vdash \exists x.P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$
$$\vdash \exists x.P(x) \Rightarrow \forall y.P(y)$$

Une preuve classique un peu plus complexe

Paradoxe (pas si paradoxal) des buveurs

Énoncé : « Il y a quelqu'un dans un bar tel que, s'il boit alors tout le monde dans le bar boit ».

Règles LK

$$P(x), P(y) \vdash P(y), \forall y.P(y)$$
$$P(x) \vdash P(y), P(y) \Rightarrow \forall y.P(y)$$
$$P(x) \vdash P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$
$$P(x) \vdash \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$
$$\vdash P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$
$$\vdash \exists x.P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$
$$\vdash \exists x.P(x) \Rightarrow \forall y.P(y)$$

Une preuve classique un peu plus complexe

Paradoxe (pas si paradoxal) des buveurs

Énoncé : « Il y a quelqu'un dans un bar tel que, s'il boit alors tout le monde dans le bar boit ».

Règles LK

$$P(x), P(y) \vdash P(y), \forall y.P(y)$$

$$P(x) \vdash P(y), P(y) \Rightarrow \forall y.P(y)$$

$$P(x) \vdash P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$

$$P(x) \vdash \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$

$$\vdash P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$

$$\dfrac{\vdash \exists x.P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y)}\ \text{cont}_{\text{right}}$$

Une preuve classique un peu plus complexe

Paradoxe (pas si paradoxal) des buveurs

Énoncé : « Il y a quelqu'un dans un bar tel que, s'il boit alors tout le monde dans le bar boit ».

▸ Règles LK

$$P(x), P(y) \vdash P(y), \forall y.P(y)$$
$$P(x) \vdash P(y), P(y) \Rightarrow \forall y.P(y)$$
$$P(x) \vdash P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$
$$P(x) \vdash \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$

$$\dfrac{\dfrac{\vdash P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\ \exists_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y)}\ \text{cont}_{\text{right}}$$

Une preuve classique un peu plus complexe

Paradoxe (pas si paradoxal) des buveurs

Énoncé : « Il y a quelqu'un dans un bar tel que, s'il boit alors tout le monde dans le bar boit ».

Règles LK

$$P(x), P(y) \vdash P(y), \forall y.P(y)$$

$$P(x) \vdash P(y), P(y) \Rightarrow \forall y.P(y)$$

$$P(x) \vdash P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$

$$\dfrac{\dfrac{\dfrac{P(x) \vdash \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}{\vdash P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\ \Rightarrow_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\ \exists_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y)}\ \text{cont}_{\text{right}}$$

Une preuve classique un peu plus complexe

Paradoxe (pas si paradoxal) des buveurs

Énoncé : « Il y a quelqu'un dans un bar tel que, s'il boit alors tout le monde dans le bar boit ».

Règles LK

$$P(x), P(y) \vdash P(y), \forall y.P(y)$$

$$P(x) \vdash P(y), P(y) \Rightarrow \forall y.P(y)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}{P(x) \vdash \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\ \forall_{\text{right}}}{\vdash P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\ \Rightarrow_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\ \exists_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y)}\ \text{cont}_{\text{right}}$$

Une preuve classique un peu plus complexe

Paradoxe (pas si paradoxal) des buveurs

Énoncé : « Il y a quelqu'un dans un bar tel que, s'il boit alors tout le monde dans le bar boit ».

▸ Règles LK

$$P(x), P(y) \vdash P(y), \forall y.P(y)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(y), P(y) \Rightarrow \forall y.P(y)}{P(x) \vdash P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\,\exists_{\mathrm{right}}}{P(x) \vdash \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\,\forall_{\mathrm{right}}}{\vdash P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\,\Rightarrow_{\mathrm{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\,\exists_{\mathrm{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y)}\,\mathrm{cont}_{\mathrm{right}}$$

Une preuve classique un peu plus complexe

Paradoxe (pas si paradoxal) des buveurs

Énoncé : « Il y a quelqu'un dans un bar tel que, s'il boit alors tout le monde dans le bar boit ».

‣ Règles LK

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x), P(y) \vdash P(y), \forall y.P(y)}{P(x) \vdash P(y), P(y) \Rightarrow \forall y.P(y)}\Rightarrow_{\text{right}}}{P(x) \vdash P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\exists_{\text{right}}}{P(x) \vdash \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\forall_{\text{right}}}{\vdash P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\Rightarrow_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\exists_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y)}\text{cont}_{\text{right}}
$$

Une preuve classique un peu plus complexe

Paradoxe (pas si paradoxal) des buveurs

Énoncé : « Il y a quelqu'un dans un bar tel que, s'il boit alors tout le monde dans le bar boit ».

Règles LK

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x), P(y) \vdash P(y), \forall y.P(y)}\text{ax}}{P(x) \vdash P(y), P(y) \Rightarrow \forall y.P(y)}\Rightarrow_{\text{right}}}{P(x) \vdash P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\exists_{\text{right}}}{P(x) \vdash \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\forall_{\text{right}}}{\vdash P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\Rightarrow_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\exists_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y)}\text{cont}_{\text{right}}
$$

Exercices en logique propositionnelle

Propositions à démontrer dans LJ et LK

1. $A \Rightarrow B \Rightarrow A$
2. $(A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$
3. $A \wedge B \Rightarrow B$
4. $B \Rightarrow A \vee B$
5. $(A \vee B) \Rightarrow (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$
6. $A \Rightarrow \bot \Rightarrow \neg A$
7. $\bot \Rightarrow A$
8. $(A \Leftrightarrow B) \Rightarrow A \Rightarrow B$
9. $(A \Leftrightarrow B) \Rightarrow B \Rightarrow A$
10. $(A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)$

Correction

Preuve (1) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{A, B \vdash A}{A \vdash B \Rightarrow A}}{\vdash A \Rightarrow B \Rightarrow A}$$

Correction

Preuve (1) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{A, B \vdash A}{A \vdash B \Rightarrow A}}{\vdash A \Rightarrow B \Rightarrow A} \Rightarrow_{\text{right}}$$

Correction

Preuve (1) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{A, B \vdash A}{A \vdash B \Rightarrow A}\Rightarrow_{\text{right}}}{\vdash A \Rightarrow B \Rightarrow A}\Rightarrow_{\text{right}}$$

Correction

Preuve (1) dans LJ/LK

Règles LJ

$$\cfrac{\cfrac{\cfrac{}{A, B \vdash A}\,\text{ax}}{A \vdash B \Rightarrow A}\,\Rightarrow_{\text{right}}}{\vdash A \Rightarrow B \Rightarrow A}\,\Rightarrow_{\text{right}}$$

Correction

Preuve (3) dans LJ/LK

▸ Règles LJ

$$A, B \vdash B$$

$$A \wedge B \vdash B$$

$$\vdash A \wedge B \Rightarrow B$$

Correction

Preuve (3) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{A, B \vdash B}{A \wedge B \vdash B}}{\vdash A \wedge B \Rightarrow B} \Rightarrow_{\text{right}}$$

Correction

Preuve (3) dans LJ/LK

▸ Règles LJ

$$\dfrac{\dfrac{A, B \vdash B}{A \wedge B \vdash B}\wedge_{\text{left}}}{\vdash A \wedge B \Rightarrow B}\Rightarrow_{\text{right}}$$

Correction

Preuve (3) dans LJ/LK

▸ Règles LJ

$$\dfrac{\dfrac{\dfrac{}{A, B \vdash B}\,\text{ax}}{A \wedge B \vdash B}\,\wedge_{\text{left}}}{\vdash A \wedge B \Rightarrow B}\,\Rightarrow_{\text{right}}$$

Correction

Preuve (4) dans LJ

▸ Règles LJ

$$\dfrac{\dfrac{B \vdash B}{B \vdash A \vee B}}{\vdash B \Rightarrow A \vee B}$$

Correction

Preuve (4) dans LJ

▸ Règles LJ

$$\dfrac{\dfrac{B \vdash B}{B \vdash A \vee B}}{\vdash B \Rightarrow A \vee B} \Rightarrow_{\text{right}}$$

Correction

Preuve (4) dans LJ

▸ Règles LJ

$$\dfrac{\dfrac{B \vdash B}{B \vdash A \vee B}\,\vee_{\text{right2}}}{\vdash B \Rightarrow A \vee B}\,\Rightarrow_{\text{right}}$$

Correction

Preuve (4) dans LJ

Règles LJ

$$\dfrac{\dfrac{\overline{B \vdash B}\ \text{ax}}{B \vdash A \vee B}\ \vee_{\text{right2}}}{\vdash B \Rightarrow A \vee B}\ \Rightarrow_{\text{right}}$$

Correction

Preuve (4) dans LK

Règles LK

$$\cfrac{\cfrac{B \vdash A, B}{B \vdash A \vee B}}{\vdash B \Rightarrow A \vee B}$$

Correction

Preuve (4) dans LK

Règles LK

$$\dfrac{\dfrac{B \vdash A, B}{B \vdash A \vee B}}{\vdash B \Rightarrow A \vee B} \Rightarrow_{\text{right}}$$

Correction

Preuve (4) dans LK

Règles LK

$$\dfrac{\dfrac{B \vdash A, B}{B \vdash A \vee B}\vee_{\text{right}}}{\vdash B \Rightarrow A \vee B}\Rightarrow_{\text{right}}$$

Correction

Preuve (4) dans LK

Règles LK

$$\dfrac{\dfrac{\overline{B \vdash A, B}\ \text{ax}}{B \vdash A \vee B}\ \vee_{\text{right}}}{\vdash B \Rightarrow A \vee B}\ \Rightarrow_{\text{right}}$$

Correction

Preuve (6) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{A, \bot \vdash \neg A}{A \vdash \bot \Rightarrow \neg A}}{\vdash A \Rightarrow \bot \Rightarrow \neg A}$$

Correction

Preuve (6) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{A, \bot \vdash \neg A}{A \vdash \bot \Rightarrow \neg A}}{\vdash A \Rightarrow \bot \Rightarrow \neg A} \Rightarrow_{\text{right}}$$

Correction

Preuve (6) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{A, \bot \vdash \neg A}{A \vdash \bot \Rightarrow \neg A}\Rightarrow_{\text{right}}}{\vdash A \Rightarrow \bot \Rightarrow \neg A}\Rightarrow_{\text{right}}$$

Correction

Preuve (6) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{\dfrac{}{A, \bot \vdash \neg A}\ \bot_{\text{left}}}{A \vdash \bot \Rightarrow \neg A}\ \Rightarrow_{\text{right}}}{\vdash A \Rightarrow \bot \Rightarrow \neg A}\ \Rightarrow_{\text{right}}$$

Correction

Preuve (7) dans LJ/LK

Règles LJ

$$\dfrac{\perp \vdash A}{\vdash \perp \Rightarrow A}$$

Correction

Preuve (7) dans LJ/LK

▸ Règles LJ

$$\frac{\bot \vdash A}{\vdash \bot \Rightarrow A}\Rightarrow_{\text{right}}$$

Correction

Preuve (7) dans LJ/LK

▸ Règles LJ

$$\dfrac{\dfrac{}{\bot \vdash A}\,\text{ax}}{\vdash \bot \Rightarrow A}\,\Rightarrow_{\text{right}}$$

Correction

Preuve (8) dans LJ

Règles LJ

$$\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A \Leftrightarrow B, A \vdash B}}{A \Leftrightarrow B \vdash A \Rightarrow B}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B}$$

Correction

Preuve (8) dans LJ

▸ Règles LJ

$$\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A \Leftrightarrow B, A \vdash B}}{A \Leftrightarrow B \vdash A \Rightarrow B}$$

$$\dfrac{A \Leftrightarrow B \vdash A \Rightarrow B}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B} \Rightarrow_{\text{right}}$$

Correction

Preuve (8) dans LJ

▸ Règles LJ

$$\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A \Leftrightarrow B, A \vdash B}}{A \Leftrightarrow B \vdash A \Rightarrow B}\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B}\Rightarrow_{\text{right}}$$

Correction

Preuve (8) dans LJ

Règles LJ

$$\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A \Leftrightarrow B, A \vdash B}\Leftrightarrow_{\text{left1}}}{A \Leftrightarrow B \vdash A \Rightarrow B}\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B}\Rightarrow_{\text{right}}$$

Correction

Preuve (8) dans LJ

▸ Règles LJ

$$\dfrac{\dfrac{\dfrac{\overline{A \vdash A}\ \text{ax} \qquad B \vdash B}{A \Leftrightarrow B, A \vdash B}\ \Leftrightarrow_{\text{left1}}}{A \Leftrightarrow B \vdash A \Rightarrow B}\ \Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B}\ \Rightarrow_{\text{right}}$$

Correction

Preuve (8) dans LJ

Règles LJ

$$
\cfrac{\cfrac{\cfrac{\cfrac{}{A \vdash A}\,\mathrm{ax} \qquad \cfrac{}{B \vdash B}\,\mathrm{ax}}{A \Leftrightarrow B, A \vdash B}\,\Leftrightarrow_{\mathrm{left1}}}{A \Leftrightarrow B \vdash A \Rightarrow B}\,\Rightarrow_{\mathrm{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B}\,\Rightarrow_{\mathrm{right}}
$$

Correction

Preuve (8) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{A \vdash B, A, B \qquad A, A, B \vdash B}{A \Leftrightarrow B, A \vdash B}}{A \Leftrightarrow B \vdash A \Rightarrow B}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B}$$

Correction

Preuve (8) dans LK

Règles LK

$$\dfrac{\dfrac{A \vdash B, A, B \qquad A, A, B \vdash B}{A \Leftrightarrow B, A \vdash B}}{\dfrac{A \Leftrightarrow B \vdash A \Rightarrow B}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B} \Rightarrow_{\text{right}}}$$

Correction

Preuve (8) dans LK

Règles LK

$$\dfrac{\dfrac{A \vdash B, A, B \qquad A, A, B \vdash B}{A \Leftrightarrow B, A \vdash B}\Rightarrow_{\text{right}}}{\dfrac{A \Leftrightarrow B \vdash A \Rightarrow B}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B}\Rightarrow_{\text{right}}}$$

Correction

Preuve (8) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{A \vdash B, A, B \qquad A, A, B \vdash B}{A \Leftrightarrow B, A \vdash B}\Leftrightarrow_{\text{left}}}{A \Leftrightarrow B \vdash A \Rightarrow B}\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B}\Rightarrow_{\text{right}}$$

Correction

Preuve (8) dans LK

Règles LK

$$\dfrac{\dfrac{}{A \vdash B, A, B}\,\text{ax} \qquad A, A, B \vdash B}{\dfrac{A \Leftrightarrow B, A \vdash B}{\dfrac{A \Leftrightarrow B \vdash A \Rightarrow B}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B}\Rightarrow_{\text{right}}}\Rightarrow_{\text{right}}}\Leftrightarrow_{\text{left}}$$

Correction

Preuve (8) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash B, A, B}\,\mathrm{ax} \qquad \dfrac{}{A, A, B \vdash B}\,\mathrm{ax}}{A \Leftrightarrow B, A \vdash B}\,\Leftrightarrow_{\mathrm{left}}}{A \Leftrightarrow B \vdash A \Rightarrow B}\,\Rightarrow_{\mathrm{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B}\,\Rightarrow_{\mathrm{right}}$$

Correction

Preuve (9) dans LJ

▸ Règles LJ

$$B \vdash B \qquad A \vdash A$$
$$A \Leftrightarrow B, B \vdash A$$
$$A \Leftrightarrow B \vdash B \Rightarrow A$$
$$\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A$$

Correction

Preuve (9) dans LJ

Règles LJ

$$\dfrac{\dfrac{\dfrac{B \vdash B \qquad A \vdash A}{A \Leftrightarrow B, B \vdash A}}{A \Leftrightarrow B \vdash B \Rightarrow A}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A} \Rightarrow_{\text{right}}$$

Correction

Preuve (9) dans LJ

Règles LJ

$$\dfrac{\dfrac{\dfrac{B \vdash B \qquad A \vdash A}{A \Leftrightarrow B, B \vdash A}}{A \Leftrightarrow B \vdash B \Rightarrow A}\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}\Rightarrow_{\text{right}}$$

Correction

Preuve (9) dans LJ

Règles LJ

$$\dfrac{\dfrac{\dfrac{B \vdash B \qquad A \vdash A}{A \Leftrightarrow B, B \vdash A}\Leftrightarrow_{\text{left2}}}{A \Leftrightarrow B \vdash B \Rightarrow A}\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}\Rightarrow_{\text{right}}$$

Correction

Preuve (9) dans LJ

Règles LJ

$$
\dfrac{\dfrac{\dfrac{\overline{B \vdash B}\;\mathrm{ax} \qquad A \vdash A}{A \Leftrightarrow B, B \vdash A}\;\Leftrightarrow_{\mathrm{left2}}}{A \Leftrightarrow B \vdash B \Rightarrow A}\;\Rightarrow_{\mathrm{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}\;\Rightarrow_{\mathrm{right}}
$$

Correction

Preuve (9) dans LJ

▸ Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,\text{ax} \qquad \dfrac{}{A \vdash A}\,\text{ax}}{A \Leftrightarrow B, B \vdash A}\,\Leftrightarrow_{\text{left2}}}{A \Leftrightarrow B \vdash B \Rightarrow A}\,\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}\,\Rightarrow_{\text{right}}$$

Correction

Preuve (9) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{B \vdash A, A, B \qquad B, A, B \vdash A}{A \Leftrightarrow B, B \vdash A}}{A \Leftrightarrow B \vdash B \Rightarrow A}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}$$

Correction

Preuve (9) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{B \vdash A, A, B \qquad B, A, B \vdash A}{A \Leftrightarrow B, B \vdash A}}{A \Leftrightarrow B \vdash B \Rightarrow A}\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}$$

Correction

Preuve (9) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{B \vdash A, A, B \qquad B, A, B \vdash A}{A \Leftrightarrow B, B \vdash A}}{A \Leftrightarrow B \vdash B \Rightarrow A}\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}\Rightarrow_{\text{right}}$$

Correction

Preuve (9) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{B \vdash A, A, B \qquad B, A, B \vdash A}{A \Leftrightarrow B, B \vdash A}\Leftrightarrow_{\text{left}}}{A \Leftrightarrow B \vdash B \Rightarrow A}\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}\Rightarrow_{\text{right}}$$

Correction

Preuve (9) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash A, A, B}\,\text{ax} \qquad B, A, B \vdash A}{A \Leftrightarrow B, B \vdash A}\,\Leftrightarrow_{\text{left}}}{A \Leftrightarrow B \vdash B \Rightarrow A}\,\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}\,\Rightarrow_{\text{right}}$$

Correction

Preuve (9) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash A, A, B}\,\text{ax} \qquad \dfrac{}{B, A, B \vdash A}\,\text{ax}}{A \Leftrightarrow B, B \vdash A}\,\Leftrightarrow_{\text{left}}}{A \Leftrightarrow B \vdash B \Rightarrow A}\,\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}\,\Rightarrow_{\text{right}}$$

Correction

Preuve (10) dans LJ

Règles LJ

$$\dfrac{\dfrac{\dfrac{B \Rightarrow A, A \vdash A \qquad B \Rightarrow A, A, B \vdash B}{A \Rightarrow B, B \Rightarrow A, A \vdash B} \qquad \dfrac{A \Rightarrow B, B \vdash B \qquad A \Rightarrow B, B, A \vdash A}{A \Rightarrow B, B \Rightarrow A, B \vdash A}}{\dfrac{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}$$

Correction

Preuve (10) dans LJ

Règles LJ

$$B \Rightarrow A, A \vdash A \qquad B \Rightarrow A, A, B \vdash B \qquad A \Rightarrow B, B \vdash B \qquad A \Rightarrow B, B, A \vdash A$$

$$A \Rightarrow B, B \Rightarrow A, A \vdash B \qquad A \Rightarrow B, B \Rightarrow A, B \vdash A$$

$$A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B$$

$$\dfrac{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)} \Rightarrow_{\textbf{right}}$$

Correction

Preuve (10) dans LJ

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow A, A \vdash A \qquad B \Rightarrow A, A, B \vdash B}{A \Rightarrow B, B \Rightarrow A, A \vdash B} \qquad \dfrac{A \Rightarrow B, B \vdash B \qquad A \Rightarrow B, B, A \vdash A}{A \Rightarrow B, B \Rightarrow A, B \vdash A}}{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)} \Rightarrow_{\textbf{right}}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)} \Rightarrow_{\textbf{right}}$$

Correction

Preuve (10) dans LJ

▸ Règles LJ

$$\dfrac{\dfrac{\dfrac{\begin{array}{cc} B \Rightarrow A, A \vdash A \qquad B \Rightarrow A, A, B \vdash B & A \Rightarrow B, B \vdash B \qquad A \Rightarrow B, B, A \vdash A \\ A \Rightarrow B, B \Rightarrow A, A \vdash B & A \Rightarrow B, B \Rightarrow A, B \vdash A \end{array}}{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}\Leftrightarrow_{\mathbf{right}}}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\Rightarrow_{\mathbf{right}}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\Rightarrow_{\mathbf{right}}$$

Correction

Preuve (10) dans LJ

‣ Règles LJ

$$
\begin{prooftree}
\AxiomC{$B \Rightarrow A, A \vdash A$}
\AxiomC{$B \Rightarrow A, A, B \vdash B$}
\RightLabel{$\Rightarrow_{\textbf{left}}$}
\BinaryInfC{$A \Rightarrow B, B \Rightarrow A, A \vdash B$}
\AxiomC{$A \Rightarrow B, B \vdash B$}
\AxiomC{$A \Rightarrow B, B, A \vdash A$}
\noLine
\BinaryInfC{$A \Rightarrow B, B \Rightarrow A, B \vdash A$}
\RightLabel{$\Leftrightarrow_{\textbf{right}}$}
\BinaryInfC{$A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B$}
\RightLabel{$\Rightarrow_{\textbf{right}}$}
\UnaryInfC{$A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)$}
\RightLabel{$\Rightarrow_{\textbf{right}}$}
\UnaryInfC{$\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)$}
\end{prooftree}
$$

Correction

Preuve (10) dans LJ

▸ Règles LJ

$$
\dfrac{
  \dfrac{
    \dfrac{}{B \Rightarrow A, A \vdash A}\,\mathsf{ax}
    \qquad
    B \Rightarrow A, A, B \vdash B
  }{A \Rightarrow B, B \Rightarrow A, A \vdash B}\,\Rightarrow_{\mathbf{left}}
  \qquad
  \begin{array}{c}
  A \Rightarrow B, B \vdash B \qquad A \Rightarrow B, B, A \vdash A \\
  A \Rightarrow B, B \Rightarrow A, B \vdash A
  \end{array}
}{
  \dfrac{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}{
    \dfrac{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\mathbf{right}}
  }\,\Rightarrow_{\mathbf{right}}
}\,\Leftrightarrow_{\mathbf{right}}
$$

Correction

Preuve (10) dans LJ

Règles LJ

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{B \Rightarrow A, A \vdash A}\,\mathrm{ax}
        \qquad
        \dfrac{}{B \Rightarrow A, A, B \vdash B}\,\mathrm{ax}
      }{A \Rightarrow B, B \Rightarrow A, A \vdash B}\,\Rightarrow_{\mathbf{left}}
      \qquad
      \begin{array}{c}
        A \Rightarrow B, B \vdash B \qquad A \Rightarrow B, B, A \vdash A \\
        A \Rightarrow B, B \Rightarrow A, B \vdash A
      \end{array}
    }{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}\,\Leftrightarrow_{\mathbf{right}}
  }{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\mathbf{right}}
}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\mathbf{right}}
$$

Correction

Preuve (10) dans LJ

▸ Règles LJ

$$
\cfrac{
\cfrac{
\cfrac{\cfrac{}{B \Rightarrow A, A \vdash A}\,\text{ax} \qquad \cfrac{}{B \Rightarrow A, A, B \vdash B}\,\text{ax}}{A \Rightarrow B, B \Rightarrow A, A \vdash B}\,\Rightarrow_{\textbf{left}}
\qquad
\cfrac{A \Rightarrow B, B \vdash B \qquad A \Rightarrow B, B, A \vdash A}{A \Rightarrow B, B \Rightarrow A, B \vdash A}\,\Rightarrow_{\textbf{left}}
}{
\cfrac{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}
}\,\Leftrightarrow_{\textbf{right}}
}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}
$$

Correction

Preuve (10) dans LJ

Règles LJ

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$B \Rightarrow A, A \vdash A$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$B \Rightarrow A, A, B \vdash B$}
\RightLabel{$\Rightarrow_{\textbf{left}}$}
\BinaryInfC{$A \Rightarrow B, B \Rightarrow A, A \vdash B$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$A \Rightarrow B, B \vdash B$}
\AxiomC{$A \Rightarrow B, B, A \vdash A$}
\RightLabel{$\Rightarrow_{\textbf{left}}$}
\BinaryInfC{$A \Rightarrow B, B \Rightarrow A, B \vdash A$}
\RightLabel{$\Leftrightarrow_{\textbf{right}}$}
\BinaryInfC{$A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B$}
\RightLabel{$\Rightarrow_{\textbf{right}}$}
\UnaryInfC{$A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)$}
\RightLabel{$\Rightarrow_{\textbf{right}}$}
\UnaryInfC{$\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)$}
\end{prooftree}
$$

Correction

Preuve (10) dans LJ

▸ Règles LJ

$$
\dfrac{
\dfrac{
\dfrac{}{B \Rightarrow A, A \vdash A}\,\text{ax}
\qquad
\dfrac{}{B \Rightarrow A, A, B \vdash B}\,\text{ax}
}{A \Rightarrow B, B \Rightarrow A, A \vdash B}\,\Rightarrow_{\textbf{left}}
\qquad
\dfrac{
\dfrac{}{A \Rightarrow B, B \vdash B}\,\text{ax}
\qquad
\dfrac{}{A \Rightarrow B, B, A \vdash A}\,\text{ax}
}{A \Rightarrow B, B \Rightarrow A, B \vdash A}\,\Rightarrow_{\textbf{left}}
}{
\dfrac{
\dfrac{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}
}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}
}\,\Leftrightarrow_{\textbf{right}}
$$

Correction

Preuve (10) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow A, A \vdash B, A \qquad B \Rightarrow A, A, B \vdash B}{A \Rightarrow B, B \Rightarrow A, A \vdash B} \qquad \dfrac{A \Rightarrow B, B \vdash A, B \qquad A \Rightarrow B, B, A \vdash A}{A \Rightarrow B, B \Rightarrow A, B \vdash A}}{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}$$

Correction

Preuve (10) dans LK

▸ Règles LK

$$B \Rightarrow A, A \vdash B, A \qquad B \Rightarrow A, A, B \vdash B \qquad A \Rightarrow B, B \vdash A, B \qquad A \Rightarrow B, B, A \vdash A$$

$$A \Rightarrow B, B \Rightarrow A, A \vdash B \qquad\qquad A \Rightarrow B, B \Rightarrow A, B \vdash A$$

$$A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B$$

$$\frac{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)} \Rightarrow_{\textbf{right}}$$

Correction

Preuve (10) dans LK

▸ Règles LK

$$B \Rightarrow A, A \vdash B, A \qquad B \Rightarrow A, A, B \vdash B \qquad A \Rightarrow B, B \vdash A, B \qquad A \Rightarrow B, B, A \vdash A$$

$$A \Rightarrow B, B \Rightarrow A, A \vdash B \qquad\qquad A \Rightarrow B, B \Rightarrow A, B \vdash A$$

$$\cfrac{\cfrac{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\ \Rightarrow_{\textbf{right}}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\ \Rightarrow_{\textbf{right}}$$

Correction

Preuve (10) dans LK

‣ Règles LK

$$
\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow A, A \vdash B, A \qquad B \Rightarrow A, A, B \vdash B}{A \Rightarrow B, B \Rightarrow A, A \vdash B} \qquad \dfrac{A \Rightarrow B, B \vdash A, B \qquad A \Rightarrow B, B, A \vdash A}{A \Rightarrow B, B \Rightarrow A, B \vdash A}}{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}\Leftrightarrow_{\textbf{right}}}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\Rightarrow_{\textbf{right}}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\Rightarrow_{\textbf{right}}
$$

Correction

Preuve (10) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow A, A \vdash B, A \qquad B \Rightarrow A, A, B \vdash B}{A \Rightarrow B, B \Rightarrow A, A \vdash B}\Rightarrow_{\textbf{left}} \qquad \dfrac{A \Rightarrow B, B \vdash A, B \qquad A \Rightarrow B, B, A \vdash A}{A \Rightarrow B, B \Rightarrow A, B \vdash A}}{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}\Leftrightarrow_{\textbf{right}}}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\Rightarrow_{\textbf{right}}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\Rightarrow_{\textbf{right}}$$

Correction

Preuve (10) dans LK

Règles LK

$$
\dfrac{
\dfrac{
\dfrac{}{B \Rightarrow A, A \vdash B, A}\,\mathsf{ax}
\qquad
B \Rightarrow A, A, B \vdash B
}{A \Rightarrow B, B \Rightarrow A, A \vdash B}\Rightarrow_{\mathbf{left}}
\qquad
\dfrac{A \Rightarrow B, B \vdash A, B \qquad A \Rightarrow B, B, A \vdash A}{A \Rightarrow B, B \Rightarrow A, B \vdash A}
}{
\dfrac{
\dfrac{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\Rightarrow_{\mathbf{right}}
}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\Rightarrow_{\mathbf{right}}
}\Leftrightarrow_{\mathbf{right}}
$$

Correction

Preuve (10) dans LK

▸ Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow A, A \vdash B, A}\,\text{ax} \qquad \dfrac{}{B \Rightarrow A, A, B \vdash B}\,\text{ax}}{A \Rightarrow B, B \Rightarrow A, A \vdash B}\,\Rightarrow_{\textbf{left}} \qquad \begin{array}{c} A \Rightarrow B, B \vdash A, B \qquad A \Rightarrow B, B, A \vdash A \\ A \Rightarrow B, B \Rightarrow A, B \vdash A \end{array}}{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}\,\Leftrightarrow_{\textbf{right}}}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}$$

Correction

Preuve (10) dans LK

Règles LK

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$B \Rightarrow A, A \vdash B, A$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$B \Rightarrow A, A, B \vdash B$}
\RightLabel{$\Rightarrow_{\textbf{left}}$}
\BinaryInfC{$A \Rightarrow B, B \Rightarrow A, A \vdash B$}
\AxiomC{$A \Rightarrow B, B \vdash A, B$}
\AxiomC{$A \Rightarrow B, B, A \vdash A$}
\RightLabel{$\Rightarrow_{\textbf{left}}$}
\BinaryInfC{$A \Rightarrow B, B \Rightarrow A, B \vdash A$}
\RightLabel{$\Leftrightarrow_{\textbf{right}}$}
\BinaryInfC{$A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B$}
\RightLabel{$\Rightarrow_{\textbf{right}}$}
\UnaryInfC{$A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)$}
\RightLabel{$\Rightarrow_{\textbf{right}}$}
\UnaryInfC{$\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)$}
\end{prooftree}
$$

Correction

Preuve (10) dans LK

Règles LK

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$B \Rightarrow A, A \vdash B, A$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$B \Rightarrow A, A, B \vdash B$}
\RightLabel{$\Rightarrow_{\text{left}}$}
\BinaryInfC{$A \Rightarrow B, B \Rightarrow A, A \vdash B$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$A \Rightarrow B, B \vdash A, B$}
\AxiomC{$A \Rightarrow B, B, A \vdash A$}
\RightLabel{$\Rightarrow_{\text{left}}$}
\BinaryInfC{$A \Rightarrow B, B \Rightarrow A, B \vdash A$}
\RightLabel{$\Leftrightarrow_{\text{right}}$}
\BinaryInfC{$A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B$}
\RightLabel{$\Rightarrow_{\text{right}}$}
\UnaryInfC{$A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)$}
\RightLabel{$\Rightarrow_{\text{right}}$}
\UnaryInfC{$\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)$}
\end{prooftree}
$$

Correction

Preuve (10) dans LK

Règles LK

$$
\dfrac{\dfrac{\dfrac{}{B \Rightarrow A, A \vdash B, A}\text{ax} \quad \dfrac{}{B \Rightarrow A, A, B \vdash B}\text{ax}}{A \Rightarrow B, B \Rightarrow A, A \vdash B}\Rightarrow_{\textbf{left}} \quad \dfrac{\dfrac{}{A \Rightarrow B, B \vdash A, B}\text{ax} \quad \dfrac{}{A \Rightarrow B, B, A \vdash A}\text{ax}}{A \Rightarrow B, B \Rightarrow A, B \vdash A}\Rightarrow_{\textbf{left}}}{\dfrac{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}{\dfrac{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\Rightarrow_{\textbf{right}}}\Rightarrow_{\textbf{right}}}\Leftrightarrow_{\textbf{right}}
$$

Exercices en logique du premier ordre

Propositions à démontrer dans $LJ_{(em)}$ et LK

1. $\forall x.P(x) \Rightarrow \exists y.P(y) \vee Q(y)$
2. $(\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))$
3. $(\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)$
4. $(\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))$
5. $(\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))$
6. $\neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)$

Correction

Preuve (1) dans LJ/LK

▸ Règles LJ

$$P(x) \vdash P(x), Q(x)$$
$$P(x) \vdash P(x) \vee Q(x)$$
$$P(x) \vdash \exists y. P(y) \vee Q(y)$$
$$\vdash P(x) \Rightarrow \exists y. P(y) \vee Q(y)$$
$$\vdash \forall x. P(x) \Rightarrow \exists y. P(y) \vee Q(y)$$

Correction

Preuve (1) dans LJ/LK

Règles LJ

$$P(x) \vdash P(x), Q(x)$$

$$P(x) \vdash P(x) \vee Q(x)$$

$$P(x) \vdash \exists y.P(y) \vee Q(y)$$

$$\dfrac{\vdash P(x) \Rightarrow \exists y.P(y) \vee Q(y)}{\vdash \forall x.P(x) \Rightarrow \exists y.P(y) \vee Q(y)}\ \forall_{\text{right}}$$

Correction

Preuve (1) dans LJ/LK

Règles LJ

$$P(x) \vdash P(x), Q(x)$$

$$P(x) \vdash P(x) \lor Q(x)$$

$$\dfrac{\dfrac{P(x) \vdash \exists y.P(y) \lor Q(y)}{\vdash P(x) \Rightarrow \exists y.P(y) \lor Q(y)} \Rightarrow_{\text{right}}}{\vdash \forall x.P(x) \Rightarrow \exists y.P(y) \lor Q(y)} \forall_{\text{right}}$$

Correction

Preuve (1) dans LJ/LK

▸ Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x), Q(x)}{P(x) \vdash P(x) \vee Q(x)}}{P(x) \vdash \exists y.P(y) \vee Q(y)}\,\exists_{\text{right}}}{\vdash P(x) \Rightarrow \exists y.P(y) \vee Q(y)}\,\Rightarrow_{\text{right}}}{\vdash \forall x.P(x) \Rightarrow \exists y.P(y) \vee Q(y)}\,\forall_{\text{right}}
$$

Correction

Preuve (1) dans LJ/LK

▸ Règles LJ

$$
\cfrac{\cfrac{\cfrac{\cfrac{P(x) \vdash P(x), Q(x)}{P(x) \vdash P(x) \vee Q(x)}\ \vee_{\text{right}}}{P(x) \vdash \exists y.P(y) \vee Q(y)}\ \exists_{\text{right}}}{\vdash P(x) \Rightarrow \exists y.P(y) \vee Q(y)}\ \Rightarrow_{\text{right}}}{\vdash \forall x.P(x) \Rightarrow \exists y.P(y) \vee Q(y)}\ \forall_{\text{right}}
$$

Correction

Preuve (1) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x), Q(x)}\text{ax}}{P(x) \vdash P(x) \vee Q(x)}\vee_{\text{right}}}{P(x) \vdash \exists y.P(y) \vee Q(y)}\exists_{\text{right}}}{\vdash P(x) \Rightarrow \exists y.P(y) \vee Q(y)}\Rightarrow_{\text{right}}}{\vdash \forall x.P(x) \Rightarrow \exists y.P(y) \vee Q(y)}\forall_{\text{right}}$$

Correction

Preuve (2) dans LJ

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{P(x) \vdash \exists x.P(x)}}{P(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))} \quad \dfrac{\dfrac{Q(x) \vdash Q(x)}{Q(x) \vdash \exists x.Q(x)}}{Q(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))}}{P(x) \lor Q(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))}}{\exists x.P(x) \lor Q(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))}}{\vdash (\exists x.P(x) \lor Q(x)) \Rightarrow (\exists x.P(x)) \lor (\exists x.Q(x))}$$

Correction

Preuve (2) dans LJ

▸ Règles LJ

$$P(x) \vdash P(x) \qquad\qquad Q(x) \vdash Q(x)$$

$$P(x) \vdash \exists x.P(x) \qquad\qquad Q(x) \vdash \exists x.Q(x)$$

$$P(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x)) \qquad Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))$$

$$P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))$$

$$\frac{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\Rightarrow_{\text{right}}$$

Correction

Preuve (2) dans LJ

▸ Règles LJ

$$P(x) \vdash P(x) \qquad\qquad Q(x) \vdash Q(x)$$

$$P(x) \vdash \exists x.P(x) \qquad\qquad Q(x) \vdash \exists x.Q(x)$$

$$P(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x)) \qquad\qquad Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))$$

$$\dfrac{\dfrac{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\exists_{\text{left}}}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\,\Rightarrow_{\text{right}}$$

Correction

Preuve (2) dans LJ

Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{P(x) \vdash \exists x.P(x)}}{P(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))} \qquad \dfrac{\dfrac{Q(x) \vdash Q(x)}{Q(x) \vdash \exists x.Q(x)}}{Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}}{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\vee_{\text{left}}}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\exists_{\text{left}}}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\,\Rightarrow_{\text{right}}
$$

Correction

Preuve (2) dans LJ

▸ Règles LJ

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{P(x) \vdash P(x)}{P(x) \vdash \exists x.P(x)}
    }{P(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))}\lor_{\text{right1}}
    \qquad
    \dfrac{\dfrac{Q(x) \vdash Q(x)}{Q(x) \vdash \exists x.Q(x)}}{Q(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))}
  }{P(x) \lor Q(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))}\lor_{\text{left}}
}{
  \dfrac{\exists x.P(x) \lor Q(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))}{\vdash (\exists x.P(x) \lor Q(x)) \Rightarrow (\exists x.P(x)) \lor (\exists x.Q(x))}\Rightarrow_{\text{right}}
}\exists_{\text{left}}
$$

Correction

Preuve (2) dans LJ

▸ Règles LJ

$$
\dfrac{
  \dfrac{
    \dfrac{P(x) \vdash P(x)}{P(x) \vdash \exists x.P(x)}\,\exists_{\text{right}}
  }{P(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\vee_{\text{right1}}
  \qquad
  \begin{array}{c} Q(x) \vdash Q(x) \\ Q(x) \vdash \exists x.Q(x) \\ Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x)) \end{array}
}{
  \dfrac{
    \dfrac{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\exists_{\text{left}}
  }{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\,\Rightarrow_{\text{right}}
}\,\vee_{\text{left}}
$$

Correction

Preuve (2) dans LJ

▸ Règles LJ

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(x) \vdash P(x)$}
\RightLabel{$\exists_{\text{right}}$}
\UnaryInfC{$P(x) \vdash \exists x.P(x)$}
\RightLabel{$\vee_{\text{right1}}$}
\UnaryInfC{$P(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))$}
\AxiomC{$Q(x) \vdash Q(x)$}
\noLine
\UnaryInfC{$Q(x) \vdash \exists x.Q(x)$}
\noLine
\UnaryInfC{$Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))$}
\RightLabel{$\vee_{\text{left}}$}
\BinaryInfC{$P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))$}
\RightLabel{$\exists_{\text{left}}$}
\UnaryInfC{$\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))$}
\RightLabel{$\Rightarrow_{\text{right}}$}
\UnaryInfC{$\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))$}
\end{prooftree}
$$

Correction

Preuve (2) dans LJ

▸ Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x)}\ \text{ax}}{P(x) \vdash \exists x.P(x)}\ \exists_{\text{right}}}{P(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\ \vee_{\text{right1}} \qquad \dfrac{\begin{array}{c} Q(x) \vdash Q(x) \\ Q(x) \vdash \exists x.Q(x) \end{array}}{Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\ \vee_{\text{right2}}}{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\ \vee_{\text{left}}}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\ \exists_{\text{left}}}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\ \Rightarrow_{\text{right}}
$$

Correction

Preuve (2) dans LJ

Règles LJ

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{P(x) \vdash P(x)}\,\text{ax}
    }{P(x) \vdash \exists x.P(x)}\,\exists_{\text{right}}
  }{P(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\vee_{\text{right1}}
  \qquad
  \dfrac{
    \dfrac{Q(x) \vdash Q(x)}{Q(x) \vdash \exists x.Q(x)}\,\exists_{\text{right}}
  }{Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\vee_{\text{right2}}
}{
  \dfrac{
    \dfrac{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\exists_{\text{left}}
  }{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\,\Rightarrow_{\text{right}}
}\,\vee_{\text{left}}
$$

Correction

Preuve (2) dans LJ

Règles LJ

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{P(x) \vdash P(x)}\,\text{ax}
}{P(x) \vdash \exists x.P(x)}\,\exists_{\text{right}}
}{P(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\vee_{\text{right1}}
\qquad
\dfrac{
\dfrac{
\dfrac{}{Q(x) \vdash Q(x)}\,\text{ax}
}{Q(x) \vdash \exists x.Q(x)}\,\exists_{\text{right}}
}{Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\vee_{\text{right2}}
}{
\dfrac{
\dfrac{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\exists_{\text{left}}
}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\,\Rightarrow_{\text{right}}
}\,\vee_{\text{left}}
$$

Correction

Preuve (2) dans LK

Règles LK

$$P(x) \vdash P(x), \exists x.Q(x) \qquad Q(x) \vdash \exists x.P(x), Q(x)$$

$$P(x) \vdash \exists x.P(x), \exists x.Q(x) \qquad Q(x) \vdash \exists x.P(x), \exists x.Q(x)$$

$$P(x) \vee Q(x) \vdash \exists x.P(x), \exists x.Q(x)$$

$$P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))$$

$$\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))$$

$$\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))$$

Correction

Preuve (2) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x), \exists x.Q(x)}{P(x) \vdash \exists x.P(x), \exists x.Q(x)} \qquad \dfrac{Q(x) \vdash \exists x.P(x), Q(x)}{Q(x) \vdash \exists x.P(x), \exists x.Q(x)}}{P(x) \vee Q(x) \vdash \exists x.P(x), \exists x.Q(x)}}{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))} \Rightarrow_{\text{right}}$$

Correction

Preuve (2) dans LK

Règles LK

$$P(x) \vdash P(x), \exists x.Q(x) \qquad\qquad Q(x) \vdash \exists x.P(x), Q(x)$$

$$P(x) \vdash \exists x.P(x), \exists x.Q(x) \qquad\qquad Q(x) \vdash \exists x.P(x), \exists x.Q(x)$$

$$P(x) \vee Q(x) \vdash \exists x.P(x), \exists x.Q(x)$$

$$\dfrac{\dfrac{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\ \exists_{\text{left}}}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\ \Rightarrow_{\text{right}}$$

Correction

Preuve (2) dans LK

▸ Règles LK

$$P(x) \vdash P(x), \exists x.Q(x) \qquad\qquad Q(x) \vdash \exists x.P(x), Q(x)$$

$$P(x) \vdash \exists x.P(x), \exists x.Q(x) \qquad\qquad Q(x) \vdash \exists x.P(x), \exists x.Q(x)$$

$$P(x) \vee Q(x) \vdash \exists x.P(x), \exists x.Q(x)$$

$$\overline{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))} \;\vee_{\text{right}}$$

$$\overline{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))} \;\exists_{\text{left}}$$

$$\overline{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))} \;\Rightarrow_{\text{right}}$$

Correction

Preuve (2) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x), \exists x.Q(x)}{P(x) \vdash \exists x.P(x), \exists x.Q(x)} \quad \dfrac{Q(x) \vdash \exists x.P(x), Q(x)}{Q(x) \vdash \exists x.P(x), \exists x.Q(x)}}{P(x) \vee Q(x) \vdash \exists x.P(x), \exists x.Q(x)} \vee_{\text{left}}}{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))} \vee_{\text{right}}}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))} \exists_{\text{left}}}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))} \Rightarrow_{\text{right}}$$

Correction

Preuve (2) dans LK

Règles LK

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{P(x) \vdash P(x), \exists x.Q(x)}{P(x) \vdash \exists x.P(x), \exists x.Q(x)}\exists_{\text{right}}
      \qquad
      \dfrac{Q(x) \vdash \exists x.P(x), Q(x)}{Q(x) \vdash \exists x.P(x), \exists x.Q(x)}
    }{P(x) \vee Q(x) \vdash \exists x.P(x), \exists x.Q(x)}\vee_{\text{left}}
  }{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\vee_{\text{right}}
}{
  \dfrac{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\Rightarrow_{\text{right}}
}\exists_{\text{left}}
$$

Correction

Preuve (2) dans LK

▸ Règles LK

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x), \exists x.Q(x)}\;\text{ax}}{P(x) \vdash \exists x.P(x), \exists x.Q(x)}\;\exists_{\text{right}} \qquad \begin{array}{c} Q(x) \vdash \exists x.P(x), Q(x) \\ Q(x) \vdash \exists x.P(x), \exists x.Q(x) \end{array}}{P(x) \vee Q(x) \vdash \exists x.P(x), \exists x.Q(x)}\;\vee_{\text{left}}}{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\;\vee_{\text{right}}}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\;\exists_{\text{left}}}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\;\Rightarrow_{\text{right}}
$$

Correction

Preuve (2) dans LK

▸ Règles LK

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{}{P(x) \vdash P(x), \exists x.Q(x)}\ \text{ax}}{P(x) \vdash \exists x.P(x), \exists x.Q(x)}\ \exists_{\text{right}}
\qquad
\dfrac{Q(x) \vdash \exists x.P(x), Q(x)}{Q(x) \vdash \exists x.P(x), \exists x.Q(x)}\ \exists_{\text{right}}
}{
\dfrac{P(x) \vee Q(x) \vdash \exists x.P(x), \exists x.Q(x)}{
\dfrac{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\ \exists_{\text{left}}
}\ \vee_{\text{right}}
}\ \vee_{\text{left}}
}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\ \Rightarrow_{\text{right}}
$$

Correction

Preuve (2) dans LK

▸ Règles LK

$$
\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x), \exists x.Q(x)}\ \text{ax}}{P(x) \vdash \exists x.P(x), \exists x.Q(x)}\ \exists_{\text{right}} \qquad \dfrac{\dfrac{}{Q(x) \vdash \exists x.P(x), Q(x)}\ \text{ax}}{Q(x) \vdash \exists x.P(x), \exists x.Q(x)}\ \exists_{\text{right}}}{\dfrac{P(x) \vee Q(x) \vdash \exists x.P(x), \exists x.Q(x)}{\dfrac{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}{\dfrac{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\ \Rightarrow_{\text{right}}}\ \exists_{\text{left}}}\ \vee_{\text{right}}}\ \vee_{\text{left}}
$$

Correction

Preuve (3) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x), \forall x.Q(x) \vdash P(x)}{\forall x.P(x), \forall x.Q(x) \vdash P(x)} \quad \dfrac{\forall x.P(x), Q(x) \vdash Q(x)}{\forall x.P(x), \forall x.Q(x) \vdash Q(x)}}{\forall x.P(x), \forall x.Q(x) \vdash P(x) \wedge Q(x)}}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash P(x) \wedge Q(x)}}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash \forall x.P(x) \wedge Q(x)}}{\vdash (\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)}$$

Correction

Preuve (3) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x), \forall x.Q(x) \vdash P(x)}{\forall x.P(x), \forall x.Q(x) \vdash P(x)} \qquad \dfrac{\forall x.P(x), Q(x) \vdash Q(x)}{\forall x.P(x), \forall x.Q(x) \vdash Q(x)}}{\forall x.P(x), \forall x.Q(x) \vdash P(x) \land Q(x)}}{(\forall x.P(x)) \land (\forall x.Q(x)) \vdash P(x) \land Q(x)}}{(\forall x.P(x)) \land (\forall x.Q(x)) \vdash \forall x.P(x) \land Q(x)}}{\vdash (\forall x.P(x)) \land (\forall x.Q(x)) \Rightarrow \forall x.P(x) \land Q(x)} \Rightarrow_{\text{right}}$$

Correction

Preuve (3) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x), \forall x.Q(x) \vdash P(x)}{\forall x.P(x), \forall x.Q(x) \vdash P(x)} \quad \dfrac{\forall x.P(x), Q(x) \vdash Q(x)}{\forall x.P(x), \forall x.Q(x) \vdash Q(x)}}{\forall x.P(x), \forall x.Q(x) \vdash P(x) \wedge Q(x)}}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash P(x) \wedge Q(x)}}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash \forall x.P(x) \wedge Q(x)}\ \forall_{right}}{\vdash (\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)}\ \Rightarrow_{right}$$

Correction

Preuve (3) dans LJ/LK

▸ Règles LJ

$$P(x), \forall x.Q(x) \vdash P(x) \qquad \forall x.P(x), Q(x) \vdash Q(x)$$

$$\forall x.P(x), \forall x.Q(x) \vdash P(x) \qquad \forall x.P(x), \forall x.Q(x) \vdash Q(x)$$

$$\dfrac{\dfrac{\dfrac{\forall x.P(x), \forall x.Q(x) \vdash P(x) \wedge Q(x)}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash P(x) \wedge Q(x)}\wedge_{\text{left}}}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash \forall x.P(x) \wedge Q(x)}\forall_{\text{right}}}{\vdash (\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)}\Rightarrow_{\text{right}}$$

Correction

Preuve (3) dans LJ/LK

‣ Règles LJ

$$
P(x), \forall x.Q(x) \vdash P(x) \qquad \forall x.P(x), Q(x) \vdash Q(x)
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\forall x.P(x), \forall x.Q(x) \vdash P(x) \qquad \forall x.P(x), \forall x.Q(x) \vdash Q(x)}{\forall x.P(x), \forall x.Q(x) \vdash P(x) \wedge Q(x)}\ \wedge_{\text{right}}}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash P(x) \wedge Q(x)}\ \wedge_{\text{left}}}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash \forall x.P(x) \wedge Q(x)}\ \forall_{\text{right}}}{\vdash (\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)}\ \Rightarrow_{\text{right}}
$$

Correction

Preuve (3) dans LJ/LK

▸ Règles LJ

$$\dfrac{\dfrac{\dfrac{P(x), \forall x.Q(x) \vdash P(x)}{\forall x.P(x), \forall x.Q(x) \vdash P(x)}\,\forall_{\text{left}} \quad \dfrac{\forall x.P(x), Q(x) \vdash Q(x)}{\forall x.P(x), \forall x.Q(x) \vdash Q(x)}}{\dfrac{\forall x.P(x), \forall x.Q(x) \vdash P(x) \wedge Q(x)}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash P(x) \wedge Q(x)}\,\wedge_{\text{left}}}\,\wedge_{\text{right}}}{\dfrac{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash \forall x.P(x) \wedge Q(x)}{\vdash (\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)}\,\Rightarrow_{\text{right}}}\,\forall_{\text{right}}$$

Correction

Preuve (3) dans LJ/LK

▸ Règles LJ

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{P(x), \forall x.Q(x) \vdash P(x)}\;\text{ax}
        }{\forall x.P(x), \forall x.Q(x) \vdash P(x)}\;\forall_{\text{left}}
        \qquad
        \dfrac{\forall x.P(x), Q(x) \vdash Q(x)}{\forall x.P(x), \forall x.Q(x) \vdash Q(x)}
      }{\forall x.P(x), \forall x.Q(x) \vdash P(x) \wedge Q(x)}\;\wedge_{\text{right}}
    }{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash P(x) \wedge Q(x)}\;\wedge_{\text{left}}
  }{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash \forall x.P(x) \wedge Q(x)}\;\forall_{\text{right}}
}{\vdash (\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)}\;\Rightarrow_{\text{right}}
$$

Correction

Preuve (3) dans LJ/LK

‣ Règles LJ

$$
\dfrac{\dfrac{\dfrac{}{P(x), \forall x.Q(x) \vdash P(x)}\,\text{ax}}{\forall x.P(x), \forall x.Q(x) \vdash P(x)}\,\forall_{\text{left}} \qquad \dfrac{\forall x.P(x), Q(x) \vdash Q(x)}{\forall x.P(x), \forall x.Q(x) \vdash Q(x)}\,\forall_{\text{left}}}{\dfrac{\dfrac{\dfrac{\forall x.P(x), \forall x.Q(x) \vdash P(x) \wedge Q(x)}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash P(x) \wedge Q(x)}\,\wedge_{\text{left}}}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash \forall x.P(x) \wedge Q(x)}\,\forall_{\text{right}}}{\vdash (\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)}\,\Rightarrow_{\text{right}}}\,\wedge_{\text{right}}
$$

Correction

Preuve (3) dans LJ/LK

Règles LJ

$$
\dfrac{
  \dfrac{
    \dfrac{}{P(x), \forall x.Q(x) \vdash P(x)}\,\text{ax}
  }{\forall x.P(x), \forall x.Q(x) \vdash P(x)}\,\forall_{\text{left}}
  \qquad
  \dfrac{
    \dfrac{}{\forall x.P(x), Q(x) \vdash Q(x)}\,\text{ax}
  }{\forall x.P(x), \forall x.Q(x) \vdash Q(x)}\,\forall_{\text{left}}
}{
  \dfrac{
    \dfrac{
      \dfrac{\forall x.P(x), \forall x.Q(x) \vdash P(x) \wedge Q(x)}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash P(x) \wedge Q(x)}\,\wedge_{\text{left}}
    }{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash \forall x.P(x) \wedge Q(x)}\,\forall_{\text{right}}
  }{\vdash (\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)}\,\Rightarrow_{\text{right}}
}\,\wedge_{\text{right}}
$$

Correction

Preuve (4) dans LJ/LK

Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{P(x), Q(x) \vdash P(x)}{P(x) \wedge Q(x) \vdash P(x)}}{\forall x.P(x) \wedge Q(x) \vdash P(x)}}{\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)} \qquad \dfrac{\dfrac{\dfrac{P(x), Q(x) \vdash Q(x)}{P(x) \wedge Q(x) \vdash Q(x)}}{\forall x.P(x) \wedge Q(x) \vdash Q(x)}}{\forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)}}{\dfrac{\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))}{\vdash (\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))}}
$$

Correction

Preuve (4) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{P(x), Q(x) \vdash P(x)}{P(x) \wedge Q(x) \vdash P(x)}}{\forall x. P(x) \wedge Q(x) \vdash P(x)}}{\forall x. P(x) \wedge Q(x) \vdash \forall x. P(x)} \quad \dfrac{\dfrac{\dfrac{P(x), Q(x) \vdash Q(x)}{P(x) \wedge Q(x) \vdash Q(x)}}{\forall x. P(x) \wedge Q(x) \vdash Q(x)}}{\forall x. P(x) \wedge Q(x) \vdash \forall x. Q(x)}}{\dfrac{\forall x. P(x) \wedge Q(x) \vdash (\forall x. P(x)) \wedge (\forall x. Q(x))}{\vdash (\forall x. P(x) \wedge Q(x)) \Rightarrow (\forall x. P(x)) \wedge (\forall x. Q(x))} \Rightarrow_{\text{right}}}$$

Correction

Preuve (4) dans LJ/LK

▸ Règles LJ

$$P(x), Q(x) \vdash P(x) \qquad\qquad P(x), Q(x) \vdash Q(x)$$

$$P(x) \wedge Q(x) \vdash P(x) \qquad\qquad P(x) \wedge Q(x) \vdash Q(x)$$

$$\forall x. P(x) \wedge Q(x) \vdash P(x) \qquad\qquad \forall x. P(x) \wedge Q(x) \vdash Q(x)$$

$$\cfrac{\cfrac{\forall x. P(x) \wedge Q(x) \vdash \forall x. P(x) \qquad \forall x. P(x) \wedge Q(x) \vdash \forall x. Q(x)}{\forall x. P(x) \wedge Q(x) \vdash (\forall x. P(x)) \wedge (\forall x. Q(x))}\wedge_{\text{right}}}{\vdash (\forall x. P(x) \wedge Q(x)) \Rightarrow (\forall x. P(x)) \wedge (\forall x. Q(x))}\Rightarrow_{\text{right}}$$

Correction

Preuve (4) dans LJ/LK

Règles LJ

$$
\dfrac{
  \dfrac{
    \dfrac{
      \begin{array}{c} P(x), Q(x) \vdash P(x) \\ P(x) \wedge Q(x) \vdash P(x) \\ \forall x. P(x) \wedge Q(x) \vdash P(x) \end{array}
    }{\forall x. P(x) \wedge Q(x) \vdash \forall x. P(x)}\ \forall_{\text{right}}
    \qquad
    \dfrac{
      \begin{array}{c} P(x), Q(x) \vdash Q(x) \\ P(x) \wedge Q(x) \vdash Q(x) \\ \forall x. P(x) \wedge Q(x) \vdash Q(x) \end{array}
    }{\forall x. P(x) \wedge Q(x) \vdash \forall x. Q(x)}
  }{\forall x. P(x) \wedge Q(x) \vdash (\forall x. P(x)) \wedge (\forall x. Q(x))}\ \wedge_{\text{right}}
}{\vdash (\forall x. P(x) \wedge Q(x)) \Rightarrow (\forall x. P(x)) \wedge (\forall x. Q(x))}\ \Rightarrow_{\text{right}}
$$

Correction

Preuve (4) dans LJ/LK

▸ Règles LJ

$$
\dfrac{
\dfrac{
\dfrac{
\begin{array}{c} P(x), Q(x) \vdash P(x) \\ P(x) \wedge Q(x) \vdash P(x) \end{array}
}{\forall x. P(x) \wedge Q(x) \vdash P(x)}\ \forall_{\text{left}}
}{\forall x. P(x) \wedge Q(x) \vdash \forall x. P(x)}\ \forall_{\text{right}}
\qquad
\begin{array}{c} P(x), Q(x) \vdash Q(x) \\ P(x) \wedge Q(x) \vdash Q(x) \\ \forall x. P(x) \wedge Q(x) \vdash Q(x) \\ \forall x. P(x) \wedge Q(x) \vdash \forall x. Q(x) \end{array}
}{
\dfrac{\forall x. P(x) \wedge Q(x) \vdash (\forall x. P(x)) \wedge (\forall x. Q(x))}{\vdash (\forall x. P(x) \wedge Q(x)) \Rightarrow (\forall x. P(x)) \wedge (\forall x. Q(x))}\ \Rightarrow_{\text{right}}
}\ \wedge_{\text{right}}
$$

Correction

Preuve (4) dans LJ/LK

Règles LJ

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{P(x), Q(x) \vdash P(x)}{P(x) \wedge Q(x) \vdash P(x)}\;\wedge_{\text{left}}
}{\forall x.P(x) \wedge Q(x) \vdash P(x)}\;\forall_{\text{left}}
}{\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)}\;\forall_{\text{right}}
\qquad
\dfrac{
\dfrac{
\dfrac{P(x), Q(x) \vdash Q(x)}{P(x) \wedge Q(x) \vdash Q(x)}
}{\forall x.P(x) \wedge Q(x) \vdash Q(x)}
}{\forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)}
}{
\dfrac{\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))}{\vdash (\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))}\;\Rightarrow_{\text{right}}
}\;\wedge_{\text{right}}
$$

Correction

Preuve (4) dans LJ/LK

▸ Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x), Q(x) \vdash P(x)}\,\mathrm{ax}}{P(x) \wedge Q(x) \vdash P(x)}\,\wedge_{\mathrm{left}}}{\forall x.P(x) \wedge Q(x) \vdash P(x)}\,\forall_{\mathrm{left}}}{\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)}\,\forall_{\mathrm{right}} \qquad \begin{array}{c} P(x), Q(x) \vdash Q(x) \\ P(x) \wedge Q(x) \vdash Q(x) \\ \forall x.P(x) \wedge Q(x) \vdash Q(x) \\ \forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x) \end{array}}{\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))}\,\wedge_{\mathrm{right}}}{\vdash (\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))}\,\Rightarrow_{\mathrm{right}}
$$

Correction

Preuve (4) dans LJ/LK

Règles LJ

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{P(x), Q(x) \vdash P(x)}\,\text{ax}
}{P(x) \wedge Q(x) \vdash P(x)}\,\wedge_{\text{left}}
}{\forall x.P(x) \wedge Q(x) \vdash P(x)}\,\forall_{\text{left}}
}{\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)}\,\forall_{\text{right}}
\quad
\dfrac{
\begin{array}{c}
P(x), Q(x) \vdash Q(x) \\
P(x) \wedge Q(x) \vdash Q(x) \\
\forall x.P(x) \wedge Q(x) \vdash Q(x)
\end{array}
}{\forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)}\,\forall_{\text{right}}
}{
\dfrac{\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))}{\vdash (\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))}\,\Rightarrow_{\text{right}}
}\,\wedge_{\text{right}}
$$

Correction

Preuve (4) dans LJ/LK

Règles LJ

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(x), Q(x) \vdash P(x)$}
\RightLabel{$\wedge_{\text{left}}$}
\UnaryInfC{$P(x) \wedge Q(x) \vdash P(x)$}
\RightLabel{$\forall_{\text{left}}$}
\UnaryInfC{$\forall x.P(x) \wedge Q(x) \vdash P(x)$}
\RightLabel{$\forall_{\text{right}}$}
\UnaryInfC{$\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)$}
\AxiomC{$P(x), Q(x) \vdash Q(x)$}
\noLine
\UnaryInfC{$P(x) \wedge Q(x) \vdash Q(x)$}
\RightLabel{$\forall_{\text{left}}$}
\UnaryInfC{$\forall x.P(x) \wedge Q(x) \vdash Q(x)$}
\RightLabel{$\forall_{\text{right}}$}
\UnaryInfC{$\forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)$}
\RightLabel{$\wedge_{\text{right}}$}
\BinaryInfC{$\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))$}
\RightLabel{$\Rightarrow_{\text{right}}$}
\UnaryInfC{$\vdash (\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))$}
\end{prooftree}
$$

Correction

Preuve (4) dans LJ/LK

▸ Règles LJ

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{}{P(x), Q(x) \vdash P(x)}\,\text{ax}}{P(x) \wedge Q(x) \vdash P(x)}\,\wedge_{\text{left}}}{\forall x.P(x) \wedge Q(x) \vdash P(x)}\,\forall_{\text{left}}}{\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)}\,\forall_{\text{right}}
\qquad
\dfrac{\dfrac{\dfrac{P(x), Q(x) \vdash Q(x)}{P(x) \wedge Q(x) \vdash Q(x)}\,\wedge_{\text{left}}}{\forall x.P(x) \wedge Q(x) \vdash Q(x)}\,\forall_{\text{left}}}{\forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)}\,\forall_{\text{right}}
}{
\dfrac{\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))}{\vdash (\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))}\,\Rightarrow_{\text{right}}
}\,\wedge_{\text{right}}
$$

Correction

Preuve (4) dans LJ/LK

‣ Règles LJ

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{P(x), Q(x) \vdash P(x)}\ \text{ax}
      }{P(x) \wedge Q(x) \vdash P(x)}\ \wedge_{\text{left}}
    }{\forall x. P(x) \wedge Q(x) \vdash P(x)}\ \forall_{\text{left}}
  }{\forall x. P(x) \wedge Q(x) \vdash \forall x. P(x)}\ \forall_{\text{right}}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{P(x), Q(x) \vdash Q(x)}\ \text{ax}
      }{P(x) \wedge Q(x) \vdash Q(x)}\ \wedge_{\text{left}}
    }{\forall x. P(x) \wedge Q(x) \vdash Q(x)}\ \forall_{\text{left}}
  }{\forall x. P(x) \wedge Q(x) \vdash \forall x. Q(x)}\ \forall_{\text{right}}
}{
  \dfrac{\forall x. P(x) \wedge Q(x) \vdash (\forall x. P(x)) \wedge (\forall x. Q(x))}{\vdash (\forall x. P(x) \wedge Q(x)) \Rightarrow (\forall x. P(x)) \wedge (\forall x. Q(x))}\ \Rightarrow_{\text{right}}
}\ \wedge_{\text{right}}
$$

Correction

Preuve (5) dans LJ/LK

▸ Règles LJ

$$P(x) \vdash P(x)$$

$$\neg P(x), P(x) \vdash \bot$$

$$\forall x.\neg P(x), P(x) \vdash \bot$$

$$\forall x.\neg P(x), \exists x.P(x) \vdash \bot$$

$$\forall x.\neg P(x) \vdash \neg(\exists x.P(x))$$

$$\vdash (\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))$$

Correction

Preuve (5) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{\neg P(x), P(x) \vdash \bot}}{\forall x.\neg P(x), P(x) \vdash \bot}}{\forall x.\neg P(x), \exists x.P(x) \vdash \bot}}{\forall x.\neg P(x) \vdash \neg(\exists x.P(x))}}{\vdash (\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))} \Rightarrow_{\text{right}}$$

Preuve (5) dans LJ/LK

Règles LJ

$$P(x) \vdash P(x)$$

$$\neg P(x), P(x) \vdash \bot$$

$$\forall x.\neg P(x), P(x) \vdash \bot$$

$$\cfrac{\cfrac{\forall x.\neg P(x), \exists x.P(x) \vdash \bot}{\forall x.\neg P(x) \vdash \neg(\exists x.P(x))}\ \neg_{\text{right}}}{\vdash (\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))}\ \Rightarrow_{\text{right}}$$

Correction

Preuve (5) dans LJ/LK

Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{\neg P(x), P(x) \vdash \bot}}{\forall x.\neg P(x), P(x) \vdash \bot}\ \exists_{\text{left}}}{\forall x.\neg P(x), \exists x.P(x) \vdash \bot}\ \neg_{\text{right}}}{\forall x.\neg P(x) \vdash \neg(\exists x.P(x))}\ \Rightarrow_{\text{right}}}{\vdash (\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))}
$$

Correction

Preuve (5) dans LJ/LK

▸ Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{\neg P(x), P(x) \vdash \bot}}{\forall x.\neg P(x), P(x) \vdash \bot}\ \forall_{\text{left}}}{\forall x.\neg P(x), \exists x.P(x) \vdash \bot}\ \exists_{\text{left}}}{\forall x.\neg P(x) \vdash \neg(\exists x.P(x))}\ \neg_{\text{right}}}{\vdash (\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))}\ \Rightarrow_{\text{right}}
$$

Correction

Preuve (5) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{\neg P(x), P(x) \vdash \bot}\neg_{\text{left}}}{\forall x.\neg P(x), P(x) \vdash \bot}\forall_{\text{left}}}{\forall x.\neg P(x), \exists x.P(x) \vdash \bot}\exists_{\text{left}}}{\forall x.\neg P(x) \vdash \neg(\exists x.P(x))}\neg_{\text{right}}}{\vdash (\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))}\Rightarrow_{\text{right}}$$

Correction

Preuve (5) dans LJ/LK

▸ Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x)}\ \text{ax}}{\neg P(x), P(x) \vdash \bot}\ \neg_{\text{left}}}{\forall x.\neg P(x), P(x) \vdash \bot}\ \forall_{\text{left}}}{\forall x.\neg P(x), \exists x.P(x) \vdash \bot}\ \exists_{\text{left}}}{\forall x.\neg P(x) \vdash \neg(\exists x.P(x))}\ \neg_{\text{right}}}{\vdash (\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))}\ \Rightarrow_{\text{right}}
$$

Correction

Preuve (6) dans LJ_{em}

Règles LJ

$$\neg P(x) \vdash \neg P(x)$$
$$\neg P(x) \vdash \exists x.\neg P(x)$$
$$\neg\exists x.\neg P(x), \neg P(x) \vdash \bot$$
$$\neg\exists x.\neg P(x) \vdash \neg\neg P(x)$$
$$\neg\exists x.\neg P(x) \vdash P(x)$$
$$\neg\exists x.\neg P(x) \vdash \forall x.P(x)$$
$$\neg\forall x.P(x), \neg\exists x.\neg P(x) \vdash \bot$$
$$\neg\forall x.P(x) \vdash \neg\neg\exists x.\neg P(x)$$
$$\neg\forall x.P(x) \vdash \exists x.\neg P(x)$$
$$\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)$$

Correction

Preuve (6) dans LJ_{em}

Règles LJ

$$\neg P(x) \vdash \neg P(x)$$

$$\neg P(x) \vdash \exists x.\neg P(x)$$

$$\neg\exists x.\neg P(x), \neg P(x) \vdash \bot$$

$$\neg\exists x.\neg P(x) \vdash \neg\neg P(x)$$

$$\neg\exists x.\neg P(x) \vdash P(x)$$

$$\neg\exists x.\neg P(x) \vdash \forall x.P(x)$$

$$\neg\forall x.P(x), \neg\exists x.\neg P(x) \vdash \bot$$

$$\neg\forall x.P(x) \vdash \neg\neg\exists x.\neg P(x)$$

$$\dfrac{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)} \Rightarrow_{\text{right}}$$

Correction

Preuve (6) dans LJ_{em}

▸ Règles LJ

$$\neg P(x) \vdash \neg P(x)$$

$$\neg P(x) \vdash \exists x.\neg P(x)$$

$$\neg\exists x.\neg P(x), \neg P(x) \vdash \bot$$

$$\neg\exists x.\neg P(x) \vdash \neg\neg P(x)$$

$$\neg\exists x.\neg P(x) \vdash P(x)$$

$$\neg\exists x.\neg P(x) \vdash \forall x.P(x)$$

$$\neg\forall x.P(x), \neg\exists x.\neg P(x) \vdash \bot$$

$$\cfrac{\cfrac{\neg\forall x.P(x) \vdash \neg\neg\exists x.\neg P(x)}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\,\text{em}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\Rightarrow_{\text{right}}$$

Correction

Preuve (6) dans LJ_{em}

Règles LJ

$$\neg P(x) \vdash \neg P(x)$$

$$\neg P(x) \vdash \exists x.\neg P(x)$$

$$\neg\exists x.\neg P(x), \neg P(x) \vdash \bot$$

$$\neg\exists x.\neg P(x) \vdash \neg\neg P(x)$$

$$\neg\exists x.\neg P(x) \vdash P(x)$$

$$\neg\exists x.\neg P(x) \vdash \forall x.P(x)$$

$$\dfrac{\dfrac{\dfrac{\neg\forall x.P(x), \neg\exists x.\neg P(x) \vdash \bot}{\neg\forall x.P(x) \vdash \neg\neg\exists x.\neg P(x)}\,\neg_{\mathrm{right}}}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\,\mathrm{em}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\,\Rightarrow_{\mathrm{right}}$$

Correction

Preuve (6) dans LJ_{em}

▸ Règles LJ

$$\neg P(x) \vdash \neg P(x)$$

$$\neg P(x) \vdash \exists x.\neg P(x)$$

$$\neg\exists x.\neg P(x), \neg P(x) \vdash \bot$$

$$\neg\exists x.\neg P(x) \vdash \neg\neg P(x)$$

$$\neg\exists x.\neg P(x) \vdash P(x)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\exists x.\neg P(x) \vdash \forall x.P(x)}{\neg\forall x.P(x), \neg\exists x.\neg P(x) \vdash \bot}\,\neg\text{left}}{\neg\forall x.P(x) \vdash \neg\neg\exists x.\neg P(x)}\,\neg\text{right}}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\,\text{em}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\,\Rightarrow\text{right}$$

Correction

Preuve (6) dans LJ_{em}

▸ Règles LJ

$$\neg P(x) \vdash \neg P(x)$$

$$\neg P(x) \vdash \exists x. \neg P(x)$$

$$\neg \exists x. \neg P(x), \neg P(x) \vdash \bot$$

$$\neg \exists x. \neg P(x) \vdash \neg\neg P(x)$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\neg \exists x. \neg P(x) \vdash P(x)}{\neg \exists x. \neg P(x) \vdash \forall x. P(x)}\ \forall_{right}}{\neg \forall x. P(x), \neg \exists x. \neg P(x) \vdash \bot}\ \neg_{left}}{\neg \forall x. P(x) \vdash \neg\neg \exists x. \neg P(x)}\ \neg_{right}}{\neg \forall x. P(x) \vdash \exists x. \neg P(x)}\ \text{em}}{\vdash \neg(\forall x. P(x)) \Rightarrow \exists x. \neg P(x)}\ \Rightarrow_{right}$$

Correction

Preuve (6) dans LJ$_{em}$

▸ Règles LJ

$$
\begin{array}{cl}
\neg P(x) \vdash \neg P(x) & \\
\neg P(x) \vdash \exists x.\neg P(x) & \\
\neg\exists x.\neg P(x), \neg P(x) \vdash \bot & \\
\neg\exists x.\neg P(x) \vdash \neg\neg P(x) & \\
\hline
\neg\exists x.\neg P(x) \vdash P(x) & \text{em} \\
\hline
\neg\exists x.\neg P(x) \vdash \forall x.P(x) & \forall_{\text{right}} \\
\hline
\neg\forall x.P(x), \neg\exists x.\neg P(x) \vdash \bot & \neg_{\text{left}} \\
\hline
\neg\forall x.P(x) \vdash \neg\neg\exists x.\neg P(x) & \neg_{\text{right}} \\
\hline
\neg\forall x.P(x) \vdash \exists x.\neg P(x) & \text{em} \\
\hline
\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x) & \Rightarrow_{\text{right}}
\end{array}
$$

Correction

Preuve (6) dans LJ_{em}

Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\neg P(x) \vdash \neg P(x) \\ \neg P(x) \vdash \exists x.\neg P(x) \\ \neg\exists x.\neg P(x), \neg P(x) \vdash \bot\end{array}}{\neg\exists x.\neg P(x) \vdash \neg\neg P(x)}\neg_{right}}{\neg\exists x.\neg P(x) \vdash P(x)}em}{\neg\exists x.\neg P(x) \vdash \forall x.P(x)}\forall_{right}}{\neg\forall x.P(x), \neg\exists x.\neg P(x) \vdash \bot}\neg_{left}}{\neg\forall x.P(x) \vdash \neg\neg\exists x.\neg P(x)}\neg_{right}}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}em}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\Rightarrow_{right}}{}
$$

Correction

Preuve (6) dans LJ_{em}

‣ Règles LJ

$$\neg P(x) \vdash \neg P(x)$$

$$
\begin{prooftree}
\AxiomC{$\neg P(x) \vdash \exists x.\neg P(x)$}
\RightLabel{$\neg_{left}$}
\UnaryInfC{$\neg\exists x.\neg P(x), \neg P(x) \vdash \bot$}
\RightLabel{$\neg_{right}$}
\UnaryInfC{$\neg\exists x.\neg P(x) \vdash \neg\neg P(x)$}
\RightLabel{em}
\UnaryInfC{$\neg\exists x.\neg P(x) \vdash P(x)$}
\RightLabel{$\forall_{right}$}
\UnaryInfC{$\neg\exists x.\neg P(x) \vdash \forall x.P(x)$}
\RightLabel{$\neg_{left}$}
\UnaryInfC{$\neg\forall x.P(x), \neg\exists x.\neg P(x) \vdash \bot$}
\RightLabel{$\neg_{right}$}
\UnaryInfC{$\neg\forall x.P(x) \vdash \neg\neg\exists x.\neg P(x)$}
\RightLabel{em}
\UnaryInfC{$\neg\forall x.P(x) \vdash \exists x.\neg P(x)$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)$}
\end{prooftree}
$$

Correction

Preuve (6) dans LJ_{em}

▸ Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg P(x) \vdash \neg P(x)}{\neg P(x) \vdash \exists x.\neg P(x)}\ \exists_{\text{right}}}{\neg\exists x.\neg P(x), \neg P(x) \vdash \bot}\ \neg_{\text{left}}}{\neg\exists x.\neg P(x) \vdash \neg\neg P(x)}\ \neg_{\text{right}}}{\neg\exists x.\neg P(x) \vdash P(x)}\ \text{em}}{\neg\exists x.\neg P(x) \vdash \forall x.P(x)}\ \forall_{\text{right}}}{\neg\forall x.P(x), \neg\exists x.\neg P(x) \vdash \bot}\ \neg_{\text{left}}}{\neg\forall x.P(x) \vdash \neg\neg\exists x.\neg P(x)}\ \neg_{\text{right}}}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\ \text{em}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\ \Rightarrow_{\text{right}}
$$

Correction

Preuve (6) dans LJ_{em}

▸ Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg P(x) \vdash \neg P(x)}\ \text{ax}}{\neg P(x) \vdash \exists x.\neg P(x)}\ \exists_{\text{right}}}{\neg\exists x.\neg P(x), \neg P(x) \vdash \bot}\ \neg_{\text{left}}}{\neg\exists x.\neg P(x) \vdash \neg\neg P(x)}\ \neg_{\text{right}}}{\neg\exists x.\neg P(x) \vdash P(x)}\ \text{em}}{\neg\exists x.\neg P(x) \vdash \forall x.P(x)}\ \forall_{\text{right}}}{\neg\forall x.P(x), \neg\exists x.\neg P(x) \vdash \bot}\ \neg_{\text{left}}}{\neg\forall x.P(x) \vdash \neg\neg\exists x.\neg P(x)}\ \neg_{\text{right}}}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\ \text{em}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\ \Rightarrow_{\text{right}}
$$

Correction

Preuve (6) dans LK

Règles LK

$$P(x) \vdash P(x)$$

$$\vdash P(x), \neg P(x)$$

$$\vdash P(x), \exists x.\neg P(x)$$

$$\vdash \forall x.P(x), \exists x.\neg P(x)$$

$$\neg\forall x.P(x) \vdash \exists x.\neg P(x)$$

$$\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)$$

Correction

Preuve (6) dans LK

Règles LK

$$P(x) \vdash P(x)$$

$$\vdash P(x), \neg P(x)$$

$$\vdash P(x), \exists x.\neg P(x)$$

$$\vdash \forall x.P(x), \exists x.\neg P(x)$$

$$\dfrac{\neg \forall x.P(x) \vdash \exists x.\neg P(x)}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)} \Rightarrow_{\text{right}}$$

Correction

Preuve (6) dans LK

Règles LK

$$P(x) \vdash P(x)$$

$$\vdash P(x), \neg P(x)$$

$$\vdash P(x), \exists x.\neg P(x)$$

$$\dfrac{\dfrac{\vdash \forall x.P(x), \exists x.\neg P(x)}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\ \neg_{\text{left}}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\ \Rightarrow_{\text{right}}$$

Correction

Preuve (6) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{\vdash P(x), \neg P(x)}}{\vdash P(x), \exists x.\neg P(x)}}{\vdash \forall x.P(x), \exists x.\neg P(x)}\ \forall_{\text{right}}}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\ \neg_{\text{left}}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\ \Rightarrow_{\text{right}}$$

Correction

Preuve (6) dans LK

Règles LK

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{\vdash P(x), \neg P(x)}}{\vdash P(x), \exists x.\neg P(x)}\ \exists_{\text{right}}}{\vdash \forall x.P(x), \exists x.\neg P(x)}\ \forall_{\text{right}}}{\neg \forall x.P(x) \vdash \exists x.\neg P(x)}\ \neg_{\text{left}}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\ \Rightarrow_{\text{right}}
$$

Correction

Preuve (6) dans LK

▸ Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{\vdash P(x), \neg P(x)}\ \neg_{right}}{\vdash P(x), \exists x.\neg P(x)}\ \exists_{right}}{\vdash \forall x.P(x), \exists x.\neg P(x)}\ \forall_{right}}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\ \neg_{left}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\ \Rightarrow_{right}$$

Correction

Preuve (6) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x)}\,\text{ax}}{\vdash P(x), \neg P(x)}\,\neg_{\text{right}}}{\vdash P(x), \exists x.\neg P(x)}\,\exists_{\text{right}}}{\vdash \forall x.P(x), \exists x.\neg P(x)}\,\forall_{\text{right}}}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\,\neg_{\text{left}}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\,\Rightarrow_{\text{right}}$$

Outil d'aide à la preuve Coq

Caractéristiques

- Développement par l'équipe Inria πr^2 ;
- Preuve de programmes fonctionnels ;
- Théorie des types (calcul des constructions inductives) ;
- Isomorphisme de Curry-Howard (objets preuves).

Implantation

- Premières versions milieu des années 80 ;
- Implantation actuelle en OCaml ;
- Preuve interactive (peu d'automatisation) ;
- En ligne de commande ou avec l'interface graphique CoqIDE ;
- Installation : `https://coq.inria.fr/`.

Logique propositionnelle

Exemples de preuves

- Implication :

```
Coq < Parameter A : Prop.
A is assumed

Coq < Goal A -> A.
1 subgoal

  ============================
   A -> A
```

Logique propositionnelle

Exemples de preuves

- Implication :

```
Coq < intro.
1 subgoal

  H : A
  ============================
   A
```

Logique propositionnelle

Exemples de preuves

- Implication :

```
Coq < assumption.
No more subgoals.

Coq < Save my_thm.
intro.
assumption.
my_thm is defined
```

Logique propositionnelle

Exemples de preuves

- Application (modus ponens) :

```
Coq < Parameters A B : Prop.
A is assumed
B is assumed

Coq < Goal (A -> B) -> A -> B.
1 subgoal

  ============================
   (A -> B) -> A -> B
```

Logique propositionnelle

Exemples de preuves

- Application (modus ponens) :

```
Coq < intros.
1 subgoal

  H : A -> B
  H0 : A
  ============================
   B

Coq < apply (H H0).
No more subgoals.
```

Logique propositionnelle

Exemples de preuves

- Connecteurs $\wedge$ et $\vee$:

```
Coq < Parameters A B : Prop.
A is assumed
B is assumed

Coq < Goal A /\ B -> A.
1 subgoal

  ============================
   A /\ B -> A
```

Logique propositionnelle

Exemples de preuves

- Connecteurs $\wedge$ et $\vee$:

```
Coq < intro.
1 subgoal

  H : A /\ B
  ============================
   A
```

Logique propositionnelle

Exemples de preuves

- Connecteurs $\wedge$ et $\vee$:

```
Coq < elim H.
1 subgoal

  H : A /\ B
  ============================
   A -> B -> A
```

Logique propositionnelle

Exemples de preuves

- Connecteurs $\wedge$ et $\vee$:

```
Coq < intros.
1 subgoal

  H : A /\ B
  H0 : A
  H1 : B
  ============================
   A

Coq < assumption.
No more subgoals.
```

Logique propositionnelle

Exemples de preuves

- Connecteurs $\wedge$ et $\vee$:

```
Coq < Parameters A B : Prop.
A is assumed
B is assumed

Coq < Goal A -> A \/ B.
1 subgoal

  ============================
   A -> A \/ B
```

Logique propositionnelle

Exemples de preuves

- Connecteurs $\wedge$ et $\vee$:

```
Coq < intro.
1 subgoal

  H : A
  ============================
   A \/ B
```

Logique propositionnelle

Exemples de preuves

- Connecteurs $\wedge$ et $\vee$:

```
Coq < left.
1 subgoal

  H : A
  ============================
   A

Coq < assumption.
No more subgoals.
```

Logique propositionnelle

Exemples de preuves

- Connecteurs ¬ :

```
Coq < Parameters A B : Prop.
A is assumed
B is assumed

Coq < Goal A -> ~A -> False.
1 subgoal

  ============================
   A -> ~ A -> False
```

Logique propositionnelle

Exemples de preuves

- Connecteurs ¬ :

```
Coq < intros.
1 subgoal

  H : A
  H0 : ~ A
  ============================
   False

Coq < apply (H0 H).
No more subgoals.
```

Exercices

Propositions à démontrer en Coq

1. $A \Rightarrow B \Rightarrow A$
2. $(A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$
3. $A \wedge B \Rightarrow B$
4. $B \Rightarrow A \vee B$
5. $(A \vee B) \Rightarrow (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$
6. $A \Rightarrow \bot \Rightarrow \neg A$
7. $\bot \Rightarrow A$
8. $(A \Leftrightarrow B) \Rightarrow A \Rightarrow B$
9. $(A \Leftrightarrow B) \Rightarrow B \Rightarrow A$
10. $(A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)$

Logique du premier ordre

Exemples de preuves

- Quantificateur ∀ :

```
Coq < Parameter E : Set.
E is assumed

Coq < Parameter P : E -> Prop.
P is assumed

Coq < Goal forall x : E, (P x) -> (P x).
1 subgoal

  ============================
   forall x : E, P x -> P x
```

Logique du premier ordre

Exemples de preuves

- Quantificateur $\forall$:

```
Coq < intros.
1 subgoal

  x : E
  H : P x
  ============================
   P x

Coq < assumption.
No more subgoals.
```

Logique du premier ordre

Exemples de preuves

- Quantificateur $\forall$:

```
Coq < Parameter E : Set.
E is assumed

Coq < Parameter a : E.
a is assumed

Coq < Parameter P : E -> Prop.
P is assumed

Coq < Goal (forall x : E, (P x)) -> (P a).
1 subgoal

  ============================
   (forall x : E, P x) -> P a
```

Logique du premier ordre

Exemples de preuves

- Quantificateur ∀ :

```
Coq < intro.
1 subgoal

  H : forall x : E, P x
  ============================
   P a

Coq < apply H.
No more subgoals.
```

Logique du premier ordre

Exemples de preuves

- Quantificateur ∃ :

```
Coq < Parameter E : Set.
E is assumed

Coq < Parameter a : E.
a is assumed

Coq < Parameter P : E -> Prop.
P is assumed

Coq < Goal (P a) -> exists x : E, (P x).
1 subgoal

  ============================
   P a -> exists x : E, P x
```

Logique du premier ordre

Exemples de preuves

- Quantificateur $\exists$:

```
Coq < intro.
1 subgoal

  H : P a
  ============================
   exists x : E, P x
```

Logique du premier ordre

Exemples de preuves

- Quantificateur $\exists$:

```
Coq < exists a.
1 subgoal

  H : P a
  ============================
   P a

Coq < assumption.
No more subgoals.
```

Logique du premier ordre

Exemples de preuves

- Quantificateur $\exists$:

```
Coq < Parameter E : Set.
E is assumed

Coq < Parameter a : E.
a is assumed

Coq < Parameter P : E -> Prop.
P is assumed

Coq < Goal (exists x : E, ~(P x)) ->
           ~(forall x : E, (P x)).
1 subgoal

  ============================
   (exists x : E, ~ P x) -> ~ (forall x : E, P x)
```

Logique du premier ordre

Exemples de preuves

- Quantificateur $\exists$:

```
Coq < intros.
1 subgoal

  H : exists x : E, ~ P x
  ============================
   ~ (forall x : E, P x)

Coq < red.
1 subgoal

  H : exists x : E, ~ P x
  ============================
   (forall x : E, P x) -> False
```

Logique du premier ordre

Exemples de preuves

- Quantificateur $\exists$:

```
Coq < intro.
1 subgoal

  H : exists x : E, ~ P x
  H0 : forall x : E, P x
  ============================
   False
```

Logique du premier ordre

Exemples de preuves

- Quantificateur $\exists$:

```
Coq < elim H.
1 subgoal

  H : exists x : E, ~ P x
  H0 : forall x : E, P x
  ============================
   forall x : E, ~ P x -> False
```

Logique du premier ordre

Exemples de preuves

- Quantificateur $\exists$:

```
Coq < intros.
1 subgoal

  H : exists x : E, ~ P x
  H0 : forall x : E, P x
  x : E
  H1 : ~ P x
  ============================
   False
```

Logique du premier ordre

Exemples de preuves

- Quantificateur $\exists$:

```
Coq < apply H1.
1 subgoal

  H : exists x : E, ~ P x
  H0 : forall x : E, P x
  x : E
  H1 : ~ P x
  ============================
   P x

Coq < apply H0.
No more subgoals.
```

Exercices

Propositions à démontrer en Coq

1. $\forall x.P(x) \Rightarrow \exists y.P(y) \vee Q(y)$
2. $(\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))$
3. $(\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)$
4. $(\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))$
5. $(\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))$
6. $\neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)$

Guide de survie du petit Coq-uin

Correspondance LJ/Coq

Logique propositionnelle		Logique du premier ordre	
Règle LJ	Tactique Coq	Règle LJ	Tactique Coq
ax	`assumption`	$\forall_{\text{right}}$	`intro`
cut	`cut`	$\forall_{\text{left}}$	`apply`
$\Rightarrow_{\text{right}}$	`intro`	$\exists_{\text{right}}$	`exists`
$\Rightarrow_{\text{left}}$	`apply`	$\exists_{\text{left}}$	`elim`
$\Leftrightarrow_{\text{right}}$	`split`		
$\Leftrightarrow_{\text{lefti}}$	`elim`		
$\wedge_{\text{right}}$	`split`		
$\wedge_{\text{left}}$	`elim`		
$\vee_{\text{right1}}$	`left`		
$\vee_{\text{right2}}$	`right`		
$\vee_{\text{left}}$	`elim`		
$\neg_{\text{right}}$	`intro`		
$\neg_{\text{left}}$	`elimtype False` + `apply`		
$\top_{\text{right}}$, $\bot_{\text{left}}$	`auto`		