Assignment -4

Q1) $y = a_0 + a_1x$

$$a_1 = \frac{n\Sigma x_i y_i - \Sigma x_i \Sigma y_i}{n\Sigma x_i^2 - (\Sigma x_i)^2} \qquad a_0 = \bar{y} - a_1\bar{x}$$

N=9

$\Sigma x_i y_i = 683$

$\Sigma x_i = 76$

$\Sigma y_i = 70$

$\Sigma x^2 = 916$

Substituting,

a_1=0.3351 -> Slope

$\bar{y}_i$=7.7778

$\bar{x}_i$=8.4444

a_0= 4.948 -> Intercept

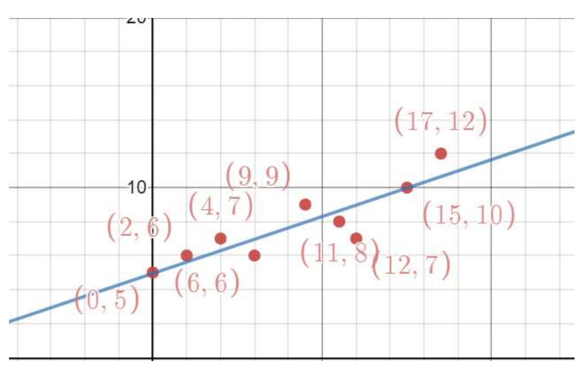
S_t= 39.5556

S_r= 8.7646

R^2=0.7784

SD= 2.2236

Standard Error= 1.11897



2. Solution

Use non-linear regression (Gauss-Newton method) to fit $y = a(1 - e^{-bx})$ to the following data set.

x	5	10	15	20	25	30	35	40	45	50
y	17	24	31	33	37	37	40	40	42	41

a) Compute a and b

$$y = f(x) = a(1-e^{-bx})$$

$$A_j = \begin{pmatrix} a_j \\ b_j \end{pmatrix}$$

Let $a_0 = 30$, $b_0 = 0.1$

Iteration1:

$$\frac{\partial f}{\partial a} = 1-e^{-bx} \qquad \frac{\partial f}{\partial b} = axe^{-bx}$$

$$Z_0 = \begin{pmatrix} \frac{\partial f_1}{\partial a} & \frac{\partial f_1}{\partial b} \\ \frac{\partial f_2}{\partial a} & \frac{\partial f_2}{\partial b} \\ \vdots & \vdots \\ \frac{\partial f_{10}}{\partial a} & \frac{\partial f_{10}}{\partial b} \end{pmatrix}$$

$$= \begin{pmatrix} 0.3935 & 90.9796 \\ 0.6321 & 110.3638 \\ 0.7769 & 100.4086 \\ 0.8647 & 81.2012 \\ 0.9179 & 61.5637 \\ 0.9502 & 44.8084 \\ 0.9698 & 31.7073 \\ 0.9817 & 21.9788 \\ 0.9889 & 14.9971 \\ 0.9933 & 10.1069 \end{pmatrix}$$

Then $Z_0^T Z_0 = \begin{pmatrix} 8 & 430 \\ 430 & 44746 \end{pmatrix}$

$$D = \begin{pmatrix} 5.1959 \\ 5.0364 \\ 7.6929 \\ 7.0601 \\ 9.4625 \\ 8.4936 \\ 10.9059 \\ 10.5495 \\ 12.3333 \\ 11.2021 \end{pmatrix}$$

$$Z_0^T D = \begin{pmatrix} 78.3 \\ 4213.4 \end{pmatrix}$$

As $Z_0^T Z_0 \Delta A = Z_0^T D$

$$\begin{pmatrix} 8 & 430 \\ 430 & 44746 \end{pmatrix} \Delta A = \begin{pmatrix} 78.3 \\ 4213.4 \end{pmatrix} \text{ where, } \Delta A = \begin{pmatrix} \Delta a \\ \Delta b \end{pmatrix}$$

Solving the two linear equations we get $\Delta a = 9.7756$ $\Delta b = 0.00021$

Now $A_1 = A_0 + \Delta A$

$$A_1 = \begin{pmatrix} 39.7756 \\ 0.10021 \end{pmatrix}$$

Approximate percentage relative error:

$$\frac{|Currrent\ iteration\ value - Previous\ iteration\ value|}{Current\ iteration\ value} \text{ x } 100$$

For a = 24.57%

For b = 0.21%

<u>Iteration 2:</u>

$a_1 = 39.7756$ $b_1 = 0.10021$

$$Z_1^T Z_1 = \begin{pmatrix} 8 & 568 \\ 568 & 78166 \end{pmatrix}$$

$$Z_1^T D = \begin{pmatrix} 4.6926 \\ -4.6944 \end{pmatrix}$$

As $Z_1^T Z_1 \Delta A = Z_1^T D$

$$\begin{pmatrix} 8 & 568 \\ 568 & 78166 \end{pmatrix} \Delta A = \begin{pmatrix} 4.6926 \\ -4.6944 \end{pmatrix} \text{ where, } \Delta A = \begin{pmatrix} \Delta a \\ \Delta b \end{pmatrix}$$

Solving the two linear equations we get $\Delta a = 1.221$ $\Delta b = -0.0089$

Now $A_2 = A_1 + \Delta A$

$A_2 = \begin{pmatrix} 40.9966 \\ 0.09139 \end{pmatrix}$

Approximate percentage relative error:

For a = 4.87%
For b = 9.65%

Iteration 3:

a_2 = 40.9966 b_2 = 0.09139

$$Z_3^T Z_3 = \begin{pmatrix} 7.2521 & 689.252 \\ 689.252 & 109381 \end{pmatrix}$$

$$Z_3^T D = \begin{pmatrix} 1.1866 \\ 55.0544 \end{pmatrix}$$

As $Z_3^T Z_3 \Delta A = Z_3^T D$

$$\begin{pmatrix} 7.2521 & 689.252 \\ 689.252 & 109381 \end{pmatrix} \Delta A = \begin{pmatrix} 1.1866 \\ 55.0544 \end{pmatrix} \text{ where, } \Delta A = \begin{pmatrix} \Delta a \\ \Delta b \end{pmatrix}$$

Solving the two linear equations we get Δa = 0.2886 Δb = -0.0013

Now $A_3 = A_2 + \Delta A$

$$A_3 = \begin{pmatrix} 41.2852 \\ 0.0901 \end{pmatrix}$$

Approximate percentage relative error:

For a = 0.7%
For b = 1.43%

Iteration 4:

a_3 = 41.2852 b_3= 0.0901

$$Z_4^T Z_4 = \begin{pmatrix} 7.2083 & 711.3827 \\ 711.3827 & 115725 \end{pmatrix}$$

$$Z_4^T D = \begin{pmatrix} 0.0025 \\ -4.3653 \end{pmatrix}$$

As $Z_2^T Z_2 \Delta A = Z_2^T D$

$$\begin{pmatrix} 7.2083 & 711.3827 \\ 711.3827 & 115725 \end{pmatrix} \Delta A = \begin{pmatrix} 0.0025 \\ -4.3653 \end{pmatrix} \text{ where, } \Delta A = \begin{pmatrix} \Delta a \\ \Delta b \end{pmatrix}$$

Solving the two linear equations we get Δa = 0.010 Δb = -0.0001

Now $A_4 = A_3 + \Delta A$

$$A_3 = \begin{pmatrix} 41.2952 \\ 0.09 \end{pmatrix}$$

Approximate percentage relative error:

For a < 1%
For b < 1%

Therefore, the regression equation was **y = 41.2952 (1-e$^{-0.09x}$)**

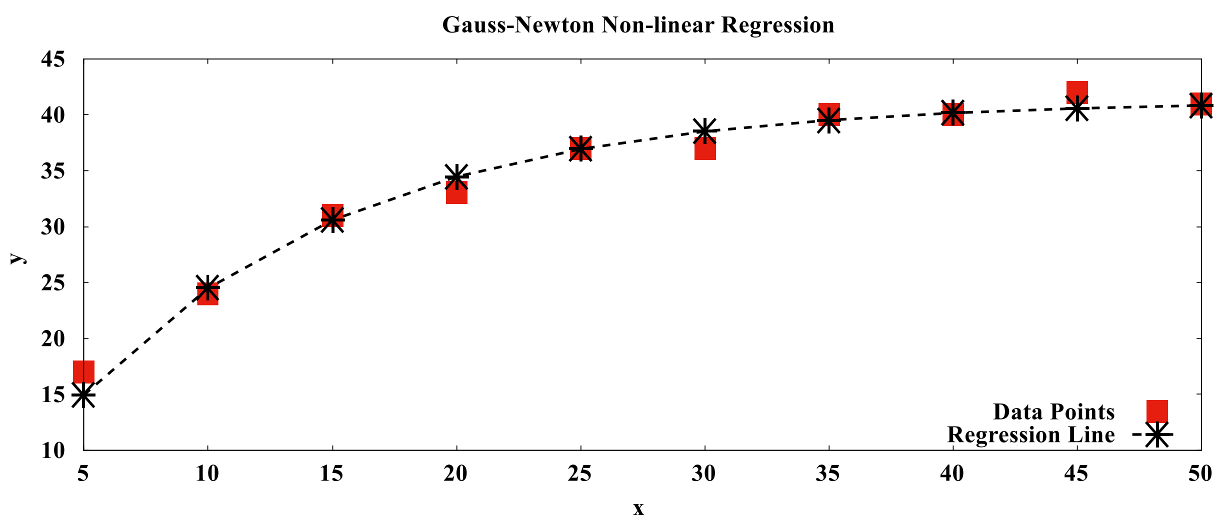
b) $S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a(1 - e^{-bxi}))^2 = 11.3458$

$S_t = \sum_{i=1}^{n} (y_i - \bar{y})^2 = 601.6$

Coefficient of determination = $r^2 = \frac{S_t - S_r}{S_t} = 0.9811$

c)

X	Y	Predicted from regression
5	17	14.9642
10	24	24.5058
15	31	30.5898
20	33	34.4691
25	37	36.9427
30	37	38.5199
35	40	39.5256
40	40	40.1669
45	42	40.5757
50	41	40.8365



Problem 3

Given:

x	1.6	2	2.5	3.2	4	4.5
f(x)	2	8	14	15	8	2

(a) Newton's Interpolating Polynomials

To find the value of $f(2.8)$ using Newton's interpolating polynomials of orders 1 through 3.

Order 1:
For order 1, we use: $x_0 = 2.5$, $x_1 = 3.2$

The divided difference is given by:
The first order polynomial is given by:

$$f_1(x) = f(x_0) + f[x_1, x_0](x - x_0)$$

$$= 14 + 1.4286(x - 2.5)$$

$$f_1(2.8) = 14 + 1.4286(2.8 - 2.5) = 14.4286$$

Order 2:
For order 2, we use: $x_0 = 2$, $x_1 = 2.5$, $x_2 = 3.2$

The divided differences are given by:

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = 12.00$$

$$f[x_2, x_1] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 1.4286$$

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = -8.8095$$

The second order polynomial is given by:

$$f_2(x) = f(x_0) + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

$$= 8 + 12.0000(x - 2) - 8.8095(x - 2)(x - 2.5)$$

$$f_2(2.8) = 8 + 12.0000(2.8 - 2) - 8.8095(2.8 - 2)(2.8 - 2.5) = 15.4857$$

Order 3:
For order 3, we use: $x_0 = 2$, $x_1 = 2.5$, $x_2 = 3.2$, $x_3 = 4$

The divided differences are given by:

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = 12.00$$

$$f[x_2, x_1] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 1.4286$$

$$f[x_3, x_2] = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = -8.75$$

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = -8.8095$$

$$f[x_3, x_2, x_1] = \frac{f[x_3, x_2] - f[x_2, x_1]}{x_3 - x_1} = -6.7857$$

$$f[x_3, x_2, x_1, x_0] = \frac{f[x_3, x_2, x_1] - f[x_2, x_1, x_0]}{x_3 - x_0} = 1.0119$$

The third order polynomial is given by:

$$f_3(x) = f(x_0) + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$

$$= 8 + 12.0000(x - 2) - 8.8095(x - 2)(x - 2.5) + 1.0119(x - 2)(x - 2.5)(x - 3.2)$$

$$f_3(2.8) = 8 + 12.0000(2.8 - 2) - 8.8095(2.8 - 2)(2.8 - 2.5) + 1.0119(2.8 - 2)(2.8 - 2.5)(2.8 - 3.2) = 15.3886$$

(b) Quadratic Splines

To develop quadratic splines for first five data points and predict $f(3.4)$ and $f(2.2)$ Let the

splines be given by:

$$\begin{aligned}
f_1(x) &= a_1x^2 + b_1x + c_1 && 1.6 \le x \le 2 \\
f_2(x) &= a_2x^2 + b_2x + c_2 && 2 \le x \le 2.5 \\
f_3(x) &= a_3x^2 + b_3x + c_3 && 2.5 \le x \le 3.2 \\
f_4(x) &= a_4x^2 + b_4x + c_4 && 3.2 \le x \le 4
\end{aligned}$$

Since there are n + 1 = 5 data points, there are n = 4 intervals, which implies that there are 3n = 12 unknown constants, which requires 12 conditions.

Function values must be equal at interior knots:

$$\begin{aligned}
a_1(2)^2 + b_1(2) + c_1 &= f(2) = 8 \\
a_2(2)^2 + b_2(2) + c_2 &= f(2) = 8 \\
a_2(2.5)^2 + b_2(2.5) + c_2 &= f(2.5) = 14 \\
a_3(2.5)^2 + b_3(2.5) + c_3 &= f(2.5) = 14 \\
a_3(3.2)^2 + b_3(3.2) + c_3 &= f(3.2) = 15 \\
a_4(3.2)^2 + b_4(3.2) + c_4 &= f(3.2) = 15
\end{aligned}$$

First and last function must pass through endpoints:

$$\begin{aligned}
a_1(1.6)^2 + b_1(1.6) + c_1 &= f(1.6) = 2 \\
a_4(4)^2 + b_4(4) + c_4 &= f(4) = 8
\end{aligned}$$

First derivatives at interior knots must be equal:

$$2a_1(2) + b_1 = 2a_2(2) + b_2$$
$$2a_2(2.5) + b_2 = 2a_3(2.5) + b_3$$
$$2a_3(3.2) + b_3 = 2a_4(3.2) + b_4$$

We are one condition short of obtaining a unique solution. For the last condition, we assume that the second derivative at the first point vanishes, i.e.

$$a_1 = 0$$

The above system of linear equations can be written in matrix form as follows:

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6.25 & 2.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6.25 & 2.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10.24 & 3.2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10.24 & 3.2 & 1 \\ 1.6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 4 & 1 \\ 1 & 0 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & -5 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6.4 & 1 & 0 & -6.4 & -1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \\ 14 \\ 14 \\ 15 \\ 15 \\ 2 \\ 8 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

This system can be solved to obtain the parameters:

$$\boxed{\begin{array}{c} b_1 = 15,\ c_1 = -22, \\ a_2 = -6,\ b_2 = 39,\ c_2 = -46 \\ a_3 = -10.8163,\ b_3 = 63.0816,\ c_3 = -76.1020, \\ a_4 = -3.2589,\ b_4 = 14.7143,\ c_4 = 1.2857 \end{array}}$$

Thus, we get the quadratic splines to be:

$$\begin{array}{ll} f_1(x) = 15x - 22 & 0.6 \le x \le 2 \\ f_2(x) = -6x^2 + 39x - 46 & 2 \le x \le 2.5 \\ f_3(x) = -10.8163x^2 + 63.0816x - 76.1020 & 2.5 \le x \le 3.2 \\ f_4(x) = -3.2589x^2 + 14.7143x + 1.2857 & 0.2 \le x \le 4 \end{array}$$

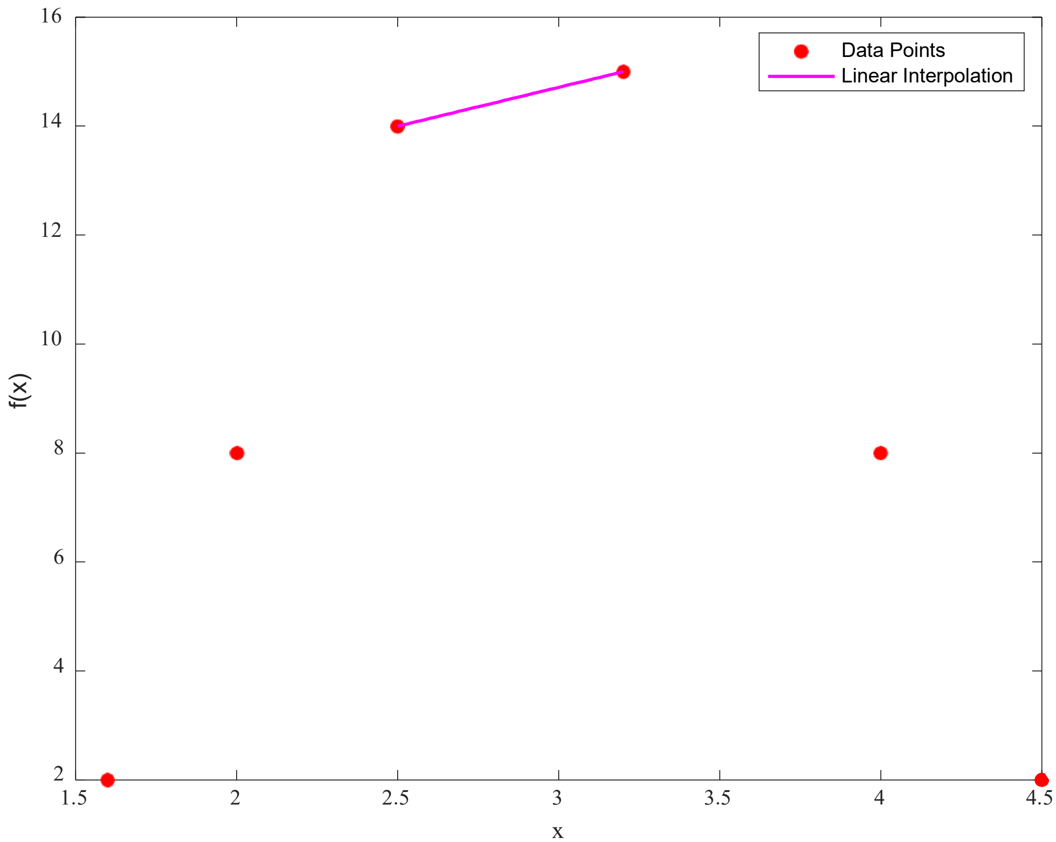
Now, since $3.2 \le 3.4 \le 4$,

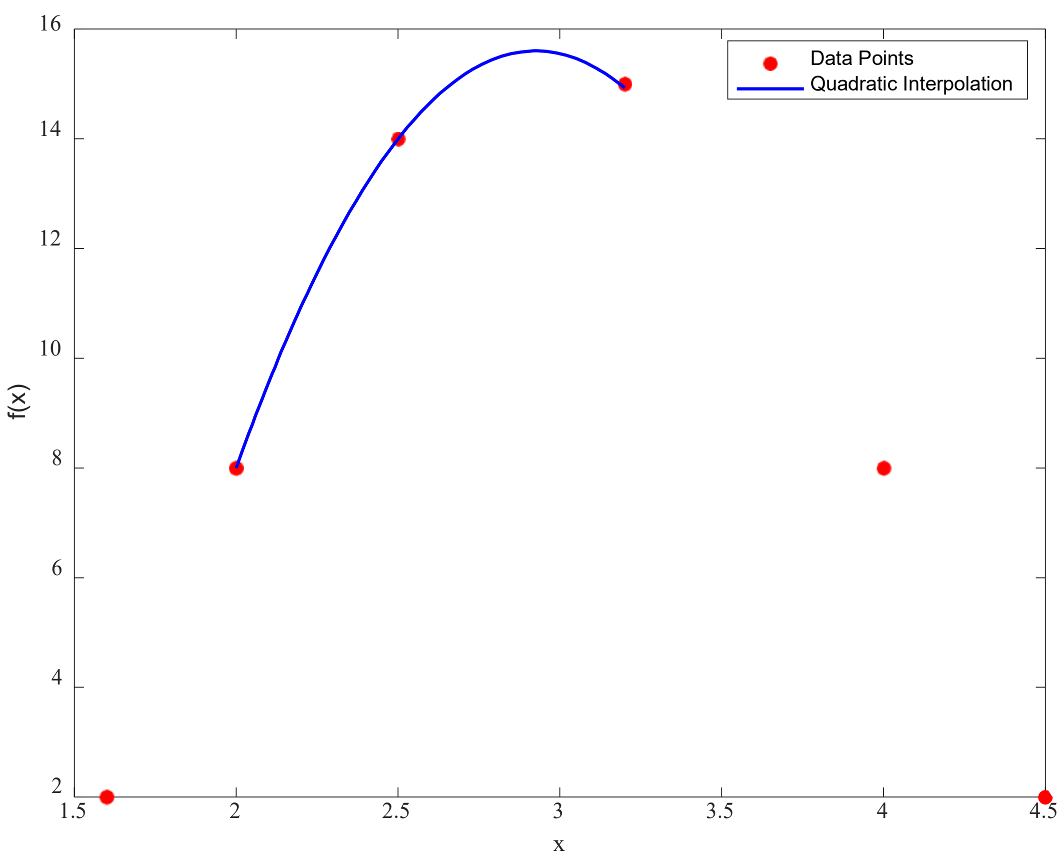
$$f(3.4) = f_4(3.4) = -3.2589(3.4)^2 + 14.7143(3.4) + 1.2857 = 13.6414$$

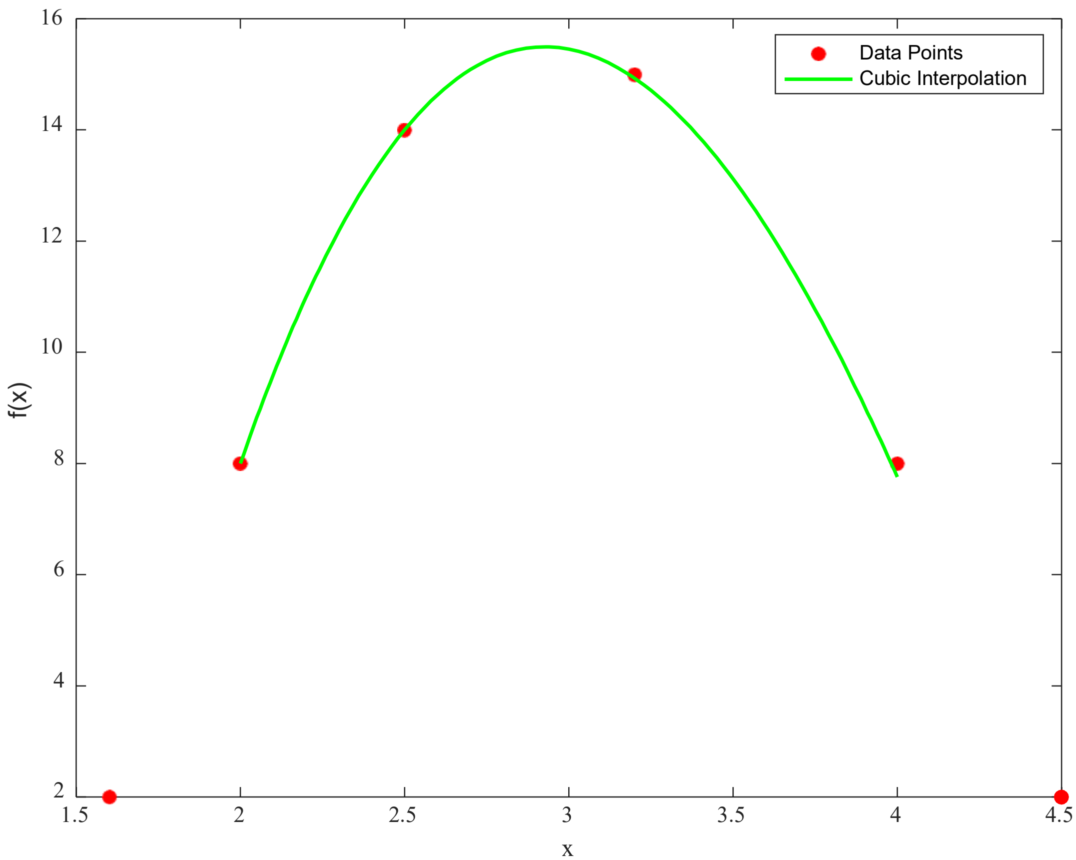
And since $2 \le 2.2 \le 2.5$

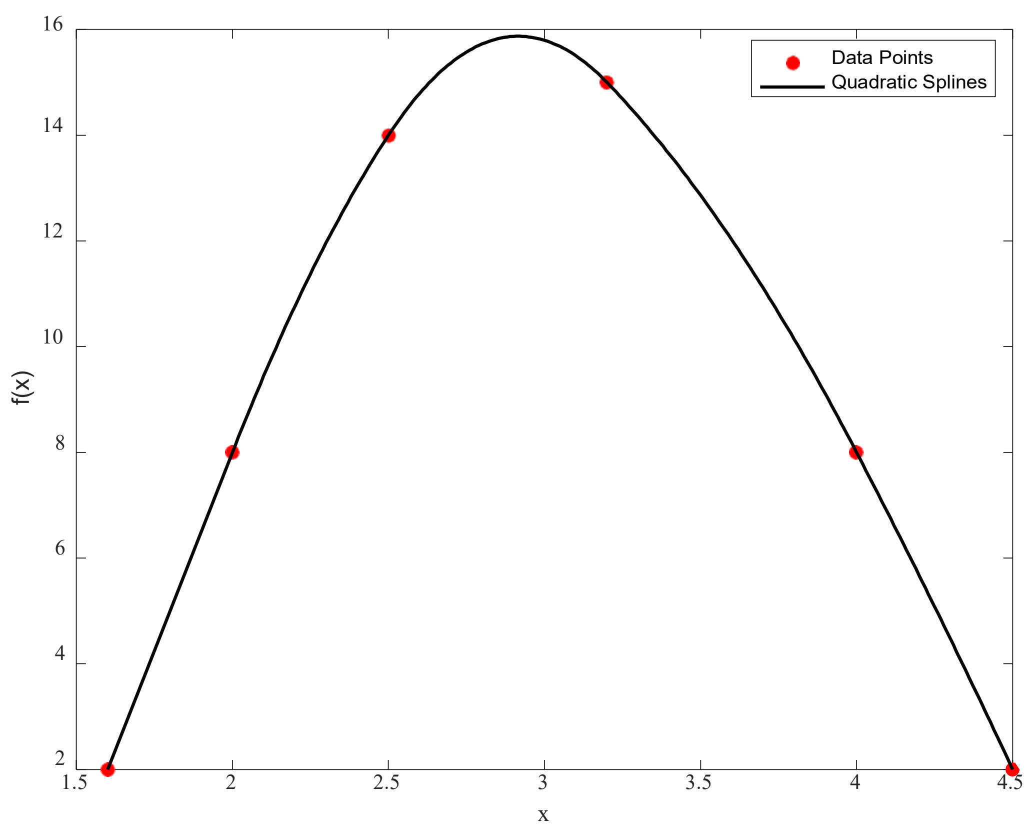
$$f(2.2) = f_2(2.2) = -6(2.2)^2 + 39(2.2) - 46 = 10.760$$

(c) Plots


Data Points
Linear Interpolation
16
14
12
10
8
6
4
2
f(x)
1.5
2
2.5
3
3.5
4
4.5
x


Data Points
Quadratic Interpolation
16
14
12
10
8
6
4
2
f(x)
1.5
2
2.5
3
3.5
4
4.5
x


16
14
12
10
8
6
4
2
1.5
2
2.5
3
3.5
4
4.5
f(x)
x
Data Points
Cubic Interpolation


16
14
12
10
8
6
4
2
1.5
2
2.5
3
3.5
4
4.5
f(x)
x
Data Points
Quadratic Splines