Assignment-3

1. Solution

a) Given linear equations

$$2x + y+z = 4$$
$$x+3y+z = 0$$
$$x-y+3z = 6$$

The given matrix can be written as A*p=B where,

$$A= \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{pmatrix} \qquad p = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad B= \begin{pmatrix} 4 \\ 0 \\ 6 \end{pmatrix}$$

The augmented matrix

$$A,B = \begin{pmatrix} 2 & 1 & 1 & 4 \\ 1 & 3 & 1 & 0 \\ 1 & -1 & 3 & 6 \end{pmatrix}$$

b) Gauss Elimination method

<u>Forward Elimination</u>

f21= a21/a11 = (1/2) = 0.5, f31 = (a31/a11) = (1/2) = 0.5

After performing forward elimination on the matrix (A,B)

$$(A,B)' = \begin{pmatrix} 2 & 1 & 1 & 4 \\ 0 & 2.5 & 0.5 & -2 \\ 0 & -1.5 & 2.5 & 4 \end{pmatrix}$$

Now, from the new matrix f32= (a32/a22) = (-1.5/2.5) = -0.6

After performing forward elimination on the new matrix

$$= \begin{pmatrix} 2 & 1 & 1 & 4 \\ 0 & 2.5 & 0.5 & -2 \\ 0 & 0 & 2.8 & 2.8 \end{pmatrix}$$

Now we have $$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2.5 & 0.5 \\ 0 & 0 & 2.8 \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 2.8 \end{pmatrix}$$

Then by doing <u>backward substitution</u> we get the unknown values as

$$2.8*z = 2.8 \quad => z = 1$$
$$(2.5*y) + (0.5*1) = -2 \quad => y = -1$$
$$(2*x) + (1*-1) + (1*1) = 4 \quad => x = 2$$

Now we have $$p = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

c) From the previous step we got the upper triangular matrix as

$$U = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2.5 & 0.5 \\ 0 & 0 & 2.8 \end{pmatrix}$$

The Lower triangular matrix is given by

$$L = \begin{pmatrix} 1 & 0 & 0 \\ f21 & 1 & 0 \\ f31 & f32 & 1 \end{pmatrix}$$

After substituting values from the previous section (b)

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & -0.6 & 1 \end{pmatrix}$$

d) Finding unknowns by using LU-decomposition method

$$Lm = b$$
$$Up = m$$

Where m= $\begin{pmatrix} m1 \\ m2 \\ m3 \end{pmatrix}$

$$Lm = b$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & -0.6 & 1 \end{pmatrix} \begin{pmatrix} m1 \\ m2 \\ m3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 6 \end{pmatrix}$$

On performing <u>forward substitution</u>

(1*m1) = 4 => m1 = 4

(0.5*4) + (1*m2) = 0 => m2 = -2

(0.5*4) + (-0.6*-2) + (1*m3) = 6 => m3 = 2.8

We got m= $\begin{pmatrix} 4 \\ -2 \\ 2.8 \end{pmatrix}$

$$Up = m$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2.5 & 0.5 \\ 0 & 0 & 2.8 \end{pmatrix} * \begin{pmatrix} p1 \\ p2 \\ p3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 2.8 \end{pmatrix}$$

On performing backward substitution

$$(2.8*p3) = 2.8 \quad => p3 = 1$$
$$(2.5*p2) + (0.5*1) = -2 \quad => p2 = -1$$
$$(2*p1) + (1*-1) + (1*1) = 4 \quad => \quad p1 = 2$$

Now we got p = $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ same as we got in section b)

e) The determinant of given matrix

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & -1 & 3 \end{vmatrix} = 2(10) - 1(2) + 1(-4) = 14$$

The given matrix can be written in the form

$$|A| = |L||U|$$

Where |L| = Product of diagonal elements of lower triangular matrix = 1

|U| = Product of diagonal elements of upper triangular matrix = 14

Therefore |A|= 1*14 = 14

We can observe that the determinant of the matrix obtained from its decomposition is same as determinant from the given matrix as whole.

f) The inverse of a matrix can be calculated using LU decomposition, by applying the method on the respective columns of the inverse matrix and the identity matrix.

$$AA^{-1} = I$$

where

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & -1 & 3 \end{pmatrix} \quad A^{-1} = \begin{pmatrix} C11 & C12 & C13 \\ C21 & C22 & C23 \\ C31 & C32 & C33 \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Finding **column 1** elements of A^{-1}

$$Lm = b$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & -0.6 & 1 \end{pmatrix} \begin{pmatrix} m1 \\ m2 \\ m3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

On performing forward substitution

$$(1*m1) = 1 \quad => m1 = 1$$
$$(0.5*1) + (1*m2) = 0 \quad => m2 = -0.5$$
$$(0.5*1) + (-0.6*-0.5) + (1*m3) = 0 \quad => m3 = -0.8$$

We got m= $\begin{pmatrix} 1 \\ -0.5 \\ -0.8 \end{pmatrix}$

$$Up = m$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2.5 & 0.5 \\ 0 & 0 & 2.8 \end{pmatrix} * \begin{pmatrix} C11 \\ C21 \\ C31 \end{pmatrix} = \begin{pmatrix} 1 \\ -0.5 \\ -0.8 \end{pmatrix}$$

On performing <u>backward substitution</u>

(2.8*C31) = -0.8 => C31 = -2/7

(2.5*C21) + (0.5*-2/7) = -0.5 => C21 = -1/7

(2*C11) + (1*-1/7) + (1*-2/7) = 1 => C11 = 5/7

Now we got column 1 as = $\begin{pmatrix} 5/7 \\ -1/7 \\ -2/7 \end{pmatrix}$

Finding **column 2** elements of A^{-1}

$$Lm = b$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & -0.6 & 1 \end{pmatrix} \begin{pmatrix} m1 \\ m2 \\ m3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

On performing <u>forward substitution</u>

(1*m1) = 0 => m1 = 0

(0.5*0) + (1*m2) = 1 => m2 = 1

(0.5*0) + (-0.6*1) + (1*m3) = 0 => m3 = 0.6

We got m= $\begin{pmatrix} 0 \\ 1 \\ 0.6 \end{pmatrix}$

$$Up = m$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2.5 & 0.5 \\ 0 & 0 & 2.8 \end{pmatrix} * \begin{pmatrix} C21 \\ C22 \\ C32 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0.6 \end{pmatrix}$$

On performing <u>backward substitution</u>

(2.8*C32) = 0.6 => C32 = 3/14

(2.5*C22) + (0.5*3/14) = 1 => C22 = 5/14

(2*p1) + (1*5/14) + (1*3/14) = 0 => C12 = -2/7

Now we got column 2 as $\begin{pmatrix} -2/7 \\ 5/14 \\ 3/14 \end{pmatrix}$

Finding **column 3** elements of A^{-1}

$$Lm = b$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & -0.6 & 1 \end{pmatrix}\begin{pmatrix} m1 \\ m2 \\ m3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

On performing <u>forward substitution</u>

(1*m1) = 0 => m1 = 0

(0.5*0) + (1*m2) = 0 => m2 = 0

(0.5*0) + (-0.6*0) + (1*m3) = 1 => m3 = 1

We got m= $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$Up = m$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2.5 & 0.5 \\ 0 & 0 & 2.8 \end{pmatrix} * \begin{pmatrix} C31 \\ C32 \\ C33 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

On performing <u>backward substitution</u>

(2.8*C33) = 1 => C33 = 5/14

(2.5*C32) + (0.5*5/14) = 0 => C32 = -1/14

(2*C31) + (1*-1/14) + (1*5/14) = 0 =>C31 = -1/7

Now we got column 3 as $\begin{pmatrix} -1/7 \\ -1/14 \\ 5/14 \end{pmatrix}$

After substituting all the column elements the inverse of the matrix, A^{-1} is

$$\begin{pmatrix} 5/7 & -2/7 & -1/7 \\ -1/7 & 5/14 & -1/14 \\ -2/7 & 3/14 & 5/14 \end{pmatrix}$$

Assignment 3

Q2) 0.8x-0.4y=41

-0.4x+0.8y-0.4z=25

-0.4y+0.8z=105

$$\begin{pmatrix} 0.8 & -0.4 & 0 \\ -0.4 & 0.8 & -0.4 \\ 0 & -0.4 & 0.8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 41 \\ 25 \\ 105 \end{pmatrix}$$

Decomposition

$e_2'=-0.5$

$e_3'=-0.67$

$f_2'=0.6$

$f_3'=0.53$

Forward Substitution

$b_1=41$

$b_2'=45.5$

$b_3'=135.48$

Backward Substitution

z=255.62

y=246.24

x=174.37

Problem 3.

Given: $x + 2y = 1$ (1)

$x - y = 4$ (2)

The true values of (x,y) of the given equations = **(3,-1).**

(a) Gauss Seidel Method-

$$\boldsymbol{x_{i+1} = 1 - 2y_i}$$

$$\boldsymbol{y_{i+1} = x_{i+1} - 4}$$

Initial guess $(x_0,y_0) = (0,0)$

i	x_i	y_i	Error in x (%)	Error in y (%)
1	1	-3	100.00	100.00
2	7	3	85.714	200.00
3	-5	-9	240.000	133.33
4	**19**	**15**	126.316	160.00

Observation: The 4th iteration gives (19,15) which is not the true solution. The error however diverges drastically.

(b) Gauss Seidel with under-relaxation method-

Given: $\lambda = 0.6$

$x_1 = 1 - 2y_0 = 1 - 2(0) = 1$

$x_1 = \lambda x_1 + (1 - \lambda)x_0 = (0.6)(1) + (1 - 0.6)(0) = 0.6$

$y_1 = x_1 - 4 = 0.6 - 4 = - 3.4$

$y_1 = \lambda y_1 + (1 - \lambda)y_0 = (0.6)(-3.4) + (1 - 0.6)(0) = - 2.04$

i	x_i	y_i	**Error in x (%)**	**Error in y (%)**
1	0.60	-2.04	100.00	**100.00**
2	3.2880	-1.2432	81.75	**64.09**
3	3.4070	-0.8531	3.494	**45.73**
4	**2.9865**	**-0.9493**	**14.08**	**10.14**

Observation: The 4th iteration gives (2.9865,-0.9493) which is closer to the true solution. The error values are converging slow.

(c) Diagonally Dominant System-

Perform elementary row operations as follows on (1) and (2):
$R_1 \leftarrow R_1 + R_2$

$$2x + y = 5 \qquad (3)$$

$R_2 \leftarrow R_1 - R_2$

$$x - y = 4 \qquad (4)$$

$$2x + y = 5$$

$$x + 2y = 1$$

$$\mathbf{a_{11}, a_{22} > a_{12}, a_{21}}$$

Taking initial guess = (0,0)

i	x_i	y_i	**Error in x** (%)	**Error in** y (%)
1	2.500	-0.750	100.00	100.00
2	2.8750	-0.9375	13.043	20.00
3	2.9688	-0.9844	3.158	4.762
4	2.9922	-0.9961	**0.783**	**1.176**

Observation: The 4th iteration converges rapidly towards the true solution with converging error.