Assignment problems solution

Assignment 2

Problem 1. Bisection method (*10 marks*)
Water is flowing in a trapezoidal channel at a rate of Q = 40 m³/s. The critical depth y for such a channel must satisfy the equation

$$0 = 1 - (Q^2B / gA_c^3)$$

where g = 9.81 m/s², A_c = the cross-sectional area (m²), and B = the width of the channel at the surface (m). For this case, the width and the cross-sectional area can be related to y by

$$B = 3 + y \quad \text{and} \quad A_c = 3y + y^2/2$$

Solve for the critical depth using bisection method. Use initial guesses of x_l = 1.0 and x_u = 3.0 and iterate until the approximate percent relative error falls below 1% or the number of iterations exceed 10.

Solution:

i) Function : $f(x,y) = \frac{1600*(3+y)}{9.81*(3y+y^2/2)^3} - 1 = 0$

ii) xr = (xl+xu)/2

for next iteration the values are

*if (f(xl)*f(xr) < 0) :*

xu = xr

xl = xl (same value)

else:

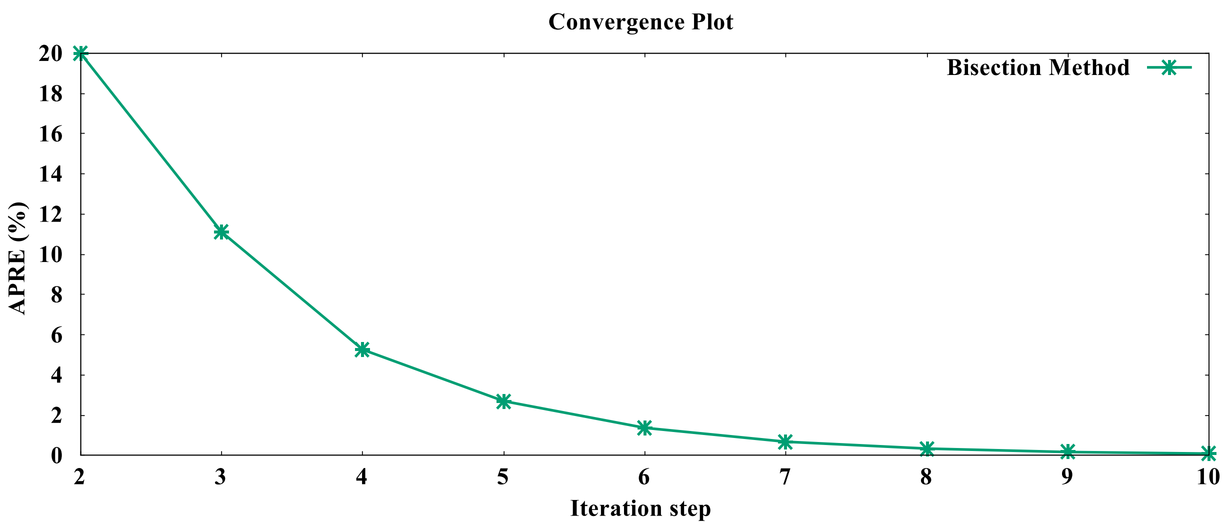
xl = xr

xu = xu (same value)

iii) Approximate percentage relative error (**w.r.t xr**):

$$\frac{|Currrent\ iteration\ root - Previous\ iteration\ root|}{Current\ iteration\ root} \times 100$$

Iterations	xl	xu	xr	f(xl)*f(xr)	APRE(%)
1	1.0000	3.0000	2.0000	8.42684	**-**
2	2.0000	3.0000	2.5000	-0.14945	**20.00**
3	2.0000	2.5000	2.2500	0.04209	**11.11**
4	2.2500	2.5000	2.3750	-0.00773	**05.26**
5	2.2500	2.3750	2.3125	-0.00171	**02.70**
6	2.2500	2.3125	2.2813	0.00157	**01.37**
7	2.2813	2.3125	2.2969	-0.00003	**00.68**
8	2.2813	2.2969	2.2891	0.00023	**00.34**
9	2.2891	2.2969	2.2930	0.00005	**00.17**
10	2.2930	2.2969	2.2949	0.00001	**00.09**



Convergence Plot
Bisection Method
APRE (%)
Iteration step
0
2
4
6
8
10
12
14
16
18
20
2
3
4
5
6
7
8
9
10

3) A) Let $x = \frac{1}{2.834}$

$f(x) = 2.843 - \frac{1}{x}$

$f^1(x) = \frac{1}{x^2}$

Take initial guess as $x_i = 0.4$

Solving by Newton- Raphson Method

X (old)	X(new)
0.4	0.26656
0.26656	0.331752
0.331752	0.351595
0.351595	0.35285
0.35285	0.35285

Taking a general form

$f(x) = N - \frac{1}{x}$

$f^1(x) = \frac{1}{x^2}$

$x_{i+1} = x_i - \frac{\left(N - \frac{1}{x_i}\right)}{\frac{1}{x_i{}^2}} = x_i(2 - Nx_i)$

B) $x = \sqrt{12.425}$

$x^2 = 12.425$

$f(x) = x^2 - 12.425$

$f^1(x) = 2x$

Take initial guess as $x_i = 3.5$

X(old)	X(new)
3.5	3.525
3.525	3.5249
3.5249	3.5249

Taking a general form

$x = \sqrt[r]{N}$

$x^r = N$

$f(x) = x^r - N$

$x_{i+1} = x_i - \frac{(x_i{}^r - N)}{rx^{r-1}}$

4) To solve this by secant method, we have to write the equation of the circle in the form of $y = f(x)$. By doing so, we can easily reach to the positive root $x = 2.464$.

   The tricky part in choosing the initial conditions for Secant method is the difference between them. If they are far apart, then this method will fail. That too, in case of a circle, if one of the initial guesses is taken beyond the circle, then the new value of x will be a complex number. So, it is crucial to choose the right values initially.

   For the Modified Secant method, the initial value should be close to the actual root and the value δ should be as small as possible. The key thing is to understand the relationship between these methods.

5. Given: $g(x) = \beta G(x) + x$.

$G(x) = 2 - x + \ln x$.

$\Rightarrow g(x) = 2\beta + x(1-\beta) + \beta[\ln(x)]$

<u>Convergence limit</u>: $|g'(x)| < 1$.

$\Rightarrow g'(x) = 1 - \beta + \frac{\beta}{x}$.

$\therefore \boxed{\left|1 - \beta + \frac{\beta}{x}\right| < 1}$

upper limit: $1 - \beta + \frac{\beta}{x} < 1$.

$\Rightarrow x > 1$. — (1).

lower limit: $-1 < 1 - \beta + \frac{\beta}{x}$.

$\Rightarrow -2 < \beta\left[\frac{1}{x} - 1\right]$ — (2)

from (1) & (2).

we get. $\boxed{\beta = 2}$ ——— (5) mark.

Also given: $x_{i+1} = g(x_i)$.

Take initial guess. $x_0 = 1$.

$\therefore x_1 = 4 - x_0 + 2\ln(x_0)$

$\boxed{x_1 = 3}$.

Error $= \frac{3-1}{3} = \boxed{0.667}$.

Tabulated Answers. $\longrightarrow$.

$x_0 = 1$ | $\cancel{\varepsilon_0}$

$x_1 = 3$ | $\varepsilon_1 = 0.667$.

$x_2 = 3.197$ | $\varepsilon_2 = 0.0617$

$x_3 = 3.127$ | $\varepsilon_3 = -0.022$.

$\boxed{x_4 = 3.153.}$ | $\boxed{\varepsilon_4 = 0.0084} < 1\%$ —— 5 mark.

Total = 10 mark.

Problem 6. Multiple roots (*10 marks*)

The function $x^3 - 2x^2 - 4x + 8$ has a double root at $x = 2$. Use **(a)** the standard Newton-Raphson, **(b)** the modified Newton-Raphson (method 1 from class notes) and **(c)** the modified Newton- Raphson (method 2 from class notes) to solve for the root at $x = 2$. Compare and discuss the rate of convergence using an initial guess of x_0 =1.2.

Solution

i) Approximate percentage relative error :

$$\frac{|Currrent\ iteration\ root - Previous\ iteration\ root|}{Current\ iteration\ root} \text{ x } 100$$

ii) True percentage relative error :

$$\frac{|True\ root - Iteration\ root|}{True\ root} \text{ x } 100$$

For **regular Newton Raphson Method**,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= xi - \frac{f(x^3 - 2x^2 - 4x + 8)}{f'(x^3 - 2x^2 - 4x + 8)}$$

Given initial value x_i = 1.2

Iteration	X	X_{i+1}	True Value	TPRE (%)	APRE (%)
1	1.2000	1.6571	2	17.14	27.59
2	1.6571	1.8370	2	08.15	09.79
3	1.8370	1.9203	2	03.99	04.34
4	1.9203	1.9605	2	01.97	02.05
5	1.9605	1.9804	2	00.98	01.00
6	1.9804	1.9902	2	00.49	00.49
7	1.9902	1.9951	2	00.24	00.25
8	1.9951	1.9976	2	00.12	00.12
9	1.9976	1.9988	2	00.06	00.06
10	1.9988	1.9994	2	00.03	00.03
11	1.9994	1.9997	2	00.02	00.02
12	1.9997	1.9998	2	00.01	00.01

For modified **Newton Raphson Method - 1**,

$$x_{i+1} = x_i - \mathrm{m}\frac{f(x_i)}{f'(x_i)}$$

$$= xi - \mathrm{m}\frac{f(x^3-2x^2-4x+8)}{f'(x^3-2x^2-4x+8)}$$

Given initial value $x_i = 1.2$ and m = 2

Iteration	X	X_{i+1}	True Value	TPRE (%)	APRE (%)
1	1.2000	2.1143	2	17.14	43.24
2	2.1143	2.0016	2	08.15	05.63
3	2.0016	2.0000	2	03.99	00.08
4	2.0000	2.0000	2	00.00.	00.00

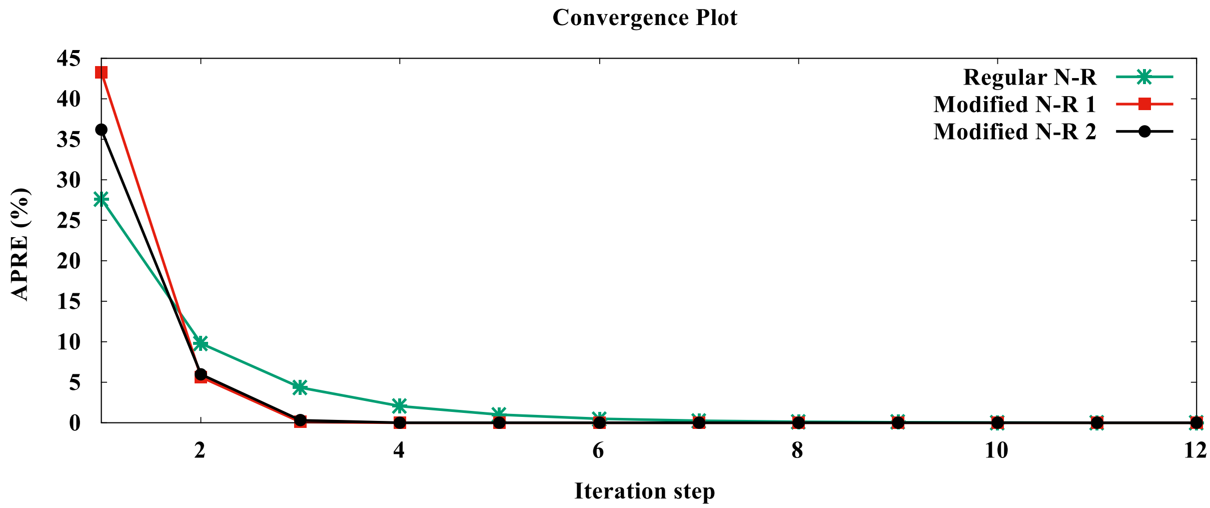
For **modified Newton Raphson Method - 2**,

$$x_{i+1} = x_i - \frac{g(x_i)}{g'(x_i)}$$

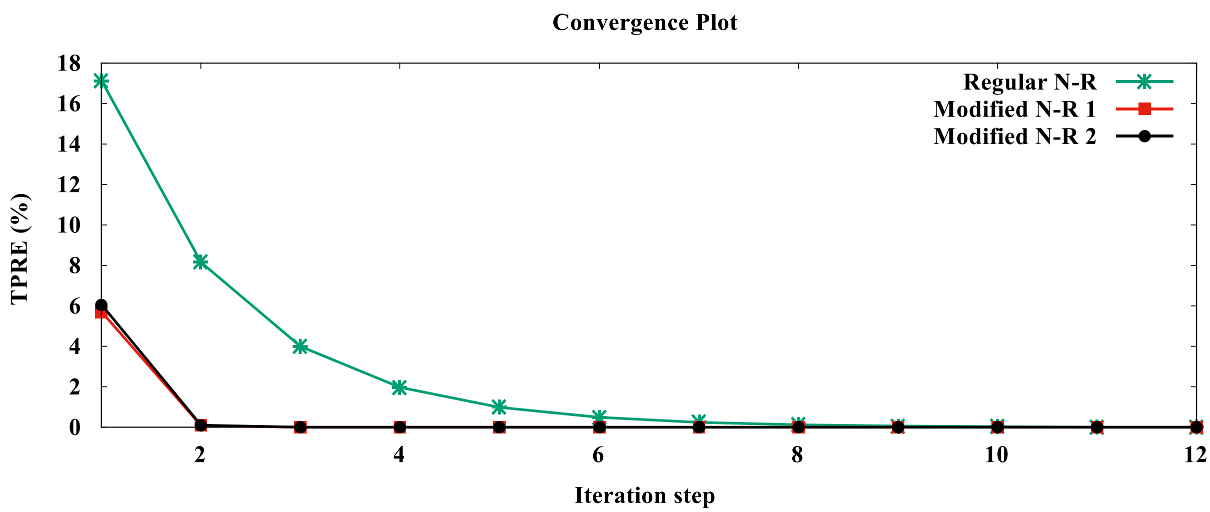
Where $g(x_i) = \frac{f(x^3-2x^2-4x+8)}{f'(x^3-2x^2-4x+8)}$

Given initial value $x_i = 1.2$

Iteration	X	X_{i+1}	True Value	TPRE (%)	APRE (%)
1	1.2000	1.8788	2	6.06	36.13
2	1.8788	1.9980	2	0.10	05.97
3	2.0061	2.0000	2	0.00	00.30
4	2.0000	2.0000	2	0.00	00.00



Convergence Plot
APRE (%)
Iteration step
Regular N-R
Modified N-R 1
Modified N-R 2



Convergence Plot
TPRE (%)
Iteration step
Regular N-R
Modified N-R 1
Modified N-R 2

7. Given: $y = -x^2 + x + 0.75$

$y + 5xy = x^2.$

(a). Fixed-point Iteration:

$$y_{i+1} = -x_i^2 + x_i + 0.75$$

$$x_{i+1} = \frac{x_i^2 - y_i}{5y_i} = \frac{1}{5}\left[\frac{x_i^2}{y_i+1} - 1\right].$$

Initial guess $(x_0, y_0) = (1.2, 1.2)$.

$\therefore x_1 = \frac{1.2^2 - 1.2}{5(1.2)} = 0.04.$

$y_1 = -1.2^2 + 1.2 + 0.75 = 0.51.$

$\varepsilon_a(x) = \frac{1.2 - 0.04}{0.04} = 29\%$

$\varepsilon_a(y) = \frac{1.2 - 0.51}{0.51} = 1.35\%$

Tabulated results:

i	x_i	y_i	$\varepsilon_a(x_i)$	$\varepsilon_a(y_i)$.
0.	1.2	1.2	–	–
1.	0.04	0.51	29	1.35.
2.	-0.1994	0.7884	1.2	0.35.
3.	-0.1899.	0.5109.	0.05	0.54.
4.	-0.1859	0.5240	0.02	0.025.
5.	-0.1868.	0.5296.	0.005	0.01.
6.	-0.1868	0.5282	3.8×10^{-5}	0.002.
7.	-0.1868	0.5282	0.0002	2×10^{-5}.

$\therefore \boxed{\begin{array}{l} x = -0.1868. \\ y = 0.5282. \end{array}}$ F-P iteration,

—— (5) mark.

(7) (a) Given: $y = -x^2 + x + 0.75$ — (1)

$x^2 = y + 5xy$ — (2)

Fixed-Point Iteration:

from (1), re-write.

$$x = (x - y + 0.75)^{1/2}$$

$$\Rightarrow x_{i+1} = \left[x_i - y_i + 0.75\right]^{1/2} \quad \text{— (3)}$$

from (2) re-write

$$y = \frac{x^2}{1+5x}$$

$$\Rightarrow y_{i+1} = \frac{x_i^2}{1+5x_i} \quad \text{— (4)}$$

$(x_0, y_0) = (1.2, 1.2)$

(3) → $x_1 = 0.8660$. $\varepsilon_a|_x = 0.27$.

(4) = $y_1 = 0.1407$. $\varepsilon_a|_y = 0.88$

Tabulated results:

i	x_i	y_i	$\varepsilon_a(x_i)$	$\varepsilon_a(y_i)$
0	1.2	1.2	–	–
1	0.8660	0.1407	0.27	0.88.
2.	1.2130	0.2130	0.40	0.48.
3.	1.325	0.2330	0.091	0.10.
4.	1.358	0.263.	0.024	0.028.
5.	1.368	0.239	0.007.	0.008.

$$\boxed{(x, y) = (1.368, 0.239)}$$ — (5) marks.

N-R Method:

(b). Given:

$$u(x,y) = y + x^2 - x - 0.75. \quad \text{— (1)}$$

$$v(x,y) = y + 5xy - x^2. \quad \text{— (2)}$$

$$x_{i+1} = -\frac{u_i \frac{\partial v_i}{\partial y} - v_i \frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x}\frac{\partial v_i}{\partial y} - \frac{\partial u_i}{\partial y}\frac{\partial v_i}{\partial x}} + x_i$$

$$y_{i+1} = y_i - \frac{v_i \frac{\partial u_i}{\partial x} - u_i \frac{\partial v_i}{\partial x}}{\frac{\partial u_i}{\partial x}\frac{\partial v_i}{\partial y} - \frac{\partial u_i}{\partial y}\frac{\partial v_i}{\partial x}}.$$

from (1) & (2).

$$\frac{\partial u}{\partial x} = 2x - 1 \qquad \frac{\partial u}{\partial y} = 1.$$

$$\frac{\partial v}{\partial x} = 5y - 2x \qquad \frac{\partial v}{\partial y} = 1 + 5x.$$

$(x_0, y_0) = (1.2, 1.2).$

$$\therefore x_1 = 1.2 - \frac{0.69(7) - 6.96(1)}{1.4(7) - 1(3.6)}.$$

$$\boxed{x_1 = 1.5435.} \qquad y_1 = 1.2 - \frac{6.96(1.4) - 0.69(3.6)}{1.4(7) - 1(3.6)}.$$

$$\varepsilon_a(x_1) = 0.223, \qquad \boxed{y_1 = 0.0290.}$$

$$\varepsilon_a(y_1) = 40.33,$$

Tabulated results $\longrightarrow$.

i	x_i	y_i	$\varepsilon_a(x_i)$	$\varepsilon_a(y_i)$
0.	1.2	1.2	–	–
1.	1.5435	0.0290.	0.22	40.33.
2.	1.3941	0.2651	0.10	0.90.
3.	1.3722	0.2630.	0.10	0.008.
4.	1.3720.	0.2482	7×10^{-5}	0.06.
5.	1.3720	0.2427.	5×10^{-7}	0.023.
6.	1.3870.	0.2407	$5\times10^{-7}\approx0$	0.083.
7.	1.3270.	0.2399.	~ 0	0.003.
8.	1.3270.	0.2396	~ 0	0.001.
9.	1.3270	0.2395	~ 0	0.0004.
10.	1.3270.	0.2395.	~ 0	1.6×10^{-4}.

$$\boxed{\begin{aligned} x &= 1.3270. \\ y &= 0.2395. \end{aligned}}$$

N-R method. — 5 mark.

Total = 10 mark.