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CH19B072

CH2061

ASSIGNMENT - I

(1) $f(x,y) = (x+y)/(x-y)$

$x = 1.5001$ $\quad$ $y = 1.4999$

x and y has 5 digits

∴ True value ⟹ 5-digit arithmetic ~~(chopping)~~

True value

$$f(1.5001, 1.4999) = \frac{1.5001 + 1.4999}{1.5001 - 1.4999} = \underline{\underline{15000}}$$

$$\therefore \text{ \% rel true error } = \frac{|15000 - \text{Appr. value}|}{15000}$$

Type of arithmetic chopping	(Approx value) $f(x,y) = \frac{x+y}{x-y}$	(Round off error → % rel) $= \frac{\lvert 15000 - \text{Appx value}\rvert}{15000}$
5 digits	$f(1.5001, 1.4999) = \underline{\underline{15000}}$	$\underline{0}$ (Taken as true value)
4 digits	$f(1.500, 1.499) = \underline{\underline{2999}}$	$\underline{\underline{80.006\%}}$
3 digits	$f(1.50, 1.49) = \underline{\underline{299}}$	$\underline{\underline{98.006\%}}$
2 digits	$f(1.5, 1.4) = \underline{\underline{29}}$	$\underline{\underline{99.806\%}}$

(2) Taylor series :

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)h^2}{2!} + \dots$$

(a) base point : $x = 0$ $\quad f(x) = e^x \quad f(0) = f'(0) = f''(0) \dots = 1$

$h = 0.5 \quad f'(x) = f''(x) = e^x$

Applying Taylor series,

$$f(0+0.5) = e^{0.5} = f(0) + f'(0)h + \frac{f''(0)h^2}{2!} + \frac{f'''(0)h^3}{3!} + \dots$$

$$\Rightarrow e^{0.5} = 1 + (0.5) + \frac{(0.5)^2}{2!} + \frac{(0.5)^3}{3!} + \dots$$

Zero-th order (only 1 term)

$e^{0.5} = \underline{1}$

First order (2 terms)

$e^{0.5} = 1 + 0.5 = \underline{1.5}$

2nd order (3 terms)

$e^{0.5} = 1 + 0.5 + \frac{(0.5)^2}{2} = \underline{1.625}$

(b) Truncation error (as true fractional relative error)

$$\varepsilon_t = \frac{|\text{Approx value} - 1.649|}{1.649}$$

0^{th} order (Approx value = 1)

$\varepsilon_t = \underline{0.393}$

1^{st} order (Approx value = 1.5)

$\varepsilon_t = \underline{0.090}$

2^{nd} order (Approx value = 1.625)

$\varepsilon_t = \underline{0.014}$

(c) $\boxed{h = 0.25}$ (Assume true value $e^{0.25} = 1.284$)

$$f(0+0.25) = e^{0.25} = f(0) + f'(0)(0.25) + f''(0)\frac{(0.25)^2}{2} + \dots$$

$$e^{0.25} = 1 + (0.25) + \frac{(0.25)^2}{2!} + \dots$$

0th order

$$e^{0.25} = \underline{\underline{1}}$$

$$\varepsilon_t = \frac{|1 - 1.284|}{1.284} = \underline{\underline{0.221}}$$

1st order

$$e^{0.25} = 1 + 0.25 = \underline{\underline{1.25}}$$

$$\varepsilon_t = \frac{|1.25 - 1.284|}{1.284} = \underline{\underline{0.026}}$$

2nd order

$$e^{0.25} = 1 + 0.25 + \frac{(0.25)^2}{2} = \underline{\underline{1.28125}}$$

$$\varepsilon_t = \frac{|1.28125 - 1.284|}{1.284} = \underline{\underline{0.002}}$$

$\boxed{h = 1}$ (Assume true value $e^1 = 2.718$)

$$f(0+1) = e^1 = f(0) + f'(0)(1) + f''(0)\frac{(1)^2}{2}$$

$$e^1 = 1 + 1 + \frac{1^2}{2!} + \dots$$

0th order

$$e^1 = \underline{\underline{1}}$$

$$\varepsilon_t = \frac{|2.718 - 1|}{2.718} = \underline{\underline{0.632}}$$

1st order

$e^1 = 1+1 = 2$

$$\varepsilon_t = \frac{|2.718 - 2|}{2.718} = 0.264$$

2nd order

$e^1 = 1+1+\frac{1}{2} = 2.5$

$$\varepsilon_t = \frac{|2.718 - 2.5|}{2.718} = 0.080$$

Tabulating ε_t for each h and each approximation,

step size (h)	True rel error (ε_t) 0th	True rel error (ε_t) 1st	True rel error (ε_t) 2nd
0.25	0.221	0.026	0.002
0.5	0.393	0.090	0.014
1	0.632	0.264	0.080

OBSERVATIONS

→ As h increases, error increases

Expl:

Error (nth appr) $= R_n = \dfrac{f^{n+1}(\xi)\, h^{n+1}}{(n+1)!}$ $\quad x < \xi < x+h$

~~Here, when~~ from relation it is clear that when h ↑, error will also ↑

→ As order of approx increases, error decreases and we are closer to true value (as more terms of Taylor series are considered)

(3).

$$f(x) = ax^3 + bx^2 + cx + d$$

$f(0) = d$ $\quad f'(0) = c$ $\quad f''(0) = 2b$ $\quad f'''(0) = 6a$

(a) base point : $x = 0$ $\quad h = 1$

Taylor series

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)h^2}{2!} + \cdots$$

$$f(1) = d + c(1) + \frac{2b(1^2)}{2!} + \frac{6a(1^3)}{3!} + \cdots$$

2nd order approximation

$$f(1) = d + c + \frac{2b}{2!} = \underline{\underline{b + c + d}}$$

Associated truncation error $= R_2 = \frac{f^3(\xi)h^3}{3!}$

ie $R_2 = \frac{f^{(3)}(\xi)}{6}$ $\quad 0 < \xi < 1$

$$R_2 = \frac{6a}{6} = \underline{\underline{a}} \quad [\text{as } f'''(x) = 6a \quad \forall x]$$

(b) Taylor series

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)h^2}{2!} + \frac{f'''(x)h^3}{3!} + \cdots$$

As $f(x)$ is a 3th degree polynomial,

$$f^{(n)}(x) = 0 \quad \forall \; n \geq 4 \text{ for any } x$$

$\therefore$ 5th term onwards ($O(h^4)$ term onwards) vanishes.

$\therefore$

$$f(x+h) = \underbrace{f(x) + f'(x)h + \frac{f''(x)h^2}{2!} + \frac{f'''(x)h^3}{3!}}$$

$f(x+h) =$ 3rd order approximation

(4) $Q = A\epsilon\sigma T^4 = kT^4$ where $k = A\epsilon\sigma = 0.76545 \times 10^{-8}$

$f(x) = kx^4$

True value

$Q(T = 650K) = k(650)^4 = 1366.376$

$f'(x) = 4kx^3$

$f''(x) = 12kx^2$

Taylor series (taking modulus)

$$|\Delta f(\tilde{x})| = \left| f'(\tilde{x})\Delta\tilde{x} + \frac{f''(\tilde{x})(\Delta\tilde{x})^2}{2!} + \cdots \right|$$

(a) $T = 650 \pm 10$ $\quad \tilde{x} = 650, \quad \Delta\tilde{x} = 10$

Actual $Q_{min} = Q(T = 640) = k(640)^4 = 1284.211$

Actual $Q_{max} = Q(T = 660) = k(660)^4 = 1452.421$

1st order approximation

$$|\Delta f(\tilde{x})| = |f'(\tilde{x})\Delta\tilde{x}| = 4k(650)^3(10) = 84.084$$

2nd order approximation

$$|\Delta f(\tilde{x})| = \left| f'(\tilde{x})\Delta\tilde{x} + \frac{f''(\tilde{x})(\Delta\tilde{x})^2}{2} \right|$$

[$f'(\tilde{x}), f''(\tilde{x}) > 0$

$\therefore$ If we use $\Delta\tilde{x} = +10$: both terms +ve (large value) → modulus

If we use $\Delta\tilde{x} = -10$: 1 term -ve, other +ve (small value)

$\therefore$ 2nd order approximation values are different for $\Delta\tilde{x} = +10$ and $\Delta\tilde{x} = -10$

We use large error, $\therefore$ use $\Delta\tilde{x} = +10$]

$$\therefore |\Delta f(\tilde{x})| = 4k(650)^3(10) + \frac{12k(650)^2(10)^2}{2} = 86.024$$

(b) $T = 650 \pm 50$ $\tilde{x} = 650$ $\Delta\tilde{x} = 50$

Actual $Q_{min} = Q(T = 600) = 992.023$

Actual $Q_{max} = Q(T = 700) = 1837.843$

1st order approximation:

$|\Delta f(\tilde{x})| = |f'(\tilde{x})\Delta\tilde{x}| = 4k(650)^3(50) = 420.423$

2nd order approximation:

$$|\Delta f(\tilde{x})| = \left| f'(x)\Delta\tilde{x} + \frac{f''(x)(\Delta\tilde{x})^2}{2} \right|$$

[Like (a), we consider only $\Delta f(\tilde{x})$ for $\Delta\tilde{x} = +50$]

$$\therefore |\Delta f(\tilde{x})| = 4k(650)^3 50 + \frac{12k(650)^2(50)^2}{2}$$

$$= 468.933$$

OBSERVATIONS

- Exact error

In first order approx, $(\Delta f(\tilde{x})) \rightarrow$ same for $+\Delta\tilde{x}$ & $-\Delta\tilde{x}$

But in second order approx, we considered only $+\Delta\tilde{x}$ as $|\Delta f(\tilde{x})|$ due to $+\Delta\tilde{x}$ is more.

In exact error, we have

+ve deviation $= |Q_{max} - Q_{actual}|$ due to $+\Delta x$

-ve deviation $= |Q_{min} - Q_{actual}|$ due to $-\Delta x$

Here also, we consider $+\Delta x$ deviation only (as it gives ~~maximum~~ more error)

↓

This can be explained in a similar way as the 2nd order approx case

∴ Exact error

(a) $|Q_{max} - Q_{actual}| = \underline{\underline{86.045}}$

(b) $|Q_{max} - Q_{actual}| = \underline{\underline{471.467}}$

- Tabulating,

	1st order appr	2nd order appr	Exact error
(a)	84.084	86.024	86.045
(b)	420.423	468.933	471.467

→ In both (a) and (b),

1st order approx $<$ 2nd order approx $<$ Exact error

Reason:

As we consider more terms of Taylor series, error becomes closer to exact error.

→ Error (a) $<$ Error (b)

Reason:

As Δx increases, error increases. This is due to Δx powers which occur in Taylor series