CH2061

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ASSIGNMENT - 3

CH19B072

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1. Gauss elimination and LU decomposition

(a) Matrix form:

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$$

(A) (3x3) (p) (3x1) (b) (3x1)

$$Ap = b$$

Pv element

(b) $\boxed{2x} + y + z = 4$ → (1) → Pivot eqn

$x + 3y + z = 0$ → (2)

$x - y + 3z = 6$ → (3)

Forward elimation

I Eliminate x from eqns (2) and (3) ⇒

(2) → (2) − (1) × 0.5

(3) → (3) − (1) × 0.5

$2x + y + z = 4$

$\boxed{2.5y} + 0.5z = -2$ → Pv eqn

$-1.5y + 2.5z = 4$

II Eliminate y from eqn (3) ⇒ ~~(3) → (3) − 2 × (−1.5 / 2.5)~~

$(3) \to (3) - (2) \times \left(\frac{-1.5}{2.5}\right)$

$2x + y + z = 4$

$2.5y + 0.5z = -2$

$2.8z = 2.8$

Backward substitution

$$2.8z = 2.8 \Rightarrow \underline{\underline{z = 1}}$$

$$2.5y + 0.5(1) = -2 \Rightarrow \underline{\underline{y = -1}}$$

$$2x + (-1) + (1) = 4 \Rightarrow \underline{\underline{x = 2}}$$

$\therefore$ soln is $P = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Note: We could apply Naive gauss elimination (without modification) because it was not an exceptional case → pivot element is not zero, coefficients are comparable and it is not ill conditioned

(c) From (b),
particularly last step of forward elimination,
we have,

$$2x + 1y + 1z = 4$$
$$0x + 2.5y + 0.5z = -2$$
$$0x + 0y + 2.8z = 2.8$$

$$\Downarrow$$

u is formed by its coefficients

$$u = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2.5 & 0.5 \\ 0 & 0 & 2.8 \end{bmatrix}$$

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For L, the $\overset{\text{wltna}}{\wedge}$ terms are actually the terms which we multiplied in forward elimination (Those terms are circled in part b) in part (b)

$$\left.\begin{aligned} l_{21} &= \frac{a_{21}}{a_{11}} = 0.5 \\ l_{31} &= \frac{a_{31}}{a_{11}} = 0.5 \end{aligned}\right\} \text{ terms which were multiplied in step I of fw elim}$$

$$l_{32} = \frac{a'_{32}}{a'_{22}} = -0.6 \;\} \text{ term multiplied in step II of fw elim}$$

other terms $\longrightarrow$ by defn of L

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & -0.6 & 1 \end{bmatrix}$$

$\therefore$

$$A = LU$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & -1 & 3 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & -0.6 & 1 \end{bmatrix}}} \underline{\underline{\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2.5 & 0.5 \\ 0 & 0 & 2.8 \end{bmatrix}}}$$

(d) $Lm = b$

$$\underset{(L)}{\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & -0.6 & 1 \end{bmatrix}} \underset{(m)}{\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}} = \underset{(b)}{\begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}}$$

<u>Forward substitution</u>

$$x' = 4$$

$$0.5(4) + y' = 0 \Rightarrow y' = -2$$

$$0.5(4) - 0.6(-2) + z' = 6 \Rightarrow z' = 2.8$$

$$\therefore\ m = \begin{bmatrix} 4 \\ -2 \\ 2.8 \end{bmatrix}$$

$$Up = m$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2.5 & 0.5 \\ 0 & 0 & 2.8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2.8 \end{bmatrix}$$

Backward substitution

$$2.8z = 2.8 \quad \Rightarrow \quad \underline{\underline{z = 1}}$$

$$2.5y + 0.5(1) = -2 \quad \Rightarrow \quad \underline{\underline{y = -1}}$$

$$2x + (-1) + (1) = 0 \quad \Rightarrow \quad \underline{\underline{x = 2}}$$

$$\therefore \text{ Solution } (p) = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

(c) Concept :

$A = LU \longrightarrow$ All are 3×3

$|A| = |L|\,|U|$

[$|L| = 1$ as its diagonal elements are 1]

$$\therefore\ \underline{\underline{|A| = |U|}}$$

Verification using given problem :

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & -1 & 3 \end{vmatrix} = \begin{array}{l} 2(3 \times 3 - (1 \times -1)) \\ -1(3 \times 1 - 1 \times 1) \\ +1(1 \times -1 - 3 \times 1) \end{array}$$

$$= \underline{\underline{14}}$$

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$$|U| = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 2.5 & 0.5 \\ 0 & 0 & 2.8 \end{vmatrix} = 2 \times 2.5 \times 2.8 = \underline{\underline{14}}$$

$\therefore$ $\underline{\underline{|A| = |U|}}$

(f) Let $A^{-1} = P \Rightarrow AP = I$

$\Rightarrow LUP = I \qquad (A = LU)$

Taking one column of P and I at a time,

$\underline{\underline{I}}$ $\quad LUP_1 = I_1$

$\rightarrow$ Fw substitution : $LM_1 = I_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & -0.6 & 1 \end{bmatrix} \begin{bmatrix} x_1' \\ y_1' \\ z_1' \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$x_1' = 1$

$0.5(1) + y_1' = 0 \Rightarrow y_1' = -0.5$

$0.5(1) - 0.6(-0.5) + z_1' = 0 \Rightarrow z_1' = -0.8$

$\rightarrow$ Bw substitution : $UP_1 = M_1$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2.5 & 0.5 \\ 0 & 0 & 2.8 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.5 \\ -0.8 \end{bmatrix}$$

$2.8 z_1 = 0.8 \Rightarrow \boxed{z_1 = -0.2857}$

$2.5 y_1 + 0.5(-0.2857) = -0.5 \Rightarrow \boxed{y_1 = -0.1429}$

$2x_1 + (-0.1429) + (-0.2857) = 1$

$\Rightarrow \boxed{x_1 = 0.7143}$

II

$LUP_2 = I_2$

Forward substitution : $LM_2 = I_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & -0.6 & 1 \end{bmatrix} \begin{bmatrix} x_2' \\ y_2' \\ z_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_2' = 0, \quad y_2' = 1, \quad z_2' = 0.6$$

Backward substitution : $UP_2 = M_2$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2.5 & 0.5 \\ 0 & 0 & 2.8 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0.6 \end{bmatrix}$$

$$\Rightarrow \boxed{z_2 = 0.2143} \quad \boxed{y_2 = 0.3571} \quad \boxed{x_2 = -0.2857}$$

III

$LUP_3 = I_3$

Forward subst : $LM_3 = I_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & -0.6 & 1 \end{bmatrix} \begin{bmatrix} x_3' \\ y_3' \\ z_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \quad x_3' = 0, \quad y_3' = 0, \quad z_3' = 1$$

Backward subst : $UP_3 = M_3$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2.5 & 0.5 \\ 0 & 0 & 2.8 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \boxed{z_3 = 0.3571} \quad \boxed{y_3 = -0.0714} \quad \boxed{x_3 = -0.1429}$$

$$\therefore \quad P = A^{-1} = \begin{bmatrix} 0.7143 & -0.2857 & -0.1429 \\ -0.1429 & 0.3571 & -0.0714 \\ -0.2857 & 0.2143 & 0.3571 \end{bmatrix}$$

2: Thomas Algorithm | Tri diagonal Matrix Algorithm

(a) $$\underbrace{\begin{bmatrix} 0.8 & -0.4 & 0 \\ -0.4 & 0.8 & -0.4 \\ 0 & -0.4 & 0.8 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 41 \\ 25 \\ 105 \end{bmatrix}}_{P}$$

(b) <u>Decomposition</u>

let $A = \begin{bmatrix} f_1 & g_1 & 0 \\ e_2 & f_2 & g_2 \\ 0 & e_3 & f_3 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.4 & 0 \\ -0.4 & 0.8 & -0.4 \\ 0 & -0.4 & 0.8 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ e_2' & 1 & 0 \\ 0 & e_3' & 1 \end{bmatrix}$ $\quad U = \begin{bmatrix} f_1 & g_1 & 0 \\ 0 & f_2' & g_2 \\ 0 & 0 & f_3' \end{bmatrix}$

(f_1, g_i remains same in this decomposition)

$e_k' = \dfrac{e_k}{f'_{k-1}}$ $\quad f_k' = f_k - e_k' g_{k-1}$

$\underline{\underline{k=2}}$

$e_2' = \dfrac{e_2}{f_1'} = \dfrac{e_2}{f_1} = -0.5$

$f_2' = f_2 - e_2' g_1 = 0.8 - (-0.5)(-0.4) = 0.6$

$\underline{\underline{k=3}}$

$e_3' = \dfrac{e_3}{f_2'} = \dfrac{-0.4}{0.6} = -0.6667$

$f_3' = f_3 - e_3' g_2 = 0.8 - (-0.6667)(-0.4) = 0.5333$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0 & -0.6667 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 0.8 & -0.4 & 0 \\ 0 & 0.6 & -0.4 \\ 0 & 0 & 0.5333 \end{bmatrix}$$

FW substitution

$$b_k' = b_k - e_k' b'_{k-1} \qquad b_1' = b_1 = \underline{\underline{41}}$$

$$b_2' = b_2 - e_2' b_1' = 25 - (-0.5)(41) = \underline{\underline{45.5}}$$

$$b_3' = b_3 - e_3' b_2' = 105 - (-0.6667)(45.5)$$

$$= \underline{\underline{135.33}}$$

BW substitution

$$z = \frac{b_3'}{f_3'} = \frac{135.33}{0.5333} = \underline{\underline{253.765}}$$

$$y = \frac{b_2' - g_2 z}{f_2'} = \frac{45.5 - (-0.4)(253.765)}{0.6}$$

$$= \underline{\underline{245.01}}$$

$$x = \frac{b_1' - g_1 y}{f_1'} = \frac{41 - (-0.4)(245.01)}{0.8}$$

$$= \underline{\underline{173.75}}$$

$$\therefore \text{solution} = \begin{bmatrix} 173.75 \\ 245.01 \\ 253.765 \end{bmatrix}$$

(c) We observe that no. of calculations is lesser in TDMA (Thomas algorithm) when compared to Gauss elimination. This is in accordance with the fact that flops (TDMA) = $O(n)$ while flops (Gauss elim) = $O(n^3)$

3. Gauss - Seidel and relaxation

(a) $x + 2y = 1$

$x - y = 4$

Normal Gauss - Seidel method :

$x_{i+1} = 1 - 2y_i$

$y_{i+1} = x_{i+1} - 4$

Use initial guess $x_0 = y_0 = 0$

[Note: % approx relative error $(x) = \left|\frac{x_i - x_{i-1}}{x_i}\right|$

% approx relative error $(y) = \left|\frac{y_i - y_{i-1}}{y_i}\right|$]

(By error, I mean approx. rel % error)

iter - 1

$x_1 = 1 - 2y_0 = 1$

$y_1 = x_1 - 4 = -3$

approx % rel error → undefined (iter-1)

Iter - 2

$x_2 = 1 - 2y_1 = 1 - 2(-3) = 7$

$y_2 = x_2 - 4 = 3$

error$(x) = \frac{7-1}{7} \times 100 = \underline{85.714\%}$

error $(y) = \frac{3--3}{3} \times 100 = \underline{200\%}$

iter - 3

$x_3 = 1 - 2y_2 = -5$

$y_3 = -5 - 4 = -9$

error $(x) = \frac{1200}{5} = \underline{240\%}$ error $(y) = \frac{1200}{9} = \underline{133.3\%}$

iter – 4

$x_4 = 1 - 2(-9) = 19$

$y_4 = 19 - 4 = 15$

$\text{error}(x) = \frac{24}{19} \times 100 = 126.3\%$

$\text{error}(y) = \frac{24}{15} \times 100 = 160\%$

Clearly, after 4 iterations it is not converging

[As error is not decreasing]

(b) $\lambda = 0.6 \rightarrow$ under-relaxation

Formula :

~~$x_{i+1} = \lambda(x_i)$~~

$x_i^{new} = \lambda x_i + (1-\lambda) x_{i-1}$

$y_i^{new} = \lambda y_i + (1-\lambda) y_{i-1}$

iteration – 1 : [Initial guess : $x_0 = 0$, $y_0 = 0$]

$x_1 = 1 - 2y_0 = 1$

$x_1^{new} = 0.6 x_1 + 0.4 x_0 = 0.6$

$y_1 = x_1^{new} - 4 = 0.6 - 4 = -3.4$

$y_1^{new} = 0.6 y_1 + 0.4 y_0 = -2.04$

error (x), error (y) → not defined

Iteration – 2

$x_2 = 1 - 2y_1^{new} = 5.08$

$x_2^{new} = 0.6 x_2 + 0.4 x_1^{new} = 3.288$

$y_2 = x_2^{new} - 4 = -0.712$

$y_2^{new} = 0.6\,y_2 + 0.4\,y_1^{new} = \underline{\underline{-1.2432}}$

$error(x) = \dfrac{3.288 - 0.6}{3.288} \times 100 = 81.75\,\%$

$error(y) = \left| \dfrac{-1.2432 - -2.04}{-1.2432} \times 100 \right| = 64.09\,\%$

iteration - 3

$x_3 = 1 - 2y_2^{new} = 3.4864$

$x_3^{new} = 0.6\,x_3 + 0.4\,x_2^{new} = \underline{\underline{3.40704}}$

$y_3 = x_3^{new} - 4 = -0.59296$

$y_3^{new} = 0.6\,y_3 + 0.4\,y_2^{new} = \underline{\underline{-0.85305}}$

$error(x) = 3.49\,\%$

$error(y) = 45.73\,\%$

iteration - 4

$x_4 = 1 - 2y_3^{new} = 2.7061$

$x_4^{new} = 0.6\,x_4 + 0.4\,x_3^{new} = \boxed{2.98648}$

$y_4 = x_4^{new} - 4 = -1.01353$

$y_4^{new} = 0.6\,y_4 + 0.4\,y_3^{new} = \boxed{-0.94934}$

$error(x) = 14.08\,\%$

$error(y) = 10.14\,\%$

∴ The solution is better than (a), where we got huge errors, but still, ~~we can do better~~. this solution is not the best.

The solution from (b) is not very converging. hence we need better answers.

(c) <u>Diagonally dominant</u>

$$|a_{ii}| > \sum_{j=1,\, j\neq i}^{n} |a_{ij}|$$

By trying various combinations, we find that if we use

$$2y + x = 1$$

$$-y + x = 4$$

We have $a_{11} = 2$ $a_{12} = 1$

$a_{21} = -1$ $a_{22} = 1$

$\therefore$ We use

$$y_{i+1} = \frac{1 - x_i}{2} \qquad x_{i+1} = 4 + y_{i+1}$$

<u>iter - 1</u> [Initial guess : $x_0 = 0$, $y_0 = 0$]

$y_1 = \frac{1-0}{2} = 0.5$

$x_1 = 4 + 0.5 = 4.5$

<u>iter - 2</u>

$y_2 = \frac{1-4.5}{2} = -1.75$ error $(y) = 128\%$

$x_2 = 4 + y_2 = 2.25$ error $(x) = 100\%$

<u>iter - 3</u>

$y_3 = \frac{1-2.25}{2} = -0.625$ error $(y) = 180\%$

$x_3 = 4 + y_3 = 3.375$ error $(x) = 33.3\%$

iter- 4

$y_4 = \frac{1 - 3.375}{2} = \boxed{-1.1875}$ error(y) = 47.3%

$x_4 = 4 + y_4 = \boxed{2.8125}$ error(x) = 20%

Here also, the solution is not much converging after 4 iterations, but it is better than (a)
[May be it will converge after a few more iterations]