CH 2061

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CH19B072

ASSIGNMENT - 5

8-11-20

1: (Integration Techniques)

$$I = \int_{-2}^{4} (1 - x - 4x^3 + 2x^5)\, dx$$

(a) Analytically:

$$I = \left[x - \frac{x^2}{2} - x^4 + \frac{x^6}{3} \right]_{-2}^{4}$$

$$= 4 - 8 - 256 + 1365.333 - (-2 - 2 - 16 + 21.333)$$

$$= \underline{\underline{1104}}$$

(b) Single application of Trapezoidal Rule:

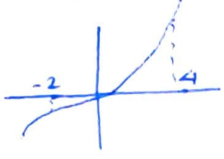
$f(x) = 1 - x - 4x^3 + 2x^5$

$a = -2$ $\qquad$ $b = 4$

$f(-2) = -29$ $\qquad$ $f(4) = 1789$

[Since both $f(-2)$ and $f(4)$ are of opposite signs, our answer is prone to errors

$f(x)$ looks like:



$\Rightarrow$ In this case Trapezoidal rule might have huge errors]

$$I = \frac{(b-a)}{2}(f(a)+f(b)) = \frac{4-(-2)}{2}[f(4)+f(-2)]$$

$$= \frac{6}{2}(1789-29) = \underline{5280}$$

(c) <u>Composite Trapezoidal rule $(n=4)$:</u>

$a = -2 \qquad b = 4$

$h = \frac{b-a}{n} = 1.5$ (step size)

$x_0 = -2 \qquad x_1 = -0.5 \qquad x_2 = 1 \qquad x_3 = 2.5 \qquad x_4 = 4$

$f(x_0) = -29 \qquad f(x_1) = 1.9375 \qquad f(x_2) = -2 \qquad f(x_3) = 131.3125$

$f(x_4) = 1789$

[Here also, the same issues may arise, but as we take more points, error is likely to be less than (a)]

I = sum of 4 trapeziums

$$= \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1}(f(x_i)) + f(x_n)\right]$$

$$= \frac{1}{2} \times 1.5 \times [f(-2) + 2f(-0.5) + 2f(1) + 2f(2.5) + f(4)]$$

$$= 0.75 \times [2022.5]$$

$$= \underline{1516.875}$$

(d) Single application of Simpson's $\frac{1}{3}$ rule :

Here, we fit a quadratic equation and approximate the required integral to the integral of the quadratic equation obtained (final expression given below)

Here, we need three points → we choose

$x_0 = -2$ (a) $x_1 = 1$ and $x_2 = 4$ (b)

~~[Plot of the function is attached.]~~

$$h = \frac{b-a}{2} = \underline{\underline{3}}$$

$$I = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$= \frac{3}{3}\left[-29 + 4 \times -2 + 1789\right]$$

$$= \underline{\underline{1752}}$$

(e) Composite Simpson's $\frac{1}{3}$ rule $(n = 4)$

$a = x_0 = -2$ $b = x_4 = 4$ $h = \frac{b-a}{n}$ $= 1.5$

$x_1 = -0.5$ $x_2 = 1$ $x_3 = 2.5$

$f(x_1) = 1.9375$ $f(x_2) = -2$ $f(x_3) = 131.3125$

Here, we have 5 points; we split the points into two halves and integrate x_0 to x_2 and x_2 to x_4 seperately using Simpson's $1/3$ rule (final expression given below)

$$I = \frac{h}{3}\left[f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{i=2,4,6}^{n-2} f(x_i) + f(x_n)\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4\left(f(x_1) + f(x_3)\right) + 2f(x_2) + f(x_4)\right]$$

$$= \frac{1.5}{3}\left[-29 + 4(1.9375 + 131.3125) + 2(-2) + 1789\right]$$

$$= \underline{\underline{1144.5}}$$

(f)

	Method	Integral	True relative error
(a)	Analytical (True)	1104 (True)	0
(b)	Trapezoidal (Single)	5280	3.7826
(c)	Trapezoidal (Multiple)	1516.875	0.37398
(d)	Simpson's $\frac{1}{3}$ (Single)	1752	0.58695
(e)	Simpson's $\frac{1}{3}$ (Multiple)	1144.5	0.036684

OBSERVATIONS:

→ Trapezoidal (single) has huge errors (as I mentioned)

→ Simpson's $\frac{1}{3}$ rule works better than Trapezoidal rule

→ For a particular rule, it is better to use composite application.

2: (Central difference approximation)

	(x_0)	(x_1)	(x_2)	(x_3)	(x_4)	(x_5)
Time (x)	0	25	50	75	100	125
$(f(x))$ Distance	0	32	58	85	92	100

(a) Velocity

$v = \frac{ds}{dt}$ → we find the derivative using central diff. method

(we take time as x and distance as $f(x)$)

$h = x_i - x_{i-1}$ (difference between successive steps of x)

$h = 25$ (From the data)

$$\boxed{f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}}$$

$$f'(25) = \frac{f(50) - f(0)}{50} = \underline{1.16}$$

$$f'(50) = \frac{f(75) - f(25)}{50} = \underline{1.06}$$

$$f'(75) = \frac{f(100) - \cancel{f(50)}\ f(50)}{50} = \underline{0.68}$$

$$f'(100) = \frac{f(125) - f(75)}{50} = \underline{0.3}$$

[In central difference method, we cannot find derivative at end points]

Acceleration

$a = \frac{d^2 s}{dt^2}$ → we find the 2nd derivative using central difference method.

$x \rightarrow$ time

$f(x) \rightarrow$ distance

$h = x_i - x_{i-1} = 25$

Centered difference

$$\boxed{f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h2}}$$

$$f''(25) = \frac{f(50) - 2f(25) + f(00)}{25 \times 25} = -9.6 \times 10^{-3}$$

$$f''(50) = \frac{f(75) - 2f(50) + f(25)}{25 \times 25} = 1.6 \times 10^{-3}$$

$$f''(75) = \frac{f(100) - 2f(75) + f(50)}{25 \times 25} = -0.032$$

$$f''(100) = \frac{f(125) - 2f(100) + f(75)}{25 \times 25} = 1.6 \times 10^{-3}$$

[In central difference method, we cannot find 2nd derivative at end points]

NOTE: For both first derivative and second derivative, I used simple centered difference (error: $O(h^2)$). We can use (error: $O(h^4)$) for more accuracy.

For finding velocity and acceleration at end points, we can use forward and backward substitutions : ($O(h^2)$ formulae)

velocity

$$f'(0) = \frac{-f(50) + 4f(25) - 3f(0)}{50} = 1.4$$

$$f'(125) = \frac{3f(125) - 4f(100) + f(75)}{50} = 0.34$$

acceleration

$$f''(0) = \cancel{2f(0) - 5f}\ \frac{-f(75) + 4f(50) - 5f(25) + 2f(0)}{625} =$$

$$= -0.0208$$

$$f''(125) = \frac{2f(125) - 5f(100) + 4f(75) - f(50)}{625} = 0.0352$$

Summary

Putting in appropriate units for velocity and acceleration:

Time (s)	Distance (km)	Velocity (km/s)	Acceleration (km/s²)
0	0	1.4	-0.0208
25	32	1.16	-9.6×10^{-3}
50	58	1.06	1.6×10^{-3}
75	85	0.68	-0.0032
100	92	0.3	1.6×10^{-3}
125	100	0.34	0.0352

(b) plots → attached at the end

3: (Different methods of Numeric Differentiation)

$f(x) = y = \sin x$

$h = \frac{\pi}{12} = x_i - x_{i-1} = 0.2618$

Required point: $x_i = \frac{\pi}{4}$

(other points used):

$x_{i-2} = \frac{\pi}{4} - 2h = \frac{\pi}{12}$ $\quad f(x_{i-2}) = 0.2588$

$x_{i-1} = \frac{\pi}{4} - h = \frac{\pi}{6}$ $\quad f(x_{i-1}) = 0.5$

$x_i = \frac{\pi}{4}$ $\quad f(x_i) = 0.7071$

$x_{i+1} = \frac{\pi}{4} + h = \frac{\pi}{3}$ $\quad f(x_{i+1}) = 0.8660$

$x_{i+2} = \frac{\pi}{4} + 2h = \frac{5\pi}{12}$ $\quad f(x_{i+2}) = 0.9659$

(First derivative)

Forward difference

$O(h)$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} = \underline{\underline{0.607}}$$

$O(h^2)$

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} = \underline{\underline{0.7196}}$$

Backward difference

~~f'(x~~ $O(h)$

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} = \underline{\underline{0.7911}}$$

$O(h^2)$

$$f'(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h} = \underline{\underline{0.7259}}$$

Centered difference

$O(h^2)$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} = \underline{\underline{0.6990}}$$

$O(h^4)$

$$f'(x_i) = \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h} = \underline{\underline{0.7069}}$$

Analytical solution:

$f'(x) = \cos x$

$f'(\pi/4) = \cos\frac{\pi}{4} = \underline{\underline{0.7071}}$

OBSERVATION

Among the various techniques $O(h^4) \rightarrow$ central approx is the best ~~se~~ method because it has error $O(h^4)$

PLOTS

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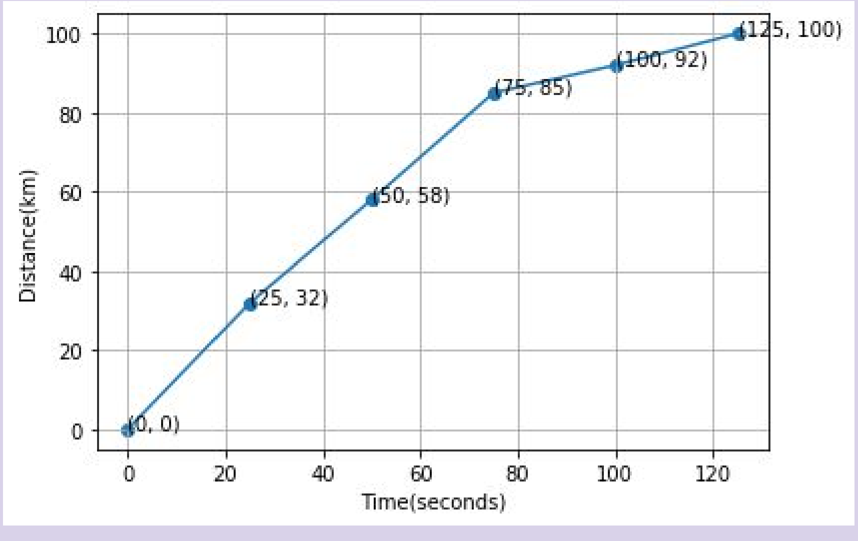
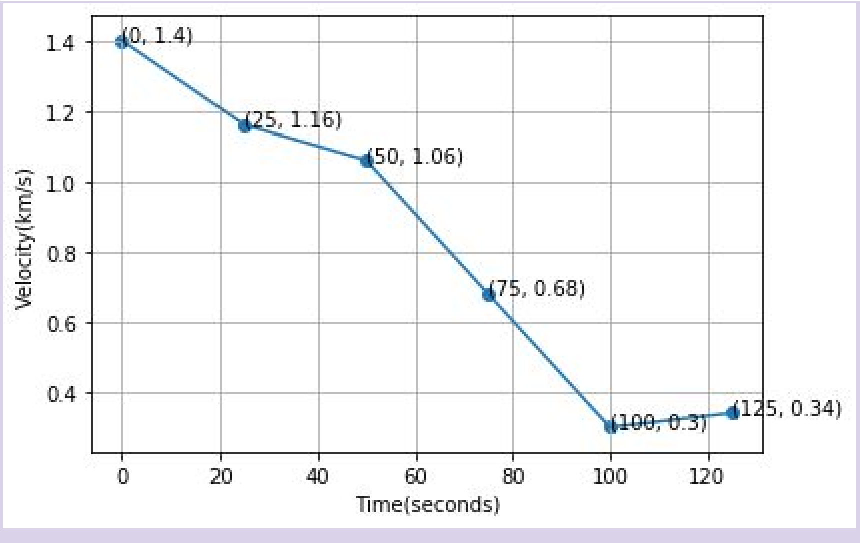
Qn 2:

Distance vs Time:



Velocity vs Time:



Acceleration vs Time: