Solutions to Problems Sheet 7

1. Given $E[X] = 5/4$, $\mathrm{var}(X) = 5/48$. ~~By Chebyshev~~,

$P(X \geq 2.5) = P(X \geq \frac{5}{4} + \frac{5}{4})$.

$\therefore P(|X-\mu| \geq 5/4) = P(X \geq 5/2 \text{ or } X \leq 0)$

$= P(X \geq 5/2)$ $\quad$ $\because P(X \leq 0) = 0$ Assumption missing in the question.

$\leq \frac{\sigma^2}{(5/4)^2}$ (by Chebyshev)

$= \frac{5/48}{25/16} = 1/15$

$P(X \geq a) \leq 15/100 \Rightarrow \frac{15}{100} = \frac{\sigma^2}{k^2} = \frac{5/48}{k^2}$

$\Rightarrow k = 5/6$.

$\therefore$ ~~$P(X \geq \mu$~~ Hence $a = \mu + 5/6 = \frac{5}{4} + \frac{5}{6} = \frac{25}{12}$.

2. Given $X \sim UNIF(0,10)$. $\therefore E[X] = \mu = 5$; $\sigma = 5/\sqrt{3}$

$P(2 \leq X \leq 8) = P(5-3 \leq X \leq 5+3)$

$\Rightarrow 3 = k\sigma \Rightarrow k = 3/\sigma \Rightarrow k = \frac{3\sqrt{3}}{5}$

$\therefore P(2 \leq X \leq 8) \geq 1 - 1/k^2 = 2/27$.

3c. Actual prob. $= \int_2^8 \frac{dx}{10} = 3/5$. which is much more than $2/27$.

3. Given $X \sim UNIF(0,4)$. $\therefore E[X] = 2$.

$P(X \geq 2) = \frac{1}{2}$. and $P(X \geq 4) = 0$.

But by Markov,

$P(X \geq 2) \leq \frac{2}{2} = 1$ & $P(X \geq 4) \leq \frac{2}{4} = 0.5$

4. Let X_i denote the weight. Given $X_i \sim UNIF(5,50)$ $i = 1, 2 \dots 100$.

$\therefore \mu = 27.5$ and $\sigma^2 = \frac{(b-a)^2}{12} = 168.75$.

To find $P(X_1 + \dots + X_{100} \geq 3000) = P(S_{100} \geq 3000)$

$$P(S_{100} \leq 3000) = P\left(\frac{\cancel{300}S_{100} - n\mu}{\sigma\sqrt{n}} \leq \frac{3000 - 100 \times 27.5}{\sqrt{168.75 \times 100}}\right)$$

$$\approx P(Z \leq 1.92) \quad (\text{by C.L.T})$$

$$= 0.9726.$$

$\therefore$ Reqd. prob. $= 1 - .9726 = 0.0274$.

5. Let $X_i \sim BER(\frac{1}{2})$, $i = 1, 2, \dots 20$.

Then $X = X_1 + \dots + X_{20} \sim BIN(20; \frac{1}{2})$

$\mu = p = \frac{1}{2}$, $\sigma^2 = p(1-p) = \frac{1}{4}$.

$$P(8 \leq X \leq 10) = P\left(\frac{8 - n\mu}{\sigma\sqrt{n}} \leq \frac{X - n\mu}{\sigma\sqrt{n}} \leq \frac{10 - n\mu}{\sigma\sqrt{n}}\right)$$

$$= P\left(-\frac{2}{\sqrt{5}} \leq Z_n \leq 0\right)$$

$$\approx \Phi(0) - \Phi\left(-\frac{2}{\sqrt{5}}\right) = 0.3145.$$

6. Given $Y = X_1 + \cdots + X_{50}$ where X_i is the service time spent for i^{th} customer and $E[X_i] = 2$, $\sigma_{X_i}^2 = 1$.

$$P(90 < Y < 100) = P\left(\frac{90 - n\mu}{\sigma\sqrt{n}} \le \frac{Y - n\mu}{\sigma\sqrt{n}} \le \frac{100 - n\mu}{\sigma\sqrt{n}}\right)$$

$$= P(-\sqrt{2} < Z_n < \sqrt{2})$$

$$\approx \Phi(\sqrt{2}) - \Phi(-\sqrt{2})$$

$$= 0.8502.$$

7. $$P(68.91 \le X_1 + \cdots + X_{25} \le 71.97)$$

$$= P(-0.68 \le Z_{25} \le 0.36)$$

$$\approx \Phi(0.36) - \Phi(-0.68)$$

$$= 0.5941$$

8. Let $S_n = X_1 + \cdots + X_n$ where X_i denotes the life time of i^{th} bulb. To ~~at~~ find n so that

$$0.95 = P(S_n \geq 60).$$

$$\therefore 0.95 = P(S_n \geq 60) = P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \geq \frac{60 - 2n}{\sqrt{n}/4}\right)$$

$$= P\left(Z_n \geq \frac{240 - 8n}{\sqrt{n}}\right)$$

$$\Rightarrow P\left(Z_n < \frac{240 - 8n}{\sqrt{n}}\right) = 0.05.$$ From the table,

$$\frac{240 - 8n}{\sqrt{n}} = \cancel{0.16}\ -1.645 \Rightarrow 1.645\sqrt{n} + 8n - 240 = 0$$

$$\Rightarrow \sqrt{n} = -5.375,\ 5.581$$

$$\Rightarrow n = 31.15.$$

$\therefore$ We need $n = 32$.

9. Let $\epsilon > 0$.

$$P(|X_n - X| \geq \epsilon) = P(|Y_n| \geq \epsilon)$$

$$= P(|Y_n - E[Y_n] + E[Y_n]| \geq \epsilon)$$

$$\leq P(|Y_n - E[Y_n]| + \tfrac{1}{n} \geq \epsilon)$$

(by triangle ineq.)

$$= P(|Y_n - E[Y_n]| \geq \epsilon - 1/n)$$

$$\leq \frac{Var(Y_n)}{(\epsilon - 1/n)^2}$$

(by Chebyshev ineq. and $k\frac{\sigma}{\sqrt{n}} = \epsilon - 1/n$)

$$= \frac{\sigma^2}{n(\epsilon - 1/n)^2} \to 0 \text{ as } n \to \infty.$$

$\therefore X_n \xrightarrow{P} X$