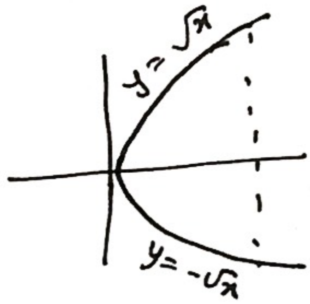
Solutions to Problem Sheet - 5

1. $$\iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = 1$$

$$\Rightarrow k \int_0^1 \int_x^1 x y^2\,dy\,dx = 1$$

$$\Rightarrow \frac{k}{3}\int_0^1 x y^3\Big|_x^1\,dx = 1 \Rightarrow k = 10.$$



2. $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_{-\sqrt{x}}^{\sqrt{x}} \tfrac{3}{4}\,dy = \frac{3}{2}\sqrt{x}$$

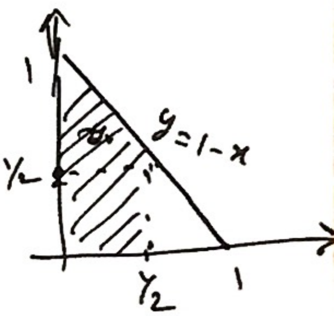
$$\therefore f_X(x) = \begin{cases} \tfrac{3}{2}\sqrt{x}, & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$|||^{ly}$ $$f_Y(y) = \begin{cases} \frac{3}{4}(1-y^2) & 0<y^2<x \\ 0 & \text{otherwise.} \end{cases}$$

3. Given $$f_{X,Y}(x,y) = \begin{cases} 2x & 0<x,y<1 \\ 0 & \text{otherwise} \end{cases}$$

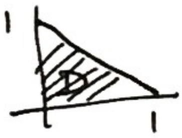
$$P(X+Y\le 1 \mid X\le \tfrac{1}{2}) = \frac{P\big((X+Y\le 1)\cap(X\le \tfrac12)\big)}{P(X\le \tfrac12)}$$

$$= \frac{\int_0^{1/2}\int_0^{1-x} 2x\,dy\,dx}{\int_0^1\int_0^{1/2} 2x\,dx\,dy} = 2/3.$$



4) $$\frac{\partial^2 F_{X,Y}}{\partial x\,\partial y}(x,y) = \begin{cases} \frac{1}{5}(6x^2+12xy) & 0 \le x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

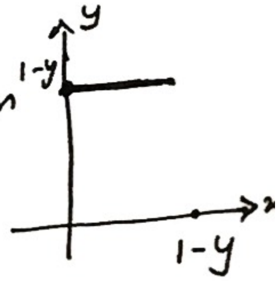
5) Given X & Y have joint PDF which is uniform on D.



$$\therefore f_{X,Y}(x,y) = \begin{cases} 2 & \text{if } (x,y) \in D \\ 0 & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^{1-y} 2\,dx = 2(1-y), \quad 0 \le y \le 1$$

$$\therefore f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{1-y}, \quad 0 \le x \le 1-y$$



(ie) $f_{X|Y}(x|y)$ has uniform distribution for each fixed y.

$$E[X|Y=y] = \int_0^{1-y} x \cdot \frac{1}{1-y}\,dx$$

$$= \frac{1-y}{2}, \quad 0 \le y \le 1.$$

$$E[X] = \int_{-\infty}^{\infty} E[X|Y=y]\, f_Y(y)\,dy$$

$$= \int_0^1 \frac{1-y}{2} \times 2(1-y)\,dy = 1/3.$$

6) $$f_{X,Y}(x,y) = f_{Y|X}(y|x) \cdot f_X(x).$$

$$= \frac{y}{2x^2} \times 24x^2 = 12y, \quad 0 < y < 2x < 1.$$

$$\therefore f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{y/2}^{1/2} 12y\,dx$$

$$= 6y(1-y), \quad 0<y<1.$$

$$\therefore f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{12y}{6y(1-y)} = \frac{2}{1-y}$$

$$\therefore f_{X|Y}(x|y) = \begin{cases} 2/1-y & 0<y<2x<1 \\ 0 & \text{otherwise.} \end{cases}$$

7) Given $X \sim EXP(1)$ & $A = \{X>1\}$.

$$f_X(x) = \begin{cases} e^{-x}, & x>0 \\ 0 & \text{otherwise} \end{cases}$$

$$P(A) = \int_1^{\infty} e^{-x}\,dx = e^{-1}.$$

$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{P(A)}, & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} = e^{-x+1}, \quad x>1$$

$$E[X|A] = \int_{-\infty}^{\infty} x\, f_{X|A}(x)\,dx$$

$$= \int_1^{\infty} x e^{-x+1}\,dx = 2.$$

$$E[X^2|A] = \int_1^{\infty} x^2 e^{-x+1}\,dx = 5$$

$$\therefore Var(X|A) = 5-4 = 1.$$

8) Given $f_{X,Y}(x,y) = \begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{xy}{6}, & 0 \le x \le 1,\ 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \frac{3y^2+y+1}{12}, \quad 0 \le y \le 2.$$

$$\therefore f_{X|Y}(x|y) = \begin{cases} \dfrac{3x^2+3y^2+2xy}{3y^2+y+1}, & 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X|Y=1] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|1)\,dx$$

$$= \int_0^1 x \cdot \frac{3x^2+3+2x}{5}\,dx = 7/12.$$

|||ly $E[X^2|Y=1] = 21/50$.

$$\text{Var}(X|Y=1) = \frac{21}{50} - \frac{49}{144} = \frac{287}{3600}.$$

$$P(X < 1/2 \,|\, Y = y) = \int_0^{1/2} \frac{3x^2+3y^2+2xy}{3y^2+y+1}\,dx$$

$$= \frac{1}{3y^2+y+1}\left[x^3 + 3y^2x + x^2y\right]_0^{1/2}$$

$$= \frac{1}{3y^2+y+1}\left[\frac{1}{8} + \frac{3}{2}y^2 + \frac{y}{4}\right]$$

$$\therefore P(X < 1/2 \,|\, Y=0) = 1/8.$$

9) Given $f_{X,Y}(x,y) = \begin{cases} 1/\pi & \text{if } (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$

$\therefore f_X(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

& $f_Y(y) = \begin{cases} \frac{2}{\pi}\sqrt{1-y^2}, & -1 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$

$\therefore f_{X|Y}(x|y) = \begin{cases} \frac{1}{2\sqrt{1-y^2}} & \text{if } -\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \\ 0 & \text{otherwise.} \end{cases}$

(ie) $f_{X|Y}(x|y) \sim UNIF(-\sqrt{1-y^2}, \sqrt{1-y^2})$

Since $f_{X|Y}(x|y) \neq f_X(x)$, X & Y are NOT indep.

10) $f_Y(y) = \int_{-1}^{1} \left(x^2 + \frac{y}{3}\right) dx = \frac{2}{3}y + \frac{2}{3}$, $0 \le y \le 1$.

$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{3x^2+y}{2y+2}, & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

where $0 \le y \le 1$. Since it depends on y, X and Y are NOT independent.