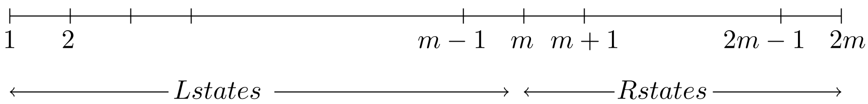
Department of Mathematics, IIT Madras
MA 2040 : Probability, Statistics and Stochastic Processes
July - Nov 2020
Solutions to Problem Set - 9

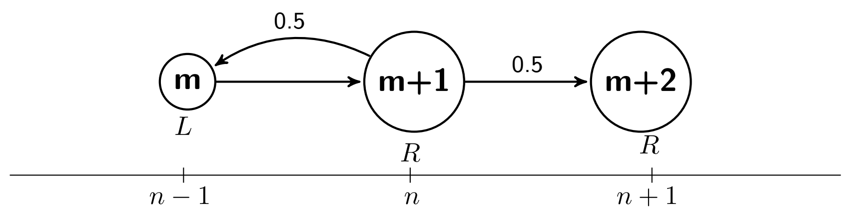
1. Note that,



We have,

$$P(X_{n+1} = R | X_n = R, X_{n-1} = L) = \frac{1}{2}. \tag{1}$$

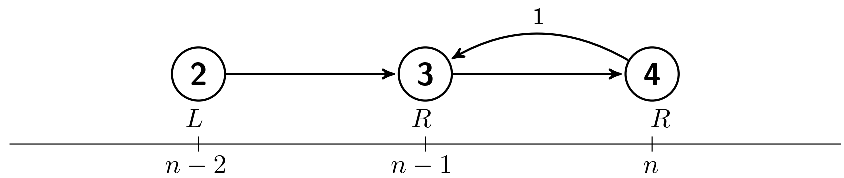
Graphically, we have



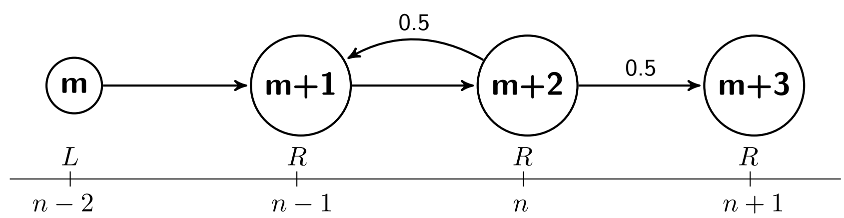
Also,

$$P(X_{n+1} = R | X_n = R, X_{n-1} = R, X_{n-2} = L) = 1. \tag{2}$$

For $m = 2$

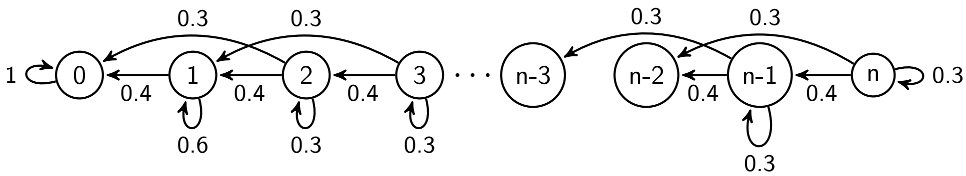


For $m = 3, 4, 5, \cdots$



From (1) and (2) , it follows that the generated sequence of signals L and R is not a Markov chain.

2. (a) Let n be the initial distance between spider and fly. State space $= \{0, 1, 2, \cdots , n\}$.
Markov chain with states as distance.



The transition probability matrix is,

$$P = R(1) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & n \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ .4 & .6 & 0 & \cdots & 0 \\ .3 & .4 & .3 & \cdots & 0 \\ 0 & .3 & .4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & .3 \end{bmatrix} \end{array}$$

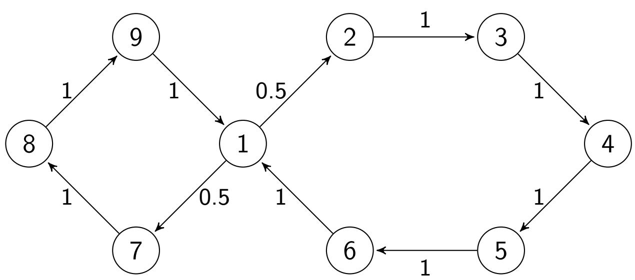
Note that

$p_{10} = P$(fly remains at its place and spider moves a step given the distance is one unit) $= .4$

$p_{11} = P$(fly moves away one unit and spider moves a unit towards fly | distance between spider and fly is one unit)$+$ P(fly reaches the position of spider and spider reaches the position of fly | distance between spider and fly is one unit) $= .3 + .3 = .6$

(b) State 0 is recurrent and all other states are transient.

3. This Markov chain can be represented as



All states are recurrent states and there is only one recurrent class; namely, the whole state space. Note that, this recurrent class can be partitioned into two subsets, $S_1 = \{2, 4, 6, 7, 9\}$ and $S_2 = \{1, 3, 5, 8\}$, and the state of the Markov chain jumps from S_1 to S_2 (with probability 1) in one time step. Therefore, this Markov chain's recurrent class has period 2.

4. (a) Recurrent states are $1, 2, 4, 5$ and 6.

Transient state is 3.

There are two recurrent classes $A(1) = A(2) = \{1, 2\}$ and $A(4) = A(5) = A(6)\{4, 5, 6\}$. Only the second recurrent class is periodic.

(b) The required probability $= \underbrace{0.2 \times 0.2 \times \cdots \times 0.2}_{n \text{ times}} = (0.2)^n$.

(c) Process stays in state 3 with probability 0.2

Process leaves state 3 with probability 0.8.

Trials up to and including the trial on which process leaves state 3 is a geometric random variable with parameter 0.8.

Hence expected number of trials $= \dfrac{1}{0.8} = \dfrac{5}{4}$.

(d) If the process goes from state 3 to state 4, then it never enters state 1. If the process goes from state 3 to 2, then it will reac state 1 with probability 1.

Hence, the probability of never entering state 1 is $= \dfrac{3}{8}$.

(e) Desired probability

$= p_{34} \cdot (p_{45}p_{56}p_{64})^3 + (p_{33})^3 p_{34} \cdot (p_{45}p_{56}p_{64})^2 + (p_{33})^6 p_{34} \cdot (p_{45}p_{56}p_{64}) + (p_{33})^9 \cdot p_{34}$

$= 0.3 + (0.2)^3 \times 0.3 + (0.2)^6 \times 0.3 + (0.2)^9 \times 0.3 = 0.3024$

The required probability may also be obtained as the $(3,4)^{th}$ element of P^{10}.

(f) $P($ process was in state 4 after the first trial | process is in state 4 after 10 trials$)$

$$= \frac{P(\text{in state 4 after 10 trials} \mid \text{in state 4 after first trial})P(\text{in state 4 after first trial})}{P(\text{in state 4 after 10 trials})}$$

$$= \frac{(p_{45}p_{56}p_{64})^3 \cdot p_{34}}{0.3024} = \frac{1 \times 0.3}{0.3024} = 0.992$$

5. (a) Transition probability matrix is given by

$$P = R(1) = \begin{bmatrix} .4 & .6 & 0 \\ .3 & .5 & .2 \\ 0 & .2 & .8 \end{bmatrix}$$

The steady-state probabilities π_j satisfy

$\pi_j = \sum_{k=1}^{m} \pi_k p_{kj}, \quad j = 1, 2, 3$

From this we obtain, $\pi_1 = .2, \pi_2 = .4, \pi_3 = .4$ Alternatively for large n, (say $n = 100$)

$$P^{100} = \begin{bmatrix} .2 & .4 & 4 \\ .2 & .4 & .4 \\ 2 & .4 & .4 \end{bmatrix}$$

, by using Python or Matlab.

which imply $r_{11}(100) = r_{21}(100) = r_{31}(100) = .2$

$r_{12}(100) = r_{22}(100) = r_{32}(100) = .4$

$r_{31}(100) = r_{32}(100) = r_{33}(100) = .4$

that is the nth step probabilities does not depends on initial state. In other words the probability of going to state 1 from state 1, state 2, and state 3 is same and is equal to .2. Probability is .4 for the system to be in state 2 after reaching steady state.

(b) Desired probability $= P(\text{transition from state 1 to 2}|\text{ chain is in state 1})\ P(\text{in state 1}) + P(\text{transition from state 2 to 3}|\text{ chain is in state 2}) P(\text{in state 2})$
$= .6 \times .2 + .2 \times .4 = .2$

(c) $P(\text{first change of state is a birth })$
$= P(\text{first change of state is a birth}|\text{ chain is in state 1}) P(\text{chain in state 1}) + P(\text{first change of state is a birth}|\text{ chain is in state 2}) P(\text{chain in state 2}) + P(\text{first change of state is a birth}|\text{ chain is in state 3}) P(\text{chain in state 3})$
$= 1 \times .2 + \frac{.2}{.2+.3} \times .4 + 0 \times .4 = .36$

(d) $P(\text{Process was in state 2} \mid \text{first transition is a birth})$

$$= \frac{P(\text{Process was in state 2 and first transition is a birth})}{P(\text{first transition is a birth})}$$

$$= \frac{P(\text{first transition is a birth — Process was in state 2})}{P(\text{first transition is birth})} \times P(\text{Process was in state 2})$$

$= \frac{.2 \times .4}{.2}$ (using 5(b))
$= .4$

(e) Similar to part $5(d)$,
$P(\text{Process was in state 2} \mid \text{first change of state is birth})$

$$= \frac{P(\text{first change of state is birth — process was in state 2})}{P(\text{First change of state is birth})} \times P(\text{Process was in state 2})$$

$= \frac{.2/(.2+.3) \times .4}{.36} = \frac{.16}{.36} = \frac{4}{9}$.

(f) $P(\text{first observed transition is a birth} \mid \text{first observed transition resulted in change of state})$

$$= \frac{P(\text{first observed transition is a birth and it resulted in change of state})}{P(\text{first observed transition resulted in change of state})}$$

$$= \frac{p_{12}\pi_1 + p_{23}\pi_3}{p_{12}\pi_1 + (p_{23} + p_{21})\pi_2 + p_{32}\pi_3}$$

$$= \frac{.6 \times .2 + .2 \times .4}{.6 \times .2 + .5 \times .4 + .2 \times .4} = 0.5$$

(g) $P(\text{first transition leads to state } 2/mid \text{ first observed transition resulted in change of state})$

$$= \frac{P(\text{first transition leads to state 2 and it also resulted in change of state})}{P(\text{first observed transition resulted in change of state})}$$

$$= \frac{p_{12}\pi_1 + p_{32}\pi_3}{p_{12}\pi_1 + (p_{23} + p_{21})\pi_2 + p_{32}\pi_3}$$

$$= \frac{.6 \times .2 + .2 \times .4}{.6 \times .2 + .5 \times .4 + .2 \times .4} = 0.5$$