

Department of Mathematics, IIT Madras
MA 2040 : Probability, Statistics and Stochastic Processes
Additional Problem Set - II

1. An internet service provider uses 50 modems to serve the needs of 1000 customers. It is estimated that at a given time, each customer will need a connection with probability 0.01, independent of the other customers.
 - (a) What is the PMF of the number of modems in use at the given time?
need a connection with a Poisson PMF.
 - (b) What is the probability that there are more customers needing a connection than there are modems? Provide an exact formula.
2. You go to a party with 500 guests (including yourself). What is the probability that exactly one other guest has the same birthday as you? (For simplicity, exclude birthdays on February 29).
3. You just rented a large house and the realtor gave you 5 keys, one for each of the 5 doors of the house. Unfortunately, all keys look identical, so to open the front door, you try them at random.
 - (a) Find the PMF of the number of trials you will need to open the door, under the following alternative assumptions:
 - (i) after an unsuccessful trial, you mark the corresponding key, so that you never try it again, and
 - (ii) at each trial you are equally likely to choose any key.
 - (b) Repeat part (a) for the case where the realtor gave you an extra duplicate key for each of the 5 doors.
4.
 - (a) A family has 5 natural children and has adopted 2 girls. Each natural child has equal probability of being a girl or a boy, independent of the other children. Find the PMF of the number of girls out of the 7 children.
 - (b) Let K be a random variable that takes, with equal probability $1/(2n+1)$, the integer values in the interval $[-n, n]$. Find the PMF of the random variable $Y = \ln X$, where $X = a^{|K|}$, and a is a positive number.
5. (St. Petersburg paradox). You toss independently a fair coin and you count the number of tosses until the first tail appears. If this number is n , you receive 2^n dollars. What is the expected amount that you will receive? How much would you be willing to pay to play this game?
6. A stock market trader buys 100 shares of stock A and 200 shares of stock B. Let X and Y be the price changes of A and B, respectively, over a certain time period. and assume that the joint PMF of X and Y is uniform over the set of integers x and y satisfying

$$-2 \leq x \leq 4, \quad -1 \leq y - x \leq 1.$$

- (a) Find the marginal PMFs and the means of X and Y .
 - (b) Find the mean of the trader's profit.
7. A class of n students takes a test consisting of m questions. Suppose that student i submitted answers to the first m_i questions.
- (a) The grader randomly picks one answer, call it (I, J) , where I is the student ID number (taking values $1, \dots, n$) and J is the question number (taking values $1, \dots, m$). Assume that all answers are equally likely to be picked. Calculate the joint and the marginal PMFs of I and J .
 - (b) Assume that an answer to question j , if submitted by student i , is correct with probability p_{ij} . Each answer gets a points if it is correct and gets b points otherwise. Calculate the expected value of the score of student i .
8. Consider four independent rolls of a 6-sided die. Let X be the number of 1s and let Y be the number of 2s obtained. What is the joint PMF of X and Y ?
9. A coin that has probability of heads equal to p is tossed successively and independently until a head comes twice in a row or a tail comes twice in a row. Find the expected value of the number of tosses.
10. Suppose that X and Y are independent, identically distributed, geometric random variables with parameter p . Show that

$$P(X = i | X + Y = n) = \frac{1}{n-1}, \quad i = 1, \dots, n-1.$$