Solutions to Problems Sheet 3.

1. Range of $X = \{0.2, 0.4, 0.5, 0.8, 1\}$

i) $P(0.25 \leq X \leq 0.75) = P(X \in \{0.4, 0.5\})$

$$= p_X(0.4) + p_X(0.5) = .2 + .2 = .4$$

ii) $$P(X = .2 \mid X < .6) = \frac{P((X = .2) \text{ and } (X < .6))}{P(X < .6)}$$

$$= \frac{P(X = .2)}{P(X < .6)} = \frac{0.1}{.1 + .2 + .2} = 0.2$$

2) Range of $X = \{0, 1, 2\}$.

$p_X(0) = 1/4 = p_X(2)$ and $p_X(1) = 1/2$.

$F_X(x) = P(X \leq x)$.

$$\therefore F_X(x) = \begin{cases} 0 & \text{if } x < 0. \\ 1/4 & \text{if } 0 \leq x < 1 \\ 3/4 & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2. \end{cases}$$

3) Given $p_X(x) = 1/2^x \quad , x = 1, 2, \cdots$

We can compute this P(X > 4) using CDF.

(ie) $P(X > 4) = 1 - P(X \leq 4) = 1 - F_X(4)$.

$$= 1 - \left[\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}\right] = 1/16$$

4) Given $X \sim GEO(p)$

$$E[X] = \sum_{x=1}^{\infty} x\, p_X(x) = \sum_{x=1}^{\infty} x(1-p)^{x-1} \cdot p$$

$$= p \sum_{x=1}^{\infty} x(1-p)^{x-1}$$

Note, $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, $|x| < 1$.

Differentiating,

$$\sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2} \quad , \quad |x| < 1.$$

$$\therefore E[X] = p \cdot \frac{1}{[1-(1-p)]^2} = p \cdot \frac{1}{p^2} = \frac{1}{p}.$$

—— x ——

5) $Y = X(X-1)(X-2)$,

Range of $Y = \{x(x-1)(x-2) : x = 0, 1, 2, 3\}$
$= \{0, 6\}$.

$p_Y(0) = p_X(0) + p_X(1) + p_X(2). = 0.7$

$p_Y(6) = p_X(3) = 0.3.$

$$\therefore p_Y(y) = \begin{cases} 0.7 & \text{if } y = 0 \\ 0.3 & \text{if } y = 6 \\ 0 & \text{otherwise.} \end{cases}$$

6) $Y = -2X + 3$.

$E[Y] = -2E[X] + 3 \Rightarrow E[X] = 1$

$Var(Y) = 4\, Var(X)$ But $Var(Y) = E[Y^2] - (E[Y])^2$
$= 9 - 1 = 8.$

$\Rightarrow Var(X) = 2.$

7. $\sum_{y=1}^{3} \sum_{x=1}^{3} k\,xy = 1 \Rightarrow k = 1/36.$

8. The marginal pmf of X is $\sum_{y=1}^{3} \frac{1}{21}(x+y)$

which equals $\frac{x+2}{7}$, $x = 1, 2.$

$|||^{ly}$ the marginal PMF of Y is $\frac{3+2y}{21}$, $y = 1, 2, 3.$

9. $p_{X|A}(1) = P(X=1 \mid X<5)$

$= \frac{P(X=1)}{P(X<5)} = \frac{1/6}{4/6} = 1/4.$

Given $A = \{X<5\}$
$P(A) = 5/6.$

$p_{X|A}(2) = p_{X|A}(3) = p_{X|A}(4) = 1/4$

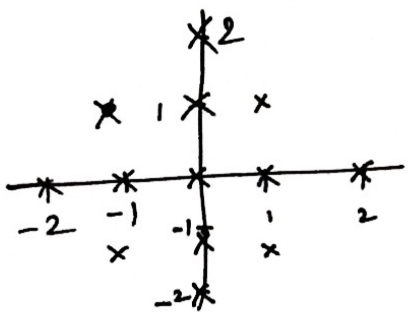
$p_{X|A}(5) = 0.$

10. i) Joint pmf is $p_{X,Y}(x,y) = \begin{cases} 1/13 & \text{if } (x,y) \in G \\ 0 & \text{otherwise.} \end{cases}$

where $G = \{(x,y) : x, y \in \mathbb{Z} \;\&\; |x|+|y| \le 2\}$

ii) $p_X(x) = \sum_y p_{X,Y}(x,y)$

$p_X(-2) = p_{X,Y}(-2,0) = 1/13;$



$|||^{ly}$ $p_X(-1) = 3/13;\; p_X(0) = 5/13$

$p_X(1) = 3/13;\; p_X(2) = 1/13.$

$p_{X|Y}(x|1) = \frac{p_{X,Y}(x,1)}{p_Y(1)} = \frac{1/13}{3/13} = 1/3;\; x = -1, 0, 1.$

iii) X & Y are NOT indep. because, $p_{X|Y}(x|y) \neq p_X(x).$