Solution Problem Sheet 11

November 20, 2020

1. A radar system uses radio waves to detect aircraft. The system receives a signal and based on the signal it needs to decide whether an aircraft is present or not. Suppose that X is the received signal, where $X = \theta + W$, where $\theta = 0$ if no aircraft is present, and $\theta = 1$ if an aircraft is present, here $W \sim Normal(0, \frac{1}{16})$. Suppose we define

$$H_0 : \text{ No aircraft is present.}$$

and

$$H_1 : \text{ An aircraft is present.}$$

(a) Write null hypothesis and alternate hypothesis in terms of possible values of θ.
Solution:

$$H_0 : \theta = 0, \;\; H_1 : \theta = 1.$$

(b) Design a level $\alpha = 0.05$ test to decide between H_0 and H_1. Find the probability of Type II error β for this test.
Solution: Note that under H_0, the observed data $X \sim Normal(0, \frac{1}{16})$ and under H_1, $X \sim Normal(1, \frac{1}{16})$. We suggest the following test:
We choose a cutoff c. If the observed value of X is greater than c, we reject H_0 and if the observed value of X is less than c, we fail to reject H_0.
To choose c, we use the significance level α of the test.

$$\alpha = P(\text{Type I error}) = P(\text{Reject } H_0 \mid H_0) = P(X > c \mid H_0) = P(W > c) = 1 - \Phi(4c).$$

Taking $\alpha = 0.05$, we see $c = \frac{1}{4} z_{0.95} = 0.412$.
Note

$$\beta = P(\text{ Type II error}) = P(\text{accept } H_0 \mid H_1) = P(X < c \mid H_0) = P(W < c-1) = \Phi(4(c-1)).$$

Using $c = 0.412$, we see $\beta = 0.0094$.

(c) If we observe $X = 0.5$, is there enough evidence to reject H_0 at significance level $\alpha = 0.05$? What about at significance level $\alpha = 0.01$?

Solution: Since $0.5 > 0.412$, we reject H_0 at significance level of $\alpha = 0.05$. Following the same procedure, we see that to reject it at significance level of $\alpha = 0.01$, we should have $c = \frac{1}{4}z_{0.99} = 0.582$. Since $0.5 < 0.582$, we fail to reject H_0 at significance level $\alpha = 0.01$.

(d) What should be the smallest possible value of α, which ensures $\beta \leq 0.05$.
Solution: Note $\beta = \Phi(4(c-1))$, hence $\beta = 0.05$, implies $c = 1 + \frac{1}{4}z_{0.05} = 1 - \frac{1}{4}z_{0.95} = 0.588$. Thus in that case Type I error is $\alpha = 1 - \Phi(4c) = 0.0094$.

2. You know batteries from company A lasts for expected time 1000 hours with standard deviation 30 hours. Now a new company B claims that their batteries lasts more than that of company A while maintaining the same standard deviation. To decide whether to buy from the of company B or not, you ask 10 customers who have purchased the product of company B to find out that their batteries last 1020 hours on average. Assume that times are normally distributed.

(a) You decide to do hypothesis testing at significance level 5%. Based on this result will you decide to purchase from company A or B?
Solution: This is a one sided hypothesis testing with

$$H_0 : \mu = 1000 \text{ vs } \quad H_1 : \mu > 1000.$$

We know $\overline{X} = 1020$ for sample size $n = 10$. We assume that $\overline{X}$ is normal distribution with standard deviaion 30.
The critical value c for test at significance level α should satisty

$$P(\overline{X} > c) = \alpha \implies P\Big(\frac{\overline{X} - 1000}{\frac{30}{\sqrt{10}}} > \frac{c - 1000}{\frac{30}{\sqrt{10}}}\Big) = \alpha.$$

Thus, $c = 1000 + \frac{30}{\sqrt{10}}z_{1-\alpha}$. Hence for $\alpha = 0.05$, we see that $c = 1015.65$, since $\overline{X} = 1020 > c = 1015.65$, you should purchase product of company B.

(b) You decide to do hypothesis testing at significance level 1%. Based on this result will you decide to purchase from company A or B?
Solution: In this case $c = 1000 + \frac{30}{\sqrt{10}}z_{0.99} = 1022.10$, since in this case $\overline{X} = 1020 < c = 1022.10$, you should purchase product of company A.

(c) Suppose that the company B claims that their batteries lasts 1040 hours with standard deviation 30 hours. If you decide to do hypothesis testing at significance level 1% and Type II error at 5%, how many customers shall you ask and what should be the critical value for accepting/rejecting the hypothesis?
Solution:

$$H_0 : \mu = 1000 \text{ vs } \quad H_1 : \mu = 1040.$$

We need to find the critical value c and number of required samples n so that we have

$$P(\overline{X} > c \mid H_0 \text{ true }) = 0.01, \quad P(\overline{X} < c \mid H_1 \text{ true }) = 0.05.$$

In other words,

$$P\Big(\frac{\overline{X} - 1000}{\frac{30}{\sqrt{n}}} > \frac{c - 1000}{\frac{30}{\sqrt{n}}}\Big) = 0.01, \quad P\Big(\frac{\overline{X} - 1040}{\frac{30}{\sqrt{n}}} < \frac{c - 1040}{\frac{30}{\sqrt{n}}}\Big) = 0.05.$$

Equivalently,

$$\frac{c - 1000}{\frac{30}{\sqrt{n}}} = 2.33, \quad \frac{c - 1040}{\frac{30}{\sqrt{n}}} = -1.65.$$

Solving for n, we see $\sqrt{n} = 2.985$, hence $n \approx 8.91$, hence $n = 9$ and $c = 1023.42$.

3. You know that robotic house cleaner from company A cleans a 1000 square-feet apartment in 30 minutes with a standard deviation of 5 minutes. You want to buy a new robotic house cleaner after your old robotic house cleaner of company A has broken down. In the store, you found out that a new company B claims that their robotic house cleaner cleans a 1000 square-feet apartment in less than 30 minutes with a standard deviation of 5 minutes. You sample 10 models of company B and found that on average their robotic house cleaner cleans a 1000 square-feet apartment in 28 minutes. You are willing to take risk and decided to do hypothesis testing at a high significance value 10% to decide. Based on this test, should you stick to company A or switch to company B ? Assume that times are normally distributed.

   Solution: This is a one sided hypothesis testing with

   $$H_0 : \mu = 30 \text{ vs } \quad H_1 : \mu < 30.$$

   We know $\overline{X} = 28$ for sample size $n = 10$. We assume that $\overline{X}$ is normal distribution with standard deviaion 5.
   The critical value c for test at significance level 0.1 should satisty

   $$P(\overline{X} < c) = 0.1 \implies P\Big(\frac{\overline{X} - 30}{\frac{5}{\sqrt{10}}} < \frac{c - 30}{\frac{5}{\sqrt{10}}}\Big) = 0.1.$$

   Thus, $\frac{c-30}{\frac{5}{\sqrt{10}}} = -1.29$, simplifying we get $c = 27.96$, since the average $\overline{X} = 28 > c$, you should stick to company A.

4. A rice manufacturer makes rice packets of expected weight 1 kg and standard deviation 10 grams. Samples are drawn hourly and checked. If the production level gets

out of sync with a statistical significance more than 5%, then the process is stopped and fixed. A sample of 20 rice packets have mean 992 grams. Should the process be stopped for adjustment?

Solution: This is a two sided hypothesis testing with

$$H_0 : \mu = 1000 \ \text{ vs } \quad H_1 : \mu \neq 1000.$$

Test statistics is $Z = \frac{\overline{X}-1000}{\frac{10}{\sqrt{20}}}$.
Thus for the test at significance level 5%, we should have c such that

$$P(|Z| < c) = 0.05 \implies c = 1.96.$$

Equivalently, $995.62 < \overline{X} < 1004.38$. Since $\overline{X}$ is outside this region, we should stop the process for adjustment.