Solutions to Problem Sheet 4

1. Given X has the pdf $f_X(x) = \begin{cases} h\,x/2 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$

$\int_0^2 f_X\,dx = 1 \Rightarrow h = 1.$

$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2/4 & \text{if } 0 \le x \le 2 \\ 1 & \text{if } x \ge 2. \end{cases}$ $\quad E[X] = \int_0^2 x f_X(x)\,dx = 4/3$

$E[X^2] = 2.$

$\Rightarrow \text{var}(X) = 2/9.$

2. Given $X \sim UNIF(0,1)$ and $Y = g(X)$ where

$g(x) = \begin{cases} 1 & x \le 1/3 \\ 2 & x > 1/3. \end{cases}$

$P(Y=1) = P(X \le 1/3) = \int_0^{1/3} dx = 1/3$

$P(Y=2) = P(X > 1/3) = 2/3.$

$\therefore E[Y] = \frac{1}{3} + \frac{4}{3} = 5/3.$

3. Given $F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{x+1}{2} & -1 \le x < 1 \\ 1 & x \ge 1 \end{cases}$

$\therefore f_X(x) = \begin{cases} 1/2 & -1 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$

$E[X] = 0$ & $E[X^2] = 1/3. \quad \therefore \text{Var}(X) = 1/3$

4. (a) & (c) from table.
(b) follows because when $X \sim N(\mu, \sigma^2)$, $Y = aX + b \sim N(a\mu + b, a^2\sigma^2)$

5. 0.3601, 1.5498, 2.0198, −0.5299 and 0.7499.

6. Given $X \sim N(10; 2)$

$P(X \leq 10) = 0.5$ & $P(8 \leq X \leq 14) = 0.8186$.

7. $\mu = np = 120$, & $\sigma = \sqrt{np(1-p)} = 6.93$

$$P(X > 100) = P(X > 100.5)$$
$$= P\left(\frac{X-\mu}{\sigma} > \frac{100.5-120}{6.93}\right)$$
$$= P(Z > -2.81) = 0.9975$$

8. Given $X \sim EXP(2)$. & $Y = 2 + 3X$. Note, $E[X] = \frac{1}{2}$ & $Var(X) = 1/4$

$$P(X > 2) = 1 - P(X \leq 2) = 1 - F_X(2)$$
$$= 1 - (1 - e^{-4}) = e^{-4}$$

$$E[Y] = 2 + 3 \times \frac{1}{2} = 7/2.$$
$$Var(Y) = 9\,Var(X) = 9/4.$$
$$P(X > 2 \mid Y < 11) = P(X > 2 \mid 2 + 3X < 11)$$
$$= P(X > 2 \mid X < 3)$$
$$= \frac{P(2 < X < 3)}{P(X < 3)} = \frac{e^{-4} - e^{-6}}{1 - e^{-6}}$$

9. A number X is chosen at random on $[2, 6]$.

Note, $P(X \in [x_1, x_2]) \propto x_2 - x_1$ and

$P(X \in [2, 6]) = 1$.

$$\therefore P(X \in [x_1, x_2]) = \begin{cases} \frac{x_2 - x_1}{6 - 2} & \text{if } 2 < x_1 < x_2 < 6 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore F_X(x) = \begin{cases} \frac{x-2}{4} & \text{if } 2 \leq x \leq 6 \\ 0 & \text{if } x < 2 \\ 1 & \text{if } x > 6. \end{cases} \Rightarrow f_X(x) = \begin{cases} 1/4 & \text{if } 2 \leq x \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

(ie). $X \sim UNIF(2, 6)$.

$$\therefore E[X] = \frac{6+2}{2} = 4.$$