Solutions to Problem Sheet - 6.

1. Given $X \sim EXP(2)$. $f_X(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0 & x < 0 \end{cases}$.

Let $Y = \sqrt{X}$

$F_Y(y) = P(\sqrt{X} \leq y) = P(X \leq y^2) = \int_0^{y^2} 2e^{-2x}\,dx$

$= -e^{-2y^2} + 1$

$\therefore f_Y(y) = F_Y'(y) = \begin{cases} 4y e^{-2y^2}, & y > 0 \\ 0 & \text{otherwise}. \end{cases}$

2. Given $X \sim UNIF(0,1)$.

$\therefore f_X(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

Let $Y = \frac{1}{X}$.

$F_Y(y) = P(Y \leq y) = P(\frac{1}{X} \leq y) = P(X \geq \frac{1}{y}) = 1 - P(X < 1/y)$

$= 1 - \int_0^{1/y} dx = 1 - \frac{1}{y}$.

$\therefore f_Y(y) = \begin{cases} 1/y^2, & \text{if } y > 1 \\ 0 & \text{otherwise}. \end{cases}$

3. Given $X, Y \sim UNIF(0,1)$. & X & Y are indep.

Let $Z = X + Y$.

$\therefore f_Z(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\,dx$.

The integrand is nonzero (equal to 1) when $0 \leq x \leq 1$ & $0 \leq z - x \leq 1$ (ie) when $0 \leq x \leq 1$ & $z - 1 \leq x \leq z$. (ie) when

$\max\{0, z-1\} \leq x \leq \min\{1, z\}$

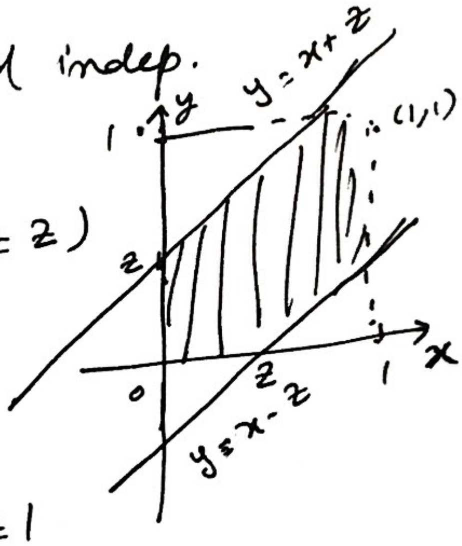
$$\therefore\ f_Z(z) = \int_{\max\{0,z-1\}}^{\min\{1,z\}} dx = \min\{1,z\} - \max\{0,z-1\}.$$

where $0 \le z \le 2$.

$f_Z(z) = 0$ ~~at~~ $x \notin [0,2]$

4) Given $X, Y \sim UNIF(0,1)$ and indep.

Let $Z = |X-Y|$.



$F_Z(z) = P(Z \le z) = P(|X-Y| \le z)$

$= $ Shaded area

$= 1 - (1-z)^2$

$$\therefore\ f_Z(z) = \begin{cases} 2(1-z) & , 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases}$$

5) Given $X, Y \sim EXP(1)$ & independent.

Let $Z = X+Y$.

$$f_X(x) = \begin{cases} e^{-x}, & x \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

~~$F_Z(z) = P(Z \le z) = P(X+Y \le z)$~~

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx.$$

$$= \int_0^z e^{-x} \cdot e^{-(z-x)} dx$$, since $f_Y(z-x) \neq 0$ only when $z - x \ge 0$ (ie) $x \le z$.

$$= e^{-z} \int_0^z dx = z e^{-z}.$$

6) Given $X \sim EXP(3)$ & $Y \sim EXP(4)$ ~~and~~ independent.

Let ~~$Z = X+Y$~~. $Z = \min\{X, Y\}$.

$$F_Z(z) = P(\min\{X,Y\} \leq z)$$
$$= 1 - P(\min\{X,Y\} > z)$$
$$= 1 - P(X > z, Y > z)$$
$$= 1 - P(X > z) \cdot P(Y > z) \quad (\because X \,\&\, Y \text{ are indep})$$
$$= 1 - e^{-3z} e^{-4z}$$
$$= 1 - e^{-7z}$$

$$\therefore f_Z(z) = \begin{cases} 7e^{-7z} & , z \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(ie) $Z \sim EXP(7)$.

7. $$\mathrm{Cov}(X, 1+2X+3X^2) = E[X+2X^2+3X^3] - E[X]E[1+2X+3X^2].$$

$$= E[X] + 2E[X^2] + 3E[X^3] - [E[X](1+2E[X]+3E[X^2])]$$

$$= 2$$

$\rho(X,Y) = 2$, since $Var(X) = 1 = Var(Y)$

8. Let $Z = 2X - Y$ & $W = X+Y$.

~~Since X & Y are indep~~ Given Z & W are indep.

$\therefore \mathrm{Cov}(Z, W) = 0$

(ie) $0 = \mathrm{Cov}(2X - Y, X+Y) \Rightarrow \mathrm{Cov}(X,Y) = 1$

$$\therefore \rho(X,Y) = \frac{1}{2 \times 3} = 1/6.$$

9. Let $X, Y \sim N(0;1)$ & indep.

$Z = 7 + X + Y$ & $W = 1 + Y$.

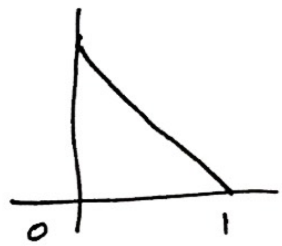
$$\begin{aligned}\text{Cov}(Z, W) &= \text{Cov}(7 + X + Y, 1 + Y) \\ &= \text{Cov}(X + Y, Y) \\ &= \text{Cov}(X, Y) + \text{Var}(Y) = \text{Var}(Y) = 1.\end{aligned}$$

$\text{Var}(Z) = \text{Var}(X + Y) = 2$

$\text{Var}(W) = \text{Var}(Y) = 1.$

$\therefore \rho(X, Y) = 1/\sqrt{2}$

10. $f_{X,Y}(x,y) = \begin{cases} 2 & y + x \leq 1,\ x, y > 0 \\ 0 & \text{otherwise.} \end{cases}$



$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^{1-x} 2\,dy = 2(1-x).$

$\therefore f_X(x) = \begin{cases} 2(1-x), & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

Illy $f_Y(y) = \begin{cases} 2(1-y), & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$

$E[X] = \int_0^1 2x(1-x)\,dx = 1/3 = E[Y].$

Illy $E[X^2] = 1/6 = E[Y^2].$ $E[XY] = \int_0^1 \int_0^{1-x} 2xy\,dx\,dy = 1/12$

$\therefore \text{Cov}(X, Y) = 1/12 - \left(\frac{1}{3}\right)^2 = -1/36$

Hence $\rho(X, Y) = \dfrac{-1/36}{1/18} = -1/2$