Solutions to Problem Sheet - 1

1. $1 = P(S) = P(\{1\}) + P(\{2\}) + \cdots + P(\{6\})$.

where $S = \{1, 2, \ldots 6\}$.

Let p be the prob. of each odd face and $2p$ be the prob. of each even face.

$\therefore \quad 9p = 1 \Rightarrow p = 1/9$.

$P(\{\text{outcome} < 4\}) = P(\{1\}) + P(\{2\}) + P(\{3\}) + \cancel{P(\{4\})}$

$= p + 2p + p = 4p = 4/9$.

2. Since $\Omega = \bigcup_{i=1}^{n} S_i$ and $S_i \cap S_j = \phi$, $(i \neq j)$,

we have $A = A \cap \Omega = A \cap \left(\bigcup_{i=1}^{n} S_i\right)$

$= \bigcup_{i=1}^{n} (A \cap S_i)$

Note $(A \cap S_i) \cap (A \cap S_j) = \phi$. $(i \neq j)$

$\therefore P(A) = P\left(\bigcup_{i=1}^{n} (A \cap S_i)\right) = \sum_{i=1}^{n} P(A \cap S_i)$

Take $S_1 = B \cup C$, $S_2 = (B \cup C)^c = B^c \cap C^c$.

Now $P(A) = P(A \cap S_1) + P(A \cap S_2)$

$$= P(A \cup (B \cup C)) + P(A \cap (B^c \cap C^c))$$

$$= P((A \cap B) \cup (A \cap C)) + P(A \cap B^c \cap C^c)$$

$$= P((A \cap B)) + P((A \cap C)) - P((A \cap B) \cap (A \cap C)) + P(A \cap B^c \cap C^c).$$

$$= P((A \cap B)) + P((A \cap C)) + P(A \cap B^c \cap C^c) - P(A \cap B \cap C).$$

—— x ——

3) $P(A \cup B) = P(A) + P(B) - P(A \cap B).$

$P(A) + P(B) - P(A \cap B) \leq 1$

4) Let $A = \{$ both dice show same number $\}$

$B = \{$ roll results in a sum not exceeding 4 $\}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = ?$$

$A = \{(1,1), (2,2), \ldots (6,6)\}$

$B = \{(1,1), (1,2), (2,1), (2,2)\}$

$A \cap B = \{(1,1), (2,2)\}$

$$\therefore P(A|B) = \frac{2/36}{4/36} = \frac{1}{2}.$$

Let $C = \{\text{atleast one die is } 6\}$

$D = \{\text{two dice land on different numbers}\}$

$$C = \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (6,5), (6,4), (6,3), (6,2), (6,1)\}.$$

~~$D = S \setminus \{ \quad S \setminus C$~~

$$D = S \setminus \{(1,1), (2,2), \ldots, (6,6)\}$$

$$\therefore P(C|D) = \frac{P(C \cap D)}{P(D)} = \frac{10/36}{30/36} = 1/3.$$

—— × ——

5) $P(\{\text{Batch being accepted}\}) = P(\{\text{all 4 are nondefective}\})$

$$= p_1 p_2 p_3 p_4$$

Where $p_i = P(\{i^{th}\text{ item is nondefective given that previous } (i-1) \text{ items are defective}\})$

$$p_1 = \frac{95}{100}; \quad p_2 = \frac{94}{99}, \quad p_3 = \frac{93}{98}, \quad p_4 = \frac{92}{97}$$

Thus the desired prob. is $\dfrac{95 \times 94 \times 93 \times 92}{100 \times 99 \times 98 \times 97}$.

—— × ——

6) $P(X \text{ winning}) = \sum_{n=1}^{\infty} P(X \text{ winning on } n^{th} \text{ attempt})$

$= \sum_{n=1}^{\infty} p_n$ where $p_n = P($x doesn't get a sum of 6 and y doesn't get a sum of 7 in the previous $(n-1)$ attempts.)

$= p^{n-1} q^{n-1} (1-p)$ where

$p = P($ not getting a sum of 6 $) = 1 - 5/36 = \frac{31}{36}$

$q = P($ not getting a sum of 7 $) = 1 - \frac{6}{36} = \frac{5}{6}$.

$$\text{Reqd. prob} = \sum_{n=1}^{\infty} \left(\frac{31}{36}\right)^{n-1} \left(\frac{5}{6}\right)^{n-1} \times \frac{5}{36}$$

$$= \frac{5}{36} \times \frac{1}{1 - \frac{155}{216}} = \frac{30}{61}$$

———X———

7) $P(A) = \frac{13}{52}$; $P(B) = \frac{4}{52}$

$P(A \cap B) = 1/52$; $P(A|B) = 1/4$.

$P(B|A) = 1/13$.

Since $P(A|B) = P(A)$, A and B are indep.

———X———

8. P(odd prime occuring in a Single roll) $= 1/3$.

P(no odd prime in the 1st 5 rolls) =
$(2/3)^5$

9. Prob(failing to accommodate everyone)
$= P($51 or 52 individuals appear$)$
$= 52\,(0.95)^{51}(0.05) + (0.95)^{52}$.

Reqd. prob. = 1 − P(failing to accom. everyone)

10) (i) Sum must be 1 . $\therefore p = 0.05$

(ii) $P(\text{it will be sunny}) = 0.2 + 0.6 = 0.8$

$\therefore$ Prob. (it wont rain for one week) $= (0.8)^7$

(iii) $P(\text{no power cut}) = 0.65$

$P(\text{atleast 1 power cut in the next 3 days}) = 1 - (0.65)^3$

≈ 0.725

(iv) $P(A) = 0.2 + 0.15 = 0.35$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.2 + 0.6} = 0.25$$

(v) $P(A^c \mid B) = 0.75$

$$P(A \mid B^c) = \frac{P(A \cap B^c)}{P(B^c)} = \frac{0.15}{0.2} = 0.75$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{4}{7}$$

$P(B^c / A) = 3/7$

$$P(B \mid A^c) = \frac{P(A^c \cap B)}{P(A^c)} = \frac{12}{13}$$

$$P(B^c \mid A^c) = \frac{P(A^c \cap B^c)}{P(A^c)} = 1/13 .$$