Algebra of Continuous Functions on a Compact set

following "Les maths en tête - Analyse" by X. Gourdon

November 28, 2025

Statement

Theorem (Exercise 8 p. 38, in both 2008 and 2020 edition)

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c) *Determine the algebra morphism from $\mathcal{C}(X, \mathbb{R})$ to $\mathbb{R}$ (i.e. the "characters").*

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so one can make sense of $1/F \in \mathcal{C}(X,\mathbb{R})$. Hence $F \cdot (1/F) = 1$ is in $\mathfrak{I}$, implying $\mathfrak{I} = \mathcal{C}(X,\mathbb{R})$, contradiction. $\square$

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b) Let us now show that the $\mathfrak{I}_s$ are precisely the maximal ideals of $\mathcal{C}(X, \mathbb{R})$.

Let $s \in X$. $\mathfrak{I}_s$ is stable under addition and multiplication by any function of $\mathcal{C}(X, \mathbb{R})$, so it is an ideal. If there exists an ideal $\mathcal{J}$ such that $\mathfrak{I}_s \subseteq \mathcal{J}$, then either $\mathcal{J} = \mathcal{C}(X, \mathbb{R})$ or by point a), there exists $t \in X$ such that $\mathcal{J} \subseteq \mathfrak{I}_t$ so that $\mathfrak{I}_s \subseteq \mathfrak{I}_t$.

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Let now $\mathfrak{I}$ be a maximal ideal. There exists by a), an $s \in X$ such that $\mathfrak{I} \subseteq \mathfrak{I}_s$. Since $\mathfrak{I}$ is maximal and $\mathfrak{I}_s \neq \mathcal{C}(X, \mathbb{R})$, one must have $\mathfrak{I} = \mathfrak{I}_s$.

c) Characters on $\mathcal{C}(X, \mathbb{R})$

Let us for simplicity denote $\mathcal{A} := \mathcal{C}(X, \mathbb{R})$ and show that any non trivial algebra morphism $\varphi : \mathcal{A} \to \mathbb{R}$ is an **evaluation map**, i.e.

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$\mathrm{Ker}\,\varphi$ is maximal so by b), there exists $s \in X$ s.t. $\boxed{\mathrm{Ker}\,\varphi = \mathfrak{I}_s}$. It follows from the first isomorphism theorem, $\mathcal{A}/\mathrm{Ker}\,\varphi \cong \mathbb{R}$, that $\mathrm{Ker}\,\varphi$ is a hyperplane/has co-dimension 1.

Note now that the evaluation map $\mathrm{Eval}_s : \begin{cases} \mathcal{A} \longrightarrow \mathbb{R} \\ f \longmapsto f(s) \end{cases}$ has the same kernel $\mathfrak{I}_s$ but any two linear forms with the same hyperplane kernel are proportional (algebra morphisms are linear in particular), so there exists $\lambda \in \mathbb{R}^*$ such that

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For $f = 1_{\mathcal{A}}$ the constant function, which is the unit in $\mathcal{A} = \mathcal{C}(X, \mathbb{R})$, one obtains

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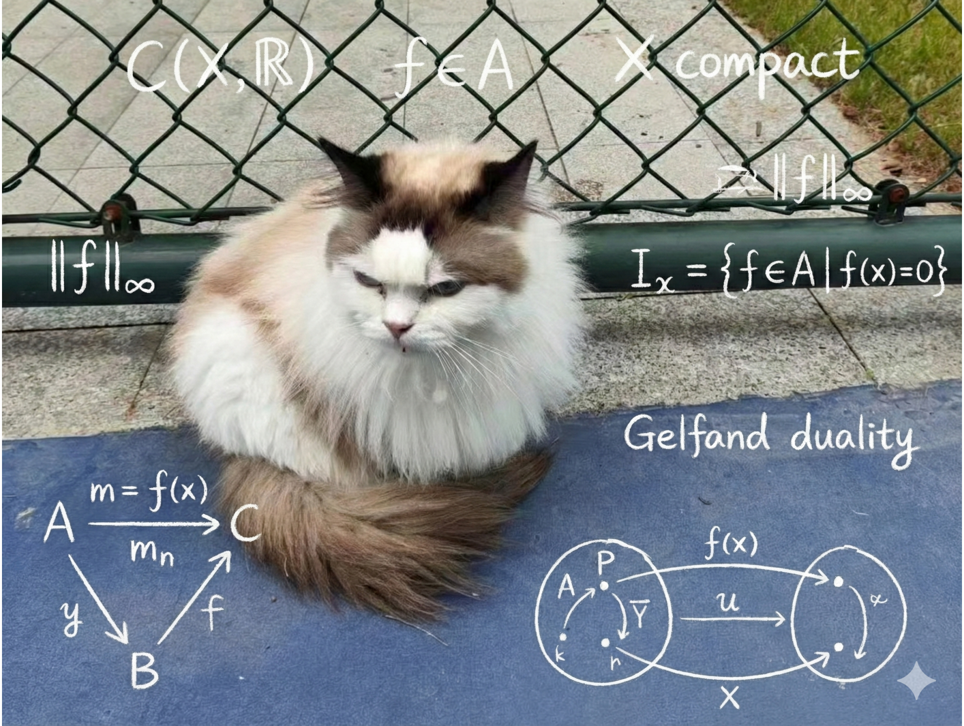
Remark : For X locally compact, the algebra $\mathcal{A} := \mathcal{C}_0(X, \mathbb{R})$ of functions which vanish at infinity has no unit, the constant function 1 is not in the algebra. But one can still write for an $f \in \mathcal{A}$ s.t. $f(s) \neq 0$

$$f^2(s) = \mathrm{Eval}_s\big(f^2\big) = \lambda\, \varphi\big(f^2\big) = \lambda\, \varphi(f)^2 = \lambda \left(\frac{f(s)}{\lambda}\right)^2 = \frac{f^2(s)}{\lambda}$$

One can still conclude that $\lambda = 1$.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