Compactness in Metric Spaces

following “Les maths en tête - Analyse” by X. Gourdon

December 4, 2025

Definition and Theorem

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Proof: $\boxed{(\Rightarrow)}$ Let $\epsilon > 0$. The family $\big(\mathcal{B}(x, \epsilon)\big)_{x \in E}$ of open subsets obviously covers $E = \bigcup\limits_{x \in E} \mathcal{B}(x, \epsilon)$ so one can extract a finite subcover:

$$\exists \ \{x_1, \cdots, x_n\} \subset E \quad \text{such that} \quad E = \bigcup_{i=1}^{n} \mathcal{B}(x_i, \epsilon)$$

Let now $(x_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$ be a Cauchy sequence.

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$$\begin{cases} \forall\ \epsilon_1 > 0,\ \exists\ N_1 \in \mathbb{N},\ \forall\ n \in \mathbb{N}\backslash[\![0, N_1]\!], \quad d\left(x_{\varphi(n)}, x\right) < \epsilon_1 \\ \forall\ \epsilon_2 > 0,\ \exists\ N_2 \in \mathbb{N},\ \forall\ (p,q) \in \left(\mathbb{N}\backslash[\![0, N_2]\!]\right)^2, \quad d\left(x_p, x_q\right) < \epsilon_2 \end{cases}$$

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Let $\epsilon > 0$. By the previous lines, with $\epsilon_1 = \frac{\epsilon}{2} = \epsilon_2$, for all $k, n \in \mathbb{N}$ with $k > N_2$ and $n > N_1$ such that $\varphi(n) > N_2$,

$$d\left(x_k, x\right) \leq d\left(x_k, x_{\varphi(n)}\right) + d\left(x_{\varphi(n)}, x\right) < \epsilon_2 + \epsilon_1 = \epsilon\,.$$

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Since E is precompact, it is covered by a finite set of balls of radius 1, one of which, that we denote $B(a_0, 1)$, must contain an infinite number of terms of $(x_n)_{n\in\mathbb{N}}$. This defines a subsequence $\left(x_{\varphi_0(n)})\right)_{n\in\mathbb{N}}$.

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$B(a_0, 1) \cap E \subset E$ itself is precompact, so it can be covered by a finite number of open balls of radius $\frac{1}{2}$, one of which, that we denote $B\left(a_1, \frac{1}{2}\right)$ contains an infinite number of terms from $\left(x_{\varphi_0(n)}\right)_{n \in \mathbb{N}}$. This defines a subsequence of the subsequence such that

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By induction, one obtains for all $p\in\mathbb{N}$ a subsequence $\left(x_{\varphi_0\circ\cdots\circ\varphi_p(n)}\right)_{n\in\mathbb{N}}$ and an open ball $B\left(a_p,\frac{1}{2^p}\right)$ such that

$$\forall\ n\in\mathbb{N},\quad x_{\varphi_0\circ\cdots\circ\varphi_p(n)}\subset\bigcap_{0\leq k\leq p}B\left(a_k,\frac{1}{2^k}\right)$$

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We have at the moment an infinite family of subsequences, let us now define a single sequence $\left(x_{\psi(n)}\right)_{n\in\mathbb{N}}$ by the so-called Cantor diagonal argument.

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We have at the moment an infinite family of subsequences, let us now define a single sequence $\left(x_{\psi(n)}\right)_{n\in\mathbb{N}}$ by the so-called Cantor diagonal argument. Set $\psi(0) := \varphi_0(0)$ so that $x_{\psi(0)} \in B(a_0,1)$. Choose then $\psi(1) \in \left\{\varphi_0 \circ \varphi_1(n) > \psi(0),\ n \in \mathbb{N}\right\}$. And for $\psi(p)$ given, choose

$$\psi(p+1) \in \left\{\varphi_0 \circ \cdots \cdot \varphi_{p+1}(n),\ n \in \mathbb{N}\right\} \cap \left\{m \in \mathbb{N},\ m > \psi(p)\right\}.$$

This sequence is Cauchy: let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that $\dfrac{1}{2^N} < \epsilon$.

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i.e. $d\left(x_{\psi(q)}, x_{\psi(p)}\right) \leq d\left(x_{\psi(q)}, a_p\right) + d\left(a_p, x_{\psi(p)}\right) < \frac{1}{2^p} + \frac{1}{2^p} \leq \frac{2}{2^{N+1}} < \epsilon.$

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Since E is complete, the Cauchy subsequence converges, so one has obtained a convergent subsequence. $\square$