The Polar Decomposition of Matrices

based on the “Les maths en tête - Algèbre” by Xavier Gourdon

October 23, 2025

Statement of the Polar Decomposition Theorem

Recall that

- a matrix $U \in \mathcal{M}_n(\mathbb{C})$ is **unitary** if $U^* U = I_n$ (finite dimension). Equivalently,

$$\forall\ (X, Y) \in (\mathbb{C}^n)^2, \quad \left\langle UX, UY \right\rangle = \left\langle X, U^* UY \right\rangle = \left\langle X, Y \right\rangle .$$

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Exercise 6 (p.249 in the 2nd Edition, 2009), or (p.261 in the 3rd Edition, 2021)

Let $A \in \mathcal{M}_n(\mathbb{C})$. Show that there exists a pair (U, H) of matrices, where U is **unitary** *and H is* **positive** *(***Hermitian***), such that* $\boxed{A = UH}$.

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If A is invertible, show that such a pair (U,H) is then **unique***.*

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$\boxed{\text{Proof:}}$ **Step 1, the square-root matrix.** if a polar decomposition $A = UH$ exists with U unitary and H self-adjoint, then

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Remark : In the abstract framework of C^*-algebras, one can make sense of a continuous function applied to a normal element, here $N := A^*A$. More precisely, the Gelfand map $\Phi : \begin{cases} \mathcal{C}(\mathrm{Sp}(N)) \longrightarrow \langle N, N^* \rangle \subseteq \mathcal{M}_n(\mathbb{C}) \\ \qquad f \quad \longmapsto \quad f(N) \end{cases}$, where $\mathrm{Sp}(N) := \{\lambda \in \mathbb{C},\ N - \lambda I_n \notin GL_n(\mathbb{C})\}$, is $*$-isomorphism. We'll see that $\mathrm{Sp}(N) \subseteq \mathbb{R}_+$, so one can make sense of $\sqrt{A^*A}$.

Step 1 (continued)

More concretely, A^*A is self-adjoint $\left((A^*A)^* = A^*A^{**} = A^*A \right)$, so it is diagonalizable in an orthonormal basis, i.e. $\exists\ P \in GL_n(\mathbb{C})$ and $D \in \mathcal{M}_n(\mathbb{C})$ diagonal such that

$$A^*A = P \cdot D \cdot P^{-1} = P \cdot \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix} \cdot P^{-1} \tag{2}$$

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$N := A^*A$ diagonalizable means that there exists a basis consisting of eigenvectors. Let $\mathcal{B} = \left(C_1, C_2, \cdots, C_n\right) \in \mathbb{C}^{n \times n}$ be the corresponding column vectors and $\left(\lambda_1, \lambda_2, \cdots, \lambda_n\right) \in \mathbb{C}^n$ such that $\boxed{\forall\ i \in [\![1, n]\!],\ N C_i = \lambda_i C_i}$.

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We interpret them as columns of a matrix P, which is thus such that if $\Xi \in \mathbb{C}^n$ are the coordinates of a vector in the new basis, those in the original one are given by $X = P\Xi$. $\mathcal{B}$ being a free and generating family means that $\Xi \longmapsto P\Xi$ is a bijection, so one can write $P^{-1}X = \Xi$. The boxed equalities can thus be rewritten $\boxed{NP = PD}$ or $\boxed{N = PDP^{-1}}$ or $\boxed{P^{-1}NP = D}$.

Step 1 (end)

"N self-adjoint" implies that it is **semi-simple**, i.e. if $E \subseteq \mathbb{C}^n$ is a stable subspace then it admits a supplementary subspace F. Here it is even better, one can take $F := E^\perp$.

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$$\forall\, X \in E,\, Y \in F, \quad \langle X, NY \rangle = \langle NX, Y \rangle = 0 \quad \text{so} \quad NY \in F.$$

If $\lambda_i = \lambda_j$, one can simply choose an orthonormal basis of the corresponding eigenspace and if $\lambda_i \neq \lambda_j$, one can show that the eigenspaces are orthogonal to each other. So ultimately, P can in fact be chosen unitary.

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So $N = P \cdot D \cdot P^{-1}$, cf. (2). Moreover N is positive: for all $X \in \mathbb{C}^n$,

$$\langle X, A^* A X \rangle = \langle AX, AX \rangle = \|AX\|^2 \geq 0.$$

In particular this is true for $X := C_i$:

$$\forall\, i \in [\![1, n]\!], \quad \langle C_i, A^* A C_i \rangle = \langle C_i, \lambda_i C_i \rangle = \lambda_i \|C_i\|^2 \geq 0$$

Hence all the eigenvalues are positive.

Step 1 (end); Step 2: if A is invertible

Define thus

$$H := P \cdot \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sqrt{\lambda_n} \end{pmatrix} \cdot P^{-1} \tag{3}$$

It is indeed such that $A^*A = H^2$. There is a unique positive such matrix.

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A first argument for the last step is the equivalence

$$\text{“}A^*A - 0I_n \quad \text{invertible iff} \quad 0 \notin \mathrm{Sp}(A^*A)\text{”}$$

Hence for all $i \in [\![1, n]\!]$, $\lambda_i \neq 0$. A second argument is the fact that if $f \circ g$ is bijective, then f is surjective and g is injective. One can also reason with the determinant.

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Step 3, if A is non-invertible. We still have H positive such that $A^* A = H^2$. Furthermore, for all $X \in \mathbb{C}^n$,

$$\|AX\|^2 = \langle AX, AX \rangle = \langle X, A^* AX \rangle = \langle X, H^2 X \rangle = \langle HX, HX \rangle = \|HX\|^2 \tag{5}$$

so $\mathrm{Ker}(A) = \mathrm{Ker}(H)$.

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so $\mathrm{Ker}(A) = \mathrm{Ker}(H)$. Since H is self-adjoint (and even positive), $\mathbb{C}^n$ can be written as the direct sum of eigenspaces or just $\boxed{\mathbb{C}^n = \mathrm{Ker}(H) \oplus \mathrm{Im}(H) = \mathrm{Ker}(H) \oplus \mathrm{Ker}(H)^\perp}$.

Step 3, constructive method

Since we are looking for U s.t. $A = UH$, we must have $\forall X \in \mathbb{C}^n$, $U\big(HX\big) = AX$.

$$U|_{\mathrm{Im}(H)} : \begin{cases} \mathrm{Im}(H) \longrightarrow \mathrm{Im}(A) \\ \quad Y \quad \longmapsto AX \text{ where } X \in \mathbb{C}^n \text{ is s.t. } Y = HX \end{cases} \quad \text{is well-defined.}$$

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A systematic choice of X can be made as follows: if we restrict H to $\mathrm{Ker}(A)^\perp$ (or to any other supplementary space to $\mathrm{Ker}(A)$), then $H|_{\mathrm{Ker}(A)^\perp} : \mathrm{Ker}(A)^\perp \longrightarrow \mathrm{Im}(H)$ is a bijection, and $U|_{\mathrm{Im}(H)} = A\, H|_{\mathrm{Ker}(A)^\perp}^{-1}$.

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Here, one can choose any isometry $J : \mathrm{Ker}(H) \longrightarrow \mathrm{Im}(A)^{\perp}$.These two spaces have the same dimension since $\mathrm{Ker}(H) = \mathrm{Ker}(A)$. Together with the rank theorem, one obtains that $\dim \mathrm{Ker}(H) = \dim \mathrm{Ker}(A) = n - \dim \mathrm{Im}(A) = \dim \big(\mathrm{Im}(A)^{\perp}\big)$.

Step 3, Constructive method, U is unitary

Let us show that

$$U : \begin{cases} \mathbb{C}^n = \mathrm{Ker}(H) \oplus \mathrm{Im}(H) \longrightarrow \mathbb{C}^n = \mathrm{Im}(A)^\perp \oplus \mathrm{Im}(A) \\ \qquad\qquad W + Y \quad \longmapsto \quad JW + A\, H|_{\mathrm{Ker}(A)^\perp}^{-1} \end{cases} \tag{6}$$

satisfies $A = UH$ (obvious) and that it is unitary:

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$$\|UZ\|^2 = \|JW + UHX\|^2 = \|JW + AX\|^2 = \|JW\|^2 + \|AX\|^2$$

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By the polarization identity (the version where the inner product is linear on the right)

$$\langle UX, UY\rangle = \frac{\|U(X+Y)\|^2 - \|U(X-Y)\|^2 - i\,\|U(X+iY)\|^2 + i\,\|U(X-iY)\|^2}{4}$$

$$= \frac{\|X+Y\|^2 - \|X-Y\|^2 - i\,\|X+iY\|^2 + i\,\|X-iY\|^2}{4} = \langle X, Y\rangle$$

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Take for example $A_p := A - \dfrac{\rho}{p+2} I_n,\ p \in \mathbb{N}$. By step 2, $\exists\ (U_p, H_p)$ s.t. $A_p = U_p H_p$.

Step 3, Topological method (continued)

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Moreover, $\mathcal{U}_n(\mathbb{C})$ lies in the unit sphere of $\mathcal{M}_n(\mathbb{C})$ endowed with the **operator norm**:

$$|||M||| := \sup_{\|X\|=1} \|MX\| \quad \text{where} \quad \|X\| := \sqrt{\sum_{i=1}^{n} \overline{x_i} x_i}$$

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Indeed, if U is unitary, then it is an isometry: for all $X \in \mathbb{C}^n$, $\|UX\| = \|X\|$ so the operator norm $|||U||| = 1$. As a closed and bounded subset of a finite dimensional vector space, $\mathcal{U}_n(\mathbb{C})$ is compact.

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By the Bolzano-Weierstraß theorem, the sequence $(U_p)_{p \geq 0} \in \mathcal{U}_n(\mathbb{C})^{\mathbb{N}}$ admits a convergent subsequence. One can express this as follows: there exists $\varphi : \mathbb{N} \to \mathbb{N}$, strictly increasing such that $\lim_{k\to\infty} U_{\varphi(k)}$ exists and is in $\mathcal{U}_n(\mathbb{C})$. Let us denote it U_∞.

Topological method (end)

- $(A_p)_{p\geq 0}$ is convergent, hence, so any of its subsequence.
- we have just chosen $\left(U_{\varphi(k)}\right)_{k\in\mathbb{N}}$ convergent, so finally

$$\exists\ H_\infty \in \mathcal{M}_n(\mathbb{C}), \quad H_{\varphi(k)} = U_{\varphi(k)}^{-1} A_{\varphi(k)} \underset{k\to\infty}{\longrightarrow} H_\infty .$$

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Finally, by continuity of the product $A = \lim_{k\to\infty} U_{\varphi(k)} H_{\varphi(k)} = U_\infty \cdot H_\infty$ with U_∞ unitary and H_∞ positive.