Algebra of Continuous Functions on a Compact set

following "Les maths en tête - Analyse" by X. Gourdon

December 5, 2025

Statement

Theorem (Exercise 8 p. 38, in both 2008 and 2020 edition)

Let (X, d) be a compact metric space and $\mathcal{C}(X, \mathbb{R})$ be the algebra of continuous functions with value in $\mathbb{R}$. For all $s \in X$, let us also denote

$$\mathfrak{I}_s := \left\{f \in \mathcal{C}(X, \mathbb{R}),\ f(s) = 0\right\}$$

a) *Let $\mathfrak{I}$ be a proper ideal of $\mathcal{C}(X, \mathbb{R})$. Show that there exists $s \in X$ such that $\mathfrak{I} \subseteq \mathfrak{I}_s$.*

b) *Give a characterization of the proper maximal ideals[a] of $\mathcal{C}(X, \mathbb{R})$.*

c) *Determine the algebra morphism from $\mathcal{C}(X, \mathbb{R})$ to $\mathbb{R}$ (i.e. the "characters").*

[a] maximality w.r.t $\subseteq$: $\mathcal{I}$ is maximal if whenever there exist $\mathcal{J}$ an ideal s.t. $\mathcal{I} \subseteq \mathcal{J}$ then $\mathcal{J} = \mathcal{I}$ or $\mathcal{J} = \mathcal{C}(X, \mathbb{R})$.

(a step in the commutative Gelfand representation theorem).

a) Inclusion in a maximal ideal

One must show that all functions in $\mathfrak{I}$ share a common zero:

$$\exists\, s \in X,\ \forall f \in \mathfrak{I}, \quad f(s) = 0$$

Proof: If by absurd, for all $s \in X$ there exists $f_s \in \mathfrak{I}$ such that $f_s(s) \neq 0$, then by continuity of that function at s,

$$\forall\, \epsilon > 0,\ \exists\, \delta > 0, \quad \forall\, x \in X,\ d(x,s) < \delta,\ \ |f_s(x) - f_s(s)| < \epsilon$$

In particular, for $\epsilon_0 := \left|\frac{f_s(s)}{2}\right| > 0$, $\exists\, \delta_0 > 0$ such that

$$\forall\, x \in \mathcal{B}(s,\delta_0), \quad f_s(x) \in \left] f_s(s) - \left|\frac{f_s(s)}{2}\right|, f_s(s) + \left|\frac{f_s(s)}{2}\right| \right[$$

This interval does not contain 0, so $f_s^2(x) > 0$ in the neighborhood $\mathcal{B}(s,\delta_0)$ of s. One obtains an open covering $X = \bigcup_{s \in X} \mathcal{B}(s,\delta_0)$ from which one can extract a finite subcover $\big(\mathcal{B}(s_i,\delta_0)\big)_{i=1}^{n}$ and define $\boxed{F := \sum_{i=1}^{n} f_{s_i}^2}$. It is in $\mathfrak{I}$ and it is strictly positive on X

so one can make sense of $1/F \in \mathcal{C}(X, \mathbb{R})$. Hence $F \cdot (1/F) = 1$ is in $\mathfrak{I}$, implying $\mathfrak{I} = \mathcal{C}(X, \mathbb{R})$, contradiction. $\square$

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b) Let us now show that the $\mathfrak{I}_s$ are precisely the maximal ideals of $\mathcal{C}(X, \mathbb{R})$.

Let $s \in X$. $\mathfrak{I}_s$ is stable under addition and multiplication by any function of $\mathcal{C}(X, \mathbb{R})$, so it is an ideal. If there exists an ideal $\mathcal{J}$ such that $\mathfrak{I}_s \subseteq \mathcal{J}$, then either $\mathcal{J} = \mathcal{C}(X, \mathbb{R})$ or by point a), there exists $t \in X$ such that $\mathcal{J} \subseteq \mathfrak{I}_t$ so that $\mathfrak{I}_s \subseteq \mathfrak{I}_t$. If by absurd $t \neq s$, then one can find a function in $\mathfrak{I}_s$ but not in $\mathfrak{I}_t$, , e.g. $x \longmapsto d(x, s)$, so the inclusion is possible only if $t = s$. Hence $\mathcal{J} = \mathfrak{I}_s$.

Let now $\mathfrak{I}$ be a maximal ideal. There exists by a), an $s \in X$ such that $\mathfrak{I} \subseteq \mathfrak{I}_s$. Since $\mathfrak{I}$ is maximal and $\mathfrak{I}_s \neq \mathcal{C}(X, \mathbb{R})$, one must have $\mathfrak{I} = \mathfrak{I}_s$.

c) Characters on $\mathcal{C}(X,\mathbb{R})$

Let us for simplicity denote $\mathcal{A} := \mathcal{C}(X,\mathbb{R})$ and show that any non trivial algebra morphism $\varphi : \mathcal{A} \to \mathbb{R}$ is an **evaluation map**, i.e.

$$\exists\, s \in X,\ \forall\, f \in \mathcal{C}(X,\mathbb{R}), \quad \varphi(f) = f(s)$$

Since φ is a non zero algebra morphism, $\mathrm{Ker}\,\varphi = \{f \in \mathcal{A},\ \varphi(f) = 0\}$ is a proper ideal of $\mathcal{A}$ and since the image $\varphi(\mathcal{A}) = \mathbb{R}$ is a field, $\mathrm{Ker}\,\varphi$ is <u>maximal</u>: indeed if there exists an ideal $\mathcal{J} \supsetneq \mathrm{Ker}\,\varphi$, then $\exists\, a \in \mathcal{J} \setminus \mathrm{Ker}\,\varphi,\ \ \varphi(a) \neq 0$. By surjectivity of φ, there exists $b \in \mathcal{A}$ such that $\varphi(a)\cdot\varphi(b) = 1$, i.e. $\varphi\big(a \cdot b - 1_{\mathcal{A}}\big) = 0$ so

$$\exists\, k \in \mathrm{Ker}\,\varphi \subset \mathcal{J}, \quad a \cdot b - 1_{\mathcal{A}} = k$$

Since $\mathcal{J}$ is an ideal, $a \cdot b \in \mathcal{J}$. Hence $1_{\mathcal{A}} = a \cdot b - k \in \mathcal{J}$ and thus $\mathcal{J} = \mathcal{A}$.

$\mathrm{Ker}\,\varphi$ is maximal so by b), there exists $s \in X$ s.t. $\boxed{\mathrm{Ker}\,\varphi = \mathfrak{I}_s}$. It follows from the first isomorphism theorem, $\mathcal{A}/\mathrm{Ker}\,\varphi \cong \mathbb{R}$, that $\mathrm{Ker}\,\varphi$ is a hyperplane/has co-dimension 1.

Note now that the evaluation map $\mathrm{Eval}_s : \begin{cases} \mathcal{A} \longrightarrow \mathbb{R} \\ f \longmapsto f(s) \end{cases}$ has the same kernel $\mathfrak{I}_s$ but any two linear forms with the same hyperplane kernel are proportional (algebra morphisms are linear in particular), so there exists $\lambda \in \mathbb{R}^*$ such that

$$\forall\, f \in \mathcal{C}(X, \mathbb{R}), \quad \mathrm{Eval}_s(f) = f(s) = \lambda\, \varphi(f)$$

For $f = 1_{\mathcal{A}}$ the constant function, which is the unit in $\mathcal{A} = \mathcal{C}(X, \mathbb{R})$, one obtains

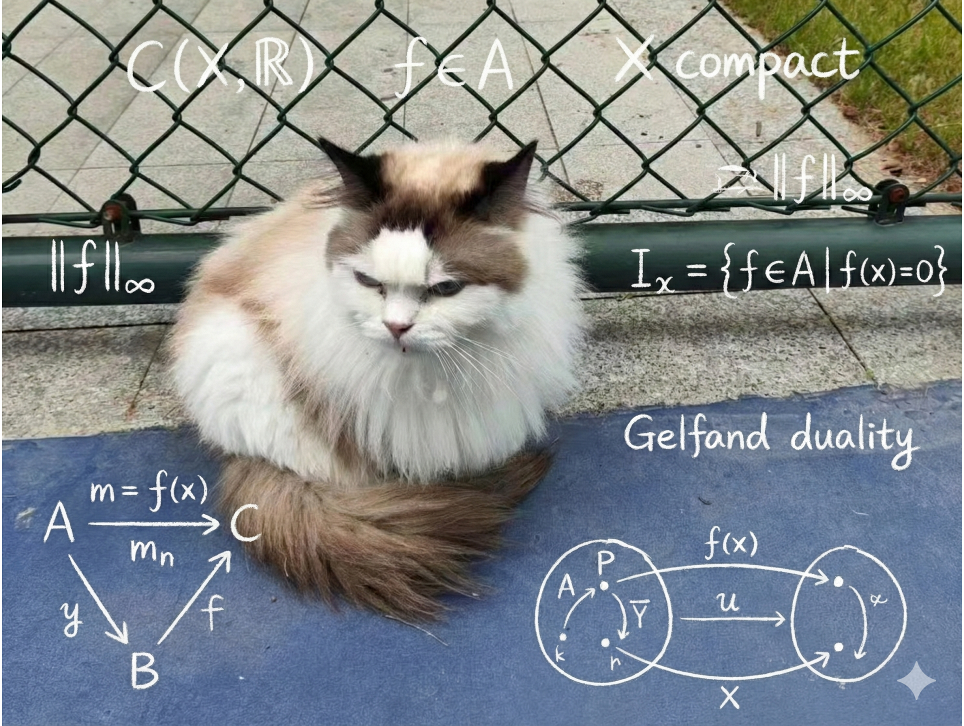
$$f(s) = 1 = \lambda\, \varphi\big(1_{\mathcal{A}}\big) = \lambda \cdot 1$$

One concludes that $\varphi = \mathrm{Eval}_s$.

Remark : For X locally compact, the algebra $\mathcal{A} := \mathcal{C}_0(X, \mathbb{R})$ of functions which vanish at infinity has no unit, the constant function 1 is not in the algebra. But one can still write for an $f \in \mathcal{A}$ s.t. $f(s) \neq 0$

$$f^2(s) = \mathrm{Eval}_s\big(f^2\big) = \lambda\, \varphi\big(f^2\big) = \lambda\, \varphi(f)^2 = \lambda \left(\frac{f(s)}{\lambda}\right)^2 = \frac{f^2(s)}{\lambda}$$

One can still conclude that $\lambda = 1$.



$C(X,\mathbb{R})$
$f \in A$
X compact
$\|f\|_\infty$
$\|f\|_\infty$
$I_x = \{f \in A \mid f(x)=0\}$
Gelfand duality
$m = f(x)$
A
C
m_n
y
f
B
$f(x)$
P
A
$\overline{Y}$
k
n
u
α
X