Compactness in Metric Spaces

following "Les maths en tête - Analyse" by X. Gourdon

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Definition and Theorem

Recall that (*E* stands for "Espace"= Space in french)

- A topological space $(E, \mathcal{T})$ is **compact** if every open cover admits a finite subcover. If it is a metric space, then we will equivalently use the **sequential characterization**: every sequence admits a convergent subsequence.
- A metric space (E, d) is **precompact** or **totally bounded** if for every $\epsilon > 0$, it can be covered by a finite number of open balls of radius ϵ.
- (E, d) is **complete** if every Cauchy sequence converges in E.

Theorem (Exercise 2 p.32)

*A metric space (E, d) is **compact** if and only if it is **precompact** and **complete**.*

Proof: $\boxed{(\Rightarrow)}$ Let $\epsilon > 0$. The family $\left(\mathcal{B}(x, \epsilon)\right)_{x \in E}$ of open subsets obviously covers $E = \bigcup_{x \in E} \mathcal{B}(x, \epsilon)$ so one can extract a finite subcover:

$$\exists \ \{x_1, \cdots, x_n\} \subset E \quad \text{such that} \quad E = \bigcup_{i=1}^{n} \mathcal{B}(x_i, \epsilon)$$

Let now $(x_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$ be a Cauchy sequence. By the sequential characterization of compactness, one can extract a convergent subsequence $\left(x_{\varphi(n)}\right)_{n\in\mathbb{N}}$. Let $x \in E$ be the limit. Since the original sequence is Cauchy, it will also converge to x:

$$\begin{cases} \forall\ \epsilon_1 > 0,\ \exists\ N_1 \in \mathbb{N},\ \forall\ n \in \mathbb{N}\backslash[\![0, N_1]\!], \quad d\left(x_{\varphi(n)}, x\right) < \epsilon_1 \\ \forall\ \epsilon_2 > 0,\ \exists\ N_2 \in \mathbb{N},\ \forall\ (p,q) \in \left(\mathbb{N}\backslash[\![0, N_2]\!]\right)^2, \quad d\left(x_p, x_q\right) < \epsilon_2 \end{cases}$$

Let $\epsilon > 0$. By the previous lines, with $\epsilon_1 = \frac{\epsilon}{2} = \epsilon_2$, for all $k, n \in \mathbb{N}$ with $k > N_2$ and $n > N_1$ such that $\varphi(n) > N_2$,

$$d\left(x_k, x\right) \leq d\left(x_k, x_{\varphi(n)}\right) + d\left(x_{\varphi(n)}, x\right) < \epsilon_2 + \epsilon_1 = \epsilon\,.$$

$\boxed{(\Leftarrow)}$ let us show that if E is precompact and complete, then it is sequentially compact[1]. Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in E.

Since E is precompact, it is covered by a finite set of balls of radius 1, one of which, that we denote $B(a_0, 1)$, must contain an infinite number of terms of $(x_n)_{n\in\mathbb{N}}$. This defines a subsequence $\left(x_{\varphi_0(n)})\right)_{n\in\mathbb{N}}$.

[1]This is equivalent to compactness in the case of metric spaces (it is a difficult theorem, cf. Lebesgue's number lemma).

$B(a_0,1)\cap E\subset E$ itself is precompact, so it can be covered by a finite number of open balls of radius $\frac{1}{2}$, one of which, that we denote $B\left(a_1,\frac{1}{2}\right)$ contains an infinite number of terms from $\left(x_{\varphi_0(n)}\right)_{n\in\mathbb{N}}$. This defines a subsequence of the subsequence such that

$$\forall\ n\in\mathbb{N},\quad x_{\varphi_0\circ\varphi_1(n)}\subset B(a_0,1)\cap B\left(a_1,\frac{1}{2}\right)$$

By induction, one obtains for all $p\in\mathbb{N}$ a subsequence $\left(x_{\varphi_0\circ\cdots\circ\varphi_p(n)}\right)_{n\in\mathbb{N}}$ and an open ball $B\left(a_p,\frac{1}{2^p}\right)$ such that

$$\forall\ n\in\mathbb{N},\quad x_{\varphi_0\circ\cdots\circ\varphi_p(n)}\subset\bigcap_{0\leq k\leq p}B\left(a_k,\frac{1}{2^k}\right)$$

We have at the moment an infinite family of subsequences, let us now define a single sequence $\left(x_{\psi(n)}\right)_{n\in\mathbb{N}}$ by the so-called Cantor diagonal argument. Set $\psi(0):=\varphi_0(0)$ so that $x_{\psi(0)}\in B(a_0,1)$. Choose then $\psi(1)\in\left\{\varphi_0\circ\varphi_1(n)>\psi(0),\ n\in\mathbb{N}\right\}$. And for $\psi(p)$ given, choose

$$\psi(p+1)\in\left\{\varphi_0\circ\cdots\cdot\varphi_{p+1}(n),\ n\in\mathbb{N}\right\}\cap\left\{m\in\mathbb{N},\ m>\psi(p)\right\}.$$

This sequence is Cauchy: let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that $\dfrac{1}{2^N} < \epsilon$.
For all $q > p > N$,

$$x_{\psi(q)} \in \left\{x_{\varphi_0 \circ \cdots \circ \varphi_q(n)},\ n \in \mathbb{N}\right\} \subset \left\{x_{\varphi_0 \circ \cdots \circ \varphi_p(n)},\ n \in \mathbb{N}\right\} \subset B\left(a_p, \frac{1}{2^p}\right)$$

i.e. $d\left(x_{\psi(q)}, x_{\psi(p)}\right) \leq d\left(x_{\psi(q)}, a_p\right) + d\left(a_p, x_{\psi(p)}\right) < \dfrac{1}{2^p} + \dfrac{1}{2^p} \leq \dfrac{2}{2^{N+1}} < \epsilon.$

Since E is complete, the Cauchy subsequence converges, so one has obtained a convergent subsequence. $\square$