Quiz 2

October 10, 2024

Task 1

Find a combination $x_1w_1 + x_2w_2 + x_3w_3$ that gives the zero vector:

$$w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

Solution:

To find a combination $x_1w_1 + x_2w_2 + x_3w_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, we set up the following augmented matrix:

$$\begin{bmatrix} 1 & 4 & 7 & | & 0 \\ 2 & 5 & 8 & | & 0 \\ 3 & 6 & 9 & | & 0 \end{bmatrix}$$

We can perform row operations to reduce this matrix:
1. **R2 = R2 - 2R1**:

$$\begin{bmatrix} 1 & 4 & 7 & | & 0 \\ 0 & -3 & -6 & | & 0 \\ 3 & 6 & 9 & | & 0 \end{bmatrix}$$

2. **R3 = R3 - 3R1**:

$$\begin{bmatrix} 1 & 4 & 7 & | & 0 \\ 0 & -3 & -6 & | & 0 \\ 0 & -6 & -12 & | & 0 \end{bmatrix}$$

3. **R3 = R3 - 2R2**:

$$\begin{bmatrix} 1 & 4 & 7 & | & 0 \\ 0 & -3 & -6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Now we have a row of zeros, indicating the system is dependent.

From $-3y_2 - 6y_3 = 0$:

$$y_2 + 2y_3 = 0 \implies y_2 = -2y_3$$

Substituting y_2 into the first equation:

$$y_1 + 4(-2y_3) + 7y_3 = 0 \implies y_1 - 8y_3 + 7y_3 = 0 \implies y_1 = y_3$$

Thus, we can express the solution as:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

where y_3 can be any scalar.

Task 2

If the rank of a 5×3 matrix A is 3, what is $\text{rref}(A)$?

Solution:

For a 5×3 matrix A with rank 3, the reduced row echelon form (rref) will have the following structure:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix contains 3 leading 1's corresponding to the rank of the matrix, and the remaining rows are filled with zeros.

Task 3

For which values of the constants c and d is

$$\begin{bmatrix} 1923813 \\ 1923815 \\ c \\ d \end{bmatrix}$$

a linear combination of

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}?$$

Solution:

To determine the values of c and d, we express the vector

$$\begin{bmatrix} 1923813 \\ 1923815 \\ c \\ d \end{bmatrix}$$

as a linear combination of the given vectors. Thus, we set up the equation:

$$x \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1923813 \\ 1923815 \\ c \\ d \end{bmatrix}$$

This leads to the following system of equations:

$$\begin{aligned} x + y &= 1923813 \quad (1) \\ x + 2y &= 1923815 \quad (2) \end{aligned}$$

Subtracting equation (1) from equation (2):

$$(x + 2y) - (x + y) = 1923815 - 1923813 \implies y = 2$$

Substituting $y = 2$ back into equation (1):

$$x + 2 = 1923813 \implies x = 1923813 - 2 = 1923811$$

Now, substituting x and y to find c and d:

$$c = x + 3y = 1923811 + 3(2) = 1923811 + 6 = 1923817$$

$$d = x + 4y = 1923811 + 4(2) = 1923811 + 8 = 1923819$$

Thus, the values of c and d are:

$$c = 1923817, \quad d = 1923819$$