

Definition

Consider an eigenvalue λ of an $n \times n$ matrix A. Then the kernel of the matrix is called the eigenspace associated with λ , denoted by E_{λ} :

$$E_{\lambda} = \ker(A - \lambda I_n) = \{ \vec{v} \in \mathbb{R}^n : A\vec{v} = \lambda \vec{v} \}$$

Problem

Find the eigenspaces of the matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. Diagonalize matrix A if you can..

At first to find eigenvalues we do: $\det(A - \lambda I) = 0 \Rightarrow (1 - \lambda)(3 - \lambda) - 8 = 0 \Rightarrow \lambda^2 - 4\lambda - 5 = 0 \Rightarrow \lambda_1 = 5, \lambda_2 = -1$

$$E_5 = \ker(A - 5I_2) = \ker\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \Rightarrow \text{by inspection } E_5 = span \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$E_{-1} = \ker(A + I_2) = \ker\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \Rightarrow span\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

so we see that our vectors we found are eigenvectors of A

The vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ form an eigenbasis for A, so A is diagonalizable, with $S = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 0 \\ -0 & -1 \end{pmatrix}$

Which of the following statements are true?..(Explain why)

- (S1) Every 2×2 matrix has an eigenvector
- (S2) if A and B are two $n \times n$ matrices such that AB = BA, and if u is an eigenvector of B, then Au is an eigenvector of B as well
- (S3) if u and v are the eigenvectors of a 2×2 matrix A, then u v is also an eigenvector of A

Solution

- (S1) The statement if false. The matrix $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ has no eigenvectors since its characteristic polynomial is $(1-\lambda)^2+1$, which cannot be factored on $\mathbb R$
- (S2) The statement is true. Assume u is an eigenvector of B and let λ be the corresponding eigenvalue. Then:

$$B(Au) = A(Bu) = A(\lambda u) = \lambda A(u)$$

Hence Au is an eigenvector of B corresponding to the eigenvalue λ

(S3) The statement is false. Let $A=\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Then $u=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the eigenvectors of A. However, $u-v=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is not an eigenvector of A since

$$A(u-v) = \begin{pmatrix} 2\\1 \end{pmatrix}$$