Quiz 3 Solutions

October 18, 2024

Task 1: Prove that $T(0) = 0$

We are asked to prove that $T(0) = 0$, where T is a linear transformation.

By the property of linear transformations, we know that:

$$T(c\mathbf{u}) = cT(\mathbf{u}) \quad \text{for all vectors } \mathbf{u} \text{ and scalars } c.$$

Now, since $0 = -0$, we can write:

$$T(0) = T(-0) = -T(0).$$

The only vector for which $\mathbf{w} = -\mathbf{w}$ is the zero vector. Therefore, $T(0) = 0$, which completes the proof.

Task 2: Verify Linearity of Transformations

1. $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x + 1$:

   To verify if this transformation is linear, we check the condition $T(0) = 0$. We have:

   $$T(0) = 0 + 1 = 1 \neq 0.$$

   Since $T(0) \neq 0$, this transformation is not linear.

2. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x) = 1.5x$:

   We need to check if this transformation satisfies the two defining properties of linearity:

   (a) **Additivity**: For all vectors $\mathbf{u}$ and $\mathbf{v}$, we need to verify that:

   $$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}).$$

   $$T(\mathbf{u} + \mathbf{v}) = 1.5(\mathbf{u} + \mathbf{v}) = 1.5\mathbf{u} + 1.5\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v}).$$

   (b) **Homogeneity**: For all scalars c, we need to verify that:

   $$T(c\mathbf{u}) = cT(\mathbf{u}).$$

   $$T(c\mathbf{u}) = 1.5(c\mathbf{u}) = c(1.5\mathbf{u}) = cT(\mathbf{u}).$$

Since both properties hold, T is a linear transformation.

3. $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 3x - y \\ y \\ x \end{pmatrix}.$$

We will check the linearity of this transformation. Let $\mathbf{u} = \binom{x_1}{y_1}$ and $\mathbf{v} = \binom{x_2}{y_2}$, and verify the two properties:

(a) **Additivity**: We need to check if:

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}).$$

$$T\left(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}\right) = \begin{pmatrix} 3(x_1 + x_2) - (y_1 + y_2) \\ y_1 + y_2 \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 - y_1 \\ y_1 \\ x_1 \end{pmatrix} + \begin{pmatrix} 3x_2 - y_2 \\ y_2 \\ x_2 \end{pmatrix} = T(\mathbf{u}) + T(\mathbf{v}).$$

(b) **Homogeneity**: We need to check if:

$$T(c\mathbf{u}) = cT(\mathbf{u}).$$

$$T\left(c\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) = \begin{pmatrix} 3(cx_1) - (cy_1) \\ cy_1 \\ cx_1 \end{pmatrix} = c\begin{pmatrix} 3x_1 - y_1 \\ y_1 \\ x_1 \end{pmatrix} = cT\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right).$$

Since both properties hold, T is a linear transformation.

4. $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x) = x + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

We check the first property $T(0) = 0$. We have:

$$T(0) = 0 + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \neq 0.$$

Since $T(0) \neq 0$, this transformation is not linear.

Task 3 Solution

Compute the orthogonal projection of $x = \binom{-6}{4}$ onto the line L spanned by $u = \binom{3}{2}$, and find the distance from x to L.

Solution:

First, we compute the projection of x onto u using the formula for the orthogonal projection:

$$x_L = \frac{x \cdot u}{u \cdot u} u$$

We begin by calculating the dot products:

$$x \cdot u = (-6)(3) + (4)(2) = -18 + 8 = -10$$
$$u \cdot u = (3)(3) + (2)(2) = 9 + 4 = 13$$

Now, substitute these values into the projection formula:

$$x_L = \frac{-10}{13} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = -\frac{10}{13} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Next, to find the vector $x_{L^\perp}$, subtract the projection from x:

$$x_{L^\perp} = x - x_L = \begin{pmatrix} -6 \\ 4 \end{pmatrix} - \left(-\frac{10}{13} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right) = \frac{1}{13} \begin{pmatrix} -48 \\ 72 \end{pmatrix}$$

The distance from x to L is the magnitude of $x_{L^\perp}$, given by:

$$\|x_{L^\perp}\| = \frac{1}{13} \sqrt{48^2 + 72^2} \approx 6.656$$