Quiz (G-9)

Linear Algebra for Informatics
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Definition

Consider an eigenvalue λ of an $n \times n$ matrix A. Then the kernel of the matrix is called the *eigenspace* associated with λ, denoted by E_λ :

$$E_\lambda = \ker(A - \lambda I_n) = \{\vec{v} \in \mathbb{R}^n : A\vec{v} = \lambda\vec{v}\}$$

Problem

Find the eigenspaces of the matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. Diagonalize matrix A if you can..

Solution

At first to find eigenvalues we do: $\det(A-\lambda I)=0 \Rightarrow (1-\lambda)(3-\lambda)-8=0 \Rightarrow \lambda^2-4\lambda-5=0 \Rightarrow \lambda_1=5, \lambda_2=-1$

$$E_5=\ker(A-5I_2)=\ker\begin{pmatrix}-4 & 2\\ 4 & -2\end{pmatrix} \Rightarrow \text{by inspection } E_5 = span\begin{pmatrix}1\\2\end{pmatrix}$$

$$E_{-1}=\ker(A+I_2)=\ker\begin{pmatrix}2 & 2\\ 4 & 4\end{pmatrix} \Rightarrow span\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}5\\10\end{pmatrix}=5\begin{pmatrix}1\\2\end{pmatrix}$$

$$\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}=(-1)\begin{pmatrix}1\\-1\end{pmatrix}$$

so we see that our vectors we found are eigenvectors of A

The vectors $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}$ form an eigenbasis for A, so A is diagonalizable, with $S=\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}$ and $B=\begin{pmatrix}5 & 0\\ 0 & -1\end{pmatrix}$

Problem

Which of the following statements are true?..(Explain why)

- **(S1)** Every 2×2 matrix has an eigenvector
- **(S2)** if A and B are two $n \times n$ matrices such that $AB = BA$, and if u is an eigenvector of B, then Au is an eigenvector of B as well
- **(S3)** if u and v are the eigenvectors of a 2×2 matrix A, then $u - v$ is also an eigenvector of A

Solution

(S1) The statement if false. The matrix $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ has no eigenvectors since its characteristic polynomial is $(1-\lambda)^2 + 1$, which cannot be factored on $\mathbb{R}$

(S2) The statement is true. Assume u is an eigenvector of B and let λ be the corresponding eigenvalue. Then :

$$B(Au) = A(Bu) = A(\lambda u) = \lambda A(u)$$

Hence Au is an eigenvector of B corresponding to the eigenvalue λ

(S3) The statement is false. Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Then $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the eigenvectors of A.
However, $u - v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is not an eigenvector of A since

$$A(u-v) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$