Quiz (G-1)

Linear Algebra for Informatics
Fall semester 2024
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Problem

$$\text{Compute } \det\begin{pmatrix} 0 & -7 & -4 \\ 2 & 4 & 6 \\ 3 & 7 & -1 \end{pmatrix} \text{ and :}$$

$$\det\left(\begin{pmatrix} 3 & 7 & -1 \\ 2 & 4 & 6 \\ 0 & -7 & -4 \end{pmatrix} \times \begin{pmatrix} 2 & 4 & 6 \\ 3 & 7 & -1 \\ 0 & -7 & -4 \end{pmatrix} \times \begin{pmatrix} 2 & 4 & 6 \\ 0 & -7 & -4 \\ 3 & 7 & -1 \end{pmatrix} \times \begin{pmatrix} -1 & -2 & -3 \\ 0 & -7 & -4 \\ 3 & 7 & -1 \end{pmatrix}\right)$$

Solution

$$\det\begin{pmatrix} 0 & -7 & -4 \\ 2 & 4 & 6 \\ 3 & 7 & -1 \end{pmatrix} \Rightarrow 0 \cdot \det\begin{pmatrix} 4 & 6 \\ 7 & -1 \end{pmatrix} - (-7) \cdot \det\begin{pmatrix} 2 & 6 \\ 3 & -1 \end{pmatrix} + (-4)\det\begin{pmatrix} 2 & 4 \\ 3 & 7 \end{pmatrix} = 0 + 7(-2-18) - 4(14-12) = -140 - 8 = -148$$

$$\text{For the second part :}$$

$$\det(ABCD) = \det(A)\det(B)\det(C)\det(D)$$

First matrix is made by swapping the 1st and 3th rows from the above matrix, (swapping two rows in the matrix multiplies det(A) by −1 so $\det(A) = 148$
Second matrix is made by swapping the 1st and 2nd rows from the previous one so $\det(B) = -148$
Third matrix is made by swapping the 2nd and 3rd rows from the previous matrix so $\det(C) = 148$
4th matrix is made by multiplying the first row of the previous matrix by (-1/2) so $\det(D) = 148 \cdot \left(-\frac{1}{2}\right) \Rightarrow -74$
so finally $\det(ABCD) = \det(A)\det(B)\det(C)\det(D) = 148 \cdot (-148) \cdot 148 \cdot (-74)$

Problem

$$\text{Compute } \det\begin{pmatrix} \pi & e & 11 \\ 3\pi & 3e & 33 \\ 12 & -7 & 2 \end{pmatrix}$$

Solution

$$\text{You can easily see that } R_2 = 3 \cdot R_1, \text{second row is dependant on the first one so.. } \det\begin{pmatrix} \pi & e & 11 \\ 3\pi & 3e & 33 \\ 12 & -7 & 2 \end{pmatrix} = 0$$

Problem

Bonus

Find the area of the interior E of the ellipse defined by the equation :

$$\left(\frac{2x-y}{2}\right)^2 + \left(\frac{y+3x}{3}\right)^2 = 1$$

Hint: The ellipse is obtained from the unit circle $X^2 + Y^2 = 1$ by linear change of coordinates..

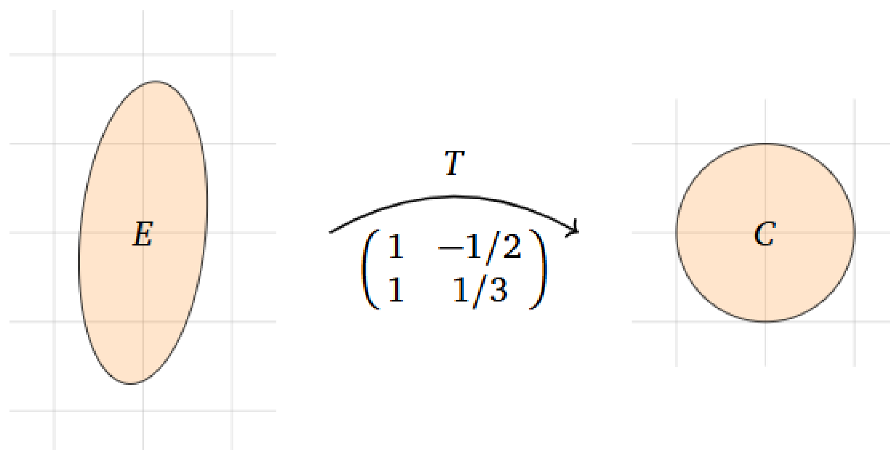
Solution

$$X = \frac{2x - y}{2}, \; Y = \frac{y + 3x}{3}$$

We can define the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2x-y}{2} \\ \frac{y+3x}{3} \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{3} \end{pmatrix} \Longrightarrow A = \begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{3} \end{pmatrix}$$



Therefore, T scales areas by a factor of $|\det(A)| = \frac{5}{6}$, The area of the unit circle is π, so

$$\pi = \mathrm{vol}(C) = \mathrm{vol}(T(E)) = |\det(A)| \cdot \mathrm{vol}(E) = \frac{5}{6}\,\mathrm{vol}(E) \Rightarrow \mathrm{vol}(E) \Longrightarrow \pi \cdot \frac{6}{5}$$