online multiplication

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idea:

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- other tapes with numbers $\tau \in [1:k]$: work tapes
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- decompose $X \in \mathbb{B}^n$ as

$$\begin{aligned} X_H &= X[n-1:i] \\ X_L &= X[i-1:0] \end{aligned}$$

Then

$$\langle x \rangle = \langle x_H \rangle \cdot 2^i + \langle x_L \rangle$$

-

$$\begin{aligned} \langle A \rangle \cdot \langle B \rangle &= \langle A_H \rangle \cdot \langle B_H \rangle \cdot 2^{2i} \\ &+ (\langle A_H \rangle \cdot \langle B_L \rangle + \langle A_L \rangle \cdot \langle B_H \rangle) \cdot 2^i \\ &+ \langle A_L \rangle \cdot \langle B_L \rangle \end{aligned}$$

2 Stating the lower bound

Ω-notation: 2 Variants for $f(n) = \Omega(g(n))$

- for almost all n:

$$\exists c > 0,\ n_0\ \forall n \geq n_0. \quad f(n) \geq c \cdot g(n)$$

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proof ideas:

- overlap argument using edge partition in the style of 'determinism versus nondeterminism'
- multiplication is not hard for all operands. E.g. multiplication with 0 is trivial. Multiplication with 2^n is a shift.
- special sequence of first operands

$$K_N \in \mathbb{B}^N \quad , \quad K_N[i] = 1 \leftrightarrow \exists r \in \mathbb{N}_0.\ i = 2^r$$

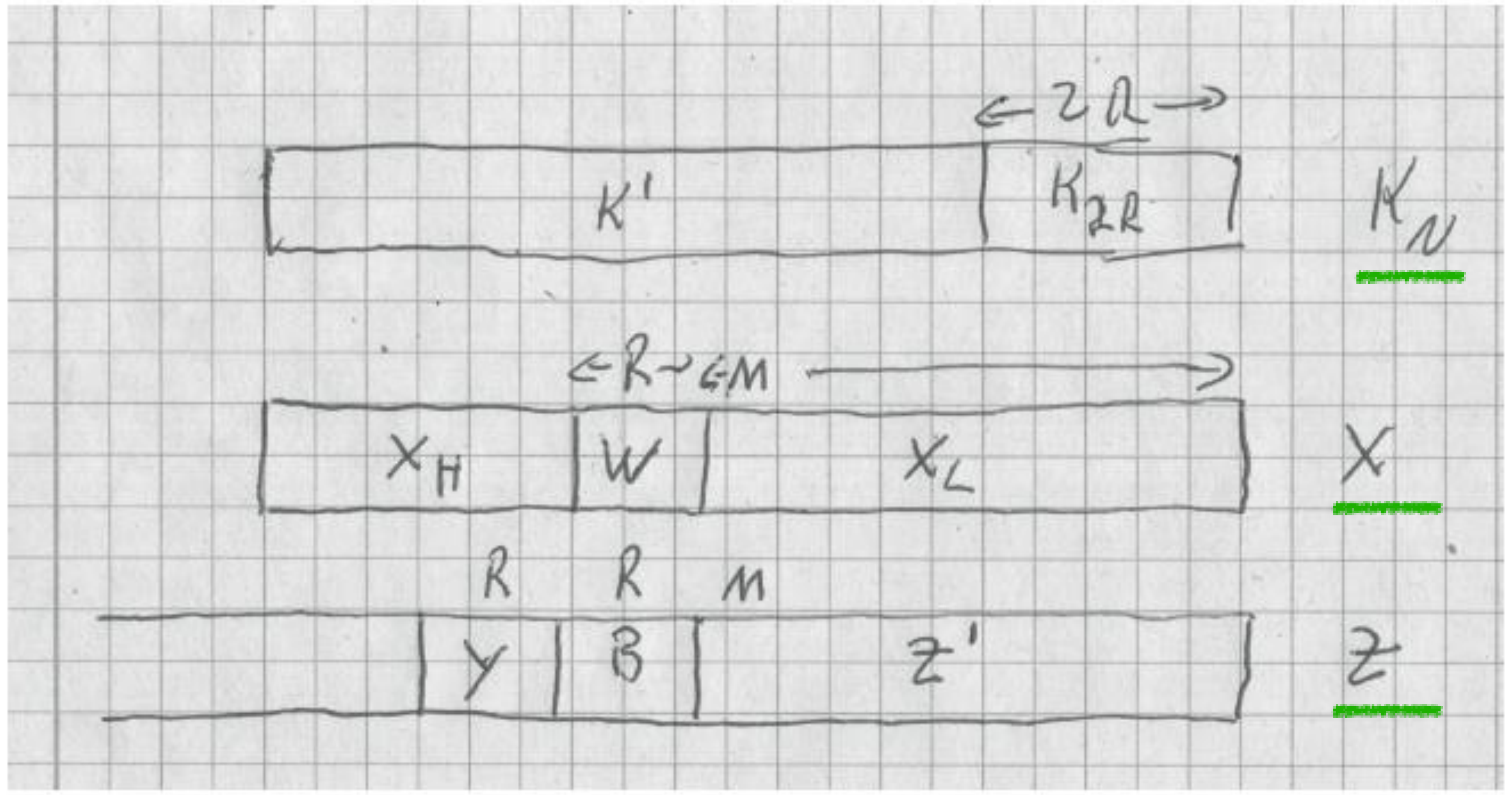
  Ones at positions which are powers of two.

$$K_N = \ldots 10000000100010110$$

$$k_N = \langle K_N \rangle = \sum_{2^r < N} 2^{2^r}$$

- 1969, 1974: argue for operands of length N about average time taken over all second operands X.
- 1982: for each N argue with *one* Kolmogorov-random operand X: short computations would allow to compress the X operand.

3 Reconstructing operand bits from result bits



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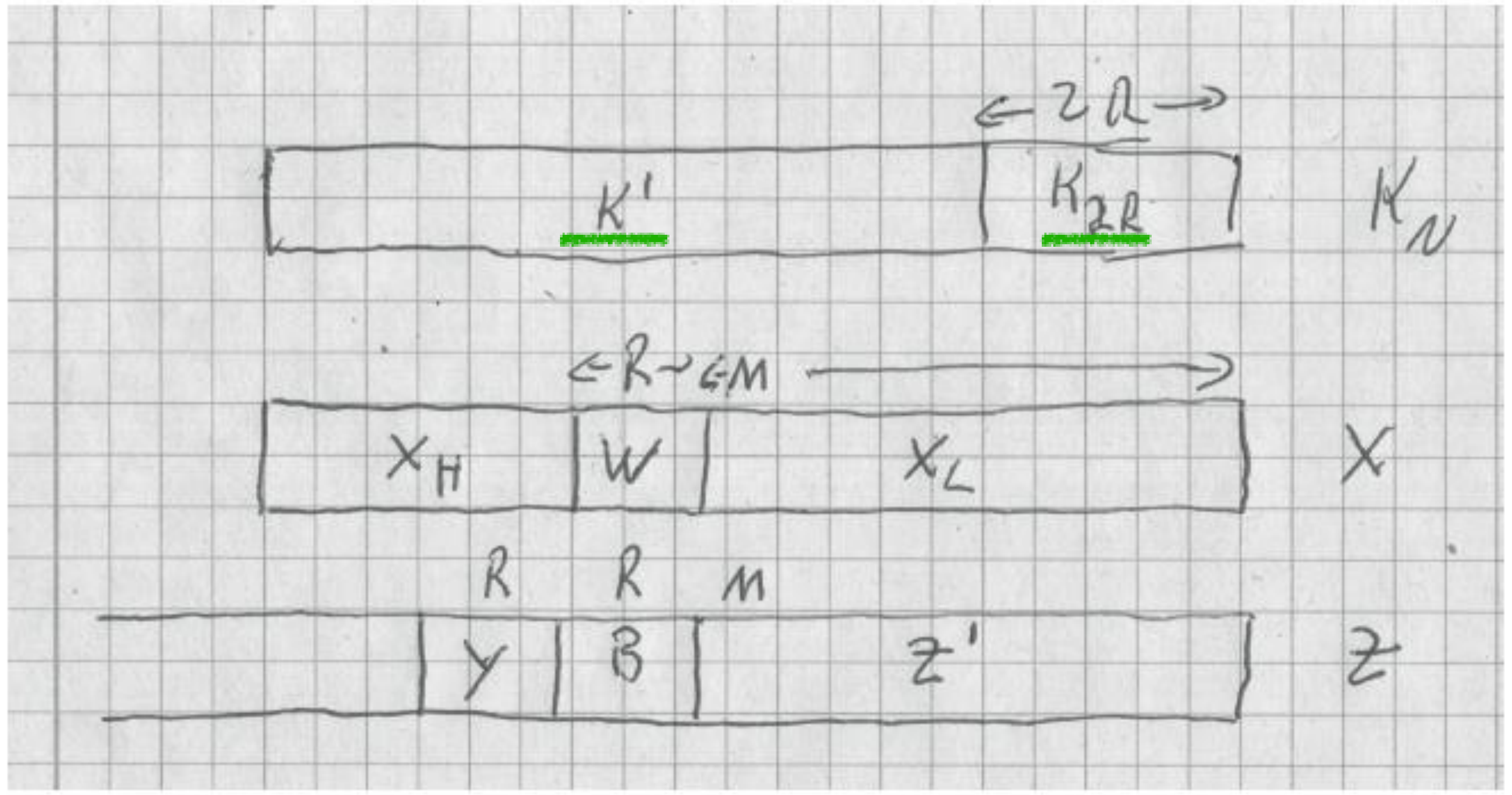

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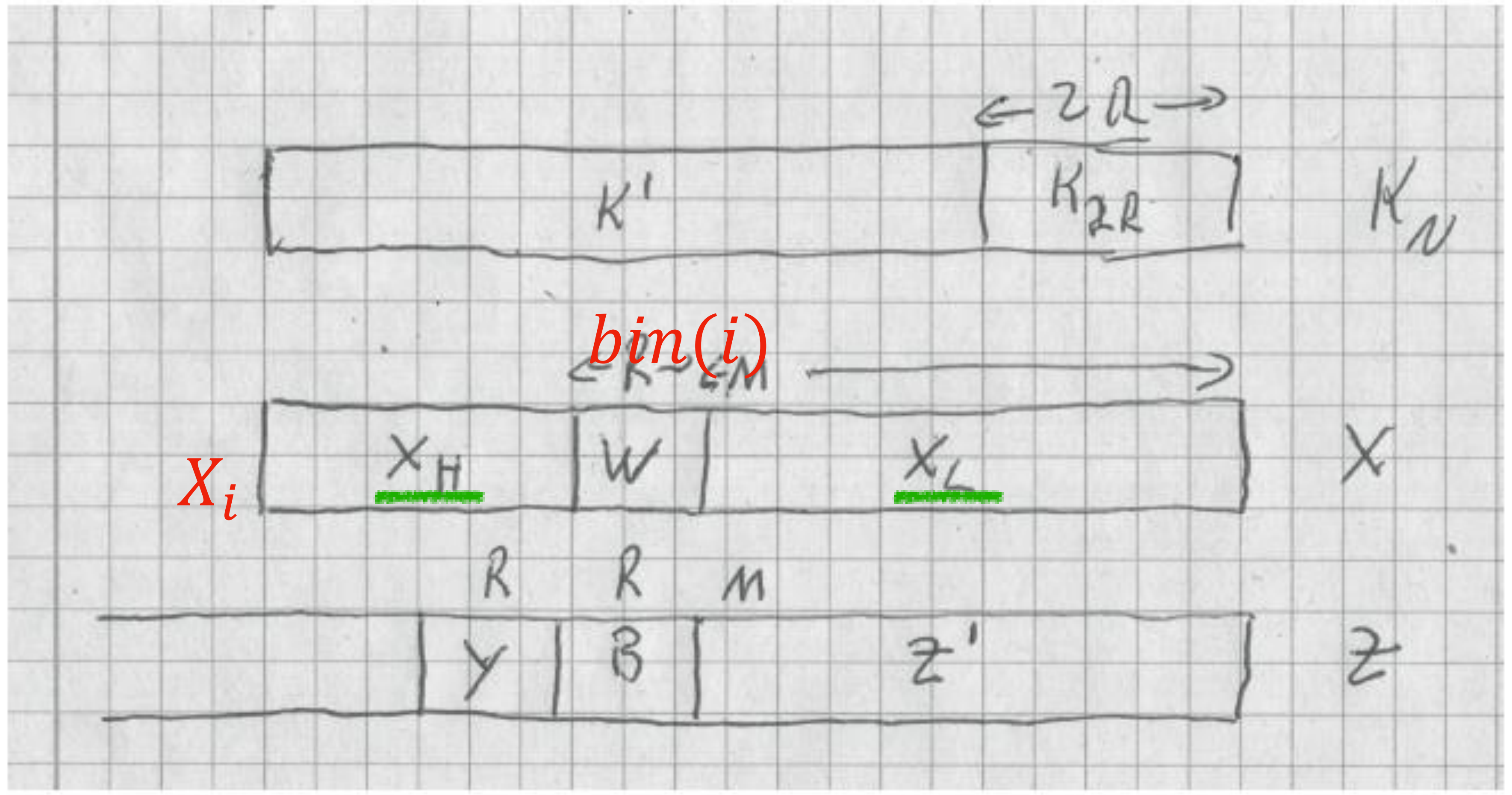

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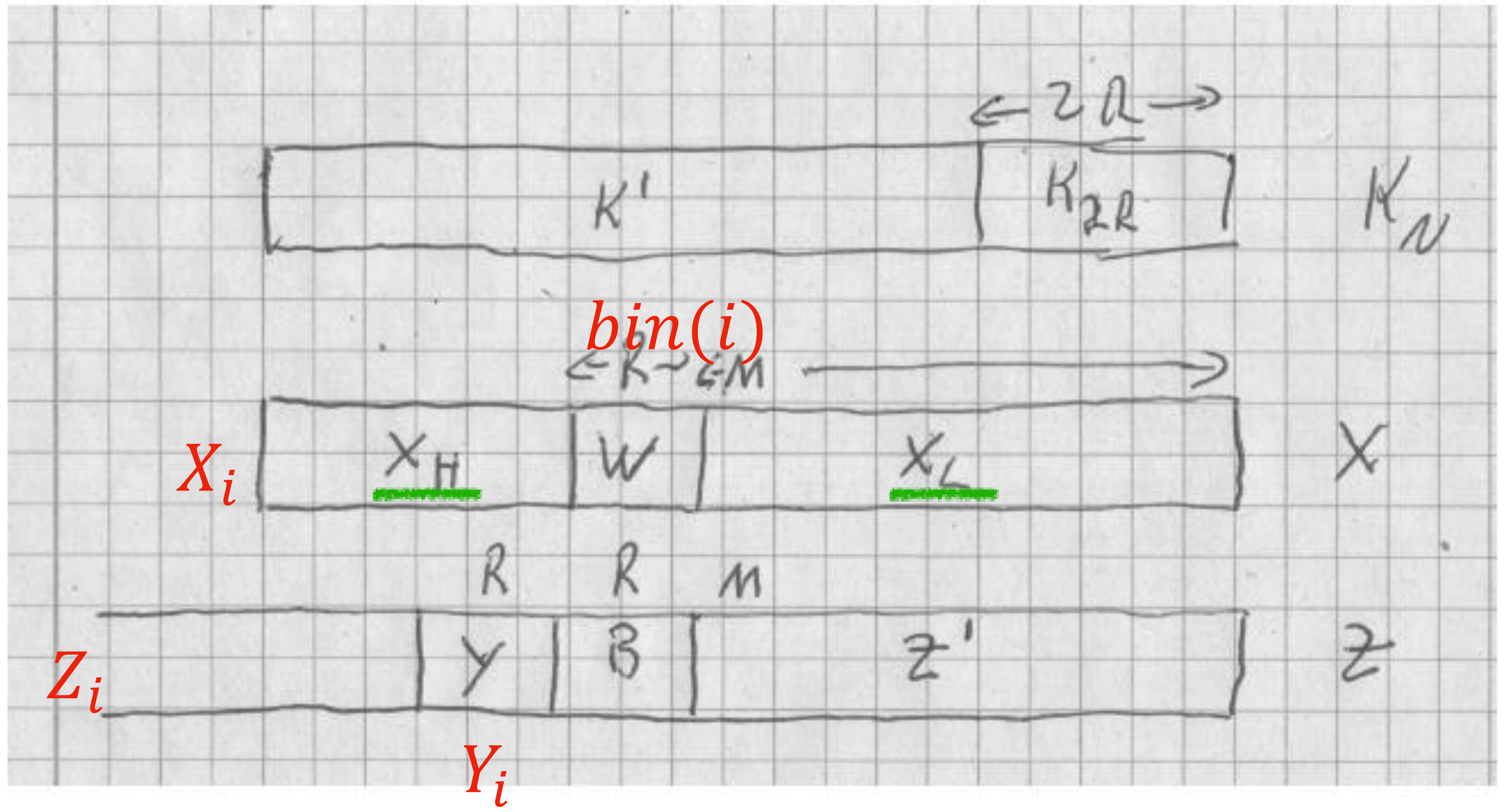

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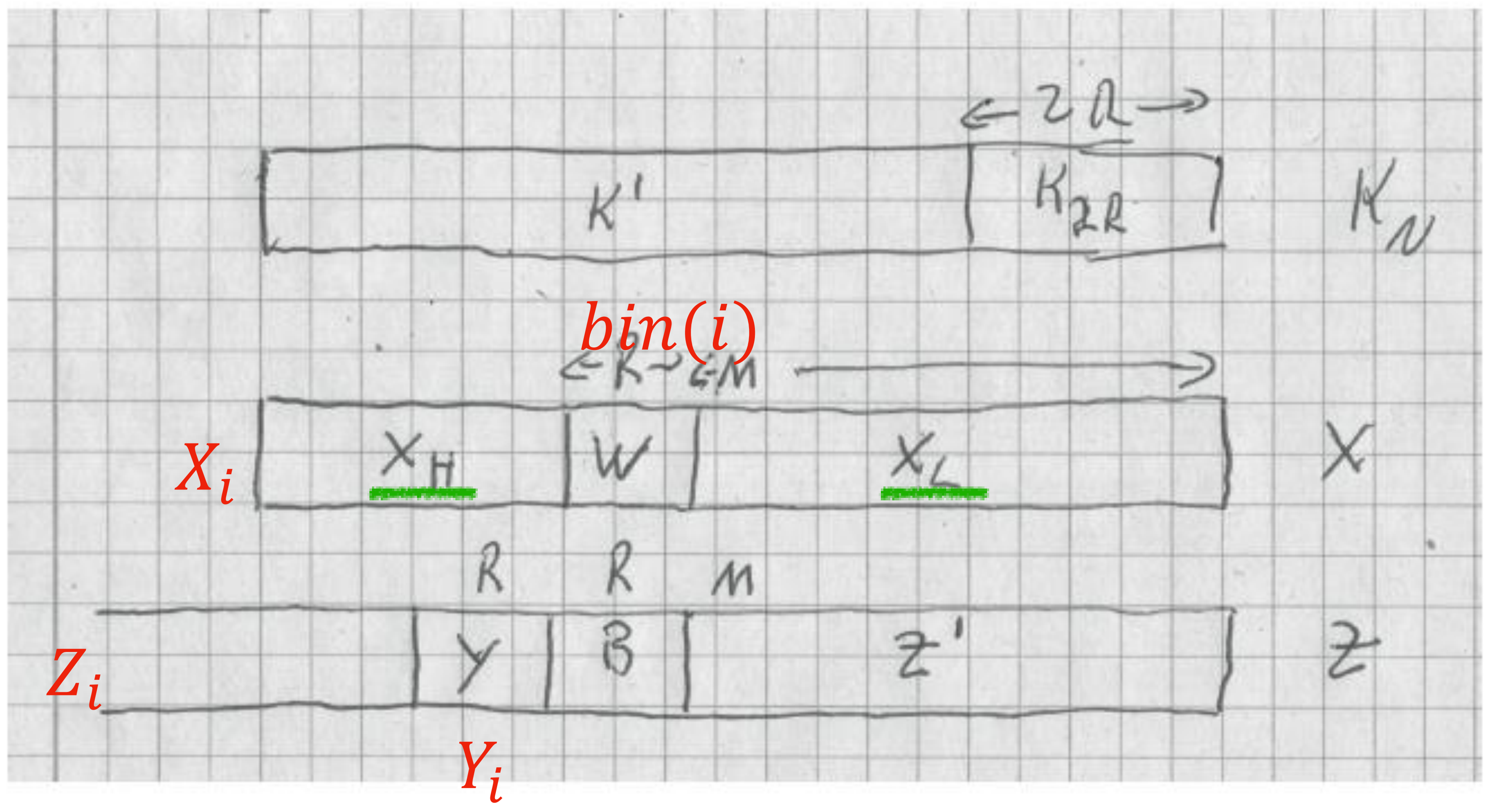

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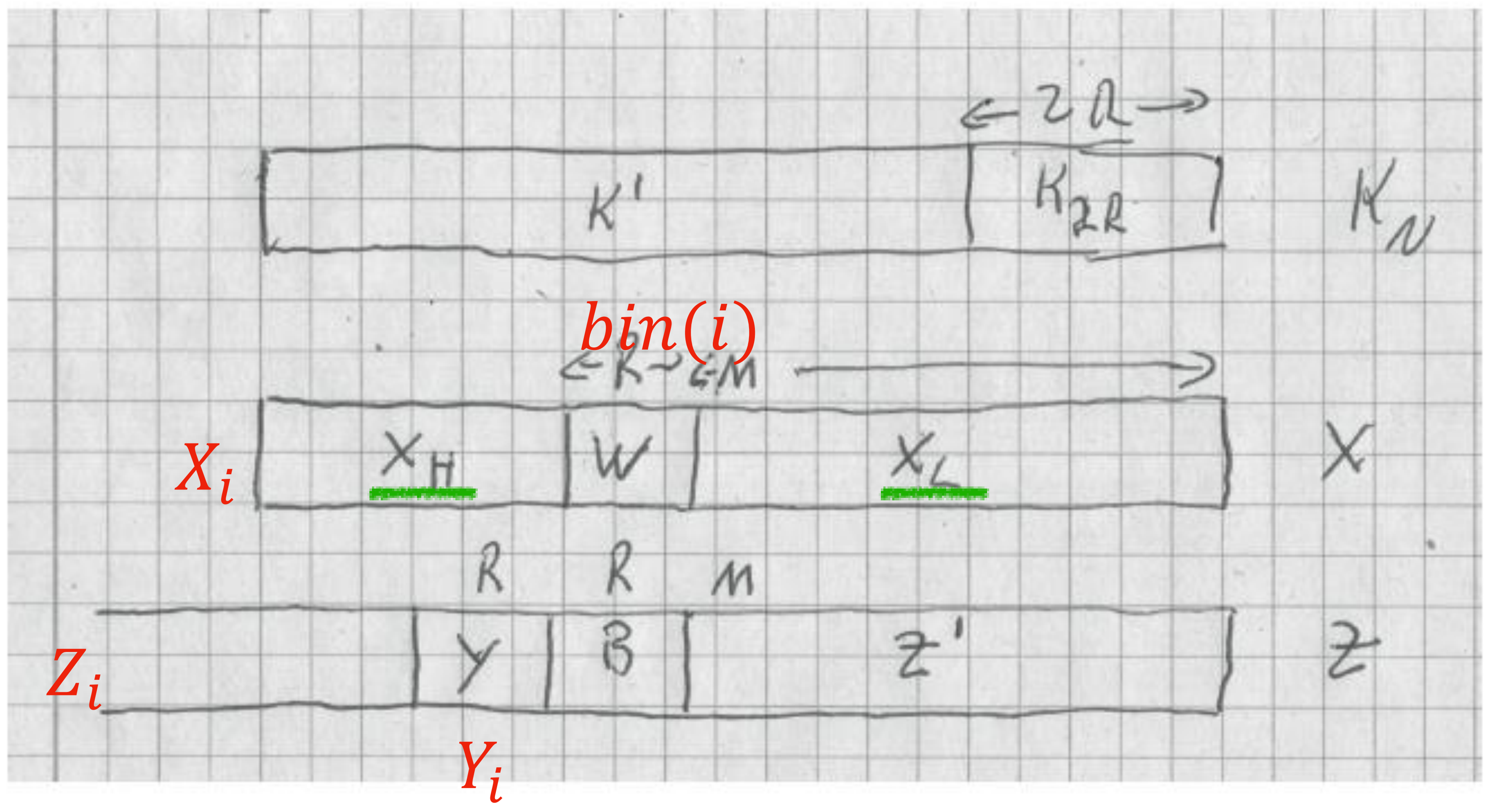

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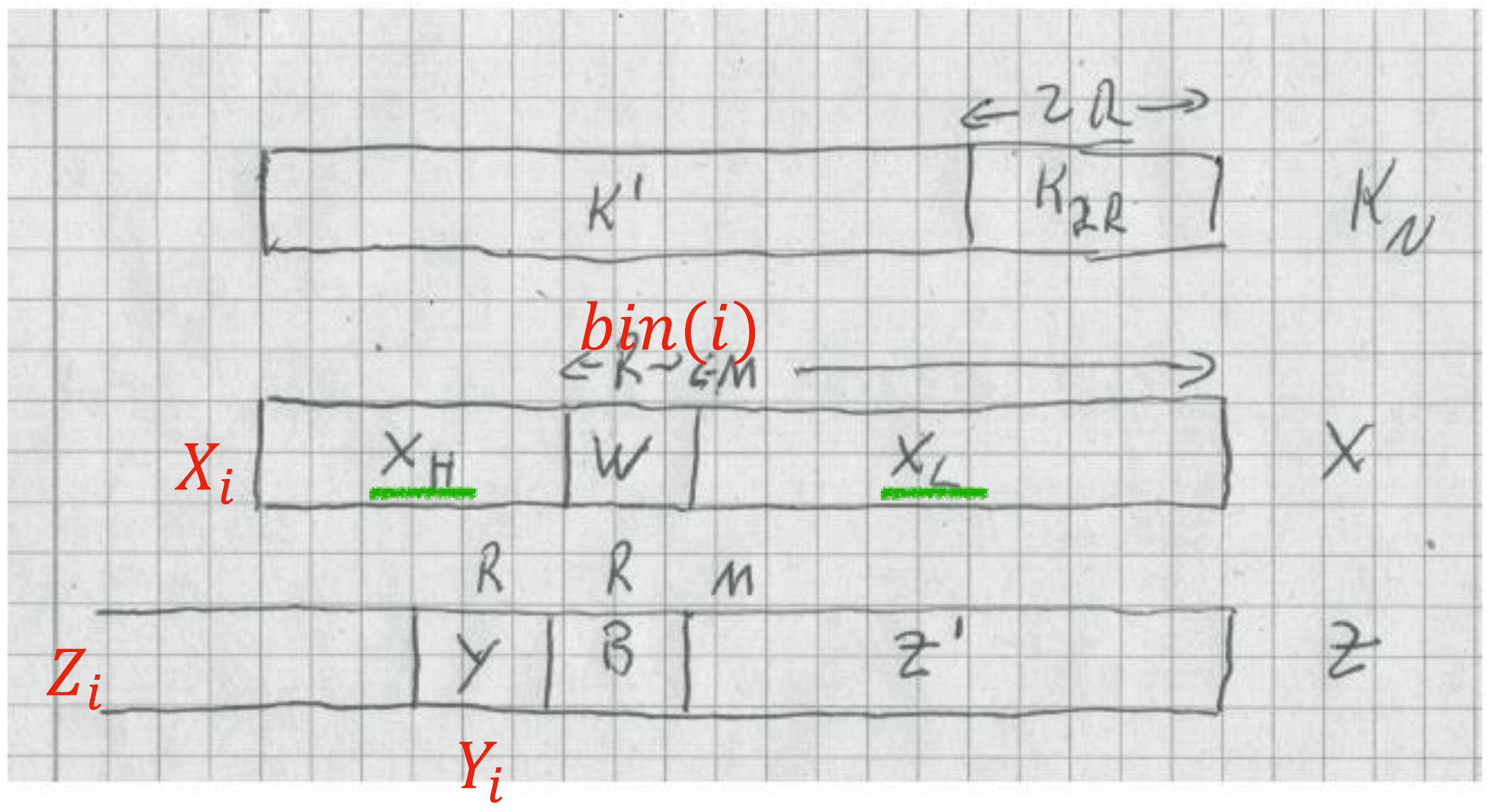

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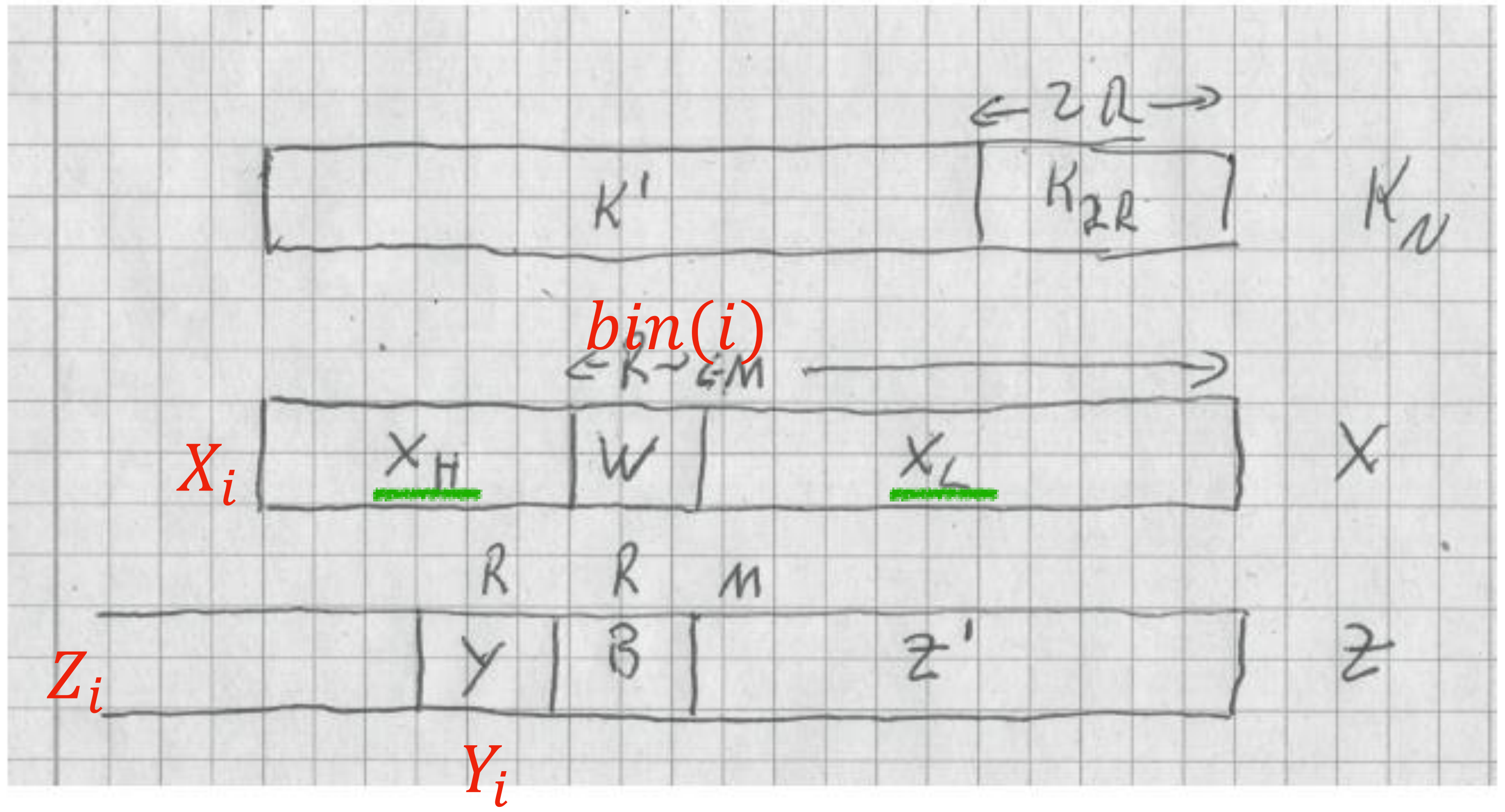

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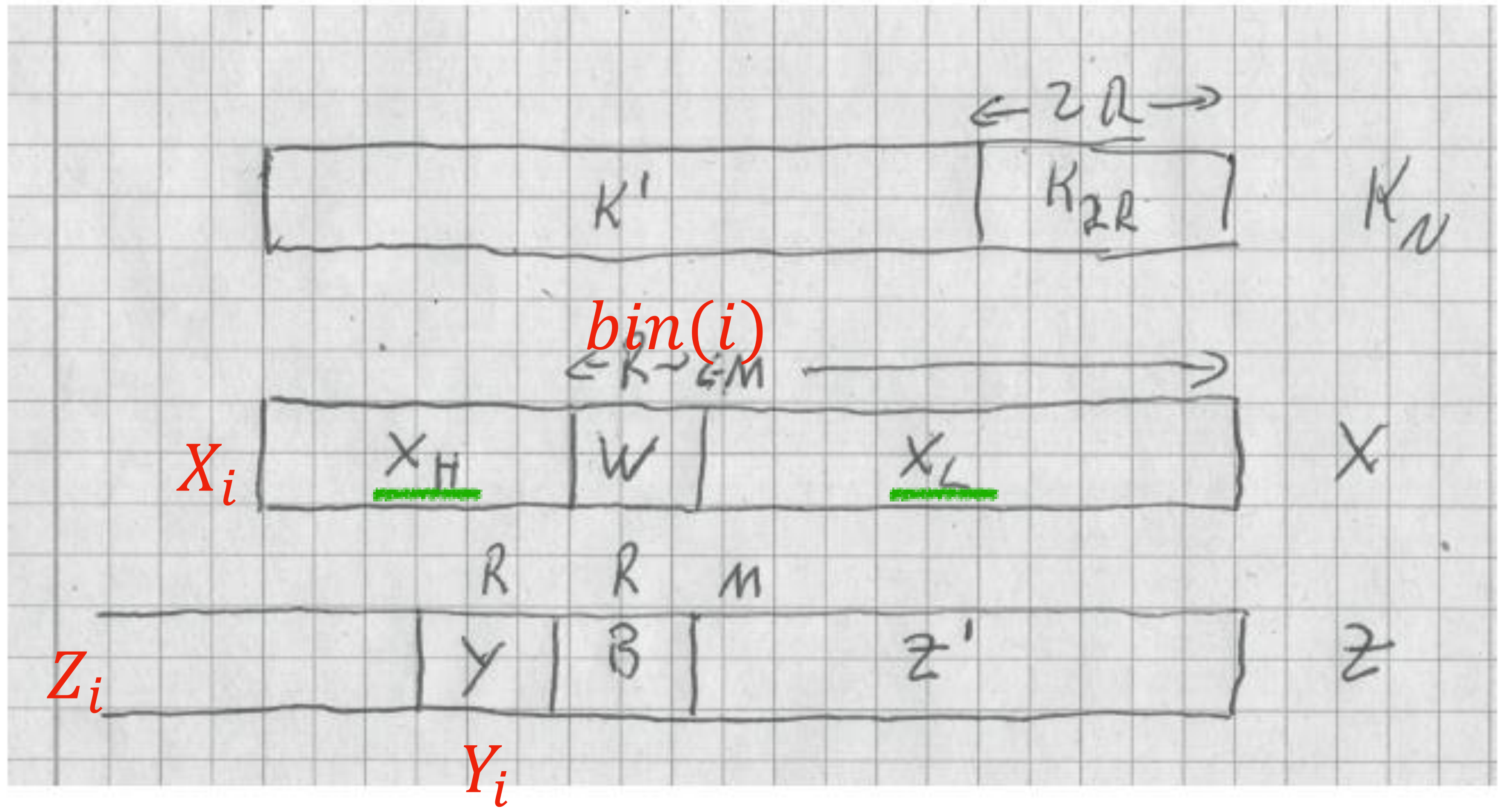

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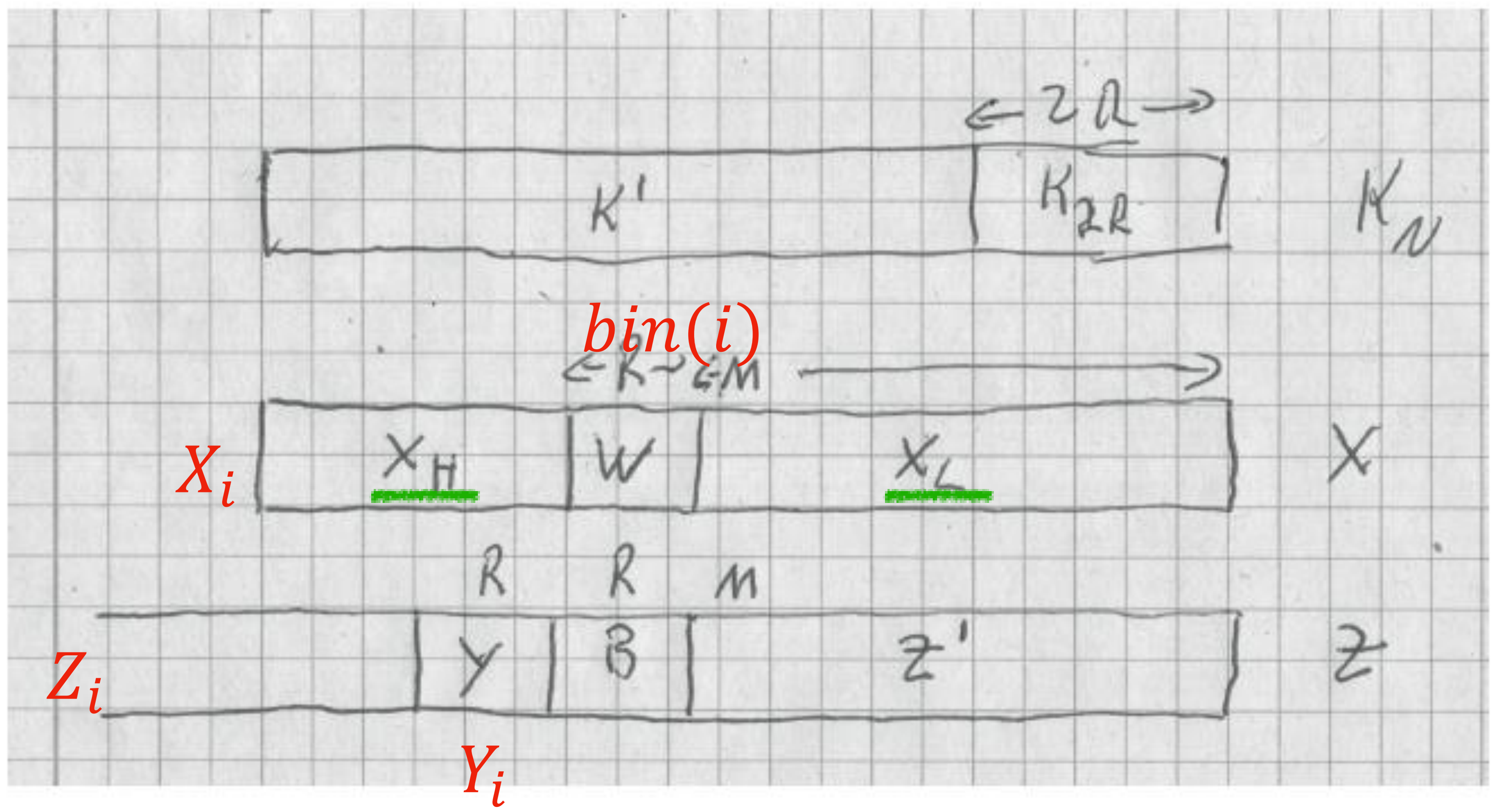

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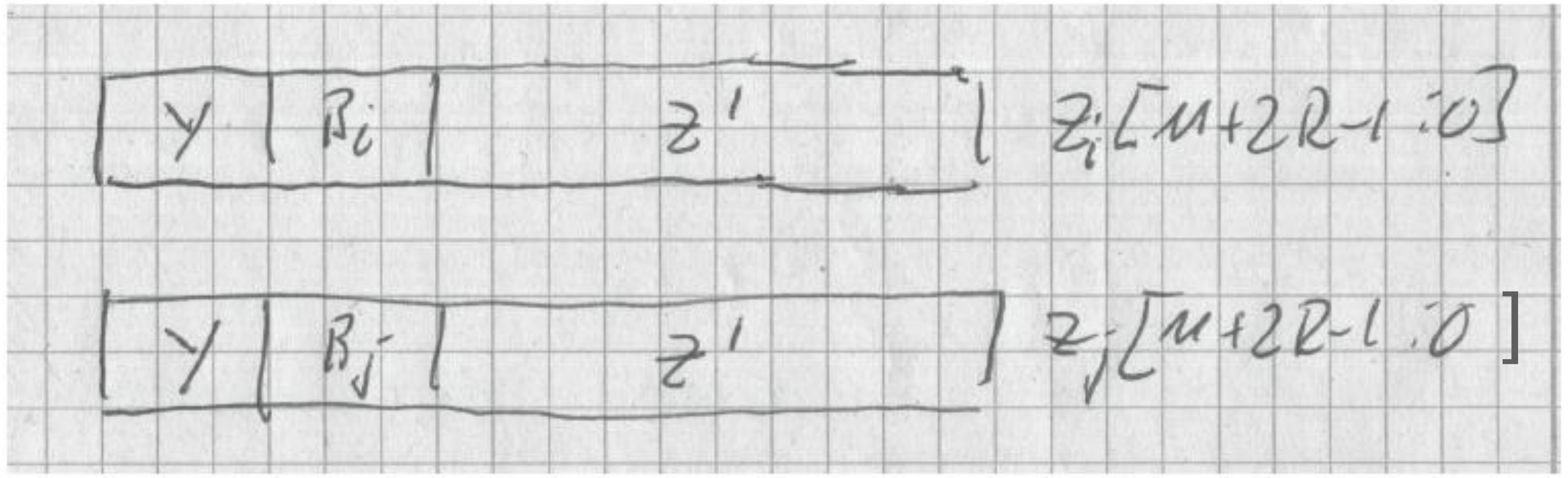

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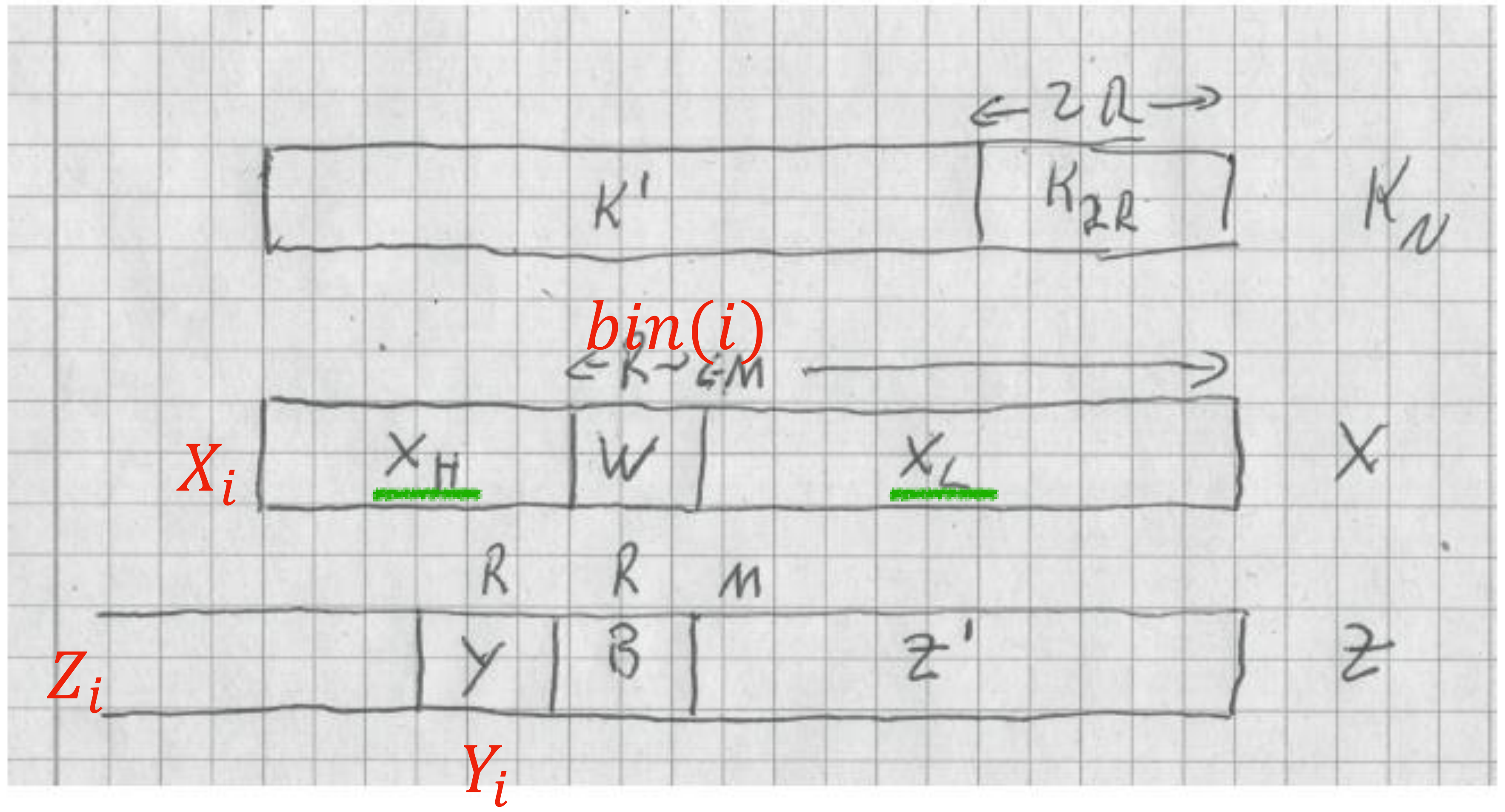

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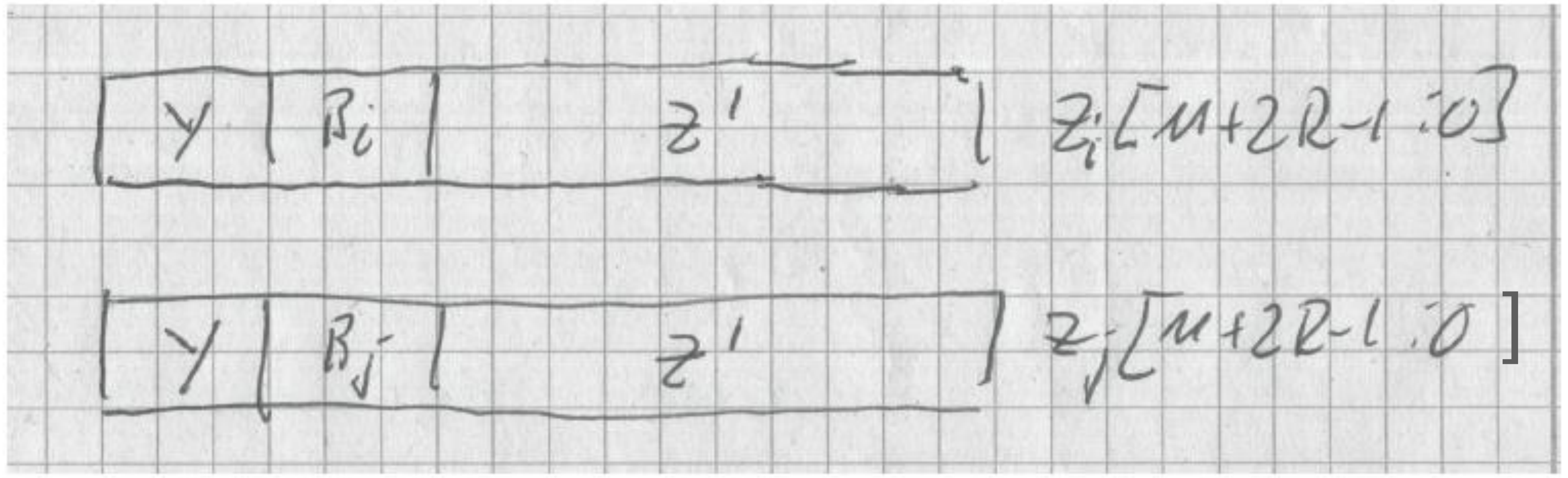

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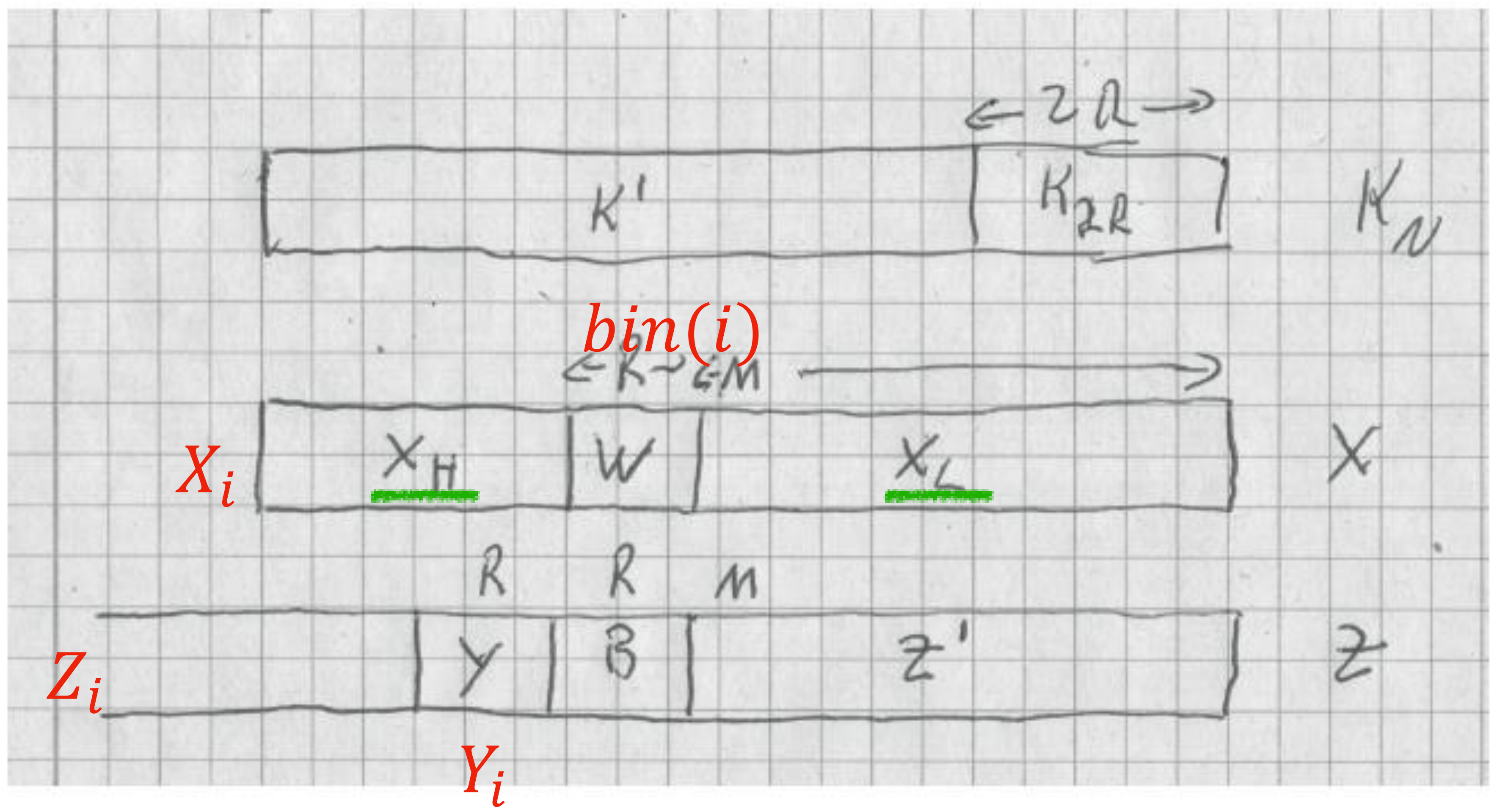

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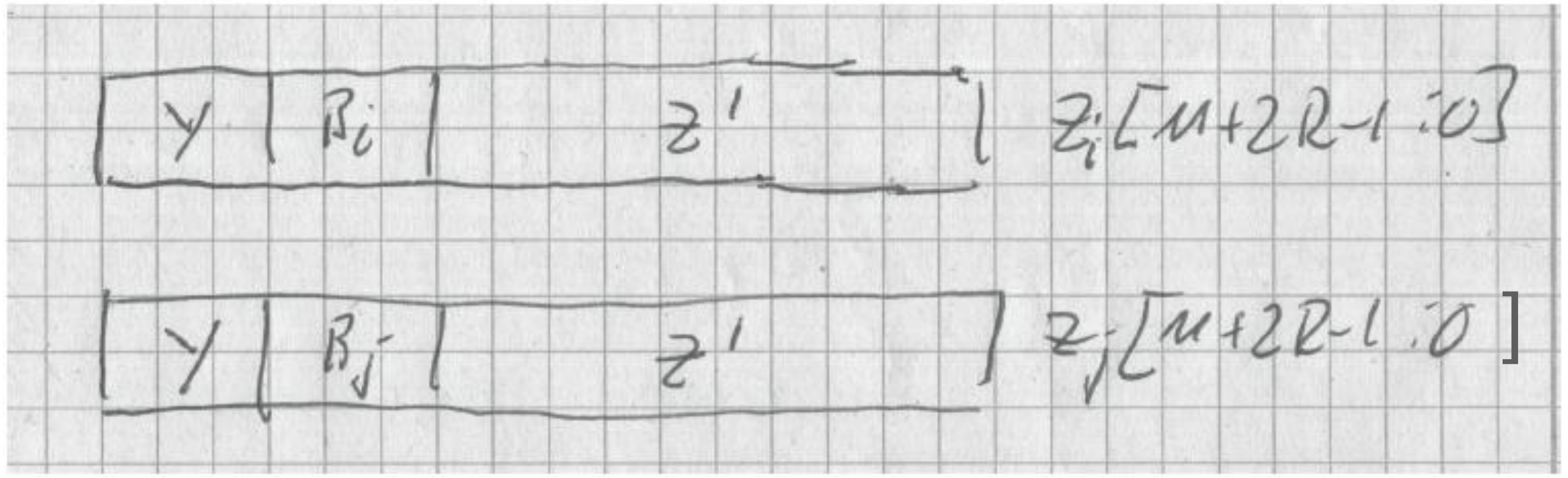

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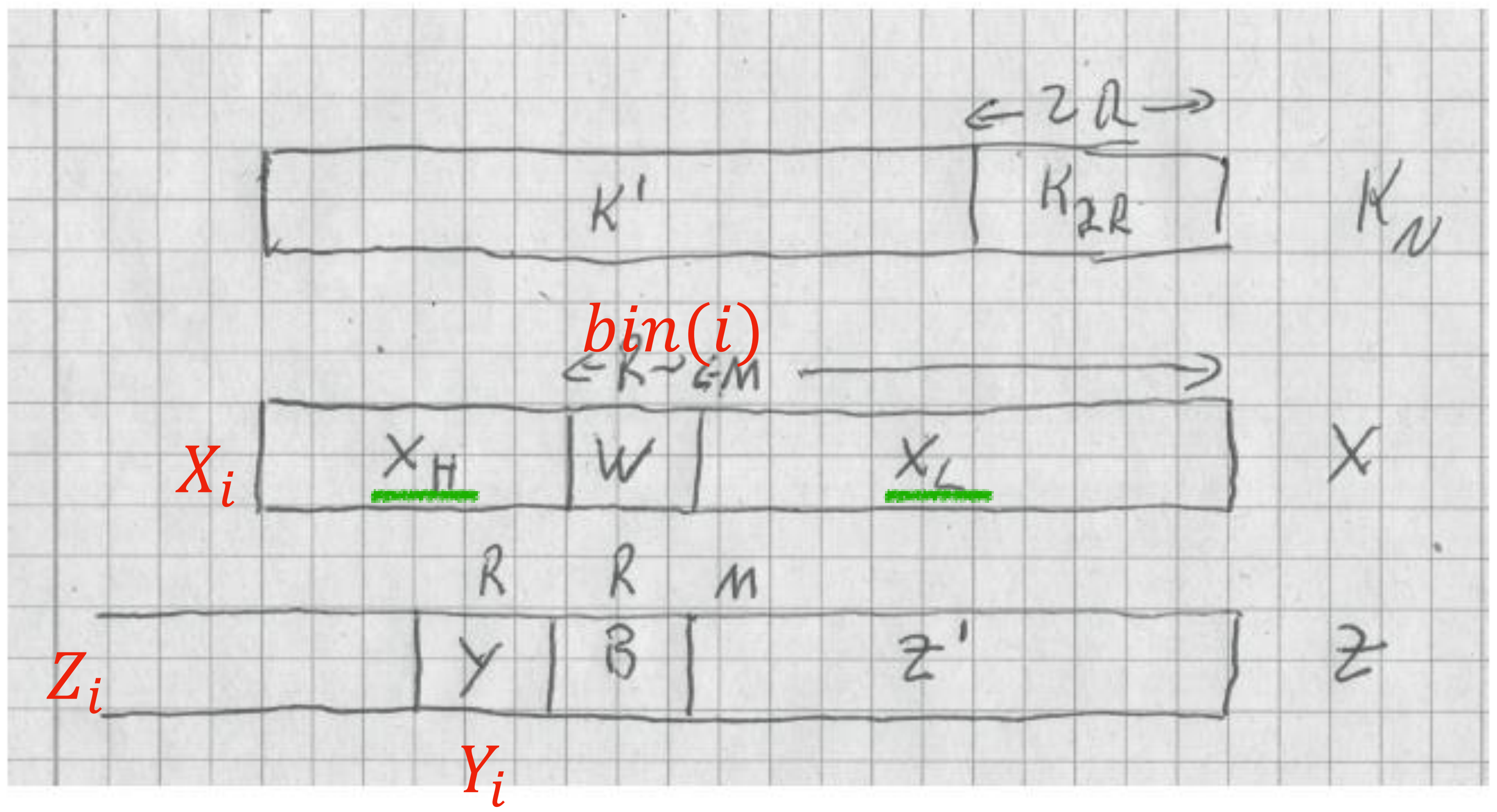

Lemma 3. *For any given $Y \in \mathbb{B}^R$ there exist at most 2 indices i such that $Y_i = Y$, i.e. there are at most 2 possibilites for W.*

- abbreviate for all i

$$x_i = \langle X_i \rangle \ , \ y_i = \langle Y_i \rangle \ , \ z_i = \langle Z_i \rangle \ , k_i = \langle K_i \rangle \ , k' = \langle K' \rangle \ , \ k_{2R} = \langle K_{2R} \rangle$$

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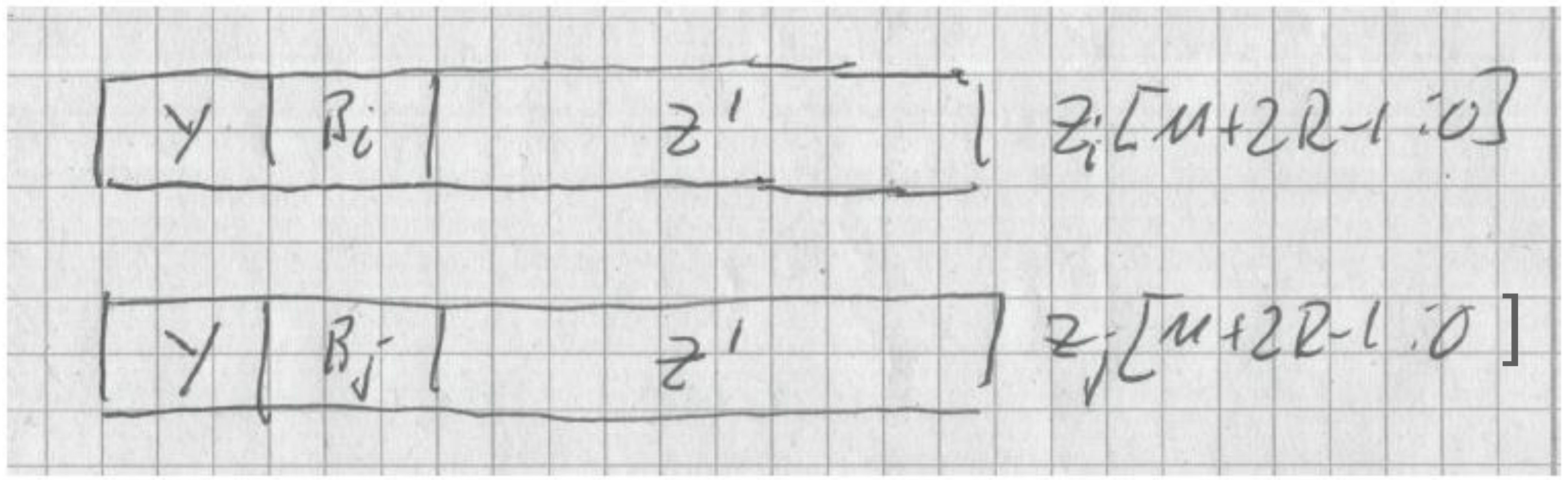

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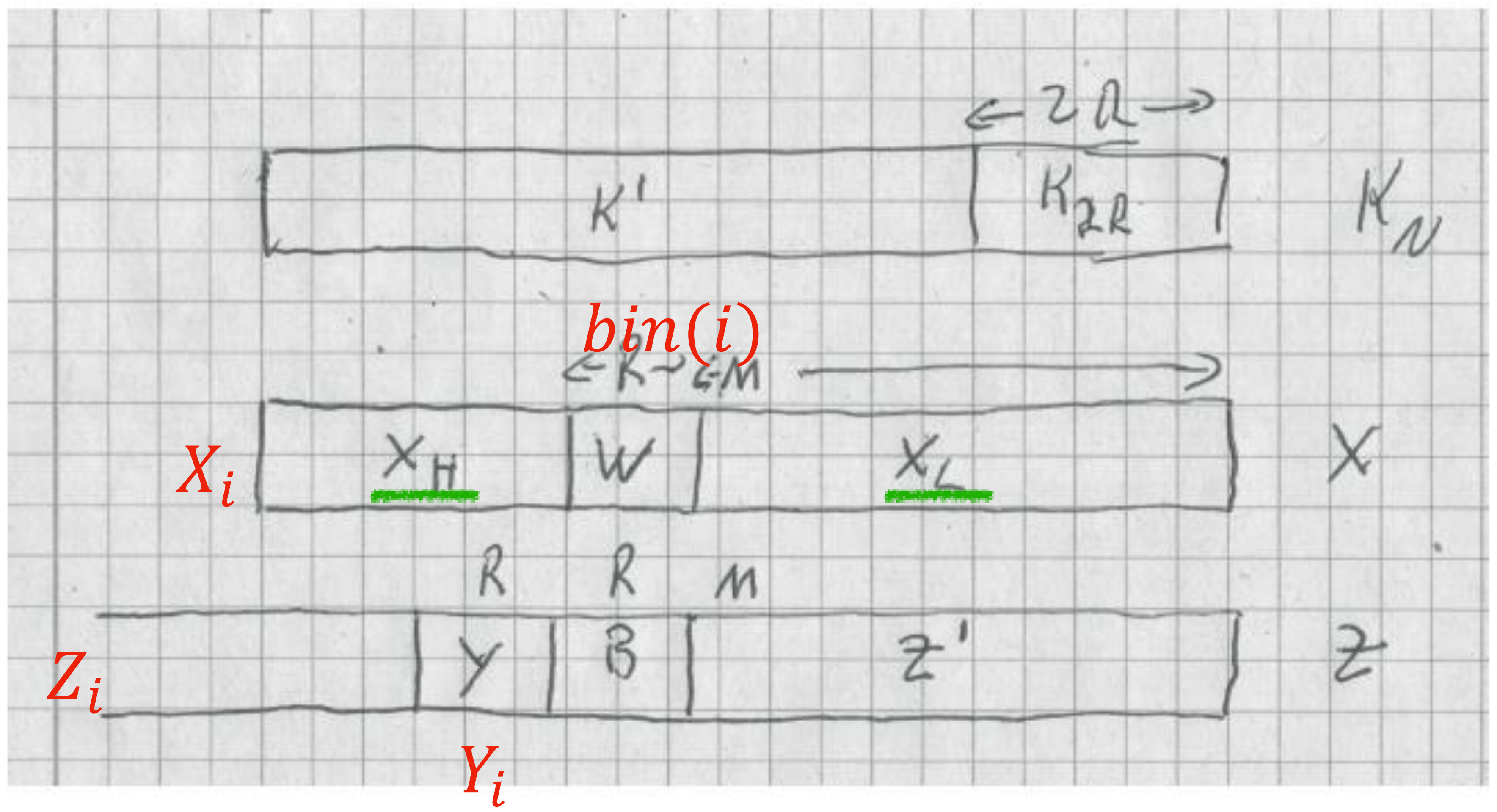

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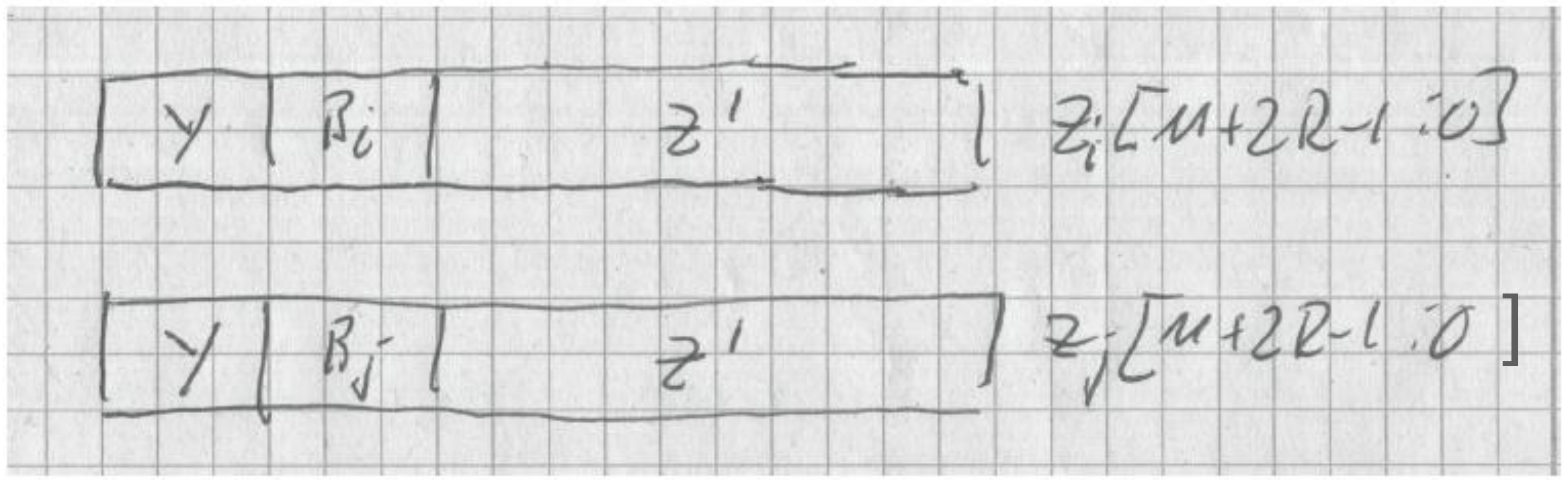

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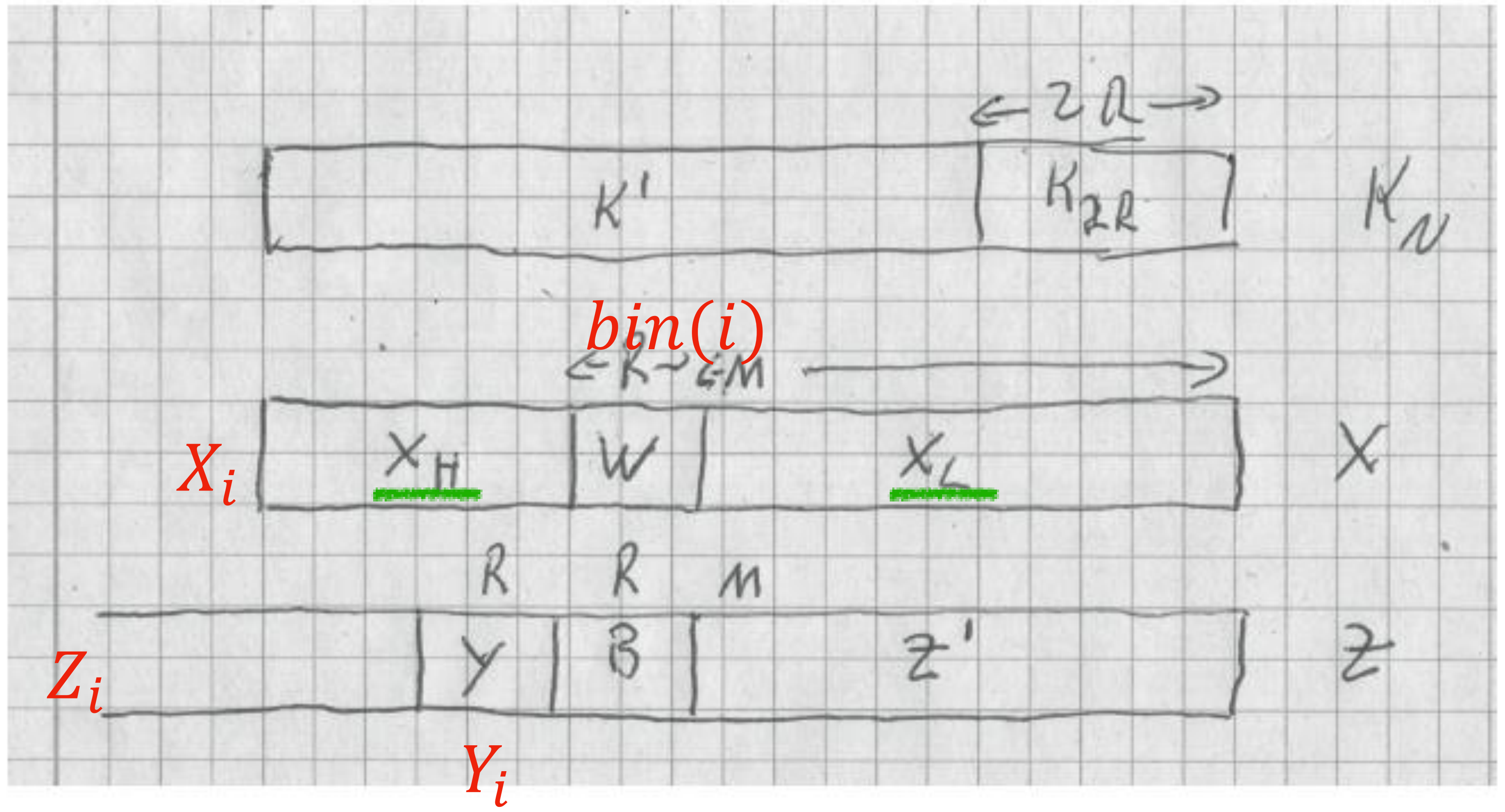

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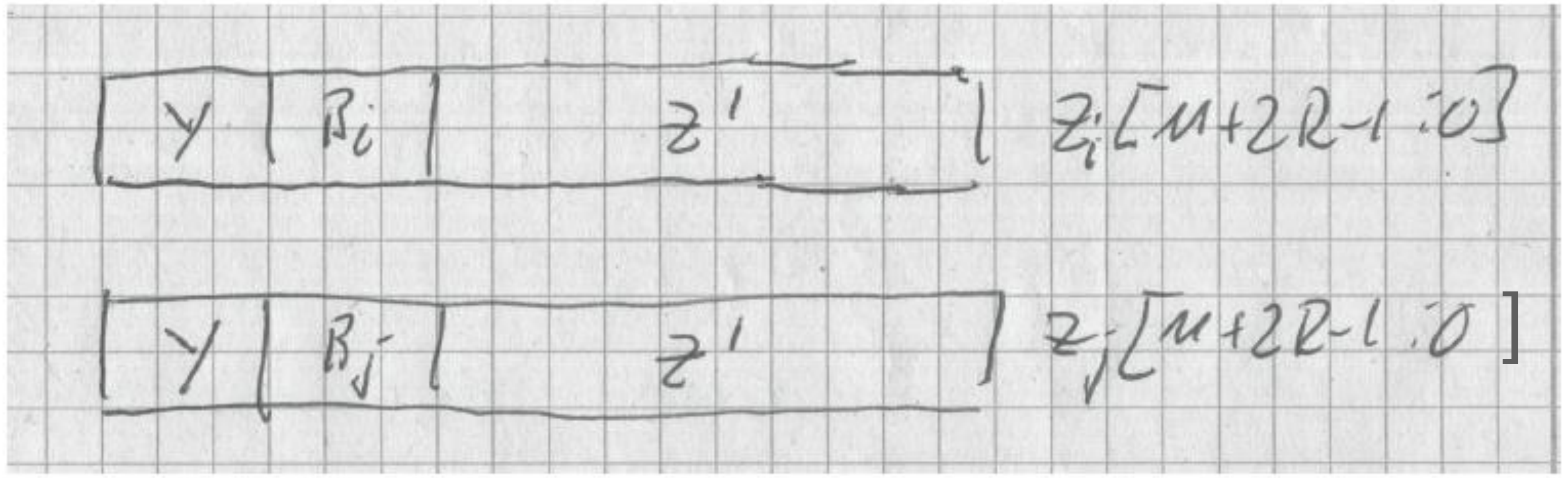

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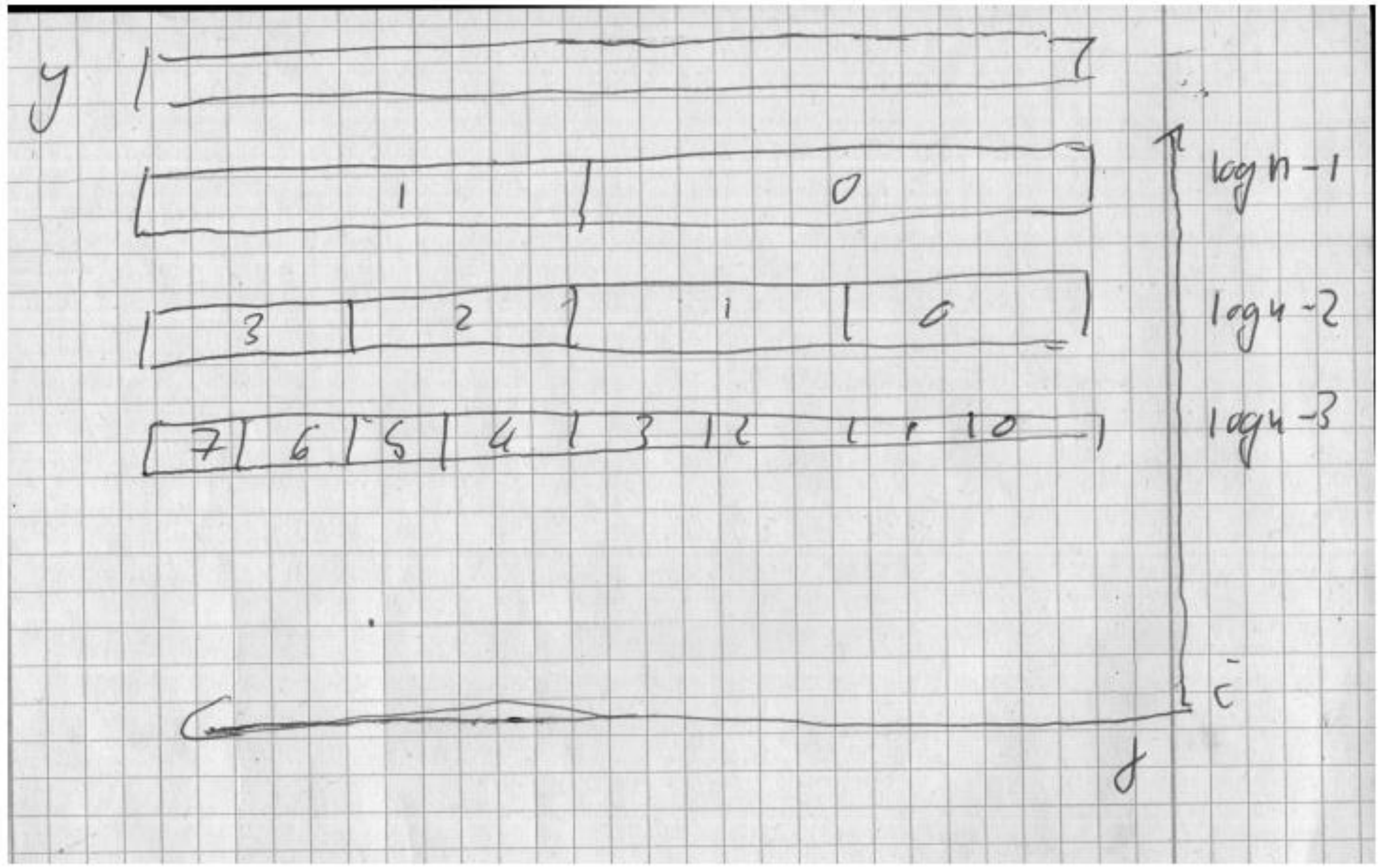
4 Overlap argument

Let

- *Mult* be $t(N)$ time bounded k-tape online TM performing the multiplication

partitioning operands: For powers of two N and $i \in [0 : \log N - 1]$ partition operands $Y \in \mathbb{B}^N$ into $N/2^i$ intervals $Y_{i,j}$ of length 2^i as shown in figure 3.

$$Y_{i,j} = [(j+1) \cdot 2^i - 1 : j \cdot 2^i] \quad \text{for } j \in [0 : N/2^i - 1]$$



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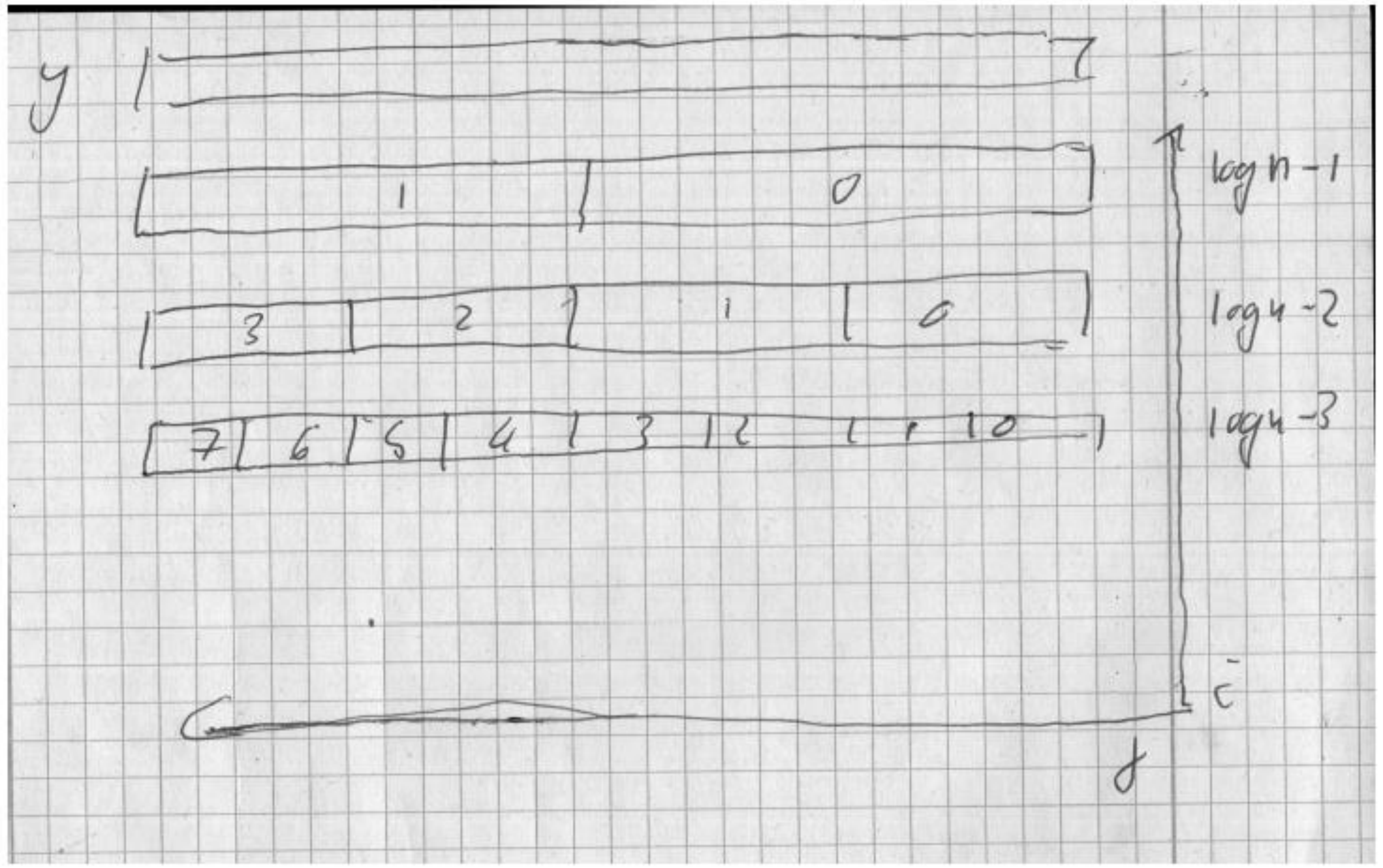

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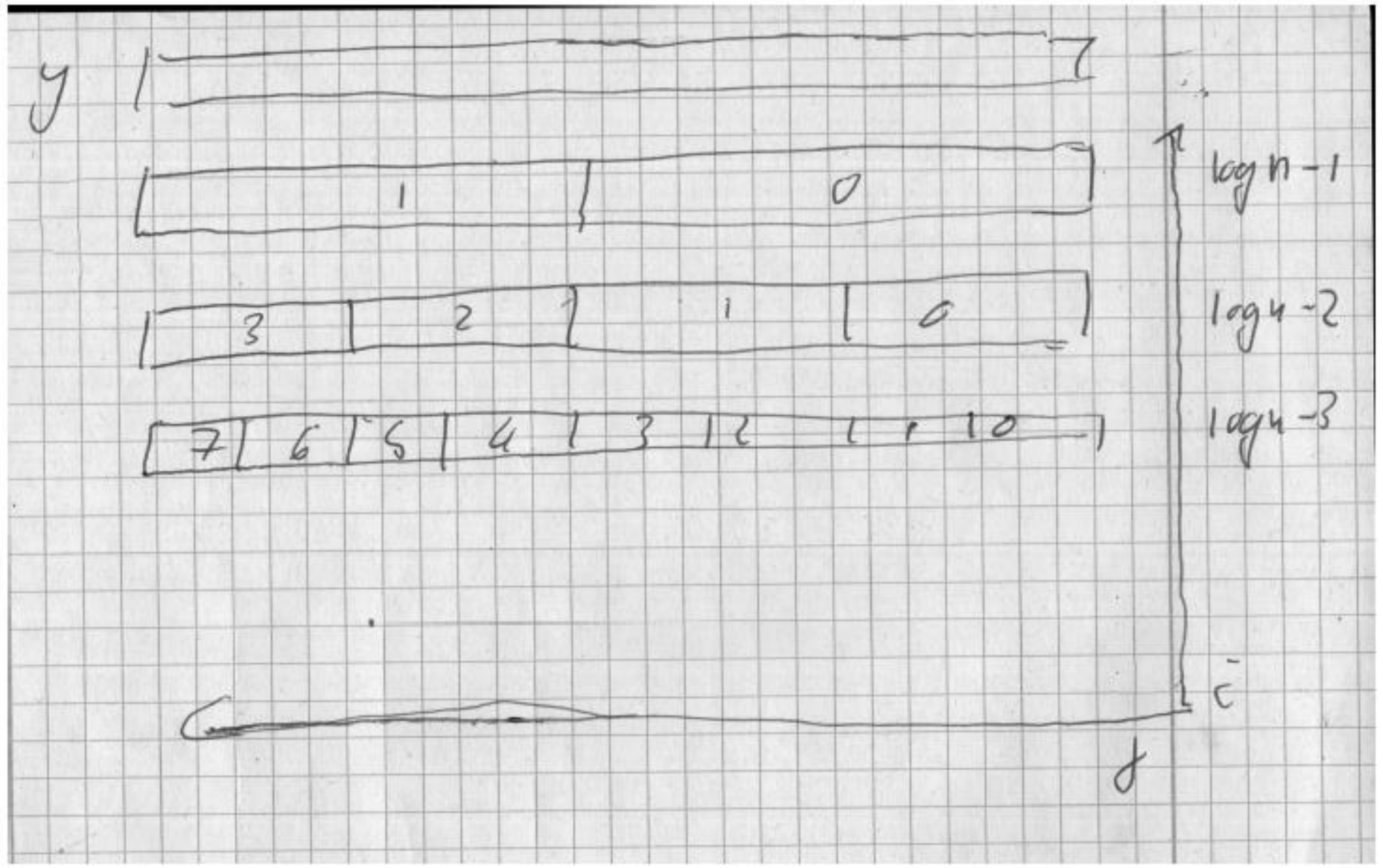

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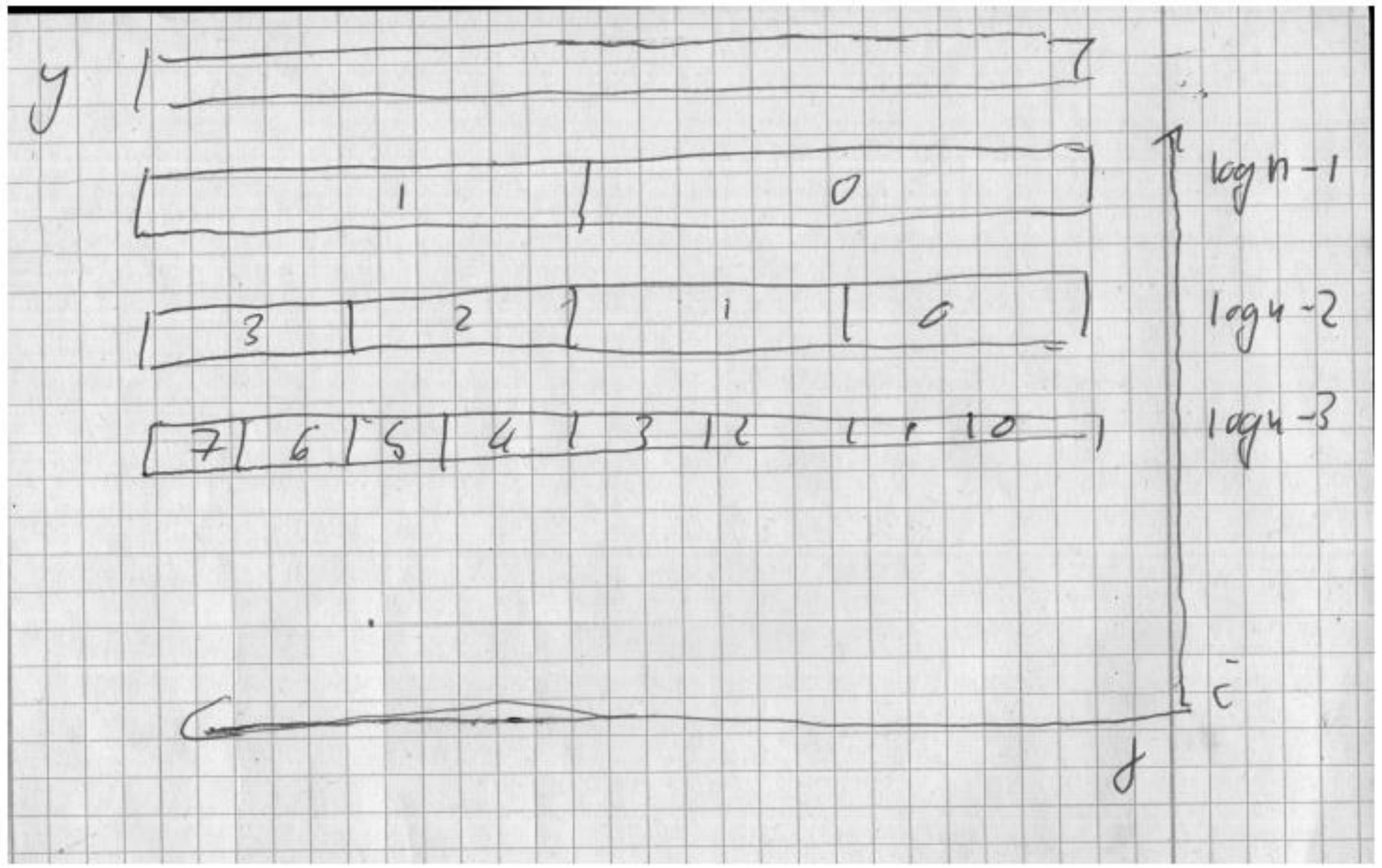
$$V = [0 : t(n)]$$

- edges $(t,t') \in E^\tau$ from t to t' iff $t' > t$ and there is a tape cell of tape τ visitend in steps t an t' but not in between.



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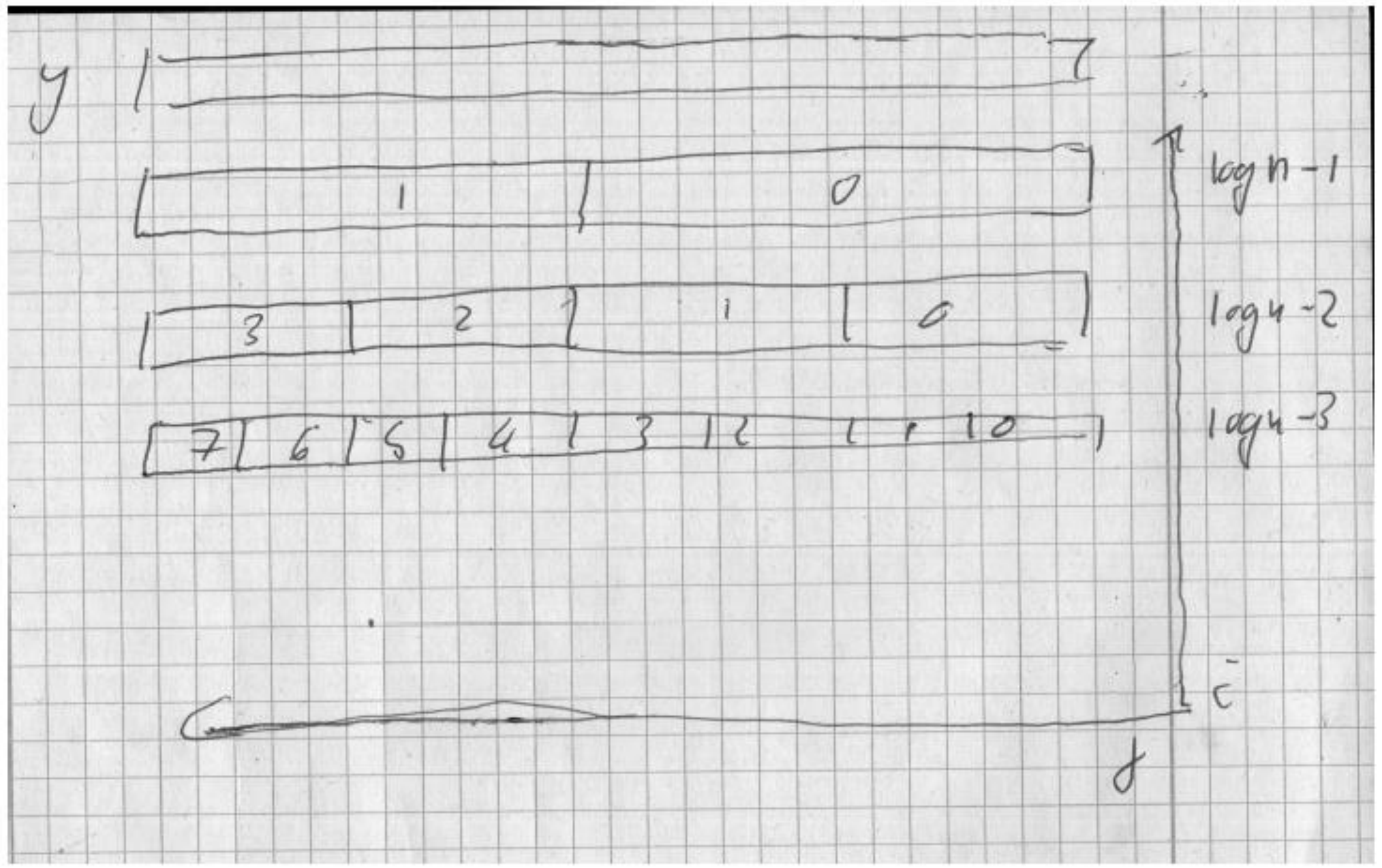

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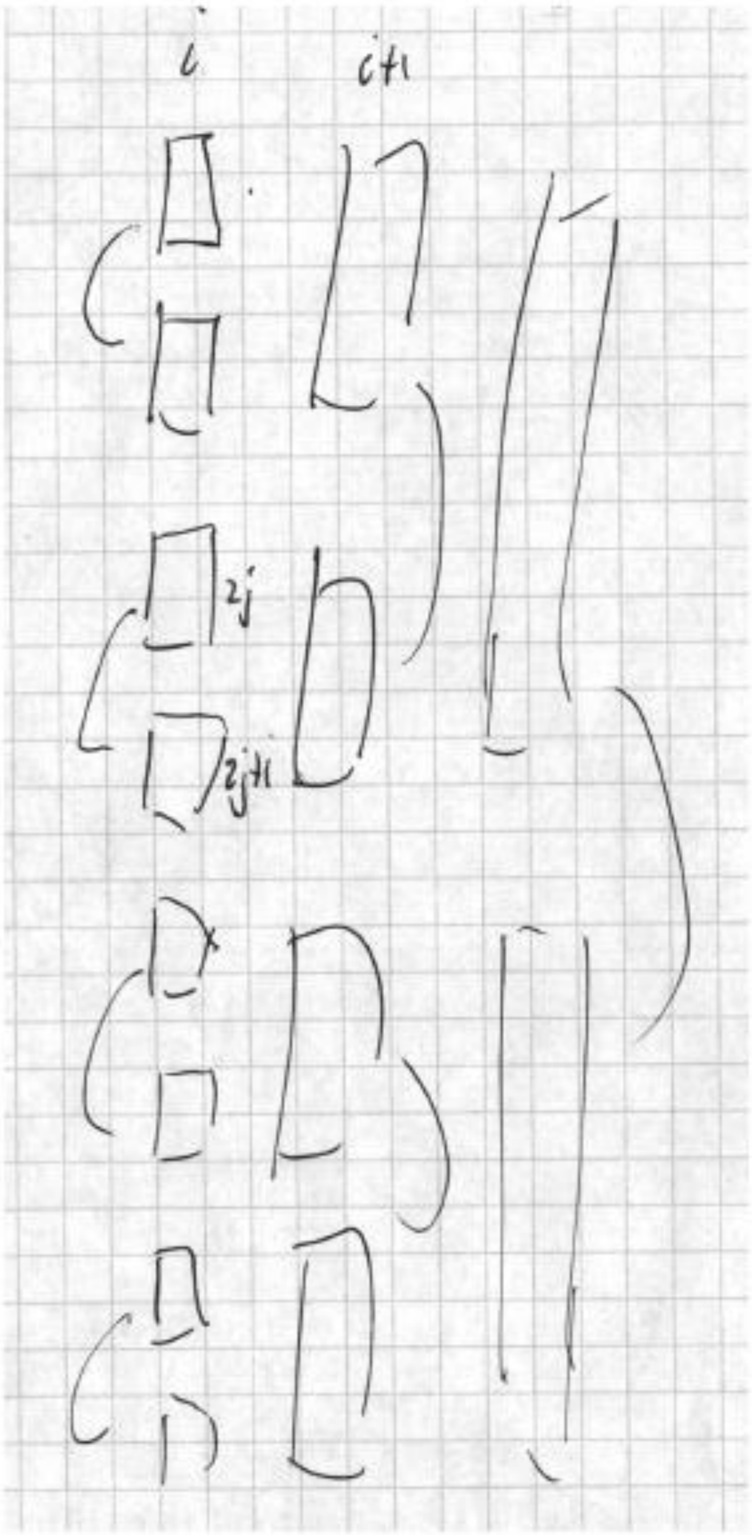

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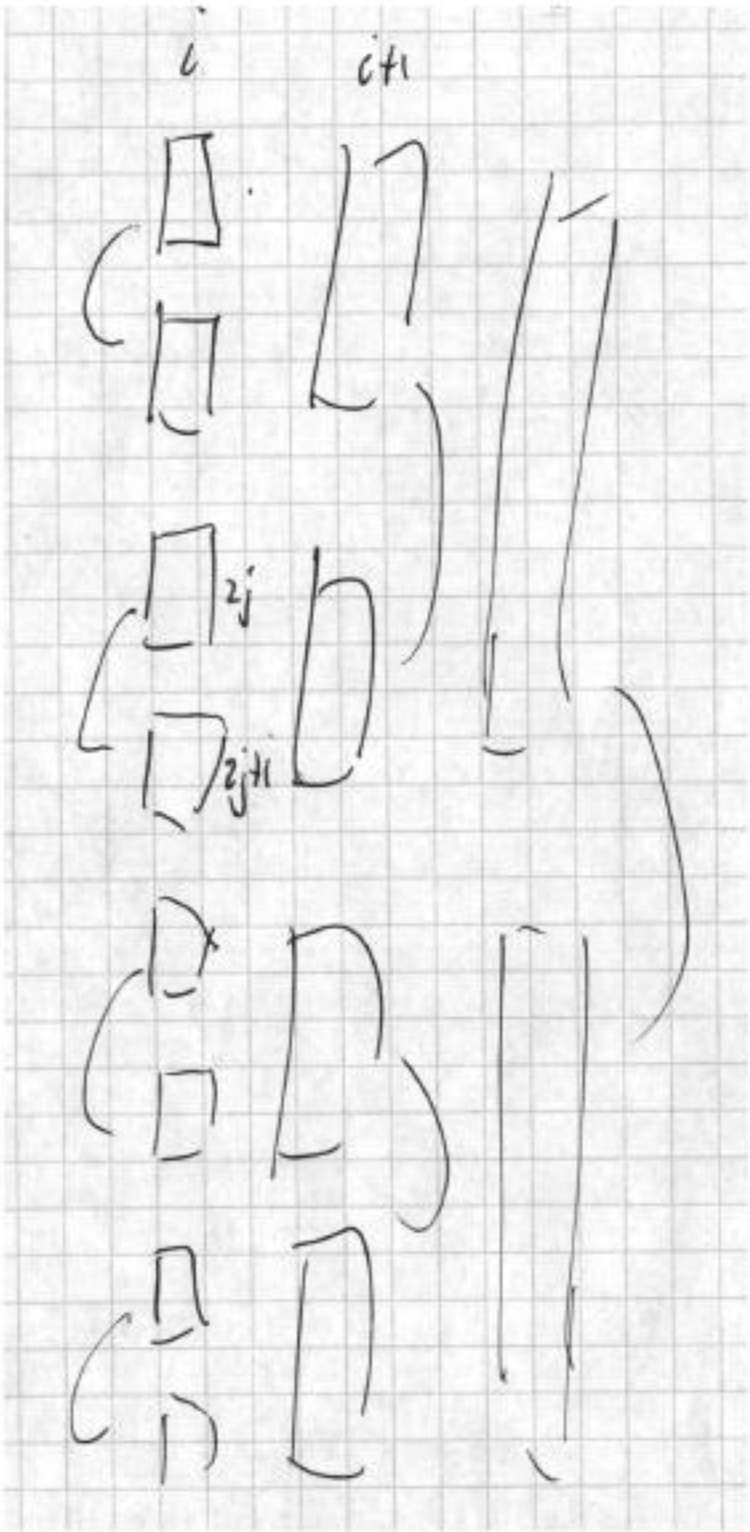

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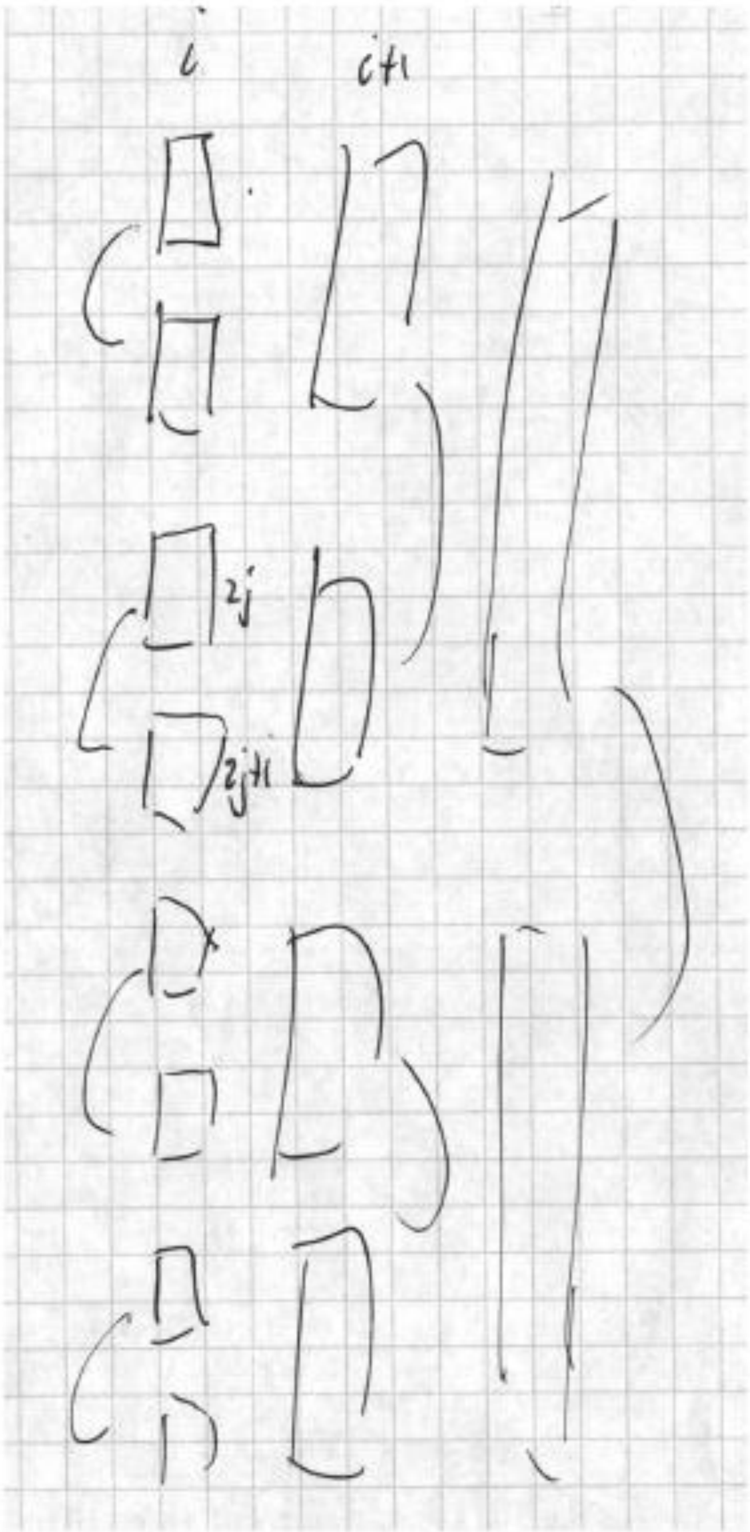

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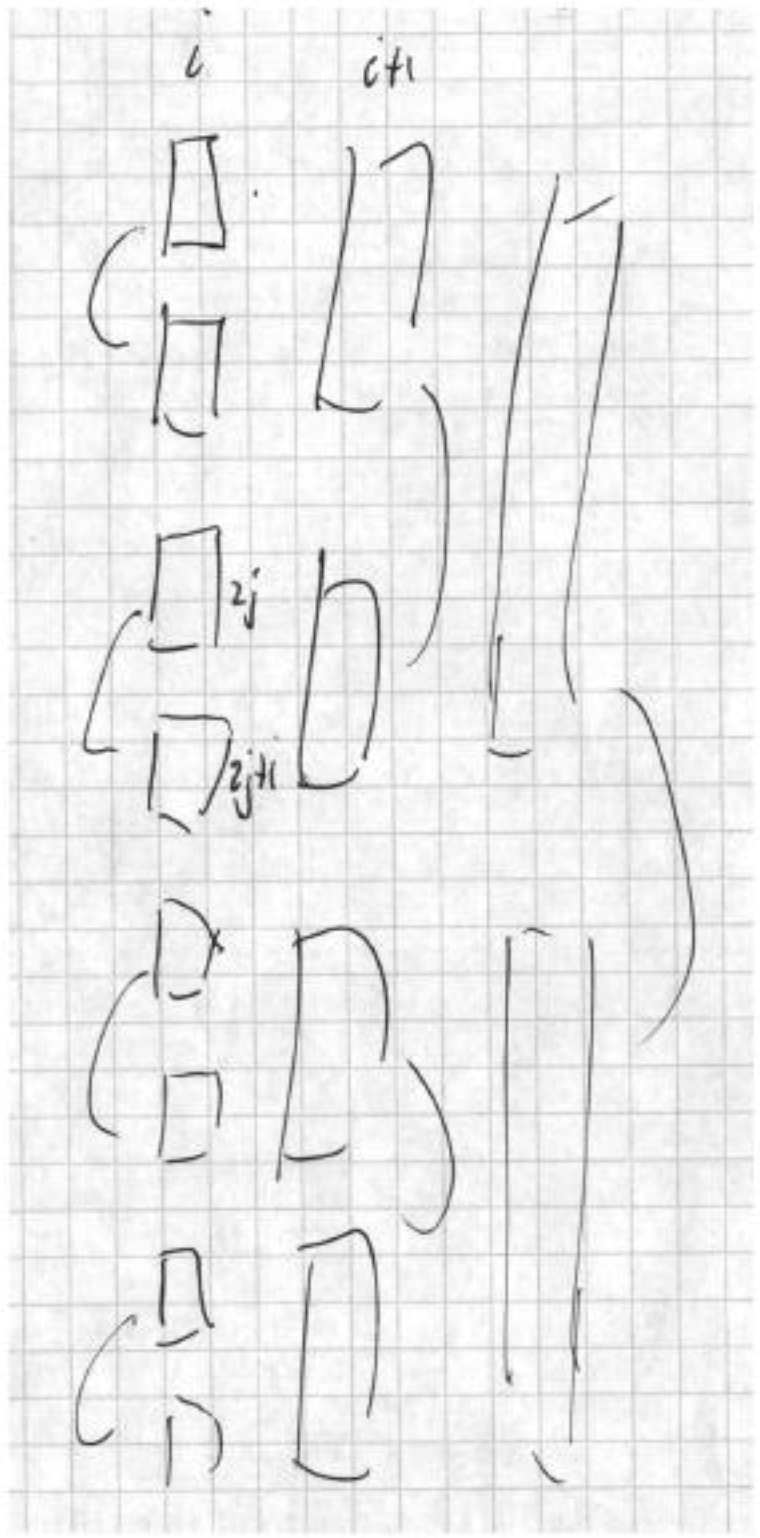

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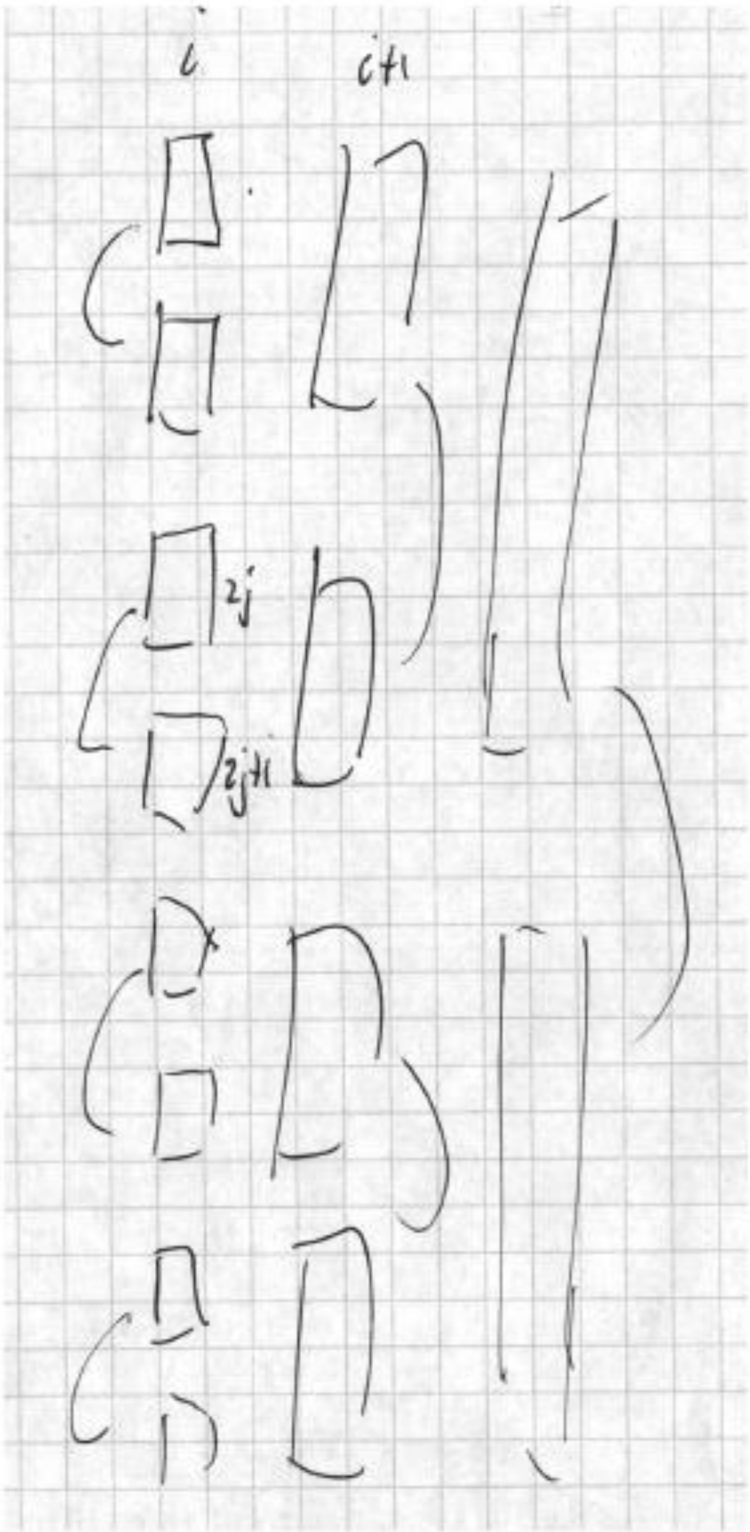

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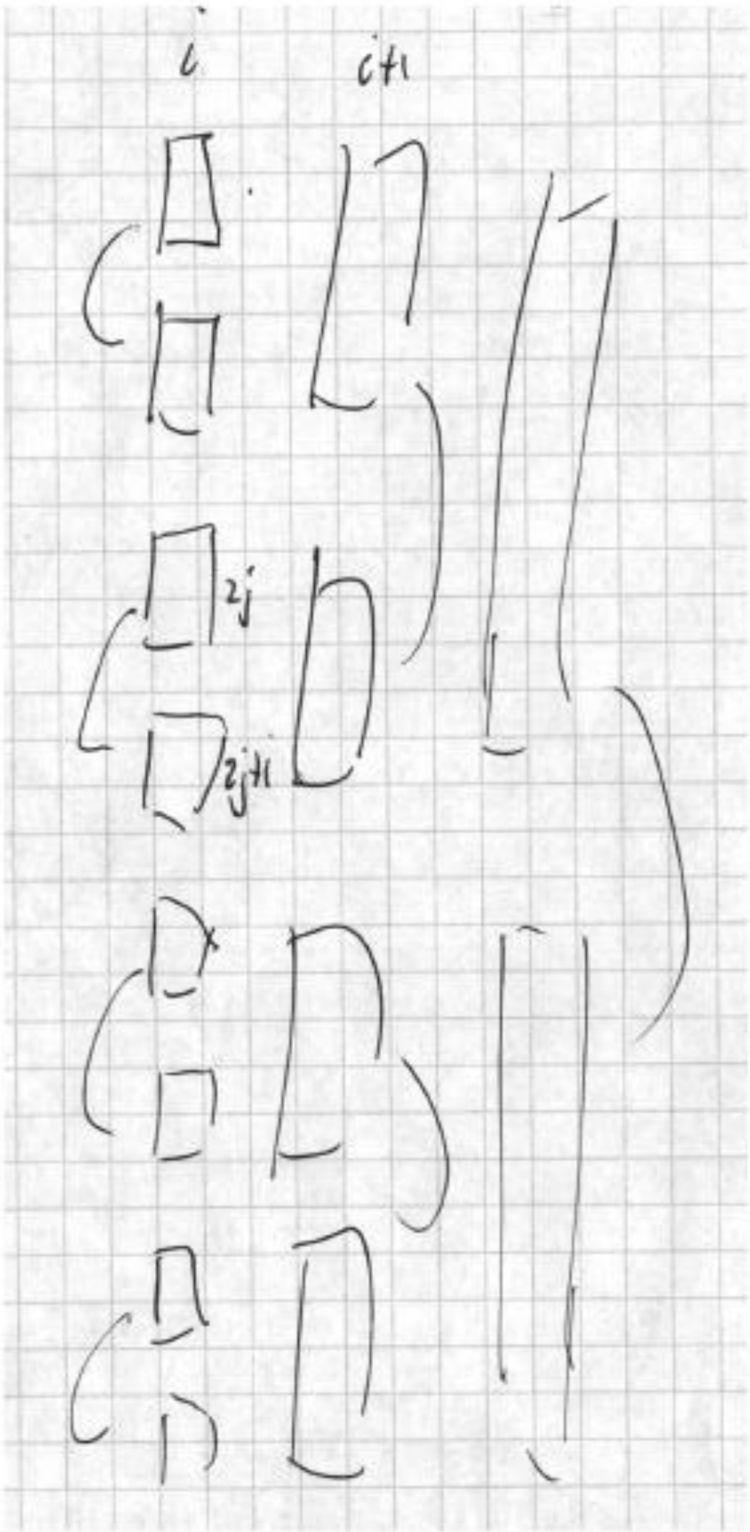

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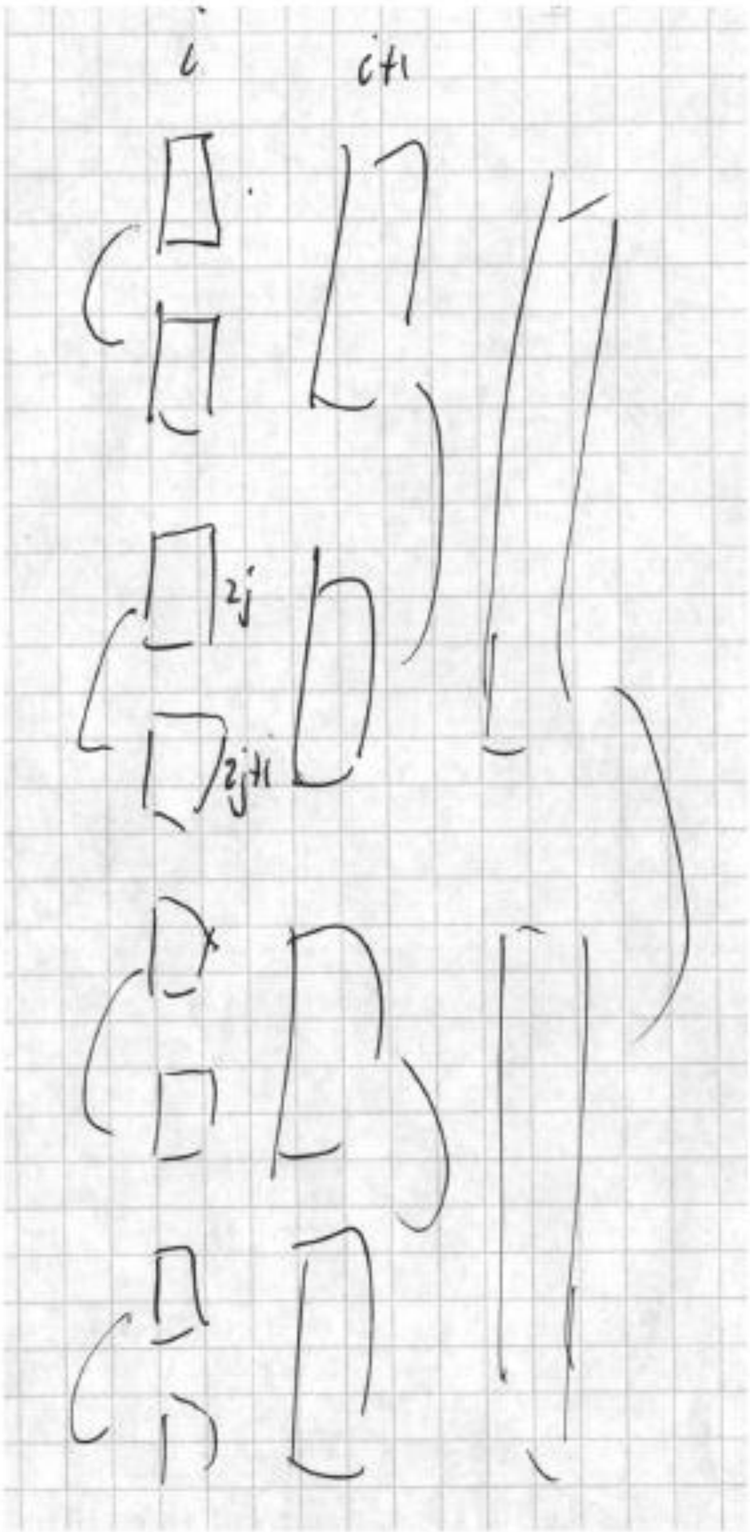

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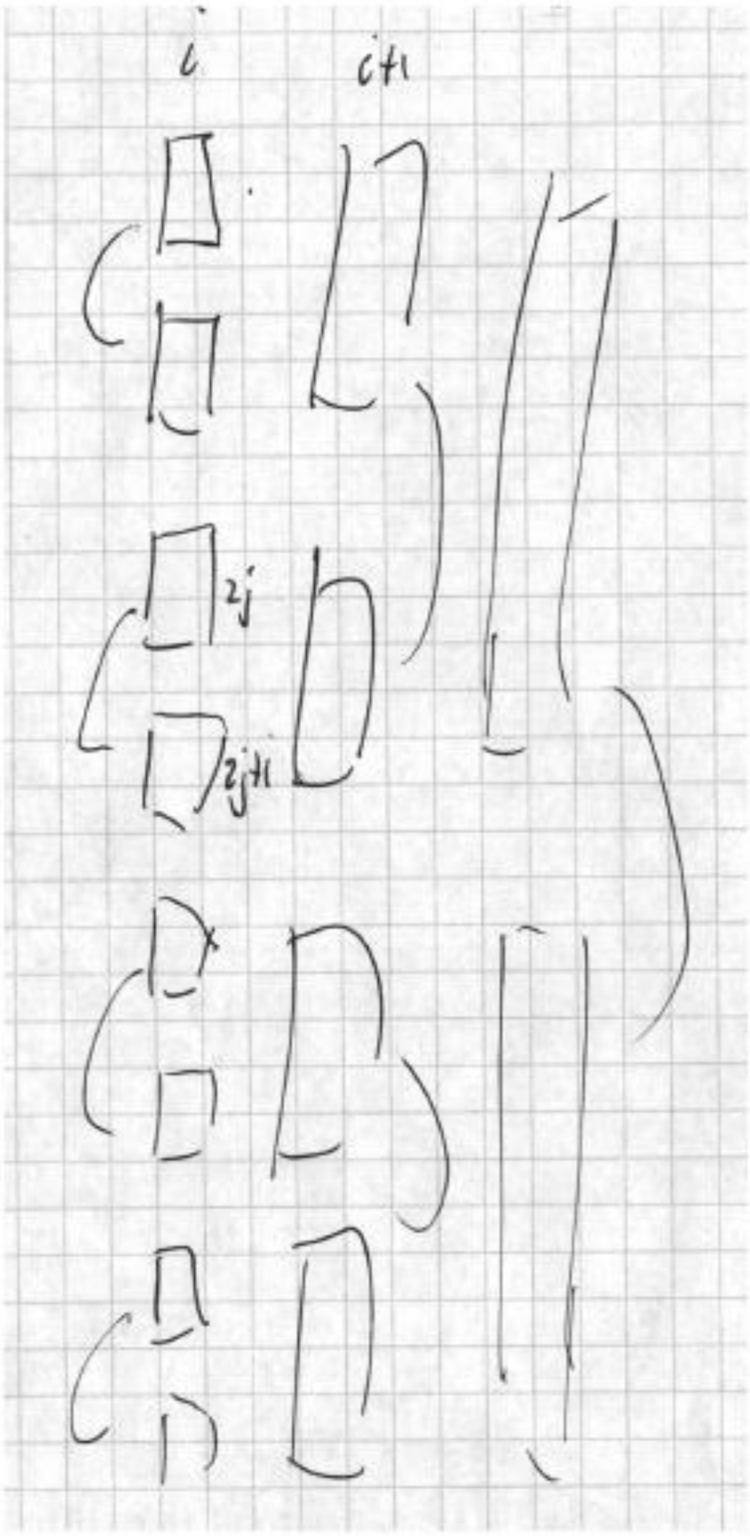

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def: overlap $\omega^{\tau}_{i,2j}$ on a tape : the portion of tape τ visited in time interval $T_{i,2j}$ and revisited in time inteval $T_{i,2j+1}$.

- on a single tape τ an edge in $E^{\tau}_{i,2j}$ can contribute at most 1 tape cell to overlap

$$|\omega^{\tau}_{i,2j}| \leq |E^{\tau}_{i,2j}| \leq T$$

result $res(i,2j)$ of time interval $T_{i,2j}$: contains for each tape τ

- the position of the overlap region $\omega^{\tau}_{i,2j}$ on the tape. It is an interval, so it suffices to specify the borders

$$O(\log(t(N)) \text{ bits}$$

- for all tapes τ inscription of the overlap region and head position at the end of time interval $T_{i,2j}$.

$$c \cdot |\omega^{\tau}_{i,2j}| + \log(t(N)) \text{ bits for some } c > 0$$

- state of $Mult$ at the end of time interval $T_{i,2j}$

$$O(1) \text{ bits}$$

- with size of overlap region $\omega_{i,2j}$ for all tapes

$$\boxed{|\omega_{i,2j}| = \sum_{\tau} |\omega^{\tau}_{i,2j}|}$$

we get

$$\boxed{|res(i,2j)| \leq c \cdot |\omega_{i,2j}| + O(\log(t(N)) \text{ bits}}$$

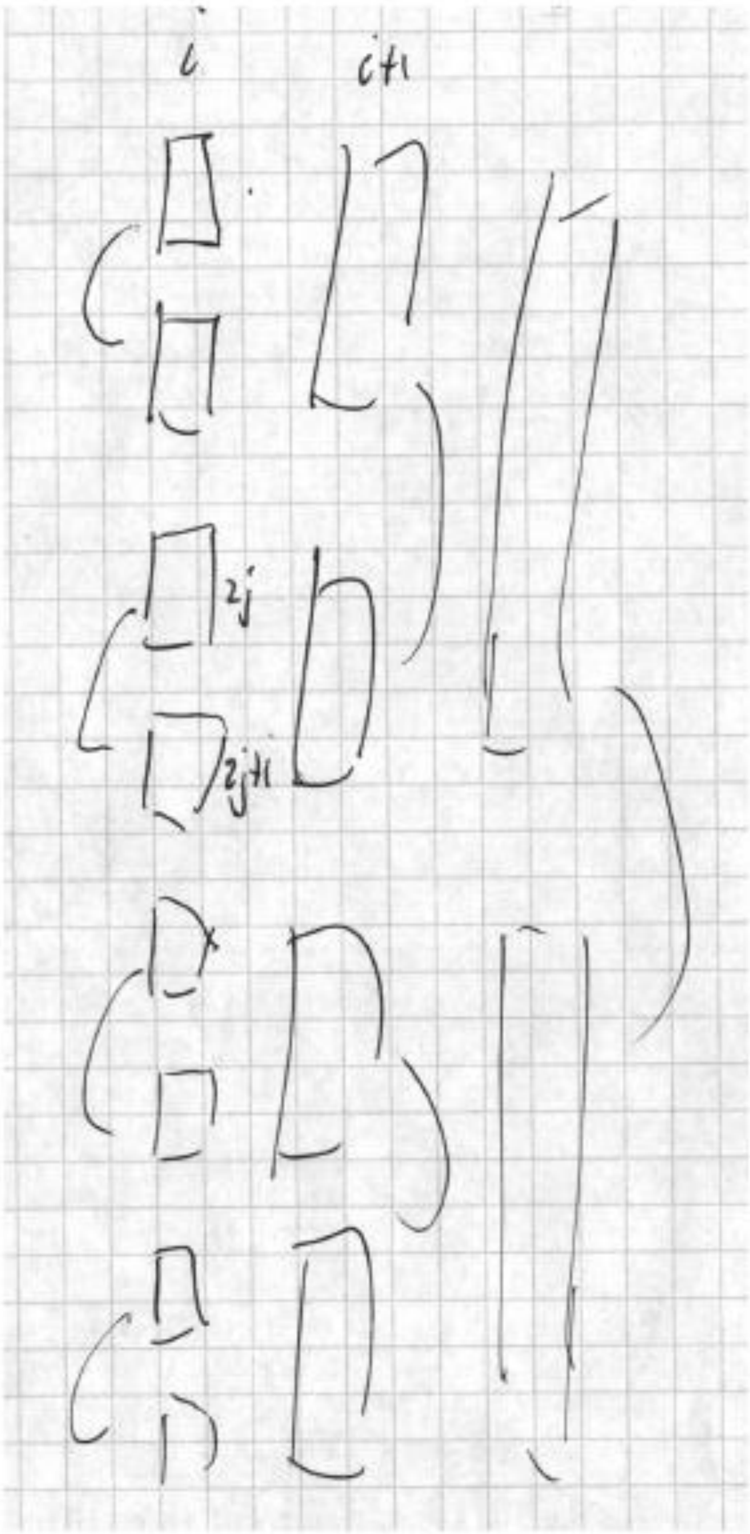

Figure 4: Partitioning edges into set $E_{i,2j}$. Edges in $E_{i,2j}$ go from steps in intervals $T_{i,2j}$ with even indices $2j$ to the next interval $T_{i,2j+1}$

$$T \geq |E^{\tau}| = \sum_i \sum_j |E^{\tau}_{i,j}|$$

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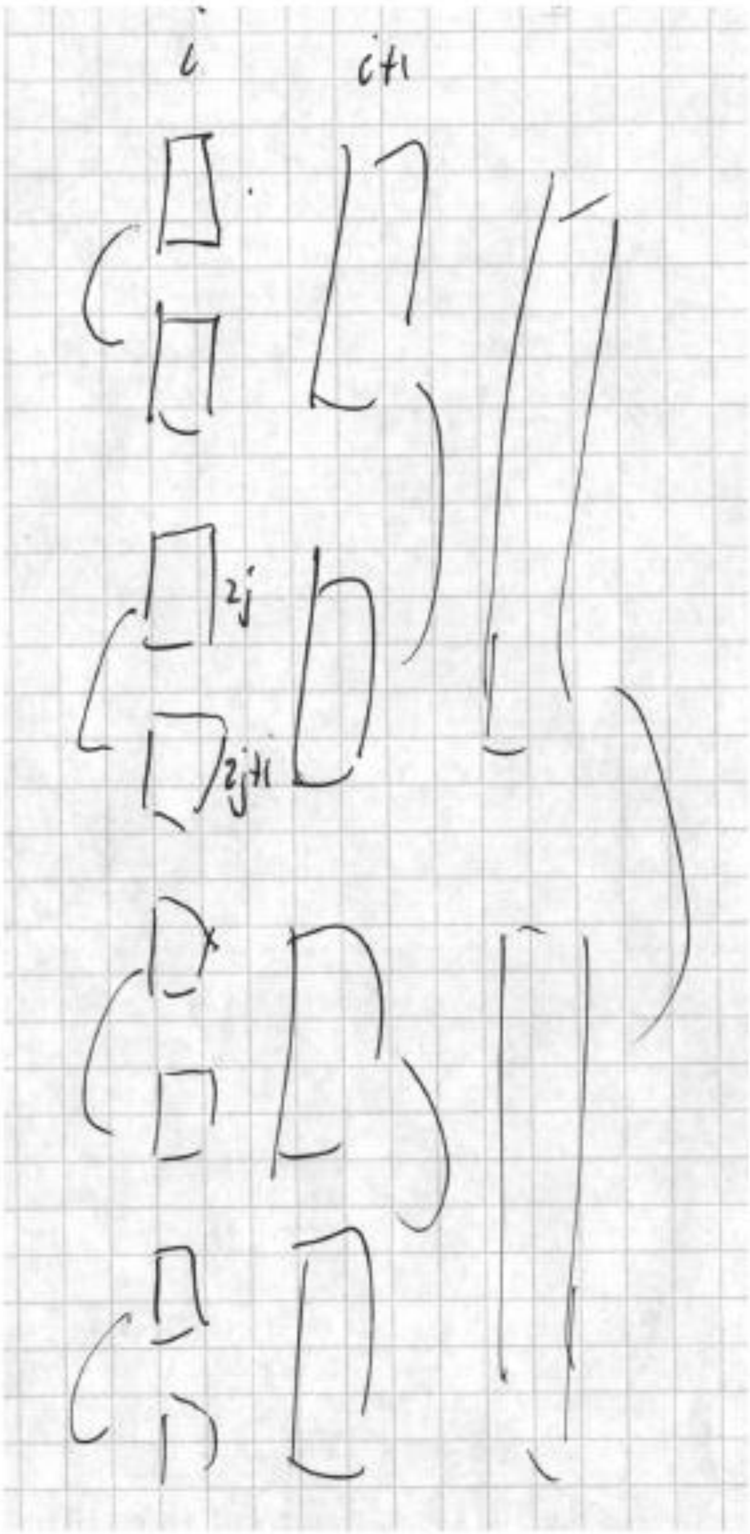
5 Incompressibility argument



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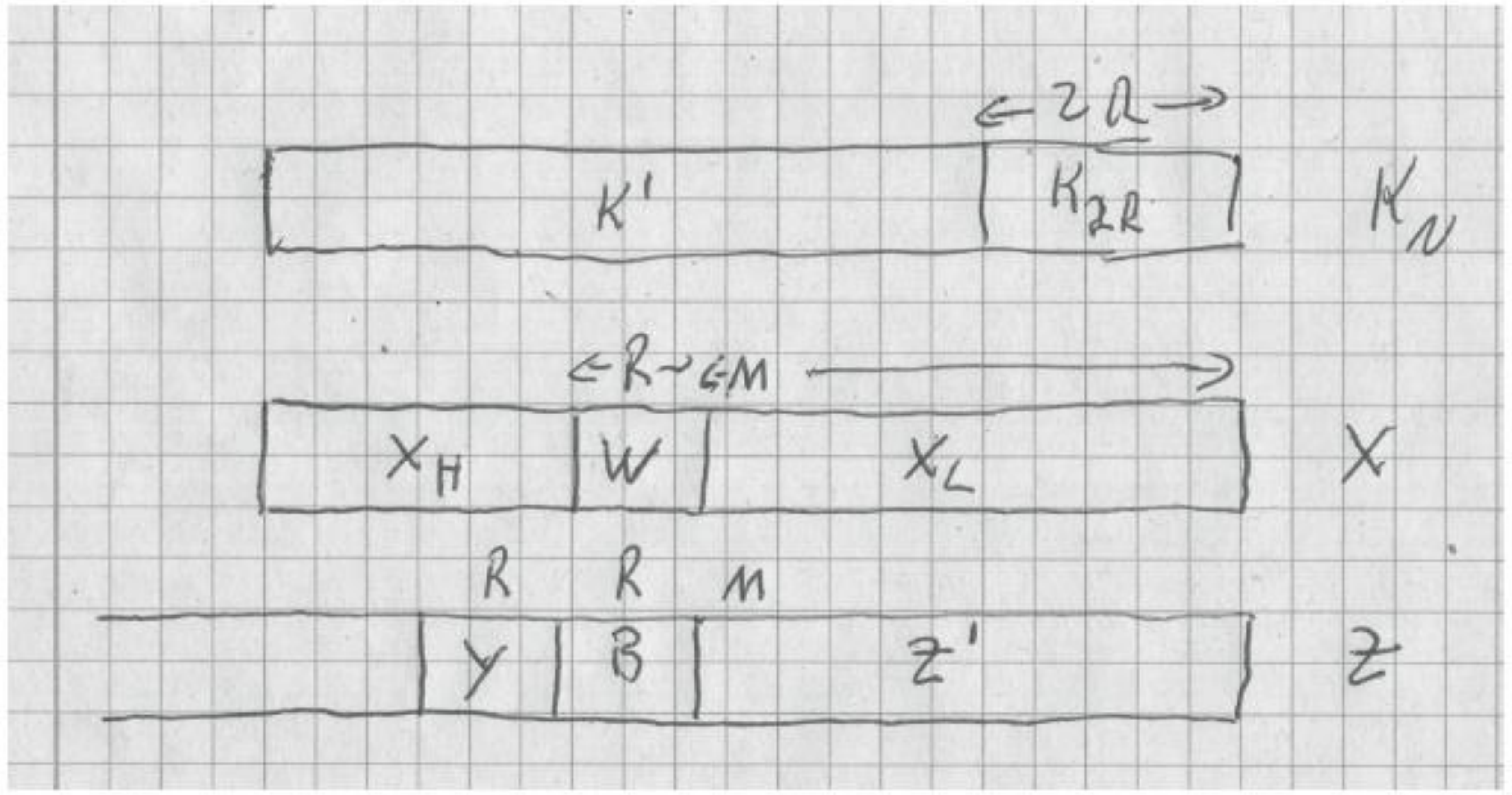
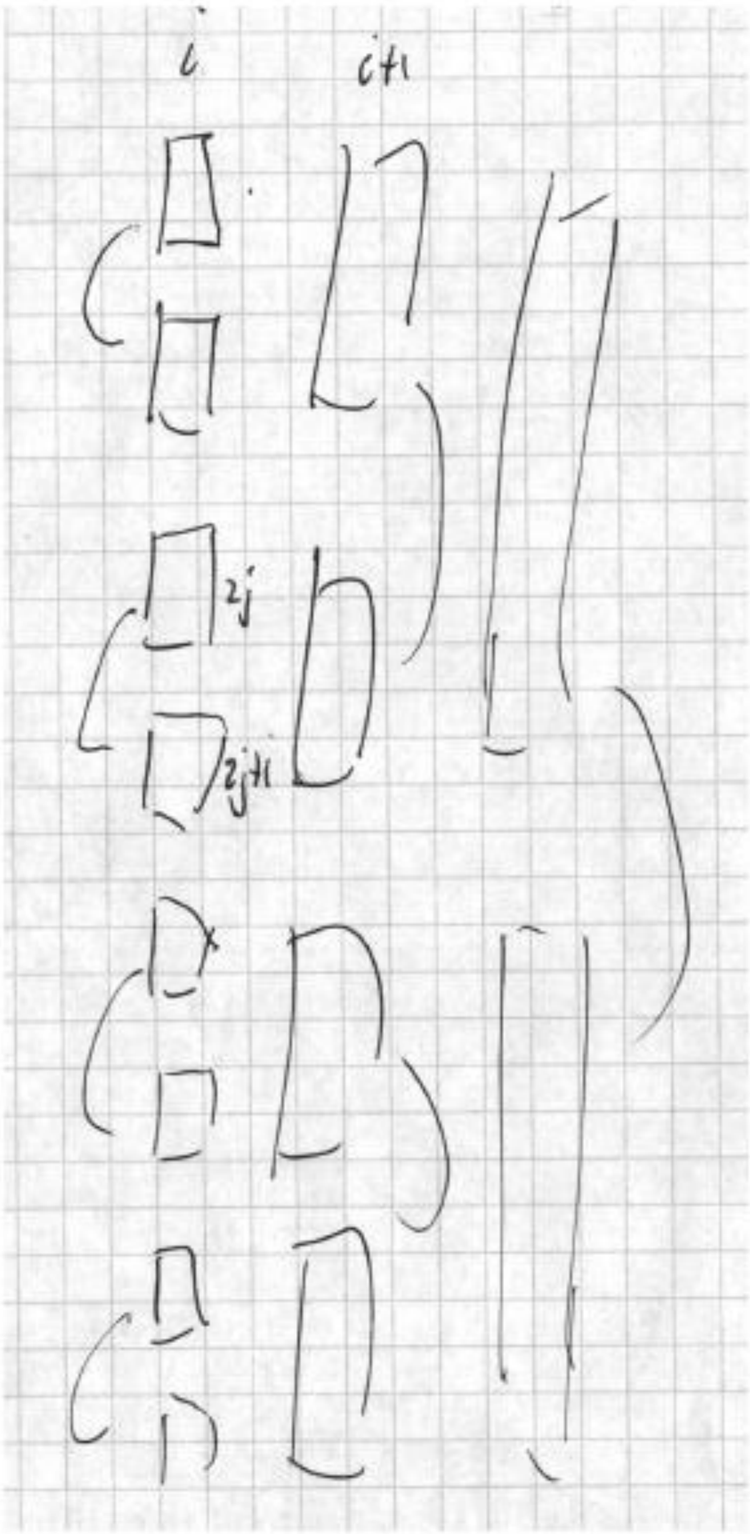


choosing Kolmogorov random input X

- choose $X \in \mathbb{B}^n$ as a Kolmogorov random string.
- let $R = 2^i$ be the length of strings $W = X_{i,j}$ and $K_{i,j}$ read in time interval $T_{i,2j}$ and of the string Y output during time interval $T_{i,2j+1}$ as shown in figure 1

5 Incompressibility argument



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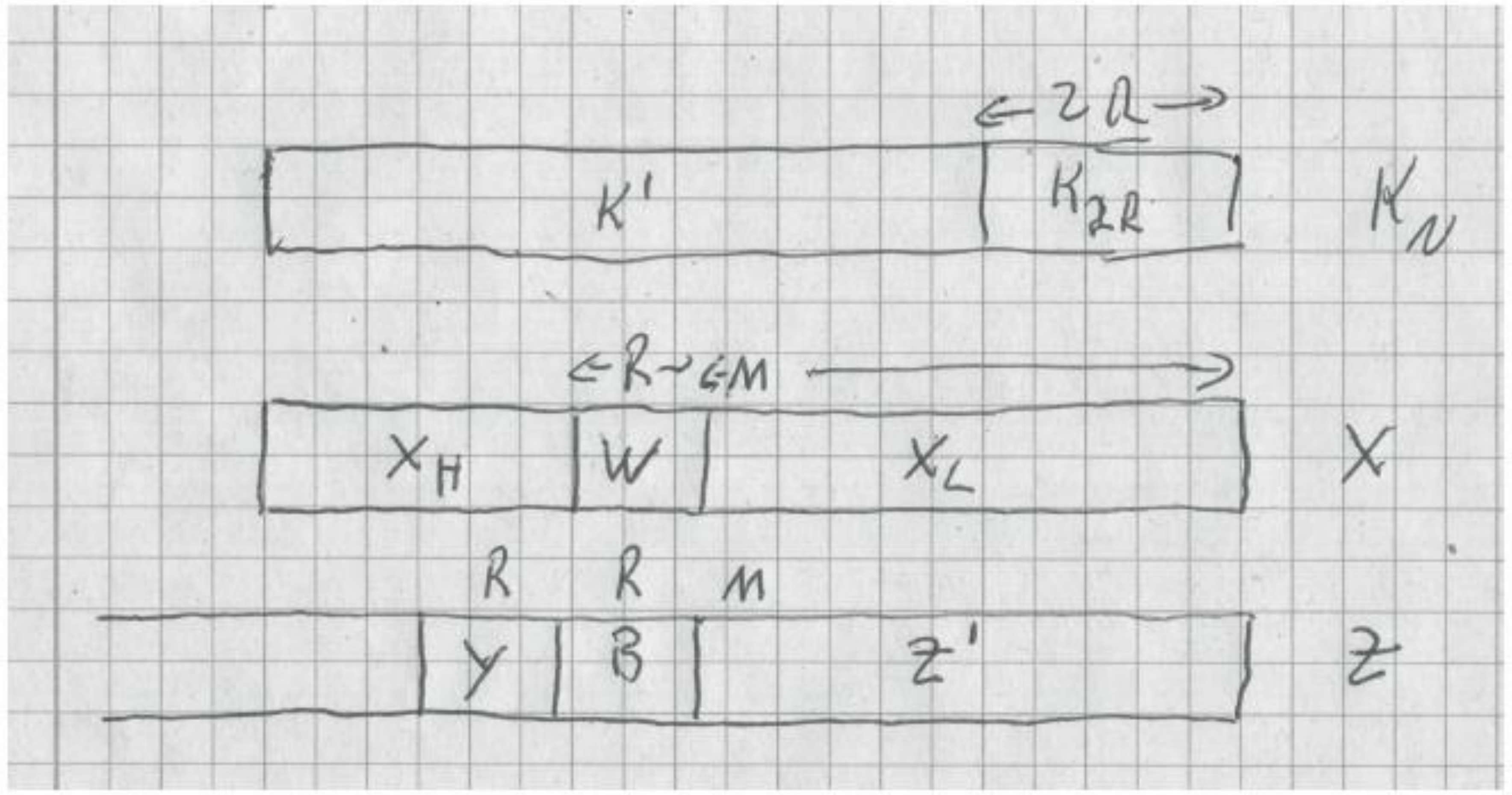

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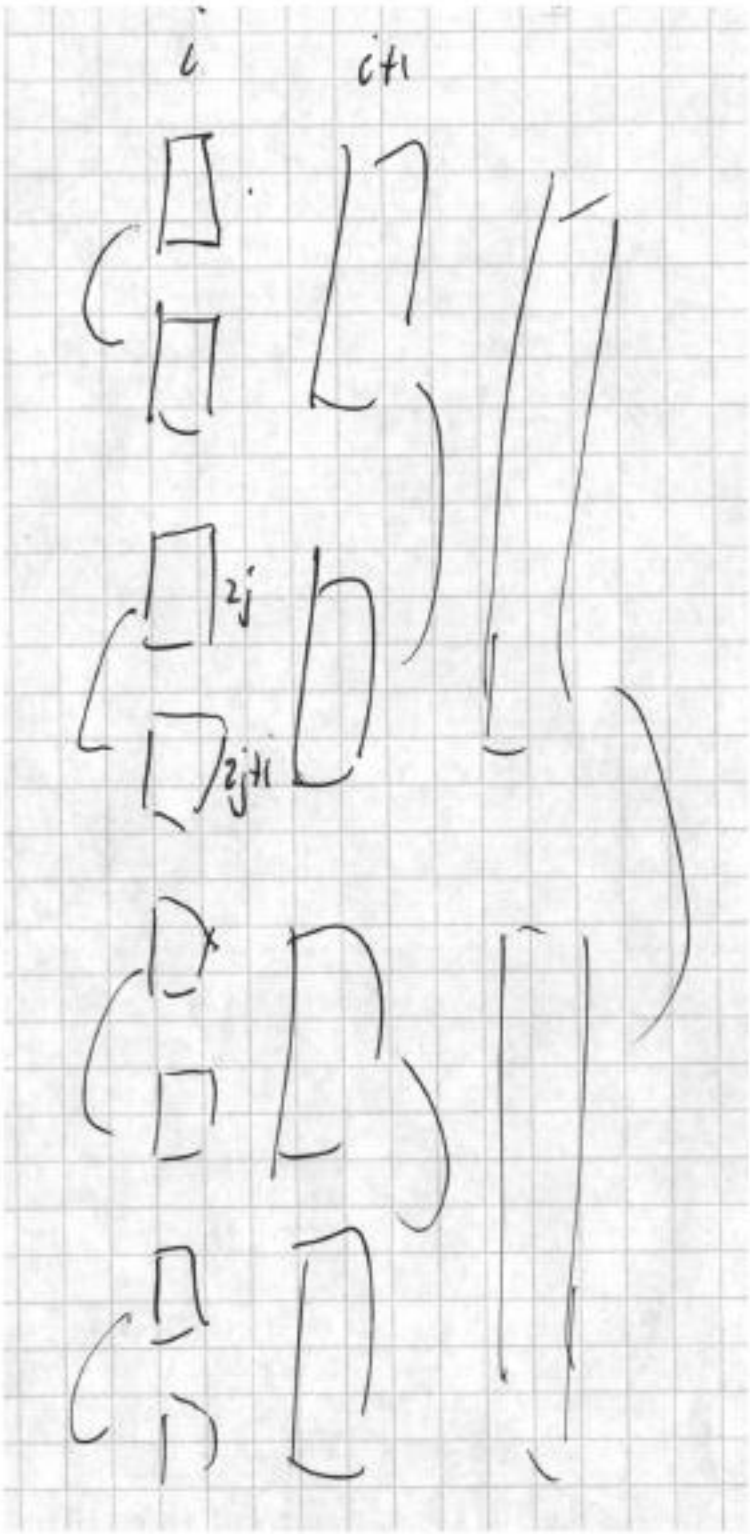
- substrings W of random strings X are locally almost random, even given the remainder $X_H X_L$ of the string

$$K(W|\, X_H X_L) \geq |Y| - O(\log(N)) = 2^i - O(\log(N))$$

shown earlier



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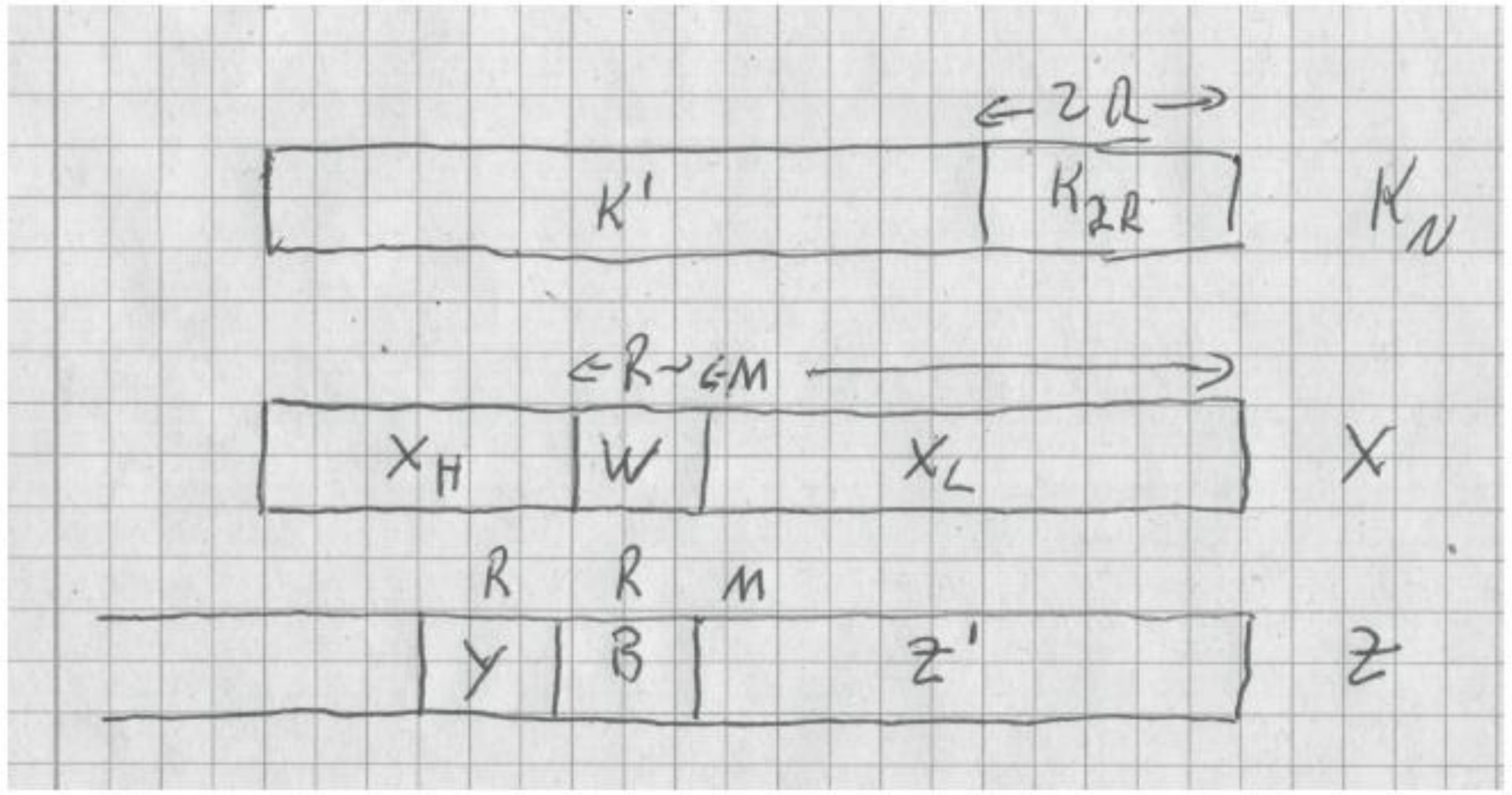
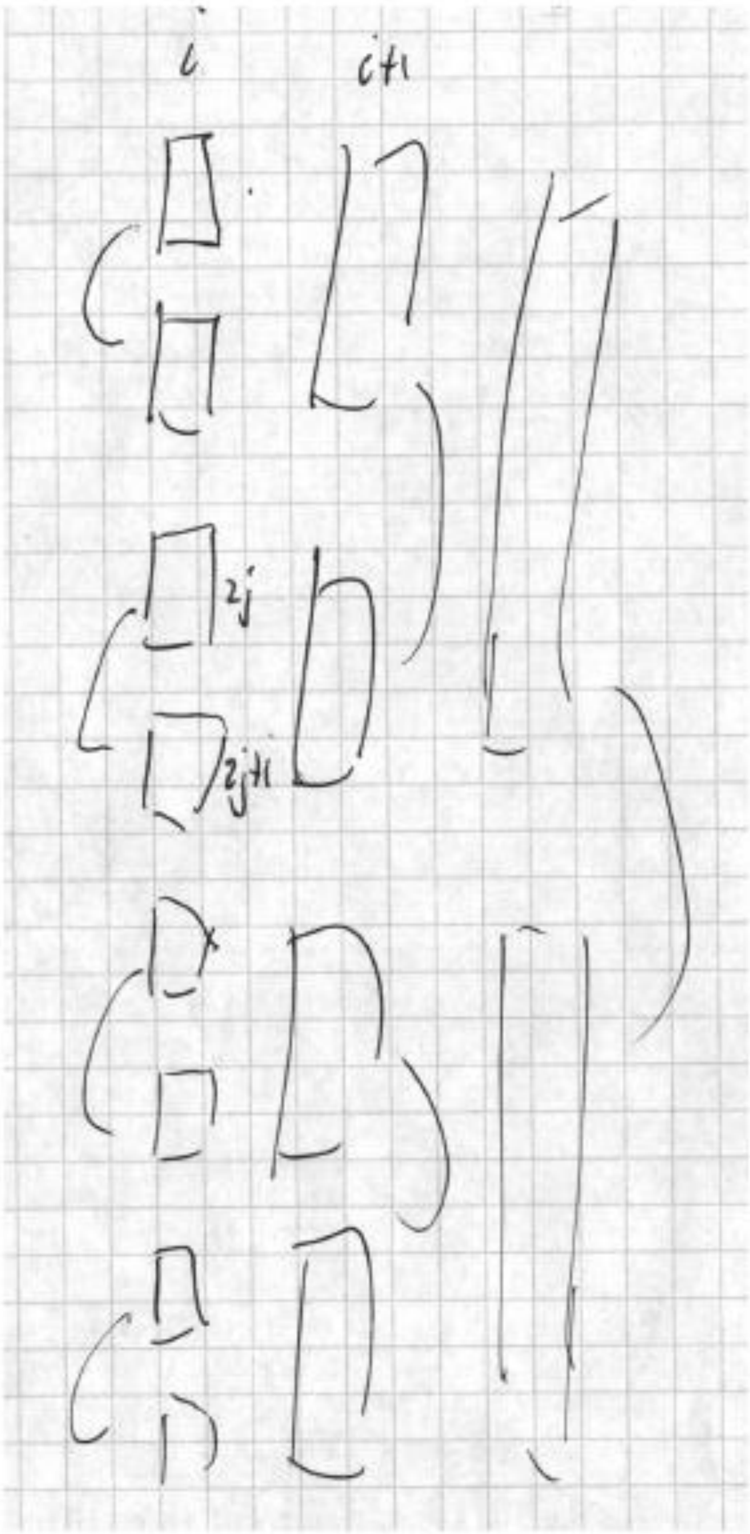
$$\boxed{K(W \mid X_H X_L) \geq |Y| - O(\log(N)) = 2^i - O(\log(N))}$$



Using $res(i,2j)$ to decode W from $X_H X_L$ with machine D

- input $I\#X_H X_L$
- with extra input $I = bin(|X_H|)' res(i,2j) b$ where $b \in \mathbb{B}$. Length:

$$|I| = |res(i,2j)| + O(\log N)$$



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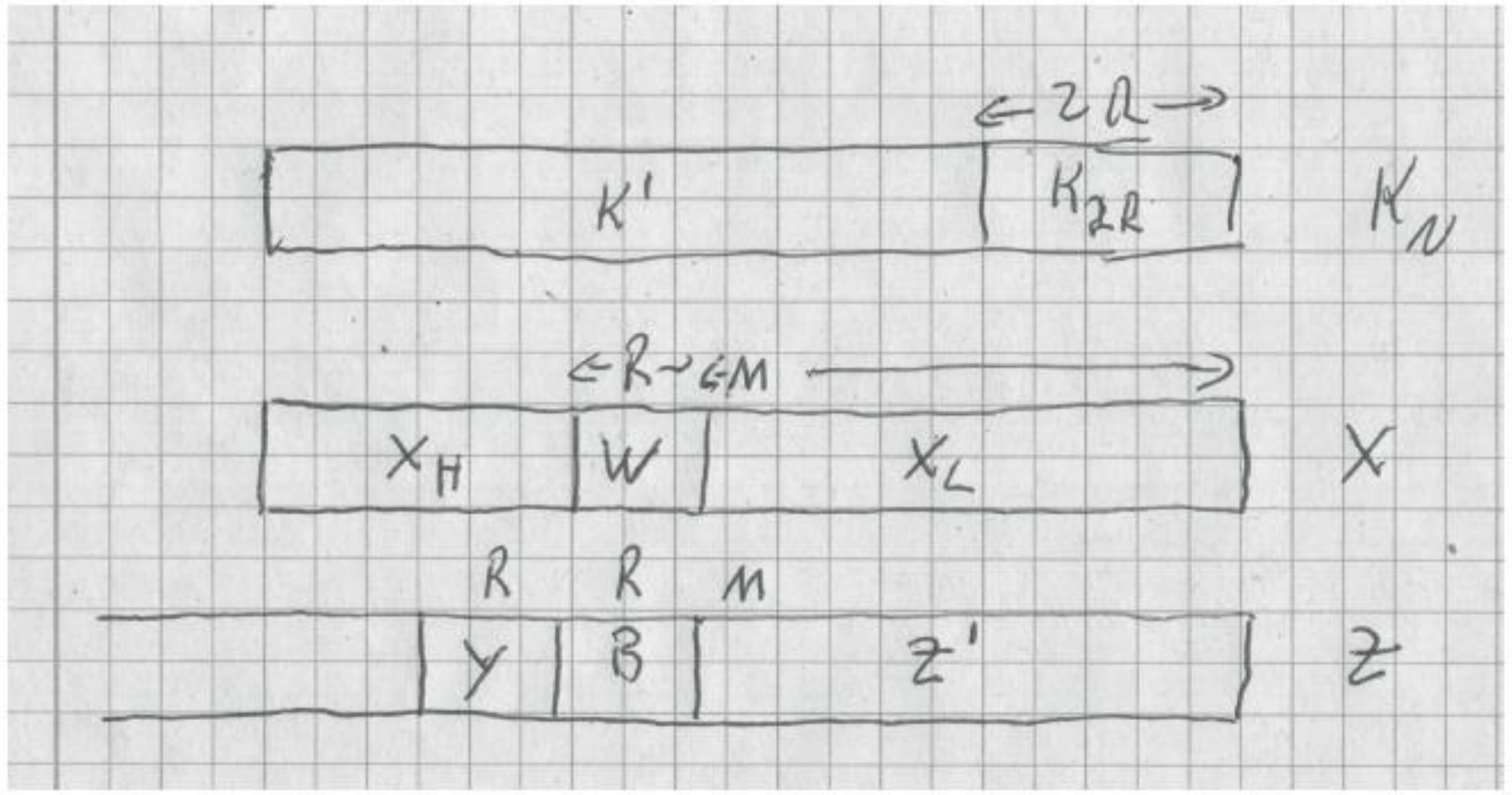

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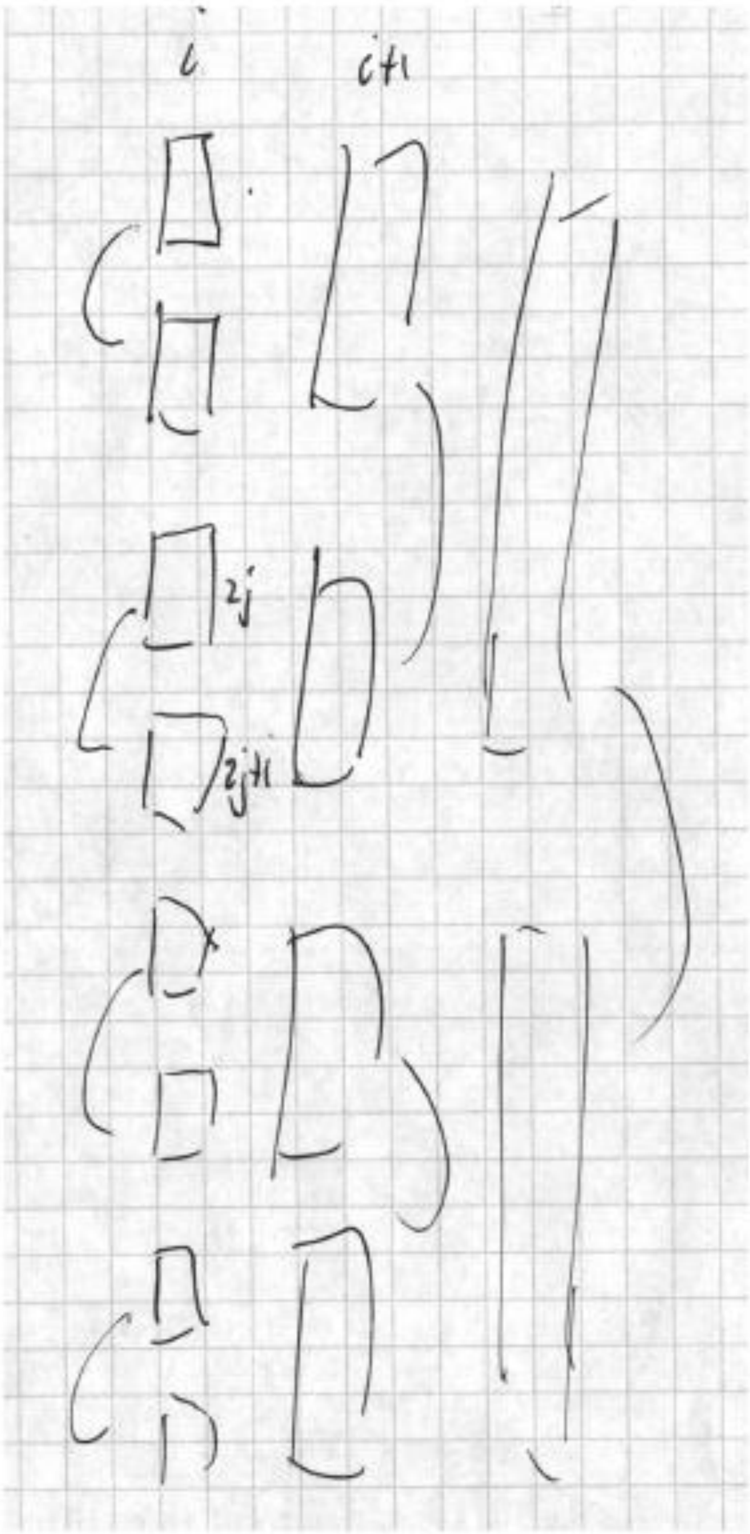
$$|I| = |res(i,2j)| + O(\log N)$$

- machine D runs $Mult$ on input X_L until the start of time interval $T_{i,2j}$ and interrupts in the configuration k at the start of this interval.
- instead of processing input W of this time interval it uses $res(i,2j$ and configuration k to construct configuration k' at the start of time interval $T_{i,2j+1}$



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$$T \geq |E^\tau| = \sum_i \sum_j |E^\tau_{i,j}|$$

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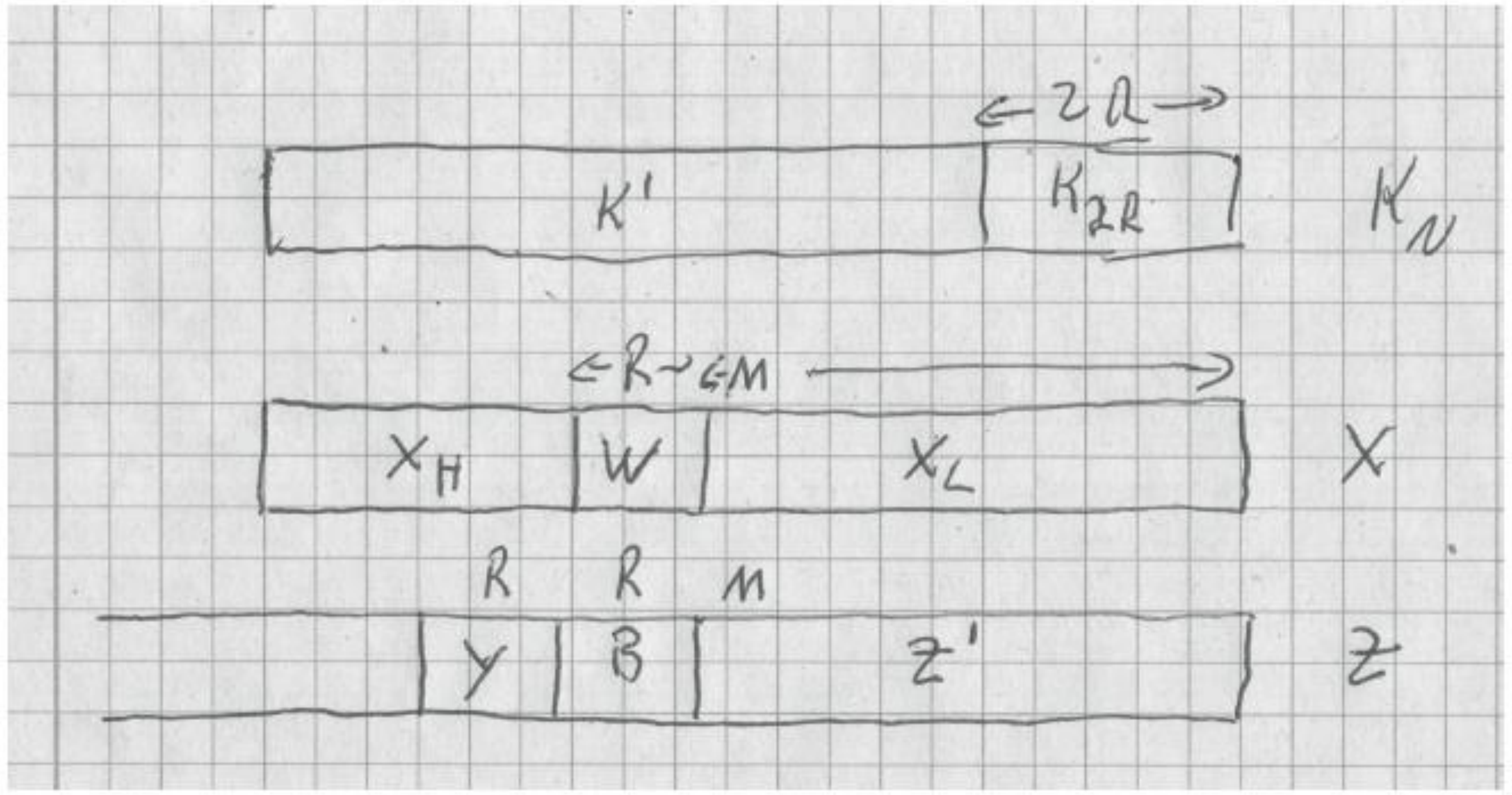

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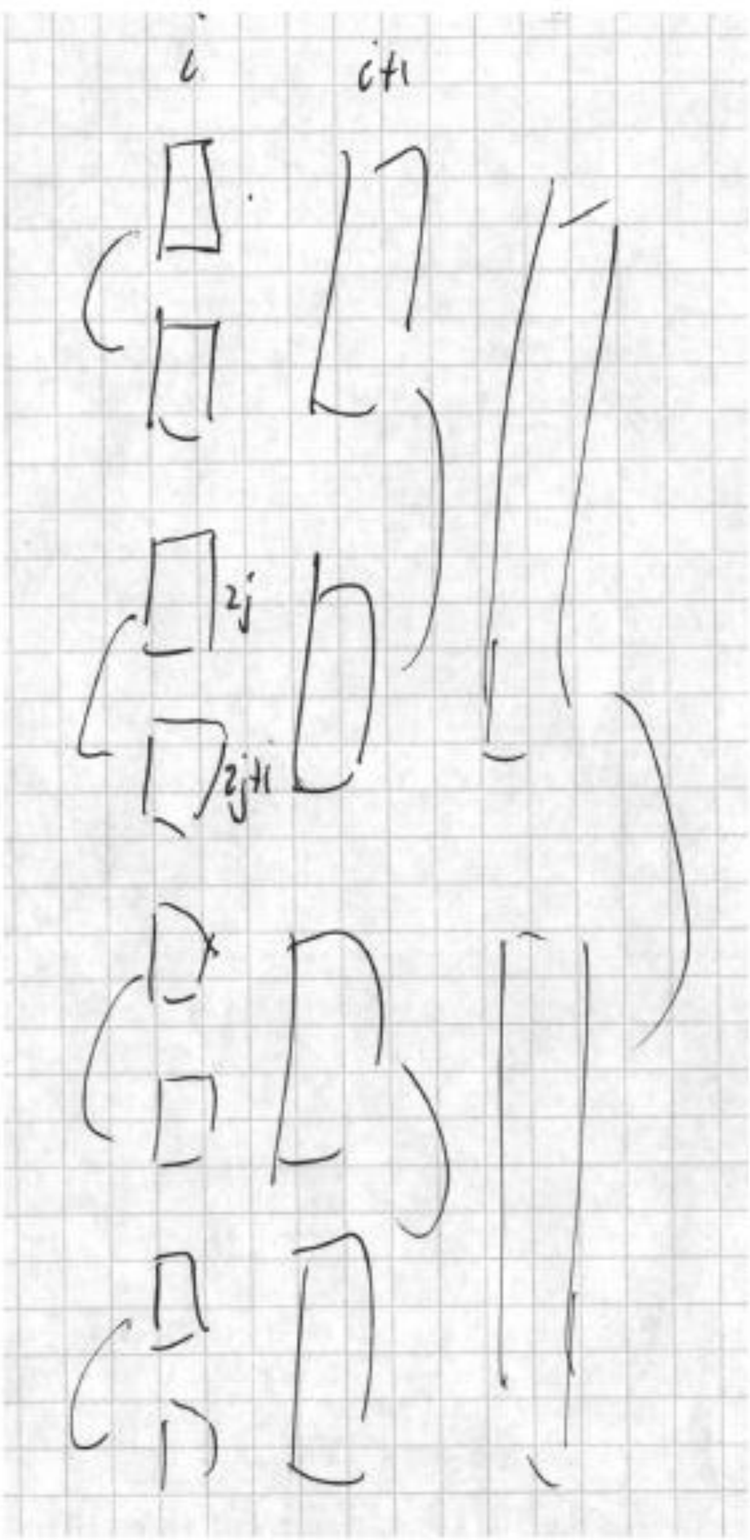
- instead of processing input W of this time interval it uses $res(i,2j$ and configuration k to construct configuration k' at the start of time interval $T_{i,2j+1}$
- continuing to act like $Mult$ it accesses bits of X_H and produces Y.
- finally it cycles through all strings W and notes which ones produce Y. By lemma 3 there are at most 2 such W. If $b = 0$ it chooses the lexicographically first one, otherwise the second one.



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$$\boxed{\begin{aligned} T &\geq |E^\tau| = \sum_i \sum_j |E^\tau_{i,j}| \\ |\omega_{i,2j}| &= \sum_\tau |\omega^\tau_{i,2j}| \\ |res(i,2j)| &\leq c \cdot |\omega_{i,2j}| + O(\log(t(N)) \text{ bits} \end{aligned}}$$

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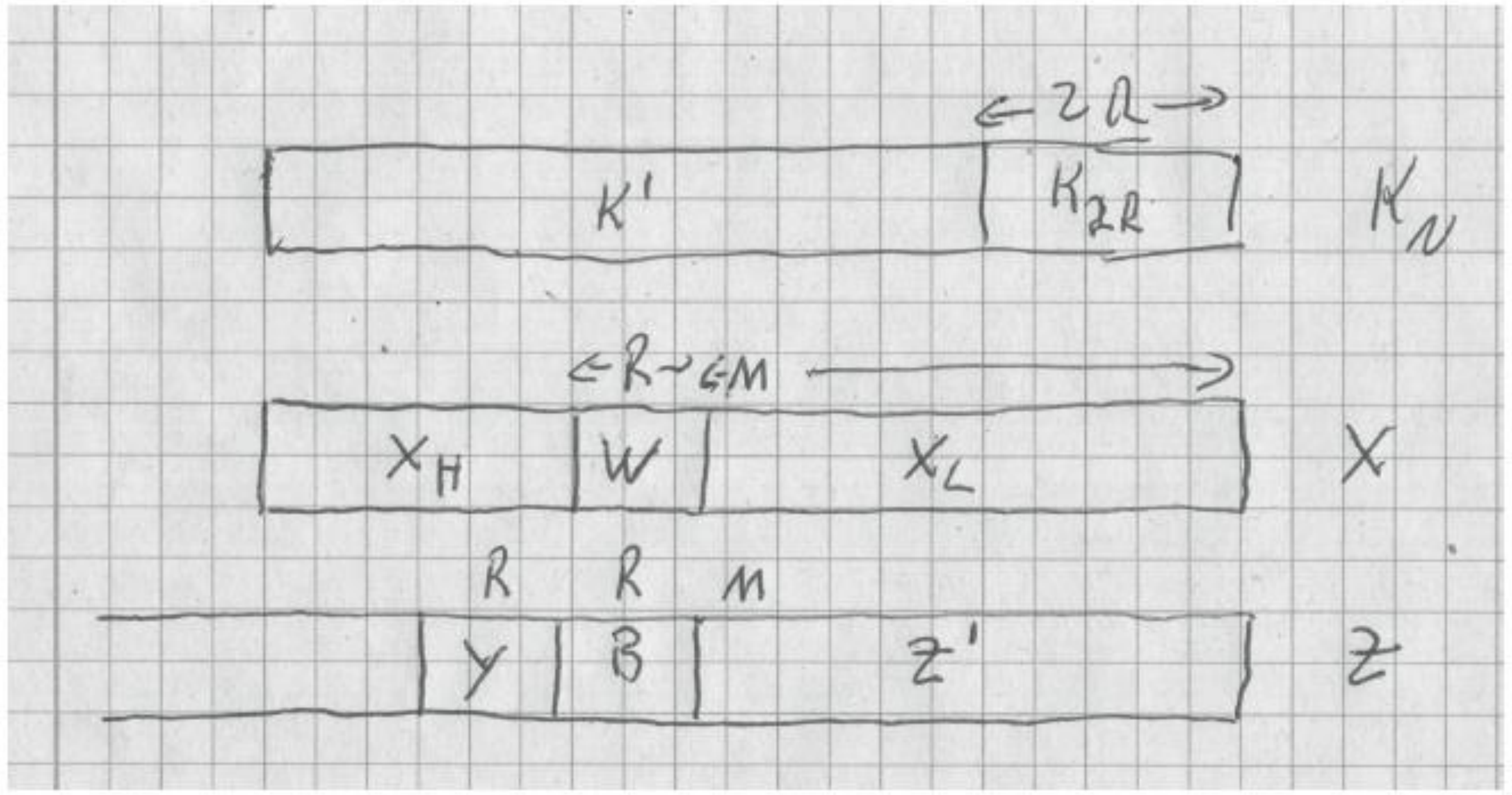

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lower bound on time

- bounding size of overlap:

$$\begin{aligned} 2^i - O(\log(N) &\leq K(Y|X_H X_L) \\ &\leq O(1) + |I| \\ &\leq |res(i,2j)| + O(\log N) \\ &\leq c \cdot |\omega_{i,2j}| + O(\log n) \end{aligned}$$

$$\boxed{2^i - C \cdot \log(N) = c \cdot \sum_\tau \omega^\tau_{i,2j}}$$

for some C, N_0 and all $N \geq N_0$

$$\boxed{\begin{array}{l} T \geq |E^\tau| = \sum_i \sum_j |E^\tau_{i,j}| \\ |\omega_{i,2j}| = \sum_\tau |\omega^\tau_{i,2j}| \\ |res(i,2j)| \leq c \cdot |\omega_{i,2j}| + O(\log(t(N)) \text{ bits} \end{array}}$$

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&\leq |res(i,2j)| + O(\log N) \\
&\leq c \cdot |\omega_{i,2j}| + O(\log n) \\
\boxed{2^i - C \cdot \log(N) = c \cdot \sum_\tau \omega^\tau_{i,2j}}
\end{aligned}$$

  for some C, N_0 and all $N \geq N_0$

- for $i \geq (\log N)/2$ holds for some $N_1 \geq N_0$ and all $N \geq N_1$

$$\begin{aligned}
2^i &\geq \sqrt{N} \\
&\geq 2C \cdot \log N \\
2^i &\geq 2^{i-1} + C \cdot \log N \\
2^i - C \cdot \log N &\geq 2^{i-1}
\end{aligned}$$

- bounding time

$$\begin{aligned}
\sum_{i \geq (\log N)/2} \sum_{2j < N/2^i} 2^{i-1} &\leq \sum_{i \geq (\log N)/2} \sum_j (2^i - C \cdot \log(N)) \\
&\leq c \cdot \sum_\tau \sum_i \sum_j \omega^\tau_{i,2j} \\
&\leq ck \cdot T(N)
\end{aligned}$$

$$\boxed{\begin{aligned} T \geq |E^\tau| &= \sum_i \sum_j |E^\tau_{i,j}| \\ |\omega_{i,2j}| &= \sum_\tau |\omega^\tau_{i,2j}| \\ |res(i,2j)| &\leq c \cdot |\omega_{i,2j}| + O(\log(t(N)) \text{ bits} \end{aligned}}$$

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- evaluating left hand side

$$\begin{aligned} \sum_{i \geq (\log N)/2} \sum_{2j < N/2^i} 2^{i-1} &= \sum_{i \geq (\log N)/2} 2^{i-1} \cdot N/2^{i+1} \\ &= \frac{1}{4} \sum_{i \geq (\log N)/2} N \\ &= \frac{1}{8} N \log N \end{aligned}$$

$$\boxed{\begin{aligned} &T \geq |E^\tau| = \sum_i \sum_j |E^\tau_{i,j}| \\ &|\omega_{i,2j}| = \sum_\tau |\omega^\tau_{i,2j}| \\ &|res(i,2j)| \leq c \cdot |\omega_{i,2j}| + O(\log(t(N)) \text{ bits} \end{aligned}}$$

lower bound on time

- bounding size of overlap:

$$\begin{aligned} 2^i - O(\log(N) &\leq K(Y|X_H X_L) \\ &\leq O(1) + |I| \\ &\leq |res(i,2j)| + O(\log N) \\ &\leq c \cdot |\omega_{i,2j}| + O(\log n) \end{aligned}$$

$$\boxed{2^i - C \cdot \log(N) = c \cdot \sum_\tau \omega^\tau_{i,2j}}$$

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- here it is:

$$\boxed{T(N) \geq \frac{1}{8ck} N \log N}$$