Kolmogorov Complexity

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(during a conference):

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how would you have defined ‘description’?

Hint: this is I2TC

1 Information content of a bit string $x \in \mathbb{B}^n$

examples:

- $x = 000000000000000000000000000000$:

$$x = 0^{30} \quad \text{or} \quad x = 0 \text{ (30 times)}$$

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- unit: not bits, as $\# \notin \mathbb{B}$. Position of # contains information.

def: binary comma u'; self delimiting strings

- for $w \in \{0,1,\#\}^*$ obtain (as usual)

$$h(0) = 00 \ , \ h(1) = 11 \ , \ h(\#) = 10$$

$$h(w[1:n]) = h(w_1)\dots h(w_n)$$

- for $u \in \mathbb{B}^*$ define

$$u' = h(bin(|u|)\#)u$$

example $u = 01$

$$\begin{aligned} bin(|u|) &= 10 \\ h(bin(|u|)) &= 1100 \\ u' &= 11001001 \end{aligned}$$

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Proof.

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Lemma 3.

$$K(1^n) = \log(n) + O(1)$$

- M_u started with $bin(n)$ prints 1^n and halts
- $u'bin(n)$ describes 1^n

$$|u'bin(n)| = \log(n) + O(1)$$

2 Random Strings

idea: a string x is random, if it is its own shortest description.

def: random strings

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- assume for all $x \in \mathbb{B}^n$

$$K(x) \leq n - 1$$

- the number N of descriptions with length $i < n$ is at most

$$\begin{aligned} N &= |\{u'v : |u'v| \leq n-1\}| \\ &\leq \sum_{i=0}^{n-1} |\mathbb{B}^i| \\ &= 2^n - 1 \\ &< |\mathbb{B}^n| \end{aligned}$$

not enough short descriptions

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3 Independence of machine model

- Definition of $K(x)$ depends on coding of 1-Tape Turing machines M_u. So we really defined

$$K(x) = K_{1-tape-TM}(x)$$

- What if we use (binary coded) C-programs instead do define $K_C(x)$?
- what if we compare $K_M(x), K_{M'}(x)$ for arbitrary machine models M and M', each capable to compute the computable functions.

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- Let $u'v$ be a shortest description of x in model M.
- let J be an interpreter of programs for M written in M', i.e. J started with $a'b$ simulates program a on input b. (Church's thesis)
- then $J'u'v$ is a description of x in model M'

$$K_{M'}(x) = |J'u'v| = O(1) + |u'v| = O(1) + K_M(x)$$

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- M_u with input $bin(n)$
  1. enumerates in lex. order $x \in \mathbb{B}^n$ and computes $K(x)$
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- $u'bin(n)$ describes a random string in $x \in \mathbb{B}^n$. Thus

$$\begin{aligned} n &\leq K(x) \\ &\leq |u'bin(n)| \\ &\leq O(1) + \log(n) \end{aligned}$$

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Here we can afford to indicate, what is given with $\# \notin \mathbb{B}$.

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example:

Lemma 7. $K(x|\overline{x}) = O(1)$

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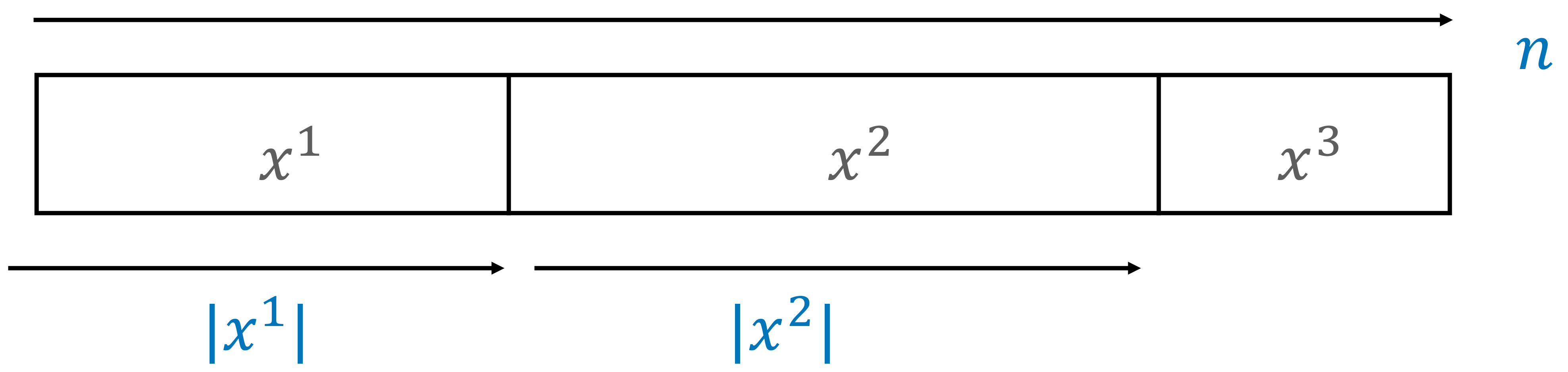

6 Substrings of random strings

Lemma 8. *Substrings of random strings are almost random, even if the remainder of the string is given. Let*

$$x = x^1x^2x^3 \in \mathbb{B}^n \text{ be random}$$

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$$K(x^2|x^1x^3) \geq |x^2| - O(\log(n))$$

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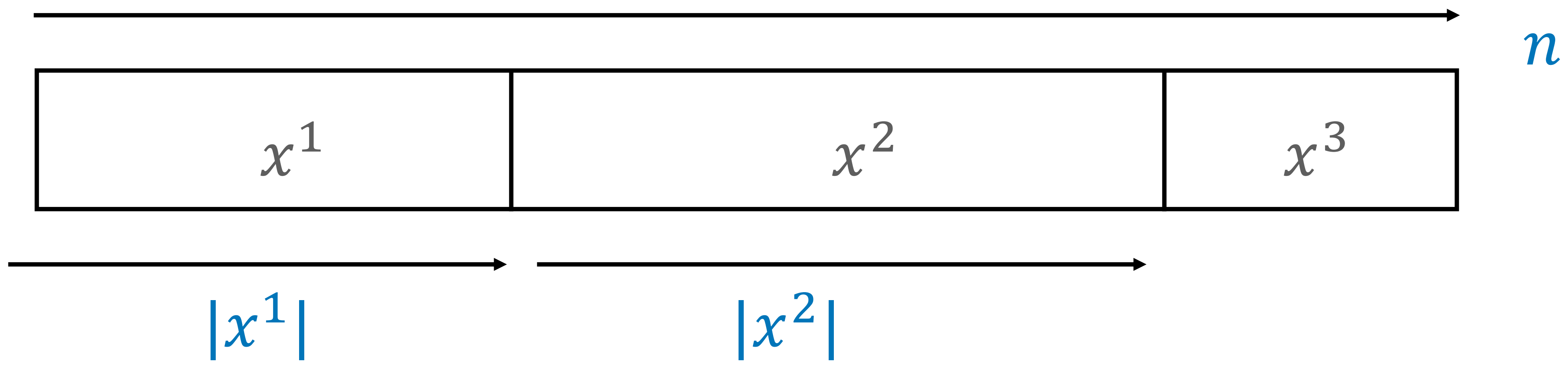

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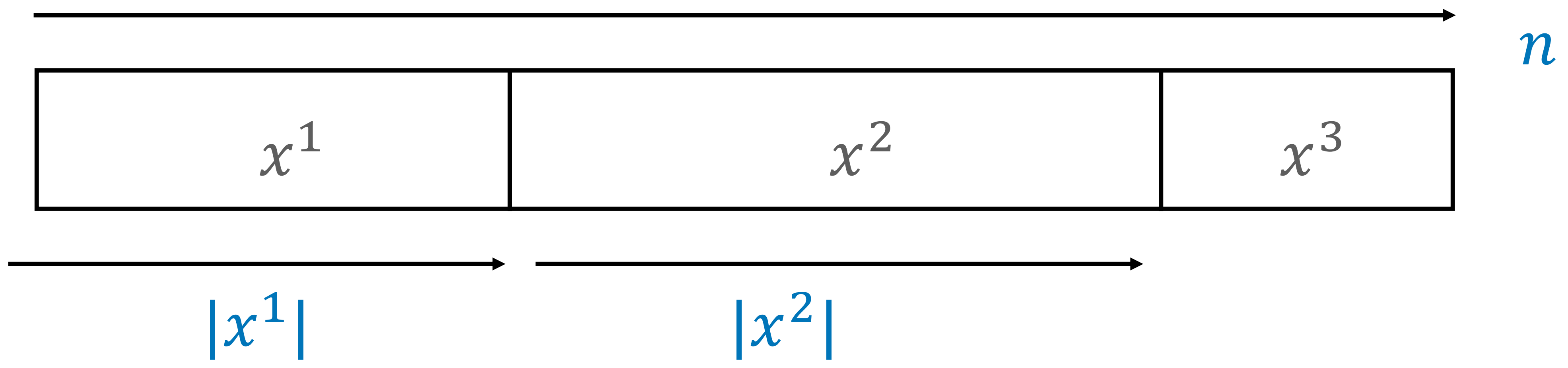

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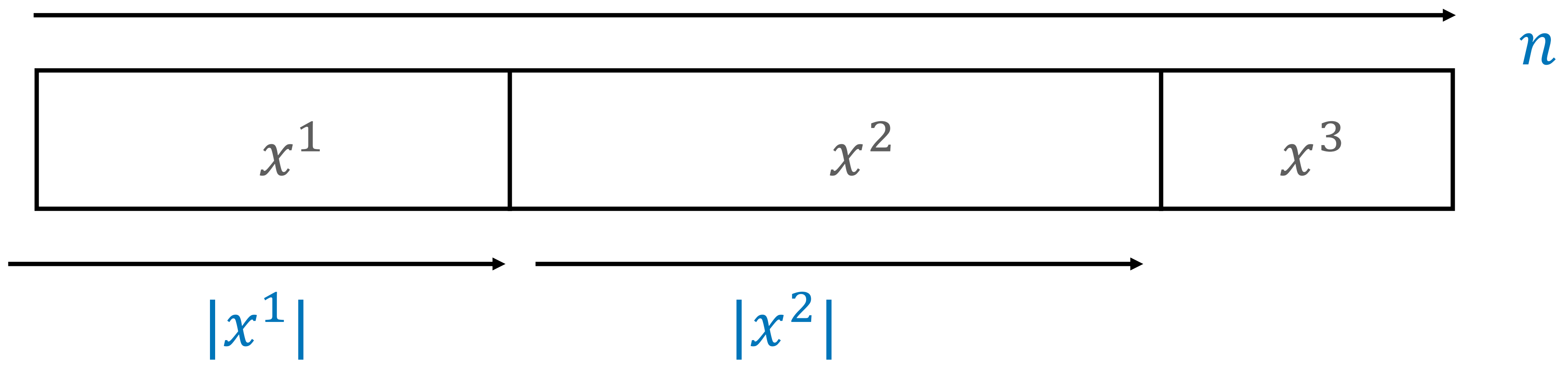

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$$\begin{aligned}
n &\leq K(x^1x^2x^3) \\
&\leq |v'u'z'bin(|x^1|)'x^1x^3| \\
&= O(1) + K(x^2|x^1x^3) + O(\log n) + |x^1x^3| \\
|x^2| - O(\log(n)) &\leq K(x^2|x^1x^3)
\end{aligned}$$

7 Crossing sequences

Hennie 1965

- exploiting communication bottleneck of 1-tape Turing machines
- can transport information across a cell boundary only in the state $z \in Z$

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Fix 1-tape TM

$$M = (Z, \Sigma, \delta, z_0, Z_A)$$

and w.l.o.g assume that computations of M end with head on left end of tape inscription.

def: crossing sequence For $w \in A^*$ and numbers i of tape cells the crossing sequence $CS(w,i)$ is the sequence of states of M started with w before its head crosses the border between tape cells i and $i+1$

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Lemma 9. *Let $u, v, x, y \in A^*$ and assume that uv and wx produce the same crossing sequences between them.*

$$CS(uv, |u|) = CS(wx, |w|)$$

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- consider head movements accepting computations of M with inputs uv and ux as shown in figures 1 and 2

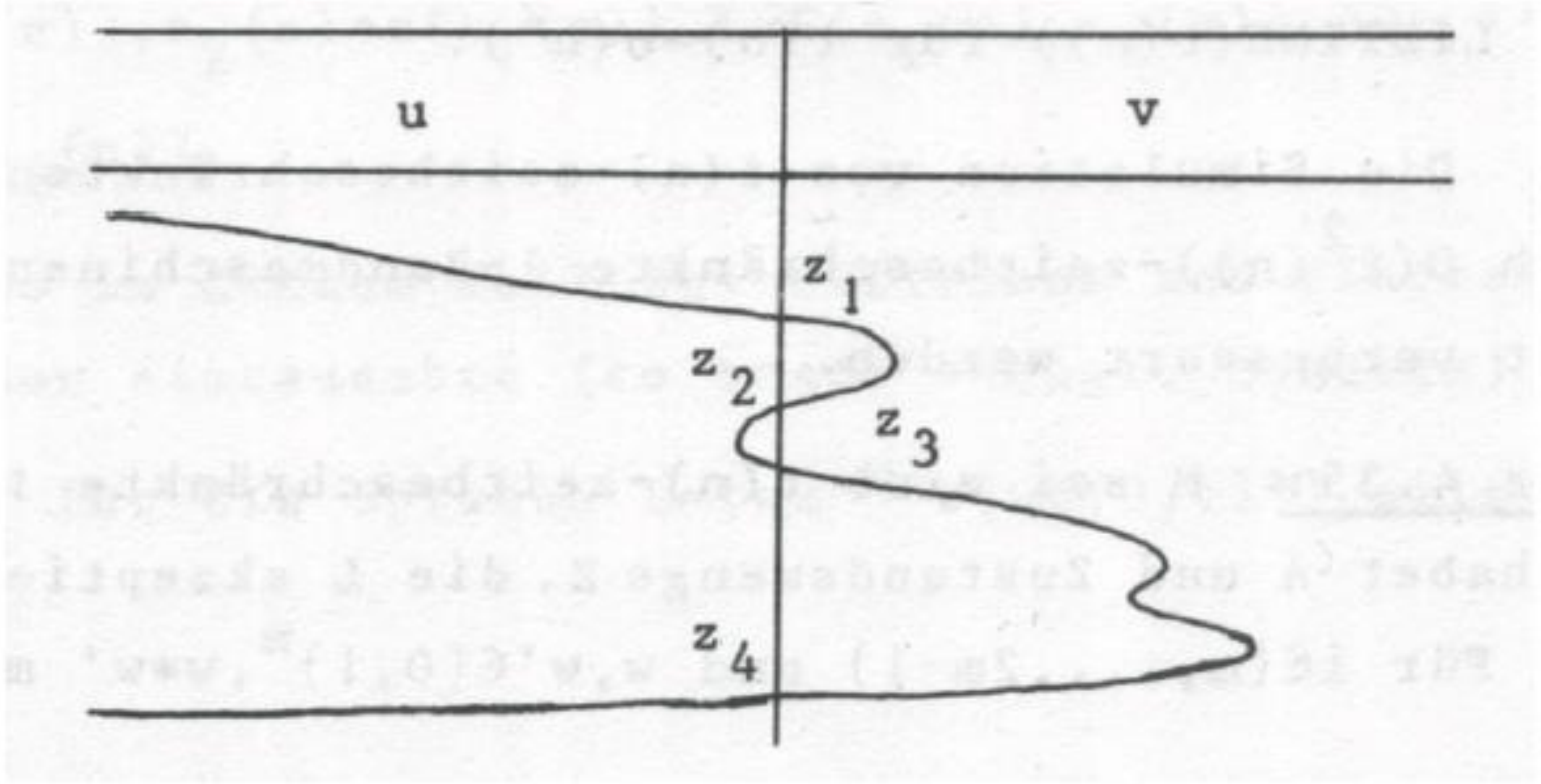

Figure 1: head movement and states of the computation with input uv. Time axis is pointing downward.

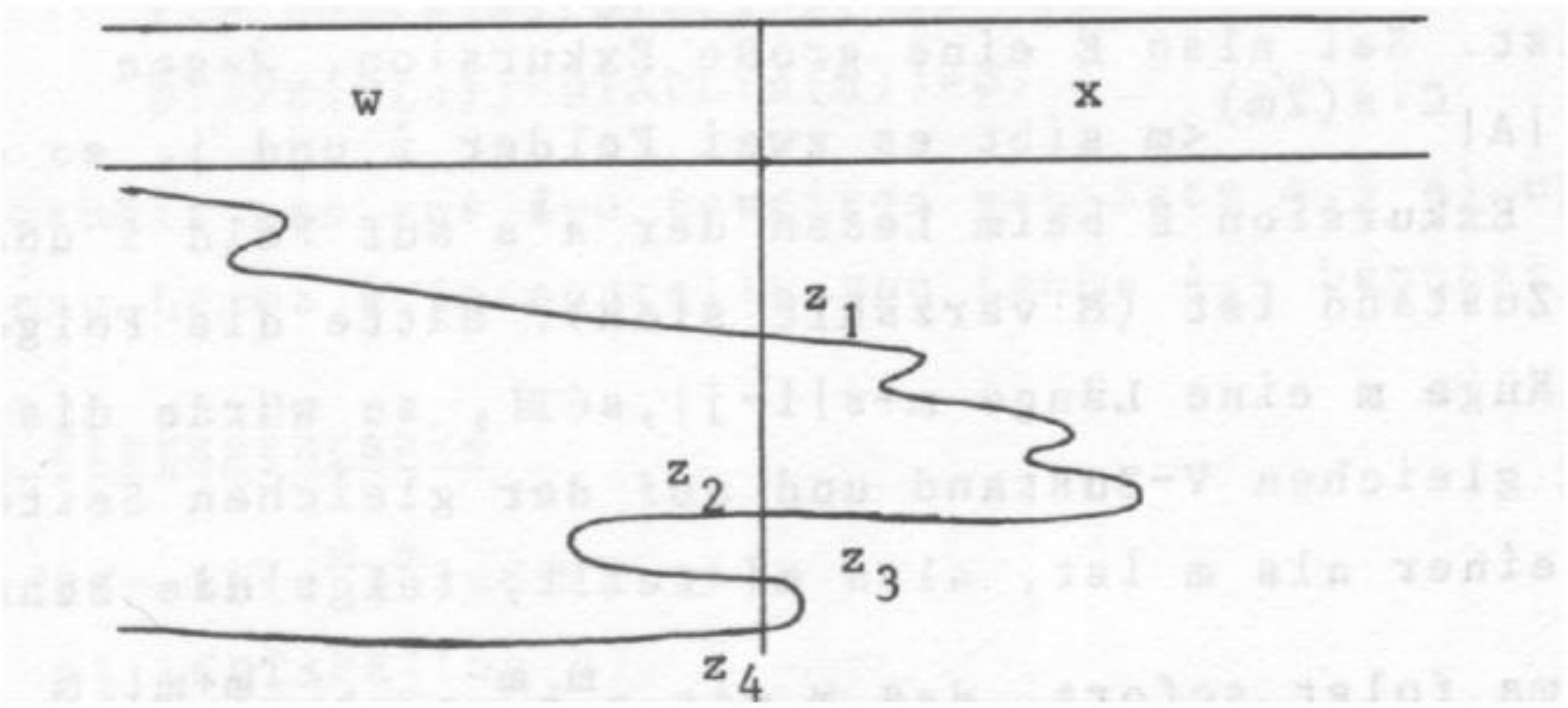

Figure 2: head movement and states of the computation with input wx

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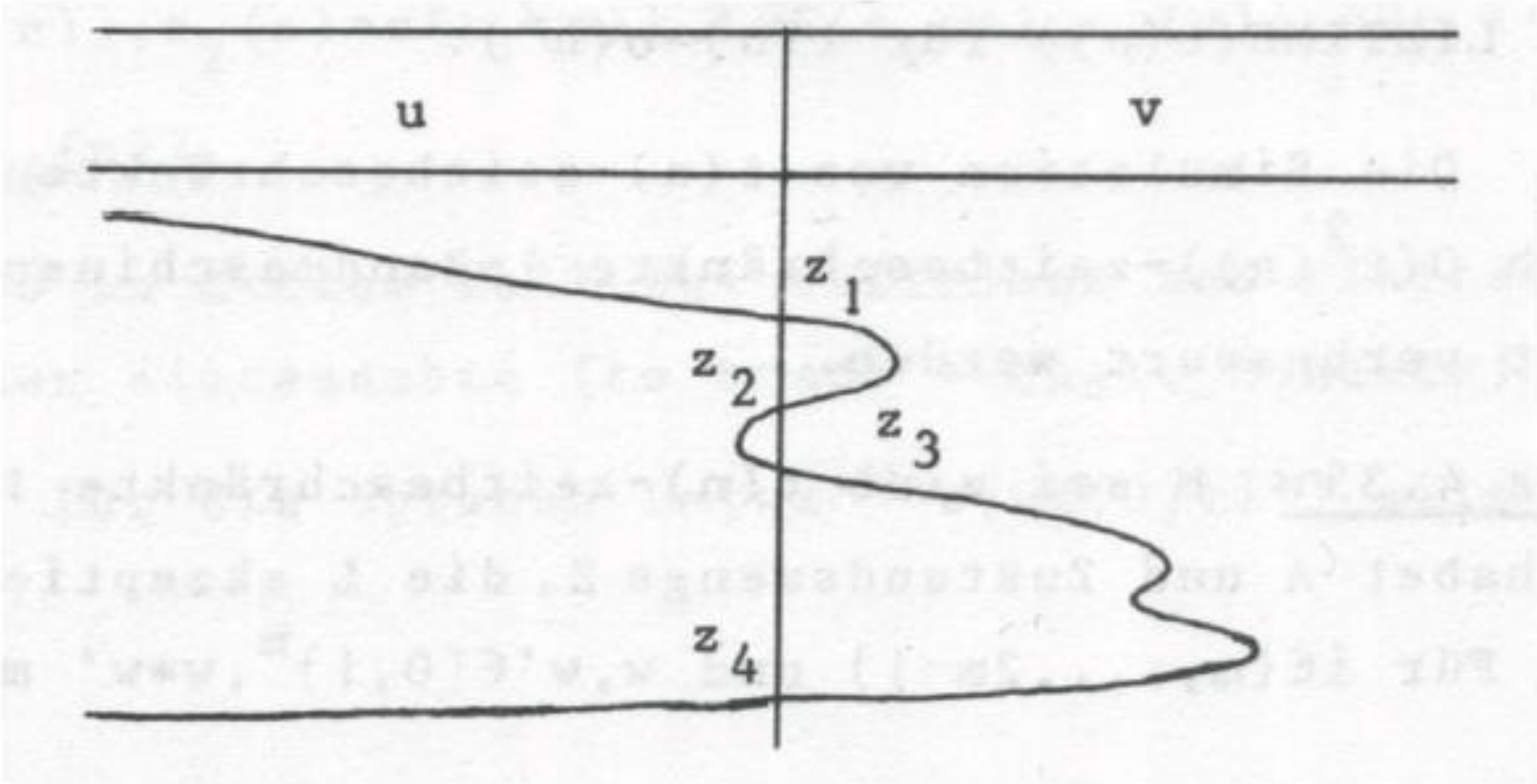
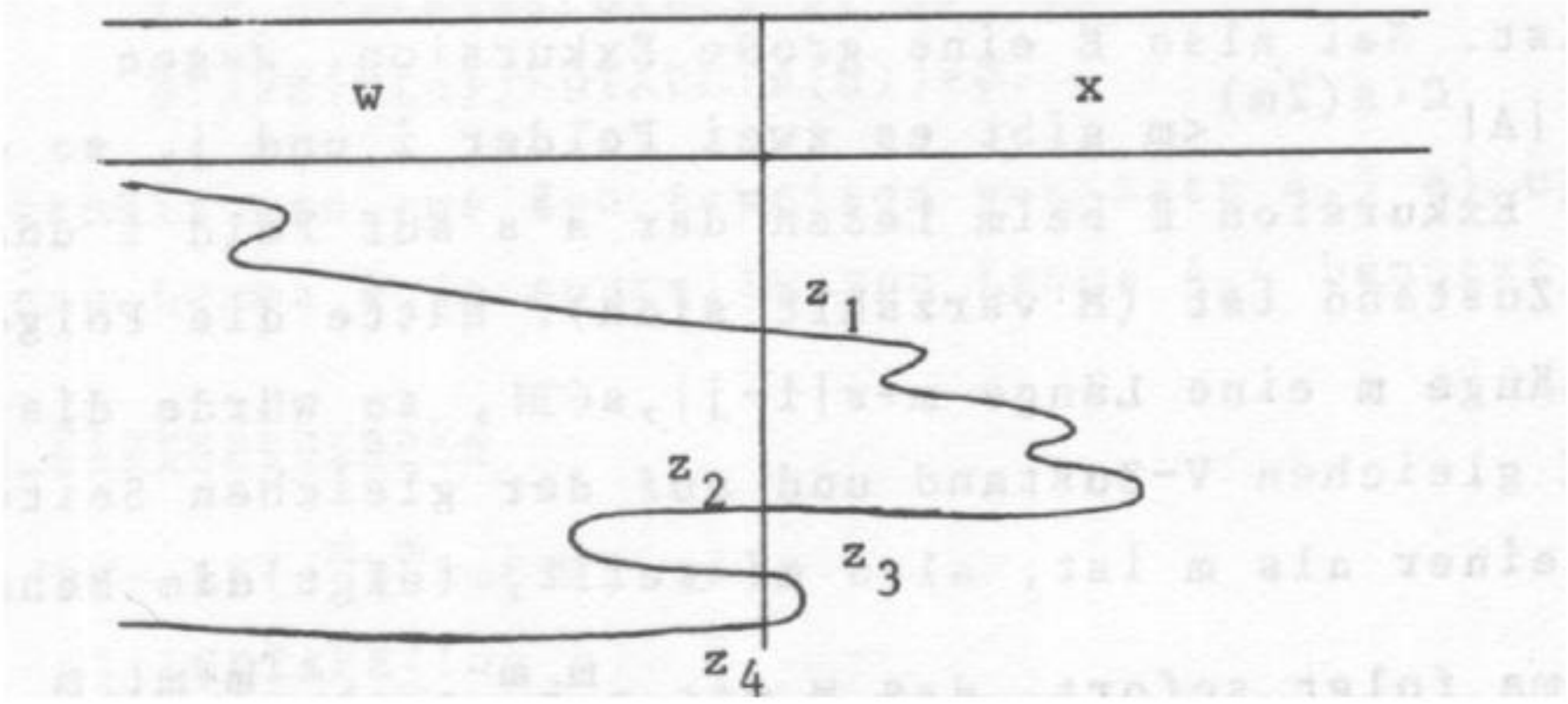
- 'cut' at border and glue together as shown in figure 3 gives an computation of ux. The decision whether to accept is done on the left side and is the same as for uv





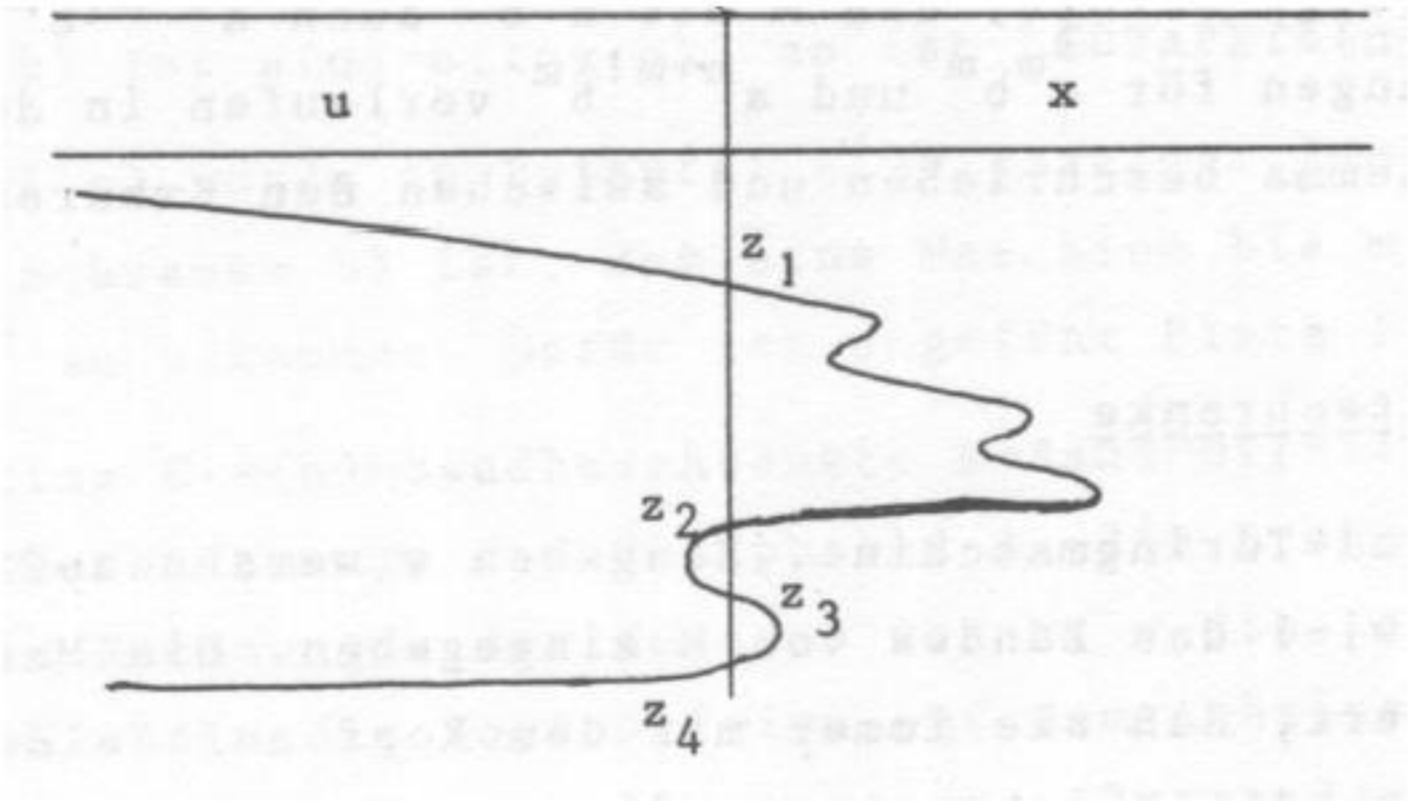


Figure 3: head movement and states of the computation with input ux

Lemma 10. *Let*

$$L = \{u\#^m u \ : \ u \in \mathbb{B}^m \ , \ m \in \mathbb{N}_0\}$$

Let M be a $t(n)$- time bounded 1-tape TM and $t(n) = o(n^2)$. Then M does not accept L

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Assume $L = L(M)$

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- $v'bin(m)'bin(i)'\tilde{CS}(u\#^m u, i)$ describes u. If u is random, we have

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- run time is at least sum of lengths of crossing sequences. Let $n = 3m$. Then run time T_n with random $u \in \mathbb{B}^m$ is

$$\begin{aligned} T_n &\geq \sum_{i=m}^{2m} |CS(u\#^m u, i)| \\ &\geq m \cdot m/(2\rho) \\ &= n^2/(18\rho) \end{aligned}$$