context free languages and pushdown automata

on the abstract side but…wait!

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def: cfg, more comfortable

$$G = (N, T, P, S)$$

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you have almost seen this

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- start symbol does not appear on right: add new start symbol S' and production $S' \to S$.
- eliminate right hand sides with more than 2 symbols: for each production

  $$P: \quad n \to a_1 \dots a_s \; s \geq 2$$

  introduce a new non terminal x and and replace p by

  $$p \to a_1 \dots a_{s-2}x \ , \ x \to a_{s-1}a_s$$

  Repeat until all right hand sides have length 2 or 1.

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- eliminate ε-rules $A \to \varepsilon$. Order set of nonterminals different from S'

$$N \setminus \{S'\} = \{n_1, \ldots, n_k\}$$

for $i = 1$ to k:
{
if $n_i \to \varepsilon$: i) drop this rule; ii) for each rule with n_i on right side add a rule where each occurrence of n_i is dropped.

$$n \to xn_i \text{ or } n \to n_i x : \quad \text{add} \quad n \to x$$

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Lemma 2. *Let G_i be the grammar after pass i of the loop. Then $L(G) = L(G_i)$.*

induction on i. $i = 0$: $G = G_0$, trivial.

$i - 1 \to i$. not completely trivial!

- Let $w \in L(G)$. By induction hypothesis there is a derivation tree for w in G_{i-1}. *It uses no productions $n_k \to \varepsilon$* for $k < i$. Uses of productions $n_i \to \varepsilon$ are replaced by G_i, thus $w \in L(G_i)$.
- Let $w \in L(G_i)$. Replace in a derivation tree for w in G_i each use of a new production by its original in G_{i-1} and $n_i \to \varepsilon$. Thus $w \in L(G_{i-1}) = L(G)$ by induction hypothesis.

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- for all chain rules between nonterminals $A \to B$ with $A, B \in N$ drop $A \to B$ and for all productions

$$B \to u : \quad \text{add} \quad A \to u$$

repeat until no chain rules between nonterminals are left.

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2 Deciding the 'word problem' $w \in L$ for context free languages:

Lemma 3. *For each cf language L there is an algorithm which decides $w \in L$ in time $O(n^3)$ for $n = |w|$.*

Proof. dynamic programing; Younger's algorithm from I2EA □

def: nondeterministic pushdown automaton (npda)

$$M = (Z, \Sigma, \Gamma, \delta, z_0, Z_A)$$

- Z finite set of states
- Σ finite input alphabet.

$$\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$$

  ε ignores current input symbol.

- Γ finite pushdown/stack alphabet

$$\Gamma_\varepsilon = \Gamma \cup \{\varepsilon\}$$

  Pushdown actions:

$$PA = \{pop\} \cup \{push\,\gamma \;:\; \gamma \in \Gamma_\varepsilon\}$$

  $push(\epsilon)$: no pop on stack

  ε ignores current top symbol of stack.

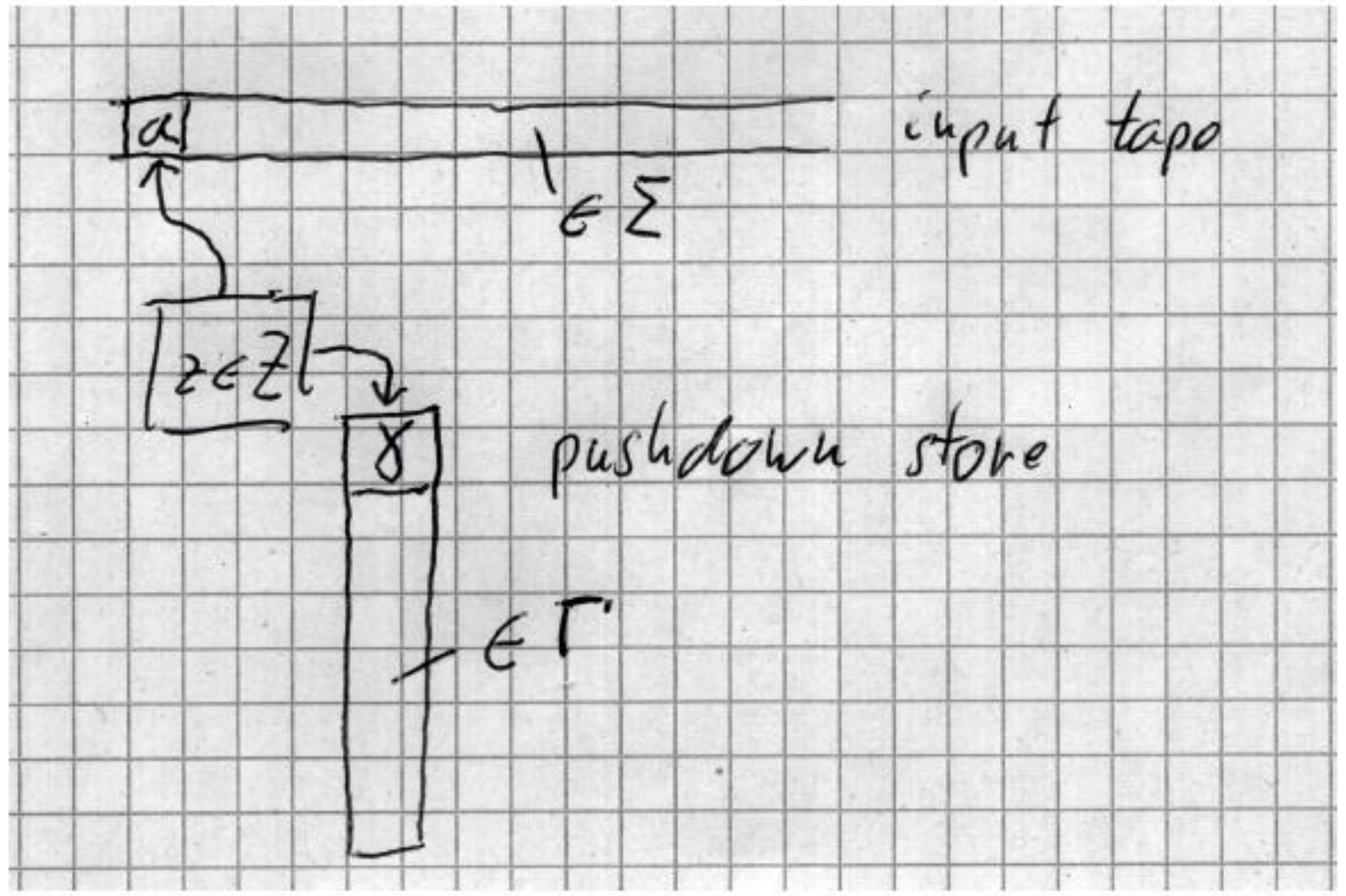
- nondeterministic transition function

$$\delta : Z \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to 2^{Z \times PA}$$

  1 step: $(z', pa) \in \delta(z, a, \gamma)$ means: if M reads input a and top of stack γ, then it can go to state z' and perform pushdown action pa. It moves the input head to the right of a (if $a = \varepsilon$ the head does not move).

- no pop ignoring (or from empty) stack: $(z', pop) \notin \delta(z, a, \epsilon)$

- $z_0 \in Z$ start state
- $Z_A \subseteq Z$ set of accepting states



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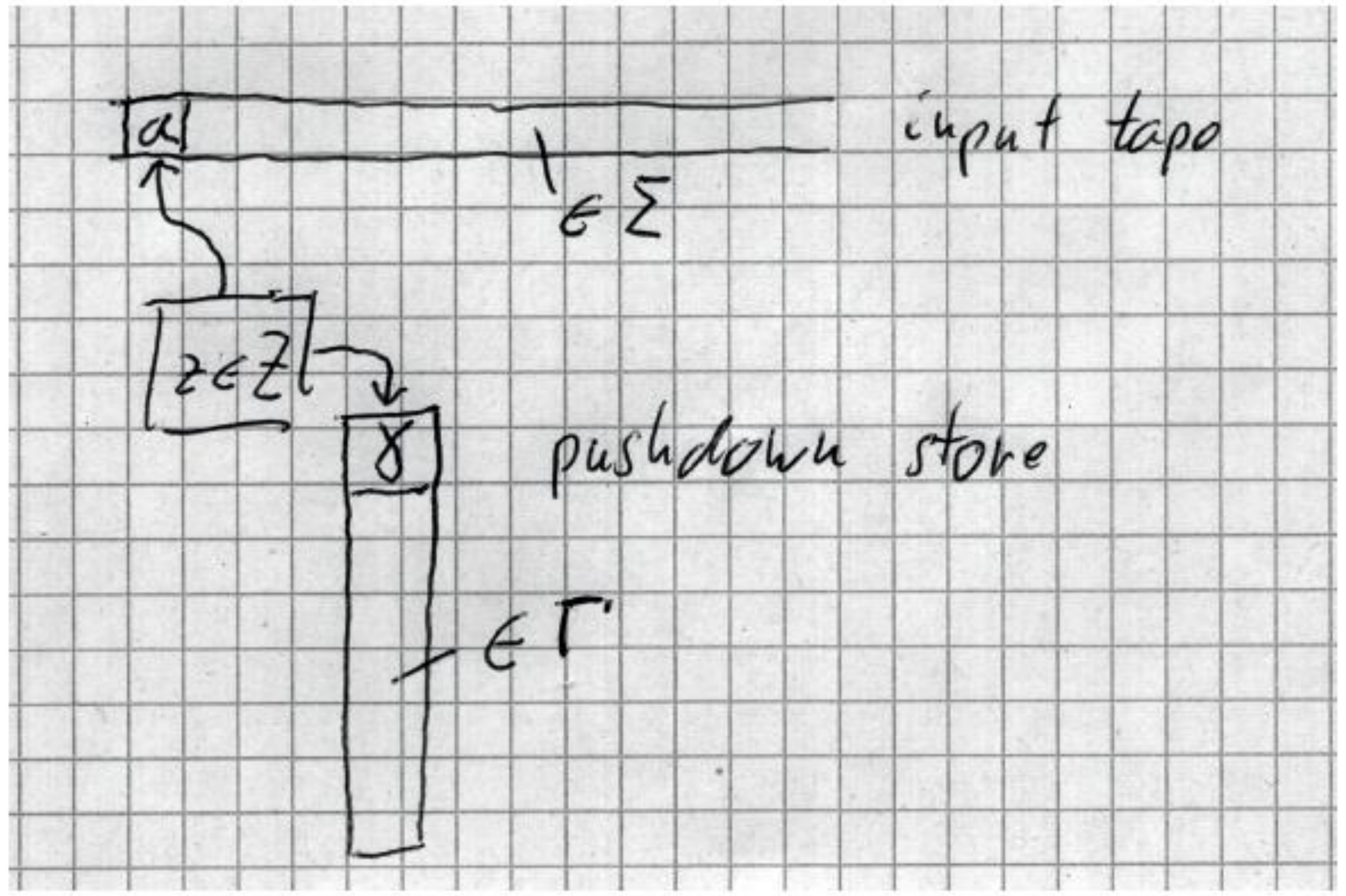

def: configurations

$$k = (z, w, p)$$

- $z \in Z$ current state
- $w \in \Sigma^*$ remaining input
- $p \in \Gamma^*$ pushdown tape/stack. From top to bottom $p_1 \dots p_k$

def: initial configuration with input w:

$$k_0 = (z_0, w, \varepsilon)$$

def: possible next/successor configuration

$$k = (z, w, p) \vdash k' = (z', w', p')$$

holds if one of the following holds

- input tape not ignored and pop

$$(z', pop) \in \delta(z, w_1, p_1) \cup \delta(z, w_1, \varepsilon) \ , \ w' = tail(w) \ , \ p' = tail(p)$$

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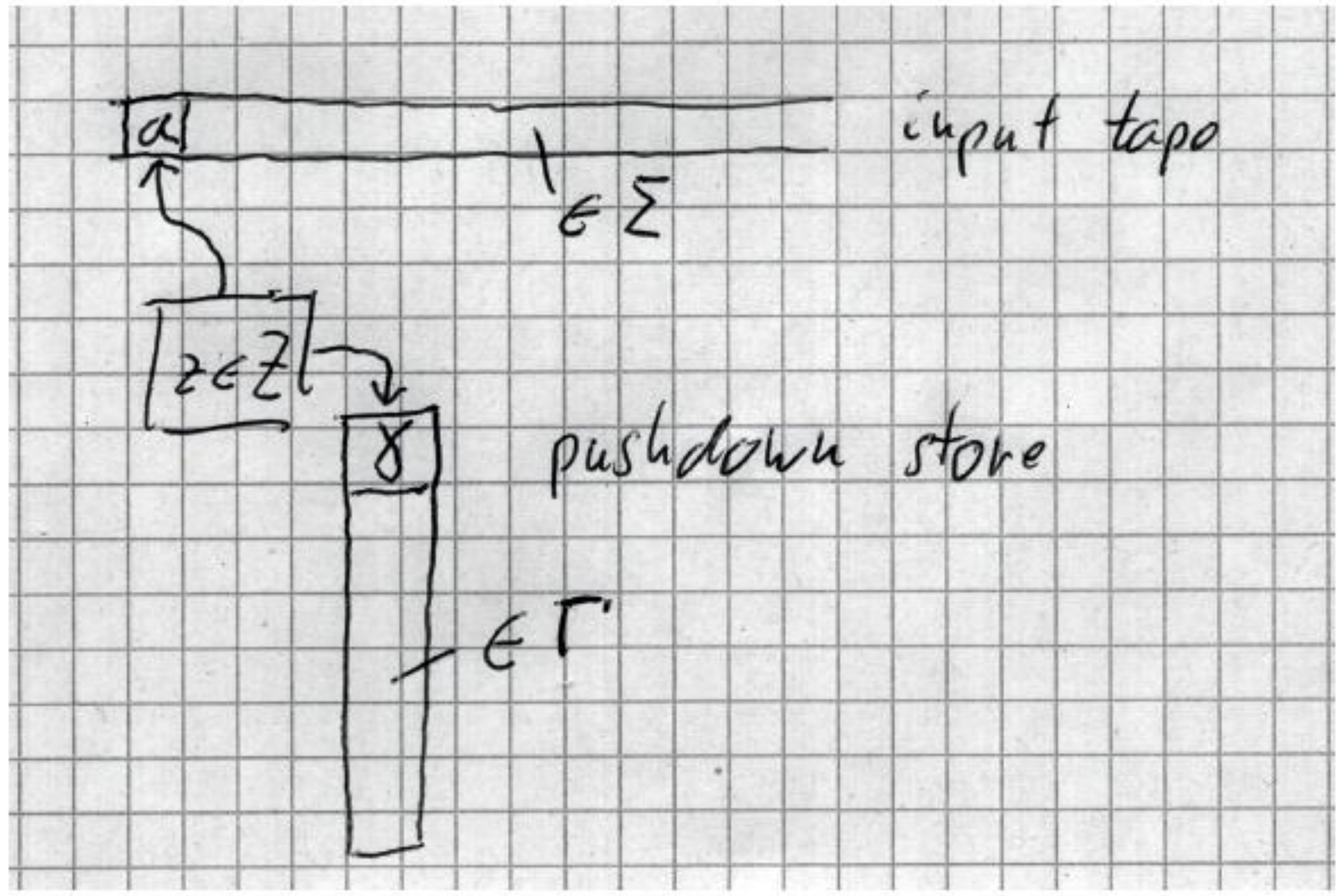

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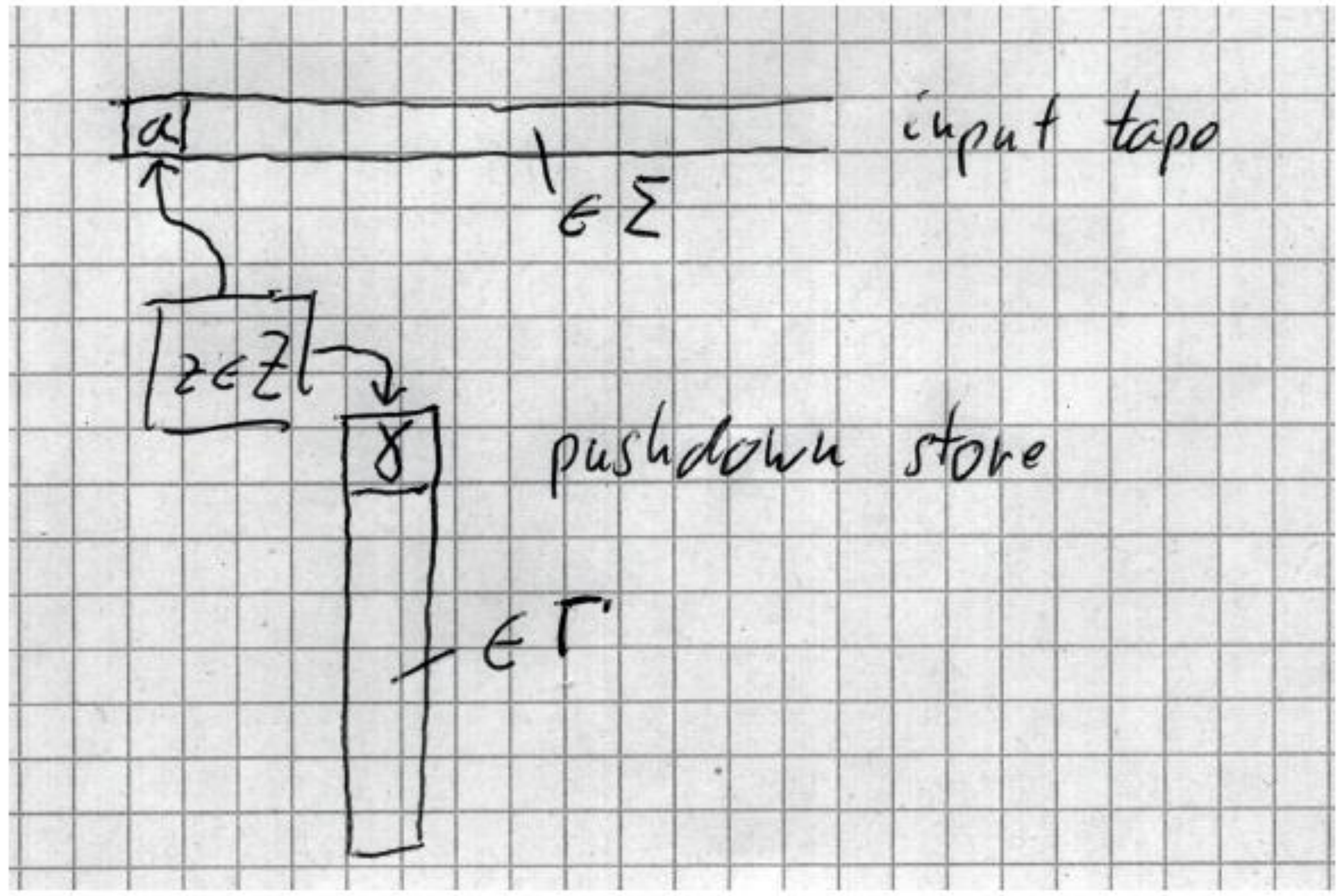

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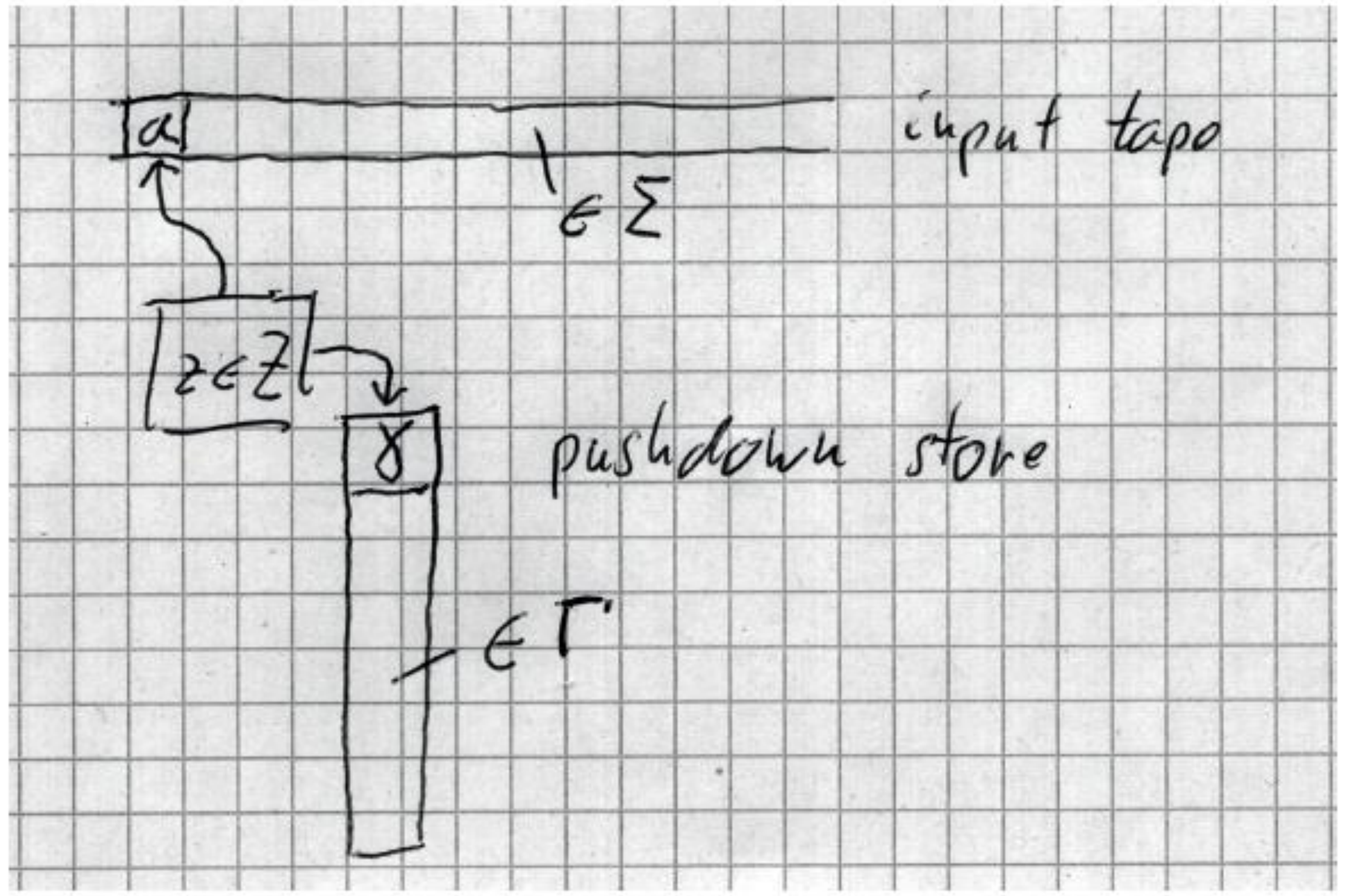


def: computation sequence $(k_0, \ldots k_m)$ with $k_i \vdash k_{i+1}$ for $i < m$. Accepting if state of k_m in Z_A and stack of k_m empty

$$k_m = (q, \varepsilon, \varepsilon)\ ,\ q \in Z_A$$

Variations: i) accept by empty stack ii) by final state in Z_A.

def: $L(M)$ M accepts w if there exists and accepting computation of M started with w.

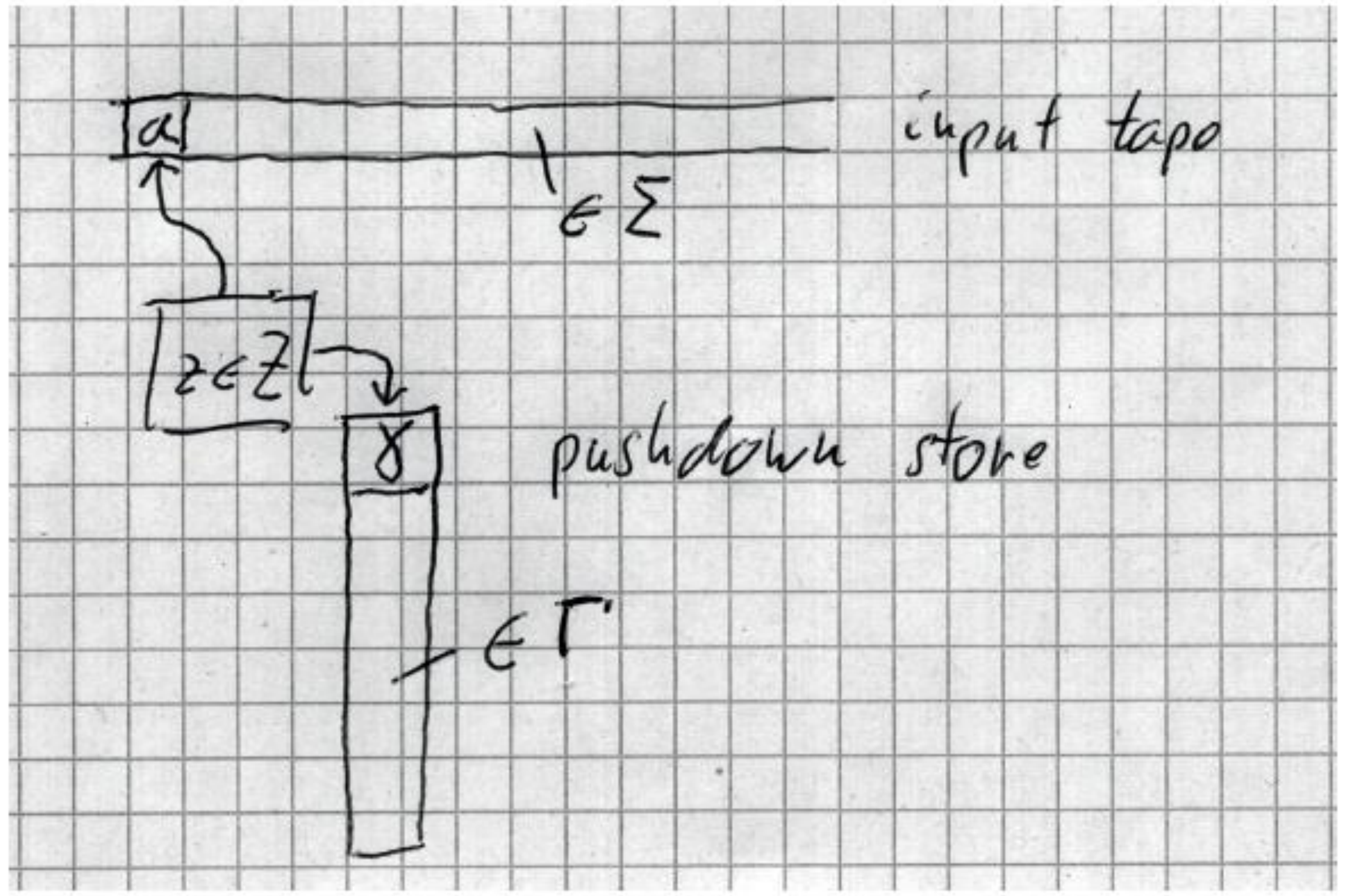
$$L(M) = \{w \in \Sigma^*\ :\ M \text{ accepts } w\}$$

example

$$\begin{aligned} w &\in A^* \\ w &= w_1 \dots w_n \\ w^R &= w_n \dots w_1 \\ L &= \{ww^R \,:\, w \in \mathbb{B}\} \end{aligned}$$

L is context free:

$$S \to 0S0 \mid 1S1 \mid \varepsilon$$



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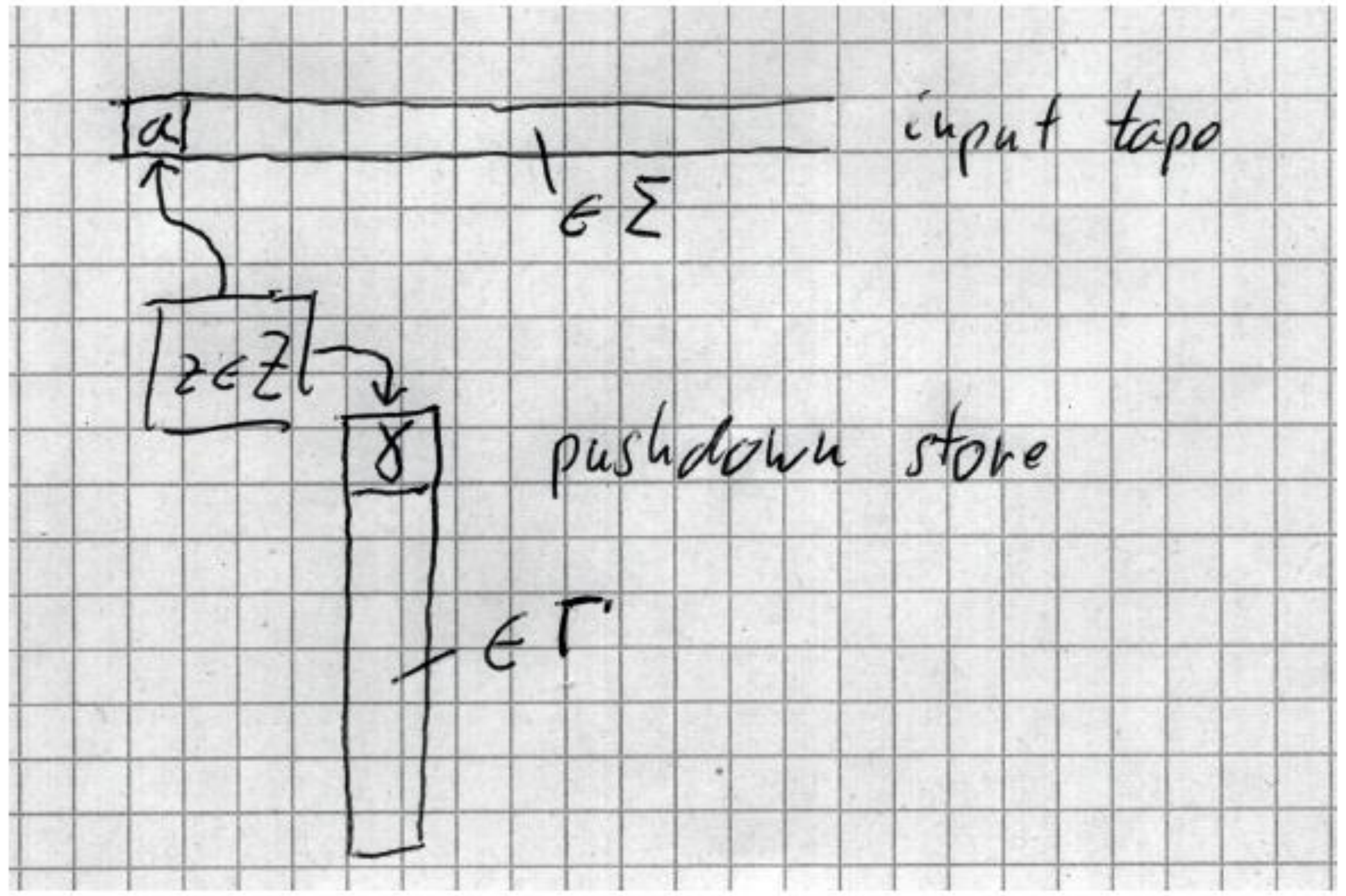

pushdown automaton for L:

$$Z = \{z, z'\} \quad , \quad z_0 = z \quad , \quad Z_A = \{z'\}$$

push w

$$\begin{aligned} (z, push\ 0) &\in \delta(z, 0, \varepsilon) \\ (z, push\ 1) &\in \delta(z, 1, \varepsilon) \end{aligned}$$

guess middle

$$(z', push\ \varepsilon) \in \delta(z, \varepsilon, \varepsilon)$$

check matching symbols

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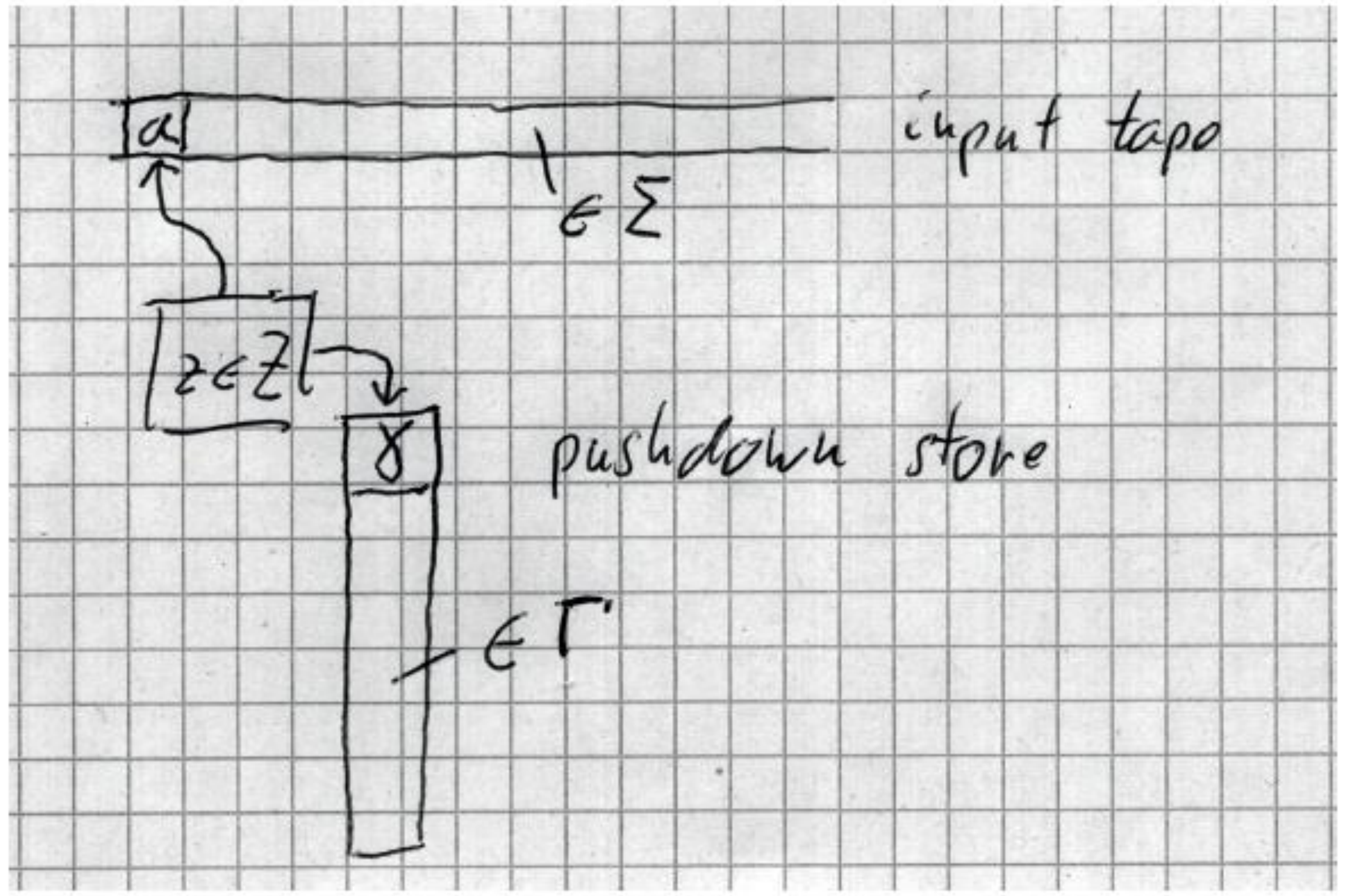

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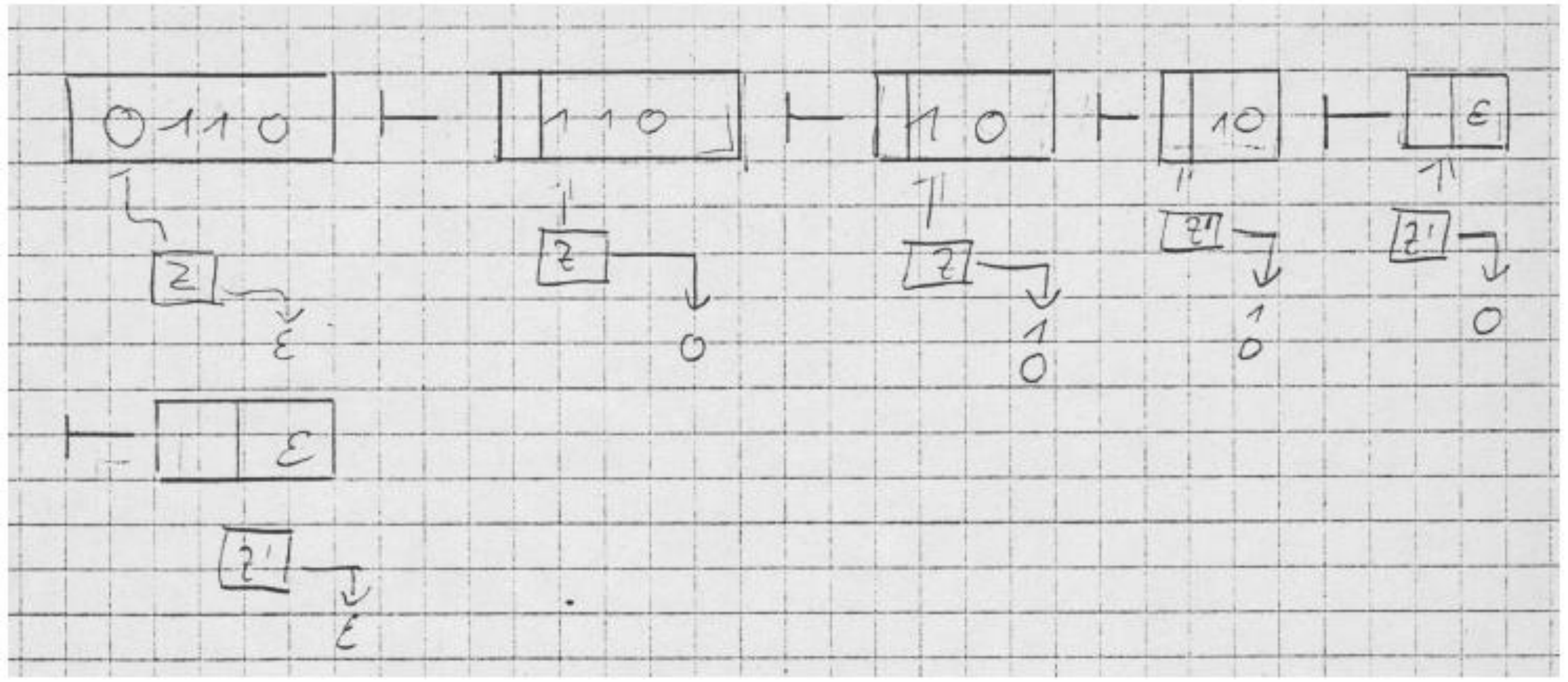

Figure 2: computation of the npda with input $w = 0110$

4 Context free languages are accepted by npda's

Lemma 5. *Let* $G = (N, T, P, S)$ *be a cf grammar. Then there is an npda* $M = (Z, \Sigma, \Gamma, \delta, z_0, Z_A)$ *with*

$$L(M) = L(G)$$

Proof in 3 steps:

1. construction of M from G
2. $w \in L(G) \rightarrow w \in L(M)$
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$$\Sigma = T \quad , \quad \Gamma = N \cup T$$

Number productions

$$P = \{P_1,\dots,P_n\}$$

production i

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$$\delta(z_0, \varepsilon, \varepsilon) = \{(r, push\ S)\}$$

2. choose a production P_i matching the top symbol $n \in N$ on stack

$$\delta(r, \varepsilon, n) = \{(r_i, push\ \varepsilon) \ : \ n_i = n\}$$

now replace $n = n_i$ on stack by right hand side $X_{i_1} \dots X_{i_{g(i)}}$ of production P_i

$$\delta(r_i, \varepsilon, n_i) = \begin{cases} \{(r_{i,g(i)}, pop)\} & g(i) \geq 1 \\ \{(r, pop)\} & g(i) = 0 \quad \text{i.e. } n_i \rightarrow \varepsilon \end{cases}$$

$$\delta(r_{i,j}, \varepsilon, \varepsilon) = \begin{cases} \{(r_{i,j-1}, push\ X_{i_j})\} & j \geq 2 \\ \{(r, push\ X_{i_1})\} & j = 1 \end{cases}$$

Effect of these transitions:

$$(r, w, n_i\alpha) \vdash^* (r, w, X_{i_1} \dots X_{i_{g(i)}}\alpha)$$

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$$\delta(z_0,\varepsilon,\varepsilon) = \{(r, push\ S)\}$$

2. choose a production P_i matching the top symbol $n \in N$ on stack

$$\delta(r,\varepsilon,n) = \{(r_i, push\ \varepsilon) \ : \ n_i = n\}$$

now replace $n = n_i$ on stack by right hand side $X_{i_1} \dots X_{i_{g(i)}}$ of production P_i

$$\delta(r_i,\varepsilon,n_i) = \begin{cases} \{(r_{i,g(i)}, pop)\} & g(i) \geq 1 \\ \{(r, pop)\} & g(i) = 0 \quad \text{i.e. } n_i \to \varepsilon \end{cases}$$

$$\delta(r_{i,j},\varepsilon,\varepsilon) = \begin{cases} \{(r_{i,j-1}, push\ X_{i_j})\} & j \geq 2 \\ \{(r, push\ X_{i_1})\} & j = 1 \end{cases}$$

Effect of these transitions:

$$(r,w,n_i\alpha) \vdash^* (r,w,X_{i_1} \dots X_{i_{g(i)}}\alpha)$$

3. cancel matching terminals $\in T$

$$\delta(r,t,t) = \{(R, pop)\}$$

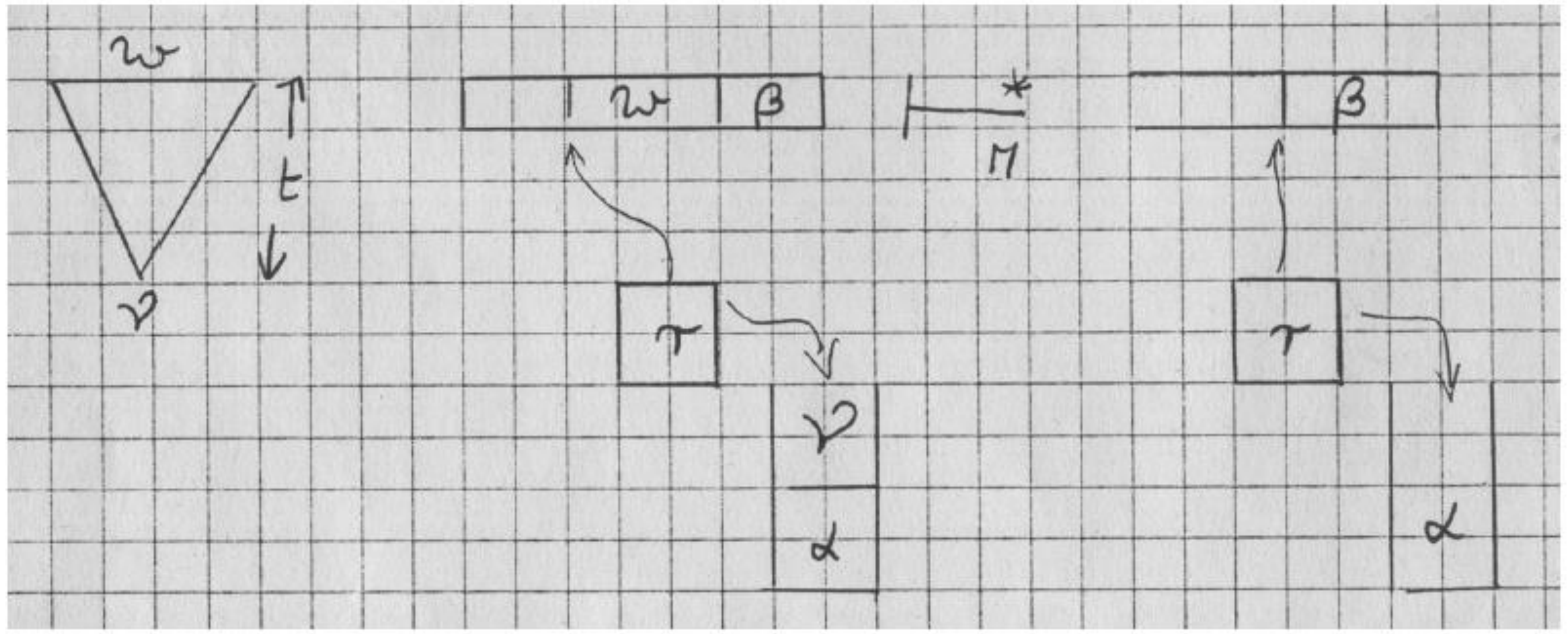
Effect

$$(r,t\beta,t\alpha) \vdash^* (r,\beta,\alpha)$$

4.2 $w \in L(G) \to w \in L(M)$:

Lemma 6. *Let $w, \beta \in T^*$ and $v \in N \cup T$ and $v \to^* w$ by a derivation tree of depth $\leq t$. Then*

$$(r, w\beta, v\alpha) \vdash^* (r, \beta, \alpha)$$



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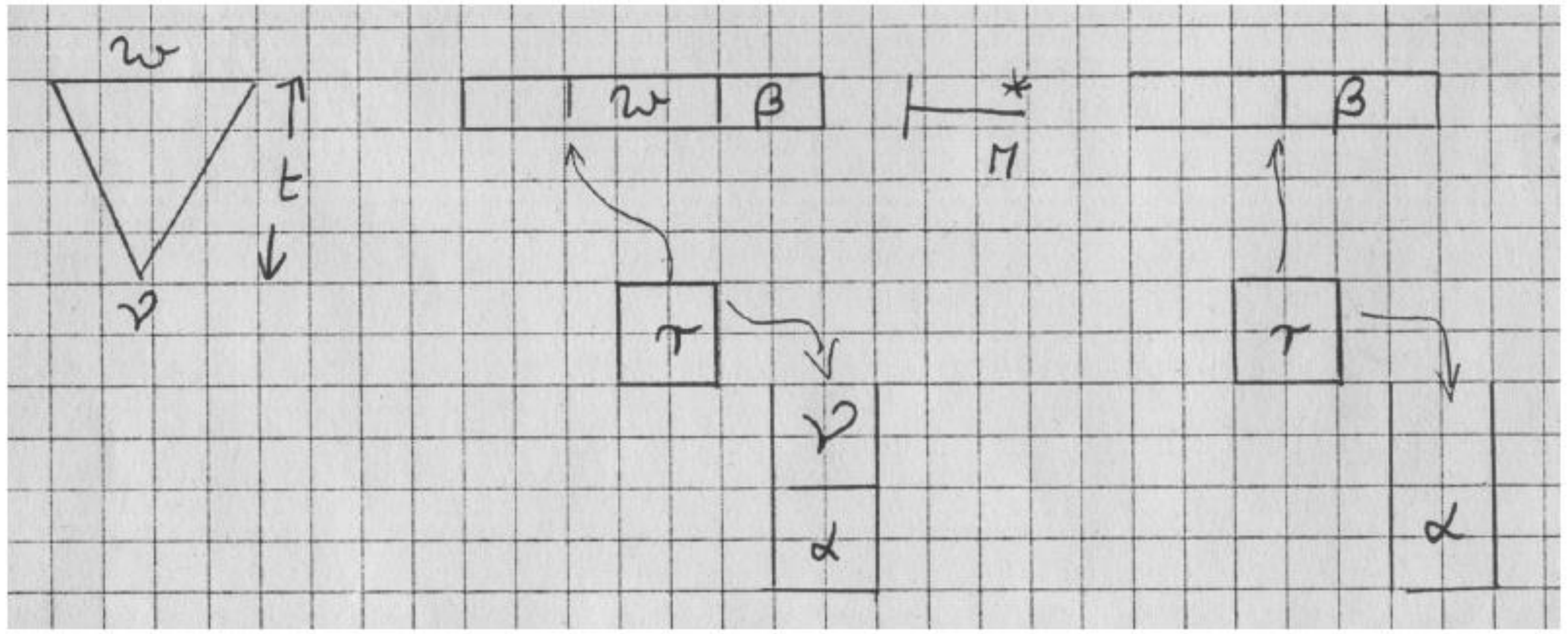
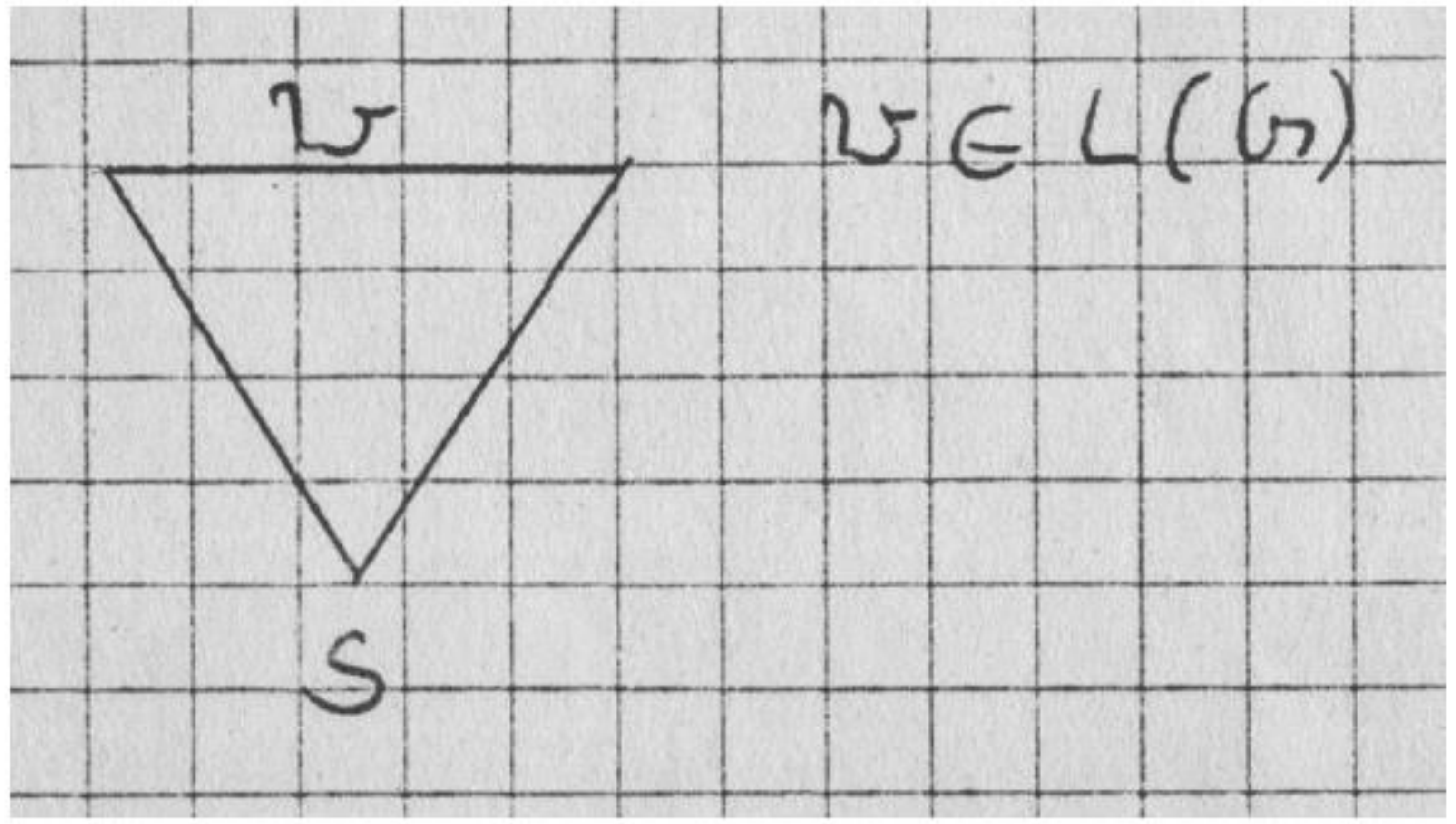


Lemma implies claim: $w \in L(G)$ implies derivation tree as in figure 4



$$\begin{aligned} (z_0, w, \varepsilon) &\vdash (r, w, S) && \text{(construction 1)} \quad \text{pushing S} \\ &\vdash^* (r, \varepsilon, \varepsilon) && \text{(lemma 6)} \end{aligned}$$

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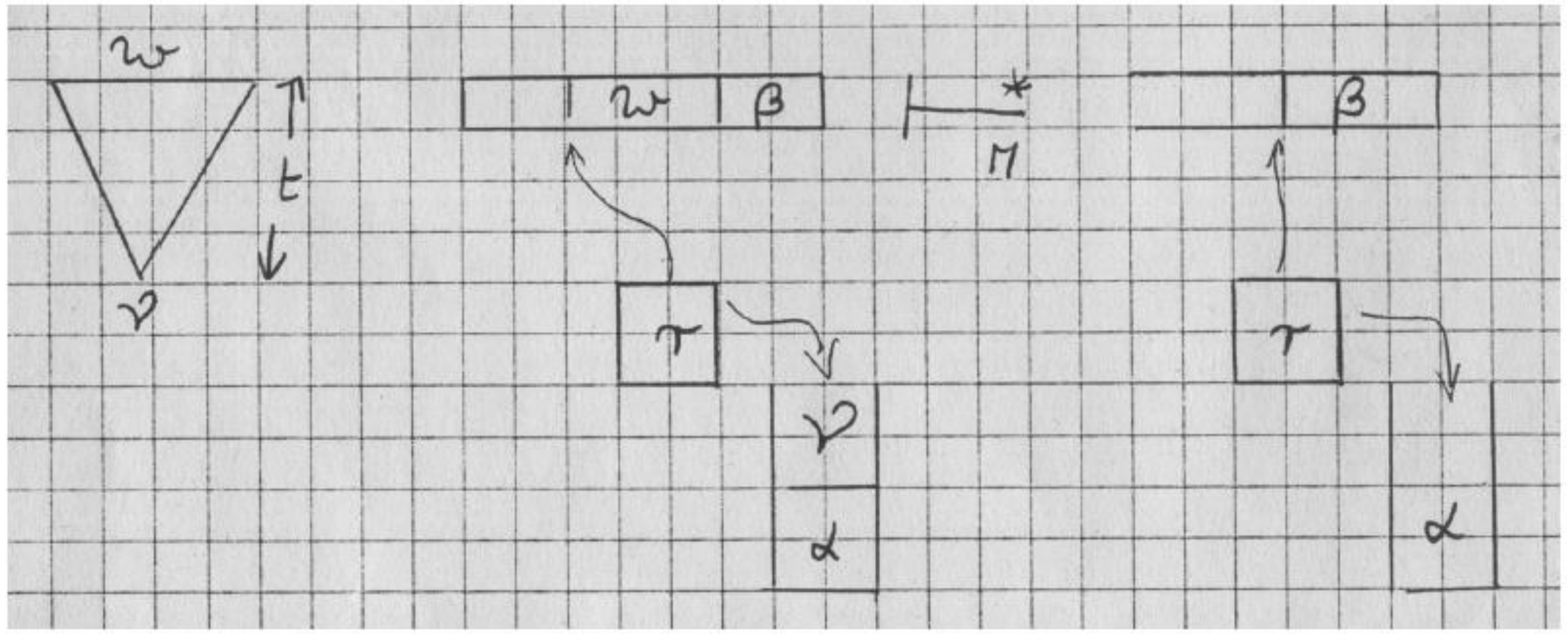

proof of lemma 6 by induction on t:

- $t = 0$: $v = w \in T$. Construction 3 $\rightarrow$ claim.

cancelling
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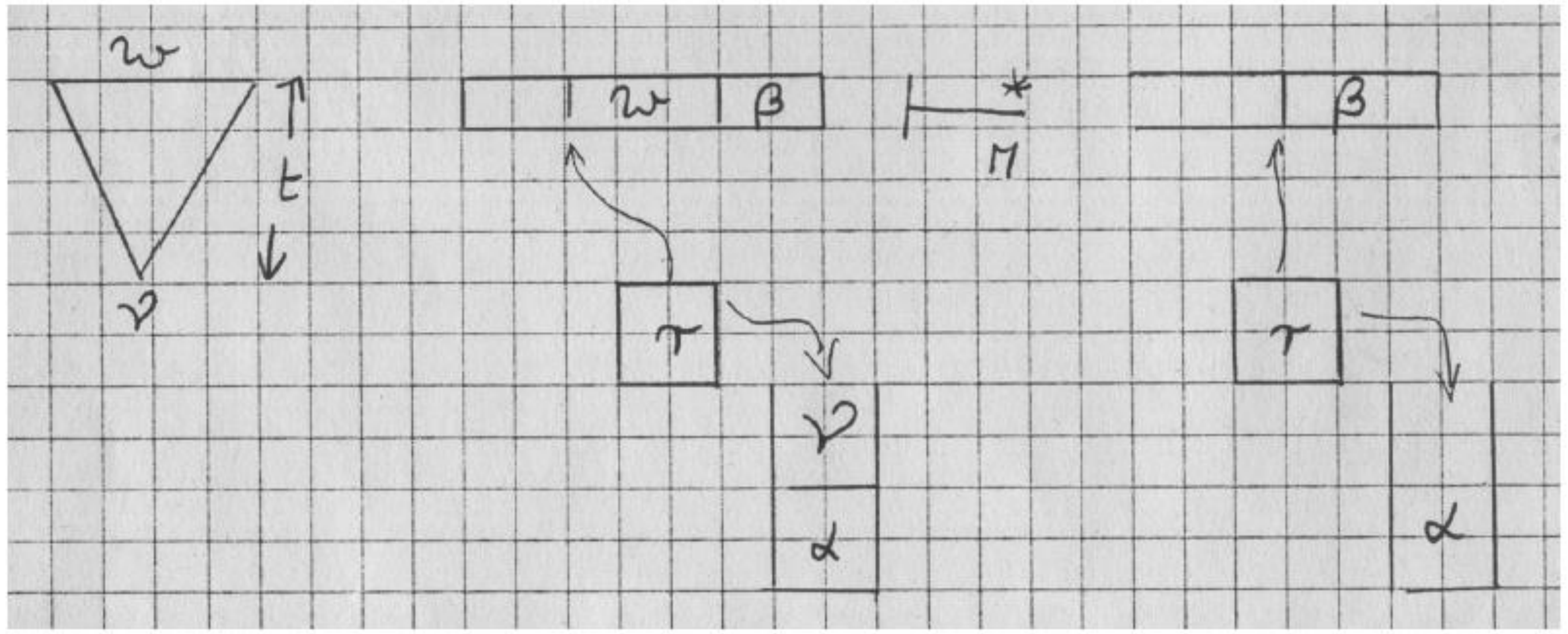

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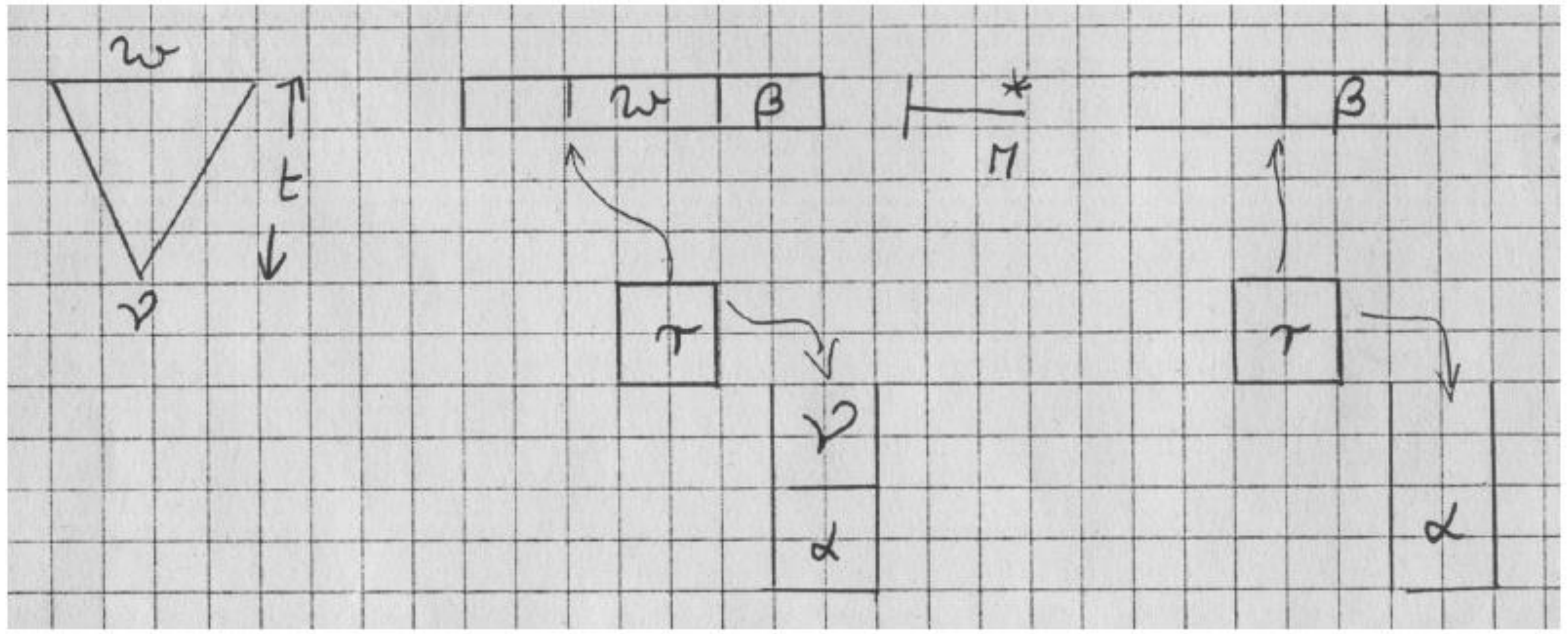

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otherwise: $n = n_i$ derivation tree as in figure 5

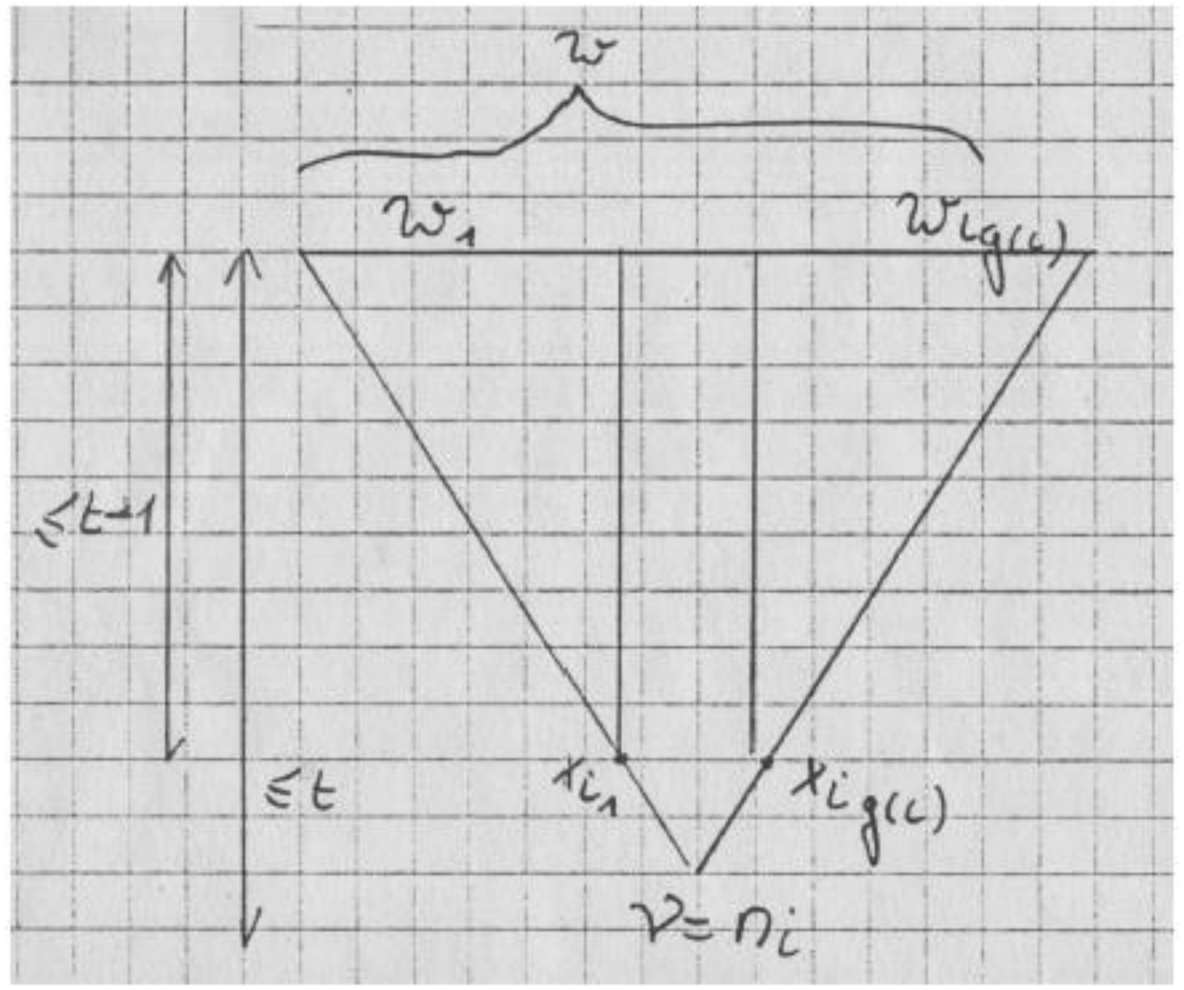

Figure 5: derivation tree of w from $v = n_i$

$$\begin{aligned}
(r, w\beta, v\alpha) &\vdash^*_{constr.2} (r, w_1 \ldots w_{g(i)}\beta, X_{i_1} \ldots X_{i_{g(i)}}\alpha) \\
&\vdash^*_{IH} (r, w_2 \ldots\ldots w_{g(i)}\beta, X_{i_2} \ldots X_{i_{g(i)}}\alpha) \\
&\ldots \\
&\vdash^*_{IH} (r, \beta, \alpha)
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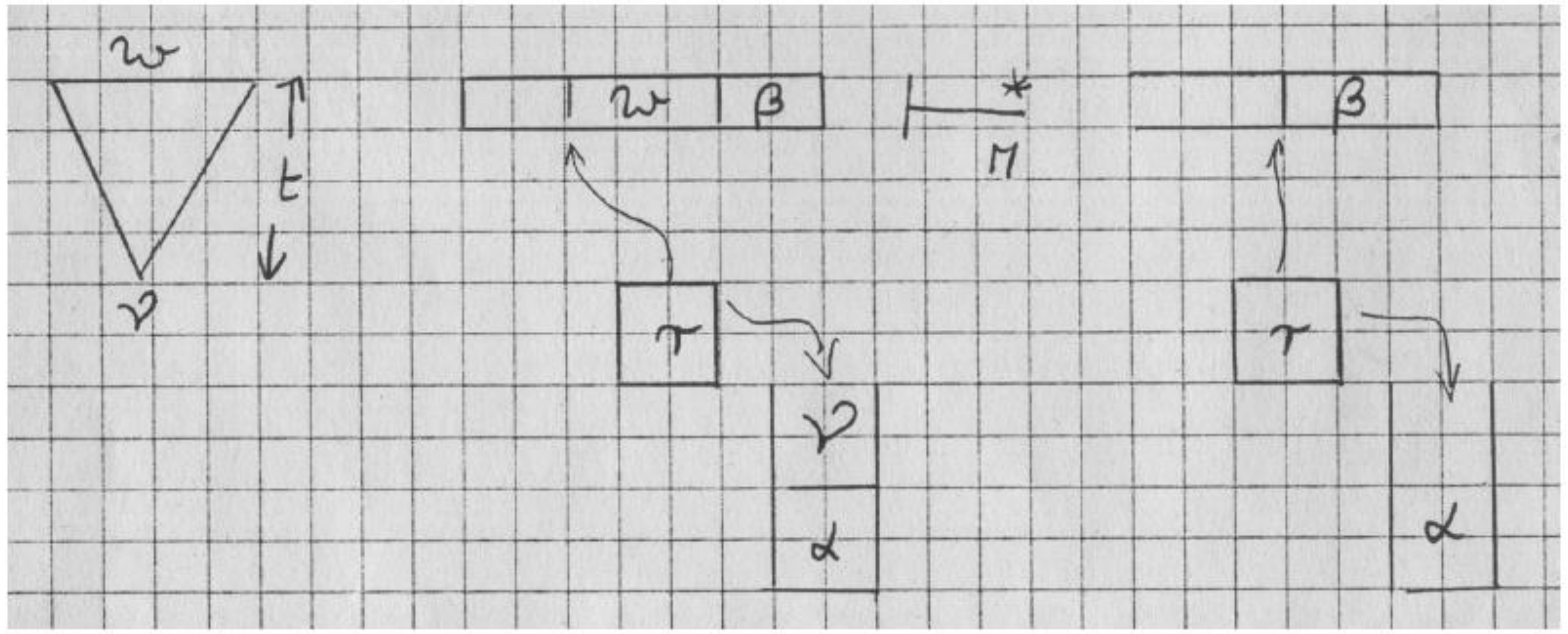

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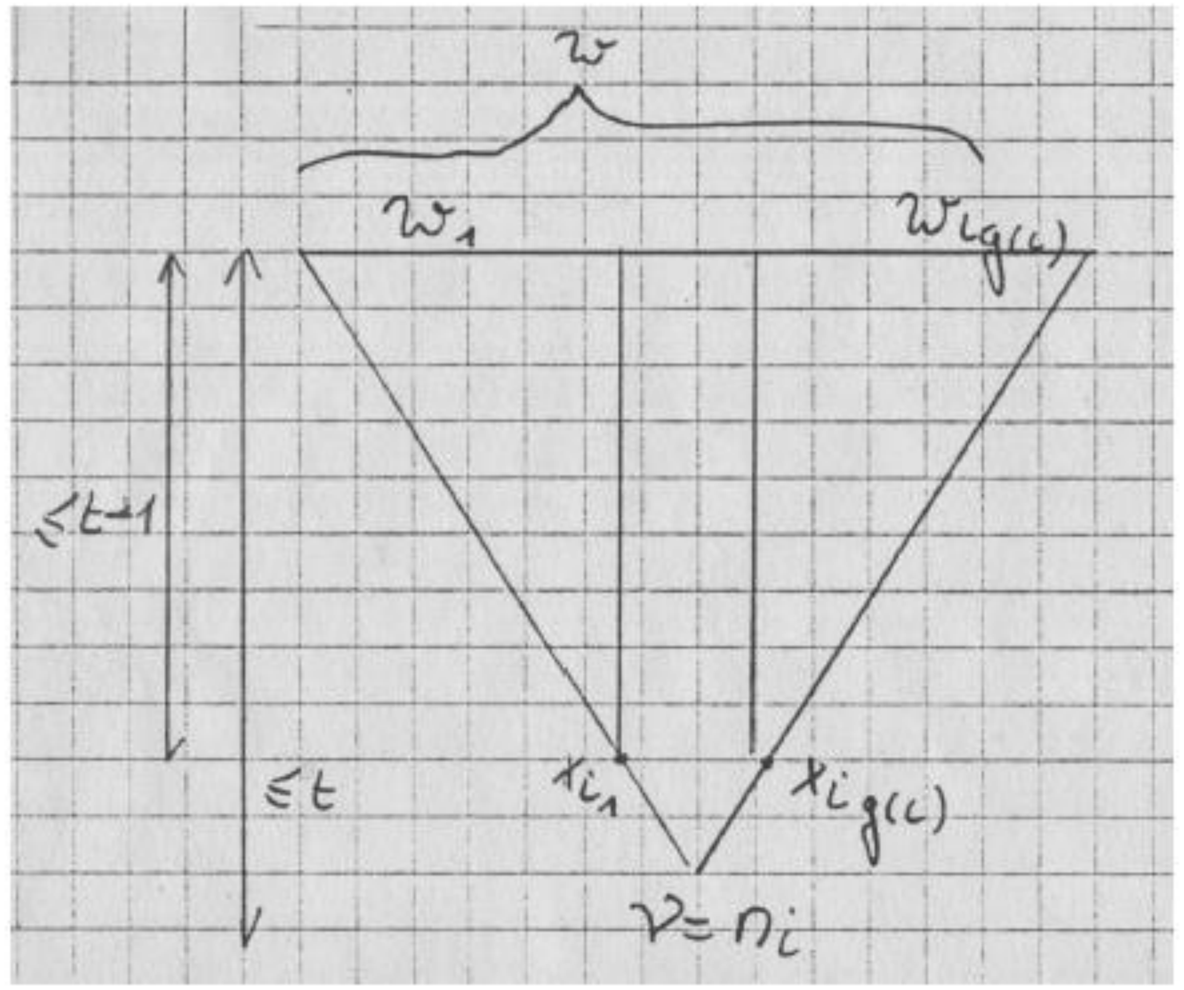

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-

$Z_A = \{z_s, r\}$

4.3 $w \in L(M) \rightarrow w \in L(G)$:

Let $w \in L(M)$. Consider accepting computation

$$R = (k_0, \dots, k_s)$$

of M started with w

$$k_0 = (z_0, w, \varepsilon) \quad , \; k_s = (r, \varepsilon, \varepsilon)$$

For

$$k \in \{k_0, \dots, k_s\} \; , \; k = (r, \mu, \boxed{p\alpha}) \; , \; p \in N \cup T$$

define $N(k)$ as first configuration after k with state r and stack α

$$N(k) = (r, \mu', \boxed{\alpha})$$

Define $\tau(k)$ as number of steps between k and $N(k)$.

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$$v \rightarrow^* \mu$$

Claim follows with $\alpha = \beta = \varepsilon$

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proof of lemma : induction on $\tau(k)$

- $\tau(k) = 1$: construction 3 $\to$

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cancelling
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special case of construction 2: $g(i) = 0$. Then

$$v \to \varepsilon \in P \ , \ \mu = \varepsilon \ , v \to^1 \mu$$

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$$g(i) \geq 1, v = n_i$$

decompose computation

$$\begin{array}{rlll}
(r, \mu\beta, v\alpha) & \vdash^{*} & k_1 = (r, \mu\beta, X_{i_1} \dots X_{i_{g(i)}}\alpha) & \text{(construction 2)} \\
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& \dots & & \\
& \vdash^{\tau(k_{g(i)-1})} & k_{g(i)} = (r, \overline{\mu_{g(i)-1}}\beta, X_{i_{g(i)}}\alpha) & \\
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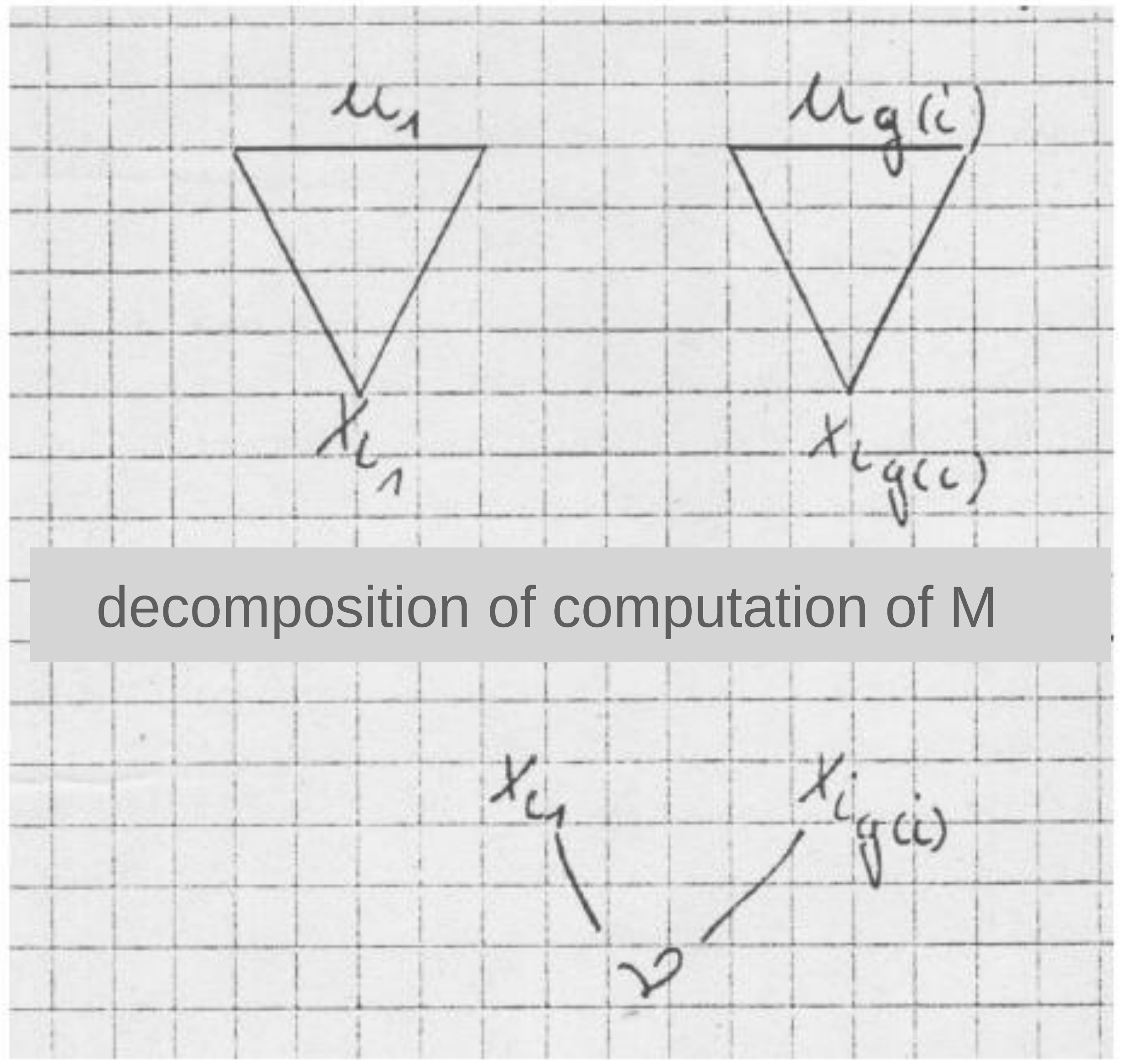
with $\tau(k_j) < \tau(k)$ for all j.



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part 1: reasonably straight forward construction
parts 2 and 3: clever induction hypothesis,
Then: bookkeeping.

decompose computation

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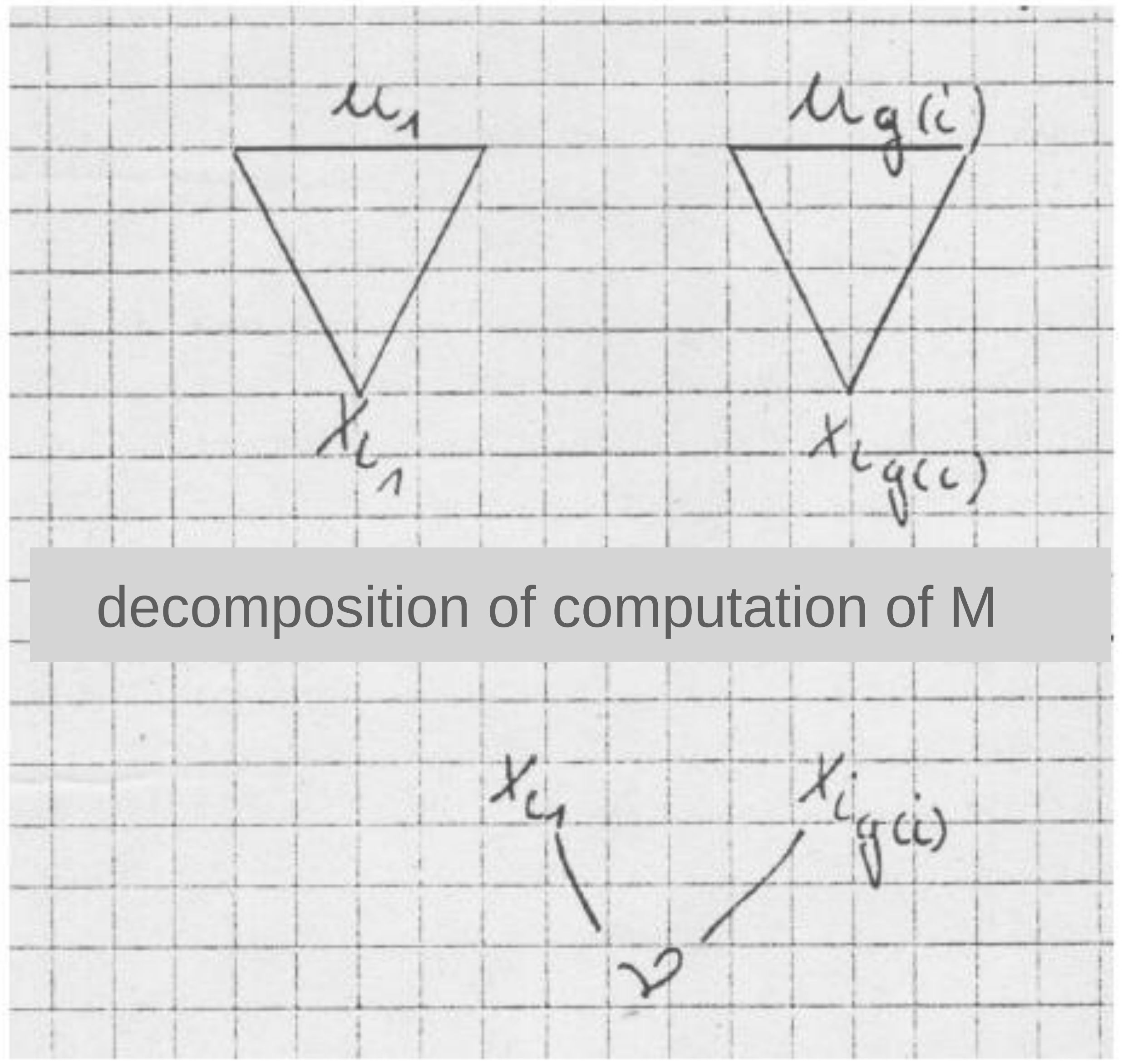

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5 Languages accepted by npda's are context free

‚computing with grammars'

Lemma 8. *Let* $M = (Z, \Sigma, \Gamma, \delta, z_0, Z_A)$ *be an npda. Then there is a context free grammar* $G = (N, T, P, S)$ *with*

$$L(M) = L(G)$$

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Proof in 3 steps:

1. construction of G
2. $x \in L(G) \rightarrow x \in L(M)$
3. $x \in L(M) \rightarrow x \in L(G)$

Notation For $n \in N$ we define the language generated by n as

$$L(n) = \{w \in T^* \ : \ n \rightarrow^* w\}$$

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5.1 Construction of G

$$\begin{aligned} T &= \Sigma \\ N &= \{\langle q, A, p\rangle \ : \ p, q \in Z \ , \ A \in \Gamma_\varepsilon\} \cup \{S\} \\ S &= S \end{aligned}$$

intention:

$$L(\langle q, A, p\rangle) = \{w \ : \ (q, w, A) \vdash^* (p, \varepsilon, \varepsilon)\}$$

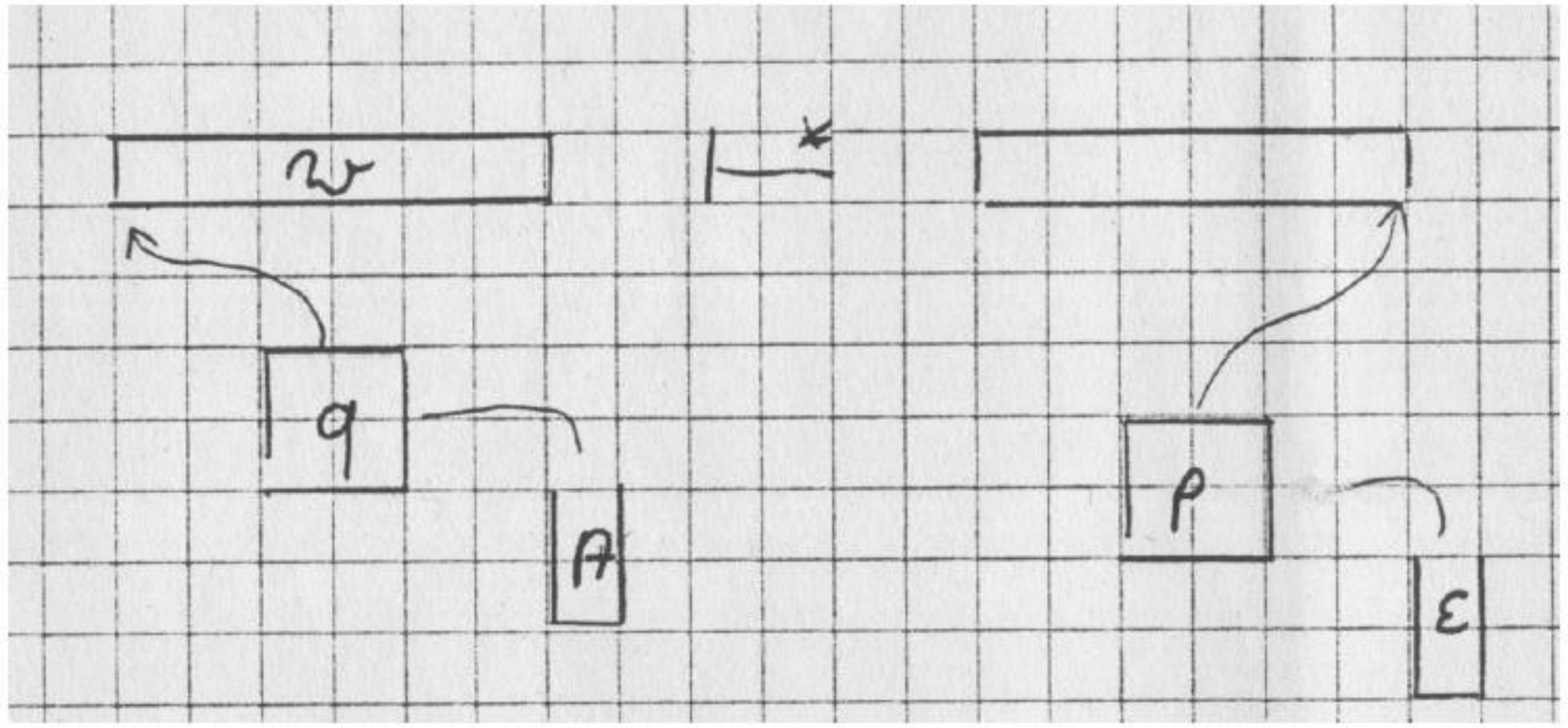

Figure 7: Symbol $\langle q, A, p\rangle$ should generate all words w such that M started in state q with input w and stack A consumes the input, empties the stack and ends in state p

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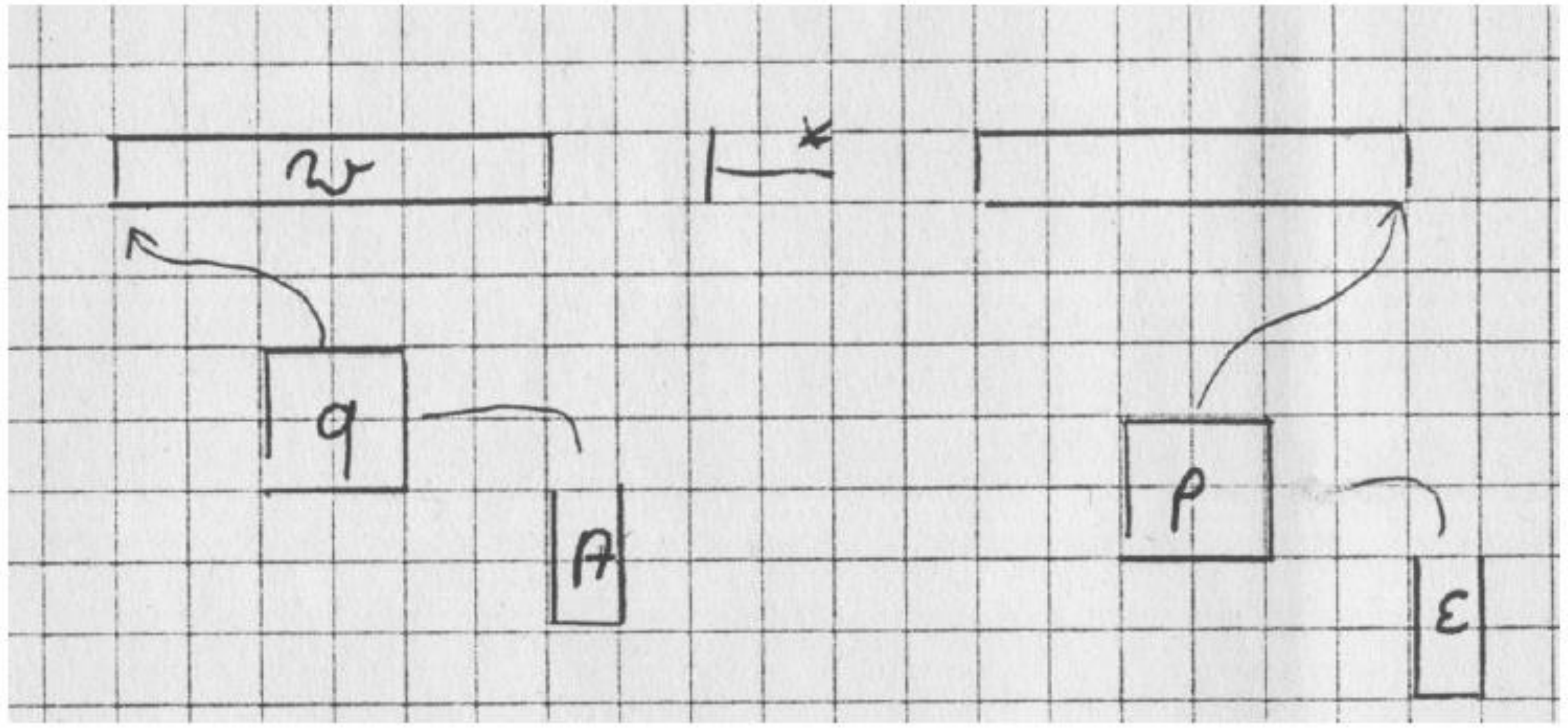

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1. for all accepting states $f \in Z_A$:

$$S \rightarrow \langle z_0, \varepsilon, f\rangle$$

2. if

$$(r, push\ a) \in \delta(q,u,A) \quad \text{with} \quad u \in \Sigma_\varepsilon \quad a \in \Gamma_\varepsilon, \quad A \in \Gamma$$

then for all $q_1, p \in Z$ intermediate and end states

$$\langle q,A,p\rangle \rightarrow u\langle r,a,q_1\rangle\langle q_1,A,p\rangle$$

3. similar; if

$$(r, push\ a) \in \delta(q,u,\varepsilon) \quad \text{with} \quad u \in \Sigma_\varepsilon \quad a \in \Gamma_\varepsilon$$

then for all $q_1, p \in Z$ and $b \in \Gamma_\varepsilon$: also possible top(stack)

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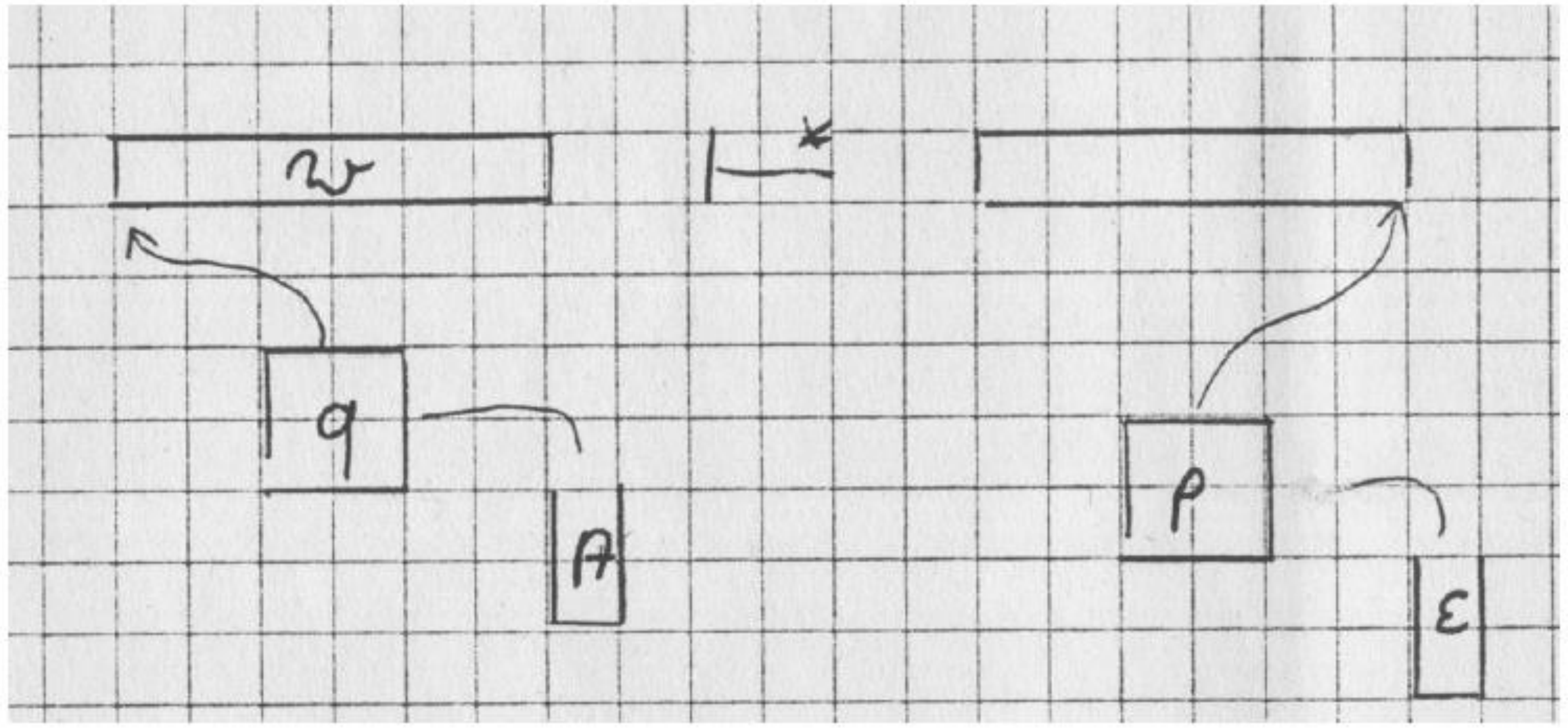

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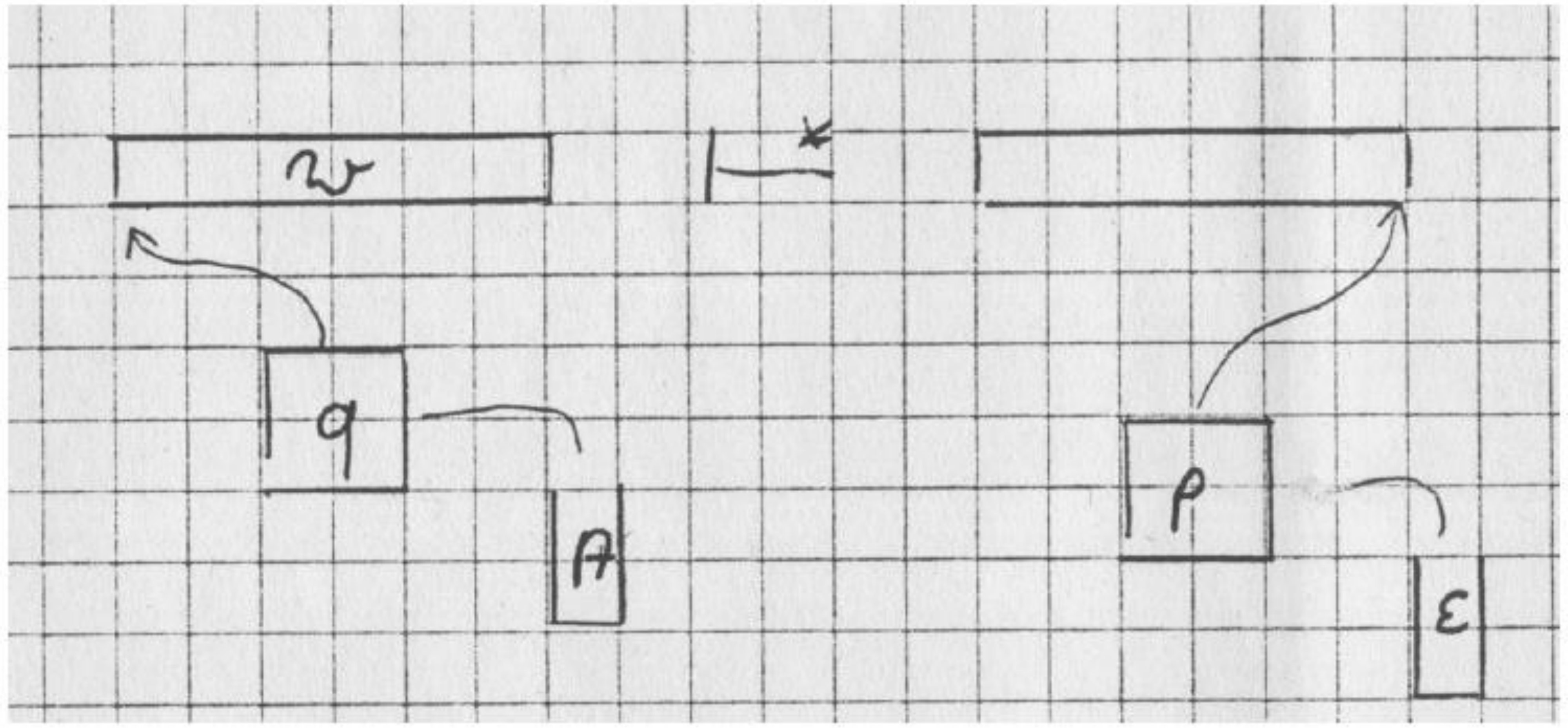

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$$\langle q,b,p\rangle \to u\langle r,\varepsilon,p\rangle$$

6. for all $q \in Z$

$$\langle q,\varepsilon,q\rangle \to \varepsilon$$

5.2 $x \in L(G) \to x \in L(M)$

Lemma 9.

$$x \in L(\langle q, A, p \rangle) \to (q, x, A) \vdash^* (p, \varepsilon, \varepsilon)$$

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Lemma 9.

$$x \in L(\langle q, A, p\rangle) \to (q, x, A) \vdash^* (p, \varepsilon, \varepsilon)$$

Proof by induction on depth t of derivation tree from $\langle q, A, p\rangle$ to x

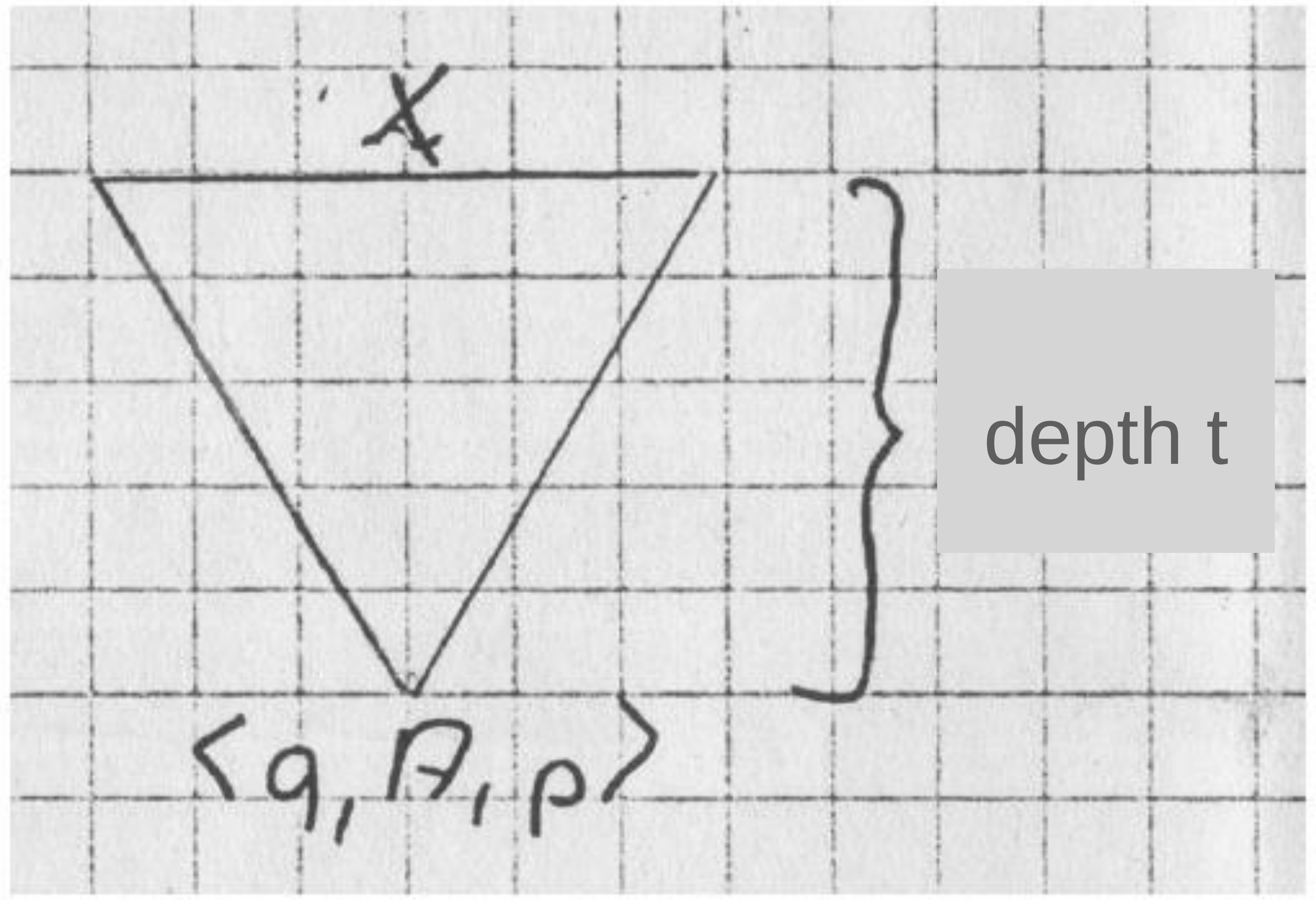

Figure 8: by induction on the depth t of the derivation tree we want to conclude $(q, x, A) \vdash^* (p, \varepsilon, \varepsilon)$.

- $t = 1$ only for construction 6.

$$\langle q, \varepsilon, q\rangle \to \varepsilon \quad , \quad \varepsilon \in T^*$$

$$p = q \ , x = A = \varepsilon$$

$$(q, x, A) = (q, \varepsilon, \varepsilon) \vdash^0 (q, \varepsilon, \varepsilon) = (p, \varepsilon, \varepsilon)$$

Lemma 9.

$$x \in L(\langle q, A, p\rangle) \to (q,x,A) \vdash^* (p,\varepsilon,\varepsilon)$$

Proof by induction on depth t of derivation tree from $\langle q, A, p\rangle$ to x

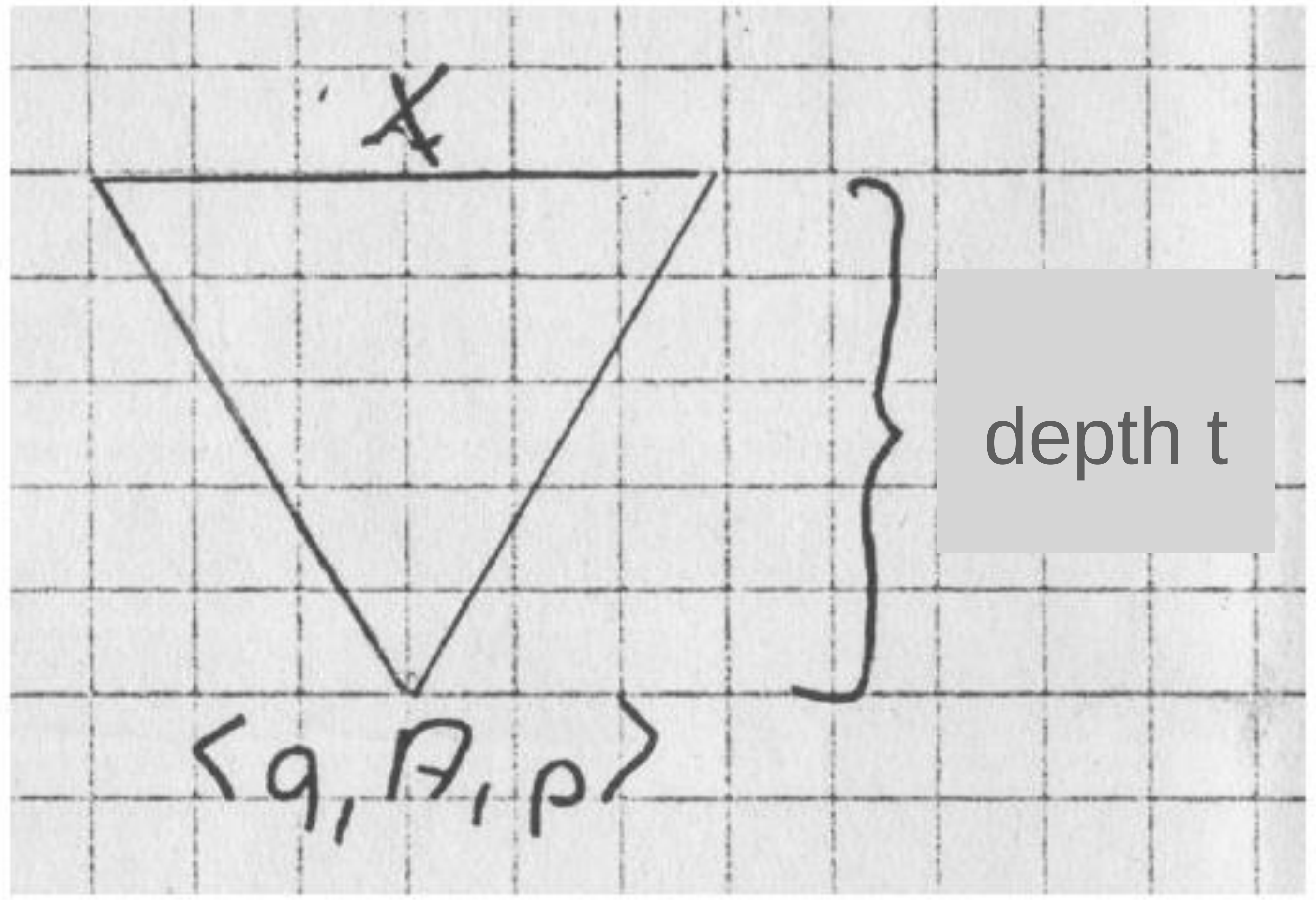

Figure 8: by induction on the depth t of the derivation tree we want to conclude $(q,x,A) \vdash^* (p,\varepsilon,\varepsilon)$.

- $t-1 \to t$

construction 2: see figure 9 push a with $A \neq \epsilon$

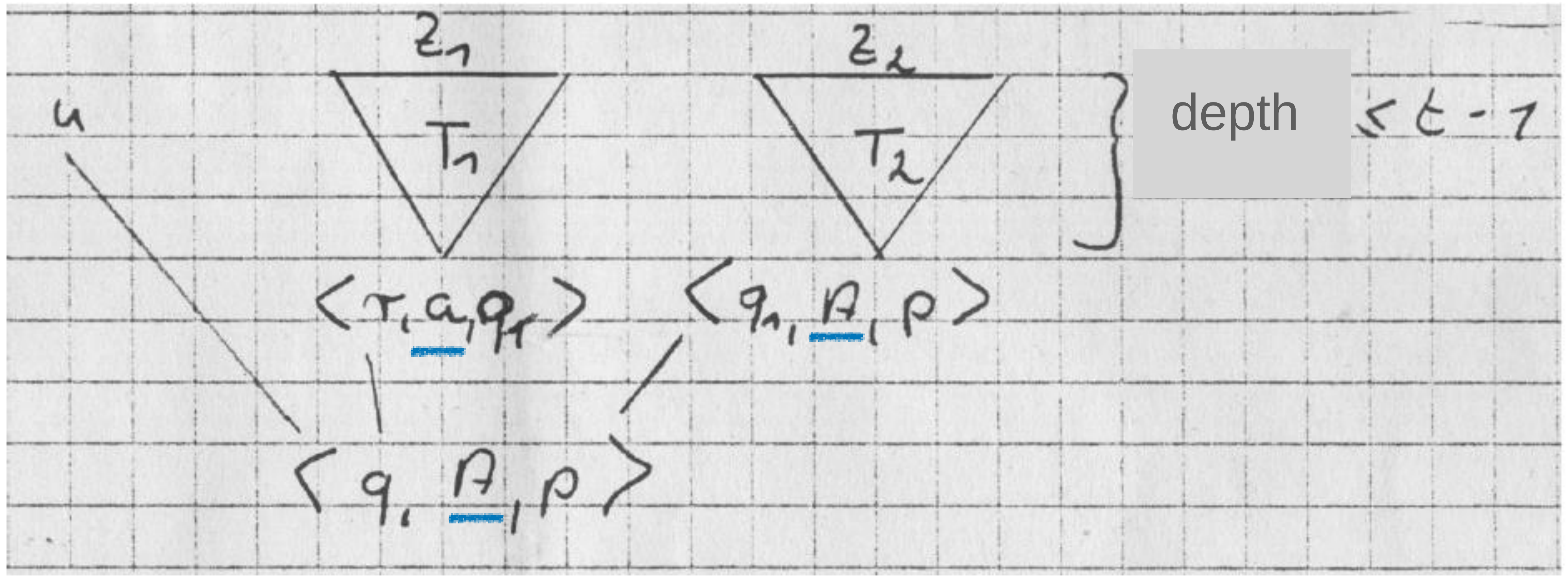

Figure 9: derivation tree for case 2 of the construction. The input word is $x = uz_1z_2$

Lemma 9.

$$x \in L(\langle q, A, p \rangle) \rightarrow (q, x, A) \vdash^* (p, \varepsilon, \varepsilon)$$

Proof by induction on depth t of derivation tree from $\langle q, A, p \rangle$ to x

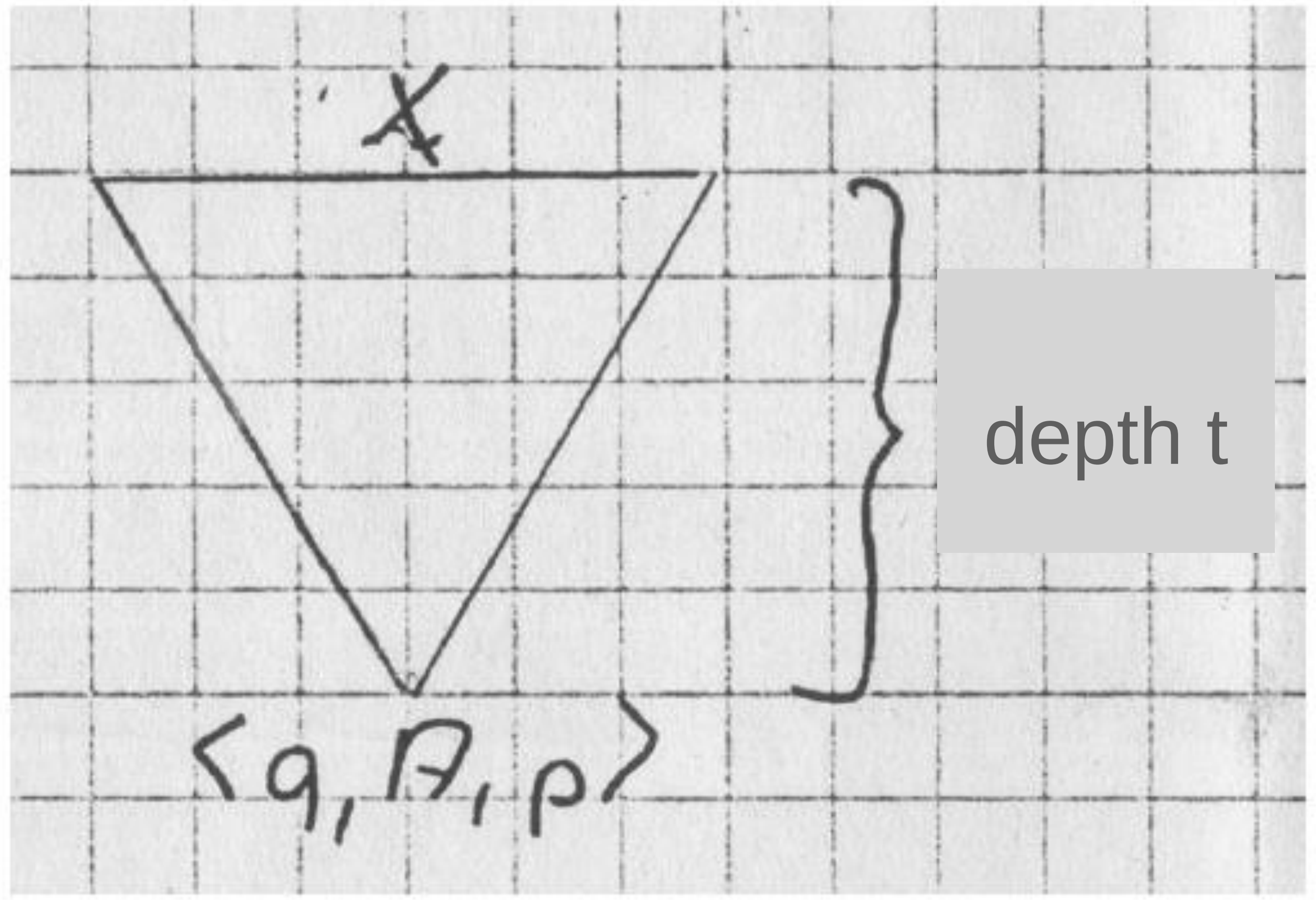

Figure 8: by induction on the depth t of the derivation tree we want to conclude $(q, x, A) \vdash^* (p, \varepsilon, \varepsilon)$.

- $t - 1 \rightarrow t$

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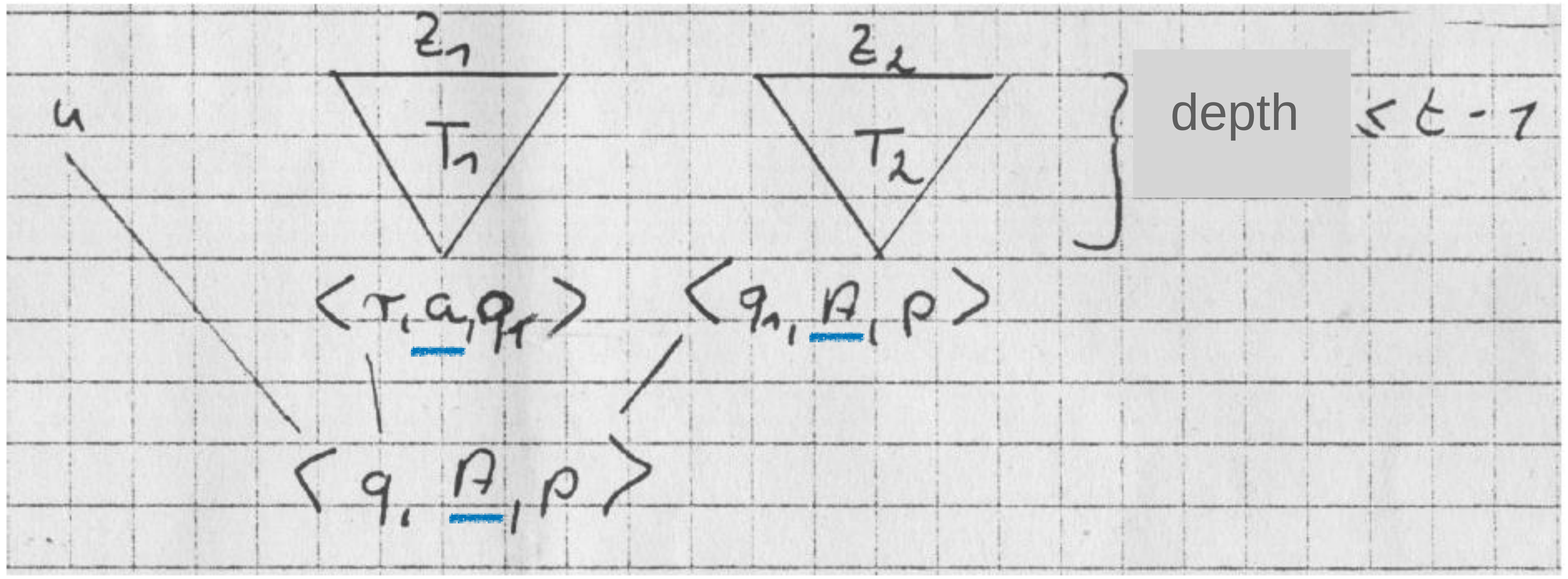

Figure 9: derivation tree for case 2 of the construction. The input word is $x = uz_1z_2$

$$\begin{array}{llll} (q, x, A) & \vdash & (r, z_1 z_2, aA) & \text{(construction of } G\text{)} \\ (r, z_1, a) & \vdash^* & (q_1, \varepsilon, \varepsilon) & \text{(induction hypothesis for } T_1\text{)} \end{array}$$

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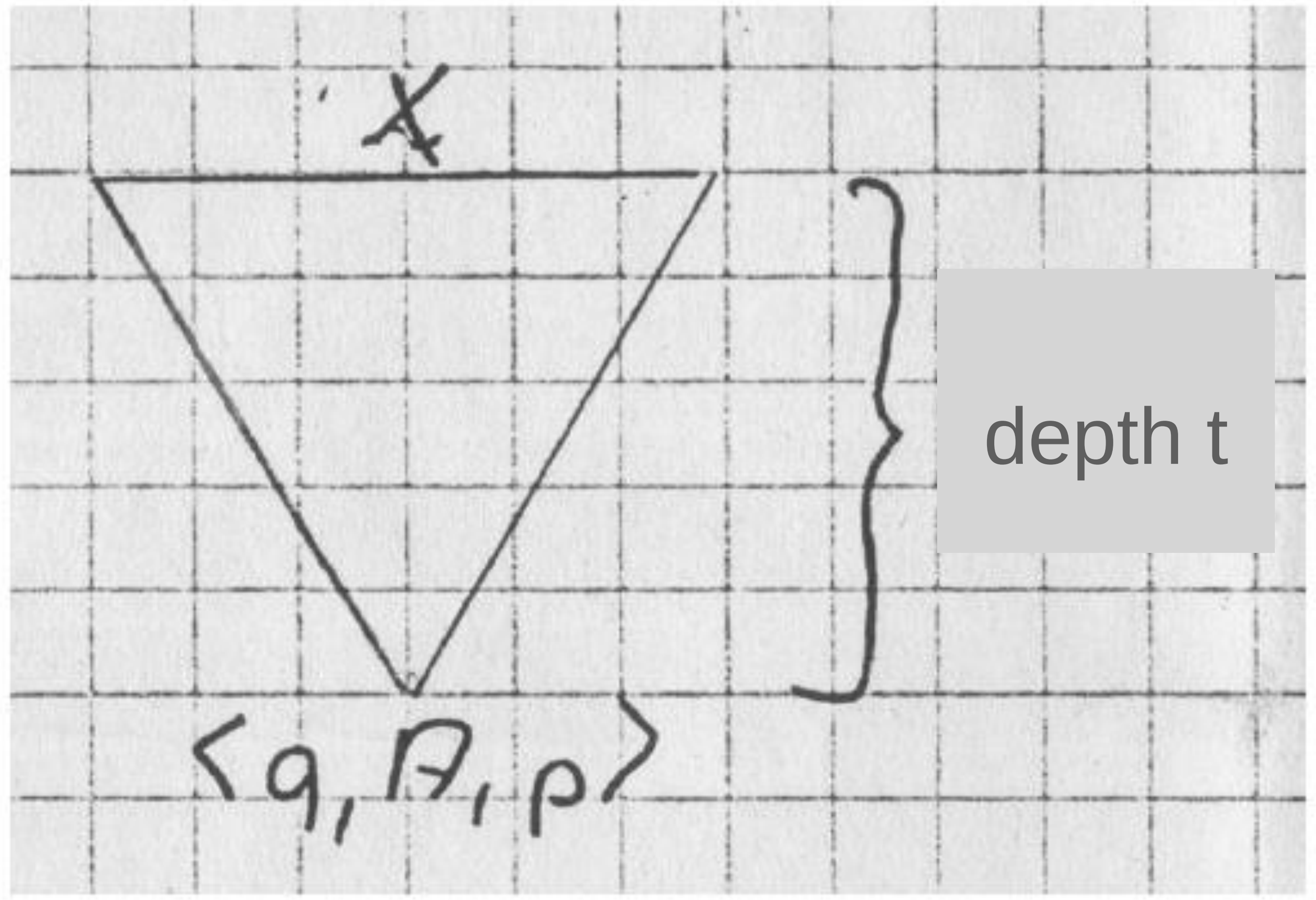
Proof by induction on depth t of derivation tree from $\langle q, A, p\rangle$ to x



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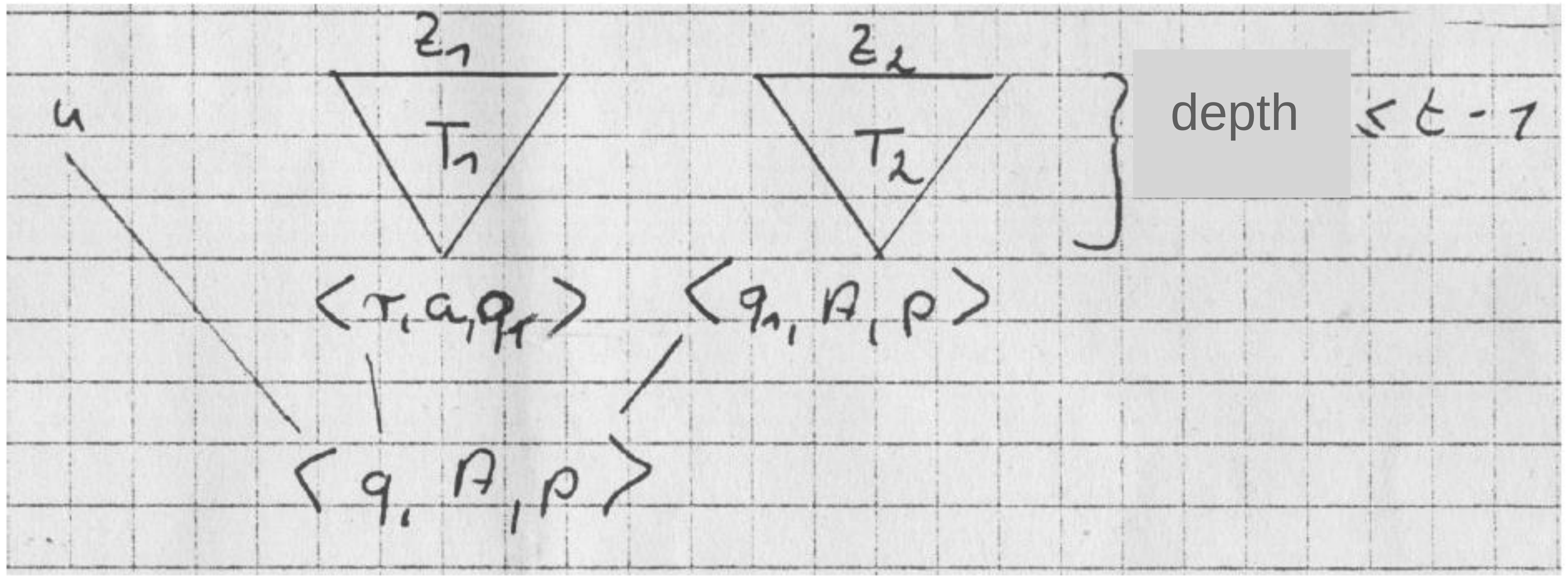

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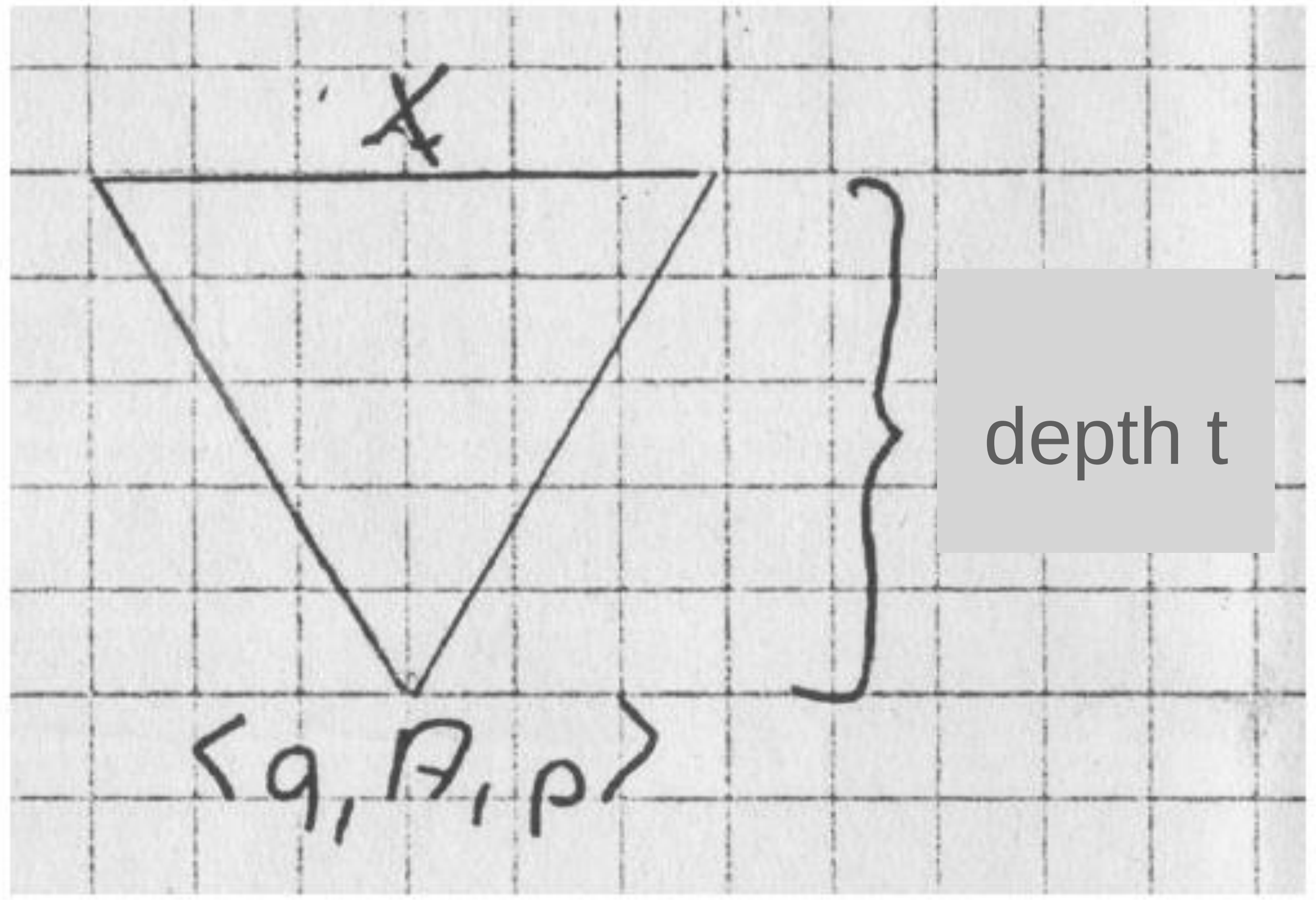

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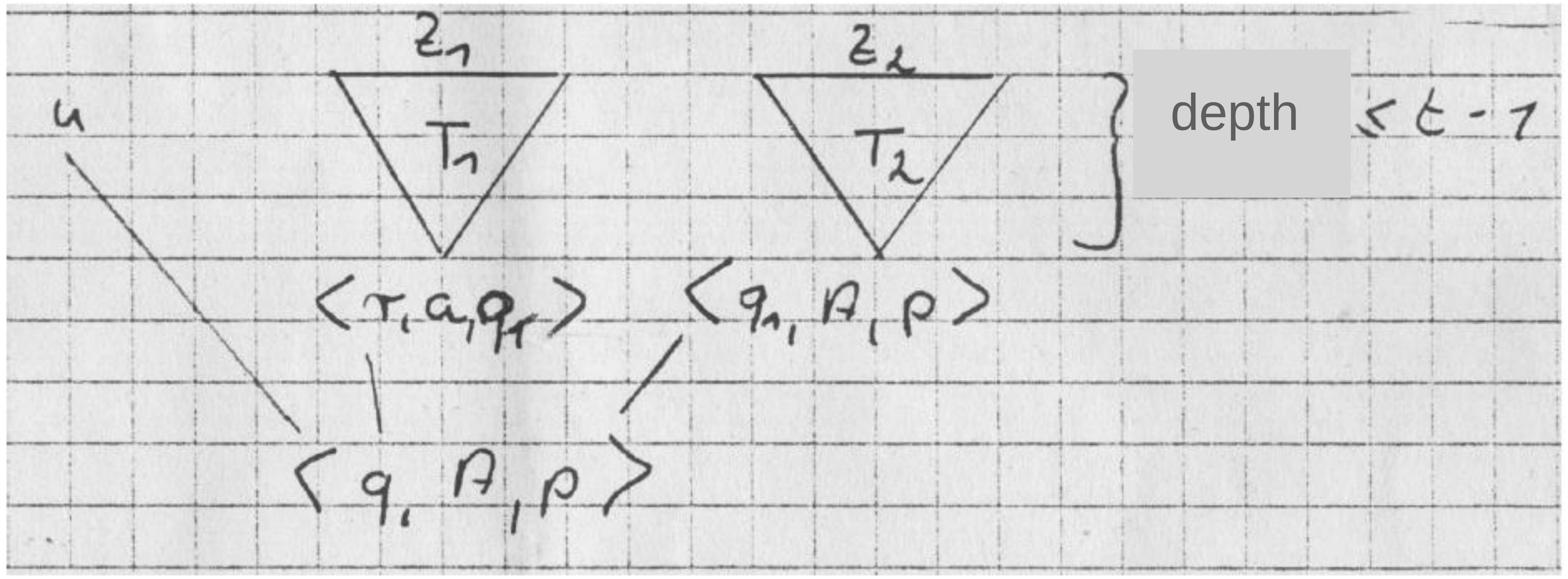

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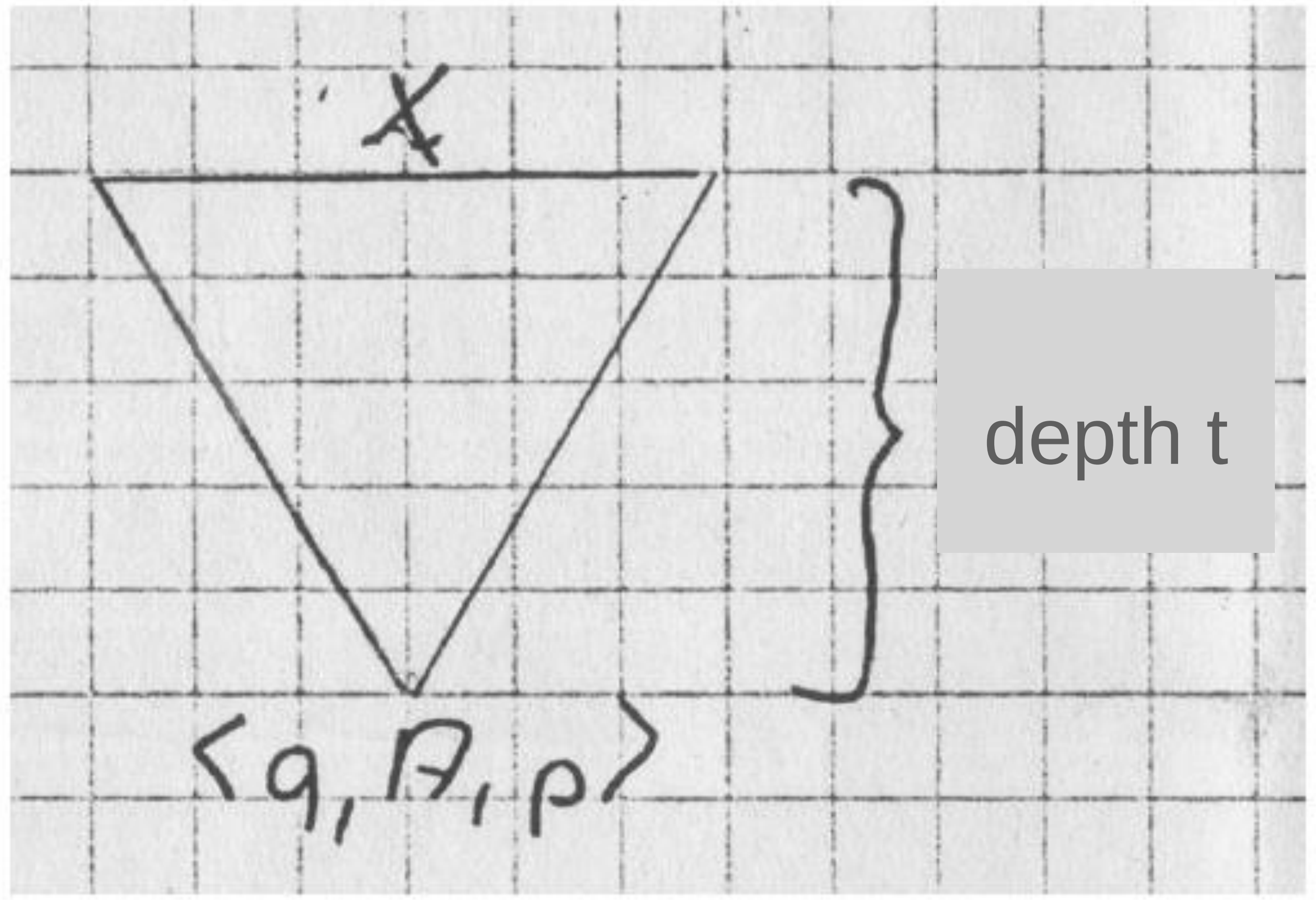
hence

$$\begin{array}{llll} (r, z_1z_2, aA) & \vdash^* & (q_1, z_2, A) & \text{(pushdown automata work this way, def of } \vdash) \\ & \vdash^* & (p, \varepsilon, \varepsilon) & \text{(induction hypothesis for } T_2) \end{array}$$

Lemma 9.

$$x \in L(\langle q, A, p\rangle) \rightarrow (q, x, A) \vdash^* (p, \varepsilon, \varepsilon)$$

Proof by induction on depth t of derivation tree from $\langle q, A, p\rangle$ to x



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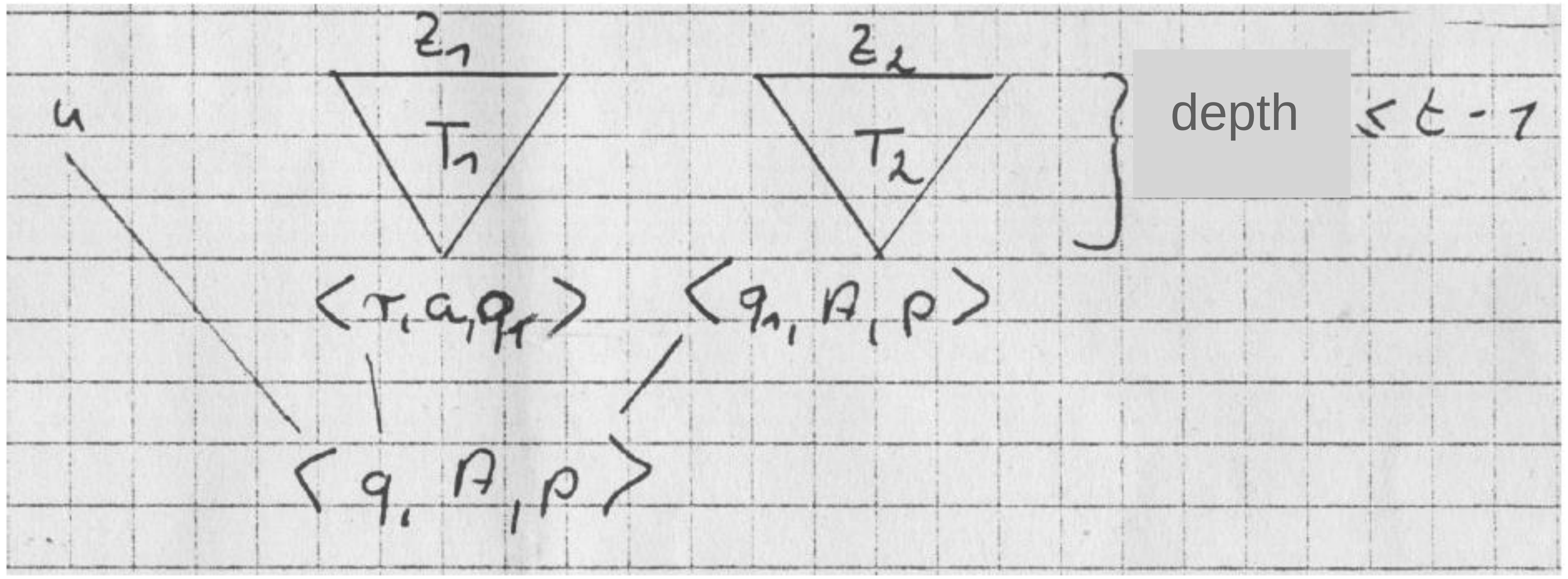
- $t-1 \rightarrow t$

construction 2: see figure 9



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construction 3: exercise

Lemma 9.

$$x \in L(\langle q, A, p\rangle) \rightarrow (q, x, A) \vdash^* (p, \varepsilon, \varepsilon)$$

Proof by induction on depth t of derivation tree from $\langle q, A, p\rangle$ to x

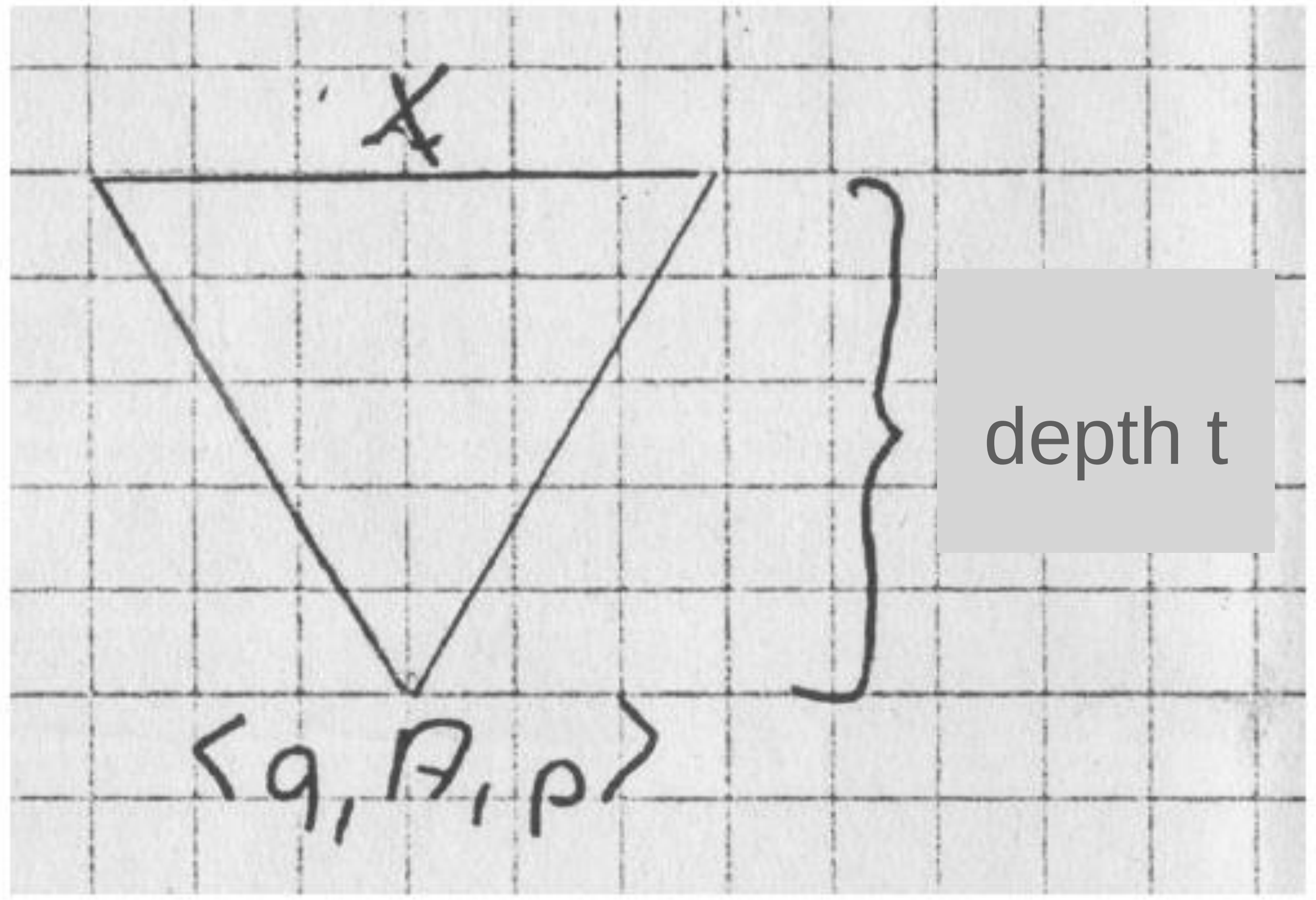

Figure 8: by induction on the depth t of the derivation tree we want to conclude $(q, x, A) \vdash^* (p, \varepsilon, \varepsilon)$.

construction 4: pop with $A \neq \epsilon$

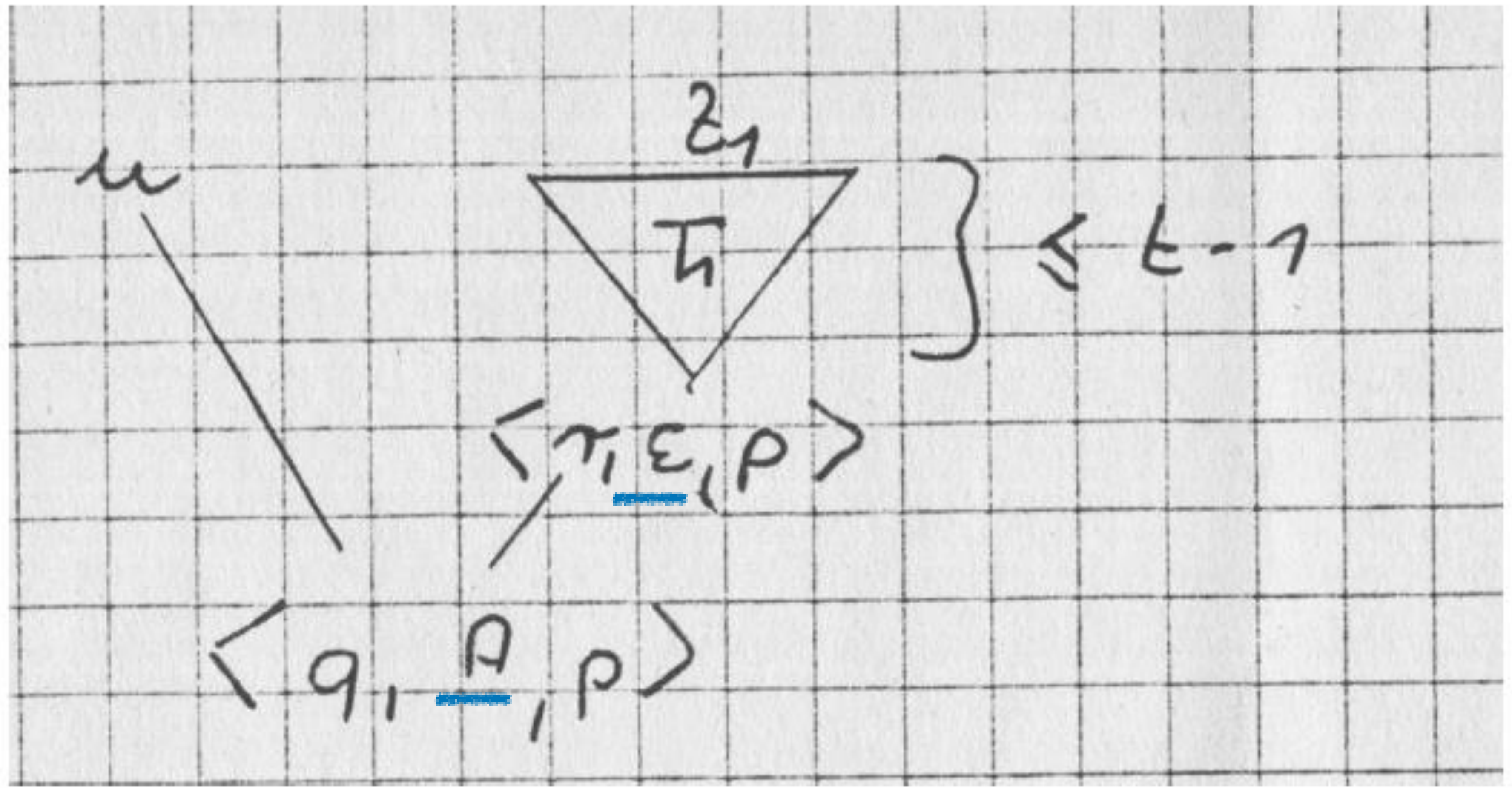

Figure 10: derivation tree for case 4 of the construction. The input word is $x = uz_1$

Lemma 9.

$$x \in L(\langle q, A, p\rangle) \rightarrow (q, x, A) \vdash^* (p, \varepsilon, \varepsilon)$$

Proof by induction on depth t of derivation tree from $\langle q, A, p\rangle$ to x

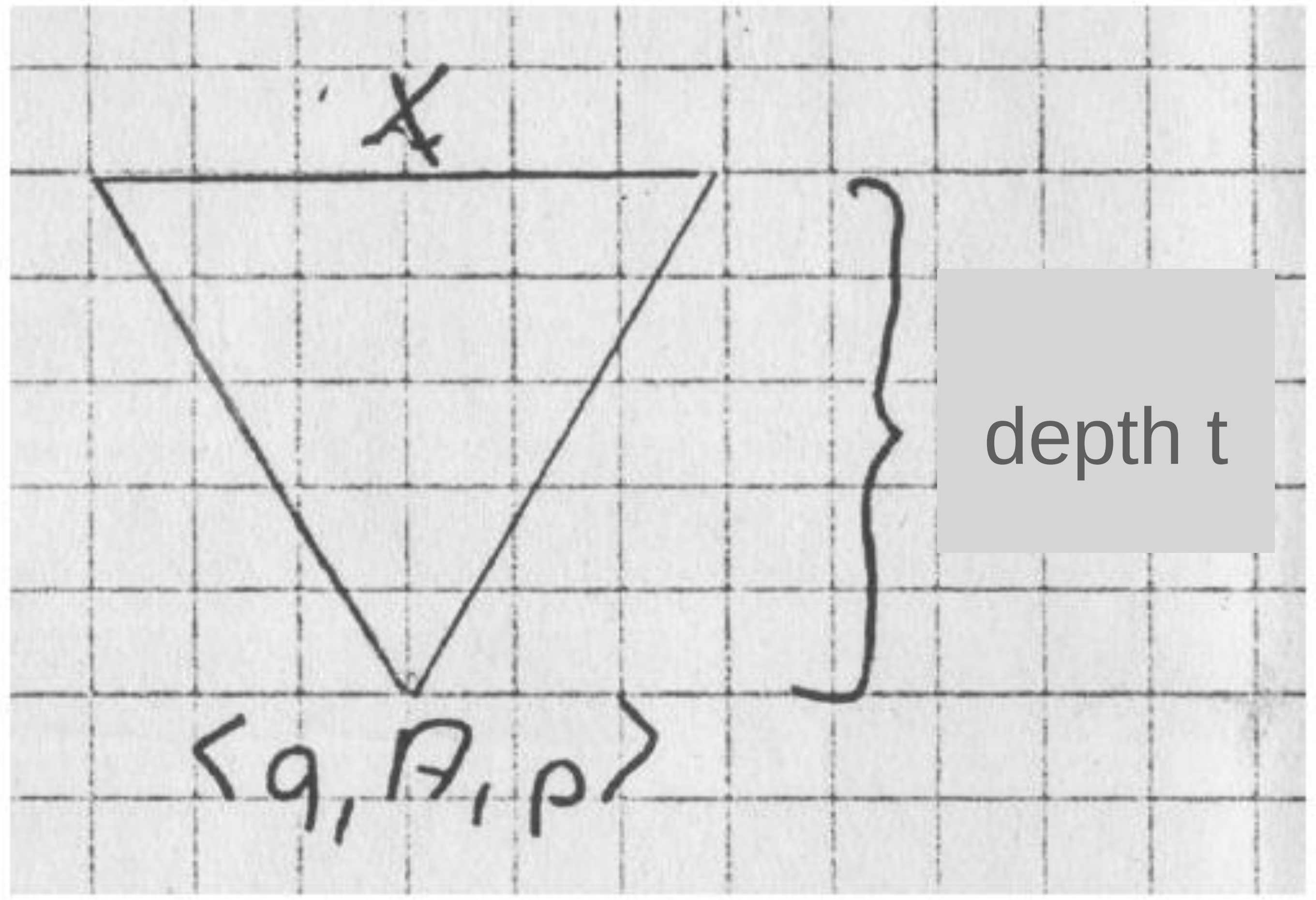

Figure 8: by induction on the depth t of the derivation tree we want to conclude $(q,x,A) \vdash^* (p,\varepsilon,\varepsilon)$.

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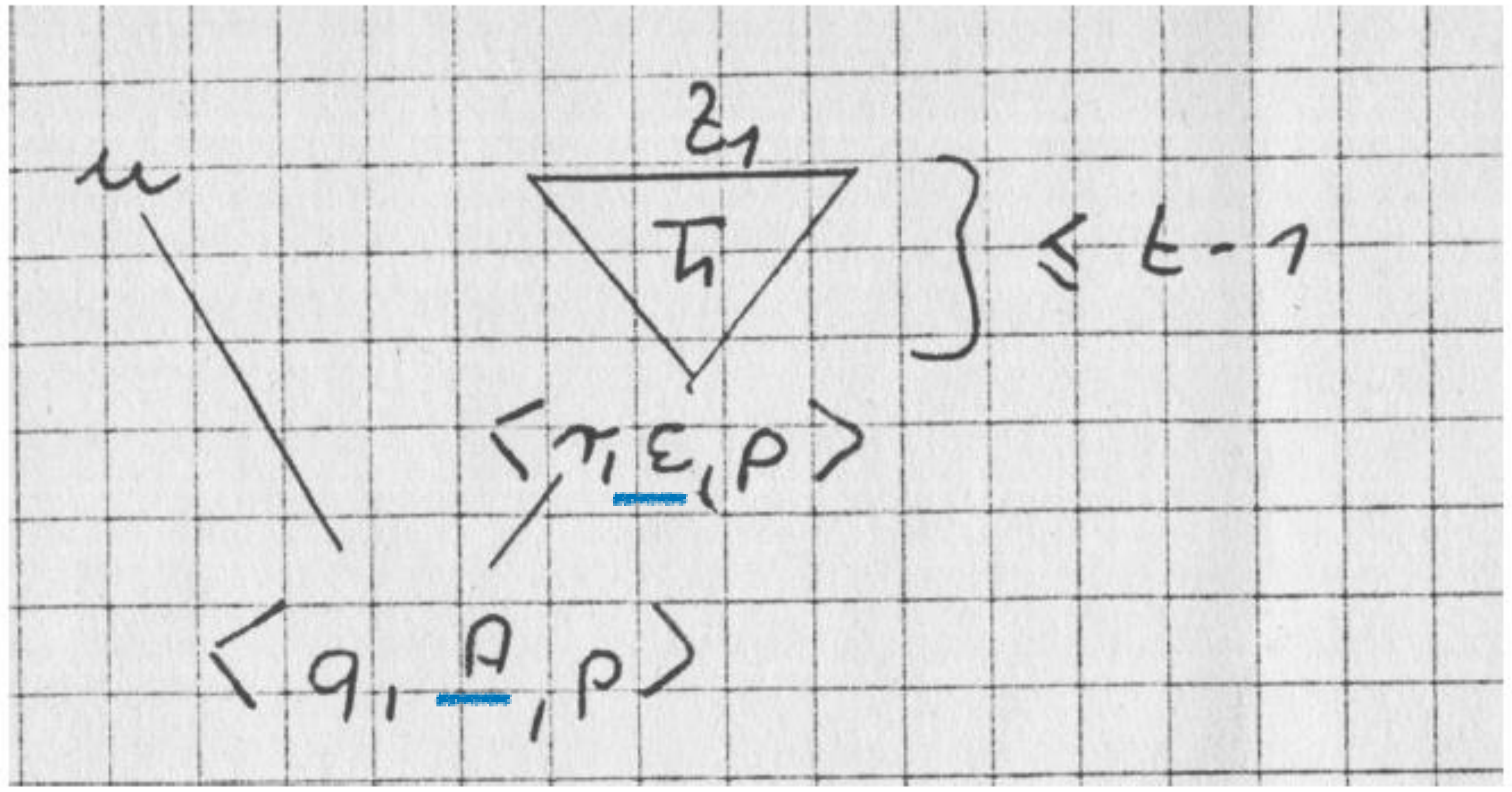


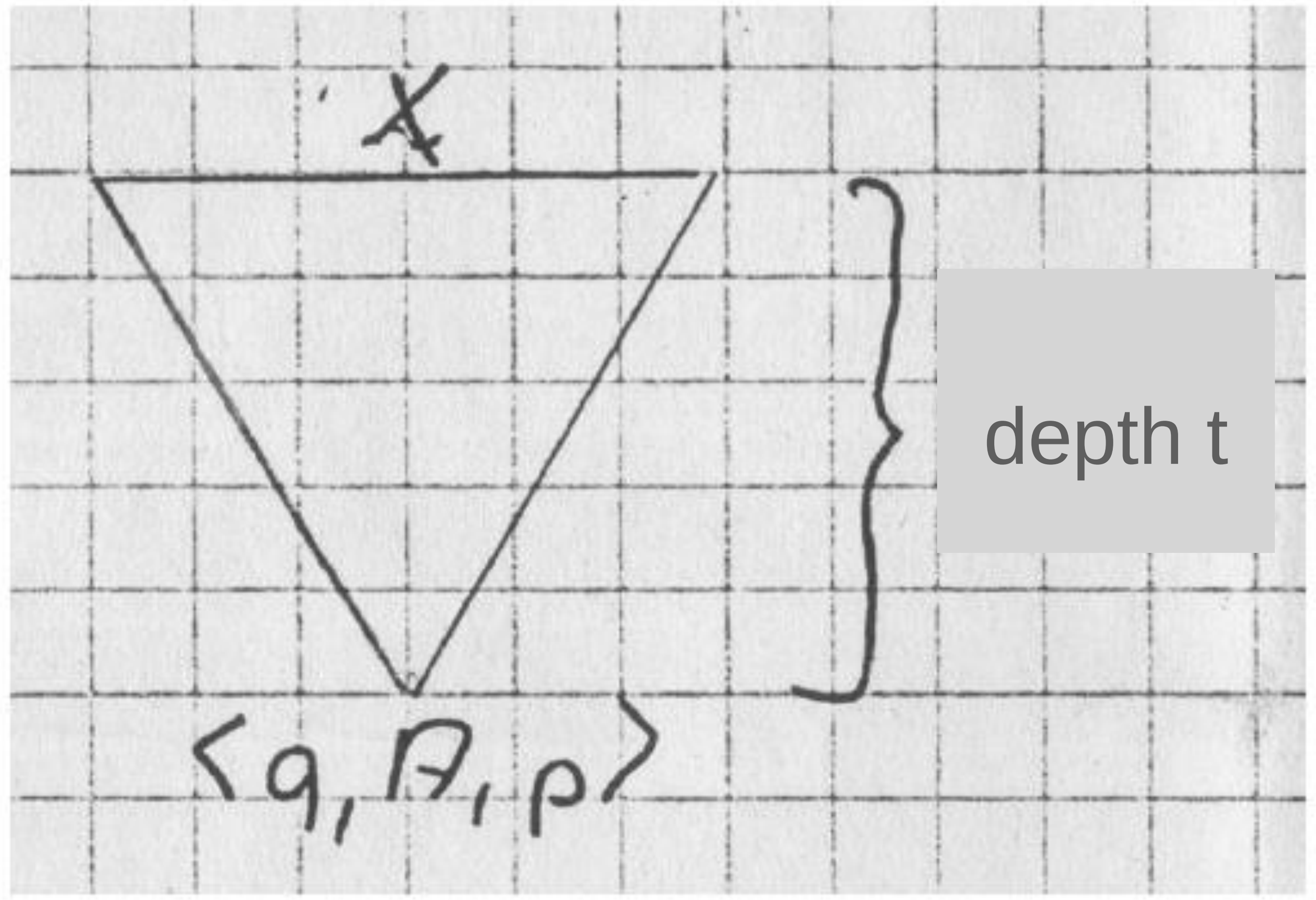
Figure 10: derivation tree for case 4 of the construction. The input word is $x = uz_1$

$$\begin{array}{rcll} (q,x,A) & \vdash & (r,z_1,\varepsilon) & \text{(construction of } G) \\ & \vdash^* & (p,\varepsilon,\varepsilon) & \text{(induction hypothesis for } T_1) \end{array}$$

Lemma 9.

$$x \in L(\langle q,A,p\rangle) \rightarrow (q,x,A) \vdash^* (p,\varepsilon,\varepsilon)$$

Proof by induction on depth t of derivation tree from $\langle q,A,p\rangle$ to x



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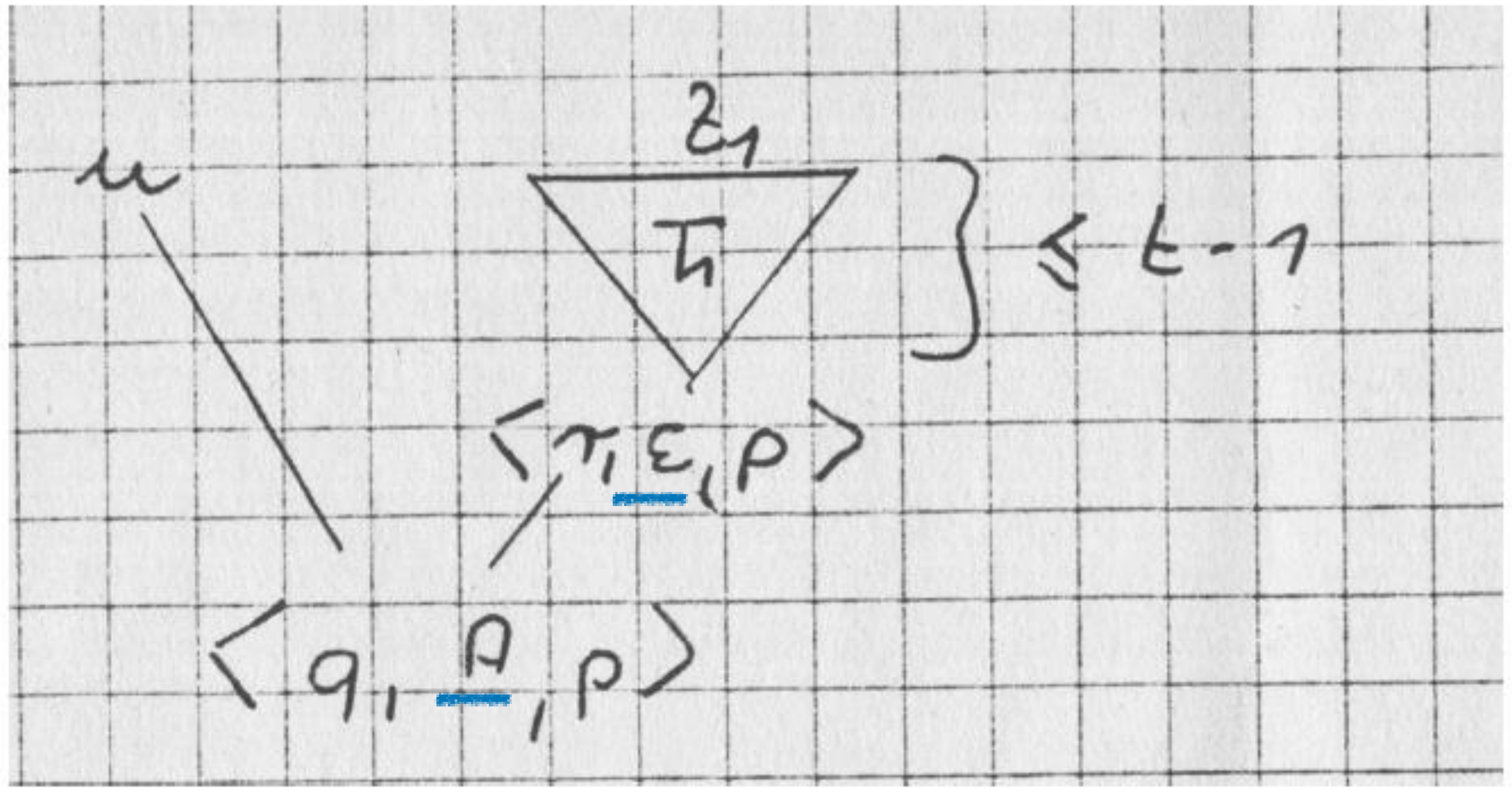

Figure 10: derivation tree for case 4 of the construction. The input word is $x = uz_1$

$$\begin{array}{lcll} (q,x,A) & \vdash & (r,z_1,\varepsilon) & \text{(construction of } G) \\ & \vdash^* & (p,\varepsilon,\varepsilon) & \text{(induction hypothesis for } T_1) \end{array}$$

~~construction 5: exercise~~

classical place for errors (although I did not find it yet)

Lemma 9.

$$x \in L(\langle q, A, p\rangle) \to (q, x, A) \vdash^* (p, \varepsilon, \varepsilon)$$

Proof by induction on depth t of derivation tree from $\langle q, A, p\rangle$ to x

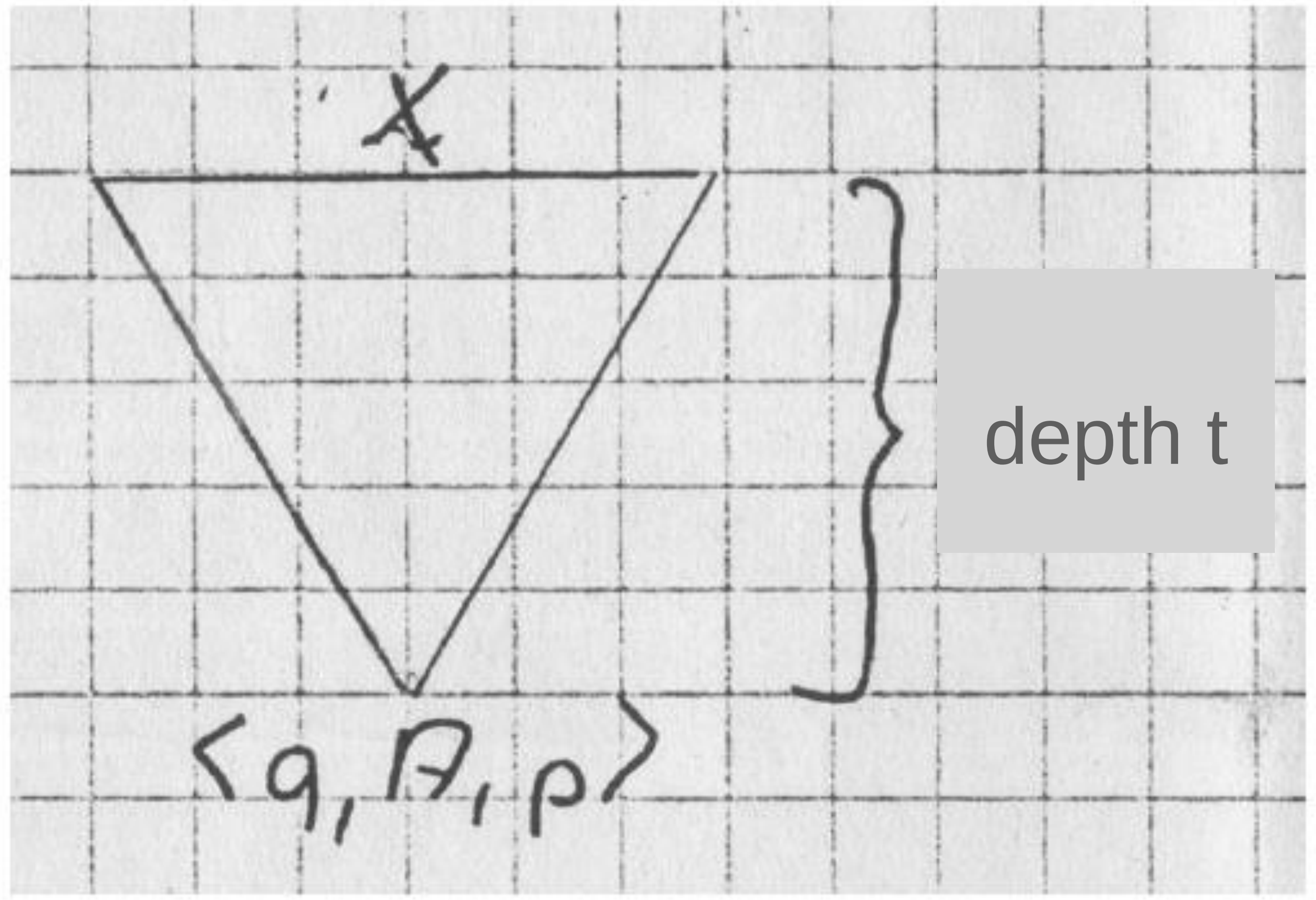

Figure 8: by induction on the depth t of the derivation tree we want to conclude $(q, x, A) \vdash^* (p, \varepsilon, \varepsilon)$.

lemma implies claim $w \in L(G) = L(S)$ has derivation tree as in figure 11 .

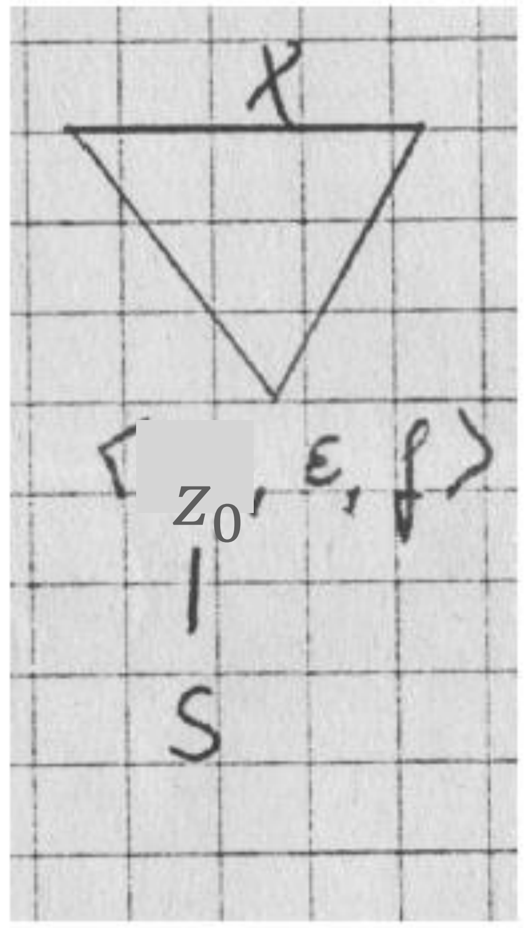

Figure 11: derivation tree for x from start symbol S

The lemma implies

$$(z_0, x, \varepsilon) \vdash^* (f, \varepsilon, \varepsilon)$$

$$f \in Z_A \to x \in L(M)$$

5.3 $x \in L(M) \rightarrow x \in L(G)$

Lemma 10. *For all $x \in \Sigma^*$ and $A \in \Gamma_\varepsilon$ and $p, q \in Z$ holds: if*

$$(q, x, A) \vdash^t (p, \varepsilon, \varepsilon)$$

then

$$\langle q, A, p \rangle \rightarrow^* x$$

5.3 $x \in L(M) \to x \in L(G)$

Lemma 10. *For all* $x \in \Sigma^*$ *and* $A \in \Gamma_\varepsilon$ *and* $p, q \in Z$ *holds: if*

$$(q,x,A) \vdash^t (p,\varepsilon,\varepsilon)$$

then

$$\langle q,A,p \rangle \to^* x$$

lemma implies claim: choose

$$q = z_0 \,,\ A = \varepsilon,\ p = f \in Z_A$$

Then

$$S \to \langle z_0, \varepsilon, f \rangle \to^* x$$

5.3 $x \in L(M) \rightarrow x \in L(G)$

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$$(q, x, A) \vdash^t (p, \varepsilon, \varepsilon)$$

then

$$\langle q, A, p \rangle \rightarrow^* x$$

proof of lemma by induction on t

- $t = 0$

$$q = p \ , \ x = \varepsilon \ , \ A = \varepsilon$$

$$\langle q, \varepsilon, q \rangle \rightarrow \varepsilon \quad \text{(construction 6)}$$

5.3 $x \in L(M) \to x \in L(G)$

Lemma 10. *For all $x \in \Sigma^*$ and $A \in \Gamma_\varepsilon$ and $p, q \in Z$ holds: if*

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then

$$\langle q, A, p \rangle \to^* x$$

- $t - 1 \to t$:

  case 1: computation starts with *push a* for some $a \in \Gamma_\varepsilon$.

  $$(q, x, A) \vdash (r, z, aA) \vdash^{t-1} (p, \varepsilon, \varepsilon) \quad , \quad x = uz \, , \, u \in \Sigma_\varepsilon$$

5.3 $x \in L(M) \rightarrow x \in L(G)$

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- $t - 1 \rightarrow t$:

  case 1: computation starts with *push a* for some $a \in \Gamma_\varepsilon$.

  $$(q, x, A) \vdash (r, z, aA) \vdash^{t-1} (p, \varepsilon, \varepsilon) \quad , \quad x = uz \ , \ u \in \Sigma_\varepsilon$$

  then

  $$(r, push\ a) \in \delta(q, u, A) \quad \text{or} \quad (r, push\ a) \in \delta(q, u, \varepsilon)$$

  in both cases by construction 2 and 3 there is production

  $$\langle q, A, p \rangle \rightarrow u \langle r, a, q_1 \rangle \langle q_1, A, p \rangle \quad \text{for all} \quad q_1 \in Z$$

5.3 $x \in L(M) \rightarrow x \in L(G)$

Lemma 10. *For all $x \in \Sigma^*$ and $A \in \Gamma_\varepsilon$ and $p, q \in Z$ holds: if*

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$$\langle q,A,p\rangle \rightarrow^* x$$

- $t-1 \rightarrow t$:

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$$\langle q,A,p\rangle \rightarrow u\langle r,a,q_1\rangle\langle q_1,A,p\rangle \quad \text{for all} \quad q_1 \in Z$$

Determining q_1: choose t_1 as first t when a is popped from stack.

$$(r,z,aA) \vdash^{t_1} (q_1,z_2,A) \vdash^{t_2} (p,\varepsilon,\varepsilon) \quad \text{with} \quad z = z_1 z_2 \; , \; t_1 + t_2 = t$$

M is pushdown automaton/definition of $\vdash$:

$$(r,z_1,a) \vdash^{t_1} (q_1,\varepsilon,\varepsilon)$$

induction hypothesis gives derivation trees T_1, T_2 of figure 12

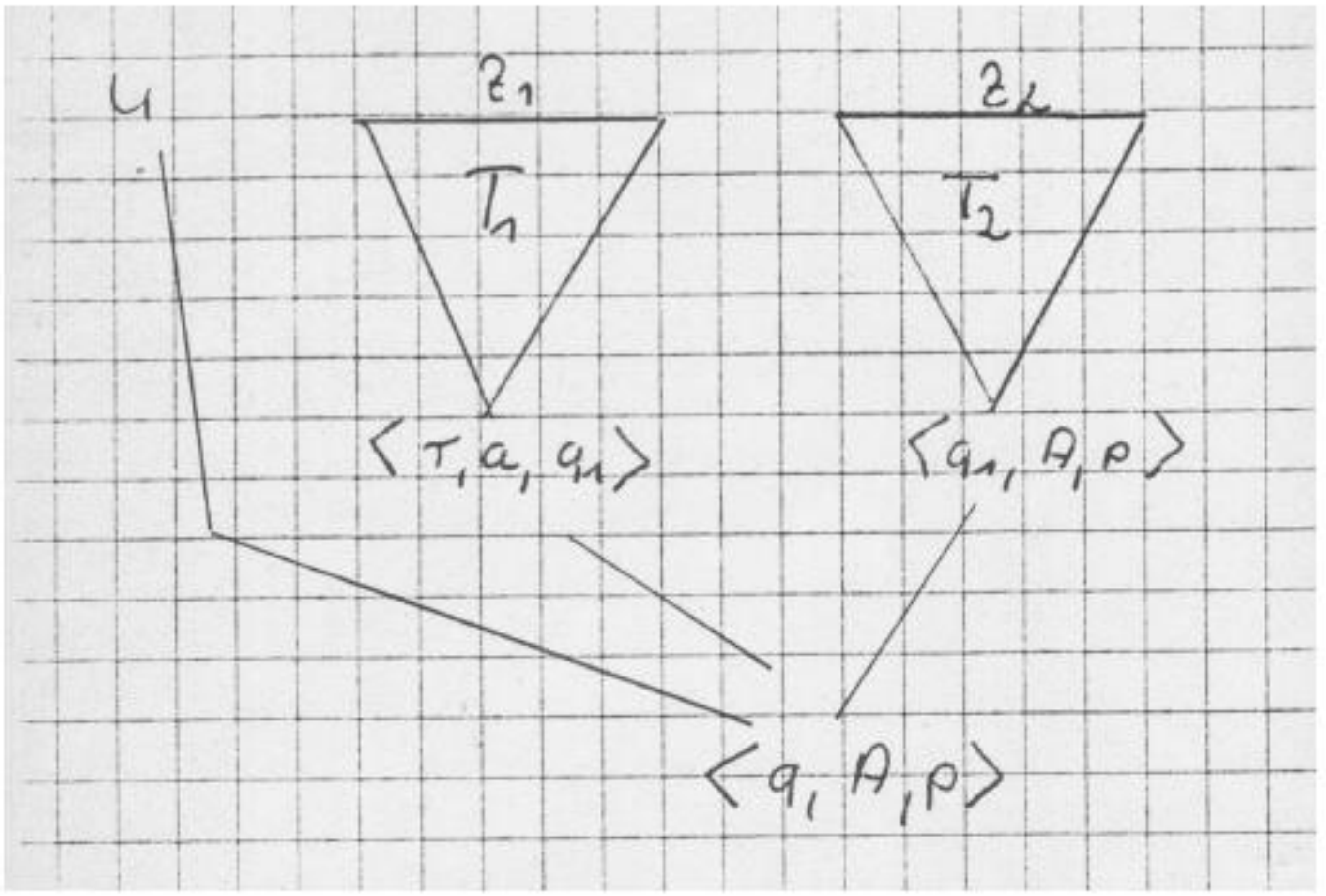

Figure 12: derivation tree for x if the computation starts with *push* a

5.3 $x \in L(M) \rightarrow x \in L(G)$

Lemma 10. *For all $x \in \Sigma^*$ and $A \in \Gamma_\varepsilon$ and $p, q \in Z$ holds: if*

$$(q,x,A) \vdash^t (p,\varepsilon,\varepsilon)$$

then

$$\langle q,A,p\rangle \rightarrow^* x$$

case 2: computation starts with pop

$$(q,x,A) \vdash (r,z,\varepsilon) \vdash^{t-1} (p,\varepsilon,\varepsilon) \quad , \quad x = uz \ , \ u \in \Sigma_\varepsilon$$

then

$$(r,pop) \in \delta(q,u,A) \quad \text{or} \quad (r,pop) \in \delta(q,u,\varepsilon)$$

in both cases by construction 4 and 5 there is production

$$\langle q,A,p\rangle \rightarrow u\langle r,\varepsilon,p\rangle$$

5.3 $x \in L(M) \rightarrow x \in L(G)$

Lemma 10. *For all $x \in \Sigma^*$ and $A \in \Gamma_\varepsilon$ and $p,q \in Z$ holds: if*

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then

$$\langle q,A,p \rangle \rightarrow^* x$$

case 2: computation starts with *pop*

$$(q,x,A) \vdash (r,z,\varepsilon) \vdash^{t-1} (p,\varepsilon,\varepsilon) \quad , \quad x = uz \text{ , } u \in \Sigma_\varepsilon$$

then

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in both cases by construction 4 ~~and 5~~ there is production

$$\langle q,A,p \rangle \rightarrow u\langle r,\varepsilon,p \rangle$$

induction hypothesis gives derivation tree T_1 of figure 13

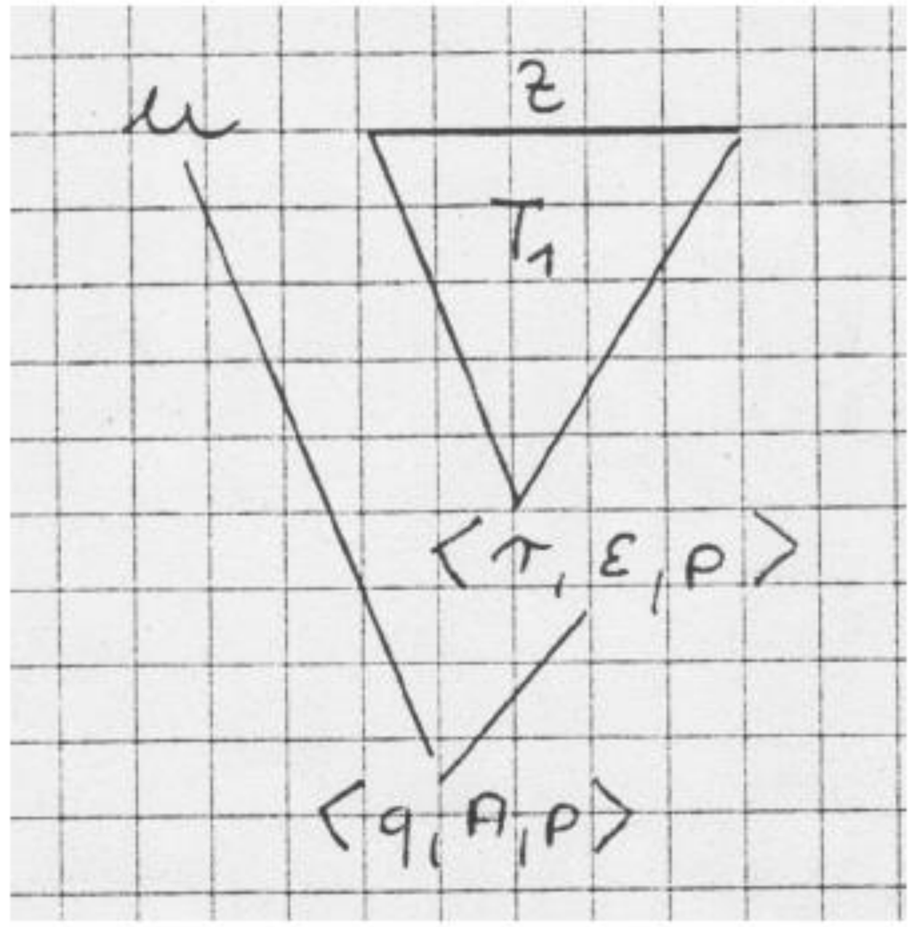

Figure 13: derivation tree for x if the computation starts with *pop*

5.3 $x \in L(M) \rightarrow x \in L(G)$

Lemma 10. *For all $x \in \Sigma^*$ and $A \in \Gamma_\varepsilon$ and $p,q \in Z$ holds: if*

$$(q,x,A) \vdash^t (p,\varepsilon,\varepsilon)$$

then

$$\langle q,A,p\rangle \rightarrow^* x$$

case 2: computation starts with *pop*

$$(q,x,A) \vdash (r,z,\varepsilon) \vdash^{t-1} (p,\varepsilon,\varepsilon) \quad , \quad x = uz \, , \, u \in \Sigma_\varepsilon$$

then

$$(r,pop) \in \delta(q,u,A) \quad \text{or} \quad (r,pop) \in \delta(q,u,\varepsilon)$$

in both cases by construction 4 ~~and 5~~ there is production

$$\langle q,A,p\rangle \rightarrow u\langle r,\varepsilon,p\rangle$$

done

remember: this is
hardcore CS
from 1962

I was 11 years old

induction hypothesis gives derivation tree T_1 of figure 13

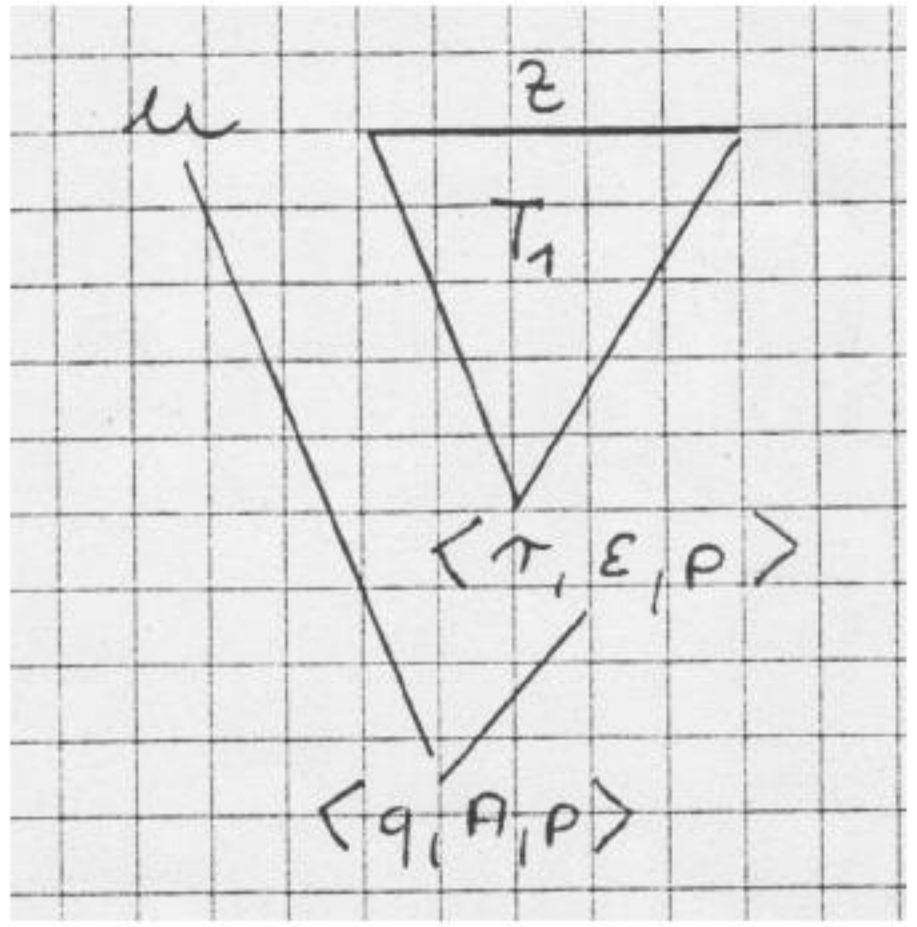

Figure 13: derivation tree for x if the computation starts with *pop*