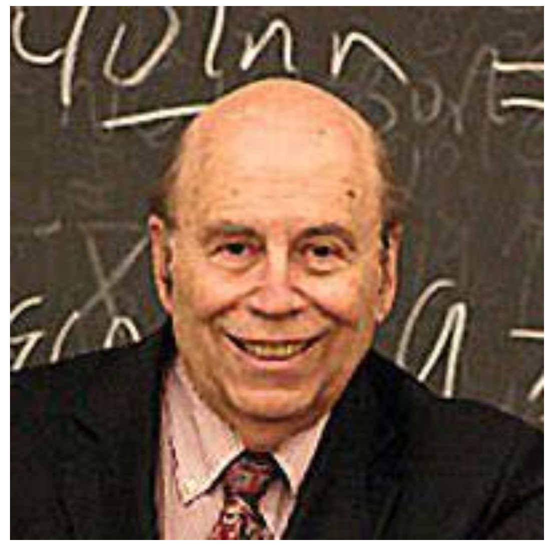
Presburger Arithmetic

a non polynomial lower bound

results of this chapter from: M.J. Fisher and M. Rabin 1973



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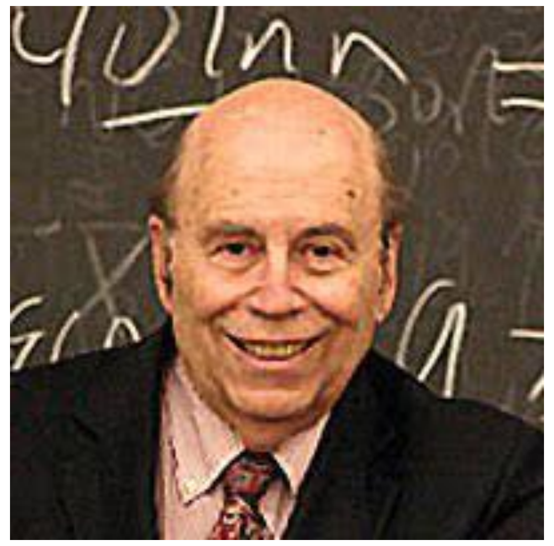

my role model for teaching

1 Background

1.1 Undecidabitlity of elementary arithmetic Z_E

review: Z_E**:**

- dealing with elements in $\mathbb{N}_0$
- Peano axioms
- predicates involving $=, +, \cdot$
- truth of predicates/statements undecidable

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the crucial predicate: Consider 1-tape TM

$$M_u = (Z, A, \delta, z_0, E)$$

and $v \in \mathbb{B}^*$. M_u started with v halts iff

$$\exists w.\ w = \$k_0\$ \ldots \$k_t\$ \quad , \quad w \in (A \cup Z \cup \{\$\})^+$$

with

1. $k_0 = B \ldots B z_0 v B \ldots B$
2. $k_i \vdash k_{i+1}$ for $i < t$
3. $|k_i| = |k_j|$ for all i, j
4. in no k_i if a $z \in Z$ first or last element
5. k_t is endconfiguraion

$$H(u\#v) :\equiv \exists w\ (1) \wedge (2) \wedge (3) \wedge (4) \wedge (5)$$

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1.2 Bounding the length of strings involved

parameters:

- for fixed machine $M = (Z, A, \delta, z_0, Z_A)$
- input size $|v| = n$
- step number $t \geq n$.

length of strings involved

- configurations with state and surrounding blanks:

$$|k_i| \leq t + 3$$

- word w with $t+1$ configurations and $t+1$ separation signs \$:

$$|w| \leq (t+4) \cdot (t+1) + 1$$

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coding strings in numbers Let

$$p > \#A + \#Z + 2 \quad \text{prime number}$$

Interpret $w \in (A \cup Z \cup \{\$\})^*$ as number representation to base p.

- $$\psi : A \cup Z \cup \{\$\} \to [1 : p-1]$$
  $$\hat{a} = \overline{\psi(a)} = 1 + \ldots + 1 \ (\psi(a) \text{ times})$$

- $$\psi(\varepsilon) = 0$$

- extend to

$$\psi : (A \cup Z \cup \{\$\})^* \to [1 : p-1]$$

$$\psi(w[s-1:0]) = \sum_{i=1}^{s-1} \psi(w_i) \cdot p^i$$

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- for fixed machine M (and variable v) all other parts of $H(u\#v)$ have length $O(1)$

Lemma 2. *For fixed machines $M = M_u$*

$$|H(u\#v)| = O(n)$$

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- Recall Hilbert's program:
- show that Z_E is decidable
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- in time $2^{2^{2^{O(n \log n)}}}$ (D.C. Oppen, ACM STOC 1973)

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proof system

$$Z_P = (\Sigma_P, L_P, A_P, S_P)$$

Lemma 3. *The language of true predicates of Z_P*

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$$L \in TIME(2^{Cn})$$

- Let $M = M_u$ be 2^{Cn}-time bounded 1-tape TM accepting L
- we show

$$L \leq_p Z_P$$

by constructing for input v with $|v| = n$ a predicate

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which is true iff $M = M_u$ started with v halts in an accepting state.

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which is true iff $M = M_u$ started with v halts in an accepting state.
- length of the computation:
$$t = 2^{Cn}$$
- size of the numbers involved

$$\begin{aligned}
\psi(w) &\leq p^{|w|} \\
&\leq 2^{(\log p)\cdot((t+4)\cdot(t+1)+1))} \quad \text{(lemma 1)} \\
&\leq 2^{(\log p)\cdot((2^{Cn}+4)\cdot(2^{Cn}+1)+1))} \\
&\leq 2^{2^{3Cn}}
\end{aligned}$$

2.2 Expressing multiplication $a \cdot b = c$ for bounded a in Z_P

Lemma 4. *There is a predicate $m_n(a,b,c)$ of Z_P and there are numbers*

$$p_n \geq 2^{2^n}$$

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proof: obviously by induction on n

$n = 1$:

$$2^{2^1} = 4$$

$$\begin{aligned} m_1(a,b,c) \quad \equiv \quad & a = 0 \wedge c = 0 \\ & \vee \left(\bigvee_{i=1}^{4} (a = 1 + \ldots + 1 \quad (i \text{ times}) \right. \\ & \left. \wedge\, c = b + \ldots + b \quad (i \text{ times})) \right. \end{aligned}$$

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$n \rightarrow n+1$**:**

Lemma 5. *For every $a \in \mathbb{N}$ there are natural numbers*

$$a_1, a_2, a_3, a_4 \leq \lfloor \sqrt{a} \rfloor$$

such that

$$a = a_1 \cdot a_2 + a_3 + a_4$$

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$$a = a_1 \cdot a_2 + a_3 + a_4$$

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2.2 Expressing multiplication $a \cdot b = c$ for bounded a in Z_P

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$$p_n \geq 2^{2^n}$$

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proof: obviously by induction on n

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unfortunately the length would grow too fast

$$|m_n| \geq 5^n$$

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length of m_n: *counting length of variables as* 1

$$L(n) = |m_n(a,b,c)|$$

then

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size of operands a If

$$m_n(a,b,c) \wedge a \leq p_n \rightarrow c = a \cdot b$$

then

$$m_{n+1}(a,b,c) \wedge a \leq p_n^2 + 2p_n \rightarrow c = a \cdot b$$

$$\begin{aligned} p_1 &= 4 \\ p_{n+1} &> p_n^2 \end{aligned}$$

Lemma 6.

$$p_n \geq 2^{2^n}$$

Proof. easy induction

2.1 EXPTIME hardness

proof system

$$Z_P = (\Sigma_P, L_P, A_P, S_P)$$

Lemma 3. *The language of true predicates of Z_P*

$$T_P = \{A \in L_P \ : \ A \text{ is true}\}$$

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proving lemma 3:

- for $|v| = n$ and $M = M_u$ obtain predicate $H_n(M,v)$ by replacing in $H(u\#v)$ every occurrence of a predicate $a = b \cdot c$ by $m_{3Cn}(a,b,c)$

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- indexing variable names with binary or decimal numers

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3 A lower bound

- time hierarchy theorem

$$P \subsetneq EXPTIME$$

- $T_P \in P$ and T_P EXPTIME-hard would imply $EXPTIME \subseteq P$

Lemma 7.

$$T_P \notin P$$

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exercise: try to derive a concrete lower bound for the run time $t(n)$ of Turing machines deciding Z_P, e.g.

$$t(n) \geq 2^{\sqrt[3]{n}}$$