Theory of Computation

G-4

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Random exercises

Is it Regular? Context Free?…

$$L = \{w \in \{0,1\}^* \mid w = \text{reverse}(w) \text{ and the length of } w \text{ is divisible by } 4\}$$

- if you think its regular, try constructing a DFA/NFA..
- if you think its context free, try constructing its grammar/NPDA..

Random exercises

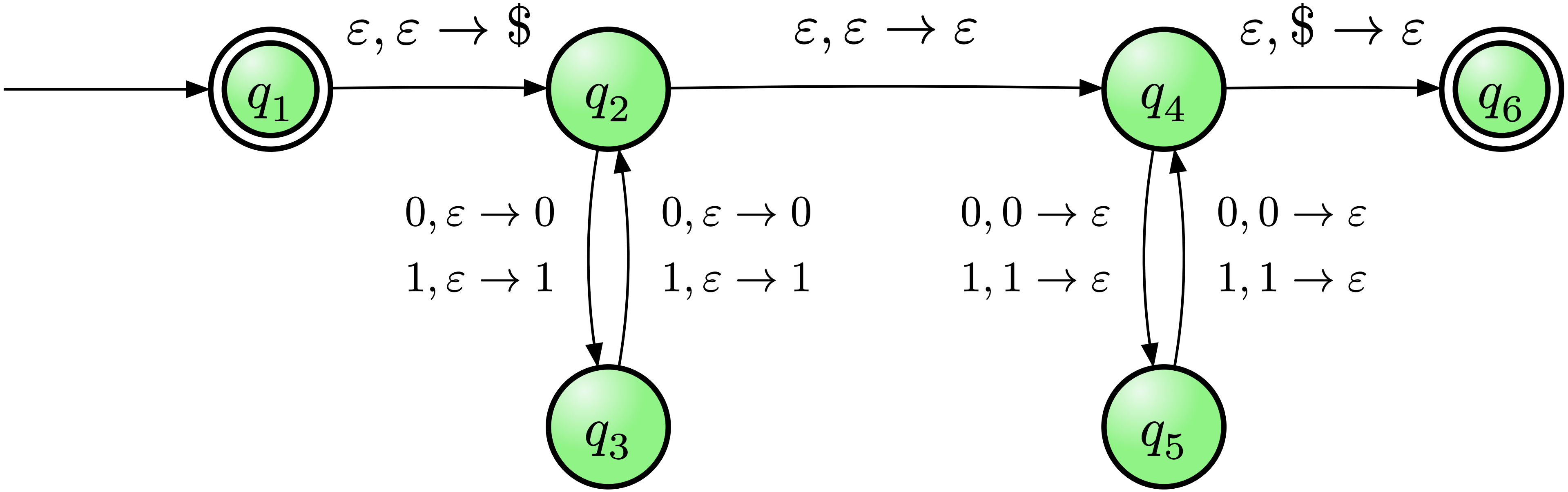
Answer: L is CFL,

$$G = (N, T, P, S),\ N = \{S\},\ T = \{0, 1\}, \tag{1}$$

starting variable S, and rules:

$$P = \{S \to 00S00 \mid 01S10 \mid 10S01 \mid 11S11 \mid \varepsilon\} \tag{2}$$

Random exercises



Random exercises

Is it regular?

Random exercises

Lets prove that L is not regular by contradiction, suppose its regular,let p be the pumping length.. consider string $s = 0^p 1^{2p} 0^p \in L$ $|s| = 4p > p$ so the conclusions of the pumping lemma must hold, thus we can split $s = xyz$ $(1) xy^i z \in L\ \forall . i \geq 0, (2) |y| > 0 \text{ and } (3)$ $|xy| \leq p$. because all of the first p symbols of s are 0s, (3) implies that x and y must only consist of 0s. Also z must consist of the rest of the 0s at the beginning, followed by $1^{2p}0^p$.

$x = 0^j, y = 0^k, z = 0^m 1^{2p} 0^p$ where $j + k + m = p$.. (2) implies that $k > 0$ so by (1) $xyyz$ must belong to L.

$$xyyz = 0^j 0^k 0^k 0^m 1^{2p} 0^p = 0^{p+k} 1^{2p} 0^p \notin L \quad \text{contradiction} \tag{3}$$

More exercises

Task 1

Can a transition function of a dfa be bijective? Explain..

More exercises

$$\delta : Z \times A \to Z \tag{4}$$

For a bijective function between two sets, the domain and codomain must have the same size(cardinality)

$$|Z \times A| = |Z| \quad \to \quad |Z| \times |A| = |Z| \tag{5}$$

$$\implies |A| = 1. \tag{6}$$

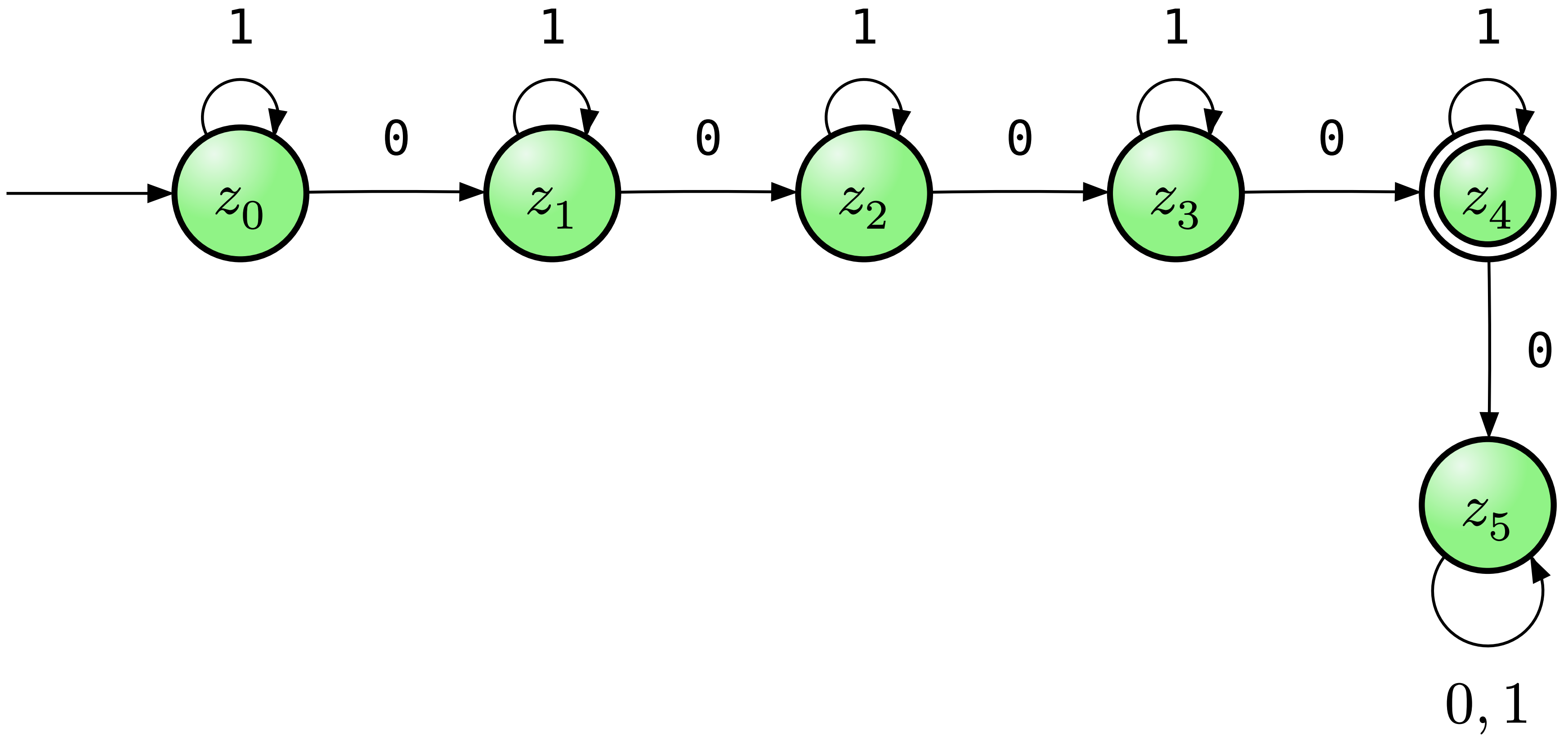
So yes, transition function δ of a dfa can be bijective, if alphabet contains just one symbol..

More exercises

Task 2

Sketch a dfa that accepts binary strings that have exactly 4 zeros

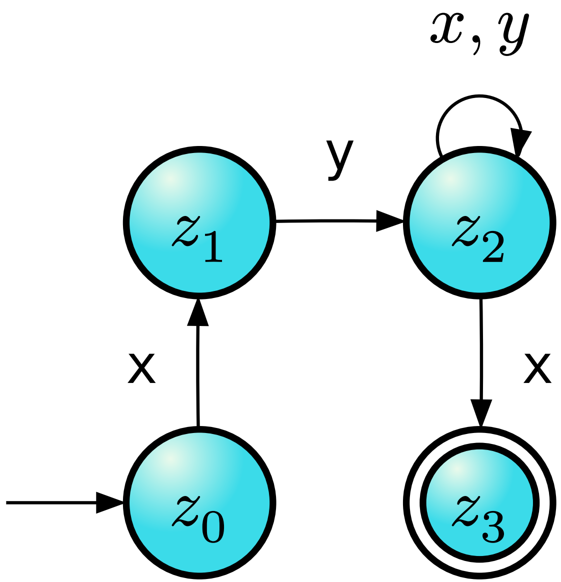
More exercises



More exercises

Task 3

Give an example of an accepting computation (by writing down the sequence of configurations)



More exercises

Accepting string can be for example xyx

Configurations:

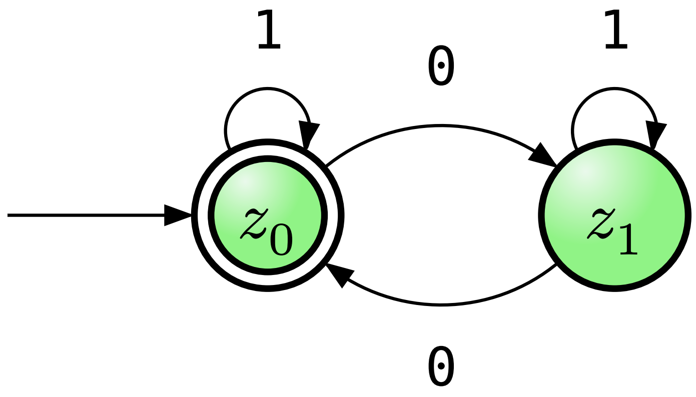
$$(z_0, xyx) \to (z_1, yx) \to (z_2, x) \to (z_3, \varepsilon)$$

thats it…

More exercises

Task 4

Describe the language accepted by DFA



More exercises

its easy to see…

$$1^*(0(1^*)0)^*1^* \tag{7}$$

More exercises

Task 5

Say $k = (z_4, 01100, s0a0)$ is the configuration of NPDA, and $(z_1, \text{push } a) \in \delta(z_4, 0, \varepsilon)$. Write down a possible successor configuration of k.

More exercises

$$k = (z_4, 01100, s0a0)$$
$$\delta(z_4, 0, \varepsilon) \to (z_1, \text{push } a) \tag{8}$$

means that if we are in the state z_4 and read 0, we push a on top of the stack, no matter what is on top..

successor configuration would look like this:

$$k_{\text{succ}} = (z_1, 1100, as0a0) \tag{9}$$

More exercises

Task 6

Let A, B be two alphabets. Does $A^* \cup B^* = (A \cup B)^*$?

More exercises

$$A^* \cup B^* = (A \cup B)^*?$$

Lets consider two alphabets: $A = \{a\}, \quad B = \{b\}$
$A^* = \{\varepsilon, a, aa, \dots\} \quad B^* = \{\varepsilon, b, bb, \dots\}$ $A^* \cup B^* = \{\varepsilon, a, b, aa, bb\dots\}$
Note that the string ab is in $(A \cup B)^*$ but not in $A^* \cup B^*$

$$A^* \cup B^* \neq (A \cup B)^*$$
$$A^* \cup B^* \subseteq (A \cup B)^*$$

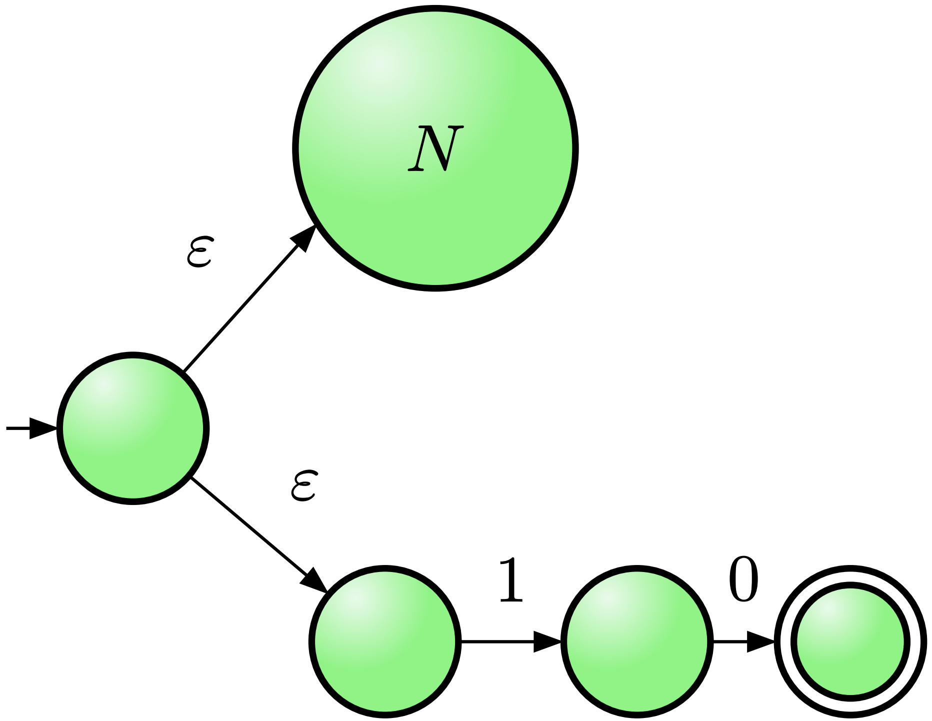
More exercises

Task 7

Say N is a given nfa that accepts the language L over the alphabet $A = \{0, 1\}$. Construct an nfa that accepts $L \cup \{10\}$

More exercises

N is our given nfa, to construct union of two nfas we can just use ε moves like that:



More exercises

Task 8

Prove that the set of integers

$$\mathbb{Z} = \{... - 2, -1, 0, 1, 2, 3...\} \tag{10}$$

is countable

More exercises

Lets try to come up with a bijective function $f : \mathbb{N} \rightarrow \mathbb{Z}$

$$f(n) = \begin{cases} \frac{n}{2} & \text{n is even} \\ -\frac{n+1}{2} & \text{n is odd} \end{cases} \tag{11}$$

$$0 \rightarrow 0,\ 1 \rightarrow -1,\ 2 \rightarrow 1,\ 3 \rightarrow -2,\ 4 \rightarrow 2 \text{ and so on ...} \tag{12}$$

More exercises

Task 9

Is every countably infinite language regular?
(provide proof for your answer)

More exercises

$$\{0^n 1^n \mid n \in \mathbb{N}_n\} \tag{13}$$

Is countable and not regular by pumping lemma.

The bijection is $n \mapsto 0^n 1^n$, thats it..

More exercises

Task 10

Find a regular expression which represents the set of strings over $\{a, b\}$ which contain the two substrings aa and bb

More exercises

$$(a \cup b)^*((aa(a \cup b)^*bb) \cup (bb(a \cup b)^*aa))(a \cup b)^*$$

that might not be intuitive at first but…

(Which of the following L is regular)

$$\{a^n b^m \mid n \geq m \vee n \leq m\}$$

(Which of the following L is regular)

$$\{a^n b^m \mid n \geq m \vee n \leq m\}$$

is the same as $a^* b^*$ so its regular..

(Which of the following L is regular)

$$\{a^n b^m \mid n > m \wedge n < m\}$$

(Which of the following L is regular)

$$\{a^n b^m \mid n > m \wedge n < m\}$$

is empty $\implies$ regular

(Which of the following L is regular)

$$\{a^n b^m \mid n \geq m \wedge n \leq m\}$$

(Which of the following L is regular)

$$\{a^n b^m \mid n \geq m \wedge n \leq m\}$$

is the same as $a^n b^n \Rightarrow$ not regular (already seen why)

(Which of the following L is regular)

$$\{a^n b^m \mid n > m \vee n < m\}$$

(Which of the following L is regular)

$$\{a^n b^m \mid n > m \vee n < m\}$$

It is the same language as $\{a^n b^m \mid n \neq m\}$
which is complement to $\{a^n b^m \mid n = m\}$.

Since regularity is **closed under complementation**, the considered language can not be regular.
(cuz it would imply that $\{a^n b^m \mid n = m\}$ is regular)

More exercises

Task 999

Define a primitive recursive function $f : \mathbb{N} \to \mathbb{N}$ that counts the number of occurences of the digit 5 in a natural number

More exercises

Code

```python
count = 0
while n > 0:
  last_digit = n % 10
  if last_digit == 5:
    count += 1
  n //= 10
print(count)
```

More exercises

At first we need some auxiliary primitive recursive functions

More exercises

- quotient $\left\lfloor \frac{x}{y} \right\rfloor : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$$quo(x, y) = \min\{m : m \leq x \wedge (m+1)y > x\}$$

$$quo(5, 2) = \ldots$$

$$\text{for } m = 0, 0 \leq 5, 1 \cdot 2 > 5 \text{ ? no}$$

$$\text{for } m = 1, 1 \leq 5, 2 \cdot 2 > 5 \text{ ? no}$$

$$\text{for } m = 2, 2 \leq 5, 3 \cdot 2 > 5 \text{ ? yes}$$

outputs `quo(5, 2) = 2`

More exercises

- remainder $rem(x,y) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$$
\begin{aligned}
rem(x,y) &= x \dot{-} (y \cdot quo(x,y)) \\
rem(17,5) &= 17 \dot{-} (5 \cdot quo(17,5)) \\
&= 17 \dot{-} 5 \cdot 3 \\
&= 2
\end{aligned}
$$

More exercises

- length(number of digits) : $\mathbb{N} \to \mathbb{N}$

  $length(x) = \min\{j : j \leq x,\ 10^{j+1} > x\} + 1$

  $length(12) = ...$

$$j = 0, 0 \leq 12 \text{ but } 10^1 > 12?$$

$$j = 1, 1 \leq 12, \text{and } 10^2 > 12$$

so length(12) outputs simply 2

More exercises

$f : \mathbb{N} \to \mathbb{N}$ is then defined as:

$$f(m) = \sum_{i=1}^{length(m)} eq(5, rem(quo(m, 10^{i-1}), 10))$$

More exercises

Example:

$$
\begin{aligned}
f(253) &= eq(5, rem(quo(253, 1), 10)) \\
&+ eq(5, rem(quo(253, 10), 10)) \\
&+ eq(5, rem(quo(253, 100), 10)) \\
&= 0 + 1 + 0 = 1
\end{aligned}
$$

More exercises

Useful book to understand μ recursive functions

click it…

Good luck on midterm..