Exercises below are your homework; they will be discussed during exercise classes. Problems marked with a (*) are more challenging.

Week 6

1. Show that following are primitive recursive
   (1) Functions $\min(x, y)$ and $\max(x, y)$.
   (2) Function $|x - y|$.

2. Show that for a primitive recursive function f, the functions of *bounded sum* and *bounded product* of f, respectively given by

$$\mathrm{bsum}_f(x_0, \dots, x_{k-1}, y) := \sum_{i=0}^{y} f(x_0, \dots, x_{k-1}, i)$$

and

$$\mathrm{bprod}_f(x_0, \dots, x_{k-1}, y) := \prod_{i=0}^{y} f(x_0, \dots, x_{k-1}, i)$$

are also primitive recursive.

3. In the lecture, up to notation, we gave following examples of primitive recursive functions
   - addition:

$$\begin{aligned} \mathrm{add}(x, 0) &= x, \\ \mathrm{add}(x, y+1) &= s(\mathrm{add}(x, y)); \end{aligned}$$

   - multiplication:

$$\begin{aligned} \mathrm{mult}(x, 0) &= 0, \\ \mathrm{mult}(x, y+1) &= \mathrm{add}(x, \mathrm{mult}(x, y)). \end{aligned}$$

However, the definition of primitive recursion operation takes as arguments a k-ary function[1] g, a $k+2$-ary function h, and returns a $k+1$-ary function f defined as follows

$$\begin{aligned} f(x_0, \dots, x_{k-1}, 0) &= g(x_0, \dots, x_{k-1}), \\ f(x_0, \dots, x_{k-1}, y+1) &= h(x_0, \dots, x_{k-1}, y, f(x_0, \dots, x_{k-1}, y)). \end{aligned}$$

The definition of multiplication and addition does not fit the format of this definition. Why? Provide the definition of add and mult so that it fits this format.

4. Show that Predicate $x \mid y$, that is a function $\mid\colon \mathbb{N}_0 \times \mathbb{N}_0 \to \{\mathrm{true}, \ \mathrm{false}\}$, given by $x \mid y = \mathrm{true}$ if and only if x divides y is primitive recursive.

[1] By k-ary function g we mean a function $g\colon \mathbb{N}_0^k \to \mathbb{N}_0$.

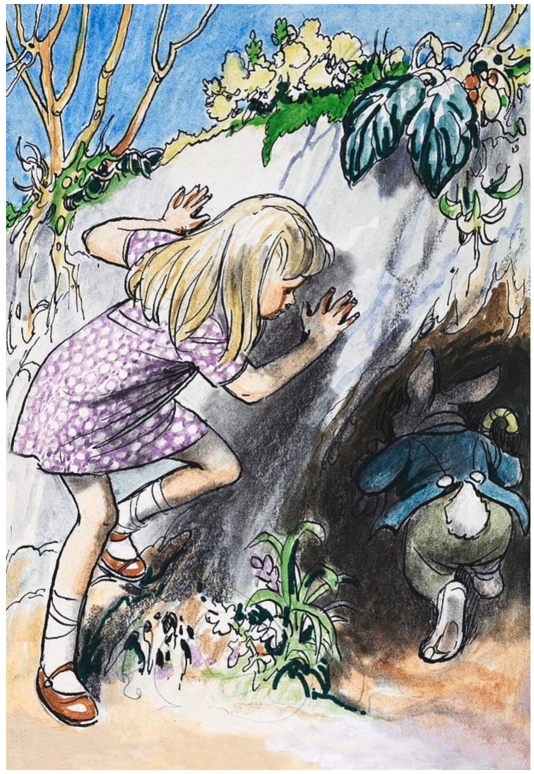

FIGURE 1. *Alice Chasing The White Rabbit*, by Philip Mendoza.

5. In Exercise 2, you showed that a primitive recursive function f, the functions of *bounded sum* and *bounded product* of f, respectively given by

$$\mathrm{bsum}_f(x_0, \ldots, x_{k-1}, y) = \sum_{i=0}^{y} f(x_0, \ldots, x_{k-1}, i),\ \mathrm{bprod}_f(x_0, \ldots, x_{k-1}, y) = \prod_{i=0}^{y} f(x_0, \ldots, x_{k-1}, i)$$

are also primitive recursive.

Now, conclude (by proving) that primitive recursive predicates are closed under *bounded quantification*. That is, show that if P on $\mathbb{N}_0^{k+1}$ is a primitive recursive predicate, then so are the predicates $\forall z \leq y(P(x_0, \ldots, x_{k-1}, z) = \text{true})$ and $\exists z \leq y(P(x_0, \ldots, x_{k-1}, z) = \text{true})$.[2]

6. (*) In the lecture we sketched a proof that not every computable function is primitive recursive.
   a) Where does this proof fail for μ-recursive functions?
   b) Where does this proof fail if we only consider total (defined everywhere) μ-recursive functions? See Figure 1.

Note that the set of μ-recursive functions, as well as its proper subset of total μ-recursive function are countably infinite.

[2]Here, $\forall z \leq y(P(x_0, \ldots, x_{k-1}, z) = \text{true})$ is an $\mathbb{N}_0^{k+1}$ predicate which is true for $(a_0, \ldots, a_{k-1}, a_k)$ if and only if $P(a_0, \ldots, a_{k-1}, a_k) = \text{true}$ for all $a_k \leq y$; and $\exists z \leq y(P(x_0, \ldots, x_{k-1}, z) = \text{true})$ is an $\mathbb{N}_0^{k+1}$ predicate which is true for $(a_0, \ldots, a_{k-1}, a_k)$ if and only if $P(a_0, \ldots, a_{k-1}, a_k) = \text{true}$ for some $a_k \leq y$.