Nondeterministic finite automata

and pumping lemma

4 Nondeterministic finite automata

formal definition: nondeterministic finite automaton (nfa) M:

$$M = (Z, A, \delta, z_s, Z_A)$$

- A finite input/tape alphabet
- Z finite set of states
- set valued transition function

  $$\delta : Z \times A \to 2^Z$$

  recall

  $$2^Z = \{A \ : \ A \subseteq Z\}$$

- $z_0 \in Z$ initial state
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1 Step:

- if automaton is in state s and reads input $a \in A$ and $s' \in \delta(s, a)$, then it can go to state s' and it moves the head 1 field to the right.

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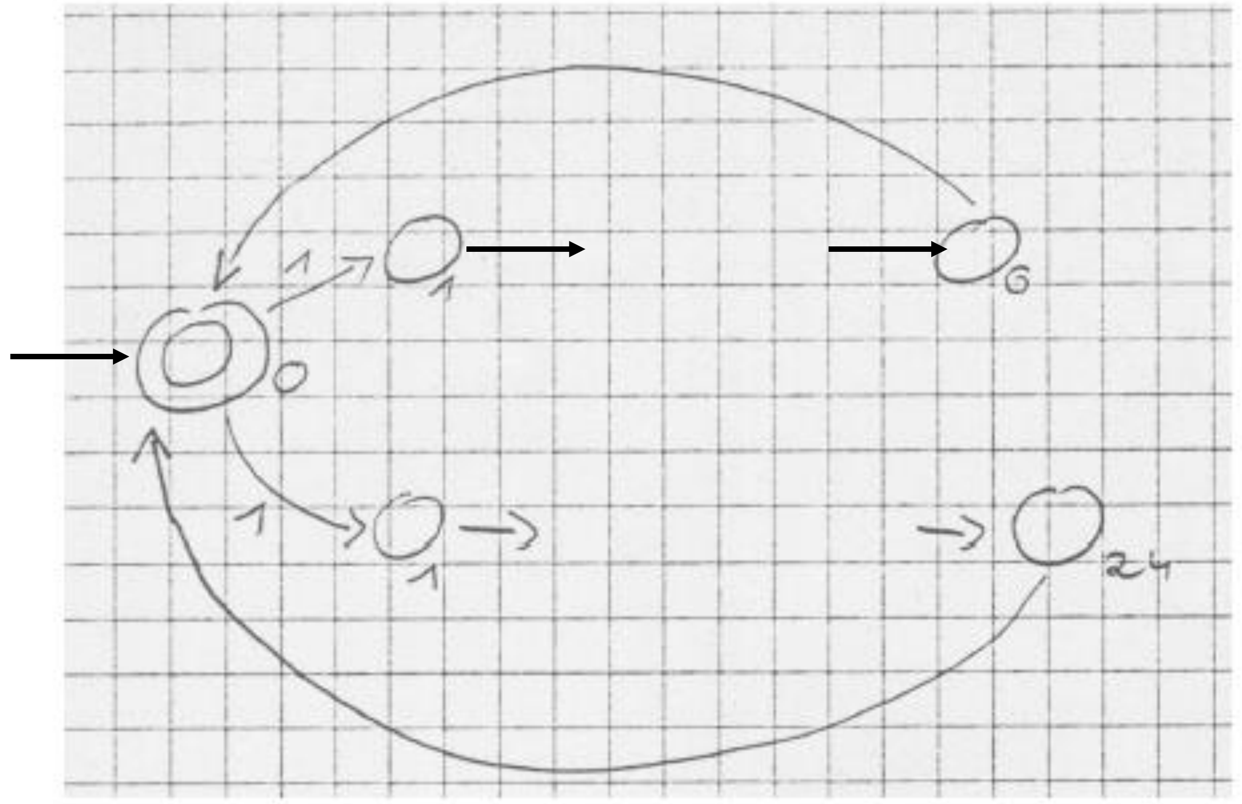

Figure 5: Example of an nda. With input 1^{14} it has accepting and rejecting computations. It accepts this input.

semantics

- set of configurations

$$K = S \times A^*$$

  where for $(s, w) \in K$

$$s: \text{ current state} \quad , \quad w: \text{ remaining input}$$

- transition relation $\vdash \subseteq K \times K$ (in general not a function)

$$k \vdash k' : \; k' \text{ is a successor configuration of } k$$

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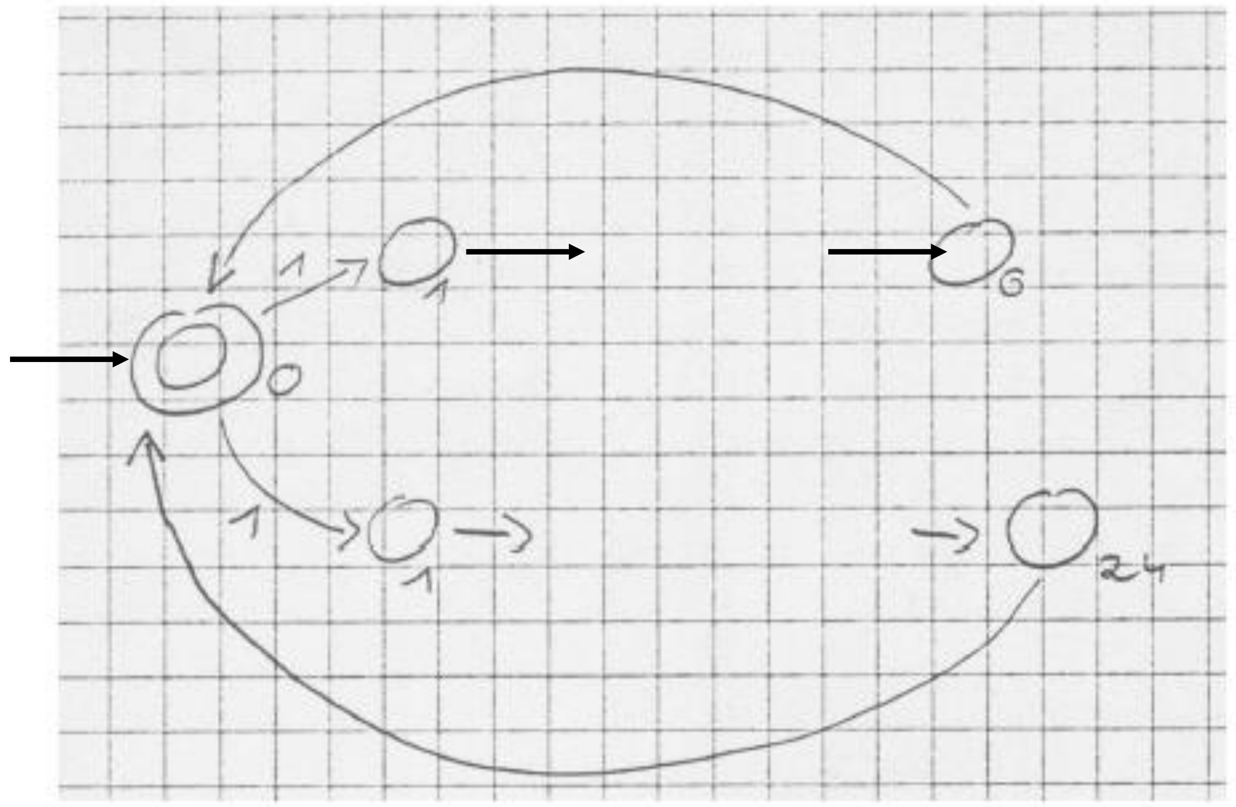

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(at a price)

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also for all future models of computation

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- $n \to n+1$

$$(z_0', w) \vdash_{M'}^{n} (Y, w_{n+1} \ldots w_s) \vdash_{M'} (Y', w_{n+2} \ldots w_s)$$

$$IH: \quad Y = \{z \in Z \,:\, (z_0, w) \vdash_M^n (z, w_{n+1} \ldots w_s)$$

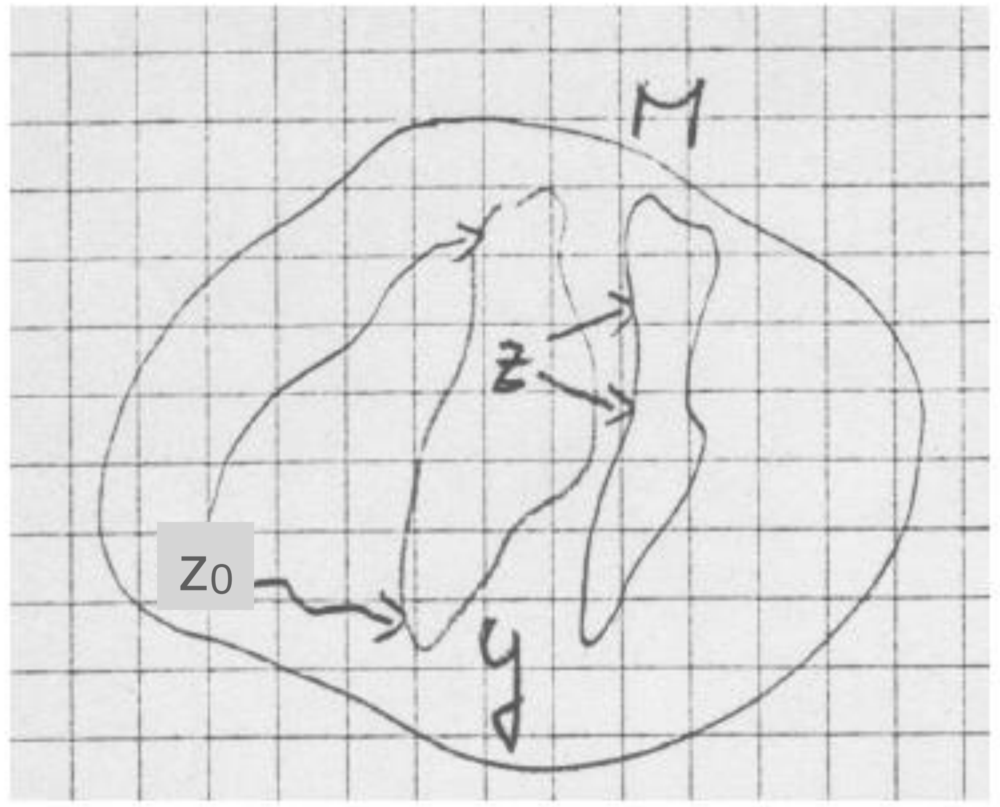

Figure 6: illustration of the power set construction

$$\begin{aligned} Y^* &= \bigcup_{z \in Y} \delta(z, w_{n+1}) \quad \text{(construction of } \delta'\text{)} \\ &= \{z \in Z \,:\, (z_0, w) \vdash_M^{n+1} (z, w_{n+2} \ldots w_s)\} \quad \text{(definition of } \vdash^{n+1}\text{)} \end{aligned}$$

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so far: A finite input/tape alphabet, Z finite set of states, set valued transition fuction

$$\delta : Z \times A \rightarrow 2^Z$$

now:

$$A_\varepsilon = A \cup \{\varepsilon\} \ , \ \delta : Z \times A_\varepsilon \rightarrow 2^Z$$

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  new initial state and next state in deterministic simulation

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OR: have you seen it already?

deterministic and nondeterministic models of computation

always

- K: set of configurations
- E: set of inputs

determinism:

- $k \in K$ and $e \in E$ uniquely determine next configuration k'
- model by transition function

$$\delta : K \times E \to K$$

- examples: hardware, MIPS, C0

graphical representation

$$k \to^{e} k' \leftrightarrow k' = \delta(k, e).$$

k' is **the** next configuration after k with input e

nondeterminism:

$$k \to^{e} k'$$

k' is **a possible** next configuration after k with input e

example: ISA + disk

- processor step: $d \to^{eev} \delta(d, eev)$
- disk step: $d \to \eta(d)$

consumes no external interrupt signal

ϵ-move!

6 Regular expressions and nfa's

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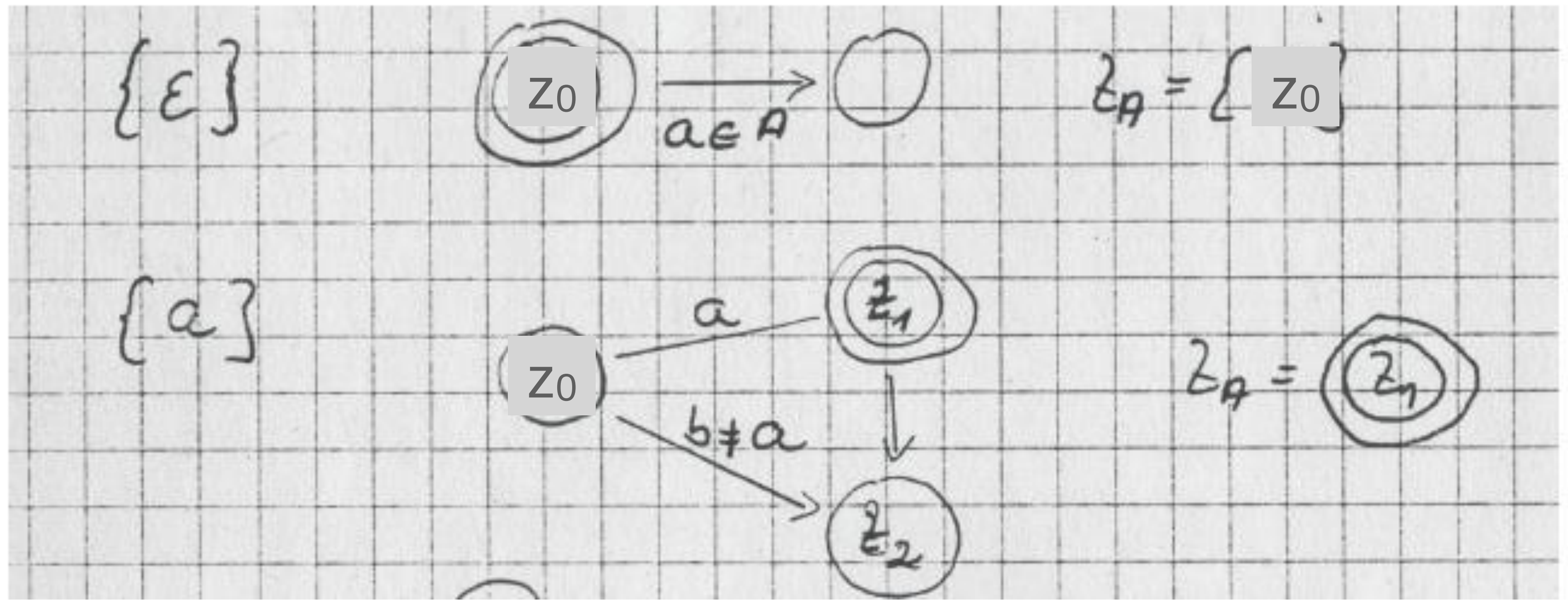

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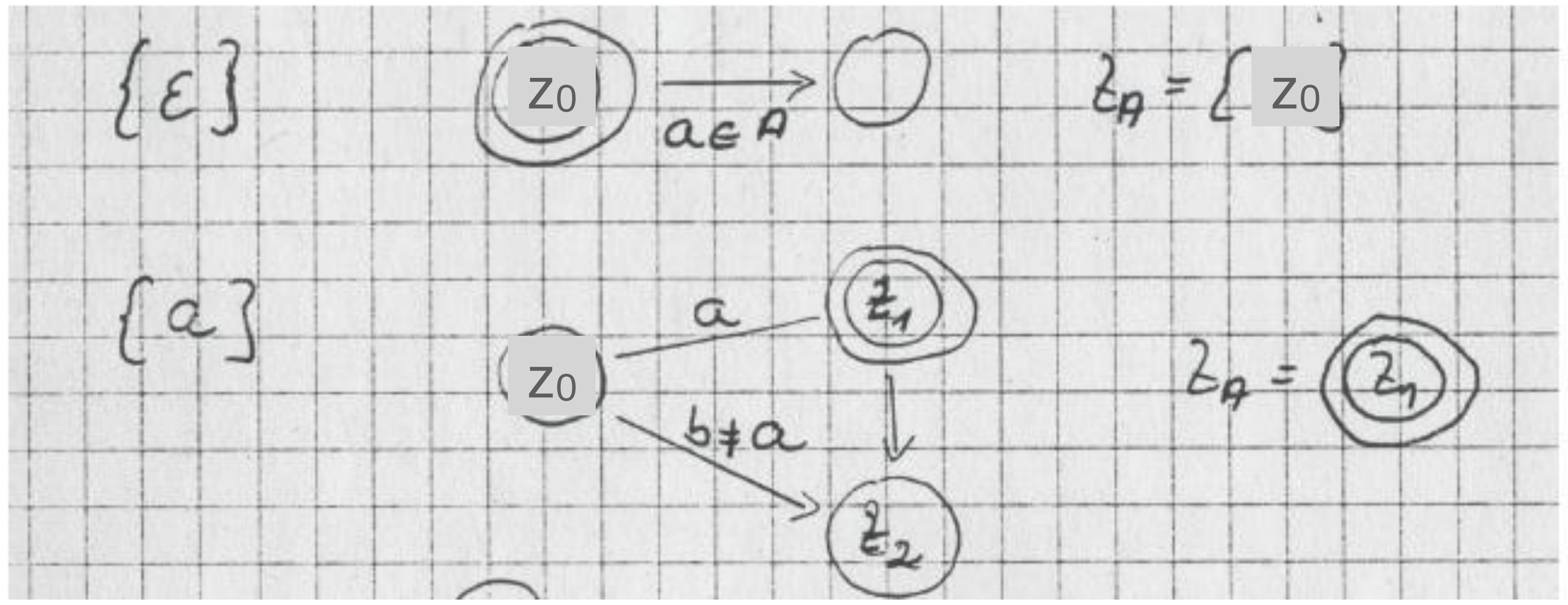

- $L(M'') = L \circ L'$:

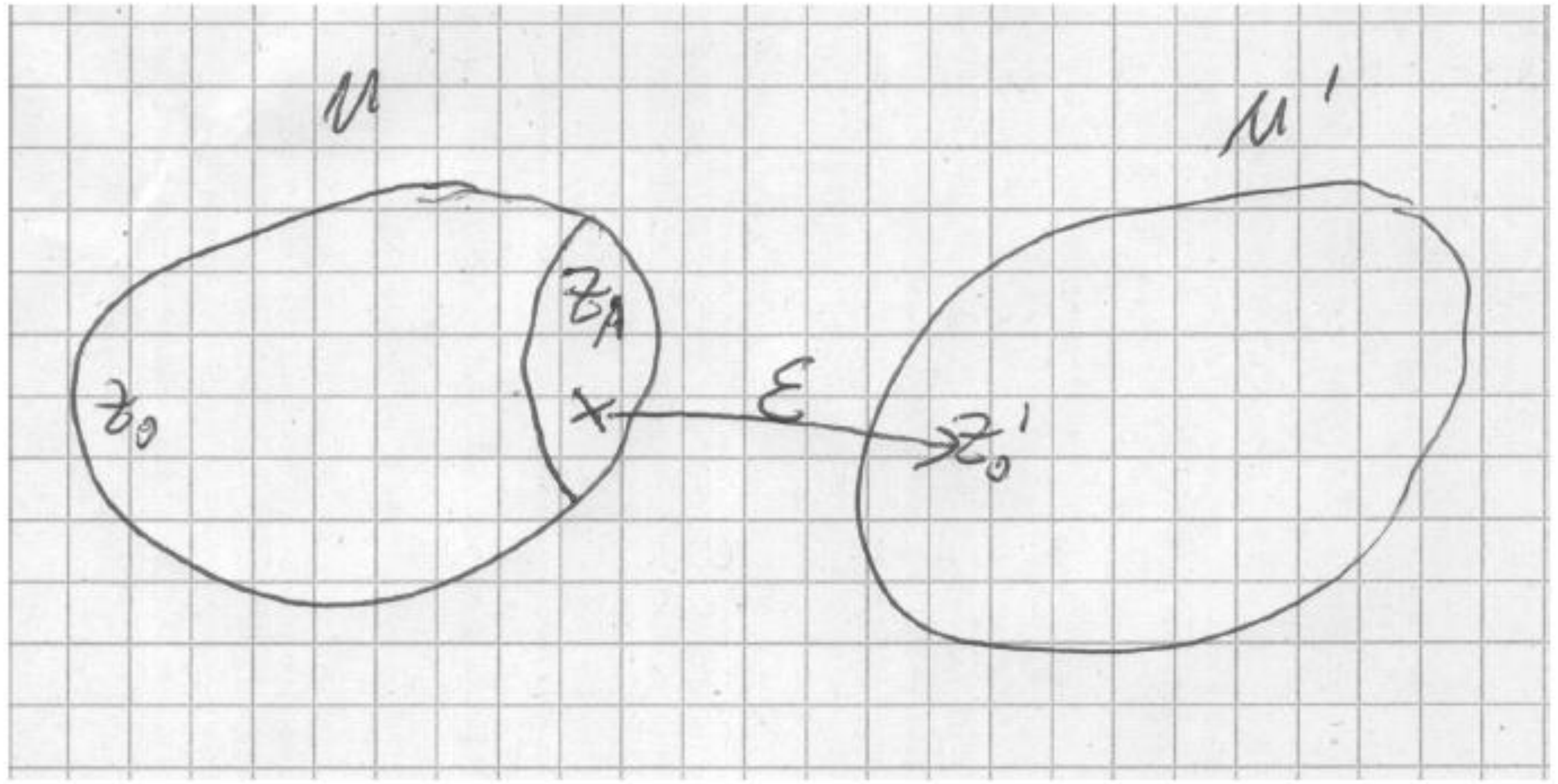

Figure 8: Automata for regular languages: $L \circ L'$

$$
\begin{aligned}
Z'' &= Z \cup Z' \\
z_0'' &= z_0 \\
Z_A'' &= Z_A' \\
\delta''(x,a) &= \begin{cases} \delta(x,a) & x \in Z \setminus Z_A \\ \delta(x,a) \cup \{z_0'\} & x \in Z_A \text{ , } a = \varepsilon \\ \delta'(x,a) & \text{otherwise} \end{cases}
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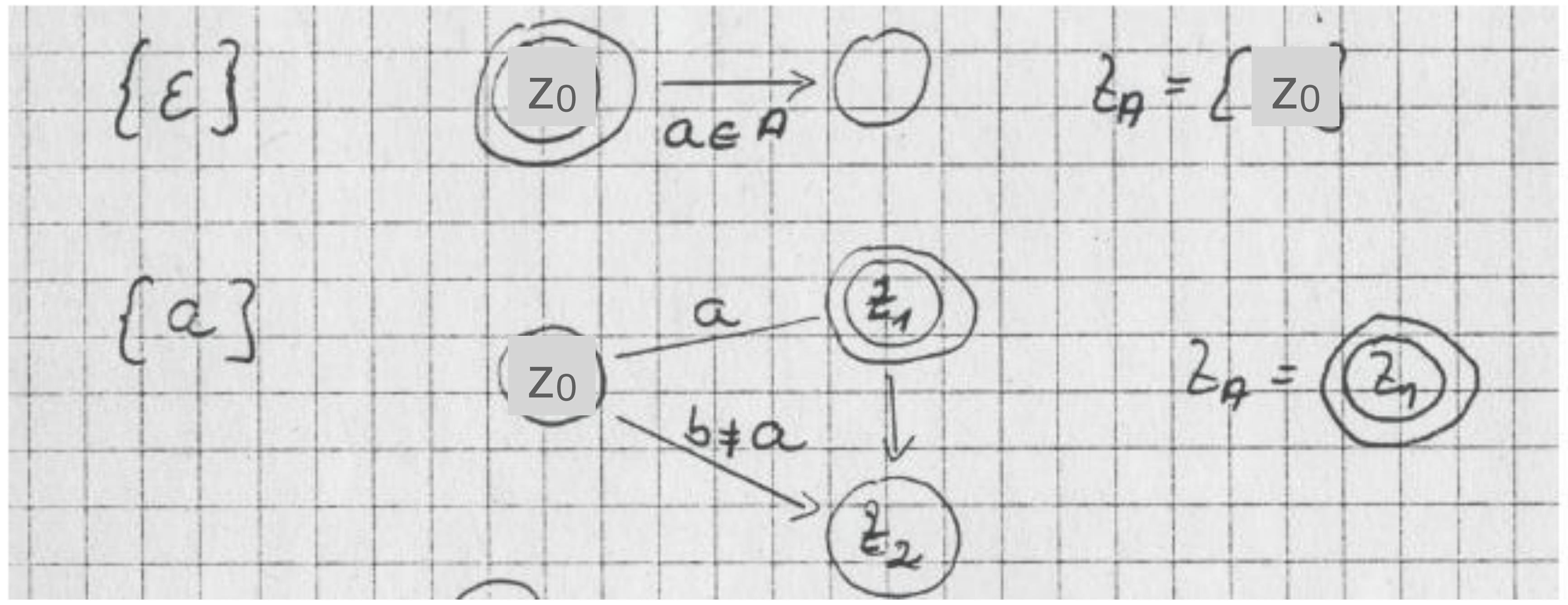


- $L(M'') = L \cup L'$

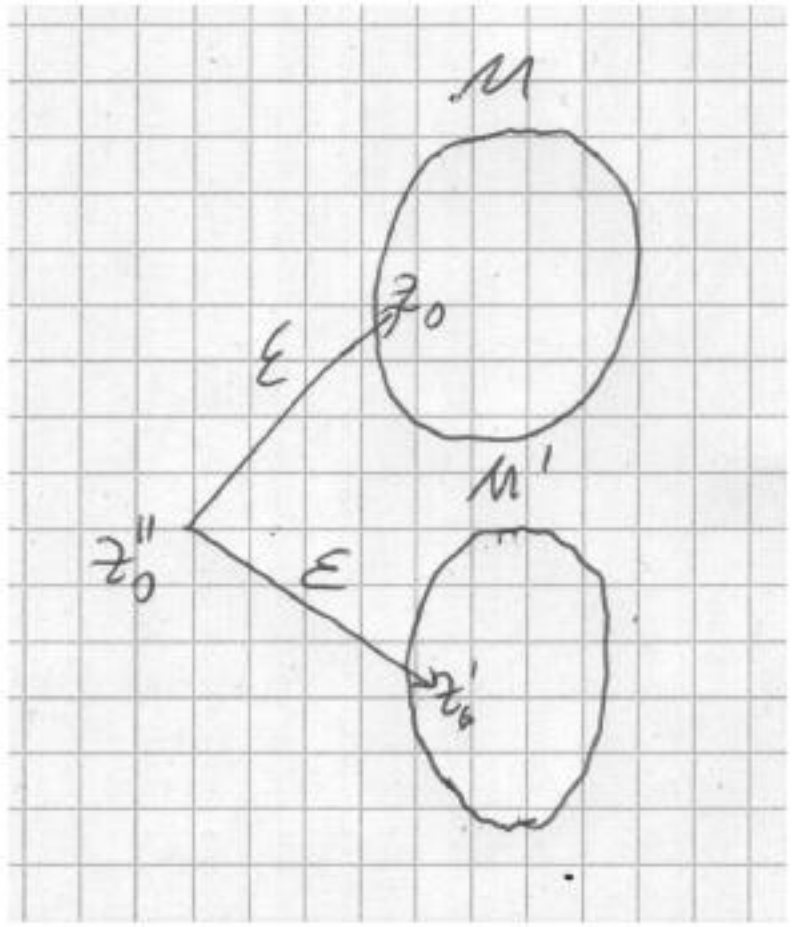


Figure 9: Automata for regular languages: $L \cup L'$

$$Z = Z \cup Z' \cup \{z_0''\} \quad \text{(new start state)}$$

$$\delta(x,a) = \begin{cases} \delta(x,a) & x \in Z \\ \delta'(x,a) & x \in Z' \\ \{z_0, z_0'\} & x = z_0'' \text{ , } a = \varepsilon \end{cases}$$

$$Z_a'' = Z_A \cup Z_A'$$

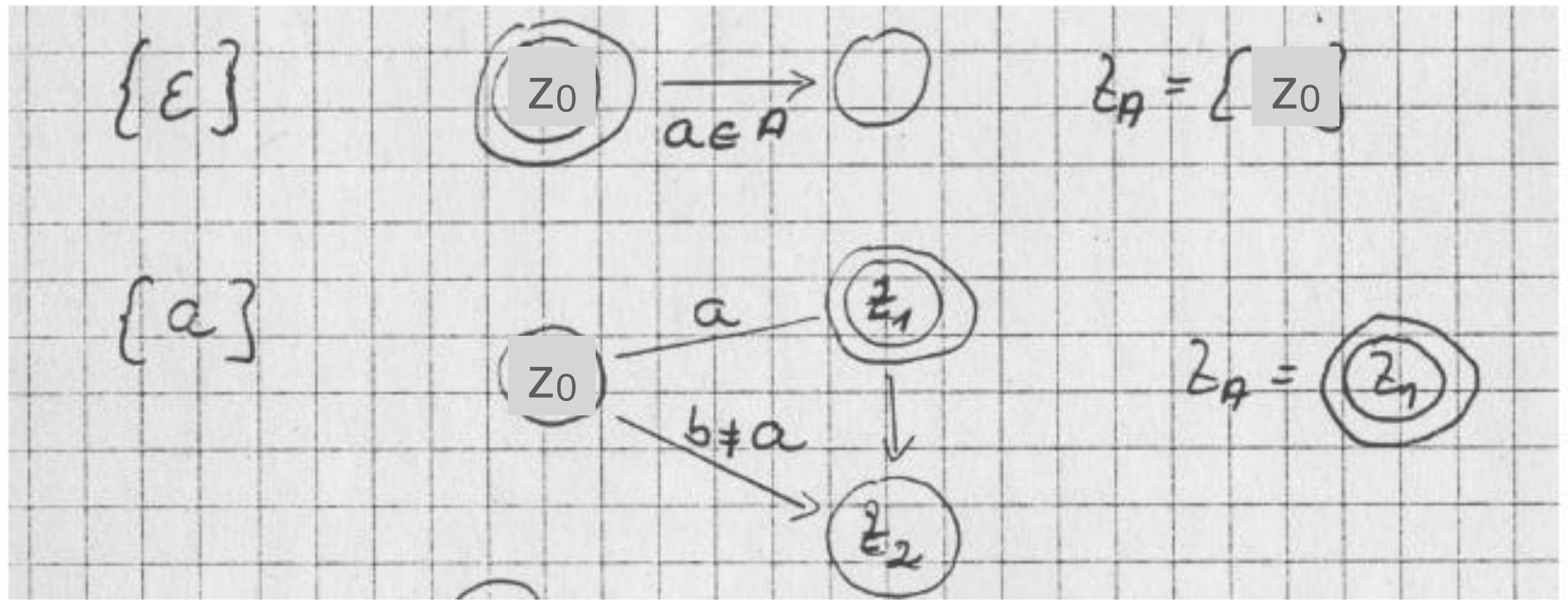
5 Regular expressions and nfa's

Lemma 4. *For every regular language expression $L \in R(A)$ there is an nfa M which accepts L:*

$$L(M) = L$$

Proof by induction over the structure of regular expressions:

- base case: $L = \{\varepsilon\}$ or $L = \{a\}$



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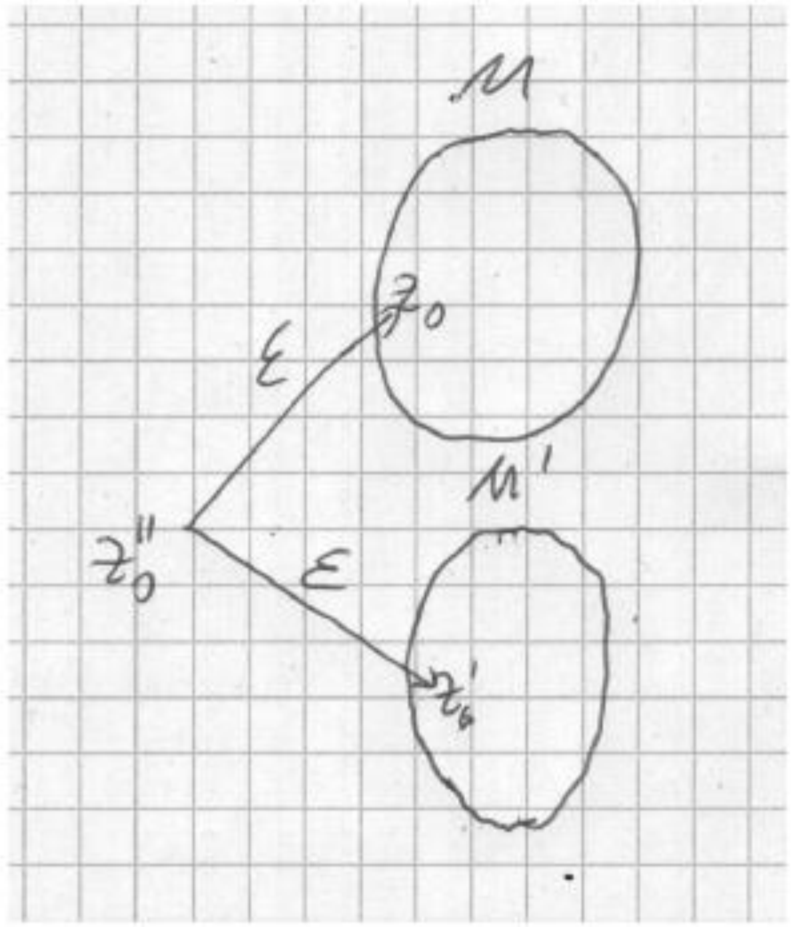

Figure 9: Automata for regular languages: $L \cup L'$

$$
\begin{aligned}
Z &= Z \cup Z' \cup \{z_0''\} \quad \text{(new start state)} \\
\delta(x,a) &= \begin{cases} \delta(x,a) & x \in Z \\ \delta'(x,a) & x \in Z' \\ \{z_0, z_0'\} & x = z_0'',\ a = \varepsilon \end{cases} \\
Z_a'' &= Z_A \cup Z_A'
\end{aligned}
$$

- $L(M'') = L^*$: exercise

7 Pumping lemma

Lemma 5. *Let $L \in R(A)$ be a regular language. Then there is a constant N such that all words $w \in L$ which are longer than N, i.e.*

$$|w| = n > N$$

can be decomposed as

$$w = uvx$$

such that

$$\begin{array}{rrcl} & v & \geq & 1 \\ & |uv| & \leq & N \\ \text{you can pump on v:} \quad \forall i \in \mathbb{N}_0. & uv^i w & \in & L \end{array}$$

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Hence

Lemma 6.

$$\{a^n b^n \ : \ n \in \mathbb{N}\} \quad \textit{is not regular}$$

why?

7 Pumping lemma

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can be decomposed as

$$w = uvx$$

such that

$$\begin{aligned} v &\geq 1 \\ |uv| &\leq N \end{aligned}$$

you can pump on v: $\forall i \in \mathbb{N}_0.\quad uv^i w \in L$

Hence

Lemma 6.

$$\{a^n b^n \;:\; n \in \mathbb{N}\} \quad \textit{is not regular}$$

proof of lemma 5:

Let $N =$ number of states of dfa M accepting L and let $n = |w| > N$. Consider computation of M started with w

$$(k_0, k_1, \ldots, k_n)$$

$$k_i = (z_i, w_{i+1} \ldots w_n)$$

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$$(k_0, k_1, \ldots, k_n)$$

$$k_i = (z_i, w_{i+1} \ldots w_n)$$

Pidgeon hole argument:

$$n > N \ \rightarrow \exists i, j > i. \ z_i = z_j$$

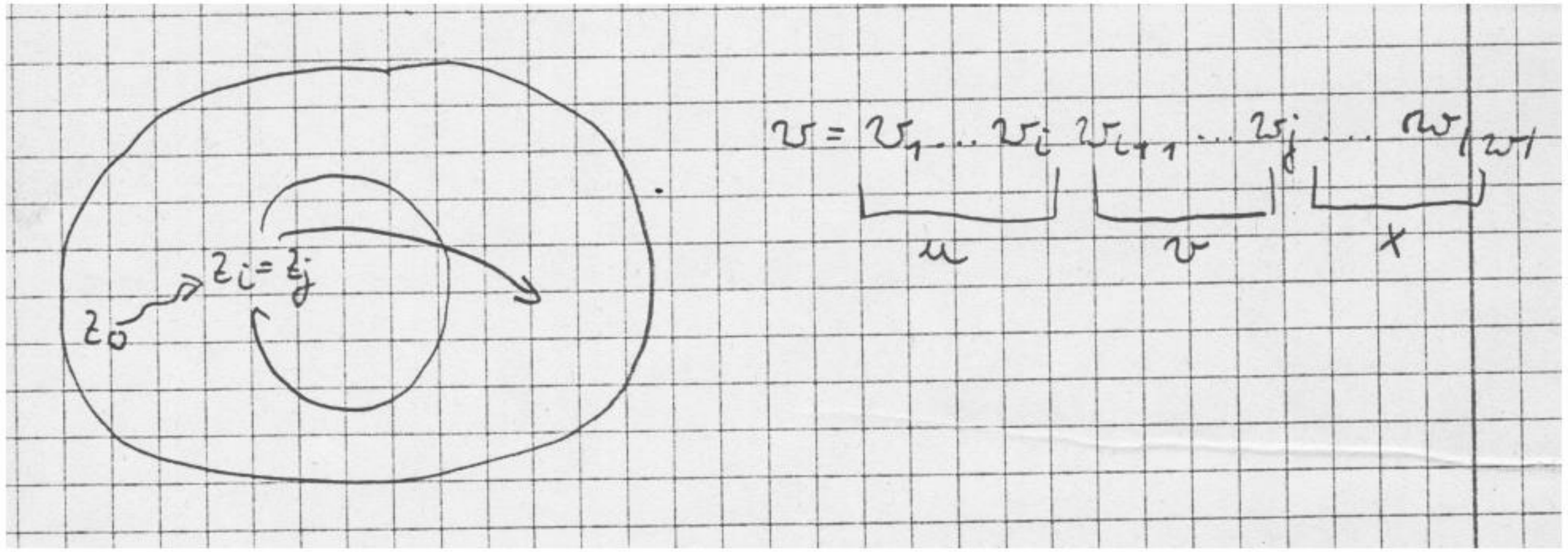

Figure 10: proof of puming lemma for regular languages