Theory of Computation

Midterm

Kutaisi International University

2025-07-18

Version A

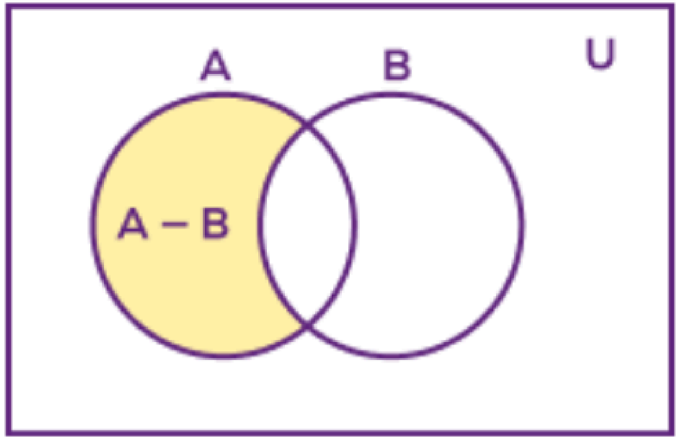
Task 1

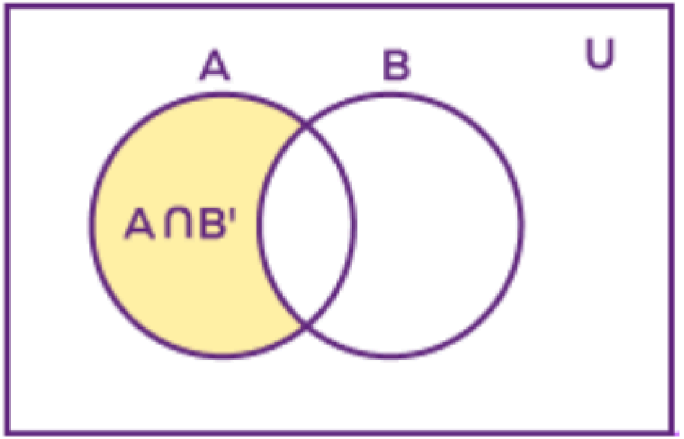
Prove that the difference of two regular languages is regular

Version A

Solution

$$L_1 \setminus L_2 = L_1 \cap \overline{L_2} \tag{1}$$





Version A

We know that if L_2 is regular, $\overline{L_2}$ is also regular by flipping accepting with non-accepting states in DFA for L_2

Intersection is regular cuz its a complement of the union of complements..

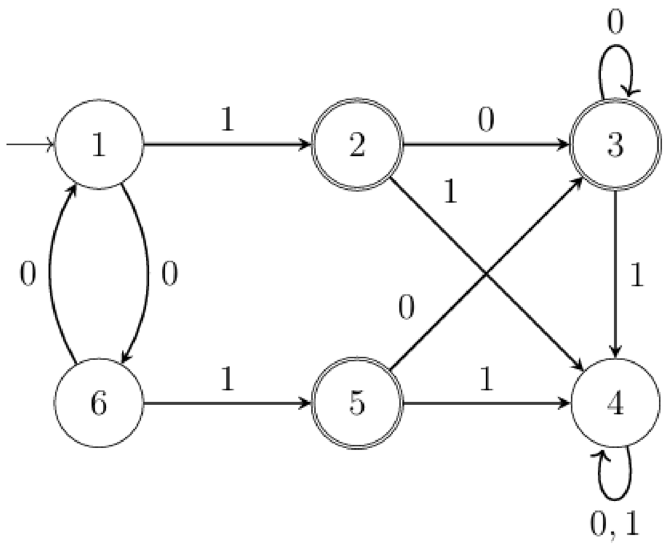
$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}} \tag{2}$$

thats it..

Version A

Task 2

The DFA $M = (Z, A, \delta, z_s, Z_A)$ is given below:



1. Give a formal description of M
2. Give a regular expression for $R(1, 4, 5)$

Version A

First is straightforward. For the second, we want words that go from state 1 to 4 without passing through 6. So, these are words that start with 11 and words that start with 10 and contain one more 1

In regular expression this will be:

$$((1 \circ 1) \circ (1 \cup 0)^*) \cup ((1 \circ 0) \circ 0^* \circ 1 \circ (1 \cup 0)^*) \tag{3}$$

Version A

Task 3

Prove that every NFA can be converted to NFA that accepts the same exact language but has a single accept state

Version A

Lets call our NFA N, new NFA N' is exactly like N except it has ε transitions from the states corresponding to the accept states of N, to a new accept state z_{accept}. $N' = \left(Z \cup \left\{z_{\text{accept}}\right\}, A, \delta', z_0, \left\{z_{\text{accept}}\right\}\right)$

$$\delta'(z, a) = \begin{cases} \delta(z, a) & \text{if } a \neq \varepsilon \text{ or } z \notin Z_A \\ \delta(z, a) \cup \left\{z_{\text{accept}}\right\} & \text{if } a = \varepsilon \text{ and } z \in Z_A \end{cases} \tag{4}$$

and $\delta'\left(z_{\text{accept}}, a\right) = \emptyset$ for each $a \in A_\varepsilon$

Version A

Task 4

Give a context-free grammar that generates:

$$\{w \mid w \in \mathbb{B}^*, \text{length of } w \text{ is odd and its middle symbol is } 0\} \tag{5}$$

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The idea is to generate 0 in the middle and then create all the possible combinations of 0s and 1s on the left and on the right so that every time you use that production rule, the length stays odd

simply:

$$S \to 0 \mid 0S0 \mid 0S1 \mid 1S0 \mid 1S1$$

Version A

Task 5

Prove that the set of primitive recursive functions that are projections is countable

$$p_i^r : \mathbb{N}_0^r \to \mathbb{N}_0 \tag{6}$$

Version A

Solution

Every projection $p_i^r : \mathbb{N}_0^r \to \mathbb{N}_0$ is uniquely specified by the pair of natural numbers (r, i), we conclude by observing that $\mathbb{N}_0 \times \mathbb{N}_0$ is countable, thats it..

Version A

Task 6

Let $G = (\{E, T, F\}, \{\times, +, (,), a\}, P, E)$ where P is given by

$$
\begin{aligned}
E &\to E + T \mid T, \\
T &\to T \times F \mid F \\
F &\to (E) \mid a
\end{aligned}
\tag{7}
$$

1. give a derivation for the string ((a))
2. give an informal description of NPDA that accepts G

Version A

First is obvious, for the second the process would look be:

1. Place E on the stack
2. Repeat the following steps
   3. if the top of stack is the variable E,pop it and nondeterministically push either $E + T$ or T
   4. if the top of stack is the variable T, pop it and nondeterministically push either $T \times F$ or F
   5. if the top of stack is the variable F, pop it and nondeterministically push either (E) or a
   6. if the top of stack is a terminal symbol read the next symbol from the input and compare it to the terminal in the stack, if they match, repeat, if they do not match, reject on this branch of the nondeterminism
   7. if the top of the stack is ε, enter the accept state. Doing so accepts the input if it has all been read..

Version A

Task 7

Fix a positive natural number n, sketch a proof that the function $f(x) = x^n$ is primitive recursive

Version A

Solution

$f(0) = 0, f(x+1) = f(x) + \sum_{i=0}^{n-1} \binom{n}{i} x^i$

Since

$$\begin{aligned} (x+1)^n - x^n &= \\ &= \sum_{i=0}^{n} \binom{n}{i} x^i \; - \; x^n \\ &= \sum_{i=0}^{n-1} \binom{n}{i} x^i \end{aligned} \tag{8}$$

Task 8

State pumping lemma for regular languages. Give an example of a language that is not regular Sketch an idea (without the proof) how we prove that the language you gave is not regular

Version A

Solution

From slides TCS-2, language $\{a^n b^n \mid n \in \mathbb{N}\}$

Version 2

Task 1

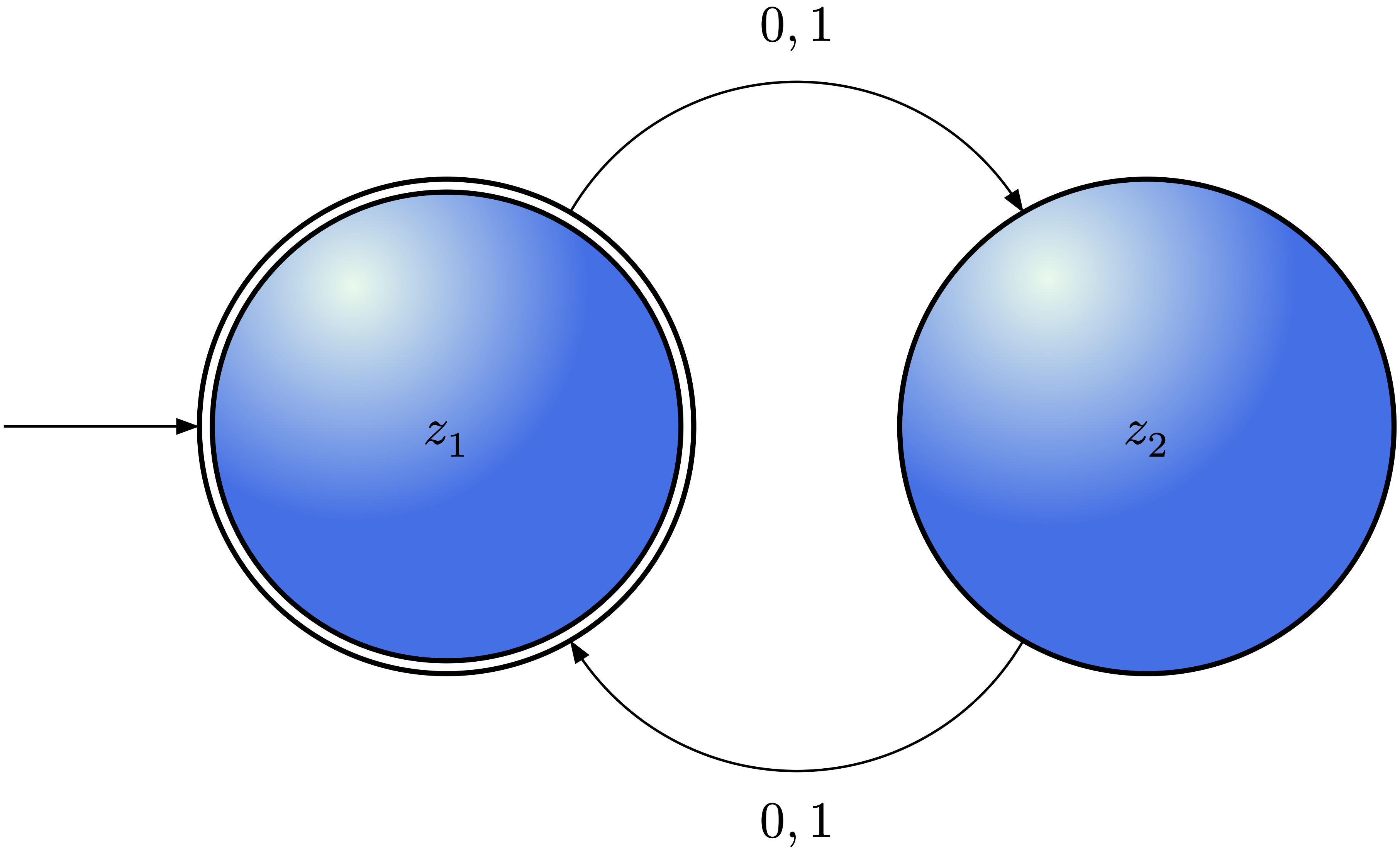
Construct a DFA that accepts a language given by the following regular expression written for the alphabet $\{0, 1\}$.

$$((0 \circ 0) \cup (0 \circ 1) \cup (1 \circ 0) \cup (1 \circ 1))^* \tag{9}$$

Version 2

If you take a closer look, these are all string of even length (including an ε). You can easily construct a DFA for this..

Version 2

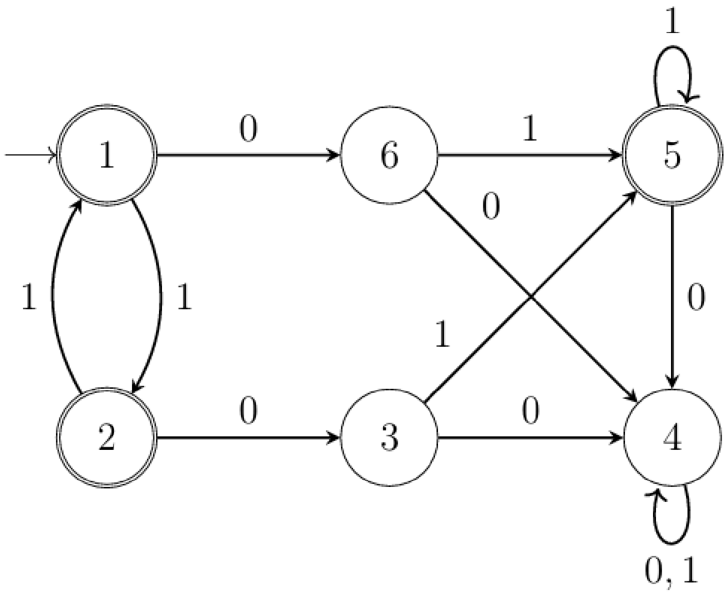


Version 2

Task 2

The DFA M is given

1. Give a formal description of M
2. Give a regular expression for $R(1,5,5)$



Version 2

First one is easy, for the second, we want words that start with odd number of 1’s, followed by 01 and no more zeros after

$$1 \circ (1 \circ 1)^* \circ (0 \circ 1) \circ 1^* \tag{10}$$

Version 2

Task 3

Construct NFA with a single accepting state, that accepts the same exact language as DFA Figure 1

Version 2

💡 **Solution**

Just add ε moves from old accepting states to the newly created single accepting state. 0 and 1 both go to empty set from the new accepting state

Version 2

Task 4

Give a context-free grammar that generates

$$\{w \mid w \in \mathbb{B}^* \text{ and } w \text{ contains at least three 1s}\} \tag{11}$$

Version 2

Solution

so the idea is to generate three 1s first and then random number of 0s and 1s between them

$$\begin{aligned} S &\to R1R1R1R \\ R &\to 0R \mid 1R \mid \varepsilon \end{aligned} \tag{12}$$

Version 2

Task 5

Prove that the set of primitive recursive functions that are constants is countable

$$c_i^r : \mathbb{N}_0^r \to \mathbb{N}_0 \tag{13}$$

Version 2

Solution

Every constant $c_i^r : \mathbb{N}_0^r \to \mathbb{N}_0$ is uniquely specified by the pair of natural numbers (r, i). We conclude by observing that $\mathbb{N}_0 \times \mathbb{N}_0$ is countable

Version 2

Task 6

Let $G = (\{R, X, S, T\}, \{a, b\}, P, R)$ where P is given by:

$$\begin{aligned} R &\to XRX \mid S, \\ S &\to aTb \mid bTa, \\ T &\to XTX \mid X \mid \varepsilon \\ X &\to a \mid b \end{aligned} \tag{14}$$

1. Give a derivation of XXX starting from T

   $$T \xrightarrow[G]{*} XXX$$

2. Give an informal description of NPDA that accepts G

Version 2

1. Place start variable R on the stack

2. Repeat the following steps
   3. if the top of stack is the variable R,pop it and nondeterministically push either XRX or S
   4. if the top of stack is the variable S, pop it and nondeterministically push either aTb or bTa
   5. if the top of stack is the variable T, pop it and nondeterministically push either XTX, X or ε
   6. if the top of stack is the variable X, pop it and nondeterministically push either a or b
   7. if the top of stack is a terminal symbol read the next symbol from the input and compare it to the terminal in the stack, if they match, repeat, if they do not match, reject on this branch of the nondeterminism
   8. if the top of the stack is ε, enter the accept state. Doing so accepts the input if it has all been read..

Version 2

Task 7

lets say $f : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ is given by

$$f(m, x) = (4m + 2x) \dot{-} 11 \tag{15}$$

Determine $\mu f(x)$, show your work..

Version 2

Solution

$$\mu f(0) = \min_m \left\{4m + 2 \cdot 0 \dot{-} 11 = 0\right\} = 0$$
$$\mu f(1) = \min_m \left\{4m + 2 \cdot 1 \dot{-} 11 = 0\right\} = 0$$

similarly $\mu f(2) = \mu f(3) = \mu f(4) = \mu f(5) = 0.$

$\mu f(i)$ is undefined for $i > 5$ cuz $4m + 2 \cdot i \dot{-} 11 > 0$

Version 2

Task 8

State pumping lemma for context free languages. Give an example of a language that is not context free. Sketch an idea(without proof) how we prove that the language is not context free..

Version 2

💡 **Solution**

From slides TCS-4, language $\{a^n b^n c^n \mid n \in \mathbb{N}\}$

Version 2