minimization of dfa’s

1 ’Historical Background’

I found 2 pages on this topic in my old lecture notes.

- great algorithm, great example, too good to leave out!
- arguments missing
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notation in that document

$$B = (Q, \Sigma, \delta, q_0, F)$$

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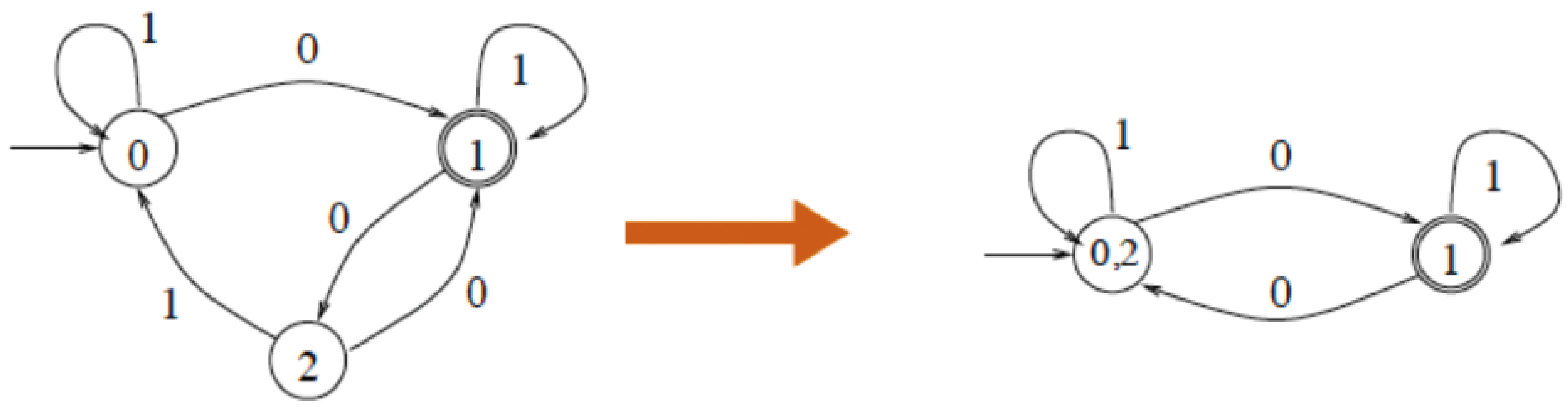
2 The problem

Given dfa M: find an automaton M' with

$$L(M) = L(M')$$

and a minimal number of states.



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3 Preliminaries

reachable states: a state $p \in Q$ of a fa is *reachable* if there is a path from the start state q_0 to p.

- find reachable states by a graph search (DFS or BFS)

- eliminate non reachable states

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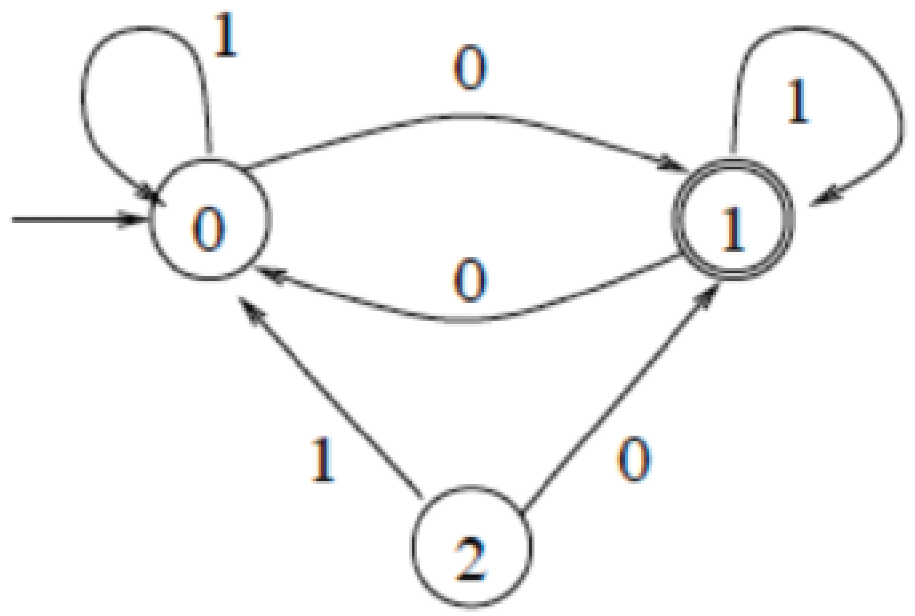

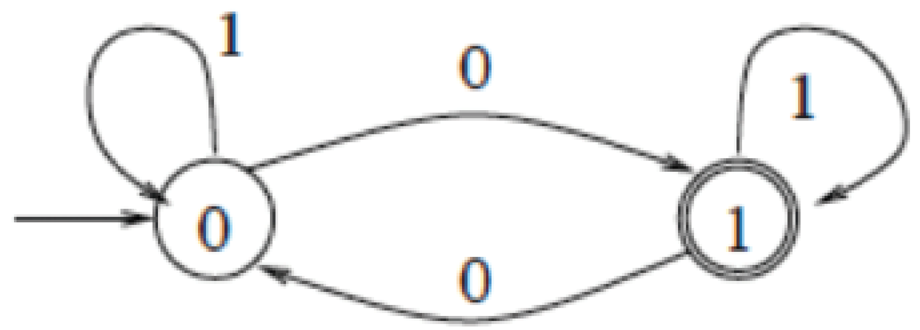

iterated transition function: transition function

$$\delta : Q \times \Sigma \to Q$$

gives iterated transition function

$$\delta^* : Q \times A^* \to Q$$

For $p \in Q, a \in \Sigma, w \in \Sigma^*$ define

$$\begin{aligned} \delta^*(p,\varepsilon) &= p \\ \delta^*(p,aw) &= \delta^*(\delta(p,a),w) \end{aligned}$$

making steps until w is consumed

4 Distiguishable states

States $p,q \in Q$ are *distinguishable* if there is $w \in \Sigma^*$ such that

$$\delta^*(p,w) \in F \quad , \quad \delta^*(q,w) \notin F$$

or

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We say: w *distinguishes* between p and q.

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Lemma 1. *Let B be a dfa in which all states are reachable. Then B is minimal iff all states are distinguishable.*

Proof: ($\Rightarrow$) This implication is easy, for if B has two undistinguishable states, one of them can be eliminated, and the transitions into this state can be changed to go to the other. This will not affect the accepted language. (Formal proof omitted.)

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- $\rightarrow$: trival. Merge non distiguishable states.
- $\leftarrow$: Let $|Q| = k$ and let B' be automaton with $|Q'| = \ell < k$. Claim:

$$L(B) \neq L(B')$$

For each $q \in Q$ let

$$\delta^*(q_0, w_q) = q$$

exists because q reachable.

pidgeon hole argument: $\ell < k \rightarrow$:

$$\exists p, q. \quad \delta'^*(q_0', w_p) = \delta'^*(q_0', w_q) = r$$

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p, q in B distinguished by v. w.l.o.g:

$$\delta^*(p,v) \in F \quad , \quad \delta^*(q,v) \notin F$$

$$w_p v \in L(B) \quad , \quad w_q v \notin L(B)$$

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in B':

$$w_p v, w_q v \in L(B') \leftrightarrow \delta'^*(r,v) \in F'$$

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Proof. exercise

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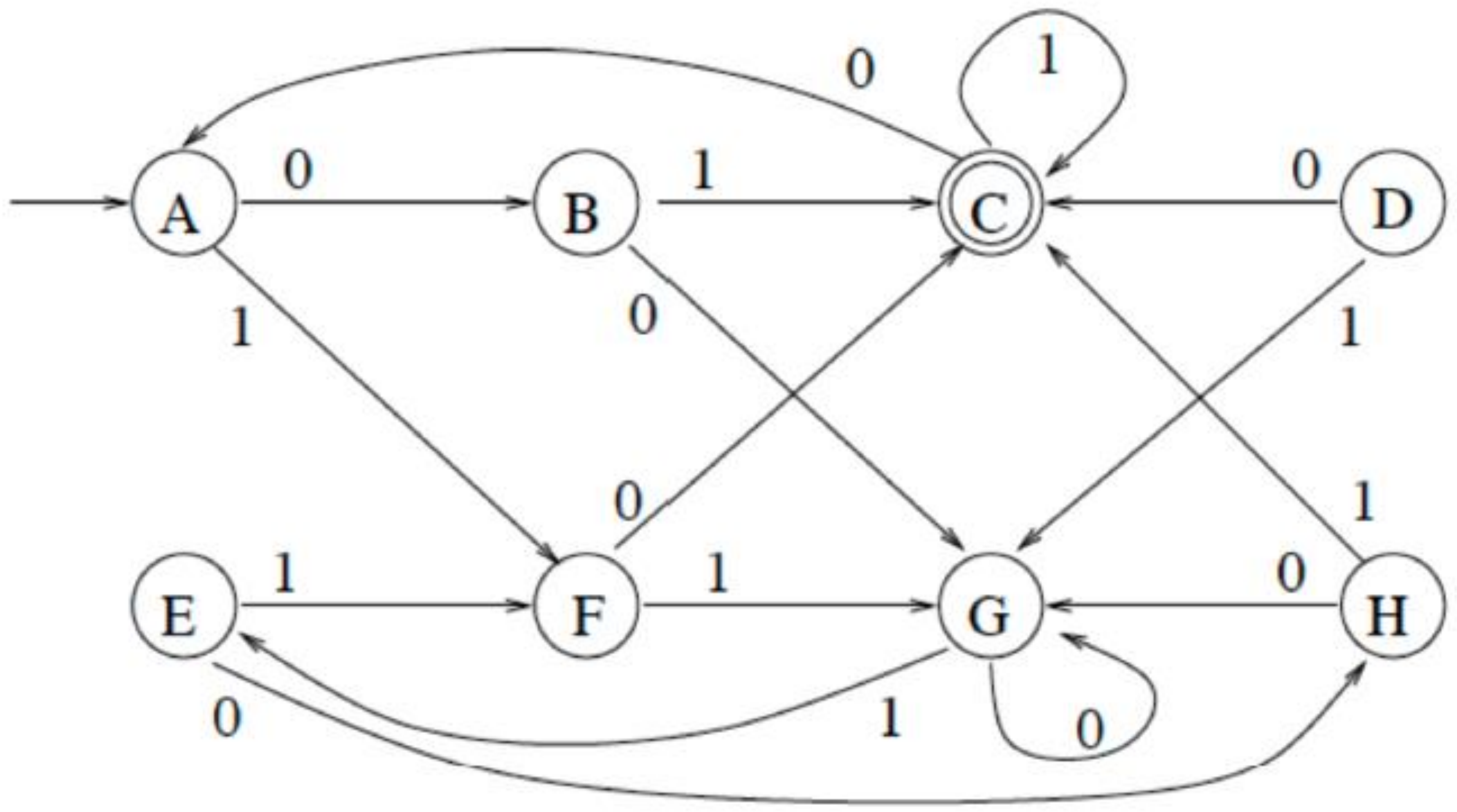
Lemma 3. *Let $\delta(p,a) = p'$ and $\delta(q,a) = q'$. Then:*

$$p', q' \textit{ distinguishable } \rightarrow p, q \textit{ distinguishable}$$

Proof. If w distinguishes p', q', then aw distinguishes p, q.

5 Minimization algorithm

running example :



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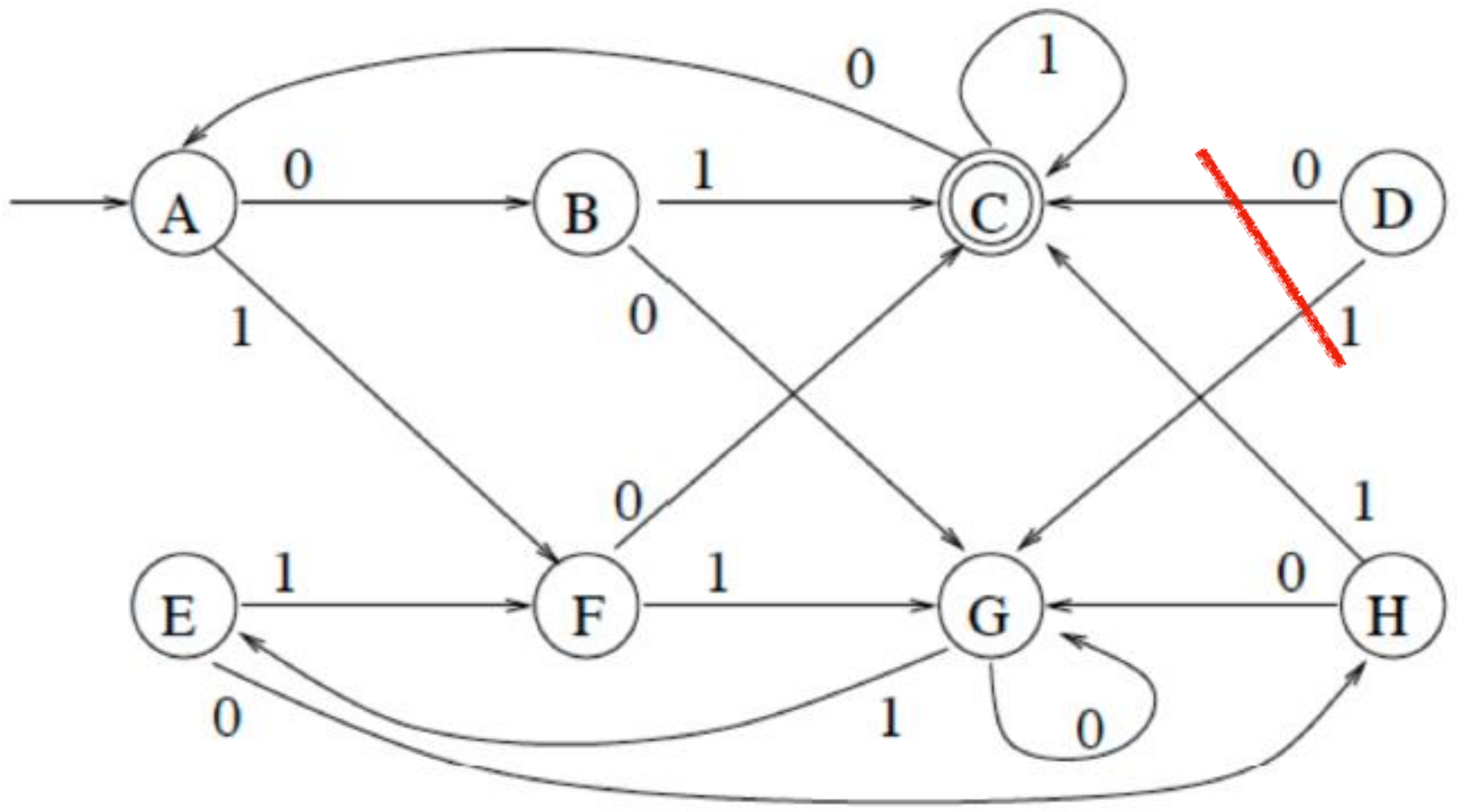

step 1: eliminate unreachable states

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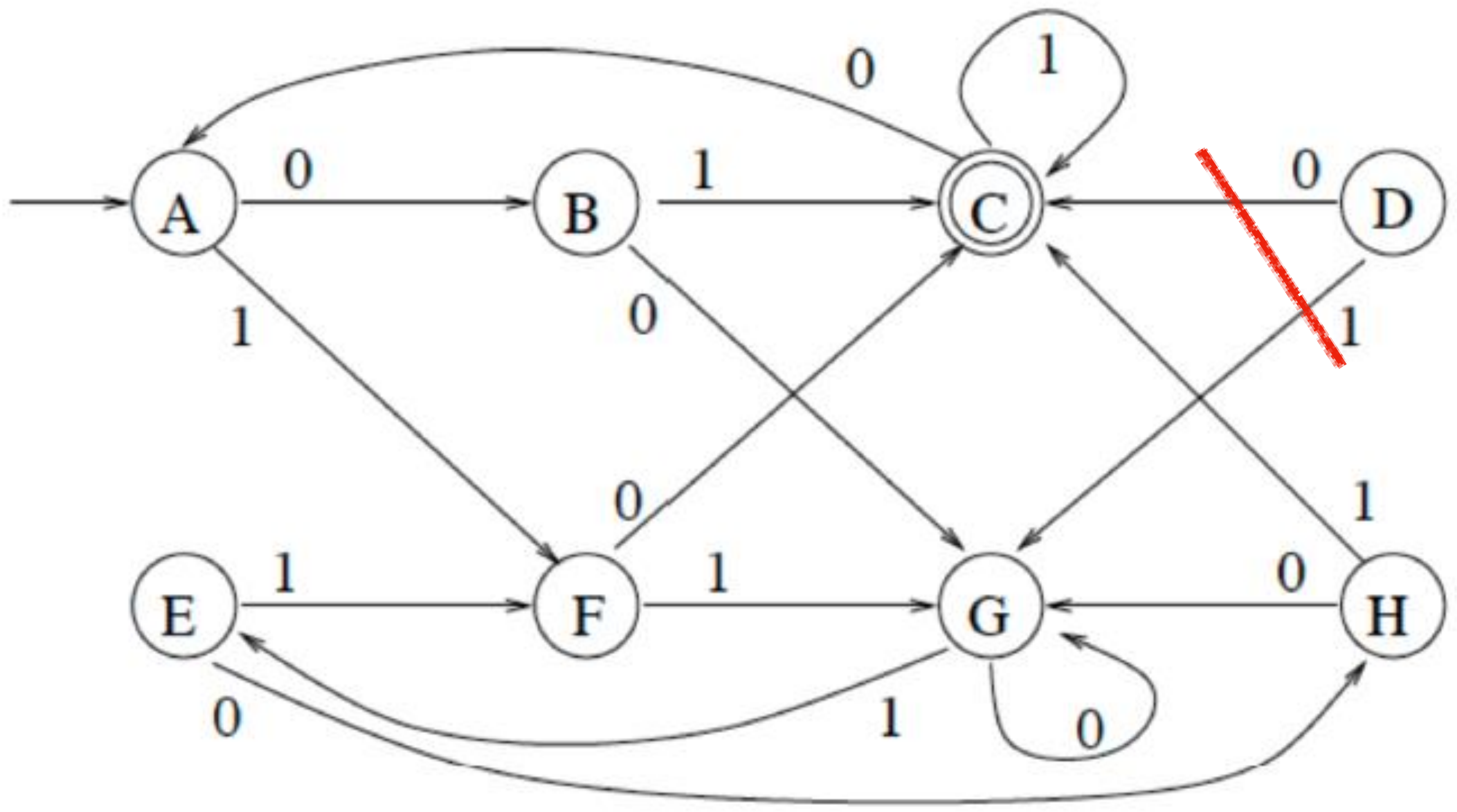


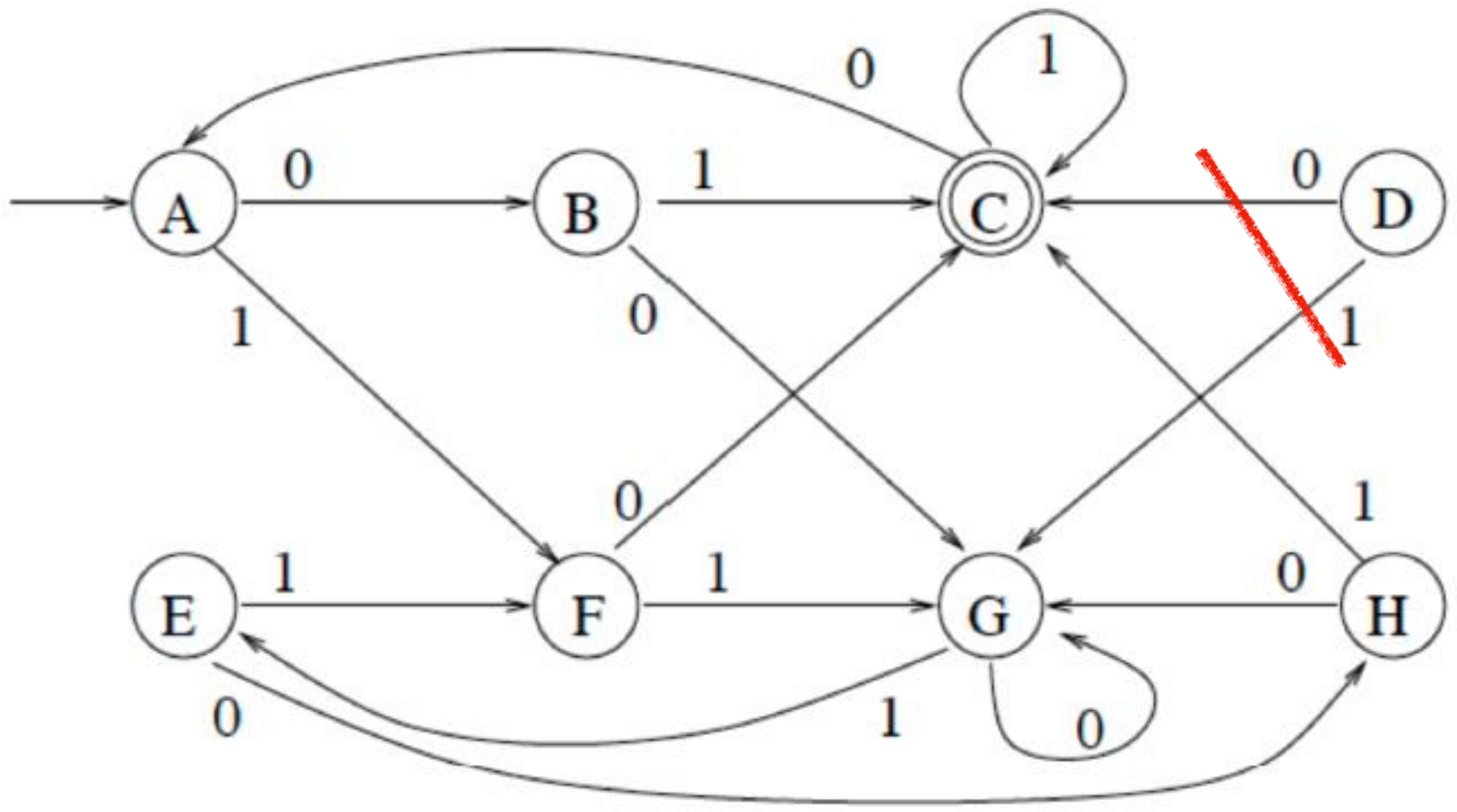
step 1: eliminate unreachable states

step 2: Mark the distinguishable pairs of states. To achieve this task, we first mark all pairs p, q, where $p \in F$ and $q \notin F$ as distinguishable.

1. initiate:

B						
C	X	X				
E			X			
F			X			
G			X			
H			X			
	A	B	C	E	F	G

running example :



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Then, we proceed as follows:

repeat
for all non-marked pairs p, q do
for each letter a do
if the pair $\delta(p, a), \delta(q, a)$ is marked then mark p, q
until no new pairs are marked

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2. e.g. $\delta(A,1) = F, \delta(B,1) = C$, mark A, B

B	X					
C	X	X				
E		X	X			
F	X	X	X	X		
G		X	X		X	
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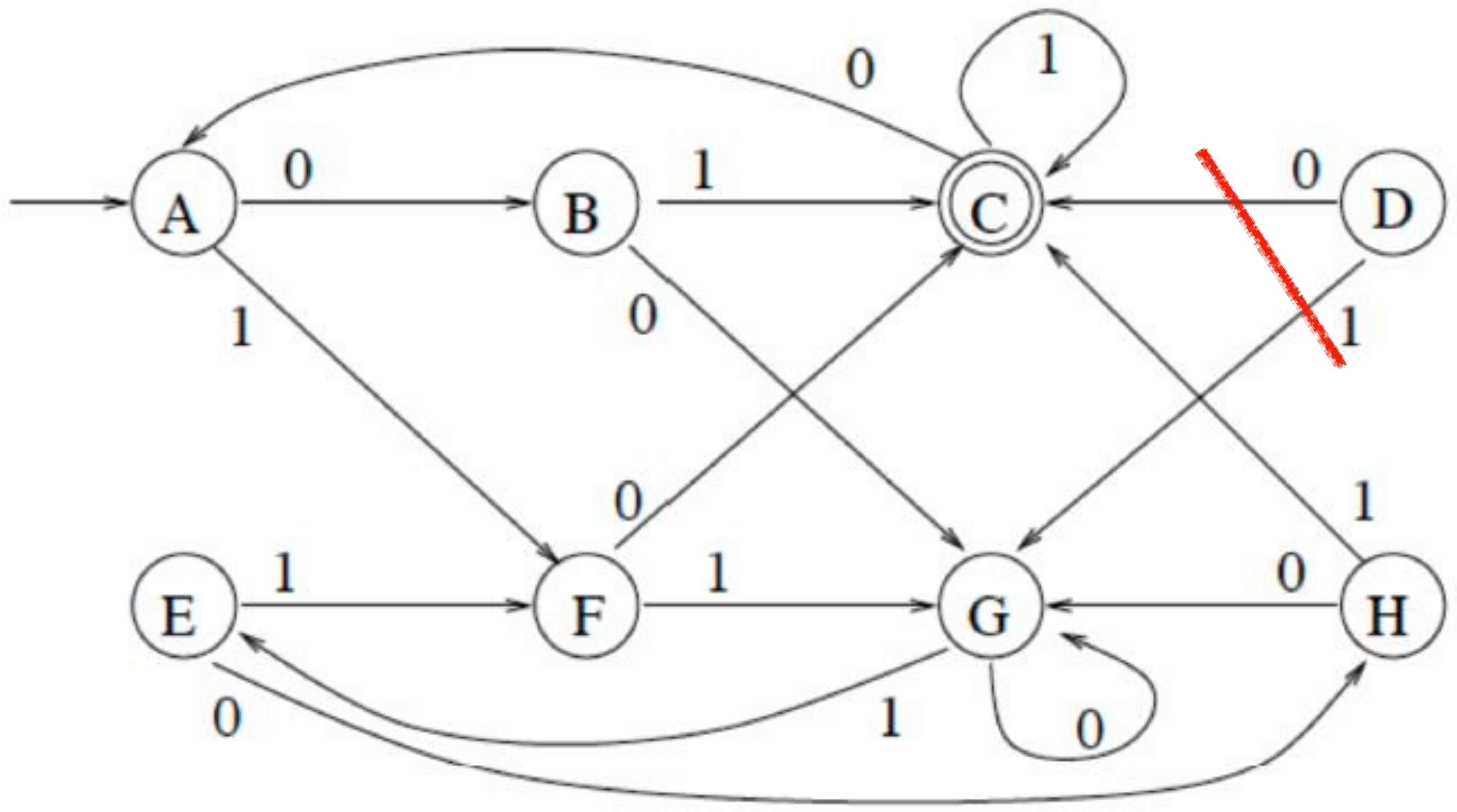
	A	B	C	E	F	G

Figure 5: after 1 iteration

running example :



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C	X	X				
E		X	X			
F	X	X	X	X		
G		X	X		X	
H	X		X	X	X	X
	A	B	C	E	F	G

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3. e.g. B,G marked, $\delta(A,0) = B, \delta(G,0) = G$, mark A,G

B	X					
C	X	X				
E		X	X			
F	X	X	X	X		
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Figure 6: after 2 iterations

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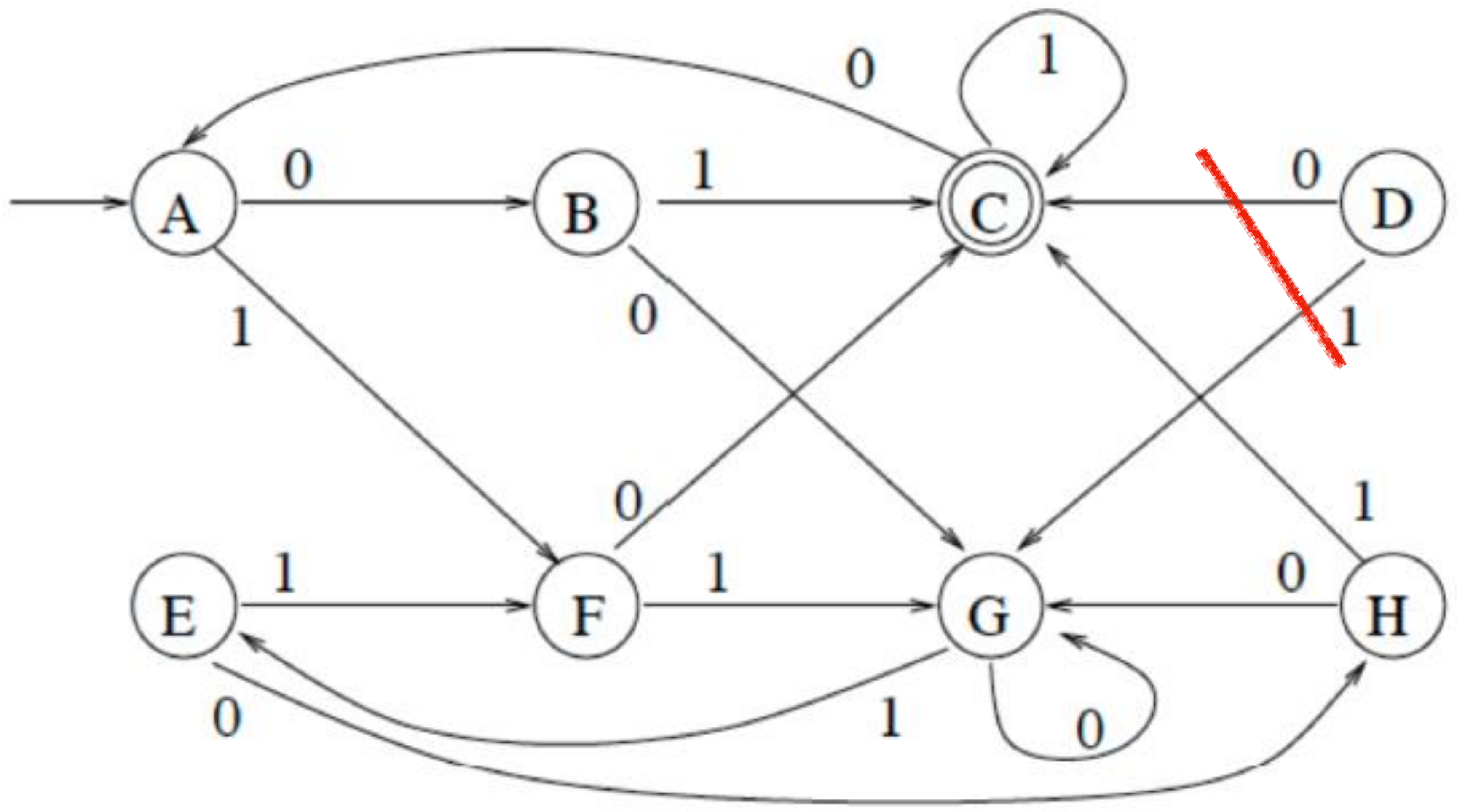

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Figure 6: after 2 iterations

4. then nothing new

Lemma 4. *for all $p,q \in Q$:*

$$p,q \text{ marked} \leftrightarrow p,q \text{ distinguishable}$$

Proof. Induction on the length of the shortest string that distinguishes p,q, using Lemma 3 . Exercise. $\square$

running example :

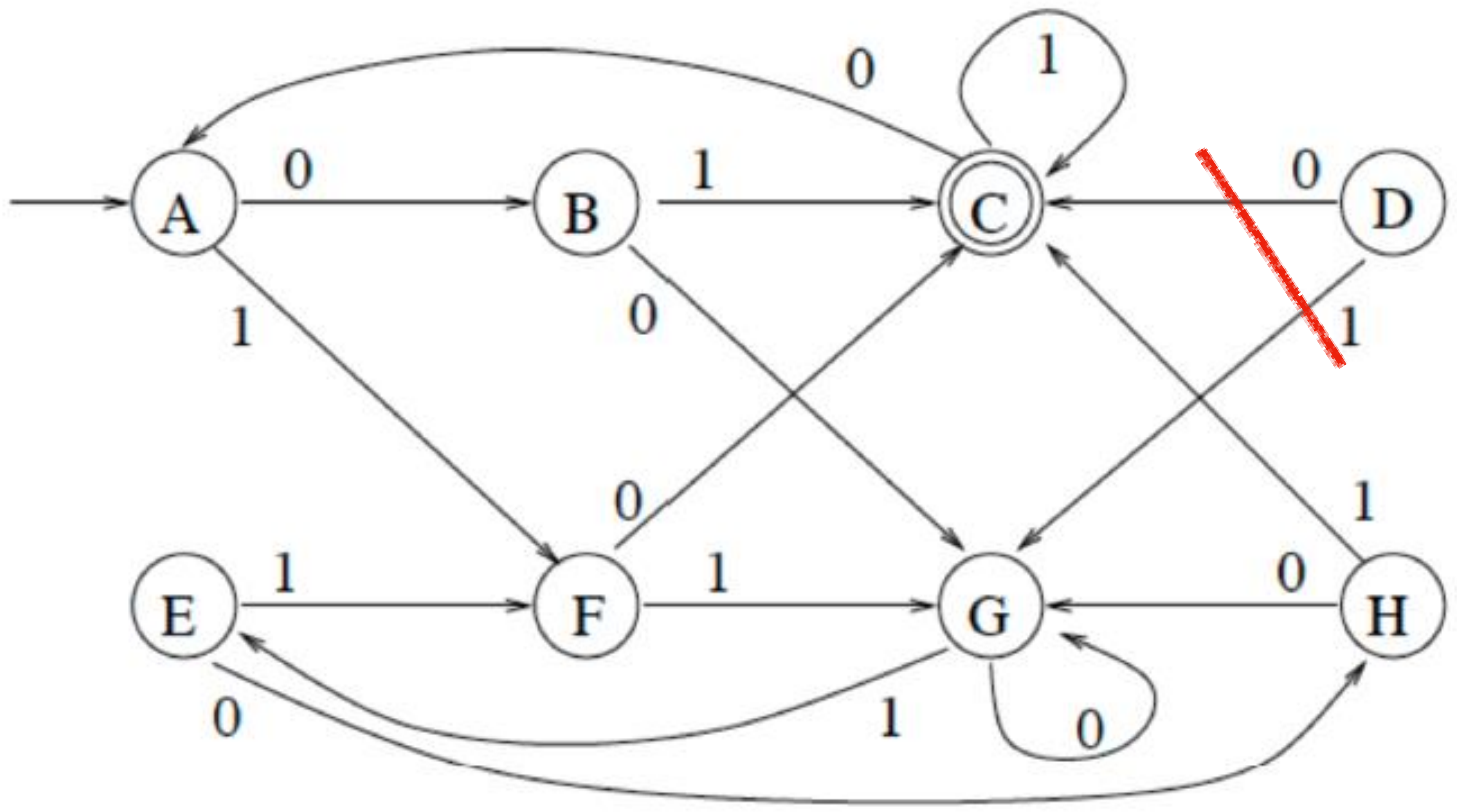

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Figure 6: after 2 iterations

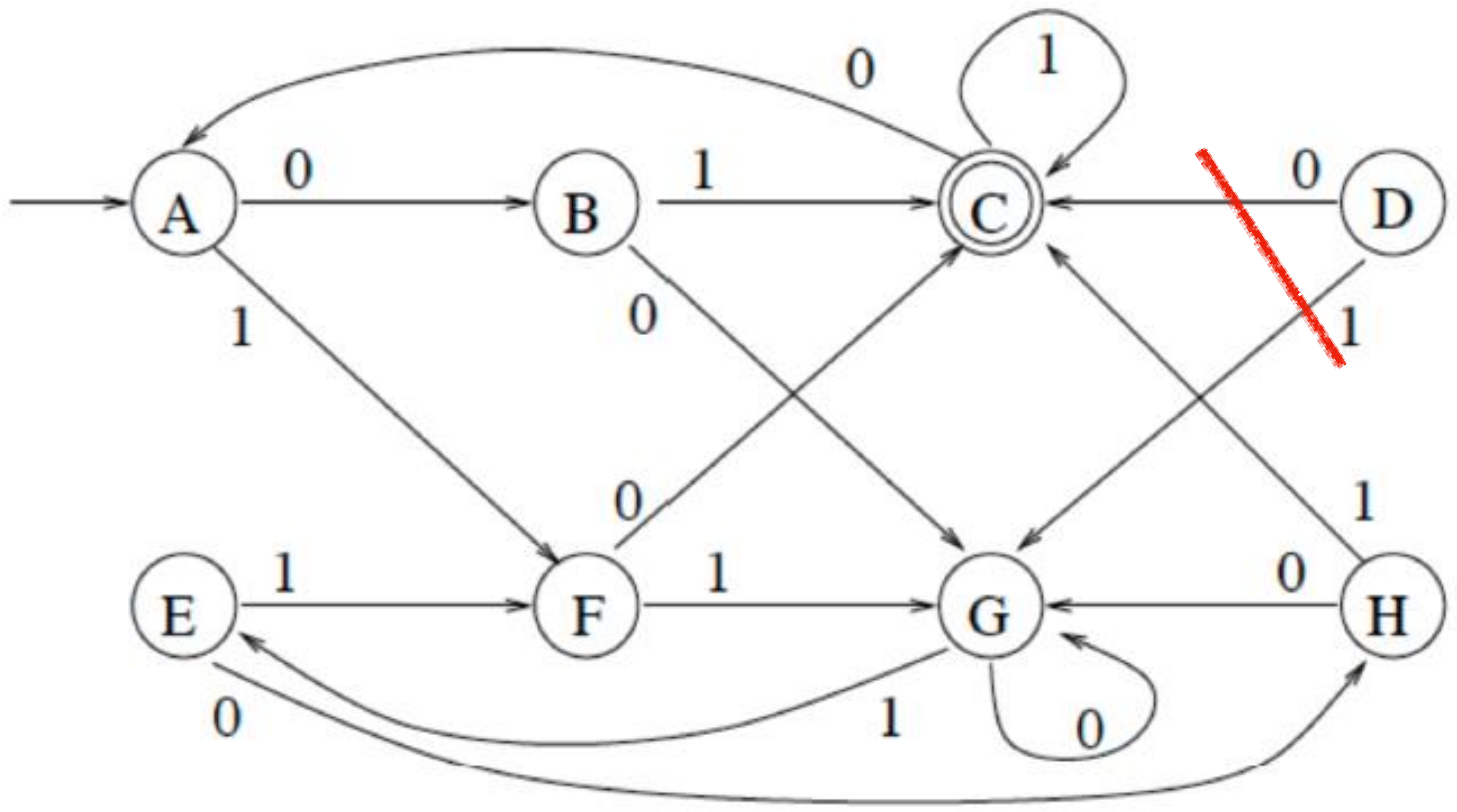
$$\hat{Q} = \{\{A,E\},\{B,H\},\{C\},\{F\},\{G\}\}$$

step 3: Construct the reduced automaton $\hat{B}$.

- determine the equivalence classes of the indistinguishability relation. For $q \in Q$

$$[q] = \{a \in Q \; : \; a,q \text{ not marked}\}$$

running example :



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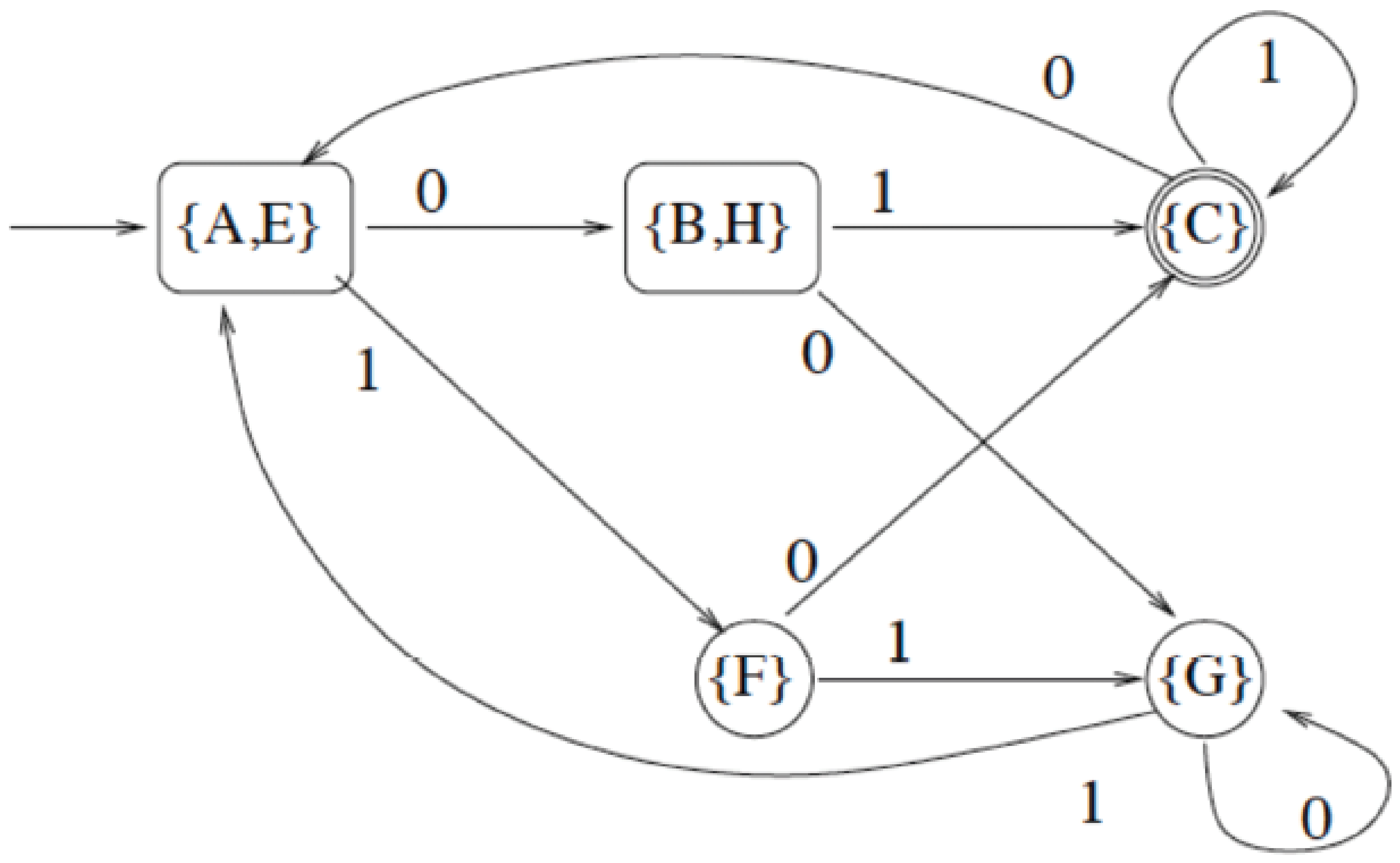

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- states

$$\hat{Q} = \{[q] \; : \; q \in Q\}$$

- initial state

$$\hat{q_0} = [q_0]$$

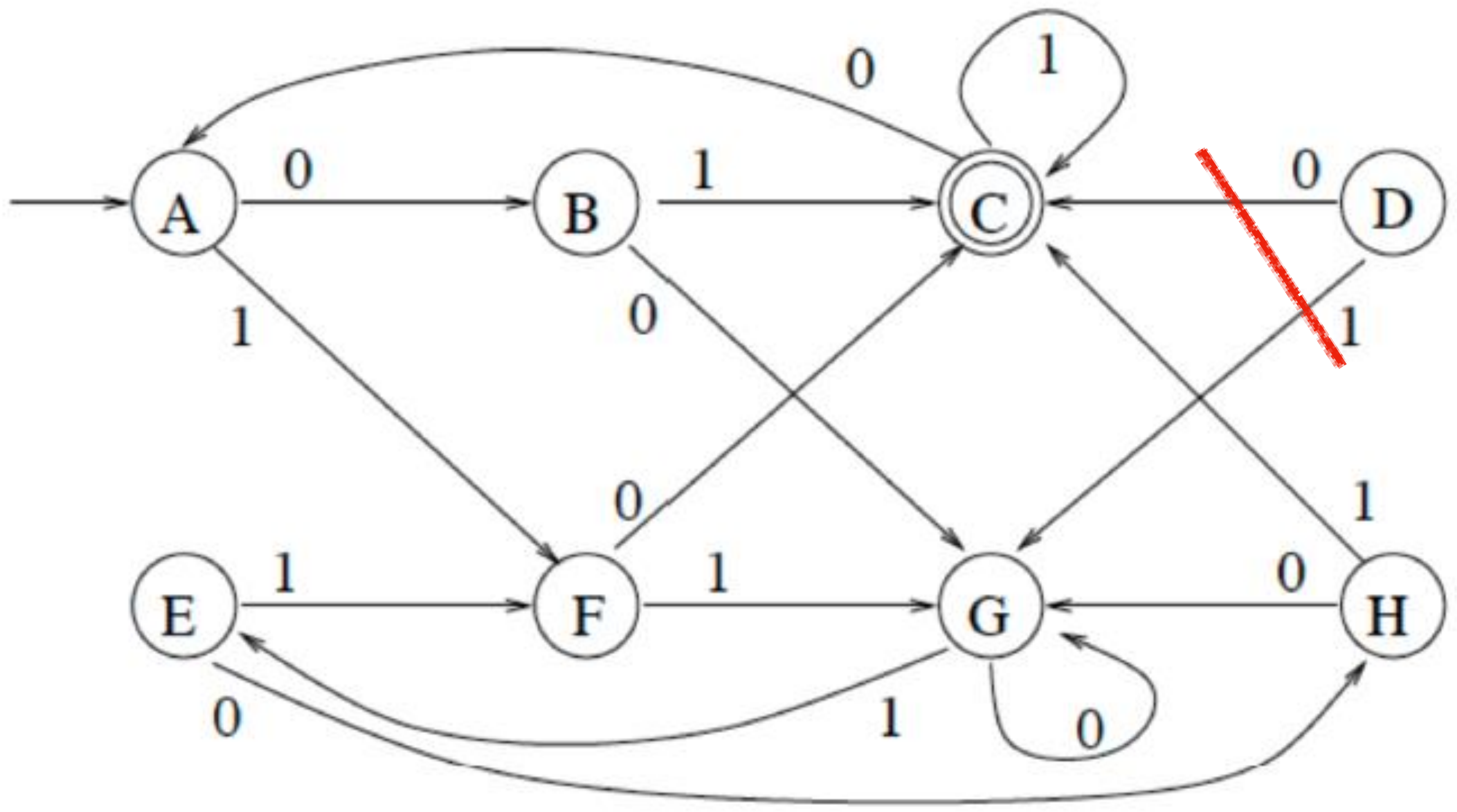
- final states

$$\hat{F} = \{[q] \; : \; q \in F\}$$

- transition function

$$\hat{\delta}([q],a) = [\delta(q,a)]$$

running example :



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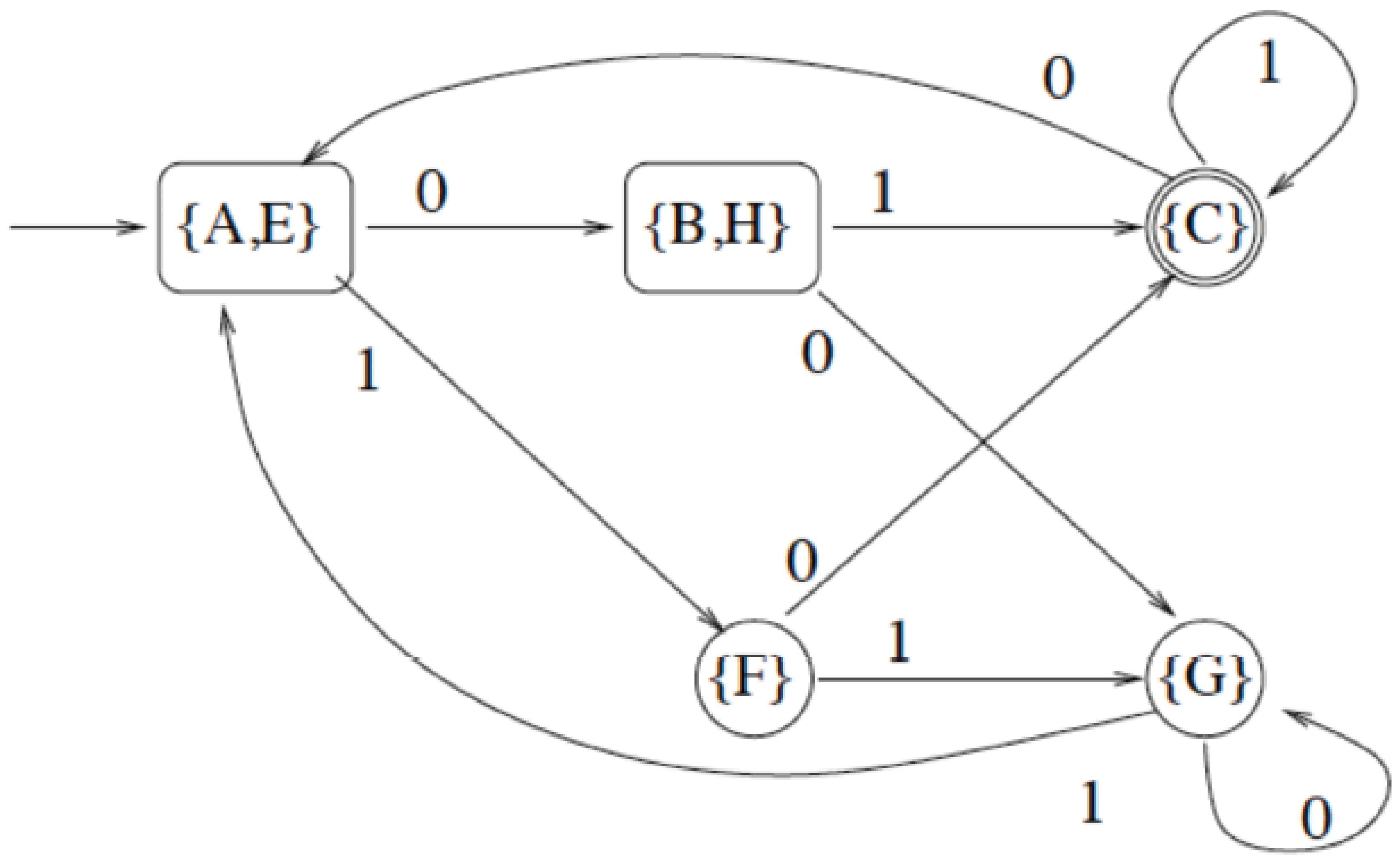


Lemma 5. *$\hat{B}$ is well defined*

Claim: definitions of $\hat{F}$ and $\hat{\delta}$ are independent of choice of $q \in [q]$

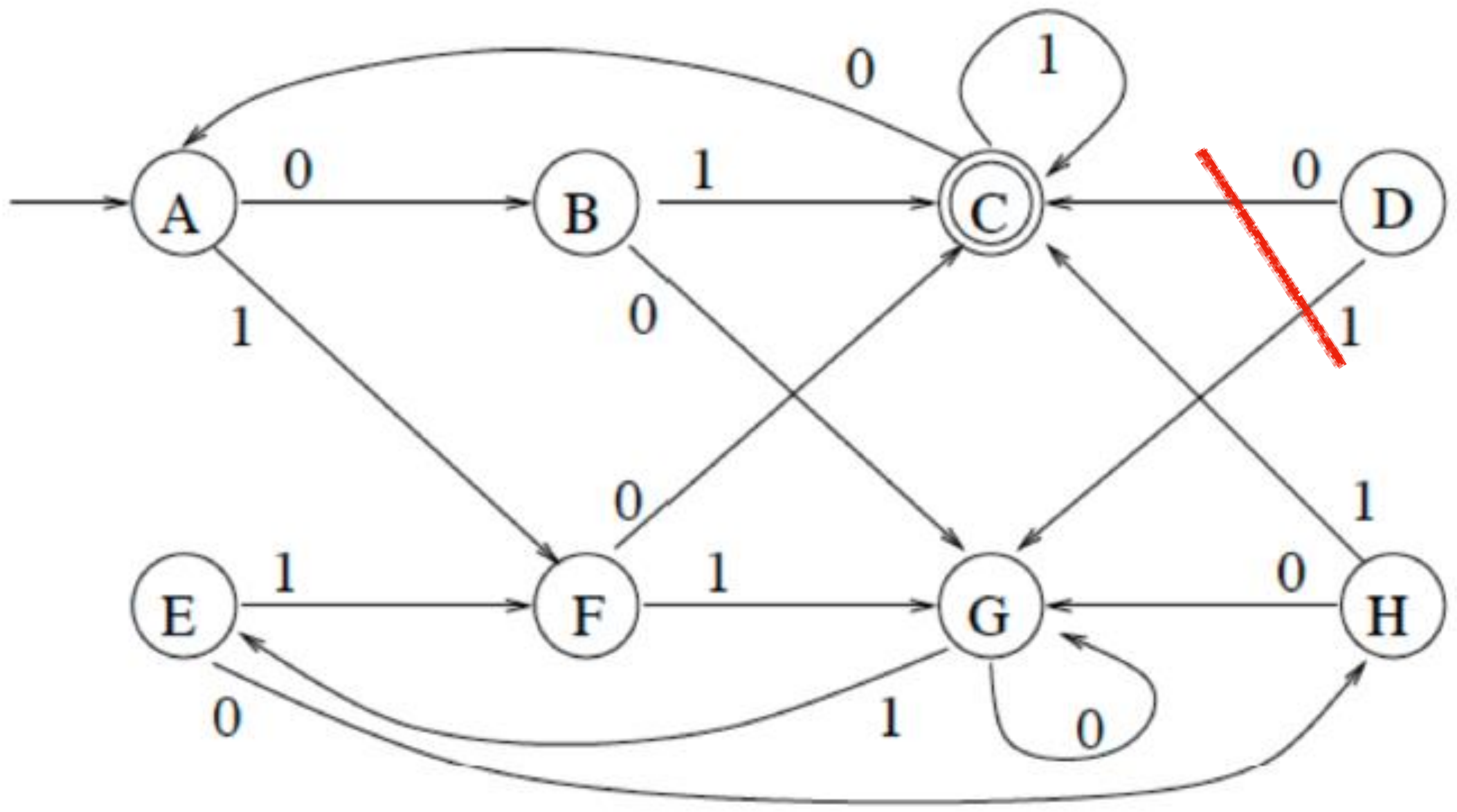
- $\hat{F}$: Let $q \in F$ and $p \notin F$. Then p, q marked in initial phase and stay marked. Hence

$$p \notin [q]$$

- $\hat{\delta}$ Let $p \in [q]$. As a does not distinguish p and q:

$$\delta(p,a) = \delta(q,a) \quad , \quad [\delta(p,a)] = [\delta(q,a)]$$

running example :



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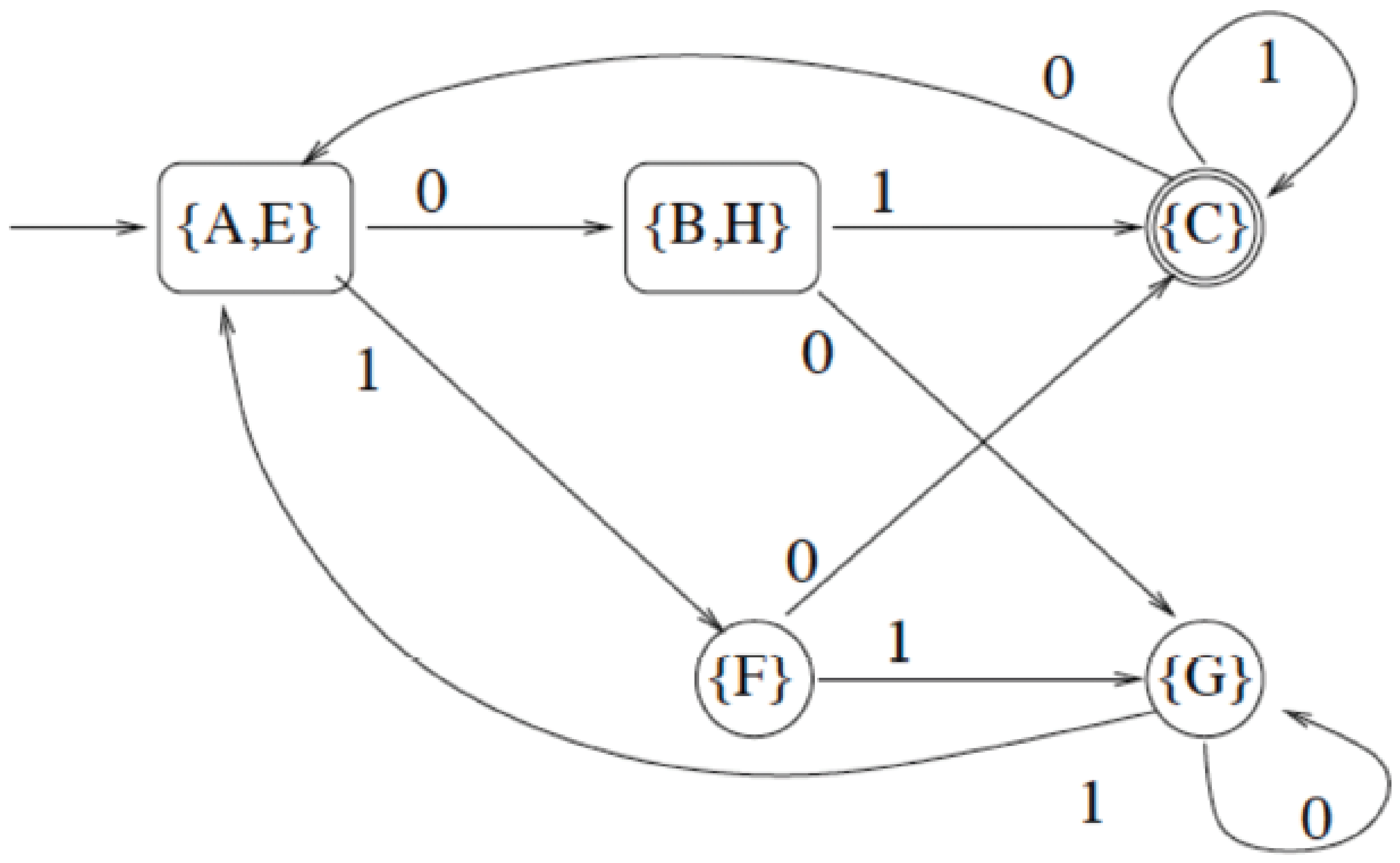

Lemma 6.

$$L(B) = L(\hat{B})$$

- Claim: for all w

$$\delta^*(q_0, w) \in \hat{\delta}^*(\hat{q}_0, w)$$

  Proof: induction on$|w|$.

-

$$\delta^*(q_0, w) \in F \ \rightarrow \hat{\delta}^*(\hat{q}_0, w) = [\delta^*(q_0, w)] \in \hat{F}$$

$$\delta^*(q_0, w) \notin F \ \rightarrow \hat{\delta}^*(\hat{q}_0, w) = [\delta^*(q_0, w)] \notin \hat{F}$$

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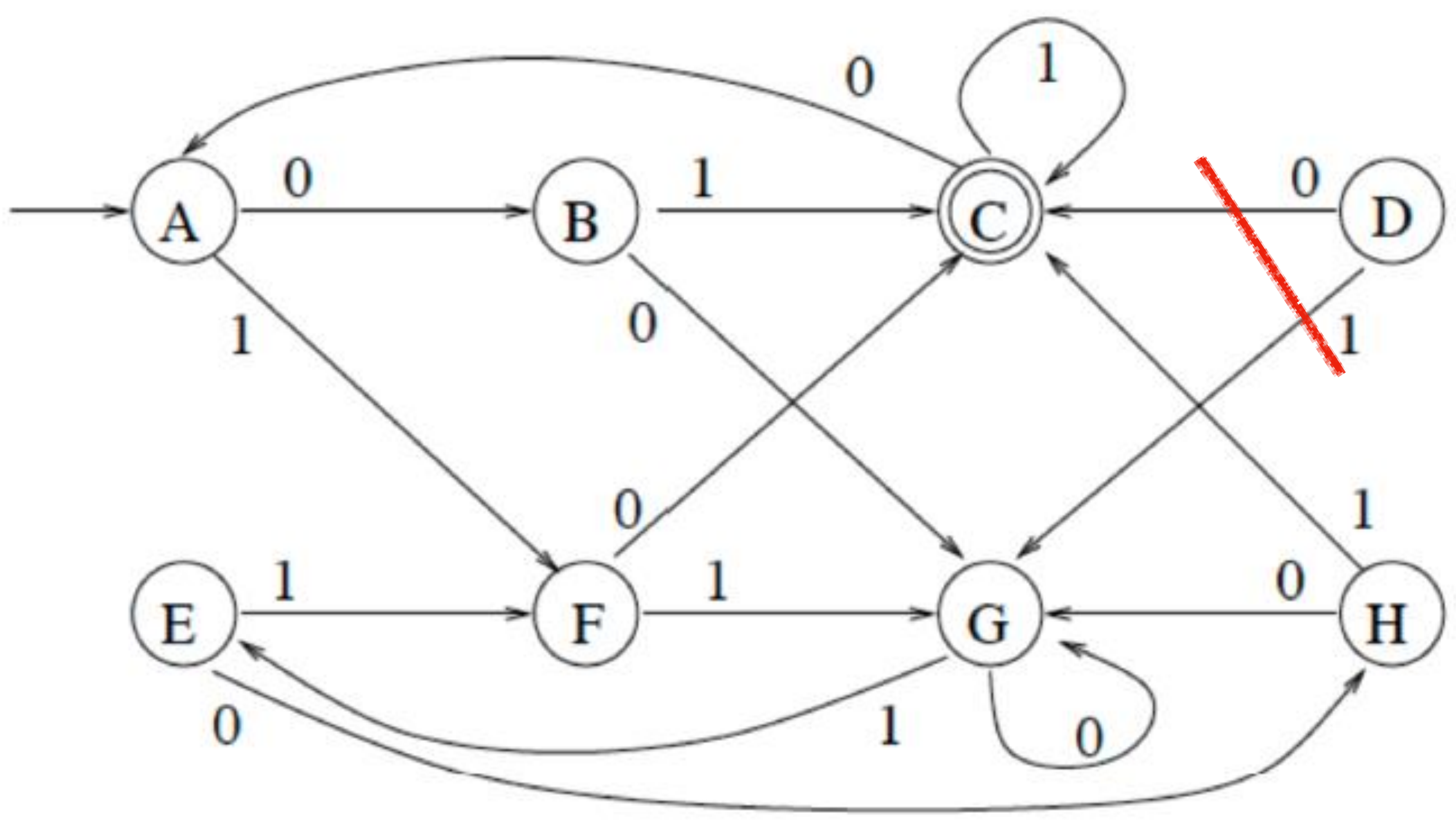

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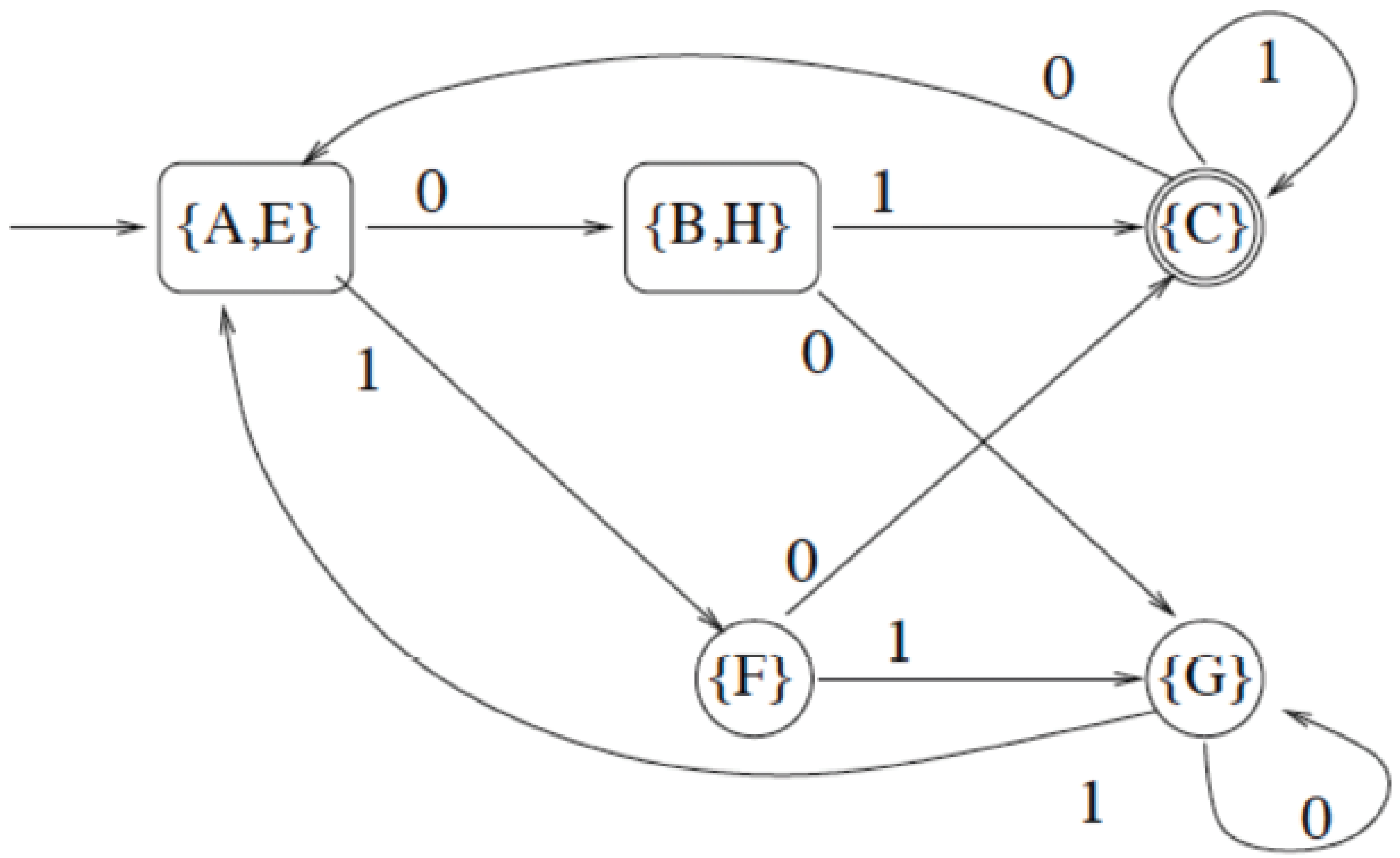

Lemma 7. *$\hat{B}$ is minimal.*

- states of $\hat{B}$ are distinguishable.
- apply lemma 1