CS 330 : Discrete Computational Structures
Spring Semester, 2014
ASSIGNMENT #7
Due Date: Tuesday, Mar 25

Suggested Reading: Rosen Section 5.3; LLM Chapter 6.1 - 6.3

These are the problems that you need to turn in. For more practice, you are encouraged to work on the other problems. **Always explain your answers and show your reasoning.**

1. **[16 Pts]** Let S defined recursively by (1) $6 \in S$ and (2) if $s \in S$ and $t \in S$, then $s + t \in S$. Let A be the set of positive integers divisible by 6. Prove that

   (a) **[8 Pts]** $A \subseteq S$ by mathematical induction.

   (b) **[8 Pts]** $S \subseteq A$ by structural induction.

   Solution:
   From the definition of A, we can get that: $A = \{6x | x \in Z^+\}$.
   (a) Prove by MI: $\forall x \in Z^+, 6x \in S$.
   In the base case: $k = 1$, $6 \times 1 = 6 \in S$ *by the basis of induction definition of* S. *So it is true for the base case.*
   Inductive: Assume that $6k \in S$ *for* $k \in N$, *prove* $6(k+1) \in S$.
   We already have $6 \in S$ *from the base case, and* $6k \in S$ *from the inductive hypothesis, then we can get* $6(k+1) = 6k + 6 \in S$ *according to the recursive definition of* S. *So the inductive step is true.*
   (b) By the base case, we know 6 is in S, and 6 is also divisible by 6. Hence by the definition of A, $6 \in A$.
   Inductive: consider $a, b \in S$. *By IH, assume that* $a, b \in A$. *Now, by inductive step of inductive definition of S,* $a + b \in S$. *We prove that* $a + b \in A$. *Since* $a, b \in A$, $a = 6x$ *and* $b = 6y$ *for* $x, y \in Z^+$. *So,* $a + b = 6x + 6y = 6(x + y)$, *where* $x + y \in Z^+$. *So* $a + b \in A$, *as required.*

2. **[30 Pts]** Let S be defined by (1) $(0,0) \in S$, and (2) if $(a,b) \in S$, then $(a+1, b+3) \in S$ and $(a+3, b+1) \in S$.

   (a) **[6 Pts]** List the elements in S produced by the first five applications of the inductive step of the definition. The basis step produces $(0,0)$.

   First application $(1,3), (3,1)$
   Second application $(2,6), (4,4), (6,2)$
   Third application $(3,9), (5,7), (7,5), (9,3)$
   Fourth application $(4,12), (6,10), (8,8), (10,6), (12,4)$
   Fifth application $(5,15), (7,13), (9,11), (11,9), (13,7), (15,5)$

(b) **[8 Pts]** Use strong induction on the number of applications of the inductive step to prove that if $(a, b) \in S$ then 4 divides $a + b$. State your inductive hypothesis.

Let $P(n)$ be the statement that "4 divides $a + b$ whenever (a, b) is defined to be in S by n applications of the recursive step in the definition of S". Base Case: $P(0)$ is true, because only $(0, 0)$ can be obtained by applying recursion zero times, and 4 divides $0 + 0 = 0$. We now assume the strong inductive hypothesis that 4 divides $a + b$ whenever (a, b) is defined to be in S by k or fewer applications of the recursive step in the definition. Let (x, y) be an element defined to be in S by $k + 1$ applications of the recursive step. Let (a, b) be the element to which the recursive step is applied in order to produce (x, y). Then either $x = a + 1$ and $y = b + 3$, or $x = a + 3$ and $y = b + 1$. Then $x + y = a + b + 1 + 3 = a + b + 4$. By the inductive hypothesis, we may assume that 4 divides $a + b$, and 4 divides 4, so 4 divides $a + b + 4$.

(c) **[8 Pts]** Now, use structural induction to prove that if $(a, b) \in S$ then 4 divides $a + b$.

This holds for the basis step, because 4 divides $0 + 0$. If 4 divides (a, b), then 4 also divides $(a + 1, b + 3)$ and $(a + 3, b + 1)$, the elements obtained from (a, b) in the recursive step, by the same argument as in the previous part.

(d) **[8 Pts]** Disprove the converse of the statement above, *i.e.*, show that if $a, b \in \mathcal{N}$, and $a + b$ is divisible by 4, it does not follow that $(a, b) \in S$. Modify the recursive definition of S to make the converse true.

The converse is not true, since it is clear that 4 divides $4 + 0 = 0 + 4 = 4$, but $(4, 0) \notin S$, $(0, 4) \notin S$.
The definition of S should be modified as follows:
(1) $(0, 0) \in S$.
(2) if $(a, b) \in S$, then $(a, b + 4) \in S$, $(a + 1, b + 3) \in S$, $(a + 2, b + 2) \in S$, $(a + 3, b + 1) \in S$, $(a + 4, b) \in S$.

3. **[8 Pts]** Rosen, Section 5.3: Exercise 44

Solution:
For the basis step we have the tree consisting of just the root, so there is one leaf and there are no internal vertices, and $l(T) = i(T) + 1$ holds. For the recursive step, assume that this relationship holds for T_1 and T_2, and consider the tree with a new root, whose children are the roots of T_1 and T_2. The new root is an internal vertex of T, and every internal vertex in T_1 or T_2 is an internal vertex of T, so $i(T) = i(T_1) + i(T_2) + 1$. Similarly, the leaves of T_1 and T_2 are the leaves of T, so $l(T) = l(T_1) + l(T_2)$. Thus we have $l(T) = l(T_1) + l(T_2) = i(T_1) + 1 + i(T_2) + 1$ by the inductive hypothesis, which equals $(i(T_1) + i(T_2) + 1) + 1 = i(T) + 1$, as desired.

4. **[8 Pts]** Give an inductive definition for the set of all palindromes over the alphabet $\{a, b, c\}$.

Solution: We will define a set P of strings, which will be the set of palindromes. First, the empty string $\lambda \in P$. Second, we place each length-one string in P: $a \in P$, $b \in P$,

$c \in P$. Then, for the recursive step: if $x \in P$ then $axa \in P$, $bxb \in P$ and $cxc \in P$. Everything defined is a palindrome, because the empty string and single character strings are palindromes, and, if x is a palindrome, putting the same character at the start and end of x produces another palindrome. Further, if p is a palindrome, it falls under this definition, since either p is one of the base cases, or the length of p is at least 2, and it has the same first and last character, so it will be built by some application of the recursive step.

5. [**12 Pts**] LLM Problem 6.4 (a) [**6 Pts**] (b) [**6 Pts**]

   Solution:

   (a) $1 \in S$; *and, if* $a \in S$ *then* $2a \in S$ *and* $3a \in S$ *and* $5a \in S$.

   (b) $1 \in T$; *and, if* $a \in T$ *then* $2 \cdot 3^2 a \in T$ *and* $3 \cdot 5a \in T$ *and* $5a \in T$. *(This is of course the same as: if* $a \in T$ *then* $18a \in T$ *and* $15a \in T$ *and* $5a \in T$.*)*

6. [**26 Pts**]

   (a) [**8 Pts**] Give an inductive definition of the set $L = \{(a,b) \mid a,b \in \mathcal{Z}, (a+b) \bmod 3 = 0\}$. Let L' be the set obtained by your inductive definition.

   (b) [**9 Pts**] Prove that $L' \subseteq L$.

   (c) [**9 Pts**] Prove that $L \subseteq L'$.

   Solution:

   (a) *Base:* $(0,0) \in L'$. *Recursive: if* $(x,y) \in L'$ *then* $(x+1, y-1) \in L'$ *and* $(x-1, y+1) \in L'$ *and* $(x, y+3) \in L'$, *and* $(x, y-3) \in L'$.

   (b) $L' \subseteq L$ *means that every ordered pair* (a,b) *produced by the inductive definition of* L' *has the property that* $a+b$ *is divisible by 3. This holds for the base case of the definition, as* $0+0 = 3(0)$. *Now let's make sure that we always stay inside* L *when we are applying the recursive step. We start with ordered pair* (a,b) *such that* $a+b$ *is divisible by 3. So,* $a+b = 3k$ *for some integer* k. *The recursive step allows us to build* $(a+1, b-1)$, $(a-1, b+1)$, $(a, b+3)$ *and* $(a, b-3)$. *Now,* $(a+1)+(b-1)$ *and* $(a-1)+(b+1)$ *both equal* $a+b$, *which is divisible by 3. Also,* $a+(b+3) = 3k+3 = 3(k+1)$, *so it is divisible by 3. Finally,* $a+(b-3) = 3k-3 = 3(k-1)$, *so it is divisible by 3. So in every case, the recursive step produces ordered pairs that satisfy membership in L.*

   (c) *if* $(a,b) \in L$ *then* $a+b = 3k$ *for some integer* k, *so* $b = 3k-a$. *So any element of L will have the form* $(a, 3k-a)$. *To reach all such ordered pairs using the inductive definition of* L', *we apply the following inductive steps:*

   (a) First, we move from $(0,0)$ *to* $(0,3k)$. *If* $k \geq 0$, *then apply* k *times the rule* $(x, y+3)$. *Otherwise, if* $k < 0$, *then apply* $-k$ *times the rule* $(x, y-3)$.

   (b) Next, we move from $(0,3k)$ *to* $(a, 3k-a)$. *If* $a \geq 0$, *then apply* a *times the rule* $(x+1, y-1)$. *Otherwise, if* $a < 0$, *then apply* $-a$ *times the rule* $(x-1, y+1)$.

   As we can reach any element of L using the rules of L', $L \subseteq L'$.

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7. **[Extra Credit]** Is your inductive definition ambiguous? In other words, are there two ways to generate the same element (a, b)? See if you can come up with an unambiguous definition for L.