CS 330 : Discrete Computational Structures
Fall Semester, 2015
ASSIGNMENT #12 **[Extra Credit]**
Due Date: Friday, Dec 11

Suggested Reading: Chapter 11.1 - 11.3 and 11.9 - 11.11 of Lehman et al.

These are the problems that you need to turn in. Always explain your answers and show your reasoning. **Spend time giving a complete solution. You will be graded based on how well you explain your answers. Just correct answers will not be enough!**

1. **[12 Pts]** How many integers between 1000 and 9999 inclusive contain (a) at least one 0 and at least one 1, (b) at least one 0, at least one 1 and at least one 2? Solve part (a) using the Inclusion-Exclusion Principle for two sets, and part (b) using the Inclusion-Exclusion Principle for three sets: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

   a. Consider the complement: integers between 1000 and 9999 that contain no zeroes, or no ones. We can count this using inclusion-exclusion: let A be the set of integers between 1000 and 9999 that contain no zeroes and let B be the set of integers between 1000 and 9999 that contain no ones. Then $A \cap B$ is the set of integers between 1000 and 9999 that contain no zeroes or ones. By inclusion-exclusion, the total is $|A| + |B| - |A \cap B|$.
   A: for each digit we have 9 choices, so $|A| = 9^4$.
   B: for the first digit, we have 8 choices (if we choose 0, then we're no longer above 1000). For the other three digits, we have 9 choices, so $|B| = 8 * 9^3$.
   $A \cap B$: We have 8 choices for each digit, so $|A \cap B| = 8^4$.
   The size of the complement is thus $9^4 + 8 * 9^3 - 8^4 = 8297$. To get the size of the original, we need to subtract the complement from the size of the universe. There are 9000 integers between 1000 and 9999, so we get $9000 - 8297$ integers between 1000 and 9999 that contain at least one zero and at least one one.

   b. Same idea as last time, but now we have C: the set of integers between 1000 and 9999 that contain no 2s. We count the complement using inclusion-exclusion:
   $|A|, |B|, |A \cap B|$ remain the same.
   $|C| = 8 * 9^3$, for the same reason as $|B|$.
   $|A \cap C| = 8^4$ for the same reason as $|A \cap B|$.
   $|B \cap C| = 7 * 8^3$, because we have 7 choices for the first digit (can't choose 0).
   $|A \cap B \cap C| = 7^4$, because we have 7 choices for each digit.
   Adding it up, we get the size of the complement is $9^4+8*9^3+8*9^3-8^4-8^4-7*8^3+7^4 = 8850$. Subtracting from 9000, we get 150 numbers between 1000 and 9999 that contain at least one 0, at least one 1, and at least one 2.

2. **[10 Pts]** If G is a simple graph with n vertices and n edges, is G connected? If *yes*, give a short justification. If *no*, give a counterexample.

   Solution:
   No. The graph with 5 vertices $\{a, b, c, d, e\}$ *and 5 edges* $\{(a, b), (b, c), (a, c), (c, d), (a, d)\}$ *is not connected.*

3. **[14 Pts]** Prove by induction that a complete binary tree of height h has 2^h leaves. Use the inductive definition of complete binary trees.

Solution:
Base Case: $H = 0$. *A binary tree of height 0 is just a single node with no children, and therefore has 1 leaf.* $1 = 2^0$, *so the base case satisfies the induction hypothesis (see below).*
Induction Hypothesis: Suppose that for some $k >= 0$, *all binary trees of height* $h = k$ *have* 2^k *leaves.*
Induction Step: Let T *be a binary tree of height* $h = k + 1$. *Then* T*'s left and right subtrees are each binary trees of height* $= k$, *and thus by the I.H. have* 2^k *leaves. The number of leaves in* T *is equal to the sum of the number of leaves in* T*'s subtrees, which must be equal to* $2^k + 2^k = 2^(k + 1)$. *Hence the hypothesis holds for* $k + 1$, *and so it's proved.*

4. **[14 Pts]** Suppose that a graph $G = (V, E)$ is connected and $|E| = |V| - 1$. Prove that G is acyclic.

   Suppose G is not acyclic. Then it has at least one cycle. Remove an edge from a cycle; if there are still cycles, repeat. The result is a connected, acyclic graph, which is by definition a tree. A tree with $|V|$ vertices has $|V| - 1$ edges. But the original graph had $|V| - 1$ vertices, and we removed some. This is a contradiction, so G must be acyclic.