CS 330 : Discrete Computational Structures
Spring Semester, 2014
ASSIGNMENT #11 SOLUTIONS **[Extra Credit]**
Due Date: Tuesday, Apr 29

Suggested Reading: Rosen Sections 6.4 - 6.5

These are the problems that you need to turn in. Always explain your answers and show your reasoning. **Spend time giving a complete solution. You will be graded based on how well you explain your answers. Just correct answers will not be enough!**

1. **[8 Pts Zhenbi]** Prove that $P(n,2)C(n-2,k-2) = C(n,k)P(k,2)$, where $n \geq k \geq 2$, by using a combinatorial proof.

   Solution:

   One way of thinking of this problem is that we have n people and we want to choose a k-person committee that has 1 chair and 1 vice chair. So the left-hand-side of the equals sign, $P(n,2)C(n-2,k-2)$, can be interpreted as: count the number of possibilities if we first choose the 1 chair and 1 vice chair, and then choose $k-2$ committee members from the $n-2$ people remaining. The right-hand-side of the equals sign, $C(n,k)P(k,2)$ can be interpreted as: count the number of possibilities if we first choose k committee members, and then choose 1 chair and 1 vice chair from those k committee members. Since the end result is the same, a count of the possible k-person committees with 1 chair and 1 vice chair, the two sides of the equals sign are, indeed, equal.

2. **[10 Pts Zhenbi]** Rosen, Section 6.4: Exercise 28

   Solution:

   (a) Suppose there is a group of n men and n women. Then, of course, there are $C(2n,2)$ ways to pick two people from the group. That's the left hand side of the equals sign. We can also evaluate the number of possibilities as follows. If we pick two people, we are either picking 2 men, 2 women, or 1 man and 1 woman. There are $C(n,2)$ ways to pick two men, $C(n,2)$ ways to pick two women, and n^2 pairs of one man and one woman. All told, this is $2C(n,2)+n^2$ possible ways to choose 2 people from $2n$ people, so the two sides of the equals sign are, indeed, equal.

   (b) We expand out the formulas below.

$$\begin{aligned}
C(2n,2) &= \frac{(2n)!}{(2n-2)!2!} \\
&= \frac{2n(2n-1)}{2!} \\
&= n(2n-1) \\
2C(n,2)+n^2 &= \frac{2(n!)}{(n-2)!2!} + n^2 \\
&= n(n-1)+n^2 \\
&= n(2n-1)
\end{aligned}$$

   Thus, $C(2n,2) = 2C(n,2)+n^2$.

3. **[8 Pts Zhenbi]** Prove, using a combinatorial argument, that

$$C(2n, r) = \sum_{k=0}^{r} [C(n, k)C(n, r-k)].$$

Solution:
Suppose we have a group of n men and n women, and we want to pick a committee of size r where $r \leq n$. There are $C(2n, r)$ ways to choose this committee, which is the lhs of the equals sign. On the other hand, we can compute the answer to this problem piecewise, by breaking each committee down into male and female members. So the total number of committees is all the committees that have 0 men and r women, 1 man and $r-1$ women, and so on, until 0 men and r women. Since there are n men and n women, one way to express this sum is with the formula

$$C(n, 0)C(n, r) + C(n, 1)C(n, r-1) + \cdots + C(n, r)C(n, 0)$$

Since this is a direct sum, we can also write it in Sigma notation. We will sum over the number of men in the committee (so k will run from 0 men to r men): $\sum_{k=0}^{r} C(n, k)C(n, r-k)$. This is the right hand side of the equals sign, so we have just proved that the two sides are equal.

4. **[16 Pts Xiyuan]** A cookie shop sells 5 different kinds of cookies. How many different ways are there to choose 24 cookies if (a) you have no restrictions? (b) you pick at least three of each? (c) you pick at least 5 chocolate chip cookies and exactly 3 oatmeal cookies? (d) you pick at least 5 oatmeal cookies and at most 4 sugar cookies?

Solution:
(a) This is a "combinations with repetitions" problem, where we pick 24 objects with repetitions from a set of 5 elements. So, $C(5+24-1, 24) = C(28, 24) = 20475$.

(b) If we first pick three of each kind, then we have picked $3 \times 5 = 15$ cookies. This leaves 9 cookies left to pick without restriction, so $C(5+9-1, 9) = C(13, 9) = 715$.

(c) 8 cookies are specified (3 oatmeal cookies and 5 chocolate chip cookies). Moreover, no more oatmeal cookies can be picked. So the problem is the same as choosing $24 - 8 = 16$ cookies without restriction from 4 types of cookies, which can be done in $C(4+16-1, 16) = C(19, 16) = 969$.

(d) If we choose the required 5 oatmeal cookies, then there are $24 - 5 = 19$ left to choose, and if there were no restriction on the sugar cookies then there would be $C(5+19-1, 19) = C(23, 19) = 8855$ ways to make the selections. If in addition we were to violate the sugar restriction by choosing at least five sugar cookies, there would be $C(5+14-1, 14) = C(18, 14) = 3060$ choices. Therefore the number of ways to make the selection without violating the restriction is $8855 - 3060 = 5795$.

5. **[5 Pts Elliott]** How many ways are there to distribute 10 identical notebooks to 6 students?

Solution:
This is a "placing identical objects into distinguishable boxes" problem and is equivalent to picking 10 objects of 6 different types, i.e., combinations with repetitions.
Therefore, the answer is $C(10+6-1, 10) = C(15, 10) = 3003$.

6. **[5 Pts Elliott]** If I have 5 bananas, 3 oranges, and 8 apples, how many ways can I distribute these to 16 friends, if each friend gets one fruit?

Solution:
In this problem, the 5 bananas, 3 oranges, and 8 apples are each sets of indistinguishable objects. The 16 friends, however, are all distinct.
If all 16 fruit were distinct, then the answer would be 16! since there would be 16 choices for the first friend, 15 choises for the next, and so on. Since some of the fruit are identical, however, we divide by $5!3!8!$*, and therefore our answer is* $\frac{16!}{5!3!8!}$.

7. **[12 Pts Xiang]** How many ways are there to pack 18 different books into 6 boxes with 3 books each if (a) all 6 boxes are sent to different addresses? (b) all 6 boxes are sent to the same address? (c) 3 of the boxes are shipped to three different addresses while 3 are left to be addressed later?

Solution:
(a)That means we distinguish these 6 boxes. So we get $\binom{18}{3} \times \binom{15}{3} \times \binom{12}{3} \times \binom{9}{3} \times \binom{6}{3}$. *That is: we pick 3 books from 18 books for the first box. Then 3 books from the rest books for the second box and so on.*

(b)That means we do not distinguish these 6 boxes. So what we do in (a) will count every possible case for $6!$ *times. So the answer for (b) is* $(\binom{18}{3} \times \binom{15}{3} \times \binom{12}{3} \times \binom{9}{3} \times \binom{6}{3})/6!$

(c) We will distinguish the three to be shipped boxes but treat the unaddressed boxed as the same. That is: $(\binom{18}{3} \times \binom{15}{3} \times \binom{12}{3} \times \binom{9}{3} \times \binom{6}{3})/3!$

8. **[12 Pts Xiang]** How many ways are there to pack 5 identical books into 5 identical boxes with no restrictions placed on how many can go in a box (some boxes can be empty)? What if the books are different?

Solution:
(a)The problem is equivalent to "how many ways do you have to split the number 5 positive integers"? The list: 5= 5 =1+4= 2+3 =1+1+3=1+2+2 =1+1+1+2=1+1+1+1+1. You have 7 ways to split 5.
Comment: This kind of problem is known as "Integer Partition problem" in number theory, which counts the number of partitions of an nature number.

(b)If the books are different:
For $5 = 5$. *We can have only 1 way to put the books. That is, put all of the 5 books into one box.*
For $5 = 1 + 4$. *We can have* $\binom{5}{1} = 5$ *ways to put the books. That is, pick 1 out if 5 books into one box and then put the rest 4 books into another box.*
For $5 = 2 + 3$. *We can have* $\binom{5}{2} = 10$ *ways to put the books. That is, pick 2 out if 5 books into one box and then put the rest 3 books into another box.*
For $5 = 1 + 1 + 3$. *We can have* $(\binom{5}{3} \times \binom{2}{1})/2! = 10$ *ways to put the books. That is, pick 3 out if 5 books into one box and then pick 1 book from the rest two book into another box. The last book will be put into a third box. Dividing by* $2!$ *is because we do not distinguish the 2 boxes with 1 book.*
For $5 = 1 + 2 + 2$. *We can have* $(\binom{5}{2} \times \binom{3}{2})/2! = 15$ *ways to put the books. Analysis similar to the above case.*
For $5 = 1+1+1+2$. *We can have* $(\binom{5}{2} \times \binom{3}{1} \times \binom{2}{1})/3! = 10$ *ways to put the books. Two books*

in one box. The rest 3 books in one box each. Dividing by 3! *is because we do not distinguish the 3 boxes with 1 book.*
For $5 = 1+1+1+1+1$. *We have to put each book in one box and that is the only way to do it.*
Sum all above up, we get $thetotalnumber = 1+5+10+10+15+10+1 = 52$.
Comment: This kind of number is known as "Bell Number", which counts the number of partitions of a set. Here we compute the 5th Bell number.

9. **[12 Pts Aaron]** How many solutions are there to the equation $x_1+x_2+x_3+x_4+x_5 = 24$, where, for all i, x_i is a non-negative integer, if (a) $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$, $x_5 \geq 5$? (b) $x_1 \geq 3$ and $x_3 \geq 3$? (c) $x_1 \geq 3$ and $x_3 < 4$?

   Solution:
   (a) $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$, $x_5 \geq 5$*?*
   This is the same as having 24 indistinguishable balls to place into 5 distinguishable bins, with certain requirements that some balls go into certain bins. At least one ball goes into bin x_1*, at least 2 balls go into bin* x_2*, and so on. In sum, 15 of the balls are already assigned locations in the five bins. So the problem simplifies to: how many ways can one place* $24-15=9$ *indistinguishable balls into 5 distinguishable bins. The answer is* $\binom{9+5-1}{9} = \binom{13}{9} = 715$.

   (b) $x_1 \geq 3$ *and* $x_3 \geq 3$*?*
   We solve this problem in the same way as we solved (a). The conditions $x_1 \geq 3$ *and* $x_3 \geq 3$ *together require that 6 of the indistinguishable balls be assigned to those bins. This leaves* $24-6=18$ *indistinguishable balls to distribute among 5 distinguishable locations. So the formula is* $\binom{18+5-1}{18} = \binom{22}{18} = 7315$.

   (c) $x_1 \geq 3$ *and* $x_3 < 4$*?*
   To solve this problem, we split it into two parts. First, to count the number of solutions where $x_1 \geq 3$*, we note that it is the same as placing* $24-3=21$ *indistinguishable balls into 5 distinguishable bins, which can be calculated by* $\binom{21+5-1}{21} = \binom{25}{21} = 12,650$. *Second, to calculate the number of choices where* $x_3 < 4$*, we take the total number of choices where* $x_1 \geq 3$ *and subtract the number of choices where* $x_1 \geq 3$ *and* $x_3 \geq 4$. *This, then, is* $\binom{25}{21} - \binom{(24-17)+5-1}{17} = \binom{25}{21} - \binom{21}{17} = 12,650 - 5985 = 6665$.

10. **[12 Pts Aaron]** How many ways can we place 10 books on a bookcase with 3 shelves if the books are (a) indistinguishable copies (b) all distinct?

    Solution:
    (a) We use the "stars and bars" counting method, with 10 books and two bars, because two bars are enough to divide the books into three separate areas. So the answer is $\binom{10+3-1}{10} = \binom{12}{10} = 66$.

    (b) Order all the books first, then count the number of ways they can be put on the shelves in that order. The number of ways to order the books is $10!$*, while the number of ways a single ordering can be put on the shelves is the answer to (a). So the answer to (b) is* $10!\binom{12}{10} = 239,500,800$.