CS 330 : Discrete Computational Structures

Spring Semester, 2014

ASSIGNMENT #5 SOLUTIONS

Due Date: Tuesday, Mar 4

Suggested Reading: Rosen Section 5.1 - 5.2; Lehman et al. Chapter 5.1 - 5.3

These are the problems that you need to turn in. For more practice, you are encouraged to work on the other problems. **Always explain your answers and show your reasoning.**

1. [**50 Pts**] Prove the following statements by mathematical induction. Clearly state your basis step and prove it. What is your inductive hypothesis? Prove the inductive step and show clearly where you used the inductive hypothesis.

   (a) [**Zhenbi Hu**] $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2))^2$, for all positive integers n.
   Solution:
   Basis step: $P(1) : 1^3 = (1(1+1)/2)^2$.
   $1^3 = 1$ *and* $(1(1+1)/2)^2 = (2/2)^2 = 1$. *So* $P(1)$ *is true.*
   Inductive step:
   Let $P(k)$ *holds true, which is:*
   $P(k) : 1^3 + 2^3 + \cdots + k^3 = (k(k+1)/2)^2$.
   To prove the inductive step we need to show that the implication $P(k) \to P(k+1)$ *is true.*
   $P(k+1) : 1^3 + 2^3 + \cdots + k^3 = ((k+1)((k+1)+2)/2)^2$.

   $$\begin{aligned}(1^3 + 2^3 + \cdots + k^3) + (k+1)^3 &= (k(k+1)/2)^2 + (k+1)^3, \textit{ by inductive hyp.}\\ &= (k+1)^2(k/2)^2 + (k+1)^2(k+1)\\ &= (k+1)^2(k^2/4 + k + 1)\\ &= (k+1)^2(k^2 + 4k + 4)/4\\ &= ((k+1)(k+2)/2)^2\end{aligned}$$

   Therefore, $P(k+1)$ *is true.*
   We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n.

   (b) [**Zhenbi Hu**] $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$, for all positive integers n.
   Solution:
   Basis step: $P(1) : 1 \cdot 1! = (1+1)! - 1$.
   $1 \cdot 1! = 1$ *and* $(1+1)! - 1 = 1$. *So* $P(1)$ *is true.*
   Inductive step:
   We assume that $P(k)$ *is true, where*
   $P(k) : 1 \cdot 1! + \cdots k \cdot k! = (k+1)! - 1$.

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We prove that $P(k+1)$ *is true, where*
$P(k+1) : 1 \cdot 1! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! = (k+2)! - 1.$

$$\begin{aligned} & 1 \cdot 1! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! \\ = & (k+1)! - 1 + (k+1) \cdot (k+1)!, \textit{ by inductive hypothesis} \\ = & (k+1)! \cdot (k+1+1) - 1 \\ = & (k+1)! \cdot (k+2) - 1 \\ = & (k+2)! - 1 \end{aligned}$$

Therefore, $P(k+1)$ *is true.*
We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n.

(c) **[Swagoto Roy]** $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = n(n+1)(n+2)(n+3)/4$, for all positive integers n.

Solution:
Basis step: $n = 1$ $P(1) : 1 \cdot 2 \cdot 3 = 1 \cdot 2 \cdot 3 \cdot 4/4$
$1 \cdot 2 \cdot 3 = 6$ *and* $1 \cdot 2 \cdot 3 \cdot 4/4 = 6$ *as well, so* $P(1)$ *holds.*
Inductive Step: Assume $P(k)$ *holds true, where*
$P(k) : 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k+1)(k+2) = k(k+1)(k+2)(k+3)/4$
To prove the inductive step we need to show that the implication $P(k) \to P(k+1)$ *is true. So, prove* $P(k+1)$ *is true, where*
$P(k+1) : 1 \cdot 2 \cdot 3 + \cdots + k(k+1)(k+2) + (k+1)(k+2)(k+3) = (k+1)(k+2)(k+3)(k+4)/4.$

$$\begin{aligned} & 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k+1)(k+2) + (k+1)(k+2)(k+3) \\ = & k(k+1)(k+2)(k+3)/4 + (k+1)(k+2)(k+3) \textit{ by inductive hypothesis} \\ = & (k+1)(k+2)(k+3)(k/4+1) \\ = & (k+1)(k+2)(k+3)(k+4)/4 \end{aligned}$$

We have completed both the basis step and the inductive step, so by the principle of mathematical induction $P(k) \to P(k+1)$, *the statement is true for every positive integer* k.

(d) **[Swagoto Roy]** $n! < n^n$, for all integers greater than 1.

Solution:
Base case: For $n = 2$, $P(2) : 2! < 2^2$.
$2! = 2$ *and* $2^2 = 4$. *Since* $2 < 4$, $P(2)$ *is true.*
Inductive step:
Suppose $P(k)$ *is true, where* $P(k) : k! < k^k$.

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Prove that $P(k+1)$ is true, where $P(k+1) : (k+1)! < (k+1)^{k+1}$.

$$\begin{aligned}(k+1)! &= (k+1)k! \\ &< (k+1)k^k, \textit{ by inductive hypothesis} \\ &< (k+1)(k+1)^k \textit{ since } k^k < (k+1)^k \\ &< (k+1)^{k+1}\end{aligned}$$

Therefore, $P(k+1)$ is true. We have completed both the basis step and the inductive step, so by the principle of mathematical induction $P(k) \to P(k+1)$, the statement is true for every positive integer k greater than 1.

(e) **[Swagoto Roy]** 6 divides $n^3 - n$, for all non-negative integers n. *Solution:*
Base case: $n = 0$, 6 divides $0^3 - 0 = 0$. True
Inductive step:
Assume $P(k)$: 6 divides $k^3 - k$, where k is a non-negative integer. We need to prove $P(k+1)$: 6 divides $(k+1)^3 - (k+1)$.

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= k^3 + 3k^2 + 2k \\ &= k^3 - k + 3k^2 + 2k + k \\ &= (k^3 - k) + (3k^2 + 3k) \\ &= (k^3 - k) + 3(k^2 + k) \\ &= (k^3 - k) + 3(k(k+1))\end{aligned}$$

Now from the inductive hypothesis, $(k^3 - k)$ must be divisible by 6. Therefore, $k^3 - k = 6\ell$, where ℓ is an integer.
It is also known the product $(k(k+1))$ is an even number since product of two consecutive integers is even. So let $(k(k+1)) = 2m$ where m is an integer.
Therefore,
$(k+1)^3 - (k+1) = (k^3 - k) + 3(k(k+1)) = 6\ell + 3(2m) = 6(\ell + m)$.
Therefore, $(k+1)^3 - (k+1)$ is divisible by 6, implying that $P(k+1)$ is true.
We have completed both the basis step and the inductive step, so by the principle of mathematical induction $P(k) \to P(k+1)$, the statement is true for every non-negative integer k.

2. **[10 Pts Zhenbi Hu]** Rosen, Section 5.1: Exercise 42

Solution: We solve this problem by performing an induction on the number of sets in the formula; or, in other words, on the size of n. First we consider the base case, then argue for an inductive step.

Base Case: $n = 1$ *When $n = 1$, the formula in question is $(A_1 - B) = (A_1) - B$, which is true, since it's the same set difference taken on both sides.*

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Inductive Step *Now suppose the formula is true with n-many "A-sets" appearing in it. We want to prove that it still holds with $n+1$-many such sets. We perform the following set-theoretic manipulations.*

$$\begin{aligned}
&(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \cap (A_{n+1} - B) \\
&= [(A_1 \cap A_2 \cap \cdots \cap A_n) - B] \cap (A_{n+1} - B) \text{ by inductive hypothesis} \\
&= [(A_1 \cap A_2 \cap \cdots \cap A_n) \cap \overline{B}] \cap (A_{n+1} \cap \overline{B}) \text{ by def of set difference} \\
&= (A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) \cap \overline{B} \cap \overline{B} \text{ by Associative laws} \\
&= (A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) \cap \overline{B} \text{ by Idemopotent laws} \\
&= (A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) - B \text{ by def of set difference}
\end{aligned}$$

Therefore, if the equality holds at n, it also holds at $n+1$.

Since the equality holds at the base case, and for an inductive step, it is true for all values of $n \geq 1$.

3. **[20 Pts Zhenbi Hu]** Let $P(n)$ be the statement that n-cent postage can be formed using just 4-cent and 7-cent stamps. Prove that $P(n)$ is true for all $n \geq 18$, using the steps below.

   (a) First, we prove $P(n)$ by regular induction. Prove (i) $P(18)$, and (ii) $P(k) \to P(k+1)$ for all $k \geq 18$.

   Solution: $P(18)$ is true because $2(7) + 4 = 18$, so two seven cent stamps with one four cent stamp make eighteen cents. So now assume $k \geq 18$, and $P(k)$ is true. We want to show $P(k+1)$ is true. If a 7-cent stamp is used to add up to k postage, we can take away that stamp and add two 4-cent stamps to produce $k-7+4\times 2 = k+1$ cents is stamps. On the other hand, if there is no 7-cent stamp used, then k must be a multiple of 4. Since $k \geq 18$, it must be the case that $k \geq 20$, so there are at least five 4-cent stamps used to add up to k. Then remove those five 4-cent stamps, and add three 7-cent stamps, to produce $k-5\times 4+3\times 7 = k+1$ using 4-cent and 7-cent stamps. In either case, $P(k+1)$ is true.

   (b) Now, we prove $P(n)$ by strong induction. Prove that $P(18)$, $P(19)$, $P(20)$ and $P(21)$ to complete the basis step.

   P(18) is true, because we can form 18 cents of postage with one 4-cent stamp and two 7-cent stamps.
   P(19) is true, because we can form 19 cents of postage with three 4-cent stamps and one 7-cent stamp.
   P(20) is true, because we can form 20 cents of postage with five 4-cent stamps.
   P(21) is true, because we can form 21 cents of postage with three 7-cent stamps.

   (c) For the inductive step, state clearly what you can assume and what you need to prove.

   The inductive hypothesis is the statement that using just 4-cent and 7-cent stamps we can form j cents postage for all j with $18 \leq j \leq k$, where we assume that $k \geq 21$.

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(d) Now, prove the inductive step and explain why this complete your proof.

We want to form $k+1$ cents of postage. Since $k-3 \geq 18$, we know that P(k-3) is true, that is, that we can form $k-3$ cents of postage. Put one more 4-cent stamp on the envelope, and we have formed $k+1$ cents of postage, as desired.

We have completed both the basis step and the inductive step, so by the principle of strong induction, the statement is true for every integer n greater than or equal to 18.

4. **[8 Pts Xiang]** Suppose $P(1)$ and $P(2)$ are true. Determine for what values of n, $P(n)$ is true if

   (a) for every positive integer k, if $P(k)$ is true then $P(k+3)$ is true.

   (b) for every positive integer k, if $P(k)$ is true then $P(k+2)$ is true.

   (a) *Solution:*
   Since $P(1)$ and $P(2)$ is true, and for every positive integer k if $P(k)$ is true then $P(k+3)$ is true, we know that for every $n = 1 + 3i$ or $n = 2 + 3i$, where $i \in \mathbb{N}$, $P(n)$ is true. Hence, $P(n)$ is true when $n \in \{1 + 3i | i \in \mathbb{N}\} \cup \{2 + 3i | i \in \mathbb{N}\}$, or $n \not\equiv 0 \pmod 3$.

   (b) *Solution:*
   Since $P(1)$ and $P(2)$ is true, and for every positive integer k if $P(k)$ is true then $P(k+2)$ is true, we know that for every $n = 1 + 2i$ or $n = 2 + 2i$, where $i \in \mathbb{N}$, $P(n)$ is true. Hence, $P(n)$ is true when $n \in \{1+2i | i \in \mathbb{N}\} \cup \{2+2i | i \in \mathbb{N}\} = \mathbb{Z}^+$.

5. **[12 Pts Xiang]** Suppose $P(n)$ is true for every positive integer n that is a power of 2. Also, suppose that $P(k+1) \to P(k)$ for all positive integers k. Now, prove that $P(n)$ is true for all positive integers.

   Solution: For any $n \in \mathbb{Z}^+$, there exist some i, such that $n \leq 2^i$. Since 2^i is a power of 2, so $P(2^i)$ is true by assumption. We now claim $P(k)$ is true, for every j such that $n \leq k < 2^i$. Suppose not, then there must be some k, $n \leq k < 2^i$, so that $P(k)$ is false. Now pick the largest of such k, we know that $P(k+1)$ must be true, or else $k+1 > k$ is a larger number than k that make $P(\bullet)$ false, which contradicts to the way we picked k. But then we will have $P(k+1)$ true, and $P(k)$ false, which contradicts to the fact that $P(k+1) \to P(k)$. Hence no such k ever exist. So $P(k)$ is true, for every k such that $n \leq k \leq 2^i$. So $P(n)$ is true.

6. **[Extra Credit]** Prove that $P(n)$ can be proven by strong induction if and only if it can be proven by regular induction. *Hint: If $P(n)$ can be proven by strong induction, strengthen the inductive hypothesis to prove it by regular induction.*

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