CS 330 : Discrete Computational Structures
Spring Semester, 2014
ASSIGNMENT #8 SOLUTIONS
Due Date: Tuesday, Apr 1

Suggested Reading: Rosen 9.1 and 9.5, LLM 9.4

These are the problems that you need to turn in. For more practice, you are encouraged to work on the other problems. **Always explain your answers and show your reasoning.**

1. **[40 Pts Xiyuan]** For each of these relations on the set of real numbers decide whether it is reflexive, anti-reflexive, symmetric, anti-symmetric and transitive. Justify your answers.

   (a) $(x, y) \in R_1$ if and only if $x + y = 5$
   Solution:

   not reflexive *because* $(1, 1) \notin R_1$.
   not antireflexive *because* $(2.5, 2.5) \in R_1$.
   symmetric *because* $x + y = y + x$, *so if* $(x, y) \in R_1$, *then* $(y, x) \in R_1$.
   not antisymmetric *because* $(5, 0) \in R_1$ *and* $(0, 5) \in R_1$.
   not transitive *because* $(1, 4) \in R_1$ *and* $(4, 1) \in R_1$, *but* $(1, 1) \notin R_1$.

   (b) $(x, y) \in R_2$ if and only if $x = 2y$
   Solution:

   not reflexive *because* $(1, 1) \notin R_2$.
   not antireflexive *because* $(0, 0) \in R_2$.
   not symmetric *because* $(2, 1) \in R_2$ *but* $(1, 2) \notin R_2$.
   antisymmetric *because* $x = 2y \Rightarrow y \neq 2x$ *except when* $x = y = 0$.
   not transitive *because* $(4, 2) \in R_2$ *and* $(2, 1) \in R_2$, *but* $(4, 1) \notin R_2$.

   (c) $(x, y) \in R_3$ if and only if $x = 1$ or $y = 1$
   Solution:

   not reflexive *because* $(2, 2) \notin R_3$.
   not antireflexive *because* $(1, 1) \in R_3$.
   symmetric *because if* $(x, y) \in R_3$, *then* $x = 1$ *or* $y = 1$,
   which implies $(y, x) \in R_3$.
   not antisymmetric *because* $(1, 4) \in R_3$ *and* $(4, 1) \in R_3$.
   not transitive *because* $(4, 1) \in R_3$ *and* $(1, 4) \in R_3$, *but* $(4, 4) \notin R_3$.

   (d) $R_4 = \mathcal{R} \times \mathcal{R}$
   Solution:

   reflexive *because* $(x, x) \in R_4$.
   not antireflexive *because* $(x, x) \in R_4$.

symmetric *because if* $(x, y) \in R_4$, *then* $(y, x) \in R_4$.
not antisymmetric *because* $(1, 2) \in R_4$, *and* $(2, 1) \in R_4$.
transitive *because if* $(x, y) \in R_4$, *and* $(x, y) \in R_4$, *then* $(x, z) \in R_4$.

2. [**18 Pts Zhenbi**] Prove that these relations on the set of all functions from $\mathcal{Z}$ to $\mathcal{Z}$ are equivalence relations. Describe the equivalence classes.

   (a) $\{(f, g) \mid f(0) = g(0) \text{ and } f(1) = g(1)\}$

   Solution: We show this relation on functions is reflexive, symmetric and transitive. This will prove it is an equivalence relation.

   Reflexive: *Since* $f(0) = f(0)$ *and* $f(1) = f(1)$ *for any function* f *from integers to integers,* (f, f) *is a member of the relation, for any* f, *so the relation is reflexive.*

   Symmetric: *Suppose* (f, g) *is in the relation. Then* $f(0) = g(0)$ *and* $f(1) = g(1)$. *Since equality is symmetric,* $g(0) = f(0)$ *and* $g(1) = f(1)$, *so* (g, f) *is in the relation also. This is true for arbitrary choice of* f, g *so the relation is symmetric.*

   Transitive: *If* (f, g) *and* (g, h) *are in the relation, then* $f(0) = g(0)$, $f(1) = g(1)$, $g(0) = h(0)$ *and* $g(1) = h(1)$. *Since equality is transitive,* $f(0) = h(0)$ *and* $f(1) = h(1)$. *Hence,* (f, h) *is in the relation for arbitrary* f, g, h *and the relation is transitive.*

   Now we consider the equivalence classes. Each equivalence class is naturally defined by an ordered pair (x, y), *and contains the functions* g *where* $g(0) = x$ *and* $g(1) = y$. *More formally, for each* $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ *there is an equivalence class:*

   $$\{g \mid g(0) = x \text{ and } g(1) = y\}$$

   (b) $\{(f, g) \mid \exists C \in \mathcal{Z}, \forall x \in \mathcal{Z}, f(x) - g(x) = C\}$

   As with (a), we prove this relation is reflexive, symmetric and transitive, which will prove it is an equivalence relation.

   Reflexive: *For any integer* a, $a - a = 0$. *So, for any function* f *from integers to integers, it is true that, for all* x, $f(x) - f(x) = 0$. *So for any* f, (f, f) *is in the relation, with* $C = 0$, *hence the relation is reflexive.*

   Symmetric: *Suppose, for any* f, g, (f, g) *is in the relation. Then there is a* C *such that, for all* x, $f(x) - g(x) = C$. *Then, for all* x, $g(x) - f(x) = -C$. *Therefore* (g, f) *is in the relation as well, and the relation is symmetric.*

   Transitive: *Suppose, for any* f, g, h, *that* (f, g) *and* (g, h) *are both in the relation. Then there are* C *and* C' *such that, for all* x, $f(x) - g(x) = C$ *and* $g(x) - h(x) = C'$. *Then, by simple algebra, for all* x, $(f(x) - g(x)) + (g(x) - h(x)) = C + C'$, *which simplifies to* $f(x) - h(x) = C + C'$. *Let* $D = C + C'$. *We have shown that, for all* x, $f(x) - h(x) = D$, *for some integer* D, *so* (f, h) *is in the relation, and therefore the relation is transitive.*

Now we consider the equivalence classes. The equivalence class for a function f is all the rigid translations of f along the y-axis. Phrased another way, start with the graph of f, and then move it up or down as far as you want, without moving it side-to-side or squishing it out of shape. This movement is a vertical rigid translation, and will give you the graph of another function g that is in $[f]$. In set builder notation, the equivalence class of any function f is:

$$[f] = \{g \mid \exists C \in \mathbb{Z}, \forall x \in \mathbb{Z}, f(x) - g(x) = C\}$$

3. [**18 Pts Zhenbi**] Show that these relations on the set of all functions from $\mathcal{Z}$ to $\mathcal{Z}$ are not equivalence relations. Argue which properties are not satisfied.

   (a) $\{(f, g) \mid f(0) = g(0) \text{ or } f(1) = g(1)\}$
   Solution: This is not an equivalence relation, lacking in transitivity. For example, let $f(0) = 0$, $f(1) = 1$, $g(0) = 0$, $g(1) = 2$, $h(0) = 3$ and $h(1) = 2$, then $(f, g) \in \mathbf{R}$ and $(g, h) \in \mathbf{R}$, but $(f, h) \notin \mathbf{R}$. $\mathbf{R}$ is clearly reflexive and symmetric.

   (b) $\{(f, g) \mid f(x) - g(x) = 1 \text{ for all } x \in \mathcal{Z}\}$
   Solution: This is not an equivalence relation, lacking in reflexivity, symmetry and transitivity. Let $f(x) = x$, $g(x) = x + 1$ and $h(x) = x + 2$. In this case $\forall x \in \mathbf{Z}, f(x) - f(x) = 0$, so $(f, f) \notin \mathbf{R}$, which means $\mathbf{R}$ is not reflexive. $(g, f) \in \mathbf{R}$, since $\forall x \in \mathbf{Z}, g(x) - f(x) = x + 1 - x = 1$, but on the other hand, $\forall x \in \mathbf{Z}, f(x) - g(x) = x - (1 + x) = -1$, so $(f, g) \notin \mathbf{R}$. So it is not symmetric. Finally, $(g, f) \in \mathbf{R}$ and $(h, g) \in \mathbf{R}$, but $(h, f) \notin \mathbf{R}$. So it is not transitive.

4. [**12 Pts Elliott**] Let R be the relation on $\mathcal{Z} \times \mathcal{Z}$ where $((a, b), (c, d) \in R$ if and only if $a - c = b - d$.

   (a) Prove that R is an equivalence relation.
   Solution: For reflexivity, $((a, b), (a, b)) \in R$ because $a - a = b - b$.
   If $((a, b), (c, d)) \in R$ then $a - c = b - d$, which also means that $c - a = d - b$, so $((c, d), (a, b)) \in R$; this tells us that R is symmetric.
   Finally, if $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$ then $a - c = b - d$ and $c - e = d - f$. So, $(a - c) + (c - e) = (b - d) + (d - f)$, implying $a - e = b - f$. Therefore, $((a, b), (e, f)) \in R$; this tells us that R is transitive.

   (b) Define a function f such that $f(a, b) = f(c, d)$ if and only if $((a, b), (c, d)) \in R$.
   Solution:
   $f(x, y) = x - y$ since $a - c = b - d$ implies $a - b = c - d$.

   (c) Define the equivalence class containing $(1, 1)$ and list 2 elements in the class.
   Solution:
   $[(1, 1)] = \{(x, x) | x \in Z^+\}$

Two elements: $(2,2),(3,3)$

(d) Describe each equivalence class. How many classes are there and how many elements in each class?

Solution:
The number of equivalence classes is countably infinite, and each class contains a countably infinite number of elements. For example: $[(a,b)] = \{(c,d) \mid a-b = c-d \text{ where } c,d \in \mathbb{Z}\}$. *Example:* $[(2,3)] = \{(1,2),(2,3),(3,4),(4,5),\ldots\}$.

5. **[4 Pts Xiang]** Describe all the equivalence classes for the relation *congruence modulo 5* over $\mathcal{Z}$, using set-builder notation. What is the equivalence class $[2]$?

Solution:
There are five equivalence classes, one for each possible remainder when an integer is divided by five. In other words:
$[0] = \{5n \mid n \in \mathbb{Z}\}$
$[1] = \{5n+1 \mid n \in \mathbb{Z}\}$
$[2] = \{5n+2 \mid n \in \mathbb{Z}\}$
$[3] = \{5n+3 \mid n \in \mathbb{Z}\}$
$[4] = \{5n+4 \mid n \in \mathbb{Z}\}$
The equivalence class $[2] = \{5n+2 \mid n \in Z\}$

6. **[8 Pts Xiang]** LLM Problem 9.10 (a)

Solution:
By performing breadth-first search on the diagram from top to bottom, we can obtain the following ordering: $8, 18, 9, 11, 10, 4, 6, 8$

7. **[Extra Credit Xiang]** LLM Problem 9.10 (b) (c) (d)

HW 8 Coms 330

1. a) $(x,y) \in R_1$ iff $xy = 4$

Reflexive: $(1,1) \notin R_1$ NO

Anti-Reflective: NO, $(2,2) \in R_1$

Symm: Yes, because $xy = yx$ so if $(x,y) \in R_1$, then $(y,x) \in R_1$

anti symm: No, because $(4,1) \in R_1$ and $(1,4) \in R_1$

transitive: NO, $(1,4) \in R_1$ and $(4,1) \in R_1$ but $(1,1) \notin R_1$

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b)

R:

A:

S:

A:

T:

Look at solution

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c)

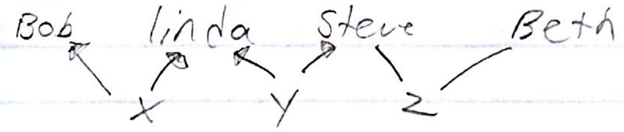
R: Trivial true since I have same BP as myself

A: NO. (x,x) have same BP

S: Yes. If x shares BP w/ y. Then y shares BP w/ x.

A: No. Because of symmetry

T: NO. If x shares BP with y and y shares BP with z. X has a chance to not be related to z.



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2. $x/y = Q$

Reflective: Yes. $2/2 = 1 \in Q$

Anti: No. Because $3/3 = 1$ $1 \in Q$

Sym. Yes. $(3,4) = 3/4$ and $(4,3) = 4/3$. $3/4$ and $4/3 \in Q$

Anit sym. No, because symmetry

Transitive: Yes. $(1,2) = 1/2$, $(2,3) = 2/3$, $(1,3) = 1/3$ all $\in Q$.

By def, is a equiv. relation because it is Reflective, Sym, and Transitive.

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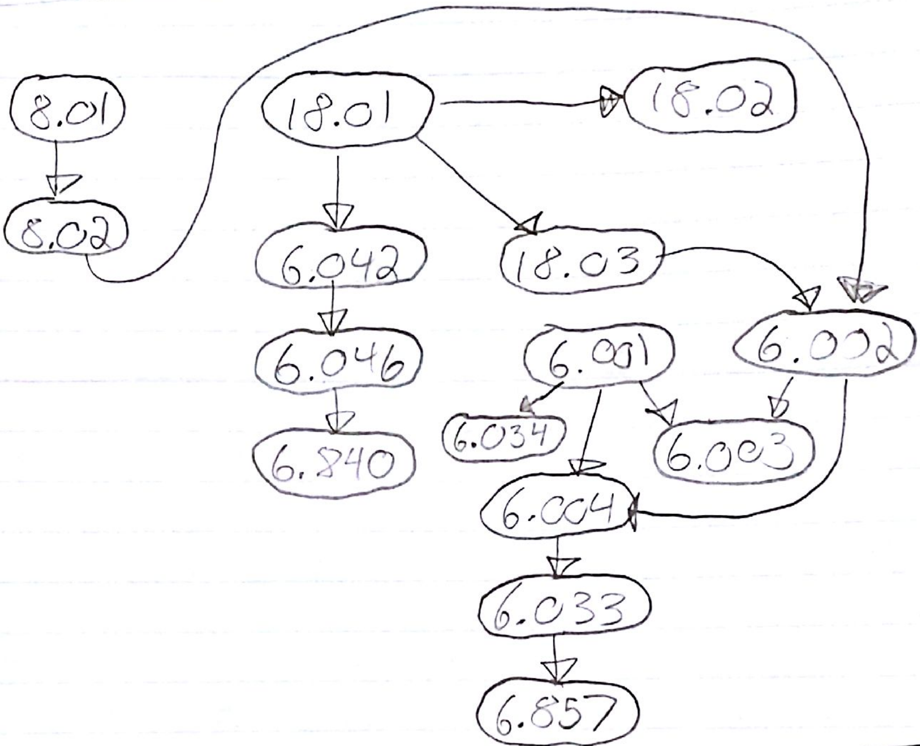
$[2]_R \{ \forall n \in R^+ \mid \frac{2}{n} \text{ or } \frac{n}{2} \}$

$[\pi]_R \{ \forall n \in R^+ \mid \frac{\pi}{n} \text{ or } \frac{n}{\pi} \}$

5. c)



4.

$[0] = \{6n \mid n \in Z\}$

$[1]$ $6n+1$ "

$[2]$ " $+2$ "

$[3]$ " $+3$ "

$[4]$ " $+4$ "

$[5]$ " $+5$ "

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