CS 330 : Discrete Computational Structures
%

Fall Semester, 2015
ASSIGNMENT #6
Due Date: Sunday, Oct 18

Suggested Reading: Rosen Section 5.2 - 5.3; Lehman et al. Chapter 5

These are the problems that you need to turn in. For more practice, you are encouraged to work on the other problems. **Always explain your answers and show your reasoning.**

1. [**10 Pts**] Rosen, Section 5.3: Exercise 8 (a), (d) **Solution**

   a. $a_1 = 2, a_{k+1} = a_k + 4$
   d. $a_1 = 1, a_{k+1} = a_k + 2k + 1$

2. [**8 Pts**] Rosen, Section 5.3: Exercise 16

   Solution:

   We have to prove : $P(n) = f_0 - f_1 + f_2 - ... - f_{2n-1} + f_{2n} = f_{2n-1} - 1$.
   Basis Step: $n = 1, P(1) = f_0 - f_1 + f_2 = 0 - 1 + 1 = 0, f_{2-1} - 1 = f_1 - 1 = 1 - 1 = 0$

   Inductive Step: Assume $P(k) = f_0 - f_1 + f_2 - ... - f_{2k-1} + f_{2k} = f_{2k-1} - 1$

   $P(k+1) = f_0 - f_1 + f_2 - ... - f_{2k-1} + f_2k - f_{2k+1} + f_{2k+2}$ {using Inductive Hypothesis}

   $= f_{2k-1} - 1 - f_{2k+1} + f_{2k+2}$

   $= f_{2k-1} - 1 - f_{2k+1} + f_{2k} + f_{2k+1}$

   $= f_{2k-1} - 1 + f_{2k}$

   $= f_{2k-1} + f_{2k} - 1$

   $= f_{2k+1} - 1$

   $= f_{2(k+1)-1} - 1$.

3. [**12 Pts**] Consider the following state machine. The machine has four states, labeled 0, 1, 2, and 3. The start state is 0. The transitions are $0 \to 1$, $1 \to 2$, $2 \to 3$, and $3 \to 0$.

   Prove that if we take n steps in the state machine we will end up in state 0 if and only if n is divisible by 4. Argue why we cannot prove the statement above by induction. Instead, we need to *strengthen the induction hypothesis*. State the strengthened hypothesis and prove it.

   Solution: Using the above statement as our inductive hypothesis, we don't know which state we're in when n is not divisible by 4, and when we're in a state other than 0, all we know about n is that n is not divisible by 4.

   Stronger inductive hypothesis: after n steps, we are:
   in state 0 if and only if $n\%4 = 0$
   in state 1 if and only if $n\%4 = 1$
   in state 2 if and only if $n\%4 = 2$
   in state 3 if and only if $n\%4 = 3$
   for some integer k.

Proof using stronger inductive hypothesis:
Base case: after 0 steps, the machine is in state 0, and 0 is a multiple of 4.
Inductive step: assume that the SIH holds after n steps. There are 4 cases:
Case 1: The machine was in state 0, and $n\%4 = 0$. After $n + 1$ steps, the machine is now in state 1, and $(n + 1)\%4 = 1 + 0 = 1$.
Case 2: The machine was in state 1, and $n\%4 = 1$. After $n + 1$ steps, the machine is now in state 2, and $(n + 1)\%4 = 1 + 1 = 2$.
Case 3: The machine was in state 2, and $n\%4 = 2$. After $n + 1$ steps, the machine is now in state 3, and $(n + 1)\%4 = 1 + 2 = 3$.
Case 4: The machine was in state 3, and $n\%4 = 3$. After $n + 1$ steps, the machine is now in state 0, and $(n + 1)\%4 = 1 + 3 = 4 = 0$.
Thus, the SIH holds in all the cases, so the inductive step is proved. As a consequence, the machine is in state 0 after n steps if and only if n is a multiple of 4.

4. [**8 Pts**] Lehman et al. Problem 5.10

Solution:
The proof is by induction. Let P(n) be the proposition that periphery length is even after n squares are placed. In the base case, P(1) is true because the periphery of a single square has length 4, which is even.
In the inductive step, assume that the periphery length is even after n squares are placed to prove that the periphery length is even after n+1 squares are placed. The (n+1)th square could share 1, 2, 3, or 4 edges with previously-placed squares.
If the new square shares 1 edge with a previously placed square, then this one edge is removed from the periphery, but three edges of the new square are added to the periphery. Overall, the periphery length increases by two and thus remains even.
If the new square shares 2 edges with previously placed squares, then these two edges are removed from the periphery, but two edges of the new square are added. The periphery length is unchanged and thus remains even.
If the new square shares 3 edges, then these three edges are removed from the periphery, but one edge is added. The periphery length decreases by two and remains even.
If the new square shares 4 edges, then these four edges are removed from the periphery and none are added. The periphery length decreases by four and remains even.
In all cases, the length of the periphery remains even. Therefore, for all $n \geq 1$, P(n) implies P(n+1) and the claim is proved by induction.

5. [**12 Pts**] A robot wanders around a 2-dimensional grid. He starts out at (0,0) and can take the following steps: (+2,-1), (-1,-1) and (0,3). Define a state machine for this problem. Then, define a Preserved Invariant and prove that the robot will never get to (2,0).

The state machine has states (x, y) where x and y are integers, and every state (x, y) has three outgoing transitions, to $(x + 2, y - 2)$, to $(x - 1, y - 1)$, and to $(x, y + 3)$.

Preserved Invariant: if the robot is in state (x, y), then $x - y$ is a multiple of 3.

Base Case: The robot starts in (0,0). 0-0=0, and 0 is a multiple of 3.

Inductive step: Assume the robot is in state (x, y), where $x - y$ is a multiple of 3. After one step, there are three states the robot could be in.
Case 1: The robot moved to $(x + 2, y - 1)$. Then $(x + 2) - (y - 1) = x - y + 3$. Since $x - y$ is a multiple of 3 (by IH), this is a multiple of 3.

Case 2: The robot moved to $(x-1, y-1)$. Then $(x-1)-(y-1) = x-y$, which is a multiple of 3 by the IH.
Case 3: The robot moved to $(x, y+3)$. Then $x-(y+3) = x-y-3$. Since $x-y$ is a multiple of 3 (by IH), this is a multiple of 3.
Thus, the invariant is preserved by the transitions.

Since the robot can only move to states (x, y) where $x-y$ is a multiple of 3, and $2-0$ is not a multiple of 3, the robot can never move to $(2, 0)$.