$a \not\approx b$ iff ≥ 2

1. Short Answers [**24 Points**]

   (a) [**12 Pts**] Is the relation $\{(a,b) \mid a/b \geq 2\}$ over the set of positive reals (i) reflexive, (ii) anti-reflexive, (iii) symmetric, (iv) anti-symmetric, ~~and (v) transitive~~? Is it (i) an equivalence relation, (ii) a partial order, or (iii) a strict partial order? Justify your answers.

i) No. $1/1 \geq 2$

ii) Yes. $a/a \not\geq 2$ so $(a,a) \notin R$

iii) No. $2/1 \geq 2$ but $1/2 \not\geq 2$.

iv) Yes. Suppose $a/b \geq 2$, then $b/a \leq \frac{1}{2} < 2$

$a \geq 2b$

$1 \geq \frac{2b}{a} \rightarrow \frac{1}{2} \geq \frac{b}{a}$

v) Yes. Suppose $\frac{a}{b} \geq 2$ and $\frac{b}{c} \geq 2$ then

$a \geq 2b$ and $b \geq 2c$

$a \geq 4c \rightarrow \frac{a}{c} \geq 4 > 2$

iii) SPO

   (b) [**4 Pts**] Give an *inductive definition* for a_i of the sequence $5, 11, 23, 47, 95, \ldots$, where a_0 is the first term.

$a_0 = 5$

$a_n = 2a_{n-1} + 1$

or $a_{n+1} = 2a_n + 1$

   ~~(c) [**8 Pts**] If A is countable and $A \cup B$ is uncountable, prove that B is uncountable.~~

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Reflexive: $a \approx a$ for all a
Symm: if $a \approx b$ then $b \approx a$
Transitive: if $a \approx b$ and $b \approx c$, then $a \approx c$

2. Equivalence Relations **[20 Points]**

Define the relation $\approx$ on the set of real numbers, where $a \approx b$ if and only if $a = n/2 + b$ for some integer n.

(a) **[8 pts]** Prove that $\approx$ is an equivalence relation.

R: $a = \frac{n}{2} + a$, true when $n = 0$. Done. ✓

S: if $a \approx b$, then $a = \frac{n}{2} + b$ and $b = \frac{-n}{2} + a$

so $b \approx a$ ✓

T: $a = \frac{n}{2} + b$ and $b = \frac{m}{2} + c$, $n, m \in \mathbb{N}$

$a = \frac{n}{2} + \frac{m}{2} + c$

$a = \frac{n+m}{2} + c$ ✓

(b) **[4 pts]** List 3 elements of the equivalence class containing 1/2. Then, formally define the equivalence class using set builder notation.

$\frac{1}{2}$ $\quad \frac{1}{2} = \frac{n}{2} + b \quad n = 1$, b has to be $\underline{0}$

$\boxed{\left\{\frac{n}{2} + \frac{1}{2} \mid n \in \mathbb{Z}\right\}}$ $\quad \frac{1}{2} = \frac{n}{2} + b \quad n = -1$, b has to be $\underline{1}$

(c) **[8 pts]** Describe the equivalence classes formally. How many equivalence classes are there, and how many elements in each equivalence class? ~~If infinite, state whether countable or uncountable. Justify your answer.~~

$\left[\frac{11}{10}\right] \approx \frac{11}{10}, \frac{16}{10}, \frac{6}{10}, \frac{25}{10}, \frac{1}{10}$

$\left\{\frac{n}{2} + \frac{11}{10} \mid n \in \mathbb{Z}\right\}$

$[0, \frac{1}{2})$ → ∞ many

for any $\mathbb{R}$ r

$[r] = \left\{\frac{n}{2} + r \mid n \in \mathbb{Z}\right\}$

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4. Mathematical Induction [**18 Points**]

Prove the statement below, by mathematical induction, for all positive integers n.

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$$

(a) [**6 Pts**] State the base case and prove it.

$n=1$

$1 \cdot 1! = (1+1)! - 1$

$1 = 1$ ✓

(b) [**4 Pts**] State what you assume and what you prove in the inductive step.

Assume $1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1$

Prove $1 \cdot 1! + 2 \cdot 2! + \ldots + (n+1)(n+1)! = (n+2)! - 1$

(c) [**8 Pts**] Now, prove the inductive step.

$= (n+1)! - 1 + (n+1)(n+1)!$

$= (n+1)!\,(1 + (n+1)) - 1$

$= (n+2)! - 1$

Prove

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5. Strong Induction [**18 Points**]

Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cent and 11-cent stamps. Prove that $P(n)$ is true for all $n \geq 21$, using the steps below.

(a) Prove $P(n)$ for all $n \geq 21$ by *regular* induction.

i. [**3 Pts**] Prove $P(21)$ to complete the basis step.

$7 \cdot 3¢ = 21¢$

ii. [**6 Pts**] Prove $P(k) \to P(k+1)$ for all $k \geq 21$.

Case 1: -1 $11¢$ $+ 4 \cdot 3¢$

Case 2: At least $7 \cdot 3¢$

$-7 \cdot 3¢$ $+ 2 \cdot 11¢$

If no $11¢$, then because $k \geq 21$, we need 7 $3¢$ stamps.

(b) Now, prove $P(n)$ for all $n \geq 21$ by *strong* induction.

i. [**3 Pts**] Prove that $P(21)$, $P(22)$ and $P(23)$ to complete the basis step.

$P(21) = 7 \cdot 3¢$

$P(22) = 2 \cdot 11¢$

$P(23) = 4 \cdot 3¢ + 11¢$

ii. [**6 Pts**] Prove that for all $k \geq 23$, if $P(j)$ is true for $21 \leq j \leq k$, then $P(k+1)$ is true.

$(k+1) - 3 \geq 21$

So, $P(k-2)$, add a $3¢$ stamp for $k+1$ ¢

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6. Structural Induction **[24 Points]**

Consider this inductive definition for a set of integers S.

Base Case: $5 \in S$.

Inductive Step: if m is in S and n is in S, then mn is in S.

Let $A = \{5^i \mid i \in \mathbb{Z}^+\}$. We prove that $A = S$.

(a) **[12 Pts]** Prove that $A \subseteq S$ using mathematical induction. In other words, prove that for all $i \in \mathbb{Z}^+$, $5^i \in S$. Prove the basis and inductive steps clearly.

Base: $i = 1$, $5^1 \in S$ ✓

- Assume $5^k \in S$, prove $5^{k+1} \in S$
- $5^k \in S$, $5 \in S$
- So, $5^k \cdot 5 = 5^{k+1} \in S$ (Def of S) ✓

(b) **[12 Pts]** Prove that $S \subseteq A$ using structural induction. In other words, prove that for all $x \in S$, $x = 5^i$ for some $i \in \mathbb{Z}^+$. Prove the basis and inductive steps clearly.

Base: $5 = 5^1 \in A$ ✓

Inductive: Assume $n, m \in A$

Prove $m \cdot n \in A$

- Since $n \in A$ so $m = 5^x$, $n = 5^y$ > pos. int.
- $m \cdot n = 5^y \cdot 5^x = 5^{y+x}$
- $y+x$ is positive int.
- $5^{y+x} \in A$ ✓

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