CS 330 : Discrete Computational Structures

Fall Semester, 2015

ASSIGNMENT #3

Due Date: Sunday, September 20

Suggested Reading: Rosen Sections 1.7 - 1.8, 2.1; Lehman et al. Chapter 1 and 4.1

These are the problems that you need to turn in. For more practice, you are encouraged to work on the other problems. Always explain your answers and show your reasoning.

1. [**5 Pts**] Prove that the average of two rational numbers is rational. Use a direct proof.

   Let a and b be two arbitrary rational numbers. From the definition of rational, we can write $a = \frac{p}{q}$ and $b = \frac{m}{n}$ for some integers p, q, m, n where $q, n \neq 0$.

   The average of a and b is $\frac{a+b}{2} = \frac{\frac{p}{q}+\frac{m}{n}}{2} = \frac{\frac{pn+mq}{qn}}{2} = \frac{pn+mq}{2qn}$. The sum of two integers is an integer, and the product of two integers is an integer, so $\frac{pn+mq}{2qn}$ is a ratio of two integers; since $q \neq 0$ and $n \neq 0$, $2qn \neq 0$. Thus, the average of a and b is rational.

2. [**6 Pts**] Prove that there is an irrational number between any rational number and irrational number.

   The average of two different numbers a and b is between a and b. Let a be a rational number and b be an irrational number. If we prove that the average of a and b is irrational, we're done. We do so by contradiction.

   Suppose the average of a and b is rational, for contradiction. Then $\frac{a+b}{2} = \frac{p}{q}$ for some integers p, q where $q \neq 0$. With a little algebra, we get $a = \frac{2p}{q} - b = \frac{2p-bq}{q}$. q and $2p-bq$ are integers, and $q \neq 0$, so a is a rational number. This contradicts our assumption that b is irrational. Thus, the average of a and b is irrational.

3. [**6 Pts**] Prove that the square root of 15 is irrational.

   Suppose, for the sake of a contradiction, that $\sqrt{15}$ is rational. Then $\sqrt{15} = \frac{p}{q}$ for some integers p, q where $q \neq 0$ and p, q have no common divisors. This gives us $15 = \frac{p^2}{q^2}$, so $p^2 = 15q^2$. This means p^2 is a multiple of 15, which means p is a multiple of 15. Since p is a multipe of 15, $p = 15r$ for some integer r. Then we can write p^2 as $(15r)^2$, so $(15r)^2 = 15q^2$. Cancel a 15 from both sides, and we have $15p^2 = q^2$. This means q^2 is a multiple of 15, so q is a multiple of 15. This contradicts that p and q have no common divisors. Thus, $\sqrt{15}$ must be irrational.

4. **[6 Pts]** Prove that if you have 25 cookies that you will distribute to 6 children, at least one child will get more than 4 cookies. What proof method did you use?

   Proof by contrapositive:
   The contrapositive statement is "If there are 6 children and each child gets at most 4 cookies, I cannot distribute 25 cookies".

   Let $c_1, c_2, \ldots c_6$ be the number of cookies the first, second, ... sixth child gets, respectively. If each child gets at most 4 cookies, $c_1 \leq 4, c_2 \leq 4, \ldots c_6 \leq 4$. The total number of cookies distributed is $c_1 + c_2 + \ldots c_6$, which is $\leq 4 + 4 + \ldots 4 = 4 \times 6 = 24$. Thus, the total number of cookies distributed cannot be 25.

5. **[6 Pts]** Prove or disprove that p is odd if and only if p^3 is odd.

   To prove that p is odd $\leftrightarrow p^3$ is odd we have to prove 2 ways:

   first: if p is odd then p^3 is odd. If p is odd we can write as p = 2k+1, where k is an integer.

   $p = 2k + 1 \Longrightarrow p^3 = (2k + 1)^3$

   $\Longrightarrow p^3 = 2k^3 + 12k^2 + 6k + 1$

   $\Longrightarrow p^3 = 2(4k^3 + 6k^2 + 3k) + 1$

   $\Longrightarrow p^3 = 2m + 1$, where m = $(4k^3 + 6k^2 + 3k)$

   $\Longrightarrow p^3$ is odd

   second: If p^3 is odd then p is odd.

   proof by contrapositive : p^3 is even if p is even.

   let p= 2h , then $p^3 = 8h^3 = 2(4h^3) = 2t$ and $t = 4h^3$

   $\Longrightarrow p^3$ is even.

   Therefore if p^3 is odd then p is odd.

6. **[6 Pts]** Prove that there exist 1000 consecutive positive numbers that are not perfect squares. Is your proof constructive or non-constructive? Explain.

   Let x be a positive integer. Then two consecutive positive perfect squares are x^2 and $(x+1)^2$. all the integers between these two are not perfect squares. Number of positive integers between x^2 and $(x+1)^2$ will be $(x+1)^2 - x^2 = x^2 + 2x + 1 - x^2 = 2x + 1$. we want 2x+1 be greater than 1000 so $2x + 1 \geq 1000 \Longrightarrow x \geq 499.5 \Longrightarrow$ so we know that there is no perfect square between $x^2 = 500^2 = 250000$ and $(x + 1)^2 = 501^2 = 251001$. So we have 1000 consecutive positive integers which are not perfect squares.

   The proof is constructive because we showed it by existing an example.

7. **[5 Pts]** Prove that if A is a subset of C and B is a subset of D, then $A \times B$ is a subset of $C \times D$.

Assuming that $(x, y) \in A \times B$, so by the definition of cartesian product we know that $x \in A$ and $y \in B$. Since $A \subseteq C$ and $B \subseteq D$ then $x \in C$ and $y \in D$ thus $(x, y) \in C \times D \Longrightarrow A \times B \subseteq C \times D$

8. [**4 Pts**] Use set builder notation to describe the following sets:
   (a) $\{6, 14, 22, \ldots\}$ (c) $\{3, 6, 11, 18, \ldots\}$

   a) $A = \{8n - 2 \mid n \in N\}$

   b) $B = \{n^2 + 2 \mid n \in Z^+\}$

9. [**6 Pts**] Prove that if A and B are non-empty sets where $A \neq B$, then $A \times B \neq B \times A$.

   proof by Contradiction Assume that $A \times B = B \times A$. let x and y be two arbitrary elements such that $x \in A$ and $y \in B$. Then we have :

   $x \in A$ and $y \in B \Longrightarrow (x, y) \in A \times B$ (by definition of Cartesian product)

   Since we know that $A \times B = B \times A$ we have $(x, y) \in B \times A$ so by definition of cartesian product we have $x \in B$ and $y \in A$ so we have A = B.

   This contradicts the given assumption that $A \neq B$. Therefore the assumption that $A \times B = B \times A$ is not true.