CS 330 : Discrete Computational Structures

Fall Semester, 2015

Assignment #4

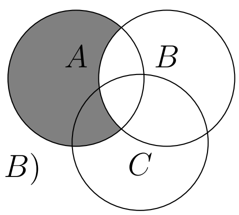
Due Date: Sunday, Sept 27

Suggested Reading: Rosen Sections 2.2 - 2.3; Lehman et al. Chapter 4.1, 4.3, 4.4

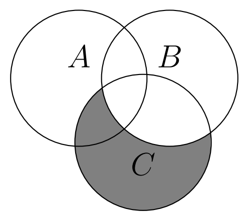
These are the problems that you need to turn in. For more practice, you are encouraged to work on the other problems. Always explain your answers and show your reasoning.

1. [**4 Pts**] Use Venn Diagrams to prove that $(A - B) \cup (C - B) = (A \cup C) - B$.

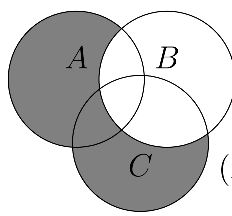
   We draw $(A - B)$ and $(C - B)$. Then we take the union.



$(A - B)$

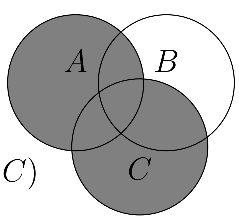


$(C - B)$

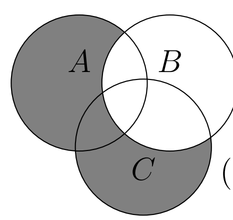


$(A - B) \cup (C - B)$

   On the other hand, below we show $(A \cup C)$ and $(A \cup C) - B$.



$(A \cup C)$



$(A \cup C) - B$

   By the diagrams, we conclude that $(A - B) \cup (C - B) = (A \cup C) - B$ as needed.

2. [**4 Pts**] Use an iff argument to prove that $(A - B) \cup (C - B) = (A \cup C) - B$. You may use logical equivalences in your proof.

   $x \in (A - B) \cup (C - B)$
   iff $(x \in A \wedge x \notin B) \vee (x \in C \wedge x \notin B)$ the definition of union and set difference
   iff $(x \in A \vee x \in C) \wedge x \notin B$ distributive
   iff $(x \in A \cup C) \wedge x \notin B$ the definition of union
   iff $x \in (A \cup C) - B$ the definition set difference

3. **[4 Pts]** Prove by contradiction that $(A - C) \cap (C - B) = \emptyset$.

   Suppose, for a contradiction, that $(A-C)\cap(C-B) \neq \emptyset$. Then $\exists x \in (A-C)\cap(C-B)$. So, $x \in (A - C)$ and $x \in (C - B)$. If $x \in (A - C)$, then $x \in A$ and $x \notin C$, but if $x \in (C - B)$, then $x \in C$ and $x \notin B$. This proves that $x \in C$ and $x \notin C$, which is a contradiction.

4. **[6 Pts]** Disprove the statements below.

   (a) If $A \cup C = B \cup C$ then $A = B$.
   Let $A = \{0\}, B = \{1\}, C = \{0, 1\}$. Then $A \cup C = \{0, 1\} = B \cup C$, but $A \neq B$.

   (b) If $A \cap C = B \cap C$ then $A = B$.
   Let $A = \{0\}, B = \{1\}, C = \emptyset$. Then $A \cap C = \emptyset = B \cap C$, but $A \neq B$.

5. **[6 Pts]** Prove that if $A \cup C = B \cup C$ and $A \cap C = B \cap C$ then $A = B$.

   Suppose $A \cup C = B \cup C$ and $A \cap C = B \cap C$, and let x be an arbitrary element of A. We need to show $x \in B$. We do this with two cases: $x \in C$ and $x \notin C$.

   If $x \in C$, then $x \in A$ and $x \in C$, so $x \in A \cap C$. Since $A \cap C = B \cap C$, $x \in B \cap C$. Therefore, $x \in B$.

   If $x \notin C$, then because $x \in A$, $x \in A \cup C$. Since $A \cup C = B \cup C$, $x \in B \cup C$. Since $x \notin C$, $x \in B$.

   This shows that $A \subseteq B$. By an identical proof with B and A exchanged, $B \subseteq A$. Thus, $B = A$.

6. **[8 Pts Peter]** Prove by subset argument (in both directions) that $\overline{A \cup B} = \overline{A} \cap \overline{B}$. You *may not* use logical equivalences in your proof.

   First, we prove that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$, by contradiction:
   Let x be an arbitrary element of $\overline{A \cup B}$. Suppose, for contradiction, that $x \in A$. Then $x \in A \cup B$, so $x \notin \overline{A \cup B}$, which is a contradiction; thus, $x \notin A$. For the same reason, $x \notin B$. So, $x \in \overline{B}$ and $x \in \overline{A}$, so $x \in \overline{A} \cap \overline{B}$. Therefore, $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

   Now, we prove that $\overline{A \cup B} \supseteq \overline{A} \cap \overline{B}$, also by contradiction.
   Let x be an arbitrary element of $\overline{A} \cap \overline{B}$. Suppose, for contradiction, that $x \in A \cup B$. Then $x \in A$ or $x \in B$. But $x \notin A$, because $x \in \overline{A}$; similarly $x \notin B$, giving a contradiction.

7. **[4 Pts Peter]** Consider the function f mapping $\mathcal{R}$ to $\mathcal{R}$, where $f(n) = 3n^2 - 8$.

   (a) Explain why f is neither one-to-one nor onto.
   f is not one-to-one because $f(-1) = f(1) = -5$.
   f is not onto because there is no real number n such that $3n^2 - 8 = -11$, because if so, then $n^2 = -1$ and the square root of -1 is not real.

   (b) Now, restrict either the domain or co-domain to make f one-to-one.
   Some possible solutions: restrict the domain to any of $\mathcal{Z}^+, \mathcal{R}^+, \emptyset, \{100\}$

(c) Then, restrict either the domain or co-domain to make f onto.

Restrict the codomain to any of the following: $\mathcal{Z}^+$,$\{n|n \geq -8, n \in \mathcal{R}\}$,$\{f(x)|x \in \mathcal{R}\}$,$\emptyset$, $\{22\}$

8. **[4 Pts Peter]** Prove that $f(n) = 3n+5$ is one-to-one, where the domain and co-domain of f is $\mathcal{Z}^+$. Show that f is not onto.

   Suppose $f(n) = f(m)$. Then $3n + 5 = 3m + 5$, so $3n = 3m$, so $n = m$. Thus, f is 1-1. f is not onto, because 2 is in the co-domain, but there is no positive integer n such that $f(n) = 2$.

9. **[4 Pts Peter]** Prove that $f(m, n) = m+n-2$ is onto, where the domain of f is $\mathcal{Z} \times \mathcal{Z}$ and the co-domain of f is $\mathcal{Z}$. Show that f is not one-to-one.

   Let x be an arbitrary element of $\mathcal{Z}$. Then $(x, 2)$ is an element of $\mathcal{Z} \times \mathcal{Z}$ and $f(x, 2) = x + 2 - 2 = x$, so f is onto.
   f is not 1-1, because $f(0, 1) = f(1, 0) = -1$.

10. **[6 Pts Peter]** Prove that $f(m, n) = (m + n, m - n)$ is one-to-one and onto, where the domain and co-domain of f is $\mathcal{R} \times \mathcal{R}$. Give the inverse function of f.

    Suppose $f(m, n) = f(x, y)$. Then $(m + n, m - n) = (x + y, x - y)$, so $m + n = x + y$ and $m - n = x - y$. So, $(m + n) + (m - n) = (x + y) + (x - y)$, so $2m = 2x$, so $m = x$. Then we can substitute in to the first equation to get $x + n = x + y$, so $n = y$. Thus, $(m, n) = (x, y)$, proving that f is 1-1.

    Let (a, b) be an arbitrary element of $\mathcal{R} \times \mathcal{R}$. Then $f(x, y) = (a, b)$. So, $x + y = a$ and $x - y = b$. Adding these together gives us $2x = a + b$, and subtracting the second from the first gives $2y = a - b$. So, $x = \frac{a+b}{2}$ and $y = \frac{a-b}{2}$. Since this gives us the input corresponding to an arbitrary output, this is in fact f^{-1}.

    To double check that this works: We can see that $(\frac{a+b}{2}, \frac{a-b}{2}) \in \mathcal{R} \times \mathcal{R}$. Evaluating $f(\frac{a+b}{2}, \frac{a-b}{2}) = (\frac{a+b}{2} + \frac{a-b}{2}, \frac{a+b}{2} - \frac{a-b}{2}) = (\frac{2a}{2}, \frac{2b}{2}) = (a, b)$, proving that f is onto and that $f(f^{-1}(a, b)) = (a, b)$.