CS 330 : Discrete Computational Structures
Spring Semester, 2014
ASSIGNMENT #10 SOLUTIONS
Due Date: Tuesday, Apr 22

Suggested Reading: Rosen Sections 6.1 - 6.3.

These are the problems that you need to turn in. Always explain your answers and show your reasoning. **Spend time giving a complete solution. You will be graded based on how well you explain your answers. Just correct answers will not be enough!**

1. [**6 Pts Xiyuan**] An ISU Computer Science shirt is sold in 8 colors, 4 sizes, collared or tee, and long sleeve or short sleeve.
(a) How many different shirts are being sold? (b) What if collared shirts only come in 5 colors and 2 sizes?

Solution:
(a) By the product rule there are $8 \times 4 \times 2 \times 2 = 128$ *different types of shirt.*
(b) There are $8 \times 4 \times 2 = 64$ *different types of tee shirts, and there are* $5 \times 2 \times 2 = 20$ *different types of collared shirts. So the total shirts is* $64 + 20 = 84$.

2. [**9 Pts Xiyuan**] (a) How many different four-letter codes can there be? (b) What if letters cannot be repeated? (c) What if, in addition, two of the letters are x and y?

Solution:
(a) By the product rule there are $26^4 = 456976$ *different four-letter codes.*
(b) If no letter is to be repeated, then there are 26 choices for the first code, 25 choices for the second, 24 choices for the third, and 23 choices for the fourth. By the product rule, the answer is $26 \times 25 \times 24 \times 23 = 358800$.
We can subtract the number of four-letter code strings without x, y *in it and also without repeating from the number of four-letter code strings without repeating. Thus the answer is* $26 \times 25 \times 24 \times 23 - 24 \times 23 \times 22 \times 21 = 103776$.

3. [**5 Pts Elliott**] How many integers between 10000 and 99999, inclusive, are divisible by 4 or 5 or 9?

Solution:
A *represents integers which are divisible by* 4 *between* 10000 *and* 99999.
B *represents integers which are divisible by* 5 *between* 10000 *and* 99999.
C *represents integers which are divisible by* 9 *between* 10000 *and* 99999.
Then the number of integers between 10000 *and* 99999 *which are divisible by* 4 *or* 5 *or* 9 *is* $|A \cup B \cup C|$.
According to the Inclusion-Exclusion Principle:
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

The number of integers which are divisible by 4: $\lfloor \frac{99999}{4} \rfloor - \lfloor \frac{10000-1}{4} \rfloor = 22,500$.

The number of integers which are divisible by 5: $\lfloor\frac{99999}{5}\rfloor - \lfloor\frac{10000-1}{5}\rfloor = 18,000$.

The number of integers which are divisible by 9: $\lfloor\frac{99999}{9}\rfloor - \lfloor\frac{10000-1}{9}\rfloor = 10,000$.

The number of integers which are divisible by 4 and 5 $(4\times5 = 20)$*:* $\lfloor\frac{99999}{20}\rfloor-\lfloor\frac{10000-1}{20}\rfloor = 4,500$.

The number of integers which are divisible by 4 and 9 $(4\times9 = 36)$*:* $\lfloor\frac{99999}{36}\rfloor-\lfloor\frac{10000-1}{36}\rfloor = 2,500$.

The number of integers which are divisible by 5 and 9 $(5\times9 = 45)$*:* $\lfloor\frac{99999}{45}\rfloor-\lfloor\frac{10000-1}{45}\rfloor = 2,000$.

The number of integers which are divisible by 4 and 5 and 9 $(4 \times 5 \times 9 = 180)$*:* $\lfloor\frac{99999}{180}\rfloor - \lfloor\frac{10000-1}{180}\rfloor = 500$.

So the answer is $22,500 + 18,000 + 10,000 - 4,500 - 2,500 - 2,000 + 500 = 42,000$.

4. [**4 Pts Elliott**] How many bit strings of length 7 either begin with two 1's or end with three 1's?

   Solution:
   The number of bit strings of length 7 that begin with two 1's: $2^5 = 32$
   The number of bit strings of length 7 that end with three 1's: $2^4 = 16$
   The number of bit strings of length 7 that contain both: $2^2 = 4$
   So by the Inclusion-Exclusion Principle, the answer is $32 + 16 - 4 = 44$ *possible bit strings of length* 7 *that either begin with two* 1*'s or end with three* 1*'s.*

5. [**6 Pts Xiyuan**] How many strings of 8 digits (a) end with an even digit? (b) have at least one repeated digit?

   Solution:
   (a) For each of the first 7 *digits it has* 10 *choices, for the last digit it has* 5 *choices. So there are :* $10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 5 = 50,000,000$.
   (b) The total number strings of 8 digits: 10^8.
   The number of 8 digit strings without repeating digits: $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 = 1,814,400$.
   So the number of 6 digits with at least one repeated digit: $10^8 - 1,814,400 = 98,185,600$.

6. [**8 Pts Zhenbi**] How many ways can 7 friends line up if Ann, Beth and Chris have to stand next to each other (a) where Ann is ahead of Beth and Beth is ahead of Chris? (b) where Ann, Beth and Chris can be in any order?

   Solution:
   (a) We can treat Ann, Beth and Chris as one person, then there are 5 people who need to be arranged, so there are $5! = 120$ *ways.*

(b) First there are $5!$ *ways to align 5 persons in a group, and there are* $3!$ *ways to order Ann, Beth and Chris. So the total is* $(5!)(3!) = 720$.

7. [**9 Pts Zhenbi**] Let A and B be sets of 5 elements and 8 elements, respectively. (a) How many different functions possible from A to B? (b) How many of these functions are one-to-one? (c) How many different relations possible from A to B?

   Solution:
   (a) Since for every element in A, it has 8 choices, there are 8^5 *different functions.*
   (b) If these functions are one-to-one, then there are 8 choices for the first, 7 choices for the second, ..., 4 choices for the last. By the product rule, the answer is $P(8,5) = 6720$.
   (c) There are $5 \times 8 = 40$ *possible pairs from A to B, the number of relations is the cardinality of the power set, which is* 2^{40}.

8. [**9 Pts Zhenbi**] In how many ways can a photographer arrange 8 people in a row from a family of 10 people, if (a) the bride and groom are in the photo, (b) the bride and groom are next to each other in the photo, (c) either the bride or the groom is in the photo, not both.

   Solution:
   (a)First we choose 6 people from 8 people not including bride and groom, there are $C(8,6) = 28$ *different ways. Then we arrange the 8 people with bride and groom in a row, there are* $8! = 40320$ *different ways. So the answer is* $28 \times 40320 = 1,128,960$.
   (b)First we can treat bride and groom as one person, there are $2! = 2$ *ways to arrange them. Then we choose 6 people from 8 people not including bride and groom, there are* $C(8,6) = 28$ *different ways. Then there are* $7! = 5040$ *ways to arrange 7 objects. So the total is* $2 \times 28 \times 5040 = 282,240$.
   (c) Since either the bride or the groom is in the photo, not both, then first we can choose 1 people from bride and groom, there are 2 different ways. Then we choose 7 other people from 8 people not including bride and groom, there are $C(8,7) = 8$ *different ways. Then we arrange the 8 chosen people in a row, there are* $8! = 40320$ *different ways. So the answer is* $2 \times 8 \times 40320 = 645,120$.

9. [**8 Pts Xiang**] A sack contains 240 movie tickets, 20 for each of 12 different movies. You want to go to the movies with four of your friends. How many tickets would you have to remove from the sack to guarantee that all of you will be able to watch the same movie? What principle did you use? What if you really wanted to go to 'Captain America'? How many tickets would you have to remove from the sack in that case?

   Solution:
   (a) $12 \times 4 + 1 = 49$. *The worst case is you first get* 4 *tickets for each of* 12 *movies. Then by Pigeonhole Principle, picking one more ticket will guarantee you have* 5 *tickets for some movie.*
   (b) $11 \times 20 + 5 = 225$, *you must assume that you have very bad luck so you get all tickets for the other 11 movies before you get 5 tickets for Captain America.*

10. [**6 Pts Xiang**] How many numbers must be selected from the set of numbers $\{2, 4, \ldots, 20\}$ to guarantee that at least one pair (from the group of selected numbers) adds up to 22? at least two pairs add up to 22? Explain.

Solution:
(a) Let's first group the set of numbers into $\{\{2,20\},\{4,18\},\{6,16\},\{8,14\},\{10,12\}\}$. *There are* 5 *pairs that can sum up to* 22. *In order to get a pair that can sum up to* 22, *the worst case is that the first 5 we pick include exactly one from each of the above mentioned pairs. Then another one more will guarantee us to have a pair. That is* $5 + 1 = 6$ *elements from the original set.*
(b) Pick one more than what is in (a), and we will get a second pair. That is 7.

11. [**3 Pts Xiang**] Suppose there are 70 students in a discrete math class. If all the students are CS, SE or CprE majors, show that there are at least 29 CS majors, at least 25 SE majors or at least 18 CprE majors in the class.

Solution:
Prove by contradiction: Suppose not. Then there are at most 28 *CS majors, at most* 24 *SE majors and at most* 17 *CprE majors, which would sum up to* $69 < 70$. *This contradicts the assumption that we have* 70 *students.*

12. [**9 Pts Xiang**] How many bit strings of length 9 contain (a) exactly three 1s? (b) at most three 1's? (c) at least three 1's?

Solution:

(a) $\binom{9}{3} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2} = 84$. *Pick three bits from nine bits and set them to* 1. *All the other bits set to* 0.

(b) $\binom{9}{0} + \binom{9}{1} + \binom{9}{2} + \binom{9}{3} = 1 + 9 + 36 + 84 = 130$. *That is: no bit set to* 1, *exactly one bit set to* 1, *two bits set to* 1, *or three bits set to* 1.

(c) $2^9 - (\binom{9}{0} + \binom{9}{1} + \binom{9}{2}) = 512 - (1 + 9 + 36) = 466$. *Count the complement, which is, we have zero, one, or two bits set to 1. This will give us* $\binom{9}{0} + \binom{9}{1} + \binom{9}{2}$. *Then subtract it from total number of possibilities, which is* 2^9.

13. [**9 Pts Aaron**] A coin is flipped nine times where each flip comes up either head or tails. How many possible outcomes contain at least five heads? Can you come up with two different ways to do this problem? How about three?

Solution: Here are three ways to solve this problem.

(a) *Count all outcomes that have at least five heads. This is equal to the number of outcomes that have exactly five heads plus the number of outcomes that have exactly six heads plus ... up to the number of outcomes that have exactly nine heads. So* $\binom{9}{5} + \binom{9}{6} + \binom{9}{7} + \binom{9}{8} + \binom{9}{9} = 256$.

(b) *We note that an outcome with five heads has exactly 4 tails, and an outcome with six heads has exactly 3 tails. So we could calculate the answer by:* $\binom{9}{4} + \binom{9}{3} + \binom{9}{2} + \binom{9}{1} + \binom{9}{0} = 256$.

(c) *"At least five heads" is the same as "at most four tails." Since heads and tails appear with equal likelihood, "at most four heads" will occur exactly as often as "at most four tails." Since those two sets partition all possible flip sequences, the size of the "at most four tails" set must be exactly half of the total possible outcomes. So the number of outcomes that have at least five heads is equal to half the total, or* $2^9/2 = 2^8 = 256$.

14. **[9 Pts Aaron]** 12 women and 10 men are on the faculty. How many ways are there to pick a committee of 7 if (a) Claire and Bob will not serve together, (b) at least one woman must be chosen, (c) at least one man and one woman must be chosen. Are there multiple ways to solve these problems? Explain.

Solution:

(a) *There are 22 people total, so a total of* $\binom{22}{7}$ *possible committees that can be formed if we don't limit the number of committees in any way. If Claire and Bob are both on the committee and we want to choose the rest of the members, we are choosing 5 other committee members from 20 people. So the answer to this question is* $\binom{22}{7} - \binom{20}{5} = 155,040$.

(b) *There are* $\binom{10}{7}$ *ways to pick 7 committee members without picking a single woman. So the answer is* $\binom{22}{7} - \binom{10}{7} = 170,424$.

(c) *As we did with (a) and (b), we subtract the "problem" cases from the total number of possibilities. There are* $\binom{12}{7}$ *ways of creating an all-women committee, and* $\binom{10}{7}$ *ways of creating an all-male committee. So there are* $\binom{12}{7} + \binom{10}{7}$ *ways of creating a one-sex-only committee. Therefore, the answer to our problem is* $\binom{22}{7} - \left(\binom{12}{7} + \binom{10}{7}\right) = 169,632$.

Another way to solve these problems: Instead of subtracting problem cases from total possibilities, we could add up the correct number of cases "from below." For example, here is an alternate way to solve (b). To count the number of ways at least one woman may be chosen, we count the number of ways exactly 1 woman (and 6 men) is chosen, plus the number of ways exactly 2 women (and 5 men) are chosen, etc., up to exactly 7 women (and 0 men) being chosen for our seven-member committee. As a formula:

$$\binom{12}{1}\binom{10}{6} + \binom{12}{2}\binom{10}{5} + \binom{12}{3}\binom{10}{4} + \binom{12}{4}\binom{10}{3} + \binom{12}{5}\binom{10}{2} + \binom{12}{5}\binom{10}{2} + \binom{12}{6}\binom{10}{1} + \binom{12}{7}\binom{10}{0} = 170,424$$

15. **[Extra Credit Aaron]** Rosen, Section 6.1: Exercise 72

16. **[Extra Credit Aaron]** Rosen, Section 6.3: Exercise 24