CS 330 : Discrete Computational Structures
Fall Semester, 2015
ASSIGNMENT #9
Due Date: Tuesday, Nov 17

Suggested Reading: Rosen Section 2.5; LLM Chapter 7.1

These are the problems that you need to turn in. For more practice, you are encouraged to work on the other problems. **Always explain your answers and show your reasoning.**

1. **[16 Pts]** Show that the following sets are countably infinite, by defining a bijection between $\mathcal{N}$ (or $\mathcal{Z}^+$) and that set. You do not need to prove that your function is bijective.

   (a) **[8 Pts]** the set of positive integers divisible by 5
   $f(n) = 5n$ maps $\mathcal{Z}^+$ to the set of positive integers divisible by 5.

   (b) **[8 Pts]** $\{1, 2, 3\} \times \mathcal{Z}$
   $f(n) = \left(n\%3 + 1, (-1)^{\lfloor \frac{n}{3} \rfloor} \lfloor \frac{n+3}{6} \rfloor\right)$ maps N to $\{1, 2, 3\} \times \mathcal{Z}$
   or equivalently,
   $f(n) = (1, \frac{n}{6})$ if $n = 0 \bmod 6$
   $f(n) = (2, \frac{n-1}{6})$ if $n = 1 \bmod 6$
   $f(n) = (3, \frac{n-2}{6})$ if $n = 2 \bmod 6$
   $f(n) = (1, -\frac{n+3}{6})$ if $n = 3 \bmod 6$
   $f(n) = (2, -\frac{n+2}{6})$ if $n = 4 \bmod 6$
   $f(n) = (3, -\frac{n+1}{6})$ if $n = 5 \bmod 6$

2. **[16 Pts]** Determine whether the following sets are countable or uncountable. Prove your answer.

   (a) **[8 Pts]** the set of real numbers with decimal representation consisting of all 1's (1.11 and 11.111... are such numbers). *Solution:*
   This set is countable. We represent the numbers in the table below. Each row is labeled by the number of 1's before the decimal for any number in that row. Each column is labeled by the number of 1's after the decimal for any number in that column. (Note that we need a column for all the numbers with infinite recurring 1's after the decimal). We now have an infinite 2-dimensional table representing all the numbers. We cannot count all the numbers by counting off each row (or each column) since each row or column is infinite. Instead, we use dovetailing by counting off each diagonal, such as $1, .\overline{1}$ *followed by* $11, 1.\overline{1}, .1$*, followed by* $111, 11.\overline{1}, 1.1, .11$*, and so on.*

	0	∞	1	2	3	4	...
0		$.\overline{1}$	.1	.11	.111	.1111	...
1	1	$1.\overline{1}$	1.1	1.11	1.111	1.1111	...
2	11	$11.\overline{1}$	11.1	11.11	11.111	11.1111	...
3	111	$111.\overline{1}$	111.1	111.11	111.111	111.1111	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

   (b) **[8 Pts]** the set of real numbers with decimal representation consisting of 1's and 9's
   Solution:

This set is uncountable. We will prove this by showing the set contains an uncountable subset. Let S be the set of real numbers in the interval $[0, 1]$ with infinite decimal representation consisting only of 1's and 9's, for example, $0.1111\overline{1}$, which has infinite number of 1's after the decimal, is included in S, but 0.1111, which has finite number of 1's after the decimal, is not included. Assume, for contradiction, that S is countable. Then there is a sequence $s_0, s_1, s_2, \ldots$ such that each s_i is an element of s, the s_i's are countable, and all elements of S appear in this countable sequence. We will write $s_i[i]$ to mean "the ith digit of the number s_i."

We diagonalize against this countable sequence by defining a new number t digit by digit, as follows, where $t[1]$ is the first digit of t after the decimal, $t[2]$ is the second digit after the decimal, and so on.

$$t[i] = \{ 1 \;\text{ if } s_i[i] = 99 \text{ if } s_i[i] = 1$$

t is a member of S because it is a number between 0 and 1 with decimal representation being an infinite sequence of 1's and 9's. However, the existence of t contradicts the claim that the sequence $s_1, s_2, s_3, \ldots$ is countable, because t differs from every s_i in the ith digit (and perhaps elsewhere also). Therefore, S is uncountable. Since a superset of an uncountable set is itself uncountable, we have proved that the set in the problem statement is uncountable.

3. **[9 Pts]** Give an example of two uncountable sets A and B such that $A \cap B$ is (a) finite, (b) countably infinite, (c) uncountably infinite.

   (a) finite
   Solution
   Let A, B be uncountable subsets of $\mathbb{R}$ such that $A = (0, 1]$ and $B = [1, 2)$.
   $A \cap B = \{1\}$
   A, B are uncountable subsets of $\mathbb{R}$. Since A and B shares one element, $A \cap B$ is finite.

   (b) countably infinite
   Solution
   Let $A = \mathbb{N} \cup (0, 1)$ and let $B = \mathbb{N} \cup (1, 2)$.
   $A \cap B = \mathbb{N}$
   A, B both include uncountable subsets of $\mathbb{R}$. So both of them are uncountable sets. $A \cap B = \mathbb{N}$ which is countably infinite.

   (c) uncountably infinite
   Solution
   Let $A = (0, 2)$ and let $B = (0, 1)$.
   $A \cap B = (0, 1) = A$
   A, B are uncountable subsets of $\mathbb{R}$. $A \cap B = B$ which is uncountably infinite.

4. **[9 Pts]** Prove that the set of functions from $\mathcal{N}$ to $\{0, 1\}$ is uncountable, by using a diagonalization argument.

   Suppose the set of functions from $\mathcal{N}$ to $\{0, 1\}$ is countable. Then we can list them all: $f_0, f_1, f_2, \ldots$. Define a new function f_ν, where $f_\nu(n) = 0$ if $f_n(n) = 1$ and $f_\nu(n) = 1$ if $f_n(n) = 0$. Thus, f_ν is not any of the functions in our list, since it differs from each function at at least one point, but it's a function from $\mathcal{N}$ to $\{0, 1\}$. This contradicts that we listed all of the functions, so the set is uncountable.