CS 330 : Discrete Computational Structures

Fall Semester, 2015

ASSIGNMENT #5

Due Date: Sunday, Oct 11

Suggested Reading: Rosen Section 5.1 - 5.2; Lehman et al. Chapter 5.1 - 5.3

These are the problems that you need to turn in. For more practice, you are encouraged to work on the other problems. **Always explain your answers and show your reasoning.**

1. **[35 Pts]** Prove the following statements by mathematical induction. Clearly state your basis step and prove it. What is your inductive hypothesis? Prove the inductive step and show clearly where you used the inductive hypothesis.

   (a) **[Ghazaleh, solution by Zhenbi Hu]** $1^3+2^3+\cdots+n^3 = (n(n+1)/2))^2$, for all positive integers n.
   Basis step: $P(1) : 1^3 = (1(1+1)/2)^2$.
   $1^3 = 1$ and $(1(1+1)/2)^2 = (2/2)^2 = 1$. So $P(1)$ is true.
   Inductive step:
   Let $P(k)$ holds true, which is:
   $P(k) : 1^3 + 2^3 + \cdots + k^3 = (k(k+1)/2)^2$.
   To prove the inductive step we need to show that the implication $P(k) \to P(k+1)$ is true.
   $P(k+1) : 1^3 + 2^3 + \cdots + k^3 = ((k+1)((k+1)+2)/2)^2$.

$$\begin{aligned}(1^3 + 2^3 + \cdots + k^3) + (k+1)^3 &= (k(k+1)/2)^2 + (k+1)^3, \text{ by inductive hyp.}\\ &= (k+1)^2(k/2)^2 + (k+1)^2(k+1)\\ &= (k+1)^2(k^2/4 + k + 1)\\ &= (k+1)^2(k^2 + 4k + 4)/4\\ &= ((k+1)(k+2)/2)^2\end{aligned}$$

   Therefore, $P(k+1)$ is true.
   We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n.

   (b) **[Ghazaleh, solution by Zhenbi Hu]** $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$, for all positive integers n.
   Basis step: $P(1) : 1 \cdot 1! = (1+1)! - 1$.
   $1 \cdot 1! = 1$ and $(1+1)! - 1 = 1$. So $P(1)$ is true.
   Inductive step:
   We assume that $P(k)$ is true, where
   $P(k) : 1 \cdot 1! + \cdots k \cdot k! = (k+1)! - 1$.
   We prove that $P(k+1)$ is true, where

$P(k+1): 1\cdot 1! + \cdots + k\cdot k! + (k+1)\cdot(k+1)! = (k+2)! - 1.$

$$\begin{aligned}
&1\cdot 1! + \cdots + k\cdot k! + (k+1)\cdot(k+1)!\\
=&(k+1)! - 1 + (k+1)\cdot(k+1)!, \text{ by inductive hypothesis}\\
=&(k+1)!\cdot(k+1+1) - 1\\
=&(k+1)!\cdot(k+2) - 1\\
=&(k+2)! - 1
\end{aligned}$$

Therefore, $P(k+1)$ is true.
We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n.

(c) [**Ghazaleh, solution by Swagoto Roy**] $1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \cdots + n(n+1)(n+2) = n(n+1)(n+2)(n+3)/4$, for all positive integers n.
Basis step: $n = 1$ $P(1): 1\cdot 2\cdot 3 = 1\cdot 2\cdot 3\cdot 4/4$
$1\cdot 2\cdot 3 = 6$ and $1\cdot 2\cdot 3\cdot 4/4 = 6$ as well, so $P(1)$ holds.
Inductive Step: Assume $P(k)$ holds true, where
$P(k): 1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \cdots + k(k+1)(k+2) = k(k+1)(k+2)(k+3)/4$
To prove the inductive step we need to show that the implication $P(k) \to P(k+1)$ is true. So, prove $P(k+1)$ is true, where
$P(k+1): 1\cdot 2\cdot 3 + \cdots + k(k+1)(k+2) + (k+1)(k+2)(k+3) = (k+1)(k+2)(k+3)(k+4)/4.$

$$\begin{aligned}
&1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \cdots + k(k+1)(k+2) + (k+1)(k+2)(k+3)\\
=&k(k+1)(k+2)(k+3)/4 + (k+1)(k+2)(k+3) \text{ by inductive hypothesis}\\
=&(k+1)(k+2)(k+3)(k/4+1)\\
=&(k+1)(k+2)(k+3)(k+4)/4
\end{aligned}$$

We have completed both the basis step and the inductive step, so by the principle of mathematical induction $P(k) \to P(k+1)$, the statement is true for every positive integer k.

(d) [**Ghazaleh, solution by Swagoto Roy**] $n! < n^n$, for all integers greater than 1.
Base case: For $n = 2$, $P(2): 2! < 2^2$.
$2! = 2$ and $2^2 = 4$. Since $2 < 4$, $P(2)$ is true.
Inductive step:
Suppose $P(k)$ is true, where $P(k): k! < k^k$.
Prove that $P(k+1)$ is true, where $P(k+1): (k+1)! < (k+1)^{k+1}$.

$$\begin{aligned}
(k+1)! &= (k+1)k!\\
&< (k+1)k^k, \text{ by inductive hypothesis}\\
&< (k+1)(k+1)^k \text{ since } k^k < (k+1)^k\\
&< (k+1)^{k+1}
\end{aligned}$$

Therefore, $P(k+1)$ is true. We have completed both the basis step and the inductive step, so by the principle of mathematical induction $P(k) \to P(k+1)$, the statement is true for every positive integer k greater than 1.

(e) **[Peter]** 6 divides $n^3 - n$, for all non-negative integers n.

We're going to do this by strong induction - instead of proving that $P(n) \rightarrow P(n+1)$, we'll prove that $P(n) \rightarrow P(n+2)$, because I think that's easier to prove. We need more than one base case, now - $P(0)$ and $P(n) \rightarrow P(n+2)$ give us all the even numbers, but doesn't tell us anything about the odd numbers. However, by proving $P(1)$ as well, we can use $P(n) \rightarrow P(n+2)$ to conclude that $P(n)$ is true for all odd numbers as well.

Base cases: $P(0)$ and $P(1)$. $0^3 - 0 = 0$, and $1^3 - 1 = 0$, and 6 divides 0.
Inductive step: Suppose 6 divides $n^3 - n$. Prove 6 divides $(n+2)^3 - (n+2)$.
$(n+2)^3 - (n+2) = n^3 + 6n^2 + 12n + 8 - n - 2 = n^3 - n + 6n^2 + 12n + 6$
6 clearly divides $6n^2 + 12n + 6$. By the inductive hypothesis, 6 divides $n^3 - n$. Since 6 divides the entire right side, 6 divides $(n+2)^3 - (n+2)$.

2. **[15 Pts Peter, solution by Zhenbi]** Let $P(n)$ be the statement that n-cent postage can be formed using just 4-cent and 7-cent stamps. Prove that $P(n)$ is true for all $n \geq 18$, using the steps below.

(a) First, we prove $P(n)$ by regular induction. Prove (i) $P(18)$, and (ii) $P(k) \rightarrow P(k+1)$ for all $k \geq 18$.

$P(18)$ is true because $2(7) + 4 = 18$, so two seven cent stamps with one four cent stamp make eighteen cents. So now assume $k \geq 18$, and $P(k)$ is true. We want to show $P(k+1)$ is true. If a 7-cent stamp is used to add up to k postage, we can take away that stamp and add two 4-cent stamps to produce $k - 7 + 4 \times 2 = k + 1$ cents is stamps. On the other hand, if there is no 7-cent stamp used, then k must be a multiple of 4. Since $k \geq 18$, it must be the case that $k \geq 20$, so there are at least five 4-cent stamps used to add up to k. Then remove those five 4-cent stamps, and add three 7-cent stamps, to produce $k - 5 \times 4 + 3 \times 7 = k + 1$ using 4-cent and 7-cent stamps. In either case, $P(k+1)$ is true.

(b) Now, we prove $P(n)$ by strong induction. Prove that $P(18)$, $P(19)$, $P(20)$ and $P(21)$ to complete the basis step.

$P(18)$ is true, because we can form 18 cents of postage with one 4-cent stamp and two 7-cent stamps.
$P(19)$ is true, because we can form 19 cents of postage with three 4-cent stamps and one 7-cent stamp.
$P(20)$ is true, because we can form 20 cents of postage with five 4-cent stamps.
$P(21)$ is true, because we can form 21 cents of postage with three 7-cent stamps.

(c) Now, state and prove the inductive step to complete the strong induction proof.

We want to form $k+1$ cents of postage. Since $k - 3 \geq 18$, we know that $P(k-3)$ is true, that is, that we can form $k - 3$ cents of postage. Put one more 4-cent stamp on the envelope, and we have formed $k+1$ cents of postage, as desired.