CS 330 : Discrete Computational Structures
Fall Semester, 2015
ASSIGNMENT #8
Due Date: Tuesday, Nov 3

Suggested Reading: Rosen 9.1 and 9.5, LLM 9.4

These are the problems that you need to turn in. For more practice, you are encouraged to work on the other problems. **Always explain your answers and show your reasoning.**

1. **[18 Pts]** For each of these relations decide whether it is reflexive, anti-reflexive, symmetric, anti-symmetric and transitive. Justify your answers. R_1 and R_2 are over the set of real numbers and R_3 is over the set of all people.

   (a) $(x, y) \in R_1$ if and only if $xy = 4$
   Solution:
   NOT reflexive because $(1, 1) \notin R_1$
   NOT anti-reflexive because $(2, 2) \in R_1$
   Symmetric because if we have $(x, y) \in R_1$ then xy = 4 = yx, so $(y, x) \in R1$.
   NOT antisymmetric because $(1, 4), (4, 1) \in R_1$
   NOT transitive because $(1, 4), (4, 1) \in R_1$ but $(1, 1) \notin R_1$.

   (b) $(x, y) \in R_2$ if and only if $x = 1$ or $y = 1$
   Solution:
   NOT reflexive because $(0, 0) \notin R_2$
   NOT anti-reflexive because $(1, 1) \in R_2$
   Symmetric because if $(x, y) \in R_2$, $x = 1$ or $y = 1$, so $(y, x) \in R_2$
   NOT antisymmetric because $(1, 4), (4, 1) \in R_2$
   NOT transitive because $(0, 1), (1, 2) \in R_2$ but $(0, 2) \notin R_2$.

   (c) $(x, y) \in R_3$ if and only if x and y share a common parent
   Solution:
   Reflexive because every person shares biological parents with themself.
   NOT anti-reflexive because people share biological parents with themself
   Symmetric because if x and y share a common parent then y and x share a common parent.
   NOT antisymmetric If two people A and B, have two kids together, C and D, then $(C, D), (D, C) \in R_3$, but $C \neq D$.
   NOT transitive because If x and y share a father and y and z share a mother, x and z may not share either.

2. **[12 Pts]** Consider relation R_4 on the set of positive real numbers where $(x, y) \in R_4$ if and only if $x/y \in \mathcal{Q}$. Decide whether it is reflexive, anti-reflexive, symmetric, anti-symmetric and transitive and show that this an equivalence relation. Describe the equivalence classes. What is the equivalence class of 2? of π? Justify your answers.

   Reflexive: Yes For any $x \in \mathcal{R}^+$, $x/x = 1$. Since $1 \in \mathcal{Q}$, $(x, x) \in R_4$.

   Anti-Reflexive: No $1/1 = 1$, which is in $\mathcal{Q}$, so $(1, 1) \in R_4$.

Symmetric: Yes If $(x, y) \in R_4$, $x/y \in \mathcal{Q}$. By the definition of rational, $x/y = m/n$ where $m, n \in \mathcal{Z}$. Since $x, y \in \mathcal{R}^+$, $x/y > 0$, so $x/y \neq 0$, so $m \neq 0$. So, $n/m \in \mathcal{Q}$, so $y/x \in \mathcal{Q}$. Thus, $(y, x) \in R_4$.

Anti-Symmetric: No $(1, 2)$ and $(2, 1)$ are in R_4, since $\frac{1}{2}$ and 2 are rational, but $1 \neq 2$.

Transitive: Yes If (x, y) and (y, z) are in R_4, then x/y and y/z are rational. The product of two rationals is rational, so x/z is rational, so $(x, z) \in R_4$.

Since R_4 is reflexive, transitive, and symmetric, it is an equivalence relation. The equivalence class of a real number x is $\{qx | q \in \mathcal{Q}^+\}$. The equivalence class of 2 is all rational numbers - for any $q \in \mathcal{Q}^+$, 2 is related to q because $2/q$ is still in $\mathcal{Q}^+$. The equivalence class of π is $\{q\pi | q \in \mathcal{Q}^+\}$. Given a rational multiple $q\pi$, π is related to $q\pi$ because $q\pi/\pi = q$, which is rational.

3. **[12 Pts]** Let R_5 be the relation on $\mathcal{Z} \times \mathcal{Z}$ where $((a, b), (c, d) \in R_5$ if and only if $a - c = b - d$.

   (a) Prove that R_5 is an equivalence relation.

   We need to show that R_5 is reflexive, transitive, and symmetric.

   Reflexive: (a, b) is related to (a, b), because $a - a = b - b$.

   Symmetric: If (a, b) is related to (c, d), then $a - c = b - d$. Negate both sides: $c - a = d - b$. So, (c, d) is related to (a, b).

   Transitive: If (a, b) is related to (c, d) and (c, d) is related to (e, f), then $a - c = b - d$ and $c - e = d - f$. Add these equations: $a - e = b - f$. So, (a, b) is related to (e, f).

   (b) Define a function f such that $f(a, b) = f(c, d)$ if and only if $((a, b), (c, d)) \in R_5$.

   Let $f(a, b) = a - b$. Then, if $f(a, b) = f(c, d)$, $a - b = c - d$, so $a - c = b - d$, so $(a, b) \sim (c, d)$. Conversely, if $(a, b) \sim (c, d)$, then $a - c = b - d$, so $a - b = c - d$, so $f(a, b) = f(c, d)$.

   (c) Define the equivalence class containing $(1, 1)$ and list 2 elements in the class.

   The equivalence class of $(1, 1)$ is all pairs (c, d) such that $1 - c = 1 - d$. In other words, all numbers such that $c = d$. In set builder notation, $[(1, 1)]R_5 = \{(x, x) | x \in \mathcal{Z}\}$.

   (d) Describe each equivalence class. How many classes are there and how many elements in each class?

   The equivalence class of (a, b) is all pairs (c, d) such that $a - c = b - d$. This can be written as $\{(c, d) | a - b = c - d\}$. $\forall n \in \mathcal{Z}$, $\{(c, d) | c - d = n\}$ is an equivalence class. There is one equivalence class, for each integer, so there are (countably) infinitely many equivalence classes. Each equivalence class contains (countably) infinitely many elements.

4. **[4 Pts]** Describe all the equivalence classes for the relation *congruence modulo 6* over $\mathcal{Z}$, using set-builder notation.

   Solution:

   $[0] = \{\ x \in Z | x = 6k, k \in Z\ \}$

   $[1] = \{\ x \in Z | x = 6k + 1, k \in Z\ \}$

   $[2] = \{\ x \in Z | x = 6k + 2, k \in Z\ \}$

   $[3] = \{\ x \in Z | x = 6k + 3, k \in Z\ \}$

   $[4] = \{\ x \in Z | x = 6k + 4, k \in Z\ \}$

$[5] = \{\ x \in Z | x = 6k + 5, k \in Z\ \}$

In general we can say:

$\forall\ m$ such that $0 \leq m \leq 5 : [m] = \{x \in Z | x = 6k + m, k \in Z\}$

5. **[4 Pts]** LLM Problem 9.9 (c)

   One possible schedule: 18.01, 18.02, 18.03, 8.01, 8.02, 6.001, 6.002, 6.003, 6.004, 6.033, 6.034, 6.042, 6.046, 6.840, 6.857