CS 330 : Discrete Computational Structures
Spring Semester, 2014
ASSIGNMENT #6 SOLUTIONS
Due Date: Tuesday, Mar 11

Suggested Reading: Rosen Section 5.2 - 5.3; Lehman et al. Chapter 5

These are the problems that you need to turn in. For more practice, you are encouraged to work on the other problems. **Always explain your answers and show your reasoning.**

1. **[10 Pts Swagoto]** Rosen, Section 5.3: Exercise 8 (a) (d)

   Solution:
   8(a) : Given $a_n = 4n - 2$ ..... *(1)*
   $n = 1, 2, 3, ...$
   So, $a_1 = 4 - 2 = 2$
   Now, $a_{n-1} = 4(n-1) - 2$
   or $a_{n-1} = 4n - 6$ ..... *(2)*
   Subtracting (2) from (1) we get
   $a_n - a_{n-1} = 4$
   Recursive definition of a_n*:*
   $a_1 = 2$,
   $a_n = a_{n-1} + 4$ *for* $n \geq 2$

   8(d) : Given $a_n = n^2$ ..... *(1)*
   $n = 1, 2, 3, ...$
   So, $a_1 = 1^2 = 1$
   Now, $a_{n-1} = (n-1)^2$
   or $a_{n-1} = n^2 - 2n + 1$ ..... *(2)*
   Subtracting (2) from (1) we get
   $a_n - a_{n-1} = 2n - 1$
   Recursive definition of a_n*:*
   $a_1 = 2$,
   $a_n = a_{n-1} + 2n - 1$ *for* $n \geq 2$

2. **[10 Pts Elliott]** Rosen, Section 5.3: Exercise 14

   Solution:
   Base step: $n = 1$
   $P(1) : f_1 f_2 - f_1^2 = (-1)^1$
   $f_1 f_2 - f_1^2 = 0 \cdot 1 - 1^2 = -1$ *and* $(-1)^1 = -1$. *So,* $P(1)$ *is true.*
   Inductive step:
   Assume that $P(n)$ *is true, where* $P(n) : f_{n-1} f_{n+1} - f_n^2 = (-1)^n$.

Prove that $P(k+1)$ *is true, where* $P(n+1): f_n f_{n+2} - f_{n+1}^2 = (-1)^{n+1}$.

$$\begin{aligned}
f_n \cdot f_{n+2} - f_{n+1}^2 &= f_n(f_n + f_{n+1}) - f_{n+1}^2 \\
&= f_n^2 + f_n f_{n+1} - f_{n+1}^2 \\
&= f_n^2 + f_{n+1}(f_n - f_{n+1}) \\
&= f_n^2 - f_{n+1}(f_{n+1} - f_n) \\
&= f_n^2 - f_{n+1} f_{n-1} \\
&= -(f_{n-1} f_{n+1} - f_n^2) \\
&= (-1)(-1)^n, \quad \textit{by induction hypothesis} \\
&= (-1)^{n+1}
\end{aligned}$$

This proves $P(n+1)$. *Since both the base case and the inductive step are proved, therefore the claim is proved by induction.*

3. **[10 Pts Zhenbi]** Rosen, Section 5.3: Exercise 16

Solution:
Base step: $n = 1$
$P(1): f_0 - f_1 + f_2 = f_1 - 1$
Since $f_0 = 0$, $f_1 = 1$ *and* $f_2 = 1$, *we get* $f_0 - f_1 + f_2 = 0 - 1 + 1 = 0$ *and* $f_1 - 1 = 1 - 1 = 0$.
So, $P(1)$ *is true.*
Inductive step:
Assume $P(n)$ *is true, where* $P(n): f_0 - f_1 + f_2 - \cdots - f_{2n-1} + f_{2n} = f_{2n-1} - 1$.
Prove $P(n+1)$ *is true, where* $P(n+1): f_0 - f_1 + f_2 - \cdots - f_{2n+1} + f_{2n+2} = f_{2n+1} - 1$.

$$\begin{aligned}
f_0 - f_1 + f_2 - \cdots - f_{2n+1} + f_{2n+2} &= f_{2n-1} - 1 - f_{2n+1} + f_{2n+2}, \quad \textit{by IH} \\
&= f_{2n-1} - 1 - f_{2n+1} + f_{2n} + f_{2n+1} \\
&= f_{2n-1} + f_{2n} - 1 \\
&= f_{2n+1} - 1
\end{aligned}$$

This proves $P(n+1)$. *Since both the base case and the inductive step are proved, therefore the claim is proved by induction.*

4. **[20 Pts Zhenbi]** Lehman et al. Problem 5.10

Solution:
The proof is by induction. Let P(n) be the proposition that periphery length is even after n squares are placed. In the base case, P(1) is true because the periphery of a single square has length 4, which is even.
In the inductive step, assume that the periphery length is even after n squares are placed to prove that the periphery length is even after n+1 squares are placed. The (n+1)th square could share 1, 2, 3, or 4 edges with previously-placed squares.

If the new square shares 1 edge with a previously placed square, then this one edge is removed from the periphery, but three edges of the new square are added to the periphery. Overall, the periphery length increases by two and thus remains even.
If the new square shares 2 edges with previously placed squares, then these two edges are removed from the periphery, but two edges of the new square are added. The periphery length is unchanged and thus remains even.
If the new square shares 3 edges, then these three edges are removed from the periphery, but one edge is added. The periphery length decreases by two and remains even.
If the new square shares 4 edges, then these four edges are removed from the periphery and none are added. The periphery length decreases by four and remains even.
In all cases, the length of the periphery remains even. Therefore, for all $n \geq 1$, P(n) implies P(n+1) and the claim is proved by induction.

5. **[10 Pts Swagoto]** Lehman et al. Problem 5.18

Solution:
Predicate P holds for certain non negative integers.
For $n = 0, 1, 2, 3, ...$ *it has been proved* $P(n) \rightarrow P(n+3)$
(a) Suppose it is proved that P(5) holds.
Then $P(5) \rightarrow P(8) \rightarrow P(11) \rightarrow P(14)....$
1. P(n) holds for all $n \geq 5$
False. For example, $P(6)$ *does not hold.*
2. P(3n) holds for all $n \geq 5$
False. For example, $P(15)$ *does not hold.*
3. P(n) holds for $n = 8, 11, 14, ...$
True (as we know $P(5) \rightarrow P(8) \rightarrow P(11) \rightarrow P(14)....$*)*
4. P(n) does not hold for $n < 5$
False. For example, we do not know P(n) holds for $n < 5$.
5. $\forall n, P(3n+5)$
True. Substitute n=0,1,2,... We know P(5),P(8),P(11)...holds.
6. $\forall n > 2, P(3n-1)$
True. Substitute n=3,4,5,... we know P(8),P(11),P(14),... holds.
7. $P(0) \rightarrow \forall n P(3n+2)$
False. For example if $n = 0$*, P(2) but P(2) not known.*
8. $P(0) \rightarrow \forall n P(3n)$
True. since it is known $P(n) \rightarrow P(n+3)$

(b) In order to conclude P(n) holds for $n \geq 5$, $P(5), P(6), P(7), P(8), P(9), ...$ *must hold true*
So the following three sequences must be proved:
a) $P(5) \rightarrow P(8) \rightarrow P(11).....$
b) $P(6) \rightarrow P(9) \rightarrow P(12).....$
and
c) $P(7) \rightarrow P(10) \rightarrow P(13).....$

1. $P(0)$ No. Not sufficient as only $P(0) \to P(3) \to P(6)$..... proved
2. $P(5)$ No. Not sufficient as only $P(5) \to P(8) \to P(11)$..... proved
3. $P(5)$ and $P(6)$ Not sufficient as only $P(5) \to P(8) \to P(11)$....., and $P(6) \to P(9) \to P(12)$..... proved
4. $P(0)$, $P(1)$, and $P(2)$ Yes. All three sequences can be proved. Sufficient.
5. $P(5)$, $P(6)$, and $P(7)$ Yes. All three sequences can be proved. Sufficient.
6. $P(2)$, $P(4)$, and $P(5)$ No. Cannot prove the sequence $P(6) \to P(9) \to P(12)$.....
7. $P(2)$, $P(4)$, and $P(6)$ Yes. All three sequences can be proved
8. $P(3)$, $P(5)$, and $P(7)$ Yes. All three sequences can be proved

6. **[20 Pts Xiyuan]** Consider the following state machine. The machine has four states, labeled 0, 1, 2 and 3. The start state is 0. The transitions are $0 \to 1$, $1 \to 2$, $2 \to 3$, and $3 \to 0$.

   Prove that if we take n steps in the state machine we will end up in state 0 if and only if n is divisible by 4. Note that we cannot prove the statement above by induction. Instead, we need to *strengthen the induction hypothesis.* State the strengthened hypothesis and prove it.

   Solution:
   We strengthen the induction hypothesis to be:
   For all $n \geq 0$, after n steps, the state machine is in state p iff $n \equiv p \pmod 4$.

   Base case *At $n = 0$, the remainder machine is in state 0, and $0 \equiv 0 \pmod 4$.*

   Inductive step *Assume our strengthened hypothesis is true after k steps. We will show it remains true at $k+1$ steps. There are four cases to consider.*

   Case 1: $k \equiv 0 \pmod 4$ *The state machine is in state 0 after k steps, so it will be in state 1 after $k+1$ steps. Since k is divisible by 4, $k+1$ has remainder 1 when divided by 4, so $k+1 \equiv 1 \pmod 4$. So the strengthened hypothesis remains true at $k+1$.*

   Case 2: $k \equiv 1 \pmod 4$ *This argument is similar. The state machine is in state 1 after k steps, so it will be in state 2 after $k+1$ steps. If k leaves remainder 1 when divided by 4, then $k+1$ leaves remainder 2 when divided by 4, so $k+1 \equiv 2 \pmod 4$. Again, the strengthened hypothesis holds at $k+1$.*

   Case 3: $k \equiv 2 \pmod 4$ *Same idea. The state machine is in state 2 after k steps, so it will be in state 3 after $k+1$ steps. If k leaves a remainder of 2 when divided by 4, then $k+1$ leaves remainder 3 when divided by 4, so $k+1 \equiv 3 \pmod 4$, as needed.*

   Case 4: $k \equiv 3 \pmod 4$ *Same idea. The state machine is in state 3 after k steps, so it will be in state 0 after $k+1$ steps. If k leaves a remainder of 3 when divided by 4, then $k+1$ is divisible by 4, so $k+1 \equiv 0 \pmod 4$, as needed.*

7. **[20 Pts Xiang]** A robot wanders around a 2-dimensional grid. He starts out at (0,0) and can take the following steps: (+1,+2), (+3,0), (-2,-1). Define a state machine for this problem. Then, define a Preserved Invariant and prove that the robot will never get to (1,1).

Solution:

(a)

Let the set of states be Z × Z. *The start state is (0, 0). The possible transitions are*

$(x, y) \to (x, y) + (u, v)$ *(1)*

where $(u, v) \in \{(+1, +2), (+3, 0), (-2, -1)\}$.

(b)

Let $P(x, y) \equiv$ "$(x + y) \mod 3 = 0$".

We claim P is a preserved invariant. To show this, we must show that if $3|(x + y)$, *and the robot moves to* $(x, y) + (u, v) = (x + u, y + v)$, *then 3 divides*

(x+u)+(y+v). *(2)*

But this value equals

(x+y)+(u+v), *(3)*

and since $3|(u + v)$

for each of the four possible moves (u,v) listed above (as is easily checked), we conclude that 3 divides both terms in the sum (3) and therefore divides the whole sum. This proves implies that 3 divides (2), completing the proof that P is preserved by transitions.

Now P holds in the start state, since $3|(0 + 0)$. *However, P does not hold for the destination state, (1,1), since* $1 + 1 = 2$ *is not a multiple of 3. Therefore, by the Invariant Principle, (1,1) is not a reachable state.*

8. **[Extra Credit]** Lehman et al. Problem 5.35