CS 330 : Discrete Computational Structures

Fall Semester, 2015

ASSIGNMENT #6

Due Date: Sunday, Oct 18

Suggested Reading: Rosen Section 5.2 - 5.3; Lehman et al. Chapter 5

These are the problems that you need to turn in. For more practice, you are encouraged to work on the other problems. **Always explain your answers and show your reasoning.**

1. **[10 Pts]** Rosen, Section 5.3: Exercise 8 (a), (d)
   Anwser 8a:

$$
\begin{aligned}
a_n &= 4n - 2 \qquad \text{for } n \text{ in the set of positive integers} \\
a_{n-1} &= 4(n-1) - 2 \\
&= 4n - 6 \\
a_1 &= 4 * 1 - 2 \\
&= 2 \\
a_n - a_{n-1} &= 4
\end{aligned}
$$

   Recursive Def of a_n:

$$
\begin{aligned}
a_1 &= 2 \\
a_n &= a_{n-1} + 4 \qquad \text{for } n \geq 2
\end{aligned}
$$

   Anwser 8d:

$$
\begin{aligned}
a_n &= n^2 \qquad \text{for } n \text{ in the set of positive integers} \\
a_{n-1} &= (n-1)^2 \\
&= n^2 - 2n + 1 \\
a_1 &= 1^2 \\
&= 1 \\
a_n - a_{n-1} &= 2n - 1
\end{aligned}
$$

   Recursive Def of a_n:

$$
\begin{aligned}
a_1 &= 1 \\
a_n &= a_{n-1} + 2n - 1 \quad \text{for } n \geq 1
\end{aligned}
$$

2. **[8 Pts]** Rosen, Section 5.3: Exercise 16
   Answer:

   The proof would be something like this, but I don't understand how the series in the question works, so I don't know how to attack it.

   Basis: $P(1) : 1^3 = 1$ and $(1(\frac{1+1}{2}))^2 = (1 * 1)^2 = 1$
   Inductive Step:
   Assume $P(k)$ where $P(k) = 1^3 + 2^3 + \cdots + k^3 = k(\frac{k+1}{2})^2$
   Prove: $P(k) \rightarrow P(k+1)$
   $P(k+1) : 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = (k+1)(\frac{k+2}{2})$

$$\begin{aligned} 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= k(\tfrac{k+1}{2})^2 + (k+1)^3 \\ &= k(\tfrac{k+1}{2})^2 \\ &= (k+1)^2(\tfrac{k^2}{4} + k + 1) \\ &= (k+1)^2\tfrac{(k^2+4k+4)}{4} \\ &= (k+1)(\tfrac{k+2}{2}) \end{aligned}$$

3. **[12 Pts]** Consider the following state machine. The machine has four states, labeled 0, 1, 2, and 3. The start state is 0. The transitions are $0 \rightarrow 1$, $1 \rightarrow 2$, $2 \rightarrow 3$, and $3 \rightarrow 0$.

   Prove that if we take n steps in the state machine we will end up in state 0 if and only if n is divisible by 4. Argue why we cannot prove the statement above by induction. Instead, we need to *strengthen the induction hypothesis.* State the strengthened hypothesis and prove it.

   Answer:

   Strengthened Hyp: $P(n)$: If we take n steps in the state machine we will end up in state s if and only if $n\%4 = s$.

   Basis: $P(0) : 0\%4 = 0$, basis holds.
   Inductive Step:
   Assume $P(k)$ where $P(k) = n\%4 = s$
   Prove: $P(k) \rightarrow P(k+1)$

   Consider four cases:
   Case 1: $k\%4 = 0$, after k steps we are in state 0 because $\frac{k}{4}$ has a remainder of 0. Since $\frac{k}{4}$ has a remainder of 0 then $\frac{k+1}{4}$ will have a remainder of 1, placing the state machine in state 1.
   Case 2: $k\%4 = 1$, after k steps we are in state 1 because $\frac{k}{4}$ has a remainder of 1. Since $\frac{k}{4}$ has a remainder of 1 then $\frac{k+1}{4}$ will have a remainder of 2, placing the state machine in state 2.
   Case 3: $k\%4 = 2$, after k steps we are in state 2 because $\frac{k}{4}$ has a remainder of 2. Since $\frac{k}{4}$ has a remainder of 2 then $\frac{k+1}{4}$ will have a remainder of 3, placing the state machine in state 3.
   Case 4: $k\%4 = 3$, after k steps we are in state 3 because $\frac{k}{4}$ has a remainder of 3. Since $\frac{k}{4}$ has a remainder of 3 then $\frac{k+1}{4}$ will again be divisible by 4 and thus have a remainder of 0, placing the state machine in state 0.

4. [**8 Pts**] Lehman et al. Problem 5.10
   Answer:

   $P(n)$: after n squares have been places the periphery length will be even.

   Basis: $P(1)$: on square has 4 sides and 4 is even. $P(1)$ holds.

   Inductive Step:
   Assume $P(k)$: after k squares have been places the periphery length will be even.
   Prove: $P(k) \rightarrow P(k+1)$

   Consider four cases:
   Case 1: $k + 1th$ square shares 1 edge with existing periphery, the periphery loses 1 edge, but gains 3. This results in an overall increase of 2 and therefore the length remains even.
   Case 2: $k + 1th$ square shares 2 edges with existing periphery, the periphery loses 2 edges, but gains 2. This results in an overall increase of 0 and therefore the length remains even.
   Case 3: $k + 1th$ square shares 3 edges with existing periphery, the periphery loses 3 edges, but gains 1. This results in an overall decrease of 2 and therefore the length remains even.
   Case 4: $k + 1th$ square shares 4 edges with existing periphery, the periphery loses 4 edges, and gains 0. This results in an overall decrease of 4 and therefore the length remains even.

5. [**12 Pts**] A robot wanders around a 2-dimensional grid. He starts out at (0,0) and can take the following steps: (+2,-1), (-1,-1) and (0,3). Define a state machine for this problem. Then, define a Preserved Invariant and prove that the robot will never get to (2,0).

   Answer:

   To prove this I'd need to find an invariant that is true for all possible moves, but not true for position $(2, 0)$. Obviously one exists, but I cannot figure out what it is.