CS 330 : Discrete Computational Structures
Spring Semester, 2014
ASSIGNMENT #12 SOLUTIONS **[Extra Credit]**
Due Date: Friday, May 2

Suggested Reading: Chapter 11.1 - 11.3 and 11.9 - 11.11 of Lehman et al.

These are the problems that you need to turn in. Always explain your answers and show your reasoning. **Spend time giving a complete solution. You will be graded based on how well you explain your answers.**

1. **[20 Pts Zhenbi]** How many integers between 1000 and 9999 inclusive contain (a) at least one 0 and at least one 1, (b) at least one 0, at least one 1 and at least one 2? Solve part (a) using the Inclusion-Exclusion Principle for two sets, and part (b) using the Inclusion-Exclusion Principle for three sets: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

Solution:
(a) There are $9(10^3)$ *numbers from 1000 to 9999.*
Let A_0 *be the set of numbers in that range that contain no 0;*
let A_1 *be the set of numbers in that range that contain no 1.*
We need to count $\overline{A_0} \cap \overline{A_1}$*, the set of numbers in that range that contain at least one 0 and at least one 1. By De Morgan's Law,* $\overline{A_0} \cap \overline{A_1}$ *is the complement of* $A_0 \cup A_1$*.*
So $|\overline{A_0} \cap \overline{A_1}| = 9(10^3) - |A_0 \cup A_1|$*.*
Now we calculate $|A_0 \cup A_1|$*, using the Inclusion-Exclusion Principle. Since* A_0 *excludes the use of the 0 digit everywhere, but allows any other digits,* $|A_0| = 9(9^3) = 9^4$*. By the exact same argument with 1 instead of 0,* $|A_1| = 8(9^3)$*. On the other hand, the set* $A_0 \cap A_1$ *prevents a 1 in the leftmost place (0 is already excluded), and 0 or 1 in all other places, so* $|A_0 \cap A_1| = 8(8^3) = 8^4$*. By the Inclusion-Exclusion Principle:*

$$\begin{aligned} |A_0 \cup A_1| &= |A_0| + |A_1| - |A_0 \cap A_1| \\ &= 9^4 + 8(9^3) - 8^4 \end{aligned}$$

Therefore, the number of integers between 1000 and 9999 with at least one 0 and at least one 1 is:

$$\begin{aligned} |\overline{A_0} \cap \overline{A_1}| &= 9(10^3) - |A_0 \cup A_1| \\ &= 9(10^3) - [9^4 + 8(9^3) - 8^4] \\ &= 9(10^3) + 8^4 - 9^4 - 8(9^3) \\ &= 703 \end{aligned}$$

(b) *We set up the problem much as we set up (a). Let* A_0 *be the numbers that contain no 0,* A_1 *the numbers that contain no 1, and* A_2 *the numbers that contain no 2. We need to count* $\overline{A_0} \cap \overline{A_1} \cap \overline{A_2}$*, the set of numbers in that range that contain at least one 0,*

at least one 1 and at least one 2. By De Morgan's Law, $\overline{A_0} \cap \overline{A_1} \cap \overline{A_2}$ *is the complement of* $A_0 \cup A_1 \cup A_2$*. So,* $|\overline{A_0} \cap \overline{A_1} \cap \overline{A_2}| = 9(10^3) - |A_0 \cup A_1 \cup A_2|$*.*
Now, we calculate $|A_0 \cup A_1 \cup A_2|$ *using the Inclusion-Exclusion Principle.*
From the argument in (a), $|A_0| = 9^4$ *and* $|A_1| = |A_2| = 8(9^3)$*. Again from the argument in (a),* $|A_0 \cap A_1| = |A_0 \cap A_2| = 8^4$*. To compute* $|A_1 \cap A_2|$*, note that we are excluding 0, 1, or 2 from the leftmost place, and 1, 2 from the other three places, so* $|A_1 \cap A_2| = 7(8^3)$*. Finally, to compute* $|A_0 \cap A_1 \cap A_2|$*, note that we are excluding 0, 1 and 2 from all four places, so we obtain* 7^4*. By the Inclusion-Exclusion Principle for Three Sets, we obtain*

$$\begin{aligned}|A_0 \cup A_1 \cup A_2| &= |A_0| + |A_1| + |A_2| - |A_0 \cap A_1| - |A_0 \cap A_2| - |A_1 \cap A_2| + |A_0 \cap A_1 \cap A_2| \\ &= 9^4 + 8(9^3) + 8(9^3) - 8^4 - 8^4 - 7(8^3) + 7^4 \\ &= 9^4 + 16(9^3) - 2(8^4) - 7(8^3) + 7^4\end{aligned}$$

Then the answer to the problem is obtained by

$$\begin{aligned}|\overline{A_0} \cap \overline{A_1} \cap \overline{A_2}| &= 9(10^3) - |A_0 \cup A_1 \cup A_2| \\ &= 9(10^3) - [9^4 + 16(9^3) - 2(8^4) - 7(8^3) + 7^4] \\ &= 9(10^3) + 2(8^4) + 2(8^3) - 9^4 - 16(9^3) - 7^4 \\ &= 150\end{aligned}$$

2. **[10 Pts Zhenbi]** Let G be a simple, undirected graph that has 6 vertices with degrees of 5, 5, 3, 3, 2, 2. How many edges does the graph G have? Justify your answer.

   Solution:
   The sum of the degrees is $5 + 5 + 3 + 3 + 2 + 2 = 20$*. An edge joins two vertices, say* v *and* w*. That single edge accounts for a degree of 1 in* v *and a degree of 1 in* w*, so the sum of degrees of the vertices will be twice the number of edges. Therefore, any graph with this degree sequence will have 10 edges. We can also prove this immediately by invoking the Handshaking Theorem, which says that the number of edges is half the total vertex degree.*

3. **[10 Pts Elliott]** If G is a simple graph with n vertices and n edges, is G connected? If *yes*, give a short justification. If *no*, give a counterexample.

   Solution:
   No, a graph with n *vertices and* n *edges is not necessarily connected. A counterexample would be the graph* $G = (V, E)$ *consisting of two* 3*-node cycle graphs. More formally,* $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ *and* $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_1), (v_4, v_5), (v_5, v_6), (v_6, v_4)\}$*. This has* 6 *vertices and* 6 *edges but the graph has two connected components* $\{v_1, v_2, v_3\}$ *and* $\{v_4, v_5, v_6\}$*.*

4. **[20 Pts Aaron]** Prove that a simple graph is a tree if and only if it is connected but removing any edge will disconnect the graph.

   Solution:
   First, suppose simple graph G *is a tree. By definition,* G *is a connected graph with no cycles. So, in particular,* G *is connected. Now suppose there exists some edge*

$e \in E(G)$ such that removing e does not disconnect G. Let v and w be the endpoints of edge e. Since the removal of e does not disconnect G, there is some path in $G - \{e\}$ from v to w. Let's call this path $\langle v, u_1, u_2, u_3, \ldots, w\rangle$. The existence of such a path means that the path $P = \langle v, u_1, u_2, \ldots, w, v\rangle$ exists in G, since w is connected to v via e. But path P is a cycle that starts and ends at v, which contradicts the fact that G has no cycles. Therefore, by contradiction, for every edge e, it must be the case that removing e disconnects G.

Now we prove the other direction. Suppose G is connected, and the removal of any edge from G will disconnect G. Then G is connected, so to show it is a tree, we only need to show that it contains no cycles. Let's suppose that there exists a cycle in G that contains the edge e. Let v and w be the endpoints of e. By the argument above, there will still be a path from v to w even if e is removed. Therefore, removal of e does not disconnect G, contrary to our assumption about the properties of G. So it must be the case that G contains no cycles. Therefore G is a tree.

5. **[20 Pts Xiyuan]** Lehman et al. Problem 11.8

   Solution:
   (a) One counterexample would be a disconnected graph $G = (V, E)$ made up of a 3-node cycle graph (a triangle) and an additional single edge. More formally, $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_1), (v_4, v_5)\}$. G has two vertices of degree 1, and all other vertices of degree 2, but it is not a line graph.

   (b) The proof for the base case, for line graphs with one edge, is correct. For the inductive step, we assume that every two-ended graph of k edges is a line graph. We need to prove that every two-ended graph of $k+1$ edges is a line graph. The proof constructs a particular two-ended graph G_{n+1} of $k+1$ edges, by adding an edge to a k-edge two-ended graph G_n, and shows that it is a line graph. The error in the proof is that this does not consider all possible $k+1$ edge two-ended graphs since not all $k+1$ edge two-ended graphs can be constructed by adding an edge to a k-edge two-ended graph. So, it does not prove that every two-ended graph of $k+1$ edges is a line graph.

6. **[20 Pts Xiang]** Lehman et al. Problem 11.11

   Solution:
   We model the situation as a bipartite matching problem as follows. The vertices of the bipartite graph is $X \cup Y$ where X is the vertices on the left side and Y is the vertices on the right. Let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of students. Define the set $Y = \{y_{ij} | 1 \le i \le 4, 1 \le j \le 20\}$. Each vertex y_{ij} corresponds to seat j in recitation i. We now define the set of edges E. For each student x_k, if the student's schedule allows her to attend recitation i, we let $(x_k, y_{ij}) \in E$ for all j such that $1 \le j \le 20$. A bipartite matching M of this graph corresponds to a scheduling of students into recitations. If $(x_k, v_{ij}) \in M$, then this means student x_k is going to occupy seat j in recitation i. Since $(x_k, v_{ij}) \in E$, clearly recitation i does not conflict with student x_k's schedule. Also, since M is a matching, there is no other student x_ℓ such that $(x_\ell, v_{ij}) \in M$, so no other student x_ℓ is also assigned to seat j in recitation i.