CS 330 : Discrete Computational Structures
**Fall Semester, 2015**
ASSIGNMENT #11 **[Extra Credit]**
**Due Date:** Sunday, Dec 6

**Suggested Reading:** Rosen Sections 6.4 - 6.5

These are the problems that you need to turn in. Always explain your answers and show your reasoning. **Spend time giving a complete solution. You will be graded based on how well you explain your answers. Just correct answers will not be enough!**

1. [**5 Pts**] Prove, using a combinatorial argument, that $C(m+n,2) = C(m,2) + C(n,2) + mn$, where $m, n \geq 2$.

   Consider choosing 2 spacemonauts from a group of $m$ martians and $n$ neptunians.
   Choose 2 from $m+n$: $C(m+n,2)$.
   Choose 2 martians, or choose 2 neptunians, or choose 1 of each: $C(m,2) + C(n,2) + mn$.
   Thus, $C(m+n,2) = C(m,2) + C(n,2) + mn$.

2. [**10 Pts**] Prove that $C(n,2k)C(2k,k) = C(n,k)C(n-k,k)$, where $n \geq 2k > 0$, by using (a) a combinatorial proof, (b) an algebraic proof.

   a. From a group of $n$ spacemonauts about to go on a daring space mission, choose $k$ for ground control, then from the remaining spacemonauts, choose $k$ to into space: $\binom{n}{k}\binom{n-k}{k}$.
   Or, choose $2k$ for the space mission, then choose $k$ of those to be on ground control (the remaining $k$ are the ones going into space): $\binom{n}{2k}\binom{2k}{k}$.
   Thus, $\binom{n}{k}\binom{n-k}{k} = \binom{n}{2k}\binom{2k}{k}$.

   b.

$$\begin{aligned}
C(n,2k)C(2k,k) &= \frac{n!}{2k!(n-2k)!}\frac{2k!}{k!k!} \\
&= \frac{n!}{(n-2k)!}\frac{1}{k!k!} \\
&= \frac{n!}{k!}\frac{1}{k!(n-2k)!} \\
&= \frac{n!}{k!(n-k)!}\frac{(n-k)!}{k!(n-2k)!} \\
&= C(n,k)C(n-k,k)
\end{aligned}$$

3. [**5 Pts**] Prove, using a combinatorial argument, that

$$C(2n,n) = \sum_{k=0}^{n}[C(n,k)]^2.$$

   *Hint: Show that both sides count the number of ways to select a committee of size $n$ from a group of $n$ men and $n$ women.*

   Selecting a committee of $n$ people from $n$ men and $n$ women is $C(2n,n)$.
   We can also break it up based on number of men on the committee. We can add up the

ways to form a committee with 0 men, 1 man, 2 men, and so on, up to n men. To form a committee with $k$ men, there are $\binom{n}{k}$ ways to choose the men, and then $\binom{n}{n-k}$ ways to choose the rest of the committee from the women. We get $\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}$.
Since $\binom{n}{k}=\binom{n}{n-k}$, this simplifies to $\sum_{k=0}^{n}\binom{n}{k}^2$. Thus, $\binom{2n}{n}=\sum_{k=0}^{n}\binom{n}{k}^2$.

4. **[7 Pts]** A cookie shop sells 5 different kinds of cookies. How many different ways are there to choose 16 cookies if (a) you have no restrictions? (b) you pick at least two of each? (c) you pick at least 4 oatmeal cookies, at least 3 sugar cookies and at most 5 chocolate chip cookies?

   a. Stars and bars: we have 5 categories of thing to choose from (4 bars) and 16 things to choose (16 stars). There are $\binom{16+5-1}{16}=\binom{20}{16}$ ways to choose this.

   b. We must pick 2 of each cookie. After we do that, we can get 6 more cookies, and we have 5 choices for each. Total: $\binom{6+5-1}{6}=\binom{10}{6}$

   c. We can count the number of ways to pick at least 4 oatmeal and at least 3 sugar, and subtract the ones with more than 5 chocolate chip.
   At least 4 oatmeal, at least 3 sugar: We have 7 cookies picked, 9 more to choose, so total $=\binom{9+5-1}{9}=\binom{13}{9}$.
   At least 4 oatmeal, at least 3 sugar, and more than 5 chocolate: More than 5 chocolate is the same as at least 6 chocolate, so we have 13 cookies chosen, so $\binom{3+5-1}{3}=\binom{7}{3}$.
   Total: $\binom{13}{9}-\binom{7}{3}$.

5. **[4 Pts]** If I have 5 bananas, 3 oranges, and 8 apples, how many ways can I distribute these to 16 friends, if each friend gets one fruit?

   *Solution:*
   *In this problem, the 5 bananas are indistinguishable, the 3 oranges are indistinguishable, and the 8 apples are indistinguishable. The 16 friends, however, are all distinct.*
   *If all 16 fruits were distinct, then the answer would be* 16! *since there would be 16 choices for the first friend, 15 for the next, and so on. Since some of the fruits are identical, however, we divide by* 5!3!8! *and thus our answer is* $\frac{16!}{5!3!8!}$

6. **[9 Pts]** How many ways are there to pack 18 different books into 6 boxes with 3 books each if (a) all 6 boxes are sent to different addresses? (b) all 6 boxes are sent to the same address? (c) 3 of the boxes are shipped to three different addresses while 3 are left to be addressed later?

   *Solution:*
   *(a) Since we are sending the boxes to different addresses, none of the boxes are indistinguishable, so we cannot swap any boxes to reduce the total number of ways to do this. Therefore, we put pick any 3 books per box until there are no more books,boxes left.*
   $C(18,3)C(15,3)C(12,9)C(9,6)C(6,3)C(3,3)=\frac{18!}{3!3!3!3!3!3!}$
   *(b) This is similar to the above, but the boxes are indistinguishable, so we divide out all the cases where the boxes can be swapped. So the answer is:*
   $\frac{18!}{3!3!3!3!3!3!6!}$
   *(c)In this case we only have three indistinguishable boxes, so we divide out those cases where the boxes are swapped. So the answer is:*
   $\frac{18!}{3!3!3!3!3!3!3!}$

7. **[5 Pts]** How many ways are there to pack 5 different books into 5 identical boxes with no restrictions placed on how many can go in a box (some boxes can be empty)? What if the books are identical?

*Solution:*
*If the books and boxes are identical, then the problem is "how many ways can 6 be written as a sum of positive integers?" The ways are:*
*5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1*
*for a total of 7 ways*

*If the books are different, then the problem breaks into the following cases, where each number represents a box containing that many books (empty boxes are left out for convenience):*
*1,1,1,1,1: There is only one way to put one book in each box.*
*2,1,1,1: There are* $\binom{5}{2}$ *ways to pick a pair of books to share a box.*
*2,2,1: There are* $\binom{5}{2}$ *ways to pick the first pair, then* $\binom{3}{2}$ *ways to pick the second pair. However, picking (A,B) then (C,D) is the same as picking (C,D) then (A,B), so this counts twice the number of solutions. We need to divide by 2 for an accurate count.*
*3,1,1:* $\binom{5}{3}$ *ways to pick the triplet.*
*3,2:* $\binom{5}{3}$ *ways to pick the triplet, then* $\binom{2}{2}$ *ways to pick the pair.*
*4,1:* $\binom{5}{4}$ *ways to pick the quintuplet.*
*5: There is only one way to put all the books in the same box.*
*The sum is 1+10+10*3/2+10+5+1=42 ways to put 5 different books into 5 identical boxes.*

8. **[5 Pts]** How many ways can we place 8 books on a bookcase with 5 shelves if the books are (a) indistinguishable copies (b) all distinct? Note that the position of the books on the shelves matter.

*Solution:*
*(a) We use the "stars and bars" counting method, with 8 books and 4 bars, because 4 bars are enough to divide the books into 5 separate areas. So the answer is* $\binom{8+5-1}{8} = \binom{12}{8} = 495$.

*(b) Order all the books first, then count the number of ways they can be put on the shelves in that order. The number of ways to order the books is* $8!$, *while the number of ways a single ordering can be put on the shelves is the answer to (a). So the answer to (b) is* $8!\binom{12}{8} = 19,958,400$.