Algebraic Geometry - *Exercise Sheet 3*

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Exercise 2

We have

$$
\begin{aligned}
&F : \mathrm{Sh}(X) \to \mathrm{Sh}_{\mathcal{B}}(X); \quad \mathcal{F} \mapsto \mathcal{F}|_{\mathcal{B}^{\mathrm{op}}} \\
&G : \mathrm{Sh}_{\mathcal{B}}(X) \to \mathrm{Sh}(X); \quad \mathcal{F} \mapsto (U \mapsto \varprojlim_{V \subset U, V \in \mathcal{B}} \mathcal{F}(V)).
\end{aligned}
$$

We need to show that there exists natural isomorphisms $\eta : F \circ G \to \mathrm{Id}_{\mathrm{Sh}_{\mathcal{B}}(X)}$ and $\xi : G \circ F \to \mathrm{Id}_{\mathrm{Sh}(X)}$. For convenience we will write $\mathcal{B}_U = \{V \in \mathcal{B} \mid V \subset U\}$, then we have

$$
F \circ G : \mathrm{Sh}_{\mathcal{B}}(X) \to \mathrm{Sh}_{\mathcal{B}}(X); \mathcal{F} \mapsto \left(U \mapsto \varprojlim_{V \in \mathcal{B}_U} \mathcal{F}(V) \right).
$$

Since $U \in \mathcal{B}$, in particular $U \in \mathcal{B}_U$, so U defines a final object in $\mathcal{B}_U$, therefore $\mathcal{F}(U)$ defines an initial in the diagram. So $\mathcal{F}(U)$ is the inverse limit up to unique isomorphism.
For the other composition, we have

$$
G \circ F : \mathrm{Sh}(X) \to \mathrm{Sh}(X); \mathcal{F} \mapsto \left(U \mapsto \varprojlim_{V \in \mathcal{B}_U} \mathcal{F}(V) \right),
$$

but now U is not in general in $\mathcal{B}$. However, $\mathcal{B}_U$ defines a cover of U by open sets, so since $\mathcal{F}$ is a sheaf $\mathcal{F}(U)$ satisfy the universal property of the following equalizer

$$
\begin{tikzcd}
\mathcal{F}(U) \arrow[r] & \prod_{V \subset U, V \in \mathcal{B}} \mathcal{F}(V) \arrow[r, shift left, "\mathrm{res}^{V_1}_{V_1 \cap V_2}"] \arrow[r, shift right, "\mathrm{res}^{V_1}_{V_1 \cap V_2}"'] & \prod_{V_1, V_2 \subset U} \mathcal{F}(V)
\end{tikzcd}
\tag{1}
$$

We claim that this universal property implies that $\mathcal{F}(U)$ satisfy the universal property of the inverse limit (in fact the two universal properties are the equivalent, but we don't need this fact) and thus $\mathcal{F}(U)$ is the inverse limit up to unique isomorphism.
To prove the claim consider $V, V' \in \mathcal{B}$ such that $V' \subset V \subset U$. Then equaliser diagram implies that the following diagram commutes

$$
\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[dr] & \mathcal{F}(V) \arrow[d, "\mathrm{res}^V_{V \cap V'}"] \\
& \mathcal{F}(V')
\end{tikzcd}
$$

So $\mathcal{F}(U)$ is a cone. Suppose K is another cone of the diagram. Then in particular the diagram

$$
\begin{tikzcd}
K \arrow[r] \arrow[d] & \mathcal{F}(V_1) \arrow[d, "\mathrm{res}^{V_1}_{V_1 \cap V_2}"] \\
\mathcal{F}(V_2) \arrow[r, "\mathrm{res}^{V_1}_{V_1 \cap V_2}"] & F(V_1 \cap V_2)
\end{tikzcd}
$$

commutes, where $V_1, V_2 \in \mathcal{B}_U$. So K is an equalizer and thus factors through $\mathcal{F}(U)$ by a unique map. These unique isomorphisms is the maps that will define the isomorphisms of sheaves (ie. natural isomorphism) that will again define the natural isomorphisms from the compositors $F \circ G$ and $G \circ F$ to the respective identities. It remains to show the neutrality requirements these unique isomorphisms are assemble into morphisms of sheaves that again assemble into natural transformations. We will fist consider the composition $G \circ F$.

Consider a pair $U' \subset U \subset X$ of open sets, and denote the inclusion by $\iota : U' \hookrightarrow U$ and fix a sheaf $\mathcal{F}$. Then $\iota^* := (G \circ F)(\mathcal{F})(\iota)$ is defined by the universal property of the inverse limit.

Let $V', V \in \mathcal{B}_{U'}$, then the following diagram commutes:

$$\begin{tikzcd}
\varprojlim_{W\in\mathcal{B}_U} \mathcal{F}(W) \arrow[r] \arrow[dr] \arrow[rr, bend left, dotted, "\iota^*"] & \mathcal{F}(V) \arrow[d] & \varprojlim_{W\in\mathcal{B}'_U} \mathcal{F}(W) \arrow[l] \arrow[dl] \\
& \mathcal{F}(V') &
\end{tikzcd} \tag{2}$$

Since $\mathcal{B}_{U'} \subset \mathcal{B}_U$, the left triangle holds for all $V, V' \in \mathcal{B}_{U'}$, so the limit over $\mathcal{B}_U$ is a cone over the diagram indexed by $\mathcal{B}_{U'}$. So by the universal property of the limit over $\mathcal{B}_{U'}$, there is a unique map a ι^*, such that the diagram commutes, for all $V, V' \in \mathcal{B}_{U'}$.

Now that we have this map, we want to show that the diagram

$$\begin{tikzcd}
\varprojlim_{W\in\mathcal{B}_U} \mathcal{F}(W) \arrow[r, "\iota^*"] \arrow[d, "\cong"] & \varprojlim_{W\in\mathcal{B}'_U} \mathcal{F}(W) \arrow[d, "\cong"] \\
\mathcal{F}(U) \arrow[r, "\mathcal{F}(\iota)"] & \mathcal{F}(U')
\end{tikzcd} \tag{3}$$

commutes, which shows that the unique isomorphism define a morphism of sheaves. We will denote the isomorphism on the left by ψ, the one on the right by ϕ, the restriction maps from $\varprojlim_{W\in\mathcal{B}_U} \mathcal{F}(W) \to \mathcal{F}(W)$ by ρ_W and from $\varprojlim_{W\in\mathcal{B}'_U} \mathcal{F}(W) \to \mathcal{F}(W)$ by ρ'_W. Then by the defining properties of ψ and ϕ,

$$\rho_W \circ \psi^{-1} = \mathrm{res}^U_W \quad \text{and} \quad \rho'_W \circ \phi^{-1} = \mathrm{res}^{U'}_W . \tag{4}$$

So since $\mathcal{F}$ is a sheaf $\rho'_W \circ \phi^{-1} \circ \mathcal{F}(\iota) = \rho_W \circ \psi^{-1}$ (it follows from the equalizer condition for the cover $\{U, U'\}$ of U.) This implies that diagram (3) commutes, by considering the diagram combined with diagram (2).

Let $\mathcal{F}, \mathcal{G} : \mathrm{Ouv}(X)^{\mathrm{op}} \to \mathrm{Sets}$ be sheaves and $\alpha : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves. Then like above $\hat{\alpha} = (G \circ F)(\alpha)$ is defined by the universal property of the inverse limit. Let $U \subset X$ open, then for all $V' \subset V$, the diagram

$$\begin{tikzcd}
\varprojlim_{V\in\mathcal{B}_U} \mathcal{F}(V) \arrow[r] \arrow[dr] \arrow[rrr, bend left, dotted, "\hat{\alpha}_U"] & \mathcal{F}(V) \arrow[r, "\alpha_V"] \arrow[d] & \mathcal{G}(V) \arrow[d] & \varprojlim_{V\in\mathcal{B}_U} \mathcal{G}(V) \arrow[l] \arrow[dl] \\
& \mathcal{F}(V') \arrow[r, "\alpha_{V'}"] & \mathcal{G}(V') &
\end{tikzcd} \tag{5}$$

without $\hat{\alpha}_U$ commutes. So $\varprojlim_{V\subset U} \mathcal{F}(V)$ is a cone over $\mathcal{G}(V)$ indexed by $\mathcal{B}_U$ and $\hat{\alpha}_U$ is the unique map such that the diagram commutes. To show that $\hat{\alpha}_U$ assemble into a morphism of sheaves we need the outer square in the following diagram to commute

$$\begin{tikzcd}
\varprojlim_{V\in\mathcal{B}_U} \mathcal{F}(V) \arrow[rrrr, "\hat{\alpha}_U"] \arrow[dd, "\iota^*"] \arrow[dr] & & & & \varprojlim_{V\in\mathcal{B}_U} \mathcal{G}(V) \arrow[dd, "\iota^*"] \arrow[dl] \\
& \mathcal{F}(V) \arrow[rr, "\alpha_V"] & & \mathcal{G}(V) & \\
\varprojlim_{V\in\mathcal{B}_{U'}} \mathcal{F}(V) \arrow[rrrr, "\hat{\alpha}_{U'}"] \arrow[ur] & & & & \varprojlim_{V\in\mathcal{B}_{U'}} \mathcal{G}(V) \arrow[ul]
\end{tikzcd}$$

We obeserve that the left and right triangles comutes by the definition of ι^* and that the top and bottum inner squares cummutes by the defention of $\hat{\alpha}$. So in particular, all paths from $\varprojlim_{V\in\mathcal{B}_U} \mathcal{F}(V)$ to $\mathcal{G}(V)$ agree and thus the outer square most commute by the universal propery of the limit at the bottum right.

The only thing that remains to show now is that the following diagram commutes

$$\begin{tikzcd}
\varprojlim_{V\in\mathcal{B}_U} \mathcal{F}(V) \arrow[r, "\hat{\alpha}_U"] \arrow[d, "\psi"] & \varprojlim_{V\in\mathcal{B}_U} G(V) \arrow[d, "\phi"] \\
\mathcal{F}(U) \arrow[r, "\alpha_U"] & \mathcal{G}(U)
\end{tikzcd}$$

Similarly to how we showed diagram (3), by combining the diagram with diagram (**??**) and use equation (4), we get that $\tau_V \circ \phi^{-1} \circ \alpha_U$ where τ_V is the restiction of the limit onto $\mathcal{G}(V)$ is the same as $\alpha_V \circ \rho_V \circ \psi^{-1}$. Since $\alpha_V \circ \rho_V$, for $V \in \mathcal{B}_U$, are the maps of the cone in diagram (**??**),

$$\alpha_V \circ \rho_V = \tau_V \circ \hat{\alpha}_U.$$

So

$$\tau_V \circ \phi^{-1} \circ \alpha_U = \tau_V \circ \hat{\alpha}_U \circ \psi^{-1}.$$

Finaly, since this holds for all $V \in \mathcal{B}_U$, the univeral propery of the inverse limit implies that we can cancele the τ_V ans thus the diaram comutes.

Exercise 3

1. To show that $\mathrm{Supp}(s)$ closed, we will prove that $U \setminus \mathrm{Supp}(s)$ is open. Suppose $x \notin \mathrm{Supp}(s)$, that is $s_x = 0$ or eqvivalently there exists $x \in V \subset U$ open such that $s|_V = 0$. But then $s_y = 0$ for all $y \in V$. Hence $V \cap \mathrm{Supp}(s) = \emptyset$, which proves that $U \setminus \mathrm{Supp}(s)$ is open and thus $\mathrm{Supp}(s)$ is closed in U.

2. For $f \in A$, $\tilde{M} := M \otimes_A A[f^{-1}] \cong M[f^{-1}]$.

   We have

$$\mathrm{Supp}(\tilde{m}) := \{x \in X \mid \tilde{m}_x \neq 0\}$$
$$V(\mathrm{Ann}_A(m)) := \{x \in X \mid \forall f \in \mathrm{Ann}_A(m) \ , \ f(x) = 0\}$$

   Like in part 1. we will consider the complement of the sets and show that they are the same. We have

$$x \notin \mathrm{Supp}(\tilde{m}) \iff \exists f \in A \ : \ x \in D(f) \ \wedge \ (\tilde{m}, D(f)) = 0$$
$$x \notin V(\mathrm{Ann}_A(m)) \iff \exists f \in \mathrm{Ann}_A(m) \ : \ f(x) \neq 0$$

   Suppose $x \notin \mathrm{Supp}(\tilde{m})$ and let $f \in \mathrm{Ann}_A(m)$ such that $f(x) \neq 0$. Then $fm = 0$ and $x \in D(f)$, so $(\tilde{m}, D(f)) = 0$ and thus $x \notin \mathrm{Supp}(\tilde{m})$.

   Conversly suppose $x \notin \mathrm{Supp}(\tilde{m})$ and let $f \in A$ such that $x \in D(f)$ and $(\tilde{m}, D(f)) = 0$. Then $f^n m = 0$ for some $n \in \mathbb{N}$, so $f^n \in \mathrm{Ann}_A(m)$. Since $x \in D(f)$, $f(x) \neq 0$ so $f \not\subseteq x$. Therefore $f^n \notin x$, since x is a prime ideal, and thus $f^n(x) \neq 0$. So $g = f^n \in \mathrm{Ann}_A(m)$ and $x \in D(g)$. Hence $x \notin V(\mathrm{Ann}_A(m))$.

   Now suppose $M = \langle m_1, ..., m_n \rangle$. Suppose $x \notin V(\mathrm{Ann}_A(M))$, that is there exists $f \in \mathrm{Ann}_A(M)$ such that $f(x) \neq 0$. So $fm = 0$ for all $m \in M$ and thus, by the argument above, $(\tilde{m}, D(f)) = 0$ and $x \in D(f)$. Hence $\tilde{M}_x = 0$ and thus $x \notin \mathrm{Supp}(\tilde{M})$.

   Conversly suppose $x \notin \mathrm{Supp}(\tilde{M})$, that is $\tilde{M}_x = 0$. So there exits $f \in A$ such that $x \in D(f)$ and for all $\tilde{m} \in M[f^{-1}]$ $(\tilde{m}, D(f)) = 0$. In particular $(\tilde{m}_i, D(f)) = 0$, so by the argument above there exists $f_i \in A$ such that $f_i m_i = 0$ and $f_i(x) \neq 0$. Let $g = \prod_i f_i$ and consider $m \in M$, then $m = \sum_i a_i m_i$, so $gm = \sum_i a_i \prod_j f_j m_i = 0$ and ths $g \in \mathrm{Ann}_A(M)$. Furthermore $f_i \notin x$, so since x is prime, $g \notin x$. Hence $x \notin \mathrm{Supp}(\tilde{M})$.

3. Let $A = \mathbb{Z}$ and $M = \bigoplus_{p \text{ prime}} \mathbb{Z}_p$. Consider $p \in \mathbb{N}$ be prime and $n \in \mathbb{Z}$ such that $p \in D(n) = \{q \in X \mid q \nmid n\}$ (ie. $p \nmid n$). Let $m = (m_i)_{i \text{ prime}}$ and $m_i = 1$, then $nm \neq 0$, since $nm_p = 0$ would imply $p \mid n$. So $p \in \mathrm{Supp}(\tilde{M})$.

   On the other hand $0 \notin \mathrm{Supp}(\tilde{M})$. Since if $0 \in D(n)$, then $n \in \langle 0 \rangle$, so $n = 0$ and thus $nm = 0$ for all $m \in M$.

   So $\mathrm{Supp}(\tilde{M}) = X \setminus \{0\}$. But $\{0\}$ is not closed in X, since the only closed set contiaining 0 is X.