Algebraic Geometry - *Exercise Sheet 1*

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Solution 1

1. Define $f \in k[X_1, ..., X_n]$ by $f(x) = \prod_y (x - y)$ for $x \in \mathbb{A}^n(k)$. Then

$$V(f) = \{x \in \mathbb{A}^n(k) \ : \ f(x) = 0\} = Z.$$

   Hence Z is closed wrt. the Zariski topology.

2. The Zariski closed subsets of $\mathbb{A}^1(k)$ are precisely the finite subsets, so the product topology on $\mathbb{A}^1(k) \times \mathbb{A}^1(k)$ is precisely the finite subsets. However

$$\{(x, -x) \ : \ x \in k\} = \{(x, y) \ : \ x + y = 0\},$$

   which is not finite (assuming k is not finite), is closed in $\mathbb{A}^2(k)$.

Solution 2

Let $f \in k[X_1, ..., X_n]$ be non-constant. Then there exists $x_1 \neq x_2 \in \mathbb{A}^n(k)$ such that $f(x) \neq f(y)$. Define $g \in k[X]$, by

$$g(x) = f(xx_1 + (1 - x)x_2),$$

then g is polynomial of one variable that is non-constant, since $g(0) = f(x_2) \neq f(x_1) = g(1)$. So since k is algebraically closed there exists $y \in k$ such that $g(y) = 0$ and thus

$$f(yx_1 - yx_2 + x_2) = 0,$$

hence $y(x_1 - x_2) + x_2$ is a zero of f.