Algebraic Geometry - *Exercise Sheet 1*

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Solution 1

1. Define $f \in k[X_1, ..., X_n]$ by $f(x) = \prod_{y \in Z}(x - y)$ for $x \in \mathbb{A}^n(k)$. Then
$$V(f) = \{x \in \mathbb{A}^n(k) \ : \ f(x) = 0\} = Z.$$
Hence Z is closed wrt. the Zariski topology.

2. The Zariski closed subsets of $\mathbb{A}^1(k)$ are precisely the finite subsets, so the product topology on $\mathbb{A}^1(k) \times \mathbb{A}^1(k)$ is precisely the finite subsets. However
$$\{(x, -x) \ : \ x \in k\} = \{(x, y) \ : \ x + y = 0\},$$
which is not finite (assuming k is not finite), is closed in $\mathbb{A}^2(k)$.

Solution 2

Let $f \in k[X_1, ..., X_n]$ be non-constant. Then there exists $x_1 \neq x_2 \in \mathbb{A}^n(k)$ such that $f(x) \neq f(y)$. Define $g \in k[X]$, by
$$g(x) = f(xx_1 + (1 - x)x_2),$$
then g is polynomial of one variable that is non-constant, since $g(0) = f(x_2) \neq f(x_1) = g(1)$. So since k is algebraically closed there exists $y \in k$ such that $g(y) = 0$ and thus
$$f(yx_1 - yx_2 + x_2) = 0,$$
hence $y(x_1 - x_2) + x_2$ is a zero of f.

Solution 3

1. By def

2. Define $g(x_1, ..., x_{n+1}) = f(x_1, ..., x_n)x_{n+1} - 1$, then
$$V(I(X) \cup \{g\}) = \mathrm{Im}(D(f)) \subseteq \mathbb{A}^{n+1}(k), \tag{1}$$
where $\iota : I(X) \hookrightarrow k[x_1, ..., x_{n+1}]$ in the obvious way.

<u>proof of (1)</u> So the inclusion ι gives $V(I(X)) = X \times k \subseteq \mathbb{A}^{n+1}(k)$ ans clearly $g(x_1, ..., x_{n+1}) = 0$ implies $x_{n+1} = f^{-1}(x_1, ..., x_n)$. So
$$\begin{aligned} V(I(X) \cup \{g\}) &= V(I(X)) \cap V(g) \\ &= \{(x_1, ..., x_n, f^{-1}(x_1, ..., x_n)) \in X \ : \ (x_1, ..., x_n) \in X \text{ and } f(x_1, ..., x_n) \neq 0\}, \end{aligned}$$
which is clearly equal to $\mathrm{Im}(D(f))$.

It remains to show that the coordinate ring of $\mathrm{Im}(D(f))$ is given by the localisation $A[f^{-1}]$.

By (1) we have that the coordinate ring is given by
$$C = k[x_1, ..., x_{n+1}]/(I(X) \cup \langle g \rangle) \cong A/\langle g \rangle$$

Define $\psi : A[f^{-1}] \to A/\langle g \rangle$ by sending $\frac{1}{f}$ to x_{n+1} and x_i to itself. This is well defined since $fx_{n+1} = 1$ in $A/\langle g \rangle$ and also clearly a homeomorphism. ψ is also clearly an isomorphism since it is bijective on the generators.

Solution 4

We will prove TFAE

1. A is nilpotent (ie. $A^n = 0$)
2. $A^2 = 0$, ie. $(a, b, c, d) \in V(I)$,
3. $\det A = 0$ and $\operatorname{tr} A = 0$, ie. $(a, b, c, d) \in V(J)$.

(3) $\implies$ (2) By the Cayley-Hamilton theorem

$$A^2 - \operatorname{tr}(A)A + \det(A)I_2 = 0.$$

By assumption $\operatorname{tr}(A) = 0$ and $\det(A) = 0$. So $A = 0$

(2) $\implies$ (1) By definition

(1) $\implies$ (3) Consider an eigenvalue λ of A, $Ax = \lambda x$. By recursively applying A we have $A^n x = \lambda x$. By assumtion $A^n = 0$ so $\lambda = 0$. This is true for any eigenvalue. The determinant is the product of the eigenvalues and the trace is the sum of them, so $\det(A) = \operatorname{tr}(A) =$.

We have $(a + d) \in J$ and $(a + d)$ has degree one, but all the polynomials in I have degree 2, so $(a + d) \notin I$. Hence $J \neq I$.

By the nullstellensatz; $\operatorname{rad}(I) = I(V(I)) = I(V(J)) = \operatorname{rad}(J)$.

Furthermore we claim $\operatorname{rad}(J) = J$, so $\operatorname{rad}(I) = J$.

We have J radical iff $k[a, b, c, d]/J$ is reduced, which is clear, since both $ad - bc$ and $a + d$ are irreducible polynomials.