Algebraic Geometry - *Exercise Sheet 5*

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Exercise 2

1. We'll denote the map by Ψ.

$$\Psi : \mathrm{Hom}(h_X, F) \to F(X); \; \eta \; \mapsto \eta_X(\mathrm{Id}_X)).$$

Let $\eta \in \mathrm{Hom}(h_X, F)$, ie. a natural transformation from h_X to F. Then for $Y, Z \in \mathcal{C}$ and $f : Y \to Z$, the following diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(Z, X) & \xrightarrow{\eta_Y} & \mathcal{F}(Y) \\ \big\downarrow{\scriptstyle h_X f} & & \big\downarrow{\scriptstyle F(f)} \\ \mathrm{Hom}_{\mathcal{C}}(Y, X) & \xrightarrow{\eta_Z} & F(Y) \end{array}$$

In particular, the diagram commutes for $f : Y \to X$. So

$$\eta_Y(f) = \eta_Y(f \circ \mathrm{Id}_X) = F(f)(\eta_X(\mathrm{Id}_X)). \tag{1}$$

So if $\xi \in \mathrm{Hom}(h_X, F)$, such that $\eta_x(\mathrm{Id}_X) = \xi_X(\mathrm{Id}_X)$, ie. $\Psi(\eta) = \Psi(\xi)$. Then $\eta_Y(f) = \xi_Y(f)$, for all $f : Y \to X$, so $\eta = \xi$. So Ψ is injective.

Furthermore, if $x \in F(X)$, we may define $\eta^x \in \mathrm{Hom}(\eta_X, F)$, by

$$\eta^x_Y(f) := F(f)(x), \qquad \forall Y \in \mathcal{C}, \; f : Y \to X.$$

Then

$$\Psi(\eta^x) = \eta^x(\mathrm{Id}_X) = F(\mathrm{Id}_X)(x) = \mathrm{Id}_X(x) = x.$$

So Ψ is also surjective. It remains to show that η^x satisfies the naturally conditions (ie. the commutative diagram above). Let $Y, Z \in \mathcal{C}$, $g : Y \to Z$, and $f : Y \to X$. Then

$$(\eta^x_Z \circ h_X g)(f) = \eta^x_Z(f \circ g) = F(f \circ g)(x) = F(g)(F(f)(x)) = F(g)(\eta^x_Y(f)).$$

Consider the case $F = h_Y$, then the Yoneda lemma implies that

$$\mathrm{Hom}(h_X, h_Y) \cong \mathrm{Hom}(X, Y).$$

So the functor $\mathcal{C} \mapsto \mathrm{Fun}(\mathbb{C}^{\mathrm{op}}, \mathrm{Sets}); X \mapsto h_X$ is fully faithful.

2. The first bijection follows immediately from part 1. So what we need to show is the second bijections.

Let $\Xi : \mathrm{Hom}_S(X, Y) \to \mathrm{Hom}(h_{X|\mathcal{D}}, h_{Y|\mathcal{D}})$ be the composition of the Yoneda map Ψ from part 1, and the restriction sub-category $\mathcal{D}$. More explicitly Xi is the composition

$$\begin{array}{ccccc} \mathrm{Hom}_S(X, Y) & \longrightarrow & \mathrm{Hom}(h_X, x_Y) & \longrightarrow & \mathrm{Hom}(h_{X|\mathcal{D}}, h_{Y|\mathcal{D}}) \\ f & \longmapsto & \eta^f & \longmapsto & \eta^f_{|\mathcal{D}} \end{array}$$

where η^f is defined (like in part 1, with $F = h_Y$) by

$$\eta^f_Z(g) = h_Y(g)(f) = f \circ g, \qquad \forall Z \in \mathcal{C}, \quad g : Z \to X,$$

and $\eta^f_{|\mathcal{D}}$ is the restriction of a natural transformation to a natural transformation between the restriction of the functors.

To show the second bijection we will construct a map $\Phi : \mathrm{Hom}(h_{X|\mathcal{D}}, h_{Y|\mathcal{D}}) \to \mathrm{Hom}(X, Y)$ inverse to Ξ.

Let $\eta \in \mathrm{Hom}(h_{X\mathcal{D}}, h_{Y\mathcal{D}})$ and $X = \bigcup_{i\in I}$ such that X_i is an affine scheme. Let

$$\eta_i := \eta_{X_i} : \mathrm{Hom}_S(X_i, X) \to \mathrm{Hom}_S(X_i, Y)$$

and $\iota_{i,j} : X_i \cap X_j \hookrightarrow X_i$, with $(\iota_{i,j})^{\#}_x = \mathrm{id} : \mathcal{O}_{X_i,x} \to \mathcal{O}_{X_i\cap X_j,x} = \mathcal{O}_{X_i,x}$, for all $x \in X_i \cap X_j$. Then, by the neutrality of η, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_S(X_i, X) \arrow[r, "\eta_i"] \arrow[d, "h_X \iota_{i,j}"] & \mathrm{Hom}_S(X_i, Y) \arrow[d, "h_Y \iota_{i,j}"] \\
\mathrm{Hom}_S(X_i \cap X_j, X) \arrow[r, "\eta_{X_i\cap X_j}"] & \mathrm{Hom}_S(X_i \cap X_j, Y)
\end{tikzcd}$$

commutes. Equivalently, for all morphisms $f : X_i \to X$, then

$$\eta_i(f) \circ \iota_{i,j} = \eta_{X_i\cap X_j}(f \circ \iota i, j) \quad \text{and} \quad (\iota_i)^{\#} \circ \eta_i(f^{\#}) = \eta_{X_i\cap X_j}\left((\iota_i)^{\#} \circ f^{\#}\right) \tag{2}$$

Let $\iota_i : X_i \hookrightarrow X$, be the inclusion, with $(\iota_i)^{\#}_x = \mathrm{id} : \mathcal{O}_{X,x} \to \mathcal{O}_{X_i,x} = \mathcal{O}_{X,x}$. Define $\Psi(\eta) : X \to Y$, for all $x \in X$, by

$$\Psi(\eta)(x) := \eta_i(\iota_i)(x) \quad \text{and} \quad \Psi(\eta)^{\#}_x := \eta_i\left((\iota_i)^{\#}_x\right), \qquad \text{if} \quad x \in X_i.$$

Clearly ι_i is a morphism of schemes over S, so, if well defined, $\Psi(\eta)$ is also a morphism of schemes over S.

It follows from equation (2) that this is well-defined. Suppose $x \in X_i \cap X_j$, then

$$\eta_i(\iota_i)(x) = (\eta_i(\iota_i) \circ \iota_{i,j})(x) = \eta_{X_i\cap X_j}(\iota_i \circ \iota_{i,j})(x) = \eta_{X_i\cap X_j}(\iota_j \circ \iota_{j,i})(x) = \eta_j(\iota_j)(x),$$

and

$$\eta_i\left((\iota_i)^{\#}_x\right) = (\iota_j)^{\#}_x \circ \eta_i\left((\iota_i)^{\#}_x\right) = \eta_{X_i\cap X_j}\left((\iota_i)^{\#}_x \circ (\iota_j)^{\#}_x\right) = \eta_j\left((\iota_j)^{\#}_x\right).$$

The first and third equality follows from $(\iota_j)^{\#}_x = \mathrm{id}$, and the second by equation (2).

It remains to show that Ψ defines an inverse of Ξ. Let $f : X \to Y$ be a morphism of schemes over S. Then for $x \in X_i$

$$\begin{aligned}
\Psi(\Xi(f))(x) &= \eta^f_{X_i}(\iota_i)(x) = (f \circ \iota_i)(x) = f(x), \quad \text{and} \\
\Psi(\Xi(f))^{\#}_x &= \eta^f_{X_i}((\iota_i)^{\#}_x) = (\iota_i)^{\#}_x \circ f^{\#}_x = f^{\#}_x.
\end{aligned}$$

Conversely, let $\eta \in \mathrm{Hom}(h_{X|\mathcal{D}}, h_{Y|\mathcal{D}})$, $g : Z \to X$, where Z affine scheme, $z \in Z$ and $i \in I$, such that $g(z) \in X_i$. Then

$$\begin{aligned}
\Xi(\Psi(\eta))(g)(z) &= \eta^{\Psi(\eta)}_Z(g)(z) \\
&= \Psi(\eta)(g(z)) \\
&= \eta_i(\iota_i)(g(z))
\end{aligned}$$

Exercise 3

We have a pull-back diagram, for each of the two fibre-products.

$$\begin{tikzcd} X \times_S Y \arrow[r, "p_X"] \arrow[d, "p_Y"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & S \end{tikzcd} \qquad \begin{tikzcd} |X| \times_{|S|} |Y| \arrow[r, "q_X"] \arrow[d, "q_Y"] \arrow[dr, phantom, "\lrcorner", very near start] & |X| \arrow[d, "|f|"] \\ |Y| \arrow[r, "|g|"] & |S| \end{tikzcd}$$

By the first diagram $|p_X| : |X \times_S Y| \to |X|$ and $|p_Y| : |X \times_S Y| \to |Y|$, such that $|f| \circ |p_X| = |g| \circ |p_Y|$. So by the universal prop of the second there is a unique map $\pi : |X \times_S Y| \to |X| \times_{|S|} |Y|$.

To show that π is surjective. Consider $x \in X$, $y \in Y$, $T = \mathrm{Spec}(k)$, for some filed k and define

$$\xi_x : T \to X; \star \to x, \quad \text{and} \quad \zeta_y : T \to Y; \star \to y,$$

with $\xi_x^\# = 0 : \mathcal{O}_{X,x} \to k$ and $\zeta_y^\# = 0 : \mathcal{O}_{Y,y} \to k$.

Then if $f(x) = g(y)$, (T, ξ_x, ζ_y), defines a cone over the first pull-back diagram and $(|T|, |\xi_x|, |\zeta_y|)$ defines a cone over the second pull-back diagram. Let $\phi_{x,y}$ and $\psi_{x,y}$ denote the unique maps determined by the universal properties of the first and second pull-backs, respectively. By the uniqueness of $\psi_{x,y}$, $\psi_{x,y} = \phi_{x,y} \circ \pi$ and $\psi_{x,y}(\star) = [(x, y)] \in |X| \times_{|S|} |Y|$, so $[(x, y)] \in \mathrm{Im}\,\pi$. Since x, y was arbitrary, such that $f(x) = g(y)$, π is surjective.