Algebraic Geometry - *Exercise Sheet 8*

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Exercise 1

1. By definition $\mathrm{QCoh}(X)$ is a full subcategory of $\mathcal{O}_X$-Mod, since a morphism in $\mathrm{QCoh}(X)$ is simply a morphism of $\mathcal{O}_X$-modules. So (assuming $\mathcal{O}_X$ is abilian) we only need to check that the kernels and cokernels of morphisms of quasi-coherent $\mathcal{O}_X$-modules are in $\mathrm{QCoh}(X)$.

   Let $f : \mathcal{M} \to \mathcal{N}$, be a morphism of quasi-coherent $\mathcal{O}_X$-modules. Suppose $X = \operatorname{Spec} A$. Then, by Corollary 11.7 in the notes, $\ker f = \widetilde{\ker(f(X))}$ and coker $\widetilde{\operatorname{coker}(f(X))}$, and are thus quasi-coherent.

   Now let $(U_i)_{i\in I}$ be a cover of X by affines, such that $\mathcal{M}_{|U_i} \cong \widetilde{M_I}$ and $\mathcal{N}_{|U_i} \cong \widetilde{N_i}$, for $\mathcal{O}_X(U_i)$-modules M_i and N_i.

   Then, by the above argument, $\ker f_{|U_i} \cong \widetilde{\ker f(U_i)}$, so $\ker f$ is quasi-coherent. Similarly, $\operatorname{coker} f_{|U_i} \cong \widetilde{\operatorname{coker} f(U_i)}$, so also $\operatorname{coker} f$ is quasi-compact.

   To show that $\mathrm{QCoh}(X)$ is closed under extensions, let

   $$0 \to \mathcal{M} \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{N} \to 0$$

   be an exact sequence in $\mathcal{O}_X$-Mod, with $\mathcal{M}, \mathcal{N} \in \mathrm{QCoh}(X)$.

   First suppose $X = \operatorname{Spec} A$, then again, by corollary 11.7,

   $$0 \to M \xrightarrow{f(X)} \mathcal{E}(X) \xrightarrow{g(X)} N \to 0$$

   is exact, with N, M A-modules. As $\mathcal{E}$ is an $\mathcal{O}_X$-module, $\mathcal{E}(X)$ is also an A-module. Applying the tildification again we get, by cor 11.7, that

   $$0 \to \widetilde{M} \xrightarrow{\widetilde{f(X)}} \widetilde{\mathcal{E}(X)} \xrightarrow{\widetilde{g(X)}} \widetilde{N} \to 0$$

   is exact. Since $\mathcal{M} \cong \widetilde{M}$, $\mathcal{N} \cong \widetilde{N}$, $f = \widetilde{f(X)}$ and $g = \widetilde{g(X)}$, we get that also $\mathcal{E} = \widetilde{\mathcal{E}(X)}$.

   If $(U_i)_{i\in I}$ is a cover of X, by affines such that $\mathcal{M}_{|U_i} \cong \widetilde{M_i}$ and $\mathcal{N}_{|U_i} \cong \widetilde{N_i}$. Then, since exactness is a local property (we could consider stalks),

   $$0 \to \mathcal{M}_{|U_i} \xrightarrow{f_{|U_i}} \mathcal{E}_{|U_i} \xrightarrow{g_{|U_i}} \mathcal{N}_{|U_i} \to 0$$

   is also exact, and thus by the argument above $\mathcal{E}_{|U_i} \cong \widetilde{\mathcal{E}(U_i)}$.

   Finally it is immediate from the fact that the functor is inclusion and that it is fully frightful, that it is exact.

2. Since the kernel and cokernel of morphisms between finally generated modules are also f.g., it is clear that the statement about abelian also hold for coherent $\mathcal{O}_X$-modules. Similarly the category of f.g. modules is closed under extensions, so also this statement holds for coherent $\mathcal{O}_X$-modules.