Algebraic Geometry *- Exercise Sheet 4*

Tor Gjone & Paul November 8, 2021

Exercise 2

1. $\{V \subset U\} \cap \{V \subset X \setminus U\}$ defines a cover of X, so by the definition of a sheaf there exists a unique $e_U \in \Gamma(X, \mathcal{O}_X)$ such that

$$e_U|_V = 1 \text{ if } V \subset U \text{ and } e_U|_V = 0 \text{ if } V \subset X \setminus U.$$

Note that U is open and $X \setminus U$ is open, so $e_U = 1$ and $e_U|_{X \setminus U} = 0$.

If $V \subseteq U$ then, by the functorialety of the restriction map $(e_U e_U)|_V = 1 \cdot 1 = 1$ and similarly of $V \subseteq X \setminus U$, $(e_U e_U)|_V = 0$. So by the uniqueness, from above, $e_U e_U = e_U$ and thus e_U is idempotent.

If $V \subset X$ is another clopen set. ie.

$$V \cap (X \setminus U)\emptyset \;\vee\; U \cap (X \setminus V) \neq \emptyset.$$

By symmetry we may wlog. assume

$$W = V \cap (X \setminus U) \neq \emptyset.$$

Since V and U are both clopen, W most also be clopen and we have $W \subseteq V$ and $W \subseteq X \setminus U$. So $e_V|_W = 1$ but $e_U|_W = 0$. So

$$\Psi : \mathrm{OC}(X) \to \mathrm{Idem}(\Gamma(X, \mathcal{O}_X));\; U \mapsto e_U$$

is an injection.

Let $e \in \mathrm{Idem}(\Gamma(X, \mathcal{O}))$. Then for $x \in X$, $e_x = 1$ or $e_x = 0$, since e_x is idempotent in $(\mathcal{O}_X)_x$ ($e_x = (e \cdot e)_x = e_x \cdot e_x$.)

Since $(\mathcal{O}_X)_x$ is a local ring e_x is a unit or $1 - e_x$ is a unit. We have

$$ae_x = 1 \implies ae_x e_x = e_x \implies e_x = ae_x = 1, \tag{1}$$
$$a(1 - e_x) = 1 \implies a(e_x - e_x^2) = e_x \implies e_x = a(0) = 0. \tag{2}$$

So $e_x = 1$ or $e_x = 0$.

Let $U = \{x \in X \mid e_x = 1\}$. The we want to show that U is clopen. Let $x \in U$, then by the definition of the stalk, there exists $V \subseteq X$ such that $x \in V$ and $e|_V = 0$. So V is a neighbourhood of x in U and thus U is open. Similarly let $x \in X \setminus U$, then $e_x = 1$, so there exists a neighbourhood $V \subseteq X$ of x such $e|_V = 1$. So $V \subset X \setminus U$. Hence U is closed and thus clopen. Clearly $e = e_U$.

2. If $V \subseteq U \cap U'$, $(e_U e_{U'})|_V = 1$, by functorialety of the restiction, like above. If $V \subseteq X \setminus (U \cap U') = (X \setminus U) \cup (X \setminus U')$, let $V_1 = (X \setminus U) \cap V$ and $V_2 = (X \setminus U') \cap V$. Then, $e_U|_{V_1} = e_{U'}|_{V_2} = 0$, so for $i = 1, 2$,

$$(e_U e_{U'})|_{V_i} = e_U|_{V_i} e_{U'}|_{V_i} = 0.$$

Since $\{V_1, V_2\}$ defines a cover of V, we can conclude that $(e_U e_{U'})|_V = 0$. Hence $e_U e_{U'} = e_{U \cap U'}$, by the uniqueness from part one.

3. If X is connected there are no proper non-empty clopen subsets of X, so by part one there are no idempotents $e \neq 0, 1$.

   Conversely if X is not connected, there exists a proper, non-empty, clopen $U \subseteq X$ and $e_U \neq 0, 1$ defines an idempotent.

   Suppose $\Gamma(X, \mathcal{O}_X) \cong R_1 \times R_2$ (where R_1 and R_2 are non-zero rings), then $(1, 0) \in \mathbb{R}_1 \times R_2$ defines an idempotent $\neq 0, 1$.

   Conversely if $e \neq 0, 1$ is an idempotent in $R = \Gamma(X, \mathcal{O}_X)$, then Re and $R(1-e)$ defines two non zero ring such that $R \cong Re \times R(1-e)$. Since e is non-zero $e \in Re$ is non-zero. Similarly since $e \neq 1$, $(1-e) \in R(1-e)$ is non-zero.

   Since Re is an ideal, it is enough to show that there is a unit element to show that Re is a ring. $e \in Re$ is a unit element, since for all $a \in R$ $(ae)e = ae$. So Re is a ring. The same argument works for $R(1-e)$, since $(1-e)(1-e) = 1 - e - e + e = 1 - e$ and thus $(1-e)$ is an idempotent.

   To show that $R \cong Re \times R(1-e)$. We define $\psi : R \to Re \times R(1-e)$ by $r \mapsto (re, r(1-e))$ and $\phi : Re \times R(1-e) \to R$ by $(a, b) \mapsto a + b$. Clearly this defines ring homomorphisms, and in fact they defines mutual inverses.

   Let $r \in R$, then $(\phi \circ \psi)(r) = re + r(1-e) = r$. Conversely let $(ae, b(1-e)) \in Re \times R(1-e)$, then

$$\begin{aligned}(\psi \circ \phi)(ae, a(1-e)) &= ((ae + b(1-e))e, (ae + b(1-e))(1-e)) \\ &= (ae + b(e-e), a(e-e) + b(1-e)) \\ &= (ae, b(1-e))\end{aligned}$$

Exercise 3

1. Let $p \in Z$ and $(a, V) \in \mathcal{F}_p$, that is $V \subseteq Z$ open and $a \in \mathcal{F}(V)$. By the definition of subspace topology there exists $\tilde{V} \subseteq X$, such that $V = \tilde{V}|_Z = i^{-1}(\tilde{V})$. So $a \in \mathcal{F}(V) = \mathcal{F}(i^{-1}(\tilde{V}))$ and thus $(a, \tilde{V}) \in (i_*\mathcal{F})_p$.

   Conversely, let $(a, V) \in (i_*\mathcal{F})_p$, then $a \in \mathcal{F}(i^{-1}(V))$. So $(a, i^{-1}(V)) \in \mathcal{F}_p$. This construction is clearly mutually inverse.

   If $p \neq Z$, then $X \setminus U$ is a neighbourhood of p and $\mathcal{F}(i^{-1}(X \setminus U)) = \mathcal{F}(\emptyset) = 0$. So $(i_*\mathcal{F})_p = 0$.

2. Since the sheafification does not change the stalks, the stalk $(j_!\mathcal{F})_p$ is given by pairs (a, V), for $p \in V \subseteq U$ and $a \in \mathcal{F}(V)$, and $(0, V)$, for $V \not\subseteq U$. Also, since U is open, $V \subseteq U$ is open in U iff V is open in X.

   Hence clearly $(j_!\mathcal{F})_p = p$ if $p \in U$ and $(j_!\mathcal{F})_p = 0$ if $p \neq U$.

3. I suffices to show the exact sequence on stalks.

   Consider $p \in U$, then

$$\begin{aligned}(j_!(j^{-1}\mathcal{F}))_p &= (j^{-1}\mathcal{F})_p, && \text{by part 2} \\ &= (j^+\mathcal{F})_p, && \text{since the stalk is preserved by sheafification} \\ &= \mathcal{F}_p, && \text{since } j \text{ is an open embedding. And} \\ (i_*(i^{-1}\mathcal{F}))_p &= 0, && \text{by part 1.}\end{aligned}$$

If $p \in Z$, then

$$
\begin{aligned}
(j_!(j^{-1}\mathcal{F}))_p &= 0, && \text{by part 2, and} \\
(i_*(i^{-1}\mathcal{F}))_p &= (i^{-1}\mathcal{F})_p, && \text{by part 1.} \\
&= (i^+\mathcal{F})_p, \\
&= \varinjlim_{p \in V \in \mathcal{O}_X} \mathcal{F}(V) && \text{colimit of colimit and since } j(p) = p, \\
&= \mathcal{F}_p && \text{by definition of stalk.}
\end{aligned}
$$

So on the stalks, the sequence has the form

$$
0 \longrightarrow \mathcal{F}_p \longrightarrow \mathcal{F}_p \longrightarrow 0 \longrightarrow 0
$$

$$
0 \longrightarrow 0 \longrightarrow \mathcal{F}_p \longrightarrow \mathcal{F}_p \longrightarrow 0
$$

where in the first line $p \in U$ and in the second $p \in Z$.

Since the maps are defined by the inclusions it is clear that these sequences are exact. ie. the map $\mathcal{F}_p \to \mathcal{F}_p$, is an isomorphism.

4. This question does not make sense.

   We have

$$
\begin{aligned}
(j_*(j^{-1}\mathcal{F}))_0 &= (j^{-1}\mathcal{F})_0 && \text{by part 1} \\
&= \varinjlim_{0 \in V \in \mathcal{O}_{\mathbb{C}}} \mathcal{F}(V) = \mathcal{F}_0
\end{aligned}
$$

   But $(j_*(j^{-1}\mathcal{F}))_x = 0$, for any point $x \neq 0$. So any map $\mathcal{F} \to j_*(j^{-1}\mathcal{F})$ cannot be injective.

   On the other hand $(j_!(j^{-1}\mathcal{F}))_0 = 0$, by part 2, and the map $j_!(j^{-1}\mathcal{F}) \to \mathcal{F}$, for part 3, is injective, but cannot be surjective, since $\mathcal{F}_x \neq 0$.