Algebraic Topology - Exercise Sheet 3

Tor Gjone (2503108) & Michele Lorenzi (3461634) November 7, 2021

Exercise 1

(a) We have

$$|X| := (\bigsqcup_{n\geq 0} X_n \times \nabla^n)/\sim,$$

where $\sim$ is generated by, for all $\alpha : [n] \to [m]$, $x \in X_m$, $t \in \nabla^n$

$$(x, \alpha_*(t)) \sim (\alpha^*(x), t).$$

So what we need to show is that the map defined on the union factors through the quotient. We have

$$\begin{aligned} f_n(\alpha^*(x))(t) &= \alpha^*(f_m)(t), && \text{since } f \text{ is a morphism in } \mathbf{sSet}, \\ &= f_m(x)(\alpha_*)(t)), && \text{by definition of } \alpha^*. \end{aligned}$$

So $\widehat{f} : |X| \to T$ defined by $(x, t) \mapsto f_n(x)(t)$, is well-defined.

(b) Let

$$\Phi : \mathrm{Hom}_{\mathbf{sSet}}(X, \mathcal{S}(T)) \to \mathrm{Hom}_{\mathbf{Top}}(|X|, T); \quad f \mapsto \widehat{f}.$$

To show that Φ is a bijection, we construct an inverse.

Let $g : |X| \to T$ and $\tilde{g} = g \circ q : \bigsqcup_{n\geq 0} X_n \times \nabla^n \to T$, where q is the quotient map defined by the equivalence relation $\sim$ above. Define $\bar{g} : X \to \mathcal{S}(T)$, by

$$\bar{g}_n(x)(t) := \tilde{g}(x, t),$$

for all $x \in X_n$ and $t \in T$. We need to show that this construction satisfy the naturally conditions of a morphism in **sSet**.

Let $\alpha : [n] \to [m]$, then for $x \in X_m$ and $t \in \nabla^n$

$$\begin{aligned} (\bar{g}_n \circ \alpha^*)(x)(t) &= \tilde{g}(\alpha^*(x), t) \\ &= \tilde{g}(x, \alpha_*(t)), && \text{since } \tilde{g} \text{ passes through } |X| \\ &= \bar{g}_m(x)(\alpha_*(t)) \\ &= (\alpha \circ \bar{g}_m)(x)(t) \end{aligned}$$

Clearly the maps Φ and $(g \mapsto \bar{g})$ are mutual inverses and thus Φ most be a bijection.

(c) We'll start by showing naturally in the first variable. Let $X, Y \in \mathbf{sSet}$ and $\phi : X \to Y$ be a morphism in **sSet**. Then we want to show that the following diagram (in which, we're suppressing the subscripts on Hom) commutes

$$\begin{array}{ccc} f \circ \phi & \qquad & \mathrm{Hom}(X, \mathcal{S}(T)) \xrightarrow{\Phi} \mathrm{Hom}(|X|, T) \\ \Big\uparrow & & {\scriptstyle \mathrm{Hom}(\phi, \mathcal{S}(T))}\Big\uparrow \qquad\qquad\qquad \Big\uparrow{\scriptstyle \mathrm{Hom}(|\phi|, \mathcal{S}(T))} \\ f & & \mathrm{Hom}(Y, \mathcal{S}(T)) \xrightarrow{\Phi} \mathrm{Hom}(|Y|, T) \end{array}$$

Or equivalently: for all $f \in \mathrm{Hom}_{\mathbf{sSet}}(Y \to \mathcal{S}(T))$,

$$\widehat{f \circ \phi} = \widehat{f} \circ |\phi| : |X| \to T,$$

where $|\phi| : |X| \to |Y|$ is defined by $|\phi|(x, t) := (\phi_n(x), t)$, for $x \in X_n$, $t \in \nabla^n$.
Let $f : Y \to \mathcal{S}(T)$, $x \in X_n$ and $t \in \nabla^n$, then

$$\begin{aligned} \widehat{f \circ \phi}(x, t) &= (f \circ \phi)_n(x)(t), && \text{by def. of } \hat{\cdot} \\ &= (f_n \circ \phi_n)(x)(t) \\ &= (f_n(\phi_n(x)))(t) \\ &= \widehat{f}(\phi_n(x), t), && \text{by def. of } \widehat{f} \\ &= (\widehat{f} \circ |\phi|)(x, t) && \text{by def. of } |\phi|. \end{aligned}$$

Naturally in the second argument is similar. Let $\psi : T \to S$ be a cnt. map. Then we want to show that the following diagram commutes

$$\begin{array}{ccc} f & \qquad & \mathrm{Hom}(X, \mathcal{S}(T)) \xrightarrow{\Phi} \mathrm{Hom}(|X|, T) \\ \Big\downarrow & & {\scriptstyle \mathrm{Hom}(X, \mathcal{S}(\psi))}\Big\downarrow \qquad\qquad\qquad \Big\downarrow{\scriptstyle \mathrm{Hom}(|X|, \mathcal{S}(\psi))} \\ \mathcal{S}(\psi) \circ f & & \mathrm{Hom}(X, \mathcal{S}(S)) \xrightarrow{\Phi} \mathrm{Hom}(|X|, S) \end{array}$$

Or equivalently: for all $f \in \mathrm{Hom}_{\mathbf{sSet}}(Y, \mathcal{S}(T))$,

$$\widehat{\mathcal{S}(\psi) \circ f} = \psi \circ \widehat{f} : |X| \to S,$$

where $\mathcal{S}(\psi) : \mathcal{S}(X) \to \mathcal{S}(Y)$ is defined by $(\mathcal{S}(\psi))_n(\xi) := \psi \circ \xi$, for $\xi \in \mathcal{S}(X)_n$.
Let $x \in X_n$, $t \in \nabla^n$ and $f \in \mathrm{Hom}_{\mathbf{sSet}}(X, \mathcal{S}(T))$

$$\begin{aligned} \widehat{\mathcal{S}(\psi) \circ f}(x, t) &= (\mathcal{S}(\psi) \circ f)_n(x)(t) \\ &= (\mathcal{S}(\psi)_n \circ f_n)(x)(t) \\ &= \mathcal{S}(\psi)_n(f_n(x))(t) \\ &= (\psi \circ f_n(x))(t), && \text{by def. of } \mathcal{S}(\psi) \\ &= (\psi \circ \widehat{f})(x)(t), && \text{by def. of } \widehat{f}. \end{aligned}$$

Exercise 2 We'll write

$$\Psi = (|p_1|, |p_2|) : |\Delta[n] \times \Delta[1]| \to |\Delta[n]| \times |\Delta[1]|.$$

To show that Ψ is a homeomorphism, we will construct an inverse Φ.
Every element on the left side can be represented by

$$((\alpha, \beta), t), ((\alpha, t), (\beta, s)) \tag{1}$$

where $\alpha : [k] \to [n]$, $\beta : [k] \to [1]$ and $t \in \nabla^k$. On the other hand, an arbitrary element on the right side can be represented by

$$((\alpha, t), (\beta, s)), \tag{2}$$

where $\alpha : [m] \to [n]$, $\beta : [l] \to [1]$, $t \in \nabla^m$ and $s \in \nabla^l$.
Then, on representatives, the map Ψ is defined by

$$((\alpha, \beta), t) \mapsto ((\alpha, t), (\beta, t)).$$

To construct Φ, we will consider some representative as in (2).
We claim that we may wlog. assume that $m \leq n$ and $l \leq 1$. Or equivalently for all representatives as in (**??**), there exists $\alpha' : [m'] \to [n]$, $\beta' : [l'] \to [1]$, $t' \in \nabla^{m'}$ and $s' \in \nabla^{l'}$ such that $m' \leq n$, $l' \leq 1$ and

$$(\alpha, t) \sim (\alpha', t') \text{ and } (\beta, s) \sim (\beta', s').$$

proof of claim. $\square$ proof

Note that $(t, s) \in \nabla^m \times \nabla^l$. By the claim $l = 0, 1$. If $l = 0$, then $k = m$ and $\nabla^m \times \nabla^0$ is already a k-simplex. So we will mostly consider the case where $l = 1$. (In the following arguments $l = 1$, but we will still write l to keep track of it) The geometrical idea of the constructions of Φ, consists of partitioning the polyhedra $\nabla^m \times \nabla^l$ into k-simplices where $k = m + l$.

Observe that the vertices of $\nabla^m \times \nabla^l$ can be parametrise by $[m] \times [l]$. Define an ordering of on $[m] \times [l]$ by $(i, j) \geq (i', j')$ iff $i + j \geq i' + j$. Let $S \subset [m] \times [l]$, such that $|S| = m + l + 1 = k + 1$, then the convex hull of S defines a k-simplex $\nabla_S \subseteq \nabla^m \times \nabla^l$. In particular, if $\xi : [k] \to [m] \times [l]$ is a strictly increasing map (wrt. the order defined above) then the convex hull of $\xi([k])$ defines a k-simplex $\nabla_\xi \subseteq \nabla^m \times \nabla^l$.

We claim that $\Xi_{m,l} = \{\nabla_\xi \mid \xi : [k] \to [m] \times [l]\}$ defines a partitioning of $\nabla^m \times \nabla^l$ into k-simplices. That is

$$\bigsqcup_{\nabla_\xi \in \Xi_{m,l}} \nabla_\xi = \nabla^m \times \nabla^l,$$

and if $\xi, \xi' \in \Xi_{m,l}$ are distinct then

$$\mathring{\nabla}_\xi \cap \mathring{\nabla}_{\xi'} = \emptyset. \tag{3}$$

proof of claim. $\square$ proof

Since $\Xi_{m,l}$ partitions $\nabla^m \times \nabla^l$, there exists $\xi \in \nabla^m \times \nabla^l$ such that $(t,s) \in \nabla_\xi$ and if $(t,s) \in \mathring{\nabla}_\xi$, then ξ is unique.

Define $\widehat{\alpha}_\xi = \alpha \circ \mathrm{pr}_1 \circ \xi : [k] \to [n]$, $\widehat{\beta}_\xi = \beta \circ \mathrm{pr}_2 \circ \xi : [k] \to [1]$ and $\hat{t} = (t,s) \in \nabla_\xi$. Then $((\widehat{\alpha}_\xi, \widehat{\beta}_\xi), \hat{t})$ defines a representeative in $|\nabla[n] \times \nabla[1]|$. We define Φ by

$$((\alpha, s), (\beta, t)) \mapsto ((\widehat{\alpha}_\xi, \widehat{\beta}_\xi), \hat{t}).$$

We still need to check that this definition is independent of the ξ, well-defined and continues. And we need to check that Φ acutualy defines an inverse of Ψ.

Suppose $\hat{t} = (t,s) \in \nabla_\xi \cap \nabla_{\xi'}$. Then, by (3), $\hat{t}$ most be contained in a face of both ∇_ξ and $\nabla_{\xi'}$.

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Exercise 3