Algebraic Topology - Exercise Sheet 4

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Exercise 1

(a)

(b) The fact that the two boundary points are mapped to the base point follows from

$$d_0(g) = \{1\} = d_1(g).$$

Let $\delta_i : [n-1] \to [n]$, be the face map such that $\delta_i^* = d_i$. Then

$$(BG)_1 \times \nabla^1 \ni (g, 0) = (g, \delta_{0*}(1)) \sim (d_0(g), 1) = (1, 1) \in (BG)_0 \times \nabla^1,$$

so $(g, 1) \in \{g\} \times \nabla^1$ is mapped to $(g, 0) = (1, 1)$ in $|BG|$. Similarly

$$(g, 1) = (g, (\delta_1)_*(1)) \sim (d_1(g), 1) = (1, 1),$$

so also $(g, 1) \in \{g\} \times \nabla^1$ is mapped to $(g, 1) = (1, 1)$ in $|BG|$.

(c) Let $g, h \in G$ and $\eta(g, h) : \nabla^2 \to |BG|$ defined by the composition

$$\{(g, h)\} \times \nabla^2 \hookrightarrow \bigcup_{n \geq 0} (BG)_n \times \nabla^n \twoheadrightarrow |BG|.$$

Like in (b), let $d_i = \delta_i^*$. Then

$$\delta_0; \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 2 \end{cases} \qquad \delta_1; \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 2 \end{cases} \qquad \delta_2; \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 1 \end{cases}$$

So for $t \in \nabla^2$, we have

$$\omega(g)(t) = (g, t) = (d_0(g, h), t) \sim ((g, h), (\delta_0)_*(t)) = (\eta(g, h)(\delta_0)_*)(t),$$

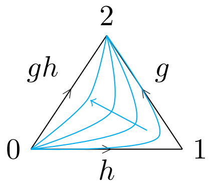
and similarly

$$\omega(h)(t) = (\eta(g, h)(\delta_2)_*)(t) \quad \text{and} \quad \omega(g \cdot h)(t) = (\eta(g, h)(\delta_1)_*)(t),$$

So ω on g, h and gh correspond to the loops defined η restricted to the three faces as illustrated in figure (1)

Also illustrated in the figure, by the cyan curves, $\eta(g, h)$ defines a homotopy from $\omega(g) \cdot \omega(h)$ to $\omega(g \cdot h)$.

Figure 1:



Exercise 2

Exercise 3

(a) Consider $S^1 = [0,1]/(0 \sim 1)$. And let $U_1 = (0,1) \subset S^1$ and $U_2 = [0,1] \setminus \{1/2\} \subset S^1$. Then $\{U_i\}$ is a cover of S^1 and we claim that $p^{-1}(U_i) \cong F \times U_i$.

In the case of $i = 1$ this is clear and for $i = 2$ we have

$$\begin{aligned} p^{-1}(U_2) &\cong (F \times [0,1/2) \sqcup F \times (1/2,1]) \,/\, ((x,0) \sim (f(x),1)) \\ &\cong_\phi (F \times [0,1/2) \sqcup F \times (1/2,1]) \,/\, ((x,0) \sim (x,1)) \\ &\cong F \times U_2 \end{aligned}$$

where $\phi = \mathrm{id}_F \times \mathrm{id} \sqcup f^{-1} \times \mathrm{id}$. ϕ is well-defined since $\mathrm{id}(x) = x = f^{-1}(f(x))$ and it is clearly a homeomorphism.

(b)

Lemma 0.1. Let $\pi : E \to [0,1]$ be a fibre bundle, the π is trivial. That is $E \cong F \times [0,1]$. (Here F is the unique fibre. Unique since $[0,1]$ is connected.)

Proof. Let $S = \{s \in [0,1] \mid \pi^{-1}([0,s]) \cong F \times [0,s]\}$. We want to show that S is both closed and open and thus equal to $[0,1]$.

We start with open. Let $s \in S$, then since π is a fiblre bundle there exists an open neigboorhood $V \ni s$, such that $\pi^{-1}(V) \cong F \times V$. Let $\epsilon > 0$, such that $I_{s,\epsilon} \subset V$ (closed ball centred at s with radius ϵ, ie. $I_{s,\epsilon} = [s-\epsilon, s+\epsilon]$.)

Since $s \in S$, there exists $f_s : \pi^{-1}([0,s]) \xrightarrow{\sim} F \times [0,s]$. And by restricting the homeomorphism assosiated to V, we have $g : \pi^{-1}(I_{s,\epsilon}) \xrightarrow{\sim} F \times I_{s,\epsilon}$.

We want to make g and f agree on the intersection, so we define a map to make up for the difference. Define $\tilde{g} : F \times I_{s,\epsilon} \to F \times I_{s,\epsilon}$ by

$$\tilde{g}(f,t) := \begin{cases} (f_t \circ g_t^{-1})(f) & \text{if } t \in [s-\epsilon, s], \\ (f_s \circ g_s^{-1})(f) & \text{if } t \in [s, s+\epsilon] \end{cases}$$

where $f_t : F \to F$ is the map f resticted to the fiber over t. $\tilde{g}$ is clearly well-defined and continues, since g is a homeomorphim m and both g and f maps the fiber over t to itself, by the definition of a fiber bundle.

We observe that $\tilde{g} \circ g|_{\pi^{-1}([s-t,s])} = f|_{\pi^{-1}([s-t,s])}$. So the following map is well defined. Define $\tilde{f} : \pi^{-1}([0, s+\epsilon]) \to F \times [0, s+\epsilon]$ by

$$x \mapsto \begin{cases} f(x) & \text{if } \pi(x) \in [0,s], \\ (\tilde{g} \circ g)(x) & \text{if } \pi(x) \in [s-\epsilon, s+\epsilon] \end{cases}$$

It is quite clear that this defines a homeomorphism, so $s + \epsilon \in S$ and thus S is open. prove that S is closed

$\square$

Let $p : E \to S^1$ be a fibre bundle with fibre F (S^1 is connected), and $U_1 = [0, 1/2] \subset S^1$ and $U_2 = ([1/2, 1]) \subset S^1$. Then, for $i = 1, 2$, $p_i = p|_{p^{-1}(U_i)} : p^{-1}(U_i) \to U_i$ defines fibre bundle over the closed interval U_i. So by the lemma above, p_i is trivial. Let $f_i : p^{-1}(U_i) \xrightarrow{\sim} F \times U_i$.

The two intervals overlap on exactly on the points $0 \sim 1$ and $1/2$. We define the difference maps $\xi_1, \xi_2 : F \to F$ by

$$\xi_1 = f_2 \circ f_1^{-1}(-, 0) \quad \text{and} \quad \xi_2 = f_1 \circ f_2^{-1}(-, 1/2).$$

Clearly both ξ_1 and ξ_2 defines homeomorphisms. We want to modify f_2 such that it agrees with f_1 over the point $1/2$. Let $\tilde{f} = (\xi_2 \times \mathrm{id}) \circ f_2$. Then by constuction $\tilde{f}$ and f_1 agrees over the point $1/2$. Furthermore $f_2 = \xi_1 \circ (f_1)_0$ (where $\cdot_0$ means over the point $0 \sim 1$), so

$$(\tilde{f})_0 = \xi_2 \circ \xi_1 \circ (f_1)_0.$$

Let $\xi = \xi_2 \circ \xi_1$, then $E \cong T_\xi$, by the map $f : E \xrightarrow{\sim} T_\xi$ defined by the compositon

$$E \xrightarrow{f_1 \sqcup_{1/2} \tilde{f}_2} F \times [0,1] \xrightarrow{quationt} T_\xi.$$