Algebraic Topology - Exercise Sheet 3

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Exercise 1

(a) We have

$$|X| := (\bigsqcup_{n\geq 0} X_n \times \nabla^n)/ \sim,$$

where $\sim$ is generated by, for all $\alpha : [n] \to [m]$, $x \in X_m$, $t \in \nabla^n$

$$(x, \alpha_*(t)) \sim (\alpha^*(x), t).$$

So what we need to show is that the map defined on the union factors through the quotient. We have

$$\begin{aligned} f_n(\alpha^*(x))(t) &= \alpha^*(f_m)(t), && \text{since } f \text{ is a morphism in } \mathbf{sSet}, \\ &= f_m(x)(\alpha_*)(t)), && \text{by definition of } \alpha^*. \end{aligned}$$

So $\widehat{f} : |X| \to T$ defined by $(x, t) \mapsto f_n(x)(t)$, is well-defined.

(b) Let

$$\Phi : \mathrm{Hom}_{\mathbf{sSet}}(X, \mathcal{S}(T)) \to \mathrm{Hom}_{\mathbf{Top}}(|X|, T); \quad f \mapsto \widehat{f}.$$

To show that Φ is a bijection, we construct an inverse.

Let $g : |X| \to T$ and $\tilde{g} = g \circ q : \bigsqcup_{n\geq 0} X_n \times \nabla^n \to T$, where q is the quotient map defined by the equivalence relation $\sim$ above. Define $\bar{g} : X \to \mathcal{S}(T)$, by

$$\bar{g}_n(x)(t) := \tilde{g}(x, t),$$

for all $x \in X_n$ and $t \in T$. We need to show that this construction satisfy the naturally conditions of a morphism in **sSet**.

Let $\alpha : [n] \to [m]$, then for $x \in X_m$ and $t \in \nabla^n$

$$\begin{aligned} (\bar{g}_n \circ \alpha^*)(x)(t) &= \tilde{g}(\alpha^*(x), t) \\ &= \tilde{g}(x, \alpha_*(t)), && \text{since } \tilde{g} \text{ passes through } |X| \\ &= \bar{g}_m(x)(\alpha_*(t)) \\ &= (\alpha \circ \bar{g}_m)(x)(t) \end{aligned}$$

Clearly the maps Φ and $(g \mapsto \bar{g})$ are mutual inverses and thus Φ most be a bijection.

(c) We'll start by showing naturally in the first variable. Let $X, Y \in \mathbf{sSet}$ and $\phi : X \to Y$ be a morphism in **sSet**. Then we want to show that the following diagram (in which, we're suppressing the subscripts on Hom) commutes

$$\begin{tikzcd} f \circ \phi & \mathrm{Hom}(X, \mathcal{S}(T)) \arrow[r, "\Phi"] & \mathrm{Hom}(|X|, T) \\ f \arrow[u, mapsto] & \mathrm{Hom}(Y, \mathcal{S}(T)) \arrow[u, "\mathrm{Hom}(\phi, \mathcal{S}(T))"] \arrow[r, "\Phi"] & \mathrm{Hom}(|Y|, T) \arrow[u, "\mathrm{Hom}(|\phi|, \mathcal{S}(T))"'] \end{tikzcd}$$

Or equivalently: for all $f \in \mathrm{Hom}_{\mathbf{sSet}}(Y \to \mathcal{S}(T))$,

$$\widehat{f \circ \phi} = \widehat{f} \circ |\phi| : |X| \to T,$$

where $|\phi| : |X| \to |Y|$ is defined by $|\phi|(x, t) := (\phi_n(x), t)$, for $x \in X_n$, $t \in \nabla^n$.
Let $f : Y \to \mathcal{S}(T)$, $x \in X_n$ and $t \in \nabla^n$, then

$$\begin{aligned} \widehat{f \circ \phi}(x, t) &= (f \circ \phi)_n(x)(t), && \text{by def. of } \hat{\cdot} \\ &= (f_n \circ \phi_n)(x)(t) \\ &= (f_n(\phi_n(x)))(t) \\ &= \widehat{f}(\phi_n(x), t), && \text{by def. of } \widehat{f} \\ &= (\widehat{f} \circ |\phi|)(x, t) && \text{by def. of } |\phi|. \end{aligned}$$

Naturally in the second argument is similar. Let $\psi : T \to S$ be a cnt. map. Then we want to show that the following diagram commutes

$$\begin{tikzcd} f \arrow[d, mapsto] & \mathrm{Hom}(X, \mathcal{S}(T)) \arrow[r, "\Phi"] \arrow[d, "\mathrm{Hom}(X, \mathcal{S}(\psi))"'] & \mathrm{Hom}(|X|, T) \arrow[d, "\mathrm{Hom}(|X|, \mathcal{S}(\psi))"] \\ \mathcal{S}(\psi) \circ f & \mathrm{Hom}(X, \mathcal{S}(S)) \arrow[r, "\Phi"] & \mathrm{Hom}(|X|, S) \end{tikzcd}$$

Or equivalently: for all $f \in \mathrm{Hom}_{\mathbf{sSet}}(Y, \mathcal{S}(T))$,

$$\widehat{\mathcal{S}(\psi) \circ f} = \psi \circ \widehat{f} : |X| \to S,$$

where $\mathcal{S}(\psi) : \mathcal{S}(X) \to \mathcal{S}(Y)$ is defined by $(\mathcal{S}(\psi))_n(\xi) := \psi \circ \xi$, for $\xi \in \mathcal{S}(X)_n$.
Let $x \in X_n$, $t \in \nabla^n$ and $f \in \mathrm{Hom}_{\mathbf{sSet}}(X, \mathcal{S}(T))$

Exercise 2

Exercise 3