Algebraic Geometry - *Exercise Sheet 1*

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Exercise 1

Let X be a finite CW-complex and Y a space such that for every base point $y \in Y$ and every number i that is less than or equal to the dimension of X the set $\pi_i(Y, y)$ is finite. Show that the set $[X, Y]$ of homotopy classes of maps from X to Y is finite.

Solution 1

We will be doing the proof in a number of steps.

- Suppose for now that we have already proved the statement for Y path-connected. Let $X^i X$, denote the path-connected components of X and similarly $Y^j \subset Y$ the path-connected components of Y. Then, by assumption, $[X^i, Y^j]$, the set of homotopy classes of maps from X^i to Y^j, is finite.

  Consider a map $f : X \to Y$, then for each $X^i \subset X$, $f|_{X^i}$ most be contained in one of the path-connected components of Y, say Y^j. So $f|_{X^i}$ defines an element in $[f|_{X^i}^{Y^j}] \in [X^i, Y^j]$.

  Define the following map

  $$\Phi : [X, Y] \to \prod_{i,j} \left([X^i, Y^j] \sqcup \{\omega\} \right); \quad f \mapsto \prod i, j [f|_{X^i}^{Y^j}],$$

  where $[f|_{X^i}^{Y^j}] = \omega$, whenever $f(X^i) \cap Y^j = \emptyset$. We want to show that this map is well-defined and injective, because if it is $[X, Y]$ most be fine, since the co-domain is a finite product of finite elements and thus finite.

  Suppose $f, g : X \to Y$ are homotopy equivalent by some homotopy H, then $H|_{X^i}$ defines a homotopy from $f|_{X^i}$ to $g|_{X^i}$. So Φ is well-defined.

  Now suppose $f, g : X \to Y$ such that $\Phi([f]) = \Phi([g])$. Then $f|_X^i$ is homotopy equivalent to $g|_{X^i}$ by some homotopy H_i for all of the path-connected compoinents of X. Clearly we may defined $H : I \times X \to Y$ by $H|_{X_i} = H_i$, which defines a homotopy between f and g.

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