Algebraic Topology - Exercise Sheet 4

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Exercise 1

(a)

(b) The fact that the two boundary points are mapped to the base point follows from

$$d_0(g) = \{1\} = d_1(g).$$

Let $\delta_i : [n-1] \to [n]$, be the face map such that $\delta_i^* = d_i$. Then

$$(BG)_1 \times \nabla^1 \ni (g, 0) = (g, \delta_{0*}(1)) \sim (d_0(g), 1) = (1, 1) \in (BG)_0 \times \nabla^1,$$

so $(g, 1) \in \{g\} \times \nabla^1$ is mapped to $(g, 0) = (1, 1)$ in $|BG|$. Similarly

$$(g, 1) = (g, (\delta_1)_*(1)) \sim (d_1(g), 1) = (1, 1),$$

so also $(g, 1) \in \{g\} \times \nabla^1$ is mapped to $(g, 1) = (1, 1)$ in $|BG|$.

(c) Let $g, h \in G$ and $\eta(g, h) : \nabla^2 \to |BG|$ defined by the composition

$$\{(g, h)\} \times \nabla^2 \hookrightarrow \bigcup_{n \geq 0} (BG)_n \times \nabla^n \twoheadrightarrow |BG|.$$

Like in (b), let $d_i = \delta_i^*$. Then

$$\delta_0; \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 2 \end{cases} \qquad \delta_1; \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 2 \end{cases} \qquad \delta_2; \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 1 \end{cases}$$

So for $t \in \nabla^2$, we have

$$\omega(g)(t) = (g, t) = (d_0(g, h), t) \sim ((g, h), (\delta_0)_*(t)) = (\eta(g, h)(\delta_0)_*)(t),$$

and similarly

$$\omega(h)(t) = (\eta(g, h)(\delta_2)_*)(t) \quad \text{and} \quad \omega(g \cdot h)(t) = (\eta(g, h)(\delta_1)_*)(t),$$

So ω on g, h and gh correspond to the loops defined η restricted to the three faces as illustrated in figure (1)

Also illustrated in the figure, by the cyan curves, $\eta(g, h)$ defines a homotopy from $\omega(g) \cdot \omega(h)$ to $\omega(g \cdot h)$.

Figure 1: