Algebraic Topology - Exercise Sheet 7

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Exercise 1

Suppose $f_n \to f$ in the compact-open topology. That is for all $K \subseteq X$ compact and for all $U \subseteq Z$ open there exists $N \in \mathbb{N}$, such that for all $n \geq N$, $f_n(K) \subseteq U$. Let $x \in X$, then $\{x\} \subseteq X$ is compact, so for all $U \ni f(x)$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $f_n(x) \in U$. So $f_n \to f$ pointwise.

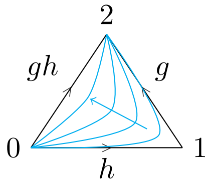

Figure 1: Graph of f_n.

Let $X = Z = [0, 1]$ and

$$f_n(x) = \begin{cases} 2nx & \text{if } x \leq (2n)^{-1} \\ 2 - 2nx & \text{if } (2n)^{-1} \leq x \leq n^{-1} \\ 0 & \text{if } n^{-1} \leq x \end{cases}$$

(see fig. (1)). Then $f_n \to f = 0$ pointwise. However $f_n([0, 1]) = [0, 1] \nsubseteq [0, 1/2)$, whereas $f([0, 1]) = \{0\} \subseteq [0, 1/2)$. So f_n does not converge to f in the compact-open topology.

Exercise 2

Since $p : E \to B$ is a Serre fibration, we have the long exact sequence

$$\ldots \pi_n(E, e) \to \pi_n(B, b) \to \pi_{n-1}(F, e) \to \pi_{n-1}(E, e) \to \ldots$$

where $e \in F = p^{-1}(b)$. Since E is contractable we have $\pi_n(E, e) = 0$, for $n \geq 0$. So the middle map is an isomorphism and thus

$$\pi_n(F, e) \cong \pi_{n+1}(B, b) \cong \pi_n(\Omega B, \star)$$

for $n \geq 0$.

Exercise 3

This exercise requires some argumentation!

Given $\sigma \in \Sigma_3$, let Ψ_σ, be the composite map defined in the exercise. Then

$$\begin{aligned} \Psi_{(0\,1)}; &\qquad [\phi, \psi] \mapsto [-\phi, \phi \circ \psi] \\ \Psi_{(0\,2)}; &\qquad [\phi, \psi] \mapsto [-\psi, -\phi] \\ \Psi_{(1\,2)}; &\qquad [\phi, \psi] \mapsto [\phi \circ \psi, -\psi] \end{aligned}$$

And if $\sigma = \sigma_1\sigma_2$, then $\Psi_\sigma = \Psi_{\sigma_1} \circ \Psi_{\sigma_2}$.