Topology II - Homology vanishing above the dimension

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All of the manifolds that we explicitly discussed admit CW-structures:

- S^n, $\mathbb{RP}^n$, $\mathbb{CP}^n$ and $\mathbb{HP}^n$,
- $V_{k,n}$, $\mathrm{Gr}_{k,n}$: CW-structures exists.

In all these cases: manifold dimension = CW-dimension

This is no coincidence: Suppose that a compact manifold M admits a CW-structure; let $x \in M$ be an interior point of a cell of top dimension (in the CW-structure). Then the open cell is is an open neighbourhood homeomorphic to $\mathbb{R}^n$ with $n =$ CW-dimension of M. Since the manifold dimension is intrinsic, $n =$ manifold dimension.

Corollary 0.1. Let M be a compact n-manifold that admits a CW-structure. Then $H_i(M; A) \cong H_i^{\mathrm{cell}}(M; A) = 0$ for $i > n$.

Warning: compact manifolds do not in general admit CW-structures. But every smooth compact manifold admits a triangulation, and hence a CW-structure.

Notation: For M an n-manifold, A an abelian group and $U \subset M$, let $H_i(M|U; A) = H_i(M, M \setminus U; A)$ denote the local i'th homology at U with values in A.

Theorem 0.2. Let M be an n-manifold, A and abelian group and K a compact subset of M. Then

(i) $H_i(M|K; A) = 0$ for $i > n$.

(ii) A class in $H_n(M|K; A)$ is zero if and only if its restriction to $H_n(M|x; A)$ is zero for all $x \in K$.

Note 0.3. M needs not be compact. But if M is compact, then $K = M$ is allowed and then both statements refer to absolute homology of M.

Proof. (The proof follows the proof of Lemma A.7 in Appendix A of Milnor-Stasheff's book "Characteristic Classes")

The proof is done in 6 steps.

(Step 1) Consider $M = \mathbb{R}^n$, K is a compact convex non-empty subset. For every $x \in K$, K can be linearly contracted onto x. Let $R > 0$ be large enough so that $K \subseteq B_R^{n-1}(x) := \{y \in \mathbb{R}^n \ : \ |x - y| \leq R\}$. Then the inclusion

$$S_{2R}^{-1}(x) := \{y \in \mathbb{R}^n \ : \ |x - y| = 2R\} \subseteq M \setminus K \subseteq M \setminus \{x\},$$

defines homotopy equivalences. So the induced map $H_i(M|K) \to H_i(M|x)$ is an isomorphism. In particular, $H_i(M|K) \cong H_i(M|x) \cong 0$ for $i > n$.

(Step 2) Let M be any n-manifold, $K = K_1 \cup K_2$ for $K_!, K_2$ compact, suppose the statements are true for K_1, K_2 and $K_1 \cap K_2$. Then the statements also hold for K. We have a long exact Mayer-Vietro's sequence for the local homology groups.

<u>Construction:</u>

$$M \setminus (K_1 \cap K_2) = (M \setminus K_1) \cup (M \setminus K_2) \quad \text{and} \quad (M \setminus K_1) \cap (M \setminus K_2) = M \setminus (K_1 \cup K_2) = M \setminus K.$$

The theorem of small subjections show that the map

$$\frac{C_*(M \setminus K_1) \oplus C_*(M \setminus K_2)}{C_*(M \setminus K)} \xhookrightarrow{\cong} C_*(M \setminus (K_1 \cap K_2))$$

is an isomorphism of all homology groups.

So the chain map

$$D := \frac{C_*(M)}{\left(\frac{C_*(M \setminus K_1) \oplus C_*(M \setminus K_2)}{C_*(M \setminus K)}\right)} \xrightarrow{\cong} \frac{C_*(M)}{C_*(M \setminus (K_1 \cap K_2))}$$

is also a quasi-isomorphism.

The so

$\square$