Commutativity of the cup product

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<u>Thm</u> Let X be a simplicial set and R a commutative ring. Then the cup product pairing on the R-cohomology of X is <u>graded-commutative</u>; for all $x \in H^n(X;R)$ and $y \in H^m(X;R)$, the relation

$$x \cup y = (-1)^{n \cdot m} \cdot y \cup x \quad \text{holds in} \quad H^{n+m}(X;R).$$

<u>Proof</u>: The proof uses another operation, the $\cup_1$-product:

$$\cup_1 : C^n(X;R) \times C^m(X;R) \longrightarrow C^{n+m-1}(X;R)$$

is defined by

$$(f \cup_1 g)(x) = \sum_{i=0}^{n-1} (-1)^{(n-i)\cdot(m+1)} f((d_i^{out})^*(x)) \cdot g((d_i^{inn})^*(x)) \qquad f : X_n \to R, \; g : X_m \to R$$

$$x \in X_{n+m+1}$$

where $d_i^{out} : [n] \longrightarrow [n+m-1]$ and $d_i^{inn} : [m] \longrightarrow [n+m-1]$ are the unique injective monotone maps with images

$$\operatorname{im}(d_i^{out}) = \{0, \ldots, i\} \cup \{i+m, \ldots, n+m-1\} \quad \text{and}$$

$$\operatorname{im}(d_i^{inn}) = \{i, \ldots, i+m\}$$

Coboundary formula for $\cup_1$:

$$\boxed{d(f \cup_1 g) = (df) \cup_1 g + (-1)^n f \cup_1 (dg) - (-1)^{n+m} f \cup g - (-1)^{(n+1)(m+1)} \cdot g \cup f.}$$

Proof: Separate slides! Now assume the coboundary formula, and let f and g be cocycles, i.e. $df = 0$, $dg = 0$.

Then
$$d(f \cup_1 g) = -(-1)^{n+m} \cdot f \cup g - (-1)^{(n+1)(m+1)} \cdot g \cup f$$

multiply by $(-1)^{n+m+1}$:
$$(-1)^{n+m+1} \cdot d(f \cup_1 g) = f \cup g - (-1)^{n \cdot m} g \cup f$$

So on the level of cohomology classes:

$$[f] \cup [g] = [f \cup g] = [(-1)^{n\cdot m} g \cup f + (-1)^{n+m+1} d(f \cup_1 g)] = (-1)^{n \cdot m} [g \cup f]$$
$$= (-1)^{n \cdot m} [g] \cup [f].$$

<u>Reinterpretation</u> of the coboundary formula:

The cup product maps can be encoded as a single morphism of cochain complexes

$$\cup : C^*(X;R) \otimes C^*(X;R) \longrightarrow C^*(X;R)$$

In dimension k:

$$\bigoplus_{n+m=k} C^n(X;R) \otimes C^m(X;R) \longrightarrow C^{n+m}(X;R)$$
$$f \otimes g \longmapsto f \cup g.$$

Leibniz formula for $f \cup g$ $\iff$ this is a cochain map

The collection of $\cup_1$-products $\cup_1 = \{\cup_1 : C^n(X,R) \otimes C^m(X,R) \longrightarrow C^{n+m-1}(X;R)\}$ define a cochain homotopy from $\cup$ (considered as a cochain map) to $\cup \circ$ twist,

where

$$\text{twist} : C^*(X;R) \otimes C^*(X;R) \longrightarrow C^*(X;R) \otimes C^*(X;R)$$
$$x \otimes y \longmapsto (-1)^{n \cdot m} y \otimes x \qquad x \in C^n(X,R), \; y \in C^m(X,R)$$