Cup product: first examples

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Some methods to determine cup products:
- directly from the definitions
- cellular approximation of the diagonal (for CW complexes) } not particularly practical
- group cohomology (for Eilenberg-MacLane spaces)
- by Poincaré duality (for manifolds) } in more detail later in the class
- de Rham cohomology (for smooth manifolds and $\mathbb{R}$-coefficients)

Example: Let X be a discrete space. Then $\mathcal{S}(X)$ is a constant simplicial set, the complex $C_*(X,\mathbb{Z})$ has the following very special form:

$$\left(\cdots \longrightarrow \underset{3}{\mathbb{Z}[X]} \xrightarrow{0} \underset{2}{\mathbb{Z}[X]} \xrightarrow{Id} \underset{1}{\mathbb{Z}[X]} \xrightarrow{0} \underset{0}{\mathbb{Z}[X]} \right)$$

Apply $\mathrm{Hom}(-,A)$ and use the freeness property of $\mathbb{Z}[X]$:

$$\mathrm{Hom}(C_*(\mathcal{S}(X),\mathbb{Z}),A) = \left(\cdots \xleftarrow{0} \underset{2}{\mathrm{map}(X,A)} \xleftarrow{Id} \underset{1}{\mathrm{map}(X,A)} \xleftarrow{0} \underset{0}{\mathrm{map}(X,A)} \right)$$

$$\Longrightarrow \quad H^n(X;A) = H^n(\mathrm{Hom}(C_*(X;\mathbb{Z}),A)) = \begin{cases} \mathrm{map}(X,A) & n=0 \\ 0 & n>0 \end{cases}$$

Now let R be a ring. Then the isomorphism

$$H^0(X;R) \cong \mathrm{map}(X,R)$$ takes the cup product to pointwise multiplication of functions.

Example: We recall the bar construction BG of a group G. This is the simplicial set defined by

$$(BG)_n = G^n, \quad \text{with face maps } d_i^*: G^n \longrightarrow G^{n-1} \quad 0 \le i \le n$$

$$d_i^*(g_1,\dots,g_n) = \begin{cases} (g_2,\dots,g_n) & \text{for } i=0 \\ (g_1,\dots,g_i\cdot g_{i+1},\dots,g_n) & \text{for } 1\le i\le n-1 \\ (g_1,\dots,g_{n-1}) & \text{for } i=n \end{cases}$$

and degeneracy maps $s_j^*: G^n \longrightarrow G^{n+1}$ for $0\le j\le n$

$$s_j^*(g_1,\dots,g_n) = (g_1,\dots,g_j,e,g_{j+1},\dots,g_n)$$

e: multiplicative unit in G.

Preview: BG is the simplicial set version of an Eilenberg-MacLane space of type $(G,1)$.

The cochain complex $C^*(BG;A)$ in low dimensions:

$$\begin{array}{ccccccc} C^0(BG;A) & \xrightarrow{d^0} & C^1(BG;A) & \xrightarrow{d^1} & C^2(BG;A) & \xrightarrow{d^2} & \cdots \\ \| & & \| & & \| & & \\ A = \mathrm{map}(\{e\},A) & \xrightarrow{0} & \mathrm{map}(G,A) & \longrightarrow & \mathrm{map}(G^2,A) & & \end{array}$$

$$a \longmapsto \{ g \mapsto d_0^*(g) - d_1^*(g) = 0 \}$$

$$\{ f: G\to A \} \longmapsto \{ (g,h) \mapsto f(d_0^*(g,h)) - f(d_1^*(g,h)) + f(d_2^*(g,h)) \}$$

$$= \{ (g,h) \mapsto f(h) - f(g\cdot h) + f(g) \}$$

Observe: $d^1 f = 0 \iff 0 = f(h) - f(gh) + f(g)$ for all $g,h\in G$

$\iff f: G\to A$ is a group homomorphism.

$$\Longrightarrow \quad H^1(BG;A) = \mathrm{Hom}(G,A)$$

Special case: $G = \mathbb{F}_2$ additive group of $\mathbb{F}_2$

$A = \mathbb{F}_2$ the field with 2 elements

$$x = [Id_{\mathbb{F}_2}] \in H^1(B\mathbb{F}_2;\mathbb{F}_2) = \mathrm{Hom}(\mathbb{F}_2,\mathbb{F}_2)$$

Set $x^n = \underbrace{x\cup\dots\cup x}_{n} \in H^n(B\mathbb{F}_2;\mathbb{F}_2)$

Prop: For all $n\ge 1$ the class $x^n \in H^n(B\mathbb{F}_2;\mathbb{F}_2)$ is represented by the cocycle $f_n: (B\mathbb{F}_2)_n = \mathbb{F}_2^{\,n} \longrightarrow \mathbb{F}_2$

$$f_n(\lambda_1,\dots,\lambda_n) = \lambda_1\cdot\dots\cdot\lambda_n = \begin{cases} 1 & \text{if } \lambda_1=\lambda_2=\dots=\lambda_n \\ 0 & \text{else.} \end{cases}$$

Moreover, $x^n \ne 0$.

Preview: In fact, x also generates $H^*(B\mathbb{F}_2,\mathbb{F}_2)$, and this graded ring is a polynomial algebra over $\mathbb{F}_2$ on the class x.

Proof: We show $x^n = [f_n]$ by induction on $n \geq 1$. $x = [Id_{\mathbb{F}_2}] = [f_1]$.
Suppose now $n \geq 2$.

$$x^n = x^{n-1} \cup x = [f_{n-1}] \cup [Id_{\mathbb{F}_2}] = [f_{n-1} \cup Id_{\mathbb{F}_2}]$$

Moreover, $(f_{n-1} \cup Id_{\mathbb{F}_2})(\lambda_1, \dots, \lambda_n) = f_{n-1}(d_{front}^*(\lambda_1, \dots, \lambda_n)) \cdot Id_{\mathbb{F}_2}(d_{back}^*(\lambda_1, \dots, \lambda_n))$

$$= f_{n-1}(d_n^*(\lambda_1, \dots, \lambda_n)) \cdot Id_{\mathbb{F}_2}((d_0^*)^{n-1}(\lambda_1, \dots, \lambda_n))$$

$d_{front} = d_n : [n-1] \longrightarrow [n]$

$d_{back} = d_0^{n-1} : [1] \longrightarrow [n]$ $\quad = f_{n-1}(\lambda_1, \dots, \lambda_{n-1}) \cdot \lambda_n = (\lambda_1 \cdots \lambda_{n-1}) \cdot \lambda_n = f_n(\lambda_1, \dots, \lambda_n)$.

It remains to show that $x^n = [f_n] \neq 0$.
In the UCT we used an evaluation homomorphism $\Phi : H^n(X; A) \longrightarrow Hom(H_n(X; \mathbb{Z}), A)$

$$[f : X_n \to A] \longmapsto \{ [\sum b_i \cdot x_i] \mapsto \sum b_i \cdot f(x_i) \}$$

We want to use the following variant for a ring R:

$$\Phi : H^n(X; R) \longrightarrow Hom(H_n(X; R), R)$$
$$[f : X_n \to R] \longmapsto \{ [\sum r_i \cdot x_i] \longmapsto \sum r_i \cdot f(x_i) \}$$

We use this for $R = \mathbb{F}_2$, $X = B\mathbb{F}_2$:

$$\Phi : H^n(B\mathbb{F}_2; \mathbb{F}_2) \longrightarrow Hom(H_n(B\mathbb{F}_2; \mathbb{F}_2), \mathbb{F}_2)$$
$$\begin{matrix} \psi \\ x^n \end{matrix} \longmapsto \Phi(x^n)$$

We show that $\Phi(x^n) \neq 0$, so in particular, $x^n \neq 0$.

We consider the element $y = \sum_{(\lambda_1, \dots, \lambda_n) \in \mathbb{F}_2^n} 1 \cdot (\lambda_1, \dots, \lambda_n) \in \mathbb{F}_2[\mathbb{F}_2^n] = \mathbb{F}_2[(B\mathbb{F}_2)_n] = C_n(B\mathbb{F}_2; \mathbb{F}_2)$

Claim: y is a cycle in $C_n(B\mathbb{F}_2; \mathbb{F}_2)$: $(\lambda_1, \dots, \lambda_n)$ and $(\lambda_1 + 1, \lambda_2, \dots, \lambda_n)$ have the same d_0^*

$(\lambda_1, \lambda_2, \dots, \lambda_n)$ and $(\lambda_1 + 1, \lambda_2 + 1, \lambda_3, \dots, \lambda_n)$ have the same d_1^*

$\vdots$

$(\lambda_1, \dots, \lambda_n)$ and $(\lambda_1, \dots, \lambda_{n-1}, \lambda_n + 1)$ have the same d_n^*

$\Rightarrow \quad dy = \sum_{i=0,\dots,n} (-1)^i \sum_{(\lambda_1, \dots, \lambda_n)} d_i^*(\lambda_1, \dots, \lambda_n)$

has every element of $\mathbb{F}_2^{n-1} = (B\mathbb{F}_2)_{n-1}$ occurring an even number of times, so this is 0 in $\mathbb{F}_2[(B\mathbb{F}_2)_{n-1}]$

So y defines a homology class $[y] \in H_n(B\mathbb{F}_2; \mathbb{F}_2)$

We calculate:

$$\Phi(x^n)[y] = \Phi[f_n](y) = \sum_{\substack{(\lambda_1, \dots, \lambda_n) \\ \in \mathbb{F}_2^n}} f_n(\lambda_1, \dots, \lambda_n)$$

So $\Phi(x^n) \neq 0$ and hence $x^n \neq 0$.

$$= \sum_{(\lambda_1, \dots, \lambda_n) \in \mathbb{F}_2^n} \lambda_1 \cdots \lambda_n = 1$$

□

Let p be an odd prime. Let $x = [Id_{\mathbb{F}_p}] \in H^1(B\mathbb{F}_p; \mathbb{F}_p)$

Then $x^n = 0$ for $n \geq 2$. The graded commutativity $y \cup z = (-1)^{n \cdot m} \cdot z \cup y$ holds for $y \in H^n(X; R)$ and $z \in H^m(X; R)$

For $n = m$ odd and $y = z \in H^m(X; R)$, this specializes to $y \cup y = -y \cup y$, or $2 \cdot y \cup y = 0$.

If 2 is also invertible in R, then $y \cup y = 0$.

So for $R = \mathbb{F}_p$ with p odd, $y \in H^n(X; \mathbb{F}_p)$, n odd, $y \cup y = 0$.

In particular $x = [Id_{\mathbb{F}_p}]$ satisfies $x^2 = 0$, hence $x^n = 0$ for $n \geq 2$.

Outlook: Define a function $h : \mathbb{F}_p \times \mathbb{F}_p \longrightarrow \mathbb{F}_p$ by $\mathbb{F}_p = \{0, 1, \dots, p-1\}$

$$h(i,j) = \begin{cases} 0 & \text{if } i+j < p \\ 1 & \text{if } i+j \geq p. \end{cases}$$

You might want to check that h is a cocycle in $C^2(B\mathbb{F}_p; \mathbb{F}_p)$. Similar (but more complicated) arguments as for $\mathbb{F}_2$ show that $y^n = [h]^n \neq 0$ in $H^{2n}(B\mathbb{F}_p; \mathbb{F}_p)$

Moreover, $H^*(B\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p[x, y] / (x^2)$ as graded rings.