Topology II - The fundamental class

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Homology of some connected compact manifolds:

$$H_n(S^n;\mathbb{Z}) \cong H_{2n}(\mathbb{CP}^n;\mathbb{Z}) \cong H_{4n}(\mathbb{HP}^n;\mathbb{Z}) \cong \mathbb{Z}$$
$$H_n(\mathbb{RP}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if} n \text{ is odd} \\ 0 & \text{if} n \text{ is even} \end{cases}$$

We will show locally that for a connected, compact, non-empty n-manifold M,

$$H_n(M;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if} M \text{ is orientable} \\ 0 & \text{if} n \text{ is not orientable} \end{cases}$$

Theorem 0.1. Let M be an orientable n-manifold and let K ve a compat subset of M. Then there is a unique class $\mu_K \in H_n(M|K)$ such that $r_x^K(\mu_K) = \mu_x$ gives local orientation in $H_n(M|x)$ for all $x \in K$.

0.1 Important special calse:

If M is it self compact, we can take $K = M$; there is then a unique class $\mu_M = [M] \in H_n(M;\mathbb{Z})$, the fundamental class such that $r_x^M[M] = \mu_x$ in $H_n(M|x)$ for all $x \in M$.

We will show later that if M is moreover connected and non-empty then $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ with generator $[M]$.

Remark 0.2. Let M be orientable, compact, connected. Let $\bar{M}$ be M with the opposite orientation. Then $[\bar{M}] = -[M]$, because both have the same image in $H_n(M|x)$ for all $x \in M$.

Remark 0.3. Compactness is necessary: Let M be an n-manifold and $U \in H_n(M;A)$ (for A some abelian group) such that $r_x^M(u) \neq 0$ in $H_n(M|x;A)$ for all $x \in M$. Then M is compact. Indeed, write $u = [x]$ for some n-cycle $x \in C_n(M;A)$. Write $x \in \sum_{\text{finite}} a_i(f_i : \nabla^n \to M)$, and set $L = \operatorname{supp}(x) = \cup f_i(\nabla^n)$, some compact subset of M.

If M is not compact, there is a point $x \in M \setminus L$. So $x \in C_n(M \setminus \{x\};A)$ is a cycle in $M \setminus \{x\}$. So $u = [x]$ has zero image in $H_n(M|x;A)$ This contradicts the assumption on u.

Proof. (following the proof of Theorem A.8 in Appendix A of Milnor-Stasheff's "Characteristic Classes")

Yet to be typed up. □

Corollary 0.4. Let M be an orientable, compact, connected n-manifold. Then for all $x \in M$, the map

$$r_x^M : H_n(M;\mathbb{Z}) \to H_n(M|x)$$

is an isomorphism. So if M is non-empty, $H_n(M;\mathbb{Z})$ is free of rank 1, and the fundamental class of the two orientations are the two generators.

Proof. To Be done □

Corollary 0.5. Let M be a compact, connected n-manifold that is not orientable. Then $H_n(M;\mathbb{Z}) = 0$.