The theorem of acyclic models

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Let A and B be simplicial abelian groups

$$\begin{tikzcd} C_*(A)\otimes C_*(B) \arrow[r, bend left, "\text{Eilenberg-Zilber}"] & C_*(A\otimes B) \arrow[l, bend left, "\text{Alexander-Whitney}"] \end{tikzcd}$$

tensor product of chain complexes — dimensionwise tensor product

Aim: The composites $AW \circ EZ : C_*(A)\otimes C_*(B) \longrightarrow C_*(A)\otimes C_*(B)$

and $EZ \circ AW : C_*(A\otimes B) \longrightarrow C_*(A\otimes B)$

are naturally chain homotopic to the identities.

We will show this for $A = \mathbb{Z}[X]$ and $B = \mathbb{Z}[Y]$ for simplicial sets X and Y; then $A\otimes B \cong \mathbb{Z}[X\times Y]$.

Prop: (Yoneda lemma). Let $\mathcal{C}$ be a category and c an object of $\mathcal{C}$. Then for every functor $F: \mathcal{C} \longrightarrow (\text{sets})$ the evaluation map

$$\mathrm{Nat}(\mathcal{C}(c,-), F) \longrightarrow F(c) \qquad \text{is bijective.}$$
$$\tau \longmapsto \tau_c(\mathrm{Id}_c)$$

Equivalently: For any element $y \in F(c)$ there is a unique natural transformation $\tau: \mathcal{C}(c,-) \longrightarrow F$ such that $\tau_c(\mathrm{Id}_c) = y$.

Proof: Injectivity: Let $\tau: \mathcal{C}(c,-) \longrightarrow F$ be any natural transformation, d any other object of $\mathcal{C}$, and $f: c \longrightarrow d$ any morphism.

Then: $\tau_d : \mathcal{C}(c,d) \longrightarrow F(d)$,

$$\tau_d(f) = \tau_d(\mathcal{C}(c,f)(\mathrm{Id}_c)) \underset{\text{naturality}}{=} F(f)(\tau_c(\mathrm{Id}_c))$$

So the value of τ at (d,f) is determined by the functoriality of F and the value at (c, Id_c).

Surjectivity: Let $y \in F(c)$ be any element. For an object d of $\mathcal{C}$ we define

$$\tau_d : \mathcal{C}(c,d) \longrightarrow F(d) \quad \text{by} \quad \tau_d(f) = F(f)(y).$$

This is indeed a natural transformation: let $g: d \longrightarrow e$ be any $\mathcal{C}$-morphism.

Then $F(g)(\tau_d(f)) = F(g)(F(f)(y)) = F(gf)(y) = \tau_e(gf) = \tau_e(\mathcal{C}(c,g)(f))$

This is naturality, so $\{\tau_d\}_{d\in\mathrm{Ob}(\mathcal{C})}$ is a natural transformation. Finally,

$$\tau_c(\mathrm{Id}_c) = F(\mathrm{Id}_c)(y) = \mathrm{Id}_{F(c)}(y) = y, \quad \text{so } \tau \text{ really evaluates to } y. \quad \square$$

Let $\mathcal{C}$ be any category and c an object of $\mathcal{C}$. We define the functor $\mathbb{Z}[\mathcal{C}(c,-)] : \mathcal{C} \longrightarrow Ab$ = category of abelian groups and homomorphisms

as the composite $\mathcal{C} \xrightarrow{\mathcal{C}(c,-)} (\text{sets}) \xrightarrow{\mathbb{Z}[-]} Ab$.

At an object d of $\mathcal{C}$, $\mathbb{Z}[\mathcal{C}(c,-)](d) = \mathbb{Z}[\mathcal{C}(c,d)]$.

Prop: (Additive Yoneda lemma). Let c be an object of the category $\mathcal{C}$ and $F: \mathcal{C} \longrightarrow Ab$ be a functor to abelian groups.

Then the evaluation map $\mathrm{Nat}(\mathbb{Z}[\mathcal{C}(c,-)], F) \longrightarrow F(c)$, $\tau \longmapsto \tau_c(1\cdot \mathrm{Id}_c)$ is bijective.

Proof: For varying objects d of $\mathcal{C}$, the bijections

$$\mathrm{Hom}_{Ab}(\mathbb{Z}[\mathcal{C}(c,d)], F(d)) \xrightarrow{\text{evaluate at generators}} \mathrm{map}(\mathcal{C}(c,d), F(d)) \quad \text{is bijective.}$$

form a bijection

$$\mathrm{Nat}_{\mathcal{C}\to Ab}(\mathbb{Z}[\mathcal{C}(c,-)], F) \longrightarrow \mathrm{Nat}_{\mathcal{C}\to \text{sets}}(\mathcal{C}(c,-), F)$$

So the proof is the composition of this bijection and the set-valued Yoneda lemma $\square$.

Def: A functor $F: \mathcal{C} \longrightarrow Ab$ is representable if there is an object c of $\mathcal{C}$ and a natural isomorphism

$$F \cong \mathbb{Z}[\mathcal{C}(c,-)].$$

Example: Let $\mathcal{C} = \text{sSets} \times \text{sSets}$ be the product of two copies of the category of simplicial sets.

Let $F : \text{sSets} \times \text{sSets} \longrightarrow Ab$ be the functor $F(X,Y) = \mathbb{Z}[X_p \times Y_q]$ $\quad (\cong \mathbb{Z}[X]_p \otimes \mathbb{Z}[Y]_q)$

This functor is also representable, namely by the object (Δ^p, Δ^q).

Indeed:
$$(\text{sSet} \times \text{sSet})((\Delta^p, \Delta^q), (X,Y)) = \text{sSet}(\Delta^p, X) \times \text{sSets}(\Delta^q, Y)$$
$$\underset{\text{Yoneda}}{\cong} X_p \times Y_q \qquad \text{(natural bijection in } X \text{ and } Y)$$

Apply free abelian groups to get an isomorphism of abelian groups
$$\mathbb{Z}[(\text{sSet} \times \text{sSets})((\Delta^p, \Delta^q), (X,Y))] \cong \mathbb{Z}[X_p \times Y_q].$$

Theorem (Acyclic models) Let $\mathcal{C}$ be a category and $F, G : \mathcal{C} \longrightarrow Ch_*$ be two functors to the category of non-negatively graded chain complexes. Let $f : F \longrightarrow G$ be a natural chain map. Suppose that:

i) The transformation $f_0 : F_0 \longrightarrow G_0$ is the zero transformation.

ii) For every $n \geq 1$, the functor $F_n : \mathcal{C} \longrightarrow Ab$ is isomorphic to a direct sum of represented functors $\mathbb{Z}[\mathcal{C}(c,-)]$ for some family of $\mathcal{C}$-objects c such that $H_n(G(c)) = 0$.

Then the f is naturally chain nullhomotopic.

Proof: For $n \geq 0$ we will construct natural transformations $s_n : F_n \longrightarrow G_{n+1}$ of functors $\mathcal{C} \longrightarrow Ab$ such that

(*) $\quad d^G_{n+1} \circ s_n + s_{n-1} \circ d^F_n = f_n$ as natural transformations of functors $F_n \longrightarrow G_n$. $\quad (s_{-1} = 0)$

The construction is inductive, starting with $s_0 = 0$, the (*) holds for $n=0$ because $f_0 = 0$.

Now suppose that $n \geq 1$ and we have constructed $s_0, \dots, s_{n-1}$ satisfying (*).

Then
$$d^G_n \circ (f_n - s_{n-1} \circ d^F_n) = d^G_n \circ f_n - d^G_n \circ s_{n-1} \circ d^F_n$$
$$\underset{f \text{ is a chain map}}{=} f_{n-1} \circ d^F_n - d^G_n \circ s_{n-1} \circ d^F_n = (f_{n-1} - d^G_n \circ s_{n-1}) \circ d^F_n$$
$$\underset{(*)}{=} s_{n-2} \circ \underbrace{d^F_{n-1} \circ d^F_n}_{=0} = 0.$$
as natural transformations of functors $F_n \longrightarrow G_{n-1}$.

In other words, the transformation $f_n - s_{n-1} \circ d^F_n : F_n \longrightarrow G_n$ takes values in the kernel of $d^G_n : G_n \longrightarrow G_{n-1}$.

By hypothesis (ii) we can assume that $F_n = \bigoplus_{i \in I} \mathbb{Z}[\mathcal{C}(c_i, -)]$ for some set I and $\mathcal{C}$-objects c_i such that moreover $H_n(G(c_i)) = 0$ for all $i \in I$.

For fixed $j \in I$, we write $x_j \in F(c_j) = \bigoplus_{i \in I} \mathbb{Z}[\mathcal{C}(c_i, c_j)]$ for the element $1 \cdot Id_{c_j}$ in the summand for $i = j$.

By the above, the element
$$f_n(x_j) - s_{n-1}(d^F_n(x_j)) \in G_n(c_j) \text{ is a cycle in the complex } G(c_j).$$

Since $H_n(G(c_j)) = 0$, this n-cycle is a boundary. So there is an element $y_j \in G(c_j)_{n+1}$ such

(*) $\quad d^G_{n+1}(y_j) = f_n(x_j) - s_{n-1}(d^F_n(x_j)).$

The additive Yoneda lemma provides a unique natural transformation $s_{n,j} : \mathbb{Z}[\mathcal{C}(c_j,-)] \longrightarrow G_{n+1}$ such that
$$\underset{\substack{\parallel \\ s_{n,j}(x_j)}}{s_{n,j}(1 \cdot Id_{c_j})} = y_j.$$

We define the desired natural transformation $s_n = \sum_{j \in I} s_{n,j} : F_n = \bigoplus_{j \in I} \mathbb{Z}[\mathcal{C}(c_j,-)] \longrightarrow G_{n+1}$

To verify the relation (*)
$$d^G_{n+1} \circ s_n + s_{n-1} \circ d^F_n = f_n$$
it suffices to show
$$d^G_{n+1} \circ s_n = f_n - s_{n-1} \circ d^F_n \quad \text{on each of the summands of } F \text{ separately.}$$

On the j-th summand, this is a relation between natural transformations of functors $\mathbb{Z}[\mathcal{C}(c_j,-)] \longrightarrow G_n$, and by the additive Yoneda lemma it suffices to check the relation at the object c_j and the element

$\ldots_{n+1} \circ \ldots_n = f_n \circ \ldots_{n+1} \circ \ldots_n$ … of the summands of … separately.

On the j-th summand, this is a relation between natural transformations of functors $\mathbb{Z}[\mathcal{C}(c_j,-)] \longrightarrow G_n$, and by the additive Yoneda lemma it suffices to check the relation at the object c_j and the element $x_j = 1 \cdot \mathrm{Id}_{c_j}$. But this is the defining relation (*) for y_j. $\square$

This completes the inductive construction, and hence the proof.

<u>Remark:</u> What I proved is only "half" of what is known as the theorem of acyclic models. The other half goes as follows:

<u>Thm:</u> Let $\mathcal{C}$ be a category, $F, G : \mathcal{C} \longrightarrow Ch_+$ two functors to the category of non-negatively graded chain complexes. Let $f_0 : F_0 \longrightarrow G_0$ be a natural transformation of functors from $\mathcal{C}$ to Ab. Suppose that for all $n \geq 1$, F_n is isomorphic to a sum of represented functors $\mathbb{Z}[\mathcal{C}(c,-)]$ for some family of $\mathcal{C}$-objects c such that $H_n(G(c)) = 0$. Then there is natural transformation $f : F \longrightarrow G$ that is f_0 in chain dimension 0.