Topology II - Stiefel manifolds

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Let $0 \leq k \leq n$. The Stiefel manifold is defined by

$$V_{k,n} = \left\{ (v_1, \dots, b_k) \in (\mathbb{R}^n)^k \ : \ \langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \right\} \tag{1}$$

$$= \text{space of orthogonal } k\text{-frames in } \mathbb{R}^n \tag{2}$$

$V_{k,n}$ comes with the subspace topology of $(\mathbb{R}^n)^k$; since $V_{k,n} \subset (S^{n-1})^4$ is a closed subset, $V_{k,n}$is compact.

Example 0.1.

$$V_{0,n} = \{\emptyset\} \text{ is a one-pint space.}$$
$$V_{1,n} = S^{n-1}$$
$$F : V_{n,n} \overset{\cong}{\leftrightarrow} O(n) : G$$

where F maps $(v_1, ..., v_n)$ to the matrix with colomns v_i and G maps A to $(Ae_1, ..., Ae_n)$ where e_i is the unit vecotr with a 0 in the i-th entry and 0's elsewhere.

The map $F : SO(n) \to V_{n-1,n}$ defined by $A \mapsto (Ae_1, ..., Ae_{n-1})$, is a continues bijection beween compact Housdorff spaces, and hence a homeomorphism.

Bijectivity: Let $(v_1, ..., v_{n-1})$ be an $(n-1)$-frame in $\mathbb{R}^n$, then the orthonogal complement of the span of $v_1, ..., v_{n-1}$ is 1-dim. So there are exactly 2 unit vectors in this complement. Exactly one of these makes $(v_1, ..., v_{n-1}, v_n)$ into an orthogonal basis of determinant $+1$.

Proposition 0.2. The space $V_{k,n}$is a manifold of dimension

$$(n-1) + (n-2) + ... + (n-k) = nk - \frac{k(k+1)}{2}.$$

Proof. By induction on k. For $k = 0$, $V_{0,n} = \{\emptyset\}$ is a 0-manifold and for $k = 1$, $V_{1,n} = S^{n-1}$ is a $(n-1)$-manifold.

Now suppose $k \geq 2$. We consider the map $\phi : S^{n-1}_+ \to O(n)$, where

$$S^{n-1}_+ = \{w \in S^{n-1} \ : \ w_1 > 0\},$$

defined be the composition

$$S^{n-1}_+ \longrightarrow \mathrm{GL}_n(\mathbb{R}) \xrightarrow{\text{Gram-Schmidt Orth.}} O(n)$$

$$O(n) \xmapsto{\quad\cong\quad} \begin{pmatrix} w_1 & 0 & \dots & 0 \\ w_2 & 1 & & 0 \\ \vdots & & \ddots & \\ w_n & 0 & & 1 \end{pmatrix}$$

Properties of ψ:

- ψ is continues
- $\psi(e_1) = \psi(1, 0, .., 0) = E_n = \text{identity matrix}$

- $\psi(w) \cdot e_1 = w$ for all $w \in S^{n-1}_+$

  Warning: There is not continues map $\psi : S^{n-1} \to O(n)$ such that $\psi(w) \cdot w = w$ for all $w \in S^{n-1}$. We define $U = \{(v_1, ..., v_k) \in V_{k,n} \ : \ v_1 \in S^{n-1}_+\}$; this is an open neighbourhood of $(e_1, ..., e_k) \in V_{k,n}$. The map $\xi : U \to S^{n-1}_+ \times V_{k-1,n-1}$ defined by

$$(v_1, ..., v_k) \mapsto (v_1, \psi(v_1)^{-1}(v_2), ..., \psi(v_1)^{-1}(v_k))$$

is a homeomorphism.

- ξ is well-defined: $\psi(v_1)^{-1}$ is an orthogonal matrix such that $\psi(v_1)^{-1}(v_1) = e_1$, since $\psi(v_1)^{-1}$ is orthogonal and $v_2, ..., v_k$ define a k-frame in $(v_1)^\perp$. So $\psi(v_1)^{-1}(v_2), ..., \psi(v_1)(v_k)$ defines a k-frame in $(e_1)^\perp = 0 \otimes \mathbb{R}^{n-1}$.

- ξ is continues

- ξ has a continues inverse:

$$S^{n-1}_+ \times C_{k-1,n-1} \to U; \qquad (v, w_1, ..., w_{k-1}) \mapsto (v, \psi(v)(0, w_1), ..., \psi(v)(0, w_{k-1})),$$

  where $(0, w_i) \in \mathbb{R}^n = \mathbb{R} \otimes \mathbb{R}^{n-1}$.

Conclution: The point $(e_1, ..., e_n) \in V_{k,n}$ has an open neighbourhood homeomorphic to $S^{n-1}_+ \times V_{k-1,n-1}$, which is a manifold of dimension

$$d = (n-1) + (n-2) + (n-3) + ... + ((n-1) - (k-1)),$$

by induction. So $(e_1, ..., e_k)$ has an open neighbourhood homeomorphic to $\mathbb{R}^d$.

Now let $(v_1, ..., v_k) \in V_{k,n}$ be any point. Complete to an orthogonal basis

$$A = (v_1, ..., v_k, v_{k+1}, ..., v_n) \in O(n).$$

Then

$$A : V_{k,n} \to V_{k,n}; \qquad (w_1, ..., w_k) \mapsto (Aw_1, ..., Aw_k)$$

is a self-homeomorphism of $V_{k,n}$ that sends $(e_1, ..., e_k)$ to $(v_1, ..., v_k)$. So also $(v_1, ..., v_k)$ has an open neighbourhood homeomorphic to $\mathbb{R}^d$. $\square$

Remark 0.3. What we really showed is that the map $V_{k,n} \to S^{n-1}$ defined by $(v_1, ..., v_k) \mapsto v_1$ is a "locally trivial fibre bundle" with fibre $V_{k-1,n-1}$.

0.1 Complex Steifel manifolds:

Let

$$V^{\mathbb{C}}_{k,n} = \{(v_1, ..., v_k) \in (\mathbb{C}^n)^k \ : \ \langle v_i, v_j \rangle \begin{cases} 1 \text{if } i = j \\ 0 \text{if } i \neq j \end{cases} \} \qquad = \text{space of (complex) } k\text{-frames in } \mathbb{C}^n$$

As in the real case, one shows that $V^{\mathbb{C}}_{k,n}$ is a compact d-manifold, where

$$d = (2n-1) + (2n-3) + ... + (2n-2k+1) = 2nk - k^2.$$

0.1.1 special case

$$\begin{aligned} V^{\mathbb{C}}_{1,n} &= \text{unit sphere in } \mathbb{C}^n = S^{2n-1} \\ V^{\mathbb{C}}_{n-1,n} &\cong \mathrm{SU}(n), \\ V^{\mathbb{C}}_{n,n} &\cong \mathrm{U}(n) \end{aligned}$$

Same induction proof, with Gram-Schmidt orthonormalization for hermitian inner product spaces; In the indutive step, you work over $S^{2n-1}_+ = \{(v_1, ..., v_n) \in S^{n-1} \ : \ \Re(v_1) > 0\}$.

0.2 Quaternion Stiefel manifolds:

$V_{k,n}^{\mathbb{H}}$ defines compact manifolds of dimention

$$(4n-1)+(4n-5)+...+(4n-4k+3)=4nk-k(2k-1).$$

0.2.1 special case

$$V_{1,n}^{\mathbb{H}} = \text{unit sphere in } \mathbb{H}^n \cong S^{4n-1},$$
$$V_{n,n}^{\mathbb{H}} = \mathrm{Sp}(n) = \{A \in M(n \times n, \mathbb{H}) \ : \ \dot{\bar{A}}A^T = \bar{A}^T \cdot A = E_n\}.$$