Modern Classical Homotopy Theory

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Dedicated to my mom and dad

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Part 2. Semi-Formal Homotopy Theory

Preface

The subject of topology can be described as the study of the category **Top** of all topological spaces and the continuous maps between them. But many topological problems, and their solutions, do not change if the maps involved are replaced with 'continuous deformations' of themselves. The equivalence relation—called *homotopy*—generated by continuous deformations of maps respects composition, so that there is a 'quotient' *homotopy category* H**Top** and a functor **Top** $\to$ H**Top**. Homotopy theory is the study of this functor. Thus homotopy theory is not entirely confined to the category H**Top**: it is frequently necessary, or at least useful, to use constructions available only in **Top** in order to prove statements that are entirely internal to H**Top**; and the homotopy category H**Top** can shed light even on questions in **Top** that are not homotopy invariant.

History. The core of the subject I'm calling 'classical homotopy theory' is a body of ideas and theorems that emerged in the 1950s and was later largely codified in the notion of a model category. This includes the notions of fibration and cofibration, CW complexes, long fiber and cofiber sequences, loop space, suspension, and so on. Brown's representability theorems show that homology and cohomology are also contained in classical homotopy theory.

One of the main complications in homotopy theory is that many, if not most, diagrams in the category H**Top** do not have limits or colimits. Thus many theorems were proved using occasionally ingenious and generally *ad hoc* constructions performed in the category **Top**. Eventually many of these constructions were codified in the dual concepts of homotopy colimit and homotopy limit, and a powerful calculus for working with them was developed. The language of homotopy limits and colimits and the techniques for manipulating them made it possible to easily state and conceptually prove many results that had previously seemed quite difficult and inscrutable.

Once the basic theory has been laid down, the most interesting and useful theorems are those that break the categorical barrier between domain and target. The basic example of such a theorem is the Blakers-Massey theorem, which compares homotopy pushout squares to homotopy pullback squares. Other excellent examples of duality-breaking theorems are the Hilton-Milnor theorem on the loop space of a wedge and Ganea's theorem (which is dual to the most important special case of the Blakers-Massey theorem). All of these results were first proved with a great deal of technical finesse but can now be established easily using homotopy pushouts and pullbacks.

The Aim of This Book. The aim of this book is to develop classical homotopy theory and some important developments that flow from it using the more modern techniques of homotopy limits and colimits. Thus homotopy pushouts and homotopy pullbacks play a central role.

The book has been written with the theory of model categories firmly in mind. As is probably already evident, we make consistent and unapologetic use of the language of categories, functors, limits and colimits. But we are genuinely interested in the homotopy theory of *spaces* so, with the exception of a brief account of the abstract theory of model categories, we work with spaces throughout and happily make use of results that are special for spaces. Indeed, the third part of the book is devoted to the development of four basic properties that set the category of spaces apart from generic model categories.

I have generally used topological or homotopy-theoretical arguments rather than algebraic ones. This almost always leads to simpler statements and simpler arguments. Thus my book attempts to upset the balance (observed in many algebraic topology texts) between algebra and topology, in favor of topology. Algebra is just one of many tools by which we understand topology. This is *not* an anti-algebra crusade. Rather, I set out hoping to find homotopy-theoretical arguments wherever possible, with the expectation that at certain points, the simplicity or clarity afforded by the standard algebraic approach would outweigh the philosophical cleanliness of avoiding it. But I ended up being surprised: at no point did I find that 'extra' algebra made any contribution to clarity or simplicity.

Omissions. This is a very long book, and many topics that were in my earliest plans have had to be (regretfully) left out. I had planned three chapters on stable homotopy, extraordinary cohomology and nilpotence and another on Goodwillie calculus. But in the book that emerged it seemed thematically appropriate to draw the line at stable homotopy theory, so space and thematic consistency drove these chapters to the cutting room floor.

Problems and Exercises. Many authors of textbooks assert that the only way to learn the subject is to do the exercises. I have taken this to heart, and so *there are no outright proofs in the book.* Instead, theorems are followed by multi-part problems that guide the readers to find the proofs for themselves. To the expert, these problems will read as terse proofs, perhaps suitable for exposition in a journal article. Reading this text, then, is a preparation for the experience of reading research articles. There are also a great many other problems incorporated into the main flow of the text, problems that develop interesting tangential results, explore applications, or carry out explicit calculations.

In addition, there are numerous exercises. These are intended to help the student develop some habits of mind that are extremely useful when reading mathematics. After definitions, the reader is asked to find examples and nonexamples, to explore how the new concept fits in with previous ideas, etc. Other exercises ask the reader to compare theorems with previous results, to test whether hypotheses are needed, or can be weakened, and so on.

Audience. This book was written with the idea that it would be used by students in their first year or two of graduate school. It is assumed that the reader is familiar with basic algebraic concepts such as groups and rings. It is also assumed that the student has had an introductory course in topology. It would be nice if that course included some mention of the fundamental group, but that is not necessary.

Teaching from This Book. This book covers more topics, in greater depth, than can be covered in detail in a typical two-semester homotopy theory or algebraic topology sequence. That being said, a good goal for a two-semester course would be to cover the high points of Parts 1 – 4 in the first semester and Parts 5 – 6 in the second semester, followed by some or all of Part 7 if time permits.

Here's some more detail.

The first semester would start with a brief (one day) introduction to the language of category theory before heading on to Part 2 to develop the basic theory of cofibrations, fibrations, and homotopy limits and colimits. Part 1 is an overview of the basics of category theory and shouldn't be covered in its own right at all; refer back to it as needed to bring in more advanced category-theoretical topics. Chapters 3 and 4, in which the category of spaces is established and the concept of homotopy is developed should be covered fairly thoroughly. Chapter 5 is on cofibrations and fibrations. The basic properties should be explored, and the mapping cylinder and its dual should be studied carefully; it's probably best to gloss over the distinction between the pointed and unpointed cases. State the Fundamental Lifting Property and the basic factorization theorems without belaboring their proofs. The fact that fiber and cofiber sequences lead to exact sequences of homotopy sets should be explored in detail. Chapter 6 is on homotopy colimits and limits. Cover homotopy pushouts in detail, appealing to duality for homotopy pullbacks, and give a brief discussion of the issues for more general diagrams. Chapter 7 is on homotopy pullback and pushout squares and should be covered in some detail. Chapters 8 and 9 offer a huge collection of topics. For the moment, only Section 8.1 (Long cofiber and fiber sequences) and perhaps Section 9.2 (on H-Spaces and co-H-spaces) are really mandatory. Other sections can be covered as needed or assigned to students as homework. Chapter 10 is a brief account of abstract model categories. It is included for 'cultural completeness' and, since it does not enter into the main flow of the text, it can be skipped in its entirety. Part 3 covers the four major special features of the homotopy theory of spaces. Chapters 11 through 14 should be covered in detail. Chapter 15 is a combination of topics and cultural knowledge. Sections 15.1 and 15.2 are crucial, but the rest can be glossed over if need be. Part 4 is where the four basic topological inputs are developed into effective tools for studying homotopy-theoretical problems. Chapters 16 through 19 should all be covered in detail. Chapter 20 contains topics which can be assigned to students as homework.

The second semester should pick up with Part 5 where we develop cohomology (and homology). Chapters 21 through 24 should be covered pretty thoroughly. Chapter 25 is a vast collection of topics, which can be covered at the instructor's discretion or assigned as homework. Part 6 is about the cohomology of fiber sequences, leading ultimately to the Leray-Serre spectral sequence, which is notoriously forbidding when first encountered. The exposition here is broken into small pieces with a consistent emphasis on the topological content. Many of the basic ideas and a nice application are covered in Chapters 26 through 29; this would be a fine place to stop if time runs out. Otherwise, Chapters 30 and 31 get to the full power of the Leray-Serre spectral sequence. This power is used in Chapter 32 to prove the Bott Periodicity Theorem. Chapter 33 is another topics chapter, which includes the cohomology of Eilenberg-Mac Lane spaces and some computations involving the homotopy groups of spheres. Finally, Part 7 covers some very fun and interesting topics: localization and completion, a discussion of the exponents of homotopy groups of spheres including a proof of Selick's theorem on the exponent of $\pi_*(S^3)$; the theory of closed classes and a dual concept known as strong resolving classes; and a proof of Miller's theorem on the space of maps from $B\mathbb{Z}/p$ to a simply-connected finite complex.

Acknowledgements. A book such as this, I have come to realize, is essentially an attempt to set down the author's point of view on his subject. My point of view has been shaped by many people, beginning with Ed Fadell, Sufian Husseini and Steve Hutt, who were my first teachers in the subject. Early in my career, my horizons were greatly expanded by conversation and collaboration with Bob Bruner and Chuck McGibbon, and even more so during my long, pleasant and fruitful collaboration with Martin Arkowitz.

At various points during the writing of this book I have turned to others for clarification or advice on certain points that escaped me. Thanks are due to Peter May, whose kind responses to my emailed questions greatly improved a chapter that is, unfortunately, no longer included in the book. The community at the website MathOverflow offered useful advice on many questions.

My thanks are also due to the students who were guinea pigs for early versions of this text. Specifically, the enthusiasm of David Arnold, Jim Clarkson, Julie Houck, Rob Nendorf, Nick Scoville, and Jason Trowbridge was inspirational. I must also thank John Martino and Jay Wood for teaching the algebraic topology sequence at Western Michigan University using early drafts of this text.

Finally, I must gratefully acknowledge the support of my family during the long writing process. Dolores was exceedingly—albeit decreasingly—patient with my nearly endless string of pronouncements that I was 'almost done', and my sanity was preserved by my son Brandon, who unknowingly and innocently forced me every day to stop working and *have fun*.

Part 1

The Language of Categories

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Chapter 1

Categories and Functors

The subject of algebraic topology is historically one of the first in which huge diagrams of functions became a standard feature. The language of category theory is intended to provide tools for understanding such diagrams, for working with them, and for studying the relations between them. In this chapter we begin to develop and make use of this powerful language.

1.1. Diagrams

Before getting to categories, let's engage in an informal discussion of diagrams. Roughly speaking, a **diagram** is a collection (possibly infinite) of 'objects' denoted A, B, X, Y, etc., and (labelled) 'arrows' between the objects, as in the examples

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

and

$$\begin{tikzcd} X_1 \arrow[rr, "f_{12}"] \arrow[dr, "f_{13}"] \arrow[dd, "h_1"'] & & X_2 \arrow[dr, "f_{24}"] \arrow[dd, "h_2"] & \\ & X_3 \arrow[rr, "f_{34}"] \arrow[dd, "h_3"] & & X_4 \arrow[dd, "h_4"] \\ Y_1 \arrow[rr, "g_{12}"] \arrow[dr, "g_{13}"'] & & Y_2 \arrow[dr, "g_{24}"] & \\ & Y_3 \arrow[rr, "g_{34}"] & & Y_4. \end{tikzcd}$$

Each arrow has a **domain** and a **target**—thus $X \xrightarrow{f} Y$ is a simple diagram with a single arrow f whose domain is X and whose target is Y. If g is another arrow with domain Y, then we can form the 'composite arrow' $g \circ f$ with domain X and target Y. The triangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

is **commutative** if $h = g \circ f$. If the diagram above is commutative, then we say that h **factors through** f and through g; we also say that h factors through Y. If X and Y are objects in a diagram, we may be able to use the various arrows and their composites to obtain many potentially distinct arrows from X to Y; for example, in the cube diagram, there are precisely 6 different composites from X_1 to Y_4. Each of these paths represents an arrow $X_1 \to Y_4$, but it can happen that different paths become, on composition, the same arrow. If it turns out that, for each pair X, Y of objects in the diagram, all of the possible composite paths from X to Y are ultimately the *same* arrow, then we say that the diagram is **commutative**.

We can expand a given (not necessarily commutative) diagram $\mathcal{D}$ by drawing as arrows all of the composites of the given arrows, as well as 'identity arrows' from each 'vertex object' to itself, which compose like identity maps. We'll refer to the expanded diagram as $\overline{\mathcal{D}}$.

Exercise 1.1. Show that $\mathcal{D}$ is commutative if and only if $\overline{\mathcal{D}}$ is commutative.

It is frequently helpful to express complicated definitions and properties in terms of diagrams.

Here's an example. Let F be a field, and let $F \subseteq E$ be a field extension. Then there is an **inclusion map** $f : F \to E$, which just carries an element $\alpha \in F$ to the same element, but thought of as being an element of E. Now an **algebraic closure** for F is an algebraic field extension $a : F \to A$ such that for any other algebraic extension f, there is a unique map $\bar{f}$ making the diagram

$$\begin{tikzcd} F \arrow[r, "a"] \arrow[d, "f"'] & A \\ E \arrow[ur, dotted, "\exists!\ \bar{f}"'] & \end{tikzcd}$$

commutative. Here you should observe that we use dotted arrows to denote arrows that we do not know exist. Also, this definition gives the algebraic closure as a solution to a 'universal problem'.

Exercise 1.2.

(a) Take some time to convince yourself that the given definition of algebraic closure actually does define what you think of as algebraic closure.

(b) The isomorphism theorems of elementary group theory can be written down in terms of diagrams. Do it!

(c) Rewrite the statement that the cube-shaped diagram above is commutative without using any diagrams at all.[1]

1.2. Categories

Informally, a category is simply a 'complete' list of all the *things* you plan to study together with a complete list of all the *allowable maps* between those things. So an algebraist might work in the category of groups and homomorphisms, while a topologist might work in the category of topological spaces and continuous functions, and a geometer could work in the category of subsets of the plane and rigid motions.

Formally, a **category** $\mathcal{C}$ consists of two things: a collection[2] $\mathrm{ob}(\mathcal{C})$, called the **objects** of $\mathcal{C}$, and, for each $X, Y \in \mathrm{ob}(\mathcal{C})$, a *set* $\mathrm{mor}_{\mathcal{C}}(X, Y)$, called the set of **morphisms** from X to Y. These are subject to the following conditions:

(1) If $X, Y, Z \in \mathrm{ob}(\mathcal{C})$, $f \in \mathrm{mor}_{\mathcal{C}}(X, Y)$, $g \in \mathrm{mor}_{\mathcal{C}}(Y, Z)$, then there is another morphism $g \circ f \in \mathrm{mor}_{\mathcal{C}}(X, Z)$ (which you should think of as the composite of f and g).[3]

(2) The composition operation is associative: $f \circ (g \circ h) = (f \circ g) \circ h$. Diagrammatically, this says that the diagram

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[dr, bend right, "g\circ h"'] & X \arrow[d, "g"] \arrow[dr, bend left, "f\circ g"] & \\ & Y \arrow[r, "f"'] & Z \end{tikzcd}$$

is commutative.

(3) For each $X \in \mathrm{ob}(\mathcal{C})$, there is a special morphism $\mathrm{id}_X \in \mathrm{mor}_{\mathcal{C}}(X, X)$ which satisfies $\mathrm{id}_X \circ f = f$ for any $f \in \mathrm{mor}_{\mathcal{C}}(W, X)$ and $g \circ \mathrm{id}_X = g$

[1]Thanks to Jason Trowbridge for this idea.

[2]The vague word 'collection' is intended to gloss over some technical set-theoretical issues. The idea is that the collection of objects should be allowed to be larger than any set, so we can't call it a set of objects. Many authors use a **class** of objects, which is a well-defined concept in set theory (or logic); but one of the go-to books on category theory (Mac Lane [**110**]) uses a set-theoretical trick to get around classes.

[3]This rule makes it meaningful to ask whether a given diagram of objects and arrows in $\mathcal{C}$ is commutative or not.

for any $g \in \mathrm{mor}_{\mathcal{C}}(X, Y)$. In diagram form:

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[dr, bend right, "f"'] & X \arrow[d, "\mathrm{id}_X"] \arrow[dr, bend left, "g"] & \\ & X \arrow[r, "g"'] & Y. \end{tikzcd}$$

Here are some simple examples to think about.

(a) The category $\mathcal{G}$ whose objects are groups and whose morphisms are group homomorphisms; also the subcategory $\text{AB}\,\mathcal{G}$ whose objects are the abelian groups.

(b) The category **Top** whose objects are topological spaces and whose morphisms are continuous functions.

(c) The category whose objects are the numbers $1, 2, 3, \ldots$ and such that there is a unique morphism $n \to m$ if n divides m and no morphisms $n \to m$ if n does not divide m.

(d) The category whose objects are the real numbers and such that there is a unique morphism $x \to y$ if $x \leq y$ and no morphism if $x > y$.

There is a lot of shorthand that is often used when confusion is unlikely. For example, we usually write $X \in \mathcal{C}$ instead of $X \in \mathrm{ob}(\mathcal{C})$; and rather than $f \in \mathrm{mor}_{\mathcal{C}}(X, Y)$, we write $f : X \to Y$.

Exercise 1.3.

(a) Give five examples of categories besides the ones already mentioned.

(b) Find a way to interpret a group G as a category with a single object.

(c) Let X be a topological space. Show how to make a category whose objects are the points of X and such that the set of morphisms from a to b is the set of all paths $\omega : [0, d] \to X$ (where $d \geq 0$) such that $\omega(0) = a$ and $\omega(d) = b$.

(d) Suppose $\mathcal{D}$ is a diagram in the sense of Section 1.1. Show that $\overline{\mathcal{D}}$ is a category.

Lots of basic mathematical ideas are 'best' expressed in the language of categories. For example: a morphism $f : X \to Y$ is an **equivalence** if there is a morphism $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. It is the usual practice to write $g = f^{-1}$ in this case.

Exercise 1.4. Show that if such a g exists, it is unique.

Problem 1.5. Let's say $X \sim Y$ if there is an equivalence $f : X \to Y$.

(a) Show that $\sim$ is an equivalence relation.

(b) Interpret ‘equivalence’ in each of the categories that have been discussed in the text so far, including the ones you found in Exercise 1.3.

Problem 1.6. Let $\mathcal{C}$ be a category and let $f : X \to Y$ be a map which has a left inverse $g : Y \to X$, and suppose g also has a left inverse. Show that f and g are two-sided inverses of each other.

Another simple—but extremely useful—idea is that of a retract. If $A, X \in \mathcal{C}$, then A is a **retract** of X if there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"'] \arrow[rr, bend left, "\mathrm{id}_A"] & X \arrow[r, "r"'] & A. \end{tikzcd}$$

If $f : A \to B$ and $g : X \to Y$, then f is a retract of g if there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}_A"] & X \arrow[r, "r"] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "j"] \arrow[rr, bend right, "\mathrm{id}_B"'] & Y \arrow[r, "s"] & B. \end{tikzcd}$$

Exercise 1.7. Whenever we use the term ‘retract’, we should be referring to the definition above, where an object A was a retract of another object X in some category. By setting up an appropriate category, show that our definition of f being a retract of g can be thought of as an instance of that general categorical definition. What is does it mean, in terms of the category $\mathcal{C}$, for two objects to be equivalent in your new category?

Hint. Obviously, f and g must be among the objects in your category!

Exercise 1.8. Find examples of retracts in algebra, topology, and other contexts.

Problem 1.9. Let $f : A \to B$ and $g : X \to Y$ be morphisms in a category $\mathcal{C}$. Assume that f is a retract of g.

(a) Show that if g is an equivalence, then f is also an equivalence.

(b) Show by example that f can be an equivalence even if g is not an equivalence.

1.3. Functors

As you have no doubt experienced, it seldom happens that any serious mathematical study is performed entirely inside a single category. For example, when Galois set out to study fields, he was forced to also work in the category of groups; it was the relationship between these two categories that led to new insights. Another algebraic example is given by group actions, in which the category of groups is studied using the category of sets; playing these two categories off of one another is how the Sylow theorems are generally proved.

A **functor** is a formalism for comparing categories; you can think of a functor intuitively as a morphism from one category to another one. Just as a homomorphism respects the algebraic structure of a group, and a continuous map respects the topological structure of a space, a functor must respect the key features of categories. Functors take objects to objects and morphisms to morphisms; they respect composition and preserve identity morphisms.

There are actually two kinds of functors—those which reverse the direction of morphisms and those which don't. A **covariant functor** $F : \mathcal{C} \to \mathcal{D}$ consists of a function

$$F : \mathrm{ob}(\mathcal{C}) \longrightarrow \mathrm{ob}(\mathcal{D})$$

and, for each $X, Y \in \mathrm{ob}(\mathcal{C})$, a function

$$F : \mathrm{mor}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{mor}_{\mathcal{D}}(F(X), F(Y)).$$

These must satisfy the following conditions:

(1) $F(g \circ f) = F(g) \circ F(f)$.

(2) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for any $X \in \mathrm{ob}(\mathcal{C})$.

Notice that if $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor and $f : X \to Y$ is a morphism in $\mathcal{C}$, then $F(f) : F(X) \to F(Y)$; thus, F carries the domain of f to the domain of $F(f)$, and similarly for the targets. In other words, $F(f)$ points in 'the same direction' as f. This is the meaning of the word 'covariant'.

Exercise 1.10.

(a) Let $\mathcal{D}$ be a diagram in a category $\mathcal{A}$, and let $\overline{\mathcal{D}}$ be the category obtained from it as in Exercise 1.3(d). Show that $\mathcal{D}$ is commutative if and only if for any two objects $X, Y \in \mathcal{D}$, $\mathrm{mor}_{\overline{\mathcal{D}}}(X, Y)$ has at most one element.

(b) Show that if you apply a covariant functor $F : \mathcal{A} \to \mathcal{B}$ to a commutative diagram in $\mathcal{A}$, the result is a commutative diagram in $\mathcal{B}$.

Problem 1.11. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Show that if $f : X \to Y$ is an equivalence in $\mathcal{C}$, then $F(f)$ is an equivalence in $\mathcal{D}$. Is it possible for $F(f)$ to be an equivalence without f being an equivalence?

Exercise 1.12. Let G be a group, and think of it as a category with one object, as in Exercise 1.3. Interpret functors $F : G \to \mathbf{Sets}$ in terms of familiar concepts in algebra.

Let's use these ideas to prove something topological.[4]

Problem 1.13. Let $i : S^1 \hookrightarrow D^2$ be the inclusion of the circle into the disk. An early success of algebraic topology concerned the existence of a continuous function $r : D^2 \to S^1$ such that the composite $r \circ i : S^1 \to S^1$ is the identity (in other words: is S^1 a retract of D^2?). The fundamental group is a covariant functor

$$\pi_1 : \mathbf{Top} \longrightarrow \mathcal{G}$$

from the category **Top** of topological spaces and continuous functions to the category $\mathcal{G}$ of groups and homomorphisms. You may have seen, and we will show later, that $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(D^2) = 0$. Using this functor, show that there can be no such function r.

The second kind of functor is just the same, except that it reverses the direction of arrows. A **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$ consists of a function

$$F : \mathrm{ob}(\mathcal{C}) \longrightarrow \mathrm{ob}(\mathcal{D})$$

and, for each $X, Y \in \mathrm{ob}(\mathcal{C})$, a function

$$F : \mathrm{mor}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{mor}_{\mathcal{D}}(F(Y), F(X)).$$

These must satisfy the following conditions:

(1) $F(g \circ f) = F(f) \circ F(g)$.
(2) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for any $X \in \mathrm{ob}(\mathcal{C})$.

In contrast to the covariant functors, if $F : \mathcal{C} \to \mathcal{D}$ is a contravariant functor and $f : X \to Y$ in $\mathcal{C}$, then $F(f) : F(Y) \to F(X)$. Thus a contravariant functor carries the domain of f to the target of $F(f)$ and carries the target of f to the domain of $F(f)$—it 'reverses the direction' of arrows.

Exercise 1.14. Let $F : \mathcal{C} \to \mathcal{D}$ be a contravariant functor.

(a) Show that if you apply F to a commutative diagram in $\mathcal{C}$, the result is a commutative diagram in $\mathcal{D}$.
(b) Show that if $f : X \to Y$ is an equivalence in $\mathcal{C}$, then $F(f)$ is an equivalence in $\mathcal{D}$.

Exercise 1.15. Show that the composite of two functors is a functor. What happens if one or both of the functors is contravariant?

Exercise 1.16. Can there be a functor $F : \mathcal{C} \to \mathcal{D}$ that is *both* covariant and contravariant? What special properties must such a functor have?

[4] I.e., interesting!

Exercise 1.17. Show that there is a universal example for contravariant functors out of a category $\mathcal{C}$. That is, show that there is a category $\mathcal{C}^{\mathrm{op}}$ and a contravariant functor $C \to \mathcal{C}^{\mathrm{op}}$ so that every other contravariant functor $\mathcal{C} \to \mathcal{D}$ has a unique factorization $\mathcal{C} \to \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, where the functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is covariant.

The category $\mathcal{C}^{\mathrm{op}}$ is known as the **opposite category** of $\mathcal{C}$. Some authors choose not to use contravariant functors at all and instead use covariant functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Let's look at some simple functors.

Exercise 1.18. Consider the categories $\mathrm{AB}\,\mathcal{G}$ of abelian groups (and homomorphisms) and **Sets** of sets (and functions). Since an abelian group is a set together with extra structure, we can define $F : \mathrm{AB}\,\mathcal{G} \to \textbf{Sets}$ by

$$G \longmapsto \boxed{G\text{, but completely forgetting the group structure}}.$$

Complete the definition of F on morphisms, and show that F is a functor.

Any functor of this kind, in which the target category is a dumbed-down version of the domain, and the functor consists of just getting dumber, is called a **forgetful functor**.

Exercise 1.19. Try to make a formal definition of 'forgetful functor'.

Exercise 1.20. Let $\mathcal{V}$ denote the category of all vector spaces (over the real numbers, say) and all linear transformations. Thus

$$\mathrm{mor}_{\mathcal{V}}(V, W) = \mathrm{Hom}_{\mathbb{R}}(V, W) = \{T : V \to W \mid T \text{ is } \mathbb{R}\text{-linear}\}.$$

(a) Define $F : \mathcal{V} \to \mathcal{V}$ by the rules

$$F(V) = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}, V) \qquad \text{and} \qquad F(f) : g \mapsto f \circ g.$$

Show that F is a covariant functor.

(b) Define $G : \mathcal{V} \to \mathcal{V}$ by the rules

$$G(V) = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \qquad \text{and} \qquad G(f) : g \mapsto g \circ f.$$

Show that G is a contravariant functor.

The functors described in Exercise 1.20 are specific examples of what are, for us, the two most important general kinds of functors.

Proposition 1.21. *Let $\mathcal{C}$ be a category, and let $A, B \in \mathrm{ob}(\mathcal{C})$.*

(a) For $f : X \to Y$, write $f^* : \mathrm{mor}_{\mathcal{C}}(Y, B) \to \mathrm{mor}_{\mathcal{C}}(X, B)$ for the function $f^* : g \mapsto g \circ f$. Then the rules

$$F(X) = \mathrm{mor}_{\mathcal{C}}(X, B) \qquad \textit{and} \qquad F(f) = f^* : F(Y) \to F(X)$$

define a contravariant functor from $\mathcal{C}$ to **Sets**.

(b) *For* $f : X \to Y$, *write* $f_* : \mathrm{mor}_{\mathcal{C}}(A,X) \to \mathrm{mor}_{\mathcal{C}}(A,X)$ *for the function* $f_* : g \mapsto f \circ g$. *Then the rules*

$$G(X) = \mathrm{mor}_{\mathcal{C}}(A,X) \quad \textit{and} \quad G(f) = f_* : G(X) \to G(Y)$$

define a covariant functor from $\mathcal{C}$ *to* **Sets**.

A functor that is constructed in either of these ways is called a **represented functor**—that is, the functor F is represented by the object B, and the functor G is represented by the object A.

Problem 1.22. Prove Proposition 1.21.

HINT. Simply generalize your work from Exercise 1.20.

Problem 1.23. Let $f : A \to B$ and $g : X \to Y$ in a category $\mathcal{C}$. Show that

$$f^* \circ g_* = g_* \circ f^* : \mathrm{mor}_{\mathcal{C}}(B,X) \longrightarrow \mathrm{mor}_{\mathcal{C}}(A,Y).$$

Problem 1.24. Let $f : A \to B$ be a morphism in the category $\mathcal{C}$.

(a) Suppose the induced map $f_* : \mathrm{mor}_{\mathcal{C}}(X,A) \to \mathrm{mor}_{\mathcal{C}}(X,B)$ is a bijection for every X. Show that f is an equivalence.

(b) Suppose the induced map $f^* : \mathrm{mor}_{\mathcal{C}}(B,X) \to \mathrm{mor}_{\mathcal{C}}(A,X)$ is a bijection for every X. Show that f is an equivalence.

HINT. Try plugging in $X = A$ and $X = B$.

1.4. Natural Transformations

Category theory was invented by Saunders Mac Lane and Samuel Eilenberg in the early 1940s, largely motivated by the desire to be precise about what is meant by (or *should* be meant by) a 'natural construction'. For many years before then, mathematicians had used the *intuitive* notion of a natural construction to mean that the construction is done in exactly the same way for all spaces, groups, or whatever. For example, for any vector space V, you can construct the dual vector space $V^* = \mathrm{Hom}(V,\mathbb{R})$; since this is done in the same way for every vector space, it is 'naturally defined' and so it will 'of course' (ha!) convert commutative diagrams to other commutative diagrams. This idea is formalized in the idea of a functor. Now how do we relate two different 'natural' constructions to one another?

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be two covariant[5] functors. A **natural transformation** $\Phi : F \to G$ is a rule that associates to each $X \in \mathrm{ob}(\mathcal{C})$ a morphism

$$\Phi_X : F(X) \longrightarrow G(X)$$

[5]They could also both be contravariant. I'll leave the formulation to you.

with the property that for any morphism $f : X \to Y$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\Phi_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\Phi_Y"] & G(Y) \end{tikzcd}$$

is commutative. The transformation Φ is called a **natural isomorphism** if for each $X \in \mathcal{C}$, the morphism $\phi_X : F(X) \to G(X)$ is an isomorphism in $\mathcal{D}$.

It is easy to find examples of natural transformations between represented functors.

Problem 1.25. Let $\phi : A \to B$ be a morphism in $\mathcal{C}$.

(a) Define two functors $\mathcal{C} \to \mathbf{Sets}$ by the rules $F(X) = \mathrm{mor}_{\mathcal{C}}(X, A)$ and $G(X) = \mathrm{mor}_{\mathcal{C}}(X, B)$. Show that $\Phi_X = \phi_* : F \to G$ is a natural transformation.

(b) Define two functors $\mathcal{C} \to \mathbf{Sets}$ by the rules $H(X) = \mathrm{mor}_{\mathcal{C}}(A, X)$ and $I(X) = \mathrm{mor}_{\mathcal{C}}(B, X)$. Show that $\Phi_X = \phi^* : I \to H$ is a natural transformation.

Exercise 1.26. This problem refers to the functors $F(V) = \mathrm{Hom}(\mathbb{R}, V)$ and $G(V) = \mathrm{Hom}(V, \mathbb{R})$ of Exercise 1.20.

(a) Show that for every vector space V, $V \cong F(V)$. Define a natural isomorphism $\Phi : F \to \mathrm{id}$.

(b) Show that for every finite-dimensional vector space V, $V \cong G(V)$. Show that there is no **natural** isomorphism $\Theta : G \to \mathrm{id}$, even if you restrict your attention just to finite-dimensional vector spaces.

In fact, the converse of Problem 1.25 is true—this is known as the *Yoneda lemma.*

Proposition 1.27. *Let $A, B \in \mathcal{C}$.*

*(a) Define functors $H, I : \mathcal{C} \to$ **Sets** by the rules $H(X) = \mathrm{mor}_{\mathcal{C}}(A, X)$ and $I(X) = \mathrm{mor}_{\mathcal{C}}(B, X)$. Then there is a bijection*

$$\{\text{natural transformations } I \to H\} \quad \longleftrightarrow \quad \mathrm{mor}_{\mathcal{C}}(A, B).$$

*(b) Define functors $F, G : \mathcal{C} \to$ **Sets** by the rules $F(X) = \mathrm{mor}_{\mathcal{C}}(X, A)$ and $G(X) = \mathrm{mor}_{\mathcal{C}}(X, B)$. Then there is a bijection*

$$\{\text{natural transformations } F \to G\} \quad \longleftrightarrow \quad \mathrm{mor}_{\mathcal{C}}(A, B).$$

Your next problem is to prove the Yoneda lemma.

Problem 1.28. Let $F, G, H, I : \mathcal{C} \to \mathbf{Sets}$ be the functors defined in Problem 1.25.

(a) If $\Phi : F \to G$ is a natural transformation, show that there is a unique map $\phi : A \to B$ such that $\Phi_X = \phi_*$ for every $X \in \mathcal{C}$.

(b) If $\Phi : I \to H$ is a natural transformation, show that there is a unique map $\phi : A \to B$ such that $\Phi_X = \phi^*$ for every $X \in \mathcal{C}$.

Notice that your proof of (b) is formally very similar to your proof of (a). Can you be precise about how the two proofs are related?

Hint. In both cases, the domain of ϕ is A, and the target is B.

Problem 1.29. Let $A, B \in \mathcal{C}$, and use them to define functors

$$F(X) = \mathrm{mor}_{\mathcal{C}}(X, A) \qquad \text{and} \qquad G(X) = \mathrm{mor}_{\mathcal{C}}(X, B).$$

(a) Suppose there is a natural isomorphism $\Phi : F \to G$. Show that $A \cong B$.

(b) Show that (a) is false without the word 'natural'—that is, make up an example where $F(X) \cong G(X)$ for all X, but where, nonetheless, $A \ncong B$.

Hint. Your category must have at least two objects; can it have exactly two objects?

(c) Prove that A and B are isomorphic if the functors $\mathrm{mor}_{\mathcal{C}}(A, ?\,)$ and $\mathrm{mor}_{\mathcal{C}}(B, ?\,)$ are naturally equivalent.

Natural Transformations in Dumber Categories. Before ending this section, we mention a wrinkle in the definition of a natural transformation. The intuitive idea of a natural transformation is that it is some construction which is done 'in the same way for all objects'. With this definition, consider the category **Rings** of all rings and their homomorphisms. For each ring R, we can define

$$\phi_R : R \longrightarrow R \qquad \text{by the rule} \qquad x \mapsto x^2.$$

This rule clearly fits into the *intuitive* idea of a natural transformation $\Phi : \mathrm{id} \to \mathrm{id}$, where $\mathrm{id} : \mathbf{Rings} \to \mathbf{Rings}$ is the identity functor. But it is not a natural transformation, *because ϕ_R is not a ring homomorphism.*

To make ϕ a natural transformation, we need to move to a category in which the maps are not required to be ring homomorphisms. One solution is to let Rings_0 be the category whose objects are rings and whose morphisms are maps of sets. Then there is a forgetful functor $F : \mathbf{Rings} \to \mathbf{Rings}_0$, and ϕ is a natural transformation from F to itself.

Thus we will sometimes find it useful to allow our natural transformations $\Phi : F \to G$ (where $F, G : \mathcal{C} \to \mathcal{D}$) to give maps $\phi_X : F(X) \to G(X)$ that are not maps in $\mathcal{D}$ but maps in some larger category that contains $\mathcal{D}$.

Equivalence of Categories. The obvious notion of equivalence of categories, where we ask for functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ whose composites are the identity functors, has proven to be more rigid than is necessary, and too rigid for many applications. Instead, a functor $F : \mathcal{C} \to \mathcal{D}$ is an **equivalence of categories** if there is a functor $G : \mathcal{D} \to \mathcal{C}$ and two natural isomorphisms

$$\mathrm{id}_{\mathcal{C}} \longrightarrow G \circ F \qquad \text{and} \qquad F \circ G \longrightarrow \mathrm{id}_{\mathcal{D}}.$$

Exercise 1.30. Show that an equivalence of categories need be neither injective nor surjective on objects.

1.5. Duality

In studying categories, you should keep your eyes open for instances of duality. The dual of a category-theoretical expression is the result of reversing all the arrows, changing each reference to a domain to refer to the target (and vice versa), and reversing the order of composition.

Exercise 1.31.

(a) The notation $f : X \to Y$ is shorthand for the sentence: 'f is a morphism with domain X and target Y.' What is the dual of this statement?

(b) Find instances of duality in the previous sections.

For example, consider **lifting problem**: you are given maps $f : A \to Y$ and $p : X \to Y$, and you would like to find a map $\lambda : A \to X$ such that $p \circ \lambda = f$. This problem is neatly expressed in the diagram

$$\begin{tikzcd} & X \arrow[d, "p"] \\ A \arrow[ur, dotted, bend left, "\lambda"] \arrow[r, "f"] & Y \end{tikzcd}$$

(when expressing problems in this way, the map you hope to find is usually dotted or dashed). The dual problem is expressed by the diagram

$$\begin{tikzcd} & V \arrow[dl, dotted, bend right, "\epsilon"'] \\ B & U. \arrow[l, "g"] \arrow[u, "q"'] \end{tikzcd}$$

This is known as an **extension problem**, because you hope to extend the map g to the 'larger' thing V.[6]

[6]Norman Steenrod, one of the architects of modern algebraic topology, used the extension and lifting problems to frame the entire subject. It can be argued that a great deal of mathematics is about lifting and extension problems. EXERCISE. *Show how the problem: 'decide whether $f : X \to Y$ is a homeomorphism' can be written in terms of extension and/or lifting problems.*

Problem 1.32. Verify that the dual of each rule for a category is also a rule for a category, and likewise for functors and natural transformations.

Because of Problem 1.32, the dual of a valid proof involving categories, functors and natural transformations is also a valid proof. Thus, the dual of every theorem of pure category theory is automatically also a theorem.

Domain- and Target-Type Objects. It often happens that an object of a category is defined in terms of certain category-theoretical properties. These properties usually give special information about the maps *out of* the object or else they give special information about the morphisms *into* that object. In the first case, we call the construction a construction of **domain type**; in the second case it is a construction of **target type**. We will sometimes refer to the results of these constructions as being objects of 'domain-type' or of 'target-type'. The distinction between 'domain-type' and 'target-type' objects or constructions is important to observe. The dual of a domain-type construction is a target-type construction and vice versa.

1.6. Products and Sums

Let $X, Y \in \mathcal{C}$. The **product** of X and Y is an object P *together with* two morphisms $\mathrm{pr}_X : P \to X$ and $\mathrm{pr}_Y : P \to Y$ (called **projections**) with the following **universal property**: if $f : Z \to X$ and $g : Z \to Y$ are any two morphisms, then there is a *unique* morphism $t : Z \to P$ so that $\mathrm{pr}_X \circ t = f$ and $\mathrm{pr}_Y \circ t = g$. This can be expressed diagrammatically as follows:

$$\begin{tikzcd}
& Z \arrow[dl, bend right, "f"'] \arrow[dr, bend left, "g"] \arrow[d, dotted, "\exists!"', "t"] & \\
X & P \arrow[l, "\mathrm{pr}_X"] \arrow[r, "\mathrm{pr}_Y"'] & Y.
\end{tikzcd}$$

Since the definition of the product P provides us with a way to understand the maps *into* P (i.e., it is the target of the hypothetical arrow), products are a target-type construction. *There is no guarantee that two given objects in a category $\mathcal{C}$ actually have a product or that there will only be one product.*

Problem 1.33. Suppose $X, Y \in \mathcal{C}$ and the objects P and Q are both products for X and Y. Show that $P \cong Q$.

Since any two products are equivalent, we often just choose one of them and denote it by $X \times Y$.

Exercise 1.34. Let $X, Y \in \mathcal{C}$, and suppose $X \times Y$ exists. Explicitly define a bijection

$$\mathrm{mor}_{\mathcal{C}}(Z, X \times Y) \xrightarrow{\cong} \mathrm{mor}_{\mathcal{C}}(Z, X) \times \mathrm{mor}_{\mathcal{C}}(Z, Y).$$

Because of this, we will generally write maps $F : Z \to X \times Y$ in the form (f, g), where $f = \mathrm{pr}_X \circ F$ and $g = \mathrm{pr}_Y \circ F$. One particularly important map is the **diagonal map**

$$\Delta : X \longrightarrow X \times X \qquad \text{defined by} \qquad \Delta = (\mathrm{id}_X, \mathrm{id}_X).$$

Exercise 1.35.

(a) Show that if one of the products $X \times (Y \times Z)$ and $(X \times Y) \times Z$ exists in $\mathcal{C}$, then so does the other, and they are isomorphic.

(b) Let $f : A \to X$ and $g : B \to Y$, and suppose that the products $A \times B$ and $X \times Y$ can be formed in $\mathcal{C}$. Give an explicit definition for the product map

$$f \times g : A \times B \longrightarrow X \times Y.$$

Suppose that $\mathcal{C}$ is a category with the property that *every* pair of objects $X, Y \in \mathcal{C}$ has a product. Then by choosing one product $X \times Y$ for each pair, we see that Exercise 1.35(b) implies that the rules $(X, Y) \mapsto X \times Y$ and $(f, g) \mapsto f \times g$ define a functor of two variables

$$? \times ? : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}.$$

Exercise 1.36. Formulate precise definitions of the product $\mathcal{C} \times \mathcal{D}$ of two categories and of a functor of two variables.

Some authors use the notation $f \times g$ to denote the map $Z \to X \times Y$ with components f and g. But this is wrong: $f \times g$ is the image of the ordered pair (f, g) under the functor $? \times ?$.

Problem 1.37. Let X and Y be objects in a category $\mathcal{C}$.

(a) Show that, for any map $f : X \to Y$, the diagram

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "f"'] & X \times X \arrow[d, "f \times f"] \\ Y \arrow[r, "\Delta"] & Y \times Y \end{tikzcd}$$

commutes.

(b) Show that the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "\Delta"] \arrow[dr, bend right, "\mathrm{id}"'] & (X \times Y) \times (X \times Y) \arrow[d, "(\mathrm{pr}_1, \mathrm{pr}_2)"] \\ & X \times Y \end{tikzcd}$$

is commutative.

It is tempting to prove these by chasing elements around, which is fine if your objects are sets. But you should prove that these diagrams commute only using the category-theoretical definitions of the maps involved.

Exercise 1.38. Explain how to view the diagonal map as a natural transformation.

Let's look at some specific examples of products.

Exercise 1.39.

(a) Show that the product of two sets X and Y is simply the ordinary cartesian product

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

(b) What is the product of two abelian groups G and H?

Let's look at the dual concept.

Problem 1.40.

(a) Formulate the (dual) definition of a **coproduct**, which is denoted $X \sqcup Y$. Coproducts are also known as (categorical) **sums**.

(b) Prove that if $X, Y \in \mathrm{ob}(\mathcal{C})$, then

$$\mathrm{mor}_{\mathcal{C}}(X \sqcup Y, B) \cong \mathrm{mor}_{\mathcal{C}}(X, B) \times \mathrm{mor}_{\mathcal{C}}(Y, B).$$

Write down the isomorphism explicitly. Thus we can (and will) describe maps $F : X \sqcup Y \to B$ with the notation (f, g), where $f : X \to B$ and $g : Y \to B$.

Hint. This is formally dual to Exercise 1.34, so you should be able to prove this by simply inverting all the arrows in your previous proof.

(c) Write down the definition of $f \sqcup g : A \sqcup B \to X \sqcup Y$.

(d) Explain how to view $\sqcup$ as a functor.

(e) The dual of the diagonal map is called the **folding map**, and we will denote it by the symbol ∇. Define it explicitly in category-theoretical language, and explain how to view it as a natural transformation.

The coproduct, being dual to the product, is a domain-type construction.

Exercise 1.41.

(a) Show that the sum of two sets X and Y is simply the disjoint union of X and Y. Conclude that $X \sqcup Y$ and $X \times Y$ are not generally equivalent.

(b) What is the folding map in the case $X = \{a, b, c\}$?

(c) What is the sum of abelian groups G and H? Construct a nice map $w : G \sqcup H \to G \times H$; what can you say about it?

Exercise 1.42.

(a) Give number-theoretical interpretations of products and sums in the category of positive integers $1, 2, 3, \ldots$ with morphisms corresponding to divisibility.

(b) Repeat (a) with the category of real numbers with morphisms corresponding to inequalities $x \le y$.

(c) Is there a category structure on the set $\mathbb{N}$ so that the categorical product is the same as the numerical product?

Larger Sums and Products. The **sum** of a set $\{X_i \mid i \in \mathcal{I}\}$ of objects in the category $\mathcal{C}$ is an object $\coprod_{\mathcal{I}} X_i$ which comes equipped with morphisms $\mathrm{in}_i : X_i \to \coprod_{\mathcal{I}} X_i$ satisfying the universal property that every collection

$$\{f_i : X_i \to Y \mid i \in \mathcal{I}\}$$

gives rise to a unique morphism $f : \coprod_{\mathcal{I}} X_i \to Y$ such that $f \circ \mathrm{in}_i = f_i$ for each $i \in \mathcal{I}$.

Dually, the **product** of the collection $\{X_i \mid i \in \mathcal{I}\}$ is an object $\prod_{\mathcal{I}} X_i$ having maps $\mathrm{pr}_i : \prod_{\mathcal{I}} X_i \to X_i$ such that for every collection of maps

$$\{g_i : W \to X_i \mid i \in \mathcal{I}\},$$

there is a unique map $g : W \to \prod_{\mathcal{I}} X_i$ such that $\mathrm{pr}_i \circ g = g_i$ for each $i \in \mathcal{I}$.

If J is some set, then we can define the J-fold sum and product of X with itself, and (if they exist) there will be a diagonal map $\Delta_J : X \to \prod_{j \in J} X$ and a fold map $\nabla_J : \coprod_{j \in J} X \to X$.

1.7. Initial and Terminal Objects

An object $\tau \in \mathcal{C}$ is called a **terminal object** if the set $\mathrm{mor}_{\mathcal{C}}(X, \tau)$ has exactly one element, no matter what $X \in \mathcal{C}$ we plug in.[7] Dually, an object $\iota \in \mathcal{C}$ is called an **initial object** if the set $\mathrm{mor}_{\mathcal{C}}(\iota, Y)$ has exactly one element, no matter what $Y \in \mathcal{C}$ we plug in.

Exercise 1.43. Find initial and terminal objects in the following contexts.

(a) The category of sets and functions.

(b) The category of topological spaces and continuous functions.

(c) The category of groups and homomorphisms.

A **pointed category** is a category $\mathcal{C}$ in which there is an object, generally[8] denoted $*$, which is simultaneously initial and terminal. If $X, Y \in \mathcal{C}$, where $\mathcal{C}$ is a pointed category, then there is a unique morphism of the form

[7] So a terminal object is a target-type concept.

[8] Though in algebraic contexts, it is often 0 or $\{1\}$ or something even more substantial.

$X \to * \to Y$; it is called the **trivial morphism**, and it will be uniformly denoted $*$.

In a pointed category, the sum of X and Y is sometimes denoted $X \vee Y$ and referred to as the **wedge** sum of X and Y.

An important example of a pointed category is the category $\mathbf{Sets}_*$ of **pointed sets**. An object of $\mathbf{Sets}_*$ is a set X with a particular point x_0 chosen and identified; it is referred to as the **basepoint** of X. A morphism from (X, x_0) to (Y, y_0) is a function $f : X \to Y$ with the additional property that $f(x_0) = y_0$. In practice, we do not give individual names to the basepoints but just call them all $*$.

Exercise 1.44. Verify that $\mathbf{Sets}_*$ is a pointed category. What is a sum in $\mathbf{Sets}_*$? What is a product?

Problem 1.45.

(a) Suppose $\mathcal{C}$ is a category with a terminal object τ, and let $X, Y \in \mathcal{C}$, and suppose that a product P for X and Y exists in $\mathcal{C}$. Show that P solves the problem expressed in the diagram

$$\begin{tikzcd}
Z \arrow[dr, dotted, "\exists!\, t"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& P \arrow[r, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_Y"'] & X \arrow[d] \\
& Y \arrow[r] & \tau.
\end{tikzcd}$$

(There is never any need to label a map into a terminal object!)

(b) Formulate and prove the dual to part (a).

Problem 1.46. Let $\mathcal{C}$ be pointed category in which products and coproducts exist for all pairs of objects.

(a) Give categorical definitions for the 'axis' maps $\mathrm{in}_X : X \to X \times Y$ and $\mathrm{in}_Y : Y \to X \times Y$.

(b) In a pointed category, there is a particularly nice morphism $w : X_1 \vee X_2 \to X_1 \times X_2$. Define it in terms of category theory, and check that the diagram

$$\begin{tikzcd}
X_1 \vee X_2 \arrow[r, "f \vee g"] \arrow[d, "w"'] & Y_1 \vee Y_2 \arrow[d, "w"] \\
X_1 \times X_2 \arrow[r, "f \times g"] & Y_2 \times Y_2
\end{tikzcd}$$

is commutative for any $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$. Why is it necessary for the category to be pointed before you can define w?

(c) State (and prove?) the dual statements.

Exercise 1.47. In what sense do the maps w constitute a natural transformation?

Problem 1.48. Show that for any $f : X \to Y$ in a pointed category $\mathcal{C}$ and $* : W \to X$, then $f \circ * = * : W \to Y$. Also show that if $* : Y \to Z$, then $* \circ f = * : X \to Z$. Conclude that the functors $\mathrm{mor}_{\mathcal{C}}(\,?\,, Y)$ and $\mathrm{mor}_{\mathcal{C}}(A, \,?\,)$ take their values in the category of pointed sets and pointed maps.

Exercise 1.49.

(a) Show that the trivial group $\{1\}$ is simultaneously initial and terminal in the category $\mathcal{G}$ of groups and homomorphisms. Show that the vector space 0 is simultaneously initial and terminal in the category of vector spaces (over $\mathbb{R}$, if you like) and linear transformations.

(b) Show that in the category $\text{AB}\,\mathcal{G}$ of abelian groups and homomorphisms, the map $w : G \vee G \to G \times H$ is an isomorphism for any G and H. Also show that the analogous statement is true in the category of vector spaces and linear transformations.

Matrix Representation of Morphisms. Since sums are domain-type constructions and products are target-type constructions, the maps from a sum to a target should be fairly easy to understand.

Problem 1.50.

(a) Show that, in any category $\mathcal{C}$, there is a natural bijection between the morphism set $\mathrm{mor}_{\mathcal{C}}(X_1 \sqcup X_2, Y_1 \times Y_2)$ and the set M of all matrices

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

with $f_{ij} \in \mathrm{mor}_{\mathcal{C}}(X_j, Y_i)$.

(b) Now suppose that $\mathcal{C}$ is a pointed category in which the canonical map $w : X \vee Y \to X \times Y$ is an isomorphism for each pair of objects $X, Y \in \mathcal{C}$. Show that composition

$$\begin{tikzcd}
\mathrm{mor}_{\mathcal{C}}(Y_1 \sqcup Y_2, Z_1 \times Z_2) \times \mathrm{mor}_{\mathcal{C}}(X_1 \sqcup X_2, Y_1 \times Y_2) \arrow[r] \arrow[d, "\cong"'] & \mathrm{mor}_{\mathcal{C}}(X_1 \sqcup X_2, Z_1 \times Z_2) \\
\mathrm{mor}_{\mathcal{C}}(Y_1 \times Y_2, Z_1 \times Z_2) \times \mathrm{mor}_{\mathcal{C}}(X_1 \sqcup X_2, Y_1 \times Y_2) \arrow[ur] &
\end{tikzcd}$$

corresponds to matrix multiplication in M.[9]

(c) Show that linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$ are in one-to-one correspondence with 2×2 matrices with real entries.

[9] EXERCISE. What exactly do I mean by 'matrix multiplication'?

Comparing Large Sums and Products. If we have a collection $\{X_i \mid i \in \mathcal{I}\}$ of objects of a pointed category $\mathcal{C}$, then we may construct a comparison map

$$w : \coprod_{\mathcal{I}} X_i \longrightarrow \prod_{\mathcal{I}} X_i$$

just as in Problem 1.46(b).

1.8. Group and Cogroup Objects

A group is a set G with a multiplication (which can be thought of as the map $\mu : G \times G \to G$ given by $(x, y) \mapsto x \cdot y$) and, for each element, an inverse (which can be thought of as the map $\nu : G \to G$ given by $x \mapsto s^{-1}$), which are required to satisfy various properties. The main observation of this section is that all of these properties can be formulated abstractly in terms of diagrams.

Let $\mathcal{C}$ be a pointed category, and let $G \in \mathcal{C}$. Then G is a **group object** if there are maps

$$\begin{array}{ll} \mu : G \times G \to G & \text{(multiplication) and} \\ \nu : G \to G & \text{(inverse)} \end{array}$$

which satisfy the following properties:

(1) (Identity) The following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "{(*,\mathrm{id}_G)}"] \arrow[dr, bend right, "\mathrm{id}_G"'] & G \times G \arrow[d, "\mu"] & G \arrow[l, "{(\mathrm{id}_G,*)}"'] \arrow[dl, bend left, "\mathrm{id}_G"] \\ & G. & \end{tikzcd}$$

(2) (Inverse) The following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "{(\nu,\mathrm{id}_G)}"] \arrow[dr, bend right, "*"'] & G \times G \arrow[d, "\mu"] & G \arrow[l, "{(\mathrm{id}_G,\nu)}"'] \arrow[dl, bend left, "*"] \\ & G. & \end{tikzcd}$$

(3) (Associativity) The following diagram commutes:

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mu \times \mathrm{id}_G"] \arrow[d, "\mathrm{id}_G \times \mu"'] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"] & G. \end{tikzcd}$$

It is sometimes useful to study objects with are not quite group objects. For example, if we drop the inverse conditions, we obtain a **monoid object**.

Exercise 1.51. Are group objects domain-type or target-type gadgets? Suppose G is a group object in $\mathcal{C}$. What conditions must you impose on a functor $F : \mathcal{C} \to \mathcal{D}$ in order to conclude that $F(G) \in \mathcal{D}$ is also a group object?

Let's check that these things are correctly named.

Exercise 1.52.

(a) Check that, in the category of pointed sets, a group object is just an ordinary group.

(b) Show that a group $G \in \mathcal{G}$ is a group object if and only if G is abelian.

(c) Write $GL_n(\mathbb{R})$ to denote the set of all $n \times n$ invertible matrices. It is a subset of $\mathbb{R}^{n^2}$, so we can give it the subspace topology. Show that matrix multiplication makes $GL_n(\mathbb{R})$ into a group object in the category of pointed topological spaces.

Exercise 1.53. You know that in the category of pointed sets and their maps, the inverse map ν for a group object G is uniquely determined by its multiplication μ. Prove that this is true for group objects in any category.

The reason group objects are so important is that they provide morphism sets with group structures.

Problem 1.54. Let G be a group object in a pointed category $\mathcal{C}$.

(a) Show that the composite map M in the diagram

$$\begin{tikzcd}
\mathrm{mor}_{\mathcal{C}}(X,G) \times \mathrm{mor}_{\mathcal{C}}(X,G) \arrow[r, "M"] \arrow[d, "\cong"'] & \mathrm{mor}_{\mathcal{C}}(X,G) \\
\mathrm{mor}_{\mathcal{C}}(X, G \times G) \arrow[ur, bend right, "\mu_*"'] &
\end{tikzcd}$$

makes $\mathrm{mor}_{\mathcal{C}}(X,G)$ into a group object in the category of pointed sets (i.e., $\mathrm{mor}_{\mathcal{C}}(X,G)$ is a group with multiplication M).

(b) Draw a diagram that shows all the maps involved in the definition of the product of $\alpha, \beta \in \mathrm{mor}_{\mathcal{C}}(X,G)$ and how they fit together. (In other words, write down explicitly what $\alpha \cdot \beta$ is.)

(c) Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Show that $f^* : \mathrm{mor}_{\mathcal{C}}(Y,G) \to \mathrm{mor}_{\mathcal{C}}(X,G)$ is a group homomorphism.

Hint. Use Problem 1.37 and part (b).

Let's think about what you have just proved. We have two functors

$$F : \mathcal{C} \longrightarrow \mathbf{Sets}_* \qquad \text{and} \qquad \text{forget} : \mathcal{G} \longrightarrow \mathbf{Sets}_*$$

which can be arranged as:

$$\begin{tikzcd} & \mathcal{G} \arrow[d, "\text{forget}"] \\ \mathcal{C} \arrow[r, "F"'] \arrow[ur, dotted, bend left] & \mathbf{Sets}_*. \end{tikzcd}$$

You have proved that the dotted arrow can be filled in, and thereby proved the following theorem.

Theorem 1.55. *If G is a group object in a pointed category $\mathcal{C}$, then the contravariant functor $F(X) = \mathrm{mor}_{\mathcal{C}}(X, G)$ factors through the forgetful functor from the category $\mathcal{G}$ of groups and homomorphisms to the category of pointed sets* $\mathbf{Sets}_*$.

We usually use the same symbol, F, for the dotted functor $\mathcal{C} \to \mathcal{G}$. It is sometimes said in this situation that F 'takes its values in the category $\mathcal{G}$'. This phrasing, though not entirely accurate, makes sense because $\mathcal{G}$ can be considered to be a subcategory of $\mathbf{Sets}_*$ (via the forgetful functor).

Exercise 1.56. Let G be a group object in a category $\mathcal{C}$. Work out the product $\mathrm{pr}_1 \cdot \mathrm{pr}_2 \in \mathrm{mor}_{\mathcal{C}}(G \times G, G)$. You should be able to express it as a specific map you already know.

Now let's dualize.

Problem 1.57. Write down the definition of a cogroup object. What is a cogroup object in the category of groups and homomorphisms? What about abelian groups and homomorphisms? What is a cogroup object in the category of pointed sets? What if you replace 'cogroup' with 'comonoid'?

By dualizing our discussion of group objects, we can immediately derive the following result.

Theorem 1.58. *If C is a cogroup object in a pointed category $\mathcal{C}$, then the covariant functor $G(Y) = \mathrm{mor}_{\mathcal{C}}(C, Y)$ takes its values in the category of groups and homomorphisms.*

Problem 1.59. Prove Theorem 1.58.

Suppose C is a cogroup object and G is a group object. Then the set $\mathrm{mor}(X, Y)$ has *two* ways to multiply. More precisely, $\mathrm{mor}_{\mathcal{C}}(C, G)$ is a group because C is a cogroup object—we'll write $\alpha\spadesuit\beta$ for this product; and $\mathrm{mor}_{\mathcal{C}}(C, G)$ is a group because G is a group object—we'll write $\alpha\heartsuit\beta$ for this product.

Problem 1.60. Show that, with the setup above, the products $\spadesuit$ and $\heartsuit$ are the same. That is, show that for any $f, g \in \mathrm{mor}_{\mathcal{C}}(C, G)$, $f\spadesuit g = f\heartsuit g$.

HINT. Write down the compositions which define $f\spadesuit g$ and $f\heartsuit g$ in a single commutative diagram. Use Problem 1.46.

In view of Problem 1.60, we will never again use suits to denote products and will be content to write $f \cdot g$, or simply fg for these products.

1.9. Homomorphisms

As you know from your study of algebra, when you are studying groups, you inevitably find yourself studying homomorphisms. Our goal in this section is to establish definitions for homomorphisms of group and cogroup objects and to prove some simple but important facts about them.

Let G and H be group objects in a pointed category $\mathcal{C}$. A map $f : G \to H$ is a **homomorphism** if the diagram

$$\begin{tikzcd} G \times G \arrow[r, "\mu_G"] \arrow[d, "f\times f"'] & G \arrow[d, "f"] \\ H \times H \arrow[r, "\mu_H"] & H \end{tikzcd}$$

commutes.[10]

Exercise 1.61. Show that when $\mathcal{C}$ is the category of pointed sets, a homomorphism of group objects is just the same as a homomorphism of groups.

Exercise 1.62. Show that if $f : G \to H$ is a homomorphism of group objects in $\mathcal{C}$, then f preserves inverses, in the sense that the diagram

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\nu_G"'] & H \arrow[d, "\nu_H"] \\ G \arrow[r, "f"] & H \end{tikzcd}$$

is commutative.

If G and H are group objects and $f : G \to H$ is some map, then we automatically get an induced map

$$f_* : \mathrm{mor}_{\mathcal{C}}(X, G) \longrightarrow \mathrm{mor}_{\mathcal{C}}(X, H)$$

From what we know already, the sets $\mathrm{mor}_{\mathcal{C}}(X, G)$ and $\mathrm{mor}_{\mathcal{C}}(X, H)$ are groups; but what can we say about the map f_*? In general, there is nothing we can say, but if f is a homomorphism of group objects, then f_* is a group homomorphism.

[10]This is actually a perfectly good definition for a monoid homomorphism.

Theorem 1.63. *If $f : G \to H$ is a homomorphism of group objects in the pointed category $\mathcal{C}$, then the induced map*

$$f_* : \mathrm{mor}_{\mathcal{C}}(X, G) \longrightarrow \mathrm{mor}_{\mathcal{C}}(X, H)$$

is a homomorphism of groups for every $X \in \mathcal{C}$.

Problem 1.64. Prove Theorem 1.63. Is the converse true?

As usual, it is up to you to formulate the duals.

Problem 1.65. Define homomorphisms of cogroup objects, and prove that they induce group homomorphisms on mapping sets.

1.10. Abelian Groups and Cogroups

Since abelian groups are especially easy to work with, we establish the notion of commutative groups and cogroups.

Abelian Objects. In any category, we can define a **twist map** for coproducts $T : X \sqcup Y \to Y \sqcup X$, which 'switches the terms'.

Problem 1.66. Write down a categorical description of T. Also define the twist map $T : X \times Y \to Y \times X$ for products. Show that both twist maps are equivalences.

A cogroup object C in a pointed category $\mathcal{C}$ is **cocommutative**, or simply **commutative**, if the diagram

$$\begin{tikzcd} & C \arrow[dl, "\phi"'] \arrow[dr, "\phi"] & \\ C \vee C \arrow[rr, "T"'] & & C \vee C \end{tikzcd}$$

is commutative.

Problem 1.67. Show that C is a cocommutative cogroup if and only if $\mathrm{mor}_{\mathcal{C}}(C, Y)$ is an abelian group for every Y.

Problem 1.68. Dualize this discussion: define a commutative group object, and verify that $\mathrm{mor}_{\mathcal{C}}(X, G)$ is abelian if and only if G is such an object.

Products of Groups. As you know, the set-theoretical product of two groups can be made into a group using coordinatewise multiplication. The same can be done with group objects. Let G and H be group objects in the pointed category $\mathcal{C}$, and denote their multiplications by μ_G and μ_H. Then

$G \times H$ can be made into a group object in $\mathcal{C}$ using the multiplication given by

$$\begin{tikzcd}
(G \times H) \times (G \times H) \arrow[r, "\mathrm{id}\times T\times \mathrm{id}"] \arrow[dr, bend right, "\mu_{G\times H}"'] & (G \times G) \times (H \times H) \arrow[d, "\mu_G \times \mu_H"] \\
& G \times H
\end{tikzcd}$$

and inverse $\nu_{G\times H} = \nu_G \times \nu_H$.

Problem 1.69.

(a) Show that $\mu_{G\times H}$ and $\nu_{G\times H}$ make $G \times H$ into a group object in $\mathcal{C}$.

(b) Show that $\mathrm{in}_1 : G \to G \times H$ and $\mathrm{in}_2 : H \to G \times H$ are homomorphisms.

(c) Show that $\mathrm{pr}_1 : G \times H \to G$ and $\mathrm{pr}_2 : G \times H \to H$ are homomorphisms.

The definition of the product in $G \times G$ involves the map $T : G \times G \to G \times G$, which was introduced in order to define commutative groups.

Problem 1.70. Show that a group object is commutative if and only if the multiplication $\mu : G \times G \to G$ is a homomorphism. What is the dual statement?

1.11. Adjoint Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and let $R : \mathcal{C} \to \mathcal{D}$ and $L : \mathcal{D} \to \mathcal{C}$ be two (covariant) functors. Then the rules

$$(X, Y) \mapsto \mathrm{mor}_{\mathcal{C}}(L(X), Y) \qquad \text{and} \qquad (X, Y) \mapsto \mathrm{mor}_{\mathcal{D}}(X, R(Y)),$$

for $X \in \mathcal{D}$ and $Y \in \mathcal{C}$ define functors

$$M, N : \mathcal{D} \times \mathcal{C} \longrightarrow \mathbf{Sets}$$

which are contravariant in the first coordinate and covariant in the second. In general, of course, there need not be any relationship between these two sets. But in many important cases, there is a natural *isomorphism* $\Phi : M \to N$; that is, for each $X \in \mathcal{D}$ and $Y \in \mathcal{C}$, there is an equivalence

$$\Phi_{X,Y} : \mathrm{mor}_{\mathcal{C}}(L(X), Y) \xrightarrow{\cong} \mathrm{mor}_{\mathcal{D}}(X, R(Y)),$$

and these equivalences respect maps between domains and maps between targets. When this occurs, we say that the functors L and R are **adjoint** to one another. More precisely, L is the **left adjoint** and R is the corresponding **right adjoint**. Whenever we introduce an adjoint pair of functors as L and R, we are making the tacit assertion that L is the left adjoint and R is the right adjoint.

We will use the notation $\widehat{\alpha} = \Phi_{X,Y}(\alpha)$. Thus, if $\alpha : LX \to Y$, then $\widehat{\alpha} : X \to RY$.

Exercise 1.71. Let $f : X \to X'$ and $g : Y \to Y'$. Write down the diagrams which must commute in order for Φ to be a natural transformation.

Problem 1.72. Let L and R be adjoint.

(a) Show that if one of the squares

$$\begin{tikzcd} LA \arrow[r, "\alpha"] \arrow[d, "Lf"'] & X \arrow[d, "g"] \\ LB \arrow[r, "\beta"] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "\widehat{\alpha}"] \arrow[d, "f"'] & RX \arrow[d, "Rg"] \\ B \arrow[r, "\widehat{\beta}"] & RY \end{tikzcd}$$

commutes, then so does the other one.

(b) Let F, G be functors. Show that $\Phi : LF \to G$ is a natural transformation if and only if $\widehat{\Phi} : F \to RG$ is a natural transformation.

Exercise 1.73. If X is a set, then we can form the **free abelian group** $F(X) = \bigoplus_{x \in X} \mathbb{Z}$. On the other hand, if G is an abelian group, then we can forget the group structure of G and just remember the underlying set $S(G)$.

(a) Show that the functors $F : \mathbf{Sets} \to \mathcal{G}$ and $S : \mathcal{G} \to \mathbf{Sets}$ are adjoint to one another. Which is the left adjoint and which is the right adjoint?

(b) Use the same scheme to express other 'free objects' that you know about in terms of adjoints.

Let's start with functors $R : \mathcal{C} \to \mathcal{D}$ and $L : \mathcal{D} \to \mathcal{C}$ which are adjoint. Taking $X = L(Y)$, we have a natural isomorphism

$$\Phi : \operatorname{mor}_{\mathcal{D}}(L(Y), L(Y)) \xrightarrow{\cong} \operatorname{mor}_{\mathcal{C}}(Y, RL(Y)).$$

Applying this to the identity $\mathrm{id}_{L(Y)}$ gives us a map $\sigma : Y \to RL(Y)$.

Problem 1.74. Show that there is a commutative diagram

$$\begin{tikzcd} \operatorname{mor}_{\mathcal{C}}(X, Y) \arrow[r, "L"] \arrow[d, equal] & \operatorname{mor}_{\mathcal{C}}(L(X), L(Y)) \arrow[d, "\Phi", "\cong"'] \\ \operatorname{mor}_{\mathcal{C}}(X, Y) \arrow[r, "\sigma_*"] & \operatorname{mor}_{\mathcal{C}}(X, RL(Y)). \end{tikzcd}$$

Thus, the effect of L on morphisms can be identified with the map σ_*, which is defined by composition of morphisms.

Exercise 1.75. Problem 1.74 shows that the maps L and σ_* are **equivalent maps**. In Section 1.2 we defined what equivalence means in categorical terms—what category are we working in here?

Dually, if we take $Y = R(X)$, then we have an isomorphism

$$\Phi : \operatorname{mor}_{\mathcal{C}}(R(X), R(X)) \longrightarrow \operatorname{mor}_{\mathcal{D}}(LR(X), X),$$

and we define $\lambda : LR(X) \to X$ to be the image of $\mathrm{id}_{R(X)}$.

Problem 1.76. Dualize the previous problem.

Problem 1.77. Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be a pair of adjoint functors. Show that there are natural maps $X \to RLX$ and $LRX \to X$ and that RX is naturally a retract of $RLRX$.

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Chapter 2

Limits and Colimits

There are two important ways to define new objects using ones you already have. The first of these is called taking the *limit* of a diagram—a limit is defined by its properties as the target of morphisms. The dual is the *colimit*, which is defined in terms of its properties as the domain of morphisms.

2.1. Diagrams and Their Shapes

We begin our discussion by revisiting diagrams from our more sophisticated category-theoretical point of view. Let's first say that two diagrams have the same **shape** if one is obtained from the other by simply reassigning the objects and morphisms but leaving the overall picture the same. For example, the diagrams

$$X \xrightarrow{f} Y \xleftarrow{g} Z \qquad \text{and} \qquad A \xrightarrow{i} B \xleftarrow{j} C$$

in the category $\mathcal{C}$ are two different diagrams with the same shape, namely $\bullet \to \star \leftarrow \circ$. This last diagram could be called the common **shape diagram** for the two example diagrams.

Now, we have seen in Exercise 1.1 that we can, without causing any trouble, take any diagram and augment it by including all composite arrows and identity arrows that are not already present in the diagram, and the result is a category. If we apply this construction to the shape diagram $\bullet \to \star \leftarrow \circ$, we obtain a category $\mathcal{I}$. The key observation is that the diagram $X \xrightarrow{f} Y \xleftarrow{g} Z$ determines a functor

$$F : \mathcal{I} \longrightarrow \mathcal{C}$$

given explicitly by

$$\begin{array}{rclcrcl} F(\bullet) & = & X, & \qquad & F(\bullet \to \star) & = & f, \\ F(\star) & = & Y, & & F(\star \leftarrow \circ) & = & g, \\ F(\circ) & = & Z, & & & & \end{array}$$

and of course F carries identities to identities. This leads us to our formal definition of a diagram: if $\mathcal{I}$ and $\mathcal{C}$ are two categories, then a **diagram** in $\mathcal{C}$ with **shape** $\mathcal{I}$ is a (covariant) functor $F : \mathcal{I} \to \mathcal{C}$.

When $\mathrm{ob}(\mathcal{I})$ is too large to be a set, serious set-theoretical questions arise when working with $\mathcal{I}$-shaped diagrams. Therefore, the usual practice is to work only with **small diagrams**—diagrams whose collection of objects is 'only' a set. In this book we will *never* work with large diagrams.

Exercise 2.1. Determine the shape of the diagram[1]

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & B \arrow[d] \\ & C \end{tikzcd}$$

if the diagram is

(a) **guaranteed** to be commutative;

(b) **not necessarily** noncommutative.

Exercise 2.2. Let G be a group, and consider it as a category with one object, as in Exercise 1.3. Interpret 'diagram with shape G' in the following categories.

(a) The category of sets and maps.

(b) The category of groups and homomorphisms.

(c) The category of topological spaces and continuous maps.

We call a shape category $\mathcal{I}$ **finite** if there is some $N \in \mathbb{N}$ such that in any composition $f_1 \circ f_2 \circ \cdots \circ f_m$ with $m > N$, at least one of the f_i is an identity map.

Exercise 2.3. Consider the group $G = \mathbb{Z}/2$ as a category with one object and two morphisms. Is G a finite shape category?

We will want to study the collection of *all* diagrams with a given shape $\mathcal{I}$. To do this, we form the **diagram category**

$$\mathcal{C}^{\mathcal{I}} = \{\text{functors } \mathcal{I} \to \mathcal{C}\}.$$

[1] In the sense of Section 1.1.

The morphisms in this category are (of course) natural transformations between functors. How many natural transformations are there from one functor to another?

Exercise 2.4.

(a) Let $\mathcal{I}$ be a small category and let $F, G \in \mathcal{C}^{\mathcal{I}}$. Express the collection of all natural transformations from F to G as a subset of some set. It follows that this collection is a set.

(b) Conclude that if $\mathcal{I}$ is small, then $\mathcal{C}^{\mathcal{I}}$ is actually a category.

(c) Find an example of a 'large' category $\mathcal{I}$ and $F, G \in \mathcal{C}^{\mathcal{I}}$ so that the collection of natural transformations $F \to G$ is not a set.

2.2. Limits and Colimits

Let's start with the diagram $X \to Y \leftarrow Z$ of the previous section. It frequently happens that we are given a diagram of this form, and we are interested in finding objects Q with maps $Q \to X$ and $Q \to Z$ such that the diagram

$$\begin{tikzcd}
Q \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& & X \arrow[d] \\
& Z \arrow[r] & Y
\end{tikzcd}$$

commutes. A particular example $P \to X$ and $P \to Z$ is a **limit** for the diagram if, for each other example $Q \to X$ and $Q \to Z$, there is a *unique* map $Q \to P$ making the diagram

$$\begin{tikzcd}
Q \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[dr, dotted] & & \\
& P \arrow[r] \arrow[d] & X \arrow[d] \\
& Z \arrow[r] & Y
\end{tikzcd}$$

commute. That is, the limit of the diagram is any object (with morphisms to X and Z) that solves the universal problem posed by the diagram above.

Exercise 2.5. Let $\mathcal{U}$ be the category whose objects are objects $Q \in \mathcal{C}$ together with maps $Q \to X$ and $Q \to Z$ making the diagram above commute;

the morphisms are maps $f : Q \to R$ such that the diagram

$$\begin{tikzcd}
X & Q \arrow[l] \arrow[r] \arrow[d, "f"] & Z \\
X \arrow[u, equal] \arrow[dr, bend right] & R \arrow[l] \arrow[r] & Z \arrow[u, equal] \arrow[dl, bend left] \\
& Y &
\end{tikzcd}$$

commutes. Show that P (with the maps to X and Z) is a limit for the diagram $X \to Y \leftarrow Z$ if and only if it is a terminal object in $\mathcal{U}$.

We can recast this definition in another way. Any object $Q \in \mathcal{C}$ gives rise to a constant functor

$$\Delta_Q : \mathcal{I} \longrightarrow \mathcal{C}$$

given by $\Delta_Q(\text{any object}) = Q$ and $\Delta_Q(\text{any morphism}) = \mathrm{id}_Q$. This defines a functor

$$\Delta : \mathcal{C} \longrightarrow \mathcal{C}^{\mathcal{I}} \qquad \text{given by} \quad \Delta(Q) = \Delta_Q,$$

which is known as the **diagonal functor**. In our example, the commutativity of the square diagram is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
Q \arrow[r, equal] \arrow[d] & Q \arrow[r, equal] \arrow[d] & Q \arrow[d] \\
X \arrow[r] & Y & Z \arrow[l]
\end{tikzcd}$$

or, in other words, the existence of a morphism (natural transformation) $\Delta_Q \to F$ in the category $\mathcal{C}^{\mathcal{I}}$. If P is the limit of the diagram, then we have a commutative diagram

$$\begin{tikzcd}
\Delta_Q \arrow[d, dashed] \\
\Delta_P \arrow[d, dashed] \\
F
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Q \arrow[r, equal] \arrow[d] & Q \arrow[r, equal] \arrow[d] & Q \arrow[d] \\
P \arrow[r, equal] \arrow[d] & P \arrow[r, equal] \arrow[d] & P \arrow[d] \\
X \arrow[r] & Y & Z. \arrow[l]
\end{tikzcd}$$

This leads us to define the **limit** of a functor $F : \mathcal{I} \to \mathcal{C}$ to be an object $P \in \mathcal{C}$ and a natural transformation $\Delta_P \to F$ such that for any other natural transformation $\Delta_Q \to F$, there is a unique map $f : Q \to P$ making

the diagram

$$\begin{tikzcd} & \Delta_P \arrow[d] \\ \Delta_Q \arrow[ur, dotted, bend left, "\Delta_f"] \arrow[r] & F \end{tikzcd}$$

commutative.

It is very common to write 'P is a limit for the diagram F' and make no mention at all of the transformation $\Delta_P \to F$.

Exercise 2.6. Consider the diagram category $\mathcal{I}$ with two objects $\bullet$ and $\star$, with only identity morphisms. Let $\mathcal{C}$ denote the category of real vector spaces and linear transformations, and let $F(\bullet) = 0$ and $F(\star) = \mathbb{R}$.

(a) Show that the limit of $F : \mathcal{I} \to \mathcal{C}$ is $\mathbb{R}$; what is the natural transformation $\Delta_{\mathbb{R}} \to F$?

(b) Let V be any nontrivial vector space. Show that there are uncountably many natural transformations $T : \Delta_V \to \Delta_{\mathbb{R}}$ making the diagram

$$\begin{tikzcd} & \Delta_{\mathbb{R}} \arrow[d] \\ \Delta_V \arrow[ur, dotted, bend left, "T"] \arrow[r] & F \end{tikzcd}$$

commutative.

(c) Show that exactly one of these is of the form Δ_f for some linear transformation $f : V \to \mathbb{R}$.

Problem 2.7. Let P be a limit for the diagram $F : \mathcal{I} \to \mathcal{C}$ (so we are given a natural transformation $L : \Delta_P \to F$). Show that there is an isomorphism

$$\mathrm{mor}_{\mathcal{C}}(Q, P) \xrightarrow{\cong} \mathrm{mor}_{\mathcal{C}^{\mathcal{I}}}(\Delta_Q, F).$$

In some categories, every diagram of this form has a limit, but in other categories (including some that are central to homotopy theory) limits need not always exist.

Exercise 2.8. Suppose that the empty diagram $\varnothing$ has a limit. Show that it is a terminal object in $\mathcal{C}$. Find a way to view a product as the limit of a diagram.

Exercise 2.9.

(a) Construct an example of a category and a diagram which has no limit.

(b) A **discrete category** is one in which the only morphisms are identities. What is the limit of a diagram whose shape is discrete?

Colimits. The discussion of limits, of course, dualizes. A **colimit** of a diagram $F : \mathcal{I} \to \mathcal{C}$ is an object $P \in \mathcal{C}$ and a morphism of diagrams $F \to \Delta_P$ that is initial among all morphisms $F \to \Delta_X$ for $X \in \mathcal{C}$. That is, given solid arrow diagram morphisms

$$\begin{tikzcd} F \arrow[d] \arrow[dr, bend left] & \\ \Delta_P \arrow[r, dotted] & \Delta_X, \end{tikzcd}$$

there exists a unique map $f : P \to X$ in $\mathcal{C}$ such that Δ_f makes the triangle commute.

Problem 2.10. Show that any two limits of a given diagram $F : \mathcal{I} \to \mathcal{C}$ are equivalent in $\mathcal{C}$. Also prove the dual statement about the uniqueness of colimits.

HINT. Let P and Q be two limits of F; use the universal property of the limit to find maps $P \to Q$ and $Q \to P$. Alternatively, construct a natural isomorphism between the functors $\mathrm{mor}_{\mathcal{C}}(\,?\,, P)$ and $\mathrm{mor}_{\mathcal{C}}(\,?\,, Q)$.

Exercise 2.11. Define the opposite $F^{\mathrm{op}} : \mathcal{I}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ of a diagram $F : \mathcal{I} \to \mathcal{C}$. Compare the limit and colimit of F^{op} to the limit and colimit of F.

Domains and Targets. Colimits C are defined in such a way that we are given information about morphisms $C \to Z$; so colimits are *domain-type* constructions. Dually, limits are defined so that we have information about morphims $Z \to L$; and hence limits are *target-type* constructions.

Exercise 2.12. Show that the colimit of the empty diagram $\varnothing \to \mathcal{C}$ is an initial object in $\mathcal{C}$. Show that the limit of the identity $\mathrm{id}_{\mathcal{C}}$ is an initial object in $\mathcal{C}$.

Exercise 2.13. Suppose $\mathcal{I}$ has an initial object, $\varnothing$, and let $F : \mathcal{I} \to \mathcal{C}$. Show that F has a limit. State and prove the dual result.

Exercise 2.14. What is the colimit of a diagram whose shape is discrete?

2.3. Naturality of Limits and Colimits

Let $F, G : \mathcal{I} \to \mathcal{C}$ be two diagrams and let $\Phi : F \to G$ be a natural transformation between them. Suppose that both diagrams have colimits in $\mathcal{C}$; call them X and Y, respectively. Then the definition implies that we have the picture

$$\begin{tikzcd} F \arrow[r] \arrow[d] & G \arrow[r] & \Delta_Y \\ \Delta_X \arrow[urr, dotted, "\exists!\,\Delta_f"'] & & \end{tikzcd}$$

for a uniquely determined morphism $f : X \to Y$. We call the unique map $f : X \to Y$ determined by this diagram the map of colimits **induced** by the map of diagrams.

Problem 2.15. Explain how a natural transformation of diagrams induces a map between the limits of those diagrams.

Problem 2.16. Let $\mathcal{C}$ be a category in which every diagram with shape $\mathcal{I}$ has a colimit. For each $F \in \mathcal{C}^{\mathcal{I}}$, choose, once and for all, a colimit for F, and call it colim F.[2]

(a) Show that the assignment $F \mapsto \operatorname{colim} F$ defines a functor $\mathcal{C}^{\mathcal{I}} \to \mathcal{C}$.

(b) Show that the functors $\mathcal{C}^{\mathcal{I}} \to \mathcal{C}$ and $\mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ given by

$$F \longmapsto \operatorname{colim} F \qquad \text{and} \qquad X \longmapsto \Delta_X$$

are adjoint to one another. Which is the right adjoint and which is the left adjoint?

You have proved the following theorem.

Theorem 2.17. *Let $\mathcal{C}$ be a category. Then the following are equivalent:*

(1) Every $\mathcal{I}$-shaped diagram in $\mathcal{C}$ has a colimit.

(2) The functor $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ has a left adjoint.

Exercise 2.18. Formulate the dual of Theorem 2.17. Do you need to prove it? Explain.

Theorem 2.19. *Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be an adjoint pair of functors.*

(a) Suppose that every diagram in $\mathcal{C}^{\mathcal{I}}$ has a colimit. Then for every $F \in \mathcal{C}^{\mathcal{I}}$, $L(\operatorname{colim} F)$ is a colimit for $L \circ F \in \mathcal{D}^{\mathcal{I}}$.

(b) Suppose that every diagram in $\mathcal{D}^{\mathcal{I}}$ has a limit. Then for every $F \in \mathcal{D}^{\mathcal{I}}$, $R(\lim F)$ is a limit for $R \circ F \in \mathcal{C}^{\mathcal{I}}$.

Problem 2.20. Prove Theorem 2.19.

2.4. Special Kinds of Limits and Colimits

Some diagram shapes are particularly useful, and their limits and colimits have special names.

[2]We consider the object colim F as coming with structure maps to the diagram F.

2.4.1. Pullback. We began our discussion of limits with a particularly simple example. This example is extremely important, and it has a special name—the pullback. In our more efficient terminology, the **pullback** of a diagram $X \to Y \leftarrow Z$ is the limit of the corresponding diagram (functor) $F : \mathcal{I} \to \mathcal{C}$, where $\mathcal{I}$ is the shape category $\bullet \to \star \leftarrow \circ$ (we will call any diagram with this shape a **prepullback** diagram). If P is a pullback for the diagram $X \to Y \leftarrow Z$, then the structure maps for P and the given diagram fit together into the commutative square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd}$$

which we call a **pullback square**.

Problem 2.21. Suppose that

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "i"'] & X \arrow[d, "j"] \\ Z \arrow[r, "z"] & Y \end{tikzcd}$$

is a pullback square. Show that if j is an equivalence, then so is i.

HINT. Use the map $j^{-1} \circ z$ to find a map $K : Z \to P$.

Exercise 2.22. Suppose that $\mathcal{C}$ has a terminal object τ. Show that products of X and Y are the same as pullbacks for the diagram $X \to \tau \leftarrow Y$.

Let's look at some specific examples.

Problem 2.23. Show that in the category of sets and functions

$$\begin{tikzcd} \{(a,b) \mid f(a) = g(b)\} \arrow[r, "\mathrm{pr}_A"] \arrow[d, "\mathrm{pr}_B"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

is a pullback square. What is the pullback of $A \to * \leftarrow B$?

Exercise 2.24.

(a) Consider the category whose objects are the integers $1, 2, 3, \ldots$ and with arrows given by divisibility. Give number-theoretical descriptions of pullbacks in this category.

(b) Repeat (a) but using the category whose objects are real numbers and whose morphisms correspond to inequalities $x \leq y$.

Exercise 2.25. Determine the pullback of the diagram

$$\{1\} \longrightarrow B \xleftarrow{f} A$$

in the category of groups and homomorphisms.

There is another kind of limit, which is closely related to pullback. It arises when you are given two morphisms $f, g : X \to Y$ and you want to study morphisms $w : W \to X$ such that $f \circ w = g \circ w$. An **equalizer** for f and g is a morphism $e : E \to X$ such that $f \circ e = g \circ e$, and for any other such map $w : W \to X$, there is a unique map $W \to E$ as in the diagram

$$\begin{tikzcd} W \arrow[d, dotted, "\exists!"'] \arrow[dr, "w"] & & \\ E \arrow[r, "e"'] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y. \end{tikzcd}$$

Exercise 2.26.

(a) What is the shape category for the equalizer?

(b) If products exist in our category, then we can form the diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d, "q"'] & Y \arrow[d, "\Delta"] \\ X \arrow[r, "(f,g)"] & Y \times Y. \end{tikzcd}$$

Show that this is a pullback square if and only if Q (with given map $q : Q \to X$) is an equalizer for f and g.

(c) Show that if pullbacks always exist in $\mathcal{C}$, then so do equalizers. Is the converse true?

2.4.2. Pushout. The pushout is dual to the pullback. More precisely, a pushout is a colimit of a **prepushout** diagram—i.e., a diagram with shape $\bullet \leftarrow \star \rightarrow \circ$. A **pushout square** is a commutative square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[d] \\ X \arrow[r] & P \end{tikzcd}$$

in which the object P (together with the maps $Z \to P$ and $X \to P$) is a colimit of the diagram $X \leftarrow Y \to Z$.

Problem 2.27. State and prove the dual of Problem 2.21.

Exercise 2.28. Work in the category **Sets** of all sets and all functions between sets. Suppose that $A \subseteq B$ and $A \subseteq C$ and that $i : A \to C$ and $p : A \to B$ are inclusion functions.

(a) Determine the pushout of the diagram $B \xleftarrow{p} A \xrightarrow{i} C$.

(b) Let $*$ denote a one-point set. Determine the pushout of the diagram $B \xleftarrow{p} A \xrightarrow{*} *$.

Problem 2.29. Let X be a topological space, and suppose $X = A \cup B$, where $A, B \subseteq X$ are closed subspaces. Show that the diagram

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

is a pushout square.

Exercise 2.30. Suppose $\mathcal{C}$ has an initial object ι, and let $X, Y \in \mathcal{C}$. Show that pushouts of diagrams of the form $X \leftarrow \iota \rightarrow Y$ are the same as coproducts $X \sqcup Y$.

Exercise 2.31.

(a) Consider the category whose objects are the integers $1, 2, 3, \ldots$ and with arrows given by divisibility. Give number-theoretical descriptions of pushouts in this category.

(b) Repeat (a) but using the category whose objects are real numbers and whose morphisms correspond to inequalities $x \leq y$.

Exercise 2.32. Determine the pushout of the diagram $A \xleftarrow{f} B \rightarrow \{1\}$ in the category of groups and homomorphisms. Compare with Exercise 2.25; what should the object you constructed here be called?

Problem 2.33. Define the dual notion of **coequalizer**, and compare coequalizers with pushouts.

Problem 2.34. Suppose $\mathcal{C}$ is a pointed category and

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "j"] & D \end{tikzcd}$$

is a pushout square in $\mathcal{C}$. Show that for any morphism $X \in \mathcal{C}$, the square

$$\begin{tikzcd} A \sqcup X \arrow[r, "f \sqcup \mathrm{id}_X"] \arrow[d, "{(g,*)}"'] & B \sqcup X \arrow[d, "{(h,*)}"] \\ C \arrow[r, "j"] & D \end{tikzcd}$$

is also a pushout square.

2.4.3. Telescopes and Towers. Telescopes and towers are the diagrams that generalize infinite ascending unions and infinite nested intersections.

A **telescope diagram** is a diagram of the form

$$X_1 \longrightarrow X_2 \to \cdots \to X_n \longrightarrow X_{n+1} \to \cdots .$$

To view this kind of diagram formally—as a functor from a shape category—let $\mathbb{N}$ be the category with objects $\{1, 2, \ldots\}$ and with a unique morphism $i \to j$ if $i \leq j$ and no morphism at all if $i > j$. Then the rule $i \mapsto X_i$ is the object part of a functor $\mathbb{N} \to \mathcal{C}$.

The colimit of such a diagram is often referred to as the **direct limit** of the diagram. Roughly speaking, it is the object X_∞ which belongs at the 'end' of the sequence.

For the dual, we reverse the arrows and look at

$$Y_1 \longleftarrow Y_2 \leftarrow \cdots \leftarrow Y_n \longleftarrow Y_{n+1} \leftarrow \cdots .$$

Formally, this is a functor $F : \mathbb{N}^{\mathrm{op}} \to \mathcal{C}$. We will call such a diagram a **tower**; a limit for a tower is frequently called an **inverse limit**.

Exercise 2.35. Reformulate the definitions of direct and inverse limits in terms of commutative diagrams.

Problem 2.36. Let $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} \cdots$ be a telescope diagram. We define the **shift map**

$$\mathrm{sh} : \textstyle\coprod_{n \geq 1} X_n \longrightarrow \coprod_{n \geq 1} X_n$$

by the formula $\mathrm{sh} = (\mathrm{in}_2 \circ f_1, \mathrm{in}_3 \circ f_2, \ldots, \mathrm{in}_{n+1} \circ f_n, \ldots)$. Show that in the pushout square

$$\begin{tikzcd}
(\coprod X_n) \sqcup (\coprod X_n) \arrow[r, "(\mathrm{sh},\mathrm{id})"] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\text{pushout}"] & \coprod X_n \arrow[d] \\
\coprod X_n \arrow[r] & P
\end{tikzcd}$$

the pushout P is a direct limit for the telescope. Formulate the dual result.

Problem 2.36 shows that colimits of telescopes can be constructed if you can construct infinite coproducts and pushouts. (If you look carefully, you will see that the pushout diagram here is secretly a coequalizer diagram.) It is possible to show that in any category $\mathcal{C}$, if you can construct arbitrary coproducts and coequalizers, then you can construct all colimits [**110**].

Problem 2.37. Let $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} \cdots$ be a telescope diagram, and let X_∞ be its colimit. If $k \leq l$, write $f_{k,l}$ for the unique map in this diagram from X_k to X_l. Let $r : \mathbb{N} \to \mathbb{N}$ be an increasing function and set up the commutative ladder

$$\begin{tikzcd}
\cdots \arrow[r] & X_n \arrow[r, "f_n"] \arrow[d, "f_{n,r(n)}"'] & X_{n+1} \arrow[r] \arrow[d, "f_{n+1,r(n+1)}"] & \cdots \\
\cdots \arrow[r] & X_{r(n)} \arrow[r, "f_{r(n),r(n+1)}"] & X_{r(n+1)} \arrow[r] & \cdots .
\end{tikzcd}$$

Show that X_∞ is also the colimit of the bottom row and that the induced map $X_\infty \to X_\infty$ is the identity map.

Exercise 2.38. Generalize the result of Problem 2.37 to other diagram shapes.

2.5. Formal Properties of Pushout and Pullback Squares

We conclude with some formal properties of pushout and pullback squares. These rules, and their homotopy-theoretical analogs, will be crucial throughout our study of homotopy theory.

Problem 2.39. Consider the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"] \\
C \arrow[r] & D.
\end{tikzcd}$$

(a) Suppose the diagram is a pushout and that f is an equivalence. Show that g is also an equivalence.

(b) Suppose f and g are both equivalences. Show that the square is a pushout.

(c) State and prove the duals of (a) and (b).

Theorem 2.40. *Consider the diagram*

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "h_1"'] \arrow[dr, phantom, "(I)"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"'] \arrow[dr, phantom, "(II)"] & A_3 \arrow[d, "h_3"] \\
B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3,
\end{tikzcd}$$

and denote the outside square by (T).

(a) If (I) and (II) are pushouts, then (T) is also a pushout.

(b) If (I) and (T) are pushouts, then (II) is also a pushout.

Problem 2.41. Prove Theorem 2.40.[3]

[3]Should break into steps.

The dual statements are also true, of course.

Theorem 2.42. *Consider the diagram*

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "h_1"'] \arrow[dr, phantom, "(I)"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"'] \arrow[dr, phantom, "(II)"] & A_3 \arrow[d, "h_3"] \\ B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3, \end{tikzcd}$$

and denote the outside square by (T).

(a) If (I) *and* (II) *are pullbacks, then* (T) *is also a pullback.*

(b) If (II) *and* (T) *are pullbacks, then* (I) *is also a pullback.*

We end the chapter with our first investigation into diagrams more complicated than pushouts and pullbacks. Consider the diagram

$$\begin{tikzcd} A_1 & A_2 \arrow[l] \arrow[r] & A_3 & & A \\ B_1 \arrow[u] \arrow[d] & B_2 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & B_3 \arrow[u] \arrow[d] & {} \arrow[r, dashed, "\text{pushout}"] & B \arrow[u] \arrow[d] \\ C_1 & C_2 \arrow[l] \arrow[r] & C_3 & & C \\ & & & & D \arrow[dl, dotted, leftrightarrow, "?"] \\ X & Y \arrow[l] \arrow[r] & Z & W. & \end{tikzcd}$$

Taking pushouts of the rows gives a prepushout diagram $A \leftarrow B \rightarrow C$. Taking pushouts of the columns gives a prepushout diagram $X \leftarrow Y \rightarrow Z$. Let D and Z be the respective pushouts. How do they compare?

Theorem 2.43. *The pushouts* D *and* W *are both colimits for the whole* 3×3 *diagram and hence are equivalent to each other.*

Problem 2.44. Suppose given maps from the spaces in the diagram of Theorem 2.43 to a space Q.

(a) Show that the given maps have unique extensions to maps from the two prepushout diagrams.

(b) Prove Theorem 2.43.

We will also need the dual result, which concerns the limits of diagrams of the form

$$
\begin{tikzcd}
A_1 \arrow[r] \arrow[d] & A_2 \arrow[d] & A_3 \arrow[l] \arrow[d] & & & A \\
B_1 \arrow[r] & B_2 & B_3 \arrow[l] & {} \arrow[r, dashed, "\text{pullback}"] & {} & B \arrow[u] \arrow[d] \\
C_1 \arrow[r] \arrow[u] & C_2 \arrow[u] & C_3 \arrow[l] \arrow[u] & & & C \\
 & & & & & D \\
X \arrow[r] & Y & Z \arrow[l] & & W. \arrow[ur, dotted, leftrightarrow, "?" description] &
\end{tikzcd}
$$

In this case, the pullbacks of the rows form a prepullback diagram $C \to A \leftarrow B$, and we let D be the pullback; likewise, we define W to be the pullback of $X \to Y \leftarrow Z$.

Theorem 2.45. *The pullbacks D and W are both limits for the whole 3×3 diagram and hence are equivalent to each other.*

Theorem 2.43 has a vast generalization. Suppose we have a diagram

$$F : \mathcal{I} \times \mathcal{J} \longrightarrow \mathcal{C}.$$

If we fix $j \in \mathcal{J}$ we obtain $F_j : \mathcal{I} \to \mathcal{C}$ given by $F_j(i) = F(i, j)$; and these diagrams are related by natural transformations corresponding to the morphisms in $\mathcal{J}$. Forming colimits over $\mathcal{I}$ yields a new diagram

$$\operatorname{colim}_{\mathcal{I}} F : \mathcal{J} \longrightarrow \mathcal{C},$$

and the colimit of this new diagram is an object $\operatorname{colim}_{\mathcal{J}} \operatorname{colim}_{\mathcal{I}} F \in \mathcal{C}$.

Theorem 2.46. *For any $F : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$,*

(a) $\operatorname{colim}_{\mathcal{J}} \operatorname{colim}_{\mathcal{I}} F \cong \operatorname{colim} F \cong \operatorname{colim}_{\mathcal{I}} \operatorname{colim}_{\mathcal{J}} F$ *and*

(b) $\lim_{\mathcal{J}} \lim_{\mathcal{I}} F \cong \lim F \cong \lim_{\mathcal{I}} \lim_{\mathcal{J}} F$.

Problem 2.47. Prove Theorem 2.46.

Part 2

Semi-Formal Homotopy Theory

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Chapter 3

Categories of Spaces

In topology, there are many theorems that are true for 'most' spaces but which fail for certain off-the-wall counterexamples. Accounting for such pathologies can require the introduction of a bewildering proliferation of hypotheses, which can obscure the fundamental content of the theorems. Our solution to this annoyance is to discard the category **Top** of all topological spaces and all continuous functions and work entirely inside a 'convenient' category of topological spaces in which no such counterexamples exist. Of course, this category cannot possibly contain *all* spaces, but we will not accept one that does not contain the 'vast majority' of spaces; in particular, it is crucial that our category of spaces contain all CW complexes. Furthermore, it should be closed under the formation of limits and colimits; and certain fundamental constructions involving mapping spaces should behave well in our category.

3.1. Spheres and Disks

We begin with spheres and disks. The n-dimensional **sphere** is the space

$$S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}.$$

This standard sphere is of course homeomorphic to any subspace of $\mathbb{R}^{n+1}$ having the form $\{x \in \mathbb{R}^{n+1} \mid |x - a| = r\}$ for any fixed center point a and radius $r > 0$. The n-sphere is the boundary of the $(n+1)$-dimensional **disk**

$$D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| \leq 1\}.$$

The 1-dimensional disk is $D^1 = [-1, 1] \subseteq \mathbb{R}$, and its boundary—the zero-sphere—is the discrete two-point space $S^0 = \{-1, 1\}$. The 0-dimensional disk D^0, which is a single point since $\mathbb{R}^0 = \{0\}$, is its own interior, and its boundary is empty. Some authors define $S^{-1} = \varnothing$, but we will restrain ourselves.

The **northern hemisphere** of S^n is the space

$$D^n_{\mathrm{N}} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1 \text{ and } x_{n+1} \geq 0\}$$

and its **southern hemisphere** is $D^n_{\mathrm{S}} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1 \text{ and } x_{n+1} \leq 0\}$.

Problem 3.1. Let $i : S^{n-1} \hookrightarrow D^n$ be the inclusion of the boundary.

(a) Show that the functions $j_{\mathrm{N}} : D^n \to D^n_{\mathrm{N}}$ and $j_{\mathrm{S}} : D^n \to D^n_{\mathrm{S}}$ given by

$$j_{\mathrm{N}} : x \longmapsto \left(x, \sqrt{1 - |x|^2}\right) \qquad \text{and} \qquad j_{\mathrm{S}} : x \longmapsto \left(x, -\sqrt{1 - |x|^2}\right)$$

are homeomorphisms.

(b) Show that the diagram

$$\begin{tikzcd} S^n \arrow[r, hook, "i"] \arrow[d, hook, "i"'] & D^{n+1} \arrow[d, "j_{\mathrm{N}}"] \\ D^{n+1} \arrow[r, "j_{\mathrm{S}}"] & S^{n+1} \end{tikzcd}$$

is a pushout square in the category **Top**.

Let $X = S^n - \{N\}$, where $N = (0, 0, \ldots, 0, 1) \in S^n$ is the **north pole** of the sphere. We can define a function $\sigma : X \to \mathbb{R}^n$ as follows:

- if $x \in X$, then $x \neq N$, and there is a unique line ℓ joining x and N;
- since the point x is 'lower' than the point N, the line ℓ crosses the plane $\mathbb{R}^n \oplus 0 \subseteq \mathbb{R}^{n+1}$ in exactly one point, which we call $\sigma(x)$.

The resulting function $\sigma : X \to \mathbb{R}^n$ is called **stereographic projection**, and it is a homeomorphism.

Problem 3.2. Prove that σ is a homeomorphism. More generally, let S be any space homeomorphic to S^n, and let $x \in S$. Show that $S - \{x\} \cong \mathbb{R}^n$.

The **unreduced suspension** of a space X is the space $\Sigma_\circ X$ which is obtained from $X \times I$ by collapsing $X \times \{0\}$ to a single point $[0]$ and also collapsing $X \times \{1\}$ to a single point $[1]$.

Problem 3.3. Show that $\Sigma_\circ S^n \cong S^{n+1}$ for each n.

3.2. CW Complexes

CW complexes are spaces that have been built from spheres and disks using colimits. We will argue later that, for our purposes, there is almost no loss of generality if we restrict our attention to CW complexes. But for now, we simply aim to establish the formalities of their step-by-step construction, study some important examples, and derive some useful basic results about them.

3.2.1. CW Complexes and Cellular Maps. To begin with, we say that a space X is a **CW complex** with **dimension zero** if and only if it has the discrete topology. Inductively, suppose X_n is a CW complex with dimension at most n. Let $\alpha_n : \coprod S^n \to X_n$ be any map from a disjoint union of copies of S^n to X_n (possibly an empty disjoint union!), and define X_{n+1} to be the pushout in the square

$$\begin{tikzcd}
\coprod S^n \arrow[r, "\coprod i"] \arrow[d, "\alpha_n"'] \arrow[dr, phantom, "\text{pushout}"] & \coprod D^{n+1} \arrow[d] \\
X_n \arrow[r, "j_n"] & X_{n+1},
\end{tikzcd}$$

where $i : S^n \hookrightarrow D^{n+1}$ is the inclusion of the boundary of the disk. Then X_{n+1}, or anything homeomorphic to X_{n+1}, is a **CW complex** with **dimension** at most $n+1$. This inductively defines what we mean by a finite-dimensional **CW complex**.

Suppose we apply this construction infinitely many times, yielding a sequence of CW complexes $X_0, X_1, \ldots, X_n, \ldots$ and maps $j_n : X_n \to X_{n+1}$ between them. Then we may form the diagram

$$X_0 \xrightarrow{j_0} X_1 \to \cdots \to X_n \xrightarrow{j_n} X_{n+1} \to \cdots .$$

The colimit X of this diagram, or anything homeomorphic to X, is called a **CW complex**. The subspace $X_n \subseteq X$ is called the n-**skeleton** of X.

The disks D^n are called the (closed) n-**cells** of X. Each n-cell of a CW complex X has a **characteristic map** $\chi : D^n \to X$ defined by the diagram

$$\begin{tikzcd}
 & D^n \arrow[d, hook] \arrow[ddr, bend left, "\chi"] & \\
\coprod_k S^n \arrow[r, "\coprod_k i"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{pushout}"] & \coprod_k D^{n+1} \arrow[d] & \\
X_n \arrow[r, "j_n"] & X_{n+1} \arrow[r] & X.
\end{tikzcd}$$

The images $\chi(\mathrm{int}(D^n))$ of the interiors of the n-cells under the characteristic maps are called **open cells** of X. A **subcomplex** of a CW complex X is a subspace $K \subseteq X$ which is a CW complex constructed by using some, but not necessarily all, of the cells used to construct X. For example, each skeleton $X_n \subseteq X$ is a subcomplex of X.

Problem 3.4. Show that any two open cells of a CW complex X are disjoint and that X is the union of its open cells.

Exercise 3.5. Is the union of any collection of open cells in a CW complex a subcomplex?

If X and Y are CW complexes and we have a commutative diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_n \arrow[r] \arrow[d] & X_{n+1} \arrow[r] \arrow[d] & \cdots \\ Y_0 \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_n \arrow[r] & Y_{n+1} \arrow[r] & \cdots, \end{tikzcd}$$

then, on forming the colimits, we obtain an induced map $f : X \to Y$. Maps of this kind, which restrict to maps $X_n \to Y_n$ for each n, are called **cellular maps**.

Exercise 3.6.

(a) Show that if $A \subseteq X$ is a subcomplex, then the inclusion $A \hookrightarrow X$ is a cellular map.

(b) Find an example of a map between CW complexes that is *not* cellular.

(c) Criticize[1] the following argument:

> *If $f : X \to Y$ is cellular, then it carries cells to cells, so the preimage of a cell of Y is a union of cells of X, and hence if $K \subseteq Y$ is a subcomplex, then $f^{-1}(K)$ is a subcomplex of X.*

Two CW complexes that are different—in the sense that the list of attaching maps are not *exactly the same*—can turn out to be homeomorphic to the same space X. We refer to these different CW complexes as being different **CW structures**, or **CW decompositions**, for the same *space* X. It is often helpful to choose a CW structure that is well suited to the work at hand. For example, a subspace A of a CW complex X may be a subcomplex of X if we choose the structure of X properly.

Exercise 3.7. Let D be the union of a finite collection of pairwise disjoint closed intervals in $\mathbb{R}$.

(a) Show that D is a CW complex.

(b) Show that $\mathbb{R}$ has a CW decomposition such that D is a subcomplex.

Exercise 3.8.

(a) Explain how to derive a CW decomposition of S^2 from a convex polyhedron.

[1] This means: decide if you believe the argument; if you do, decide if it is complete, and fill in any missing details; if you do not, explain where the reasoning is faulty. Do you believe the conclusion? Prove it or find a counterexample, accordingly.

(b) Find a CW decomposition of S^n having exactly two cells of each dimension $k \leq n$.

(c) Find a CW decomposition of S^n with a grand total of two cells.

(d) Describe the subcomplexes of S^n in each structure.

3.2.2. Some Topology of CW Complexes. The topology of a CW complex is entirely determined by its characteristic maps.

Theorem 3.9. *A CW complex X has the unique largest topology so that all of its characteristic maps $\chi : D^n \to X$ are continuous.*

In other words, the map $(\chi_i) : \coprod_{i \in \mathcal{I}} D^{n_i} \to X$, where the cells of X are indexed by $i \in \mathcal{I}$, is a quotient map.[2]

Problem 3.10. Let X be a CW complex.

(a) Let $f : X \to Y$. Show that f is continuous if and only if $f|_{X_n}$ is continuous for each n. Conclude that it suffices to prove Theorem 3.9 under the assumption that X is finite-dimensional.

(b) Let $f : X_n \to Y$. Show that f is continuous if and only if $f|_{X_{n-1}}$ is continuous and $f \circ \chi$ is continuous for all characteristic maps χ of n-cells.

(c) Prove Theorem 3.9.

Problem 3.11. Let X be a CW complex, and let $A \subseteq X$ be a subcomplex.

(a) Show that A is a closed subspace.

(b) Show that X/A, with the quotient topology, inherits the structure of a CW complex from X in such a way that the quotient map $q : X \to X/A$ is cellular.

We say that a CW complex X is **finite** if it has only finitely many cells. For CW complexes, finiteness is intimately bound up with compactness.

Problem 3.12. Show that every finite CW complex is compact.

The converse is also true. To see this, we will show that an infinite sequence of points from different open cells in X cannot have a limit point in X. If X is an infinite CW complex, then we may choose points $x_1, x_2, \ldots$ so that no two points are in the interior of the same cell.

The basis of our proof the is following construction for building closed subspaces of a CW complex.

[2]A map $f : X \to Y$ is called a **quotient map** if for any $g : Y \to Z$, g is continuous if and only if $g \circ f$ is continuous.

Problem 3.13. Let $\{\chi_i : D^{n_i} \to X\}$ be an arbitrary collection of characteristic maps for the CW complex X. Choose numbers $0 \le r_i < 1$ for each i, and let $E_i = \chi_i(r_i \cdot D^{n_i})$ denote the image of the subdisk of radius r_i. Show that $\bigcup_i E_i$ is closed in X.

Problem 3.14. Let X be a CW complex and let $\{x_i\}$ be as above.

(a) If x is a limit point of the sequence $\{x_i\}$, then $x \in X_n - X_{n-1}$ for some n. Show that x must be a limit point of the subsequence $\{x_i\} \cap (X_n - X_{n-1})$.

(b) Show that the subsequence cannot have a limit point.

Problem 3.14 proves the following very powerful statement.

Theorem 3.15. *If X is a CW complex, then a subset $A \subseteq X$ is compact if and only if it is closed and is contained in a finite subcomplex of X.*

Theorem 3.16. *Prove Theorem 3.15.*

3.2.3. Products of CW Complexes. Let $\{\chi_i : D^{n_i} \to X \mid i \in \mathcal{I}\}$ and $\{\chi_j : D^{m_j} \to Y \mid j \in \mathcal{J}\}$ be the complete lists of characteristic maps for the CW complexes X and Y. Then for each $i \in \mathcal{I}$ and $j \in \mathcal{J}$, we have the product map

$$\chi_i \times \chi_j : D^{n_i} \times D^{m_j} \longrightarrow X \times Y.$$

It is not hard to see that $D^n \times D^m \cong D^{n+m}$, so it makes sense to ask whether these maps could be the characteristic maps for a CW structure on $X \times Y$.

Problem 3.17. Show that there are homeomorphisms making the diagram

$$\begin{tikzcd}
(S^{n-1} \times D^m) \cup (D^n \times S^{m-1}) \arrow[r, "\cong"] \arrow[d] & S^{n+m-1} \arrow[d] \\
D^n \times D^m \arrow[r, "\cong"] & D^{n+m}
\end{tikzcd}$$

commutative.

Problem 3.18. Define maps $\chi_{i,j} : D^{n_i+m_j} \to X \times Y$ to be the composites

$$D^{n_i+m_j} \xrightarrow{\cong} D^{n_i} \times D^{m_j} \xrightarrow{\chi_i \times \chi_j} X \times Y$$

and let $(X \times Y)_k = \bigcup_{n_i+m_j \le k} \chi_{i,j}(D^{n_i+m_j})$. Show that for each k the square

$$\begin{tikzcd}
\coprod_{n_i+m_j=k} S^{k-1} \arrow[r] \arrow[d] & \coprod_{n_i+m_j=k} D^k \arrow[d, "\chi_{i,j}"] \\
(X \times Y)_{k-1} \arrow[r] & (X \times Y)_k
\end{tikzcd}$$

commutes and that $X \times Y = \bigcup_k (X \times Y)_k$.

This problem suggests that the product $X \times Y$ inherits the structure of a CW complex from its factors. The only question is whether or not the squares in Problem 3.18 are pushout squares. In any case, these squares show that the set $X \times Y$ has a unique topology which makes it into a CW complex with the given cells—let us call that complex the **CW product** of X and Y, which we will denote $X \times_{\mathrm{CW}} Y$. The identity map of sets defines **comparison maps**

$$\xi : X \times_{\mathrm{CW}} Y \longrightarrow X \times Y \qquad \text{and} \qquad \xi^{-1} : X \times Y \longrightarrow X \times_{\mathrm{CW}} Y$$

between the CW product and the categorical product (given by the formula $(x, y) \mapsto (x, y)$). Is the CW product $X \times_{\mathrm{CW}} Y$ of X and Y homeomorphic to the ordinary categorical product of X and Y?

Problem 3.19. Let X and Y be two CW complexes.

(a) Show that ξ is continuous.

(b) Show that ξ is a homeomorphism if X and Y are finite CW complexes.

The general question of the topology of products of CW complexes in the category **Top** is actually quite delicate. Dowker [**56**] has shown that if the complexes involved have more than countably many cells, then the comparison map $\xi : X \times_{\mathrm{CW}} Y \to X \times Y$ need not be a homeomorphism. One of the desirable features of the category of spaces in which we will do the vast majority of our work is that, there, the CW product agrees with the categorical product (see Section 3.4).

3.3. Example: Projective Spaces

In this section we'll build CW decompositions for the projective spaces over the fields $\mathbb{R}$, $\mathbb{C}$ and over the 'skew field' $\mathbb{H}$ of quaternions. These spaces are central examples in homotopy theory, and they feature prominently in its applications to other fields.

Projective spaces are topologized as quotient spaces of spheres, and the development is essentially identical for each of the three families. To give a unified account, we write $\mathbb{F}$ to denote any of $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and we let d be the dimension of $\mathbb{F}$ as a real vector space; thus $d = 1, 2$ or 4. Since $\mathbb{F}$ is a normed $\mathbb{R}$-algebra without zero divisors, the sphere $S^{d-1} \subseteq \mathbb{F}$ is a group under multiplication.[3]

3.3.1. Projective Spaces. Temporarily disregarding the topology, the n-dimensional **projective space** over $\mathbb{F}$ is the set

$$\mathbb{F}\mathrm{P}^n = \{\ell \mid \ell \text{ is a 1-dimensional vector subspace of } \mathbb{F}^{n+1}\}$$

[3] EXERCISE. Explain this sentence.

of lines in $\mathbb{F}^{n+1}$ (for $\mathbb{F} = \mathbb{H}$, we use the structure of $\mathbb{H}^n$ as a *right* $\mathbb{H}$-module). We topologize $\mathbb{F}\mathrm{P}^n$ by giving it the largest topology for which the function

$$q_n : S^{(n+1)d-1} \longrightarrow \mathbb{F}\mathrm{P}^n \qquad \text{given by} \qquad q_n : x \mapsto \mathrm{span}_{\mathbb{F}}(x)$$

is continuous; in other words, we topologize $\mathbb{F}\mathrm{P}^n$ as a quotient space of $S^{(n+1)d-1}$.

Problem 3.20. The group $S^{d-1} \subseteq \mathbb{F}$ acts on $S^{(n+1)d-1} \subseteq \mathbb{F}^{n+1}$ by coordinatewise (left) multiplication. Show that $\mathbb{F}\mathrm{P}^n$ is homeomorphic to the space $S^{(n+1)d-1}/S^{d-1}$ of orbits of the action.

This means, in particular, that $\mathbb{R}\mathrm{P}^n$ is the result of identifying each point $x \in S^n$ with its **antipode** $-x \in S^n$.

Exercise 3.21. What is $\mathbb{R}\mathrm{P}^1$? Do you recognize $\mathbb{R}\mathrm{P}^2$?

We will write $[x_1, \ldots, x_{n+1}] \in \mathbb{F}\mathrm{P}^n$ to denote the equivalence class of $(x_1, \ldots, x_{n+1}) \in S^{(n+1)d-1}$. Thus the quotient map $q_n : S^{nd+(d-1)} \to \mathbb{F}\mathrm{P}^n$ is given by the formula $(x_1, \ldots, x_{n+1}) \mapsto [x_1, \ldots, x_{n+1}]$. The description of points in $\mathbb{F}\mathrm{P}^n$ in this way is called representation by **homogeneous coordinates**.

The inclusions $\mathbb{F}^n \to \mathbb{F}^{n+1}$ given by $x \mapsto (x, 0)$ respect the action of S^{d-1}, and so they give rise to commutative diagrams

$$\begin{tikzcd}
S^{(n+1)d-1} \arrow[r] \arrow[d, "q_n"'] & S^{(n+2)d-1} \arrow[d, "q_{n+1}"] \\
\mathbb{F}\mathrm{P}^n \arrow[r] & \mathbb{F}\mathrm{P}^{n+1}
\end{tikzcd}$$

where the horizontal maps are given by the formulas $x \mapsto (x, 0)$ and $[x] \to [x, 0]$. These maps fit into a commutative ladder

$$\begin{tikzcd}
S^{d-1} \arrow[r] \arrow[d, "q_0"'] & S^{2d-1} \arrow[r] \arrow[d, "q_2"] & \cdots \arrow[r] & S^{nd-1} \arrow[r] \arrow[d, "q_{n-1}"] & S^{(n+1)d-1} \arrow[r] \arrow[d, "q_n"] & \cdots \\
\mathbb{F}\mathrm{P}^0 \arrow[r] & \mathbb{F}\mathrm{P}^1 \arrow[r] & \cdots \arrow[r] & \mathbb{F}\mathrm{P}^{n-1} \arrow[r] & \mathbb{F}\mathrm{P}^n \arrow[r] & \cdots .
\end{tikzcd}$$

The colimit of the bottom row of the ladder is the **infinite-dimensional projective space** $\mathbb{F}\mathrm{P}^\infty$. The colimit of the top is the **infinite-dimensional sphere** S^∞.

3.3.2. Cellular Decomposition of $\mathbb{F}\mathrm{P}^n$.

We will explicitly describe extremely efficient CW decompositions for projective spaces. We begin by identifying a useful nd-dimensional cell inside of $S^{(n+1)d-1}$.

Problem 3.22. Inside of $S^{nd+(d-1)} \subseteq \mathbb{F}^{n+1}$ is the subset

$$E = \{(x_1, \ldots, x_{n+1}) \in S^{nd+(d-1)} \mid x_{n+1} \in \mathbb{R}_{\geq 0}\}.$$

Let $D^{nd} \subseteq \mathbb{F}^n$ be the standard nd-dimensional disk, and define $f : D^{nd} \to E$ by the formula $f(x) = (x, \sqrt{1 - |x|^2})$.

(a) Show that each $x \in S^{nd+(d-1)}$ with $x_{n+1} \neq 0$ is equivalent to exactly one element of $E \subseteq S^{nd+(d-1)}$.

(b) Show that f is a homeomorphism.

(c) Show that the boundary of E is $S^{dn-1} \subseteq \mathbb{F}^n \oplus 0 \subseteq \mathbb{F}^n \oplus \mathbb{F}^1$.

Problem 3.23. Referring to the action of S^{d-1} on $S^{nd+(d-1)}$, let's say $x \sim y$ if and only if x and y are in the same orbit.

(a) Show that every x is $\sim$-equivalent to a point in E; conclude that the quotient map $q_n|_E : E \to \mathbb{F}\mathrm{P}^n$ is surjective.

(b) Show that if $x_{n+1} \neq 0$, then x is equivalent to a **unique** point in the interior of E; conclude that $q_n|_{\mathrm{int}(E)}$ is injective.

These problems contain all the hard work in the construction of our CW decompositions of $\mathbb{F}\mathrm{P}^n$.

Theorem 3.24. *For every n, $\mathbb{F}\mathrm{P}^n$ is the pushout in the square*

$$\begin{tikzcd} S^{nd-1} \arrow[r, "q_{n-1}"] \arrow[d, "i"'] \arrow[dr, phantom, "\text{pushout}"] & \mathbb{F}\mathrm{P}^{n-1} \arrow[d] \\ D^{nd} \arrow[r] & \mathbb{F}\mathrm{P}^n. \end{tikzcd}$$

Consequently $\mathbb{F}\mathrm{P}^n$ has a CW decomposition of the form

$$\mathbb{F}\mathrm{P}^n = * \cup D^d \cup D^{2d} \cup D^{3d} \cup \cdots \cup D^{nd}$$

and if $m \leq n$, then $\mathbb{F}\mathrm{P}^m$ is a subcomplex of $\mathbb{F}\mathrm{P}^n$.

Problem 3.25. Prove Theorem 3.24.

Problem 3.26. Show that $\mathbb{F}\mathrm{P}^1 \cong S^d$.

3.4. Topological Spaces

We will need to talk about pointed topological spaces and unpointed topological spaces; and sometimes we will prove results that work equally well in both contexts. Therefore, we will use the following notation:

- $\mathcal{T}_\circ$ will denote our category of unpointed spaces,
- $\mathcal{T}_*$ is our category of pointed spaces,[4] and
- $\mathcal{T}$ is either one.

[4]Which will be defined in Section 3.6.

3.4.1. Mapping Spaces. Some of the main properties that we require involve the space of maps from one space to another. If X and Y are two topological spaces, then we can study the set

$$\mathrm{mor}_{\mathbf{Top}}(X,Y) = \{f : X \to Y \mid f \text{ is continuous}\}$$

of all continuous maps from X to Y. It is customary to give this set the compact-open topology, and this is indeed the 'correct' topology for a great many spaces. In the next section, we will articulate a theorem that says (in part) that $\mathrm{mor}_{\mathbf{Top}}(X,Y)$ can be given a topology that serves our purposes.

A great deal of the structure of mapping spaces is revealed by the study of three very important kinds of maps between them. Like any collection of functions, $\mathrm{mor}_{\mathbf{Top}}(X,Y)$ automatically comes with an **evaluation map**

$$@ : \mathrm{mor}_{\mathbf{Top}}(X,Y) \times X \longrightarrow Y$$

given by the formula $(f,x) \mapsto f(x)$. Function composition defines a function

$$\circ : \mathrm{mor}_{\mathbf{Top}}(X,Y) \times \mathrm{mor}_{\mathbf{Top}}(Y,Z) \longrightarrow \mathrm{mor}_{\mathbf{Top}}(X,Z)$$

explicitly given by $(f,g) \mapsto g \circ f$. Finally, we have a bijection

$$\alpha : \mathrm{mor}_{\mathbf{Top}}(X \times Y, Z) \longrightarrow \mathrm{mor}_{\mathbf{Top}}(X, \mathrm{mor}_{\mathbf{Top}}(Y,Z))$$

given by the formula $\alpha(f) : x \mapsto \boxed{y \mapsto f(x,y)}$. The maps f and $\alpha(f)$ are frequently referred to as being **adjoint** to one another.

Problem 3.27. Prove that α is a bijection.

3.4.2. The Category of Unpointed Spaces. The following theorem asserts that there is a category which is good enough for our work.

Theorem 3.28. *There is a category $\mathcal{T}_\circ$ whose objects are topological spaces*[5] *and whose morphisms are*

$$\mathrm{mor}_{\mathcal{T}_\circ}(X,Y) = \mathrm{mor}_{\mathbf{Top}}(X,Y),$$

which contains every locally compact Hausdorff space and which has the following properties:

(a) The set $\mathrm{mor}_{\mathcal{T}_\circ}(X,Y)$ has a topology giving it the structure of a space, denoted $\mathrm{map}_\circ(X,Y)$, such that for all $X,Y,Z \in \mathcal{T}_\circ$,
  - *(1) $\mathrm{map}_\circ(X,Y) \in \mathcal{T}_\circ$,*
  - *(2) the evaluation map $@ : \mathrm{map}_\circ(X,Y) \times X \to Y$ is continuous,*
  - *(3) the composition map $\circ : \mathrm{map}_\circ(X,Y) \times \mathrm{map}_\circ(Y,Z) \to \mathrm{map}_\circ(X,Z)$ is continuous, and*
  - *(4) the function $\alpha : \mathrm{map}_\circ(X\times Y,Z) \to \mathrm{map}_\circ(X,\mathrm{map}_\circ(Y,Z))$ is a homeomorphism.*

(b) Every diagram of spaces in $\mathcal{T}_\circ$ has

[5] But not *all* topological spaces!

(1) a colimit in $\mathcal{T}_\circ$ and

(2) a limit in $\mathcal{T}_\circ$;

*furthermore, the forgetful functor $\mathcal{T}_\circ \to$ **Sets** respects limits.*

(c) If $X, Y \in \mathcal{T}_\circ$ are CW complexes, then the CW product $X \times_{\mathrm{CW}} Y$ is equal to the categorical product $X \times Y$ in $\mathcal{T}_\circ$.

Starting in the early 1960s, a number of authors attempted to establish 'convenient' categories of topological spaces for homotopy theory. J. Milnor [**132**] showed that the category of spaces homotopy equivalent to a CW complex has many nice properties, but it is not closed under the formation of mapping spaces. Steenrod argued persuasively in favor of the category **CG** of Hausdorff compactly generated spaces [**161**]. A space X is **compactly generated** if a map $f : X \to Y$ is continuous if and only if for every continuous $K \to X$ with K compact, the composition $K \to X \to Y$ is continuous. But there is a problem with **CG**, namely, a quotient of a space in **CG** is not necesarily in **CG**. A short time later McCord [**121**] realized that this problem disappears if 'Hausdorff' is replaced with 'weak Hausdorff'.[6] Thus our category may be taken to be the category of weak Hausdorff compactly generated spaces; Vogt [**174**] has shown that there are many such categories, some of which have additional useful properties.

The proof of Theorem 3.28 belongs entirely to point-set topology, so we will simply take it for granted. You should be aware, however, that in the category $\mathcal{T}_\circ$, the various categorical constructions, such as mapping spaces, products, etc., may be given slightly different topologies than those described in an introductory topology course. For example, the product of two compactly generated spaces need not be compactly generated, but there is a comparison map

$$X \times_{\mathbf{CG}} Y \longrightarrow X \times Y$$

where $X \times_{\mathbf{CG}} Y$ is the set $X \times Y$ with the topology determined by the maps $K \to X \times Y$ with K compact. It turns out that in the category $\mathcal{T}_\circ$, the space $X \times_{\mathbf{CG}} Y$ is the categorical product, and the same set with the classical 'product topology' is not.

If this kind of tampering with definitions makes you uneasy, consider that, for example, the usual product topology is defined precisely to make $X \times Y$ into the categorical product in the category **Top** of all spaces and not for any other overarching reason.

The canonical isomorphism $\mathrm{map}_\circ(X \times Y, Z) \xrightarrow{\cong} \mathrm{map}_\circ(X, \mathrm{map}_\circ(Y, Z))$ is known as the **exponential law**.[7] The reason for this terminology lies

[6] A space X is **weak Hausdorff** if whenever $f : K \to X$ and K is compact and Hausdorff, $f(X) \subseteq X$ is closed in X.

[7] Also called **currying** by computer scientists.

in an alternative notation for mapping spaces: $\mathrm{map}_\circ(X,Y) = X^Y$.[8] Using this notation, the exponential law reads $Z^{X\times Y} = (Z^Y)^X$. This rule plays a crucial role throughout homotopy theory.

We'd better check that the category $\mathcal{T}_\circ$ contains all CW complexes, since if it doesn't, then it is not good enough for us.

Problem 3.29. Show that every CW complex is compactly generated and Hausdorff, and conclude that every CW complex is an object in $\mathcal{T}_\circ$.

In view of Theorem 3.28(c), we never again use the notation $X \times_{\mathrm{CW}} Y$ that we established in Section 3.2.3.

The following lemma is frequently useful in cell-by-cell construction of homotopies.

Lemma 3.30. *Let X be a CW complex, and give $I = [0,1]$ the standard CW decomposition with two zero cells and one 1-cell. Show that in the CW product decomposition,*

$$(X \times I)_{n+1} = (X \times 0) \cup (X_n \times I) \cup (X \times 1) \ \subseteq\ X \times I.$$

Problem 3.31.

(a) Explicitly describe the CW structure of $X \times I$ in terms of the CW structure of X.

(b) Prove Lemma 3.30.

Problem 3.32. Suppose $A, X, Y \in \mathcal{T}_\circ$ and $A \subseteq X$. Show that $X/A \in \mathcal{T}_\circ$ and $X \times Y \in \mathcal{T}_\circ$.

Problem 3.33. Maps $f : X \to Y$ and $h : Y \to Z$ in $\mathcal{T}_\circ$ induce functions

$$f^* : \mathrm{map}_\circ(Y,Z) \longrightarrow \mathrm{map}_\circ(X,Z) \quad \text{and} \quad h_* : \mathrm{map}_\circ(X,Y) \longrightarrow \mathrm{map}_\circ(X,Z)$$

given by $f^*(g) = g\circ f$ and $h_*(g) = h\circ g$. Show that f^* and h_* are continuous.

Hint. Express the function you are interested in as a composition of the map $\circ$ with another function.

Mapping Spaces, Colimits and Limits. We construct new spaces from old ones using colimits and limits. If we hope to understand spaces of maps, then we should try to get a handle on the behavior of the mapping space functors with respect to limits and colimits.

We write $\mathrm{colim}_\circ$ and $\lim_\circ$ to denote colimits and limits formed in the category $\mathcal{T}_\circ$.

[8] In the category of sets and their functions, this is how Cantor defined exponentiation of transfinite cardinals!

Problem 3.34.

(a) Define two functors $F, G : \mathcal{T}_\circ \to \mathcal{T}_\circ$ by the rules

$$F(\,?\,) = \mathrm{map}_\circ(\,?\, \times Y, Z) \qquad \text{and} \qquad G(\,?\,) = \mathrm{map}_\circ(\,?\,, \mathrm{map}_\circ(Y, Z)).$$

Show that the exponential law defines a natural equivalence between these two functors. That is, use the exponential law to define a natural transformation $\Phi : F \to G$ and demonstrate that for every space $X \in \mathcal{T}_\circ$, Φ_X is an equivalence.

(b) Define functors $P, M : \mathcal{T}_\circ \to \mathcal{T}_\circ$ by the rules

$$P(\,?\,) = \,?\, \times Y \qquad \text{and} \qquad M(\,?\,) = \mathrm{map}_\circ(Y, \,?\,).$$

Show P and M are adjoint functors.

(c) Conclude that $\mathrm{map}_\circ(Y, \,?\,)$ commutes with limits and that $\,?\, \times Y$ commutes with colimits.

This leaves us wondering about the functor $\mathrm{map}_\circ(X, \,?\,)$.

Theorem 3.35.

(a) If $Y \in \mathcal{T}_\circ$ and $F : \mathcal{I} \to \mathcal{T}_\circ$, then the canonical comparison map

$$\mathrm{map}_\circ(\mathrm{colim}_\circ F, Y) \longrightarrow \lim_\circ \mathrm{map}_\circ(F, Y)$$

is a homeomorphism.[9]

(b) If $X \in \mathcal{T}_\circ$ and $F : \mathcal{I} \to \mathcal{T}_\circ$, then the canonical comparison map

$$\mathrm{map}_\circ(X, \lim_\circ F) \longrightarrow \lim_\circ \mathrm{map}_\circ(X, F)$$

is a homeomorphism.[10]

Problem 3.36.

(a) Show that the problems

$$\begin{tikzcd}
X \times A \arrow[r] \arrow[d] & X \times B \arrow[d] \arrow[ddr, bend left] & \\
X \times C \arrow[r] \arrow[drr, bend right] & X \times D \arrow[dr, dotted] & \\
 & & Y
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
X \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
 & \mathrm{map}_\circ(D, Y) \arrow[r] \arrow[d] & \mathrm{map}_\circ(C, Y) \arrow[d] \\
 & \mathrm{map}_\circ(B, Y) \arrow[r] & \mathrm{map}_\circ(A, Y)
\end{tikzcd}$$

are equivalent.

[9] $\mathrm{map}_\circ(F, Y)$ denotes the composite functor $\mathrm{map}_\circ(\,?\,, Y) \circ F$.

[10] $\mathrm{map}_\circ(X, F)$ denotes the composite functor $\mathrm{map}_\circ(X, \,?\,) \circ F$.

(b) Deduce that if

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a pushout square, then

$$\begin{tikzcd} \mathrm{map}_\circ(D,Y) \arrow[r] \arrow[d] & \mathrm{map}_\circ(C,Y) \arrow[d] \\ \mathrm{map}_\circ(B,Y) \arrow[r] & \mathrm{map}_\circ(A,Y) \end{tikzcd}$$

is also a pullback square.

(c) Generalize the argument to prove Theorem 3.35.

3.5. The Category of Pairs

Suppose $A \subseteq X$ and $B \subseteq Y$. We will sometimes find ourselves interested *only* in those maps $f : X \to Y$ with the additional property that $f(A) \subseteq B$. These can be written as commutative diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ X \arrow[r] & Y, \end{tikzcd}$$

but they are more typically written in the form $f : (X,A) \to (Y,B)$, and (X,A) and (Y,B) are called **pairs** of spaces. The set

$$\begin{aligned} \mathrm{map}_{(2)}((X,A),(Y,B)) &= \{f : (X,A) \to (Y,B)\} \\ &= \{f : X \to Y \mid f(A) \subseteq B\} \end{aligned}$$

is clearly a subset of $\mathrm{map}_\circ(X,Y)$; thus it inherits a topology from $\mathrm{map}_\circ(X,Y)$. There is a category of pairs, which we will denote $\mathcal{T}_{(2)}$.

Proposition 3.37. *The space* $\mathrm{map}_{(2)}((X,A),(Y,B))$ *is in* $\mathcal{T}_\circ$.

Problem 3.38. Prove Proposition 3.37 by showing that

$$\begin{tikzcd} \mathrm{map}_{(2)}((X,A),(Y,B)) \arrow[r] \arrow[d] & \mathrm{map}_\circ(X,Y) \arrow[d] \\ \mathrm{map}_\circ(A,B) \arrow[r] & \mathrm{map}_\circ(A,Y) \end{tikzcd}$$

is a pullback square.

There is an inclusion of categories $\mathcal{T}_\circ \to \mathcal{T}_{(2)}$ given by $X \mapsto (X,\varnothing)$; and there is a forgetful functor $\mathcal{T}_{(2)} \to \mathcal{T}_\circ$ given by $(Y,B) \mapsto Y$.

Problem 3.39. Show that the functors $X \mapsto (X, \varnothing)$ and $(Y, B) \mapsto Y$ are an adjoint pair of functors. Which is the left adjoint and which is the right adjoint?

Any map $f : X \to Y$ such that $f(X) \subseteq B$ is automatically an element of $\mathrm{map}_{(2)}((X, A), (Y, B))$; the space of all such maps is of course (almost) identically equal to $\mathrm{map}_\circ(X, B)$, and we won't belabor the distinction.[11] Thus, we consider space $\mathrm{map}_{(2)}((X, A), (Y, B))$ as the *pair*

$$\Bigg(\underbrace{\mathrm{map}_{(2)}((X, A), (Y, B))}_{\text{big space}},\ \underbrace{\mathrm{map}_\circ(X, B)}_{\text{subspace}} \Bigg).$$

It is customary to define the 'product' of pairs (X, A) and (Y, B) to be the pair

$$\Bigg(\underbrace{X \times Y}_{\text{big space}},\ \underbrace{A \times Y \cup X \times B}_{\text{subspace}} \Bigg).$$

However, as you can easily check, this is *not* the categorical product in the category of pairs.[12] The real importance of this construction is that it fits into the exponential law. In 'abstract' category theory, constructions like this have come to be denoted by $\otimes$, since the tensor product plays the same role in an exponential law for R-modules. But this notation seems to be a bit of a stretch in our context, because it applies a typically algebraic notation to spaces. So we'll use some new notation for the 'product' pair:

$$(X, A) \boxplus (Y, B) = (X \times Y, X \times B \cup A \times Y).$$

This notation was handy, not in use for anything that I have seen, and evokes Figure 3.1, which shows the location of $X \times B \cup A \times Y$ inside of $X \times Y$.

With these preliminaries, you can generalize some of the results of Theorem 3.28 to maps of pairs.

Problem 3.40.

(a) Show that the exponential law holds for maps of pairs.

(b) Show that the exponential law is a natural isomorphism for pairs.

(c) Show that the composition function $\circ$ is well-defined and continuous for maps of pairs.

[11] Exercise. What is the distinction?

[12] Exercise. Check it! What is the product in the category of pairs?

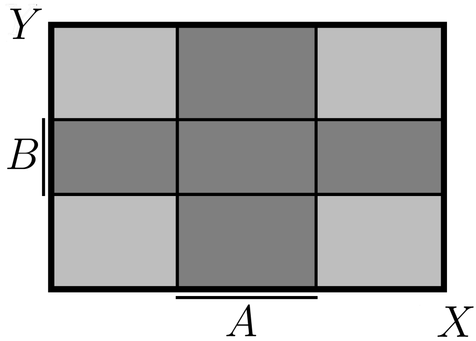

Figure 3.1. Product of pairs

Relative CW Complexes and CW Pairs. A **relative CW complex** is a pair (X, A) that is built according to the procedure of Section 3.2, but starting with '(-1)-skeleton' A, and no conditions are imposed on A (except that it should be in $\mathcal{T}_\circ$). Thus $X_0 = A \sqcup D$ where D is discrete, and X_{n+1} is built from X_n by a pushout square

$$\begin{tikzcd} \coprod S^n \arrow[r, "\coprod i"] \arrow[d, "\alpha_n"'] \arrow[dr, phantom, "\text{pushout}"] & \coprod D^{n+1} \arrow[d] \\ X_n \arrow[r, "j_n"] & X_{n+1}. \end{tikzcd}$$

Finally, X is the colimit of the telescope $\cdots \to X_n \to X_{n+1} \to \cdots$. If A is itself a CW complex, then we call (X, A) a **CW pair**.

Any argument that works by induction on the cells of a CW complex works equally well for relative CW complexes, provided the required statement can be established for the space A.

3.6. Pointed Spaces

Far and away, the most important kinds of pairs for us are those in which $A = \{x_0\}$ is just a single point of X and $B = \{y_0\}$ is a single point of Y; these special pairs are called **pointed spaces**, and the points $x_0 \in X$ and $y_0 \in Y$ are called the **basepoints** of X and Y. We usually denote the basepoint by $*$, no matter what space it is in. Maps of the form $f : (X, *) \to (Y, *)$ are called **pointed maps** (or **based maps**). When we work with pointed spaces, we almost always suppress the pair notation. Thus, we will simply write X for a pointed space, and we will know that it has a basepoint and that the basepoint is denoted $*$. We write $\mathcal{T}_*$ for the category of all pointed spaces $(X, *)$ such that the unpointed space $X \in \mathcal{T}_\circ$.

Pointed CW Complexes. In pointed CW complexes, we usually assume that the basepoint is a vertex—a point of the 0-skeleton X_0. Then the CW skeleta X_n are also given the same basepoint, so the inclusions $X_n \hookrightarrow X$ are pointed maps.

In a pointed *relative* CW complex, we take the basepoint $*$ to be a point of A, so that the inclusion $A \hookrightarrow X$ is a pointed map.

3.6.1. Pointed Mapping Spaces. When we work in $\mathcal{T}_*$, we use a simplification of the mapping space notation: instead of $\mathrm{map}_{(2)}((X, *), (Y, *))$, we write

$$\mathrm{map}_*(X, Y) = \{f : (X, *) \to (Y, *)\}$$

and call it the **space of pointed maps** from X to Y.

Exercise 3.41. According to the conventions established in Section 3.5, the mapping space $\mathrm{map}_*(X, Y)$ is a pointed space; what is the basepoint?

There is a way to go from pairs to pointed spaces: simply collapse the subspace to a point—i.e., replace (X, A) with $(X/A, *)$. Notice that the natural quotient map $q : (X, A) \to (X/A, *)$ is a map of pairs.

Theorem 3.42. *The map* $q^* : \mathrm{map}_*(X/A, Y) \to \mathrm{map}_{(2)}((X, A), (Y, *))$ *is a homeomorphism.*

Corollary 3.43. *The functor* $(X, A) \mapsto X/A$ *is left adjoint to the inclusion* $\mathcal{T}_* \hookrightarrow \mathcal{T}_{(2)}$.

Problem 3.44. Show that the induced map

$$q^* : \mathrm{map}_*(X/A, Y) \longrightarrow \mathrm{map}_{(2)}((X, A), (Y, *))$$

is a homeomorphism.

HINT. Use Theorem 3.35.

3.6.2. Products of Pointed Spaces. If we use the construction from the previous section, we find that the 'product' of a pair of pointed spaces is not a pointed space:

$$(X, *) \boxplus (Y, *) = (X \times Y, X \times * \cup * \times Y).$$

The second term of this pair is the space obtained from the disjoint union of X and Y by identifying their basepoints; it is called the **wedge** of X and Y and is denoted $X \vee Y$. We are led to two related, but very different, kinds of products for pointed spaces.

(1) The categorical product of pointed spaces X and Y is the ordinary product $X \times Y$, with basepoint $* \times *$.

(2) The **smash product** of pointed space X and Y is the result of collapsing the wedge to a point: $X \wedge Y = (X \times Y)/(X \vee Y)$.

Problem 3.45.

(a) Show that $X \vee Y$ is the categorical sum of X and Y in the category $\mathcal{T}_*$.

(b) Show that the pointed space $X \times Y$ is the categorical product of X and Y in the category $\mathcal{T}_*$.

(c) Show that the rule $X \mapsto X \wedge A$ is a functor. Show that the three-variable functors $(X \wedge Y) \wedge Z$ and $X \wedge (Y \wedge Z)$ are naturally equivalent.

When these constructions are applied to CW complexes, they return CW complexes.

Problem 3.46. Suppose X and Y are pointed CW complexes. Show that $X \vee Y$ is a subcomplex of $X \times Y$. Explicitly relate the cells of $X \vee Y$ and $X \wedge Y$ to the cells of X, Y and $X \times Y$.

3.6.3. The Category of Pointed Spaces. We want to say that the pointed version of Theorem 3.28 is valid in the category $\mathcal{T}_*$, but this requires some thought to formulate properly. It is not hard to formulate and verify the pointed versions of this theorem, except for the fourth part of (a). For this we have a natural homeomorphism

$$\alpha : \mathrm{map}_{(2)}((X,*) \boxplus (Y,*), (Z,*)) \to \mathrm{map}_{(2)}((X,*), \mathrm{map}_{(2)}((Y,*),(Z,*))).$$

The problem we have is that the domain in the first mapping space is not a pointed space—it is the pair $(X \times Y, X \vee Y)$. The way out is to collapse the wedge $X \vee Y$ to a point and use Problem 3.44. This gives us a new map

$$\begin{tikzcd}
\mathrm{map}_*(X \wedge Y, Z) \arrow[d, "q^*"', "\cong"] \arrow[dr, bend left, "\widetilde{\alpha}"] & \\
\mathrm{map}((X,*) \boxplus (Y,*), (Z,*)) \arrow[r, "\alpha", "\cong"'] & \mathrm{map}_*(X, \mathrm{map}_*(Y,Z)).
\end{tikzcd}$$

The pointed version of Theorem 3.28(a)(4) asserts that the map

$$\widetilde{\alpha} : \mathrm{map}_*(X \wedge Y, Z) \longrightarrow \mathrm{map}_*(X, \mathrm{map}_*(Y,Z))$$

is a homeomorphism. Now that we know what we mean, we can make the following assertion.

Theorem 3.47. *The category $\mathcal{T}_*$ of pointed spaces in $\mathcal{T}_\circ$ has the following properties:*

(a) For any $X, Y, Z \in \mathcal{T}_*$,
  (1) $\mathrm{map}_*(X,Y) \in \mathcal{T}_*$,
  (2) the evaluation map $@ : \mathrm{map}_*(X,Y) \wedge X \to Y$ is continuous,
  (3) the composition map $\circ : \mathrm{map}_*(X,Y) \wedge \mathrm{map}_*(Y,Z) \to \mathrm{map}_*(X,Z)$ is continuous, and

(4) *the function* $\alpha : \mathrm{map}_*(X \wedge Y, Z) \to \mathrm{map}_*(X, \mathrm{map}_*(Y, Z))$ *is a homeomorphism.*

(b) *Every diagram of spaces in* $\mathcal{T}_*$ *has*
  (1) *a colimit in* $\mathcal{T}_*$ *and*
  (2) *a limit in* $\mathcal{T}_*$;

furthermore, the forgetful functor $F : \mathcal{T}_* \to \mathcal{T}_\circ$ *respects limits.*

(c) *If* $X, Y \in \mathcal{T}_*$ *are pointed CW complexes, then the CW product* $X \times_{\mathrm{CW}} Y$ *is equal to the categorical product* $X \times Y$.

A diagram $F : \mathcal{I} \to \mathcal{T}_*$ of pointed spaces gives rise, by forgetting basepoints, to a diagram $F_- : \mathcal{I} \to \mathcal{T}_\circ$. Let $T_\mathcal{I} : \mathcal{I} \to \mathcal{T}_\circ$ be the trivial diagram with $T_\mathcal{I}(i) = *$ for all $i \in \mathcal{I}$; the inclusions of the basepoints give rise to a morphism of diagrams $T_\mathcal{I} \to F_-$.

Exercise 3.48. Show that the limit of $T_\mathcal{I}$ is simply $*$.

The limit of the pointed diagram F is the pointed space $\lim_* F$ defined by the commutative diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d, equal] & \lim_* F \arrow[d, equal] \\ \lim_\circ T_\mathcal{I} \arrow[r] & \lim_\circ F_- \end{tikzcd}$$

in $\mathcal{T}_\circ$. The colimit of $T_\mathcal{I} : \mathcal{I} \to \mathcal{T}_\circ$ can be more complicated than a single point, so the definition of the pointed colimit of F is slightly more complicated: it is the pointed space $\mathrm{colim}_* F$ defined by the pushout square

$$\begin{tikzcd} \mathrm{colim}_\circ T_\mathcal{I} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & \mathrm{colim}_\circ F_- \arrow[d] \\ * \arrow[r] & \mathrm{colim}_* F \end{tikzcd}$$

in $\mathcal{T}_\circ$.

3.7. Relating the Categories of Pointed and Unpointed Spaces

It sometimes happens that we have a problem in $\mathcal{T}_\circ$ and we would like to study it using functors that are defined on $\mathcal{T}_*$, or vice versa. To do this, we need to clarify the relationship between these two categories. The categories $\mathcal{T}_\circ$ and $\mathcal{T}_*$ are related to one another in two ways.

Given $X \in \mathcal{T}_\circ$, we form a pointed space

$$X_+ = X \sqcup *, \qquad \text{with basepoint} \quad *$$

by adding in a disjoint basepoint. Going the other way, a pointed space Y may be made into an unpointed space Y_- by simply forgetting the basepoint.

Problem 3.49. Show that the rules $X \mapsto X_+$ and $Y \mapsto Y_-$ are the object parts of an adjoint pair of functors

$$\mathcal{T}_\circ \longrightarrow \mathcal{T}_* \qquad \text{and} \qquad \mathcal{T}_* \longrightarrow \mathcal{T}_\circ.$$

Which is the left adjoint and which is the right adjoint?

3.7.1. Various Pointed and Unpointed Products. There is another kind of product, midway between the ordinary product and the smash product, which is useful enough to be given its own name and notation. If $X \in \mathcal{T}_*$ and $Y \in \mathcal{T}_\circ$, then the **half-smash product** of X with Y is the pointed space

$$X \rtimes Y = \frac{X \times Y}{* \times Y}.$$

The construction $X \rtimes Y$ is clearly functorial in both variables. We will feel free to form the half-smash of two pointed spaces, though formally, $X \rtimes Y$ is really $X \rtimes Y_-$.

Let's investigate the interaction of the functors $X \mapsto X_+$ and $Y \mapsto Y_-$ with products, smash product, half-smash product and mapping spaces.

Exercise 3.50. Assuming X and Y are CW complexes, express $X \rtimes Y$ as a CW complex.

Problem 3.51.

(a) Work out $X_+ \wedge Y_+$ and $X \wedge Y_+$ in $\mathcal{T}_*$.

(b) Express $(X \times Y)_-$ in terms of X_- and Y_-. Also write out $(X \times Y)_+$ in terms of X_+ and Y_+.

(c) Repeat (b) for coproducts.

Pointed Versions of an Unpointed Map. There is another, nonnatural, approach to 'pointing' unpointed maps. Let $f : X \to Y$ be a map in $\mathcal{T}_\circ$. If we want to think of f as a pointed map, we can choose a basepoint $x_0 \in X$; but then in order for f to be a pointed map, we are forced to choose $y_0 = f(x_0)$ as the basepoint of Y. Let's call the resulting pointed map f_{x_0}. Thus the map f corresponds to a huge collection of pointed maps, one for each point $x_0 \in X$.

3.7.2. Some Mixed Adjunctions. We have seen that mapping spaces and products are adjoint for pointed spaces and that pointed mapping spaces and smash products are adjoint in $\mathcal{T}_*$. The half-smash is part of an adjoint pair involving both pointed and unpointed spaces.

Problem 3.52. Show that the exponential law implies that there is a natural isomorphism

$$\mathrm{map}_*(X \rtimes Y, Z) \cong \mathrm{map}_*(X, (\mathrm{map}_\circ(Y, Z), *))$$

for $X, Y, Z \in \mathcal{T}_*$. Conclude that the functors $? \rtimes Y$ and $(\mathrm{map}_\circ(\,?\,, Z), *)$ are an adjoint pair.

Now we can derive some useful results about smashes and pushouts.

Proposition 3.53. *Consider the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

in $\mathcal{T}_*$.

(a) If the diagram is a pushout square, then so are

$$\begin{tikzcd} A \rtimes X \arrow[r] \arrow[d] & B \rtimes X \arrow[d] \\ C \rtimes X \arrow[r] & D \rtimes X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \wedge X \arrow[r] \arrow[d] & B \wedge X \arrow[d] \\ C \wedge X \arrow[r] & D \wedge X \end{tikzcd}$$

for any space $X \in \mathcal{T}_\circ$ (for the first square) or $X \in \mathcal{T}_*$ (for the second one).

(b) If the original diagram is a pullback square, then so are

$$\begin{tikzcd} (\mathrm{map}_\circ(X, A), *) \arrow[r] \arrow[d] & (\mathrm{map}_\circ(X, B), *) \arrow[d] \\ (\mathrm{map}_\circ(X, C), *) \arrow[r] & (\mathrm{map}_\circ(X, D), *) \end{tikzcd}$$

for any space $X \in \mathcal{T}_\circ$, and

$$\begin{tikzcd} \mathrm{map}_*(X, A) \arrow[r] \arrow[d] & \mathrm{map}_*(X, B) \arrow[d] \\ \mathrm{map}_*(X, C) \arrow[r] & \mathrm{map}_*(X, D) \end{tikzcd}$$

for any space $X \in \mathcal{T}_*$.

Problem 3.54. Prove Proposition 3.53 using Theorem 2.19.

Problem 3.55. Show that if

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a pushout square in $\mathcal{T}$, then

$$\begin{array}{ccc} A \times X & \longrightarrow & B \times X \\ \downarrow & & \downarrow \\ C \times X & \longrightarrow & D \times X \end{array}$$

is also a pushout square in $\mathcal{T}$.

(a) Prove it for $\mathcal{T}_\circ$.

(b) Show that the pushout in $\mathcal{T}_\circ$ of $* \leftarrow * \rightarrow *$ is $*$; conclude that the forgetful functor $\mathcal{T}_* \to \mathcal{T}_\circ$ respects pushout. Derive the result for $\mathcal{T}_*$.

Corollary 3.56. *There is a natural equivalence*

$$X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z)$$

in the category $\mathcal{T}_*$.

Problem 3.57. Prove Corollary 3.56.

3.8. Suspension and Loop

In this section we define two fundamental constructions that we'll use almost constantly in this book: the suspension and the loop space.

3.8.1. Suspension. We defined the smash product in the last section, and now we study a particular case of considerable importance: the (reduced) **suspension** of X is $\Sigma X = S^1 \wedge X$.

Problem 3.58.

(a) Using the identification $S^1 \cong I/\{0,1\}$, show that

$$\Sigma X \cong \frac{X \times I}{(X \times \{0\}) \cup (X \times \{1\}) \cup (\{*\} \times I)}.$$

(b) If X is a pointed space, then the inclusion of the basepoint $* \hookrightarrow X$ induces $I \cong \Sigma_\circ(*) \to \Sigma_\circ X$. Show that $\Sigma X \cong \Sigma_0 X/I$.

(c) Suppose X is a CW complex. Describe a CW decomposition of ΣX in terms of the given one for X.

We will often use the identification of Problem 3.58, because it gives us a handy notation for points in ΣX; a typical point can be written as $[x,t]$, the equivalence class of $(x,t) \in X \times I$.

Exercise 3.59. For this problem, we use the spheres $S^n \subseteq \mathbb{R}^{n+1}$ centered at $(\frac{1}{2}, 0, \ldots, 0)$, radius $\frac{1}{2}$ and basepoint at the origin $\mathbf{0}$.

(a) For $\mathbf{x} \in \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^{n+1}$ there is a well-defined circle in the unique 2-plane containing the points $\mathbf{0}, \mathbf{x}$ and $\mathbf{e}_{n+1}$ which has the segment joining $\mathbf{0}$ to $\mathbf{x}$ as a diameter. Find a constant speed parametrization $\alpha_{\mathbf{x}} : I \to \mathbb{R}^{n+1}$ of this circle.

(b) Define $\phi : \Sigma S^n \to S^{n+1}$ by $\phi([\mathbf{x}, t]) = \alpha_{\mathbf{x}}(t)$. Show that ϕ is a homeomorphism $\Sigma S^n \cong S^{n+1}$.

HINT. Draw the picture in the cases $n = 0$ and $n = 1$.

The (reduced) **cone** on a pointed space is the space $CX = X \wedge I$ (we use $1 \in I$ as the basepoint of the interval I). The (reduced) **cylinder** on X is the space $X \rtimes I$ (here the basepoint of I is immaterial). The cone CX comes with a natural inclusion map

$$\mathrm{in}_0 : X \hookrightarrow CX \qquad \text{given by} \qquad x \mapsto [x, 0].$$

There are two natural inclusions $\mathrm{in}_0, \mathrm{in}_1 : X \hookrightarrow X \rtimes I$, given by $\mathrm{in}_0(x) = [x, 0]$ and $\mathrm{in}_1(x) = [x, 1]$.

Exercise 3.60. Draw pictures of cylinders, cones, and the inclusion maps so that you understand why they are named as they are.

Problem 3.61. Show that there is a pushout square

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X. \end{tikzcd}$$

It is up to you to precisely define the maps in this square.

Problem 3.62.

(a) Show that for any $X \in \mathcal{T}_\circ$, $(X_+)_- \cong X \vee S^0$ in $\mathcal{T}_\circ$.

(b) Show that if $X \in \mathcal{T}_*$ and $X \neq *$, then $(X_-)_+ \ncong X \vee S^0$ in $\mathcal{T}_*$.

3.8.2. Loop Spaces. A special case of pointed mapping spaces that is very important is the set of pointed maps from S^1 to X; this particular mapping space is denoted $\Omega(X)$ and is called the **loop space** of X.

Problem 3.63. According to your work in Problems 3.40 and 3.44,

$$\mathrm{map}_*(X, \Omega(Y)) \cong \mathrm{map}_*(\mathcal{Q}, Y)$$

for some pointed space $\mathcal{Q}$. Give an explicit description of the space $\mathcal{Q}$. Use your answer to identify $\mathrm{map}_*(S^n, \Omega(S^m))$ with the space of maps between two familiar spaces.

3.9. Additional Problems and Projects

Project 3.64. Prove Theorem 3.28.

Project 3.65. Work through Dowker's example of CW complexes X and Y so that $X \times Y \neq X \times_{CW} Y$.

Problem 3.66. Show that for $0 \leq k \leq n$, there is a homeomorphism

$$\mathrm{map}_*(S^n, X) \cong \Omega^k(\mathrm{map}_*(S^{n-k}, X)),$$

where Ω^k indicates that the loop space functor has been applied k times.

Problem 3.67. Let X be any space, and let $* \in X$.

(a) Describe $\mathrm{map}_\circ(*, X)$. Explain how to view the evaluation function @ as a special case of the composition function $\circ$.

(b) Determine the mapping spaces

$$\mathrm{map}(S^0, X) \qquad \text{and} \qquad \mathrm{map}_*(S^0, X).$$

In other words, give a complete description of these spaces in terms of X and not including any mapping spaces.

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Chapter 4

Homotopy

Topology can be described as the study of all continuous maps between topological spaces and the composition of those maps with one another: it is the study of the category **Top** (or, for us, the study of $\mathcal{T}_\circ$ and $\mathcal{T}_*$). This is an essentially intractable problem because, for starters, there will generally be uncountably many maps from one space to another. We can simplify the problem by restricting our attention to $\mathcal{T}$ and studying equivalence classes of maps under an equivalence relation called homotopy.

The intuitive idea of homotopy is that we should consider f and g to be equivalent to one another if f can be 'continuously deformed' into g. Homotopy respects composition, so there are 'quotient categories' $\mathrm{H}\mathcal{T}_\circ$ and $\mathrm{H}\mathcal{T}_*$, known as the *homotopy categories* of $\mathcal{T}_\circ$ and $\mathcal{T}_*$, respectively. There are also 'quotient functors' $\mathrm{Ho} : \mathcal{T} \to \mathrm{H}\mathcal{T}$ which carry each map to its equivalence class. Homotopy theory should be regarded as the study of these functors.

4.1. Homotopy of Maps

In this section we give two definitions of homotopy, which are equivalent in our categories $\mathcal{T}$, and develop some basic machinery for manipulating homotopies.

4.1.1. The Deformation Approach. Imagine deforming the function f to the function g starting at time $t = 0$ and finishing up at time $t = 1$. You will have produced a family of functions $f_t \in \mathrm{map}_\circ(X, Y)$ with $f_0 = f$ and $f_1 = g$. To make our concept of deformation precise, we need to explain what is meant by a *continuous* deformation. We need functions that are nearby in 'time' to be nearby as functions. That is, we need the rule $t \mapsto f_t$ to define a continuous function $I \to \mathrm{map}_\circ(X, Y)$.

A continuous function $\omega : I \to Z$ is called a **path** in Z from $\omega(0)$ to $\omega(1)$, and so we say that two maps $f, g \in \mathrm{map}_\circ(X, Y)$ are **homotopic** if there is a path $H : I \to \mathrm{map}_\circ(X, Y)$ from f to g. The path H is called a **homotopy** from f to g. We write $H : f \simeq g$ to indicate that H is a homotopy from f to g; the notation $f \simeq g$ simply means that f is homotopic to g, without mentioning any particular homotopy. Homotopies of unpointed maps are sometimes called **free homotopies**.

We define homotopy for pointed maps in the same way. If $f, g \in \mathrm{map}_*(X, Y)$, then we say that $f \simeq g$ if there is a path from f to g in $\mathrm{map}_*(X, Y)$. There is an interesting technicality here: since $\mathrm{map}_*(X, Y)$ is a pointed space whose basepoint is the constant map $*$, any pointed map $\omega : I \to \mathrm{map}_*(X, Y)$ must be a homotopy of g to $* = \omega(1)$ (we use $1 \in I$ as the basepoint). Since we want to be able to define homotopies between nontrivial maps, we are forced to use *free* paths in $\mathrm{map}_*(X, Y)$, and we do this within $\mathcal{T}_*$ by defining a pointed homotopy $H : f \simeq g$ to be a pointed map

$$H : I_+ \longrightarrow \mathrm{map}_*(X, Y)$$

with $H(0) = f$ and $H(1) = g$.

Exercise 4.1. Define homotopy for maps in the category $\mathcal{T}_{(2)}$ of pairs.

Next we show that homotopy is an equivalence relation and begin to study the sets of equivalence classes.

Problem 4.2. Let X be a topological space, and let $x, y \in X$. Say that $x \sim y$ if x and y are in the same path component of X (i.e., if there is a path $\alpha : I \to X$ with $\alpha(0) = x$ and $\alpha(1) = y$).

(a) Show that $\sim$ is an equivalence relation.

(b) Conclude that homotopy is an equivalence relation in $\mathcal{T}_\circ$ and in $\mathcal{T}_*$.

Since homotopy is an equivalence relation, it divides the sets $\mathrm{map}_\circ(X, Y)$ and $\mathrm{map}_*(X, Y)$ into equivalence classes, called **homotopy classes** of maps. For $X, Y \in \mathcal{T}_\circ$, we write

$$\langle X, Y \rangle = \{\text{free homotopy classes of maps } X \to Y\}$$

(there is no standard notation for this set); and for $X, Y \in \mathcal{T}_*$ we write

$$[X, Y] = \{\text{pointed homotopy classes of maps } X \to Y\},$$

which is very much the standard notation. We denote the free homotopy class of a map $f : X \to Y$ in $\mathcal{T}_\circ$ by $\langle f \rangle \in \langle X, Y \rangle$ and write $[f] \in [X, Y]$ for the pointed homotopy class of $f \in \mathrm{map}_*(X, Y)$.

Exercise 4.3. The set $[X, Y]$ is a pointed set—what is the basepoint?

For $X \in \mathcal{T}$, we denote by $\pi_0(X)$ the set of path components of X. In terms of *sets*, this is simply the set $X/\sim$, where $\sim$ is the equivalence relation of Problem 4.2. If X is a pointed space, then $\pi_0(X)$ is a **pointed set**: $\pi_0(X)$ has a basepoint, namely the equivalence class of the basepoint $* \in X$.

Problem 4.4. Let $f : X \to Y$ and $g : Y \to Z$ be two maps in $\mathcal{T}_*$.

(a) Show that if $x \sim x' \in X$, then $f(x) \sim f(x')$ in Y.

(b) Define $\pi_0(f)$ to make π_0 a covariant functor from $\mathcal{T}_*$ to the category $\mathbf{Sets}_*$ of pointed sets.

(c) Show that if $f \simeq g$ in $\mathcal{T}$, then $\pi_0(f) = \pi_0(g)$.

Any functor F satisfying the homotopy invariance property $F(f) = F(g)$ when $f \simeq g$ is called a **homotopy functor**. A weaker, but still very useful, condition for a functor $F : \mathcal{T} \to \mathcal{T}$ is that it **respects homotopy** in the sense that it carries homotopic maps to homotopic maps.

Exercise 4.5. Show that if F respects homotopy and G is a homotopy functor, then $G \circ F$ is a homotopy functor.

Problem 4.6.

(a) Show that the functors $? \times Z$, $? \rtimes ?$, $? \wedge Z$ and $\mathrm{map}_*(\,?\,,\,?\,)$ respect homotopy in $\mathcal{T}_*$.

(b) Show that the functors $? \times Z$, $Z \rtimes ?$ and $\mathrm{map}_\circ(\,?\,,\,?\,)$ respect homotopy in $\mathcal{T}_\circ$.

Problem 4.7.

(a) Interpret the homotopy set $[\,?\,,\,?\,]$ in terms of π_0.

(b) Is $[\,?\,,\,?\,]$ functorial? Is it a homotopy functor? Keep in mind that there are two variables.

Exercise 4.8. Interpret π_0 in terms of $[\,?\,,\,?\,]$.

4.1.2. Adjoint Definition of Homotopy. While the deformation point of view is conceptually clear and frequently useful, it is more common to use a definition which views the rule $(x, t) \mapsto f_t(x)$ as a single function, rather than as a parametrized family of functions. The two approaches are equivalent to one another in the categories $\mathcal{T}_\circ$ and $\mathcal{T}_*$ because of the exponential law, but there are spaces X and Y in **Top** for which the two notions of homotopy disagree.[1]

[1]This is an example of the kind of pathology that we avoid by working inside the categories $\mathcal{T}$. For spaces not in $\mathcal{T}$, the notion of homotopy given here, and not the one given in terms of paths, is the standard one.

A **free** (or **unpointed**) **homotopy** from $f : X \to Y$ to $g : X \to Y$ in $\mathcal{T}_\circ$ is a map $H : X \times I \to Y$ making the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{in}_0"] \arrow[dr, bend right, "f"'] & X \times I \arrow[d, dotted, "H"] & X \arrow[l, "\mathrm{in}_1"'] \arrow[dl, bend left, "g"] \\ & Y & \end{tikzcd}$$

commutative. If there is a homotopy from f to g, then we say that f and g are **homotopic** and write $f \simeq g$; if we want to indicate that a specific map H is a homotopy from f to g, we write $H : f \simeq g$.

If $f, g : X \to Y$ in $\mathcal{T}_*$, then we require our deformation to be a deformation *through pointed maps*: that is, we require that f_t be a pointed map for each $t \in I$. A **pointed homotopy** $H : f \simeq g$ is a map $H : X \rtimes I \to Y$ in $\mathcal{T}_*$ making the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{in}_0"] \arrow[dr, bend right, "f"'] & X \rtimes I \arrow[d, dotted, "H"] & X \arrow[l, "\mathrm{in}_1"'] \arrow[dl, bend left, "g"] \\ & Y & \end{tikzcd}$$

commutative.

Problem 4.9. Show that the definitions of free and pointed homotopy in terms of cylinders are equivalent to those given in Section 4.1.1.

Problem 4.10. Using the definition of this section, show that

(a) homotopy is an equivalence relation and

(b) if $f \simeq \bar{f}$ and $g \simeq \bar{g}$, then $g \circ f \simeq \bar{g} \circ \bar{f}$

in both the pointed and unpointed contexts.

4.1.3. Homotopies of Paths. Homotopy classes of paths play an important role in many contexts, including complex analysis and geometry, not to mention homotopy theory.

Exercise 4.11. Show that if X is path-connected, then any two paths are freely homotopic. If the paths are pointed, then they are pointed homotopic.

This exercise seems to suggest that the homotopy theory of paths in spaces is entirely trivial. The concept becomes meaningful when we restrict our homotopies even further. If α and β are paths in X from x_0 to x_1, then a **path homotopy** $H : \alpha \simeq \beta$ is a homotopy H with the additional property that $H|_{0\times I}$ is constant at x_0 and $H|_{1\times I}$ is constant at x_1. In other words, H is a path homotopy if each $\alpha_t = H|_{I\times t}$ is a path from x_0 to x_1.

If $\alpha, \beta : I \to X$ are paths in X with $\alpha(1) = \beta(0)$, then we may **concatenate** them, resulting in a new path $\alpha * \beta : I \to X$ given by the formula

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & \text{if } t \in [0, \frac{1}{2}], \\ \\ \beta(2t-1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

We also define the **reverse path** $\overleftarrow{\alpha} : I \to X$ by $\overleftarrow{\alpha}(t) = \alpha(t-1)$.

Problem 4.12.

(a) Suppose there are path homotopies $\alpha_1 \simeq \alpha_2$ and $\beta_1 \simeq \beta_2$. Show that $\alpha_1 * \beta_1 \simeq \alpha_2 * \beta_2$, assuming the concatenation is defined.

(b) Show that $\alpha * \overleftarrow{\alpha} \simeq *$ is path homotopic to the constant path at $\alpha(0)$.

Hint. For (b), consider the path which follows α from time 0 to time $2t$, then sits still until time $2 - 2t$, after which it follows $\overleftarrow{\alpha}$ until time 1.

Reparametrization of Paths. One way to define new paths in terms of old ones is to reparametrize them. Explicitly, let $p : I \to I$ be any path from 0 to 1, and let $\alpha : I \to X$. Then the **reparametrization** of α by p is the path $\alpha \circ p : I \to X$.

Problem 4.13. Let $p : I \to I$ be any path from 0 to 1.

(a) Show that p is path homotopic to id_I.

(b) Show that $\alpha \circ p$ is path homotopic to α.

Hint. Look ahead to Section 4.2.1.

Problem 4.14.

(a) Show that the paths $(\alpha * \beta) * \gamma$ and $\alpha * (\beta * \gamma)$ are path homotopic.

(b) Let α be a path in X from x to y, and let $\boxed{x}$ and $\boxed{y}$ be the constant paths at x and at y, respectively. Show that there are path homotopies

$$\alpha * \boxed{y} \simeq \alpha \simeq \boxed{x} * \alpha.$$

Hint. The paths in question are reparametrizations of each other.

Exercise 4.15. Show that the paths $(\alpha * \beta) * \gamma$ and $\alpha * (\beta * \gamma)$ are equal if and only if all three paths are constant.

4.1.4. Composing and Inverting Homotopies. Since we can interpret homotopies as paths, the homotopy theory of paths immediately gives a corresponding theory for homotopies. We compose two homotopies $H : f \simeq g$ and $K : g \simeq h$, considered as paths in $\mathrm{map}_*(X, Y)$, by concatenating them as in Section 4.1.3, and so on.

From the adjoint point of view, the **concatenation** of homotopies $H : f \simeq g$ and $K : g \simeq h$ is the homotopy $H * K$ given by the formula

$$(H * K)(x,t) = \begin{cases} H(x, 2t) & \text{if } t \in [0, \frac{1}{2}], \\ K(x, 2t-1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

There is a **trivial homotopy**, which we will refer to as the **constant homotopy**, from $f : X \to Y$ to itself, namely $\boxed{f} : X \times I \to Y$ given by $\boxed{f}(x,t) = f(x)$.[2] The **reverse** of the homotopy $H : X \times I \to Y$ is a homotopy $\overleftarrow{H} : X \times I \to Y$ given by $\overleftarrow{H}(x,t) = H(x, 1-t)$.

If two homotopies $f \simeq g$, considered as paths in $\mathrm{map}(X, Y)$, are path homotopic, then we say that the homotopies are **homotopic**. If we consider the homotopies as maps $H, K : X \times I \to Y$, then a **homotopy-of-homotopies** from H to K is a homotopy

$$J : (X \times I) \times I \longrightarrow Y$$

from H to K such that $J(x, 0, s) = f(x)$ and $J(x, 1, s) = g(x)$ for all s.

Problem 4.16. Define reparametrization of homotopies, and show that there is always a homotopy-of-homotopies from H to any reparametrization of H.

Problem 4.17. Let $H : f \simeq g$ be a homotopy.

(a) Show that $H * \boxed{g} \simeq H \simeq \boxed{f} * H$.

(b) Show that there is a homotopy-of-homotopies $H * \overleftarrow{H} \simeq \boxed{f}$.

4.2. Constructing Homotopies

Now we take a break from the general theory and look at some examples to see how to construct homotopies. We start with straight-line homotopies, an extremely simple, but still very useful, approach to constructing homotopies. On the face of it, these simple homotopies can only be used in convex spaces, which are indistinguishable from the single point $*$ from the point of homotopy; but we'll see that straight-line homotopies can be the foundation of nontrivial—even nonintuitive—homotopies. Specifically, we take a dry run at the cellular approximation theorem and prove a 'standard form' result for maps between half-smash products which will be the key, later, to identifying the E_2-term of the Leray-Serre spectral sequence.

[2] Thus $\boxed{f} = f \circ \mathrm{pr}_X$.

4.2.1. Straight-Line Homotopy. If $H : X \times I \to Y$ is a homotopy from $f(x)$ to $g(x)$, then for each $x \in X$, the restriction of H to $\{x\} \times I$ defines a path $H_x : I \to X$ from $f(x) \to g(x)$. If we want to show that f and g are homotopic, the simplest thing to do is to let H_x be the straight line segment joining $f(x)$ to $g(x)$. Of course, this is impossible unless Y has some kind of linear structure so that there is a concept of 'line segment' in Y.

Piecewise Linear Paths. For points x, y in an $\mathbb{R}$-vector space V, we call any linear combination $(1-t)x+ty$ with $t \in I$ a **convex combination** of x and y. The function $\alpha : t \mapsto (1-t)x + ty$ parametrizes the unique constant speed path (with domain $I = [0, 1]$) starting at x and ending at y, called the **linear** path from x to y. A subset $Y \subseteq V$ is called **convex** if every convex combination of points in Y is also in Y.[3]

Problem 4.18. Let $Y \subseteq \mathbb{R}^n$ be convex.

(a) Let $X = \{*\}$, and let $f, g : X \to Y$ be defined by $f(*) = x$ and $g(*) = y$. Show that $f \simeq g$.

(b) Now let X be any space, and let $f, g : X \to Y$. Show that $f \simeq g$. What is $\langle X, \mathbb{R}^n \rangle$?

(c) Choose a point $a \in Y$ to be the basepoint of Y. Let X be a pointed space; what is $[X, Y]$?

If X is a 1-dimensional CW complex and $Y \subseteq V$ (where V is an $\mathbb{R}$-vector space) then, we say that a function $f : X \to Y$ is **piecewise linear** if the composite $I \xrightarrow{\chi} X \xrightarrow{f} V$ of f with each characteristic map χ is linear.

You are certainly aware that the image of a curve can be disconcertingly large. The Hahn-Mazurkowicz theorem [**89**] guarantees that any path-connected and locally path-connected compact space Y is the image of a continuous function $I \to Y$. Thus D^n could very well be the image of a path $I \to \mathbb{R}^n$. However, you can show that such a path cannot be piecewise linear.

Problem 4.19. Show that if X is a finite 1-dimensional CW complex, a piecewise linear function $f : X \to \mathbb{R}^n$ cannot contain a nonempty open set of $\mathbb{R}^n$.

HINT. Using either algebra or analysis, show that no countable collection of lines in $\mathbb{R}^2$ can contain a nontrivial open set.

Every map from a 1-dimensional CW complex to a convex space is homotopic to a piecewise linear map.

[3]Note that V itself is automatically convex.

Problem 4.20. Let $f : X \to Y$ where X is a 1-dimensional CW complex, let V be an $\mathbb{R}$-vector space, and let $Y \subseteq V$ be convex.

(a) Show that there is a unique piecewise linear function $f_{PL} : X \to Y$ such that $f_{PL}|_{X_0} = f|_{X_0}$.

(b) Show that $f_{PL} : X \to Y$.

(c) Show that $f \simeq f_{PL}$ by a homotopy $X \times I \to Y$ that is constant on X_0.

4.2.2. Pushing a Map off of a Cell. It sometimes happens that we have a map $f : X \to Z \cup D^n$, and we hope to find a map $\phi : X \to Z$ making the diagram

$$\begin{tikzcd} & Z \arrow[d, "i"] \\ X \arrow[r, "f"] \arrow[ur, dotted, bend left, "\phi"] & Z \cup D^n \end{tikzcd}$$

commute up to homotopy. That is, we want to find ϕ such that $i \circ \phi \simeq f$. We will make a preliminary investigation of this question here and take up the question again in Chapter 12.

If Z is a retract of Q, then we have maps $i : Z \to Q$ and $r : Q \to Z$ such that $r \circ i = \mathrm{id}_Z$. If, in addition, the composition $i \circ r : Q \to Q$ is homotopic to id_Q, then we say that Z is a **deformation retract** of Q. If a homotopy $H : \mathrm{id}_Q \simeq i \circ r$ can be found whose restriction to $Z \times I$ is the constant homotopy $\boxed{i}$, then Z is a **strong deformation retract** of Q.

Problem 4.21. Let $Y = Z \cup D^n$.

(a) Show that for any $x \in \mathrm{int}(D^n)$, S^{n-1} is a strong deformation retract of $D^n - \{x\}$.

HINT. If you're stumped, try the special case $n = 2$, $x = 0$.

(b) Show that for any $y \in \mathrm{int}(D^n) \subseteq Z \cup D^n$, Z is a strong deformation retract of $Q = Z \cup D^n - \{y\}$.

Now we give a simple criterion for when the map f can be pushed off the disk D^n. This result is the underlying principle in the solution to many compression problems.

Proposition 4.22 (Compression principle)**.** *Let $A \subseteq X$, and consider the commutative diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Z \arrow[d, "i"] \\ X \arrow[r, "f"] \arrow[ur, dotted] & Z \cup D^n. \end{tikzcd}$$

If there is a map $g : X \to Y$ such that $\mathrm{int}(D^n) \not\subseteq g(X)$ and a homotopy $H : f \simeq g$ that is constant on A, then the dotted arrow may be filled in with *a map ϕ so that the upper triangle is commutative and there is a homotopy $H : i \circ \phi \simeq f$ that is constant on A.*

Problem 4.23. Prove Proposition 4.22.

4.2.3. Pushing a Path off the Disk. Now we specialize to the extremely special case of paths in $\mathbb{R}^n$. In this case, we have the diagram

$$\begin{tikzcd} \{0,1\} \arrow[r] \arrow[d] & \mathbb{R}^n - \mathrm{int}(D^n) \arrow[d] \\ I \arrow[r, "f"] \arrow[ur, dotted] & \mathbb{R}^n \end{tikzcd}$$

and we wonder if the dotted arrow can be filled in to make the upper triangle commute and the lower triangle commute up to a homotopy constant on the endpoints. That is, can we use a path homotopy to push any path off of the open unit disk?

To apply Proposition 4.22, we must inquire about the existence of a map g homotopic to f such that $\mathrm{int}(D^n) \not\subseteq g(I)$.

Exercise 4.24. Show that for $n = 1$, this can be done if and only if $f(0)$ and $f(1)$ have the same sign.

For $n > 1$, the problem is more interesting. To prove every such path in $\mathbb{R}^n$ can be pushed off of D^n we will need to use the **Lebesgue Number Lemma**.[4] We'll use Lebesgue's lemma in a form specially suited to analyzing paths.

Corollary 4.25. *If $\{U_i \mid i \in \mathcal{I}\}$ is a cover of any space X and $\alpha : I \to X$, then there is an $m \in \mathbb{N}$ large enough that for each $0 \leq k < m$ there is an $i \in \mathcal{I}$ such that $\alpha\left(\left[\frac{k}{m}, \frac{k+1}{m}\right]\right) \subseteq U_i$.*

Problem 4.26. Prove Corollary 4.25.

Now we return to the problem of pushing a path in $\mathbb{R}^n$ out of the open unit disk. Consider the cover of $\mathbb{R}^n$ by the sets

$$\mathcal{U} = \{\text{open disks contained in } \mathbb{R}^n - D^n\} \qquad \text{and} \qquad V = \mathrm{int}(2 \cdot D^n).$$

Using Corollary 4.25, we decompose I into intervals of length $\frac{1}{m}$, each of which is carried by f into either a set in $\mathcal{U}$ or into V, or both. Let $K \subseteq I$ be the union of those intervals that map into sets in $\mathcal{U}$ and let L be the union of those intervals that map into V; write $f_L : L \to \mathbb{R}^n$ for the restriction of f to L.

Problem 4.27. Let $u : \mathbb{R}^n \to I$ such that $u(x) = 1$ if $|x| \geq \frac{2}{3}$ and $u(x) = 0$ if $|x| \leq \frac{1}{3}$; then define $v(x) = 1 - u(x)$.

[4]See [**138**], for example.

(a) Show that f_L is homotopic in V to a piecewise linear map $g_L : L \to \mathbb{R}^n$ such that $g_L(\frac{k}{m}) = f(\frac{k}{m})$ for all $\frac{k}{m} \in L$.

(b) Show that the map $\phi : I \to \mathbb{R}^n$ defined by the formula

$$g(t) = u(f(t)) \cdot f(t) + v(f(t)) \cdot g_L(t)$$

is well-defined and continuous.

HINT. Check it on each subinterval.

(c) Show that g is path homotopic to f and that $\mathrm{int}(D^n) \not\subseteq g(I)$.

(d) Show that the dotted arrow ϕ exists.

4.2.4. Cellular Approximation for 1-Dimensional Domains. A map $f : X \to Y$ between CW complexes may or may not be a cellular map; but even if it is not, we can ask if it is *homotopic* to a cellular map. You will show that if X is 1-dimensional, then every map f is homotopic to a cellular map. It is true for CW complexes of all dimensions, including ∞, and the proof of the general statement is based on the same ideas; you'll prove it in Chapter 12 when we have a bit more machinery.

Theorem 4.28. *Let $f : X \to Y$ be a map from one CW complex to another. Assume that X is 1-dimensional and that $f(X_0) \subseteq Y_0$. Then f is homotopic, by a homotopy constant on X_0, to another map g with $g(X) \subseteq Y_1$.*

We start the proof by reducing to a manageable special case.

Problem 4.29.

(a) Show that it suffices to prove Theorem 4.28 in the special case in which $X = I$ and Y is a finite complex.

(b) Show that it suffices to show that in the diagram

$$\begin{tikzcd} \{0,1\} \arrow[rr] \arrow[d, "i"'] & & Z \arrow[d, "i"] \\ I \arrow[urr, dotted] \arrow[rr, "f"] & & Z \cup D^n, \end{tikzcd}$$

a dotted arrow may be filled in so that the upper triangle commutes and in the lower triangle commutes up to a homotopy constant on A.

Write $\chi : D^n \to Y$ for the characteristic map of the distinguished n-cell in $Y = Z \cup D^n$. The map χ identifies $\mathrm{int}(D^n)$ with a convex subset of $\mathbb{R}^n$, which means that we can use straight-line homotopies for maps whose images are entirely contained in $\mathrm{int}(D^n)$.

We'll need to patch together the straight-line homotopy that we perform inside of $\mathrm{int}(D^n)$ with the constant homotopy outside of D^n. To ease the transition between the two, we use the function $h : Y \to I$ defined by

$$h(\chi(x)) = |x| \qquad \text{for } x \in \operatorname{int}(D^n)$$

and $h(y) = 1$ for all other $y \in Y$. Now consider the open cover of $Z \cup D^n$ by $W = h^{-1}\left(\frac{2}{3}, 1\right]$, $\mathcal{U} = \left\{\text{open disks in } h^{-1}\left(\frac{1}{3}, 1\right)\right\}$ and $V = h^{-1}\left[0, \frac{2}{3}\right)$.

Exercise 4.30.

(a) Show that h is continuous and conclude that these sets are open in $Z \cup D^n$.

(b) Write down a continuous function $u : Z \cup D^n \to I$ that is constant at 1 for $y \in W$ and constant at 0 for $y \in h^{-1}([0, \frac{1}{3}))$. Write $v = 1 - u$.

Now consider a path $f : I \to Y$ with $f(\{0,1\}) \subseteq Z_0$. Corollary 4.25 tells us that there is an $n \in \mathbb{N}$ such that $f\left(\left[\frac{k}{n}, \frac{k+1}{n}\right]\right)$ is contained in W or in a set in $\mathcal{U}$ or in V for each k. Let L be the union of intervals that map into V.

Problem 4.31.

(a) Show that the restriction $f_L : L \to V$ of f is homotopic in V, by a homotopy constant on L_0, to a piecewise linear map $g_L : L \to V$.

(b) Show that the map $g : I \to Y$ given by

$$g(t) = u(f(t)) \cdot f(t) + v(f(t)) \cdot g_L(t)$$

is well-defined and continuous.

(c) Prove Theorem 4.28.

4.2.5. Maps of Products. Now we shift our attention to another problem involving homotopy and spheres. This example will play a crucial role in the proof of Theorem 31.1 much later in the book.

Among the maps $f : S^n \rtimes X \to S^n \rtimes Y$ are those of the form $d \rtimes \phi$, where $d : S^n \to S^n$ and $\phi : X \to Y$. There are, of course, many maps which do not have this form, even up to homotopy. What do we have to know about f in order to conclude that it is homotopic to a map which decomposes this way?

Lemma 4.32. *Suppose $f : S^n \rtimes X \to S^n \rtimes Y$ fits into the commutative diagram*

$$\begin{tikzcd} S^n \rtimes X \arrow[r, "f"] \arrow[d, "\mathrm{pr}_1"'] & S^n \rtimes Y \arrow[d, "\mathrm{pr}_1"] \\ S^n \arrow[r, "d"] & S^n \end{tikzcd}$$

where $d : S^n \to S^n$ is a pointed map. Then, for each $u \in S^n$ other than the basepoint, there is a commutative square

$$\begin{tikzcd} X \arrow[r, "\phi_u"] \arrow[d, "\mathrm{in}_u"'] & Y \arrow[d, "\mathrm{in}_{d(u)}"] \\ S^n \rtimes X \arrow[r, "f"] & S^n \rtimes Y \end{tikzcd}$$

and $f \simeq d \rtimes \phi_u$.

Problem 4.33.

(a) Show that it suffices to prove $f \simeq d \rtimes \phi_0$.

(b) Let $q : D^n \to S^n$ be the quotient map. Show that the formula

$$H([q(u), x], t) = [d(q(u)), \phi_{(1-t)u}(x)]$$

defines a pointed homotopy $H : f \simeq d \rtimes \phi_0$. Be sure to pay careful attention to the basepoint.

Exercise 4.34. Generalize Lemma 4.32 to apply to maps $S^n \rtimes X \to Z \rtimes Y$ covering $d : S^n \to Z$. What can you say about maps $W \rtimes X \to Z \rtimes Y$ that cover $d : W \to Z$?

4.3. Homotopy Theory

Homotopy of maps is an equivalence relation on the morphisms of $\mathcal{T}$. Since homotopy respects composition, there are quotient categories $\mathrm{H}\mathcal{T}$, known as the homotopy categories, with the same objects but whose morphisms are homotopy classes of maps. There are also canonical 'quotient functors' $\mathrm{Ho} : \mathcal{T} \to \mathrm{H}\mathcal{T}$, and homotopy theory is the study of these functors.

4.3.1. The Homotopy Category. The **pointed homotopy category** is the category $\mathrm{H}\mathcal{T}_*$ whose objects and morphisms are given by

$$\begin{aligned} \mathrm{ob}(\mathrm{H}\mathcal{T}_*) &= \mathrm{ob}(\mathcal{T}_*), \\ \mathrm{mor}_{\mathrm{H}\mathcal{T}_*}(X, Y) &= [X, Y]. \end{aligned}$$

The **unpointed homotopy category** $\mathrm{H}\mathcal{T}_\circ$ is defined analogously; and we use $\mathrm{H}\mathcal{T}$ for statements that are equally valid in either category.

The assignment $\mathrm{Ho} : \mathcal{T} \to \mathrm{H}\mathcal{T}$ given by

$$\mathrm{Ho} : X \longmapsto X \qquad \text{and} \qquad \mathrm{Ho} : f \longmapsto [f]$$

is a functor $\mathrm{Ho} : \mathcal{T} \to \mathrm{H}\mathcal{T}$. This quotient functor is the 'universal example' for all homotopy functors, in the sense that all other homotopy functors can be factored, uniquely, through Ho.

Problem 4.35.

(a) Verify that $\mathrm{H}\mathcal{T}$ is actually a category and that Ho is a homotopy functor.

(b) Show that a functor $F : \mathcal{T} \to \mathcal{C}$ is a homotopy functor if and only if there is a functor $G : \mathrm{H}\mathcal{T} \to \mathcal{C}$ making the triangle

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "F"] \arrow[dr, "\mathrm{Ho}"'] & & \mathcal{C} \\ & \mathrm{H}\mathcal{T} \arrow[ur, "G"'] & \end{tikzcd}$$

commute. Show that if G exists, it is unique.

Exercise 4.36. Is it fair to call Ho a forgetful functor?

Two spaces X and Y are **homotopy equivalent** in $\mathcal{T}$ if they are equivalent—in the sense of category theory—in the homotopy category $\mathrm{H}\mathcal{T}$. We write $X \simeq Y$ to indicate X is homotopy equivalent to Y. Homotopy equivalence is an equivalence relation on the collection $\mathrm{ob}(\mathcal{T}) = \mathrm{ob}(\mathrm{H}\mathcal{T})$, so $\mathrm{ob}(\mathcal{T})$ is partitioned into equivalence classes, called **homotopy types**. Thus it is sometimes said that homotopy equivalent spaces are 'of the same homotopy type' or 'have the same homotopy type'.

Exercise 4.37.

(a) Write out explicitly in terms of maps, spaces and homotopies exactly what it means for two spaces to be homotopy equivalent.

(b) Let $T = \{(0,0), (\frac{1}{n}, 0) \mid n \in \mathbb{N}\} \subseteq \mathbb{R}^2$ and let $U = T \cup (0 \times I) \subseteq \mathbb{R}^2$. Let T have basepoint $(0,0)$ and let U have basepoint $(0,1)$. Show that the projection $\mathrm{pr}_1 : U \to T$ is a pointed map that is an unpointed homotopy equivalence but not a pointed homotopy equivalence.

Problem 4.38.

(a) Show that if F is a homotopy functor and $X \simeq Y$, then $F(X) \cong F(Y)$.

(b) Show that if $X \simeq Y$ in $\mathcal{T}_*$, then $\Sigma X \simeq \Sigma Y$ and $\Omega X \simeq \Omega Y$ in $\mathcal{T}_*$.

(c) Show that there is a natural equivalence

$$[X \wedge Y, Z] \xrightarrow{\cong} [X, \mathrm{map}_*(Y, Z)]$$

of three-variable functors from $\mathrm{H}\mathcal{T}_*$ to $\mathbf{Sets}_*$.

Homotopy Commutative Diagrams. A diagram $F : \mathcal{I} \to \mathcal{T}_*$ is **homotopy commutative** if the composite diagram $\mathrm{Ho} \circ F : \mathcal{I} \to \mathrm{H}\mathcal{T}_*$ is commutative. Homotopy commutative diagrams are frequently described as 'commuting up to homotopy'.

Exercise 4.39. Give an example of a diagram that is homotopy commutative but not commutative.

It is common to refer to homotopy commutative diagrams as 'commutative diagrams'—i.e., to blur the distinction between the category $\mathcal{T}_*$ and the homotopy category $\mathrm{H}\mathcal{T}_*$. When we wish to be clear that a diagram is commutative (and not just homotopy commutative), we will say that it is **strictly commutative** or commutative 'on the nose'.[5]

4.3.2. Contractible Spaces and Nullhomotopic Maps. If $X \simeq *$, then X is said to be a **contractible** space. Similarly, a map $f : X \to Y$ that is homotopic to the constant map $* : X \to Y$ is called a **nullhomotopic** map, or a **trivial** map. A homotopy H from f to $*$ is sometimes called a **nullhomotopy** of f. Maps that are not nullhomotopic are sometimes called **essential** maps.

Exercise 4.40.

(a) Is there a space such that $X \not\simeq *$ in $\mathcal{T}_*$ but X_- is contractible in $\mathcal{T}_\circ$?

(b) Show that $\mathbb{R}^n$ is contractible.

Problem 4.41. Show that the following are equivalent for $Y \in \mathcal{T}_*$:

(1) $Y \simeq *$ in $\mathcal{T}_*$,

(2) $[X, Y] = *$ for all $X \in \mathcal{T}_*$,

(3) $[Y, Z] = *$ for all $Z \in \mathcal{T}_*$.

Prove corresponding results for the category $\mathcal{T}_\circ$.

Problem 4.42.

(a) Show that if $X \simeq *$, then X is path-connected.

(b) Show that if $X \simeq *$, then $\Omega X \simeq *$ and $\Sigma X \simeq *$.

(c) Find an example of a noncontractible space $X \in \mathcal{T}_*$ whose loop space is contractible.

It is possible, but much more difficult, to find noncontractible spaces whose suspension is contractible; you will do this in Problem 19.38.

There are several useful criteria for deciding whether a map is nullhomotopic.

Problem 4.43. Let $f : X \to Y$. Show that the following are equivalent:

(1) f is nullhomotopic,

[5] Or, perhaps, 'nasally commutative'?

(2) there is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, bend right, "\mathrm{in}_0"'] & & Y \\ & CX, \arrow[ur, bend right, "e"'] & \end{tikzcd}$$

(3) f factors (up to homotopy) through a contractible space.

The special case $X = S^n$ will be used continually.

Problem 4.44. Show that a map $\alpha : S^n \to X$ is homotopic to $*$ if and only if it factors through D^{n+1}—that is, if and only if there is a map $\bar{\alpha} : D^{n+1} \to Y$ such that the diagram

$$\begin{tikzcd} S^n \arrow[rr, "\alpha"] \arrow[dr, bend right, "i"'] & & Y \\ & D^{n+1} \arrow[ur, bend right, "\bar{\alpha}"'] & \end{tikzcd}$$

is strictly commutative.

For $X \in \mathcal{T}_*$, the space of pointed paths

$$\mathcal{P}(X) = \mathrm{map}_*(I, X)$$

(using 1 as the basepoint of I) is known as the **path space** on X. It is equipped with an evaluation map $@_0 : \mathcal{P}(X) \to X$ given by $@_0(\omega) = \omega(0)$.

Problem 4.45. Show that $\mathcal{P}(X)$ is contractible.

Problem 4.46. Let $f : X \to Y$ in $\mathcal{T}_*$, and show that the following are equivalent:

(1) f is nullhomotopic,

(2) there is a lift λ in the diagram

$$\begin{tikzcd} & \mathcal{P}(Y) \arrow[d, "@_0"] \\ X \arrow[ur, dotted, "\lambda"] \arrow[r, "f"] & Y. \end{tikzcd}$$

Now we correlate our two criteria. Let $f : X \to Y$. You have shown how to construct, from a nullhomotopy $H : f \simeq *$, maps $e : CX \to Y$ extending f and $\lambda : X \to \mathcal{P}(Y)$ lifting f.

Problem 4.47. Show that there is a map $CX \to \mathcal{P}(Y)$ making the diagram

$$\begin{tikzcd} X \arrow[r, "\lambda"] \arrow[d, "\mathrm{in}_0"'] & \mathcal{P}(Y) \arrow[d, "@_0"] \\ CX \arrow[r, "e"] \arrow[ur, dotted] & Y \end{tikzcd}$$

strictly commutative.

Problem 4.48. Show that there is a pullback square

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & \mathcal{P}(X) \arrow[d, "@_0"] \\ * \arrow[r] & X. \end{tikzcd}$$

The trivial map $\Omega X \xrightarrow{*} X$ is nullhomotopic for two different reasons. First of all, it *is* the trivial map, so the constant homotopy will do the job. On the other hand, the map factors through the contractible space $\mathcal{P}(X)$, and the contraction $H : \mathcal{P}(X) \rtimes I \to \mathcal{P}(X)$ gives rise to a nullhomotopy $K : \Omega X \rtimes I \to X$.

Problem 4.49.

(a) Write out an explicit formula for the homotopy K.

(b) The homotopy K factors, uniquely, through a map $e : C\Omega X \to \mathcal{P}(X)$. Construct the diagram

$$\begin{tikzcd} \Omega X \arrow[r, "\mathrm{in}_0"] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & C\Omega X \arrow[r, "e"] \arrow[d] & \mathcal{P}(X) \arrow[d, "@_0"] \\ * \arrow[r] & \Sigma\Omega X \arrow[r, "\lambda"] & X \end{tikzcd}$$

and write out an explicit formula for the map λ.

(c) The exponential law gives an explicit homeomorphism

$$\mathrm{map}_*(\Sigma\Omega X, X) \cong \mathrm{map}_*(\Omega X, \Omega X).$$

Show that under this bijection, λ corresponds to the identity map $\mathrm{id}_{\Omega X} \in \mathrm{map}_*(\Omega X, \Omega X)$.

4.4. Groups and Cogroups in the Homotopy Category

In order to get algebra involved in our topology, we need to find some group objects and cogroup objects in the category $\mathrm{H}\mathcal{T}_*$.

Let's start by studying the space S^1, considered as the quotient of I by the relation $0 \sim 1$ and using the equivalence class of $\{0, 1\}$ as the basepoint

$*$. If we identify the point $\frac{1}{2}$ with $*$, the resulting space is homeomorphic to a wedge of two circles; thus the quotient map q is a map

$$S^1 \longrightarrow S^1 \vee S^1.$$

It is very important to be clear about this map by specifying exactly the homeomorphism from the quotient to the wedge. On the face of it, ϕ is a map $S^1 \to A \vee B$, where A is the quotient of $[0, \frac{1}{2}]$ by the relation $0 \sim \frac{1}{2}$ and B is the quotient of $[\frac{1}{2}, 1]$ by the relation $\frac{1}{2} \sim 1$.

Problem 4.50. Show that the functions

$$\alpha : A \longrightarrow S^1 \qquad \text{and} \qquad \beta : B \longrightarrow S^1,$$

given by $\alpha : [t] \mapsto [2t]$ and $\beta : [t] \mapsto [2t-1]$, are homeomorphisms.

Now we can define our map ϕ to be the composite

$$\begin{tikzcd} S^1 \arrow[rr, "\phi"] \arrow[dr, bend right, "q"'] & & S^1 \vee S^1 \\ & A \vee B. \arrow[ur, bend right, "\alpha\vee\beta"'] & \end{tikzcd}$$

We can also reverse the orientation of the circle by flipping it over, giving a map $\nu : S^1 \to S^1$.

Exercise 4.51. Write down $\nu([t])$ for $[t] \in S^1 = I/\sim$.

The maps ϕ and ν exhibit the circle as the most important cogroup object in the category $\mathrm{H}\mathcal{T}_*$.

Theorem 4.52. *The maps ϕ and ν make S^1 into a cogroup object in $\mathrm{H}\mathcal{T}_*$.*

Problem 4.53. Prove Theorem 4.52. Is S^1 a cogroup object in $\mathcal{T}_\circ$?

Hint. If $\omega : S^1 \to X$, then the composite $I \to S^1 \to X$ is a path; look over Section 4.1.3.

Once we have one cogroup object in hand, we can construct many more, as the next problem shows.

Problem 4.54. Let $X \in \mathcal{T}_*$.

(a) Show that if A is a cogroup object in $\mathrm{H}\mathcal{T}_*$, then $X \wedge A$ is also a cogroup. Describe the structure maps $\phi_{X\wedge A}$ and $\nu_{X\wedge A}$ for $X \wedge A$ in terms of the maps ϕ_A and ν_A that define the cogroup structure for A.

(b) What can you say about $X \wedge A$ if A is a commutative cogroup?

(c) Show that for any space X, the suspension ΣX is a cogroup object in $\mathrm{H}\mathcal{T}_*$. Write down explicit formulas for $\alpha + \beta$ and $-\alpha$ for $\alpha, \beta \in [\Sigma X, Y]$.

(d) Show that S^n is a cogroup object in $\mathrm{H}\mathcal{T}_*$ for all $n \geq 1$.

Hint. Use the functoriality of $X \wedge \,?\,$.

Problem 4.55.

(a) Write down the composition of maps which defines $1_{S^1} \cdot 1_{S^1} \in [S^1, S^1]$. We'll refer to this map as $\mathbf{2} : S^1 \to S^1$.

(b) Write down the composition of maps which defines $f \cdot f \in [S^1, X]$.

(c) We have $\mathbf{2}^* : [S^1, X] \to [S^1, X]$. Show that $\mathbf{2}^*(f) = f \cdot f$.

Theorem 4.52 does not guarantee that $\mathbf{2}^*$ is a homomorphism, and in fact it isn't if $[S^1, X]$ is not abelian.

Abelian Cogroup Objects. Since abelian groups are so much easier to deal with than general groups, it will be very helpful to be able to recognize abelian cogroup objects. The most important example of an abelian cogroup object in $\mathrm{H}\mathcal{T}_*$ is S^2.

Theorem 4.56. *The two-sphere S^2 is a cocommutative cogroup object in the homotopy category $\mathrm{H}\mathcal{T}_*$.*

Give the standard 2-sphere $S^2 = \{x \in \mathbb{R}^3 \mid |x| = 1\}$ the basepoint $* = (1, 0, 0)$. Let $R_\theta : S^2 \to S^2$ be rotation by θ radians about the x-axis, which is a pointed map.

Problem 4.57.

(a) Show that $R_\theta \simeq \mathrm{id}_{S^2}$ in $\mathcal{T}_*$.

(b) Show that the diagram

$$\begin{tikzcd} S^2 \arrow[rr, "R_\pi"] \arrow[d, "\phi"'] & & S^2 \arrow[d, "\phi"] \\ S^2 \vee S^2 \arrow[r, "T"] & S^2 \vee S^2 \arrow[r, "(R_\pi, R_\pi)"] & S^2 \vee S^2 \end{tikzcd}$$

is strictly commutative.

(c) Prove Theorem 4.56.

Using Theorem 4.56 we obtain a vast collection of abelian cogroup objects in $\mathrm{H}\mathcal{T}_*$.

Corollary 4.58. *For any space X, the double suspension $\Sigma^2 X$ is a cocommutative cogroup object in $\mathrm{H}\mathcal{T}_*$. In particular S^n is a cocommutative cogroup for $n \geq 2$.*

There are cocommutative cogroup objects that are not double suspensions, but even these are retracts of double suspensions.

Group Objects in the Homotopy Category. We close this section with a brief overview of the dual results.

Problem 4.59. Let X be a cogroup object in $\mathcal{T}_*$.

(a) Show that $\mathrm{map}_*(X, Y)$ is a group object in $\mathcal{T}_*$ for any space Y, and if X is a cocommutative cogroup object, then $\mathrm{map}_*(X, Y)$ is a commutative group object.

(b) Show that if Y is a group object in $\mathcal{T}_*$, then $\mathrm{map}_*(X, Y)$ is a group object, and if Y is commutative, then so is $\mathrm{map}_*(X, Y)$.

Theorem 4.60. *For any space X,*

(a) ΩX is a group object in $\mathrm{H}\mathcal{T}_*$ and

(b) $\Omega^2 X$ is a commutative group object in $\mathrm{H}\mathcal{T}_*$.

Problem 4.61. Prove Theorem 4.60, including explicitly writing out the multiplication $\Omega X \times \Omega X \to \Omega X$.

4.5. Homotopy Groups

Now we come to our first, and arguably most important, specific collection of functors—the **homotopy groups** of a pointed space $X \in \mathcal{T}_*$. For $n \geq 0$, we define

$$\pi_n(X) = [S^n, X],$$

and if $f : X \to Y$, then $\pi_n(f) = f_* : \pi_n(X) \to \pi_n(Y)$. Proposition 1.21 assures us that these rules do in fact define covariant functors. On the face of it, these functors take their values in the category of pointed sets, but the results of the last section imply that they can be given more algebraic structure.

Theorem 4.62.

(a) For $n \geq 1$, the functor π_n takes its values in the category $\mathcal{G}$ of groups and homomorphisms.

(b) For $n \geq 2$, the functor π_n takes its values in the category $\mathrm{AB}\,\mathcal{G}$.

The statements made in Theorem 4.62 require some clarification. We have a forgetful functor from the category $\mathcal{G}$ of groups to the category $\mathbf{Sets}_*$ of pointed sets, and so we may set up the lifting problems

$$\begin{tikzcd}
& \mathrm{AB}\,\mathcal{G} \arrow[d, "\text{forget}"] \\
& \mathcal{G} \arrow[d, "\text{forget}"] \\
\mathcal{T}_* \arrow[r, "\pi_n"] \arrow[uur, dotted, "\pi_n"] \arrow[ur, dotted, "\pi_1"] & \mathbf{Sets}_*.
\end{tikzcd}$$

The theorem asserts that the first lifted version of π_n (which we will also call π_n) does exist and that the second, through $\text{AB}\,\mathcal{G}$, exists if $n \geq 2$.

Problem 4.63. Prove Theorem 4.62.

Theorem 4.62 leaves open an interesting question: sure the lifts exist, but are they unique? Could it be that the *set* $\pi_n(X)$ has several different group structures, each of which is part of a different functor $\mathcal{T}_* \to \mathcal{G}$ which, after forgetting the group structure, is π_n?

Problem 4.64.

(a) Show that there is a lift $\mathcal{T}_* \to \mathcal{G}$ of the functor $[A, ?\,] : \mathcal{T}_* \to \mathbf{Sets}_*$ if and only if A is a cogroup object in $\text{H}\mathcal{T}_*$. Show that $[A, ?\,]$ lifts to $\text{AB}\,\mathcal{G}$ if and only if A is a commutative cogroup object.

(b) Show that if $\phi, \theta : A \to A \vee A$ are two different (i.e., not homotopic) comultiplications, then the functors $[A, ?\,]^\phi, [A, ?\,]^\theta : \mathcal{T}_* \to \mathcal{G}$, which use the structure maps ϕ and θ, respectively, to define the group structure, are different functors.

(c) Conclude that the lift $\pi_n : \mathcal{T}_* \to \mathcal{G}$ is unique if and only if S^n has exactly one comultiplication $S^n \to S^n \vee S^n$.

Later, you will show that for $n > 1$, S^n has exactly one comultiplication and that S^1 has infinitely many comultiplications.

We now have two definitions of the functor $\pi_0 : \mathcal{T}_* \to \mathbf{Sets}_*$, the one defined above in terms of S^0 and the one defined in Section 3.1 using path components.

Exercise 4.65.

(a) Show that the two definitions of the functor π_0 are naturally equivalent. Are they equal?

(b) Show that S^0 is not a co-H-space, and deduce that π_0 does *not* factor through $\mathcal{G}$.

(c) Intepret the groups $\pi_n(\mathrm{map}_*(X, Y))$ and $\pi_n(\mathrm{map}_\circ(X, Y))$ as sets of the form $[A, B]$, with no mapping spaces or homotopy groups involved.

Problem 4.66. Show that for any space $X \in \mathcal{T}_*$, there is a wedge of spheres $W = \bigvee_\alpha S^{n_\alpha}$ and a map $w : W \to X$ such that the induced map

$$w_* : \pi_n(W) \longrightarrow \pi_n(X)$$

is surjective for every n.

We finish with a nice corollary of Theorem 4.28. A space $X \in \mathcal{T}_*$ is called **simply-connected** if $\pi_0(X) = *$ and $\pi_1(X) = 0$.

Corollary 4.67. *Show that if $Y \in \mathcal{T}_*$ is a CW complex with basepoint $* \in Y_0$, then the inclusion $Y_1 \hookrightarrow Y$ induces a surjection $\pi_1(Y_1) \to \pi_1(Y)$.*

Problem 4.68.

(a) Prove Corollary 4.67.

(b) Show that if X has a CW structure with $X_1 = *$, then X is simply-connected.

(c) Show that S^n is simply-connected for $n > 1$.

4.6. Homotopy and Duality

As we progress, we will find ourselves proving various results about homotopies, and we will of course want to keep our eyes open for results that can be dualized. But what is the dual of a homotopy? To answer this, we need to describe homotopies using category-theoretical language.

Let Y be a space, and consider the mapping space $Y^I = \mathrm{map}_\circ(I, Y)$. This is called the (unpointed) **path space** of Y. If you choose a point $t \in I$, then evaluation at t defines a function $@_t : Y^I \to Y$.

Problem 4.69.

(a) Show that $@_t : Y^I \to Y$ is a homotopy equivalence.

(b) Show that homotopies $H : X \times I \to Y$ from f to g correspond by the exponential law to functions $K : X \to Y^I$ such that $@_0 \circ K = f$ and $@_1 \circ K = g$.

The purpose of this discussion is to clearly explain why homotopies, thought of as maps $X \to Y^I$, are dual to homotopies defined in terms of cylinders.

To achieve this, we add another layer of abstraction. For a space Y, we define a **path object** for Y to be a space, which we'll denote $\mathrm{Path}(Y)$ even though it is not unique, together with a map $e : \mathrm{Path}(Y) \to Y \times Y$. We'll write $e = (e_0, e_1)$, and both components are required to be homotopy equivalences. Here it is as a diagram:

$$\begin{tikzcd} & \mathrm{Path}(Y) \arrow[dl, bend right, "e_0"', "\simeq"] \arrow[d, "e"] \arrow[dr, bend left, "e_1", "\simeq"'] & \\ Y & Y \times Y \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & Y. \end{tikzcd}$$

The dual notion is that of a **cylinder object**, which should fit into the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{in}_0"] \arrow[dr, bend right, "i_0"', "\simeq"] & X \sqcup X \arrow[d, "i"] & X \arrow[l, "\mathrm{in}_1"'] \arrow[dl, bend left, "i_1", "\simeq"'] \\ & \mathrm{Cyl}(X). & \end{tikzcd}$$

Problem 4.70.

(a) Show that Y^I, together with the maps $e_0 = @_0$ and $e_1 = @_1$, constitutes a path object for Y in $\mathcal{T}_\circ$ and in $\mathcal{T}_*$.

(b) Show that $X \times I$, together with the maps $i_0 = \mathrm{in}_0$ and $i_1 = \mathrm{in}_1$, constitutes a cylinder object for X in $\mathcal{T}_\circ$.

(c) Show that $X \rtimes I$ is a cylinder object for $X \in \mathcal{T}_*$.

We call Y^I the **standard path object** on Y and $X \times I$ (or $X \rtimes I$) the **standard cylinder** on X. Now we define a **left homotopy** to be a function $H : \mathrm{Cyl}(X) \to Y$ such that the diagram

$$\begin{tikzcd} X \sqcup X \arrow[d, "(i_0,i_1)"'] \arrow[dr, bend left, "(f,g)"] & \\ \mathrm{Cyl}(X) \arrow[r, "H"] & Y \end{tikzcd}$$

commutes, where $\mathrm{Cyl}(X)$ is any cylinder object for X. Dually, a **right homotopy** is a map $K : X \to \mathrm{Path}(Y)$ such that

$$\begin{tikzcd} & Y \times Y \\ X \arrow[ur, bend left, "(f,g)"] \arrow[r, "K"] & \mathrm{Path}(Y) \arrow[u, "(p_0,p_1)"'] \end{tikzcd}$$

commutes, where $\mathrm{Path}(Y)$ is any path object for Y.

Exercise 4.71.

(a) Show that if we choose $\mathrm{Cyl}(X) = X \times I$ with $i = (\mathrm{in}_0, \mathrm{in}_1)$, then a left homotopy is the same thing as an ordinary homotopy.

(b) Do the same for $\mathrm{Path}(Y) = Y^I$ with $e = (@_0, @_1)$.

Problem 4.72. Let $f, g : X \to Y$, and suppose you are given a path object and a cylinder object, not necessarily the standard ones. Show that if f and g are either right homotopic or left homotopic, then they are homotopic in the standard sense.

Unfortunately, the converse of this problem is not true with the definition we have given so far.

Problem 4.73.

(a) Show that (with our definition) the space X, together with the folding map $\nabla : X \sqcup X \to X$, is a cylinder object for X. What does it take for two maps to be left homotopic using this particular cylinder object?

(b) Show that (with our definition) the space Y, together with the diagonal map $\Delta : Y \to Y \times Y$ is a path object for Y. What does it take for two maps to be right homotopic using this particular path object?

The solution to the difficulties raised in Problem 4.73 is to impose additional restrictions on the maps $e : \mathrm{Path}(Y) \to Y$ and $i : X \sqcup X \to X$. Specifically, we'll require e to be a fibration and i to be a cofibration. Once this is done, it will turn out that, for nice spaces, ordinary homotopy of maps is equivalent to left homotopy and to right homotopy, using any path or cylinder objects you choose. We will return to these ideas in Chapter 10.

4.7. Homotopy in Mapping Categories

The category $\mathcal{T}$ gives rise to the morphism category $\mathrm{map}(\mathcal{T})$ whose objects are maps $A \to X$ in $\mathcal{T}$ and whose morphisms are commutative squares. The objects and morphisms of $\mathrm{map}(\mathcal{T})$ have many more 'moving parts' than those of $\mathcal{T}$, and consequently $\mathrm{map}(\mathcal{T})$ has a richer structure than an ordinary category, and its detailed study is much more finicky. In this section, we will define a notion of homotopy for morphisms in $\mathrm{map}(\mathcal{T})$ and develop a tiny fraction of the resulting homotopy theory. In this theory there are three competing notions of homotopy equivalence for maps; each has something to recommend it, but there are significant differences between them.

For clarity in this discussion, we'll specialize and work in $\mathrm{map}(\mathcal{T}_\circ)$, but the corresponding theory is the same for $\mathrm{map}(\mathcal{T}_*)$. All that is required to make the change is to replace products $X \times I$ with half-smash products $X \rtimes I$.

4.7.1. The Category of Maps. The category of maps in $\mathcal{T}_\circ$, denoted $\mathrm{map}(\mathcal{T}_\circ)$, is the category whose objects are maps $f : X \to Y$ in $\mathcal{T}_\circ$ and whose morphisms $\alpha : f \to g$ are (strictly) commutative squares

$$\begin{tikzcd} A \arrow[r, "\alpha^d"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ X \arrow[r, "\alpha_t"] & Y \end{tikzcd}$$

in $\mathcal{T}_\circ$. The notation α^d and α_t is intended to indicate the 'domain' and 'target' parts of the morphism α; we'll sometimes use the notation $\alpha = (\alpha^d, \alpha_t) : f \to g$ to indicate the components of such a morphism.[6]

We can define homotopy of morphisms in this category. The **cylinder** on a map $f \in \operatorname{map}(\mathcal{T}_\circ)$ is the map $f \times \mathrm{id}_I : X \times I \to Y \times I$; there are inclusions

$$\mathrm{in}_0 = (\mathrm{in}_0, \mathrm{in}_0) : f \to f \times \mathrm{id}_I \qquad \text{and} \qquad \mathrm{in}_1 = (\mathrm{in}_1, \mathrm{in}_1) : f \to f \times \mathrm{id}_I.$$

A **homotopy** of morphisms $\alpha, \beta : f \to g$ in $\operatorname{map}(\mathcal{T}_\circ)$ is a morphism $H : f \times \mathrm{id}_I \to g$ making the diagram

$$\begin{tikzcd} f \arrow[r, "\mathrm{in}_0"] \arrow[dr, bend right, "\alpha"'] & f \times \mathrm{id}_I \arrow[d, "H"] & f \arrow[l, "\mathrm{in}_1"'] \arrow[dl, bend left, "\beta"] \\ & g & \end{tikzcd}$$

commute. Unwinding the definition a bit, we see that a homotopy $H : \alpha \simeq \beta$ is a pair of homotopies $(H^d : \alpha^d \simeq \beta^d,\ H_t : \alpha_t \simeq \beta_t)$ that are compatible in the sense that the diagram

$$\begin{tikzcd} A \times I \arrow[r, "H^d"] \arrow[d, "f \times \mathrm{id}_I"'] & B \arrow[d, "g"] \\ X \times I \arrow[r, "H_t"] & Y \end{tikzcd}$$

is strictly commutative. We will refer to H_t as a **homotopy under** H^d and to H^d and H_t together as a pair of **coherent homotopies**.

A morphism $\alpha : f \to g$ is a **homotopy equivalence** in $\operatorname{map}(\mathcal{T}_\circ)$ if there is a morphism $\beta : g \to f$ and homotopies (of morphisms in $\operatorname{map}(\mathcal{T}_\circ)$)

$$H : \alpha \circ \beta \simeq g \qquad \text{and} \qquad K : \beta \circ \alpha \simeq f.$$

In the following problem you will establish some useful basic properties of homotopies in mapping categories.

[6]We'll reserve the word 'map' for morphisms in $\mathcal{T}_\circ$ and use 'morphism' for these 'maps of maps'.

Problem 4.74. Let $\alpha : f \to g$, with the notation above.

(a) Suppose H_t is a homotopy under H^d and K_t is a homotopy under K^d. Show that $H_t * K_t$ is a homotopy under $H^d * K^d$.

(b) Let H_t be a homotopy under H^d, and let $\widetilde{H}_t$ and $\widetilde{H}^d$ be the results of applying the same reparametrization to both homotopies. Show that $\widetilde{H}_t$ is a homotopy under $\widetilde{H}^d$.

(c) Suppose $\alpha = (\alpha^d, \alpha_t)$ is homotopic to $\beta = (\alpha^d, \beta_t)$ under the constant homotopy $\boxed{\alpha^d}$ and β is homotopic to $\gamma = (\gamma^t, \gamma_d)$ under H^d. Show that α is homotopic to γ under H^d (and similarly in the reverse order).

4.7.2. Weaker Notions of Homotopy Equivalence for Maps. Homotopy equivalence of maps as we have defined it is the gold standard notion of equivalence. But for many applications, it is too much to ask for and much more than is needed. Here we introduce two alternate notions that are frequently useful.

A morphism $\alpha : f \to g$ in $\mathrm{map}(\mathcal{T}_\circ)$ is a **pointwise homotopy equivalence** if α^d and α_t are both homotopy equivalences in $\mathcal{T}_\circ$. If we only require the square

$$\begin{tikzcd} A \arrow[r, "\alpha^d", "\simeq"'] \arrow[d, "f"'] & B \arrow[d, "g"] \\ X \arrow[r, "\alpha_t", "\simeq"'] & Y \end{tikzcd}$$

to be *homotopy* commutative, then the homotopy classes of f and g are pointwise equivalent in the category $\mathrm{mor}(\mathrm{H}\mathcal{T})$, so we say that α is a **pointwise equivalence** in $\mathrm{H}\mathcal{T}$.

Exercise 4.75.

(a) Show that if $f \simeq g : X \to Y$, then f and g are equivalent in $\mathrm{H}\mathcal{T}$.

(b) Find an example of a morphism in $\mathrm{map}(\mathcal{T})$ that is an equivalence in $\mathrm{H}\mathcal{T}$ but is not a pointwise homotopy equivalence.

(c) Find a morphism in $\mathrm{map}(\mathcal{T})$ that is a pointwise homotopy equivalence but not a homotopy equivalence of maps.

Hint. Look at Exercise 4.37.

(d) Generalize all three definitions to define concepts of homotopy equivalent diagrams.

Maps that are equivalent in $\mathrm{H}\mathcal{T}_*$ produce 'the same' map on sets of homotopy classes.

Problem 4.76. Suppose $f : A \to B$ and $g : X \to Y$ are equivalent in $\mathrm{H}\mathcal{T}_*$, and let Z be any space. Show that the diagrams

$$\begin{tikzcd}
{[A,Z]} & {[B,Z]} \arrow[l, "(\alpha^d)^*"'] \\
{[X,Z]} \arrow[u, "f^*"] & {[Y,Z]} \arrow[l, "(\alpha_t)^*"'] \arrow[u, "g_*"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
{[Z,A]} \arrow[r, "(\alpha^d)_*"] \arrow[d, "f_*"'] & {[Z,B]} \arrow[d, "g_*"] \\
{[Z,X]} \arrow[r, "(\alpha_t)_*"] & {[Z,Y]}
\end{tikzcd}$$

are commutative and that the horizontal maps are isomorphisms.

We say two maps f and g are **pointwise homotopy equivalent** if there is a chain of pointwise homotopy equivalences $f = f_0 \leftarrow f_1 \to f_2 \leftarrow \cdots \to f_n = g$.

Problem 4.77. Show that if $f \simeq g : X \to Y$, then f and g are pointwise homotopy equivalent.

4.7.3. Spaces under A or over B. We will make use of two other categories of maps: the category of **spaces under** A, which is denoted $A{\downarrow}\mathcal{T}_\circ$; and the category of **spaces over** B, denoted $\mathcal{T}_\circ{\downarrow}B$. The objects of $A{\downarrow}\mathcal{T}_\circ$ are maps $f : A \to X$ in $\mathcal{T}_\circ$, and morphisms from $\alpha : f \to g$ are commutative triangles

$$\begin{tikzcd}
& A \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X \arrow[rr, "\alpha"] & & Y.
\end{tikzcd}$$

If $\alpha : X \to Y$ is a map in $\mathcal{T}_\circ$ such that the diagram above commutes, then we'll say that α is a **map under** A.[7]

The category $\mathcal{T}_\circ{\downarrow}B$ of spaces over B is defined dually: the objects are maps $X \to B$ and the morphisms are maps $\alpha : X \to Y$ that fit into commutative triangles

$$\begin{tikzcd}
X \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& B; &
\end{tikzcd}$$

α is said to be a map **over** B.

Homotopy for Maps under A. There is a forgetful functor $A{\downarrow}\mathcal{T}_\circ \to \mathcal{T}_\circ$ which takes $f : A \to X$ to X and carries a map $\alpha : X \to Y$ (under A) to the same map α, but forgetting the commutativity of the triangle. There is an isomorphic copy of the category $A{\downarrow}\mathcal{T}_\circ$ contained in $\mathrm{map}(\mathcal{T}_\circ)$; it is the

[7]Maybe it should be called an A-**morphism**?

category of all maps of the form $A \to X$; and morphisms are squares of the form

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "f"'] & A \arrow[d, "g"] \\ X \arrow[r, "\alpha"] & Y. \end{tikzcd}$$

To define the **cylinder** on a map $f : A \to X$, it is easiest to go outside of the category $A\downarrow\mathcal{T}_\circ$; the **external cylinder** on f is the map

$$f \times \mathrm{id}_I : A \times I \longrightarrow X \times I.$$

If you prefer to stay within the category $A\downarrow\mathcal{T}_\circ$, define the **internal cylinder** to be the map $\mathrm{Cyl}(f)$ defined by the pushout diagram

$$\begin{tikzcd} A \times I \arrow[r, "\mathrm{pr}_1"] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & A \arrow[d, "\mathrm{Cyl}(f)"] \\ X \times I \arrow[r, "p"] & P \end{tikzcd}$$

with inclusion maps $i_0 = p \circ \mathrm{in}_0$ and $i_1 = p \circ \mathrm{in}_1 : X \to P$.

Conceptually, a homotopy in $A\downarrow\mathcal{T}_\circ$ from α to β is a deformation through maps under A. We can make this precise using the external cylinder: a **homotopy** from α to β under A is a homotopy in $\mathrm{map}(\mathcal{T}_\circ)$ of the special form

$$H = \left(\boxed{\mathrm{id}_A}, H_t\right) : f \times \mathrm{id} \simeq g \circ \mathrm{pr}_A$$

such that $H \circ \mathrm{in}_0 = \alpha^t$ and $H \circ \mathrm{in}_1 = \beta^t$. That is, it is a map $H_t : X \times I \to Y$ making the diagram

$$\begin{tikzcd} & A \times I \arrow[dl, "f\times\mathrm{id}"'] \arrow[dr, "g\circ\mathrm{pr}_1"] & \\ X \times I \arrow[rr, "H_t"] & & Y \end{tikzcd}$$

commutative. In terms of the internal cylinder, a homotopy is nothing more than a morphism $H : \mathrm{Cyl}(f) \to g$ such that the diagram

$$\begin{tikzcd} f \arrow[r, "i_0"] \arrow[dr, bend right, "\alpha"'] & \mathrm{Cyl}(f) \arrow[d, "H"] & f \arrow[l, "i_1"'] \arrow[dl, bend left, "\beta"] \\ & g & \end{tikzcd}$$

is commutative.

Exercise 4.78. Show that the two definitions of homotopy in $A\downarrow\mathcal{T}_\circ$ are equivalent.

A **homotopy equivalence** in $A{\downarrow}\mathcal{T}_\circ$ is a morphism $\alpha : f \to g$ with a homotopy inverse $\beta : g \to f$. A **pointwise homotopy equivalence** in $A{\downarrow}\mathcal{T}_\circ$ is a map $\alpha : X \to Y$ under A which is a homotopy equivalence in $\mathcal{T}_\circ$.

Exercise 4.79. Interpret the results of Problem 4.74 in terms of the categories $A{\downarrow}\mathcal{T}_\circ$ and $\mathcal{T}_\circ{\downarrow}B$.

Problem 4.80. Show that an inclusion $i : A \hookrightarrow X$ is a strong deformation retract if and only if the morphism

$$\begin{tikzcd} & A \arrow[dl, "\mathrm{id}_A"'] \arrow[dr, "i"] & \\ A \arrow[rr, "i"] & & X \end{tikzcd}$$

is a homotopy equivalence in $A{\downarrow}\mathcal{T}_\circ$.

Homotopy for Maps over B. The situation here is exactly the same; the only real difference is that the construction of the cylinder of a map $f : X \to B$ is much simpler: it is the composition

$$\begin{tikzcd} X \times I \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{Cyl}(f)"'] & X \arrow[dl, "f"] \\ B. & \end{tikzcd}$$

The inclusions $\mathrm{in}_0, \mathrm{in}_1 : X \to X \times I$ are maps over B, and so we can define a homotopy $H : \alpha \to \beta$ for two morphisms $\alpha, \beta : f \to g$ to be a morphism $H : \mathrm{Cyl}(f) \to g$ such that the diagram

$$\begin{tikzcd} f \arrow[r, "\mathrm{in}_0"] \arrow[dr, bend right, "\alpha"'] & \mathrm{Cyl}(f) \arrow[d, "H"] & f \arrow[l, "\mathrm{in}_1"'] \arrow[dl, bend left, "\beta"] \\ & g & \end{tikzcd}$$

commutes. A **homotopy equivalence** in $\mathcal{T}{\downarrow}B$ is a morphism $\alpha : f \to g$ with a homotopy inverse in $\mathcal{T}{\downarrow}B$.

4.7.4. Pushouts and Pullbacks as Functors.

We conclude by interpreting pushout and pullback as functors between categories of maps.

Problem 4.81. Let $\phi : A \to B$ be any map.

(a) Show that pushout defines a functor $\mathrm{PO}_\phi : A{\downarrow}\mathcal{T}_\circ \to B{\downarrow}\mathcal{T}_\circ$.

(b) Show that pullback defines a functor $\mathrm{PB}_\phi : \mathcal{T}_\circ{\downarrow}B \to \mathcal{T}_\circ{\downarrow}A$.

(c) Show that the functors PB_ϕ and PO_ϕ respect homotopy, in the sense that if $\alpha \simeq \beta$, then $\mathrm{PB}_\phi(\alpha) \simeq \mathrm{PB}_\phi(\beta)$.

HINT. Show they carry cylinders to cylinders.

(d) Show that if f and g are homotopy equivalent in $\mathcal{T}_\circ{\downarrow}B$, then $\phi^* f$ and $\phi^* g$ are homotopy equivalent in $\mathcal{T}_\circ{\downarrow}A$.

4.7.5. Maps into CW Pairs, Triples, etc. The notion of a pair has an obvious generalization, which we will not use, except here in this section. An ***n*-ad** is an ordered tuple

$$(X; X_1, X_2, \ldots, X_{n-1}),$$

where each X_i is a subspace of X. A map

$$f : (X; X_1, X_2, \ldots, X_{n-1}) \to (Y; Y_1, Y_2, \ldots, Y_{n-1})$$

of n-ads is simply a map $f : X \to Y$ in $\mathcal{T}_\circ$ such that $f(X_i) \subseteq Y_i$ for each i. We'll write

$$\mathrm{map}_{(n)}((X; X_1, X_2, \ldots, X_{n-1}), (Y; Y_1, Y_2, \ldots, Y_{n-1})) \subseteq \mathrm{map}_\circ(X, Y)$$

for the space of n-ad maps; it is given the subspace topology. An n-ad $(X; X_1, X_2, \ldots, X_{n-1})$ is a **compact** n-ad if X and each X_i are compact; it is a **CW *n*-ad** if X is a CW complex and each X_i is a subcomplex.

Problem 4.82.

(a) Explain how to view the space of maps between n-ads as an n-ad.

(b) Define homotopy for maps of n-ads.

Since we have a notion of homotopy, we can define homotopy equivalence of n-ads. J. Milnor [**132**] studied the spaces of n-ad maps involving n-ads homotopy equivalent to CW n-ads.

Theorem 4.83 (Milnor). *Let $(X, X_1, \ldots, X_{n-1})$ be homotopy equivalent in $\mathcal{T}_{(n)}$ to a CW n-ad. Then for any compact n-ad $(C; C_1, \ldots, C_{n-1})$, the space of maps*

$$\mathrm{map}_{(n)}((C; C_1, \ldots, C_{n-1}), (X; X_1, \ldots, X_{n-1}))$$

is also homotopy equivalent to a CW n-ad.

The proof of this theorem is almost entirely point-set-theoretical, so we take it for granted and leave its proof as a project. Its importance for us is that it implies that the loop space of a CW complex is homotopy equivalent to a CW complex.

Corollary 4.84. *If X is a pointed CW complex, then ΩX is homotopy equivalent in $\mathcal{T}_*$ to a pointed CW complex.*

Project 4.85. Prove Theorem 4.83 (see [**132**, Thm. 3]).

4.8. Additional Problems

Problem 4.86. Show that if $F : \mathcal{T} \to \mathcal{T}$ respects homotopy and $X \simeq Y$, then $F(X) \simeq F(Y)$.

Problem 4.87. Suppose $X \simeq Y$. Show that

(a) $\mathrm{map}_*(A, X) \simeq \mathrm{map}_*(A, Y)$ and $\mathrm{map}_\circ(A, X) \simeq \mathrm{map}_\circ(A, Y)$,

(b) $A \wedge X \simeq A \wedge Y$, $A \rtimes X \simeq A \rtimes Y$, and $A \times X \simeq A \times Y$.

Problem 4.88. Suppose X is a path-connected, noncontractible space. Is it possible that $\Omega X \simeq X$? Is it possible that $\Omega^2 X \simeq X$? Suppose you know that $\Omega^k X \simeq X$; what can you conclude about X?

Problem 4.89. Topologize the set $\mathrm{mor}(f, g)$ of morphisms $f \to g$ by expressing it as the pullback in a diagram involving ordinary mapping spaces. Show that homotopies of morphisms in $\mathrm{map}(\mathcal{T})$ correspond to paths in $\mathrm{mor}(f, g)$.

Problem 4.90. Show that if $f : I \to \mathbb{R}^n$ with $|f(0)|, |f(1)| \leq 1$, then there is a path homotopy from f to another path g with $|g(t)| \leq 1$ for all t.

Problem 4.91. Let $f : I \to \mathbb{R}^n$ and suppose $f(I) \cap C = \varnothing$ for some closed set $C \subseteq \mathbb{R}^n$. Show that f is path homotopic in $\mathbb{R}^n - C$ to a piecewise linear path g.

Problem 4.92. Let $f : X \to Y$ in $\mathcal{T}_*$. Show that f is a homotopy equivalence if and only if the induced map $f_* : [K, X] \to [K, Y]$ is bijective for all spaces $K \in \mathcal{T}_*$.

Problem 4.93.

(a) Show that if X is compact, there is a nullhomotopic map $S^n \to X$ that is surjective.

(b) If X is finite and $\alpha : S^n \to X$, then there is a map $\beta : S^n \to X$ such that $\beta \simeq \alpha$ and β is surjective.

(c) Show that if X and Y are finite complexes, then every map $f : X \to Y$ is homotopic to a surjective map.

(d) In general, what do you need to know about two CW complexes X and Y before you can conclude that every map $f : X \to Y$ is homotopic to a surjective map?

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Chapter 5

Cofibrations and Fibrations

Let $i : A \hookrightarrow X$ in $\mathcal{T}$, and suppose $f : X \to Y$ with $f \circ i = *$. Since the quotient X/A is the pushout of $* \leftarrow A \to X$, there is a unique factorization of f through a map $X/A \to Y$. Now put on your homotopy-theorist's hat, and ask: can something similar be done if we relax the condition $f \circ i = *$ to $f \circ i \simeq *$?

The answer is yes (though we must give up on uniqueness) provided the inclusion map $i : A \to X$ is a **cofibration**—a map with the property that every homotopy $A \times I \to Y$ from $f \circ i$ to some other map factors through a homotopy $X \times I \to Y$ of f to some other map. It turns out that most nicely behaved inclusion maps are cofibrations and—even better—every map is pointwise homotopy equivalent to a cofibration. Thus the power of the theory of cofibrations can be brought to bear on any map.

There is an entirely analogous dual situation. A map $p : E \to B$ is called a **fibration** if every homotopy $X \to B^I$ of $p \circ f$ to some other map factors through a homotopy $X \to E^I$ of f to some other map. Again, every map is pointwise homotopy equivalent to a fibration, so the homotopy-minded can generally apply theorems about fibrations to any map at all.

In Chapter 10 we will outline an axiomatic approach to homotopy theory, called **model categories**. This approach has made it possible to apply the ideas and techniques of homotopy theory to other areas of mathematics (e.g. algebra and algebraic geometry). A model category is a category with three special kinds of morphisms: cofibrations, fibrations and weak equivalences, which must satisfy a number of properties that hold in our topological situation. Anything we prove using these formal properties will automatically hold in all other homotopy theories.

However, we will also prove and apply two results (Proposition 5.18 and Theorem 5.20) about cofibrations that depend on special properties of the category of spaces. But since we have the wider perspective in mind, you should always prefer a 'formal' proof to one that applies only to spaces.

Homotopy theory as a whole, and this book in particular, tends to prefer to work with pointed spaces. But in this chapter it has seemed easiest to begin with a careful development in the unpointed context and come to the pointed cofibrations and fibrations later.

5.1. Cofibrations

In this first section, we define cofibrations in a variety of ways.

5.1.1. The Homotopy Extension Property. Suppose we are given maps $i : A \to X$ and $f : X \to Y$ in $\mathcal{T}_\circ$ and a homotopy $H_A : A \times I \to Y$ from $f \circ i$ to some other map $A \to Y$. An **extension** of the homotopy H_A to X is a homotopy $H : X \times I \to Y$ from f to some other map $X \to Y$ such that $H_A = H|_{A\times I}$. This is illustrated in the diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\mathrm{in}_0"'] & X \arrow[d, "\mathrm{in}_0"'] \arrow[ddr, bend left, "f"] & \\ A \times I \arrow[r, "i \times \mathrm{id}_I"] \arrow[drr, bend right, "H_A"'] & X \times I \arrow[dr, dotted, "H"] & \\ & & Y. \end{tikzcd}$$

If, for fixed f, *every* homotopy $H_A : A \times I \to Y$ extends to a homotopy $H : X \times I \to Y$, we say that the map $i : A \to X$ has the **homotopy extension property** with respect to f. If i has the homotopy extension property with respect to *all* maps out of X, then i is called a **cofibration**.

The homotopy extension property may also be expressed from the deformation point of view using the diagram

$$\begin{tikzcd} A \arrow[r, "H_A"] \arrow[d, "i"'] & Y^I \arrow[d, "@_0"] \\ X \arrow[r, "f"] \arrow[ur, dotted] & Y. \end{tikzcd}$$

Exercise 5.1. Show that the two diagrams express equivalent conditions.

Problem 5.2. Show that a composition of cofibrations is a cofibration.

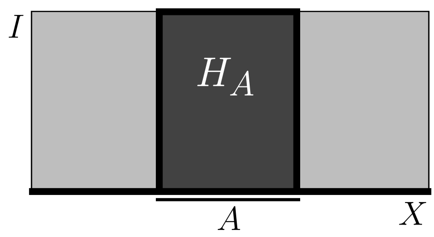

Figure 5.1. Extending a homotopy

Finally, we have a pictorial description which can help build some intuition for cofibrations. The inclusion $A \hookrightarrow X$ is a cofibration if any function defined on the darkly shaded portion of Figure 5.1 may be extended to the whole rectangle (i.e., the whole cylinder $X \times I$). No guarantees are made about the behavior of the extended homotopy outside the dark region, other than continuity.

Cofibrations allow you to replace homotopy commutative squares with strictly commutative ones.

Problem 5.3. Suppose the solid arrow part of the diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, dotted, "\gamma"'] \arrow[d, bend left, "g"] \\ C \arrow[r, "j"] & D \end{tikzcd}$$

is homotopy commutative. Show that if i is a cofibration, then g can be replaced with a homotopic map $\gamma : B \to D$ making the square strictly commutative.

Problem 5.4. Show that the inclusion $\varnothing \to X$ is a cofibration.

In general, if the unique map from the initial object to X is a cofibration, then X is called **cofibrant**. In the current context, this condition doesn't carry any weight, since all spaces are cofibrant.

5.1.2. Point-Set Topology of Cofibrations. Every cofibration in $\mathcal{T}_\circ$ is equivalent to the inclusion map of a closed subspace which is a neighborhood retract.

Problem 5.5.

(a) Show that every homeomorphism is a cofibration.

(b) Show that a cofibration $i : A \to X$ is an embedding. That is, show that the map $j : A \to i(A)$ obtained from i by restricting the target is a homeomorphism.

(c) Show that if $i : A \to X$ is an embedding, then i is a cofibration if and only if the inclusion $i(A) \hookrightarrow X$ is a cofibration.

(d) Show that if $i : A \hookrightarrow X$ is a cofibration, then there is an open set $U \subseteq X$ containing A such that A is a retract of U.

(e) Show that if $i : A \hookrightarrow X$ is a cofibration, then A is a closed subset of X.

HINT. Remember that all spaces in $\mathcal{T}_\circ$ are weak Hausdorff.

(f) Show that the inclusions $\{0\} \hookrightarrow \{0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\}$ and $\mathbb{Q} \hookrightarrow \mathbb{R}$ are not cofibrations. Find explicit homotopies that cannot be extended.

It can be conceptually useful to be able to package all the technical language in the definition of a cofibration in the slogan 'well-behaved inclusion map'. On the formal side, though, Problem 5.5(e) is the most important part for us. In the next section, we will prove Theorem 5.20, which in the category **Top** only applies to *closed* cofibrations; but since every cofibration in $\mathcal{T}_\circ$ is closed, we can dispense with this hypothesis.

Exercise 5.6. The definition of cofibration makes sense in the category **Top** of all topological spaces. Find an example of a cofibration $i : A \to X$ in **Top** whose image is not closed.

5.1.3. Two Reformulations. The concept of cofibration can be expressed in a variety of ways, and different formulations can be more useful than others in certain situations. We derive two formal reinterpretations by formally mucking around with diagrams.

Given a map $i : A \to X$, we define a space T, a map f and a homotopy H_A by completing the prepushout diagram $A \times I \leftarrow A \to X$ to the pushout square

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\mathrm{in}_0"'] \arrow[dr, phantom, "\text{pushout}"] & X \arrow[d, "f"] \\ A \times I \arrow[r, "H_A"] & T. \end{tikzcd}$$

This presents a homotopy extension problem that serves as a 'universal example' for the cofibration property: if you can extend *this* homotopy, then you can extend *any* homotopy.

Problem 5.7. Show that i is a cofibration if and only if the identity id_T can be extended to a homotopy $H : X \times I \to T$.

The diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\mathrm{in}_0"'] & X \arrow[d, "\mathrm{in}_0"'] \arrow[ddr, bend left, "f"] & \\
A \times I \arrow[r, "i \times \mathrm{id}_I"] \arrow[drr, bend right, "H_A"'] & X \times I \arrow[dr, dotted, "H"] & \\
& & Y
\end{tikzcd}$$

used to define cofibrations should remind you of the definition of the pushout.

Proposition 5.8. *Let $i : A \hookrightarrow X$ in $\mathcal{T}_\circ$. Then the following are equivalent:*

(1) *i is a cofibration,*

(2) *the inclusion $T \hookrightarrow X \times I$ has a retraction $r : X \times I \to T$.*

Corollary 5.9. *If $i : A \hookrightarrow X$ is a cofibration in $\mathcal{T}_\circ$, then for any space Y, the inclusion $A \times Y \hookrightarrow X \times Y$ is also a cofibration.*

Problem 5.10. Prove Proposition 5.8 and derive Corollary 5.9.

Hint. Consider the case $Y = T$.

Here are two extremely important families of cofibrations.

Problem 5.11.

(a) Show that the inclusion $i : S^n \hookrightarrow D^{n+1}$ is a cofibration.

(b) Show that the inclusions $\mathrm{in}_0, \mathrm{in}_1 : A \to A \times I$ are cofibrations.

5.1.4. Cofibrations and Pushouts. One of the most crucial properties of cofibrations is that they are preserved by the formation of pushouts.

Theorem 5.12. *Let $i : A \to X$ be a cofibration, and suppose the square*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] \arrow[dr, phantom, "\text{pushout}"] & B \arrow[d, "j"] \\
X \arrow[r] & Y
\end{tikzcd}$$

is a pushout square. Then $j : B \to Y$ is also a cofibration.

Problem 5.13. Prove Theorem 5.12 by studying the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & B \arrow[r] \arrow[d, "j"] & Z^I \arrow[d, "@_0"] \\
X \arrow[r] \arrow[urr, dotted] & Y \arrow[r] \arrow[ur, dotted] & Z.
\end{tikzcd}$$

Corollary 5.14. *If $A \hookrightarrow X$ is a cofibration and $T = X \cup (A \times I)$, then the inclusion $A \times I \hookrightarrow T$ is a cofibration.*

Problem 5.15. Prove Corollary 5.14.

It follows immediately from Theorem 5.12 that the inclusion $X \hookrightarrow T$ is a cofibration, regardless of the nature of $A \hookrightarrow X$.

Corollary 5.16. *If $B \hookrightarrow Y$ and $A \hookrightarrow X$ are cofibrations, then $A \times B \hookrightarrow (A \times Y) \cup (X \times B)$ is a cofibration.*

Problem 5.17. Prove Corollary 5.16.

HINT. Use Corollary 5.9.

5.2. Special Properties of Cofibrations of Spaces

We establish a second pair of equivalent conditions by clever algebraic manipulation of the cylinder coordinate in $\mathrm{Cyl}(X) = X \times I$. Results proved using these characterizations are not automatically valid in all abstract homotopy theories.

5.2.1. The Power of a Parametrized Cylinder. We give two alternative characterizations of cofibrations, this in terms of homotopies together with real-valued functions.

Proposition 5.18. *Show that $i : A \hookrightarrow X$ in $\mathcal{T}_\circ$ is a cofibration if and only if the subspace $T \subseteq X \times I$ is a strong deformation retract of $X \times I$.*

If i is a cofibration, then $T \hookrightarrow X \times I$ has a retraction $r : X \times I \to T$, which we write in the form $r(x,t) = (r_1(x,t), r_2(x,t))$, where $r_1 : X \times I \to X$ and $r_2 : X \times I \to I$.

Problem 5.19. Use the formula $H((x,t),s) = (r_1(x,st), (1-s)t + sr_2(x,t))$ to prove Proposition 5.18.

A subspace $A \subseteq X$ is a (strong) **neighborhood deformation retract** if there is an open set $U \subseteq X$ containing A and a strong deformation retraction $r : U \to A$. The inclusion $A \hookrightarrow X$ being a cofibration is *almost* equivalent to A being a strong neighborhood deformation retract.

Theorem 5.20 (Strøm[1]). *The inclusion $i : A \hookrightarrow X$ in $\mathcal{T}_\circ$ is a cofibration if and only if there is function $\phi : X \to I$ such that $A = \phi^{-1}(0)$ is a strong neighborhood deformation retract of $U = \phi^{-1}([0,1))$.*

An inclusion $A \hookrightarrow X$ of a closed subspace in the category **Top** satisfying the conditions of Theorem 5.20 is called a **closed cofibration**. Some authors refer to a pair (X, A) in which $i : A \hookrightarrow X$ is a closed cofibration as an **NDR pair**. By Problem 5.5(e), all cofibrations in $\mathcal{T}_\circ$ are inclusions

[1]No relation!

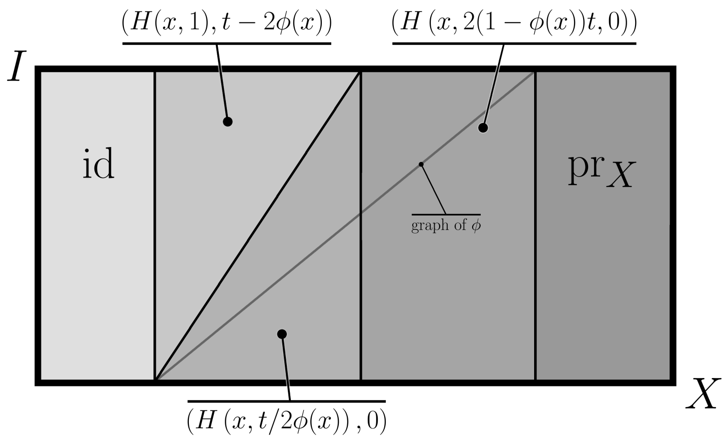

Figure 5.2. Formula for a retraction

of closed subsets, so for us Theorem 5.20 is a complete characterization of cofibrations.

Problem 5.21.

(a) Suppose $i : A \hookrightarrow X$ is a cofibration, and let $r = (r_1, r_2) : X \times I \to T$ be the retraction guaranteed by Proposition 5.8. Show that
- $U = \{x \in X \mid r_1(x, 1) \in A\}$,
- $H = \mathrm{pr}_1 \circ r|_{U \times I}$,
- $\phi(x) = \sup_{t \in I} |t - r_2(x, t)|$

satisfy the conditions of Theorem 5.20.

(b) Now suppose the set U, the deformation H and the function ϕ are given, and define a retraction $r : X \times I \to T$ as indicated in Figure 5.2. Verify that r is continuous and satisfies the conditions of Theorem 5.20.

5.2.2. Mapping Spaces into Cofibrations. Finally we study what happens when we put a (domain-type) cofibration in the target position of a mapping space. Formally, nothing good can come of this, so the proof must involve special properties of cofibrations of *spaces*. It should come as no surprise that we have to invoke Theorem 5.20.

Theorem 5.22. *If $i : A \to X$ is a cofibration in $\mathcal{T}_\circ$ and Q is compact, then the induced maps*

$$\mathrm{map}_\circ(Q, A) \longrightarrow \mathrm{map}_\circ(Q, X) \qquad \textit{and} \qquad \mathrm{map}_*(Q, A) \longrightarrow \mathrm{map}_*(Q, X)$$

are also cofibrations.

Problem 5.23. Let $i : A \to X$ be a cofibration in $\mathcal{T}_\circ$. Given a function $\phi : X \to I$ as in Theorem 5.20, define $\theta : \mathrm{map}_\circ(Q, X) \to I$ by

$$\theta(f) = \max\{\phi \circ f(Q) \subseteq I\},$$

and use it to prove Theorem 5.22.

5.2.3. Products and Cofibrations. Next we show that the map that arises when we form the 'product' of two pairs of spaces is a cofibration. The proof relies on Theorem 5.20, so it is a special feature of the category of spaces, and you should not expect it to hold in other homotopy theories.

Proposition 5.24. *If $A \to X$ and $B \to Y$ are cofibrations, then*

$$(A \times Y) \cup (X \times B) \longrightarrow X \times Y$$

is also a cofibration.

Let $u : X \to I$ and $H : X \times I \to X$ be the maps guaranteed by Theorem 5.20 because $A \to X$ is a cofibration, and likewise let $v : Y \to I$ and $K : Y \times I \to Y$ be maps which show that $B \to Y$ is a cofibration.

Problem 5.25. Show that the maps

$$\psi(x, y) = \min\{\phi(x), \theta(y)\} \qquad \text{and}$$

$$J(x, y, t) = \begin{cases} \left(H(x,t), K\left(y, t\frac{\phi(x)}{\theta(y)}\right)\right) & \text{if } \theta(y) \geq \phi(x), \\ \left(H\left(x, t\frac{\theta(y)}{\phi(x)}\right), K(y,t)\right) & \text{if } \phi(x) \geq \theta(y) \end{cases}$$

satisfy the conditions of Theorem 5.20, and so prove Proposition 5.24.

Proposition 5.24 implies that a product of cofibrations is also a cofibration.

Corollary 5.26. *If $A \to X$ and $B \to Y$ are cofibrations, then $A \times B \to X \times Y$ is a cofibration.*

Problem 5.27. Prove Corollary 5.26.

Proposition 5.24 is very useful in constructing homotopies. Suppose $f, g : X \to Y$ are homotopic by a homotopy H, and there is a homotopy-of-homotopies from $H_A = H|_{A \times I}$ to some other homotopy $K : A \times I \to Y$. If $A \hookrightarrow X$ is a cofibration, then Proposition 5.24 allows us to find the extension indicated in Figure 5.3. Thus we can sometimes construct a new homotopy from f to g which restricts to K on $A \times I$.

This same technique can be used more generally: if f is homotopic to ϕ and g is homotopic to γ by homotopies constant on A, then there is a homotopy from ϕ to γ that restricts to K.

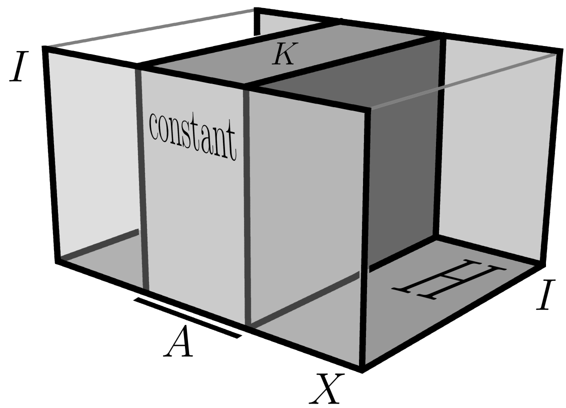

Figure 5.3. Extending a homotopy-of-homotopies

5.3. Fibrations

Fibrations are the dual concept to cofibrations, and our discussion of them is largely parallel to, but briefer than, our discussion of cofibrations.

5.3.1. Dualizing Cofibrations. To see the duality clearly, we rewrite the homotopy extension property with respect to f using the cylinder object notation, like so:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "i_0"'] & X \arrow[d, "i_0"] \arrow[ddr, bend left, "f"] & \\
\mathrm{Cyl}(A) \arrow[r, "\mathrm{Cyl}(i)"] \arrow[drr, bend right, "H_A"'] & \mathrm{Cyl}(X) \arrow[dr, dotted, "H"] & \\
 & & Y.
\end{tikzcd}$$

Since dualization reverses arrows and replaces cylinder objects with path objects, the dual property—called the **homotopy lifting property** with respect to f—is the condition set out by the diagram

$$\begin{tikzcd}
X \arrow[dr, dotted, "K"] \arrow[drr, bend left, "K_B"] \arrow[ddr, bend right, "f"'] & & \\
 & \mathrm{Path}(E) \arrow[r] \arrow[d, "p_0"] & \mathrm{Path}(B) \arrow[d, "p_0"] \\
 & E \arrow[r, "p"] & B.
\end{tikzcd}$$

If for every right homotopy K_B there is a right homotopy K making the diagram commute, then we say that p has the **homotopy lifting property** with respect to f. If it has the homotopy lifting property with respect to *every* map f, then p is a **fibration**.

This is a fine definition, but what does it mean? We are given a map $f : X \to E$, and hence the composite map $\phi = p \circ f : X \to B$; we are also given a homotopy K_B from ϕ to some other map γ. If p is a fibration, then a new homotopy K from f to some other map g such that $p \circ g = \gamma$ can be found. This may be expressed in the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{in}_0"'] & E \arrow[d, "p"] \\ \mathrm{Cyl}(X) \arrow[r, "H_B"'] \arrow[ur, dotted, "H"] & B. \end{tikzcd}$$

The idea here is that you have a *partial lift* of H—that is, the map f lifts the part of H along the bottom level of the cylinder. The question is whether or not that partial lift can be extended to a full lift. This is why the defining property of fibrations is called the **homotopy lifting property**.

When $p : E \to B$ is a fibration, we say that B is the **base** of the fibration and E is its **total space**. For $b \in B$, the space $p^{-1}(b) \subseteq E$ is called the **fiber** of p over b.[2]

Exercise 5.28. Express the fiber F of $p : E \to B$ over $b \in B$ as a pullback.

Fibrations can be used to modify a homotopy commutative square, rendering it strictly commutative.

Problem 5.29. Suppose that the solid arrow part of the square

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, dotted, "\phi"'] \arrow[d] & E \arrow[d, "p"] \\ Y \arrow[r] & B \end{tikzcd}$$

is homotopy commutative. Show that if $p : E \to B$ is a fibration, then f can be replaced with a homotopic map $\phi : X \to E$ making the square strictly commutative.

Lifting Functions. The dual of Problem 5.7 offers a useful point of view on fibrations. It also plays an important role in the detailed point-set level

[2]It can (and will) be shown that if B is path-connected, then any two fibers are homotopy equivalent to each other.

study of fibrations. The dual of the space T is simply the pullback Ω_p in the diagram

$$\begin{tikzcd}
E^I \arrow[dr, dotted, "q"] \arrow[drr, bend left, "p_*"] \arrow[ddr, bend right, "@_0"'] & & \\
& \Omega_p \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & B^I \arrow[d, "@_0"] \\
& E \arrow[r, "p"] & B.
\end{tikzcd}$$

The commutativity of the diagram defines a unique map $q : E^I \to \Omega_p$.

Problem 5.30. Show that $p : E \to B$ is a fibration if and only if the map $q : E^I \to \Omega_p$ has a section $\lambda : \Omega_p \to E^I$ such that $q \circ \lambda = \mathrm{id}_{\Omega_p}$.

The map λ in Problem 5.30 is called a **lifting function**. Suppose we are given a path $\omega : I \to B$ and we wish to lift to a path $\alpha : I \to E$ such that $p \circ \alpha = \omega$. Certainly it must be true that the point $e = \alpha(0)$ must satisfy $p(e) = \omega(0)$. A lifting function assigns—in a continuous way—to each path $\omega : I \to B$ and each $e \in E$ such that $p(e) = \omega(0)$ a definite choice of path $\alpha : I \to E$ such that $p \circ \alpha = \omega$.

5.3.2. Some Examples. Let's get a feel for some kinds of maps that turn out to be fibrations.

We need some notation to describe the mutilation and manipulation of paths. Given paths $\alpha, \beta : I \to Y$ with $\alpha(1) = \beta(0)$, we define a **parametrized concatenation** $*_{[s]}$ for $s \in I$ by the rule

$$(\alpha *_{[s]} \beta)(t) = \begin{cases} \alpha((1+s)t) & \text{if } t \leq \frac{1}{1+s}, \\ \\ \beta((1+s)t - 1) & \text{if } t \geq \frac{1}{1+s}. \end{cases}$$

Exercise 5.31. What are $\alpha *_{[0]} \beta$ and $\alpha *_{[1]} \beta$? What is $(\alpha *_{[s]} \beta)(1)$?

Problem 5.32.

(a) Let $\alpha, \beta \in Y^I$ with $\alpha(1) = \beta(0)$. Show that there is a lift $\bar\beta$ such that $\bar\beta(0) = \alpha$ and the diagram

$$\begin{tikzcd}
& Y^I \arrow[d, "@_1"] \\
I \arrow[ur, dotted, bend left, "\bar\beta"] \arrow[r, "\beta"] & Y
\end{tikzcd}$$

is strictly commutative

(b) Show that the evaluation map $@_1 : Y^I \to Y$ is a fibration in $\mathcal{T}_\circ$.

(c) Determine the fiber over $y \in Y$.

(d) Show that if Y is path-connected, then any two fibers are homotopy equivalent to each other.

Problem 5.33. Show that the projection $\mathrm{pr}_B : B \times F \to B$ is a fibration. What is the fiber over b?

Problem 5.33 implies that for any space X, the trivial map $X \to *$ is a fibration; this is sometimes expressed by saying that every space is a **fibrant** space.

5.3.3. Pullbacks of Fibrations. The fact that the pushout of a cofibration is again a cofibration is one of the most powerful tools in the study of cofibrations. The dual statement for fibrations is also true, and also extremely useful.

Theorem 5.34. *Suppose the square*

$$\begin{tikzcd} P \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\text{pullback}"] & E \arrow[d, "p"] \\ A \arrow[r] & B \end{tikzcd}$$

is a pullback square and that p is a fibration. Then q is also a fibration.

Problem 5.35. Prove Theorem 5.34.

Here is an easy and important application.

Problem 5.36.

(a) Show that the evaluation map $@_0 : \mathcal{P}(X) \to X$ is a fibration, no matter what X is.

(b) Determine the fiber of $@_0$, and show that if X is path-connected, any two fibers are homotopy equivalent.

(c) Show by example that if X is not path-connected, then there can be fibers which are not homotopy equivalent.

5.4. Factoring through Cofibrations and Fibrations

One of the foundations of homotopy theory is the fact that every map $f : X \to Y$ is, in a functorial way, pointwise homotopy equivalent to a cofibration and, dually, pointwise homotopy equivalent to a fibration. These equivalences make it possible to apply the theory of cofibrations and fibrations to the study of *all* maps.

5.4.1. Mapping Cylinders. Let $f : X \to Y$ be any map in $\mathcal{T}_\circ$, and define $M_f = Y \cup_f (X \times I)$ to be the pushout in the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{in}_0"'] \arrow[dr, phantom, "\text{pushout}"] & Y \arrow[d] \\ X \times I \arrow[r] & M_f . \end{tikzcd}$$

The space M_f is called the **mapping cylinder** of f.

Problem 5.37.

(a) Show that there is a pushout square

$$\begin{tikzcd} X \sqcup X \arrow[r, "f \sqcup \mathrm{id}_X"] \arrow[d, "(\mathrm{in}_0, \mathrm{in}_1)"'] \arrow[dr, phantom, "\text{pushout}"] & Y \sqcup X \arrow[d, "(r, j)"] \\ X \times I \arrow[r] & M_f . \end{tikzcd}$$

(b) Conclude that the maps $j : X \to M_f$ given by $j(x) = (x, 1)$ and $r : Y \to M_f$ given by $y \mapsto (y, 0)$ are cofibrations.[3]

(c) Show that M_f and Y are homotopy equivalent spaces.

HINT. Find a retraction $M_f \to Y$.

(d) Show that the diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, equal] & M_f \arrow[d, "\simeq"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is strictly commutative.

(e) Show that this whole discussion, including the homotopy $r \circ q \simeq \mathrm{id}_{M_f}$, is natural. Carefully state what categories and functors are involved.

You have proved the following theorem.

Theorem 5.38. *Every map $f : X \to Y$ in $\mathcal{T}_\circ$ fits into a strictly commutative diagram*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, equal] & M_f \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in which j is a cofibration and q is a homotopy equivalence in $\mathcal{T}_\circ$. Furthermore, the construction is functorial on the category $\mathrm{map}(\mathcal{T}_\circ)$.

[3] It is notationally convenient to view Y as $Y \times 0$ in this diagram.

The process of replacing a nasty map f with a homotopy equivalent cofibration is often referred to as 'converting f to a cofibration'.

5.4.2. Converting a Map to a Fibration. We can also convert any map $f : X \to Y$ into a fibration. Let E_f be the pullback in the diagram

$$\begin{tikzcd} E_f \arrow[r, "g"] \arrow[d, "r"'] \arrow[dr, phantom, "\text{pullback}"] & Y^I \arrow[d, "@_0"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

so that

$$E_f = \{(x, \alpha) \in X \times Y^I \mid \alpha(0) = f(x)\}$$

and q and g are the projections on the first and second coordinates, respectively. We define a map $p : E_f \to Y$ by the formula $p(x, \alpha) = \alpha(1)$.

The map $p : E_f \to Y$ is a fibration that is pointwise homotopy equivalent to f, as can be proved by dualizing to the argument just given for the mapping cylinder; but we take another approach. We need to show that there is a lift K of the homotopy H in the diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "\mathrm{in}_0"'] & E_f \arrow[d, "p"] \\ A \times I \arrow[r, "H"'] \arrow[ur, dotted, "K"] & Y. \end{tikzcd}$$

The map h has the form $h(a) = (h_0(a), \omega_a)$, where $h_0 = H|_{A\times 0} : A \to Y$ and $\omega_a : I \to Y$ with $\omega_a(0) = h_0(a)$. Write $\tau_a : I \to Y$ for the path $t \mapsto H(a, t)$.

Problem 5.39. Use the parametrized concatenation to write down an explicit homotopy K, and so prove that $p : E_f \to Y$ is a fibration.

HINT. Note that $\tau_a(0) = \omega_a(1)$.

Problem 5.40.

(a) Show that $r : E_f \to X$ is a homotopy equivalence whose inverse $i : X \to E_f$ is given by $x \mapsto \left(x, \boxed{f(x)}\right)$, where $\boxed{f(x)}$ is the constant path at $f(x)$.

HINT. Gently shrink the path coordinate of E_f down to a constant path.

(b) Show that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, equal] \\ E_f \arrow[r, "p"'] & Y \end{tikzcd}$$

is strictly commutative.

(c) Show that the whole discussion is natural, including the maps r and i and the homotopy $\mathrm{id}_{E_f} \simeq i \circ r$.

You have proved that every map f can be converted to a fibration.

Theorem 5.41. *Every map $f : X \to Y$ in $\mathcal{T}_\circ$ fits into a strictly commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"', "\simeq"] & Y \arrow[d, equal] \\ E_f \arrow[r, "p"] & Y \end{tikzcd}$$

in which p is a fibration and i is a homotopy equivalence in $\mathcal{T}_\circ$. Furthermore, the entire construction is functorial on the category of maps in $\mathcal{T}_\circ$.

The process of replacing a map with a homotopy equivalent fibration is referred to a 'converting f to a fibration'.

5.4.3. Two Double Factorizations.

The factorizations of f

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "i"'] \arrow[dr, "f" description] & M_f \arrow[d, "q"] \\ E_f \arrow[r, "p"'] & Y \end{tikzcd}$$

are very nice, but we have not made any attempt to impose conditions on the maps q and i. In some situations, it is very useful to know that these factorizations can be made so that q is a fibration and i is a cofibration.

Theorem 5.42 (Strøm)**.** *Every map $f : X \to Y$ in $\mathcal{T}_\circ$ has factorizations*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "i"'] \arrow[dr, "f" description] & \overline{M}_f \arrow[d, "q"] \\ \overline{E}_f \arrow[r, "p"'] & Y \end{tikzcd}$$

in which

- *i and j are cofibrations,*
- *p and q are fibrations, and*
- *i and q are homotopy equivalences.*

The plan of our proof is to find the factorizations in the diagram

$$
\begin{tikzcd}
X \arrow[rr, "\simeq"] \arrow[dr, "\text{cof}", "\simeq"'] \arrow[dd, "\text{cof}"'] & & E_f \arrow[dd, "\text{fib}"] \\
& \overline{E}_f \arrow[ur, "\text{fib}"', "\simeq"] \arrow[dl, "\text{cof}"'] \arrow[dr, "\text{fib}"] & \\
\overline{M}_f \arrow[rr, "\text{fib}", "\simeq"'] & & Y.
\end{tikzcd}
$$

Problem 5.43. Explain how the diagram implies Theorem 5.42.

Modified Mapping Cylinders. To establish these factorizations, we introduce two variants of the mapping cylinder construction.

First of all, given a subspace $A \subseteq Z$, write

$$Q(A, Z) = (A \times 0) \cup (Z \times (0, 1]) \subseteq Z \times I.$$

The projection on the first coordinate defines a function $p : Q(A, Z) \to Z$.

Problem 5.44.

(a) Show that $p : Q(A, Z) \to Z$ is a fibration.

(b) Show that if A is a strong deformation retract of Z, then $A \times 0$ is a strong deformation retract of $Q(Z, A)$.

(c) Show that if A is a strong deformation retract of Z, then the inclusion $A \times 0 \hookrightarrow Q(Z, A)$ is a cofibration.

The next variant is both more straightforward and more tricky than the first. We start with a fibration $p : E \to B$ and form the mapping cylinder $q : M_p \to B$, which we know is a homotopy equivalence. We might hope that q would also be a fibration, and in fact it is in many cases.

Problem 5.45. Let ℓ be a lifting function for p, and construct a (possibly discontinuous) lifting function λ for q.

To guarantee the continuity of λ, we modify the topology of the mapping cylinder M_p slightly, resulting in a new space Z_p having the same underlying set. We give Z_p the smallest topology such that

- $q : Z_p \to B$ is continuous,
- the function $r : Z_p \to I$ given by $r([e, t]) = t$ and $r([b]) = 0$ for $e \in E, b \in B$ is continuous, and
- each 'rectangle' $U \times (a, b)$ is open, where $0 < a < b \leq 1$ and $U \subseteq E$ is open.

Problem 5.46.

(a) Show that $q : Z_p \to Y$ is a homotopy equivalence.

(b) Show that the function λ you found in Problem 5.45 is a lifting function for $q : Z_p \to B$.

(c) Show that the inclusion $E \hookrightarrow Z_p$ is a cofibration.

Now you are equipped to construct the factorizations of $f : X \to Y$ necessary to establish Theorem 5.42.

Problem 5.47.

(a) Show that if $f : X \to Y$, then Z is a strong deformation retract of E_f.

(b) Prove Theorem 5.42.

5.5. More Homotopy Theory in Categories of Maps

In Section 4.7 we set up the beginnings of the homotopy theory of the categories map($\mathcal{T}$) with the warning that it is quite a bit more subtle than the homotopy theory of $\mathcal{T}$ and with the assurance that we won't need to delve too deeply into it. In this section we establish everything we need concerning the homotopy theory of mapping categories.

5.5.1. Mapping Cylinders in Mapping Categories. To construct the mapping cylinder of a morphism $\alpha = (\alpha^d, \alpha_t) : f \to g$ in map($\mathcal{T}_\circ$), first form the mapping cylinders M_{α^d} and M_{α_t} in $\mathcal{T}_\circ$. The naturality of the mapping cylinder construction in $\mathcal{T}_\circ$ gives us the factorization

$$\begin{tikzcd}
A \arrow[rr, "\alpha^d"] \arrow[dr, "j_A"'] \arrow[dd, "f"'] & & B \arrow[dd, "g"] \\
& M_{\alpha^d} \arrow[ur, "q_B"'] \arrow[dd] & \\
X \arrow[rr, "\alpha_t"] \arrow[dr, "j_X"'] & & Y \\
& M_{\alpha_t}. \arrow[ur, "q_Y"'] &
\end{tikzcd}$$

The map $M_{\alpha^d} \to M_{\alpha_t}$ is the **mapping cylinder** of α, denoted M_α. These mapping cylinders enjoy the same formal factorization and homotopy equivalence properties in map($\mathcal{T}_\circ$) as the ordinary mapping cylinders do in $\mathcal{T}_\circ$.

Problem 5.48. Let $\alpha : f \to g$ in $\mathrm{map}(\mathcal{T}_\circ)$.

(a) Show that there is a pushout diagram

$$\begin{tikzcd} f \arrow[r, "\alpha"] \arrow[d, "\mathrm{in}_0"'] & g \arrow[d] \\ f \times I \arrow[r] & M_\alpha. \end{tikzcd}$$

(b) Show that there are inclusions $f \to M_\alpha$ and $g \to M_\alpha$ and a deformation retraction $M_\alpha \to g$ in $\mathrm{map}(\mathcal{T}_\circ)$.

We will need mapping cylinders in the category $A{\downarrow}\mathcal{T}_\circ$, and we give a conceptual definition: the mapping cylinder of $\alpha : f \to g$ is the pushout in the diagram

$$\begin{tikzcd} f \arrow[r, "\alpha"] \arrow[d, "\mathrm{in}_0"'] \arrow[dr, phantom, "\text{pushout}"] & g \arrow[d] \\ \mathrm{Cyl}(f) \arrow[r] & M_\alpha \end{tikzcd}$$

in the category $A{\downarrow}\mathcal{T}_\circ$ (which means that we must use the internal cylinder).

Exercise 5.49. Give a hands-on point-set topological construction of M_α.

Problem 5.50. Show that if $\alpha : f \to g$ in $A{\downarrow}\mathcal{T}_\circ$, there are inclusions $f \to M_\alpha$, $g \to M_\alpha$ and a deformation retraction $M_\alpha \to g$ in $A{\downarrow}\mathcal{T}_\circ$.

The construction of mapping cylinders in the category $\mathcal{T}_\circ{\downarrow}B$ as a pushout involving the cylinder on f is entirely analogous.

5.5.2. Homotopy Inverses for Pointwise Equivalences. A pointwise homotopy equivalence $\alpha : f \to g$ may or may not have a homotopy inverse in $\mathrm{map}(\mathcal{T})$. The difficulty is that the homotopies that demonstrate that α^d and α_t are homotopy equivalences need not be compatible with each other. But we can impose our will on these homotopies, forcing them into compatibility, provided f and g are cofibrations.

We specialize now to the category $A{\downarrow}\mathcal{T}_\circ$; and for simplicity, we will write α instead of α_t (since $\alpha^d = \mathrm{id}_A$).

Exercise 5.51.

(a) Show that if $A = \varnothing \in \mathcal{T}_\circ$, then $A{\downarrow}\mathcal{T}_\circ$ is isomorphic to $\mathcal{T}_\circ$.

(b) Show that if $A \in \mathcal{T}_*$, then the categories $(A_-){\downarrow}\mathcal{T}_\circ$ and $A{\downarrow}\mathcal{T}_*$ are canonically isomorphic.

In view of Exercise 5.51, we'll lose absolutely no generality if we work entirely in the category $A{\downarrow}\mathcal{T}_\circ$.

Lemma 5.52. *Consider the strictly commutative diagram*

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, shift left, "\alpha"] \arrow[rr, shift right, "\beta"'] & & Y. \end{tikzcd}$$

Suppose f is a cofibration and that we are given a homotopy $H : \alpha \simeq \beta$. If there is a homotopy-of-homotopies from $H_A = H \circ (f \times \mathrm{id}_I)$ to some other homotopy K_A, then there is a homotopy $K : \alpha \simeq \beta$ such that $K_A = K \circ (f \times \mathrm{id}_I)$.

Problem 5.53. Prove Lemma 5.52.

HINT. Show that $(X \times 0) \cup (A \times I) \cup (X \times 1) \hookrightarrow X \times I$ is a cofibration.

Exercise 5.54. Can you prove that $(X \times 0) \cup (A \times I) \cup (X \times 1) \hookrightarrow X \times I$ is a cofibration using only formal properties? Or is this a special fact about spaces?

If a morphism $\alpha : f \to g$ is a homotopy equivalence in $A{\downarrow}\mathcal{T}_\circ$, then $\alpha : X \to Y$ must be a homotopy equivalence in $\mathcal{T}_\circ$. But what about the converse? If $\alpha : X \to Y$ is a map under A and it is a homotopy equivalence in $\mathcal{T}_\circ$, must it be a homotopy equivalence in $A \downarrow \mathcal{T}_\circ$? The difficulty is that we will be given a map $\beta : X \to Y$ and a homotopy $H : X \times I \to X$ from $\beta \circ \alpha$ to id_X, but we don't know that β is a map under A, and even if it is, we can't be sure that the restriction $H_A = H|_{A \times I}$ of the homotopy will be a constant homotopy.

Nevertheless, if the maps f and g are cofibrations, then we can modify the given map and homotopies, forcing them to be constant on A.

Theorem 5.55. *If $f : A \to X$ and $g : A \to Y$ are cofibrations and $\alpha : f \to g$ is such that $\alpha : X \to Y$ is a homotopy equivalence in $\mathcal{T}_\circ$, then α is a homotopy equivalence in $A{\downarrow}\mathcal{T}_\circ$.*

We begin the proof of the theorem by reducing to a special case.

Problem 5.56.

(a) Show that there is a map $\gamma : Y \to X$ which is a left homotopy inverse for α in $\mathcal{T}_\circ$ and which has the additional property that the diagram

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \arrow[ll, "\gamma"'] \end{tikzcd}$$

is commutative.

(b) Show that Theorem 5.55 is equivalent to the following: every morphism $\alpha : f \to f$ whose restriction $\alpha : X \to X$ is homotopic in $\mathcal{T}_\circ$ to the identity has a left homotopy inverse in $A{\downarrow}\mathcal{T}_\circ$.

Now we are prepared to complete the proof.

Problem 5.57. Let $f : A \to X$ be a cofibration, and let $\alpha : f \to f$ in $A{\downarrow}\mathcal{T}_\circ$ be such that $\alpha : X \to X$ is homotopic in $\mathcal{T}_\circ$ to id_X. Let $H : \mathrm{id}_X \simeq \alpha$ in $\mathcal{T}_\circ$, and write $H_A : A \times I \to X$ for the restriction of H to $A \times I$.

(a) Show there is a homotopy $J : X \times I \to X$ extending $\overleftarrow{H}_A$ from id_X to another map $\beta : X \to X$ under A.

(b) Show that β is a left homotopy inverse to α in $A{\downarrow}\mathcal{T}_\circ$, thereby proving Theorem 5.55.

Hint. Find a homotopy $\beta \circ \alpha \simeq \alpha$ under $\overleftarrow{H}_A$.

Corollary 5.58. *Let $A \hookrightarrow X$ be a cofibration and a homotopy equivalence. Then A is a strong deformation retract of X.*

Corollary 5.59. *If $f : A \to X$ is a homotopy equivalence, then $A \subseteq M_f$ is a strong deformation retract of M_f.*

Problem 5.60. Prove Corollary 5.58 and Corollary 5.59.

Homotopy Equivalence in $\mathcal{T}_\circ{\downarrow}B$. The dual result is also true, by a dual proof.

Theorem 5.61. *If $f : X \to B$ and $g : Y \to B$ are fibrations and $\alpha : f \to g$ is a pointwise homotopy equivalence in $\mathcal{T}_\circ{\downarrow}B$, then α is a homotopy equivalence in $\mathcal{T}_\circ{\downarrow}B$.*

Exercise 5.62. Carry out the dual proof.

Corollary 5.63. *If $p : E \to B$ is a fibration and a homotopy equivalence, then there is a section $s : B \to E$ and a homotopy $s \circ p \simeq \mathrm{id}_E$ over B.*

5.6. The Fundamental Lifting Property

We will refer to the main theorem governing the interaction of fibrations and cofibrations as the *Fundamental Lifting Property.* It subsumes and supersedes all other lifting and extension results, though many of these 'corollaries' are actually used in its proof.

Theorem 5.64 (Fundamental Lifting Property)**.** *Suppose that i is a cofibration and p is a fibration in the strictly commutative diagram*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] \arrow[ur, dotted] & B \end{tikzcd}$$

of spaces in $\mathcal{T}_\circ$. Then the dotted arrow can be filled in to make the diagram strictly commutative if either

- *i is a homotopy equivalence or*
- *p is a homotopy equivalence.*

The proof of Theorem 5.64 occupies the bulk of the section. Some consequences are discussed in the last two subsections. In particular, we establish a relative version of the homotopy lifting property.

Exercise 5.65. Show that Theorem 5.64 implies the homotopy lifting and homotopy extension properties for fibrations and cofibrations, respectively.

5.6.1. The Case i is a Homotopy Equivalence. In this section, we work under the general hypotheses of Theorem 5.64, together with the special one that $i : A \to X$ is a homotopy equivalence.

Problem 5.66.

(a) Show that there is a strong deformation retraction $r : X \to A$ and a function $u : X \to I$ such that $A = u^{-1}(0)$.

(b) Let $D : X \times I \to X$ be a homotopy from $i \circ r$ to id_X that is constant on A. Show that the rule

$$H(x,t) = \begin{cases} f\left(D\left(x, \frac{t}{u(x)}\right)\right) & \text{if } t < u(x), \\ f(x) & \text{if } t \geq u(x) \end{cases}$$

defines a homotopy $H : X \times I \to B$.

(c) Show that there is a lift K in the diagram

$$\begin{tikzcd} X \arrow[r, "g\circ r"] \arrow[d, "\mathrm{in}_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"] \arrow[ur, dotted, "K"] & B. \end{tikzcd}$$

(d) Show that the function $\phi(x) = K(x, u(x))$ may be used to fill in the dotted arrow in the diagram of Theorem 5.64.

5.6.2. Relative Homotopy Lifting. One consequence of this first part of Theorem 5.64 is a useful 'relative' homotopy lifting property; indeed, we will use the relative homotopy lifting property in the proof of the second part of Theorem 5.64. The situation is that we want to lift a homotopy H, and we are given a lift defined on a subset $A \subseteq X$. Generally it may not be possible to find a lift of H that agrees with the given partially lifted homotopy, but if $A \hookrightarrow X$ is a cofibration, then such an extension does exist.

Corollary 5.67. *Suppose $i : A \hookrightarrow X$ is a cofibration and $p : E \to B$ is a fibration. Then in the commutative diagram*

$$\begin{tikzcd}
(X \times 0) \cup (A \times I) \arrow[r, "\phi \cup K_A"] \arrow[d, "i"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "H"] \arrow[ur, dotted] & B
\end{tikzcd}$$

the dotted arrow can be filled in to make the diagram strictly commutative.

The property of Corollary 5.67 is sometimes called the CHEP, short for **covering homotopy extension property**.

Problem 5.68. Prove Corollary 5.67.

5.6.3. The Case p is a Homotopy Equivalence. Now we return to the proof of Theorem 5.64. Assume the general situation of that theorem and the special hypothesis that p is a homotopy equivalence.

Problem 5.69.

(a) Show that p and id_B are homotopy equivalent in $\mathcal{T}_\circ\!\downarrow\! B$. Conclude that p has a section $s : B \to E$ and a homotopy $F : s \circ p \simeq \mathrm{id}_E$ over B.

(b) Let $T = (X \times 0) \cup (A \times I)$; and define $G : T \to E$ by the rule $G(a, t) = F(g(a), t)$ for $t > 0$ and $G(x, 0) = s \circ f(x)$. Also define $H : X \times I \to B$ by $H(x, t) = f(x)$. Show that the diagram

$$\begin{tikzcd}
T \arrow[r, "G"] \arrow[d] & E \arrow[d, "p"] \\
X \times I \arrow[r, "H"] \arrow[ur, dotted] & B
\end{tikzcd}$$

commutes, and conclude that the dotted arrow can be filled in.

(c) Show that the map $\phi : X \to E$ defined by $\phi(x) = K(x, 1)$ may be used to fill in the dotted arrow in the diagram of Theorem 5.64.

You have proved the Fundamental Lifting Property!

5.6.4. Mutual Characterization of Fibrations and Cofibrations. The Fundamental Lifting Property has a converse, which characterizes cofibrations and fibrations in terms of the existence of dotted arrows in diagrams of the form

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"] \\
C \arrow[r] \arrow[ur, dotted] & D.
\end{tikzcd}$$

Let $\mathcal{M}$ be a collection of maps. We say that g has the **right lifting property** with respect to the collection $\mathcal{M}$ if whenever $f \in \mathcal{M}$, the dotted arrow can be filled in to make the diagram commute. Dually, f has the **left lifting property** with respect to $\mathcal{M}$ if in any such diagram with $g \in \mathcal{M}$, the diagonal arrow exists.

Cofibrations that are also homotopy equivalences are sometimes called **trivial cofibrations** or **acyclic cofibrations**; likewise, a fibration that is also a homotopy equivalence may be called a **trivial fibration** or an **acyclic fibration**.

Theorem 5.70.

(a) *A map $i : A \to X$ is a cofibration if and only if it has the left lifting property with respect to all acyclic fibrations.*

(b) *A map $p : E \to B$ is a fibration if and only if it has the right lifting property with respect to all acyclic cofibrations.*

(c) *A map $i : A \to X$ is an acyclic cofibration if and only if it has the right lifting property with respect to all fibrations.*

(d) *A map $p : E \to B$ is an acyclic fibration if and only if it has the left lifting property with respect to all cofibrations.*

Problem 5.71. Prove Theorem 5.70.

5.6.5. Some Consequences of the Mutual Characterization. The mutual characterization of cofibrations and fibrations makes it easy to prove certain properties about them. We'll show here that cofibrations that are homotopy equivalences are preserved by pushouts, and dually for fibrations; also, the classes of cofibrations and fibrations are closed under retracts in the category $\mathrm{map}(\mathcal{T}_\circ)$.

Corollary 5.72. *If the square*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r] & D \end{tikzcd}$$

is a pushout square and f is an acyclic cofibration, then so is g. Dually, if the square is a pullback and g is an acyclic fibration, then so is f.

Problem 5.73. Prove Corollary 5.72.

Problem 5.74.

(a) Show that if q is a retract of a fibration p, then q is also a fibration.

(b) Show that if j is a retract of a cofibration i, then j is also a cofibration.

We end by showing that the 'composition' of infinitely many cofibrations is a cofibration.

Problem 5.75. Consider the diagram $\cdots \to X_{(n)} \to X_{(n+1)} \to \cdots$ and write X for its colimit. Show that if all the maps in the diagram are cofibrations, then the induced map $f : X_{(1)} \to X$ is also a cofibration. Also state and prove the dual.

Problem 5.76. Let $L, R : \mathcal{C} \to \mathcal{D}$ be an adjoint pair of functors.

(a) Show that the lifting problems

$$\begin{tikzcd} LA \arrow[r] \arrow[d, "L(i)"'] & E \arrow[d, "f"] \\ LX \arrow[r] \arrow[ur, dotted, "\lambda" description] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r] \arrow[d] & RE \arrow[d, "R(f)"] \\ X \arrow[r] \arrow[ur, dotted, "\widetilde{\lambda}" description] & RB \end{tikzcd}$$

are equivalent in the sense that λ exists if and only if $\widetilde{\lambda}$ exists.

(b) Now specialize to $L, R : \mathcal{T}_\circ \to \mathcal{T}_\circ$, and show that if R and L both respect homotopy, then L preserves cofibrations if and only if R preserves fibrations.

5.7. Pointed Cofibrations and Fibrations

Now we develop a corresponding theory of cofibrations and fibrations in the category $\mathcal{T}_*$ of pointed spaces. The basic theory is formally identical to that for unpointed spaces, but when it comes to proving the pointed analog of the Fundamental Lifting Property, we need to call on the unpointed version. This forces us to restrict our attention to cofibrations between 'well-pointed' spaces, since such maps are both pointed and unpointed cofibrations. Fortunately, even with this restriction, the theorem holds in almost every case we will ever encounter.

The notions of cofibration and fibration apply equally well in the pointed context, provided we use the pointed cylinder $\mathrm{Cyl}(X) = X \rtimes I$. Thus we say that $i : A \to X$ is a **pointed cofibration** if for any map $f : X \to Y$ and any homotopy $H_A : A \rtimes I \to Y$, the dotted arrow in the diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\mathrm{in}_0"'] & X \arrow[d, "\mathrm{in}_0"] \arrow[ddr, bend left, "f"] & \\ A \rtimes I \arrow[r, "i \rtimes \mathrm{id}_I"] \arrow[drr, bend right, "H_A"'] & X \rtimes I \arrow[dr, dotted, "H" description] & \\ & & Y \end{tikzcd}$$

can be filled in to make the diagram strictly commutative. A **pointed fibration** is a map $p : E \to B$ satisfying the **pointed homotopy lifting property**: for any map $f : X \to B$ and any homotopy $H_B : X \rtimes I \to B$, the dotted arrow in the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{in}_0"'] & E \arrow[d, "p"] \\ X \rtimes I \arrow[r, "H_B"] \arrow[ur, dotted] & B \end{tikzcd}$$

can be filled in to make the diagram commute.

Exercise 5.77. Formulate the definitions of pointed cofibrations and pointed fibrations in terms of path spaces.

The pointed notions of cofibrations, fibrations and homotopy equivalences are not too far from their pointed counterparts.

Theorem 5.78. *Let $p : E \to B$ and $i : A \to X$ in $\mathcal{T}_*$.*

(a) If p is a fibration in $\mathcal{T}_*$, then $p_- : E_- \to B_-$ is a fibration in $\mathcal{T}_\circ$.

(b) If $i_- : A_- \to X_-$ is a cofibration in $\mathcal{T}_\circ$, then i is a cofibration in $\mathcal{T}_*$.

Problem 5.79. Prove Theorem 5.78.

Exercise 5.80.

(a) Find a pointed cofibration that is not an unpointed cofibration and an unpointed fibration that is not a pointed fibration.

(b) What distinguishes a lifting function for a pointed fibration from a lifting function for an unpointed fibration?

In Section 5.8.2 we'll see that the converses of the statements in Theorem 5.78 hold if the basepoints of the spaces involved are sufficiently well behaved.

Pointed cofibrations and fibrations are preserved by pointed pushouts and pullbacks, respectively.

Theorem 5.81. *Show that in the commutative square*

$$\begin{tikzcd} A \arrow[r, "q"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ C \arrow[r, "p"] & D \end{tikzcd}$$

(a) if the square is a pushout in $\mathcal{T}_*$ and i is a pointed cofibration, then j is a pointed cofibration,

(b) if the square is a pullback in $\mathcal{T}_*$ and p is a pointed fibration, then q is a pointed fibration.

Problem 5.82. Prove Theorem 5.81.

Problem 5.83. Show that a retract of a pointed cofibration is a pointed cofibration and dually, a retract of a pointed fibration is a pointed fibration.

Some Examples. The detection of pointed cofibrations can be reduced to universal examples.

Proposition 5.84. *Let $i : A \to X$ in $\mathcal{T}_*$, and write $T = X \cup_i A \rtimes I$. Show that i is a pointed cofibration if and only if the homotopy $A \rtimes I \hookrightarrow T$ can be extended to $X \rtimes I \to T$.*

Here are some applications.

Problem 5.85.

(a) Show that $\mathrm{in}_0, \mathrm{in}_1 : X \to X \rtimes I$ are pointed cofibrations.

(b) Show that $(\mathrm{in}_0, \mathrm{in}_1) : X \vee X \to X \rtimes I$ is a pointed cofibration.

Hint. Show that the test spaces T are homeomorphic to $X \rtimes I$.

5.8. Well-Pointed Spaces

At this point, we would like to prove theorems in the pointed category parallel to the Fundamental Lifting Principle and the mutual characterization of cofibrations and fibrations. But there is a major problem that makes the obvious modifications of these theorems *false* in $\mathcal{T}_*$.

The problem is that if X is not well-pointed, then $A \hookrightarrow A \vee X$ is not a cofibration. This seemingly minor annoyance propagates, with the effect that the pointed mapping cylinder construction on a map $f : X \to Y$ loses its *raison d'etre*: the inclusion $X \hookrightarrow M_f$ is not a cofibration! Fortunately, this serious problem disappears if we impose a fairly mild condition on our pointed spaces: they should be *well-pointed*.

5.8.1. Well-Pointed Spaces. A space $X \in \mathcal{T}_*$ is **well-pointed** if the inclusion $* \to X$ of the basepoint is a cofibration in $\mathcal{T}_\circ$.[4] We sometimes write $\mathcal{W}_*$ for the full subcategory of $\mathcal{T}_*$ whose objects are the well-pointed spaces.

Exercise 5.86. Show that every pointed CW complex is well-pointed.

Problem 5.87. Let A and B be any two well-pointed spaces.

(a) Show that $A \vee B \hookrightarrow A \times B$ is both a pointed and an unpointed cofibration.

(b) Show that the inclusion $A \hookrightarrow A \vee B$ is both a pointed and an unpointed cofibration. Is it necessary that *both* spaces be well-pointed?

[4]That is, we require $*_- \to X_-$ to be a cofibration in $\mathcal{T}_\circ$.

(c) Show that $A \vee B$ and $A \times B$ are well-pointed.

(d) Show that $(\mathrm{in}_0, \mathrm{in}_1) : A \vee A \to A \rtimes I$ is both a pointed and an unpointed cofibration.

HINT. Consider the map $(A \times 0) \cup (* \times I) \cup (A \times 1) \hookrightarrow A \times I$.

(e) Conclude that $\mathrm{in}_0, \mathrm{in}_1 : A \to A \rtimes I$ are unpointed cofibrations.

(f) Recall that the cone on A is $CA = A \wedge I$. Show that $\mathrm{in}_0 : A \hookrightarrow CA$ is both a pointed and an unpointed cofibration.

Mapping Cylinders of Well-Pointed Spaces. Now we come to one of the crucial differences between the pointed and the unpointed categories. The mapping cylinder construction can be carried out for any map $f : X \to Y$ in the category $\mathcal{T}_*$, but unless the target Y is well-pointed, the resulting inclusion $X \hookrightarrow M_f$ will not be a pointed cofibration.

The **pointed mapping cylinder** is the space M_f constructed as the pushout in the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{in}_0"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{pushout}"] & X \rtimes I \arrow[d] \\ Y \arrow[r] & M_f. \end{tikzcd}$$

We'll work almost exclusively with the pointed mapping cylinder in this text, so we won't introduce different notation for it.

Problem 5.88. Show that $Y \to M_f$ is a pointed homotopy equivalence.

Problem 5.89.

(a) Show that the inclusion $X \hookrightarrow M_f$ is a cofibration if Y is well-pointed.

(b) Show that if Y is not well-pointed, then the inclusion $X \hookrightarrow M_f$ need not be a cofibration.

Problem 5.90. Suppose X is well-pointed. Show that for any $f : X \to Y$, the inclusion $Y \hookrightarrow M_f$ is both a pointed and an unpointed cofibration.

Lest you think that the well-pointed hypothesis is just one of several ways to produce reasonable pointed mapping cylinders, consider the following.

Exercise 5.91. Show that the following conditions on Y are equivalent:

(1) For every $f : X \to Y$ in $\mathcal{T}_*$, the inclusion $X \hookrightarrow M_f$ is a pointed cofibration.

(2) Y is well-pointed.

Mapping Spaces of Well-Pointed Spaces. To make use of the beneficial properties of well-pointed spaces, we need to know when our constructions produce well-pointed spaces. This is generally easy to do when the constructions are of domain-type; but when we apply target-type constructions to cofibrations, we cannot expect anything to come easily. Nevertheless, Theorem 5.20 and its consequences are powerful enough to give us almost everything we could want.

Proposition 5.92. *If X is well-pointed and Q is compact, then the pointed spaces* $(\mathrm{map}_\circ(Q, X), *)$ *and* $\mathrm{map}_*(Q, X)$ *are well-pointed.*

Corollary 5.93. *If X is well-pointed, then so are ΩX and X^I.*

Problem 5.94. Prove Proposition 5.92 and Corollary 5.93.

5.8.2. Cofibrations and Fibrations of Well-Pointed Spaces. If we restrict our attention to well-pointed spaces, then the notions of fibration, cofibration and homotopy equivalence become even more closely bound together.

Theorem 5.95. *Let $f : X \to Y$ be a map of well-pointed spaces in $\mathcal{T}_*$. Then*

(a) f is a cofibration in $\mathcal{T}_*$ if and only if f_- is a cofibration in $\mathcal{T}_\circ$,

(b) f is a fibration in $\mathcal{T}_*$ if and only if f_- is a fibration in $\mathcal{T}_\circ$, and

(c) f is a homotopy equivalence in $\mathcal{T}_*$ if and only if f_- is a homotopy equivalence in $\mathcal{T}_\circ$.

The proof of Theorem 5.95(a) is quite technical, as it makes use of a characterization of cofibrations between well-pointed spaces along the lines of Theorem 5.20. Rather than present the details, we'll take it for granted and leave it to the interested reader to work through the proof.[5]

Problem 5.96.

(a) Show that if f_- is an unpointed fibration, then f has the right lifting property with respect to pointed cofibrations of well-pointed spaces.

(b) Show that if f has the right lifting property with respect to pointed cofibrations of well-pointed spaces, then f is a retract of a pointed fibration.

HINT. $\overline{E}_f$ is well-pointed.

(c) Prove Theorem 5.95(b).

Problem 5.97. Prove Theorem 5.95(c).

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[5]Consult [**164**].

5.8.3. Double Factorizations. Constructions analogous to those in Section 5.4 show that every pointed map is pointwise homotopy equivalent to a pointed fibration and every pointed map *between well-pointed spaces* is pointwise homotopy equivalent to a pointed cofibration. If we restrict our attention to cofibrations of well-pointed spaces, then we also have a pointed version of Theorem 5.42.

We have already defined and studied the pointed mapping cylinder. Interestingly, we do not need to modify the construction of E_f, except to give it the basepoint $(*, \boxed{*})$.

Theorem 5.98. *Let $f : X \to Y$ in $\mathcal{T}_*$. Then in the square*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "i"'] \arrow[dr, "f"] & M_f \arrow[d, "q"] \\ E_f \arrow[r, "p"'] & Y \end{tikzcd}$$

- *i and q are pointed homotopy equivalences,*
- *p is a pointed fibration, and*
- *if Y is well-pointed, then j is a pointed cofibration.*

Problem 5.99. Prove Theorem 5.98.

We can again improve our factorizations so that in both cases we factor f into a cofibration followed by a fibration. But, as usual in the pointed category, we must restrict our attention to well-pointed spaces.

Theorem 5.100 (Strøm)**.** *Every map $f : X \to Y$ of well-pointed spaces in $\mathcal{T}_*$ has factorizations*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "i"'] \arrow[dr, "f"] & \overline{M}_f \arrow[d, "q"] \\ \overline{E}_f \arrow[r, "p"'] & Y \end{tikzcd}$$

in which

- *all four spaces are well-pointed,*
- *i and q are pointed (hence unpointed) homotopy equivalences,*
- *i and j are unpointed (and hence pointed) cofibrations, and*
- *p and q are pointed (hence unpointed) fibrations.*

Some of this does not require well-pointed spaces.

Problem 5.101. Show that the factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, bend right, "i"'] & & Y \\ & \overline{E}_f \arrow[ur, bend right, "p"'] & \end{tikzcd}$$

of f into an acyclic unpointed (hence pointed) cofibration i followed by a pointed (hence unpointed) fibration p holds for *all* maps, not just the maps between well-pointed spaces.

The proof of Theorem 5.100 is a straightforward modification of the proof in the unpointed case. That is, it reduces to the case f is a pointed homotopy equivalence and this is resolved using the constructions defined in Section 5.4.3. We give the space $Q(A, X)$ the basepoint $(*, 0)$ and the space Z_p the basepoint $(*, 1)$.

Problem 5.102. Show that if A, B, E and X are well-pointed, then so are $Q(X, A)$ and Z_p.

Problem 5.103. Let $A \subseteq X$.

(a) Show that $i : A \hookrightarrow Q(A, X)$ is a (pointed) cofibration.

(b) Show that $p : Q(A, X) \to X$ is a pointed fibration.

(c) Show that if there is a function $f : X \to I$ with $* \in f^{-1}(0) \subseteq A$, then both i and p are pointed homotopy equivalences.

Problem 5.104. Assume the hypotheses of Theorem 5.100.

(a) Show that $\overline{M}_f$ and $\overline{E}_f$ are well-pointed.

(b) Show that it suffices to prove Theorem 5.100 in the case $f : X \hookrightarrow Y$ is a pointed homotopy equivalence.

(c) Prove Theorem 5.100.

5.8.4. The Fundamental Lifting Property. We finish this section by proving the pointed version of the Fundamental Lifting Property and deriving the mutual characterization of cofibrations and fibrations.

Theorem 5.105 (Fundamental Lifting Property)**.** *Suppose that $i : A \to X$ is a pointed cofibration of well-pointed spaces and that p is a pointed fibration. Then in the strictly commutative diagram*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"] \arrow[ur, dotted] & B, \end{tikzcd}$$

the dotted arrow can be filled in to make the diagram strictly commutative if either

- *i is a homotopy equivalence or*
- *p is a homotopy equivalence.*

Problem 5.106. Prove Theorem 5.105.

There is a pointed version of our mutual characterization (Theorem 5.70) of cofibrations and fibrations, but it requires that we work only with well-pointed spaces.

Theorem 5.107. *Suppose A, B, E and X are well-pointed.*

(a) A map $i : A \to X$ is a cofibration in $\mathcal{T}_*$ if and only if it has the left lifting property with respect to all acyclic fibrations in $\mathcal{W}_*$.

(b) A map $p : E \to B$ is a fibration if and only if it has the right lifting property with respect to all acyclic cofibrations in $\mathcal{W}_*$.

(c) A map $i : A \to X$ is an acyclic cofibration if and only if it has the right lifting property with respect to all fibrations in $\mathcal{W}_*$.

(d) A map $p : E \to B$ is an acyclic fibration if and only if it has the left lifting property with respect to all cofibrations in $\mathcal{W}_*$.

Problem 5.108.

(a) Show that in Theorem 5.107(a) and (c), i is a retract of $A \hookrightarrow \overline{M}_i$.

(b) Show that in Theorem 5.107(b) and (d), p is a retract of $\overline{E}_p \to B$.

(c) Prove Theorem 5.107.

Hint. The technique of Problem 5.96(b) and its dual are helpful.

Here's a nice consequence.

Problem 5.109. Let $L, R : \mathcal{W}_* \to \mathcal{W}_*$ be an adjoint pair of functors that respect homotopy. Show that L preserves cofibrations in $\mathcal{W}_*$ if and only if R preserves fibrations in $\mathcal{W}_*$.

Exercise 5.110. Find examples of such adjoint pairs.

5.9. Exact Sequences, Cofibers and Fibers

Cofibrations and fibrations are used to bring exact sequences into topology, and exact sequences are among the most important tools for making explicit calculations in homotopy theory.

5.9.1. Exact Sequences in Homotopy Theory. A sequence of pointed sets $A \to B \to C$ is **exact** if the kernel of $B \to C$ (i.e., the set of all $b \in B$ that map to $* \in C$) is precisely equal to the image of $A \to B$. Most often we deal with exact sequences in the category $\mathcal{G}$ of groups and homomorphisms. The most basic example of an exact sequence of groups is

$$N \longrightarrow G \longrightarrow G/N,$$

where N is a normal subgroup of G and the maps are the ones you expect. See Section A.2 for more information about exact sequences.

There are sequences of this kind in the category $\mathcal{T}_*$ of pointed spaces: those of the form $A \to X \to X/A$ on the domain side and $F \to E \to B$, where $F = p^{-1}(*)$ for targets. Dare we hope that the induced sequences

$$[X/A, Y] \longleftarrow [X, Y] \longleftarrow [A, Y]$$

and

$$[X, F] \longrightarrow [X, E] \longrightarrow [X, B]$$

of pointed sets should be exact?

Theorem 5.111.

(a) Let $i : A \to X$ be a cofibration in $\mathcal{T}_*$, and let $q : X \to X/A$ be the canonical quotient map. Then for any space $Y \in \mathcal{T}_*$, the sequence

$$[A, Y] \xleftarrow{i^*} [X, Y] \xleftarrow{q^*} [X/A, Y]$$

of pointed sets is exact.

(b) Let $p : E \to B$ be a fibration and let $j : F \to E$ be the inclusion of the fiber $F = p^{-1}(*) \subseteq E$. Then for any space $X \in \mathcal{T}_*$, the sequence

$$[X, F] \xrightarrow{j_*} [X, E] \xrightarrow{p_*} [X, B]$$

is exact.

Problem 5.112. Use the hypotheses and notation of Theorem 5.111.

(a) Show that if $f : X \to Y$, then $f \circ i \simeq *$ if and only if there is a map $g : X \to Y$ such that $g \simeq f$ and $g(A) = *$.

(b) Prove Theorem 5.111.

5.9.2. The Cofiber of a Map. Since the induced map $X \to X/A$ behaves well in the homotopy category if $A \hookrightarrow X$ is a cofibration, it makes sense to study a map $f : X \to Y$ by examining the quotient of the inclusion $X \hookrightarrow M_f$. The (standard) **cofiber** of a map $f : X \to Y$ of well-pointed spaces is the map $Y \to C_f$ obtained as the composite of $Y \hookrightarrow M_f$ with the quotient map $M_f \to M_f/X$. The space C_f is also called the **mapping cone** of f.

Thus the standard cofiber is often written $C_f = Y \cup_f CX$. The cofiber of f is not just the space C_f; it is the map $X \to C_f$. However, the usual practice is to be sloppy and refer to C_f as the cofiber.

Problem 5.113. Show that there are pushout squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{in}_0"'] \arrow[dr, phantom, "\text{pushout}"] & X \arrow[d] \\ CA \arrow[r] & C_f \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "j"] \arrow[d, "\mathrm{in}_0"'] \arrow[dr, phantom, "\text{pushout}"] & M_f \arrow[d] \\ CA \arrow[r] & C_f \end{tikzcd}$$

where $\mathrm{in}_0 : A \hookrightarrow CA$ is the standard inclusion into the cone $CA = A \rtimes I$.

It is possible to convert a map to a cofibration in many different ways,[6] though the mapping cylinder construction is the standard method. No matter what construction is used, the map to the resulting quotient space is referred to as a cofiber for the given map. It is natural to ask how different the corresponding cofibers can be. We will show in Problem 6.89 that cofibers arising from two different constructions must be pointwise homotopy equivalent to each other.

Problem 5.114. What is the cofiber of the unique map $S^1 \to *$? Describe the cofiber of $X \to *$ in general.

Given any map $f : A \to X$ (of well-pointed spaces), we get a cofiber $X \to C_f$; together these form a sequence of maps $A \to X \to C_f$. Any such sequence, or any sequence that is pointwise equivalent in $\mathrm{H}\mathcal{T}_*$ to such a sequence, is called a **cofiber sequence**.

Problem 5.115. Show that if $A \xrightarrow{i} B \xrightarrow{q} C$ is a cofiber sequence in $\mathcal{T}_*$, then for any $Y \in \mathcal{T}_*$ the sequence

$$[A, Y] \xleftarrow{i^*} [B, Y] \xleftarrow{q^*} [C, Y]$$

is an exact sequence of pointed sets.

5.9.3. The Fiber of a Map. The (standard) **homotopy fiber** of a map $f : X \to Y$ in $\mathcal{T}_*$ is the map $F \to X$ obtained by composing the fiber inclusion $F \hookrightarrow E_f$ with the standard homotopy equivalence $E_f \to X$. Two different methods of converting f to a fibration will yield different homotopy fibers; later (in Problem 6.89) you will show that any two homotopy fibers for a given map f are homotopy equivalent. Thus it makes sense to talk about *the* homotopy fiber of a map $f : X \to Y$.

It is also common to leave out the word 'homotopy' when discussing homotopy fibers. Thus we may speak of the fiber of an arbitrary map $f : X \to Y$.

[6]For example, if the map is a cofibration already, why do anything to it?

Problem 5.116. Determine the homotopy fiber of the unique map $* \to X$.

Any sequence of the form $F \to X \to Y$, where $F \to X$ is the homotopy fiber of $X \to Y$, or any sequence homotopy equivalent to such a sequence, is called a **fibration sequence** or a **fiber sequence**.

Problem 5.117. Let $F \xrightarrow{j} X \xrightarrow{p} Y$ be a fibration sequence in $\mathcal{T}_*$. Show that for any $A \in \mathcal{T}_*$, the sequence

$$[A, F] \xrightarrow{j^*} [A, X] \xrightarrow{p^*} [A, Y]$$

is an exact sequence of pointed sets.

5.9.4. Cofibers of Maps out of Contractible Spaces. If $i : A \to X$ is a cofibration and $A \simeq *$, then Problem 5.115 guarantees that the map $q^* : [X/A, Y] \to [X, Y]$ induced by the quotient map $q : X \to X/A$ is surjective. But in fact much more is true!

Proposition 5.118. *Let $i : A \hookrightarrow X$ be a cofibration in $\mathcal{T}_*$ and let $q : X \to X/A$ be the quotient map. If A is contractible, then the induced map*

$$q^* : [X/A, Y] \longrightarrow [X, Y]$$

is a bijection for all Y.

Problem 5.119.

(a) Show that if $f : X \to Y$ in $\mathcal{T}_*$, there is another map $\phi : X \to Y$ such that $\phi \simeq f$ and $\phi(A) = *$.

(b) Suppose $H_A : A \times I \to Y$ is a homotopy in $\mathcal{T}_*$ from the constant map $*$ to itself. Show that H_A factors as in the diagram

$$\begin{tikzcd}
A \times I \arrow[rr, "H_A"] \arrow[dr, bend right] & & Y \\
& \Sigma A. \arrow[ur, bend right, "h"'] &
\end{tikzcd}$$

(c) Show that h corresponds under the exponential law to a nullhomotopic loop in $\mathrm{map}_*(A, Y)$; conclude that there is a homotopy-of-homotopies J from H_A to the constant homotopy $\boxed{*} : A \times I \to Y$.

(d) Let $\mathbb{I} = (X \times 0) \cup (A \times I) \cup (X \times 1)$, and show that $\mathbb{I} \hookrightarrow X \times I$ is a cofibration.

(e) Now let $f, g : X/A \to Y$ and suppose $H : q^*(f) \simeq q^*(g)$; let H_A be the restriction of H to $A \times I$. Show that there is a commutative diagram

$$\begin{tikzcd}
\mathbb{I} \times 0 \arrow[r] \arrow[d] & \mathbb{I} \times I \arrow[d] \arrow[ddr, bend left, "(f\circ\mathrm{pr}_1)\cup J\cup(g\circ\mathrm{pr}_1)"] & \\
(X \times I) \times 0 \arrow[r] \arrow[drr, bend right, "H\circ\mathrm{pr}_1"'] & (X \times I) \times I \arrow[dr, dotted] & \\
 & & Y,
\end{tikzcd}$$

where J is as in part (b).

(f) Prove Proposition 5.118.

Corollary 5.120. *If $A \hookrightarrow X$ is a cofibration and $A \simeq *$, then $X \to X/A$ is a homotopy equivalence.*

Problem 5.121. Prove Corollary 5.120.

Problem 5.122. Show that if $X \in \mathcal{T}_*$ is well-pointed, then $X \rtimes I \simeq X \times I$ and $\Sigma X \simeq \Sigma_\circ X$ in $\mathcal{T}_*$.

Problem 5.123. Formulate and prove the analogous result for fibrations with contractible base.

Problem 5.124. A **graph** is a 1-dimensional CW complex; a **tree** is a connected graph without loops.

(a) Show that every connected graph G has a **spanning tree**: i.e., a maximal tree $T \subseteq G$.

(b) Show that a spanning tree must contain every vertex of G.

(c) Show that every tree is contractible.

(d) Show that every graph is homotopy equivalent to a wedge $\bigvee S^1$.

5.10. Mapping Spaces

We conclude this chapter by studying the effect of mapping space functors on fibration and cofibration sequences. These results are essentially formal: the exponential law enables us to rewrite the diagrams that define fibrations in ways that allow us to make use of the hypotheses.

5.10.1. Unpointed Mapping Spaces. We begin by working with unpointed mapping spaces.

Theorem 5.125. *If $p : E \to B$ is a fibration with fiber F, then for any space X, the induced map $p_* : \mathrm{map}_\circ(X, E) \longrightarrow \mathrm{map}_\circ(X, B)$ is a fibration with fiber $\mathrm{map}_\circ(X, F)$.*

Problem 5.126.

(a) Show that the lifting problems

$$\begin{tikzcd} Q \times \{0\} \arrow[r, "f"] \arrow[d, "j"'] & \mathrm{map}_\circ(X, E) \arrow[d, "p_*"] \\ Q \times I \arrow[r, "g"] \arrow[ur, dotted] & \mathrm{map}_\circ(X, B) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Q \times \{0\} \times X \arrow[r, "\widehat{f}"] \arrow[d, "j \times \mathrm{id}_Q"'] & E \arrow[d, "p"] \\ Q \times I \times X \arrow[r, "\widehat{g}"] \arrow[ur, dotted] & B, \end{tikzcd}$$

where $\widehat{f}$ and $\widehat{g}$ are the adjoints of f and g, respectively, are equivalent to each other.

(b) Prove Theorem 5.125.

Corollary 5.127. *If $X \to Y \to Z$ is a fibration sequence in $\mathcal{T}_\circ$, then*

$$\mathrm{map}_\circ(A, X) \longrightarrow \mathrm{map}_\circ(A, Y) \longrightarrow \mathrm{map}_\circ(A, Z)$$

is a fibration sequence.

Problem 5.128. Prove Corollary 5.127.

The statement of our next theorem may seem a bit strange because of the appearance of a pointed mapping space. But you should remember that if $A \subseteq X$ in $\mathcal{T}_\circ$, then the quotient X/A is naturally a pointed space with basepoint $[A]$.

Theorem 5.129. *If $j : A \to X$ is a cofibration in $\mathcal{T}_\circ$, then for any pointed space Y, the induced map $j^* : \mathrm{map}_\circ(X, Y_-) \longrightarrow \mathrm{map}_\circ(A, Y_-)$ is a fibration whose fiber over the constant map $A \xrightarrow{*} Y$ is $\mathrm{map}_*(X/A, Y)$.*

Problem 5.130. Show that a map $\phi : Q \times I \to \mathrm{map}_\circ(X, Y)$ makes the diagram

$$\begin{tikzcd} Q \times \{0\} \arrow[r, "f"] \arrow[d, "\mathrm{in}_0"'] & \mathrm{map}_\circ(X, Y) \arrow[d, "j^*"] \\ Q \times I \arrow[r, "g"] \arrow[ur, dotted, "\phi" description] & \mathrm{map}_\circ(A, Y) \end{tikzcd}$$

commute if and only if its adjoint $\widehat{\phi}$ makes the diagram

$$\begin{tikzcd} Q \times \{0\} \times A \arrow[r] \arrow[d] & Q \times \{0\} \times X \arrow[d] \arrow[ddr, bend left, "\widehat{f}"] & \\ Q \times I \times A \arrow[r] \arrow[drr, bend right, "\widehat{g}"'] & Q \times I \times X \arrow[dr, dotted, "\widehat{\phi}" description] & \\ & & Y \end{tikzcd}$$

commute. Deduce Theorem 5.129.

Exercise 5.131. There is a curious thing happening here: in all of these statements, the conclusion is that the induced map is a *fibration*—neither is a cofibration! Is this a failure of duality?

5.10.2. Pointed Maps into Pointed Fibrations. Now we consider analogous problems for pointed mapping spaces. To use the pointed Fundamental Lifting Property, we'll need to know that smash products with well-pointed spaces preserve cofibrations.

Problem 5.132. Suppose Q is well-pointed. Show that if $A \to X$ is a cofibration of well-pointed spaces, then $A \wedge Q \to X \wedge Q$ is also a cofibration of well-pointed spaces.

Hint. Determine the pushout of $A \wedge Q \leftarrow (X \times *) \cup (A \times Q) \to X \times Q$.

Theorem 5.133. *If Q is well-pointed and $p : E \to B$ is a fibration of well-pointed spaces, then the induced map*

$$p_* : \mathrm{map}_*(Q, E) \longrightarrow \mathrm{map}_*(Q, B)$$

is an unpointed fibration. If the mapping spaces happen to be well-pointed (i.e., if Q is compact), then $(p_*)_-$ is a pointed fibration.

5.10.3. Applications. Theorem 5.125 and its pointed analog imply that evaluation maps tend to be fibrations.

Problem 5.134. Let $x \in X$.

(a) Using the identification $\mathrm{map}_\circ(\{x\}, Y) \cong Y$ of Problem 3.67, show that

$$\begin{tikzcd} \mathrm{map}_\circ(X,Y) \arrow[r, "i^*"] \arrow[d, equal] & \mathrm{map}_\circ(\{x\},Y) \arrow[d, "\cong"', "@_x"] \\ \mathrm{map}_\circ(X,Y) \arrow[r, "@_x"] & Y \end{tikzcd}$$

commutes, and so identify $@_x$ with i^*.

(b) Show that if the inclusion $i : \{x\} \hookrightarrow X$ is a cofibration, then $@_x$ is a fibration.

The most important applications are to evaluation maps involving spaces of paths.

Problem 5.135.

(a) Show that the map $@_{0,1} : X^I \to X \times X$ given by $f \mapsto (f(0), f(1))$ is a fibration.

(b) Show that the map $@_0 : \mathcal{P}(X) \to X$ given by $f \mapsto f(0)$ is a fibration.

(c) Reprove Problem 5.32(b).

Determine the fibers of $@_{0,1}$ and $@_0$.

5.11. Additional Topics, Problems and Projects

5.11.1. Homotopy Equivalences in $A \downarrow \mathcal{T} \downarrow B$. Now we'll prove a theorem analogous to Theorem 5.55, but in the category $A \downarrow \mathcal{T} \downarrow B$. The original theorem was a key element in the proof of the Fundamental Lifting Property; now we turn the tables and use the Fundamental Lifting Property (repeatedly) in the proof of the generalization.

Theorem 5.136. *Consider the diagram*

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X \arrow[rr, "f", "\simeq"'] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B. & \end{tikzcd}$$

If i and j are cofibrations and p and q are fibrations, then f has a homotopy inverse in the category $A \downarrow \mathcal{T} \downarrow B$.

Problem 5.137. Show that Theorems 5.55 and 5.61 are special cases of Theorem 5.136.

Let's lay out what we mean by homotopy in the category $A \downarrow \mathcal{T} \downarrow B$. It will be easiest to simply define a homotopy from f to g to be a homotopy $H : X \times I \to Y$ in $\mathcal{T}$ making the diagram

$$\begin{tikzcd} & A \times I \arrow[dl, "i\times \mathrm{id}_I"'] \arrow[dr, "\boxed{j}"] & \\ X \times I \arrow[rr, "H"] \arrow[dr, "\boxed{p}"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

commute.

Exercise 5.138. Give a definition of homotopy in $A \downarrow \mathcal{T} \downarrow B$ by defining the cylinders on an object.

We will need to work with homotopies $H : X \times I \to Y$ that are not homotopies in $A \downarrow \mathcal{T} \downarrow B$ and use the power of fibrations and cofibrations to bring them to heel. Each such homotopy gives rise, by composition, to two homotopies $H_A : A \times I \to Y$ and $H_B : X \times I \to B$ which are **compatible** with one another in the sense that the square

$$\begin{tikzcd}
A \times I \arrow[r, "H_A"] \arrow[d, "i \times \mathrm{id}_I"'] & Y \arrow[d, "q"] \\
X \times I \arrow[r, "H_B"'] & B
\end{tikzcd}$$

commutes. Two homotopies-of-homotopies K_A, K_B are a **compatible pair** if the square

$$\begin{tikzcd}
A \times I \times I \arrow[r, "K_A"] \arrow[d, "i \times \mathrm{id}_{I \times I}"'] & Y \arrow[d, "q"] \\
X \times I \times I \arrow[r, "K_B"'] & B
\end{tikzcd}$$

is commutative. Thus the pair (K_A, K_B) is a homotopy through pairs of compatible homotopies.

Problem 5.139. Suppose i is a cofibration and q is a fibration.

(a) Show that every compatible pair of homotopies comes from a homotopy $H : X \times I \to Y$.

(b) Show that every compatible pair of homotopies-of-homotopies comes from a homotopy-of-homotopies $K : X \times I \times I \to Y$.

Problem 5.140.

(a) Show that if H_A and H_B are compatible, then so are $\overleftarrow{H}_A$ and $\overleftarrow{H}_B$.

(b) Show that there are compatible homotopies-of-homotopies from $\overleftarrow{H}_A * H_A$ and $\overleftarrow{H}_B * H_B$ to the corresponding constant homotopies.

The rest of the proof is exactly analogous to the proof of Theorem 5.55. First we find a homotopy inverse in the category $A \downarrow \mathcal{T} \downarrow B$.

Problem 5.141. Consider the diagram of Theorem 5.136.

(a) Show that f has a homotopy inverse g_A in $A \downarrow \mathcal{T}$.

(b) By studying the square

$$\begin{tikzcd}
Y \cup (A \times I) \arrow[r, "g_A \cup \boxed{j}"] \arrow[d] & X \arrow[d, "p"] \\
Y \times I \arrow[r, "\boxed{q}"'] & B,
\end{tikzcd}$$

show that f has a homotopy inverse that is a morphism in $A \downarrow \mathcal{T} \downarrow B$.

Problem 5.142.

(a) Show that in the situation of Theorem 5.136, it suffices to show that if $f \simeq \mathrm{id}_X$ in $\mathcal{T}_\circ$, then f has a left homotopy inverse in $A \downarrow \mathcal{T} \downarrow B$.

(b) Let $H : X \times I \to X$ be a homotopy $\mathrm{id}_X \simeq f$, with corresponding homotopies H_A and H_B.

(c) Show that there is a homotopy $J : X \times I \to X$ from id_X to some other map g having compatible pair $\overleftarrow{H}_A$ and $\overleftarrow{H}_B$.

(d) Finish the proof of Theorem 5.136 by showing that $g \circ f \simeq \mathrm{id}_X$ in $A \downarrow \mathcal{T} \downarrow B$.

5.11.2. Comparing Pointed and Unpointed Homotopy Classes. For any two spaces $X, Y \in \mathcal{T}_*$, the forgetful functor induces a comparison map $[X, Y] \to \langle X_-, Y_- \rangle$. Let's study this map and see how different these homotopy sets can be.[7]

Problem 5.143.

(a) Let $A \to X$ be a cofibration in $\mathcal{T}_\circ$. Show that $A_+ \to X_+ \to X/A$ is a cofiber sequence in $\mathcal{T}_*$.

(b) Show that if $A \to X$ in $\mathcal{T}_\circ$ is an unpointed cofibration, then $A_+ \to X_+$ is a pointed cofibration of well-pointed spaces in $\mathcal{T}_*$.

Problem 5.144. Let X be a well-pointed space. Then after forgetting the basepoints and attaching disjoint ones, the inclusion $* \to X$ of the basepoint becomes the pointed map $*_+ \to X_+$.

(a) Show that the cofiber of this map is X itself.

(b) Show that there is a short exact sequence

$$* \leftarrow \pi_0(Y) \longleftarrow \langle X, Y \rangle \longleftarrow [X, Y] \leftarrow *$$

of pointed sets.

Hint. Show that $*_+ \to X_+$ has a retraction $X_+ \to *_+$.

(c) Conclude that if Y is path-connected, then the map $[X, Y] \to \langle X, Y \rangle$ is surjective.

It is possible to find examples in which $[X, Y] \to \langle X, Y \rangle$ is neither injective nor surjective (see Problem 12.45).

The Track of an Unpointed Homotopy. Let $f, g : X \to Y$ in $\mathcal{T}$, and let $H : X \times I \to Y$ be an *unpointed* homotopy from f to g (more precisely, H is a homotopy in $\mathcal{T}_\circ$ from f_- to g_-). The **track** of H is the pointed map $I \to Y$ given by $[t] \mapsto H(*, t)$. If f and g happen to be pointed maps, then

[7] We'll generally write $\langle X, Y \rangle$ instead of the more correct $\langle X_-, Y_- \rangle$.

the track of H can be interpreted as a pointed loop $S^1 \to Y$, where we view S^1 as the quotient $I/\{0,1\}$.

Problem 5.145. Let f, g and h be maps between well-pointed spaces in $\mathcal{T}_*$.

(a) If $f \simeq g$ in $\mathcal{T}_\circ$ with track τ and τ is path homotopic to ω, then there is a homotopy $H : f \to g$ with track ω.

(b) If $f \simeq g$ with track τ and $g \simeq h$ with track ω, then $f \simeq h$ with track $\tau * \omega$.

Problem 5.146. Let $\alpha, \beta : X \to Y$, where X is well-pointed and Y is a group object in $\mathcal{T}_*$.

(a) Show that for any $\tau : S^1 \to Y$, there is a homotopy $K_\tau : \beta \simeq \beta$ with track τ.

(b) Show that if there is an unpointed homotopy $H : \alpha \simeq \beta$, then there is another homotopy $K : \alpha \simeq \beta$ whose track is a nullhomotopic loop in Y.

(c) Show that $[X, Y] \to \langle X, Y\rangle$ is bijective.

Problem 5.147. Show that $\pi_1(S^1) \to \langle S^1, S^1\rangle$ is bijective.

Exactness for Unpointed Homotopy Classes.

Problem 5.148.

(a) Show that it makes sense to describe a sequence of sets $A \to B \to C$ as exact as long as C is a pointed set.

(b) Show that if $A \to X$ is a cofibration in $\mathcal{T}_\circ$ and $Y \in \mathcal{T}_\circ$ is path-connected, then

$$\langle X/A, Y\rangle \longleftarrow \langle X, Y\rangle \longleftarrow \langle A, Y\rangle$$

is exact. Formulate a corresponding result for unpointed fibrations.

Hint. First figure out how to make $\langle X/A, Y\rangle$ pointed!

5.11.3. Problems.

Problem 5.149. Show that the standard homotopy fiber of $i : A \hookrightarrow X$ in $\mathcal{T}_*$ is the mapping space

$$\mathrm{map}_{(3)}((I; \{0\}, \{1\}), (X; A, *)).$$

(See Section 4.7.5 for the definition.)

Problem 5.150.

(a) Show that $S^1 \times S^1 = (S^1 \vee S^1) \cup_\alpha D^2$.

(b) Determine the attaching map $\alpha \in \pi_1(S^1 \vee S^1)$ and show that its suspension $\Sigma\alpha \in \pi_2(S^2 \vee S^2)$ is trivial.

(c) Determine the homotopy type of $\Sigma(S^1 \times S^1)$.

Hint. View S^1 as $I/\{0,1\}$ and $S^1 \times S^1$ as a quotient of $I \times I$.

Project 5.151. Is there a map $f : X \to Y$ of pointed spaces that cannot be converted to a cofibration?

Problem 5.152. Let $X \in \mathcal{T}_*$ and $Y \in \mathcal{T}_\circ$.

(a) Show that $X \rtimes Y = X \wedge Y_+$.

(b) Show that $S^1 \rtimes Y_+ \simeq \Sigma(Y \vee S^1)$ in $\mathcal{T}_*$.

Hint. Draw a picture!

(c) Conclude that $\Sigma X \rtimes Y \simeq \Sigma X \vee \Sigma(X \wedge Y)$.

(d) Conclude that $S^m \rtimes S^n \simeq S^m \vee S^{n+m}$.

Problem 5.153. Use lifting functions to show that the sequence obtained by applying a mapping space functor to a cofiber sequence, or to a fiber sequence, is a fibration sequence.

Problem 5.154. Show that if λ_0 and λ_1 are both lifting functions for a fibration p, then there is a homotopy $\lambda_0 \simeq \lambda_1$ *through lifting functions.* In other words, the space

$$\mathrm{Lifting}(p) = \{\lambda \mid \lambda \text{ is a lifting function for } p\} \subseteq \mathrm{map}_\circ(\Omega_p, E^I)$$

is path-connected.

Hint. For each $s \in I$, define λ_s by using λ_0 to lift the restriction of the path to $[0, s]$ and using λ_1 to lift the rest.

Problem 5.155. Show that $Y^I \to Y \times Y$, $\mathcal{P}(Y) \to Y$ (and so on) are pointed fibrations; show also that $\mathcal{P}(Y) \to Y$ is a pointed homotopy equivalence.

Problem 5.156. Show that if X is CW complex and $K \subseteq X$ is a subcomplex, then $K \hookrightarrow X$ is a cofibration.

Hint. First show that $K \hookrightarrow K \cup X_n$ is a cofibration for each n.

Problem 5.157. Show that if X is a CW complex, then no matter what basepoint $x \in X$ is chosen (whether it is in X_0 or not), the pointed space (X, x) is well-pointed.

Exercise 5.158. The unpointed mapping cylinder of the quotient map $S^1 \to \mathbb{R}\mathrm{P}^1$ is a famous space—what is it? What is the pointed mapping cylinder?

Problem 5.159. Show that in Theorem 5.64 the lift ϕ is unique up to homotopy under A if p is a homotopy equivalence and that it is unique up to a homotopy over B if i is a homotopy equivalence.

Problem 5.160. Show that if $f : X \to Y$ is surjective, then X has a topology so that f is a fibration in **Top**. Is it true in $\mathcal{T}_\circ$?

Problem 5.161. The **free loop space** on a pointed space X is the mapping space $\Lambda(X) = \mathrm{map}_\circ(S^1, X)$. This is a pointed space whose basepoint is the constant map to $* \in X$. Show that there is a pullback square

$$\begin{tikzcd} \Lambda(X) \arrow[r] \arrow[d, "@_*"'] \arrow[dr, phantom, "\text{pullback}"] & X^I \arrow[d, "@_{0,1}"] \\ X \arrow[r, "\Delta"] & X \times X. \end{tikzcd}$$

Show that $@_*$ is a fibration, and determine its fiber.

Problem 5.162. Show that if f and g are either both fibrations or both cofibrations, then a pointwise homotopy equivalence $\alpha : f \to g$ in $\mathrm{map}(\mathcal{T}_\circ)$ is actually a homotopy equivalence in $\mathrm{map}(\mathcal{T}_\circ)$.

HINT. Reduce it to the case $A{\downarrow}\mathcal{T}_\circ$ or $\mathcal{T}_\circ{\downarrow}B$.

Problem 5.163. Show that the unpointed mapping cone defines a functor $\mathrm{map}(\mathcal{T}_\circ) \to \mathcal{T}_*$.

Project 5.164. Prove Theorem 5.95(a) (See [**164**] for guidance.)

Problem 5.165. Show that if $f : X \to Y$ has a left homotopy inverse, then X is a retract of the mapping cylinder M_f.

Problem 5.166. Let $p : E \to B$ in $\mathcal{T}_*$. Show that the following are equivalent:

(1) p has the right lifting property for all acyclic pointed cofibrations of well-pointed spaces;

(2) p_- is an unpointed fibration.

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Chapter 6

Homotopy Limits and Colimits

The colimit is one of the principal tools—in any category—for constructing new objects from old ones. Since we are studying homotopy theory, we will frequently find ourselves comparing two diagrams using a pointwise homotopy equivalence. Unfortunately, pointwise homotopy equivalences of diagrams do not necessarily induce homotopy equivalences of colimits, and so the objects defined by colimits of diagrams defined up to homotopy are not well-defined, even up to homotopy type.

But if the diagrams F and G are sufficiently nice, then a pointwise homotopy equivalence $F \to G$ *does* induce a homotopy equivalence of colimits. It turns out that Every diagram is pointwise equivalent to a nice one, and we define the homotopy colimit of F to be the categorical colimit of a pointwise equivalent nice diagram $\overline{F}$. This is well-defined up to homotopy equivalence, and it can be made functorial. Thus we produce a functor hocolim : $\mathcal{T}^{\mathcal{I}} \to \mathcal{T}$ which is the best possible homotopy invariant approximation to colim, in the sense that it has the following properties:

(1) there is a natural comparison map $\xi : \operatorname{hocolim} F \to \operatorname{colim} F$;

(2) if $F \to G$ is a pointwise homotopy equivalence of diagrams, then the induced map $\operatorname{hocolim} F \to \operatorname{hocolim} G$ is a homotopy equivalence;

(3) any other functor $\mathcal{L} : \mathcal{T}^{\mathcal{I}} \to \mathcal{T}$ satisfying (1) and (2) factors through hocolim.

In category-theoretical language, this means that hocolim is a *Kan extension*.

The development of the theory for pointed and unpointed spaces is roughly parallel, the only difference being that in the pointed context, we must impose well-pointedness conditions to be able to apply the pointed Fundamental Lifting Principle and use the pointed mapping cylinder. But for the most part, there is absolutely no distinction between the two cases; thus in many sections we work in the category $\mathcal{T}$, standing for either $\mathcal{T}_\circ$ or $\mathcal{T}_*$.

The dual problem arises for limits of diagrams, and the framework for solving it dualizes, giving a theory of homotopy limits.

6.1. Homotopy Equivalence in Diagram Categories

Let $\mathcal{I}$ be a small category, and consider the category $\mathcal{T}^{\mathcal{I}}$ of $\mathcal{I}$-shaped diagrams in $\mathcal{T}$. Thus the objects of $\mathcal{T}^{\mathcal{I}}$ are functors $F : \mathcal{I} \to \mathcal{T}$ and the morphisms are natural transformations $F \to G$. For example, if $\mathcal{I}$ is the category $\bullet \to \star$, then $\mathcal{T}^{\mathcal{I}}$ is isomorphic to the category $\mathrm{map}(\mathcal{T})$ of maps in $\mathcal{T}$. In Section 4.7 we defined three different notions of homotopy equivalence in the category $\mathrm{map}(\mathcal{T})$, and these definitions generalize to all diagram categories.

Pointwise Homotopy Equivalence. The simplest idea is what we will call pointwise homotopy equivalence of diagrams. A map of diagrams $\phi : F \to G$ is a **pointwise homotopy equivalence** if for each object $i \in \mathcal{I}$, the map $\phi(i) : F(i) \to G(i)$ is a homotopy equivalence in $\mathcal{T}$.

If ϕ is a pointwise homotopy equivalence, then each $\phi(i)$ has a homotopy inverse in $\mathcal{T}$, but no claims are made about how those homotopy inverses are related to one another or about the homotopies that are implicit in the assertion that they are homotopy inverses. Indeed, we have seen examples (with $\mathcal{I}$ the category $\bullet \to \star$) of pointwise homotopy equivalences without a pointwise homotopy inverse, though these difficulties evaporated when the diagrams (maps) involved were cofibrations.

To get some control over these homotopy inverses and homotopies, we need all the homotopies to be compatible with one another. To be precise about this, we define homotopy for morphisms of diagrams; this definition is formally identical to the one we've already studied in Section 4.7 for the category $\mathrm{map}(\mathcal{T})$.

Homotopy in the Category of Diagrams. We begin by defining the cylinder functors

$$\mathrm{Cyl}(X) = \begin{cases} X \rtimes I & \text{for } X \in \mathcal{T}_*, \\ X \times I & \text{for } X \in \mathcal{T}_\circ, \end{cases}$$

which come with natural inclusions $\mathrm{in}_0, \mathrm{in}_1 : X \to \mathrm{Cyl}(X)$. The functoriality of the cylinder construction allows us to define the cylinder on a diagram $F : \mathcal{I} \to \mathcal{T}$ by setting

$$\mathrm{Cyl}(F)(i) = \mathrm{Cyl}(F(i)) \qquad \text{for } i \in \mathcal{I}.$$

The functor $\mathrm{Cyl}(F)$ comes with natural 'inclusion transformations'

$$\mathrm{in}_0 : F \to \mathrm{Cyl}(F) \qquad \text{and} \qquad \mathrm{in}_1 : F \to \mathrm{Cyl}(F).$$

Problem 6.1. Show that $\mathrm{Cyl}(\operatorname{colim} F) = \operatorname{colim} \mathrm{Cyl}(F)$.

Now we can define a homotopy between two diagram maps (i.e., natural transformations) in the obvious way: a **homotopy** between $\phi_0, \phi_1 : F \to G$ is a natural transformation

$$H : \mathrm{Cyl}(F) \longrightarrow G$$

such that the diagram of natural transformations

$$\begin{tikzcd} F \arrow[r, "\mathrm{in}_0"] \arrow[dr, bend right, "\phi_0"'] & \mathrm{Cyl}(F) \arrow[d, "H"] & F \arrow[l, "\mathrm{in}_1"'] \arrow[dl, bend left, "\phi_1"] \\ & G & \end{tikzcd}$$

commutes. In terms of spaces, a diagram homotopy is a big collection of homotopies $H_i : \phi_0(i) \simeq \phi_1(i)$, one for each object $i \in \mathcal{I}$, that are compatible with one another in the sense that for each $\alpha : i \to j$ in $\mathcal{I}$, the diagram

$$\begin{tikzcd} \mathrm{Cyl}(F(i)) \arrow[r, "\mathrm{Cyl}(F(\alpha))"] \arrow[d, "H(i)"'] & \mathrm{Cyl}(F(j)) \arrow[d, "H(j)"] \\ G(i) \arrow[r, "G(\alpha)"] & G(j) \end{tikzcd}$$

is commutative. This leads us to *the* nice property of diagram homotopy.

Proposition 6.2. *Let $\phi_0, \phi_1 : F \to G$, and let $H : \phi_0 \simeq \phi_1$. Then the maps $f_0, f_1 : \operatorname{colim} F \to \operatorname{colim} G$ induced by ϕ_0 and ϕ_1, respectively, are homotopic in $\mathcal{T}$.*

Problem 6.3. Prove Proposition 6.2.

HINT. The map of diagrams H induces a map J of colimits.

Now that we have a definition of homotopy, we can define homotopy equivalence: a map of diagrams $\phi : F \to G$ is a **diagram homotopy equivalence** if there is a $\theta : G \to F$ such that $\theta \circ \phi \simeq \mathrm{id}_F$ and $\phi \circ \theta \simeq \mathrm{id}_G$.

Exercise 6.4. Show that every diagram homotopy equivalence is a pointwise homotopy equivalence.

The following result is an instant corollary of Proposition 6.2.

Corollary 6.5. *If $\phi : F \to G$ is a diagram homotopy equivalence, then the induced map $f : \operatorname{colim} F \to \operatorname{colim} G$ is a homotopy equivalence in $\mathcal{T}$.*

Exercise 6.6. Prove it!

Pointwise Equivalence in the Homotopy Category. There is yet another way to apply the idea of homotopy equivalence to diagrams. Let $\mathrm{Ho} : \mathcal{T} \to \mathrm{H}\mathcal{T}$ be the standard functor from the topological category $\mathcal{T}$ to its homotopy category $\mathrm{H}\mathcal{T}$, defined in Section 4.3. It may be that we have two diagrams $F, G : \mathcal{I} \to \mathcal{T}$ that are connected by a collection $\{\phi_i \mid i \in \mathcal{I}\}$ of maps that make the diagrams

$$\begin{tikzcd} F(i) \arrow[r, "F(\alpha)"] \arrow[d, "\phi_i"'] & F(j) \arrow[d, "\phi_j"] \\ G(i) \arrow[r, "G(\alpha)"] & G(j) \end{tikzcd}$$

(for $\alpha : i \to j$ in $\mathcal{I}$) commute only up to homotopy. The collection $\{\phi_i\}$ does not constitute a morphism of diagrams in $\mathcal{T}^{\mathcal{I}}$, but they do define a morphism $\phi : \mathrm{Ho} \circ F \to \mathrm{Ho} \circ G$. If this ϕ is a pointwise equivalence—i.e., if each of the maps ϕ_i is a homotopy equivalence—then we say that F and G are **pointwise equivalent in $\mathrm{H}\mathcal{T}$**. This is a much weaker notion of equivalence for diagrams than the ones defined above, and we will not use it at all in this chapter. In the next chapter and later chapters it will take on much greater significance.

6.2. Cofibrant Diagrams

We have defined two (apparently different) notions of homotopy equivalence of diagrams. The first one, *pointwise homotopy equivalence*, is conceptually simple, and the second one, *diagram homotopy equivalence*, has the extremely nice property that a diagram homotopy equivalence induces a homotopy equivalence of colimits. This leads us to ask: *is every pointwise homotopy equivalence $\phi : F \to G$ automatically a homotopy equivalence of diagrams?*

Exercise 6.7. Find an example of a pointwise homotopy equivalence that is not a diagram homotopy equivalence.

HINT. Take for granted that $S^n \not\simeq *$; what is the pushout of the diagram $CX \leftarrow X \to CX$?

Now that our optimism has been appropriately dampened, we change our point of view and ask: *what do we need to know about F and G before we can conclude that every pointwise homotopy equivalence $F \to G$ is a diagram homotopy equivalence?* We have some hope here, because we have already found useful answers to this question in the categories $A{\downarrow}\mathcal{T}$ and $\mathcal{T}{\downarrow}B$: the two notions of equivalence coincide for morphisms between cofibrations (for $A{\downarrow}\mathcal{T}$) and for morphisms between fibrations (for $\mathcal{T}{\downarrow}B$).

6.2.1. Cofibrant Diagrams. A diagram map $\pi : X \to Y$ in $\mathcal{T}^{\mathcal{I}}$ is a **pointwise fibration** if for each $i \in \mathcal{I}$ the map $\pi(i) : X(i) \to Y(i)$ is a fibration in $\mathcal{T}$. A diagram $F \in \mathcal{T}^{\mathcal{I}}$ is **cofibrant** if for every $\phi : F \to Y$ and every $\pi : X \to Y$ that is both a pointwise homotopy equivalence and a pointwise fibration, the dotted morphism in

$$\begin{tikzcd} & X \arrow[d, "\pi"] \\ F \arrow[r, "\phi"] \arrow[ur, dotted, bend left] & Y \end{tikzcd}$$

can be found to make the triangle commute in $\mathcal{T}^{\mathcal{I}}$.

Problem 6.8. Let $\phi : X \to Y$ be a morphism of diagrams. Show that there is a factorization

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, bend right, "\varepsilon"'] & & Y \\ & \widetilde{X} \arrow[ur, bend right, "\pi"'] & \end{tikzcd}$$

of ϕ such that

(1) ε has a diagram homotopy inverse $\overline{\varepsilon} : \widetilde{X} \to X$ such that $\overline{\varepsilon} \circ \varepsilon = \mathrm{id}_X$,

(2) π is a pointwise fibration, and

(3) all this structure is functorial.

Hint. Use Theorem 5.41.

Cofibrant diagrams answer our question: a pointwise homotopy equivalence between cofibrant diagrams must be a diagram homotopy equivalence.

Theorem 6.9. *If F and G are cofibrant and $\phi : F \to G$ is a pointwise homotopy equivalence, then ϕ is a diagram homotopy equivalence.*

Problem 6.10.

(a) Show that if F is a cofibrant diagram, then in every diagram

$$\begin{tikzcd} & X \arrow[d, "\theta"] \\ F \arrow[r, "\phi"] \arrow[ur, dotted, bend left] & Y, \end{tikzcd}$$

of morphisms in $\mathcal{T}^{\mathcal{I}}$, where θ is a pointwise homotopy equivalence, the dotted arrow can be filled in making the triangle commute up to diagram homotopy in $\mathcal{T}^{\mathcal{I}}$.

(b) Prove Theorem 6.9.

Exercise 6.11. Is the converse to Problem 6.10(a) true?

In Section 5.1 we mentioned in passing that the (obvious) fact that for every $X \in \mathcal{T}_\circ$, the map $\varnothing \to X$ is a cofibration could be expressed by saying that every space $X \in \mathcal{T}_\circ$ is **cofibrant**. Applying the mutual characterization of cofibrations and fibrations, we see that a space $X \in \mathcal{T}_\circ$ is cofibrant if and only if whenever $p : E \to B$ is a fibration and a homotopy equivalence, there is a lift in the square

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dotted] & B. \end{tikzcd}$$

Notice that, since $\varnothing$ is the initial object of $\mathcal{T}_\circ$, the upper triangle does not actually impose any restrictions on our lift.

In the category $(\mathcal{T}_\circ)^{\mathcal{I}}$, the constant diagram at $\varnothing$ is the initial object, and so we see that if we adopt the pointwise fibrations as the 'fibrations' in the diagram category, then our notion of cofibrant diagram is formally identical with the concept for spaces. There is a major difference, though: every space is cofibrant, but many diagrams are not.

6.2.2. An Instructive and Important Example. Let's consider the example in which the shape category $\mathcal{I}$ is the prepushout category $\star \leftarrow \bullet \rightarrow \circ$. Are there reasonable conditions that we can impose on such a diagram to ensure that it is cofibrant?

Going back to the definition, we see that we must consider the lifting problems described by the diagram

$$\begin{tikzcd}
& & B \arrow[rrr, dotted, "\lambda_B"] \arrow[ddd, equal] & & & X_B \arrow[ddd, "\pi_B"] \\
& A \arrow[ur] \arrow[rrrr, dashed, "\lambda_A"] \arrow[dl] \arrow[ddd, equal] & & & X_A \arrow[ur, "\beta"'] \arrow[dl, "\gamma"] \arrow[ddd, "\pi_A"] & \\
C \arrow[rrr, dotted, "\lambda_C"] \arrow[ddd, equal] & & & X_C \arrow[ddd, "\pi_C"] & & \\
& & B \arrow[rrr, "\phi_B"] & & & Y_B \\
& A \arrow[ur] \arrow[rrrr, "\phi_A"] \arrow[dl] & & & Y_A \arrow[ur] \arrow[dl] & \\
C \arrow[rrr, "\phi_C"] & & & Y_C & &
\end{tikzcd}$$

in which the diagram morphism $\pi : X \to Y$ is a pointwise homotopy equivalence and a pointwise fibration.

Problem 6.12.

(a) Show that, once λ_A has been found, the existence of λ_B and λ_C is equivalent to the existence of lifts in the squares

$$\begin{tikzcd}
A \arrow[r, "\beta\circ\lambda_A"] \arrow[d] & X_B \arrow[d, "\pi_B"] \\
B \arrow[r, "\phi_B"] \arrow[ur, dotted] & Y_B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "\gamma\circ\lambda_A"] \arrow[d] & X_C \arrow[d, "\pi_C"] \\
C \arrow[r, "\phi_C"] \arrow[ur, dotted] & Y_C.
\end{tikzcd}$$

(b) Show that the diagram $C \leftarrow A \to B$ is cofibrant if $A \to B$ and $A \to C$ are both cofibrations (of well-pointed spaces in $\mathcal{T}_*$).

Now that we can easily recognize cofibrant diagrams, we can show that every prepushout diagram F is pointwise homotopy equivalent to a cofibrant diagram.

Problem 6.13. Let F be the prepushout diagram $C \xleftarrow{f} A \xrightarrow{g} B$, and let $\overline{F}$ be the diagram $M_g \leftarrow A \to M_f$.

(a) Show that $\overline{F}$ is cofibrant (assuming A, B and C are well-pointed for the $\mathcal{T}_*$ case).

(b) Construct a pointwise homotopy equivalence $\overline{F} \to F$.

(c) Describe in detail the sense in which the construction of the diagram morphism $\overline{F} \to F$ is functorial.

Exercise 6.14. Does a similar analysis work to identify the cofibrant diagrams with other shapes $\mathcal{I}$? What are the properties of the prepushout shape that make Problem 6.12 work?

Exercise 6.15. Suppose $C \leftarrow A \rightarrow B$ is cofibrant. Must $A \to B$ and $A \to C$ be cofibrations?

6.2.3. Cofibrant Replacements of Diagrams. If we are given a diagram $F : \mathcal{I} \to \mathcal{T}$ that is not cofibrant, then we can hope to replace it, as you did in Problem 6.13, with a pointwise homotopy equivalent diagram that is cofibrant. A pointwise homotopy equivalence $\overline{F} \to F$ where $\overline{F}$ is cofibrant is called a **cofibrant replacement** of F. For many shape categories, cofibrant replacement can be done functorially, but it is often more convenient to use some other *ad hoc* replacement. For example, if we recognize that a given diagram is already cofibrant, why modify it at all?

Since we will make use of *ad hoc* cofibrant replacements of diagrams, it will be useful to have some information about them: does a map of diagrams induce a map between their replacements? What do we need to know about a functor $\Phi : \mathcal{T} \to \mathcal{T}$ to know that the induced functor $\Phi_* : \mathcal{T}^{\mathcal{I}} \to \mathcal{T}^{\mathcal{I}}$ carries cofibrant diagrams to cofibrant diagrams?

First we consider lifting a morphism of diagrams to a morphism of cofibrant replacements.

Problem 6.16. Let $\phi : F \to G$ be a diagram morphism, and let $\overline{F} \to F$ and $\overline{G} \to G$ be any two cofibrant replacements.

(a) Show that there is a morphism of diagrams $\overline{\phi} : \overline{F} \to \overline{G}$ which is compatible with ϕ in the sense that the square

$$\begin{tikzcd} \overline{F} \arrow[r, dotted, "\overline{\phi}"] \arrow[d] & \overline{G} \arrow[d] \\ F \arrow[r, "\phi"] & G \end{tikzcd}$$

commutes up to diagram homotopy.

(b) Show that if ϕ is a pointwise homotopy equivalence, then $\overline{\phi}$ is a diagram homotopy equivalence.

(c) Show that if $\overline{F}$ and $\widetilde{F}$ are two cofibrant replacements for F, then there is a diagram homotopy equivalence $\overline{F} \xrightarrow{\simeq} \widetilde{F}$.

Now we ask which functors preserve cofibrant diagrams.

Theorem 6.17. *Let L, R be an adjoint pair of functors $\mathcal{T} \to \mathcal{T}$, and suppose R preserves fibrations and homotopy equivalences. Then if F is a cofibrant diagram, then so is $L \circ F$.*

Problem 6.18. Let L and R be an adjoint pair.

(a) Show that the lifting problems

$$\begin{tikzcd} & A \arrow[d, "f"] \\ LX \arrow[ur, dotted, "\lambda"] \arrow[r, "g"] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & RA \arrow[d, "R(f)"] \\ X \arrow[ur, dotted, "\widetilde{\lambda}"] \arrow[r, "\widetilde{g}"] & RB \end{tikzcd}$$

are equivalent in the sense that λ exists if and only if $\widetilde{\lambda}$ exists.

(b) Prove Theorem 6.17.

Corollary 6.19.

(a) If $F : \mathcal{I} \to \mathcal{T}_*$ is cofibrant, then so are $F \rtimes A$ and $F \wedge A$ and $F \times A$.

(b) If $F : \mathcal{I} \to \mathcal{T}_\circ$ is cofibrant, then so is $F \times A$.

Problem 6.20. Prove Corollary 6.19.

6.3. Homotopy Colimits of Diagrams

In this section, we define the **homotopy colimit** for diagrams $F : \mathcal{I} \to \mathcal{T}$ that have cofibrant replacements. We'll prove in the next section that for many of the most useful shape categories $\mathcal{I}$, every diagram (or almost every diagram) has a (functorial) cofibrant replacement.

6.3.1. The Homotopy Colimit of a Diagram. We begin by defining the homotopy colimit of a diagram in an *ad hoc* manner. We'll come to the more systematic categorical point of view later.

A space X is a **homotopy colimit** for a diagram $F : \mathcal{I} \to \mathcal{T}$ if $X \simeq \operatorname{colim} \overline{F}$, where $\overline{F} \to F$ is a cofibrant replacement of F. Note that the diagram map $\overline{F} \to F$ induces a comparison map $\operatorname{hocolim} F \to \operatorname{colim} F$.

It is important to understand that the cofibrant replacement $\overline{F}$ is not unique, and so the homotopy colimit of F is not uniquely determined by the definition; rather, the definition identifies a long list of spaces which qualify as homotopy colimits of F.

Problem 6.21.

(a) Show that any two homotopy colimits for the diagram F are homotopy equivalent.

(b) Show that, once a cofibrant replacement $\overline{F} \to F$ has been chosen, there are well-defined homotopy classes $F(i) \to \operatorname{hocolim} F$ for each $i \in \mathcal{I}$.

Since the homotopy colimit of a diagram is well-defined up to homotopy type, we can speak of *the* homotopy colimit of F.

Problem 6.22. Compare the homotopy colimit of $* \leftarrow X \rightarrow *$ in $\mathcal{T}$ with the colimit of $* \leftarrow X \rightarrow *$ in $\mathrm{H}\mathcal{T}$.

6.3.2. Induced Maps of Homotopy Colimits. If $F \rightarrow G$ is a map of diagrams, then the formal properties of colimits yield a unique map $\operatorname{colim} F \rightarrow \operatorname{colim} G$. By contrast, the homotopy colimit of F is not (usually) the categorical colimit of F, and so a map of diagrams does not give rise to a map of homotopy colimits in the same purely formal way.

Problem 6.16 implies a kind of limited naturality to the construction.

Problem 6.23. Let $\phi : F \rightarrow G$ be a map of diagrams; let X be a homotopy colimit for F, and let Y be a homotopy colimit for G. Show that ϕ induces a map $\Phi : X \rightarrow Y$ making the diagram

$$\begin{tikzcd}
X \arrow[r, "\Phi"] \arrow[d, "\xi_F"'] & Y \arrow[d, "\xi_G"] \\
\operatorname{colim} F \arrow[r, "\operatorname{colim} \phi"] & \operatorname{colim} G
\end{tikzcd}$$

commute up to homotopy.

The map Φ is not uniquely determined by ϕ, even up to homotopy. We should think of a map of diagrams ϕ as inducing a *set* of maps between homotopy colimits rather than a single map. However these maps are closely related to one another.

Problem 6.24. Let $\phi : F \rightarrow G$ be a diagram map and choose cofibrant approximations

$$\overline{F} \longrightarrow F, \qquad \widetilde{F} \longrightarrow F, \qquad \overline{G} \longrightarrow G, \qquad \text{and} \qquad \widetilde{G} \longrightarrow G.$$

Show that there is a diagram of diagram morphisms

$$\begin{tikzcd}
\widetilde{F} \arrow[rrr, "\widetilde{\phi}"] \arrow[dr] \arrow[dd, "\simeq"'] & & & \widetilde{G} \arrow[dl] \arrow[dd, "\simeq"] \\
& F \arrow[r, "\phi"] & G & \\
\overline{F} \arrow[rrr, "\overline{\phi}"] \arrow[ur] & & & \overline{G} \arrow[ul]
\end{tikzcd}$$

which commutes up to diagram homotopy. Conclude that any two induced maps of homotopy colimits are pointwise equivalent in $\mathrm{H}\mathcal{T}$.

6.3.3. Example: Induced Maps Between Suspensions. Before proceeding to the categorical point of view, we pause briefly to study the maps induced by diagram morphisms between the homotopy colimits of diagrams of the form $* \leftarrow X \rightarrow *$.

Problem 6.25.

(a) Show that S^1 is a homotopy colimit of the diagram $* \leftarrow S^0 \rightarrow *$.

(b) Find all the maps $S^1 \to S^1$ induced by the diagram map

$$\begin{tikzcd} * \arrow[d] & S^0 \arrow[l] \arrow[r] \arrow[d, "\mathrm{id}"] & * \arrow[d] \\ * & S^0 \arrow[l] \arrow[r] & *. \end{tikzcd}$$

(c) Find all the maps induced by the diagram

$$\begin{tikzcd} * \arrow[d] & S^0 \arrow[l] \arrow[r] \arrow[d] & * \arrow[d] \\ S^1 & * \arrow[l] \arrow[r] & S^1. \end{tikzcd}$$

You showed in Problem 6.22 that the homotopy pushout of $* \leftarrow X \rightarrow *$ is the suspension ΣX. The morphism

$$\begin{tikzcd} * \arrow[d] & X \arrow[l] \arrow[r] \arrow[d, "f"] & * \arrow[d] \\ * & Y \arrow[l] \arrow[r] & * \end{tikzcd}$$

of prepushout diagrams induces maps $\Sigma X \to \Sigma Y$ of the homotopy pushouts, but which maps are induced and which are not? To be able to compare maps to one another, we choose the standard suspensions as our homotopy colimits, so that each of them is a union of two cones:

$$\Sigma X = C_+X \cup C_-X \qquad \text{and} \qquad \Sigma Y = C_+Y \cup C_-Y.$$

With these choices, we can see that the complete list of all cofibrant replacements of the given diagram includes the diagrams

$$\begin{tikzcd} C_-X \arrow[d, "Cf"'] & X \arrow[l] \arrow[r] \arrow[d, "f"] & C_+X \arrow[d, "Cf"] \\ C_-Y & Y \arrow[l] \arrow[r] & C_+Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C_+X \arrow[d, "Cf"'] & X \arrow[l] \arrow[r] \arrow[d, "f"] & C_-X \arrow[d, "Cf"] \\ C_-Y & Y \arrow[l] \arrow[r] & C_+Y. \end{tikzcd}$$

The map induced by the first replacement is Σf, but the second diagram induces $-\Sigma f$. Thus, as long as there is a map $f : X \to Y$ whose suspension Σf does not have order 2, we will have an example where more than one homotopy class of maps is induced by a given map of prepushout diagrams.

Project 6.26. Investigate the collection $\mathcal{S}(f)$ of all homotopy classes of maps $\Sigma X \to \Sigma Y$ induced on homotopy colimits by the diagram morphism

$$\begin{tikzcd} * \arrow[d] & X \arrow[l] \arrow[r] \arrow[d, "f"] & * \arrow[d] \\ * & Y \arrow[l] \arrow[r] & *. \end{tikzcd}$$

Can there be induced maps other than $\pm\Sigma f$? Show that $\mathcal{S}(\mathrm{id}_X)$ and $\mathcal{S}(\mathrm{id}_Y)$ are groups under composition and that they act on $\mathcal{S}(f)$.

6.3.4. The Functorial Approach to Homotopy Colimits. Sometimes the freedom of using any cofibrant replacement we like is outweighed by the usefulness of well-defined and natural induced maps. We conclude this section by constructing a homotopy colimit functor $\mathrm{hocolim} : \mathcal{T}^{\mathcal{I}} \to \mathcal{T}$ and verifying that it is—up to natural isomorphism—the unique best homotopy invariant approximation to the colimit.

We suppose in this section that we have been given, or have found, a functor $\textsc{cof} : \mathcal{T}^{\mathcal{I}} \longrightarrow \mathcal{T}^{\mathcal{I}}$ and a natural transformation $\xi : \textsc{cof} \to \mathrm{id}$ so that for any $F \in \mathcal{T}^{\mathcal{I}}$, the map $\xi_F : \textsc{cof}(F) \to F$ is a cofibrant replacement.[1] Then the homotopy colimit *functor* $\mathrm{hocolim} : \mathcal{T}^{\mathcal{I}} \to \mathcal{T}$ is defined by setting

$$\mathrm{hocolim}\, F = \mathrm{colim}\, \textsc{cof}(F)$$

and for $\phi : F \to G$, defining

$$\mathrm{hocolim}(\phi) : \mathrm{hocolim}\, F \longrightarrow \mathrm{hocolim}\, G$$

to be the induced map $\mathrm{colim}(\textsc{cof}(\phi))$ of categorical colimits. For $F : \mathcal{I} \to \mathcal{T}$, the comparison $\xi : \mathrm{hocolim}\, F \to \mathrm{colim}\, F$ is the induced map of colimits $\mathrm{colim}(\xi_F) : \mathrm{colim}\, \textsc{cof}(F) \to \mathrm{colim}\, F$.

Theorem 6.27. *The transformation* $\xi : \mathrm{hocolim} \to \mathrm{colim}$ *satisfies the following properties:*

(a) If $F \to G$ is a pointwise homotopy equivalence of diagrams, then the induced map $\mathrm{hocolim}\, F \to \mathrm{hocolim}\, G$ *is a homotopy equivalence.*

(b) If $\mathcal{L} : \mathcal{T}^{\mathcal{I}} \to \mathcal{T}$ is any other functor satisfying (a) and $\zeta : \mathcal{L} \to$ colim *is any natural transformation, then there is a unique natural transformation* $\mathrm{Ho} \circ \mathcal{L} \to \mathrm{Ho} \circ \mathrm{hocolim}$ *making the diagram*

$$\begin{tikzcd} & \mathrm{Ho} \circ \mathrm{hocolim} \arrow[d, "\xi"] \\ \mathrm{Ho} \circ \mathcal{L} \arrow[ur, bend left] \arrow[r] & \mathrm{Ho} \circ \mathrm{colim} \end{tikzcd}$$

commute in $(\mathrm{H}\mathcal{T})^{\mathcal{I}}$.

[1] In the next section you will construct such functors for many important shape categories $\mathcal{I}$.

Problem 6.28.

(a) Prove that hocolim is homotopy invariant in the sense of Theorem 6.27(a).

(b) Show that for any diagram $F : \mathcal{I} \to \mathcal{T}$, the square

$$\begin{tikzcd} \mathcal{L}(\mathrm{COF}(F)) \arrow[r, "\zeta"] \arrow[d, "\mathcal{L}(\xi)"'] & \operatorname{colim} \mathrm{COF}(F) \arrow[d, "\operatorname{colim} \xi"] \\ \mathcal{L}(F) \arrow[r, "\zeta"] & \operatorname{colim} F \end{tikzcd}$$

commutes.

(c) Show that $\mathcal{L}(\xi)$ is a homotopy equivalence, and complete the proof of Theorem 6.27.

We'll write $\operatorname{hocolim} F = \operatorname{colim}(\mathrm{COF}(F))$ and call this the **standard homotopy colimit** of F.

6.4. Constructing Cofibrant Replacements

In this section we'll construct functorial cofibrant replacements for diagrams defined on *simple* shape categories. Simple categories allow inductive construction and study of diagrams, and many of the most important diagram shapes, including the prepushout and the telescope, are simple.

The vast majority of the work here is to find an easily checked condition guaranteeing that a diagram (defined on a simple category) is cofibrant. Once we have this, it is a simple matter to construct (inductively) the desired cofibrant replacements.

6.4.1. Simple Categories. A partially ordered set P can be viewed as the set of objects of a small category $\mathcal{I}$ having morphism sets

$$\mathrm{mor}_{\mathcal{I}}(x, y) = \begin{cases} \{x \to y\} & \text{if } x \le y, \\ \varnothing & \text{otherwise.} \end{cases}$$

Let $\mathcal{I}$ be a category corresponding to a partially ordered set in this way, and define a function $d(i) : \mathcal{I} \to \mathbb{N} \cup \{\infty\}$ by the rule

$$\sup\{n \mid \text{there is } x_0 \to x_1 \to \cdots \to x_n = i \text{ of nonidentity maps in } \mathcal{I}\}.$$

We say that $\mathcal{I}$ is a **simple category** if $d(i) < \infty$ for all $i \in \mathcal{I}$.

Exercise 6.29.

(a) Show that the categories $\star \leftarrow \bullet \to \diamond$ and $\mathbb{N}$ are simple.

(b) Show that a simple category $\mathcal{I}$ is the union of an increasing sequence $\cdots \subseteq \mathcal{I}_n \subseteq \mathcal{I}_{n+1} \subseteq \cdots$ of subcategories such that the only morphism in $\mathcal{I}_n$ with domain in $\mathcal{I}_n - \mathcal{I}_{n-1}$ is the identity.

(c) Show that there are no nonidentity self-maps in $\mathcal{I}$.

(d) Call an object $i \in \mathcal{I}$ a **root** of a simple category $\mathcal{I}$ if the only maps into i are identities. Show that $\mathcal{I}$ contains at least one root.

(e) Show that $\mathcal{I}_0$ contains only roots.

(f) Show that if $\mathcal{I}$ is a simple category, then $\mathcal{I}$ has a filtration in which $\mathcal{I}_0$ is precisely the set of all roots of $\mathcal{I}$.

(g) Write $\mathbb{N}$ for the category $0 \to 1 \to \cdots \to n \to n+1 \to \cdots$. Show that a small category $\mathcal{I}$ is simple if and only if the function $d : \mathcal{I} \to \mathbb{N}$ is the object part of a functor such that if $i \to j$ is not an identity morphism, then $d(i \to j)$ is not an identity morphism.

(h) Show that every subcategory of a simple category is also simple.

From now on if $\mathcal{I}$ is a simple category, then we write

$$\mathcal{I}_n = \{i \in \mathcal{I} \mid d(i) \leq n\},$$

so that $\mathcal{I}_0$ is precisely the collection of all the roots of $\mathcal{I}$.

Extension and Lifting of Functors. Simple categories are defined to make it possible to construct diagrams and diagram morphisms by induction. For an object i in a simple category $\mathcal{I}$, let $\mathcal{I}_{<i}$ denote of the full subcategory of $\mathcal{I}$ with objects $\{j \in \mathcal{I} \mid j < i\}$. If $F : \mathcal{I} \to \mathcal{C}$ is a diagram in the category $\mathcal{C}$, write $F_{<i} : \mathcal{I}_{<i} \to \mathcal{C}$ for the composite functor $\mathcal{I}_{<i} \to \mathcal{I} \xrightarrow{F} \mathcal{T}$.

For each $n \in \mathbb{N}$, define $F_n : \mathcal{I} \to \mathcal{C}$ to be the composition $\mathcal{I}_n \hookrightarrow \mathcal{I} \xrightarrow{F} \mathcal{C}$. The colimit of a diagram F on a simple category can be computed from the colimits of the restrictions F_n.

Problem 6.30. Let $F : \mathcal{I} \to \mathcal{C}$, where $\mathcal{I}$ is a simple category. For each n, define $T(n) = \operatorname{colim} F_n$.

(a) Construct maps $T(n) \to T(n+1)$, yielding a telescope diagram T.

(b) Show that $\operatorname{colim} T = \operatorname{colim} F$.

We also define $\widehat{F}_n : \mathcal{I}_{n+1} \to \mathcal{C}$ by setting

$$\widehat{F}_n|_{\mathcal{I}_n} = F_n \qquad \text{and} \qquad \widehat{F}_{n+1}(i) = \operatorname{colim} F_{<i}.$$

Problem 6.31. Let $\mathcal{I}$ be a simple category, let $\mathcal{C}$ is a category in which all colimits exist (i.e., $\mathcal{C}$ is cocomplete), and let $F : \mathcal{I} \to \mathcal{C}$.

(a) Construct natural comparison maps $\xi_i : \operatorname{colim} F_{<i} \to F(i)$.

(b) Show that restriction induces isomorphisms

$$\operatorname{mor}_{\mathcal{C}^{\mathcal{I}_{n+1}}}(\widehat{F}_n, G) \xrightarrow{\cong} \operatorname{mor}_{\mathcal{C}^{\mathcal{I}_n}}(F_n, G_n)$$

for any $G : \mathcal{I}_{n+1} \to \mathcal{C}$.

(c) Show that $\operatorname{colim} F_n \cong \operatorname{colim} \widehat{F}_n$.

The functors $\widehat{F}_n$ are used to interpolate between F_n and F_{n+1} in the following extension and lifting problem. We are given diagram morphisms

$$\begin{tikzcd} & X \arrow[d, "\pi"] \\ F \arrow[r, "\phi"] \arrow[ur, dotted, "\theta"] & Y \end{tikzcd}$$

in $\mathcal{C}^{\mathcal{I}}$, where $\mathcal{C}$ is cocomplete and we want to fill in the dotted arrow.

Problem 6.32. Show that the morphism θ exists if and only if there are morphisms $\theta_n : F_n \to X_n$ satisfying $(\theta_{n+1})_n = \theta_n$ for each n.

Now let's lay out exactly what is required in order to construct a morphism θ_{n+1} extending a given θ_n.

Problem 6.33. Suppose the morphism $\theta_n : F_n \to X_n$ has been constructed.

(a) Show there is a diagram morphism θ_{n+1} extending θ_n if and only if there is a lift in the diagram

$$\begin{tikzcd} \widehat{F}_n \arrow[r, "\bar{\theta}_n"] \arrow[d] & X \arrow[d] \\ F_{n+1} \arrow[r] \arrow[ur, dotted] & Y. \end{tikzcd}$$

(b) Show that the extension problem of part (a) is equivalent to the collection of independent problems

$$\begin{tikzcd} \operatorname{colim} F_{<i} \arrow[r] \arrow[d] & X(i) \arrow[d] \\ F(i) \arrow[r] \arrow[ur, dotted] & Y(i) \end{tikzcd}$$

for each $i \in \mathcal{I}_{n+1} - \mathcal{I}_n$.

The Cone to a Simple Category. It is especially easy to study diagrams of pointed spaces when the shape category $\mathcal{I}$ is simple and has exactly one root. Fortunately, even if the category we are given has more than one root, we can always enlarge it slightly so that it does have exactly one root. Define $\mathcal{I}_\star$ by adding one more object, $\star$, which we take to be initial: there is a unique morphism $\star \to i$ for each $i \in \mathcal{I}_\star$.

Problem 6.34. Show that $\mathcal{I}_\star$ is a simple category with exactly one root.

Now if we have a functor $F : \mathcal{I} \to \mathcal{C}$, we would like to extend F to a functor $F_\star : \mathcal{I}_\star \to \mathcal{C}$. If $\mathcal{C}$ has an initial object ι, then $F : \mathcal{I} \to \mathcal{C}$ can be extended to $\mathcal{I}_\star$ by setting $F_\star(\star) = \iota$.

Problem 6.35. Show that $F_\star$ is a functor and that restriction of functors defines a bijection

$$\mathrm{mor}_{\mathcal{C}^{\mathcal{I}_\star}}(F_\star, G) \longrightarrow \mathrm{mor}_{\mathcal{C}^{\mathcal{I}}}(F, G|_{\mathcal{I}}).$$

Conclude that $\mathrm{colim}\, F \cong \mathrm{colim}\, F_\star$.

6.4.2. Recognizing Cofibrant Diagrams. Now we establish our recognition principle for cofibrant diagrams.

Theorem 6.36. *Let $F : \mathcal{I} \to \mathcal{T}$ where $\mathcal{I}$ is a simple category, and assume that $F(i)$ is well-pointed for each i if $\mathcal{T} = \mathcal{T}_*$. If, for each i, the comparison map $\xi_i : \mathrm{colim}\, F_{<i} \to F(i)$ is a cofibration in $\mathcal{T}$, then F is a cofibrant diagram.*

The well-pointedness condition for diagrams in $\mathcal{T}_*$ is needed so that we can apply the pointed version of the Fundamental Lifting Property. To prove Theorem 6.36, we have to solve the lifting problem

$$\begin{tikzcd} & X \arrow[d, "\pi"] \\ F \arrow[r, "\phi"'] \arrow[ur, dotted, bend left, "\theta"] & Y, \end{tikzcd}$$

in which π is a pointwise fibration and a pointwise homotopy equivalence.

Problem 6.37. Use the framework of Section 6.4.1 to prove Theorem 6.36.

It turns out that the condition of Theorem 6.36 completely characterizes the unpointed cofibrant diagrams.

Problem 6.38. Let $\mathcal{I}$ be a simple category.

(a) Show that for any map $f : X \to Y$ in $\mathcal{T}_\circ$ and any $i \in \mathcal{I}$, there is a morphism of diagrams $\Phi_{f,i} : S \to T$ such that $\Phi_{f,i}(j)$ is an identity map in $\mathcal{T}$ for all $j \neq i$ and $\Phi_{f,i}(i) = f$.

Hint. If you are stumped, specialize to $\mathcal{I} = \mathbb{N}$.

(b) Prove the converse of Theorem 6.36.

Project 6.39. Can you characterize cofibrant pointed diagrams?

A Common Misconception. The technique used to build a cofibrant replacement for a prepushout diagram is to replace all the maps with pointwise homotopy equivalent cofibrations. The slogan 'replace all maps with cofibrations' is sometimes conflated with the actual procedure 'replace the diagram with a cofibrant one'.

Problem 6.40. Let $\mathcal{E}$ be the category $\bullet \rightrightarrows \star$. Show that a constant diagram $\Delta_X : \mathcal{E} \to \mathcal{T}$ is not cofibrant.

The simpler slogan works fine if the shape category $\mathcal{I}$ is **tree-like**, that is, if each object $i \in \mathcal{I}$ that is not a root has a unique predecessor, which we may as well refer to as $i-1$.[2]

Exercise 6.41. Show that the prepushout and telescope categories are tree-like.

Problem 6.42. Let $F : \mathcal{I} \to \mathcal{T}$; for $\mathcal{T} = \mathcal{T}_*$, assume that $F(i)$ is well-pointed for each $i \in \mathcal{I}$.

(a) Show that if $\mathcal{I}$ is tree-like, then $\operatorname{colim} F_{<i} = F(i-1)$.

(b) Show that F is cofibrant if and only if every map $F(i \to j)$ is a cofibration.

6.4.3. Colimits of Well-Pointed Spaces. In order to give a unified construction of the pointed and unpointed cofibrant replacement of diagrams, we need to know that a pointed colimit of well-pointed spaces is well-pointed.

Let $\mathcal{I}$ be a simple category and let $F : \mathcal{I} \to \mathcal{T}_*$ be a diagram of well-pointed spaces. Extend F to $F_\star : \mathcal{I}_\star \to \mathcal{T}_*$ by setting $F(\star) = *$. Write $F_\circ$ for the composition of $F_\star$ with the forgetful functor $\mathcal{T}_* \to \mathcal{T}_\circ$.

Proposition 6.43. *Let $\mathcal{I}$ be a simple category, and let $F : \mathcal{I} \to \mathcal{T}_*$ be a diagram of well-pointed spaces. If $F_\circ$ is cofibrant in $\mathcal{T}_\circ^{\mathcal{I}}$, then*

(a) $F_\star$ is a cofibrant diagram in $\mathcal{T}_*$ and

(b) $\operatorname{colim}_* F = \operatorname{colim}_* F_\star$ is a well-pointed space.

Problem 6.44. Let $\mathcal{I}$ be a simple category and let $F : \mathcal{I} \to \mathcal{T}_\circ$ be a cofibrant diagram.

(a) Show that the problems

$$\begin{tikzcd} \operatorname{colim} \widehat{F}_n \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \operatorname{colim} F_{n+1} \arrow[r] \arrow[ur, dotted] & B \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \widehat{F}_n \arrow[r] \arrow[d] & \Delta_E \arrow[d, "\Delta_p"] \\ F_{n+1} \arrow[r] \arrow[ur, dotted] & \Delta_B \end{tikzcd}$$

in $\mathcal{T}$ and $\mathcal{T}^{\mathcal{I}}$, respectively, are equivalent.[3]

(b) Conclude that $\operatorname{colim} F_n \to \operatorname{colim} F_m$ is a cofibration in $\mathcal{T}_\circ$ for $0 \le n \le m \le \infty$.

(c) Prove Proposition 6.43.

HINT. Show that $\operatorname{colim}_\circ F_\circ = (\operatorname{colim}_* F_\star)_-$.

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[2]But be careful: $i - 1 = j - 1$ does not imply $i = j$.

[3]$\Delta_Y : \mathcal{I}_{n+1} \to \mathcal{C}$ is the **constant diagram** given by $\Delta_Y(i) = Y$ and $\Delta_Y(i \to j) = \mathrm{id}_Y$.

6.4.4. Existence of Cofibrant Replacements. Now that we know how to recognize a cofibrant diagram, we are equipped to build functorial cofibrant replacements.

Theorem 6.45. *If $\mathcal{I}$ is a simple category, then there is a functor that assigns to every diagram $F : \mathcal{I} \to \mathcal{T}$ (of well-pointed spaces when $\mathcal{T} = \mathcal{T}_*$) a cofibrant replacement $\overline{F} \to F$.*

The idea of the proof is to define cofibrant replacements $\overline{F}_n : \mathcal{I}_n \to \mathcal{T}$ for F_n inductively, starting with the identity transformation $\overline{F}_0 \to F_0$. Suppose now that a pointwise homotopy equivalence $\overline{F}_n \to F_n$ has been defined so that $\overline{F}_n$ satisfies the conditions of Theorem 6.36. For $i \in \mathcal{I}_{n+1} - \mathcal{I}_n$ define $\overline{F}_{n+1}(i)$ to be the mapping cylinder M_{ξ_i}, in $\mathcal{T}$, of the comparison map $\xi_i : \operatorname{colim} \overline{F}_{<i} \to F(i)$.

Problem 6.46. Finish the construction of the functor $\overline{F}$ and the transformation $\overline{F} \to F$, and use it to prove Theorem 6.45.

Exercise 6.47. Show that every diagram $F : \mathcal{I} \to \mathcal{T}_*$ may be replaced using a pointwise *unpointed* homotopy equivalence $\widetilde{F} \to F$ from a diagram of well-pointed spaces.

6.5. Examples: Pushouts, 3×3s and Telescopes

In this section we give explicit constructions for certain special homotopy colimits. Where our constructions differ from the standard approach implicit in Theorem 6.45, it is to reduce the number of parameters needed to describe the points in the space, with the aim of making the homotopy colimits conceptually simpler.

For the homotopy pushout, our construction is merely a reparametrization of the standard functorial version developed in Section 6.3.4. Using this construction, it becomes clear that a homotopy pushout can be constructed by converting only one of the maps in the prepushout to a cofibration. We show how the suspension of a space and the cofiber of a map can be expressed in terms of homotopy colimits; one consequence is the homotopy invariance of the cofiber.

Our approach to telescope diagrams reduces the complexity considerably: the standard construction involves infinitely many parameters from the interval I, but the one we give here has only one parameter, indexed on $[0, \infty)$. We briefly consider 3×3 diagrams and look at some very simple examples in an attempt to develop some intuition for them.

For definiteness, we'll work entirely with pointed spaces in this section.

6.5.1. Homotopy Pushouts. The homotopy colimit of a diagram with shape $\star \leftarrow \bullet \rightarrow \diamond$ is called the **homotopy pushout** of the diagram. This is probably the most frequently used kind of homotopy colimit.

Simplifying the Homotopy Pushout Construction. You have already made a study of cofibrant approximations for prepushout diagrams $C \leftarrow A \rightarrow B$ in Section 6.2.2, resulting in the standard cofibrant approximation

$$M_g \longleftarrow A \longrightarrow M_f$$

for a prepushout diagram $C \xleftarrow{g} A \xrightarrow{f} B$ of well-pointed spaces. For clarity, we use $s \in I$ to index the cylinder in M_f and $t \in I$ to index the cylinder in M_g, so that the standard homotopy colimit of our diagram is the quotient space

$$(C \sqcup (A \rtimes I) \sqcup (A \rtimes I) \sqcup B)/\sim$$

where $\sim$ is the equivalence relation generated by the nontrivial relations

$$(a, t = 0) \sim g(a) \in C, \quad (a, t = 1) \sim (a, s = 1) \quad \text{and} \quad (a, s = 0) \sim f(a) \in B.$$

A bit of reflection will reveal that there is no need for two indices to describe this space.

Problem 6.48.

(a) Introduce an equivalence relation $\sim$ on the space $C \sqcup (A \rtimes I) \sqcup B$ by setting $(a, 0) \sim g(a)$ and $(a, 1) \sim f(a)$. Show that the quotient space

$$M(f, g) = (C \sqcup (A \rtimes I) \sqcup B)/\sim$$

is a homotopy pushout for the diagram $C \leftarrow A \rightarrow B$. Is $M(f, g)$ equal to the standard homotopy colimit? Is it homeomorphic to the standard homotopy colimit?

(b) Show that the pushout of $C \leftarrow A \hookrightarrow M_f$ is a homotopy colimit for $C \leftarrow A \rightarrow B$.

The space $M(f, g)$ is often referred to as the **double mapping cylinder** of the maps f and g. The result of Problem 6.48(b) implies that, in order to construct a homotopy pushout, it suffices to convert just one of the two maps to a cofibration.

Proposition 6.49. *Suppose $i : A \hookrightarrow B$ is a cofibration in the prepushout diagram F given by*

$$C \xleftarrow{\;f\;} A \xhookrightarrow{\;i\;} B.$$

Then the comparison map $\xi : \operatorname{hocolim}_* F \rightarrow \operatorname{colim}_* F$ is a homotopy equivalence.

This implies that the pushout of a homotopy equivalence by a cofibration is a homotopy equivalence.

Corollary 6.50. *If, in the pushout square*

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "f"', "\simeq"] \arrow[dr, phantom, "\text{pushout}"] & B \arrow[d, "g"] \\ C \arrow[r, "j"] & D, \end{tikzcd}$$

i is a cofibration and f is a homotopy equivalence, then g is also a homotopy equivalence.

Problem 6.51. Use the hypotheses and notation of Proposition 6.49.

(a) Show that if $f : A \to B$ is a cofibration, then the canonical morphism from $A \hookrightarrow M_f$ to f is a homotopy equivalence in $A\downarrow\mathcal{T}_*$.

(b) Show that the diagram morphism

$$\begin{tikzcd} C \arrow[d, equal] & A \arrow[l] \arrow[r] \arrow[d, equal] & M_f \arrow[d, "\simeq"] \\ C & A \arrow[l] \arrow[r] & B \end{tikzcd}$$

is a homotopy equivalence of diagrams.

(c) Show that there is a pointwise homotopy equivalence from $M_f \leftarrow A \to M_i$ to $C \leftarrow A \to M_i$ that induces a homotopy equivalence on pushouts.

(d) Prove Proposition 6.49.

(e) Prove Corollary 6.50.

Exercise 6.52. In what sense, if any, is the identification in Proposition 6.49 natural?

Important Examples of Homotopy Pushouts. Let's look at some special kinds of prepushout diagrams and determine their homotopy pushouts.

Problem 6.53. Show that if $f : A \to B$ is a homotopy equivalence, then the homotopy pushout D of $C \leftarrow A \to B$ is homotopy equivalent to C.

HINT. Identity maps are cofibrations.

Problem 6.54. Show that if either A or B is well-pointed, then $A \vee B$ is a homotopy pushout for $B \leftarrow * \to A$. What is the homotopy pushout when neither space is cofibrant?

Problem 6.55.

(a) Show that ΣA is the homotopy pushout of the diagram $* \leftarrow A \to *$.

(b) Show that the homotopy colimit of $* \leftarrow A \xrightarrow{f} B$ is the cofiber of f (thought of as a space).

Homotopy Invariance of Cofibers. In Section 5.9 we defined the cofiber of a map $f : A \to B$ as a *map* and claimed—without proof—that different methods of converting f to a cofibration yield pointwise homotopy equivalent cofibers. Our task now is to establish the following invariance property of cofibers.

Proposition 6.56. *Pointwise homotopy equivalent maps have pointwise homotopy equivalent cofibers.*

Corollary 6.57. *Homotopic maps have pointwise homotopy equivalent cofibers.*

We prove Proposition 6.56 by expressing cofibers as induced maps between homotopy pushouts.

Problem 6.58. Let $f : A \to B$ and consider the prepushout diagram F given by $* \leftarrow A \xrightarrow{f} B$ and its cofibrant replacement $C \leftarrow \overline{A} \to \overline{B}$.

(a) Show that in the pushout square

$$\begin{tikzcd} \overline{A} \arrow[r] \arrow[d] & \overline{B} \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

the map $\overline{B} \to D$ is a cofiber for f.

(b) Show that each cofiber of f is pointwise homotopy equivalent to one that arises in this way.

(c) Show that the map $\overline{B} \to D$ is an induced map between homotopy pushouts, and conclude that any two cofibers of f are pointwise homotopy equivalent.

(d) Prove Proposition 6.56 and derive Corollary 6.57.

6.5.2. Telescopes. A **telescope diagram** of pointed spaces is a functor $X : \mathbb{N} \to \mathcal{T}_*$, i.e., a diagram of the form

$$X_{(1)} \xrightarrow{f_1} X_{(2)} \xrightarrow{f_2} X_{(3)} \to \cdots \to X_{(n)} \xrightarrow{f_n} X_{(n+1)} \to \cdots .$$

Our criterion for cofibrant diagrams again applies.

Problem 6.59. Show that if each map in a telescope diagram is a cofibration of well-pointed spaces, then it is cofibrant.

To construct the standard cofibrant replacement of telescope diagrams, use the inductive construction indicated in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \overline{X}_{(n-1)} \arrow[r, "\overline{f}_{n-1}"] \arrow[d, "j_{n-1}"'] & \overline{X}_{(n)} \arrow[r, dotted, "\overline{f}_n"] \arrow[d, "j_n"'] & M_{f_n \circ j_n} \arrow[d, dotted] & \\
\cdots \arrow[r] & X_{(n-1)} \arrow[r, "f_{n-1}"] & X_{(n)} \arrow[r, "f_n"] & X_{(n+1)} \arrow[r, "f_{n+1}"] & \cdots .
\end{tikzcd}$$

That is, define $\overline{X}_{(n+1)} = M_{f_n \circ j_n}$ and use the standard maps.

In the standard construction, each mapping cylinder must be indexed by a new parameter, t_n, so that in the end, points in the homotopy colimit are described by an infinite collection of parameters. This profusion of parameters can be brought under control if we use a different construction.

For each n, write $M(n)$ for the mapping cylinder on f_n indexed on the interval $[n, n+1]$; explicitly, $M(n)$ is the pushout in the square

$$\begin{tikzcd}
X_{(n)} \arrow[r, "\mathrm{in}_{n+1}"] \arrow[d, "f_n"'] \arrow[dr, phantom, "\text{pushout}"] & X_{(n)} \rtimes [n, n+1] \arrow[d] \\
X_{(n+1)} \arrow[r] & M(n).
\end{tikzcd}$$

Then define $\overline{X}_{(n)} \to \overline{X}_{(n+1)}$ inductively using the (categorical) pushout squares

$$\begin{tikzcd}
 & X_{(n)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & \overline{X}_{(n)} \arrow[d, "\phi_n"] \\
X_{(n+1)} \arrow[r, hook] & M(n) \arrow[r] & \overline{X}_{(n+1)}.
\end{tikzcd}$$

Problem 6.60. Construct a cofibrant replacement

$$\begin{tikzcd}
\cdots \arrow[r] & \overline{X}_{(n)} \arrow[r, "\phi_n"] \arrow[d] & \overline{X}_{(n+1)} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & X_{(n)} \arrow[r, "f_n"] & X_{(n+1)} \arrow[r] & \cdots
\end{tikzcd}$$

and describe the colimit of the top row, which is the homotopy colimit of the bottom row.

Let's look at some cases where the determination of the homotopy colimit of a telescope diagram is easy.

Problem 6.61. Consider the telescope diagram $X : \mathbb{N} \to \mathcal{T}_*$.

(a) Show that if each map $X_{(n)} \to X_{(n+1)}$ is a homotopy equivalence, then each canonical homotopy class $X_{(n)} \to \operatorname{hocolim}_* X$ is a homotopy equivalence.

(b) Show that if each map $X_{(n)} \to X_{(n+1)}$ is nullhomotopic, then the homotopy colimit $\operatorname{hocolim}_* X$ is contractible.

Finally, we investigate the homotopy type of an infinite-dimensional CW complex as a function of its skeleta.

Problem 6.62. Let X be a CW complex with skeleta X_n. The skeleta of X form a telescope diagram $X_0 \to X_1 \to \cdots \to X_n \to X_{n+1} \to \cdots$. Show that X is a homotopy colimit for this diagram.

The Small Object Argument. The **small object argument** is a term that describes theorems that assert that a functor of the form $\operatorname{mor}_{\mathcal{C}}(K, ?\,)$ commutes with certain colimits—colimit in the target!

Problem 6.63. Let $\cdots \to X_{(n)} \to X_{(n+1)} \to \cdots$ be any telescope diagram, and let X be the homotopy colimit, constructed according to the procedure just described.

(a) Show that there is a continuous function $t : X \to [0, \infty)$ such that $X_{(n)} = t^{-1}([0, n])$.

(b) Show that if $f : K \to X$ with K compact, then f factors

$$\begin{tikzcd} K \arrow[rr, "f"] \arrow[dr, bend right] & & X \\ & X_{(n)} \arrow[ur, bend right] & \end{tikzcd}$$

for some n.

(c) Show that if K is compact, then the map $\operatorname{colim}_n[K, X_{(n)}] \to [K, X]$ is bijective.

Exercise 6.64. Show that in the situation of Problem 6.63, the map

$$\operatorname{colim} \operatorname{map}_*(K, X_{(n)}) \longrightarrow \operatorname{map}_*(K, X)$$

is bijective. Is it a homeomorphism?

6.5.3. 3×3 Diagrams. Next we consider diagrams which have the shape of the strictly commutative diagram

$$\begin{tikzcd}
(\star,\circ) & (\bullet,\circ) \arrow[l] \arrow[r] & (\circ,\circ) \\
(\star,\bullet) \arrow[u] \arrow[d] \arrow[ur, phantom, "II"] \arrow[dr, phantom, "III"] & (\bullet,\bullet) \arrow[l] \arrow[r] \arrow[u] \arrow[d] \arrow[ur, phantom, "I"] \arrow[dr, phantom, "IV"] & (\circ,\bullet) \arrow[u] \arrow[d] \\
(\star,\star) & (\bullet,\star) \arrow[l] \arrow[r] & (\circ,\star).
\end{tikzcd}$$

We'll refer to diagrams with this shape as 3×3 **diagrams**.[4] A 3×3 diagram F decomposes into four commutative square diagrams, which we refer to as $F_I, F_{II}, F_{III}, F_{IV}$, according to the usual precalculus convention for quadrants.

We begin our analysis of the homotopy colimits of 3×3 diagrams by determining the standard homotopy colimits of commutative squares.

Problem 6.65. Consider the commutative square

$$\begin{tikzcd}
A_0 \arrow[r, "f_0"] \arrow[d, "g_0"'] & A_1 \arrow[d, "g_1"] \\
B_0 \arrow[r, "f_1"] & B_1.
\end{tikzcd}$$

The maps f_0, f_1 induce a map $f : M_{g_1} \to M_{g_2}$, and likewise the maps g_0, g_1 induce a map $g : M_{f_0} \to M_{f_1}$.

(a) Show that the mapping cylinder of f is homeomorphic to the mapping cylinder of g and that both are homeomorphic to the standard homotopy colimit of the square.

(b) In the 3×3 diagram, let $\overline{F}(\bullet,\bullet) = F(\bullet,\bullet)$, and define

$$\begin{aligned}
\overline{F}(\circ,\circ) &= \operatorname{hocolim}_*(F_I),\\
\overline{F}(\star,\circ) &= \operatorname{hocolim}_*(F_{II}),\\
\overline{F}(\star,\star) &= \operatorname{hocolim}_*(F_{III}),\\
\overline{F}(\circ,\star) &= \operatorname{hocolim}_*(F_{IV}),
\end{aligned}$$

and define

$$\overline{F}(\circ,\bullet), \quad \overline{F}(\bullet,\circ), \quad \overline{F}(\star,\bullet) \qquad \text{and} \qquad \overline{F}(\bullet,\star)$$

to be the mapping cylinders of the maps from $F(\bullet,\bullet)$. Show that it is possible to define $\overline{F}$ on morphisms and find a natural transformation $\overline{F} \to F$ that is a cofibrant replacement for F.

[4]This shape category is the product of the prepushout diagram with itself.

Exercise 6.66. Letting $\mathcal{I}$ be the 3×3 diagram above, define a function on the objects of $\mathcal{I}$ by setting

$$b(i) = \boxed{\text{the number of } \bullet \text{ entries in } i}.$$

(a) Define a diagram F by setting $F(\bullet, \bullet) = \{0\}$ and for $i \neq (\bullet, \bullet)$

$$F(i) = [b(i) - 2, 2 - b(i)]$$

and letting all maps be the inclusions of subintervals. Draw the homotopy colimit of F using the construction described above.

(b) Now define $G(i) = [-b(i), b(i)]$ and let all maps be bijective linear maps between intervals. Draw the homotopy colimit of G.

Problem 6.67.

(a) Determine the homotopy colimit of the diagram

$$\begin{tikzcd} * & * \arrow[l] \arrow[r] & * \\ * \arrow[u] \arrow[d] & S^0 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & * \arrow[u] \arrow[d] \\ * & * \arrow[l] \arrow[r] & *. \end{tikzcd}$$

(b) What happens if S^0 is replaced with S^n, or even a general space X?

6.6. Homotopy Limits

The theory of homotopy limits is parallel—and dual—to our painstaking development of homotopy colimits. Therefore we will be content to cover only the high points of the theory, confident that the nooks and crannies can be understood by straightforward adaptations in our earlier work.

We'll do the construction in the category of unpointed spaces first, because that is much more straightforward and more clearly dual to what we've done. Then we'll have an account of the changes—all related to navigating around the well-pointedness condition—that have to be made for pointed spaces.

6.6.1. Fibrant Diagrams of Unpointed Spaces. Instead of pointwise fibrations, we now focus on pointwise *cofibrations*. A morphism $F \to G$ of $\mathcal{I}$-shaped diagrams is a **pointwise cofibration** if for each $i \in \mathcal{I}$, the map $F(i) \to G(i)$ is a cofibration in $\mathcal{T}_\circ$.

Every morphism of diagrams in $\mathcal{T}_\circ$ may be factored through a pointwise homotopy equivalent pointwise cofibration.

Problem 6.68. Show that every morphism $\phi : F \to G$ of diagrams in $\mathcal{T}_\circ$ has a factorization

$$\begin{tikzcd} & \widetilde{G} \arrow[dr, bend left, "\pi"] & \\ F \arrow[ur, bend left, "\theta"] \arrow[rr, "\phi"] & & G \end{tikzcd}$$

of ϕ such that

(1) θ is a pointwise cofibration,

(2) π is a diagram homotopy equivalence,

(3) there is a morphism $\overline{\pi} : G \to \widetilde{G}$ such that $\overline{\pi} \circ \pi = \mathrm{id}_G$, and

(4) all this structure is functorial.

We say that a diagram $F : \mathcal{I} \to \mathcal{T}_\circ$ is **fibrant** if in any diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & F \\ B \arrow[ur, dotted, bend right] & \end{tikzcd}$$

in which the diagram morphism $A \to B$ is both a pointwise cofibration and a pointwise homotopy equivalence, the dotted arrow can be filled in to make the triangle commute in $\mathcal{T}_\circ^{\mathcal{I}}$.

Problem 6.69. If F is a fibrant diagram, then in any triangle

$$\begin{tikzcd} A \arrow[r] \arrow[d] & F \\ B \arrow[ur, dotted, bend right] & \end{tikzcd}$$

in which the $A \to B$ is a pointwise homotopy equivalence, the dotted arrow can be filled in to make the triangle commute up to diagram homotopy in $\mathcal{T}_*^{\mathcal{I}}$.

Certain functors $\mathcal{T}_\circ \to \mathcal{T}_\circ$, when applied to fibrant diagrams, return new fibrant diagrams.

Problem 6.70. Formulate the duals of Theorem 6.17 and Corollary 6.19.

A **fibrant replacement** for a diagram F is a pointwise homotopy equivalence $F \to \overline{F}$ in which $\overline{F}$ is fibrant. Fibrant replacements, if they exist, enjoy a certain measure of naturality.

Problem 6.71. Let $\phi : F \to G$ be a diagram map, and let $F \to \overline{F}$, $G \to \overline{G}$ be fibrant replacements.

(a) Show that there is a diagram map $\overline{\phi} : \overline{F} \to \overline{G}$ making the diagram

$$\begin{tikzcd} F \arrow[r, "\phi"] \arrow[d] & G \arrow[d] \\ \overline{F} \arrow[r, "\overline{\phi}"] & \overline{G} \end{tikzcd}$$

commute up to diagram homotopy.

(b) Show that if ϕ is a pointwise homotopy equivalence, then $\bar{\phi}$ is a diagram homotopy equivalence.

(c) Show that any two fibrant replacements $F \to \overline{F}$ and $F \to \widetilde{F}$ are homotopy equivalent in $\mathcal{T}_\circ^{\mathcal{I}}$.

6.6.2. Homotopy Limits. It follows from Problem 6.71 that the homotopy type of the categorical limit of a fibrant replacement $\overline{F}$ for F is independent of the choice of $\overline{F}$. We say that a space X is a **homotopy limit** of the diagram F if it is homotopy equivalent to the categorical limit of a fibrant replacement $\overline{F}$ for F.

Induced Maps Between Homotopy Limits. The (however limited) naturality properties of fibrant replacements translate into corresponding naturality for homotopy limits.

Problem 6.72. Let $\phi : F \to G$ be a diagram map, and let $F \to \overline{F}$, $G \to \overline{G}$ be fibrant approximations.

(a) Let X be a homotopy limit for F and let Y be a homotopy limit for G. Show that the map $\phi : F \to G$ induces a map of spaces $f : X \to Y$.

(b) Show that any two induced maps $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are pointwise homotopy equivalent to one another; i.e., there is a homotopy commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "\simeq"'] & Y_1 \arrow[d, "\simeq"] \\ X_2 \arrow[r, "f_2"] & Y_2. \end{tikzcd}$$

6.6.3. Existence of Fibrant Replacements. We will show that every diagram $F : \mathcal{I} \to \mathcal{T}_\circ$ whose shape is the opposite of a simple category can be replaced with a pointwise equivalent fibrant one.

A category $\mathcal{I}$ is called an **opposite-simple category** if its opposite category $\mathcal{I}^{\mathrm{op}}$ is a simple category (we're just going to use this admittedly clunky terminology in this section).

Since our study of direct categories in Section 6.4.1 was purely category-theoretical, the statements and proofs dualize to give a corresponding theory for opposite-simple categories.

Exercise 6.73. Write out the theory of opposite-simple categories.

The argument proving Theorem 6.45 dualizes, giving us our theorem on the existence of fibrant replacements of unpointed diagrams.

Theorem 6.74. *If $\mathcal{I}$ is an opposite-simple category, then there is a functor* $\mathrm{FIB} : \mathcal{T}_\circ^{\mathcal{I}} \to \mathcal{T}_\circ^{\mathcal{I}}$ *which comes equipped with a comparison transformation* $\xi : \mathrm{id} \to \mathrm{FIB}$ *such that* $\xi : F \to \mathrm{FIB}(F)$ *is a fibrant replacement for each* $F \in \mathcal{T}_\circ^{\mathcal{I}}$.

Problem 6.75. Prove Theorem 6.74.

The Functorial Approach to Homotopy Limits. The maps of homotopy limits induced by a diagram map are not unique, but if we agree to use a functorial fibrant replacement, which we will denote by $\mathrm{FIB} : \mathcal{T}_\circ^{\mathcal{I}} \to \mathcal{T}_\circ^{\mathcal{I}}$, then we can define a homotopy limit *functor* by the rule

$$\mathrm{holim}_\circ F = \lim_\circ \mathrm{FIB}(F).$$

The replacement map $F \to \mathrm{FIB}(F)$ induces a natural comparison map $\xi : \lim_\circ F \to \mathrm{holim}_\circ F$. The homotopy limit functor has a nearly universal property that is very similar to the property that defines the ordinary categorical limit of diagrams.

Theorem 6.76. *The transformation* $\xi : \lim_\circ \to \mathrm{holim}_\circ$ *satisfies the following:*

(a) if $F \to G$ is a pointwise homotopy equivalence of diagrams, then the induced map $\mathrm{holim}_\circ F \to \mathrm{holim}_\circ G$ *is a homotopy equivalence, and*

(b) if $\mathcal{L} : \mathcal{T}_\circ^{\mathcal{I}} \to \mathcal{T}_\circ$ *is any other functor satisfying* (a) *and* $\zeta : \lim_\circ \to \mathcal{L}$ *is any natural transformation, then there is a unique natural transformation* $\mathrm{Ho} \circ \mathrm{holim}_\circ \to \mathrm{Ho} \circ \mathcal{L}$ *making the diagram*

$$\begin{tikzcd}
\mathrm{Ho} \circ \lim_\circ \arrow[r, "\mathrm{Ho}(\zeta)"] \arrow[d, "\mathrm{Ho}(\xi)"'] & \mathrm{Ho} \circ \mathcal{L} \\
\mathrm{Ho} \circ \mathrm{holim}_\circ \arrow[ur, bend right] &
\end{tikzcd}$$

commute in $(\mathrm{H}\mathcal{T}_\circ)^{\mathcal{I}}$.

Problem 6.77. Prove Theorem 6.76.

Exercise 6.78. Explain how Theorem 6.76 gives canonical homotopy classes $\mathrm{holim}_\circ F \to F(i)$ for each $i \in \mathcal{I}$.

6.6.4. Homotopy Limits of Pointed Spaces. There are two obstructions to carrying out this program in the category of pointed spaces. First of all, the homotopy invariance of homotopy limits of unpointed spaces is founded on Problem 6.68, which makes use of the mapping cylinder construction. As we saw when we were proving Theorem 5.98, the mapping cylinder of $f : X \to Y$ in $\mathcal{T}_*$ only works well when the X and Y are well-pointed. Secondly, the recognition principle for fibrant diagrams ultimately rests on the Fundamental Lifting Property, and a glance at Theorem 5.105 will remind you that the pointed version is only valid for cofibrations $i : A \to X$ *between well-pointed spaces.*

To suit the category of pointed spaces, we must make two changes in the definition of fibrant diagrams. First, we use a less restrictive lifting property, requiring extensions only in diagrams of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & F \\ B \arrow[ur, dotted] & \end{tikzcd}$$

in which $A \to B$ is a pointwise cofibration in $\mathcal{T}_*$ *and* A and B are diagrams of well-pointed spaces. Second, we put an added restriction on fibrant diagrams, requiring them to be diagrams of well-pointed spaces. If we write $\mathcal{W}_* \subseteq \mathcal{T}_*$ for the subcategory of well-pointed spaces, then we are saying that fibrant diagrams must be diagrams in $\mathcal{W}_*^{\mathcal{I}}$.

With this definition, the definition and construction of homotopy limits of diagrams of pointed spaces proceeds exactly as before.

A pointwise homotopy equivalence $\xi : F \to \overline{F}$ in $\mathcal{T}_*^{\mathcal{I}}$ is a **fibrant replacement** if $\overline{F}$ is fibrant. A morphism $\phi : F \to G$ of diagrams can be—what is the dual of 'covered'? undermined?—by a map $\overline{\phi} : \overline{F} \to \overline{G}$ making the square

$$\begin{tikzcd} F \arrow[r, "\phi"] \arrow[d] & G \arrow[d] \\ \overline{F} \arrow[r, "\overline{\phi}"] & \overline{G} \end{tikzcd}$$

commute up to diagram homotopy in $\mathcal{T}_*^{\mathcal{I}}$. It follows as in Problem 6.71 that any two fibrant replacements are homotopy equivalent in $\mathcal{T}_*^{\mathcal{I}}$.

It follows that homotopy type of the limit $\lim_* \overline{F}$ depends only on the diagram F and not the choice of fibrant replacement $\overline{F}$, so we define

$$\operatorname{holim}_* F = \lim_* \overline{F} \qquad \text{where} \qquad F \to \overline{F} \text{ is a fibrant replacement.}$$

A map $\phi : F \to G$ induces maps of homotopy limits, and any two induced maps are pointwise equivalent in $\mathrm{H}\mathcal{T}_*$.

This is, of course, just empty theory unless we can prove the existence of fibrant replacements. We can easily recognize fibrant diagrams whose shape is an opposite-simple category.

Theorem 6.79. *If $\mathcal{I}$ is an opposite-simple category, then a diagram $F : \mathcal{I} \to \mathcal{W}_*$ is fibrant if each comparison map $\xi_i : F(i) \to \lim F_{>i}$ is a fibration.*

Now that we know what fibrant diagrams are like, we can build them.

Theorem 6.80. *Let $\mathcal{I}$ be an opposite-simple category. There is a functor $\mathrm{FIB} : \mathcal{W}_*^{\mathcal{I}} \to \mathcal{W}_*^{\mathcal{I}}$ and a natural transformation $\xi : \mathrm{id} \to \mathrm{FIB}$ such that $\xi : F \to \mathrm{FIB}(F)$ is a fibrant replacement.*

Problem 6.81. Prove Theorems 6.79 and 6.80.

Exercise 6.82. Is the converse of Theorem 6.79 valid?

6.6.5. Special Cases: Maps, Pullbacks, 3×3s and Towers. Now we take a look at some special examples of homotopy limits.

Homotopy Pullbacks. To find fibrant replacements for prepullback diagrams, we simply replace both maps with fibrations using the procedure we developed in Chapter 5. However, we can construct the homotopy pullback of a diagram by converting only one of the maps to a fibration.

Proposition 6.83. *If one of the maps in $Y \to Z \leftarrow X$ is a fibration, then the pullback is a homotopy pullback for the diagram.*

Problem 6.84. Prove Proposition 6.83.

Here are some simple and important examples.

Problem 6.85. Determine the homotopy pullbacks of the following diagrams:

(a) $* \to X \leftarrow *$,

(b) $X \to * \leftarrow Y$.

Exercise 6.86. Study the maps induced by the diagram

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & X \arrow[d, "f"] & * \arrow[l] \arrow[d] \\
* \arrow[r] & Y & *. \arrow[l]
\end{tikzcd}$$

The homotopy fiber of a map can be thought of as a homotopy pullback.

Proposition 6.87. *Let $f : X \to Y$ and define F to be the homotopy pullback of the prepullback diagram $* \to Y \xleftarrow{f} X$. Then the canonical homotopy class* $\mathrm{hocolim}_* F \to X$ *is the homotopy fiber of f.*

Corollary 6.88. *Pointwise homotopy equivalent maps have pointwise homotopy equivalent fibers.*

Problem 6.89. Prove Proposition 6.87 and Corollary 6.88.

Homotopy Limits of Towers. The dual of a telescope diagram is generally called a **tower**. It is a diagram whose shape

$$1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots \leftarrow n \leftarrow n+1 \leftarrow \cdots$$

is the opposite category $\mathbb{N}^{\mathrm{op}}$.

Problem 6.90. Dualize the construction of Section 6.5.2 to give an efficient model for the homotopy limit of a tower.

3×3 **Diagrams.** The product $\mathcal{I}$ of a prepullback diagram with itself is the category

$$\begin{tikzcd}
(\star, \circ) \arrow[r] \arrow[d] & (\bullet, \circ) \arrow[d] & (\circ, \circ) \arrow[l] \arrow[d] \\
(\star, \bullet) \arrow[r] & (\bullet, \bullet) & (\circ, \bullet) \arrow[l] \\
(\star, \star) \arrow[r] \arrow[u] & (\bullet, \star) \arrow[u] & (\circ, \star). \arrow[l] \arrow[u]
\end{tikzcd}$$

We refer to a diagram $F : \mathcal{I} \to \mathcal{T}_*$ as a 3×3 **diagram**.

Problem 6.91. Dualize the procedure detailed in Section 6.5.3 to obtain fibrant replacements for 3×3 diagrams.

6.7. Functors Applied to Homotopy Limits and Colimits

Any functor $L : \mathcal{C} \to \mathcal{D}$ that is a left adjoint must preserve colimits and, dually, any right adjoint preserves limits. That is, if $F : \mathcal{I} \to \mathcal{C}$ has colimit X, then there is a natural map $\operatorname{colim} L \circ F \to L(\operatorname{colim} F)$, and if L is a left adjoint, this map is an isomorphism in $\mathcal{D}$, and the comparison map $R(\lim F) \to \lim R \circ F$ is an isomorphism in $\mathcal{C}$ if R is a right adjoint.

In this section we ask which functors $\mathcal{T} \to \mathcal{T}$ preserve *homotopy* colimits and, dually, which preserve homotopy limits. This is nearly the same as asking which functors preserve cofibrant diagrams and which preserve fibrant diagrams, and this is the approach we will take. Since cofibrant diagrams are defined in terms of homotopy equivalence, colimits and cofibrations, it is reasonable to look at left and right adjoints that respect homotopy.

6.7.1. The Unpointed Case. We begin in the more straightforward unpointed case.

Proposition 6.92. *Let $L, R : \mathcal{T}_\circ \to \mathcal{T}_\circ$ be an adjoint pair of covariant functors, both respecting homotopy. Show that the following are equivalent:*

(1) L carries cofibrations to cofibrations;

(2) R carries fibrations to fibrations.

Problem 6.93. Prove Proposition 6.92.

HINT. Use the Fundamental Lifting Property.

Now we can establish our criteria for preservation of homotopy colimits and limits.

Theorem 6.94. *Let $L, R : \mathcal{T}_\circ \to \mathcal{T}_\circ$ be an adjoint pair of covariant functors that respect homotopy and satisfy the equivalent conditions of Proposition 6.92. Then:*

(a) L preserves homotopy colimits of diagrams $\mathcal{I} \to \mathcal{T}_\circ$, and

(b) R preserves homotopy limits of diagrams $\mathcal{I} \to \mathcal{T}_\circ$.

It is worth pointing out that Theorem 6.94 applies to all homotopy colimits (and limits), not just those of diagrams over simple (or opposite-simple) categories.

Problem 6.95. Prove Theorem 6.94.

HINT. Use Theorem 6.17.

If $F : \mathcal{I} \to \mathcal{T}_\circ$, and let $X, Y \in \mathcal{T}_\circ$, then define functors

$$F \times Y : \mathcal{I} \longrightarrow \mathcal{T}_\circ \qquad \text{and} \qquad \mathrm{map}_\circ(X, F) : \mathcal{I} \longrightarrow \mathcal{T}_\circ$$

by the rules $(F \times Y)(i) = F(i) \times Y$ and $\mathrm{map}_\circ(Y, F)(i) = \mathrm{map}_\circ(X, F(i))$.[5]

Corollary 6.96. *There are homotopy equivalences*

$$\mathrm{hocolim}_\circ(F \times Y) \simeq (\mathrm{hocolim}_\circ F) \times Y$$

and

$$\mathrm{holim}_\circ(\mathrm{map}_\circ(X, F)) \simeq \mathrm{map}_\circ(X, (\mathrm{holim}_\circ F)).$$

[5]Strictly speaking, $F \times Y = (\ ?\ \times Y) \circ F$ and $\mathrm{map}_\circ(X, F) = \mathrm{map}_\circ(X, \ ?\ ) \circ F$.

6.7.2. The Pointed Case. Now we deal with the pointed case.

Theorem 6.97. *Let $L, R : \mathcal{T}_* \to \mathcal{T}_*$ be an adjoint pair of functors that both respect homotopy.*

(a) If R preserves fibrations, then L commutes with homotopy colimits.

(b) If L preserves cofibrations and both L and R preserve well-pointed spaces, then R commutes with homotopy limits.

Problem 6.98. Prove Theorem 6.97.

Problem 6.99. Show that the hypotheses of Theorem 6.97 hold for the adjoint pairs

(a) $\mathrm{map}_*(A, ?)$ and $? \wedge A$ if A is compact and well-pointed, and

(b) $(\mathrm{map}_\circ(A, ?), *)$ and $? \rtimes A$ if A is compact.

Problem 6.100. Let $A \in \mathcal{T}_*$ be compact and well-pointed.

(a) Show that $\mathrm{map}_*(A, \mathrm{holim}_* F) \simeq \mathrm{holim}_* \mathrm{map}_*(A, F)$.

(b) Show that $\mathrm{hocolim}_*(F \wedge A) \simeq (\mathrm{hocolim}_* F) \wedge A$.

Problem 6.101. Show that $\mathrm{hocolim}_* F \times A \simeq (\mathrm{hocolim}_* F) \times A$.

6.7.3. Contravariant Functors. If $K : \mathcal{T}_1 \to \mathcal{T}_2$ is contravariant, then it doesn't even make sense to ask if K preserves homotopy colimits or limits. It is conceivable that K could carry colimits to limits, or vice versa, and this does happen.

Theorem 6.102. *Let $K : \mathcal{T}_1 \to \mathcal{T}_2$ be a contravariant functor that respects homotopy and carries categorical colimits to categorical limits and cofibrations to fibrations. Then for any direct category $\mathcal{I}$ and any $F : \mathcal{I} \to \mathcal{T}_1$,*

(a) the induced map $K \circ F \to K \circ \overline{F}$ of diagrams is a fibrant replacement, and

(b) there is a diagram

$$\begin{array}{ccc} K(\mathrm{colim}\, F) & \longrightarrow & K(\mathrm{hocolim}\, F) \\ \cong \downarrow & & \downarrow \cong \\ \lim K \circ F & \longrightarrow & \mathrm{hocolim}\, K \circ F. \end{array}$$

Problem 6.103. Assume the notation and hypotheses of Theorem 6.102.

(a) Show that if F is a cofibrant diagram, then $K \circ F$ is a fibrant diagram.

(b) Derive Theorem 6.102.

Problem 6.104.

(a) Show that the hypotheses of Theorem 6.102 hold for the functor $\mathrm{map}_*(?, Y)$.

(b) Show that $\mathrm{holim}_* \mathrm{map}_*(F, Y) \simeq \mathrm{map}_*(\mathrm{hocolim}_* F, Y)$.

6.8. Homotopy Colimits of More General Diagrams

In this section we discuss, somewhat informally, the construction of homotopy colimits (and limits) for diagrams whose shape category is not simple. The results of this section are not used in the main body of the rest of the text, so we'll package most of the hard work in the form of projects.

A functor $\lambda : \mathcal{T}^{\mathcal{I}} \to \mathcal{T}$ should be considered a **homotopy colimit** functor if it comes equipped with a natural transformation $\xi : \lambda \to \operatorname{colim}$ and satisfies the conditions

- if $F \to G$ is a pointwise homotopy equivalence of diagrams, then the induced map $\lambda(F) \to \lambda(G)$ is a homotopy equivalence, and
- if $\mathcal{L} : \mathcal{T}^{\mathcal{I}} \to \mathcal{T}$ is a homotopy invariant functor equipped with a natural transformation $\zeta : \mathcal{L} \to \operatorname{colim}$, then there is a unique natural transformation $\mathrm{Ho} \circ \mathcal{L} \to \mathrm{Ho} \circ \lambda$ making the diagram

$$\begin{tikzcd} & \mathrm{Ho} \circ \lambda \arrow[d, "\xi"] \\ \mathrm{Ho} \circ \mathcal{L} \arrow[ur, bend left] \arrow[r] & \mathrm{Ho} \circ \operatorname{colim} \end{tikzcd}$$

  commute in $(\mathrm{H}\mathcal{T})^{\mathcal{I}}$.

Dually, λ is a homotopy limit functor if it satisfies the conditions of Theorem 6.76.

We have explicitly constructed homotopy colimits by first replacing a diagram F with a pointwise homotopy equivalent cofibrant diagram and forming its colimit. This approach works for diagrams whose shape $\mathcal{I}$ is a *direct* category. For other shape categories, one has to replace the given shape $\mathcal{I}$ with a 'homotopy equivalent' category $N(\mathcal{I})$ called the **nerve** of $\mathcal{I}$.

A small category $\mathcal{I}$ is **direct** if for some ordinal α there is a functor $\mathcal{I} \to \lambda$ taking every nonidentity morphism in $\mathcal{I}$ to a nonidentity morphism in α. This is a nontrivial generalization of simple categories: in a simple category, there is at most one morphism $j \to i$; this need not be the case in a direct category.

The recognition principle for cofibrant diagrams defined on direct categories is analogous to the condition we have established for simple diagrams. The difference is that, in order to account for the multiplicity of maps $j \to i$, the subcategory $\mathcal{I}_{<i}$ must be replaced with $\mathcal{I} \downarrow i$. (Recall that the objects of $\mathcal{I} \downarrow i$ are morphisms $f : j \to i$ and the morphisms $\alpha : f \to g$ are commutative

triangles

$$\begin{tikzcd} j \arrow[rr, "\alpha^d"] \arrow[dr, "f"'] & & k \arrow[dl, "g"] \\ & i & \end{tikzcd}$$

in $\mathcal{I}$.)

Project 6.105. Let $\mathcal{I}$ be a direct category, and let $F : \mathcal{I} \to \mathcal{T}$ (assume that F takes its values in well-pointed spaces if $\mathcal{T} = \mathcal{T}_*$). For $i \in \mathcal{I}$, define $\partial_i F : \mathcal{I} \downarrow i \to \mathcal{T}$ by the rules

$$\partial_i F \left(j \xrightarrow{f} i \right) = F(j) \qquad \text{and} \qquad \partial_i F \left(f \xrightarrow{\alpha} g \right) = F \left(j \xrightarrow{\alpha^d} k \right).$$

Construct maps $\xi_i : \operatorname{colim} \partial_i F \to F(i)$, and show that if each ξ_i is a cofibration, then F is cofibrant. Then define a cofibrant replacement functor for diagrams defined on direct categories, and show that diagrams defined on direct categories have homotopy colimits.

For categories that are not direct, we replace the diagram F with its composite with the 'nerve', which is a kind of cofibrant replacement for categories. Explicitly, $N(\mathcal{I})$ is the category whose objects are ordered lists $(f_1, f_2, \ldots, f_n)$ of composable morphisms in $\mathcal{I}$ and the morphisms are extensions of lists:

$$(f_1, f_2, \ldots, f_n) \longrightarrow (f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_m).$$

There is a functor $\nu : N(\mathcal{I}) \to \mathcal{I}$ given by

$$\nu : (f_1, f_2, \ldots, f_n) \longmapsto \operatorname{Target}(f_n);$$

the morphism displayed above is sent to $g_m \circ g_{m-1} \circ \cdots \circ g_1$.

Project 6.106. Show that $N(\mathcal{I})$ is a direct category, so that we can define $\mathcal{T}^{\mathcal{I}} \to \mathcal{T}$ by the rule $F \mapsto \operatorname{hocolim}(F \circ \nu)$. Show that this rule defines a homotopy colimit functor for $\mathcal{I}$-diagrams.

The unpointed homotopy colimit of the trivial diagram $T_{\mathcal{I}} : \mathcal{I} \to \mathcal{T}_\circ$ given by $T_{\mathcal{I}}(*) = *$ can have interesting topology, and this topology encodes useful information about the shape category $\mathcal{I}$. This space is denoted

$$B\mathcal{I} = \operatorname{hocolim}_\circ T_{\mathcal{I}}$$

and is called the **classifying space** of $\mathcal{I}$. (See Project 15.90 for some interesting classifying spaces.)

Problem 6.107.

(a) Show that for any shape category $\mathcal{I}$, $\operatorname{hocolim}_* T_{\mathcal{I}} \simeq *$.

(b) Show that if $\mathcal{I}$ is simple, $B\mathcal{I} \simeq *$.

Project 6.108. Dualize the discussion above.

More on the Small Object Argument. The 'long telescopes'—categories of (large) ordinals—are important examples of categories that are direct but not simple. The small object argument can be applied to colimits of long telescopes; for very long telescopes, large objects can be small.

Project 6.109. Let $Y = \operatorname{colim} F$, where $F : \alpha \to \mathcal{T}$, and let X be a CW complex. Show that if the cofinality of α is greater than the cardinality of the collection of cells of X, then every map $f : X \to Y$ factors (on the nose!) through $F(\beta)$ for some $\beta < \alpha$.

6.9. Additional Topics, Problems and Projects

6.9.1. Rigidifying Homotopy Morphisms of Diagrams. Suppose we are given two diagrams $F, G : \mathcal{I} \to \mathcal{T}_*$ and a diagram morphism $\phi : \mathrm{Ho} \circ F \to \mathrm{Ho} \circ G$, where $\mathrm{Ho} : \mathcal{T}_* \to \mathrm{H}\mathcal{T}_*$ is the functor defined in Section 4.3. If there is a morphism $f : F \to G$ of diagrams in $\mathcal{T}_*$ such that $\phi = L(f)$, then we say that f is a **rigidification** of ϕ.

Problem 6.110.

(a) Find an example of a morphism ϕ that cannot be rigidified.

(b) Show that if $\mathcal{I}$ is a simple category and F is cofibrant, then ϕ can be rigidified.

6.9.2. Homotopy Colimits versus Categorical Colimits. The homotopy colimit of a diagram $F : \mathcal{I} \to \mathcal{T}$ is *almost* the colimit of the diagram $\mathrm{Ho} \circ F : \mathcal{I} \to \mathrm{H}\mathcal{T}$.

Problem 6.111. Let X be a homotopy colimit of the diagram $F : \mathcal{I} \to \mathcal{T}$. Being a topological space, X can be considered as an object of $\mathcal{T}$ or of $\mathrm{H}\mathcal{T}$.

(a) Show that there is a map $j : \mathrm{Ho} \circ F \to \Delta_X$ of diagrams in $\mathrm{H}\mathcal{T}$ with the following nice property:

($\star$) if $Y \in \mathrm{H}\mathcal{T}$ and there is a diagram map $g : \mathrm{Ho} \circ F \to \Delta_Y$, then there is a map $f : X \to Y$ such that

$$\begin{tikzcd} \mathrm{Ho} \circ F \arrow[rr] \arrow[dr] & & \Delta_Y \\ & \Delta_X \arrow[ur, dotted, "\Delta_f"'] & \end{tikzcd}$$

commutes in the category $(\mathrm{H}\mathcal{T})^{\mathcal{I}}$.

(b) Show that if $\mathrm{Ho} \circ F$ has a colimit Q, then Q is a retract (in $\mathrm{H}\mathcal{T}$) of X.

The space X would be the categorical pushout if the map f were unique.

Problem 6.112. Show by example that the map f need not be unique, even up to homotopy.[6]

6.9.3. Homotopy Equivalence in Mapping Categories. The results of Section 5.5.2 extend to the full mapping category $\mathrm{map}(\mathcal{T})$. You were asked to do this by hand in the previous chapter, but now we can apply our understanding of diagram homotopy equivalence together with the identification $\mathrm{map}(\mathcal{T}) = \mathcal{T}^{\bullet\to\star}$.

Problem 6.113. Let f and g be maps in $\mathcal{T}$ (between well-pointed spaces if $\mathcal{T} = \mathcal{T}_*$). Show that a pointwise homotopy equivalence $\alpha : f \to g$ in $\mathrm{map}(\mathcal{T})$ is a homotopy equivalence of maps if both f and g are fibrations or if both f and g are cofibrations.

6.9.4. Problems and Projects.

Problem 6.114. Using Theorem 4.83 and Problem 5.149, show that the homotopy fiber of a map $f : X \to Y$ between pointed CW complexes is homotopy equivalent in $\mathcal{T}_*$ to a CW complex.

Problem 6.115. Show that if $A \hookrightarrow X$ is a cofibration in $\mathcal{T}_\circ$ and Y is contractible, then any map $A \to Y$ can be extended to a map $X \to Y$.

Project 6.116. Under what conditions do adjoint functors $L : \mathcal{T}_\circ \to \mathcal{T}_*$, $R : \mathcal{T}_* \to \mathcal{T}_\circ$ or $L : \mathcal{T}_* \to \mathcal{T}_\circ$, $R : \mathcal{T}_\circ \to \mathcal{T}_*$ preserve homotopy limits or colimits?

Problem 6.117. Show that if $\mathcal{I}$ and $\mathcal{J}$ are direct, so is $\mathcal{I} \times \mathcal{J}$.

Problem 6.118.

(a) Suppose D is the homotopy pushout of the diagram $C \leftarrow A \to B$. What is the homotopy pushout of $C \wedge X \leftarrow A \wedge X \to B \wedge X$?

(b) Let $f : A \to B$. What is the cofiber of $f \wedge \mathrm{id}_X : A \wedge X \to B \wedge X$? What is the cofiber of $\Sigma^n f$?

(c) What is the cofiber of $A \times Y \to X \times Y$?

(d) What is the homotopy pullback of $\Omega C \to \Omega D \leftarrow \Omega B$, where B, C and D are well-pointed?

Problem 6.119. Suppose A is a retract of X, and determine the homotopy limit and homotopy colimit of the diagram

$$\cdots \to A \to X \to A \to X \to A \to X \to \cdots .$$

Problem 6.120. The (pointed) **mapping torus** $T(f)$ of a self-map $f : X \to X$ is the quotient space $X \rtimes I/\sim$, where the equivalence relation $\sim$ is determined by $(x, 1) \sim (f(x), 0)$. Express $T(f)$ as the homotopy colimit of a diagram.

[6] You may assume that there exists a space X such that $\Sigma X \not\simeq *$.

Problem 6.121. Determine the homotopy colimit of the diagram

$$\begin{tikzcd}
* & * \arrow[l] \arrow[r] & * \\
C \arrow[u] \arrow[d] & A \arrow[l] \arrow[r] \arrow[u] \arrow[d] & B \arrow[u] \arrow[d] \\
* & * \arrow[l] \arrow[r] & *.
\end{tikzcd}$$

Problem 6.122. Show that if X is the homotopy colimit of $F : \mathbb{N} \to T_*$, then X is the colimit of $L \circ F : \mathbb{N} \to \mathrm{H}\mathcal{T}_*$.

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Chapter 7

Homotopy Pushout and Pullback Squares

In this chapter, we'll develop techniques for manipulating and computing homotopy pushouts and pullbacks. The central idea is to study not just the space $\operatorname{hocolim}_* F$, but the square containing the homotopy colimit and its defining data. Such squares are called *homotopy pushout squares*, and there is a powerful calculus for constructing, combining and analyzing them. Dually, we study homotopy pullback squares.

Since homotopy limits and colimits are categorical limits and colimits of related diagrams, many of the properties of limits and colimits remain true for homotopy colimits and limits. The purpose of this chapter is to set down these key properties and derive some important consequences of them.

7.1. Homotopy Pushout Squares

If we are given a strictly commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D, \end{tikzcd}$$

then we may find a cofibrant replacement $\overline{C} \leftarrow \overline{A} \to \overline{B}$ for its prepushout part $C \leftarrow A \to B$. The induced map $\xi : \overline{D} \to D$ from the categorical pushout completes our cofibrant replacement to a cubical diagram

$$
\begin{tikzcd}
\overline{A} \arrow[rr] \arrow[dr] \arrow[dd] & & \overline{B} \arrow[dr] \arrow[dd] & \\
& \overline{C} \arrow[rr] \arrow[dd] & & \overline{D} \arrow[dd, "\xi"] \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& C \arrow[rr] & & D,
\end{tikzcd}
$$

in which the top face is a categorical pushout square and $\overline{D}$ is the homotopy colimit of the diagram $C \leftarrow A \to B$. We call ξ a *comparison map* because it allows us to compare the space D with the homotopy colimit of $C \leftarrow A \to B$.

Exercise 7.1. Explain why the map ξ exists. In what sense is it unique?

Problem 7.2. Let $\widetilde{C} \leftarrow \widetilde{A} \to \widetilde{B}$ be another cofibrant replacement of the diagram $C \leftarrow A \to B$, and let $\widetilde{D} \to D$ be its comparison map. Show that $\overline{D} \to D$ is a homotopy equivalence if and only if $\widetilde{D} \to D$ is a homotopy equivalence.

This is all very nice, but strictly commutative squares are too much to ask for in day-to-day homotopy theory. We need to compare squares that are merely *homotopy commutative* to pushout squares. Suppose, then, that the given square is homotopy commutative. Just as before, we may find a cofibrant replacement $\overline{C} \leftarrow \overline{A} \to \overline{A}$ (with pushout $\overline{D}$) for $C \leftarrow A \to B$ and use it to form the diagram

$$
\begin{tikzcd}
\overline{A} \arrow[rr] \arrow[dr] \arrow[dd] & & \overline{B} \arrow[dr] \arrow[dd] & \\
& \overline{C} \arrow[rr] \arrow[dd] & & \overline{D} \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \text{no map!} \\
& C \arrow[rr] & & D.
\end{tikzcd}
$$

This time, however, we cannot expect there to be a map $\overline{D} \to D$ making the diagram commute, because the bottom square is only commutative up to homotopy: the composites

$$\overline{A} \longrightarrow \overline{B} \longrightarrow D \qquad \text{and} \qquad \overline{A} \longrightarrow \overline{C} \longrightarrow D$$

are homotopic to each other, but not equal (as far as we know). It is possible to compare D with $\overline{D}$, though, because the map $\overline{A} \to \overline{B}$ is a cofibration.

Problem 7.3. Show that there is a map $\phi : \overline{B} \to D$ so that the diagram

$$\begin{tikzcd}
\overline{A} \arrow[rr] \arrow[dr] \arrow[dd] & & \overline{B} \arrow[dr] \arrow[dd] \arrow[dddr, bend left, "\phi"] & \\
& \overline{C} \arrow[rr] \arrow[dd] & & \overline{D} \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr, dashed] & \\
& C \arrow[rr] & & D
\end{tikzcd}$$

commutes up to homotopy and the solid arrow part of the diagram is strictly commutative. Explain why the map ϕ yields a comparison map $\xi : \overline{D} \to D$, and discuss the uniqueness or nonuniqueness of ξ.

Exercise 7.4. Consider the diagram

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] \arrow[dd, equal] & & CX \arrow[dr] \arrow[dd] \arrow[dddr, bend left, "\phi"] & \\
& CX \arrow[rr] \arrow[dd] & & \Sigma X \\
X \arrow[rr] \arrow[dr] & & * \arrow[dr, dashed] & \\
& * \arrow[rr] & & \Sigma X,
\end{tikzcd}$$

and notice that all maps $\phi : CX \to \Sigma X$ are homotopic, since the domain is contractible. Determine the comparison map ξ in the two cases

(1) $\phi = *$, the constant map, and

(2) $\phi : CX \to \Sigma X$ is the quotient map which collapses the base of the cone.

Show that if $\Sigma X \not\simeq *$, then these comparison maps are not homotopic, or even homotopy equivalent, to one another.

Problem 7.5. Let $\overline{C} \leftarrow \overline{A} \to \overline{B}$ be your favorite cofibrant replacement for $C \leftarrow A \to B$. Suppose there is a map $\phi : B \to D$ homotopic to the composition $\overline{B} \to B \to D$ which induces a homotopy equivalence $\xi : \overline{D} \to D$. Let $\widetilde{C} \leftarrow \widetilde{A} \to \widetilde{B}$ be another cofibrant replacement for $C \leftarrow A \to B$. Show that there is a map $\widetilde{B} \to D$ inducing a homotopy equivalence $\widetilde{\xi} : \widetilde{D} \to D$.

Now we are prepared to make our definition: a homotopy commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a **homotopy pushout square** if for any cofibrant replacement $\overline{C} \leftarrow \overline{A} \rightarrow \overline{B}$ of $C \leftarrow A \rightarrow B$, there is a map $\phi : \overline{B} \rightarrow D$ homotopic to the composite $\overline{B} \rightarrow B \rightarrow D$ so that the solid arrow part of the homotopy commutative diagram

$$\begin{tikzcd} \overline{A} \arrow[rr] \arrow[dr] \arrow[dd] & & \overline{B} \arrow[dr] \arrow[dd] \arrow[ddr, bend left, "\phi"] & \\ & \overline{C} \arrow[rr] \arrow[dd] & & \overline{D} \arrow[dd, dotted, "\xi"] \\ A \arrow[rr] \arrow[dr] & & B \arrow[dr, dashed] & \\ & C \arrow[rr] & & D \end{tikzcd}$$

is strictly commutative and the induced comparison map $\xi : \overline{D} \rightarrow D$ is a homotopy equivalence.

This definition has been written to emphasize that the property of being a homotopy pushout square does not depend on making a particular choice of cofibrant replacement. But, as a practical matter, you should keep in mind that, because of Problem 7.5, it suffices to check that the comparison map is a homotopy equivalence just for your favorite cofibrant replacement.

Problem 7.6. Show that if

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a homotopy pushout square, then D is the homotopy pushout of the diagram $C \leftarrow A \rightarrow B$ and the maps $B \rightarrow D$ and $C \rightarrow D$ are equivalent in $\mathrm{H}\mathcal{T}_*$ to the canonical homotopy classes guaranteed by Problem 6.21(b).

We end this section with some important examples.

Problem 7.7.

(a) Show that the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

is a homotopy pushout square.

(b) Prove that $A \to B \to C$ is a cofiber sequence if and only if the square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ * \arrow[r] & C \end{tikzcd}$$

is a homotopy pushout square.

Proposition 7.8. *Consider the homotopy commutative square*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g", "\simeq"'] & B \arrow[d, "f"] \\ C \arrow[r] & D. \end{tikzcd}$$

Show that the square is a homotopy pushout square if and only if f is a homotopy equivalence.

Problem 7.9. Prove Proposition 7.8.

Later, in Problem 19.38, we will find spaces A such that $A \not\simeq *$ but $\Sigma A \simeq *$. If A is such a space, then the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & * \arrow[d, "f"] \\ * \arrow[r] & * \end{tikzcd}$$

is a homotopy pushout square in which f is a homotopy equivalence, but g is not.

7.2. Recognition and Completion

In order to make use of homotopy pushout squares, it is crucial to be able to recognize them, and it is important to be able to build new ones. We establish a simple recognition principle that allows us to say when a categorical pushout square is also a homotopy pushout square, and we show that any prepushout diagram $C \leftarrow A \to B$ can be completed to a homotopy pushout square.

7.2.1. Recognition. If the maps $A \to B$ and $A \to C$ are both cofibrations in the pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & B \arrow[d] \\ C \arrow[r] & D, \end{tikzcd}$$

then $C \leftarrow A \to B$ is its own cofibrant replacement, and the induced map from the homotopy pushout to D is the identity, which means that the square is a homotopy pushout square.

In fact, this square is much better than an ordinary run of the mill homotopy pushout square. We call a strictly commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

a **strong homotopy pushout square** if there is a cofibrant replacement $\overline{C} \leftarrow \overline{A} \to \overline{B}$ so that the induced map $\overline{D} \to D$ of colimits is a homotopy equivalence.

Problem 7.10.

(a) Show that a strong homotopy pushout square is an ordinary homotopy pushout square.

(b) Find an example of an ordinary homotopy pushout square that is not a strong homotopy pushout square.

(c) Show that if a square is a strong homotopy pushout square, then for *every* cofibrant replacement $\widetilde{C} \leftarrow \widetilde{A} \to \widetilde{B}$, the induced map $\widetilde{D} \to D$ is a homotopy equivalence.

Requiring both maps to be cofibrations is more than we need: a pushout square in which *just one* of the maps is a cofibration is a strong homotopy pushout square.

Theorem 7.11. *If, in the pushout square above, either $A \to B$ or $A \to C$ is a cofibration, then the square is a strong homotopy pushout square.*

Problem 7.12. Prove Theorem 7.11.

Exercise 7.13. Find an example of a strong homotopy pushout square in which none of the maps is a cofibration. Can you find an example in which the square is also a categorical pushout square?

7.2.2. Completion. Now we show that any set of prepushout data can be completed to a homotopy pushout square.

Theorem 7.14. *For any prepushout diagram $C \leftarrow A \to B$ there is a space D and maps $B \to D$ and $C \to D$ so that the square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a homotopy pushout square. This completion is unique up to pointwise equivalence of diagrams in the homotopy category: if $B \to \widetilde{D}$ and $C \to \widetilde{D}$ also complete the given diagram to a homotopy pushout square, then there is a homotopy commutative cube

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] \arrow[dd, "\simeq"'] & & B \arrow[dr] \arrow[dd, "\simeq"' near end] & \\ & C \arrow[rr] \arrow[dd, "\simeq"' near start] & & \widetilde{D} \arrow[dd, "\simeq"] \\ A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\ & C \arrow[rr] & & D. \end{tikzcd}$$

Problem 7.15.

(a) Consider the homotopy commutative cube

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] \arrow[dd, "\simeq"'] & & B \arrow[dr, dashed] \arrow[dd, "\simeq" near end] & \\ & C \arrow[rr] \arrow[dd, "\simeq"' near start] & & D \arrow[dd, "\simeq"] \\ A' \arrow[rr] \arrow[dr] & & B' \arrow[dr, dashed] & \\ & C' \arrow[rr] & & D' \end{tikzcd}$$

whose vertical maps are homotopy equivalences. Suppose the solid arrow part is strictly commutative. Show that if the top face is a homotopy pushout square, then the bottom face is also a homotopy pushout square.

(b) Prove Theorem 7.14.

The result of Problem 7.15(a) will be superseded later by Theorem 7.36, which is both simpler to state and more powerful.

The completion theorem can be extended to maps of prepushout diagrams.

Problem 7.16. Show that a homotopy commutative diagram

$$\begin{tikzcd} C \arrow[d] & A \arrow[l] \arrow[r] \arrow[d] & B \arrow[d] \\ Y & W \arrow[l] \arrow[r] & X \end{tikzcd}$$

can be completed to a homotopy commutative cube

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[dr] \arrow[dd] & \\ & C \arrow[rr] \arrow[dd] & & D \arrow[dd] \\ W \arrow[rr] \arrow[dr] & & X \arrow[dr] & \\ & Y \arrow[rr] & & Z \end{tikzcd}$$

in which the top and bottom faces are homotopy pushout squares and the map $D \to Z$ is an induced map of homotopy pushouts.

Exercise 7.17. Find an example of a prepushout diagram that cannot be completed to a strong pushout square.

7.3. Homotopy Pullback Squares

In this section we give the basic definitions and theory of homotopy pullback squares. Since this is precisely dual to the discussion of homotopy pushout squares in the previous two sections, this section will have the flavor of a quick summary rather than a detailed study.

We assume throughout this section that any pointed diagrams are diagrams of *well-pointed* spaces.

We begin by comparing a homotopy commutative square to a pullback diagram. If we are given the homotopy commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D, \end{tikzcd}$$

then we find a fibrant replacement $\overline{C} \to \overline{D} \leftarrow \overline{B}$ with pullback $\overline{A}$ and use it to form the homotopy commutative diagram

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, dashed] \arrow[dd, dotted, "\zeta"'] \arrow[dddr, bend right, "\theta"] & & B \arrow[dr] \arrow[dd] & \\
& C \arrow[rr] \arrow[dd] & & D \arrow[dd] \\
\overline{A} \arrow[rr] \arrow[dr] & & \overline{B} \arrow[dr] & \\
& \overline{C} \arrow[rr] & & \overline{D}.
\end{tikzcd}$$

If there is a map $\theta : A \to \overline{C}$ making the solid arrow part of the diagram strictly commutative, then there is an induced comparison map $\xi : A \to \overline{A}$ from A to the homotopy pullback $\overline{A}$ of $C \to D \leftarrow B$.

The square is a **homotopy pullback square** if for every fibrant replacement, a map θ can be found so that the induced comparison map $\zeta : A \to \overline{A}$ is a homotopy equivalence. A strictly commutative square is a **strong homotopy pullback square** if the comparison map $A \to \overline{A}$ is a homotopy equivalence when θ is taken to be the given composite $A \to C \to \overline{C}$.

As with homotopy pushout squares, you really only need to check these conditions for your favorite fibrant replacement.

Problem 7.18. Show that if, for your favorite fibrant replacement, there exists a θ whose induced comparison map ζ is a homotopy equivalence, then there are such choices for every fibrant replacement.

Now we look at some important examples.

Problem 7.19.

(a) Show that

$$\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & X
\end{tikzcd}$$

is a homotopy pullback square.

(b) Show that $F \to E \to B$ is a fibration sequence if and only if

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & E \arrow[d] \\
* \arrow[r] & B
\end{tikzcd}$$

is a homotopy pullback square.

Problem 7.20. Consider the square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"] & B \arrow[d, "f", "\simeq"'] \\ C \arrow[r] & D. \end{tikzcd}$$

(a) Show that the square is a homotopy pullback square if and only if g is a homotopy equivalence.

(b) Find an example of a homotopy pullback square in which g is a homotopy equivalence while f is not.

We finish by addressing the recognition and completion problems.

Theorem 7.21.

(a) If in the pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & B \arrow[d, "f"] \\ C \arrow[r] & D \end{tikzcd}$$

the map f is a fibration, then the square is a strong homotopy pullback square.

(b) Given a prepullback diagram $C \to D \leftarrow B$, there is a space A and maps $A \to B$, $A \to C$ making the square

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HPB}"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

a homotopy pullback square.

Problem 7.22. Prove Theorem 7.21. Discuss the uniqueness of the square in part (b), and show by example that there are prepullback diagrams that cannot be completed to strong homotopy pullback squares.

7.4. Manipulating Squares

In this section we establish the basic formal rules for working with homotopy pushout and pullback squares. We will accumulate a vast collection of applications—with varying levels of importance—of these results in the next two chapters.

7.4.1. Composition of Squares. We begin by adapting Theorems 2.40 and 2.42, which concern categorical pushout and pullback squares, to homotopy pushout and pullback squares.

Theorem 7.23. *Consider the commutative diagram*

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "h_1"'] \arrow[dr, phantom, "(I)"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"'] \arrow[dr, phantom, "(II)"] & A_3 \arrow[d, "h_3"] \\ B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3, \end{tikzcd}$$

and denote the outside square by (T).

(a) If (I) and (II) are homotopy pushout squares, then (T) is also a homotopy pushout square.

(b) If (I) and (T) are homotopy pushout squares, then (II) is also a homotopy pushout square.

(c) If (I) and (II) are homotopy pullback squares, then (T) is also a homotopy pullback square.

(d) If (II) and (T) are homotopy pullback squares, then (I) is also a homotopy pullback square.

Problem 7.24. Prove Theorem 7.23.

7.4.2. 3×3 **Diagrams.** Next we prove the homotopy theoretic analog of Theorems 2.43 and 2.45 and derive some nice consequences. Let us consider the diagram F:

$$\begin{tikzcd} A_1 & A_2 \arrow[l] \arrow[r] & A_3 \\ B_1 \arrow[u] \arrow[d] & B_2 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & B_3 \arrow[u] \arrow[d] \\ C_1 & C_2 \arrow[l] \arrow[r] & C_3 \end{tikzcd}$$

and its cofibrant replacement $\overline{F}$:

$$\begin{tikzcd} \overline{A}_1 & \overline{A}_2 \arrow[l] \arrow[r] & \overline{A}_3 \\ \overline{B}_1 \arrow[u] \arrow[d] & \overline{B}_2 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \overline{B}_3 \arrow[u] \arrow[d] \\ \overline{C}_1 & \overline{C}_2 \arrow[l] \arrow[r] & \overline{C}_3, \end{tikzcd}$$

which we construct following the algorithm detailed in Section 6.5.3. Then $Q = \operatorname{colim} \overline{F}$ is the homotopy colimit of the given diagram F.

Problem 7.25.

(a) Show that the rows of $\overline{F}$ are cofibrant replacements for the rows of F. Conclude that the pushouts $\overline{A}, \overline{B}$ and $\overline{C}$ of the rows of $\overline{F}$ are homotopy pushouts of the rows of F.

(b) Show that the induced maps $\overline{C} \leftarrow \overline{A} \rightarrow \overline{B}$ are cofibrations.

Hint. Show that $\overline{C}$ and $\overline{B}$ are mapping cylinders.

(c) Show that with these maps the square

$$\begin{tikzcd} \overline{A} \arrow[r] \arrow[d] & \overline{B} \arrow[d] \\ \overline{C} \arrow[r] & \operatorname{colim} \overline{F} \end{tikzcd}$$

is a homotopy pushout square.

(d) Verify that all of your arguments are purely formal, and hence dualizable.

You have proved the following theorem.

Theorem 7.26. *Consider the diagrams*

$$\begin{tikzcd} A_1 & A_2 \arrow[l] \arrow[r] & A_3 \\ B_1 \arrow[u] \arrow[d] & B_2 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & B_3 \arrow[u] \arrow[d] \\ C_1 & C_2 \arrow[l] \arrow[r] & C_3 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A_1 \arrow[r] \arrow[d] & A_2 \arrow[d] & A_3 \arrow[l] \arrow[d] \\ B_1 \arrow[r] & B_2 & B_3 \arrow[l] \\ C_1 \arrow[r] \arrow[u] & C_2 \arrow[u] & C_3, \arrow[l] \arrow[u] \end{tikzcd}$$

and write Q to denote either the homotopy colimit of the first diagram or the homotopy limit of the second diagram.

(a) A choice of induced maps between the homotopy pushouts of the rows in the first diagram gives a prepushout diagram $\overline{A} \leftarrow \overline{B} \rightarrow \overline{C}$ whose maps are induced maps. Likewise, the homotopy pushouts of the columns fit into another prepushout diagram $\overline{X} \leftarrow \overline{Y} \rightarrow \overline{Z}$. The squares

$$\begin{tikzcd} \overline{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & \overline{A} \arrow[d] \\ \overline{C} \arrow[r] & \overline{Q} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \overline{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & \overline{X} \arrow[d] \\ \overline{Z} \arrow[r] & \overline{Q} \end{tikzcd}$$

are both homotopy pushout squares. In particular, the homotopy pushouts of $\overline{A} \leftarrow \overline{B} \rightarrow \overline{C}$ and $\overline{X} \leftarrow \overline{Y} \rightarrow \overline{Z}$ are homotopy equivalent to one another.

(b) A choice of induced maps between the homotopy pullbacks of the rows in the first diagram gives a prepullback diagram $\overline{A} \rightarrow \overline{B} \leftarrow \overline{C}$ whose maps are induced maps. Likewise, the homotopy pullbacks of the columns fit

into another prepushout diagram $\overline{X} \to \overline{Y} \leftarrow \overline{Z}$. The squares

$$\begin{tikzcd} \overline{Q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{HPB}"] & \overline{A} \arrow[d] \\ \overline{C} \arrow[r] & \overline{B} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \overline{Q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{HPB}"] & \overline{X} \arrow[d] \\ \overline{Z} \arrow[r] & \overline{Y} \end{tikzcd}$$

are both homotopy pullback squares. In particular, the homotopy pullbacks of $\overline{A} \to \overline{B} \leftarrow \overline{C}$ and $\overline{X} \to \overline{Y} \leftarrow \overline{Z}$ are homotopy equivalent to one another.

Here are some nice simple examples.

Problem 7.27. Determine the homotopy colimits of the diagrams

$$\begin{tikzcd} * & * \arrow[l] \arrow[r] & * \\ * \arrow[u] \arrow[d] & X \arrow[l] \arrow[r] \arrow[u] \arrow[d] & * \arrow[u] \arrow[d] \\ * & * \arrow[l] \arrow[r] & * \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, equal] & X \arrow[r, equal] & X \\ X \arrow[u, equal] \arrow[d, equal] & * \arrow[l] \arrow[r] \arrow[u] \arrow[d] & X \arrow[u, equal] \arrow[d, equal] \\ X \arrow[r, equal] & X \arrow[r, equal] & X. \end{tikzcd}$$

What are the duals?

Corollary 7.28. *Consider the map of prepushout diagrams*

$$\begin{tikzcd} C \arrow[d, "h"'] & A \arrow[l] \arrow[r, "h"] \arrow[d] & B \arrow[d, "g"] \\ Y & W \arrow[l] \arrow[r] & X \end{tikzcd}$$

with induced map of homotopy pushouts $D \to Z$. Show that the cofiber of $D \to Z$ is the homotopy pushout of the diagram $C_h \leftarrow C_f \to C_g$ that results from choosing induced maps between the cofibers of the columns.

Problem 7.29. Prove Corollary 7.28.

Our final application of Theorem 7.26 in this section shows that certain combinations of homotopy pushout squares are again homotopy pushout squares, and similarly for homotopy pullbacks.

Lemma 7.30. *Consider the diagrams*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z. \end{tikzcd}$$

(a) If they are both homotopy pushout diagrams, then so is

$$\begin{tikzcd} A \vee W \arrow[r] \arrow[d] & B \vee X \arrow[d] \\ C \vee Y \arrow[r] & D \vee Z. \end{tikzcd}$$

(b) If they are both homotopy pullback diagrams, then so is

$$\begin{tikzcd} A \times W \arrow[r] \arrow[d] & B \times X \arrow[d] \\ C \times Y \arrow[r] & D \times Z. \end{tikzcd}$$

Problem 7.31. Prove Lemma 7.30.

Problem 7.32.

(a) Determine the homotopy pushout of the diagram $B \xleftarrow{*} A \xrightarrow{f} C$.
HINT. The trivial map $A \to B$ can be viewed as $A \vee * \to * \vee B$.

(b) Determine the homotopy pullback of the diagram $B \xrightarrow{*} A \xleftarrow{f} C$.

(c) What are the fiber and cofiber of a trivial map?

7.4.3. Application of Functors. Let's extend Theorem 6.17 to homotopy pushout and pullback squares.

Proposition 7.33. *Let $L, R : \mathcal{T}_1 \to \mathcal{T}_2$ be an adjoint pair of homotopy functors between one of our categories of topological spaces and another.*

(a) If L preserves cofibrations, then L preserves homotopy pushouts.

(b) If R preserves fibrations and well-pointed spaces, then R preserves homotopy pullbacks.

Problem 7.34. Prove Proposition 7.33.

Here are the expected applications.

Problem 7.35. Consider the homotopy commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D. \end{tikzcd}$$

(a) Show that if it is a homotopy pushout square, then so are

$$\begin{tikzcd} A \times X \arrow[r] \arrow[d] & B \times X \arrow[d] \\ C \times X \arrow[r] & D \times X, \end{tikzcd} \qquad \begin{tikzcd} A \rtimes X \arrow[r] \arrow[d] & B \rtimes X \arrow[d] \\ C \rtimes X \arrow[r] & D \rtimes X \end{tikzcd}$$

and

$$\begin{tikzcd}
A \wedge X \arrow[r] \arrow[d] & B \wedge X \arrow[d] \\
C \wedge X \arrow[r] & D \wedge X
\end{tikzcd}$$

for any space X.

(b) Show that if the original diagram is a homotopy pullback square, then

$$\begin{tikzcd}
\mathrm{map}_\circ(X, A) \arrow[r] \arrow[d] & \mathrm{map}_\circ(X, B) \arrow[d] \\
\mathrm{map}_\circ(X, C) \arrow[r] & \mathrm{map}_\circ(X, D)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{map}_*(X, A) \arrow[r] \arrow[d] & \mathrm{map}_*(X, A) \arrow[d] \\
\mathrm{map}_*(X, A) \arrow[r] & \mathrm{map}_*(X, A)
\end{tikzcd}$$

are also homotopy pullback squares.

7.5. Characterizing Homotopy Pushout and Pullback Squares

We show that the property of being a homotopy pushout square is preserved by pointwise equivalences in the homotopy category, and similarly for homotopy pullback squares. This implies much simpler and more conceptual characterizations of homotopy pushout and pullback squares.

Theorem 7.36. *Consider the homotopy commutative cube*

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd, "\simeq"'] & & B \arrow[dr] \arrow[dd, "\simeq", near start] & \\
& C \arrow[rr, crossing over] & & D \arrow[dd, "\simeq"] \\
A' \arrow[rr] \arrow[dr] & & B' \arrow[dr] & \\
& C' \arrow[rr] \arrow[from=uu, crossing over, "\simeq", near end] & & D'
\end{tikzcd}$$

in which the vertical maps are homotopy equivalences. Then

(a) the top square is a homotopy pushout square if and only if the bottom square is a homotopy pushout square, and

(b) the top square is a homotopy pullback square if and only if the bottom square is a homotopy pullback square.

Problem 7.37. Prove Theorem 7.36.

Hint. Flatten the cube into a square planar diagram subdivided into nine smaller squares. Use Theorem 7.23.

Because of Theorem 7.36, we can give a new, and conceptually much nicer, characterization of homotopy pushout and pullback squares.

Corollary 7.38.

(a) *A homotopy commutative square is a homotopy pushout square if and only if it is pointwise equivalent in the homotopy category to a pushout square in which all four maps are cofibrations.*

(b) *Dually, a homotopy commutative square is a homotopy pullback square if and only if it is pointwise equivalent in the homotopy category to a pullback square in which all four maps are fibrations.*

Problem 7.39. Prove Corollary 7.38.

7.6. Additional Topics, Problems and Projects

7.6.1. Cartesian and Cocartesian Cubes. Squares are only the first (or second) step in a long list of interesting diagrams. Let's write $\mathcal{C}_n$ for the n-cube diagram $(\star \to \bullet)^n$. These diagrams are simple, and we call the category $(\mathcal{C}_n)_{<\bullet^n}$ the **punctured** n-cube diagram. An n**-cube diagram** is a functor $\mathcal{C}_n \to \mathcal{T}$ and a **punctured** n-cube diagram is the functor $F^\circ : (\mathcal{C}_n)_{<\bullet^n} \to \mathcal{T}$; F is called **cocartesian** if the comparison map $\operatorname{hocolim} F^\circ \to F(\bullet^n)$ is a homotopy equivalence.

Problem 7.40. Show that a cocartesian 2-cube is just a strong homotopy pushout square.

The cube diagram F is **strongly cocartesian** if its restriction $F_{i,j}$ to each 2-dimensional face is cocartesian.

Problem 7.41. Find a diagram that is cocartesian but not strongly cocartesian.

Project 7.42. Define and study cartesian cubes.

7.6.2. Problems.

Problem 7.43. Suppose $f : X \to Y$ is **idempotent** up to homotopy (i.e., $f \circ f \simeq f$), and let F be its homotopy fiber. Show there is a fibration sequence of the form $F \to F \to F$.

Problem 7.44. Show that if the bottom face and all of the sides of a commutative cube are homotopy pullbacks, then the top is a homotopy pullback and dualize.

Project 7.45. Write P_n for the diagram $(\star \leftarrow \bullet \to \diamond)^n$ (so that P_2 is what we have called a 3×3 diagram).

(a) Explicitly construct functorial cofibrant replacements for diagrams $F : P_n \to \mathcal{T}$.

(b) Let $F : P_n \to \mathcal{T}$, and let G be the prepushout diagram

$$\operatorname{hocolim} F|_{\star\times P_{n-1}} \leftarrow \operatorname{hocolim} F|_{\bullet\times P_{n-1}} \to \operatorname{hocolim} F|_{\circ\times P_{n-1}}.$$

Show that $\operatorname{hocolim} G = \operatorname{hocolim} F$.

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Chapter 8

Tools and Techniques

In this chapter, we'll develop a number of powerful homotopy-theoretical tools. Most of these have to do with establishing long cofiber and fiber sequences and studying the resulting long exact sequences of homotopy sets. We also study homotopy colimits (and limits) of diagrams defined on certain product categories $\mathcal{I} \times \mathcal{J}$, and we conclude with some basic homotopy theory of group actions.

8.1. Long Cofiber and Fiber Sequences

When we apply the functor $[\,?\,, Q]$ to a cofiber sequence $X \to Y \to Z$ in $\mathcal{T}_*$, the result is an exact sequence $[X, Q] \leftarrow [Y, Q] \leftarrow [Z, Q]$ of pointed sets. Furthermore, any given map may be converted into a cofibration, and so it can be inserted into a cofiber sequence. In view of Problem 7.7, this is accomplished by forming the strong homotopy pushout diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dotted] \arrow[dr, phantom, "\text{HPO}"] & Y \arrow[d, "j"] \\ CX \arrow[r, dotted] & Z. \end{tikzcd}$$

We can hardly be restrained from asking: since this can be done with *any* map, why not repeat the process with the map j?

8.1.1. The Long Cofiber Sequence of a Map. Here is the diagram that results from cheerful iteration:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dotted] \arrow[dr, phantom, "\text{pushout}"] & Y \arrow[r, dotted] \arrow[d, "j"] \arrow[dr, phantom, "\text{pushout}"] & CY \arrow[d, dotted] & & \\
CX \arrow[r, dotted] & Z \arrow[r, "\partial"] \arrow[d, dotted] \arrow[dr, phantom, "\text{pushout}"] & X^{(2)} \arrow[r, dotted] \arrow[d, "f^{(2)}"] \arrow[dr, phantom, "\text{pushout}"] & CX^{(2)} \arrow[d, dotted] & \\
& CZ \arrow[r, dotted] & Y^{(2)} \arrow[r, "j^{(2)}"] \arrow[d, dotted] \arrow[dr, phantom, "\text{pushout}"] & Z^{(2)} \arrow[d, "\partial^{(2)}"] & \cdots \\
& & CY^{(2)} \arrow[r, dotted] & X^{(3)} & \cdots \\
& & & \vdots & \text{etc.,}
\end{tikzcd}$$

where the spaces $X^{(2)}, Y^{(2)}, Z^{(2)}$ and the maps $f^{(2)}$, $j^{(2)}$, etc., are yet to be determined. All the maps in the diagram, except possibly f and $CX \to Z$, are cofibrations, and the central zigzag of labeled solid arrows is a long cofiber sequence; we'll refer to it as the **long cofiber sequence** of f.

Since the suspension ΣA is the union of two cones, a map $f : A \to B$ induces maps $\phi : \Sigma A \to \Sigma B$, among which are $\pm\Sigma f$. The identification of ϕ depends on which cones we consider to be the top cones and which are the bottom cones. If the map carries the top to the top, then it is Σf; but if it carries the top cone C_+A to the bottom cone C_-B and the bottom to the top, then $\phi = -\Sigma f$. Since our construction involves repeated attachment of cones, we cannot possibly identify our maps without deciding on a convention for deciding which cone is the top cone.

Convention: *The most recently attached cone is the top cone.*

We begin our investigation of the long cofiber sequence with the simplest possible nontrivial example.

Problem 8.1.

(a) Show that if $f = \mathrm{id}_{S^0}$, then the long cofiber sequences has the form

$$S^0 \xrightarrow{\mathrm{id}} S^0 \xrightarrow{j} I \xrightarrow{q} \Sigma S^0 \xrightarrow{\phi} S(1),$$

for some space $S(1)$.

(b) Show that there is a homotopy equivalence $z : S(1) \to \Sigma S^0$.

(c) Determine the composite $\Sigma S^0 \xrightarrow{\phi} S(1) \xrightarrow{z} \Sigma S^0$.

(d) Determine the first five spaces and the first four maps in the long cofiber sequence for $f = \mathrm{id}_X$.

Theorem 8.2. *For any map $f : X \to Y$ with cofiber Z, there is a functorial long cofiber sequence of the form*

$$X \xrightarrow{f} Y \xrightarrow{j} Z \xrightarrow{\partial} \Sigma X \xrightarrow{-f} \Sigma Y \to \cdots \to \Sigma^n X \xrightarrow{(-1)^n \Sigma^n f} \Sigma^n Y \to \cdots .$$

Problem 8.3.

(a) Argue that the sequence is functorial.

(b) Using the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d] & X \arrow[d, "f"] \\ X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d] \\ Y \arrow[r, "\mathrm{id}_Y"] & Y, \end{tikzcd}$$

construct homotopy equivalences $\Sigma X \to X^{(2)}$ and $Y^{(2)} \to \Sigma Y$; these are the maps we'll use to identify the homotopy types of $X^{(2)}$ and $Y^{(2)}$.

(c) Show that, under these identifications, $f^{(2)}$ corresponds to $-\Sigma f$.

(d) Prove Theorem 8.2.

Corollary 8.4. *Let $f : X \to Y$ with cofiber Z, and let Q be any space. Then there is a long exact sequence of the form*

$$\cdots \leftarrow [\Sigma^n X, Q] \xleftarrow{(-1)^n \Sigma^n f^*} [\Sigma^n Y, Q] \xleftarrow{(-1)^n \Sigma^n j^*} [\Sigma^n Z, Q] \leftarrow [\Sigma^{n+1} X, Q] \leftarrow \cdots$$

in which all the terms involving suspensions are groups, and the maps between them are homomorphisms; those involving more than one suspension are abelian groups.

The naturality of these long sequences can be used to test whether a map is nontrivial.

Problem 8.5. Let $f : X \to Y$ and extend this map to the long cofiber sequence

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \longrightarrow \Sigma Z \to \cdots .$$

(a) Show that if f is trivial, then there is a map $\Sigma X \to Z$ such that $\Sigma X \to Z \to \Sigma X$ is homotopic to $\mathrm{id}_{\Sigma X}$.

HINT. Express the condition in the form of a commutative square.

(b) Suppose you have a functor $F : \mathrm{H}\mathcal{T}_* \to \mathrm{AB}\,\mathcal{G}$ such that $F(Z) = 0$ and $F(\Sigma X) \neq 0$. Show that $f \not\simeq *$.

8.1.2. The Long Fiber Sequence of a Map. The construction of the long cofiber sequence of $f : X \to Y$ is based only on the formal properties of cofibers and pushout squares, so we can dualize the whole thing. Starting with the map $f : X \to Y$ and repeatedly forming fibers, we obtain the diagram

$$\begin{tikzcd}
\text{...etc.} & \vdots & & \\
\cdots & \Omega F \arrow[r, dotted] \arrow[d, "-\Omega i"'] \arrow[dr, phantom, "\text{pullback}"] & \mathcal{P}(\Omega Y) \arrow[d, dotted] & \\
\cdots & \Omega X \arrow[r, "-\Omega f"] \arrow[d, dotted] \arrow[dr, phantom, "\text{pullback}"] & \Omega Y \arrow[r, dotted] \arrow[d, "\partial"] \arrow[dr, phantom, "\text{pullback}"] & \mathcal{P}(X) \arrow[d, dotted] \\
& \mathcal{P}(F) \arrow[r, dotted] & F \arrow[r, "i"] \arrow[d, dotted] \arrow[dr, phantom, "\text{pullback}"] & X \arrow[d, "f"] \\
& & \mathcal{P}(Y) \arrow[r, dotted] & Y
\end{tikzcd}$$

and derive the dual theorem.

Theorem 8.6. *For any map f, there is a functorial long fiber sequence of the form*

$$\cdots \to \Omega^n X \xrightarrow{(-1)^n \Omega^n f} \Omega^n Y \to \cdots \to \Omega Y \xrightarrow{\partial} F \xrightarrow{i} X \xrightarrow{f} Y.$$

Corollary 8.7. *Let $f : X \to Y$ with fiber F, and let Q be any space. Then there is a long exact sequence*

$$\cdots \to [Q, \Omega^n X] \xrightarrow{(-1)^n \Omega^n f_*} [Q, \Omega^n Y] \to \cdots \to [Q, F] \xrightarrow{i_*} [Q, X] \xrightarrow{f_*} [Q, Y].$$

The sets involving at least one loop space are groups, and the maps between them are homomorphisms. The ones involving at least two loops are abelian groups.

Problem 8.8. Prove Theorem 8.6 and Corollary 8.7.

Hint. Use Problem 8.1.

Theorem 8.6 yields a long exact sequence of homotopy groups for a fiber sequence.

Corollary 8.9. *Let $F \to E \to B$ be a fiber sequence. Then there is a functorial long exact sequence*

$$\cdots \to \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \pi_{n-1}(E) \to \cdots .$$

(If $n \geq 1$, then these sets are groups, and if $n \geq 2$, they are abelian.)

Problem 8.10. Prove Corollary 8.9.

A **section** of a map $p : E \to B$ is a map $s : B \to E$ such that $p \circ s = \mathrm{id}_B$. In other words, s is a right inverse of p; the term section is typically used when p is a fibration.

Problem 8.11. Let $p : E \to B$ be a fibration with fiber F. Show that if p has a section, then $\Omega E \simeq \Omega B \times \Omega F$.

Problem 8.12. Formulate and prove the dual to Problem 8.5.

8.2. The Action of Paths in Fibrations

The fibers over two points in the same path component of the base of a fibration are homotopy equivalent, since they are pullbacks of a fibration by homotopic maps. In this section we will use the homotopy lifting property to use paths in the base to build homotopy equivalences between the fibers, called *admissible maps.* Path-homotopic paths yield homotopic equivalences, so, if the base is simply-connected, the admissible maps provide canonical equivalences between the fibers. For non-simply-connected bases, the situation is not hopeless. If we plan to study our fibration with a homotopy functor h, it may happen that the fibration is *orientable* with respect to h, which means that the induced maps $h(f)$ are equal for all admissible maps.

8.2.1. Admissible Maps. Let $p : E \to B$ be a fibration and let $\omega : I \to B$ be a path from a to b, and consider the diagram

$$\begin{tikzcd} F_a \arrow[r, hook] \arrow[d, "\mathrm{in}_0"'] & E \arrow[d, "p"] \\ I \times F_a \arrow[r, "\omega \circ \mathrm{pr}_1"'] \arrow[ur, dotted, "\Phi"] & B. \end{tikzcd}$$

Since p is a fibration and in_0 is a cofibration, the dotted arrow can be filled in by a map $\Phi : I \times F_a \to E$ making both triangles commute. Now we define

$$\phi_\omega : F_a \longrightarrow F_b \qquad \text{by the rule} \qquad \phi_\omega(x) = \Phi(x, 1).$$

The definition of a fibration guarantees that the map Φ exists, but it does not say anything about it being unique, even up to homotopy. So, how much does the map ϕ_ω change if we choose a different Φ? What if we choose a different path?

Proposition 8.13. *Let $\omega, \tau : I \to B$ be two paths in B from a to b, and let $\phi_\tau, \phi_\omega : F_a \to F_b$ be any two maps obtained by the construction above. If τ and ω are path homotopic, then $\phi_\tau \simeq \phi_\omega$ in $\mathcal{T}_\circ$.*

Corollary 8.14. *If ϕ_ω and θ_ω are maps obtained from our construction from the same path ω, then $\phi_\omega \simeq \theta_\omega$ in $\mathcal{T}_\circ$.*

Corollary 8.15. *If B is simply-connected, then for any $a, b \in B$ there is a unique homotopy class of admissible maps $\phi_{a,b} : F_a \to F_b$.*

Problem 8.16.

(a) Prove Proposition 8.13 by studying the diagram

$$\begin{tikzcd}
(F_a \times I \times 0) \cup (F_a \times \{0,1\} \times I) \arrow[rr] \arrow[d, "i"'] & & E \arrow[d, "p"] \\
F_a \times I \times I \arrow[rr, "H \circ \mathrm{pr}_{2,3}"] & & B,
\end{tikzcd}$$

where $H : \omega \simeq \tau$.

(b) Derive Corollaries 8.14 and 8.15.

Write $\mathcal{P}(B, a, b)$ to denote the space of all paths in B from a to b, and $\pi_1(B, a, b)$ for the set of path homotopy classes of such paths. We have shown that the rule $\omega \mapsto \phi_\omega$ defines a function

$$\pi_1(B, a, b) \longrightarrow \langle F_a, F_b \rangle.$$

This function is not necessarily one-to-one, nor onto. But the maps in $\langle F_a, F_b \rangle$ that are in the image of this map are particularly important because they can be understood in terms of the homotopy theory of B; a map $f : F_a \to F_b$ which is (freely) homotopic to ϕ_ω for some path $\omega \in \pi_1(B, a, b)$ is called an **admissible map** for the fibration p.

Next we show that the construction of admissible maps from paths carries concatenation of paths to composition of functions. Since every path has a reverse, this implies that all admissible maps are homotopy equivalences.

Proposition 8.17. *If $\omega, \tau : I \to B$ can be concatenated, then $\phi_{\omega * \tau} \simeq \phi_\omega \circ \phi_\tau$.*

Problem 8.18. Prove Proposition 8.17.

Let $p : E \to B$ be a fibration with simply-connected base B. Because of Corollary 8.15 each pair of points $a, b \in B$ gives rise to a unique homotopy class $\phi_{a,b} : F_a \to F_b$ from the fiber over a to the fiber over b.

Problem 8.19. Show that $\phi_{b,c} \circ \phi_{a,b} = \phi_{a,c}$ for any $a, b, c \in B$.

In the special case $a = b$, our paths are loops in B based at a, and so we have a function $\pi_1(B, a) \to \langle F_a, F_a \rangle$. Let

$$\mathcal{E}(F_a) = \{\langle f \rangle \mid f : F_a \to F_a \text{ is a homotopy equivalence}\} \subseteq \langle F_a, F_a \rangle.$$

Problem 8.20. Show that $\mathcal{E}(F_a)$ is a group under composition; show also that our map $\pi_1(B,a) \to \langle F_a, F_a\rangle$ factors through a group homomorphism $\pi_1(B,a) \to G(F_a)$.

This discussion shows that $\pi_1(B,a)$ acts (in the homotopy category) on the fiber F_a. Actions of this kind are very important, and they show up all over topology and geometry; they are often referred to as **holonomy**.

Exercise 8.21. Let F be a space, and let $f : F \to F$ be a homotopy equivalence. Is there a fibration $p : E \to B$ with fiber F such that f is an admissible map for f?

The Action of the Fundamental Groupoid. To describe these phenomena from a more global point of view, we introduce two categories, $\Pi(B)$ and $\mathcal{F}$, defined as follows. The objects of $\Pi(B)$ are the points of B, and the morphisms from a to b are path homotopy classes of paths from a to b: $\mathrm{mor}_{\Pi(B)}(a,b) = \pi_1(B,a,b)$. The category $\Pi(B)$ is called the **fundamental groupoid** of B. The objects of $\mathcal{F}$ are the spaces F_a for $a \in B$, and $\mathrm{mor}_{\mathcal{F}}(F_a, F_b) = \langle F_a, F_b\rangle$.

Problem 8.22. Show that the rule $\omega \mapsto \phi_\omega$ defines a functor $\alpha : \Pi(B) \to \mathcal{F}$.

8.3. Every Action Has an Equal and Opposite Coaction

A map $f : X \to Y$ gives rise to long exact sequences that are mostly exact sequences of (abelian) groups. But the terms at the end are only pointed sets, and the exactness is very weak: it could be, for example, that the kernel K of $f : S \to T$ is very large, but $f|_{S-K} : (S - K) \to (T - \{*\})$ is a bijection. In this section, we'll see that some of the algebraic structure can be pushed back one more step, giving us useful information about the injectivity of the 'connecting map' $\partial : C_f \to \Sigma X$.

Exercise 8.23. Let $\mathcal{C}$ be an arbitrary pointed category, and suppose G is a grouplike object in $\mathcal{C}$. Define what it means for G to act on the object $X \in \mathcal{C}$. Then dualize to give a definition of a *coaction* of a cogroup object C on X.

8.3.1. Coactions in Cofiber Sequences.

Let $i : A \to X$ be a cofibration in $\mathcal{T}_*$ and consider the diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\mathrm{in}_0"'] \arrow[dr, phantom, "\text{pushout}"] & X \arrow[r, "q"] \arrow[d, "j"'] \arrow[dr, phantom, "\text{pushout}"] & X/A \arrow[d, "\mathrm{in}_1"] \\ CA \arrow[r] & X \cup CA \arrow[r, "b"] & X/A \vee \Sigma A \end{tikzcd}$$

in which both squares are strong homotopy pushouts. Let $e : X \cup CA \to X/A$ be the collapse map, which is a homotopy equivalence because $i : A \hookrightarrow X$ is a cofibration. Then we define

$$\beta = b \circ e^{-1} : X/A \longrightarrow X/A \vee \Sigma A,$$

which is a well-defined homotopy class but not a well-defined map. If we apply a functor of the form $[\,?\,, Y]$, the homotopy class β induces a map which may be naturally identified with

$$\beta^* : [X/A, Y] \times [\Sigma A, Y] \longrightarrow [X/A, Y].$$

Note that, although $[X/A, Y]$ is only a pointed set (as far as we know), $[\Sigma A, Y]$ is naturally a group.

Problem 8.24.

(a) Show that if this construction is applied to the cofibration $A \hookrightarrow CA$, the resulting coaction $\beta : \Sigma A \to \Sigma A \vee \Sigma A$ is simply the comultiplication of the cogroup object ΣA.

(b) Show that the homotopy class β is a coaction of the cogroup object ΣA on X/A in the category $\mathcal{T}_*$.

(c) Show that β^* is an action of the group $[\Sigma A, Y]$ on the pointed set $[X/A, Y]$.

It is common to use exponential notation for the action of $[\Sigma A, Y]$ on $[X/A, Y]$; thus the element $\beta^*(u, \delta) \in [X/A, Y]$ is generally written u^δ. This action is natural in two ways.

Problem 8.25.

(a) Suppose $f : Y \to Z$. Show that $f_*(u^\delta) = (f_*(u))^{f_*(\delta)}$.

(b) Suppose the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] & Y \end{tikzcd}$$

is strictly commutative, with induced map $h : X/A \to Y/B$ of cofibers. Show that $h^*(u^\delta) = (h^*(u))^{(\Sigma f)^*(\delta)}$.

The most important property of this action, however, is that it can be used to express a useful exactness property in the pointed set end of the long cofiber sequence.

Theorem 8.26. *Let $A \xrightarrow{i} X \xrightarrow{j} B$ be a cofiber sequence in $\mathcal{T}_*$. Then for any $Y \in \mathcal{T}_*$ the natural action $[B, Y] \times [\Sigma A, Y] \longrightarrow [B, Y]$ has the property that $j^*(u) = j^*(v)$ if and only if there is a $\delta \in [\Sigma A, Y]$ such that $v = u^\delta$.*

Exercise 8.27. Show that it suffices to prove Theorem 8.26 for sequences in which i is a cofibration, $B = X/A$ and q is the canonical quotient map.

Let $A \xrightarrow{i} X \xrightarrow{q} X \cup CA \xrightarrow{\partial} \Sigma A \to \cdots$ be the beginning of the long cofiber sequence of f, and consider the action of $[\Sigma A, Y]$ on $[X/A, Y]$.

Problem 8.28. Show that if $u = v^{\delta}$ in $[X/Y, Y]$, then $q^*(u) = q^*(v)$.

Problem 8.29. Let $u, v \in [X/A, Y]$ such that $q^*(u) = q^*(V) \in [X, Y]$.

(a) Show that there is a map $w : X \cup CA \to Y$ such that $w \simeq v$ and $j \circ w = u \circ q$.

(b) Prove Theorem 8.26.

It is important to understand what Theorem 8.26 does *not* tell us. If the sequence $[X, Y] \leftarrow [X/A, Y] \leftarrow [\Sigma A, Y]$ were an exact sequence of *groups*, then the preimage of each element $\alpha \in [X, Y]$ would be a coset of the image of $[\Sigma A, Y]$. Thus, two preimages would have the same number of elements, and if the map $[X/A, Y] \leftarrow [\Sigma A, Y]$ were trivial, then the preimages would all be singletons. These properties need not hold in the semi-algebraic situation of the proposition.

Nevertheless, we can sometimes obtain the stronger conclusion of genuine injectivity.

Problem 8.30. Let $A \xrightarrow{i} X \xrightarrow{q} B$ be a cofiber sequence, and suppose $[\Sigma A, Y] = *$. Show that $[X \cup CA, Y] \to [X, Y]$ is injective.

8.3.2. A Diagram Lemma. It sometimes happens that we are given a (homotopy) commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[dl, dotted] \\ C \arrow[r] & D, \end{tikzcd}$$

and we want to find a dotted arrow that will make both triangles commute up to homotopy.

Exercise 8.31. Give an example of a square in which no such map exists.

HINT. Take for granted that there are noncontractible spaces.

Even though these maps do not generally exist, there are some special situations in which they can be found.

Problem 8.32. Let $f : X \to Y$. Show that the following are equivalent:

(1) the map $\Omega f : \Omega X \to \Omega Y$ has a homotopy section $g : \Omega Y \to \Omega X$,

(2) the map $f_* : [\Sigma Z, X] \to [\Sigma Z, Y]$ is surjective for every space Z.

Proposition 8.33. *Suppose that there is a map $\alpha : Z \to A$ such that the top row in the diagram*

$$\begin{tikzcd} Z \arrow[r, "\alpha"] & A \arrow[r, "j"] \arrow[d, "g"'] & B \arrow[d, "h"] \arrow[dl, dotted, "\xi"] \\ & C \arrow[r, "f"'] & D \end{tikzcd}$$

is a cofiber sequence, and suppose also that the map $f : C \to D$ satisfies the conditions of Problem 8.32. Then the following are equivalent:

(1) *the composite $Z \to A \to C$ is trivial,*

(2) *the dotted arrow $\xi : B \to C$ can be filled in so that both triangles commute up to homotopy.*

One implication is trivial: if ξ exists, then $Z \to A \to C$ must be trivial.

Problem 8.34. Now suppose that the composite $Z \to A \to C$ is trivial.

(a) Show that there is a map $\zeta : B \to C$ making the upper left triangle commute up to homotopy. Describe the set of all such maps.

(b) Show that there is a $\delta \in [\Sigma Z, D]$ such that $h \simeq (f \circ \zeta)^{\delta}$.

(c) Finish the proof of Proposition 8.33.

8.3.3. Action of ΩY on F. We'll briefly discuss the dual construction. Given a fibration $p : E \to B$ with fiber F, we build the diagram

$$\begin{tikzcd} F \times \Omega B \arrow[r, "a"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\text{pullback}"] & \widetilde{F} \arrow[r] \arrow[d, "j"'] \arrow[dr, phantom, "\text{pullback}"] & \mathcal{P}(B) \arrow[d, "@_0"] \\ F \arrow[r, "i"] & E \arrow[r, "p"] & B \end{tikzcd}$$

in which both squares are strong homotopy pullback squares. The space $\widetilde{F}$ is, by definition,

$$\widetilde{F} = \{(x, \omega) \mid x \in E, \omega \in \mathcal{P}(B) \text{ and } \omega(0) = p(x)\},$$

and there is a canonical equivalence $e : F \xrightarrow{\simeq} \widetilde{F}$, given by $x \mapsto (x, *)$. Thus we may define a homotopy class

$$\alpha = e^{-1} \circ a : F \times \Omega B \longrightarrow F.$$

Problem 8.35.

(a) Show that α is an action of ΩB on F in the category $\mathrm{H}\mathcal{T}_*$.

(b) Show that if this construction is applied to the fibration $\mathcal{P}(B) \hookrightarrow B$, the resulting coaction $\alpha : \Omega B \times \Omega B \to \Omega B$ is simply the multiplication of the group object ΩB.

(c) Show that if this construction is applied to the cofiber sequence $A \hookrightarrow CA$, the resulting coaction $\beta : \Sigma A \to \Sigma A \vee \Sigma A$ is simply the comultiplication of the cogroup ΣA.

(d) Show that α induces an action of $[X, \Omega B]$ on $[X, F]$.

(e) Write out and verify the naturality properties of the action.

We use the exponential notation for this action as well: $\alpha_*(u, \delta) = u^\delta$.

Theorem 8.36. *Let $A \xrightarrow{i} Y \xrightarrow{j} B$ be a cofiber sequence in $\mathcal{T}_*$. Then for any $X \in \mathcal{T}_*$ the natural action*

$$[X, \Omega B] \times [X, F] \longrightarrow [X, F]$$

has the property that $i_*(u) = i_*(v)$ if and only if there is a $\delta \in [X, \Omega B]$ such that $v = u^\delta$.

Problem 8.37. Prove Theorem 8.36.

Exercise 8.38. What is the connection between the action of ΩB on F and admissible maps?

8.4. Mayer-Vietoris Sequences

If X is the homotopy pushout of $C \leftarrow A \to B$, then we would be well within our rights to ask for a description of $[X, Q]$ in terms of $[A, Q]$, $[B, Q]$, etc. The Mayer-Vietoris sequence offers such a description: a long cofiber sequence involving the spaces A, $B \vee C$ and X (and their suspensions). The technique, being completely formal, is easily dualized, so that we can describe a homotopy pullback in terms of the other three spaces in its defining square.

Theorem 8.39. *Consider the commutative square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ C \arrow[r, "g"] & D. \end{tikzcd}$$

(a) If the square is a homotopy pushout square, then there is a cofiber sequence $B \vee C \xrightarrow{(v,g)} D \xrightarrow{\partial} \Sigma A$.

(b) If the square is a homotopy pullback square, then there is a fiber sequence $\Omega D \xrightarrow{\partial} A \xrightarrow{(f,u)} B \times C$.

Problem 8.40. Use the diagram

$$\begin{tikzcd} C \arrow[d] & * \arrow[l] \arrow[r] \arrow[d] & B \arrow[d] \\ C & A \arrow[l] \arrow[r] & B \end{tikzcd}$$

to prove Theorem 8.39.

This cofiber sequence is known as the **Mayer-Vietoris cofiber sequence**. The dual sequence is not as frequently used, but it makes sense to call it the **Mayer-Vietoris fiber sequence**.

Problem 8.41. Extend the Mayer-Vietoris sequence to the right to get the long sequence

$$B \vee C \xrightarrow{(j,g)} D \xrightarrow{\partial} \Sigma A \xrightarrow{\theta} \Sigma B \vee \Sigma C \to \cdots .$$

Identify the map θ in terms of the maps f, g, i and j from the original square.

If the space A in the upper left corner is a suspension, then we can start our Mayer-Vietoris sequence one step to the left.

Problem 8.42. Show that the following are equivalent:

(1) there is a cofiber sequence $\Sigma X \xrightarrow{f-u} B \vee C \xrightarrow{(v,g)} D$,

(2) there is a homotopy pushout square

$$\begin{tikzcd} \Sigma X \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ C \arrow[r, "g"] & D. \end{tikzcd}$$

(3) State and prove the dual statement.

Hint. Find maps (not homotopy classes!) $CX \to A$ and $CX \to B$ that give a map of prepushout squares

$$\begin{tikzcd} CX \arrow[d] & X \arrow[l] \arrow[r] \arrow[d] & CX \arrow[d] \\ C & * \arrow[l] \arrow[r] & B \end{tikzcd}$$

which induces $f - u$.

8.5. The Operation of Paths

If $X \in \mathcal{T}_\circ$, then we can choose our favorite point $x_0 \in X$ and so make X into a pointed space. How does the pointed homotopy type of X depend on the choice of basepoint? More precisely, let $x_0, x_1 \in X$ and write $X(0)$ for X with basepoint x_0 and $X(1)$ for X with basepoint x_1. How can we compare the homotopy sets $[A, X(0)]$ and $[A, X(1)]$ for $A \in \mathcal{T}_*$?

Suppose $A \in \mathcal{T}_*$ is well-pointed, and let $\alpha : I \to X$ be a path from x_0 to x_1. Now if $f : A \to X(0)$ is a pointed map, then we can view α as an unpointed homotopy of f restricted to the basepoint of A. Since A is well-pointed, this inclusion is a cofibration, and so the homotopy α may be extended to an unpointed homotopy $H : A \times I \to X$.

Problem 8.43. Let $\alpha : I \to X$ be a path from x_0 to x_1, and let $H : A \times I \to X$ be an extension of α.

(a) Show that the rule $a \mapsto H(a, 1)$ defines a pointed map $g : A \to X(1)$.

(b) Let K be another extension of the path α to a homotopy $A \times I \to X$, and let $h : A \to X(1)$ be the resulting pointed map. Show that $g \simeq h$ in $\mathcal{T}_*$.

Now we have a rule taking a path α and a map $g : X(0) \to Y$ to a homotopy class of maps $\alpha * g : X(1) \to Y$. We want to show that this rule only depends on the path homotopy class of α and the pointed homotopy class of g.

Problem 8.44.

(a) Show that if α and β are path homotopic and if $g \simeq h : X(0) \to Y$ are homotopic in $\mathcal{T}_*$, then $\alpha * g \simeq \beta * h$ in $\mathcal{T}_*$.

(b) Show that the map $\alpha * \,? : [A, X(0)] \to [A, X(1)]$ is an isomorphism of pointed sets.

Problem 8.45. Suppose $h : X \to X$ is (freely) homotopic to id_X, by a homotopy $H : \mathrm{id}_X \simeq h$. Show that $h_*(f) = \alpha * f$, where α is the track of $*$ under H.

Problem 8.46. Suppose x_0 and x_1 lie in the same path component of X. Show that the maps $\Omega f : \Omega(X, x_0) \to \Omega Y$ and $\Omega f : \Omega(X, x_1) \to \Omega Y$ are pointwise homotopy equivalent.

Exercise 8.47. Find an example to show that the path component hypothesis in Problem 8.46 is necessary.

Problem 8.48. Show that if $f : X \to Y$ is a pointed map that is an unpointed homotopy equivalence, then $f_* : [K, X] \to [K, Y]$ is a bijection for every well-pointed space K.

Problem 8.48 gives another way to prove that a pointed map $f : X \to Y$ of pointed spaces is a pointed homotopy equivalence if and only if f_- is an unpointed homotopy equivalence.

Problem 8.49. Show that the map $[X, Y] \to \langle X, Y\rangle$ is a bijective if Y is simply-connected.

Hint. Look at Problem 5.144.

8.6. Fubini Theorems

Let $F : \mathcal{I}\times\mathcal{J} \to \mathcal{T}$. For each $i \in \mathcal{I}$ we may form the diagram $F|_{i\times\mathcal{J}} : \mathcal{J} \to \mathcal{T}$, and for each $j \in \mathcal{J}$, we may form the diagram $F|_{\mathcal{I}\times j} : \mathcal{I} \to \mathcal{T}$. Then, using the functor $\mathrm{hocolim}_*$, we define

$$\mathrm{hocolim}_{\mathcal{I}} F : j \mapsto \mathrm{hocolim}_* F|_{\mathcal{I}\times j} \quad \text{and} \quad \mathrm{hocolim}_{\mathcal{J}} F : i \mapsto \mathrm{hocolim}_* F|_{i\times\mathcal{J}}.$$

Thus $\mathrm{hocolim}_{\mathcal{I}} F \in (\mathcal{T})^{\mathcal{J}}$ and $\mathrm{hocolim}_{\mathcal{J}} F \in (\mathcal{T})^{\mathcal{I}}$, so we can form their homotopy colimits, resulting in three spaces:

$$\mathrm{hocolim}_{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{J}} F), \qquad \mathrm{hocolim}\, F \qquad \text{and} \qquad \mathrm{hocolim}_{\mathcal{J}}(\mathrm{hocolim}_{\mathcal{I}} F).$$

An analogy can be made between integration and colimits (both of which are generalized sums). The first big theorem on integration on a product is Fubini's theorem, which says that one can evaluate the integral over a product using iterated integrals; the analogous statement for homotopy colimits suggests that the three spaces above should be homotopy equivalent to one another.

We state our 'Fubini theorem' for general homotopy colimits, but you will only prove the special case in which $\mathcal{I}$ and $\mathcal{J}$ are simple; and this is the only case we will use in this text.

Theorem 8.50. *Let $\mathcal{I}$ and J be any shape categories, and let $F : \mathcal{I} \times \mathcal{J} \to \mathcal{T}$. Then there are (natural) homotopy equivalences*

$$\mathrm{hocolim}_{\mathcal{I}}(\mathrm{hocolim}_{\mathcal{J}} F) \simeq \mathrm{hocolim}_* F \simeq \mathrm{hocolim}_{\mathcal{J}}(\mathrm{hocolim}_{\mathcal{I}} F).$$

Our proof of the special case depends on a lemma.

Lemma 8.51. *Let $\mathcal{I}$ and J be any shape categories (not necessarily simple).*

(a) If $F : \mathcal{I} \times \mathcal{J} \to \mathcal{T}$ is a cofibrant diagram, then the functors

$$\mathrm{colim}_{\mathcal{I}} F : \mathcal{J} \longrightarrow \mathcal{T} \qquad \textit{and} \qquad \mathrm{colim}_{\mathcal{J}} F : \mathcal{I} \longrightarrow \mathcal{T}$$

are cofibrant.

(b) If $F : \mathcal{I} \times \mathcal{J} \to \mathcal{T}$ is cofibrant and $\mathcal{I}$ is simple, then for each $i \in \mathcal{I}$, the diagram $F|_{i\times\mathcal{J}}$ is cofibrant.

Because of the symmetry of the statement, we may as well focus on the functor $G = \operatorname{colim}_{\mathcal{I}} F : \mathcal{J} \to \mathcal{T}$. For this, we let $p : E \to B$ be a map of pointed $\mathcal{J}$-shaped diagrams that is a pointwise homotopy equivalence and pointwise fibration and try to find a lift in

$$\begin{tikzcd} & E \arrow[d, "p"] \\ G \arrow[ur, dotted] \arrow[r, "\Phi"] & B. \end{tikzcd}$$

Write $\overline{G}$, $\overline{E}$ and $\overline{B}$ for the $\mathcal{I} \times \mathcal{J}$-shaped diagrams given by $\overline{G}(i, j) = G(j)$, and so on. There is a natural map of $\mathcal{I} \times \mathcal{J}$-diagrams $F \to \overline{G}$, and it figures in the lifting problem

$$\begin{tikzcd} & & \overline{E} \arrow[d, "p"] \\ F \arrow[r] \arrow[urr, dotted] & \overline{G} \arrow[r, "\Phi"] \arrow[ur, dotted] & \overline{B}. \end{tikzcd}$$

Problem 8.52.

(a) Show that the lift $F \to \overline{E}$ exists, and from it construct the desired lift $G \to E$, thereby proving Lemma 8.51(a).

(b) Prove Lemma 8.51(b).

(c) Prove Theorem 8.50 in the special case $\mathcal{I}$ and $\mathcal{J}$ are simple.

Let $F \to G$ be a map of $\mathcal{I}$-shaped diagrams, and let $T_{\mathcal{I}} : \mathcal{I} \to \mathcal{T}$ be the trivial diagram with $T_{\mathcal{I}}(i) = *$ for all $i \in \mathcal{I}$. Then we may consider

$$T_{\mathcal{I}} \longleftarrow F \longrightarrow G$$

as an $\mathcal{I} \times \mathcal{J}$-shaped diagram, where $\mathcal{J}$ is the prepushout diagram $\star \leftarrow \bullet \rightarrow \diamond$. Applying Theorem 8.50 to this diagram gives the following.

Corollary 8.53. *Let $F \to G$ be a morphism of $\mathcal{I}$-diagrams. Then there is an $\mathcal{I}$-diagram C and a morphism $G \to C$ such that for each $i \in \mathcal{I}$, the sequence*

$$F(i) \longrightarrow G(i) \longrightarrow C(i)$$

is a cofiber sequence and each $C(i \to j)$ is an induced map of cofibers.

(a) In $\mathcal{T}_*$, the induced sequence

$$\operatorname{hocolim}_* F \longrightarrow \operatorname{hocolim}_* G \longrightarrow \operatorname{hocolim}_* C$$

is a cofiber sequence.

(b) In $\mathcal{T}_\circ$, the induced maps fit into a strong homotopy pushout square

$$\begin{tikzcd}
\operatorname{hocolim}_\circ F \arrow[r] \arrow[d] & \operatorname{hocolim}_\circ G \arrow[d] \\
B\mathcal{I} \arrow[r] & \operatorname{hocolim}_\circ C.
\end{tikzcd}$$

Corollary 8.53 gives us what we need to compare pointed and unpointed homotopy colimits.

Corollary 8.54. *Let $F_* : \mathcal{I} \to \mathcal{T}_*$, and write $F_\circ$ for the composite of F with the forgetful functor $\mathcal{T}_* \to \mathcal{T}_\circ$. There is a cofiber sequence*

$$(B\mathcal{I})_+ \longrightarrow (\operatorname{hocolim}_\circ F_\circ)_+ \longrightarrow \operatorname{hocolim}_* F_*$$

of pointed spaces.

Problem 8.55. Prove Corollaries 8.53 and 8.54.

Problem 8.56. Show that $\operatorname{hocolim}_\circ F_\circ \simeq (\operatorname{hocolim}_* F_*)_-$ in $\mathcal{T}_\circ$ if $\mathcal{I}$ is simple.

8.7. Iterated Fibers and Cofibers

We have suggested that the fiber and cofiber of a map can be usefully interpreted as a measure of the deviation of the map from being a homotopy equivalence. In this section we produce spaces that measure the deviation of a commutative square from being a strong homotopy pushout square or a strong homotopy pullback square.

Consider the strictly commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\
C \arrow[r, "g"] & D.
\end{tikzcd}$$

The induced maps between the cofibers of the rows and those between the cofibers of the columns fit together to form the (strictly) commutative 3×3 diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[r] \arrow[d, "v"] & C_f \arrow[d] \\
C \arrow[r, "g"] \arrow[d] & D \arrow[r] \arrow[d] & C_g \arrow[d, dotted] \\
C_u \arrow[r] & C_v \arrow[r, dotted] & \boxed{???}.
\end{tikzcd}$$

How can we fill in the bottom right corner? There are two equally reasonable choices: the cofiber of $C_f \to C_g$ and the cofiber of $C_u \to C_v$. But which one should we choose?

If the original square is a strong homotopy pushout square, then the induced maps $C_f \to C_g$ and $C_u \to C_v$ are both homotopy equivalences, so their cofibers are contractible. This suggests that perhaps the two iterated cofibers always have the same homotopy type. That this is generally true is the content of the following theorem, which is sometimes called the **Cohen-Moore-Neisendorfer lemma**.

Theorem 8.57. *The cofibers of the maps $C_f \to C_g$ and $C_u \to C_g$ are homotopy equivalent. Writing Q for this common space, there are induced maps $C_g \to Q$ and $C_v \to Q$ making the square*

$$\begin{tikzcd} D \arrow[r] \arrow[d] & C_g \arrow[d] \\ C_v \arrow[r] & Q \end{tikzcd}$$

commutative. Dually, the fibers of $F_f \to F_g$ and $F_u \to F_v$ are homotoppy equivalent to a single space Q, and there is a commutative square

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & F_u \arrow[d] \\ F_f \arrow[r] & A. \end{tikzcd}$$

One excellent way to prove that two things are equivalent is to show that a single procedure constructs each of them. You'll prove Theorem 8.57 by showing that each of these cofibers is in fact the homotopy colimit of a certain 3×3 diagram.

Problem 8.58.

(a) Find a 3×3 diagram whose homotopy colimit is the space Q.

(b) Prove Theorem 8.57.

The space Q of Theorem 8.57 is known as the **iterated cofiber** of the square; dually, the fiber of the induced map of fibers is called the **iterated fiber** of the square. The idea that the iterated cofiber (or fiber) is a measure of how far the given square is from being a homotopy pushout (or pullback) square dovetails nicely with our previous suggestion about fibers and cofibers measuring the deviation of a map from being a homotopy equivalence.

Problem 8.59. Let P denote the homotopy pushout of $C \leftarrow A \rightarrow B$. Use the diagram morphism

$$\begin{tikzcd}
* & * \arrow[l] \arrow[r] & * & & * & * \arrow[l] \arrow[r] & * \\
* \arrow[u] \arrow[d] & \overline{A} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \overline{B} \arrow[u] \arrow[d] & \longrightarrow & * \arrow[u] \arrow[d] & A \arrow[l] \arrow[r] \arrow[u] \arrow[d] & B \arrow[u] \arrow[d] \\
* & \overline{C} \arrow[l] \arrow[r] & P & & * & C \arrow[l] \arrow[r] & D
\end{tikzcd}$$

to show that there is a cofiber sequence $P \to D \to Q$, where Q is the iterated cofiber of the square. State and prove the dual result.

Problem 8.60. Show that for any composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ there is a fiber sequence $F_f \to F_{g\circ f} \to F_g$ of homotopy fibers and a cofiber sequence $C_f \to C_{g\circ f} \to C_g$ of cofibers.

Problem 8.61. If $f : X \to Y$, the **graph** of f is the map $\mathrm{Graph}(f) : X \to X \times Y$ given by $x \mapsto (x, f(x))$. Determine the homotopy fiber of $\mathrm{Graph}(f)$.[1]

8.8. Group Actions

There is a vast theory of spaces with group actions. In this section we will introduce some of the basic notions of the theory, put it into a more categorical context, and then move to a homotopy-theoretical version.

8.8.1. G-Spaces and G-Maps. If a group G acts on a space $X \in \mathcal{T}_\circ$ (so that each $g : X \to X$ is a map in $\mathcal{T}_\circ$), then we call X a **G-space**. A map $f : X \to Y$ from one G-space to another is said to be **equivariant** if it commutes with the G-action: $f(g \cdot x) = g \cdot f(x)$ for every $x \in X$ and every $g \in G$ (such maps are also called **G-maps**).

A group G can be considered as a category with a single object $\star$ and morphism set $\mathrm{mor}(\star, \star) = G$. The diagram category $\mathcal{T}_\circ^G$ may be identified with the category whose objects are G-spaces in $\mathcal{T}_\circ$ and whose morphisms are G-maps.

Problem 8.62.

(a) Show that a G-action on $X \in \mathcal{T}_\circ$ is simply a functor $A : G \to \mathcal{T}_\circ$ with $A(\star) = X$.

(b) Show that a G-map is simply a map of diagrams.

[1] This problem came up in conversation with Ran Levi, who attributed it to Fred Cohen.

An action of a group G on a space X gives rise to an equivalence relation: two points in X are equivalent if they are in the same orbit. We may form the quotient of X by this relation, resulting in the **orbit space** X/G. Dually (in a sense to be made clear presently), we may consider the subspace of points of X that are fixed under the action and arrive at the **space of fixed points** X^G.

Problem 8.63.

(a) Show that the orbit space of the action A is $\operatorname{colim}_\circ A$ and that the space of fixed points of the action is $\lim_\circ A$.

(b) Show that the orbit space and fixed point space constructions define functors $O, F : \mathcal{T}_\circ^G \to \mathcal{T}_\circ$.

(c) Let $V : \mathcal{T}_\circ \to \mathcal{T}_\circ^G$ be the functor that assigns to X the same space with the trivial G-action. Show that
  (1) O is left adjoint to V and
  (2) V is left adjoint to F.

We say that an action of the group G on a CW complex X is a **cellular action** if $g : X \to X$ is a cellular map for each $g \in G$.

Problem 8.64. Show that if G acts cellularly, then for each n the action of G permutes the open n-cells of X.

Problem 8.64 does not imply that an element of g that sends each cell to itself must be acting trivially. It could be that g carries each cell to itself but acts nontrivially on the points in the cell.

Exercise 8.65. Find and example of a G-action on a CW complex that sends each cell to itself but does not act as the identity within the cells.

If G acts cellularly on the CW complex X with the additional property that g can carry a cell to itself only if it is the identity on that cell, then we call X a **G-CW complex**.

Problem 8.66. Show that if X is a G-CW complex, then the orbit space X/G inherits the structure of a CW complex from X and the quotient map $X \to X/G$ is a cellular map.

Exercise 8.67. Give an example of a cellular action for which the quotient X/G does not inherit a CW structure.

Construction of G-Cellular Maps. Next we show that G-maps can be constructed skeleton-by-skeleton. Suppose we have a G-map $f_n : X_n \to Y$ and we want to extend it to a G-map $f_{n+1} : X_{n+1} \to Y$. Since the action of G is cellular, it induces an action of G on the $(n+1)$-cells of X, so a lift defined on an $(n+1)$-cell of X determines the lift on all the other cells in its orbit.

Problem 8.68. Let X be a G-CW complex, and suppose $f_n : X_n \to Y$ is a G-map. If the stabilizer of the cell D^{n+1} is contained in the (pointwise) stabilizer of $f_n(D^{n+1})$, then the following are equivalent:

(a) f_n can be extended to a non-G-map $\phi : X_n \cup_\alpha D_1^{n+1} \to Y$,

(b) the composite $S^n \xrightarrow{\alpha} X_n \xrightarrow{f_n} Y$ is trivial, and

(c) f_n can be extended to a G-map defined on $X_n \cup \bigcup_{g\in G} g \cdot D^{n+1}$.

8.8.2. Homotopy Theory of Group Actions. Actions of groups on spaces can be rather nasty and lead to pathological and uninformative fixed point and orbit spaces. The solution is to use cofibrant or fibrant replacements for the given action (i.e., diagram) to form the **homotopy orbit** space and the **homotopy fixed point** space, which are, of course, the homotopy colimit and homotopy limit of the diagram.

We have shown that every diagram defined on a simple (or opposite-simple) category has a cofibrant (or fibrant) replacement, but we have not even considered the problems for other kinds of shape categories.

Exercise 8.69. Show that a nontrivial group G is neither a simple category nor an opposite-simple category, so our theorems guaranteeing the existence of cofibrant and fibrant replacements are moot.

Recognizing Cofibrant G-Actions. An action of a group G on a set X is **free** if for every $x \in X$ and every $g \neq 1 \in G$, $g \cdot x \neq x$. You will show in Section 16.5 that for every group G, there is a contractible free G-CW complex EG. These spaces play an extremely important role in the homotopy theory of groups actions.

Problem 8.70. Let X be a G-CW complex, and suppose the space EG exists. Show that the following are equivalent:

(1) there is a G-map $X \to EG$,

(2) X is a free G-CW complex.

Now we can start to see some necessary conditions for cofibrant G-spaces.

Problem 8.71. Suppose the space EG exists, and consider the diagram

$$\begin{tikzcd} & EG \arrow[d, "p"] \\ X \arrow[ur, dotted, bend left] \arrow[r] & * \end{tikzcd}$$

of G-maps, where X is a G-space. Show that if a G-space X is a cofibrant diagram, then the action of G on X must be free.

In fact, all free G-CW complexes *are* cofibrant.

Theorem 8.72. *A free G-CW complex is a cofibrant diagram in $(\mathcal{T}_\circ)^G$.*

Problem 8.73. Let $p : E \to B$ be a fibration and a homotopy equivalence which is a G-map, and consider the lifting problem

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dotted, "\phi"] \arrow[r, "f"'] & B \end{tikzcd}$$

in $(\mathcal{T}_\circ)^G$. Assume that X is a free G-CW complex.

(a) Show that the restriction of f to X_0 has a lift $\phi_0 : X_0 \to E$.

(b) Suppose that ϕ_n has been constructed lifting $f|_{X_n}$. Show that there is a G-map $\phi_{n+1} : X_{n+1} \to E$ lifting $f|_{X_{n+1}}$.

Hint. Use Problem 8.68.

(c) Show that X is cofibrant, thereby proving Theorem 8.72.

Constructing Cofibrant Replacements. Now that we can identify cofibrant G-actions, we are poised to construct cofibrant replacements for G-actions, assuming the existence of a contractible free G-CW complex EG.

The construction involves the **diagonal action** of G on the product $X \times Y$ of two G spaces, given by the formula $g \cdot (x, y) = (g \cdot x, g \cdot y)$.

Problem 8.74.

(a) Show that if G acts cellularly on X and Y, then (using the standard CW decomposition for $X \times Y$) G acts cellularly on $X \times Y$.

(b) Show that if either action is free, then the diagonal action is free.

(c) Show that if one of the spaces is a free G-CW complex, then the product is a free G-CW complex.

Now we can find cofibrant replacements for G-CW complexes.

Theorem 8.75. *Let G be a group acting cellularly on the CW complex X. Assume that there is a contractible CW complex EG on which G acts freely and cellularly. Then the projection $EG \times X \to X$ is an unpointed cofibrant replacement for the given action of G on X.*

Problem 8.76. Prove Theorem 8.75.

Theorem 8.75 does not establish the existence of cofibrant replacements for all G-spaces. Such replacements do exist, at least for G-spaces which are CW complexes, because every CW complex Y with a G-action has a **G-CW replacement**: an equivariant homotopy equivalence $X \to Y$ where X is a G-CW complex.

Problem 8.77. Show that if Y has a G-CW replacement, then Y has a cofibrant replacement (assuming that EG exists).

The **homotopy orbit space** of a G-CW complex is the ordinary orbit space $(EG \times X)/G$; it is denoted $EG \times_G X$ and is called the **Borel construction** on X. Once the space EG has been decided upon, the construction is functorial in X. We can even make the construction of EG depend functorially on G, so that the homotopy orbit space is a functor of G as well.

Of particular importance is the special case $X = *$ with the only possible action: the trivial one. The unpointed homotopy orbit space is simply the orbit space EG/G, which is denoted BG.

Problem 8.78. Determine the (unpointed) homotopy orbit space of the space X with the trivial G-action.

8.8.3. Homotopy Colimits of Pointed G-Actions. An action of a group G on a space $X \in \mathcal{T}_*$ must respect the basepoint, so the action cannot be free. Thus we are forced to find a new criterion for cofibrant G-actions. Luckily, we do not have to look too far.

Proposition 8.79. *Let X be a pointed G-CW complex. If the action of G on $X - \{*\}$ is free, then X is a cofibrant G-shaped diagram.*

Corollary 8.80. *If X is a CW complex with a cellular G-action, then the map $p : EG \ltimes X \to X$ is a pointed cofibrant replacement.*

To show that X is cofibrant, you need to show that there is a lift in any diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[r, "f"'] \arrow[ur, dotted, bend left, "\lambda"] & B \end{tikzcd}$$

in which p is a G-equivariant pointed homotopy equivalence and pointed fibration. The value of λ on $* \in X$ is forced to be $* \in E$.

Problem 8.81.

(a) Assuming that X is as in Proposition 8.79, build the lift step-by-step, just as in the proof of Theorem 8.72.

(b) Derive Corollary 8.80.

Problem 8.82. Let $X \in \mathcal{T}_*$ be a space with a trivial action of the group G. Show that $\operatorname{hocolim}_* X = BG \ltimes X$.

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Chapter 9

Topics and Examples

This chapter is devoted to introducing a wide variety of interesting and important topics and examples. Many of these ideas will be used frequently later in the book; others will show themselves only occasionally.

9.1. Homotopy Type of Joins and Products

In this section we introduce a construction called the **join** of two spaces. Joins are defined as homotopy pushouts, but surprisingly they appear quite frequently in the study of homotopy fibers of maps. We determine the homotopy type of the join in terms of smash products and suspensions, and we show how the join figures in the splitting of $\Sigma(X \times Y)$ as a wedge sum of other spaces. Then we turn to the study of the homotopy type of a product of two mapping cones, which leads us to the Whitehead product.

9.1.1. The Join of Two Spaces. The **join** of X and Y is defined up to homotopy type as the homotopy pushout in the square

$$\begin{tikzcd} X \times Y \arrow[r, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_Y"'] \arrow[dr, phantom, "\mathrm{HPO}"] & X \arrow[d] \\ Y \arrow[r] & X * Y. \end{tikzcd}$$

We define it up to homeomorphism as the *standard* homotopy pushout of the diagram $Y \leftarrow X \times Y \rightarrow X$.

This construction can be carried out in either the pointed category $\mathcal{T}_*$ or the unpointed category $\mathcal{T}_\circ$. The classical notion is the unpointed one, but we'll need both. When we need to specify, we'll use the terms 'unpointed join' or 'pointed join'.

Exercise 9.1.

(a) Show that the unpointed join $X * Y$ is homeomorphic to the quotient of $X \times I \times Y$ by the equivalence relation given by

$$(x, 1, y) \sim (x, 1, y'), \qquad\qquad (x, 0, y) \sim (x', 0, y)$$

for any $x, x' \in X$ and $y, y' \in Y$.

(b) Show that the unpointed join can also be described as the quotient of the set

$$X \;\cup\; \{t \cdot x + (1-t) \cdot y \mid x \in X, y \in Y, t \in I\} \;\cup\; Y$$

by the equivalence relation

$$1 \cdot x + 0 \cdot y \sim x \in X \qquad \text{and} \qquad 0 \cdot x + 1 \cdot y \sim y \in Y.$$

(c) Find an inclusion $X * Y \hookrightarrow CX \times CY$.

(d) How does the pointed join differ?

This description of the join given in Exercise 9.1(b) explains its name: $X * Y$ is the union of line segments joining each point $x \in X$ to each point $y \in Y$.

Problem 9.2. Let $X, Y \in \mathcal{T}_*$.

(a) Show that the square

$$\begin{tikzcd} X \vee Y \arrow[r, "(\mathrm{id}_X, *)"] \arrow[d, "(*, \mathrm{id}_Y)"'] & X \arrow[d] \\ Y \arrow[r] & * \end{tikzcd}$$

is a homotopy pushout square. Is it a categorical pushout square?

(b) Show that there is a homotopy equivalence $X * Y \xrightarrow{\simeq} \Sigma(X \wedge Y)$.

Exercise 9.3. Are the spaces $X * Y$ and $\Sigma(X \wedge Y)$ homeomorphic?

9.1.2. Splittings of Products. The suspension of a product splits into a wedge of simpler spaces.

Problem 9.4.

(a) Show that the maps $X \to X * Y$ and $Y \to X * Y$ (in the homotopy pushout square that defines the join) are nullhomotopic.

(b) Use a Mayer-Vietoris sequence to express the space $\Sigma(X \times Y)$ as a wedge of other spaces.

(c) Identify the maps

$$\Sigma(\mathrm{in}_X) : \Sigma(X) \hookrightarrow \Sigma(X \times Y) \qquad \text{and} \qquad \Sigma(\mathrm{pr}_X) : \Sigma(X \times Y) \to \Sigma X$$

and the maps $\Sigma(\mathrm{pr}_X)$ and $\Sigma(\mathrm{pr}_Y)$ in terms of your answer to part (b).

(d) Determine the cofibers of $\mathrm{pr}_Y : X \times Y \to Y$ and $\mathrm{pr}_X : X \times Y \to X$.

The formula you derived in Problem 9.4(b) shows that products **split** as a wedge after one suspension. Some spaces require more than one suspension to split, and still others never split.

Problem 9.5. Using Problem 9.4(b), write down a splitting of the n-fold product $\Sigma(X_1 \times X_2 \times \cdots \times X_n)$.

It is easy enough to dualize this. The smash product is the cofiber of $X \vee Y \to X \times Y$, so the dual is the fiber of $X \vee Y \to X \times Y$. One of the early notations for the smash product was the musical 'sharp' symbol: $X\sharp Y$. Authors who used this notation were naturally led to use the 'flat' symbol $\flat$ for the dual operation, and so you will sometimes see the homotopy fiber of $X \vee Y \to X \times Y$ denoted $X\flat Y$ (and even referred to as the 'flat product'). Both of these notations have fallen out of use. Eventually, you will determine the homotopy type of this fiber in terms of the basic constructions we already know, so we won't need any permanent notation for this space.

9.1.3. Products of Mapping Cones. In Chapter 3 we worked out an explicit CW decomposition for a product of CW complexes. This ultimately rested on the homeomorphism $D^n \times D^m \cong D^{n+m}$, which can be generalized to products of cones.

Problem 9.6. Show that there is a homeomorphism $C(A*B) \xrightarrow{\cong} CA \times CB$ that fits into a commutative square

$$\begin{tikzcd}
A * B \arrow[r, equal] \arrow[d] & A * B \arrow[d] \\
C(A * B) \arrow[r, "\cong"] & CA \times CB.
\end{tikzcd}$$

This suggests that we should be able to generalize our decomposition of products of CW complexes to products of mapping cones. Let $f : A \to X$ and $g : B \to Y$; inside the product $C_f \times C_g$ we have a subspace $T(f, g)$ defined by the pushout square

$$\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & X \times C_g \arrow[d] \\
C_f \times Y \arrow[r] & T(f, g).
\end{tikzcd}$$

We will determine precisely how $C_f \times C_g$ is constructed from $T(f, g)$.

Problem 9.7. Let $f : A \to X$ and let $g : B \to Y$.

(a) Show that the inclusion $T(f, g) \hookrightarrow C_f \times C_g$ is a cofibration. What is its cofiber?

(b) Show that $T(\mathrm{id}_A, \mathrm{id}_B) = A * B$.

(c) Let $f : A \to *$ and $g : B \to *$; then show $T(f, g) = \Sigma A \vee \Sigma B$.

Here is our main result.[1]

Proposition 9.8. *Let $f : A \to X$ and let $g : B \to Y$. Then there is a cofiber sequence*

$$A * B \xrightarrow{w} T(f,g) \hookrightarrow C_f \times C_g$$

which is functorial in both f and g. More precisely, $C_f \times C_g$ is homeomorphic to the standard cofiber of $A * B \to T(f, g)$.

Exercise 9.9. Write out explicitly what it means for the sequence to be functorial in f and g. What categories are involved? What functors?

Problem 9.10. Consider the diagram

$$\begin{tikzcd}
A \times Y \arrow[rr] \arrow[dd] \arrow[dr] & & CA \times Y \arrow[dd] \arrow[dr] & & \\
& A \times C_g \arrow[rr, crossing over] \arrow[dd] & & T(\mathrm{id}_A, g) \arrow[r] \arrow[dd] & CA \times C_g \arrow[dd] \\
X \times Y \arrow[rr, crossing over] \arrow[dr] & & C_f \times Y \arrow[dr] & & \\
& X \times C_g \arrow[rr] & & T(f,g) \arrow[r] & C_f \times C_g.
\end{tikzcd}$$

(a) Show that the rightmost square is a pushout square.

(b) Show that there is a pushout square

$$\begin{tikzcd}
A * B \arrow[r] \arrow[d, "w"'] \arrow[dr, phantom, "\text{pushout}"] & CA \times CB \arrow[d] \\
T(f,g) \arrow[r] & C_f \times C_g.
\end{tikzcd}$$

(c) Prove Proposition 9.8.

---

[1]This approach is due to Don Stanley [**155**].

9.1.4. Whitehead Products. For $f : A \to *$ and $g : B \to *$ our decomposition asserts the existence of a cofiber sequence

$$A * B \xrightarrow{w} \Sigma A \vee \Sigma B \longrightarrow \Sigma A \times \Sigma B,$$

and we understand all the spaces in the sequence. The map $w : A*B \to \Sigma A \vee \Sigma B$ in this sequence is the universal example of a (generalized) Whitehead product. The **generalized Whitehead product** is a binary operation

$$[\Sigma A, X] \times [\Sigma B, X] \longrightarrow [A * B, X]$$

taking maps $\alpha : \Sigma A \to X$ and $\beta : \Sigma B \to X$ is the map $[\alpha, \beta]$ and returning the composition

$$\begin{tikzcd} A * B \arrow[rr, "{[\alpha,\beta]}"] \arrow[dr, "w"'] & & X \\ & \Sigma A \vee \Sigma B. \arrow[ur, "{(\alpha,\beta)}"'] & \end{tikzcd}$$

Thus in particular $w = [\mathrm{id}_A, \mathrm{id}_B]$.

Problem 9.11. Let $\alpha \in [\Sigma A, X]$ and $\beta \in [\Sigma B, X]$.

(a) Show that the Whitehead product $[\alpha, \beta]$ is well-defined up to homotopy.

(b) Show that the Whitehead product of $\alpha : \Sigma A \to X$ and $\beta : \Sigma B \to X$ is trivial if and only if the map $(\alpha, \beta) : \Sigma A \vee \Sigma B \to X$ extends (up to homotopy) to a map $\Sigma A \times \Sigma B \to X$.

Whitehead products were first defined for homotopy classes $\alpha, \beta \in \pi_*(X)$. Let's look at that case in more detail.

Problem 9.12.

(a) Show that the Whitehead product defines a homomorphism

$$\pi_n(X) \times \pi_m(X) \longrightarrow \pi_{n+m-1}(X).$$

(b) Show that in the case $n = m = 1$, $[\alpha, \beta]$ is the commutator $\alpha\beta\alpha^{-1}\beta^{-1}$.

HINT. View $S^1 \times S^1$ as a square with parallel sides identified.

9.2. H-Spaces and co-H-Spaces

There are many spaces which have part of the structure needed to make them group objects in $\mathrm{H}\mathcal{T}_*$, but not all of it; such spaces are called *H-spaces.* The 'H' stands for Heinz Hopf, who made an early and very influential study of these spaces. The dual—spaces that are not quite cogroup objects—is called, unimaginatively, *co-H-spaces.*

9.2.1. H-Spaces. A space X is called an **H-space** if it comes equipped with a map $\mu : X \times X \to X$, called its **multiplication**, that makes the diagram

$$\begin{tikzcd} X \vee X \arrow[d] \arrow[dr, bend left, "\nabla"] & \\ X \times X \arrow[r, "\mu"] & X \end{tikzcd}$$

commute up to homotopy. This diagram expresses the condition that the basepoint $* \in X$ should be, up to homotopy, a multiplicative identity element for μ. A map $f : X \to Y$ from one H-space to another is an **H-map** (or a **homomorphism** of H-spaces) if the square

$$\begin{tikzcd} X \times X \arrow[r, "f \times f"] \arrow[d, "\mu"'] & Y \times Y \arrow[d, "\mu"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes up to homotopy.

Problem 9.13. Show that if X is a well-pointed H-space, then X has a multiplication in which $* \in X$ is a multiplicative identity on the nose:

$$\mu(x, *) = \mu(*, x) = x$$

for all $x \in X$.

Problem 9.14. Let X be an H-space.

(a) Show that $[A, X]$ has a natural multiplication with $*$ as multiplicative unit.

(b) Show that if $f : X \to Y$ is an H-map, then the natural transformation $f_* : [\,?\,, X] \to [\,?\,, Y]$ respects multiplication.

Problem 9.15. Show that X is an H-space if and only if in any diagram of the form

$$\begin{tikzcd} A \vee B \arrow[r, "(f,g)"] \arrow[d] & X \\ A \times B \arrow[ur, dotted] & \end{tikzcd}$$

the dotted arrow can be filled in to make the diagram commute up to homotopy. Can it be filled in to commute on the nose?

Problem 9.16. Show that if X is an H-space, then every Whitehead product $[\alpha, \beta] \in \pi_*(X)$ is trivial.

Problem 9.17. Show that if X is a retract of Y (up to homotopy) and Y is an H-space, then X is also an H-space. Is there necessarily an H-map from X to Y or from Y to X?

Problem 9.18. Show that if X and Y are H-spaces and $f : X \to Y$ is an H-map, then the homotopy fiber F is also an H-space.

Problem 9.19. Suppose X and Y are H-spaces.

(a) Show that $X \times Y$ is an H-space. Explicitly write down the multiplication $\mu_{X\times Y}$ in terms of μ_X and μ_Y.

HINT. How do you define multiplication on the product of two groups? Write that down in diagram form.

(b) Show that the maps $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ are H-maps.

(c) Show that the natural isomorphism $[A, X\times Y] \to [A, X]\times[A, Y]$ respects multiplication.

Algebraic Bells and Whistles. H-spaces have only the bare minimum algebraic structure and so are very far from being group objects in $\mathrm{H}\mathcal{T}_*$. Many H-spaces actually have a bit more structure, even if they are not grouplike.

An H-space X is called **associative** if the square

$$\begin{tikzcd} X \times X \times X \arrow[r, "\mathrm{id}_X\times\mu"] \arrow[d, "\mu\times\mathrm{id}_X"'] & X \times X \arrow[d, "\mu"] \\ X \times X \arrow[r, "\mu"] & X \end{tikzcd}$$

commutes up to homotopy. An associative H-space is a **monoid object** in $\mathrm{H}\mathcal{T}_*$.

A space X is called a **commutative** H-space if the diagram

$$\begin{tikzcd} X \times X \arrow[rr, "T"] \arrow[dr, "\mu"'] & & X \times X \arrow[dl, "\mu"] \\ & X & \end{tikzcd}$$

commutes up to homotopy (where T is the twist map $(x, y) \mapsto (y, x)$).

Problem 9.20. Let X be an H-space.

(a) Show that if X is associative, then the functor $[\,?\,, X]$ takes its values in the category of monoids and their homomorphisms.

(b) Show that if X is commutative, then $[\,?\,, X]$ takes its values in the category of abelian monoids.

Exercise 9.21. If X is a homotopy retract of an H-space Y, then X is also an H-space. What can you say about X if Y is commutative? Or if Y is associative?

Problem 9.22. Show that an H-space X is commutative if and only if the functor $[\,?\,, X]$ takes its values in the category of pointed sets with commutative multiplications.

9.2.2. Co-H-Spaces. Now we turn to the dual notion. A **co-H-space** is a space X with a map $\phi : X \to X \vee X$ making the diagram

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, bend right, "\Delta"'] & X \vee X \arrow[d] \\ & X \times X \end{tikzcd}$$

commute up to homotopy.

Exercise 9.23. Formulate definitions for commutative and associative co-H-spaces.

Problem 9.24. Show that if X is a co-H-space, then $[X, \,?\,]$ takes its values in the category of pointed sets with unital multiplications.

Problem 9.25.

(a) Formulate and prove the dual of Problem 9.15.

(b) Show that a retract of a co-H-space is also a co-H-space.

Problem 9.26. Let A be a co-H-space and let X, Y be any two spaces. Show that the projections $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ induce a multiplicative isomorphism

$$[A, X \times Y] \longrightarrow [A, X] \times [A, Y].$$

Problem 9.27. Suppose X and Y are co-H-spaces.

(a) Show that $X \vee Y$ is a co-H-space. Explicitly write down the comultiplication $\phi_{X \vee Y}$ in terms of ϕ_X and ϕ_Y.

HINT. How do you define multiplication on the product of two groups? Write that down in diagram form.

(b) Show that the natural isomorphism $[X \times Y, Z] \to [X, Z] \times [Y, Z]$ respects multiplication.

Problem 9.28. Show that a co-H-space A is cocommutative if and only if the functor $[A, \,?\,]$ takes its values in the category of pointed sets with commutative multiplications.

9.2.3. Maps from Co-H-Spaces to H-Spaces. When A is a co-H-space and X is an H-space, then the set $[A, X]$ inherits two multiplications: one from A and one from X. How do they compare?

Problem 9.29. Suppose A is a co-H-space and X is an H-space. Use the diagram

$$\begin{tikzcd} A \vee A \arrow[r, "\alpha\vee\beta"] \arrow[d, "\mathrm{in}"'] & X \vee X \arrow[d, "\mathrm{in}"] \\ A \times A \arrow[r, "\alpha\times\beta"] & X \times X \end{tikzcd}$$

to show that the two multiplications on $[A, X]$ are the same.

Much more is true about $[A, X]$, but to get at it we need a different approach, called the **Eckmann-Hilton argument**. Let X be an H-space with multiplication μ and let A be a co-H-space with comultiplication ϕ. Since the product on $[A, X]$ is induced by μ and ϕ, we see that if $f : B \to A$ and $g : X \to Y$, then the induced maps

$$[B, X] \xleftarrow{f^*} [A, X] \xrightarrow{g_*} [A, Y]$$

are *both* homomorphisms (of multiplicative pointed sets).

Problem 9.30. Let A be a co-H-space with comultiplication ϕ and let X be an H-space with multiplication μ.

(a) Show that the product $[A, X] \times [A, X] \to [A, X]$ is a homomorphism.

(b) Show that the 'interchange formula' $(\alpha\beta)(\gamma\delta) = (\alpha\gamma)(\beta\delta)$ holds for all $\alpha, \beta, \gamma, \delta \in [A, X]$.

HINT. Use Problem 1.23

(c) Use the interchange formula to show that multiplication on $[A, X]$ is associative and commutative.

This has some easy implications for the homotopy groups of H-spaces.

Problem 9.31. Let X be an H-space.

(a) Show that $\pi_1(X)$ is an abelian group.

(b) Show that if $\alpha : S^n \to S^m$, then the induced map $\alpha^* : \pi_m(X) \to \pi_n(X)$ is a homomorphism.

Problem 9.32. Let X be an H-space with multiplication $\mu : X \times X \to X$. The **shear map** $s : X \times X \to X \times X$ is the map $s = (\mathrm{pr}_1, \mu)$. Using the canonical identification $\pi_*(X \times X) \cong \pi_*(X) \times \pi_*(X)$, identify the induced maps μ_* and s_*.

Problem 9.33.

(a) Show that if A is a co-H-space or if X is an H-space, then $\mathrm{map}_*(A, X)$ is an H-space.

(b) Show that if A is a co-H-space or if X is an H-space, the exponential law isomorphism $[\,?\wedge A, X] \xrightarrow{\cong} [\,?\,, \mathrm{map}_*(A, X)]$ respects multiplication.

(c) Show that if A and B are both co-H-spaces, then $A \wedge B$ is a cocommutative co-H-space.

This is a new proof that S^n has the structure of a cocommutative co-H-space for $n \geq 2$.

9.3. Unitary Groups and Their Quotients

In this section we introduce some more important examples, starting with the **unitary groups**. We'll use 'unitary groups' as a blanket term for the groups of inner product preserving $\mathbb{F}$-linear transformations $V \to V$. These groups may be identified with groups of matrices, which gives them a topology and makes them *topological* groups. The unitary group on a vector space V acts transitively on the set of k-dimensional subspaces of V, as well as on the ordered lists of k orthonormal vectors in V (called k-frames), and by virtue of the orbit-stabilizer theorem, these sets inherit a topology. We conclude by constructing explicit CW decompositions of these spaces.

Recall that $\mathbb{F}$ stands for any of $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and that $d = \dim_{\mathbb{R}}(\mathbb{F})$.

9.3.1. Orthogonal, Unitary and Symplectic Groups. Let V be a finite-dimensional $\mathbb{F}$-vector space with an inner product, denoted $\langle\,?\,,\,?\,\rangle$. We write

$$G(V) = \{T : V \xrightarrow{\cong} V \mid \langle T(x), T(y)\rangle = \langle x, y\rangle \text{ for all } x, y \in V\}$$

for the group of invertible linear transformations that preserve the inner product.

Since V is finite-dimensional, we may find an orthonormal basis for V and identify $G(V)$ with a subset of the collection $M_{n\times n}(\mathbb{F})$ of all $n \times n$ matrices with entries in $\mathbb{F}$.

Problem 9.34. Show that $A \in M_{n\times n}(\mathbb{F})$ is in $G(V)$ if and only if its columns form an orthonormal set of vectors in V.

Problem 9.35. If V is finite-dimensional over $\mathbb{R}$ or $\mathbb{C}$, then we may define the determinant of $T : V \to V$ to be the determinant of the corresponding matrix in $M_{n\times n}(\mathbb{F})$.[2]

[2]It is standard linear algebra that this is well-defined.

(a) Show that if $T \in G(V)$, then $|\det(T)| = 1$.

(b) Show that $G_1(V) = \{T \in G(V) \mid \det(T) = 1\}$ is a subgroup of $G(V)$.

(c) Show that if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, there are homeomorphisms $G(V) \cong S^{d-1} \times G_1(V)$. Are they group isomorphisms?

When $\mathbb{F} = \mathbb{C}$, $G(V)$ is called the **unitary group** of the vector space V and is denoted $U(V)$; the subgroup $G_1(V)$ is the **special unitary group**, denoted $SU(V)$. For $\mathbb{F} = \mathbb{R}$, $G(V)$ is called the **orthogonal group** of V and is denoted $O(V)$; and $G_1(V)$ is called the **special orthogonal group**, denoted $SO(V)$. The group $G(V)$ is called the **symplectic group** when $\mathbb{F} = \mathbb{H}$ and denoted $Sp(V)$; there is no 'special' symplectic group.

If $V = \mathbb{F}^n$, the group $G(V)$ may be abbreviated $G(n)$; correspondingly, you will often encounter $U(n)$, $O(n)$ and $Sp(n)$, as well as their 'special' counterparts $SU(n)$, $SO(n)$.

Embedding $G(X)$ into $G(V)$. If X is a subspace of the inner-product space V, there exists a unique subspace Y, called the **complement** of X, such that $V = X + Y$ and $X \perp Y$. Then every $T \in G(X)$ gives rise to a transformation $\widetilde{T} : V \to V$ defined by the formula

$$\widetilde{T}(x + y) = T(x) + y.$$

Problem 9.36. Suppose $V = X + Y$, where $X \perp Y$.

(a) Show that $V \cong X \oplus Y$.

(b) Show that $\widetilde{T} \in G(V)$.

(c) Show that the map $G(X) \to G(V)$ given by $T \mapsto \widetilde{T}$ is an injective homomorphism.

Because of Problem 9.36, we can always treat $G(X)$ as a subgroup of $G(V)$, and we will do so without comment in the future. Specializing to $\mathbb{F}^n \oplus 0 \hookrightarrow \mathbb{F}^{n+1}$, we have canonical inclusions $U(n) \to U(n+1)$, and so on.

Quotients of Unitary Groups. In Section 3.3 we defined the projective space $\mathbb{F}\mathrm{P}^n$ as the space of all 1-dimensional $\mathbb{F}$-subspaces of $\mathbb{F}^{n+1}$. At least on the face of it, there is nothing special about the 1-dimensional subspaces of $\mathbb{F}^{n+1}$. Why not consider the space of k-dimensional subspaces of $\mathbb{F}^{n+k}$? The **Grassmannian** of d-planes in $\mathbb{F}^{n+k}$ is denoted

$$\mathrm{Gr}_k(\mathbb{F}^{n+k}) = \{k\text{-dimensional } \mathbb{F}\text{-subspaces of } \mathbb{F}^{n+k}\}.$$

Note that $\mathbb{F}\mathrm{P}^n = \mathrm{Gr}_1(\mathbb{F}^{n+1})$, at least as sets; we have yet to define a topology on $\mathrm{Gr}_k(\mathbb{F}^{n+k})$.

Problem 9.37. Let $G(\mathbb{F}^{n+k})$ act on $\mathrm{Gr}_k(\mathbb{F}^{n+k})$ by the rule $T \cdot V = T(V)$.

(a) Show that the action is transitive.

(b) Determine the stabilizer of $V = \operatorname{span}\{\mathbf{e}_{n+1}, \mathbf{e}_{n+2}, \ldots, \mathbf{e}_{n+k}\}$.

(c) Show that there is a bijection $G(\mathbb{F}^n \oplus \mathbb{F}^k)/G(\mathbb{F}^n) \times G(\mathbb{F}^k) \to \mathrm{Gr}_k(\mathbb{F}^{n+k})$.

The quotient map $S^{(n+1)d-1} \to \mathbb{F}\mathrm{P}^n$ is given by $v \mapsto \operatorname{span}\{v\}$. Thus the preimage of a line ℓ is $\ell \cap S^{(n+1)d-1}$. This may be interpreted in may ways, but we choose to view the preimage of ℓ as the set of all orthonormal bases of ℓ. To generalize, we need a space whose points are orthonormal sets of k vectors in $\mathbb{F}^{n+k}$. A **k-frame** in $\mathbb{F}^{n+k}$ is an $(n+k) \times k$ matrix $[v_1, v_2, \ldots, v_k]$ with orthonormal columns. The **Stiefel manifold** of k-frames in $\mathbb{F}^{n+k}$ is denoted

$$V_k(\mathbb{F}^{n+k}) = \{k\text{-frames in } \mathbb{F}^{n+k}\}.$$

A k-frame may thus be interpreted as an inner-product preserving linear map $\mathbb{F}^k \to \mathbb{F}^{n+k}$.

Problem 9.38. Show that $G(\mathbb{F}^{n+k})$ acts transitively on $V_k(\mathbb{F}^{n+k})$ and determine the stabilizer of the frame $[\mathbf{e}_{n+1}, \mathbf{e}_{n+2}, \ldots, \mathbf{e}_{n+k}]$.

Now we have the generalization of the quotient map $S^{(n+1)d-1} \to \mathbb{F}\mathrm{P}^n$.

Problem 9.39. Show that the rule

$$[v_1, v_2, \ldots, v_k] \longmapsto \operatorname{span}\{v_1, v_2, \ldots, v_k\}$$

defines a surjective function $V_k(\mathbb{F}^{n+k}) \to \mathrm{Gr}_k(\mathbb{F}^{n+k})$. Show that the preimage of $V \in \mathrm{Gr}_k(\mathbb{F}^{n+k})$ may be identified with $G(V)$.

9.3.2. Topology of Unitary Groups and Their Quotients. Using the orbit-stablizer theorem, we have established bijections

$$G(\mathbb{F}^n \oplus \mathbb{F}^k)/(G(\mathbb{F}^n) \times G(\mathbb{F}^k)) \xrightarrow{\cong} \mathrm{Gr}_k(\mathbb{F}^{n+k})$$

and

$$G(\mathbb{F}^n \oplus \mathbb{F}^k)/G(\mathbb{F}^n) \xrightarrow{\cong} V_k(\mathbb{F}^{n+k})$$

(writing $G(\mathbb{F}^n)$ for $G(\mathbb{F}^n \oplus 0)$ and $G(\mathbb{F}^k)$ for $G(0 \oplus \mathbb{F}^k)$). Once we have established a topology on $G(V)$, we will be able to topologize the Grassmann and Stiefel manifolds as quotient spaces.

Since $M_{n\times n}(\mathbb{F}) \cong \mathbb{F}^{n^2}$, it has a natural topology, and we give $G(V)$ the subspace topology. On the face of it, this topology could vary depending on the choice of basis. But in fact the topology on $G(V)$ is independent of the basis used to define it.

Proposition 9.40. *The topology of $G(V)$ is independent of the choice of basis.*

Problem 9.41. Prove Proposition 9.40 in two ways.

(1) If V is an inner product space over $\mathbb{F}$, then V has a metric and hence a topology. Then $G(V) \subseteq \operatorname{map}(V, V)$ inherits a topology.

(2) Show that an inner-product preserving isomorphism $T : V \to W$ induces a homeomorphism of $G(V)$ (with topology defined by basis $\mathcal{B}$) with $G(W)$ (with topology defined by basis $T(\mathcal{B})$).

Now our unitary groups are unambiguously topological spaces.

Problem 9.42. Let V be an n-dimensional inner product space over $\mathbb{F}$.

(a) Show that $G(V)$ is a compact Hausdorff topological space, so $G(V)$, with the identity transformation as basepoint, is a space in $\mathcal{T}_*$.

(b) Show that $G(V)$ is a topological group under composition.

Just as we topologized $\mathbb{F}\mathrm{P}^n$ as a quotient of $S^{(n+1)d-1}$, we can topologize $\mathrm{Gr}_d(\mathbb{F}^{n+d})$ as a quotient of $V_d(\mathbb{F}^{n+d})$.

Problem 9.43. Show that $V_d(\mathbb{F}^{n+d}) \to \mathrm{Gr}_d(\mathbb{F}^{n+d})$ is a quotient map.

9.3.3. Cellular Structure for Unitary Groups. We will exhibit an explicit CW decomposition for the unitary groups. The method is to construct maps $Q_n \to G(n)$, where Q_n has a known cellular structure. This gives us some cells in $G(n)$, and we produce more cells by multiplying them together; a certain collection of these 'product cells' is our decomposition.

The Map $Q_n \to G(n)$. Let W be an $\mathbb{F}$-inner product space. Given a subspace $V \subseteq W$ and a scalar $z \in S^{d-1}$, define $T_{z,V} : W \to W$ by the rule

$$T_{z,V}(x) = \begin{cases} z \cdot x & \text{if } x \in V, \\ x & \text{if } x \in V^{\perp}. \end{cases}$$

Problem 9.44. Show that the function $T : S^{d-1} \times \mathrm{Gr}_k(W) \to G(W)$ given by $T : (z, V) \mapsto T_{z,V}$ is well-defined and continuous.

For our present purposes, we are interested in the composite

$$\begin{tikzcd} S^{d-1} \times S^{nd-1} \arrow[rr, "\alpha"] \arrow[dr, "\mathrm{id}\times\mathrm{span}"'] & & G(\mathbb{F}^n) \\ & S^{d-1} \times \mathrm{Gr}_1(\mathbb{F}^n) \arrow[ur, "T"'] & \end{tikzcd}$$

where $\mathrm{span} : v \mapsto \mathrm{span}(v)$.

Problem 9.45.

(a) Let $Q_n = \mathrm{Im}(\alpha) \subseteq G(n)$. Show that Q_n is compact and Hausdorff, and conclude that $S^{d-1} \times S^{nd-1} \to Q_n$ is a quotient map.

(b) Show that there is a commutative ladder

$$\begin{tikzcd}
\cdots \arrow[r] & S^{d-1} \times S^{nd-1} \arrow[r] \arrow[d] & S^{d-1} \times S^{(n+1)d-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & G(n) \arrow[r] & G(n+1) \arrow[r] & \cdots .
\end{tikzcd}$$

Now we want to define cellular structures on Q_n so that the inclusions $Q_n \hookrightarrow Q_{n+1}$ are cellular. Let $E \subseteq S^{d-1} \times S^{(n+1)d-1}$ be the subspace of all (z, v) where the $(n+1)^{\text{st}}$ coordinate of v is a nonnegative real number (we view $S^{(n+1)d-1}$ as a subspace of $\mathbb{F}^{n+1}$).

Problem 9.46.

(a) Show that $E \cong D^{(n+1)d-1}$ and that its boundary lies in $S^{d-1} \times S^{nd-1}$.

(b) Show that the square

$$\begin{tikzcd}
S^{(n+1)d-2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & E \arrow[d] \\
Q_n \arrow[r] & Q_{n+1}
\end{tikzcd}$$

is a pushout square.

(c) Conclude that Q_n has a CW decomposition

$$\begin{aligned}
Q_{n+1} &= Q_n \cup D^{(n+1)d-1} \\
&= * \cup D^{d-1} \cup D^{2d-1} \cup \cdots \cup D^{nd-1} \cup D^{(n+1)d-1}.
\end{aligned}$$

In fact, the spaces Q_n can be identified explicitly.

Problem 9.47.

(a) Show that $Q_{n+1} = \mathbb{R}\mathrm{P}^n_+$ if $\mathbb{F} = \mathbb{R}$.

(b) Show that $Q_{n+1} = \Sigma(\mathbb{C}\mathrm{P}^n_+)$ if $\mathbb{F} = \mathbb{C}$.

Project 9.48. What can you say about Q_n if $\mathbb{F} = \mathbb{H}$?

Cellular Structure for $G(n)$. Now we are ready to establish the CW structure on the unitary groups. Write $\chi_i : D^{id-1} \to G(n)$ for the composition of the characteristic map of the $(id-1)$-cell of Q_n with the inclusion $Q_n \hookrightarrow G(n)$.

For any ordered subset $I \subseteq \{1, \ldots, n\}$, write $|I| = \sum_{i \in I}(di - 1)$, and define a map $\chi_I : D^{|I|} \to G(n)$ as the composite

$$\begin{tikzcd}
D^{|I|} \arrow[r, "\chi_I"] \arrow[d, "\cong"'] & G(n) \\
\prod_{i \in I} D^{id-1} \arrow[r, "\prod \chi_i"] & \prod_I G(n). \arrow[u, "\mu"']
\end{tikzcd}$$

Theorem 9.49. *The maps χ_I are the characteristic maps for a CW decomposition of $G(n)$.*

Problem 9.50.

(a) Show that each map χ_I is injective on its interior.

(b) Show that images of the interiors are pairwise disjoint and cover $G(n)$.

(c) Show that χ_I carries the boundary of $D^{|I|}$ into the union of the cells of dimension less than $|I|$.

It follows from Problem 9.50 that the set $G(n)$ has a topology for which the maps χ_I define a CW structure. We'll write $G(n)_{CW}$ for the set $G(n)$ with this topology. Since each map χ_I is continuous, there is a comparison map $\xi : G(n)_{CW} \to G(n)$.

Problem 9.51. Prove Theorem 9.49 by showing that ξ is a homeomorphism.

A similar approach can be used to construct a CW decomposition for Stiefel manifolds.

Project 9.52. For $I \subseteq \{1, 2, \ldots, n+k\}$, write $\bar{\chi}_I : D^{|I|} \to V_k(\mathbb{F}^{n+k})$ for the composite $D^{|I|} \xrightarrow{\chi_I} G(n+k) \to V_d(\mathbb{F}^{n+k})$. Show that the maps $\bar{\chi}_I$ for $\{1, 2, \ldots, k\} \subseteq I$ are the characteristic maps for a CW decomposition of $V_k(\mathbb{F}^{n+k})$.

Cellular Structure for $SU(n)$. We would like to have a corresponding CW decomposition for the special unitary group $SU(n) \subseteq U(n)$, and the first place to look is among the subcomplexes of the structure we have just constructed.

Exercise 9.53. Show that $Q_n \cap SU(n) = \{\mathrm{id}\}$, so $SU(n)$ cannot be a subcomplex in the structure defined above.

HINT. What is the determinant of an element of Q_n?

Evidently, we have to try something new, but we will make use of the same basic plan. We will construct a map $\gamma : \Sigma\mathbb{C}\mathrm{P}^{n-1} \to SU(n)$, giving some cells of $SU(n)$. The products of these cells will give a cellular decomposition.

Problem 9.54. Consider $S^1 \cong U(1) \subseteq U(n)$, induced, as usual, by the standard inclusion $\mathbb{C}^1 \hookrightarrow \mathbb{C}^n$.

(a) Show that the commutator[3] map $[\,?\,,\,?\,] : U(1) \times U(n) \to U(n)$ is given by the formula $(z, A) \mapsto T_{z,\mathbb{C}^1} \cdot T_{\bar{z},A(\mathbb{C}^1)}$.

[3]We use $[A, B] = ABA^{-1}B^{-1}$.

(b) Show that $[\,?\,,\,?\,]$ factors as in the diagram

$$\begin{tikzcd} U(1)\times U(n) \arrow[r, "{[?,?]}"] \arrow[d] & U(n) \\ \Sigma\mathbb{C}\mathrm{P}^{n-1} \arrow[r, "\gamma"] & SU(n). \arrow[u] \end{tikzcd}$$

(c) Show that γ is an embedding.[4]

(d) Show that in the diagram

$$\begin{tikzcd} \Sigma\mathbb{C}\mathrm{P}^{n-1} \arrow[r] \arrow[d, "\gamma"'] & \Sigma\mathbb{C}\mathrm{P}^{n} \arrow[r] \arrow[d, "\gamma"'] & S^{2n+1} \arrow[d, "g"] \\ SU(n) \arrow[r] & SU(n+1) \arrow[r] & V_1(\mathbb{C}^{n+1}), \end{tikzcd}$$

where the top row is a cofiber sequence, the map g is a homeomorphism.

Since we have a standard CW decomposition of $\mathbb{C}\mathrm{P}^n$, we have a corresponding decomposition for its suspension $\Sigma\mathbb{C}\mathrm{P}^n$, and this gives us cells $\chi_{2i-1} : D^{2i-1} \to SU(n+1)$ for $2 \le i \le n+1$. For an ordered subset $I \subseteq \{2, 3, \ldots, n+1\}$, we write $|I| = \sum_{i\in I} 2i-1$ and define 'product cells' $\chi_I : D^{|I|} \to SU(n+1)$ as before:

$$\begin{tikzcd} D^{|I|} \arrow[r, "\chi_I"] \arrow[d, equal] & SU(n+1) \\ \prod_I D^{2i-1} \arrow[r, "\prod_I \chi_i"] & \prod_i SU(n+1). \arrow[u, "\mu"'] \end{tikzcd}$$

Theorem 9.55. *The maps χ_I for $I \subseteq \{2, 3, \ldots, n+1\}$ are the characteristic maps of a CW decomposition for $SU(n)$.*

Project 9.56. Prove Theorem 9.55.

Cellular Structure for $\mathrm{Gr}_k(\mathbb{F}^{n+k})$. We finish this section by briefly indicating the standard cellular structure on the Grassmannian manifolds. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For each $V \in \mathrm{Gr}_k(\mathbb{F}^{n+k})$, we form the sequence of subspaces

$$(V \cap \mathbb{F}^0) \subseteq (V \cap \mathbb{F}^1) \subseteq \cdots \subseteq (V \cap \mathbb{F}^{n+k}).$$

Problem 9.57. Show $\dim(V \cap \mathbb{F}^j) \le \dim(V \cap \mathbb{F}^{j+1}) \le \dim(V \cap \mathbb{F}^j) + 1$.

Since $\dim(V \cap \mathbb{F}^{n+k}) = \dim(V) = k$, it follows from Problem 9.57 that there are precisely k integers j for which $\dim(V\cap\mathbb{F}^{j-1}) < \dim(V\cap\mathbb{F}^j)$. Write

[4] That is, the map $\Sigma\mathbb{C}\mathrm{P}^{n-1} \to \mathrm{Im}(\gamma)$ obtained by restricting the target is a homeomorphism.

$j(V) = (j_1, j_2, \ldots, j_k)$ for the list of these integers, in increasing order; this is called the **Schubert symbol** of V. For a given $J = (j_1, j_2, \ldots, j_k)$, write

$$E(J) = \{V \in \mathrm{Gr}_k(\mathbb{F}^{n+k}) \mid j(V) = J\}$$

and define $|J| = \sum_{i=1}^{k}(j_i - i)$.

Theorem 9.58.

(a) For any $J = (j_1, j_2, \ldots, j_k)$, $E(J) \cong \mathrm{int}(D^{|J|})$, and

(b) the open cells $E(J)$ for all $J \subseteq \{1, 2, \ldots, n+k\}$ give $\mathrm{Gr}_k(\mathbb{F}^{n+k})$ the structure of a CW complex.

Project 9.59. Prove Theorem 9.58 (see [**134**] for guidance).

9.4. Cone Decompositions

CW complexes are comparatively easy to analyze because they are built in steps using a simple procedure, formation of cofibers, and simple spaces, wedges of spheres. In this section, we consider the construction of spaces by iterated cofibers using spaces other than spheres.

9.4.1. Cone Decompositions. A map $f : X \to Y$ is called a **principal cofibration** if there is a space A and a map $A \to X$ such that the sequence $A \to X \to Y$ is a cofiber sequence. Because of the coaction of ΣA on Y, these maps are particularly easy to work with on the domain side.

Exercise 9.60. Show that the suspension of any map is a principal cofibration.

If f is not a principal cofibration, then we may try to understand it by expressing it—up to homotopy equivalence—as a composition of a finite number of principal cofibrations. More precisely, we may try to find a homotopy commutative diagram of the form

$$\begin{tikzcd}
A_0 \arrow[d] & A_1 \arrow[d] & & A_{n-1} \arrow[d] & \\
X_{(0)} \arrow[r] \arrow[d, "\simeq"'] & X_{(1)} \arrow[r] & \cdots \arrow[r] & X_{(n-1)} \arrow[r] & X_{(n)} \arrow[d, "\simeq"] \\
X \arrow[rrrr, "f"] & & & & Y
\end{tikzcd}$$

in which each sequence $A_s \to X_{(s)} \to X_{(s+1)}$ is a cofiber sequence. Such a diagram is called a **cone decomposition** of f with **length** n. We consider $X \xrightarrow{\mathrm{id}_X} X$ to be a cone decomposition of id_X with length 0.

The notion of cone decomposition and cone length makes sense in the unpointed category as well; the only difference is that one should use unpointed cones.

It can be useful to think of a cone decomposition of f as a recipe for building Y from X using the basic pieces $A_0, A_1, \ldots, A_{n-1}$. Sometimes it happens that we want to exert some control over the pieces involved by requiring that the spaces A_k be chosen from a predetermined collection $\mathcal{A}$. An **$\mathcal{A}$-cone decomposition** of f is an ordinary cone decomposition in which each of the spaces A_s is in the collection $\mathcal{A}$. It makes no difference other than to simplify notation to assume that $\mathcal{A}$ is closed under homotopy equivalence: if $A \in \mathcal{A}$ and $B \simeq A$, then $B \in \mathcal{A}$.

The **cone length** of f with respect to the collection $\mathcal{A}$ is

$$L_{\mathcal{A}}(f) = \inf\{\text{length}(\mathcal{D}) \mid \mathcal{D} \text{ is an } \mathcal{A}\text{-cone decomposition of } f\}$$

(where, as usual, $\inf(\varnothing) = \infty$).

Problem 9.61.

(a) Show that $L_{\mathcal{A}}(f) = L_{\mathcal{A}}(g)$ if f and g are pointwise equivalent in $\mathrm{H}\mathcal{T}_*$.

(b) Show that $L_{\mathcal{A}}(f \circ g) \le L_{\mathcal{A}}(f) + L_{\mathcal{A}}(g)$.

(c) Show that in the homotopy pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\text{HPO}"] & B \arrow[d, "g"] \\ C \arrow[r] & D, \end{tikzcd}$$

$L_{\mathcal{A}}(g) \le L_{\mathcal{A}}(f)$.

In fact, the $\mathcal{A}$-cone length of maps can be characterized as the largest invariant satisfying the properties you established in Problem 9.61.

Project 9.62. Suppose $\mathcal{L}$ is a numerical invariant of homotopy classes that satisfies the properties in Problem 9.61. Show that $\mathcal{L}(f) \le L_{\mathcal{A}}(f)$ for all f. Thus $L_{\mathcal{A}}$ is the largest such numerical invariant.

We write $\Sigma\mathcal{A} = \{\Sigma A \mid A \in \mathcal{A}\}$ and say that $\mathcal{A}$ is **closed under suspension** if $\Sigma\mathcal{A} \subseteq \mathcal{A}$. Similarly, write $\mathcal{A}^{\vee} = \{\text{finite wedges of spaces in } \mathcal{A}\}$, and say that $\mathcal{A}$ is **closed under wedges** if $A^{\vee} \subseteq \mathcal{A}$.

Problem 9.63.

(a) Show that if $\mathcal{A}$ is closed under suspension, then $L_{\mathcal{A}}(\Sigma f) \le L_{\mathcal{A}}(f)$ for any map f.

(b) Show that if $\mathcal{A}$ is closed under wedges, then for any maps f and g, $L_{\mathcal{A}}(f \vee g) \le \max\{L_{\mathcal{A}}(f), L_{\mathcal{A}}(g)\}$.

(c) Now assume that $\mathcal{A}$ is closed under wedges and suspension, and consider the diagram

$$\begin{tikzcd} C \arrow[d, "h"'] & A \arrow[l] \arrow[r] \arrow[d, "f"] & B \arrow[d, "g"] \\ Y & W \arrow[l] \arrow[r] & X \end{tikzcd}$$

with induced map of homotopy pushouts $\phi : D \to Z$. Show that

$$L_{\mathcal{A}}(\phi) \leq L_{\mathcal{A}}(f) + \max\{L_{\mathcal{A}}(g), L_{\mathcal{A}}(h)\}.$$

Long Cone Decompositions. It can be useful to apply these ideas to maps which do not have finite length $\mathcal{A}$-cone decompositions. In fact, some maps with infinite cone length are better than others. A **long $\mathcal{A}$-cone decomposition** of $f : X \to Y$ is a homotopy commutative diagram

$$\begin{tikzcd} X_{(0)} \arrow[r] \arrow[d, "\simeq"'] & X_{(\infty)} \arrow[d, "\simeq"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where $X_{(0)} \to X_{(\infty)}$ is the induced map from $X_{(0)}$ to the (homotopy) colimit of a cofibrant telescope diagram

$$X_{(0)} \to X_{(1)} \to \cdots \to X_{(s)} \to X_{(s+1)} \to \cdots$$

in which each $X_{(s)} \to X_{(s+1)}$ is the cofiber of a map $A_s \to X_{(s)}$ with $A_s \in \mathcal{A}$.

Not every map will have a long cone decomposition.

Problem 9.64. Let $f : X \to Y$, and suppose there is a space Q for which $\mathrm{map}_*(X, Q) \simeq *$ and $\mathrm{map}_*(A, Q) \simeq *$ for every $A \in \mathcal{A}$.

(a) Show that if f has an $\mathcal{A}$-cone decomposition (finite or infinite), then $\mathrm{map}_*(Y, Q) \simeq *$.

(b) Show that Q does not have an $\mathcal{A}$-cone decomposition (finite or infinite).

Problem 9.65. Show that if Y is a connected CW complex, then $X \ltimes Y$ has a long cone decomposition with respect to the collection $\{\Sigma^n X \mid n \geq 0\}$.

Cone Length of Spaces. Cone length of maps makes sense in both the pointed and the unpointed categories, and the difference between the two is mainly negligible, since it involves replacing unreduced cones with reduced ones. The distinction takes on a bit more significance when we define the cone length of spaces.

The **cone length** (with respect to the collection $\mathcal{A}$) of a pointed space $X \in \mathcal{T}_*$ is defined by $\mathrm{cl}_{\mathcal{A}}(X) = L_{\mathcal{A}}(* \to X)$. In other words, it is the length of the unique map from the initial object of $\mathcal{T}_*$ to X.

Exercise 9.66. Let X be a CW complex, and let $\mathcal{W}$ be the collection of all wedges of spheres. Show that $L_{\mathcal{W}}(X_n \hookrightarrow X) \le \dim(X) - n$. Give an example to show that the inequality can be strict.

Problem 9.67. Show that if $X \simeq Y$, then $\mathrm{cl}_{\mathcal{A}}(X) = \mathrm{cl}_{\mathcal{A}}(Y)$.

Some properties of spaces in $\mathcal{A}$ are inherited by all spaces with finite $\mathcal{A}$-cone length.

Problem 9.68. Let $Z \in \mathcal{T}_*$ be a space such that $\mathrm{map}_*(A, Z) \simeq *$ for all $A \in \mathcal{A}$, and let $f : X \to Y$.

(a) Show that if $\mathrm{cl}_{\mathcal{A}}(X) < \infty$, then $\mathrm{map}_*(X, Z) \simeq *$.

(b) What can you say about $\mathrm{map}_*(Y, Z)$ if $L_{\mathcal{A}}(f) < \infty$? What if you only know that f has a long cone decomposition?

In the unpointed category, it is natural to define $\mathrm{cl}_{\mathcal{A}}(X)$ to be the unpointed length of the unique map $\varnothing \to X$. A cone decomposition for this map necessarily has the form

$$\begin{tikzcd} \varnothing \arrow[d, "\alpha_{-1}"] & A_0 \arrow[d, "\alpha_0"] & A_1 \arrow[d, "\alpha_1"] & & \\ \varnothing \arrow[r] & * \arrow[r] & X_{(1)} \arrow[r] & \cdots \arrow[r] & X. \end{tikzcd}$$

Since the first step is always the same, the standard practice is to disregard it in counting up the cone length—hence the indexing in the diagram.[5] Thus we define

$$\mathrm{cl}_{\mathcal{A}}(X) = L_{\mathcal{A}}(\varnothing \to X) - 1$$

when $X \in \mathcal{T}_\circ$. This normalization is consistent with our already-established concept of dimension for CW complexes.

Exercise 9.69. Show that if X is a path-connected CW complex, then the unpointed cone length of X with respect to the collection of all wedges of spheres is bounded above by $\dim(X)$.

Note that an unpointed cone decomposition of X actually provides X with a basepoint, namely the composition from $X_{(0)} = *$ to X.

Problem 9.70. Let $\mathcal{A}$ be a collection of cofibrant pointed spaces, and let $\mathcal{A}_-$ be the result of forgetting all the basepoints. Show that the pointed cone length $\mathrm{cl}_{\mathcal{A}}(X)$ is equal to the unpointed cone length $\mathrm{cl}_{\mathcal{A}_-}(X_-)$.

Exercise 9.71. Is the cone length of $X \in \mathcal{T}_\circ$ equal to the cone length of $X_+ \in \mathcal{T}_*$?

[5]Note also that this makes the idea that S^{-1} should be $\varnothing$ seem less silly.

Boundary Maps of Cone Decompositions. A (long) cone decomposition of space X can be embellished by forming cofibers, like so:

$$\begin{tikzcd}
& A_{s-1} \arrow[d] & A_s \arrow[d, dotted] \arrow[dd, bend left, "\partial_s"] & A_{s+1} \arrow[d, dashed] & \\
\cdots \arrow[r, dashed] & X_{(s-1)} \arrow[r] \arrow[d, dashed] & X_{(s)} \arrow[r, dotted] \arrow[d] & X_{(s+1)} \arrow[r, dashed] \arrow[d, dotted] & \cdots \\
& \Sigma A_{s-2} & \Sigma A_{s-1} & \Sigma A_s, &
\end{tikzcd}$$

giving rise to composite maps $\partial_s : A_s \to \Sigma A_{s-1}$ for each s. We call these maps the **boundary maps** of the decomposition. We will see several times in our later work that boundary maps encode useful information about the space X.

9.4.2. Cone Decompositions of Products. Next we consider the cone length of cartesian products and smash products. For cartesian products it is most natural to work with unpointed cone decompositions, and for smash products we use pointed ones.

Decomposing a Smash Product. Without loss of generality, we can assume that X is the categorical colimit of a telescope

$$* \longrightarrow X_{(1)} \longrightarrow X_{(2)} \to \cdots \to X_{(s)} \longrightarrow X_{(s+1)} \to \cdots ,$$

in which each map is the inclusion of $X_{(s)}$ into the mapping cone of a map $A_s \to X_{(s)}$ (and similarly for Y). Then inside of $X \wedge Y$, we have the subspaces $X_{(s)} \wedge Y_{(t)}$ and, more importantly, the subspaces

$$D_n = \bigcup_{s+t=n} X_{(s)} \wedge Y_{(t)}.$$

If the subspaces $X_{(s)}$ and $Y_{(t)}$ were the CW skeleta of X and Y, then D_n would be the n-skeleton of $X \wedge Y$ in the product CW structure. This suggests that the inclusion $D_n \hookrightarrow D_{n+1}$ might be a principal cofibration, so that

$$D_0 \longrightarrow D_1 \to \cdots \to D_n \longrightarrow D_{n+1} \to \cdots$$

would be a (long) cone decomposition for $X \wedge Y$. This is in fact the case.

Proposition 9.72. *There is a cofiber sequence*

$$\bigvee_{s+t=n} A_s * B_t \longrightarrow D_n \longrightarrow D_{n+1}.$$

Problem 9.73. Prove Proposition 9.72 by showing that the inclusion

$$\big(X_{(s)} \wedge Y_{(t-1)}\big) \cup \big(X_{(s-1)} \wedge Y_{(t)}\big) \hookrightarrow X_{(s)} \wedge Y_{(t)}$$

is a principal cofibration.

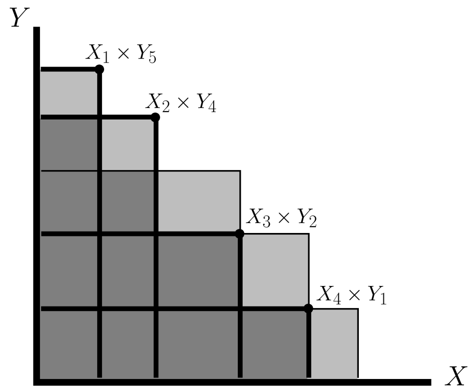

Figure 9.1. Cone decomposition of a product

Corollary 9.74. *Suppose $\mathcal{A}$ is closed under wedges and joins, and suppose X and Y have (long) $\mathcal{A}$-cone decompositions. Then*

(a) $X \wedge Y$ has a (long) cone decomposition, and

(b) $\mathrm{cl}_{\mathcal{A}}(X \wedge Y) \leq \mathrm{cl}_{\mathcal{A}}(X) + \mathrm{cl}_{\mathcal{A}}(Y)$.

Problem 9.75. Prove Corollary 9.74.

Decomposing a Cartesian Product. A similar scheme can be used to give a cone decomposition for a cartesian product. This time we start with cone decompositions having the form $\varnothing \to * \to X_{(1)} \to \cdots \to X$ and $\varnothing \to * \to Y_{(1)} \to \cdots \to Y$ and define

$$D_n = \bigcup_{s+t=n} X_{(s)} \times Y_{(t)},$$

where s or t could be -1 . It can be very helpful to visualize these sets as indicated in Figure 9.1.

To proceed, we need to think about the empty set.

Exercise 9.76. What is $X \times \varnothing$? What is the cone $C\varnothing$? What is $X * \varnothing$?

Now we can confidently establish our decomposition for products.

Proposition 9.77. *For each n, there is a (homotopy) pushout square*

$$\begin{tikzcd} \coprod_{s+t=n} A_i * B_j \arrow[r] \arrow[d] & \coprod_{s+t=n} C(A_i * B_j) \arrow[d] \\ D_n \arrow[r] & D_{n+1}, \end{tikzcd}$$

where s or t could be -1.

Corollary 9.78. *Let X and Y be pointed and path-connected spaces, and suppose that the cone decompositions $\varnothing \to * \to X_{(1)} \to \cdots \to X$ and $\varnothing \to * \to Y_{(1)} \to \cdots \to Y$ are the result of forgetting the basepoints in pointed cone decompositions for X and Y. Assume that $\mathcal{A}$ is closed under wedges and joins. Then there are cofiber sequences*

$$\bigvee_{s+t=n} A_i * B_j \longrightarrow D_n \longrightarrow D_{n+1},$$

so that $X \times Y$ has a (long) cone decomposition, and

$$\mathrm{cl}_{\mathcal{A}}(X \times Y) \leq \mathrm{cl}_{\mathcal{A}}(X) + \mathrm{cl}_{\mathcal{A}}(Y).$$

Exercise 9.79. What can you say about the cone length of a product of spaces that are not necessarily path-connected?

9.4.3. Boundary Maps for Products. We have just seen that if X and Y have (long) cone decompositions with boundary maps $\partial_s^X : A_s \to \Sigma A_{s-1}$ and $\partial_t^Y : B_t \to \Sigma B_{t-1}$, respectively, then $X \times Y$ has an induced (long) cone decomposition $D_0 \to D_1 \to \cdots \to D_{n-1} \to D_n \to \cdots$. What are the boundary maps for this cone decomposition?

Problem 9.80.

(a) Determine the homotopy type of D_n/D_{n-1}. Show that the boundary maps for $X \times Y$ have the form

$$\partial_n : \bigvee_{i+j=n} A_i * B_j \longrightarrow \bigvee_{s+t=n-1} \Sigma A_s \wedge \Sigma B_t.$$

(b) For each i and j, write $W_{i,j} = (\Sigma A_i \wedge \Sigma B_{j-1}) \vee (\Sigma A_{i-1} \wedge \Sigma B_j)$. Show that there is a (strictly) commutative square

$$\begin{tikzcd} A_i * B_j \arrow[r, hook] \arrow[d, "d_{ij}"'] & \bigvee_{i+j=n} A_i * B_j \arrow[d, "\partial_n"] \\ W_{i,j} \arrow[r, hook] & \bigvee_{s+t=n-1} \Sigma A_s \wedge \Sigma B_t. \end{tikzcd}$$

Hint. For part (b), show that you can restrict attention to $X_i \times Y_j$.

Problem 9.80 implies that understanding the boundary map ultimately boils down to understanding the maps $A_i * B_j \to W_{i,j}$ for various i and j.

Theorem 9.81. *The boundary map is* $\mathrm{id}_{\Sigma A_i} \wedge \partial_j^Y + \partial_i^X \wedge \mathrm{id}_{\Sigma B_j}$.[6]

The space $A_i * B_j$ appears in this cone decomposition because of Proposition 9.8, where it appeared in the form

$$A_i * B_j = (CA_i \times B_j) \cup (A_i \times CB_j) \subseteq CA_i \times CB_j.$$

[6] To be perfectly pedantic, it is $\mathrm{in}_1 \circ (\mathrm{id}_{\Sigma A_i} \wedge \partial_j^Y) + \mathrm{in}_2 \circ (\partial_i^X \wedge \mathrm{id}_{\Sigma B_j})$.

We also know from Problem 9.2 that

$$A_i * B_j \simeq \Sigma(A_i \wedge B_j) \cong \Sigma A_i \wedge B_j \cong A_i \wedge \Sigma B_j.$$

Problem 9.82. Work inside of $X_i \wedge Y_j$.

(a) Show that the map d_{ij} factors like so:

$$\begin{tikzcd}
(CA_i \times B_j) \cup (A_i \times CB_j) \arrow[d] \arrow[dr, "d_{ij}"] & \\
\frac{(CA_i\times B)}{(A_i\times B_j)} \vee \frac{(A_i\times CB_j)}{(A_i\times B_j)} \arrow[r] \arrow[d, equal] & (\Sigma A_i \wedge \Sigma B_{j-1}) \vee (\Sigma A_{i-1} \wedge \Sigma B_j) \\
(\Sigma A_i \rtimes B_j) \vee (A_i \ltimes \Sigma B_j) \arrow[r] & (\Sigma A_i \wedge B_j) \vee (A_i \wedge \Sigma B_j). \arrow[u, "\alpha\vee\beta"']
\end{tikzcd}$$

HINT. Refer to Proposition 9.8 and its proof.

(b) The map from the upper left to lower right in the diagram of part (a) can be identified with a map

$$A_i * B_j \longrightarrow (A_i * B_j) \vee (A_i * B_j).$$

Show that this map is the suspension co-H-structure you derived in Problem 9.2 by making the identification $A_i * B_j \simeq \Sigma(A_i \wedge B_j)$. Conclude that the boundary map is precisely $\alpha + \beta$.

(c) Show that the map α is determined by the diagram

$$\begin{tikzcd}
CA_i \times B_j \arrow[r] \arrow[dd] & (X_{(i-1)} \cup CA_i) \times Y_{(j)} \arrow[d] \\
& \Sigma A_i \rtimes Y(j) \arrow[d] \\
\Sigma A_i \wedge B_j \arrow[r, "\alpha"] & \Sigma A_i \wedge \Sigma B_{j-1},
\end{tikzcd}$$

and conclude that $\alpha = \mathrm{id}_{\Sigma A_i} \wedge \partial_{B_j}$.

(d) Carry out the analogous identification $\beta = \partial_{A_i} \wedge \mathrm{id}_{\Sigma B_j}$ and finish the proof of Theorem 9.81.

9.4.4. Generalized CW Complexes. A CW complex is a space with a very special kind of long cone decomposition, in which each space A_i must be a wedge of spheres in dimension i.

We say that the colimit of a (long) cone decomposition with respect to the collection $\mathcal{W}$ of all wedges of spheres is a **generalized CW complex**.

It makes sense to talk about subcomplexes of generalized CW complexes.

Problem 9.83. Show that if $A \subseteq X$ is a generalized CW subcomplex, then the inclusion $i : A \hookrightarrow X$ is a cofibration.

Problem 9.84. Show that a product of generalized CW complexes is also a generalized CW complex.

Later we will show that every generalized CW complex is homotopy equivalent to a genuine CW complex.

9.5. Introduction to Phantom Maps

If X is the colimit of the telescope diagram

$$X_{(0)} \longrightarrow X_{(1)} \longrightarrow X_{(2)} \to \cdots \to X_{(n)} \longrightarrow X_{(n+1)} \to \cdots,$$

then for any $Y \in \mathcal{T}_*$, the maps $X \to Y$ are completely determined by the compositions $X_{(n)} \to X \to Y$. But what about the *homotopy classes* of maps? If $f, g : X \to Y$ and $f|_{X_{(n)}} \simeq g|_{X_{(n)}}$ for each n, does it follow that $f \simeq g$?

The case in which the diagram is the telescope of CW skeleta of X and $g = *$ is of particular importance. We say that $f : X \to Y$ is a **phantom map** if $f|_{X_n} \simeq *$ for all n and write $\mathrm{Ph}(X, Y) \subseteq [X, Y]$ for the set of all homotopy classes of phantom maps from X to Y. Clearly the trivial map is a phantom map, but are there nontrivial phantom maps? We are ready to set up a good deal of the basic theory of phantom maps in preparation for later attacks on this question.

9.5.1. Maps out of Telescopes. Let X be the homotopy colimit of the telescope diagram $X_{(0)} \to X_{(1)} \to X_{(2)} \to \cdots \to X_{(n)} \to X_{(n+1)} \to \cdots$. Because maps from homotopy colimits are not uniquely determined by the diagram, we first show that the condition $f|_{X_{(n)}} \simeq g|_{X_{(n)}}$ for each n actually makes sense.

Exercise 9.85. Let $f, g : X \to Y$.

(a) Show that there are maps $j_n : X_{(n)} \to X$ induced from the diagram and that they are unique up to homotopy equivalence of maps.

Hint. They are induced maps between homotopy colimits of diagrams.

(b) Suppose j_n and $\widetilde{j}_n$ are two such induced maps. Show that $f \circ j_n \simeq g \circ j_n$ if and only if $f \circ \widetilde{j}_n \simeq g \circ \widetilde{j}_n$.

This shows that the question is well-defined in that it does not depend on which choice of map $j_n : X_{(n)} \to X$ we choose. So choose your favorite induced maps $j_n : X_{(n)} \to X$ and define

$$j = (j_n) : \bigvee_1^\infty X_{(n)} \longrightarrow X.$$

To simplify notation, we write $W = \bigvee_1^\infty X_{(n)}$, so that j is a map $W \to X$. Write $\Theta_X : X \to C_j$ for the cofiber of j.

Problem 9.86. Let X be a CW complex, and consider it as the (homotopy) colimit if its skeleta, and let $\Theta_X : X \to C_j$ be the resulting map.

(a) Show that Θ_X is a phantom map.

(b) Show that $f|_{X_n} \simeq *$ for all n if and only if there is a homotopy factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, bend right, "\Theta_X"'] & & Y \\ & C_j. \arrow[ur, dotted] & \end{tikzcd}$$

Conclude that there are nontrivial phantom maps out of X if and only if $\Theta_X \not\simeq *$.

Because of Problem 9.86, the map $\Theta_X : X \to C_j$ is called the **universal phantom map** out of X.[7]

Let us study these maps in more detail. The **shift map** is the map $\mathrm{sh} : W \to W$ given by

$$\begin{tikzcd} X_{(n)} \arrow[r, "i_n"] \arrow[d, hook] & X_{(n+1)} \arrow[d, hook] \\ W \arrow[r, "\mathrm{sh}"] & W, \end{tikzcd}$$

where $i_n : X_{(n)} \hookrightarrow X_{(n+1)}$ is the map from the telescope diagram.

Problem 9.87.

(a) Show that there is a homotopy pushout square

$$\begin{tikzcd} W \vee W \arrow[r, "\nabla"] \arrow[d, "(\mathrm{sh}, \mathrm{id}_W)"'] & W \arrow[d] \\ W \arrow[r, "j"] & X. \end{tikzcd}$$

(b) Show that $C_j \simeq \Sigma W$. Conclude that there is a cofiber sequence

$$W \xrightarrow{j} X \xrightarrow{\Theta_X} \Sigma W \xrightarrow{q} \Sigma W \xrightarrow{\Sigma j} \Sigma X \longrightarrow \cdots .$$

Hint. You can convert ∇ to a cofibration by replacing W with $W \rtimes I$.

The existence of nontrivial phantom maps out of X boils down to whether or not Θ_X is trivial, so let's see what the triviality of Θ_X implies.

[7] Actually, since the factorization is not unique, the 'uni' part of the term 'universal' is undeserved—Θ_X is the **versal** phantom map out of X.

Problem 9.88. Show that if $\Theta_X \simeq *$, then Σj has a section up to homotopy.

9.5.2. Inverse Limits and $\lim^1$ for Groups. We will find a formula for $\mathrm{Ph}(X, Y)$ in terms of the spaces X_n and Y. This formula will involve certain group-theoretic functors which we will define and study.

Problem 9.89. Identify $[\Sigma W, Y] = [\bigvee \Sigma X_{(n)}, Y] \cong \prod[\Sigma X_{(n)}, Y]$.

(a) Determine the induced map $q^* : \prod[\Sigma X_{(n)}, Y] \to \prod[\Sigma X_{(n)}, Y]$.

HINT. Since the domain is ΣW, it suffices to determine the restrictions $\Sigma X_{(n)} \to W$ for each n. Follow the cones!

(b) Determine the action of $\prod[\Sigma X_{(n)}, Y]$ on $\prod[\Sigma X_{(n)}, Y]$.

In view of Problem 9.89(b), we can identify $\mathrm{Ph}(X, Y)$ with the set of orbits of the action of $\prod[\Sigma X_n, Y]$ on $\prod[\Sigma X_n, Y]$. This orbit set is an existing algebraic construction. To describe the algebraic context, we begin with an explicit construction for the limit of a tower

$$A_1 \xleftarrow{p_2} A_2 \xleftarrow{p_3} A_3 \longleftarrow \cdots$$

of sets. Form the product $\prod A_n$ and define the shift map

$$\mathrm{sh} : \prod A_n \longrightarrow \prod A_n$$

by the formula $\mathrm{sh}(a_1, a_2, \ldots) = (p_2(a_2), p_3(a_3), \ldots)$.

Problem 9.90.

(a) Show that $L = \{(a_1, a_2, \ldots) \mid p_n(a_n) = a_{n-1} \text{ for all } n\}$, with the obvious maps $L \to A_n$, is a limit for the tower.

(b) Show that if the tower is a tower of groups and homomorphisms, then L is also a group, and it is the limit, in the category $\mathcal{G}$, of the tower.

(c) Show that if it is a tower of abelian groups, then $L = \ker(\mathrm{id}_{\prod A_n} - \mathrm{sh})$.

Our next functor is most naturally defined for towers of abelian groups. In this case, the tower gives rise to the map

$$\sigma = \big(\mathrm{id}_{\prod A_n} - \mathrm{sh}\big) : \prod A_N \longrightarrow \prod A_n,$$

and from there to the exact sequence

$$0 \to \ker(\sigma) \longrightarrow \prod A_n \xrightarrow{\ \sigma\ } \prod A_n \longrightarrow \mathrm{coker}(\sigma) \to 0.$$

We have already identified the kernel of σ: it is the limit of the tower. The cokernel is also important, and it is known as $\lim^1$ of the tower. The usual practice is to omit the maps in the tower from the notation, so that we write $\lim A_n$ and $\lim^1 A_n$ for these groups, even though they depend crucially on the maps involved.

When the groups are not abelian, we can still define $\lim^1 A_n$. First define an action of $\prod A_n$ on itself by the rule

$$(a_1, \ldots, a_n, \ldots)^{(b_1, \ldots, b_n, \ldots)} = (b_1^{-1} a_1 p_2(b_2), \ldots, b_n^{-1} a_n p_{n+1}(b_{n+1}), \ldots).$$

Then $\lim^1 A_n$ is the orbit set

$$\lim^1 A_n = \left(\prod A_n\right) / (\text{action}).$$

Exercise 9.91. Show that if the groups A_n are abelian, then the two definitions of $\lim^1 A_n$ agree.

Problem 9.92.

(a) Show that if each group A_n is finite, then $\lim^1 A_n = *$.

(b) Show that if each group A_n is a compact Hausdorff topological group and the homomorphisms $A_n \to A_{n-1}$ are continuous, then $\lim^1 A_n = *$.

Now we have a formula for the set of phantom maps from X to Y.

Theorem 9.93. *For any CW complex X*, $\mathrm{Ph}(X, Y) \cong \lim^1 [\Sigma X_n, Y]$.

9.5.3. Mapping into a Limit. This entire discussion is easily dualized. If Y is the limit of a tower $\cdots \to Y_n \to Y_{n-1} \to \cdots$ of fibrations, then we can ask about the maps $f : X \to Y$ such that each composition $X \to Y \to Y_n$ is trivial. If we write $P = \prod_{n=1}^{\infty} Y_n$, then map f has this property if and only if the composition $X \xrightarrow{f} Y \xrightarrow{k} P$ is trivial, which is to say, if and only if there is a lift up to homotopy in the diagram

$$\begin{tikzcd} & F_k \arrow[d] \\ X \arrow[ur, dotted, bend left] \arrow[r, "f"] & Y. \end{tikzcd}$$

Problem 9.94.

(a) Show that $F_k \simeq \prod_{n=1}^{\infty} \Omega Y_n$.

(b) Determine the action of ΩP on F_k.

(c) Show that there is an exact sequence of pointed sets

$$* \to \lim^1 [X, \Omega Y_n] \longrightarrow [X, Y] \longrightarrow \lim [X, Y_n] \to *.$$

Problem 9.95. Show that if each Y_n is simply-connected and has finite homotopy groups and if X is a connected finite complex, then $[X, Y] \to \lim [X, Y_n]$ has trivial kernel.

9.6. G. W. Whitehead's Homotopy Pullback Square

The loop space functor defines maps $\Omega : [X, Y] \to [\Omega X, \Omega Y]$. This is an important construction, but it is very hard to study from this point of view because, unless we unpack the definition, it is just an abstract 'procedure'. But if we do unpack the definition, then we are likely to wind up doing a lot of point-set topology that may turn out to be beside the point. Therefore, we rarely study the map Ω directly but instead study its composition with the exponential law isomorphism. This map is entirely equivalent to Ω, and it has the virtue that it is induced by a map of spaces.

Problem 9.96. Let $\lambda : \Sigma\Omega X \to X$ be the map adjoint to $\mathrm{id} : \Omega X \to \Omega X$. Show that there is a commutative diagram

$$\begin{tikzcd}
{[X,Y]} \arrow[r, "\Omega"] \arrow[d, equal] & {[\Omega X, \Omega Y]} \arrow[d, "\cong"] \\
{[X,Y]} \arrow[r, "\lambda^*"] & {[\Sigma\Omega X, Y].}
\end{tikzcd}$$

Conclude that $f \simeq \Omega g$ for some g if and only if ϕ, the adjoint of f, is in the image of λ^*.

The premise of our discussion so far has been that by replacing the 'procedure' Ω with the induced map λ^*, we should be able to obtain useful homotopy-theoretical information about it. We make a start on this by proving something first observed by G. W. Whitehead: the map λ features in a very natural homotopy pullback square.

Theorem 9.97. *There is a strong homotopy pullback square*

$$\begin{tikzcd}
\Sigma\Omega X \arrow[r] \arrow[d, "\lambda"'] \arrow[dr, phantom, "\mathrm{HPB}"] & X \vee X \arrow[d, "\mathrm{in}"] \\
X \arrow[r, "\Delta"] & X \times X.
\end{tikzcd}$$

You will prove Theorem 9.97 in the next problem, but first let's establish some notation. If $\omega : I \to X$ is a path and $[a, b] \subseteq I$, then we write $\omega_{[a,b]} : I \to X$ for the restriction of ω to $[a, b]$, linearly reparametrized to the interval I. Explicitly, $\omega_{[a,b]}(t) = \omega((1 - t)a + tb)$.

Problem 9.98.

(a) Convert the map $\mathrm{in} : X \vee X \hookrightarrow X \times X$ to a fibration $E \to X \times X$; explain how to express points in E as pairs of paths (ω_1, ω_2) in X; what relations must be satisfied by these pairs?

(b) Let P be the pullback of E over Δ. Let $P_1 \subseteq P$ be the subspace of those pairs with $\omega_1(1) = *$, and let $P_2 \subseteq P$ be the space of pairs with

$\omega_2(1) = *$. Show that P_1 and P_2 are contractible. What is the homotopy type of $P_1 \cap P_2$?

(c) Show that the inclusions $P_1 \cap P_2 \hookrightarrow P_1$ and $P_1 \cap P_2 \hookrightarrow P_2$ are cofibrations. What is the homotopy type of P?

(d) Define a map $\delta : \Sigma\Omega X \to P$ by the formula

$$\delta([x,t]) = \begin{cases} (\overleftarrow{\omega}_{[0,t]}, \omega_{[t,2t]}) & \text{if } t \le \frac{1}{2}, \\ (\overleftarrow{\omega}_{[2t-1,t]}, \omega_{[t,1]}) & \text{if } t \ge \frac{1}{2}. \end{cases}$$

Show that the diagram

$$\begin{tikzcd} \Sigma\Omega X \arrow[r, "\delta"] \arrow[dr, bend right, "\lambda"'] & P \arrow[r] \arrow[d] & E \arrow[d] \\ & X \arrow[r, "\Delta"] & X \times X \end{tikzcd}$$

is commutative.

(e) Finish the proof of Theorem 9.97 by showing that δ is a homotopy equivalence.

Hint. It is an induced map of homotopy pushouts.

9.7. Lusternik-Schnirelmann Category

It seems that ‘category’ was a go-to term in the early part of the twentieth century, so that in addition to—and preceding—a category as a bunch of objects and morphisms, we also have Baire category in analysis and Lusternik-Schnirelmann category (generally and confusingly referred to simply as ‘category’ or ‘L-S category’) in homotopy theory. The original definition of Lusternik-Schnirelmann category of a space X was made in terms of open covers of X. A **Lusternik-Schnirelman cover** of X is an open cover $X = U_1 \cup \cdots \cup U_n$ such that each $U_k \hookrightarrow X$ is nullhomotopic; then the L-S category of X is the number of sets in the smallest L-S cover of X.

This idea was introduced to aid in the study of analytical problems: the main theorem of Lusternik and Schnirelmann is that if $f : M \to \mathbb{R}$ where M is a compact manifold, then the number of critical points of f is at least the L-S category of X. But it turns out that L-S category is a homotopy invariant, so it has been deeply studied via homotopy theory.

9.7.1. Basics of Lusternik-Schnirelmann Category. When they introduced it, Lusternik and Schnirelmann defined category in terms of open covers. For some purposes, closed covers are more convenient, and the two notions agree for nice enough spaces. We will actually use a third definition, due to G. W. Whitehead, that is better suited to homotopy-theoretical analysis and which is equivalent to the other two for nice enough spaces.

A pointed space X can be viewed as the pair $(X, *)$ in $\mathcal{T}_{(2)}$, and in this context we can form the 'product' pair $X \boxplus X = (X \times X, X \vee X)$, and more generally

$$(X, *)^{\boxplus k} = (X^k, T^k(X)),$$

where $T^k(X) \subseteq X^k$ is a certain subspace, called the k-fold **fat wedge** of X with itself.

Exercise 9.99.

(a) Write down the set $T^k(X)$ explicitly.

(b) Determine the quotient $X^k/T^k(X)$.

The **Lusternik-Schnirelmann category** of $X \in \mathcal{T}_*$, denoted $\mathrm{cat}(X)$, is the least integer n for which there is a map λ making the diagram

$$\begin{tikzcd} & T^{n+1}(X) \arrow[d] \\ X \arrow[ur, dotted, "\lambda"] \arrow[r, "\Delta"] & X^{n+1} \end{tikzcd}$$

commute up to homotopy (where Δ is the diagonal map).

Problem 9.100.

(a) Show that $\mathrm{cat}(X) = 0$ if and only if $X \simeq *$.

(b) Show that $\mathrm{cat}(X) \le 1$ if and only if X is a co-H-space.

We know that a retract of a co-H-space is also a co-H-space. This generalizes nicely.

Proposition 9.101. *If X is a homotopy retract of Y, then* $\mathrm{cat}(X) \le \mathrm{cat}(Y)$.

It follows that if X and Y are homotopy equivalent spaces, then they have the same Lusternik-Schnirelmann category.

Corollary 9.102. *If $X \simeq Y$, then* $\mathrm{cat}(X) = \mathrm{cat}(Y)$.

Problem 9.103.

(a) Show that $X \mapsto T^k(X)$ is a functor that respects homotopy.

(b) Prove Proposition 9.101 and derive Corollary 9.102.

Finally, we establish the fundamental relation between Lusternik-Schnirelmann category and cone length.

Theorem 9.104. *For any space X,* $\mathrm{cat}(X) \le \mathrm{cl}(X)$.

Problem 9.105.

(a) Show that it suffices to prove that if $f : A \to X$ is a cofibration of well-pointed spaces, then $\mathrm{cat}(C_f) \leq \mathrm{cat}(X) + 1$.

(b) Now let $Y = C_f$ as in part (a), and let $\mathrm{cat}(X) = n$. Show that there is a homotopy H from $\Delta_{n+1} : Y \to Y^{n+1}$ to a map D such that $D(X) \subseteq T^{n+1}(X) \subseteq T^{n+1}(Y)$.

(c) Show that there is a homotopy $K : \mathrm{id}_Y \simeq j$, where $j(CA) = *$.

(d) Prove Theorem 9.104.

The Category of a Map. Lusternik-Schnirelmann category can be generalized from spaces to maps. We say that the **Lusternik-Schnirelmann category** of a map $f : X \to Y$ is the least n for which there is a lift, up to homotopy, in the diagram

$$\begin{tikzcd} & & T^{n+1}(Y) \arrow[d] \\ X \arrow[r, "f"] \arrow[urr, dotted] & Y \arrow[r, "\Delta"] & Y^{n+1}. \end{tikzcd}$$

If $f : X \to Y$ is the inclusion of a subspace, then $\mathrm{cat}(f)$ is sometimes written $\mathrm{cat}_Y(X)$, the category of X *in* Y.

Problem 9.106.

(a) Show that $\mathrm{cat}(X) = \mathrm{cat}(\mathrm{id}_X)$.

(b) Show that $\mathrm{cat}(f) \leq \mathrm{cat}(X)$ and $\mathrm{cat}(f) \leq \mathrm{cat}(Y)$.

(c) What can you say about $\mathrm{cat}(f \circ g)$?

Point-Set Topology and Lusternik-Schnirelmann Category. As we have mentioned, Lusternik-Schnirelmann category was first defined in terms of open covers of X. Other authors found that it was convenient to use closed covers instead, and it was only later that G. W. Whitehead introduced his diagram-theoretic approach. How do these different definitions compare?

Problem 9.107. Show that if $\mathrm{cat}(X) \leq n$, then $X = X_0 \cup X_1 \cup \cdots \cup X_n$ where each $X_k \subseteq X$ is a closed set and there is a homotopy $H_k : X \times I \to X$ with $H(X_k \times \{1\}) = *$.

Project 9.108. We have three competing definitions of Lusternik-Schnirelmann category: the one we have given above, one in terms of open covers, and one in terms of closed covers. What topological conditions must you impose on a space X in order for these (or some of these) definitions to agree?

9.7.2. Lusternik-Schnirelmann Category of CW Complexes. The L-S category of CW complexes is most easily studied because open covers can be replaced by covers by subcomplexes, whose inclusion maps are cofibrations.

Theorem 9.109. *Let $f : X \to Y$ where X is a CW complex. Then* $\operatorname{cat}(f) \leq n$ *if and only if X has a CW decomposition and a cover $X = A_0 \cup A_1 \cup \cdots \cup A_n$ by subcomplexes such that each $f|_{A_k} \simeq *$.*

We'll take Theorem 9.109 for granted for now. You'll be asked to prove it later in Project 12.50. This theorem is used to prove the following important properties of the Lusternik-Schnirelmann category of maps out of CW complexes.

Theorem 9.110. *If X is a CW complex and $f : X \to Y$, then* $\operatorname{cat}(f) \leq n$ *if and only if f factors through a space with cone length $\leq n$.*

If $f : X \to Y$ as in Theorem 9.110, then Theorem 9.109 provides a cover $X = A_0 \cup A_1 \cup \cdots \cup A_n$ by subcomplexes. Define

$$\overline{X}(k) = X \cup CA_0 \cup CA_1 \cup \cdots \cup CA_k$$

for $k \leq n$ and write $\overline{X} = \overline{X}(n)$.

Problem 9.111.

(a) Show that f factors, on the nose, through $\overline{X}$.

(b) Show by induction that $\operatorname{cl}(\overline{X}(k)) \leq k$.

(c) Prove Theorem 9.110.

The construction of the space $\overline{X}$ from a cover of X has another very interesting consequence.

Theorem 9.112. *If X is a CW complex, then* $\operatorname{cat}(X) \leq \operatorname{cl}(X) \leq \operatorname{cat}(X)+1$.

Problem 9.113.

(a) Show that $\overline{X} \simeq X \vee (\bigvee_k \Sigma A_k)$.

(b) Prove Theorem 9.112.

Next we look at maps out of homotopy pushouts.

Theorem 9.114. *Suppose in the homotopy commutative diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "HPO"] & B \arrow[d] \arrow[ddr, bend left, "\beta"] & \\
C \arrow[r] \arrow[drr, bend right, "\gamma"'] & X \arrow[dr, "f"] & \\
& & Y
\end{tikzcd}$$

the square is a homotopy pushout of CW complexes (or spaces homotopy equivalent to CW complexes). Then

$$\operatorname{cat}(f) \leq \operatorname{cat}(\beta) + \operatorname{cat}(\gamma) + 1.$$

Problem 9.115.

(a) Show that it suffices to prove Theorem 9.114 for a pushout square in which all the maps are inclusions of subcomplexes of CW complexes.

(b) Prove Theorem 9.114.

HINT. Adapt Problem 9.63.

9.7.3. The Ganea Criterion for L-S Category. Tudor Ganea developed yet another criterion for detecting the L-S category of sufficiently nice spaces X. We'll restrict our attention to CW complexes.

Given a fibration $p : E \to B$ with fiber F, build a map $E \cup CF \to B$ as the induced map of pushouts in the cube

$$\begin{tikzcd}
F \arrow[rr] \arrow[dr] \arrow[dd] & & CF \arrow[dr] \arrow[dd] & & \\
& E \arrow[rr] \arrow[dd, "p"] & & E \cup CF \arrow[r] \arrow[dd] & G(E) \arrow[dd, "G(p)"] \\
* \arrow[rr] \arrow[dr] & & * \arrow[dr] & & \\
& B \arrow[rr, equal] & & B \arrow[r, equal] & B
\end{tikzcd}$$

and convert it to a fibration to obtain the map $G(p) : G(E) \to B$ with fiber $G(F)$. This is known as the **Ganea construction** on the fibration p. Later (in Section 18.1.2) we will determine the homotopy type of $G(F)$ and study this construction in more detail.

For now, though, we are only interested in the construction for its relation to Lusternik-Schnirelmann category. For this, we start with the path space fibration $@_0 : \mathcal{P}(B) \to B$ and apply the Ganea construction iteratively to obtain a sequence of fibrations $p_n : G_n(B) \to B$.

Theorem 9.116. *If X is a CW complex and $f : X \to B$, then $\operatorname{cat}(f) \leq n$ if and only if there is a lift in the diagram*

$$\begin{tikzcd}
& G_n(B) \arrow[d, "p_n"] \\
X \arrow[r, "f"] \arrow[ur, dotted, bend left] & B,
\end{tikzcd}$$

where p_n is the n^{th} Ganea fibration for B.

Suppose $\mathrm{cat}(f) = n$. Then Theorem 9.110 gives us a cover $X = A_0 \cup A_1 \cup \cdots \cup A_n$ by subcomplexes with $f|_{A_k} \simeq *$ for each k. Write $X(k) = A_0 \cup A_1 \cup \cdots \cup A_k$.

Problem 9.117.

(a) Show that there is a lift $\lambda_0 : X(0) \to G_0(B)$ of $f|_{X(0)}$.

(b) Suppose you have a lift λ_k of $f|_{X(k)}$. Show that f is homotopic to a map g_{k+1} such that $g_{k+1}(A_{k+1}) = *$ and that λ_k is homotopic to a lift $\ell : X(k) \to G_k(B)$ of $g_{k+1}|_{X(k)}$.

(c) By applying the Fundamental Lifting Property in the diagram

$$\begin{tikzcd}
X(k) \cap A_{k+1} \arrow[rr] \arrow[dr] \arrow[dd] & & CF \arrow[dr] \arrow[dd] & \\
& X(k) \arrow[rr] \arrow[dd] & & E \cup CF \arrow[dd] \\
A_{k+1} \arrow[rr] \arrow[dr] & & * \arrow[dr] & \\
& X(k+1) \arrow[rr] & & B,
\end{tikzcd}$$

show that g_{k+1} lifts through $E \cup CF \to B$.

(d) Prove Theorem 9.116.

Problem 9.118. Show that for any space B and any $n \geq 1$, the map $\Omega p_n : \Omega(G_n(B)) \to \Omega B$ has a section.

The first Ganea fibration can be easily identified, and the identification clarifies the relation between the Ganea construction and Lusternik-Schnirelmann category.

Problem 9.119. Show that $p_1 : G_1(B) \to B$ is pointwise homotopy equivalent to the map $\lambda : \Sigma\Omega B \to B$ adjoint to $\mathrm{id}_{\Omega B}$.

In Section 9.6 you showed that λ features in a homotopy pullback square involving the map $X \vee X \hookrightarrow X \times X$ that we use to define spaces with L-S category 1. This relationship generalizes.

Project 9.120. Show that there are homotopy pullback squares

$$\begin{tikzcd}
G_n(B) \arrow[r] \arrow[d, "p_n"'] \arrow[dr, phantom, "HPB"] & T^{n+1}(B) \arrow[d] \\
B \arrow[r, "\Delta"] & B^{n+1}
\end{tikzcd}$$

for each n.

The Category of a Mapping Cone. If we have a map $\alpha : A \to X$ and $\mathrm{cat}(X) = n$, then we know that the mapping cone C_α satisfies

$$\mathrm{cat}(C_\alpha) \le \mathrm{cat}(X) + 1,$$

and we would like to find some way of telling whether the inequality is strict. Set up the diagram

$$\begin{tikzcd} G_n(A) \arrow[r, "G_n(\alpha)"] \arrow[d] & G_n(X) \arrow[r, "G_n(j)"] \arrow[d] & G_n(C_\alpha) \arrow[d, "p_n"] \\ A \arrow[r, "\alpha"] & X \arrow[r, "j"] \arrow[u, bend left, "\sigma"] & C_\alpha \end{tikzcd}$$

where σ is the section guaranteed by Theorem 9.116.

Problem 9.121.

(a) Show that there is a map $h : A \to F_n(C_\alpha)$ making the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "h"'] & X \arrow[r, "j"] \arrow[d, "G_n(j)\circ\sigma"'] & C_\alpha \arrow[d, equal] \\ F_n(C_\alpha) \arrow[r] & G_n(C_\alpha) \arrow[r, "p_n"] & C_\alpha \end{tikzcd}$$

commute up to homotopy.

(b) Show that if $G_n(j) \circ \sigma \circ \alpha \simeq *$, then $\mathrm{cat}(C_\alpha) \le n$.

(c) Show that if $*$ is among the possibilities for the map h, then $\mathrm{cat}(C_\alpha) \le n$.

The map h in Problem 9.121(a) is not unique; we'll write $\mathscr{H}(\alpha)$ for the complete set of all such maps. The maps $h \in \mathscr{H}(\alpha)$ are related to the vast and fundamentally disorganized theory of **Hopf invariants**. We will touch on Hopf invariants—of various kinds—again several times in this book.

9.7.4. Category and Products. The intimate relationship between cone length and category means that we can use our already-proved formula for the cone length of a product to derive a formula for the L-S category of a product.

Theorem 9.122. *For any two spaces X and Y,* $\mathrm{cat}(X \times Y) \le \mathrm{cat}(X) + \mathrm{cat}(Y)$.

Problem 9.123. Prove Theorem 9.122.

The Ganea 'Conjecture'. Our results for $Y = S^k$ imply the inequalities

$$\mathrm{cat}(X) \le \mathrm{cat}(X \times S^k) \le \mathrm{cat}(X) + 1.$$

This (and other things) prompted Ganea to ask: is it always true that $\mathrm{cat}(X \times S^k) = \mathrm{cat}(X) + 1$? We'll say that a space X **satisfies the Ganea**

condition if $\mathrm{cat}(X \times S^k) = \mathrm{cat}(X) + 1$ for all $k \geq 1$. Ganea's question[8] was a major motivation for much of the study of L-S category from the 1970s through the late 1990s, when counterexamples were finally found by Iwase [**98**]. In this section we will examine the phenomenon responsible for all known instances of the failure of the Ganea condition.

We want a criterion for $\mathrm{cat}(X \times S^k) \leq n$. Consider the subspace

$$\overbrace{G_n(X) \times \{*\} \cup G_{n-1}(X) \times S^r}^{Q_n(X \times S^r)} \subseteq G_n(X) \times S^r,$$

which we will generally refer to simply as Q_n.

Problem 9.124.

(a) Show that $\mathrm{cl}(Q_n) \leq n$.

(b) Show that $Q_n \to G_n(X) \times S^r$ is surjective on $[\Sigma A, ?\,]$ for any space A.
HINT. Use Problem 9.25(a).

(c) Show that $\mathrm{cat}(X \times S^r) \leq n$ if the composite map

$$\begin{tikzcd} Q_n \arrow[rr] \arrow[dr] & & X \times S^r \\ & G_n(X) \times S^r \arrow[ur, "p_n \times \mathrm{id}_{S^r}"'] & \end{tikzcd}$$

has a section.

Suppose we have a space $X = W \cup_\alpha CA$ for some map $\alpha : A \to W$, where $\mathrm{cat}(X) = n + 1 > \mathrm{cat}(W)$. Thus the attachment of the cone has actually increased the category, and we know that all of the maps $h \in \mathscr{H}(\alpha)$ must be nontrivial.

Now we note that S^r is the mapping cone of the unique and trivial map $t : S^{r-1} \to *$, so we may apply the functorial construction of Section 9.1.3 to obtain the diagram

$$\begin{tikzcd} A * S^{r-1} \arrow[r] \arrow[d] & T(f,t) \arrow[r] \arrow[d] & Z \times S^r \arrow[d, "\sigma \times \mathrm{id}_{S^r}"] \arrow[dl, dotted] \\ F_{n-1}(X) * S^{r-1} \arrow[r] & T(j,t) \arrow[r] & G_n(X) \times S^r. \end{tikzcd}$$

Now we show that X will not satisfy the Ganea condition if the attaching map α has a 'Hopf invariant' h which vanishes after being suspended some number of times.

[8]Often referred to as **Ganea's conjecture**, though he only asked the question and did not offer an opinion about the answer.

Proposition 9.125. *If even one of the maps $h \in \mathscr{H}(\alpha)$ satisfies $\Sigma^r h \simeq *$, then*
$$\operatorname{cat}(X \times S^r) = n < \operatorname{cat}(X) + 1.$$

Problem 9.126.

(a) Show that $T(j,t) = Q_n$.

(b) Show that the left map is $\Sigma^r h$, where $h : A \to F_n(X)$.

(c) Prove Proposition 9.125.

Later we will study Hopf invariants in more detail, and it will emerge that there are maps $\alpha : S^n \to S^m$ whose Hopf invariants become trivial after suspension.

9.8. Additional Problems and Projects

Problem 9.127. Let X be an H-space with product $\Omega\mu$. Show that $\Omega\mu$ gives ΩX the structure of an H-space.

We already know that ΩX is an H-space with the multiplication given by concatenation of paths. Is it possible that ΩX has two distinct products?

Problem 9.128. Let τ denote the trivial path.

(a) Show that $L : \Omega X \to \Omega X$ given by $\omega \mapsto \omega * \tau$ and $R : \Omega X \to \Omega X$ given by $\omega \mapsto \tau * \omega$ are homotopic to $\mathrm{id}_{\Omega X}$.

(b) Show that if X is an H-space with multiplication $\mu : X \times X \to X$, then the products $\Omega\mu : \Omega X \times \Omega X \to \Omega X$ and $? * ? : \Omega X \times \Omega X \to \Omega X$ are homotopic to one another.

Project 9.129. Dualize the ideas of Section 9.4. Start by defining a principal fibration to be a map $f : X \to Y$ which fits into a fiber sequence $X \to Y \to B$. Define the fiber length of a map and a space, etc.

Project 9.130. Find formulas for $L_{\mathcal{A}}(f \times g)$. You will need to impose conditions on the collection $\mathcal{A}$.

Problem 9.131. Suppose $\Sigma : [X, X] \to [\Sigma X, \Sigma X]$ is onto.

(a) Explain why $[\Sigma X, \Sigma X]$ is a group.

(b) Show that if $f : \Sigma X \to \Sigma X$, then the induced maps
$$f_* : [\Sigma X, \Sigma X] \longrightarrow [\Sigma X, \Sigma X] \qquad \text{and} \qquad f^* : [\Sigma X, \Sigma X] \longrightarrow [\Sigma X, \Sigma X]$$
are both homomorphisms.

(c) Show that if $[\Sigma X, \Sigma X] \cong \mathbb{Z}$, then $\mathrm{id}_{\Sigma X}$ is a generator for $[\Sigma X, \Sigma X]$.

(d) Show that composition makes $[\Sigma X, \Sigma X]$ into a ring isomorphic to $\mathbb{Z}$.

Project 9.132. Prove the Lusternik-Schnirelmann theorem relating L-S category to critical points.

Problem 9.133. The **symmetric square** of a space X is the pushout in

$$\begin{tikzcd} X \vee X \arrow[r] \arrow[d, "\text{fold}"'] & X \times X \arrow[d] \\ X \arrow[r] & X^{[2]}. \end{tikzcd}$$

Show that X is an H-space if and only if the inclusion $X \hookrightarrow X^{[2]}$ has a right homotopy inverse.

Problem 9.134. Let $f : A \to X$ and let $g : B \to Y$.

(a) Determine the cofiber of the inclusion $X \times Y \hookrightarrow T(f, g)$.

(b) Let Q be the cofiber of $X \times Y \hookrightarrow C_f \times C_g$. Show that there is a cofiber sequence $A * B \to (\Sigma A \rtimes Y) \vee (X \ltimes \Sigma B) \to Q$.

(c) Let R be the cofiber of $X \wedge Y \hookrightarrow C_f \wedge C_g$, and show that there is a cofiber sequence $A * B \to (\Sigma A \wedge Y) \vee (X \wedge \Sigma B) \to R$.

Problem 9.135.

(a) Show that a CW complex X is a co-H-space if and only if it is a retract of a suspension.

(b) Show that if CW complexes X and Y are co-H-spaces, then $X \wedge Y$ is a cocommutative co-H-space.

Problem 9.136. Let $F \to E \to B$ be a fibration sequence. Show that

(a) Show that $\mathrm{cat}(E) + 1 \leq (\mathrm{cat}(i) + 1)(\mathrm{cat}(p) + 1)$.

(b) Show that if either $\mathrm{cat}(i) = 0$ or $\mathrm{cat}(p) = 0$, then equality holds.

Problem 9.137. Consider the evaluation map $@_{0,1} : X^I \to X \times X$ given by $@_{0,1}(\omega) = (\omega(0), \omega(1))$. The pullback by $\Delta : X \to X \times X$ is the **free loop space** on X, denoted $\Lambda(X)$.

(a) Show that $@_{0,1}$ is a fibration with fiber $\Omega(X)$.

(b) Show that $\Lambda(X) \to X$ has a section, and conclude that in the sequence

$$\Omega X \xrightarrow{\partial} \Omega X \longrightarrow \Lambda(X) \longrightarrow X$$

the connecting map ∂ is trivial.

(c) Show that $\mathrm{cat}(\Lambda(X)) \geq \mathrm{cat}(\Omega X)$.

Problem 9.138. The **reduced diagonal** map $\overline{\Delta}_n : X \to X^{\wedge n}$ is the composition

$$\begin{tikzcd} X \arrow[rr, "\overline{\Delta}_n"] \arrow[dr, bend right, "\Delta"'] & & X^{\wedge n} \\ & X^n, \arrow[ur, bend right, "q"'] & \end{tikzcd}$$

where $X^{\wedge n} = \overbrace{X \wedge X \wedge \cdots \wedge X}^{n}$ and q is the canonical quotient map. The **weak category** of X, denoted $\mathrm{wcat}(X)$, is the the least integer n for which $\overline{\Delta}_{n+1} \simeq *$.

(a) Show that $\mathrm{wcat}(X) \leq \mathrm{cat}(X)$.

(b) Show that if $A \to X \to Y$ is a cofiber sequence, then

$$(\mathrm{wcat}(X) + 1) \leq (\mathrm{wcat}(A) + 1)(\mathrm{wcat}(Y) + 1).$$

Problem 9.139. Suppose $f : A \to X$ is a map of connected CW complexes, and suppose that $\Sigma f : \Sigma A \to \Sigma X$ is a homotopy equivalence.

(a) Show that if Q is a connected CW complex, then $f \wedge \mathrm{id}_Q : A \wedge Q \to X \wedge Q$ is a homotopy equivalence.

Hint. First prove it by induction for Q finite. For the infinite case, express Q as a colimit of its finite subcomplexes.

(b) Show that if $g : B \to Y$ is a map of connected CW complexes that suspends to a homotopy equivalence, then $f \wedge g$ is a homotopy equivalence.

(c) Show that $\mathrm{wcat}(A) \leq \mathrm{wcat}(X)$.

(d) Show that if $\Sigma X \simeq *$, then $\mathrm{wcat}(X) \leq 1$.[9]

Carefully evaluate the extent to which the connectedness hypotheses are necessary.

[9] Such spaces do exist.

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Chapter 10

Model Categories

Homotopy theorists like to axiomatize their objects of study. It is a useful exercise first of all because it helps to focus attention on the key features of the theory and isolates the technical issues. It is also useful because it allows us to apply the techniques we develop in other contexts. Also, it has a historical basis in the seminal work of Eilenberg and Steenrod.

The theory we have described so far was essentially developed in the 1950s by Barratt, Puppe and others. In the late 1960s, Quillen (and Kan and others) set about trying to axiomatize homotopy theory. In Quillen's view, the key feature of the category $\mathcal{T}_*$ is that there are three special kinds of morphisms: cofibrations, fibrations and weak equivalences[1] satisfying certain formal properties. A (closed) **model category** is a category $\mathcal{M}$ in which these conditions hold, and a great deal of the theory we have developed so far is valid in *any* model category.

The basic theorem is a construction of a **homotopy category** $\mathrm{H}\mathcal{M}$ and a functor $\mathrm{Ho} : \mathcal{M} \to \mathrm{H}\mathcal{M}$, which is initial among all functors that invert the weak equivalences. It is fairly common to have a category $\mathcal{M}$ and a class of maps $\mathcal{W}$ which we wish to invert; but doing it willy-nilly leads to serious set-theoretical problems. Model categories solve these problems, when the class $\mathcal{W}$ can be taken to be the weak equivalences of a model structure.

We do not propose to develop the entire abstract theory of model categories here; rather we present here a kind of guide to the essential points of the basic theory in the hope that it will clarify some of the key features of the 'nonabstract' theory that we are developing.

[1] It is one of the crucial insights that weak equivalences are used here and not homotopy equivalences.

Accordingly, the problems here may be more challenging; and substantial results are proposed as projects. Projects, remember, are intended to be major problems that may very well require some independent inquiries into the literature.

10.1. Model Categories

A **model category** is a category $\mathcal{M}$ which has three special classes of morphisms, i.e., **cofibration**, **fibration** and **weak equivalence**, that satisfy the following five properties:

(CM1) Every diagram $F : \mathcal{I} \to \mathcal{M}$ has both a colimit and a limit.[2]

(CM2) In any diagram of the form

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Z \\ & Y, \arrow[ur, "h"'] & \end{tikzcd}$$

if any two of the maps f, g and h are weak equivalences, then so is the third.

(CM3) If f is a retract (in the category $\mathrm{mor}(\mathcal{M})$) of g, and g is a weak equivalence, then so is f; similarly, the classes of cofibrations and of fibrations are closed under retracts.

(CM4) In a square of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & E \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

in which j is a cofibration and p is a fibration, the dotted arrow can be filled in if either j or p is a weak equivalence.

(CM5) Every morphism $f : X \to Y$ has factorizations

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "i"'] \arrow[dr, "f" description] & \overline{Y} \arrow[d, "q"] \\ \overline{X} \arrow[r, "p"] & Y \end{tikzcd}$$

in which: i is a cofibration and a weak equivalence and p is a fibration; and j is a cofibration and q is a fibration and a weak equivalence.

[2] I.e., the category $\mathcal{M}$ should be **complete** and **cocomplete**.

For us, a diagram is a functor $F : \mathcal{I} \to \mathcal{M}$, where $\mathcal{I}$ is a *small* category. Some authors do study 'large' diagrams, but they do not assume that such diagrams have limits or colimits. In some expositions of model categories, axiom (CM1) is limited only to diagrams defined on *finite* shape categories. Hovey [**91**] assumes that the factorizations in (CM5) are natural, since this is the case in all known examples. Axiom (CM4) is precisely the Fundamental Lifting Property.

A morphism that is both a weak equivalence and a cofibration is called an **acyclic cofibration**, and **acyclic fibration** is defined similarly (these are also known as **trivial** cofibrations and fibrations). Two objects $X, Y \in \mathcal{M}$ are called **weakly equivalent** if there is a sequence of morphisms

$$X \longrightarrow W_0 \longleftarrow W_1 \to \cdots \leftarrow W_n \longrightarrow Y$$

each of which is a weak equivalence. The Fundamental Lifting Property actually characterizes the cofibrations and the fibrations.

Problem 10.1.

(a) Given the classes of weak equivalences and fibrations in $\mathcal{M}$, show that $i : A \to X$ is a cofibration if and only if in any diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

in which p is an acyclic fibration, the lift exists.

HINT. Factor i as $A \to \overline{X} \to X$; show $A \to X$ is a retract of $A \to \overline{X}$.

(b) Show that if the weak equivalences and the cofibrations in $\mathcal{M}$ are given, then p is a fibration if and only if the lift exists whenever i is an acyclic cofibration.

Problem 10.2. Let $\mathcal{M}$ be a model category, and let $A, B \in \mathcal{M}$. Show that $A \downarrow \mathcal{M}$ and $\mathcal{M} \downarrow B$ are model categories.

This applies, in particular, to $* \downarrow \mathcal{M}$, where $*$ is the terminal object in $\mathcal{M}$. This is $\mathcal{M}_*$, the **pointed model category** associated to $\mathcal{M}$.

Theorem 10.3 (Strøm)**.** *The category* **Top** *is a model category in which the weak equivalences are the homotopy equivalences, the fibrations are the Hurewicz fibrations and the cofibrations are the closed cofibrations.*

We have already proved the vast majority of this theorem, albeit in the category $\mathcal{T}_\circ$, which is slightly different from **Top**. According to Problem 10.2, the pointed category $\mathcal{T}_* = * \downarrow \mathcal{T}_\circ$ automatically inherits a model structure from $\mathcal{T}_\circ$. However, this automatic structure does *not* reflect pointed homotopy theory as we have developed it.

Exercise 10.4.

(a) What is the difference between the homotopy theory of $* \downarrow \mathcal{T}_\circ$ and the homotopy theory of $\mathcal{T}_*$?

(b) In fact, it appears very difficult, if not impossible, to put the homotopy theory of $\mathcal{T}_*$ entirely into the framework of model categories. Why?

The structure given by Theorem 10.3 should be referred to as the **Hurewicz model structure** (or the **Strom model@Strøm model structure**), since it is based, ultimately, on the Hurewicz definition of a fibration (and it was worked out by Strøm in the paper [**164**]). In Chapter 15 we will define another model category structure on the categories of topological spaces, called the **Serre model structure** because it is based on a notion of fibration first defined by J.-P. Serre. We will argue later that for most day-to-day homotopy theory, the two structures are essentially indistinguishable.

To show that model categories are not simply an empty exercise, we offer an example of a model category that has nothing to do with topology.

Project 10.5. Let R be a ring (with unit), and let $\mathbf{Chain}_R$ denote the category of nonnegatively graded chain complexes over R. Define

(1) $f : C \to D$ to be a weak equivalence if $H_*(f)$ is an isomorphism;

(2) $f : C \to D$ to be a cofibration if each map $C_k \to D_k$ is injective and has an R-projective cokernel,

(3) $f : C \to D$ to be a fibration if each map $C_k \to D_k$ is surjective.

Show that this defines a model structure on $\mathbf{Chain}_R$. (See Section A.4 for definitions.)

Fibrant and Cofibrant Objects. Any model category $\mathcal{M}$ has both an initial object $\varnothing$ and a terminal object $*$ (which are the same if $\mathcal{M}$ is a **pointed model category**). An object X in a model category $\mathcal{M}$ is called **cofibrant** if the map $\varnothing \to X$ is a cofibration; dually, X is called **fibrant** if $X \to *$ is a fibration.

Problem 10.6. Show that every object $X \in \mathcal{M}$ is weakly equivalent to a fibrant object and that if X is cofibrant, then the weakly equivalent object can be chosen to be cofibrant too. State and prove the dual result, and conclude that every object in X is weakly equivalent to an object which is simultaneously fibrant and cofibrant.

Problem 10.7. Show that an object $X \in \mathcal{M}$ is cofibrant if and only if in any diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[r, "f"] \arrow[ur, dotted, bend left] & B \end{tikzcd}$$

in which p is a fibration and a weak equivalence, the lift exists. Dually, show that Y is fibrant if and only if in any diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & Y \\ X \arrow[ur, dotted, bend right] & \end{tikzcd}$$

in which i is a cofbration, the extension exists.

Duality in Model Categories. We have seen a great many instances of duality in our work so far in this book. Some of it is straightforward categorical duality, but other ideas and constructions seem to come in dual pairs, for example: cofibrations and fibrations; suspensions and loop spaces; homotopy colimit and homotopy limit; and so on. This extension of the concept of duality into homotopy theory is known as **Eckmann-Hilton duality**, and they are best explained by a formal dualization that can be done in any model category.

We dualize a statement about pure category theory by simply interchanging domains and targets. Since the axioms of a category are permuted by this operation, the dual of the proof of any such statement proves the dual statement. In a model category, dualization also requires us to interchange the words 'fibration' and 'cofibration'.

Problem 10.8.

(a) Show that the dual of each axiom for a model category is also an axiom, and conclude that the dual of every theorem about model categories is also a theorem.

(b) Show that it is equivalent to say that if $\mathcal{M}$ is a model category, then so is its opposite category $\mathcal{M}^{\mathrm{op}}$.

This very precise definition clarifies the extent of the Eckmann-Hilton duality: statements that follow from the model category axioms automatically dualize; and statements proved using special properties of the objects of $\mathcal{M}$ may dualize, but not automatically. Therefore, it is very useful to keep close track, when studying homotopy theory, of results that are essentially formal consequences of the axioms and which are proved using special properties of the category of spaces.

10.2. Left and Right Homotopy

In this section we study the two approaches to homotopy that are available in a model category. A **cylinder object** for an object X to be an object $\mathrm{Cyl}(X)$ that fits into a diagram of the form

$$\begin{tikzcd} X \sqcup X \arrow[d, "i"'] \arrow[dr, bend left, "\nabla"] & \\ \mathrm{Cyl}(X) \arrow[r, "p"'] & X, \end{tikzcd}$$

in which i is a cofibration and p is a weak equivalence; we do not assume that $\mathrm{Cyl}(X)$ has been constructed from X in any particular way, or that it is functorial. Dually, a **path object** for an object Y is an object $\mathrm{Path}(Y)$ which figures in a diagram of the form

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[dr, bend right, "\Delta"'] & \mathrm{Path}(Y) \arrow[d, "q"] \\ & Y \times Y, \end{tikzcd}$$

where q is a weak equivalence and j is a cofibration. These should look very familiar to you, as we introduced these ideas in Section 4.6 when we discussed duality and homotopy.

Problem 10.9. Show that in any model category, every object has at least one cylinder object and at least one path object.

In a general model category it does not follow automatically that homotopy in terms of cylinders is the same as homotopy in terms of path spaces.

Let $f, g : X \to Y$ in a model category $\mathcal{M}$. Then f and g are **left homotopic** if there is a cylinder object $\mathrm{Cyl}(X)$ and a map $H : \mathrm{Cyl}(X) \to Y$ making the diagram

$$\begin{tikzcd} X \sqcup X \arrow[d] \arrow[dr, bend left, "(f{,}g)"] & \\ \mathrm{Cyl}(X) \arrow[r, "H"] & Y \end{tikzcd}$$

commutative. Dually, f and g are **right homotopic** if there is a path object $\mathrm{Path}(Y)$ and a map $K : X \to \mathrm{Path}(Y)$ making the diagram

$$\begin{tikzcd} X \arrow[r, "K"] \arrow[dr, bend right, "(f{,}g)"'] & \mathrm{Path}(Y) \arrow[d, "q"] \\ & Y \times Y \end{tikzcd}$$

commutative. Maps f and g are **homotopic** if they are simultaneously left and right homotopic.

We presented homotopy of maps $X \to Y$ of spaces as an equivalence relation on $\mathrm{map}(X, Y)$. But in a general model category, left and right homotopy can be different relations, and neither needs to be an equivalence relation. If the domains and/or targets are well-behaved, then the two notions do coincide, and they are equivalence relations. In this context, 'well-behaved' means fibrant (for targets) or cofibrant (for domains).

Project 10.10. Let $f, g : X \to Y$ in $\mathcal{M}$.

(a) Show that if X is cofibrant and f is left homotopic to g, then f is also right homotopic to g.

(b) Show that if X is cofibrant, then left homotopy is an equivalence relation on $\mathrm{mor}_{\mathcal{M}}(X, Y)$.

Dually, if X is fibrant and f is right homotopic to g, then f is also left homotopic to g and right homotopy is an equivalence relation on $\mathrm{mor}_{\mathcal{M}}(X, Y)$. It follows that if X is cofibrant and Y is fibrant, then we have a well-defined and ambidextrous notion of homotopy on $\mathrm{mor}_{\mathcal{M}}(X, Y)$, which we simply call **homotopy**. The set of homotopy classes of morphisms from X to Y is denoted $[X, Y]$.

In view of these results, it makes sense to focus on those objects that are simultaneously fibrant and cofibrant. Moreover, because of Problem 10.6 we know that every object is weakly equivalent to such an object.

Homotopy Equivalence. Simultaneously fibrant and cofibrant objects X and Y are **homotopy equivalent** if there are morphisms

$$f : X \longrightarrow Y \qquad \text{and} \qquad g : Y \longrightarrow X$$

and homotopies $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Problem 10.11. Show that if X and Y are simultaneously fibrant and cofibrant, then they are weakly equivalent if and only if they are homotopy equivalent.

Homotopy Invariant Functors. If $\mathcal{M}$ is a model category and $F : \mathcal{M} \to \mathcal{C}$, then we say that F is **homotopy invariant** if whenever f is a weak equivalence in $\mathcal{M}$, $F(f)$ is an isomorphism in $\mathcal{C}$.

Problem 10.12. Let $F : \mathcal{M} \to \mathcal{C}$ be a homotopy invariant functor. Show that if $f, g : X \to Y$ are left homotopic or right homotopic (or both), then $F(f) = F(g)$ are the same morphism in $\mathcal{C}$.

HINT. Use the universal example $f = i_0 : X \to \mathrm{Cyl}(X)$ and $g = i_1 : X \to \mathrm{Cyl}(X)$.

10.3. The Homotopy Category of a Model Category

Let $\mathrm{H}\mathcal{M}_{\mathrm{CF}}$ be the category whose objects are the simultaneously fibrant and cofibrant objects of $\mathcal{M}$ and whose morphisms are the sets $[X, Y]$ of homotopy classes of morphisms from X to Y.

Problem 10.13. Show that $\mathrm{H}\mathcal{M}_{\mathrm{CF}}$ is a category and that there is a functor $Q : \mathcal{M} \to \mathrm{H}\mathcal{M}_{\mathrm{CF}}$ such that $Q(X)$ is weakly equivalent in $\mathcal{M}$ to X for all $X \in \mathcal{M}$.

The **homotopy category** of $\mathcal{M}$ is the category $\mathrm{H}\mathcal{M}$ defined by

$$\begin{aligned} \mathrm{ob}(\mathrm{H}\mathcal{M}) &= \mathrm{ob}(\mathcal{M}), \\ \mathrm{mor}_{\mathrm{H}\mathcal{M}}(X, Y) &= [Q(X), Q(Y)]. \end{aligned}$$

There is a 'quotient functor' $\mathrm{Ho} : \mathcal{M} \to \mathrm{H}\mathcal{M}$ given by $\mathrm{Ho}(X) = X$ and $\mathrm{Ho}(f) = [Q(f)]$.

Theorem 10.14. *The functor* $\mathrm{Ho} : \mathcal{M} \to \mathrm{H}\mathcal{M}$ *is homotopy invariant, and if* $F : \mathcal{M} \to \mathcal{C}$ *is any other homotopy invariant functor, then there is a unique extension* Φ *in the diagram*

$$\begin{tikzcd} & \mathcal{M} \arrow[dl, "\mathrm{Ho}"'] \arrow[dr, "F"] & \\ \mathrm{H}\mathcal{M} \arrow[rr, dotted, "\exists!\,\Phi"] & & \mathcal{C}. \end{tikzcd}$$

Interestingly, the statement of Theorem 10.14 is concerned only with the weak equivalences and makes no mention of the fibrations and cofibrations. One profitable, and nonhomotopy-theoretical, point of view on the theory of model categories is that it is an elaborate apparatus for construction *localizations* of categories.

Suppose $\mathcal{C}$ is a category and we have a class of morphisms $\mathcal{W}$ in $\mathcal{C}$; and suppose we would like the morphisms in $\mathcal{W}$ to be isomorphisms. We can achieve this by looking for functors $F : \mathcal{C} \to \mathcal{D}$ with the property that $F(w)$ is an isomorphism in $\mathcal{D}$ for each $w \in \mathcal{W}$. The **localization** of $\mathcal{C}$ is an initial object in the category of all such functors; it is sometimes written $L : \mathcal{C} \to \mathcal{W}^{-1}\mathcal{C}$. Thus, Theorem 10.14 asserts that if $\mathcal{W}$ can be taken as the class of weak equivalences in some model category structure on $\mathcal{C}$, then the localization $\mathcal{W}^{-1}\mathcal{C}$ exists.

10.4. Derived Functors and Quillen Equivalence

A relation called **Quillen equivalence** has emerged as the 'correct' notion of equivalence for model categories. In this section we set down the basic definitions and theorems.

10.4.1. Derived Functors. Let $F : \mathcal{M} \to \mathcal{C}$ be a functor, where $\mathcal{M}$ is a model category (we do not assume that $\mathcal{C}$ is a model category). Then we may ask if F is homotopy invariant, which is equivalent to asking whether there is a functor Φ making the diagram

$$\begin{tikzcd} & \mathcal{M} \arrow[dl, "\mathrm{Ho}"'] \arrow[dr, "F"] & \\ \mathrm{H}\mathcal{M} \arrow[rr, dotted, "\Phi"] & & \mathcal{C} \end{tikzcd}$$

commute. Suppose there is no such functor—what then? Rather than giving up, we relax the requirement $\Phi \circ \mathrm{Ho} = F$ and ask if there is a natural transformation $\Phi \circ \mathrm{Ho} \to F$. There are probably many such functors Φ, and they form a category whose morphisms are natural transformations. If this category has a terminal object, we call it the **left derived functor** of F and we denote it $\mathrm{DER}_L(F) : \mathrm{H}\mathcal{M} \to \mathcal{N}$.

Problem 10.15. Show that the left derived functor of F, if it exists, is unique (up to natural isomorphism).

Project 10.16. Show that if $F(f)$ is an isomorphism for all weak equivalences $f : A \to B$ with A and B cofibrant, then F has a left derived functor.

Problem 10.17. Show that if X is cofibrant, then $\mathrm{DER}_L(F)(X) \to F(X)$ is an equivalence.

If $F : \mathcal{M} \to \mathcal{N}$ is a functor from one model category to another, then we say that F has a **total left derived functor** $\mathrm{H}\mathcal{M} \to \mathrm{H}\mathcal{N}$ if the composite

$$\mathcal{M} \xrightarrow{F} \mathcal{N} \xrightarrow{h} \mathrm{Ho}\mathcal{N}$$

has a left derived functor, denoted $\mathbf{L}(F)$.

We can of course dualize and obtain a concept of a right derived functor $\mathrm{DER}_R(F)$ and a total right derived functor $\mathbf{R}(F)$.

10.4.2. Quillen Equivalence of Model Categories. Now we define the appropriate notion of equivalence between model categories.

Theorem 10.18. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, and let*

$$L : \mathcal{M} \longrightarrow \mathcal{N} \qquad \text{and} \qquad R : \mathcal{N} \longrightarrow \mathcal{M}$$

be an adjoint pair of functors. If L preserves cofibrations and R preserves fibrations, then the total derived functors

$$\mathbf{L}(L) : \mathrm{H}\mathcal{M} \longrightarrow \mathrm{H}\mathcal{N} \qquad \text{and} \qquad \mathbf{R}(R) : \mathrm{H}\mathcal{N} \longrightarrow \mathrm{H}\mathcal{M}$$

exist and are adjoint to one another.

Project 10.19. Prove Theorem 10.18.

Problem 10.20. Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, and let

$$L : \mathcal{M} \longrightarrow \mathcal{N} \qquad \text{and} \qquad R : \mathcal{N} \longrightarrow \mathcal{M}$$

be an adjoint pair of functors. Show that L preserves cofibrations and trivial cofibrations if and only if R preserves fibrations and trivial fibrations.

If L and R satisfy the conditions of Problem 10.20, then they are said to be a **Quillen adjunction**.

Problem 10.21. Show that if L and R are a Quillen adjunction, then they both preserve weak equivalences.

A Quillen adjunction $L : \mathcal{M} \to \mathcal{N}$ and $R : \mathcal{N} \to \mathcal{M}$ may satisfy an additional property: for any $X \in \mathcal{M}$ and $Y \in \mathcal{N}$, a morphism $L(X) \to Y$ is a weak equivalence in $\mathcal{N}$ if and only if its adjoint $X \to R(Y)$ is a weak equivalence in $\mathcal{M}$. If L and R satisfy this condition, then the adjunction is called a **Quillen equivalence**.

Project 10.22. Show that if L and R are a Quillen equivalence, then their total derived functors are an equivalence of the categories $\mathrm{H}\mathcal{M}$ and $\mathrm{H}\mathcal{N}$. Conclude that any homotopy-theoretical question will have the same answer in both categories.

10.5. Homotopy Limits and Colimits

The theory of homotopy limits and colimits can be fit into the framework of model categories. If $\mathcal{I}$ is a simple category, then $\mathcal{M}^{\mathcal{I}}$ can be given the structure of a model category. Then it is possible to show that the functor $\mathrm{colim} : \mathcal{M}^{\mathcal{I}} \to \mathcal{M}$ has a total left derived functor,

$$\mathbf{L}(\mathrm{colim}) : \mathrm{H}(\mathcal{M}^{\mathcal{I}}) \longrightarrow \mathrm{H}\mathcal{M},$$

and this is the homotopy colimit. We'll see that this construction is precisely what we did in our development.

Because of duality, it suffices to work only with homotopy colimits. Note that $\mathrm{H}(\mathcal{M}^{\mathcal{I}})$ is isomorphic in the two cases, since the weak equivalences are the same.

10.5.1. A Model Structure for Diagram Categories.

We call a morphism $\phi : F \to G$ of $\mathcal{I}$-diagrams in the model category $\mathcal{M}$ a **pointwise fibration** if for each $i \in \mathcal{I}$, the morphism $\phi_i : F(i) \to G(i)$ is a fibration in $\mathcal{M}$; similarly, ϕ is a **pointwise weak equivalence** if each map ϕ_i is a weak equivalence in $\mathcal{M}$.

Because of Problem 10.1 if the pointwise fibrations and pointwise weak equivalences are part of a model structure on $\mathcal{M}^{\mathcal{I}}$, then they completely determine the cofibrations (via axiom (CM4)).

Theorem 10.23. *If $\mathcal{I}$ is a simple category, then there is a class of cofibrations making $\mathcal{M}^{\mathcal{I}}$ into a model category whose fibrations are the pointwise fibrations and whose weak equivalences are the pointwise weak equivalences.*

Project 10.24. Prove Theorem 10.23.

Which objects are cofibrant in $\mathcal{M}^{\mathcal{I}}$? The answer shouldn't surprise you.

Problem 10.25. Show that $F : \mathcal{I} \to \mathcal{M}$ is cofibrant if and only if each map $\operatorname{colim} F_{<i} \to F(i)$ is a cofibration in $\mathcal{M}$.

10.5.2. Homotopy Colimit. Now we have the following situation:

$$\begin{tikzcd} \mathcal{M}^{\mathcal{I}} \arrow[r, "\text{colim}"] \arrow[d, "\text{Ho}"'] & \mathcal{M} \arrow[d, "\text{Ho}"] \\ \text{H}(\mathcal{M}^{\mathcal{I}}) \arrow[r, dotted] & \text{H}\mathcal{M}, \end{tikzcd}$$

which is precisely the setup for defining a total left derived functor. So we define the **homotopy colimit** to be the total left derived functor

$$\text{hocolim} = L(\text{colim}) : \text{H}(\mathcal{M}^{\mathcal{I}}) \longrightarrow \text{H}\mathcal{M},$$

assuming it exists. Note that the very definition of homotopy colimit includes the fundamental homotopy invariance property.

Problem 10.26. Show that if the homotopy colimit exists, then pointwise weak equivalences $F \to G$ induce categorical equivalences of homotopy colimits.

Theorem 10.27. *If $\mathcal{I}$ is a direct category, then* hocolim *exists and it is left adjoint to* $\text{Ho}(\Delta) : \mathcal{M} \to \mathcal{M}^{\mathcal{I}}$.

Problem 10.28.

(a) Use Theorem 10.18 to prove Theorem 10.27.

(b) Show that the rule $F \mapsto \operatorname{colim} \overline{F}$, where $\overline{F}$ is a cofibrant replacement for F, defines a total left derived functor for colim, thereby proving Theorem 10.27.

Part 3

Four Topological Inputs

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Chapter 11

The Concept of Dimension in Homotopy Theory

We are fundamentally interested in the homotopy theory of CW complexes. CW complexes are not simply topological spaces: they have extra structure conferred upon them by virtue of their step-by-step construction. They are filtered by their skeleta, and it makes sense to talk about the dimension of a CW complex. In this section, we explore the uses of dimension in homotopy theory.

We have seen in Problem 4.92 that in order to decide whether or not a map $f : X \to Y$ is a homotopy equivalence, it is sufficient to show that for every space K, the induced map $f_* : [K, X] \to [K, Y]$ is bijective. But we may not be able to check this condition for all spaces K; perhaps we can only check it for CW complexes, or for CW complexes of dimension at most n. This leads to the concept of *n-equivalence*.

There is a related notion which estimates how nearly contractible a space is. A space X is contractible if and only if $[K, X] = *$ for all K. If we only know that $[K, X] = *$ for all CW complexes of dimension at most n, then we say that X is *n-connected*.

In this chapter we explore the relation between n-equivalence and n-connectivity. We establish five different reformulations of the concept of n-equivalence and use them to prove the celebrated J. H. C. Whitehead theorem: a map which induces isomorphisms $f_* : \pi_k(X) \to \pi_k(Y)$ for all k induces bijections $f_* : [K, X] \to [K, Y]$ for *all* CW complexes K.

11.1. Induction Principles for CW Complexes

The step-by-step construction of CW complexes makes it possible to study them by induction on their skeleta. Straightforward induction gives results about all *finite-dimensional* CW complexes but does not provide information about the infinite-dimensional ones. This section contains some technical results that facilitate the jump from finite-dimensional to infinite-dimensional.

11.1.1. Attaching One More Cell. If X is a CW complex and $K \subseteq X$ is a subcomplex, then it may happen that $K_0 = X_0$, that is, that the 0-skeleta coincide. If they are the same, then it may be that $X_1 = K_1$, and so on. If K is a *proper* subcomplex, then there must be some n for which $K_n \neq X_n$. If n is the smallest dimension for which $X_n \neq K_n$, then we can build a new subcomplex $L \subseteq X$ by attaching to K one of the n-cells of X that it does not contain. This proves the following.

Lemma 11.1. *If X is a CW complex and $K \subseteq X$ is a proper subcomplex, then there is another subcomplex L such that*

(a) $K \subsetneq L \subseteq X$ and

(b) $L = K \cup_\lambda D^n$ for some map $\lambda : S^{n-1} \to K$.

The proofs of many statements about CW complexes make essential use of Lemma 11.1, in the following way. We wish to show that some property is true of X, so we let $K \subseteq X$ be a subcomplex which is maximal with that property,[1] and we hope to show that $K = X$. So we assume that K is a proper subcomplex of X and find a slightly larger subcomplex L using Lemma 11.1. Using the close connection between K and L, we then prove that the property holds for L, contradicting the maximality of K, and thereby proving that X has the desired property. We'll refer to this kind of argument as a proof by **CW induction**.

11.1.2. Composing Infinitely Many Homotopies. Our second induction principle addresses the following question: suppose that for each n you have a homotopy $H_n : f_n \simeq f_{n+1}$. Can we piece these homotopies together to obtain a homotopy from f_1 to some other map, g?

Proposition 11.2. *Let $f : X \to Y$, where X is a CW complex. Suppose there is an infinite sequence of maps $f_n : X \to Y$ (for $n \geq 0$) and homotopies $H_n : f_n \simeq f_{n+1}$ which satisfy the following conditions:*

- *$f = f_1$,*
- *there is a function $z : \mathbb{N} \to \mathbb{N}$ such that for each $m \geq z(n)$, $H_m|_{X_n \times I}$ is the constant homotopy at a certain function g_n.*

[1]The main tool for proving the existence of maximal gadgets is Zorn's lemma.

Then the function $g : X \to Y$ defined by $g|_{X_n} = g_n$ is continuous and homotopic to f.

Problem 11.3. Reparametrize $H_0 : f \simeq f_0$ from $t = 0$ to $t = \frac{1}{2}$; call the reparametrized homotopy J_0. More generally, let J_n be the reparametrization of H_n over the interval $[1 - \frac{1}{n+1}, 1 - \frac{1}{n+2}]$.

(a) Show that the map g defined in Proposition 11.2 is continuous.

(b) Show that the homotopies J_n for $n \geq 0$ glue together to give a continuous function $\widetilde{J} : X \times [0, 1) \to Y$.

(c) Show that $\widetilde{J}$ can be extended to a homotopy $J : f \simeq g$.

As an application of infinite concatenation of homotopies we establish a criterion for the contractibility of a CW complex.

Proposition 11.4. *Let X be a connected pointed CW complex. If there is a pointed homotopy $H : \mathrm{id}_X \simeq f$ where $f(X_n) \subseteq X_{n-1}$ for $n \geq 1$, then $X \simeq *$.*

Problem 11.5. Let X be a connected pointed CW complex.

(a) Show that id_X is homotopic in $\mathcal{T}_*$ to a cellular map $g : X \to X$ such that $g(X_0) = *$.

(b) Show that under the hypotheses of Proposition 11.4, $\mathrm{id}_X \simeq \phi$ in $\mathcal{T}_*$, where $\phi(X_n) \subseteq X_{n-1}$ for all $n \geq 1$ and $\phi(X_0) = *$.

(c) Show that $\phi \simeq *$ and derive Proposition 11.4.

11.2. n-Equivalences and Connectivity of Spaces

In this section, we introduce a system of approximations to the notion of homotopy equivalence, called n-equivalences. There is an analogous collection of measures of the triviality of spaces, called n-connectivity.

11.2.1. n-Equivalences. An unpointed map $f : X \to Y$ may be made pointed by choosing a basepoint $x \in X$ and considering it as a map $(X, x) \to (Y, f(x))$, which we denote by f_x. Thus each unpointed map f gives rise to a huge collection $\{f_x \mid x \in X\}$ of pointed maps. We say that an unpointed map f between *nonempty spaces* is an n-**equivalence** if for every $x \in X$ the induced map

$$(f_x)_* : [K, X] \longrightarrow [K, Y]$$

is an isomorphism for every pointed CW complex K with $\dim(K) < n$ and is a surjection if $\dim(K) \leq n$. A *pointed* map f is an n-equivalence if the unpointed map $f_- : X_- \to Y_-$ (that results from forgetting the basepoints) is an n-equivalence of unpointed spaces.

An ∞**-equivalence** is a map which is an n-equivalence for each n. Such a map is also called a **weak homotopy equivalence** or simply a **weak equivalence**.

We would like to make weak equivalence into an equivalence relation between spaces, and it is tempting to say that X and Y are weakly equivalent if there is a weak equivalence $X \to Y$. But the existence of a weak equivalence $X \to Y$ does not imply the existence of a weak equivalence $Y \to X$, so this relation would not be symmetric. Instead, we let $\sim$ be the smallest equivalence relation on spaces such that if there is a weak equivalence $X \to Y$, then $X \sim Y$.

Exercise 11.6.

(a) Suppose we were to allow the empty space as a domain in our definition of n-equivalence. For each n, determine all spaces Y for which the unique map $f : \varnothing \to Y$ would be an n-equivalence.

(b) Show that if $f : X \to Y$ is an n-equivalence, then the induced map $f_* : \pi_k(X, x) \to \pi_k(Y, f(x))$ is an isomorphism for $k < n$ and a surjection for $k = n$, no matter what basepoint $x \in X$ is chosen.

(c) Criticize the following argument:

Because the functor $\langle K, X\rangle$ is naturally equivalent to $[K_+, X]$, a map $f : X \to Y$ is an n-equivalence if and only if the induced map $f_* : \langle K, X\rangle \to \langle K, Y\rangle$ is a bijection for $\dim(K) < n$ *and a surjection for* $\dim(K) = n$.

(d) Show that $X \sim Y$ if and only if there is a finite sequence of weak equivalences $X \to X_1 \leftarrow X_2 \to X_4 \leftarrow \cdots \to X_{n-1} \leftarrow X_n \to Y$.

This definition comes with a daunting list of conditions to verify: we need to check the infinitely many maps f_x and their induced maps on infinitely many functors $[K, ?\,]$. But this list can be drastically reduced and ultimately (for many spaces) made finite.

Problem 11.7. Let $f : X \to Y$.

(a) Show that if X and Y are path-connected, then f is an n-equivalence if and only if, for your favorite basepoint $x \in X$, the induced maps $(f_x)_* : [K, X] \to [K, Y]$ in $\mathcal{T}_*$ are isomorphisms for every pointed CW complex K with $\dim(K) < n$ and surjections for $\dim(K) \leq n$.

Hint. Use Problem 8.44.

(b) Show that in general, f is an n-equivalence if and only if $\pi_0(f)$ is a bijection and f satisfies the condition in part (a) for your favorite basepoint in each path component of X.

Problem 11.8. Let $f : X \to Y$ and $g : Y \to Z$. Suppose two of the three maps f, g and $g \circ f$ are n-equivalences (with $n \leq \infty$). What can you say about the third map?

Exercise 11.9. Criticize the following argument:

> *If $f : X \to Y$ is a homotopy equivalence in $\mathcal{T}_\circ$, then by choosing a basepoint $x \in X$, we get a map $f_x : (X, x) \to (Y, f(x))$ which is a homotopy equivalence in $\mathcal{T}_*$. It follows that f_x induces isomorphisms on all functors $[K, ?\,]$ so f is a weak homotopy equivalence.*

Problem 11.10. Show that if f and g are pointwise equivalent in $\mathrm{H}\mathcal{T}_\circ$, then f is an n-equivalence if and only if g is an n-equivalence.

11.2.2. Connectivity of Spaces. A space $X \in \mathcal{T}_\circ$ is **n-connected** if for every choice of basepoint $x \in X$, $[K, (X, x)] = *$ for all pointed CW complexes K with $\dim(K) \leq n$; we say that X is **∞-connected**, or **weakly contractible**, if it is n-connected for all n. We define connectivity of pointed spaces by forgetting that they are pointed.

Exercise 11.11. Show that 0-connected is synonymous with path-connected.

A 1-connected space is also called **simply-connected**.

Here are some useful reformulations of the concepts of n-connectivity.

Problem 11.12. Show that the following are equivalent:

(1) X is n-connected,

(2) $X \to *$ is an $(n+1)$-equivalence,

(3) $* \to X$ is an n-equivalence,

(4) $\pi_k(X, x) = *$ for all $k \leq n$ and all $x \in X$.

Hint. For (4) $\Rightarrow$ (1), show by induction on $m \leq n$ that a map $K \to X$ must factor through K/K_m.

Problem 11.13. Show that if $X \simeq Y$ in $\mathcal{T}_\circ$, then X is n-connected if and only if Y is n-connected.

The **connectivity** of a space X is the greatest n for which X is n-connected; we'll use the notation $\mathrm{conn}(X) = n$ to mean that X is n-connected but not $(n+1)$-connected. Connectivity is a homotopy invariant of spaces that measures how close X is to being the trivial space $*$. Sometimes n-equivalences are referred to as *n-connected maps.*

Problem 11.14.

(a) Suppose two of the three spaces in the fibration sequence $F \to E \to B$ are n-connected. What can you say about the third space?

(b) Suppose X is n-connected and Y is m-connected. What is the connectivity of $X \times Y$?

(c) How do the connectivities of X and ΩX compare?

Exercise 11.15. Criticize the following argument:

> *Since* $[X, Y]$ *is just a homotopy group of the space of maps* $\mathrm{map}_*(X, Y)$*, if* X *is* n*-dimensional and* Y *is* n*-connnected, then* $\mathrm{map}_*(X, Y)$ *is weakly contractible.*

11.3. Reformulations of n-Equivalences

In this section, we establish five alternate characterizations of n-equivalences. The first of these reduces the burden from studying $[K, X] \to [K, Y]$ for all K with $\dim(K) \leq n$ to just checking this for $K = S^k$ for $k \leq n$. The importance of the other new statements is analogous to the importance of the Fundamental Lifting Property: they guarantee the existence of maps satisfying various properties.

Interestingly, the definition of n-equivalence, which is fundamentally a property of unpointed maps, makes use of homotopy functors of pointed spaces. Can the concept be defined entirely within $\mathcal{T}_\circ$? As you read over Theorem 11.16, think about which parts, if any, do not involve pointed homotopies.

Theorem 11.16. *Let* $f : X \to Y$ *be a map in* $\mathcal{T}_\circ$ *such that* $\pi_0(f)$ *is onto, and let* $n \leq \infty$*. The following six statements, in three groups of two, are equivalent:*

(1) (a) For every $x \in X$*, the map* $(f_x)_* : \pi_k(X, x) \to \pi_k(Y, f(x))$ *is an isomorphism for* $k < n$ *and a surjection for* $k = n$*.*

(b) f *is an* n*-equivalence.*

(2) (a) In any strictly commutative diagram of the form

$$\begin{tikzcd}
S^k \arrow[rr, hook] \arrow[dr, "\alpha_1"] \arrow[dd, "\mathrm{in}_1"'] & & D^{k+1} \arrow[dl, dotted, "\beta_1"'] \arrow[dd, "\mathrm{in}_1"] \\
& X \arrow[dd, "f"] & \\
(D^{k+1} \times 0) \cup (S^k \times I) \arrow[rr, hook] \arrow[dr, "\beta_0 \cup H_{S^k}"'] & & D^{k+1} \times I \arrow[dl, dotted, "H_{D^{k+1}}"] \\
& Y &
\end{tikzcd}$$

with $k < n$*, the dotted arrows can be filled in to make the whole diagram strictly commutative.*

(b) *If A is any space and B is obtained from A by attaching cells of dimension at most n,*[2] *then in the strictly commutative diagram*

$$\begin{tikzcd}
A \arrow[rr, hook] \arrow[dr, "\alpha_1"] \arrow[dd, "\mathrm{in}_1"'] & & B \arrow[dl, dotted, "\beta_1"'] \arrow[dd, "\mathrm{in}_1"] \\
& X \arrow[dd, "f"] & \\
(B \times 0) \cup (A \times I) \arrow[rr, hook] \arrow[dr, "\beta_0 \cup H_A"'] & & B \times I \arrow[dl, dotted, "H_B"] \\
& Y &
\end{tikzcd}$$

the dotted arrows can be filled in so that the entire diagram commutes.

(3) (a) *In any strictly commutative diagram of the form*

$$\begin{tikzcd}
S^k \arrow[r, "\alpha_1"] \arrow[d, hook] & X \arrow[d, "f"] \\
D^{k+1} \arrow[r, "\beta_0"'] \arrow[ur, dotted, "\beta_1"] & Y
\end{tikzcd}$$

with $k < n$, the dotted arrow can be filled in so that the upper triangle commutes on the nose and the lower triangle commutes up to a homotopy which is constant on S^k.

(b) *If A is any space and B is obtained from A by attaching cells of dimension at most n, then in the strictly commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, hook] & X \arrow[d, "f"] \\
B \arrow[r, "\beta"'] \arrow[ur, dotted, "\lambda"] & Y
\end{tikzcd}$$

the dotted arrow can be filled in so that the upper triangle commutes on the nose, and the lower triangle commutes up to a homotopy which is constant on A.

One very useful consequence of the theorem is that the connectivity of a map can be determined by computing the connectivity of its fiber.

Corollary 11.17. *If $f : X \to Y$ is a map of path-connected pointed spaces, then f is an n-equivalence if and only if the homotopy fiber F is $(n-1)$-connected.*

[2] Thus B is a *relative* CW complex with dimension at most n.

Problem 11.18. Use Theorem 11.16 to prove Corollary 11.17.

Theorem 11.16(2)(b) is known as the **Homotopy Extension Lifting Property**, often abbreviated HELP.

Overview of the Proof of Theorem 11.16. The structure of this proof is a bit complicated; the plan is illustrated by the diagram

$$\begin{tikzcd}
1(b) \arrow[dd, Rightarrow] & & 2(b) \arrow[dl, Rightarrow] \arrow[dd, Rightarrow, bend left] \\
& 3(b) \arrow[ul, Rightarrow] \arrow[dd, Rightarrow] & \\
1(a) \arrow[rr, Rightarrow] & & 2(a) \arrow[dl, Rightarrow] \arrow[uu, Rightarrow, bend left] \\
& 3(a). \arrow[ul, Rightarrow] &
\end{tikzcd}$$

Since in each of the three parts, the '(a) statement' is a special case of the '(b) statement', the downward implications are trivially valid. We first prove that parts (1)(a), (2)(a) and (3)(a) imply one another cyclically. Then we show that (2)(a) and (2)(b) are equivalent. Finally we show that (2)(b) implies (3)(b) and (3)(b) implies (1)(b); since (1)(b) obviously implies (1)(a), this completes the proof.

11.3.1. Equivalence of the (a) Parts.

To prove that Theorem 11.16(1)(a) implies Theorem 11.16(2)(a) we have to be able to recognize the inclusion of a sphere into a disk when we see it.

Problem 11.19.

(a) Show there is a homeomorphism $u : D^{k+1} \to (D^{k+1} \times 0) \cup (S^k \times I)$ making the diagram

$$\begin{tikzcd}
S^k \arrow[r, "\mathrm{id}"] \arrow[d, "i"'] & S^k \arrow[d, "\mathrm{in}_1"] \\
D^{k+1} \arrow[r, "u"] & (D^{k+1} \times 0) \cup (S^k \times I)
\end{tikzcd}$$

strictly commutative.

(b) Show that there is a homeomorphism $v : D^{k+1} \to D^k \times I$ making the diagram

$$\begin{tikzcd} S^k \arrow[r] \arrow[d] & (D^k \times 0) \cup (S^{k-1} \times I) \cup (D^k \times 1) \arrow[d] \\ D^{k+1} \arrow[r, "v"] & D^k \times I \end{tikzcd}$$

strictly commutative.

Now we are prepared to prove that part 1(a) implies part 2(a).

Problem 11.20. Take Theorem 11.16(1)(a) as known.

(a) Show that α_1 extends to a map $\xi : D^{k+1} \to X$.

(b) Explain how the maps β_0, H_{S^k} and ξ define a map $Q : S^{k+1} \to Y$.

(c) Show that there is a map $R : S^{k+1} \to X$ such that $f \circ R \simeq Q$.

(d) Let $\beta_1 : D^{k+1} \to X$ be the composition

$$\begin{tikzcd} D^{k+1} \arrow[rr, "\beta_1"] \arrow[dr, bend right, "c"'] & & Y \\ & D^{k+1} \vee S^{k+1}, \arrow[ur, bend right, "(\xi, -R)"'] & \end{tikzcd}$$

where c is the map that collapses $\frac{1}{2}S^k$. Show that the map $S^{k+1} \to Y$ defined by the maps β_0, H_{S^k} and β_1 extends to a map $D^{k+2} \to Y$.

(e) Why does this prove Theorem 11.16(1)(a) implies Theorem 11.16(2)(a)?

The other two implications are comparatively straightforward.

Problem 11.21.

(a) Prove that Theorem 11.16(2)(a) implies Theorem 11.16(3)(a).

(b) Prove that Theorem 11.16(3)(a) implies Theorem 11.16(1)(a).

11.3.2. Equivalence of Parts (2)(a) and (2)(b). Since part (2)(b) obviously implies part (2)(a), we just need to show the reverse implication.

The proof makes use of our first induction principle for CW complexes. For the Zorn lemma portion of the proof, let $\mathcal{P}$ be the set of all triples (U, β_U, H_U) where U is a (relative) subcomplex of L such that $A \subseteq U \subseteq B$

and the maps β_U and H_U fit into the strictly commutative diagram

$$\begin{tikzcd}
A \arrow[rr, hook] \arrow[dr, "\alpha_1"] \arrow[dd, "\mathrm{in}_1"'] & & U \arrow[dl, "\beta_U"'] \arrow[dd, "\mathrm{in}_1"] \\
& X \arrow[dd, "f"] & \\
(U \times 0) \cup (A \times I) \arrow[rr, hook] \arrow[dr, "\beta_0|_U \cup H_A"'] & & U \times I \arrow[dl, "H_U"] \\
& Y. &
\end{tikzcd}$$

Define a partial order on $\mathcal{P}$ by setting $(U, \beta_U, H_U) \leq (V, \beta_V, H_V)$ if $U \subseteq V$ and the diagram

$$\begin{tikzcd}
& U \arrow[dl, "\beta_U"'] \arrow[dr] \arrow[dd, "\mathrm{in}_1"] & \\
X \arrow[dd, "f"'] & & V \arrow[ll, "\beta_V"'] \arrow[dd, "\mathrm{in}_1"] \\
& U \times I \arrow[dl, "H_U"'] \arrow[dr] & \\
Y & & V \times I \arrow[ll, "H_V"']
\end{tikzcd}$$

is strictly commutative.

Problem 11.22.

(a) Show that $\mathcal{P} \neq \varnothing$.

(b) Show that every chain $\cdots \leq (U, \beta_U, H_U) \leq (V, \beta_V, H_V) \leq \cdots$ of elements of $\mathcal{P}$ has an upper bound (Z, β_Z, H_Z).

(c) Use Zorn's lemma to show there is a complex maximal with the property that Theorem 11.16(2)(b) holds for the inclusion $j_M : A \hookrightarrow M$.

If $M = B$, we are done, so let us assume that M is a proper subcomplex. Then we can use Lemma 11.1 to find another subcomplex $N = M \cup_\lambda D^{k+1}$ (with $k < n$), which is the pushout in

$$\begin{tikzcd}
S^k \arrow[r, "\lambda"] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & M \arrow[d] \\
D^{k+1} \arrow[r, "\chi"] & N.
\end{tikzcd}$$

Problem 11.23. Use the diagram

$$\begin{tikzcd}
A \arrow[rr, hook] \arrow[dr, "\alpha_1"] \arrow[dd, "\mathrm{in}_1"'] & & M \arrow[dl, "\beta_K"'] \arrow[dd, "\mathrm{in}_1"] & S^k \arrow[l, "\lambda"'] \arrow[r] \arrow[dd] & D^{k+1} \arrow[dd] \\
& X \arrow[dd, "f"] & & & \\
(N \times 0) \cup (A \times I) \arrow[rr, hook] \arrow[dr, "\beta_0|_N \cup H_A"'] & & (N \times 0) \cup (M \times I) \arrow[dl, "\beta_0|_N \cup H_M"] & (D^{k+1} \times 0) \cup (S^k \times I) \arrow[l] \arrow[r] & D^{k+1} \times I \\
& Y & & &
\end{tikzcd}$$

to show that there are maps β_N and H_N so that $(N, \beta_N, H_N) > (M, \beta_M, H_M)$. Deduce that Theorem 11.16(2)(a) implies Theorem 11.16(2)(b).

11.3.3. Proof that Part (2)(b) Implies Part (3)(b). We have to show that the dotted arrow in the square

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "f"] \\
B \arrow[r, "\beta"] \arrow[ur, dotted] & Y
\end{tikzcd}$$

can be filled in so that the upper triangle commutes and the lower triangle commutes up to homotopy under A.

Problem 11.24. Prove Theorem 11.16(3)(b) by applying part (2)(b) to the diagram

$$\begin{tikzcd}
A \arrow[rr, hook] \arrow[dr] \arrow[dd, "\mathrm{in}_1"'] & & B \arrow[dl, dotted] \arrow[dd, "\mathrm{in}_1"] \\
& X \arrow[dd, "f"] & \\
(B \times 0) \cup (A \times I) \arrow[rr, hook] \arrow[dr, "\beta \cup H"'] & & B \times I \arrow[dl, dotted] \\
& Y, &
\end{tikzcd}$$

where $H = \boxed{f \circ \alpha}$ is the constant homotopy at $f \circ \alpha$.

11.3.4. Proof that Part (3)(b) Implies Part (1)(b). Now we apply 3(b) to show that f is an n-equivalence.

Problem 11.25.

(a) Take $A = *$ and $B = K$ to prove surjectivity.

(b) Take $A = (K \times 0) \cup (* \times I) \cup (K \times 1)$ and $B = K \times I$ to prove injectivity.

11.4. The J. H. C. Whitehead Theorem

One of the most important consequences of Theorem 12.34 is that a weak homotopy equivalence $f : X \to Y$ induces isomorphisms $f_* : [K, X] \to [K, Y]$ for *all* CW complexes, not just the finite ones. This is known as the **J. H. C. Whitehead theorem**.

Theorem 11.26 (J. H. C. Whitehead)**.** *If $f : X \to Y$ is an ∞-equivalence in $\mathcal{T}_\circ$, then for any $x \in X$, the induced map*

$$f_* : [K, (X, x)] \longrightarrow [K, (Y, f(x))]$$

is a bijection for all CW complexes K.

Problem 11.27. Prove the Whitehead theorem.

Hint. Look to Section 11.3.4 for inspiration.

This has an extremely important corollary: to determine whether a map of connected pointed CW complexes is a homotopy equivalence, it suffices to check that it induces isomorphisms on homotopy groups.

Corollary 11.28. *If $f : X \to Y$ is a map of connected pointed CW complexes, then the following are equivalent:*

(1) $f_* : \pi_*(X) \to \pi_*(Y)$ is an isomorphism,

(2) f is a homotopy equivalence in $\mathcal{T}_*$.

Problem 11.29. Prove Corollary 11.28.

Exercise 11.30.

(a) Show that a weakly contractible CW complex is contractible.

(b) Are there any spaces that are weakly contractible but not contractible?

11.5. Additional Problems

Problem 11.31. Let X be path-connected, and suppose that every map $S^k \to X$ with $k \leq n$ is freely homotopic to a constant. Show that X is n-connected.

Problem 11.32. Let $S^n \to S^m \to B$ be a fibration sequence with $n < m$. Prove that $\Omega B \simeq S^n \times \Omega S^m$.

Problem 11.33. Suppose $f : X \to Y$ is an n-equivalence in $\mathcal{T}_*$. What can you say about Ωf? What can you say about $f_* : \mathrm{map}_*(A, X) \to \mathrm{map}_*(A, Y)$ if A is a finite-dimensional CW complex?

Problem 11.34. Suppose f and g are pointwise n-equivalent, in the sense that there is a commutative square

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "\beta"] & Y \end{tikzcd}$$

in $\mathcal{T}_\circ$ in which α and β are n-equivalences. If f is an m-equivalence, then what can you say about g?

Problem 11.35. Let $f : X \to Y$ in $\mathcal{T}_*$ such that $f_* : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for $k \leq n$ and Y is a connected CW complex.

(a) Show that if $\dim(Y) \leq n$, then Y is a retract, up to homotopy, of X.

(b) Show that if X is also a CW complex and both $\dim(X), \dim(Y) < n$, then f is a homotopy equivalence.

Problem 11.36. Let $f : X \to Y$ be a map of pointed path-connected spaces.

(a) Show that f is a weak homotopy equivalence if and only if its homotopy fiber F_f is weakly contractible.

(b) Show that if the cofiber C_f is contractible, then $\Sigma f : \Sigma X \to \Sigma Y$ is a homotopy equivalence.

We will see later (in Problem 19.38) that there are maps f which are not even 2-equivalences but whose cofibers are contractible.

Problem 11.37. Let X be a path-connected H-space. We have shown that $[A, X]$ has a multiplication whose unit is the trivial map. We have not shown that this multiplication is associative or that elements of $[A, X]$ have inverses.

(a) Show that, under the identification $\pi_*(X \times X) \cong \pi_*(X) \times \pi_*(X)$, the map $\mu_* : \pi_*(X) \times \pi_*(X) \to \pi_*(X)$ is given by $\mu_*(\alpha, \beta) = \alpha + \beta$.

(b) Show that the **shear map** $s : X \times X \to X \times X$ given by $s : (x_1, x_2) \mapsto (x_1, x_1x_2)$ is a weak homotopy equivalence.

(c) Show that for any CW complex A, if $\alpha, \beta \in [A, X]$, then there exist unique $\xi, \zeta \in [A, X]$ such that $\alpha \cdot \xi = \beta$ and $\zeta \cdot \alpha = \beta$.

In the terminology of abstract algebra, parts (c) and (d) assert that if A is a CW complex, then $[A, X]$ is an **(algebraic) loop**.

Problem 11.38.

(a) Show that if L is a loop, then every element $\alpha \in L$ has a left inverse and a right inverse. Show that if L is associative, then the left and right inverses are equal, and give an example to show that they need not be equal in general.

(b) Let X be a connected CW complex and an H-space. Show that X has a left inverse $\nu_L : X \to X$ and a right inverse $\nu_R : X \to X$. Show that if X is homotopy associative, then $\nu_R \simeq \nu_L$.[3]

[3]There are H-spaces for which $[A, X]$ is a nonassociative algebraic loop with distinct left and right inverses.

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Chapter 12

Subdivision of Disks

Since CW complexes are built, ultimately, from collections of cells, interesting (or even mundane) properties of cells can have significant consequences for homotopy theory. In this chapter we explore the uses and implications of a simple observation: an n-cell can be subdivided into *smaller* n-cells.

This basic fact is used with the compactness of cells and the Lebesgue Number Lemma. Together, they imply that if X has an open cover $\{U_\alpha\}$ and $f : D^n \to X$, then D^n can be decomposed into smaller cells, each of which is mapped by f entirely into at least one of the sets U_α. This kind of decomposition can sometimes be used to extend a desirable property shared by the sets U_α to all of X.

We apply this idea, in dimensions 1 and 2, to prove the Seifert-Van Kampen theorem, which is a powerful tool for computing fundamental groups. We introduce (finite) simplicial complexes in order to facilitate the subdivision in higher dimensions. Using them, we establish the Cellular Approximation Theorem and show that, for some diagrams, a pointwise n-equivalence of diagrams induces an n-equivalence of homotopy colimits.

12.1. The Seifert-Van Kampen Theorem

We begin with an important theorem for the computation of fundamental groups. The proof uses the subdivision of squares into smaller squares, which is nice because it is easy to visualize.

If we are given a homotopy pushout square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\mathrm{HPO}"] & B \arrow[d] \\ C \arrow[r] & D, \end{tikzcd}$$

it is reasonable to ask how $\pi_1(D)$ relates to the fundamental groups of the spaces A, B and C. If we follow our collective noses and simply apply π_1 to the diagram, we obtain

$$\begin{tikzcd} \pi_1(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout of groups}"] & \pi_1(B) \arrow[d] \arrow[ddr, bend left] & \\ \pi_1(C) \arrow[r] \arrow[drr, bend right] & P \arrow[dr, "\xi"] & \\ & & \pi_1(D), \end{tikzcd}$$

where P is the pushout, in the category $\mathcal{G}$ of groups, of the prepushout diagram $\pi_1(C) \leftarrow \pi_1(A) \rightarrow \pi_1(B)$. The best thing we could possibly hope for is that the comparison map ξ is an isomorphism. The Seifert-Van Kampen theorem asserts that this is almost always true!

We need to be careful about the distinction between pointed and unpointed homotopy pushouts here. If the spaces in the diagram are well-pointed, then the canonical map from the unpointed homotopy pushout to the pointed one is a pointed homotopy equivalence, and we can use either construction. But if the spaces are not well-pointed, then we'll have to use the unpointed homotopy pushout.

Theorem 12.1 (Seifert-Van Kampen theorem)**.** *If the square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\mathrm{HPO}"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

in $\mathcal{T}_*$ is (after forgetting basepoints) a homotopy pushout square in $\mathcal{T}_\circ$ and A is path-connected, then the comparison map $\xi : P \to \pi_1(D)$ is an isomorphism.

To build the pushout P in the category of groups, first form the **free product** $\pi_1(B) * \pi_1(C)$, which is the set of all 'words' $x_1 x_2 \cdots x_n$ where each x_i is in either G or in H; if x_i and x_{i+1} are in the same group, then their product is defined, and we are allowed to replace $x_1 x_2 \cdots x_i x_{i+1} \cdots x_n$ with the shorter product $x_1 x_2 \cdots (x_i x_{i+1}) \cdots x_n$.

Exercise 12.2. Show that the free product of two groups is very poorly named: it is the sum in the category of groups.

Now we find the smallest normal subgroup $\overline{N} \subseteq G * H$ containing all elements of the form $f_*(x)g_*(x^{-1})$ where $x \in N$. Then the pushout P is the quotient group

$$G *_N H = (G * H)/\overline{N},$$

which is called the **amalgamated free product**[1] of G and H over N. Thus the Seifert-Van Kampen theorem is often expressed in the form

$$\pi_1(D) \cong \pi_1(B) *_{\pi_1(A)} \pi_1(C).$$

Exercise 12.3.

(a) Check that $G *_N H$ is the pushout of the diagram $G \leftarrow N \to H$ in $\mathcal{G}$.

(b) Determine the pushout of $\{1\} \leftarrow H \to G$ in the category $\mathcal{G}$.

To prove Theorem 12.1, we replace the space D with the standard homotopy pushout (i.e., the *unpointed* double mapping cylinder) $C \xleftarrow{g} A \xrightarrow{f} B$, which we construct as the quotient

$$\overline{D} = ((C \times 0) \cup (A \times I) \cup (B \times 1))/\sim$$

where $(a, 0) \sim (g(a), 0)$ and $(a, 1) \sim (f(a), 1)$; give $\overline{D}$ the basepoint $(*, \frac{1}{2})$. Let $U_0 \subseteq \overline{D}$ be the set of those points with $t \leq \frac{2}{3}$ and let $U_1 \subseteq \overline{D}$ be the set with $t \geq \frac{1}{3}$. Thus U_0 and U_1 are open, cover $\overline{D}$, and there are pointed maps

$$U_1 \longrightarrow B, \quad U_0 \longrightarrow C, \qquad \text{and} \qquad U_0 \cap U_1 \longrightarrow A$$

which are (unpointed) homotopy equivalences.

Problem 12.4. Show that to prove Theorem 12.1 it is enough to show that the map

$$\xi : \pi_1(U_0) *_{\pi_1(U_0 \cap U_1)} \pi_1(U_1) \longrightarrow \pi_1(\overline{D})$$

is an isomorphism.

Since we now are working with an open cover of $\overline{D}$, we can apply the Lebesgue Number Lemma, which will allow us to decompose a path in $\overline{D}$ into a concatenation of small paths, each of which lies entirely in one or the other of U_0 or U_1.

Problem 12.5. Let $\alpha : I \to \overline{D}$ with $\alpha(0) = \alpha(1) = *$, so that α represents a class in $\pi_1(\overline{D})$. Divide I into n subintervals, $I_1, I_2, \ldots, I_n$ with endpoints $\frac{k}{n}$ for $k = 0, 1, \ldots, n$.

(a) Show that if $n \in \mathbb{N}$ is large enough, then α carries each subinterval entirely into U_0 or U_1 (or both).

[1] *Painful but standard terminology!*

(b) Identify those endpoints $\frac{k}{n}$ such that $\alpha(I_{k-1})$ and $\alpha(I_k)$ do not both lie in U or in V; let $0 = x_0 < x_1 < \cdots < x_m = 1$ be this set of points. Show that each $x_k \in U_0 \cap U_1$.

Here's some helpful notation for paths and loops. We will find ourselves in possession of a paths $\alpha : I \to \overline{D}$ but wanting loops in $\overline{D}$. If we decide, *once and for all*, on our favorite paths $\xi_0, \xi_1 : I \to U_0 \cap U_1$ from $*$ to $\alpha(0)$ and $\alpha(1)$, respectively, then we can form the loop

$$\alpha^\circ = \xi_0 * \alpha * \overleftarrow{\xi_1}.$$

If we have not agreed on the paths ξ_0 and ξ_1, then the notation should not be used, since it suggests that α° depends only on α. Note also that it makes sense to use this notation for paths that are defined on intervals besides I; but we do assume that the loop α° is defined on I, which might require some conceptually easy but algebraically annoying reparametrization to write down explicitly.

Problem 12.5 gives us a subdivision of I into nonuniform intervals $J_k = [x_{k-1}, x_k]$. Write β_k for the restriction of α to J_k. Decide—once and for all—on your favorite paths $\xi_k : I \to U_0 \cap U_1$ from $*$ to x_k, and write $\beta_k^\circ = \xi_{k-1} * \beta_k, * \overleftarrow{\xi_k}$.

Problem 12.6.

(a) Show that $[\alpha] = [\beta_1^\circ * \beta_2^\circ * \cdots * \beta_m^\circ] \in \pi_1(\overline{D})$.

(b) Conclude that $\xi : P \to \pi_1(\overline{D})$ is surjective.

This is already good enough to prove something useful.

Corollary 12.7. *If $f : X \to Y$ with Y is simply-connected and X is path-connected, then the mapping cone C_f is simply-connected. In particular, if X is path-connected, then ΣX is simply-connected.*

Problem 12.8. Prove Corollary 12.7.

To prove that our comparison map $\xi : P \to \pi_1(\overline{D})$ is injective, suppose

$$\xi([\alpha_1] \cdot [\alpha_2] \cdots [\alpha_n]) = [\alpha_1 * \alpha_2 * \cdots * \alpha_n] = [*] \in \pi_1(\overline{D}).$$

We have to show $[\alpha_1] \cdot [\alpha_2] \cdots [\alpha_n] = 1 \in P$. Each nonidentity element of P has an expression as a product in which no factor is an identity element and no two consecutive factors are from the same group; we'll call such an expression a **standard form** for the element (it's not generally unique). We will assume that the word $[\alpha_1] \cdot [\alpha_2] \cdots [\alpha_n]$ is in standard form.

Let $H : I \times I \to \overline{D}$ be a path homotopy from $\alpha_1 * \alpha_2 * \cdots * \alpha_n$ to the constant path $*$. By the Lebesgue Number Lemma, there is some integer $r > 0$ so that if we subdivide $I \times I$ into squares of length $\frac{1}{nr}$, H carries each small square entirely into U_0 or entirely into U_1 (or both).

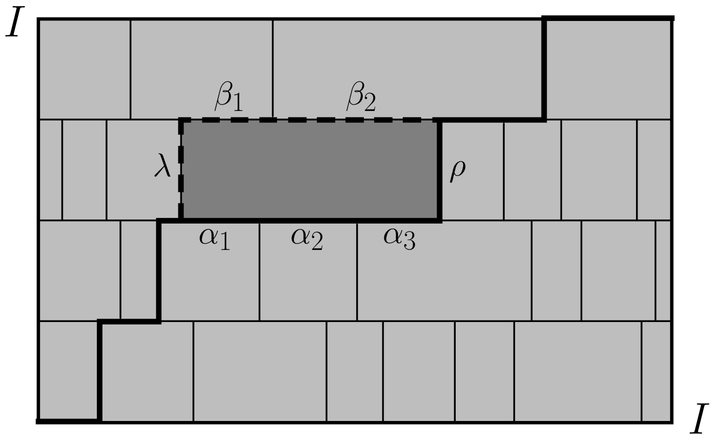

Figure 12.1. Paths in a subdivided square

Problem 12.9.

(a) Show that, by grouping adjacent squares in each row, the square $I \times I$ can be decomposed into rectangles with height $\frac{1}{nr}$ in such a way that horizontally adjacent rectangles map into different sets U_i.

(b) Show that each corner of each rectangle maps into $U_0 \cap U_1$.

For each corner point (s, t) of the decomposition you built in Problem 12.9, choose—once and for all—a path in $U_0 \cap U_1$ from $*$ to $H(s, t)$; if it happens that $H(s, t) = *$, use the constant path.

Having made these choices, each edge β of the decomposition corresponds, in a well-defined way, to a loop β°, either in U_0 or in U_1. Now consider drawing a path from $(0, 0)$ to $(1, 1)$ along the edges of the decomposition, as in Figure 12.1. For simplicity, we consider only those paths that never move down and never move left. Such a path amounts to a list of loops, some in U_0 and some in U_1; in other words, such a path identifies an element of the amalgamated product P.

Problem 12.10.

(a) Show that if $\beta_1(1) = \beta_2(0)$, then $[\beta_1^\circ * \beta_2^\circ] = [(\beta_1 * \beta_2)^\circ]$ in $\pi_1(\overline{D})$.

(b) Show that the element identified by traveling along the bottom to $(1, 0)$ and then up to $(1, 1)$ is equal to $[\alpha_1] \cdot [\alpha_2] \cdots [\alpha_n]$ in P.

(c) Show that the element identified by traveling up from $(0, 0)$ to $(0, 1)$ and then along the top to $(1, 1)$ is $[*]$.

Now we'll show that the elements of P identified by any two paths along the edges of the decomposition are equal in P. This boils down to studying the paths on the opposite sides of a single rectangle (as in Figure 12.1).

Problem 12.11.

(a) Show that $[\lambda^\circ] \cdot [\beta_1^\circ] \cdots [\beta_k^\circ] = [\alpha_1^\circ] \cdots [\alpha_l^\circ] \cdot [\rho^\circ]$ in P.

(b) Finish the proof of Theorem 12.1.

Exercise 12.12. Describe the sense in which the isomorphism of Theorem 12.1 is natural.

Problem 12.13. Generalize Theorem 12.1 to unions $X = \bigcup_{i \in \mathcal{I}} U_i$, where, for any $i \neq j \in \mathcal{I}$, $U_i \cap U_j \subseteq V$, a fixed path-connected open subset of X.

Here's a simple application of the Seifert-Van Kampen theorem.

Problem 12.14. Show that if the basepoints $* \in A$ and $* \in B$ have contractible neighborhoods, then $\pi_1(A \vee B) = \pi_1(A) * \pi_1(B)$. Generalize to arbitrary wedges.

Every Group Is a Fundamental Group. In group theory, a **presentation** of a group is an expression $G \cong \langle X | R \rangle$, where X is a set and R is a subset of $F(X)$, the free group on X. The notation means that there is a map $X \to G$, such that the extension $F(X) \to G$ is surjective and its kernel is $\overline{R}$, the smallest normal subgroup of $F(X)$ containing the set R. Very often, the set X is taken to be a subset of the group G and the map $X \to G$ is simply the inclusion. The elements of X are called **generators** for G and the elements of R are called **relations** among those generators.

Exercise 12.15.

(a) Write down $G *_N H$ in terms of generators and relations.

(b) Express $\langle X \mid R \rangle$ as a pushout of groups.

(c) Show that every group has a presentation.

We finish by showing that every group G arises as the fundamental group of a 2-dimensional CW complex.

Problem 12.16. Let $G = \langle X \mid R \rangle$.

(a) Let $W = \bigvee_{x \in X} S^1$, and show that $\pi_1(W) \cong F(X)$.

(b) Let $V = \bigvee_{r \in R} S^1$, and find a map $\rho : V \to W$ such that $\mathrm{Im}(r_*)$ is the subgroup of $F(X)$ generated by the elements of R.

(c) Then consider the homotopy pushout square

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & W \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

and show that $\pi_1(X) \cong G$.

12.2. Simplices and Subdivision

To apply this kind of argument in higher dimensions, we need to be able to decompose disks into smaller disks. We will introduce some new models for disks (and spheres), called *simplices* (and their boundaries). They have three crucial properties: they are homeomorphic to disks; they can be easily subdivided into smaller simplices; and linear maps from simplices are defined and determined by their values on the vertices of the simplex.

12.2.1. Simplices and Their Boundaries. We say that an ordered list $\{x_0, x_1, \dots, x_k\}$ of $k+1$ points in $\mathbb{R}^m$ is in **general position** if the collection of differences $\{x_1 - x_0, x_2 - x_0, \dots, x_k - x_0\}$ is a linearly independent set of vectors. The k-**simplex** spanned by the points $x_0, x_1, \dots, x_k \in \mathbb{R}^m$ in general position (with $k \leq m$) is the set

$$\Delta(x_0, x_1, \dots, x_k) = \left\{ \sum t_i x_i \;\middle|\; t_i \in I, \sum t_i = 1 \right\}.$$

The points x_i are called the **vertices** of the simplex. The **standard k-simplex** is $\Delta^k = \Delta(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{k+1}) \subseteq \mathbb{R}^{k+1}$, where the $\mathbf{e}_i$ are the standard unit vectors. The **boundary** of the simplex is

$$\partial\Delta(x_0, x_1, \dots, x_k) = \left\{ \sum t_i x_i \;\middle|\; t_i \in I, \sum t_i = 1, \text{ and at least one } t_i = 0 \right\}.$$

A map $\ell : \Delta(x_0, x_1, \dots, x_k) \to \mathbb{R}^n$ is called **linear** if $\ell(\sum t_i x_i\,) = \sum t_i \ell(x_i)$.

Exercise 12.17.

(a) Draw the standard simplices Δ^0, Δ^1, Δ^2 and their boundaries.

(b) Show that if $\{x_0, x_1 \dots, x_k\}$ is in general position, then there is a unique linear homeomorphism $\Delta^k \xrightarrow{\cong} \Delta(x_0, x_1 \dots, x_k)$ preserving the order of the vertices.

Exercise 12.18.

(a) Show that $\{x_0, x_1, \dots, x_k\}$ is in general position if and only if the set of cyclic differences $\{x_1 - x_0, x_2 - x_1, \dots, x_k - x_{k-1}\}$ is linearly independent.

(b) Show that if $\{x_0, x_1, \dots, x_k\}$ is in general position, then so is any reordering of the set.

The simplex $\Delta(x_0, x_1, \ldots, x_k)$ contains many **subsimplices**: every subset $S \subseteq \{x_0, x_1, \ldots, x_k\}$ gives rise to a simplex $\Delta(S) \subseteq \Delta(x_0, x_1, \ldots, x_k)$.

Exercise 12.19. How many j-dimensional subsimplices does a k-dimensional simplex have?

We will use simplices and their boundaries as models for disks and spheres.

Problem 12.20. Suppose $\{x_0, x_1, \ldots, x_k\}$ are in general position.

(a) Prove that there is a commutative square

$$\begin{tikzcd} \partial\Delta(x_0, x_1, \ldots, x_k) \arrow[r, "\cong"] \arrow[d] & S^{k-1} \arrow[d] \\ \Delta(x_0, x_1, \ldots, x_k) \arrow[r, "\cong"] & D^k. \end{tikzcd}$$

(b) Show that if $U \subseteq \mathbb{R}^n$ is convex, then any map $\{x_0, x_1, \ldots, x_k\} \to U$ extends to a unique linear map $\Delta(x_0, x_1, \ldots, x_k) \to U$.

The first part of Problem 12.20 shows that the simplex and its boundary are suitable models for disks and spheres. The second part gives the basic property that makes them so useful: it is easy to specify maps from simplices into convex spaces.

12.2.2. Finite Simplicial Complexes. A (finite) **abstract simplicial complex** is a list K of finitely many simplices such that

- every subsimplex of a simplex in K is also in K and
- if two simplices intersect, then their intersection is a subsimplex of each of them.

We write $|K|$ for the space that results from taking the union of the simplices in K; such a space, or anything homeomorphic to such a space, will also be called a **simplicial complex**, or a **polyhedron**. A homeomorphism $\alpha : |K| \xrightarrow{\cong} X$ is said to give the space X the **structure** of a simplicial complex.

If $\alpha : |K| \xrightarrow{\cong} X$ is a simplicial structure on X, then each element $\Delta(x_0, x_1, \ldots, x_n) \in K$ gives rise, by composition with the canonical homeomorphism $\Delta^n \xrightarrow{\cong} \Delta(x_0, x_1, \ldots, x_n)$, to a map $\Delta^n \to X$, which we call an n-**simplex** of X. The **dimension** of X is the maximum of the dimensions of its simplices.

Problem 12.21. Suppose $\alpha : |K| \xrightarrow{\cong} X$ gives X the structure of an n-dimensional simplicial complex.

(a) Show that X has the largest topology such that each simplex $\sigma : \Delta^n \to X$ is continuous.

(b) Show that if X has the structure of an n-dimensional finite simplicial complex, then X is a finite n-dimensional CW complex.

Problem 12.21 suggests another approach to defining simplicial complexes. Instead of starting with a list of simplices in $\mathbb{R}^n$, we can start with a space and look for a collection of maps $\Delta^n \to X$ that define the topology on X and which are related to one another in a way that encodes the two requirements that we've made on an abstract simplicial complex. We'll come back to this with a vengence in Section 15.6.

A **subcomplex** $L \subseteq K$ of a simplicial complex is a simplicial complex each of whose simplices is also a simplex of K. The union of the simplices of K is a subspace of $\mathbb{R}^n$, which is generally denoted $|K|$ and called the **geometric realization** of K.

Problem 12.22. Show that the boundary of a simplex is a subcomplex.

A **piecewise linear** map $f : |K| \to \mathbb{R}^n$ is one whose restriction to each simplex is linear. If X is a simplicial complex with structure map $\alpha : |K| \to X$, then we say that a map $f : X \to \mathbb{R}^n$ is linear if $f \circ \sigma$ is linear.

Problem 12.23.

(a) Let X be a simplicial complex with 0-simplices $X_0 = \{x_1, \ldots, x_m\}$. Show that any map $f_0 : X_0 \to \mathbb{R}^n$ extends to a unique piecewise linear map $f : X \to \mathbb{R}^n$.

(b) Let $U \subseteq \mathbb{R}^n$ be a convex open subspace, and let X be a finite simplicial complex with $\dim(X) < n$. Show that no piecewise linear map $f : X \to U$ can be surjective.

Hint. Look back at Problem 4.19.

12.2.3. Barycentric Subdivision. Now we have established two of the three main properties of simplices: they are models for disks, and their linear maps into convex spaces are entirely determined by the values on the vertices. It remains to discuss subdivision of simplices. The **barycenter** of the simplex $\sigma = \Delta(x_0, x_1, \ldots, x_k)$ is the point

$$b_\sigma = \frac{1}{k+1} \sum x_i.$$

Note that if $\sigma = \Delta(x)$, then $b_\sigma = \sigma$. A **barycentric subsimplex** of $\Delta(x_0, x_1, \ldots, x_k)$ is $\Delta(b_{\tau_0}, b_{\tau_1}, \ldots, b\tau_j)$ where $\tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_j$ is an increasing sequence of subsimplices of $\Delta(x_0, x_1, \ldots, x_k)$.

Exercise 12.24. Draw all the barycentric subsimplices of Δ^2 and Δ^3.

For any abstract simplicial complex K, we write $\mathrm{sd}(K)$ for the complete list of all the barycentric subsimplices of K.

Problem 12.25. Show that $\mathrm{sd}(K)$ is an abstract simplicial complex and $|\,\mathrm{sd}(K)| = |K|$.

Now let X be a finite simplicial complex with structure map $\alpha : |K| \to X$. Since $|\,\mathrm{sd}(K)| = |K|$, the same map gives X a different simplicial structure $\mathrm{sd}(\alpha) : |\,\mathrm{sd}(K)| \to \mathrm{sd}(X)$.

Exercise 12.26. Draw the second barycentric subdivision $\mathrm{sd}^2(\Delta^2)$.

Problem 12.27. Show that if X is a finite simplicial complex and $\epsilon > 0$, then there is some n for which each simplex in $\mathrm{sd}^n(X)$ has diameter less than ϵ.

HINT. It is enough to prove it for a single simplex.

12.3. The Connectivity of $X_n \hookrightarrow X$

How much of an equivalence is the inclusion $X_n \hookrightarrow X$? This ultimately boils down to determining the connectivity of $X \hookrightarrow X \cup D^{n+1}$. On the face of it, this is an intractible problem, since cell attachment is a domain-type construction and connectivity of maps is a target-type concept. But you have already made some headway on this question in Section 4.2.4, and now that we have higher-dimensional simplices available, we can resolve the general problem using the same basic ideas.

Proposition 12.28. *If X is a CW complex, then the inclusion $X_n \hookrightarrow X$ is an n-equivalence.*

The hard work in the proof of Proposition 12.28 can be shoved into the following lemma.

Lemma 12.29. *For any space X, the inclusion $X \hookrightarrow X \cup D^{n+1}$ is an n-equivalence.*

Problem 12.30. Use Lemma 12.29 to prove Proposition 12.28.

HINT. Use CW induction.

In light of Theorem 11.16(3)(a) and Problem 12.20(a), to prove Lemma 12.29, it is enough to show that in the diagram

$$\begin{tikzcd} \partial\Delta^k \arrow[r, "\alpha_1"] \arrow[d] & X \arrow[d] \\ \Delta^k \arrow[r, "\beta_0"'] \arrow[ur, dotted] & X \cup D^{n+1} \end{tikzcd}$$

with $k \leq n$, the dotted arrow can be filled in so that the top triangle commutes on the nose and the bottom triangle commutes up to a homotopy that is constant on $\partial\Delta^k$.

Problem 12.31. Show that Lemma 12.29 follows from the following statement:

> *For any α_1 and β_0 as in the square above, β_0 is homotopic, by a homotopy constant on $\partial\Delta^k$, to a map β such that $\mathrm{int}(D^n) \not\subseteq \mathrm{Im}(\beta)$.*

Problem 12.32. In the situation of Lemma 12.29, let $U = X \cup (D^{n+1} - \frac{1}{3}D^{n+1})$ and $V = \frac{2}{3}D^{n+1}$.

(a) Show that there are functions $u, v : X \cup D^{n+1} \to I$ such that

$$U = u^{-1}(0,1], \quad V = v^{-1}(0,1], \quad \tfrac{1}{3}D^{n+1} \subseteq v^{-1}(1),$$

and $u + v : X \to [0,2]$ is the constant function at 1.

(b) Show that there is a subdivision $\mathrm{sd}^n(\Delta^k)$ of Δ^k in which β_0 carries each simplex either entirely into U or entirely into V.

Let L_U be the union of those k-simplices in the decomposition of Problem 12.32(b) which map into U, and let L_V be the union of the k-simplices which map into V; write β_U and β_V for the restrictions of β to L_U and L_V. To summarize, we have the morphism of prepushouts

$$\begin{tikzcd} L_U \arrow[d, "\beta_U"] & L_U \cap L_V \arrow[l] \arrow[r] \arrow[d, "\beta_{U\cap V}"] & L_V \arrow[d, "\beta_V"] \\ U & U \cap V \arrow[l] \arrow[r] & V \end{tikzcd}$$

which induces the map $\beta_0 : \Delta^k \to X \cup D^n$ on pushouts.

Identify the interior of D^n with a convex subset of $\mathbb{R}^n$, so that we may meaningfully talk about convex combinations of points in $\mathrm{int}(D^n)$. Let $\ell : L_V \to X \cup D^n$ be the piecewise linear map determined by the values of f_V on the vertices of L_V. Then define $\phi : \Delta^k \to X \cup D^n$ by the formula

$$\phi(x) = u(f(x))f(x) + v(f(x))\ell(x).$$

Problem 12.33.

(a) Show that ϕ is homotopic to f by a homotopy that is constant on $\partial\Delta^k$.

(b) Show that $D^n \not\subseteq \phi(\Delta^k)$ and prove Lemma 12.29.

12.4. Cellular Approximation of Maps

The **Cellular Approximation Theorem** asserts that every map between CW complexes is homotopic to a cellular map.

Theorem 12.34 (Cellular Approximation). *Let $f : X \to Y$ be a map of CW complexes, and assume that $f|_A$ is cellular, where $A \subseteq X$ is a subcomplex. Then there is a map $g : X \to Y$ such that*

(a) $f \simeq g$,

(b) g is cellular, and

(c) $g|_A = f|_A$.

If f is a pointed map, then so is g and the homotopy $f \simeq g$ is pointed.

Problem 12.35.

(a) Show that f is homotopic to a map f_0 with $f(X_0) \subseteq Y_0$.

(b) Suppose $f \simeq f_n : X \to Y$ with $f|_{X_n}$ a cellular map. Then use the diagram

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & Y_n \arrow[r] & Y_{n+1} \arrow[d] \\ X_{n+1} \arrow[r] \arrow[urr, dotted] & X \arrow[r, "f_n"] & Y \end{tikzcd}$$

to show that f_n is homotopic, by a homotopy constant on X_n to a map $f_{n+1} : X \to Y$ such that $(f_{n+1})|_{X_{n+1}}$ is cellular.

(c) Prove Theorem 12.34.

In Section 4.5, we asked whether the lifts $\pi_n : \mathcal{T}_* \to \mathcal{G}$ of $\pi_n : \mathcal{T}_* \to \mathbf{Sets}_*$ are unique and showed that there was one lift for each co-H-structure on S^n. We now have the tools to enumerate the co-H-structures on S^n.

Problem 12.36.

(a) Show that for $n \geq 2$, S^n has exactly one co-H-structure.

(b) Classify the co-H-structures on S^1. That is, establish a bijection between the set of co-H-structures on S^1 and some set of homotopy classes.

12.5. Homotopy Colimits and n-Equivalences

Finally, we use the subdivision technique to prove a useful result generalizing the fact that a pointwise homotopy equivalence of prepushout diagrams induces a homotopy equivalence of their homotopy pushouts.

12.5.1. Homotopy Pushouts.

First, let's look at homotopy pushouts.

Proposition 12.37. *Let $\phi : F \to G$ be the diagram morphism*

$$\begin{tikzcd} Y \arrow[d] & W \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\ C & A \arrow[l] \arrow[r] & B. \end{tikzcd}$$

Assume that each of the vertical maps is an n-equivalence, with $n \leq \infty$.[2] *Then the induced map $f : Z \to D$ of unpointed homotopy pushouts is also an n-equivalence.*

Corollary 12.38. *If $f : X \to Y$ is a weak homotopy equivalence of well-pointed spaces, then $\Sigma f : X \to Y$ is a weak homotopy equivalence.*

Exercise 12.39.

(a) Show that if the spaces involved are well-pointed, then Proposition 12.37 holds for pointed homotopy colimits.

(b) Show that it suffices to verify Proposition 12.37 for your favorite induced map.

Since we can choose any homotopy colimit construction we like, let's use the standard double mapping cylinder. According to Theorem 11.16, we have to show that in any diagram of the form

$$\begin{tikzcd} \partial\Delta^k \arrow[r, "\beta"] \arrow[d] & Z \arrow[d, "f"] \\ \Delta^k \arrow[r, "A"'] \arrow[ur, dotted] & D \end{tikzcd}$$

with $k < n$, the dotted arrow can be filled in so that the upper triangle commutes on the nose and the lower triangle commutes up to a homotopy which is constant on $\partial\Delta^k$.

Problem 12.40.

(a) Show that Z and D can be decomposed into pairs of open subsets

$$Z = Z_1 \cup Z_2 \quad \text{and} \quad D = D_1 \cup D_2$$

so that $f(Z_1) \subseteq D_1$, $f(Z_2) \subseteq D_2$ and the three restricted maps

$$f_1 : Z_1 \longrightarrow D_1, \quad f_2 : Z_2 \longrightarrow D_2 \quad \text{and} \quad f_{12} : Z_1 \cap Z_2 \longrightarrow D_1 \cap D_2$$

are n-equivalences.

(b) Show that Δ^k can be subdivided into simplices so small that each of them is mapped either entirely into D_1 or entirely into D_2.

(c) Let K be the union of all the simplices that map into D_1 and let L be the union of all the simplices that map into D_2. Show that the dotted

[2] So ϕ could be described as a **pointwise n-equivalence**.

arrow in the diagram

$$\begin{tikzcd}
(K \cap L) \cap \partial\Delta^k \arrow[r] \arrow[d] & Z \arrow[d, "f"] \\
(K \cap L) \arrow[r] \arrow[ur, dotted] & D
\end{tikzcd}$$

can be filled in so that the top triangle commutes and the bottom triangle commutes up to a homotopy constant on $(K \cap L) \cap \partial\Delta^k$.

(d) Show that the dotted arrow in the diagram

$$\begin{tikzcd}
(K \cap L) \cup (K \cap \partial\Delta^k) \arrow[r] \arrow[d] & Z \arrow[d, "f"] \\
K \arrow[r] \arrow[ur, dotted] & D
\end{tikzcd}$$

can be filled in so that the top triangle commutes and the bottom triangle commutes up to a homotopy constant on $(K \cap L) \cup (K \cap \partial\Delta^k)$.

(e) Prove Proposition 12.37.

(f) Derive Corollary 12.38.

Proposition 12.37 implies that the pushout of an n-equivalence is also an n-equivalence.

Problem 12.41. Consider the square

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"'] \\
C \arrow[r] & D.
\end{tikzcd}$$

(a) Show that if the square is a homotopy pushout square and f is an n-equivalence, then g is also an n-equivalence.

(b) Suppose the square is a homotopy pushout square and the spaces A, B and C are n-connected. Show that D is also n-connected.

12.5.2. Telescope Diagrams. Now we tackle the analogous question for telescope diagrams.

Proposition 12.42. *Consider the map of telescope diagrams*

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "f_0"] & X_1 \arrow[r] \arrow[d, "f_1"] & \cdots \arrow[r] & X_m \arrow[r] \arrow[d, "f_m"] & X_{m+1} \arrow[r] \arrow[d, "f_{m+1}"] & \cdots \\
Y_0 \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_m \arrow[r] & Y_{m+1} \arrow[r] & \cdots ,
\end{tikzcd}$$

with induced map $f : X \to Y$ of unpointed homotopy colimits. If each map f_m is an n-equivalence, then f is also an n-equivalence.

To prove Proposition 12.42, we consider the diagram

$$\begin{tikzcd} \partial\Delta^k \arrow[r, "\beta"] \arrow[d] & X \arrow[d, "f"] \\ \Delta^k \arrow[r, "A"'] \arrow[ur, dotted] & Y \end{tikzcd}$$

with $k < n$ and show that the dotted arrow can be filled in, subject to the usual restrictions.

Problem 12.43. Use the small object argument (see Problem 6.63) to show that there is an m large enough that the dashed maps in the diagram

$$\begin{tikzcd} \partial\Delta^k \arrow[rr, "\beta"] \arrow[dr, dashed] \arrow[dd] & & X \arrow[dd, "f"] \\ & X_m \arrow[ur] \arrow[dd, "f_m"] & \\ \Delta^k \arrow[rr, "A"] \arrow[ur, dotted] \arrow[dr, dashed] & & Y \\ & Y_m \arrow[ur] & \end{tikzcd}$$

can be filled in so that the diagram commutes. Then prove Proposition 12.42.

Problem 12.44. Continue to work with the setup of Proposition 12.42.

(a) Show that if the connectivity of f_m is at least n for all sufficiently large m, then f is an n-equivalence.

(b) Show that if the connectivity of X_m goes to ∞ as $m \to \infty$, then X is weakly contractible.

12.6. Additional Problems and Projects

Problem 12.45. Show that

$$[S^1, (S^1 \vee S^1) \sqcup (S^1 \vee S^1)] \longrightarrow \langle S^1, (S^1 \vee S^1) \sqcup (S^1 \vee S^1) \rangle$$

is neither injective nor surjective.

Problem 12.46.

(a) Show that $f : X \to Y$ is a phantom map if and only if the composite $K \to X \to Y$ is trivial for any finite-dimensional CW complex K.[3]

[3] Thus we can define phantom maps out of non-CW complexes.

(b) Show that the composition of any two phantom maps between CW complexes is trivial.

Problem 12.47. Show that every generalized CW complex is homotopy equivalent to a genuine CW complex of the same dimension.

Problem 12.48. Consider the square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r] & D. \end{tikzcd}$$

(a) Show that if it is a homotopy pullback square and g is an n-equivalence, then f is also an n-equivalence.

(b) Suppose that it is a homotopy pullback square and the spaces B, C and D are n-connected; show that A is $(n-1)$-connected.

Problem 12.49. Show that a pointwise n-equivalence of prepullback diagrams induces an $(n-1)$-equivalence of homotopy pullbacks.

Project 12.50. Prove Theorem 9.109.

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Chapter 13

The Local Nature of Fibrations

The underlying theorem of this chapter is that a map which is locally a fibration (or a weak fibration) is actually a fibration (or a weak fibration). This theorem was first proved by Hurewicz and was later given its definitive form by Dold.

We recast this theorem as a rule for manipulating diagrams, called the First Cube Theorem. The First Cube Theorem states that if the top and bottom faces of a commutative cube

$$\begin{tikzcd}
W \arrow[rr] \arrow[dd] \arrow[dr] & & X \arrow[dd] \arrow[dr] & \\
& Y \arrow[rr] \arrow[dd] & & Z \arrow[dd] \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& C \arrow[rr] & & D
\end{tikzcd}$$

are strong homotopy pushout squares and the back and left faces are strong homotopy pullback squares, then the front and right faces are also homotopy pullback squares. We call such a commutative cube—in which the lateral faces are homotopy pullbacks and the top and bottom are homotopy pushouts—a **Mather cube**.

To prove the First Cube Theorem, we need to work with weak fibrations instead of fibrations. Like ordinary fibrations, weak fibrations are sufficient to define homotopy pullbacks. But they have an advantage over ordinary fibrations: a map homotopy equivalent in $\mathcal{T}_\circ{\downarrow}B$ to a weak fibration is also a weak fibration.

13.1. Maps Homotopy Equivalent to Fibrations

We have shown that a pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & B \arrow[d, "p"] \\ C \arrow[r] & D \end{tikzcd}$$

in which p is a fibration is a strong homotopy pullback square. But any map $f : X \to B$ that is homotopy equivalent, in the category $\mathcal{T}_\circ{\downarrow}B$, to a fibration also suffices for the construction of strong homotopy pullback squares. Such maps are called *weak fibrations*. In this section, we establish the properties of weak fibrations that we'll use later to prove the First Cube Theorem.

13.1.1. Weak Fibrations. We have studied, in Sections 4.7 and 5.5, the basic constructions of homotopy theory in the category $\mathcal{T}_\circ{\downarrow}B$. In particular, there is a notion of homotopy of morphisms, and hence a concept of homotopy equivalence of maps. A **weak fibration** is a map $p : E \to B$ which is homotopy equivalent in $\mathcal{T}_\circ{\downarrow}B$ to a fibration.

Exercise 13.1. Criticize the following arguments:

(1) *Any map $f : X \to Y$ can be converted to a fibration, yielding the strictly commutative square*

$$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[d, "f"'] & E_f \arrow[d, "p"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

in which p is a fibration. Thus every map is homotopy equivalent to a fibration, proving that every map is a weak fibration.

(2) *If $f : X \to A$ is homotopy equivalent to a fibration $p : E \to B$ in* $\mathrm{map}(\mathcal{T}_\circ)$*, then f is a weak fibration, because if $q : \widetilde{E} \to A$ is the pullback in the diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \widetilde{E} \arrow[r] \arrow[d, "q"] \arrow[dr, phantom, "\text{pullback}"] & E \arrow[d, "p"] \\ & A \arrow[r] & B, \end{tikzcd}$$

the horizontal maps are homotopy equivalences, so f is equivalent in $\mathcal{T}_\circ{\downarrow}A$ to the fibration q.

Exercise 13.2.

(a) Show that every fibration is a weak fibration.

(b) Show that the restriction of the projection $\mathrm{pr}_1 : S^1 \times I \to S^1$ to the subspace $S^1 \cup * \times I \subseteq S^1 \times I$ is a weak fibration but not a fibration.

The class of weak fibrations with base B is the smallest collection of maps that contains fibrations and is closed under homotopy equivalence in $\mathcal{T}_\circ{\downarrow}B$. It follows that to verify that a given map is a weak fibration, it suffices to show that it is homotopy equivalent in $\mathcal{T}_\circ{\downarrow}B$ to a weak fibration. Even better, we can simply verify that a particular universal example is a homotopy equivalence of maps.

Problem 13.3.

(a) Show that if f is a strong deformation retract of g in $\mathcal{T}_\circ{\downarrow}B$, then f is also a weak fibration.

(b) Suppose g is a weak fibration and $\alpha : f \to g$ is a morphism in $\mathcal{T}_\circ{\downarrow}B$. Show that the mapping cylinder M_α is also a weak fibration.

(c) Convert $f : X \to B$ to a fibration, yielding the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & E_f \arrow[d, "p"] \\ B \arrow[r, equal] & B. \end{tikzcd}$$

Show that f is a weak fibration if and only if this square is a homotopy equivalence in $\mathcal{T}_\circ{\downarrow}B$.

13.1.2. Homotopy Pullbacks and Weak Fibrations. Weak fibrations share many of the nice properties of fibrations. In particular, pulling back a weak fibration results in a strong homotopy pullback square.

Proposition 13.4. *If in the pullback square*

$$\begin{tikzcd} P \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\text{pullback}"] & X \arrow[d, "f"] \\ A \arrow[r] & B \end{tikzcd}$$

the map f is a weak fibration, then

(a) g is also a weak fibration, and

(b) the square is a strong homotopy pullback square.

Problem 13.5. Prove Proposition 13.4.

HINT. Use Problem 4.81.

13.1.3. Weak Homotopy Lifting. Let's say that a map $p : E \to B$ has the **weak homotopy lifting property** if in any diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathrm{in}_0"'] & E \arrow[d, "p"] \\ A \times I \arrow[r, "H"] \arrow[ur, dotted] & B \end{tikzcd}$$

in which $H|_{A\times[0,\frac{1}{2}]}$ is a constant homotopy, the dotted arrow can be filled in to make the diagram commute on the nose.

Let $p : E \to B$. Recall from Problem 5.30 the space

$$\Omega_p = \{(\omega, e) \mid \omega \in B^I, e \in E, \omega(0) = e\}.$$

The map p induces a function $\widetilde{p} : E^I \to E(p)$ given by $\omega \mapsto (p \circ \omega, \omega(e))$, and a **lifting function** for p is a section of $\widetilde{p}$.

Now we'll adapt these ideas to weak fibrations. Inside of Ω_p, we have the subspace $\widetilde{\Omega}_p = \{(\omega, e) \mid \omega \in B^I$ is constant on $[0, \frac{1}{2}], e \in E, \omega(0) = e\}$, and we say that p has a **weak lifting function** if there is lift in the diagram

$$\begin{tikzcd} & E^I \arrow[d, "\widetilde{p}"] \\ \widetilde{\Omega}_p \arrow[r, hook] \arrow[ur, dotted, bend left] & \Omega_p. \end{tikzcd}$$

The following characterization of weak fibrations won't come as a surprise.

Proposition 13.6. *If $p : E \to B$, the following are equivalent:*

(a) $p : E \to B$ is a weak fibration,

(b) p has the weak homotopy lifting property,

(c) p has a weak lifting function.

Problem 13.7. Prove Proposition 13.6.

13.2. Local Fibrations Are Fibrations

Suppose you have a function $p : E \to B$ and you wonder whether it is a fibration or not. If you have an open cover $B = \bigcup_\alpha U_\alpha$ together with lots of information about the individual subspaces U_α, then it makes sense to look at preimages $E_\alpha = p^{-1}(U_\alpha)$ and the restricted maps $p_\alpha : E_\alpha \to U_\alpha$. These maps are the pullbacks in the diagrams

$$\begin{tikzcd} E_\alpha \arrow[r] \arrow[d, "p_\alpha"'] \arrow[dr, phantom, "\text{pullback}"] & E \arrow[d, "p"] \\ U_\alpha \arrow[r] & B, \end{tikzcd}$$

so if p were a fibration, then each p_α would also be a fibration. But what about the reverse implication? If you only know that each individual map $p_\alpha : E_\alpha \to U_\alpha$ is a fibration, can you conclude that the original map p is a fibration? If the cover $\{U_\alpha\}$ satisfies a fairly mild condition, then the reverse implication is perfectly valid.

A **partition of unity** subordinate to a locally finite open cover $\{U_\alpha\}$ of X is a collection of functions $\tau_\alpha : X \to [0,1]$ such that

(1) for each $x \in X$, the (finite!) sum $\sum_\alpha \tau_\alpha(x) = 1$ and

(2) for each α, the **support** $\overline{\tau_\alpha^{-1}((0,1])}$ of τ_α is contained in U_α.

An open cover $\{U_\alpha\}$ of a space X is called **numerable** if there is a partition of unity subordinate to some refinement of the cover. For many familiar spaces, every cover is numerable.[1]

Theorem 13.8 (Hurewicz). *Suppose $p : E \to B$ is a map and $B = \bigcup_\alpha U_\alpha$ is a numerable open cover. Then the following are equivalent:*

(1) each pullback map $p_\alpha : E_\alpha \to U_\alpha$ is a fibration,

(2) $p : E \to B$ is a fibration.

We already know that (2) implies (1). For the converse, we assume (1) so that each map p_α has a lifting function λ_α. To show that p is a fibration, one uses the partition of unity to cobble the partial lifting functions λ_α together to give a full-fledged lifting function λ for p. Working through the extremely technical details here would be a major interruption in the flow of ideas; and since the method is not used anywhere else in this book, we will simply take it for granted. A very thorough exposition of the proof can be found in [**154**].

Many authors refer to a major paper of Dold [**52**] for local-to-global results about fibrations (and other things), but this particular theorem was proved by Hurewicz eight years earlier [**93**]. Dold refocused attention from properties of the space (paracompact) to properties of the cover[2] (numerable) and his work has served as the foundation for a great many local-to-global theorems.

[1] This is the case, for example, if X is paracompact. According to a theorem of H. Miyazaki, every CW complex is paracompact [**136**].

[2] This idea is present in Hurewicz's paper.

In the paper [**52**] where Dold gave his approach to Theorem 13.8, he also introduced weak fibrations and proved a corresponding local-to-global result for them.

Theorem 13.9 (Dold)**.** *Suppose $p : E \to B$ is a map and $B = \bigcup_\alpha U_\alpha$ is a numerable open cover. Then the following are equivalent:*

(1) each pullback map $p_\alpha : E_\alpha \to U_\alpha$ is a weak fibration,

(2) $p : E \to B$ is a weak fibration.

It is worth pointing out explicitly that Theorems 13.8 and 13.9 may be applied for *any* open cover $B = \bigcup_\alpha U_\alpha$ if B is paracompact (e.g., if B is a CW complex).

Project 13.10. Work through proofs of Theorems 13.8 and 13.9.

13.3. Gluing Weak Fibrations

The First Cube Theorem recasts Theorem 13.9 as a homotopy-theoretic statement involving homotopy pullbacks and pushouts. This section is devoted to establishing most of the technical issues that arise in the proof.

13.3.1. Tabs and Glue. The basic construction is a variant of the double mapping cylinder construction in which the given data is

$$Y \xleftarrow{g} W_1 \underset{\simeq}{\xleftarrow{\gamma}} W_0 \xrightarrow{f} X$$

in which γ is a homotopy equivalence. We form mapping cylinders on $W_0 \to X$ and $W_1 \to Y$, which we visualize as 'tabs' on X and Y. These tabs are glued together, using the mapping cylinder of map γ as the 'glue'.

Begin by constructing (nonstandard) unpointed mapping cylinders (indexed on parameter s) of f and g, respectively, constructed as the pushouts in the squares

$$\begin{tikzcd} W_0 \arrow[r, "\mathrm{in}_1"] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & W_0 \times [\frac{1}{3}, 1] \arrow[d] \\ X \arrow[r] & \widetilde{M}_f \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} W_1 \arrow[r, "\mathrm{in}_0"] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & W_1 \times [0, \frac{2}{3}] \arrow[d] \\ Y \arrow[r] & \widetilde{M}_g. \end{tikzcd}$$

Now let $G = \overline{M}_\gamma \times [\frac{1}{3}, \frac{2}{3}]$, where $W_0 \to \overline{M}_\gamma \to W_1$ is a factorization of γ into a cofibration followed by an acyclic fibration.[3] Define

$$Q(f, \gamma, g) = \widetilde{M}_g \cup G \cup \widetilde{M}_f,$$

[3] Recall that, as a set, $\overline{M}_f$ is the ordinary mapping cylinder, indexed on $t \in [0, 1]$.

where we identify $A_0 \times [\frac{1}{3}, \frac{2}{3}] \subseteq G$ with $A_0 \times [\frac{1}{3}, \frac{2}{3}] \subseteq \widetilde{M}_f$, and likewise, we identify $A_1 \times [\frac{1}{3}, \frac{2}{3}] \subseteq M_\phi$ with $A_1 \times [\frac{1}{3}, \frac{2}{3}] \subseteq \widetilde{M}_g$. This is the colimit of the diagram

$$\begin{tikzcd}[column sep=small] & W_1 \times \{\frac{1}{3}\} \times \{0\} \arrow[dl] \arrow[dr] & & W_0 \times \{\frac{2}{3}\} \times \{1\} \arrow[dl] \arrow[dr] & \\ \widetilde{M}_g & & \overline{M}_\gamma & & \widetilde{M}_f. \end{tikzcd}$$

This space comes with a function $\sigma : Q \to I$ that simply returns the s-value of a point in Q. Write

$$\widehat{Y} = \sigma^{-1}([0, \tfrac{1}{2}]), \qquad \widehat{W} = \sigma^{-1}(\tfrac{1}{2}) \qquad \text{and} \qquad \widehat{X} = \sigma^{-1}([\tfrac{1}{2}, 1]).$$

We'll refer to the diagram $\widehat{Y} \leftarrow \widehat{W} \to \widehat{X}$ as the **tab-and-glue** prepushout diagram constructed from the given data.

Problem 13.11. Show that the tab-and-glue diagram $\widehat{Y} \leftarrow \widehat{W} \to \widehat{X}$ is cofibrant.

Exercise 13.12. Show that a morphism of data

$$\begin{tikzcd} Y \arrow[d] & W_1 \arrow[l] \arrow[d] & W_0 \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\ C & A_1 \arrow[l] & A_0 \arrow[l] \arrow[r] & B \end{tikzcd}$$

gives rise, naturally, to a morphism of tab-and-glue prepushout diagrams.

The most important thing for us, though, is that the tab-and-glue construction can be related to the standard cofibrant replacement by double mapping cylinders. Given a prepushout diagram $C \leftarrow A \to B$, we construct the double mapping cylinder

$$D = (C \cup (A \times I) \cup B) \sim$$

which we index by the parameter s. Taking the s-coordinate gives a continuous function $\tau : D \to I$. The standard cofibrant replacement for $C \leftarrow A \to B$ is $\overline{C} \leftarrow A \to \overline{B}$, where

$$\overline{C} = \tau^{-1}([0, \tfrac{1}{2}]) \qquad \text{and} \qquad \overline{B} = \tau^{-1}([\tfrac{1}{2}, 1])$$

and we identify A with $A \times \{\frac{1}{2}\}$.

Problem 13.13. Show that the morphism of data

$$\begin{tikzcd} Y \arrow[d] & W_1 \arrow[l] \arrow[d] & W_0 \arrow[l, "\gamma"'] \arrow[r] \arrow[d] & X \arrow[d] \\ C & A \arrow[l] & A \arrow[l, "\mathrm{id}_A"] \arrow[r] & B \end{tikzcd}$$

gives rise, naturally, to a morphism

$$\begin{tikzcd}
\widehat{Y} \arrow[d] & \widehat{W} \arrow[l] \arrow[r] \arrow[d] & \widehat{X} \arrow[d] \\
\overline{C} & A \arrow[l] \arrow[r] & \overline{B}
\end{tikzcd}$$

of cofibrant replacements, and hence to an induced map $Q \to \overline{D}$ of (homotopy) pushouts.

13.3.2. Gluing Weak Fibrations with Tabs. The tab-and-glue construction is designed in such a way that when the vertical maps are fibrations and the squares are pullbacks, then the induced map of categorical pushouts is a weak fibration. Consider the diagram

$$\begin{tikzcd}
Y \arrow[d] & W_1 \arrow[l, "g"'] \arrow[d] \arrow[l, phantom, "\text{pullback}", shift right=6ex] & W_0 \arrow[l, "\gamma"', "\simeq"] \arrow[r, "f"] \arrow[d] \arrow[r, phantom, "\text{pullback}", shift right=6ex] & X \arrow[d] \\
C & A \arrow[l, "v"] & A \arrow[l, "\mathrm{id}_A"'] \arrow[r, "u"'] & B,
\end{tikzcd}$$

in which $Y \to C$ and $X \to B$ are fibrations and $A \to B$ and $A \to C$ are cofibrations. Let $\widehat{Y} \leftarrow \widehat{W_0} \to \widehat{X}$ be the tab-and-glue construction for the top row and consider the induced morphism

$$\begin{tikzcd}
\widehat{Y} \arrow[d] & \widehat{W_0} \arrow[l] \arrow[r] \arrow[d] & \widehat{X} \arrow[d] \\
\overline{C} & A \arrow[l] \arrow[r] & \overline{B}
\end{tikzcd}$$

to the standard double mapping cylinder construction for the bottom row. We want to show that the induced map $Q \to \overline{D}$ of categorical (and homotopy) pushouts is a weak fibration.

Problem 13.14.

(a) Suppose that in the square

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, "q"'] \arrow[dr, phantom, "\text{pullback}"] & Y \arrow[d, "p"] \\
A \arrow[r, "i"] & B
\end{tikzcd}$$

the maps p and q are fibrations and i is a cofibration. Using the diagram

$$\begin{tikzcd}
M_j \arrow[r, hook] \arrow[d] & Y \times I \arrow[r] \arrow[d] & Y \arrow[d] \\
M_i \arrow[r, hook] & B \times I \arrow[r] & B,
\end{tikzcd}$$

show that $M_j \to M_i$ is a fibration.

(b) Suppose $f : X \to Y$ fits into a commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f", "\simeq"'] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

in which p and q are fibrations. Factoring f into a cofibration followed by a fibration yields the diagram

$$\begin{tikzcd} X \arrow[rr, "\phi", "\simeq"'] \arrow[dr, "p"'] & & \overline{M}_f \arrow[dl, "\theta"] \\ & B, & \end{tikzcd}$$

and p is a deformation retract of θ in the category $\mathcal{T} \downarrow B$.

HINT. Use Theorem 5.136.

Problem 13.15. We'll use the notation of Section 13.3.1 in referring to subspaces of Q and $\overline{D}$ and write $\widetilde{M}_v = \tau^{-1}([0, \frac{2}{3}])$ and $\widetilde{M}_u = \tau^{-1}([\frac{1}{3}, 1])$.

(a) Show that there is a deformation retraction of $\sigma^{-1}([0, \frac{2}{3}]) \to \widetilde{M}_v$ onto $\widetilde{M}_g \to \widetilde{M}_u$ in $\mathcal{T}_\circ \downarrow \widetilde{M}_v$.

(b) Show that there is a deformation retraction of $\sigma^{-1}([\frac{1}{3}, 1]) \to \widetilde{M}_u$ onto $\widetilde{M}_f \to \widetilde{M}_u$ in $\mathcal{T}_\circ \downarrow \widetilde{M}_u$.

(c) Show that $Q \to \overline{D}$ is a weak fibration and that the squares in the diagram

$$\begin{tikzcd} \widehat{Y} \arrow[r] \arrow[d] & \widehat{Z} \arrow[d] & \widehat{X} \arrow[l] \arrow[d] \\ \overline{C} \arrow[r] & \overline{D} & \overline{B} \arrow[l] \end{tikzcd}$$

are categorical pullback squares.

13.4. The First Cube Theorem

The First Cube Theorem is a homotopy-theoretic form of Theorem 13.9. For topological spaces, the union of a collection of spaces is the prototypical example of a colimit, and the union of two spaces with specified intersection is the basic pushout. Our homotopy-theoretical version concerns the formation of homotopy pushouts (unions) of homotopy pullback squares (fibrations and pullbacks).

Theorem 13.16 (First Cube Theorem)**.** *If, in the strictly commutative cube*

$$\begin{tikzcd}
W \arrow[rr] \arrow[dr] \arrow[dd] & & X \arrow[dr] \arrow[dd] & \\
& Y \arrow[rr] \arrow[dd] & & Z \arrow[dd] \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& C \arrow[rr] & & D,
\end{tikzcd}$$

the top and bottom faces are strong homotopy pushout squares, $Z \to D$ is an induced map of homotopy pushouts, and the back and left faces are strong homotopy pullback squares, then the cube is a Mather cube.

From the cube in Theorem 13.16, build the strictly commutative diagram

$$\begin{tikzcd}
W \arrow[rrrrr] \arrow[drr] \arrow[dd] & & & & & X \arrow[dr] \arrow[dd] & \\
& & Y \arrow[rrrr] \arrow[ddd] & & & & Z \arrow[ddd] \\
\widetilde{W} \arrow[rrr] \arrow[dr] \arrow[ddd] & & & \widehat{X} \arrow[rr] \arrow[dr] \arrow[ddd] & & \widetilde{X} \arrow[ddr] \arrow[ddd] & \\
& \widehat{Y} \arrow[rrr] \arrow[dr] \arrow[ddd] & & & \widehat{Z} \arrow[drr, dotted] \arrow[ddd] & & \\
& & \widetilde{Y} \arrow[rrrr] \arrow[ddd] & & & & \widetilde{Z} \arrow[ddd] \\
A \arrow[rrr] \arrow[dr] & & & \overline{B} \arrow[rr] \arrow[dr] & & B \arrow[ddr] & \\
& \overline{C} \arrow[rrr] \arrow[dr] & & & \overline{D} \arrow[drr, dotted] & & \\
& & C \arrow[rrrr] & & & & D,
\end{tikzcd}$$

in which

(1) the vertical factorizations of the form $Q \to \widetilde{Q} \to R$ are factorizations into a homotopy equivalence followed by a fibration,
(2) the horizontal factorizations are of a cofibration followed by a homotopy equivalence, and
(3) the spaces $\widehat{Z}$ and $\overline{D}$ are the categorical pushouts, and the dotted arrows are the induced maps.

We do not assume any particular construction has been made for the factorizations.

Problem 13.17.

(a) Show that in the small cube, the top and bottom faces are strong homotopy pushout squares.
(b) Show that in the small cube, the back and left faces are strong homotopy pullback squares.
(c) Show that to prove Theorem 13.16, it suffices to show that in the small cube, the front and right faces are strong homotopy pullback squares.

We will prove Theorem 13.16 by using the tab-and-glue construction to find cofibrant replacements for the prepushout parts of the top and bottom squares in the small cube in which

- the map $\widehat{Z} \to \overline{D}$ is a weak fibration and
- the front and right faces are categorical pullback squares.

Notice that according to our construction, these prepushout diagrams are already cofibrant.

Assume that we have a cube in which the horizontal maps on the bottom are cofibrations and the maps $Y \to C$ and $X \to B$ are fibrations. Form the categorical pullbacks in the squares to obtain the strictly commutative diagram

$$\begin{tikzcd}
Y \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & W_1 \arrow[l] \arrow[d] & W \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d] & W_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & X \arrow[d] \\
C & A \arrow[l] \arrow[r, equal] & A \arrow[r, equal] & A \arrow[r] & B
\end{tikzcd}$$

in which the left and right squares are categorical pullbacks and the vertical maps, except for $W \to A$, are fibrations.

Problem 13.18.

(a) Show that there is a strictly commutative diagram

$$\begin{tikzcd} Y \arrow[d] \arrow[dr, phantom, "v"] & W_1 \arrow[l] \arrow[d] \arrow[dr, phantom, "\phi"] & W_0 \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "u"] & X \arrow[d] \\ C & A \arrow[l] \arrow[r, equal] & A \arrow[r] & B \end{tikzcd}$$

in which all the vertical maps are fibrations and ϕ is a homotopy equivalence in $\mathcal{T}_\circ \downarrow A$.

(b) Prove Theorem 13.16.

Project 13.19. Generalize Theorem 13.16, if possible, to apply to pointwise fibrations between diagrams with other diagram shapes.

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Chapter 14

Pullbacks of Cofibrations

Formally, absolutely nothing can be said about the pullback of a cofibration, because cofibrations are domain-type maps and pullbacks are target-type constructions. In this chapter we dive back into the point-set topology of cofibrations and show that in a pullback square

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "q"'] \arrow[dr, phantom, "\text{pullback}"] & B \arrow[d, "p"] \\ C \arrow[r, "i"] & D \end{tikzcd}$$

of topological spaces in which p is a fibration and i is a cofibration, not only is q a fibration (which is formal and easy), but j is a cofibration!

This is extremely interesting because it so flagrantly violates the distinction between domain-type and target-type concepts. Using it, we prove the extraordinarily powerful Second Cube Theorem, which says, roughly, that the pullback of a strong homotopy pushout square by a fibration is another strong homotopy pushout square.

14.1. Pullbacks of Cofibrations

In this first section, we prove the main point-set theoretical result of the chapter.

Theorem 14.1. *Let $p : E \to B$ be a fibration and let $i : A \hookrightarrow B$ be a cofibration in $\mathcal{T}_\circ$. Then in the pullback square*

$$\begin{tikzcd}
E_A \arrow[r, "j"] \arrow[d, "q"'] \arrow[dr, phantom, "\text{pullback}"] & E \arrow[d, "p"] \\
A \arrow[r, "i"] & B
\end{tikzcd}$$

the map $j : E_A \to E$ is also a cofibration in $\mathcal{T}_\circ$.

To prove this pullback result, we need yet another characterization of cofibrations. As mentioned, this result is not formal, so we will have to do some real topology to prove it.

Lemma 14.2. *A map $i : A \to X$ is a cofibration if and only if there is a map $u : X \to I$ such that $u(A) = \{0\}$ and a homotopy $H : X \times I \to X$ such that the diagram*

$$\begin{tikzcd}
A \arrow[r, "\mathrm{in}_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i\times\mathrm{id}"] \arrow[ddr, bend left, "\boxed{i}"] & \\
X \arrow[r, "\mathrm{in}_0"] \arrow[drr, bend right, "\mathrm{id}_X"'] & X \times I \arrow[dr, "H"] & \\
& & X
\end{tikzcd}$$

commutes and $H(x,t) \in A$ whenever $t > u(x)$.

Problem 14.3. Recall (from Proposition 5.8) that i is a cofibration if and only if there is a retraction $r : X \times I \to X \cup (A \times I)$.

(a) Supposing i is a cofibration, write $r(x,t) = (r_1(x,t), r_2(x,t))$, and define $u(x) = \sup\{t - r_2(x,t) \mid t \in I\}$. Find a homotopy $H : X \times I \to X$ so that, together, H and u satisfy the conditions of Lemma 14.2.

(b) Now suppose that u and H satisfy those conditions. Show that the function

$$r(x,t) = \begin{cases} (H(x,t), 0) & \text{if } t \le u(x), \\ (H(x,t), t - u(x)) & \text{if } t \ge u(x) \end{cases}$$

defines a retraction of $X \times I \to X \cup (A \times I)$.

Now we turn our attention to the proof of Theorem 14.1. Since $i : A \hookrightarrow B$ is a cofibration, we can find a map $u : B \to I$ and a homotopy $H : B \times I \to B$ satisfying the conditions of Lemma 14.2. Consider the diagram

$$\begin{tikzcd}
E \arrow[rr, "\mathrm{id}"] \arrow[d, "\mathrm{in}_0"'] & & E \arrow[d, "p"] \\
E \times I \arrow[r, "p\times \mathrm{id}"] \arrow[urr, dotted, "J"] & B \times I \arrow[r, "H"] & B.
\end{tikzcd}$$

Since p is a fibration, the lift J exists. Let $v = u \circ p : E \to I$ and define $K : E \times I \to E$ by the rule

$$K(x,t) = \begin{cases} J(x,t) & \text{if } t \leq v(x), \\ J(x, v(x)) & \text{if } t \geq v(x). \end{cases}$$

Problem 14.4. Prove Theorem 14.1 by showing the functions K and v satisfy the conditions of Lemma 14.2.

Problem 14.5. Show that Theorem 14.1 is equally valid in $\mathcal{T}_*$, provided all the spaces involved are well-pointed.

14.2. Pullbacks of Well-Pointed Spaces

Problem 14.5 brings up the question: is the pullback of a diagram of well-pointed spaces also well-pointed? To answer our question, we need to establish a 'two-thirds' property for compositions of cofibrations: if the composition $i \circ f$ of a map f with a cofibration i is a cofibration, then f must be a cofibration.

We need another utterly unexpected local-to-global theorem of Dold showing that a map that is 'locally a cofibration' is a cofibration [**53**].

Theorem 14.6. *Let $i : A \hookrightarrow X$, let $V \subseteq X$ and let $\{U_j \mid j \in \mathcal{J}\}$ be a numerable open cover of X.*

(a) If there is a map $\tau : X \to I$ such that $A \cap V \subseteq \tau^{-1}((0,1]) \subseteq V$, then $i_V : A \cap V \to V$ is a cofibration.

(b) If each of the maps $i_j : A \cap U_j \hookrightarrow U_j$ defined by the categorical pullback squares

$$\begin{tikzcd}
A \cap U_j \arrow[r] \arrow[d, "i_j"'] \arrow[dr, phantom, "\text{pullback}"] & A \arrow[d, "i"] \\
U_j \arrow[r] & X
\end{tikzcd}$$

is a cofibration, then i is a cofibration.

Since the proof amounts to a great deal of very tricky manipulations involving numerable covers, we'll take it for granted and proceed to the proof of Proposition 14.9.

A **halo** around a closed subspace $A \subseteq X$ is an open set U containing A such that there is a function $\tau : X \to I$ with $\tau(A) = \{1\}$ and $\tau(X - U) = \{0\}$.

Corollary 14.7. *Suppose $i : A \hookrightarrow X$ is a cofibration and U is a halo around A in X. Then $j : A \hookrightarrow U$ is also a cofibration.*

Problem 14.8. Prove Corollary 14.7.

Proposition 14.9. *Consider the commutative triangle*

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & X \\ & B \arrow[ur, "i"'] & \end{tikzcd}$$

in which i is a cofibration. Then α is a cofibration if and only if β is a cofibration.

Problem 14.10.

(a) Show that B has a halo U and a retraction $r : U \to B$.

(b) Prove Proposition 14.9 using the diagram

$$\begin{tikzcd} A \arrow[rrr] \arrow[d] & & & Y^I \arrow[d] \\ B \arrow[r] & U \arrow[r] & B \arrow[r] & Y. \end{tikzcd}$$

Finally, we use Proposition 14.9 to prove that a pullback of well-pointed spaces is well-pointed.

Theorem 14.11. *Suppose the square*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\text{pullback}"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a pullback square and $p : B \to D$ is a fibration. If B, C and D are well-pointed, then so is A.

Problem 14.12. Prove Theorem 14.11.

14.3. The Second Cube Theorem

The Second Cube Theorem turns Theorem 14.1 into a powerful tool for manipulating homotopy pullbacks and pushouts. As a warm-up exercise, we'll analyze the pullbacks of a fibration over a very strong kind of homotopy pushout square. Suppose $X = A \cup B$, where A and B are closed in X, so that X is the categorical pushout in the diagram

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X. \end{tikzcd}$$

We further assume that each map in this square is a cofibration, so the square is a strong homotopy pushout square. Now let $p : E \to X$ be a fibration, and form the pullbacks

$$\begin{tikzcd} E_{A\cap B} \arrow[r] \arrow[d, "p_{A\cap B}"'] \arrow[dr, phantom, "PB"] & E \arrow[d, "p"] \\ A \cap B \arrow[r] & X, \end{tikzcd} \qquad \begin{tikzcd} E_A \arrow[r] \arrow[d, "p_A"'] \arrow[dr, phantom, "PB"] & E \arrow[d, "p"] \\ A \arrow[r] & X, \end{tikzcd} \qquad \begin{tikzcd} E_B \arrow[r] \arrow[d, "p_B"'] \arrow[dr, phantom, "PB"] & E \arrow[d, "p"] \\ B \arrow[r] & X. \end{tikzcd}$$

We can assemble these squares into a commutative cube:

$$\begin{tikzcd} E_{A\cap B} \arrow[rr] \arrow[dr] \arrow[dd] & & E_A \arrow[dr] \arrow[dd] & \\ & E_B \arrow[rr] \arrow[dd] & & E \arrow[dd] \\ A \cap B \arrow[rr] \arrow[dr] & & A \arrow[dr] & \\ & B \arrow[rr] & & X. \end{tikzcd}$$

Problem 14.13. Show that the square

$$\begin{tikzcd} E_{A\cap B} \arrow[r] \arrow[d] & E_A \arrow[d] \\ E_B \arrow[r] & E \end{tikzcd}$$

is a strong homotopy pushout square.

The Second Cube Theorem generalizes this simple argument to cubes in which the base is a strong homotopy pushout square and the sides are strong homotopy pullback squares. Here is the statement.

Theorem 14.14 (Second Cube Theorem)**.** *Suppose that in the strictly commutative cube*

$$\begin{tikzcd}
W \arrow[rr] \arrow[dd] \arrow[dr] & & X \arrow[dd] \arrow[dr] & \\
& Y \arrow[rr] \arrow[dd] & & Z \arrow[dd] \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& C \arrow[rr] & & D
\end{tikzcd}$$

the bottom is a strong homotopy pushout square in $\mathcal{T}_\circ$ and the sides are strong homotopy pullback squares in $\mathcal{T}_\circ$. Then the top square is also a strong homotopy pushout square in $\mathcal{T}_\circ$.

Theorem 14.14 asserts simply that if the bottom of a cube is a strong homotopy pushout and the sides are strong homotopy pullbacks, then the cube is a Mather cube.

Exercise 14.15. What well-pointedness conditions must you impose in order for the corresponding statement in $\mathcal{T}_*$ to hold?

Problem 14.16.

(a) Show that the given cube is pointwise homotopy equivalent to a commutative cube with the same base but in which all the vertical maps are fibrations and all vertical squares are categorical pullbacks.

(b) Show that the given cube is pointwise homotopy equivalent to a commutative cube whose base is a categorical pushout in which all four maps are cofibrations, all the vertical maps are fibrations, and all vertical squares are categorical (hence homotopy) pullbacks.

(c) Prove Theorem 14.14.

Exercise 14.17. Can we get by with letting one or more of the faces of the cube be a simple homotopy pushout or pullback?

Project 14.18. Generalize Theorem 14.14 to other diagram shapes.

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Chapter 15

Related Topics

In this section we introduce and study a variety of topics related to, or based on, the four main topological inputs to homotopy theory. Most of these, such as the sections on locally trivial bundles and group actions, are very important in the application of homotopy theory to interesting problems and will play important roles in some of our later work. Serre fibrations, quasifibrations and the simplicial approach to homotopy theory are also important to know about, but they will be used only in later ancillary topics and not in the main flow of the book.

Section 15.2, on covering spaces, contains the single most important computation in all of homotopy theory: the fundamental group of S^1 is isomorphic to $\mathbb{Z}$, generated by id_{S^1}.

15.1. Locally Trivial Bundles

Fiber bundles are a special kind of highly structured fibration, and many important ideas and concepts outside of homotopy theory are nicely described in terms of them. Their rich structure leads to a powerful theory, but it also muddied the waters so that it was decades after their introduction before Serre and Hurewicz were finally able to distill from them the basic properties that are used to define fibrations. Thus fiber bundles were the original concept and they were studied long before fibrations were defined. In this section we'll study an intermediate notion, the *locally trivial bundles*.

15.1.1. Bundles and Fibrations. The definition begins with an extremely simple notion: a **trivial bundle** over B with fiber F is any map $p : E \to B$

which fits into a commutative triangle

$$\begin{tikzcd} B \times F \arrow[rr, "\cong"] \arrow[dr, "\mathrm{pr}_1"'] & & E \arrow[dl, "p"] \\ & B. & \end{tikzcd}$$

A map $p : E \to B$ is a **locally trivial bundle** with fiber F if there is an open cover $B = \bigcup_\alpha U_\alpha$ so that in each pullback square

$$\begin{tikzcd} E_\alpha \arrow[r] \arrow[d, "p_\alpha"'] \arrow[dr, phantom, "\text{pullback}"] & E \arrow[d, "p"] \\ U_\alpha \arrow[r] & B \end{tikzcd}$$

the map $p_\alpha : E_\alpha \to U_\alpha$ is a trivial bundle with fiber F.

Problem 15.1. Show that a locally trivial bundle over a paracompact base B is a fibration.

In the same way, if $p : E \to B$ is a locally trivial bundle and the open cover $B = \bigcup_\alpha U_\alpha$ is numerable, then p is a fibration. Such bundles are sometimes referred to as **numerable bundles**.

A **bundle map** from one locally trivial bundle to another is a strictly commutative square

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "g"] & B_2. \end{tikzcd}$$

Thus there is a category of locally trivial bundles, and an isomorphism of bundles is simply an invertible morphism in this category.

Exercise 15.2. Consider the map p from Möbius strip M to the circle S^1 given by projection onto the median circle. Show that p is a locally trivial bundle that is not a trivial bundle.

Problem 15.3.

(a) Show that the pullback of a locally trivial bundle by a map g is again a locally trivial bundle.

(b) Show that the map from the total space of a pullback of $p : E \to B$ to E restricts to homeomorphisms between fibers.

The converse result is true, too: a bundle map which restricts to homeomorphisms on fibers is a pullback square.

Problem 15.4. Consider the bundle map

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "g"] & B_2. \end{tikzcd}$$

(a) Show that if the restriction of f to each fiber is a homeomorphism, then the square is a pullback square.

(b) Show that if, in addition, g is a homeomorphism, then the square is a bundle equivalence.

HINT. Let $\mathrm{Homeo}(X)$ denote the subspace of $\mathrm{map}_\circ(X, X)$ comprising all the homeomorphisms. Show that the map $\nu : \mathrm{Homeo}(X) \to \mathrm{Homeo}(X)$ given by $\nu(f) = f^{-1}$ is a homeomorphism.

Some authors require all bundle maps to induce homeomorphisms on fibers. This results in a different category of bundles, which is a subcategory of the one we have defined. We have adopted the more general definition because we will need to study maps between bundles having different fibers.

15.1.2. Example: Projective Spaces. We defined and studied the projective spaces in Section 3.3. They are topologized as quotients using the map $q_n : S^{(n+1)d-1} \to \mathbb{F}\mathrm{P}^n$ which takes a point $x \in S^{(n+1)d-1}$ to its equivalence class $[x]$ (under the coordinatewise multiplication action of S^{d-1}). Now we'll show that q_n is a locally trivial fiber bundle.

Problem 15.5.

(a) Show that the sets $U_i = \{[x] \mid x_i \neq 0\}$ for $i = 1, 2, \ldots, n+1$ form an open cover of $\mathbb{F}\mathrm{P}^n$.

(b) Show that if $[x] \in U_i$, then there is a unique point in $\widetilde{x} \in [x]$ such that $\widetilde{x}_i \in \mathbb{R}^+$.

(c) Show that the pullback map p_i in the diagram

$$\begin{tikzcd} E_i \arrow[r] \arrow[d, "p_i"'] \arrow[dr, phantom, "\text{pullback}"] & S^{(n+1)d-1} \arrow[d, "q_n"] \\ U_i \arrow[r] & \mathbb{F}\mathrm{P}^n \end{tikzcd}$$

is a trivial bundle over U_i.

(d) Conclude that $q_n : S^{(n+1)d-1} \to \mathbb{F}\mathrm{P}^n$ is a locally trivial bundle, and hence a fibration, with fiber S^{d-1}.

Again and again, the study of projective spaces has provided deep insights into homotopy theory. For example, since $\mathbb{F}\mathrm{P}^1 = S^d$, we have fibration sequences that can be used to find valuable information about the low-dimensional homotopy groups of spheres.

Problem 15.6. Show that there are fibration sequences

$$S^1 \longrightarrow S^3 \longrightarrow S^2 \qquad \text{and} \qquad S^3 \longrightarrow S^7 \longrightarrow S^4.$$

The map $S^3 \to S^2$ is generally denoted η and is called the **Hopf map**. The map $S^7 \to S^4$ is also called a **Hopf map**[1] and is denoted ν. Most of the maps $S^{n+k} \to S^n$ for small k (say $k < 20$) are related to the Hopf maps in one way or another.

Problem 15.7. What can you say about the homotopy types of the spaces ΩS^4 and ΩS^2?

15.2. Covering Spaces

Covering maps are an extremely special and rigid kind of fiber bundle. Their rigidity makes it possible to give an essentially complete description of all of them in terms of the fundamental group of the base. Conversely, coverings provide a useful approach to the computation of fundamental groups.

Let $p : \widetilde{X} \to X$ be a map, and let $U \subseteq X$ be open. We say that U is **evenly covered** if there is a homeomorphism $U \times D \xrightarrow{\cong} p^{-1}(U)$, where D is a discrete set,[2] making the diagram

$$\begin{tikzcd} U \times D \arrow[rr, "\cong"] \arrow[dr, "\mathrm{pr}_1"'] & & p^{-1}(U) \arrow[dl, "p|_{p^{-1}(U)}"] \\ & U & \end{tikzcd}$$

commute. We say that p is a **covering** if every point $x \in X$ has an evenly covered neighborhood U.

Every covering map $p : \widetilde{X} \to X$ is a locally trivial bundle, and hence, if X is paracompact, a fibration.

15.2.1. Unique Lifting.

Since a covering $p : \widetilde{X} \to X$ is a fibration, every path in the base can be lifted to a path in $\widetilde{X}$. But much more is true.

[1] But not *the* Hopf map!

[2] Notice that $X \coprod X = X \times S^0$, and, more generally, if we give our indexing set $\mathcal{I}$ the discrete topology, then $\coprod_{i\in\mathcal{I}} X \cong X \times \mathcal{I}$.

Theorem 15.8. *If $p : \widetilde{X} \to X$ is a covering, then in any diagram*

$$\begin{tikzcd} * \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\ I \times I \arrow[r, "H"] \arrow[ur, dotted] & X \end{tikzcd}$$

there is a unique lift.

Corollary 15.9. *If $p : \widetilde{X} \to X$ is a covering, then*

(a) in any diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & \widetilde{X} \arrow[d, "p"] \\ I \arrow[r, "\alpha"] \arrow[ur, dotted] & X \end{tikzcd}$$

there is a unique lift and

(b) p is a fibration, even if X is not paracompact.

Problem 15.10.

(a) Show that the lift is unique if the covering p is a trivial bundle.

(b) Find a cover $\{U_i \mid i \in \mathcal{I}\}$ of X by open sets that are evenly covered. Show that there is an n large enough that for each $k, l \in \{0, \ldots, n-1\}$, the image $\alpha([\frac{k}{n}, \frac{k+1}{n}] \times [\frac{l}{n}, \frac{l+1}{n}])$ is completely contained in at least one of the sets U_i.

(c) Prove Theorem 15.8.

(d) Prove Corollary 15.9.

Theorem 15.8 is the fundamental result underlying the main properties and uses of covering spaces. For example, it implies that lifts of maps out of path-connected spaces are uniquely determined by their value on a single point.

Corollary 15.11. *Let $p : \widetilde{X} \to X$ be a covering, and let Z be path-connected. If there is a lift in the diagram*

$$\begin{tikzcd} * \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\ Z \arrow[r, "f"] \arrow[ur, dotted] & X, \end{tikzcd}$$

then it is unique.

Problem 15.12.

(a) Prove Corollary 15.11.

HINT. Draw a path in Z from $*$ to z.

(b) Show that if $p : \widetilde{X} \to X$ is a covering, then the only lifts of constant paths in X are constant paths in $\widetilde{X}$.

15.2.2. Coverings and the Fundamental Group. Next we explore the relationship between the coverings of X and the fundamental group $\pi_1(X)$.

Let $p : \widetilde{X} \to X$ be a covering map with fiber F. A loop in X can be thought of as a path $\alpha : I \to X$ with $\alpha(0) = \alpha(1) = *$. According to Corollary 15.9, the path α has a unique lift $\bar{\alpha}$ such that $\bar{\alpha}(0) = *$. Thus we get a well-defined function

$$L : \Omega X \longrightarrow F \qquad \text{given by} \qquad L(\alpha) = \bar{\alpha}(1) \in F.$$

What if α and β represent the same class in $\pi_1(X)$? That is, what if there is a path homotopy $H : \alpha \simeq \beta$?

Problem 15.13. Let $p : \widetilde{X} \to X$ be a covering, and suppose $\alpha, \beta : I \to X$ are path homotopic loops in X. Let $\bar{\alpha}$ and $\bar{\beta}$ be the lifts starting at $* \in \widetilde{X}$.

(a) Show that the lifts $\bar{\alpha}$ and $\bar{\beta}$ are path homotopic.

(b) Show that $\bar{\alpha}(1) = \bar{\beta}(1)$.

Because of Problem 15.13, we have a well-defined map $\pi_1(X) \to F$.

Problem 15.14.

(a) Show that if $\widetilde{X}$ is path-connected, then $\pi_1(X) \to F$ is surjective.

(b) Show that if $\widetilde{X}$ is simply-connected, then $\pi_1(X) \to F$ is injective.

A covering $p : \widetilde{X} \to X$ in which $\widetilde{X}$ is simply-connected is called a **universal cover** for X. We have shown that if $p : \widetilde{X} \to X$ is a universal cover with fiber F, then there is a bijection $\pi_1(X) \to F$.

How do we recover the group structure?

15.2.3. Lifting Criterion. To deduce the group structure of $\pi_1(X)$ from a universal cover $p : \widetilde{X} \to X$, we begin by establishing a criterion for lifting a map in X through p.

Theorem 15.15. *Let Z be path-connected, and let $p : \widetilde{X} \to X$ be a covering. The lifting problem*

$$\begin{tikzcd} * \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\ Z \arrow[r, "f"] \arrow[ur, dotted] & X \end{tikzcd}$$

has a solution (which is unique) if and only if $f_*(\pi_1(Z)) \subseteq p_*(\pi_1(E))$.

Corollary 15.16. *Any two universal covers for X are isomorphic in $\mathcal{T}_* \downarrow X$.*

The uniqueness is what makes this theorem easy to prove: we can write down the only possible candidate for a lift and then check that it satisfies the properties we want.

Problem 15.17.

(a) Using lifting of paths, describe exactly what λ *must* be.

(b) Suppose $f_*(\pi_1(Z)) \subseteq p_*(\pi_1(E))$. Show that if $\alpha : I \to Z$ is a loop, then the unique lift $\bar{\alpha}$ is a loop also.

(c) Finish the proof of Theorem 15.15 by showing that the function λ of part (a) is well-defined and continuous.

Let $p : \widetilde{X} \to X$ be a universal covering with fiber F. For each $x \in F$, let $i_x : * \to \widetilde{X}$ be the map given by $i_x(*) = x$. Now consider the lifting problem

$$\begin{tikzcd} * \arrow[r, "i_x"] \arrow[d] & \widetilde{X} \arrow[d, "p"] \\ \widetilde{X} \arrow[r, "p"'] \arrow[ur, dotted, "d_x"] & X \end{tikzcd}$$

According to Theorem 15.15 the problem has a unique solution, which we'll write $d_x : \widetilde{X} \to \widetilde{X}$. This is called a **deck transformation** of p. The set $\mathrm{Deck}(p)$ of all deck transformations is in natural bijective correspondence with F, which is in bijective correspondence with $\pi_1(X)$. Thus we have a bijective map

$$\delta : \pi_1(X) \longrightarrow \mathrm{Deck}(p).$$

Theorem 15.18. *The set* $\mathrm{Deck}(p)$ *is a group under composition, and the map* $\delta : \pi_1(X) \to \mathrm{Deck}(p)$ *is an isomorphism of groups.*

Problem 15.19. Prove Theorem 15.18

The connection between covering spaces and the fundamental group goes much deeper.

Project 15.20. A subgroup $H \subseteq \pi_1(X)$ acts, via the isomorphism δ, on $\widetilde{X}$. Show that p induces a covering $p_H : \widetilde{X}/H \to X$, and develop a Galois-type theory relating covering spaces of X with subgroups of $\pi_1(X)$.

15.2.4. The Fundamental Group of S^1. We put the theory of covering spaces to work and establish the single most important computation in all of homotopy theory: the group $\pi_1(S^1)$.

Theorem 15.21. $\pi_1(S^1) \cong \mathbb{Z} \cdot [\mathrm{id}_{S^1}]$.

Problem 15.22.

(a) Show that the map $\exp(t) = e^{2\pi i t}$ defines a covering $\mathbb{R} \to S^1$.

(b) Prove Theorem 15.21.

Problem 15.23. Determine all the homotopy groups of S^1.

The fundamental groups of the other spheres are much more simple.

Problem 15.24. Show that S^n is simply-connected if $n \geq 2$.

Problem 15.25. Show that the quotient map $q_n : S^n \to \mathbb{R}\mathrm{P}^n$ is a covering map and determine $\pi_1(\mathbb{R}\mathrm{P}^n)$ for all n.

Problem 15.26. For which values of k is $\pi_k(\mathbb{R}\mathrm{P}^n) \cong \pi_k(S^n)$?

Problem 15.27. Show that $\pi_2(S^2) \cong \mathbb{Z}$. Can you find a generator?

15.3. Bundles Built from Group Actions

Projective spaces are obtained by forming orbit spaces of group actions, and the associated quotient maps turned out to be locally trivial bundles. We'll see that very often the quotient maps associated to group actions are bundles.

15.3.1. Local Sections for Orbit Spaces. We begin with an example of an orbit map that is not a bundle.

Exercise 15.28. Let the group S^1 act on the disk D^2 by rotation (or by multiplication, if we consider $D^2 \subseteq \mathbb{C}$). Let Z be the orbit space and show that the quotient map $D^2 \to Z$ is not a locally trivial bundle.

Evidently orbit quotient maps are locally trivial bundles only for actions satisfying some extra condition.

Problem 15.29. Let X be a G-space. Show that the following are equivalent:

(1) the quotient $X \to X/G$ is a locally trivial bundle,

(2) each point of X/G has a neighborhood U whose inclusion map lifts as in the diagram

$$\begin{tikzcd} & X \arrow[d] \\ U \arrow[ur, dotted, bend left] \arrow[r, "\mathrm{in}"] & X/G. \end{tikzcd}$$

A lift as in part (2) is called a **local section**.

If G acts on X, then any subgroup $H \subseteq G$ also acts on X and so gives rise to an orbit space X/H. There is a natural map from X/H to X/G.

Exercise 15.30. Show that there is a natural map $q : X/H \to X/G$.

Proposition 15.31. *Let X be a space with a free G-action, and suppose X/G has an open cover $\{U_\alpha\}$ by sets having local sections. If $H \subseteq G$, then the induced map $q : X/H \to X/G$ is a locally trivial bundle with fiber G/H.*

Problem 15.32. In the situation of Proposition 15.31, let $p : X \to X/G$ be the quotient map.

(a) Prove Proposition 15.31 holds under the assumption that $p : X \to X/G$ is a trivial bundle.

(b) Let $U \subseteq X/G$ be an open set with a local section $\sigma : U \to X$. Show that $q^{-1}(U) \to U$ is a trivial bundle with fiber G/H.

(c) Prove Proposition 15.31.

Let's put this to use and derive a previously unsuspected fibration.

Problem 15.33. Consider the action of $S^3 \subseteq \mathbb{H}$ on $S^{4n+3} \subseteq \mathbb{H}^{n+1}$, and note that $S^1 \subseteq S^3$ (it is $\mathbb{C} \cap S^3$). Show that there is a fiber sequence of the form

$$S^2 \longrightarrow \mathbb{C}\mathrm{P}^{2n+1} \xrightarrow{\ p\ } \mathbb{H}\mathrm{P}^n$$

(better, p is a locally trivial bundle with fiber S^2).

Hint. Your work in Problem 15.5 provides all the local sections you need.

The difficult part of applying Proposition 15.31 is finding an open cover of the base having local sections. This problem can be reduced to finding a single local section when X is itself a topological group and $G \subseteq X$ is a subgroup.

Proposition 15.34. *Let X be a topological group, let $G \subseteq X$ be a subgroup and let $p : X \to X/G$ be the quotient map to the orbit space. Then the following are equivalent:*

(1) There is an open set $U \subseteq X/G$ with a local section $U \to X$ of p.

(2) X/G has an open cover $X/H = \bigcup U_\alpha$ where each U_α has a local section $\sigma_\alpha : U_\alpha \to X$.

Problem 15.35.

(a) Show that X acts on X and X/G by *right* multiplication and that the quotient $X \to X/G$ is equivariant with respect to this action.

(b) Show that if $x \in X$, then $U \cdot x$ is an open set in X/G.

(c) Prove Proposition 15.34.

15.3.2. Stiefel Manifolds and Grassmannians. We introduced Stiefel manifolds as generalizations of spheres and Grassmannians as generalizations of projective spaces. We pursue that analogy here by showing that the natural quotient maps $V_k(\mathbb{F}^{n+k}) \to \mathrm{Gr}_k(\mathbb{F}^{n+k})$ are locally trivial fiber bundles.

The construction of local sections in these examples depends on the **Gram-Schmidt** process, which takes an $(n+k) \times k$ matrix $[x_1 \cdots x_k]$ with linearly independent columns and returns a matrix $[v_1 \cdots v_k]$ with orthonormal columns (i.e., a k-frame in $\mathbb{F}^{n+k}$) having the same span. If we write $\mathcal{B}_k(\mathbb{F}^{n+k})$ for the set of matrices $[x_1 \cdots x_k]$ with linearly independent columns, then the process defines a continuous function

$$\mathrm{GS} : \mathcal{B}_k(\mathbb{F}^{n+k}) \longrightarrow V_k(\mathbb{F}^{n+k}).$$

This map has the properties

- if $\mathrm{GS}([x_1 \cdots x_k]) = [v_1 \cdots v_k]$, then
$$\mathrm{Span}(\{x_1, \ldots, x_j\}) = \mathrm{Span}(\{v_1, \ldots, v_j\})$$
for each $j \leq k$ and
- if $x_{j+1} \perp \mathrm{Span}(\{x_1, \ldots, x_j\})$, then $v_{j+1} = \frac{1}{|x_{j+1}|} \cdot x_{j+1}$.

Theorem 15.36. *Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Each of the quotient maps in the triangle*

$$\begin{tikzcd} G(\mathbb{F}^{n+k}) \arrow[rr] \arrow[dr] & & V_k(\mathbb{F}^{n+k}) \arrow[dl] \\ & \mathrm{Gr}_k(\mathbb{F}^{n+k}) & \end{tikzcd}$$

is a fiber bundle.

Problem 15.37.

(a) Show that the set $U = \{V \subseteq \mathbb{F}^{n+k} \mid V + (0 \oplus \mathbb{F}^n) = \mathbb{F}^{n+k}\}$ is an open neighborhood of $\mathbb{F}^k \oplus 0$ in $\mathrm{Gr}_k(\mathbb{F}^{n+k})$.

(b) Let $\mathrm{pr}_V : \mathbb{F}^{n+k} \to V$ be the orthogonal projection onto V, and show that
$$\lambda(V) = \mathrm{GS}([\mathrm{pr}_V(e_1) \cdots \mathrm{pr}_V(e_k) \,|\, e_{k+1} \cdots e_{k+n}])$$
is a local section on U.

(c) Next, let $V \subseteq V_k(\mathbb{F}^{n+k})$ be the preimage of U. Show that the rule
$$\ell([x_1 \cdots x_k]) = \mathrm{GS}([x_1 \cdots x_k \,|\, e_{k+1} \cdots e_{n+k}])$$
defines a local section on V.

(d) Prove Theorem 15.36.

(e) What are the fibers of these bundles?

Problem 15.38. Show that the determinant $\det : U(n) \to S^1$ is a fiber bundle with fiber $SU(n)$.

15.4. Some Theory of Fiber Bundles

There is a vast theory of fiber bundles. We touch on some of the fundamentals here.

15.4.1. Transition Functions. Let $p : E \to B$ be a locally trivial bundle. Suppose that we are given an open cover $B = \bigcup_\alpha U_\alpha$ and specific trivializations of the pullbacks $p_\alpha : E_\alpha \to U_\alpha$ for each α. If $U_\alpha \cap U_\beta \neq \varnothing$, then there are two trivializations over $U_\alpha \cap U_\beta$, and we may compare them by studying the diagram

$$\begin{tikzcd}
(U_\alpha \cap U_\beta) \times F \arrow[rr, "(\mathrm{in}_1, \bar{g}_{\alpha\beta})"] \arrow[dr] \arrow[ddr, bend right, "\mathrm{pr}_1"'] & & (U_\alpha \cap U_\beta) \times F \arrow[dl] \arrow[ddl, bend left, "\mathrm{pr}_1"] \\
& E_\alpha \cap E_\beta \arrow[d] & \\
& U_\alpha \cap U_\beta &
\end{tikzcd}$$

where $\bar{g}_{\alpha\beta} : (U_\alpha \cap U_\beta) \times F \to F$ restricts to homeomorphisms on each subspace of the form $x \times F$. As in Problem 15.4, write

$$\mathrm{Homeo}(F) = \{g : F \to F \mid g \text{ is a homeomorphism}\}$$

for the subspace of $\mathrm{map}_\circ(F, F)$ comprising the self-homeomorphisms of F, which is a topological group under composition. Then the adjoints of the maps $\bar{g}_{\alpha\beta}$ are maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Homeo}(F),$$

called the **transition functions** for the locally trivial bundle p.

Problem 15.39. Let $p : E \to B$ be a locally trivial bundle, and let $B = \bigcup_\alpha U_\alpha$.

(a) Show that p is a trivial bundle if and only if the trivializations $\phi_\alpha : U_\alpha \times F \to E_\alpha$ can be chosen so that the transition functions are $g_{\alpha\beta}(x) = \mathrm{id}_F$ for all x, α and β.

(b) Give an example of a trivial bundle which has nonidentity transition functions.

The transition functions satisfy certain compatibility conditions.

Proposition 15.40. *The transition functions for a locally trivial bundle $p : E \to B$ satisfy:*

(a) $g_{\beta\gamma} \circ g_{\alpha\beta} = g_{\alpha\gamma}$,

(b) $g_{\alpha\alpha} = \mathrm{id}_F$,

(c) $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$,

for any α, β *and* γ.

Problem 15.41. Prove Proposition 15.40.

Recovering a Bundle from Its Transition Functions. The space B has been decomposed into the open sets U_α. To reconstruct B from the sets U_α, we simply take the union. Categorically speaking, the sets U_α, and their various (finite) intersections, form a category, and the inclusions of subsets defines a diagram whose colimit is B.

Similarly, the total space E is decomposed, up to homeomorphism over B, into the spaces $U_\alpha \times F$. The transition functions give the data needed to glue these pieces together to obtain E.

Problem 15.42. Explain how to reconstruct the map $p : E \to B$ from the following data:

- the space B and the G-space F,
- the cover $B = \bigcup_\alpha U_\alpha$, and
- the transition functions $g_{\alpha\beta}$.

15.4.2. Structure Groups. An action of a topological group G on F is adjoint to a continuous homomorphism $h : G \to \mathrm{Homeo}(F)$. If G acts on the fiber F of a locally trivial bundle, then it may happen that the dotted arrow

$$\begin{tikzcd} & G \arrow[d, "h"] \\ U_\alpha \cap U_\beta \arrow[ur, dotted, bend left] \arrow[r, "g_{\alpha\beta}"] & \mathrm{Homeo}(F) \end{tikzcd}$$

can be filled in with a continuous map. If this can be done for each α and β in such a way that the properties of Proposition 15.40 are satisfied, then we say that G is a **structure group** for p, and we refer to the maps $U_\alpha \cap U_\beta \to G$ as transition functions.

Exercise 15.43. Give an example in which the dotted arrow exists but the properties of Proposition 15.40 are not satisfied.

A **fiber bundle** is a locally trivial bundle with an identified structure group G; we'll also call them G-bundles. A **bundle map** from one G-bundle to another is a commutative square

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "g"] & B_2 \end{tikzcd}$$

in which the restrictions to each fiber are G-equivariant. An invertible bundle map is a **bundle isomorphism**.

Exercise 15.44. Show that every locally trivial bundle with fiber F is a fiber bundle with structure group $\mathrm{Homeo}(F)$.

Problem 15.45. Show that the pullback of a G-bundle is again a G-bundle.

Exercise 15.46. Formulate a definition for fiber bundle maps between bundles with different structure groups G and H.

15.4.3. Change of Fiber and Principal Bundles. Now suppose $p : E \to B$ is a fiber bundle with fiber F and structure group G. Let Q be some other space, and suppose we have a homomorphism $\xi : G \to \mathrm{Homeo}(Q)$ (i.e., an action of G on Q). Using Problem 15.42, we may form a new bundle over B with fiber Q using the following data:

- the space B and the G-space Q,
- the cover $B = \bigcup_\alpha U_\alpha$, and
- the transition functions $\xi \circ g_{\alpha\beta}$.

This procedure is called **change of fiber**.

Exercise 15.47. Show that if the action of G on Q is trivial, then the resulting bundle is the map $\mathrm{pr}_1 : B \times Q \to B$.

Thus we see that a given collection of transition functions determines a wide variety of bundles with various fibers. Among these, there is a preferred basic example: the one whose fiber is the group G and where the action is simply left multiplication. This is called the **associated principal bundle** over B with fiber G.

Problem 15.48.

(a) A principal bundle has a section if and only if it is trivial.

(b) Show that a map of one principal G-bundle to another must be a homeomorphism on fibers.

(c) Show that a map of bundles need not be a homeomorphism on fibers.

Changing the fiber starting with a principal bundle has a cleaner description in terms of the **Borel construction**. If we write

$$E \times_G F = E \times F/(ge, f) \sim (e, gf),$$

then we have a map $\pi : E \times_G F \to B$ given by $(e, f) \mapsto p(e)$.

Problem 15.49. Show that π is isomorphic to the fiber bundle obtained from p by the change of fiber technique.

What we have done so far is to break the problem of classifying all G-bundles into two steps:

(1) listing the principal G-bundles (up to equivalence) and

(2) listing the G-spaces (up to isomorphism).

We will see in Section 16.5.5 that the first of these has a useful answer in terms of homotopy theory.

15.5. Serre Fibrations and Model Structures

This section is a very brief overview of an introduction to a vast collection of ideas. We'll state some theorems and offer only a pointer to the literature instead of a proof or proof outline. Still, anything labelled 'Problem' should be doable.

We are interested in homotopy equivalences of spaces, and so we introduced n-equivalences to quantify how close a map is to being homotopy equivalences. But the connectivity of a map is just as much—even more!—an estimate of how close it is to being a *weak* homotopy equivalence. Furthermore, for CW complexes, there is no difference between weak homotopy equivalences and genuine ones. All of this suggests that it is no great loss, and perhaps a bit of a simplification, to forget about genuine homotopy equivalences and to concentrate instead on weak homotopy equivalences.

In this section we will introduce and study a notion of fibration, called **Serre fibrations** after J.-P. Serre, which is well-suited to lifting problems involving CW complexes. The Serre fibrations and weak homotopy equivalences, together with a collection that could reasonably be called **Serre cofibrations**, define a new and different model category structure on $\mathcal{T}_\circ$. This new structure is indistinguishable from the one we have already constructed, so long as you work only with CW complexes. It has an advantage, though, in that the corresponding model category structure on $\mathcal{T}_*$ is consistent with the existing homotopy theory of pointed CW complexes.

15.5.1. Serre Fibrations.

Since we are mostly interested in CW complexes, why ask for the homotopy lifting property with respect to all spaces? If we only require it for CW complexes, we have defined Serre fibrations.

Problem 15.50. Let $p : E \to B$. Show that the following are equivalent:

(1) p has the left lifting property with respect to $\mathrm{in}_0 : D^n \hookrightarrow D^n \times I$ for every n.

(2) p has the left lifting property with respect to $\mathrm{in}_0 : X \hookrightarrow X \times I$ for every CW complex X.

A map satisfying the equivalent conditions of Problem 15.50 is called a **Serre fibration**. The **fiber** of a Serre fibration $p : E \to B$ over $b \in B$ is simply the preimage $p^{-1}(B)$, and we call the resulting sequence $F \to E \to B$ a **Serre fibration!sequence**.

Many of the nice properties of genuine fibrations are shared by Serre fibrations and Serre fibration sequences, at least when they are used as the targets for maps out of CW complexes. In particular, we have an analog of the Fundamental Lifting Property.

Theorem 15.51. *Let $p : E \to B$ be a Serre fibration, and let $i : A \hookrightarrow X$, where X has been built from A by attaching cells. If either i or p is a weak homotopy equivalence, then the lifting problem*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p", "\sim"'] \\ X \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

has a solution.

Problem 15.52. Let $p : E \to B$ and $i : A \to X$ be as in Theorem 15.51.

(a) Use CW induction to reduce the proof of Theorem 15.51 to the case $i : S^n \hookrightarrow D^{n+1}$.

(b) Suppose there is a map ℓ such that in the diagram

$$\begin{tikzcd} S^n \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ D^{n+1} \arrow[r] \arrow[ur, "\ell"] & B \end{tikzcd}$$

the upper triangle is strictly commutative and the lower triangle commutes up to homotopy constant on S^n. Show that ℓ can be replaced with a homotopic map λ making the whole diagram commute on the nose.

(c) Prove Theorem 15.51.

Here are some other useful properties.

Problem 15.53. Let $p : E \to B$ be a Serre fibration with fiber F.

(a) If $A \hookrightarrow X$, where X has been obtained from A by attaching cells, then the relative homotopy lifting problem

$$\begin{tikzcd} (X \times \{0\}) \cup (A \times I) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \times I \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

has a solution.

HINT. Use Problem 11.19.

(b) Show that a pullback of a Serre fibration is also a Serre fibration.

(c) Show that $\Omega p : \Omega E \to \Omega B$ is also a Serre fibration.

(d) Show that if $F \to E \to B$ is a Serre fibration sequence in $\mathcal{T}_*$, then for any CW complex X, the sequence

$$[X, F] \longrightarrow [X, E] \longrightarrow [X, B]$$

of pointed sets is exact.

Comparing Serre and Hurewicz Fibrations. Serre fibrations can be roughly described as maps that are 'weakly homotopy equivalent to fibrations'. Given a map $p : E \to B$, let $b \in B$, $e \in p^{-1}(b)$, and write $P_b = p^{-1}(b)$ for the 'quasifiber' of p over b. Then convert p to a genuine Hurewicz fibrations, resulting in the diagram

$$\begin{tikzcd} P_b \arrow[r, "j"] \arrow[d, "\xi_b"'] & E \arrow[r, "p"] \arrow[d, "\simeq"] & B \arrow[d, equal] \\ F_b \arrow[r] & \overline{E} \arrow[r, "q"] & B \end{tikzcd}$$

which defines comparison maps $\xi_b : P \to F_b$.

Theorem 15.54. *If $p : E \to B$ is a Serre fibration, then for any $b \in B$, the comparison map $\xi_b : P_b \to F_b$ is a weak homotopy equivalence.*

Project 15.55. Prove Theorem 15.54.

Exercise 15.56. Find an example of a map $p : E \to B$ for which each comparison $\xi_b : P_b \to F_b$ is a weak homotopy equivalence but p is not a Serre fibration.

Local Serre Fibrations Are Serre Fibrations. There is an analog for Serre fibrations of Hurewicz's theorem, but it works for any open cover, and it is much easier to prove.

Theorem 15.57. *Let $p : E \to B$, and suppose B has an open cover $\{U_i \mid i \in \mathcal{I}\}$ such that in each pullback square*

$$\begin{tikzcd} E_i \arrow[r] \arrow[d, "p_i"'] \arrow[dr, phantom, "\text{pullback}"] & E \arrow[d, "p"] \\ U_i \arrow[r] & B \end{tikzcd}$$

the map p_i is a Serre fibration. Then p is a Serre fibration.

To prove Theorem 15.57, we have to find a lift in the diagram

$$\begin{tikzcd} D^n \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ D^n \times I \arrow[r, "f"] \arrow[ur, dotted] & B. \end{tikzcd}$$

The construction is a fairly straightforward adaptation of the technique we used to prove Proposition 12.37.

Problem 15.58.

(a) Show that $D^n \times I$ can be decomposed into products of smaller disks with smaller intervals, each of which is mapped by f entirely into (at least) one of the sets U_i.

(b) Let $\mathcal{P}$ be the set of lifts $\lambda_K : K \to E$ making the square

$$\begin{tikzcd} K \cap (D^n \times \{0\}) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ K \arrow[r, "f|_K"'] \arrow[ur, "\lambda_K"] & B \end{tikzcd}$$

commute, where $K \subseteq D^n \times I$ is a subcomplex. Define a partial order on $\mathcal{P}$, and show that $\mathcal{P}$ has a maximal element.

(c) Prove Theorem 15.57.

15.5.2. The Serre-Quillen Model Structure. Theorem 15.51 suggests that the Serre fibrations participate in a new model structure on $\mathcal{T}_\circ$.

Theorem 15.59 (Quillen)**.** *The inclusions $i : A \hookrightarrow X$ of relative CW complexes, the Serre fibrations and the weak homotopy equivalences give $\mathcal{T}_\circ$ the structure of a model category.*

In this structure, every space is fibrant. On the domain side, a map $i : A \to X$ is a cofibration if and only if there is a lift in every square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

in which $p : E \to B$ is a Serre fibration and a weak homotopy equivalence. It is possible to completely determine such maps.

Theorem 15.60. *A map $i : A \to X$ is a cofibration if and only if it is a retract of a generalized relative CW complex.*

Corollary 15.61. *The cofibrant spaces are precisely the retracts of generalized CW complexes, and hence are homotopy equivalent to CW complexes.*

Since every object in a model category has a cofibrant replacement, Theorem 15.59 implies that every space X has a **cellular replacement**: a weak homotopy equivalence $\overline{X} \to X$, where $\overline{X}$ is a CW complex. (We will prove this directly in Section 16.1.)

Project 15.62. Work through proofs of Theorems 15.59 and 15.60. (The article [**57**] is a good reference.)

This information is enough to determine the homotopy category of the Serre-Quillen structure.

Corollary 15.63. *The homotopy category is equivalent to the category whose objects are CW complexes and whose morphisms are* $\langle X, Y \rangle$.

Is there any difference between the Serre-Quillen homotopy category and the Hurewicz homotopy category?

Exercise 15.64. Criticize this argument:

> *Since CW complexes are simultaneously fibrant and cofibrant in both the Serre and the Hurewicz structures, the homotopy categories can be given the exact same description: the objects are CW complexes and the morphisms are ordinary homotopy classes of maps. In other words, though the model structures may be different, their homotopy categories are isomorphic.*

The Pointed Serre-Quillen Structure. Every model category $\mathcal{M}$ has a corresponding pointed model category $* \downarrow \mathcal{M}$, in which the pointed cofibrations, fibrations and weak equivalences are the morphisms which, after forgetting the basepoint, are cofibrations, fibrations and weak equivalences in $\mathcal{M}$. This is the case for the Hurewicz model structure which we described in Theorem 10.3, but the resulting pointed homotopy theory was not the one we wanted: the homotopy equivalences were not pointed homotopy equivalences, for example (they are pointed maps f such that f_- is a homotopy equivalence in $\mathcal{T}_\circ$).

The pointed Serre-Quillen structure, though, *does* capture the notion of pointed weak homotopy equivalence that we want. This is because the objects that are simultaneously fibrant and cofibrant—the CW complexes—are automatically well-pointed. In this context, we can dispense with 'well-pointed' hypotheses, replacing them with the theoretically more satisfying requirement that spaces should be cofibrant. The fact that the 'correct' pointed category $\mathcal{T}_*$ has a full-fledged model structure that is so transparently related to the unpointed structure is one reason that many people prefer the Serre-Quillen approach.

Homotopy Colimits and Limits in the Quillen-Serre Structure. The theory of homotopy colimits and limits that we developed works in essentially the same way in any model category. In the Quillen-Serre structure, of course, they are well-defined up to the weak equivalences in the structure, namely the weak homotopy equivalences.

We have corresponding notions of ordinary and strong **Quillen-Serre homotopy pushout squares** (which is just a homotopy pushout in the Quillen-Serre structure). Explicitly, a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a strong homotopy pushout square if the comparison map $\xi : P \to D$ from the homotopy pushout P of $C \leftarrow A \to B$ is a *weak* homotopy equivalence. Dually, the square is a **strong Quillen-Serre pullback square** if the comparison map $\xi : A \to Q$ from A to the homotopy pullback of $C \to D \leftarrow B$ is a weak homotopy equivalence.

15.6. The Simplicial Approach to Homotopy Theory

In Section 12.2.2 we defined *finite* simplicial complexes, which was sufficient for our needs at the time. But there is no serious difficulty in defining and working with simplicial complexes having infinitely many cells; indeed, we'll see that every CW complex is homotopy equivalent to a (possibly infinite) simplicial complex and that the homotopy theory of simplicial complexes has a very fruitful combinatorial abstraction.

15.6.1. Simplicial Complexes. We suggested, following Problem 12.21, that a simplicial structure on a space X can be thought of as a collection of maps $\Delta^n \to X$ satisfying various conditions (note that we no longer mention finiteness). Now we will explore this idea in some detail.

Let's look at the standard simplices Δ^n. Write $\mathbf{n}$ for the ordered set $\{0, 1, \ldots, n\}$, and for each ordered subset $I \subseteq \mathbf{n}$, write

$$\Delta(I) = \Delta(e_{i_0}, e_{i_1}, \ldots, e_{i_k}) \subseteq \Delta^n.$$

The ordering of I yields an embedding $\Delta^k \to \Delta^n$. Special among these are the $(n-1)$-dimensional subsimplices, called the **faces** of Δ^n. We write

$$\begin{tikzcd} \Delta^{n-1} \arrow[rr, "\partial_i"] \arrow[dr, "\cong"'] & & \Delta^n \\ & \Delta(\mathbf{n} - \{i\}) \arrow[ur] & \end{tikzcd}$$

and call ∂_i the i^{th} **face map**.

Problem 15.65.

(a) Show that the embeddings $\Delta^k \to \Delta^n$ corresponding to subsimplices are composites of face maps. Can you find a standard form?

(b) Show that if $i < j$, then $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$.

An n-dimensional **singular simplex** of a space X is a continuous map $\sigma : \Delta^n \to X$. The i^{th} **face** of σ is the composite $\sigma \circ \partial_i : \Delta^{n-1} \to X$, and a **subsimplex** of σ is the composite of σ with any collection of face maps.

A **simplicial structure** on a space X is a collection K of singular simplices $\sigma : \Delta^n \to X$ such that

- each $\sigma \in K$ is an embedding,
- the map $(\sigma) : \coprod_{\sigma \in K} \Delta^{n_\sigma} \to X$ is a quotient map, and
- if $\sigma(\Delta^n) \cap \tau(\Delta^m) \neq \varnothing$, then σ and τ have a common subsimplex $\rho : \Delta^k \to X$ such that $\sigma(\Delta^n) \cap \tau(\Delta^m) = \rho(\Delta^k)$.

We say that X is a **simplicial complex** if it has a simplicial structure.

Suppose X has a simplicial structure K, and group the simplices of X according to dimension:

$$K_0, K_1, \ldots, K_n, K_{n+1}, \ldots.$$

These sets (whose union is K) have complex relationships between them, since certain simplices in K_n have faces in K_m, and so on. These relationships are entirely captured by the **face maps**

$$d_i : K_n \longrightarrow K_{n-1}$$

induced by the face maps $\partial_i : \Delta^{n-1} \to \Delta^n$.

The topology of X is entirely determined by the combinatorics of K and its face maps. To see this, we define an **abstract simplicial complex** to be a list of sets $K_0, K_1, \ldots, K_n, \ldots$ equipped with maps $d_i : K_n \to K_{n-1}$ for $0 \le i \le n$ satisfying the relation $d_j \circ d_i = d_i \circ d_{j-1}$ if $i < j$.

Exercise 15.66. Show that if K is a simplicial structure on a space X, then K is an abstract simplicial complex.

We can view any abstract simplicial complex as a recipe for building a space by gluing simplices together. We define the **geometric realization** of K to be the space

$$|K| = \coprod_n \coprod_{\sigma \in K_n} \Delta^n_\sigma \Big/ \sim,$$

where the equivalence relation $\sim$ is generated by the simple equivalences

$$\partial_i(x) \in \Delta^n_\sigma \qquad \text{is equivalent to} \qquad x \in \Delta^{n-1}_{d_i(\sigma)}.$$

If K is a simplicial structure on a space X, then we have a map $\phi : |K| \to X$ whose restriction to Δ^n_σ is just the map σ.

Theorem 15.67. *If K defines a simplicial structure on X, then the map $\phi : |K| \to X$ is a homeomorphism.*

Problem 15.68. Prove Theorem 15.67 by showing that ϕ is a bijective quotient map.

Project 15.69. Define the barycentric subdivision of an abstract simplicial complex, and hence of a simplicial complex.

Simplicial Maps. Let X and Y be two simplicial complexes. A **simplicial map** $f : X \to Y$ is a map with the property that for each simplex $\sigma \subseteq X$, there is a simplex $\tau \subseteq Y$ and a commutative diagram

$$\begin{tikzcd} \sigma \arrow[r, "f_\sigma"] \arrow[d] & \tau \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in which the map f_σ is **linear**.

Problem 15.70. Show that a simplicial map is entirely determined by its values on the vertices of X.

Exercise 15.71. Give an example of a map from one simplicial complex to another that is not homotopic to a simplicial map.

HINT. Try maps $S^1 \to S^1$.

This exercise shows that a straightforward translation of the Cellular Approximation Theorem is bound to be false. We can get around the problem be restructuring X before attempting to approximate our map.

Theorem 15.72 (Simplicial Approximation). *Let $f : X \to Y$ be any map where Y is a simplicial complex and X is a finite simplicial complex. Then for some n the same map $f : \mathrm{sd}^n X \to Y$ is homotopic to a simplicial map.*

We will not use Theorem 15.72 in this book, so we'll leave its proof as a project. You should be annoyed by the restriction that the domain must be finite; in fact, this restriction can be removed, at the expense of allowing subdivisions other than barycentric ones. The problem is that with infinitely many simplices, the n^{th} simplex may require n subdivisions, and this forces us to subdivide the whole complex infinitely many times. However, with some cleverness, it it possible to subdivide (nonbarycentically) each simplex just enough to find a homotopy from a given map to a simplicial map.

Project 15.73.

(a) Prove Theorem 15.72.

(b) Show by example that the finiteness hypothesis on X cannot be removed.

(c) Carefully formulate and prove a variant on the theorem that is valid for infinite domains.

(d) Suppose $f, g : X \to Y$ are simplicial maps that are homotopic. Is there a simplicial homotopy between them?

Simplicial Complexes versus CW Complexes. We have seen that a finite simplicial complex is a special kind of CW complex.

Problem 15.74. Show that every simplicial complex is a CW complex.

The reverse is also true: every CW complex is homotopy equivalent to a simplicial complex. This goes to show that the homotopy category of the category of spaces with the Serre model structure can be taken to have as its objects the simplicial complexes. Simplicial complexes have the drawback that they require many more cells (simplices) than CW complexes and the advantage that they are completely described by combinatorial data.

Problem 15.75.

(a) Invent a mapping cylinder construction for simplicial complexes. That is, give the mapping cylinder of a simplicial map a simplicial structure.

(b) Give the cone on a simplicial complex a simplicial structure.

(c) Give the mapping cone of a simplicial map a simplicial structure.

(d) Show that every CW complex is homotopy equivalent to a simplicial complex.

15.6.2. The Functorial Viewpoint. Suppose $f : X \to Y$ is a simplicial map of simplicial complexes (with structures K and L). It could well be that the restriction of f to X_0 is not injecitive, and consequently the images of an n-simplex σ of X is, topologically, an m-simplex of Y with $m < n$. Thus f does not induce maps $K_n \to L_n$.

However, the composite $f \circ \sigma$ is clearly a perfectly nice *singular* simplex of Y. Thus if we hope to study simplicial maps from the combinatorial point of view, we are forced to consider the 'degenerate' singular simplices of X.

Every order-preserving map $\mathbf{n} \to \mathbf{m}$, with no requirements about the relationship between n and m, induces a unique linear map $\Delta^n \to \Delta^m$. Among these are the face maps, but there are also new maps, including the maps

$$\sigma_i : \Delta^n \longrightarrow \Delta^{n-1} \qquad \text{for } 0 \leq i < n$$

that collapse together the vertices i and $i + 1$.

Problem 15.76.

(a) Show that every map $\Delta^n \to \Delta^m$ that preserves the order of the vertices is a composite of the maps ∂_i and σ_j for various i and j.

(b) Work out relations for various composites involving the maps ∂_i and s_j for various i and j.

Though the list of relations you found in Problem 15.76 was historically the first definition of simplicial sets, it is far more efficient to define a category whose composition rule encodes all these maps and their relations. First redefine $\mathbf{n}$ to be the category

$$0 \to 1 \to \cdots \to n.$$

Then we define the category Δ by setting

$$\mathrm{ob}(\Delta) = \{\mathbf{n} \mid n \geq 0\} \qquad \text{and} \qquad \mathrm{mor}_\Delta(\mathbf{n}, \mathbf{m}) = \mathrm{Functors}(\mathbf{n}, \mathbf{m}).$$

A **simplicial set** is a contravariant functor $\Delta \to$ **Sets**. Simplicial sets are Δ-shaped diagrams in the category **Sets**, which means that they are the objects of a category, which is denoted s**Sets**.

A simplex σ in a simplicial set is called **degenerate** if it can be written $\tau \circ s_i$ for some other simplex τ. For simplicial complexes, a simplex $\sigma : \Delta^n \to X$ is degenerate if and only if it has a factorizaion $\Delta^n \xrightarrow{\sigma_i} \Delta^{n-1} \to X$.

Problem 15.77.

(a) Show that a simplex $\sigma : \Delta^n \to X$ is degenerate if and only if it is not injective on its vertices.

(b) Suppose K is a simplicial structure on a space X. Then we may add to K all the degenerate simplices in X, resulting in a list of simplices

$$\widehat{K}_0, \widehat{K}_1, \ldots, \widehat{K}_n, \ldots$$

with face maps $d_i : \widehat{K}_n \to \widehat{K}_{n-1}$ and **degeneracy maps** $s_j : \widehat{K}_n \to \widehat{K}_{n+1}$. Show that $\widehat{K}$ may be viewed as a simplicial set.

(c) Show more generally that every abstract simplicial complex K can be extended to a unique simplicial set $\widehat{K} : \Delta^{\mathrm{op}} \to$ **Sets** such that K_n is precisely the set of nondegenerate simplices in $\widehat{K}(\mathbf{n})$.

(d) Show that maps of simplicial complexes correspond bijectively to natural transformations of simplicial sets.[3]

Finally, let us consider singular simplices in a general space X. Define a new category $\Delta_{\mathrm{top}} \subseteq \mathcal{T}$ by $\mathrm{ob}(\Delta_{\mathrm{top}}) = \{\Delta^n \mid n \geq 0\}$ and

$$\mathrm{mor}_{\Delta_{\mathrm{top}}}(\Delta^n, \Delta^m) = \{\text{linear maps respecting the order of vertices}\}.$$

[3] Remember that a simplicial set is a functor.

It is easy to see that the categories Δ and Δ_{top} are isomorphic (small) categories.

Problem 15.78.

(a) Show that $\Delta \cong \Delta_{\text{top}}$.

(b) Show that the rule $S_\bullet(X) = \text{mor}_{\mathcal{T}}(\,?\,, X) : \Delta_{\text{top}} \to s\mathbf{Sets}$ defines a functor $\mathcal{T}_\circ \to s\mathbf{Sets}$ assigning to each space X a simplicial set $S_\bullet(X)$.

The simplicial set $S_\bullet(X)$ is known as the **singular simplical set** determined by X. There is no trouble in extending the notion of geometric realization from abstract simplicial complexes to simplicial sets.[4] Geometric realization and the singular simplicial set are very nicely related to one another.

Theorem 15.79. *The functors $S_\bullet(\,?\,) : \mathcal{T}_\circ \to s\mathbf{Sets}$ and $|\,?\,| : s\mathbf{Sets} \to \mathcal{T}_\circ$ are an adjoint pair.*

Problem 15.80. Prove Theorem 15.79.

Model Structure on sSets. There is a way to define fibrations, cofibrations and weak equivalences on the category $s\mathbf{Sets}$ in such a way that the combinatorial model category is Quillen equivalent to the Quillen-Serre structure on spaces.

Theorem 15.81. *The functors $S_\bullet$ and $|?|$ are a Quillen equivalence between the Serre structure on $\mathcal{T}$ and the standard model category structure on sSets.*

Project 15.82. Work through a proof of Theorem 15.81 (see the book [**70**]).

Theorem 15.81 implies that the formal homotopy theory of simplicial sets is equivalent to that of spaces (in the Quillen-Serre structure). The combinatorial nature of $s\mathbf{Sets}$ is frequently used to make explicit constructions that would be tricky to execute using spaces.

Finally, we mention a far-reaching abstraction of the concept of a simplical complex. A **simplicial object** in a category $\mathcal{C}$ is a contravariant functor $\Delta \to \mathcal{C}$; the category of simplicial objects in $\mathcal{C}$ is denoted $s\mathcal{C}$. Thus the category of simplicial sets is just the special case $\mathcal{C} = \mathbf{Sets}$; it is customary to denote a simplicial object using notation such as $X_\bullet$, $Y_\bullet$, etc.

15.7. Quasifibrations

In this section we'll introduce and briefly study quasifibrations. Quasifibrations are weaker than Serre fibrations but are still good enough to produce

[4] Geometric realization is an example of the category-theoretic construction called a **coend**.

long exact sequences of homotopy groups, so they are useful for computations.

Given a map $p : E \to B$, let $b \in B$, $e \in p^{-1}(b)$, and write $P_b = p^{-1}(b)$ for the 'quasifiber' of p over b. Then convert p and the inclusion $i : P_b \hookrightarrow E$ to genuine Hurewicz fibrations, resulting in the diagram

$$\begin{tikzcd}
 & P_b \arrow[r, "i"] \arrow[d, "\simeq"] \arrow[dd, bend right=40, "\xi_b"'] & E \arrow[r, "p"] \arrow[d, equal] & B \arrow[d, equal] \\
F_j \arrow[r] \arrow[d, "\zeta_b"'] & \overline{P} \arrow[r, "j"] \arrow[d] & E \arrow[r, "p"] \arrow[d, "\simeq"] & B \arrow[d, equal] \\
\Omega B \arrow[r] & F_b \arrow[r] & \overline{E} \arrow[r, "q"] & B.
\end{tikzcd}$$

This defines comparison maps $\xi_b : P \to F_b$.

Problem 15.83.

(a) Show that for $n \geq 1$, $\xi_b : \pi_n(P_b) \to \pi_n(F_b)$ is an isomorphism if and only if $\zeta_b : \pi_n(F_j) \to \pi_n(\Omega B)$ is an isomorphism.

(b) Show that if ξ_b is a weak homotopy equivalence, then ζ_b is a weak homotopy equivalence.

We say that $p : E \to B$ is a **quasifibration** if for every $b \in B$, the comparison map $\zeta_b : F_j \to \Omega B$ is a weak homotopy equivalence. Theorem 15.54 implies that every Serre fibration is a quasifibration.

Another way to view quasifibrations is to study the diagram

$$\begin{tikzcd}
\Omega B \arrow[r] \arrow[d] \arrow[dr, phantom, "(H)PB"] & \mathcal{P}(E, e) \arrow[d] \\
P_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & E \arrow[d] \\
\{b\} \arrow[r] & B.
\end{tikzcd}$$

The top square is a categorical, and homotopy, pullback square; the bottom square is a categorical pullback square. For p to be a quasifibration, the composite squares (one for each $b \in B$) should be homotopy pullback squares.

Recognizing Quasifibrations. Perhaps unsurprisingly by now, there is a very useful local-to-global theorem for quasifibrations. The statement is complicated by the fact that a pullback of a quasifibration need not be a quasifibration.

Theorem 15.84. *Let $p : E \to B$. If B has an open cover $\{U_i \mid i \in \mathcal{I}\}$ such that*

- *the pullback map $p : p^{-1}(U_i) \to U_i$ is a quasifibration*[5] *for each $i \in \mathcal{I}$ and*
- *for each $x \in U_i \cap U_j$, there is a $k \in \mathcal{I}$ so that $x \in U_k \subseteq U_i \cap U_j$,*

then p is a quasifibration.

Project 15.85. Prove Theorem 15.84 (consult [**55**] or [**12**]).

15.8. Additional Problems and Projects

Problem 15.86. Let X and Y be well-pointed, and consider the homotopy fiber F of the collapse map $q : X \vee Y \to Y$.

(a) Build a Mather cube involving F over the square

$$\begin{tikzcd} * \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & X \arrow[d] \\ Y \arrow[r] & X \vee Y \end{tikzcd}$$

by pulling back from the map $\mathcal{P}(Y) \to Y$.

(b) Determine the homotopy type of F in terms of the spaces X, Y, and basic operations.

(c) Show that q has a (homotopy) section, and deduce that $\Omega(X \vee Y) \simeq \Omega X \times \Omega F$.

Problem 15.87. Suppose $p : \widetilde{X} \to X$ is a universal covering. Show that $p_* : \pi_k(\widetilde{X}) \to \pi_k(X)$ is an isomorphism for all $k \geq 2$.

Problem 15.88.

(a) Show that the quotient $S^\infty \to \mathbb{C}\mathrm{P}^\infty$ is a fiber bundle.

(b) Show that $\Omega\mathbb{C}\mathrm{P}^\infty \simeq S^1$. Determine all the homotopy groups of $\mathbb{C}\mathrm{P}^\infty$.

Problem 15.89.

(a) Determine the homotopy fiber of the fold map $X \vee X \to X$.

(b) What does this tell you in the special case $\mathbb{C}\mathrm{P}^\infty \vee \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$.

Project 15.90. The **simplex category** of a simplicial complex K is the category $\mathcal{S}_K$ whose objects are the simplices of K and whose morphisms are the inclusions of simplices. The 'identity' diagram $\iota : \mathcal{S}_K \to \mathcal{T}_\circ$ carries the abstract object σ in $\mathcal{S}_K$ to the topological subspace $\sigma \subseteq K$ and the abstract morphism $\sigma \to \tau$ in $\mathcal{S}_K$ to the inclusion $\sigma \hookrightarrow \tau$ of subspaces of K.

[5]Such an open set is generally referred to as a **distinguished** open set.

(a) Show that ι is a cofibrant diagram.

(b) Show that $B\mathcal{S}_K = K$.

(c) What is the homotopy colimit of the trivial diagram $\mathcal{S}_K \to \mathcal{T}_\circ$ that takes each σ to $*$?

Let $\mathcal{S}_U$ be the simplex category of U (defined in Project 15.90). If $f : U \to V$ is a simplicial map of simplicial complexes, then we may define a diagram

$$\Theta_f : \mathcal{S}_V \longrightarrow \mathcal{T}_\circ$$

by the rule $\Theta_f(\sigma) = f^{-1}(\sigma)$.

Theorem 15.91. *For any simplicial map $f : U \to V$, the diagram Θ_f is cofibrant, and*

$$\operatorname{hocolim}_\circ \Theta_f = \operatorname{colim}_\circ \Theta_f = U.$$

Problem 15.92. Prove Theorem 15.91.

Part 4

Targets as Domains, Domains as Targets

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Chapter 16

Constructions of Spaces and Maps

In this chapter we use the basic topological inputs to do a variety of things, all of which fall loosely under the heading of construction of spaces or construction of maps.

First we prove the existence of cellular replacements. To apply our results about CW complexes to other spaces, we need to be able to approximate an arbitrary space by a CW complex. An n-**skeleton** of a space X is an n-equivalence $K \to X$ in which K is an n-dimensional CW complex. When $n = \infty$, this is simply a weak homotopy equivalence $K \to X$ from a CW complex K, usually called a **CW replacement** for X. We show that every space X has a CW replacement and hence has n-skeleta for each n.

Cellular replacements can be used to detect connectivity: a space X is n-connected if and only if it has a CW replacement with a trivial n-skeleton. This simple criterion is applied to determine the connectivity of a great many domain-type constructions, such as smash products, joins, wedges, etc., of both maps and spaces.

On the target side, we construct **Postnikov sections**, which are n-equivalences $X \to P$ where $\pi_k(P) = 0$ for $k > n$ (P is n-anticonnected).

Finally we show that for any topological group G, there is a free contractible G-space EG. With a little topological tinkering, it can be arranged that the quotient map from EG to its orbit space is a fiber bundle. Using it, we show that any topological group G is homotopy equivalent to a loop space, by showing that there is a fibration sequence $G \to EG \to BG$.

16.1. Skeleta of Spaces

An n-**skeleton** for a space X is an n-equivalence $X_n \to X$, where X_n is a CW complex of dimension at most n. We'll establish some basic properties of skeleta and prove that every space has a CW replacement, which implies that it has n-skeleta for every n.

16.1.1. Formal Properties of Skeleta.

Our interest in skeleta stems from certain properties that flow directly from the definition and not from any particular construction of them. To emphasize this, we derive these properties first and address the existence question later.

Exercise 16.1.

(a) Explain, in the language of an introductory topology course, what exactly is a 0-skeleton for a space X? Show that every space X has a 0-skeleton.

(b) Define a functor $Z : \mathcal{T} \to \mathcal{T}$ and a natural transformation $Z \to \mathrm{id}$ so that for every $X \in \mathcal{T}$, the map $Z(X) \to X$ is a 0-skeleton.

(c) Give examples to show that n-skeleta are not unique.

(d) Show that if X is already a CW complex, then any CW replacement $\overline{X} \to X$ must be a homotopy equivalence.

(e) Show that if $X_n \to X$ is an n-skeleton for X, then for any $k \leq n$, the composite $(X_n)_k \hookrightarrow X_n \to X$ is a k-skeleton of X.

Maps of spaces can be covered by maps of n-skeleta. When $n = \infty$, these maps between skeleta are unique up to homotopy, and it follows that any two ∞-skeleta are homotopy equivalent to one another.

Theorem 16.2. *Let $f : X \to Y$, and let $i : X_n \to X$ be an n-skeleton for X and let $j : Y_m \to Y$ be m-skeleta for Y. If $n \leq m$, then there is a map $g : X_n \to Y_m$ such that the diagram*

$$\begin{tikzcd} X_n \arrow[r, "g"] \arrow[d, "i"'] & Y_m \arrow[d, "j"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes up to homotopy.[1] *If $n < m$ or if $n = m = \infty$, then any two choices $g, g' : X_n \to Y_m$ are homotopic to each other.*

Corollary 16.3. *If $\overline{X}$ and $\widetilde{X}$ are two CW replacements for the same space X, then $\overline{X} \simeq \widetilde{X}$.*

[1] Can it be chosen to commute on the nose?

Problem 16.4. Prove Theorem 16.2 and Corollary 16.3.

HINT. Use Corollary 11.28.

Exercise 16.5. Give an example to show that the uniqueness statement of Theorem 16.2 is not valid for $n = m < \infty$.

Theorem 16.2 gives useful information when applied to $f = \mathrm{id}_X$.

Problem 16.6. Show that if $i : K \to X$ and $j : L \to X$ are both n-skeleta, then for any covariant homotopy functor $F : \mathcal{T}_* \to \mathbf{Sets}_*$,

$$\mathrm{Im}(F(K) \to F(X)) = \mathrm{Im}(F(L) \to F(X)).$$

Formulate a dual statement for contravariant functors.

Pointed and Unpointed Spaces. The concept of n-equivalence is most easily defined in the unpointed category $\mathcal{T}_\circ$. In the pointed category, we forget the basepoints and use the definition for unpointed spaces.

We have shown that $f : X \to Y$ is an n-equivalence if and only if its restriction to each path-component is an n-equivalence. Also, if $f : X \to Y$ is a map of *path-connected* pointed spaces, then f is an n-equivalence if and only if the induced map $f_* : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for $k < n$ and a surjection for $k = n$.

Problem 16.7. Show that a path-connected space $X \in \mathcal{T}_*$ has a pointed n-skeleton $X_n \to X$ if and only if $X_- \in \mathcal{T}_\circ$ has an unpointed n-skeleton.

16.1.2. Construction of n-Skeleta. The construction of n-skeleta is essentially a follow-your-nose affair: given an n-skeleton $X_n \to X$, we construct an $(n+1)$-skeleton in two steps, first attaching $(n+1)$-cells to kill the kernel of $\pi_n(X_n) \to \pi_n(X)$ and then attaching more $(n+1)$-cells to ensure that the map $\pi_{n+1}(X_{n+1}) \to \pi_{n+1}(X)$ is surjective.

There are essentially two ways to carry out this plan: either use a very inefficient method with no choices involved and obtain a very large, functorial n-skeleton or make choices and build comparatively small but nonfunctorial skeleta. We'll go through the more *ad hoc* construction in detail and leave the functorial approach to you.

The Efficient Approach. We'll prove more than just the existence of n-skeleta: we show that every n-skeleton can be extended to an m-skeleton for $m \geq n$.

Theorem 16.8 (Cellular Replacement)**.** *Let $X_n \to X$ be an n-skeleton of the space X. Then for any m with $n \leq m \leq \infty$, there is an m-skeleton*

$X_m \to X$ such that $(X_m)_n = X_n$ and the triangle

$$\begin{tikzcd} X_n \arrow[rr] \arrow[dr] & & X_m \arrow[dl] \\ & X & \end{tikzcd}$$

is strictly commutative.

Corollary 16.9. *Every space X has a cellular replacement.*

Exercise 16.10. Show that it suffices to prove Theorem 16.2 for a path-connected space $X \in \mathcal{T}_*$.

Let $X \in \mathcal{T}_*$ be path-connected, and let $i : X_n \to X$ be an n-skeleton for X. Write $K = \ker(\pi_n(X_n) \to \pi_n(X))$ and for each element $\alpha \in K$, choose a map $f_\alpha : S^n \to X_n$ in that homotopy class. Then we define Y_{n+1} to be the $(n+1)$-dimensional CW complex in the pushout square

$$\begin{tikzcd} \coprod_{\alpha\in K} S^n \arrow[r] \arrow[d, "(f_\alpha)"'] \arrow[dr, phantom, "\text{pushout}"] & \coprod_{\alpha\in K} D^{n+1} \arrow[d] \\ X_n \arrow[r] & Y_{n+1}. \end{tikzcd}$$

Problem 16.11.

(a) Show that there is a map $j_{n+1} : Y_{n+1} \to X$ whose induced map $\pi_k(Y_n) \to \pi_k(X)$ is an isomorphism for $k \leq n$.

(b) Show that there is a wedge $W = \bigvee S^{n+1}$ and a map $W \to X$ so that the induced map $\pi_{n+1}(W) \to \pi_{n+1}(X)$ is surjective.

(c) Deduce Theorem 16.8 for $m < \infty$.

Repeatedly applying the case $m < \infty$ of Theorem 16.8 produces the strictly commutative diagram

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & X_{n+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & X_{n+r} \arrow[r] \arrow[d] & X_{n+r+1} \arrow[r] \arrow[d] & \cdots \\ X \arrow[r, equal] & X \arrow[r, equal] & \cdots \arrow[r, equal] & X \arrow[r, equal] & X \arrow[r, equal] & \cdots \end{tikzcd}$$

in which the top and bottom rows are cofibrant telescope diagrams. Let $\overline{X}$ be the colimit of the top row so that the induced map of (homotopy) colimits is a map $\overline{X} \to X$.

Problem 16.12.

(a) Show that $\overline{X}$ is a CW complex.

(b) Finish the proof of Theorem 16.8.

(c) Prove Corollary 16.9.

Theorem 16.2 comes close to saying that CW replacement is functorial. But it does not quite say that, because the map g is only well-defined up to homotopy and because of the many choices made in the construction of $\overline{X}$ and $\overline{Y}$.

Exercise 16.13. Show that by making choices of CW replacements $\overline{X} \to X$ for each X, our construction can be used to define a functorial CW replacement whose target is the homotopy category $\mathrm{H}\mathcal{T}$.

The Functorial Construction. We end this section by outlining a functorial CW replacement in the category $\mathcal{T}$. This has the advantage of being functorial, but also the disadvantage of requiring infinitely many cells of all dimensions, even when the original space is a finite CW complex!

Problem 16.14. Modify the proof of Theorem 16.8 by replacing the set K with the set $\widehat{K} = \{\alpha \in \mathrm{map}_\circ(S^n, X_n) \mid i_n \circ \alpha \simeq *\}$.

(a) Check that the proof works equally well with this modified construction.

(b) Find a functorial way to extend a map $f : X \to Y$ to a strictly commutative square

$$\begin{tikzcd} \overline{X} \arrow[r, "\bar{f}"] \arrow[d] & \overline{Y} \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

(that is, verify that $\bar{g} \circ \bar{f} = \overline{g \circ f}$).

This construction is extremely inefficient.

Exercise 16.15. Show that if X is a nontrivial CW complex, then $\overline{X}$ has uncountably many n-cells for each n.

16.2. Connectivity and CW Structure

The connectivity of a space X is defined in terms of maps into X, which means that it is a target-type concept. Therefore it is comparatively easy to keep track of what happens to connectivity when target-type constructions (e.g., products and loop spaces) are applied to a space. By contrast, the effect of domain-type constructions on connectivity is far from obvious. After we characterize connectivity in terms of CW structure, though, determining the effect of domain-type constructions on connectivity is as easy as keeping track of the cells. This is a bit of a miracle, and can be traced back to one of the four basic topological inputs, namely the subdivision of disks.

16.2.1. Cells and n-Equivalences. An n-equivalence identifies the low-dimensional part of the domain with that of the target. More precisely, an n-equivalence can be approximated by a map of CW complexes which is the *identity* on n-skeleta.

Theorem 16.16. *Let $f : X \to Y$, and let $\overline{X} \to X$ be a CW replacement for X. Then the following are equivalent:*

(1) f is an n-equivalence,

(2) Y has a CW replacement $\overline{Y} \to Y$ such that $\overline{X} \subseteq \overline{Y}$ is a subcomplex with $\overline{Y}_n = \overline{X}_n$ and the diagram

$$\begin{tikzcd} \overline{X} \arrow[r, hook] \arrow[d] & \overline{Y} \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is strictly commutative.

Problem 16.17. Prove Theorem 16.16.

Applying Theorem 16.16 to the map $* \to X$ gives our cellular characterization of connectivity.

Corollary 16.18. *A space X is n-connected if and only if X has a CW replacement $\overline{X} \to X$ with $\overline{X}_n = *$.*

Problem 16.19. Prove Corollary 16.18.

16.2.2. Connectivity and Domain-Type Constructions. Now that we understand how the connectivity of a space is reflected in its cells, we can estimate the connectivity of spaces built from other spaces by domain-type constructions. These connectivity estimates will be used over and over in the following chapters.

Problem 16.20. Let X be $(n-1)$-connected and let Y be $(m-1)$-connected.

(a) What can you say about the connectivity of $X \times Y$?

(b) What can you say about the connectivity of $X \wedge Y$?

(c) What can you say about the connectivity of ΣX?

(d) What can you say about the connectivity of $X * Y$?

(e) What can you say about the connectivity of $X \rtimes Y$?

Hint. Use the standard CW structure on a product and on a quotient.

When working with the join, it is sometimes more convenient to express connectivity in the form $(k-2)$.

Exercise 16.21. Suppose X is $(n-2)$-connected and Y is $(m-2)$-connected; then what can you say about the connectivity of $X * Y$?

Finally, we find lower bounds for the connectivity of a smash product of maps.

Problem 16.22. Let $f : A \to X$ and $g : B \to Y$.

(a) What can you say about the connectivity of $f \wedge \mathrm{id}_B$?

(b) What can you say about the connectivity of $f \wedge g$?

What does this tell you about the connectivity of Σf?

HINT. Your answers will depend on the connectivity of the spaces A and B, not just the connectivities of f and g.

We end with another important connectivity estimate.

Problem 16.23. Suppose A is $(a-1)$-connected and B is $(b-1)$-connected. How highly connected is the inclusion $A \vee B \hookrightarrow A \times B$?

This connectivity estimate implies that a space whose dimension is small in comparison to its connectivity must be a co-H-space.

Problem 16.24. Show that if X is an $(n-1)$-connected CW complex with $\dim(X) < 2n$, then X is a co-H-space.

We have shown that X is a co-H-space if and only if $\mathrm{cat}(X) \leq 1$. Problem 16.24 generalizes to give an upper bound for the Lusternik-Schnirelmann category in terms of dimension and connectivity.

Problem 16.25. Let X be an $(n-1)$-connected and d-dimensional CW complex. Show that $\mathrm{cat}(X) \leq d/n$.

Since we know from Problem 9.135 that a co-H-space is a homotopy retract of a suspension, Problem 16.24 shows that there is a space Y such that X is a homotopy retract of ΣY. It does not assert any relation between the cells of Y and the cells of X—indeed, it does not even claim that Y is a CW complex! But it does suggest a question: are the attaching maps of the cells in X actually the suspensions of other maps? Is X itself a suspension?

16.3. Basic Obstruction Theory

Let X be a CW complex and let $A \subseteq X$ be a subcomplex, and consider the extension problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[dl, dotted, "\phi"] \\ Y. & \end{tikzcd}$$

It is natural to try to construct the desired map by induction on the skeleta, or the cells, of X. The inductive step of such a construction reduces to the case $X = W \cup D^n$, where we have $f : W \to Y$ and we wish to extend f to a map $\phi : X \to Y$ making the diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "i"] \arrow[dr, bend right, "f\circ i"'] & W \arrow[r, "j"] \arrow[d, "f"] & X \arrow[dl, dotted, "\exists?\,\phi"] \\ & Y & \end{tikzcd}$$

commute.

Problem 16.26. Show that ϕ exists if and only if $f \circ i = 0 \in \pi_{n-1}(Y)$.

The element $f \circ i \in \pi_{n-1}(Y)$ is called the **obstruction** to extending the map f. If $f \circ i = 0$, then we say that the obstruction **vanishes**, or that there is **no obstruction** to extending the map. This technique is called **obstruction theory**.

Obstruction theory can be used very effectively to study the maps from a CW complex into a space whose homotopy groups vanish in large dimensions.

Theorem 16.27. *Let Y be a space with $\pi_k(Y) = 0$ for all $k > n$, and let X be a CW complex. Suppose $A \subseteq X$ is a subcomplex with $X_{n+1} \subseteq A \subseteq X$. Then for any map $f : A \to Y$, the dotted arrow in*

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "f"'] & & X \arrow[dl, dotted, "\phi"] \\ & Y & \end{tikzcd}$$

can be filled in so that the triangle is strictly commutative.

The proof is by CW induction.

Problem 16.28.

(a) Let $\mathcal{P}$ denote the set of pairs (U, f_U), where U is a subcomplex of X with $A \subseteq U \subseteq X$ and $f_U : U \to Y$ extends f. Define a partial order on the set $\mathcal{P}$, and show that $\mathcal{P}$ contains a maximal element.

(b) Show that if $U \neq X$, then (U, f_U) is not maximal, and thereby prove Theorem 16.27.

Corollary 16.29. *If $i : A \to X$ is an n-equivalence between CW complexes and $\pi_k(Y) = 0$ for all $k > m$, then the induced map $i^* : [X, Y] \to [A, Y]$ is*

(a) bijective if $n > m$ and

(b) injective if $n = m$.

Problem 16.30. Prove Corollary 16.29.

Hint. Use Theorem 16.16.

We are now equipped to derive a result guaranteeing that certain spaces are H-spaces, roughly dual to Problem 16.24.

Problem 16.31. Show that if X is an $(n-1)$-connected CW complex and $\pi_k(X) = 0$ for $k \geq 2n-1$, then X is an H-space.

Exercise 16.32. Is Problem 16.31 dual to Problem 16.24?

16.4. Postnikov Sections

In Section 16.1 we repeatedly attached cells to construct a CW complex weakly equivalent to a given space. Now we use the same basic construction, but this time, we use it to mutilate the given space, rendering trivial all of its homotopy groups above a specified dimension. This construction produces spaces to which we can apply our simple obstruction theory.

Let X be any space, and let $n \geq 1$. A map $p : X \to Q$ is called an n^{th} **Postnikov approximation** if

- $p_* : \pi_k(X) \to \pi_k(Q)$ is an isomorphism for $k \leq n$ and
- $\pi_k(Q) = 0$ for $k > n$.

Theorem 16.33. *For any space X and any $n \geq 1$, there is an n^{th} Postnikov approximation $p : X \to Q$, and it is unique up to pointwise weak homotopy equivalence of maps.*

Problem 16.34. Prove the existence part of Theorem 16.33 by inductively constructing a sequence of spaces Q_j and maps $X \to Q_j$ (with $j \geq n$) such that

(1) $\pi_k(X) \to \pi_k(Q_j)$ is an isomorphism for $k \leq n$,
(2) $\pi_k(Q_j) = 0$ for $n < k \leq j$, and
(3) Q_j is obtained from Q_{j-1} by attaching $(j+1)$-dimensional cells.

Start the inductive construction by taking $X \to Q_n$ to be the identity map $X \to X$.

Problem 16.35. Show that if X is a CW complex, then it has Postnikov sections that are also CW complexes.

Now we study the naturality properties of Postnikov sections, which leads, naturally, to some conclusions about their uniqueness.

Problem 16.36. Let $X \to Y$ be any map with $\pi_k(Y) = 0$ for $k \geq n$, and let $X \to Q$ be the particular Postnikov approximation you constructed in Problem 16.34.

(a) Show that there is a map $Q \to Y$ making the diagram

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ Q \arrow[rr] & & Y \end{tikzcd}$$

strictly commutative.

(b) Prove the uniqueness part of Theorem 16.33.

Functorial Postnikov Sections. If we eliminate all choices from our construction, we can define a functor that produces Postnikov sections.

Problem 16.37. Adapt the method of Problem 16.14 to construct functorial Postnikov sections, which we will denote $P_n(X)$.

Problem 16.38. Let X and Y be CW complexes.

(a) Show $P_n(X)$ and $P_n(Y)$ are CW complexes.

(b) Show that if $f : X \to Y$ is an $(n+1)$-equivalence, then $P_n(X) \to P_n(Y)$ is a homotopy equivalence.

(c) Show that if $m \geq n$, then the map $P_n(X) \to P_m(P_n(X))$ is a homotopy equivalence.

Problem 16.39.

(a) Show that there are maps $P_{n+1}(X) \to P_n(X)$ making the diagram

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ P_{n+1}(X) \arrow[rr] & & P_n(X) \end{tikzcd}$$

strictly commutative.

(b) What can you say about the homotopy fiber of $P_{n+1}(X) \to P_n(X)$?

(c) Show that the natural map $X \to \operatorname{holim} P_n(X)$ is a weak homotopy equivalence.

Connective Covers. We have studied the universal cover of a space X, which is a map $\widetilde{X} \to X$ where $\widetilde{X}$ is simply-connected and whose induced map is an isomorphism on homotopy groups π_n for $n \geq 2$. Using Postnikov sections, we can generalize this to an n-**connected cover** for each n. Define $X\langle n\rangle$ to be the homotopy fiber in the fiber sequence

$$X\langle n\rangle \longrightarrow X \longrightarrow P_n(X).$$

Problem 16.40. Show that $X\langle n\rangle$ is n-connected and that the induced map $\pi_k(X\langle n\rangle) \to \pi_k(X)$ is an isomorphism for $k > n$.

16.5. Classifying Spaces and Universal Bundles

Every loop space is an H-space, but what about the converse: if X is an H-space, is there another space Y such that $X \simeq \Omega Y$? This is an extremely delicate question, and its answer is bound up with the extent to which the multiplication in X is associative. In this section, we will show that all topological groups—which are perfectly associative—are homotopy equivalent to loop spaces. This is done by constructing a locally trivial fiber bundle $G \to EG \to BG$ in which the space EG is weakly contractible.

We can construct a free and weakly contractible G-space $\mathscr{E}(G)$ by repeated application of a simple and essentially 'obvious' construction. Then we obtain a quotient map $p : \mathscr{E}(G) \to \mathscr{B}(G)$ from $\mathscr{E}(G)$ to its orbit space $\mathscr{B}(G)$. To show that p is a locally trivial fiber bundle, we simply have to find an open cover of $\mathscr{B}(G)$ by open sets having local sections, and this is easy to do, too. But then we find ourselves in a difficult situation: it is unclear whether or not our local sections are continuous!

In [**129**] J. Milnor found the way out. He altered the topologies of the spaces slightly—resulting in a new space EG and a new orbit space BG with the same underlying sets—in such a way as to force the continuity of the local sections while preserving the crucial properties: (1) EG has a continuous G-action and (2) $EG \simeq *$. This is done using a technique similar to the modification of the mapping cylinder topology used in Section 5.4.3.[2]

16.5.1. The Simple Construction. We are hoping to find a contractible free G-space, but we'll be content for now to find a G-equivariant embedding $i : E \hookrightarrow \widehat{E}$ of E into another free G-space such that i is nullhomotopic (but probably not G-nullhomotopic).

The obvious thing to do is to embed E into its (unreduced) cone and G does act on CE in such a way that $E \hookrightarrow CE$ is G-equivariant. But CE is not a *free* G-space, since the cone point is fixed by the action.

Continuing to blunder our way to a solution, we make the action free in the most brute force possible way: by forming the space $G \times CE$ and letting G act by multiplication on the first coordinate. We'd like to include E into $G \times CE$ using the map $e \mapsto (1, [0, e])$, but then the inclusion $E \hookrightarrow G \times CE$ is not G-equivariant. The solution to this final difficulty is to (brute) force

[2] I suspect that this topology-switching trick was first used by Milnor in making this construction.

the two actions to agree by forming the pushout in the square

$$
\begin{tikzcd}
G \times E \arrow[r, "\mathrm{id}_G \times \mathrm{in}_0"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\text{pushout}"] & G \times CE \arrow[d, "q"] \\
E \arrow[r, "i"] & \widehat{E}.
\end{tikzcd}
$$

The action $\widehat{\mu}$ of G on $\widehat{E}$ can be clearly described as the induced map of categorical pushouts in the cube

$$
\begin{tikzcd}
G \times G \times E \arrow[rr] \arrow[dr, "\mathrm{id}_G \times \mu"] \arrow[dd, "\mu \times \mathrm{id}_E"'] & & G \times G \times CE \arrow[dr, "\mathrm{id}_G \times q"] \arrow[dd, "\mu \times \mathrm{id}_{CE}" near end] & \\
& G \times E \arrow[rr, crossing over] & & G \times \widehat{E} \arrow[dd, "\widehat{\mu}"] \\
G \times E \arrow[rr] \arrow[dr, "\mu"] & & G \times CE \arrow[dr] & \\
& E \arrow[rr] \arrow[from=uu, crossing over, "\mu" near start] & & \widehat{E}.
\end{tikzcd}
$$

Points in $\widehat{E}$ are equivalence classes of points in $G \times CE$, so they are equivalence classes $[g, t, e]$ where $g \in G$, $t \in I$ and $e \in E$. It is sometimes convenient to use the alternate notation

$$[g, t, e] = tg + (1 - t)e.$$

Note that in this notation $0g + 1e$ is identified with $e \in E$ and $1g + 0e$ is identified with $g \in G \times *$, where $*$ is the cone point in CE.

Problem 16.41.

(a) Show that $\widehat{\mu} : G \times \widehat{E} \to \widehat{E}$ is a free G-action.

(b) Show that inclusion $E \hookrightarrow \widehat{E}$ is a G-map.

(c) Show that $E \hookrightarrow \widehat{E}$ is nullhomotopic.

(d) Show that $E \hookrightarrow \widehat{E}$ is a cofibration (in $\mathcal{T}_\circ$).

To build a free contractible G-space, start with $\mathscr{E}_0(G) = G$, which is a G-space with action given by the multiplication $\mu : G \times \mathscr{E}_0(G) \to \mathscr{E}_0(G)$. Then inductively define $\mathscr{E}_{n+1}(G) = \widehat{\mathscr{E}_n}(G)$ and let $\mathscr{E}_\infty(G)$ be the (homotopy) colimit of the telescope diagram $\cdots \to \mathscr{E}_n(G) \to \mathscr{E}_{n+1}(G) \to \cdots$.

Problem 16.42. Show that $\mathscr{E}_\infty(G)$ is a contractible free G-space.

Exercise 16.43. What is the role of associativity in this construction?

Identification of $\widehat{E}$ and $\mathscr{E}_n(G)$. The spaces $\widehat{E}$ and $\mathscr{E}_n(G)$ are not actually terribly mysterious: they can be described explicitly in terms of basic homotopy-theoretical constructions.

Problem 16.44.

(a) Show that the map $\mu : G \times E \to E$ is pointwise homotopy equivalent to $\mathrm{pr}_2 : G \times E \to E$.

(b) Determine the homotopy type of $\widehat{E}$ in terms of E and G.

(c) Show that the identification in (b) is homeomorphism, not just a homotopy equivalence.

It follows from Problem 16.44 that $\mathscr{E}_n(G) \cong \overbrace{E * E * \cdots * E}^{n+1 \text{ terms}}$. Points in this iterated join can be written in the form $t_0 g_0 + t_1 g_1 + \cdots + t_n g_n$ where each $t_i \in I$ and $\sum t_i = 1$ and G acts by the rule

$$g \cdot (t_0 g_0 + t_1 g_1 + \cdots + t_n g_n) = t_0(gg_0) + t_1(gg_1) + \cdots + t_n(gg_n).$$

16.5.2. Fixing the Topology. If we write $\mathscr{B}_n(G)$ for the space of orbits of the G-space $\mathscr{E}_n(G)$, then we have a natural quotient map

$$p_n : \mathscr{E}_n(g) \longrightarrow \mathscr{B}_n(G),$$

and it would be very nice if this map were a fibration. In fact, there is good evidence that p_n should be a fiber bundle.

Problem 16.45. Let $U_i = \{t_0 g_0 + t_1 g_1 + \cdots + t_n g_n \in \mathscr{E}_n(G) \mid t_i \neq 0\}$ and let $V_i = p_n(U_i)$.

(a) Show that $V_i \subseteq \mathscr{B}_n(G)$ is open and that the square

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & \mathscr{E}_n(G) \arrow[d] \\ V_i \arrow[r] & \mathscr{B}_n(G) \end{tikzcd}$$

is a pullback square.

(b) Show that each orbit $b = [t_0 g_0 + t_1 g_1 + \cdots + t_n g_n] \in V_i$ contains a unique element of the form

$$\sigma_i(b) = t_0 g_0 + t_1 g_1 + \cdots + t_i \cdot 1_G + \cdots + t_n g_n \in U_i.$$

(c) Show that $\{V_i \mid i = 0, 1, \ldots, n\}$ is a numerable open cover of $\mathscr{B}_n(G)$.

It seems that we can simply apply Proposition 15.31 to conclude that p_n is a fiber bundle. Unfortunately, the maps σ_i may not be continuous! We can get around this problem by slightly modifying the topology. Define new spaces and new orbit maps

$$p_n : E_n(G) \longrightarrow B_n(G)$$

(which are exactly the same as the old ones if we forget the topologies) by giving the *set* $\mathscr{E}_n(G)$ the coarsest topology (fewest open sets) so that the functions

- $\tau_i : \mathscr{E}_n(G) \to I$ given by $\tau_i(\sum t_i g_i) = t_i$ and
- $\gamma_i : \tau_i^{-1}((0,1]) \to G$ given by $\gamma_i(\sum t_i g_i) = g_i$

for $0 \leq i \leq n$ are continuous; the orbit set $B_n(G)$ inherits the quotient topology from $E_n(G)$.

Now that we have changed the topology, we may have lost some (or all) of the desirable properties we have already established for this construction.

Exercise 16.46. Are $E_n(G)$ and $G * G * \cdots * G$ still homeomorphic? Are they homotopy equivalent?

Whatever you find out about these questions, the most important properties have not been sacrificed.

Problem 16.47. (a) Show that $E_n(G) \hookrightarrow E_{n+1}(G)$ is a nullhomotopic cofibration and that $E_{n+1}(G)$ is a free G-space.

(b) Show that the maps $\sigma_i : V_i \to E_n(G)$ are continuous local sections, so that $p_n : E_n(G) \to B_n(G)$ is a fiber bundle with fiber G.

You have proved the existence of universal bundles.

Theorem 16.48 (Milnor)**.** *Let G be a topological group. Then there is a contractible free G-space EG such that the quotient to the orbit space BG is a numerable fiber bundle $p : EG \to BG$ with fiber G.*

This implies that every topological group is homotopy equivalent to a loop space.

Corollary 16.49. *If G is a topological group, then $G \simeq \Omega(BG)$.*

As nice as it is to have a statement for all topological groups, we will mostly use the following corollary.

Corollary 16.50. *If G is a discrete group, then there is a contractible free G-CW complex EG.*

We find a G-CW replacement, following the procedure of Section 16.1.2.

Problem 16.51. Let E be the space built by Milnor's construction.

(a) Show that E, as constructed, is homotopy equivalent to a CW complex.

(b) Find a G-equivariant 0-skeleton $G \to E$.

(c) Suppose you have found a G-equivariant n-skeleton $X_n \to E$; write $K = \ker(\pi_n(X_n) \to \pi_n(E))$ and for each $\alpha \in K$, choose a map $f_\alpha : S^n \to E$ representing α. Then let Y_n be the pushout in

$$\begin{tikzcd}
\coprod_{(g,\alpha)\in G\times K} S^n \arrow[rr] \arrow[d, "(g\circ f_\alpha)"'] & \text{pushout} & \coprod_{G\times K} D^{n+1} \arrow[d] \\
X_n \arrow[rr] & & Y_{n+1},
\end{tikzcd}$$

where we use the G-action on X_n to interpret $g \in G$ as a map $g : X_n \to X_n$. Show that Y_n is a free G-CW complex and there is an induced G-map $Y_n \to E$ which induces isomorphisms on π_k for $k \le n$.

(d) Finish the proof of Corollary 16.50.

Corollary 16.50 gives us the free contractible G-CW complex EG required for the construction for homotopy colimits of G-CW complexes outlined in Section 8.8.

16.5.3. Using EG for EH. Before Milnor's construction, many groups had been shown to have principal bundles with contractible total spaces by relating them to other groups known to have them.

Here is the basic principle.

Problem 16.52. Suppose $H \subseteq G$. Show that EG can serve as EH and that there is an induced map $BH \to BG$, which is a locally trivial fiber bundle with fiber G/H.

Cyclic Groups. One simple but important special case is the free action of S^1 on S^∞ and the resulting principal fiber bundle $S^\infty \to \mathbb{C}\mathrm{P}^\infty$. Since every finite cyclic group $\mathbb{Z}/k$ may be embedded in S^1 (as the k^{th} roots of unity), we have fiber bundles

$$p : S^\infty \longrightarrow B\mathbb{Z}/k$$

for each k. We'll use this construction to learn valuable information about the cellular structure of $B\mathbb{Z}/k$.

Problem 16.53. Consider the $\mathbb{Z}/k$-action on S^∞; we give S^∞ a cellular structure compatible with the action.

(a) Suppose the structure is already defined so that $E_{2n-1} \cong S^{2n-1}$ with a free $\mathbb{Z}/k$-CW action. Define $(E\mathbb{Z}/k)_{2n}$ to be the pushout in

$$\begin{tikzcd} G \times S^{2n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & G \times D^{2n} \arrow[d] \\ S^{2n-1} \arrow[r] & (E\mathbb{Z}/k)_{2n}. \end{tikzcd}$$

The action of $g \in G$ is to permute the indices on the $2n$-cells and to act on their boundaries in the way already defined. Check that this is a free cellular action.

(b) Now we have embeddings $f_g : S^{2n} \to (E\mathbb{Z}/k)_{2n}$ carrying the southern hemisphere to $g \times D^{2n}$ and the northern hemisphere to $(g+1) \times D^{2n}$. Attach cells by all these maps to obtain $(E\mathbb{Z}/k)_{2n+1}$, and show that $(E\mathbb{Z}/k)_{2n+1} \cong S^{2n+1}$.

Now we give S^∞ the cellular structure it inherits as the colimit of the telescope diagram $\cdots \to S^{2n-1} \to S^{2n+1} \to \cdots$. This makes S^∞ a free $\mathbb{Z}/k$-CW complex, and so it endows the orbit space $B\mathbb{Z}/k = (S^\infty)/(\mathbb{Z}/k)$ with a CW structure.

Problem 16.54. Show that in this structure, $B\mathbb{Z}/k$ has a single cell in each dimension $n \geq 0$.

Problem 16.55. Show that $B\mathbb{Z}/2 = \mathbb{R}\mathrm{P}^\infty$, $BS^1 = \mathbb{C}\mathrm{P}^\infty$ and $BS^3 = \mathbb{H}\mathrm{P}^\infty$.

Unitary and Compact Lie Groups. A much more wide-ranging example comes from the classifying bundle for $U(n)$. To find a classifying bundle for $U(n)$, we turn to the principal $U(n)$-bundles

$$p : V_k(\mathbb{C}^{n+k}) \longrightarrow \mathrm{Gr}_k(\mathbb{C}^{n+k})$$

we studied in Section 15.3.2.

Problem 16.56.

(a) Show that there are fiber bundles

$$S^{2n-1} \longrightarrow V_k(\mathbb{C}^{n+k}) \longrightarrow V_{k-1}(\mathbb{C}^{n+k})$$

for all $k \geq 2$.

(b) Show that $V_1(\mathbb{C}^{k+1}) \cong S^{2k+1}$ and determine the connectivity of $V_k(\mathbb{C}^{n+k})$.

(c) Now consider the ladder

$$\begin{tikzcd} \cdots \arrow[r] & V_k(\mathbb{C}^{n+k}) \arrow[r] \arrow[d] & V_k(\mathbb{C}^{n+k+1}) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & \mathrm{Gr}_k(\mathbb{C}^{n+k}) \arrow[r] & \mathrm{Gr}_k(\mathbb{C}^{n+k+1}) \arrow[r] & \cdots . \end{tikzcd}$$

Show that its colimit $p : V_k(\mathbb{C}^\infty) \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ is a principal $U(k)$-bundle and $V_k(\mathbb{C}^\infty) \simeq *$.

Problem 16.56 gives the required bundle for the group $U(k)$. If $G \subseteq U(k)$, then the restriction of the $U(k)$-action to G makes $V_k(\mathbb{C}^\infty)$ into a free G-space, and the quotient map

$$V_k(\mathbb{C}^\infty) \longrightarrow V_k(\mathbb{C}^\infty)/G$$

is again a fiber bundle. Thus infinite-dimensional Stiefel manifolds can serve as EG for any G which can be embedded in $U(k)$ for some k. This is the case, for example, for any compact Lie group.

16.5.4. Discrete Abelian Torsion Groups. In the previous discussion, it emerged that an inclusion of groups $H \hookrightarrow G$ induces a fiber bundle $BH \to BG$. Now we show that the BH built by the Milnor construction may be naturally viewed as a subcomplex of BG. We use this to prove that if G is an abelian torsion group, then BG is the (pointed) homotopy colimit of BH for $H \subseteq G$ finite.

Problem 16.57. Let G be a discrete group, and let EG be the space constructed by the Milnor construction.

(a) Show that EG is a simplicial complex, whose n-simplices are in bijective correspondence with ordered lists $[g_0, g_1, \ldots, g_n]$ of elements of G, and that the action of $g \in G$ is simply to multiply each vertex g_i by g.

(b) If $H \subseteq G$, then there is a natural inclusion $EH \hookrightarrow EG$. Show that if σ and τ are simplices in EH, then $\sigma \sim \tau$ in EG if and only if $\sigma \sim \tau \in EH$. Conclude that the induced map $BH \to BG$ is the inclusion of a subcomplex.

(c) Show that if G is a torsion abelian group, then every simplex in BG is the image of a simplex in BH for $H \subseteq G$ finite.

Fix a discrete group G, and write $\mathcal{F}_G$ for the category whose objects are finite subgroups of G and whose morphisms are the inclusions of subgroups. Then the classifying space functor restricts to a diagram

$$\beta : \mathcal{F}_G \longrightarrow \mathcal{T} \qquad \text{given by} \qquad H \mapsto BH.$$

Naturality gives a collection of maps $BH \to BG$ compatible with those in the diagram, and hence a comparison map

$$\xi : \operatorname{colim} \beta \longrightarrow BG.$$

Notice that since $B(\{1\}) = *$, we can view this diagram and these maps as lying either in $\mathcal{T}_\circ$ or $\mathcal{T}_*$.

If the group G is an abelian torsion group, then we can say much more than this.

Proposition 16.58. *If G is an abelian torsion group and $\beta : \mathcal{F}_G \to \mathcal{T}_\circ$ is the diagram defined above, then*

(a) β is cofibrant and

(b) the comparison map $\xi : \operatorname{colim}_\circ \beta \to BG$ is a homeomorphism.

It follows that BG is the (pointed or unpointed) homotopy colimit of β.

Problem 16.59.

(a) Show that if G is an abelian torsion group, then any finite set of elements in G generates a *finite* subgroup of G.[3]

(b) Prove Proposition 16.58.

16.5.5. What do Classifying Spaces Classify? We have used the term **classifying space** to describe the quotient BG which is the base of the principal G-bundle with contractible total space EG. This terminology suggests that these spaces can be used to classify something. In this section we give a brief overview of the elements of this theory.

Theorem 16.60. *Let $p : E \to B$ be a fiber bundle, let $f, g : X \to B$ and form the pullback bundles*

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} g^*E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \arrow[r, "g"] & B. \end{tikzcd}$$

If $f \simeq g : X \to B$ (in $\mathcal{T}_\circ$), then the pullback bundles $f^*E \to X$ and $g^*E \to X$ are equivalent.

Now suppose we are interested in principal G-bundles over a space X, and we are given another principal G-bundle $p : E \to B$. There is a fairly simple criterion guaranteeing that all bundles over X are pullbacks from p.

Theorem 16.61. *If X is an n-dimensional CW complex and E is n-connected ($n \leq \infty$), then every bundle is equivalent to $f^*E \to X$ for some map $f : X \to B$.*

Corollary 16.62. *For any topological group G and any CW complex X, there is a bijection*

$$\langle X, BG \rangle \longleftrightarrow \{\textit{equivalence classes of } G\textit{-bundles over } X\}.$$

Thus the bundles over X are **classified** by maps into BG. It follows that understanding the set of equivalence classes of principal G-bundles over X, which is quite a mysterious set on the face of it, is equivalent to understanding the homotopy set $\langle X, B \rangle$, which we have tools for studying.

[3]A group (not necessarily abelian) with this property is called a **locally finite** group.

Project 16.63. Work through proofs of these results.

16.6. Additional Problems and Projects

Problem 16.64. Show that if X is a CW complex that is n-connected for every n, then $X \simeq *$.

Problem 16.65. What is the connectivity of $\Omega^n \Sigma^m X$?

Problem 16.66. Suppose A is $(a-1)$-connected. Then what is the connectivity of $\bigvee_{i \in \mathcal{I}} A \hookrightarrow \prod_{i \in \mathcal{I}} A$ (where the product is the weak product)?

Exercise 16.67. Criticize the following argument:

> *If G is a discrete group and X is an $(n-1)$-connected G-CW complex, then X has a CW replacement with a trivial $(n-1)$-skeleton. Since G permutes the cells of a G-CW complex, X/G inherits a CW structure from X in which the k-cells are in bijective correspondence with the equivalence classes of k-cells of X. In particular, the $(n-1)$-skeleton of X/G is trivial, and so X/G is $(n-1)$-connected.*

Project 16.68. For $k \in \mathbb{N}$, write S_k for the n-sphere in $\mathbb{R}^{n+1}$ with radius $\frac{1}{k}$ and center $(0, 0, \ldots, \frac{1}{k})$. The n-dimensional **Hawaiian earring** is the union $\mathcal{H}^n = \bigcup_{k=1}^{\infty} S_k$. Find a CW replacement for $\mathcal{H}^n$. (See [**20**] for ideas.)

Problem 16.69. Estimate the connectivity of the maps

$$f \vee g, \quad f \times g, \quad f * g \qquad \text{and} \qquad f \rtimes g$$

in terms of the connectivities of f, g and the spaces involved.

Problem 16.70. Suppose $f : \Sigma A \to B$ and $g : A \to \Omega B$ are adjoint. How do their connectivities compare?

Problem 16.71. Let X and Y be CW complexes. Show that if $f : X \to Y$ is an n-equivalence, then $P_n(X) \to P_n(Y)$ is a homotopy equivalence.

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Chapter 17

Understanding Suspension

We will show that the suspension map $\sigma : X \to \Omega\Sigma X$ adjoint to the identity $\Sigma X \to \Sigma X$ can be built by a domain-type procedure called the James construction, which produces the free topological monoid on X. The James construction enables us to understand the suspension operation $[X, Y] \to [\Sigma X, \Sigma Y]$; this understanding leads to the Freudenthal Suspension Theorem and a variety of computations.

We establish the existence and properties of Eilenberg-Mac Lane spaces and use them to settle the more delicate 1-dimensional case of the Freudenthal Suspension Theorem.

17.1. Moore Paths and Loops

Concatenation of paths endows the loop space ΩX with the structure of a homotopy-associative, but not strictly associative, H-space. In this section, we show that ΩX is homotopy equivalent, by an H-map, to the strictly associative H-space $\Omega_M X$ of measured loops. Later in this chapter, we'll use measured loops and paths to show that the James construction on X, which is extremely easy to compare to strictly associative H-spaces, is (weakly) homotopy equivalent to the loop space on ΣX.

17.1.1. Spaces of Measured Paths. The space of free **Moore paths** (also called **measured paths**) in a space X is the space

$$M(X) = \left\{(\alpha, a) \mid \alpha : [0, \infty) \to X \text{ and } \alpha|_{[a,\infty)} \text{ is constant}\right\},$$

with the topology it inherits as a subspace of $\mathrm{map}_\circ([0,\infty), X) \times [0,\infty)$. The **initial point** of the Moore path (α, a) is $\alpha(0) \in X$, and its **endpoint** is $\alpha(a)$. If (α, a) and (β, b) are Moore paths in X and if the endpoint of (α, a) is equal to the initial point of (β, b), then we may **concatenate** them by setting $(\alpha, a) \star (\beta, b) = (\alpha \star \beta, a + b)$, where

$$\alpha \star \beta(t) = \begin{cases} \alpha(t) & \text{if } t \leq a, \\ \beta(t-a) & \text{if } t \geq a. \end{cases}$$

Since this notion of concatenation does not rescale the paths, we'll refer to this as **rigid concatenation**.

The first thing to do is to show that the space of Moore paths is, homotopically speaking, equivalent to the ordinary path space. We'll use the evaluation map

$$@_{0,\mathrm{end}} : M(X) \to X \times X \qquad \text{given by} \qquad @_{0,\mathrm{end}}(\alpha, a) = (\alpha(0), \alpha(a)),$$

and $@_{\mathrm{end}} : M(X) \to X$ by $@_{\mathrm{end}}(\alpha, a) = \alpha(a)$.

Exercise 17.1. Show that $@_{\mathrm{end}}$ is continuous and deduce that $@_{0,\mathrm{end}}$ is continuous.

Proposition 17.2. *The rule* $(\alpha, a) \mapsto \boxed{t \mapsto \alpha(at)}$ *defines a homotopy equivalence*

$$\begin{tikzcd} M(X) \arrow[rr] \arrow[dr, "@_{0,\mathrm{end}}"'] & & X^I \arrow[dl, "@_{0,1}"] \\ & X \times X & \end{tikzcd}$$

in the category $\mathcal{T} \downarrow X \times X$ of spaces over $X \times X$.

Corollary 17.3. *The map $@_{0,\mathrm{end}}$ is a weak fibration.*

Problem 17.4. Prove Proposition 17.2 and Corollary 17.3.

Exercise 17.5. Investigate the relatinship between rigid concatenation in $M(X)$ and ordinary concatenation in X^I.

The space of **based Moore paths** in a pointed space X is the subspace of $M(X)$ comprising all Moore paths that end at the basepoint:

$$\mathcal{P}_M(X) = \{(\alpha, a) \mid \alpha([a,\infty)) = *\} \subseteq M(X).$$

The space of **Moore loops** on X, denoted $\Omega_M(X)$, is the fiber of the **evaluation map** $@_0 : \mathcal{P}_M(X) \to X$ given by $(\alpha, a) \mapsto \alpha(0)$.

Problem 17.6.

(a) Show that $@_0 : \mathcal{P}_M(X) \to X$ is a weak fibration.

(b) Show that the rule $(\alpha, a) \mapsto \boxed{t \mapsto \alpha(at)}$ defines homotopy equivalences

$$\mathcal{P}_M(X) \xrightarrow{\simeq} \mathcal{P}(X) \qquad \text{and} \qquad \Omega_M(X) \xrightarrow{\simeq} \Omega X.$$

Since the space of Moore loops on X is homotopy equivalent to ΩX, it can be given the structure of an H-space. But the advantage of the Moore loop space is that the multiplication is strictly associative.

Problem 17.7. Show that rigid concatenation of Moore loops gives $\Omega_M(X)$ the structure of a topological monoid.

It follows from Problem 17.6 that for any space A the induced map $[A, \Omega_M X] \to [A, \Omega X]$ is a bijection; but since these homotopy sets are groups, we'd really like the map to be a group isomorphism. To prove this, you simply need to verify that the map $\Omega_M X \to \Omega X$ is an H-map.

Problem 17.8. Show that the homotopy equivalence $\Omega_M X \to \Omega X$ is an H-map.

17.1.2. Composing Infinite Collections of Homotopies. A left homotopy $H : X \times I \to Y$ is adjoint to a right homotopy $\widehat{H} : X \to Y^I$. If we choose to index our homotopy on the interval $[0, a]$, then the adjoint is a map $X \to \mathrm{map}_\circ([0, a], Y)$, which can be interpreted as an eventually constant right homotopy $\widehat{H} : X \to M(Y)$, given explicitly by

$$\widehat{H}(x) = \Big(\boxed{t \mapsto H(x, \min\{t, a\})}, a\Big).$$

The rigid concatenation of an infinite sequence of homotopies $H_0 : f_0 \simeq f_1$, $H_1 : f_1 \simeq f_2$ and so on defines a function $H : X \times [0, \infty) \to Y$. We can interpret the adjoint $X \to \mathrm{map}([0, \infty), Y)$ as a map to $M(X)$ if for each $x \in X$, the (long) path $\widetilde{H}|_{x \times [0,\infty)}$ is eventually constant.

So let's suppose there is a continuous function $z : X \to [0, \infty)$ such that for each $x \in X$, the restriction $H|_{x \times [z(x),\infty)}$ is constant. Then the rule

$$\widehat{H}(x) = (H|_{x \times [0,\infty)}, z(x))$$

defines a lift in the diagram

$$\begin{tikzcd} & M(Y) \arrow[d] \\ X \arrow[ur, bend left, "\widehat{H}"] \arrow[r, "H"] & \mathrm{map}_\circ([0,\infty), Y). \end{tikzcd}$$

Exercise 17.9. Show that the map $\widehat{H}$ is continuous.

The map $\widehat{H}$ is a 'Moore homotopy' from $H_1|_{X\times 0}$ to ... *what?* Define $X_{(n)} = z^{-1}([0,n]) \subseteq X$ and write $g_n = H|_{X_{(n)}\times n} : X_{(n)} \to Y$. These maps fit together to give a map

$$\begin{tikzcd}
\cdots \arrow[r] & X_{(n)} \arrow[r] \arrow[drr, bend right, "g_n"'] & X_{(n+1)} \arrow[r] \arrow[dr, bend left, "g_{n+1}"] & \cdots \\
& & & Y
\end{tikzcd}$$

from the telescope diagram to Y. If X happens to be the colimit of the telescope diagram, then this data defines a unique map $g : X \to Y$, which we hope to interpret as the end of the homotopy.

Problem 17.10. Using the notation and setup from this discussion, assume that X is the colimit of the diagram $\cdots \to X_{(n)} \to X_{(n+1)} \to \cdots$, and show that in the diagram

$$\begin{tikzcd}
& & Y^I \arrow[r] & Y^{[0,1)} \arrow[d, "\cong"] \\
X \arrow[urr, dotted, "H"] \arrow[r] & \mathcal{P}_M(Y) \arrow[rr] & & Y^{[0,\infty)}
\end{tikzcd}$$

the lift H exists and it is a (right) homotopy from f to g.

Exercise 17.11. Compare your work here with the approach to infinite composition of homotopies given in Section 11.1.2.

17.2. The Free Monoid on a Topological Space

The James construction builds the free topological monoid $J(X)$ generated by X. In this section we construct $J(X)$ and establish its fundamental properties.

17.2.1. The James Construction. Let $X \in \mathcal{T}_*$ and consider the n-fold products X^n for $n \geq 0$. We consider $X^n \subseteq X^{n+1}$ by identifying the point $(x_1, x_2, \ldots, x_n) \in X^n$ with the point $(x_1, x_2, \ldots, x_n, *) \in X^{n+1}$. The union of all of the X^n is X^∞, the set of all *finite* sequences of points of X. Formally, X^∞ is the colimit of the telescope diagram

$$X^0 \longrightarrow X^1 \longrightarrow X^2 \to \cdots \to X^n \longrightarrow X^{n+1} \to \cdots .$$

It is sometimes known as the **weak infinite product**.

Problem 17.12. Show that if X is well-pointed, then all the maps in this diagram are cofibrations, so the colimit X^∞ is also a homotopy colimit for the telescope diagram.

Now we define equivalence relations on X^n: two points $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are equivalent if and only if, after deleting all entries $*$, they are exactly the same list of elements in exactly the same order. For example, if $*, a, b \in X$, then

$$(*, a, *, *, b, *, *, a, *, b) \sim (*, *, *, a, *, b, a, *, b, *, *) \quad \text{in} \quad X^n.$$

We let $J^n(X)$ denote the set of equivalence classes of points in X^n; we of course have quotient maps $q_n : X^n \to J^n(X)$, which figure in the commutative ladder

$$\begin{tikzcd}
\cdots \arrow[r] & X^n \arrow[r] \arrow[d, "q_n"] & X^{n+1} \arrow[r] \arrow[d, "q_{n+1}"] & \cdots \\
\cdots \arrow[r] & J^n(X) \arrow[r] & J^{n+1}(X) \arrow[r] & \cdots .
\end{tikzcd}$$

The **James construction** on X, denoted $J(X)$, is the colimit of the bottom row; thus we have a natural map $q : X^\infty \to J(X)$. We'll write $[x_1, \ldots, x_n]$ for the equivalence class of $(x_1, \ldots, x_n)$ in any of the spaces $J^m(X)$ for $n \leq m \leq \infty$.

Problem 17.13.

(a) Show that q is a quotient map.

(b) Show that this construction is functorial and that it respects homotopy.

Hint. First show that $X \mapsto X^n$ respects homotopy.

17.2.2. The Algebraic Structure of the James Construction. Next we show that the James construction builds the free topological monoid generated by X.

Problem 17.14. Show that the map $\mu_n : X \times J^n(X) \to J^{n+1}(X)$ given by $(x, [x_1, \ldots, x_n]) \mapsto [x, x_1, \ldots, x_n]$ is a quotient map.

We use the maps μ_n to fit the spaces $J^n(X)$ into a pushout square.

Problem 17.15.

(a) Show that there is a categorical pushout square

$$\begin{tikzcd}
(X \times J^{n-1}(X)) \cup (* \times J^n(X)) \arrow[r, "i"] \arrow[d, "\mu_{n-1} \cup \mathrm{pr}_2"'] & X \times J^n(X) \arrow[d, "\mu_n"] \\
J^n(X) \arrow[r, hook] & J^{n+1}(X).
\end{tikzcd}$$

Hint. Let P be the pushout, and show that $P \to J^{n+1}(X)$ is a bijection.

(b) Prove that if X is well-pointed, then each inclusion $J^n(X) \to J^{n+1}(X)$ is a cofibration, and conclude that the squares in part (a) are homotopy pushout squares.

(c) Show that $J^{n+1}(X)/J^n(X) \cong X^{\wedge(n+1)}$.

(d) Show that if X is a CW complex, then $J(X)$ is a generalized CW complex.

Exercise 17.16. What must be true in order for $J(X)$ to be a CW complex?

Finally, we justify the assertion that $J(X)$ is the free topological monoid generated by the set X. The multiplication is the map

$$\mu : J(X) \times J(X) \longrightarrow J(X)$$

given by the rule $\mu([x_1, \ldots, x_n], [y_1, \ldots, y_m]) = [x_1, \ldots, x_n, y_1, \ldots, y_m]$.

Proposition 17.17. *Let $X \in \mathcal{T}_*$.*

(a) Show that μ makes $J(X)$ into a topological monoid.

(b) Show that a pointed map $f : X \to M$ from X to a topological monoid M extends to a unique monoid map $\phi : J(X) \to M$ and if f is continuous, then so is ϕ.

(c) If $f \simeq g : X \to M$ in $\mathcal{T}_*$, then the extensions $\phi, \gamma : J(X) \to M$ are homotopic through homomorphisms.

Problem 17.18. Prove Proposition 17.17.

Hint. Show that $((x_1, \ldots, x_n), (y_1, \ldots, y_m)) \mapsto (x_1, \ldots, x_n, y_1, \ldots, y_m)$ defines a continuous function $X^\infty \times X^\infty \to X^\infty$. For part (c), find a right homotopy that is a homomorphism.

Corollary 17.19. *If X is a topological monoid, then the inclusion $j : X \to J(X)$ has a left inverse τ.*

Problem 17.20. Prove Corollary 17.19.

The James Construction and Weak Equivalences. We will prove a powerful theorem identifying the homotopy type of the James construction $J(X)$, but it requires X to be a CW complex. To apply this result to other spaces, we need to know that the James construction carries weak equivalences to weak equivalences.

Problem 17.21. Show that if $f : X \to Y$ is an m-equivalence between well-pointed spaces, then $J^n(X) \to J^n(Y)$ is an m-equivalence for all $n \leq \infty$.

17.3. Identifying the Suspension Map

Suspension defines a natural transformation $\Sigma : [\,?\,,\,?\,] \to [\Sigma\,?\,,\Sigma\,?\,]$. We also have a natural isomorphism $\alpha : [\Sigma A, \Sigma X] \xrightarrow{\cong} [A, \Omega\Sigma X]$ (coming from the exponential law). Writing $S = \alpha \circ \Sigma$, we have a commutative diagram of functors and natural transformations

$$\begin{tikzcd}
{[A,X]} \arrow[r, "\Sigma"] \arrow[d, equal] & {[\Sigma A, \Sigma X]} \arrow[d, "\alpha", "\cong"'] \\
{[A,X]} \arrow[r, "S"] & {[A, \Omega\Sigma X].}
\end{tikzcd}$$

The diagram shows that the study of the natural transformation Σ is equivalent to the study of the transformation S.

Problem 17.22. Show that $S = \sigma_*$, where $\sigma(x) = \boxed{t \mapsto [x,t]}$.

The **suspension map** σ corresponds, via the inclusion $\Omega\Sigma X \hookrightarrow \Omega_M\Sigma X$, to the map

$$\sigma_M : X \longrightarrow \Omega_M X \qquad \text{given by} \qquad \sigma_M : x \mapsto \left(\boxed{t \mapsto [x,t]}, 1\right),$$

which we call the **Moore suspension** map. By Proposition 17.17(b), there is a unique continuous homomorphism $e_X : J(X) \to \Omega_M(X)$ of topological monoids making the diagram

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, equal] & J(X) \arrow[d, "e_X"] \\
X \arrow[r, "\sigma_M"] & \Omega_M\Sigma X
\end{tikzcd}$$

strictly commutative.

Problem 17.23. Show that the rule $X \mapsto e_X$ defines a natural transformation $J(\,?\,) \to \Omega_M(\,?\,)$.

Theorem 17.24 (James). *If $X \in \mathcal{T}_*$ is a connected CW complex, the homomorphism*

$$e_X : J(X) \longrightarrow \Omega_M\Sigma X$$

is a homotopy equivalence.

Corollary 17.25. *If X is a well-pointed space, then e_X is a weak equivalence.*

Problem 17.26.

(a) Show that if $f : X \to Y$ is a weak homotopy equivalence, then so is $\Omega_M\Sigma X \to \Omega_M\Sigma Y$.

(b) Use Theorem 17.24 to prove Corollary 17.25.

Now we fix a pointed connected CW complex X and write J and J^n for $J(X)$ and $J^n(X)$. The proof of Theorem 17.24 amounts to a detailed study of the strictly commutative cube

$$\begin{tikzcd}
X \times J \arrow[rr, "\mathrm{in}_0 \times \mathrm{id}_J"] \arrow[dr, "\mu"] \arrow[dd, "\mathrm{pr}_1"'] & & CX \times J \arrow[dr, "\nu"] \arrow[dd, "\mathrm{pr}_1"] & \\
& J \arrow[rr, crossing over, "\xi"] & & T \arrow[dd, "q"] \\
X \arrow[rr, "\mathrm{in}_0"] \arrow[dr] & & CX \arrow[dr] & \\
& * \arrow[rr] \arrow[from=uu, crossing over] & & \Sigma X,
\end{tikzcd}$$

in which the top and bottom squares are both categorical and homotopy pushouts and q is the induced map of pushouts.

Problem 17.27. Let X be a connected pointed CW complex.

(a) Show that the squares

$$\begin{tikzcd}
J \arrow[d] & X \times J \arrow[l, "\mu"'] \arrow[r, "\mathrm{in}_0 \times \mathrm{id}_J"] \arrow[d, "\mathrm{pr}_1"'] & CX \times J \arrow[d, "\mathrm{pr}_1"] \\
* & X \arrow[l] \arrow[r, "\mathrm{in}_0"] & CX
\end{tikzcd}$$

are strong homotopy pullback squares.

(b) Show that the squares

$$\begin{tikzcd}
J \arrow[r] \arrow[d] & T \arrow[d, "q"] & CX \times J \arrow[l, "\nu"'] \arrow[d, "\mathrm{pr}_1"] \\
* \arrow[r] & \Sigma X & CX \arrow[l]
\end{tikzcd}$$

are strong homotopy pullback squares.

Is the connected hypothesis necessary in Problem 17.27? It is very instructive to work out an explicit example.

Exercise 17.28. Study this cube diagram for $X = S^0$. What is J? What is T? Explicitly describe the map $q : T \to S^1$. Is the square of Problem 17.27(a) a homotopy pullback square?

Now we study the space T and the map q in greater detail. Since the map μ is surjective, so is ν, and therefore each point of T is the equivalence class $\langle [x,t], [x_1, \ldots, x_n] \rangle$ of a point $([x,t], [x_1, \ldots, x_n]) \in J$.

Problem 17.29.

(a) Show that each point in T, except $*$, has a unique expression in the form $\langle [x,t],[x_1,\ldots,x_n]\rangle$ with $t<1$.

(b) Show that q is given by $q(\langle [x,t],[x_1,\ldots,x_n]\rangle) = [x,t]$.

(c) Show that $\xi : J \to T$ is given by $\xi([x_1,\ldots,x_n]) = \langle *, [x_1,\ldots,x_n]\rangle$.

(d) Show that ν is a quotient map.

(e) Show that T is the quotient space $(CX \times J)/\sim$, where $\sim$ is the equivalence relation given by

$$([x,0],[x_1,\ldots,x_n]) \sim (*,[x,x_1,\ldots,x_n]).$$

We will show that T is contractible by constructing a filtration $\cdots \subseteq T_n \subseteq T_{n+1} \subseteq \cdots \subseteq T$ and showing that the identity map is homotopic to a map $d : T \to T$ with $d(T_n) \subseteq T_{n-1}$ for each $n > 0$ and $d(T_0) = *$. Then a simple adaptation of the method of Problem 11.4 will show that T is contractible.

The filtration is defined by setting $T_n = \nu(CX \times J_n) \subseteq T$. Define a homotopy $H : T \times I \to T$ by the rule

$$H(([x,t],[x_1,\ldots,x_n]),s) = ([x,(1-s)t+s],[x_1,\ldots,x_n]).$$

We write $d = H|_{T\times 1}$; this map is given explicitly by the formula

$$d : ([x_0,t],[x_1,\ldots,x_n]) \mapsto ([x_1,0],[x_2,\ldots,x_n]).$$

Problem 17.30.

(a) Show that H is a continuous pointed homotopy $\mathrm{id}_T \simeq d$.

(b) Show that $H(T_n \times I) \subseteq T_n$ and that $d(T_n) \subseteq T_{n-1}$.

(c) Show that T is the colimit of the diagram $\cdots \to T_n \to T_{n+1} \to \cdots$.

(d) Show that d^∞, the infinite composite of d with itself, is constant and homotopic to id_T.

Since T is contractible and the square

$$\begin{tikzcd} J \arrow[r] \arrow[d] & T \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

is a strong homotopy pullback, we deduce that $J \simeq \Omega\Sigma X$. But we want to know even more: the homotopy equivalence should be none other than the map e_X. To show this, we study the front square of our cube in still more detail. Since $T \simeq *$, the map $q : T \to \Sigma X$ is nullhomotopic, so it factors through the path fibration $\mathcal{P}_M\Sigma X \to \Sigma X$. Each choice of nullhomotopy $q \simeq *$ gives rise to a diagram of the form

$$\begin{tikzcd} J \arrow[r, "\xi"] \arrow[d, "f"'] & T \arrow[d, "u"] \\ \Omega_M\Sigma X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & \mathcal{P}_M(\Sigma X) \arrow[d, "@_0"] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

in which f is a homotopy equivalence. We will use the nullhomotopy that comes from the contraction of T we found in Problem 17.30.

Problem 17.31. Let $u : T \to \mathcal{P}_M(\Sigma X)$ be the lift of q corresponding to $H^{*\infty}$. In this problem, we'll use the notation $\omega_x : I \to \Sigma X$ for the loop $\omega_x(t) = [x, t]$.

(a) Show that $u(\langle *, [x_1, \ldots, x_n]\rangle) = \omega_{x_1} \star \cdots \star \omega_{x_n}$.

(b) Prove Theorem 17.24 by showing that the map $J \to \Omega_M\Sigma X$ is e_X.

Our analysis has shown that the suspension map σ is equivalent to both the Moore suspension σ_M and to the inclusion $j : X \hookrightarrow J(X)$. The situation is summarized in the strictly commutative diagram

$$\begin{tikzcd} & X \arrow[dl, bend right, "j"'] \arrow[d, "\sigma_M"] \arrow[dr, bend left, "\sigma"] & \\ J(X) \arrow[r, "e", "\simeq"'] & \Omega_M\Sigma X & \Omega\Sigma X \arrow[l, "\simeq"] \end{tikzcd}$$

in which both of the horizontal maps are H-maps.

17.4. The Freudenthal Suspension Theorem

Our motivation in this chapter is to develop an understanding of the suspension transformation. We have shown that it is enough to understand the map $j : X \to J(X)$. The explicit topological construction of $J(X)$ gives valuable information about the connectivity of j.

Problem 17.32. Let $Y \in \mathcal{T}_*$ be an $(n-1)$-connected well-pointed space.

(a) If Y is a CW complex, then $J(Y)$ inherits the structure of a generalized CW complex in which each $J^k(Y)$ is a subcomplex. Show that $J^k(Y)$ contains all cells in $J(Y)$ of dimension at most $n(k+1) - 1$.

(b) Show that the suspension map $\sigma : Y \hookrightarrow \Omega\Sigma Y$ is a $(2n-1)$-equivalence, whether Y is a CW complex or not.

The connectivity of σ determines the behavior of the suspension transformation for low-dimensional domains.

Theorem 17.33 (Freudenthal Suspension Theorem)**.** *Let $Y \in \mathcal{T}_*$ be an $(n-1)$-connected well-pointed space and let X be a pointed CW complex. Then*

(a) if $\dim(X) < 2n-1$, then $\Sigma : [X, Y] \to [\Sigma X, \Sigma Y]$ is an isomorphism, and

(b) if $\dim(X) = 2n-1$, then $\Sigma : [X, Y] \to [\Sigma X, \Sigma Y]$ is surjective.

The special case in which X is a sphere is particularly important.

Corollary 17.34. *Let Y be an $(n-1)$-connected space. Then*

(a) $\Sigma : \pi_k(Y) \to \pi_{k+1}(\Sigma Y)$ is an isomorphism for $k < 2n-1$, and

(b) $\Sigma : \pi_{2n-1}(Y) \to \pi_{2n}(\Sigma Y)$ is onto.

Problem 17.35. Prove Theorem 17.33 and Corollary 17.34.

The Connectivity of a Suspension. We have already shown, using cellular replacements, that $\operatorname{conn}(\Sigma X) \geq \operatorname{conn}(X) + 1$. Now we are prepared to show that this is an equality, at least for simply-connected spaces.

Problem 17.36. Show that if X is simply-connected, then $\operatorname{conn}(\Sigma X) = \operatorname{conn}(X) + 1$. Does your argument work for path-connected spaces X that are not simply-connected?

17.5. Homotopy Groups of Spheres and Wedges of Spheres

Using the Freudenthal Suspension Theorem, we can bootstrap up from the fundamental computation $\pi_1(S^1) \cong \mathbb{Z} \cdot \mathrm{id}$ and determine the groups $\pi_k(S^n)$ for $k \leq n$.

Theorem 17.37. *For all $n \geq 1$, the suspension $\Sigma : \pi_n(S^n) \to \pi_{n+1}(S^{n+1})$ is an isomorphism, and hence*

$$\pi_n(S^n) = \begin{cases} \mathbb{Z} \cdot [\mathrm{id}_{S^n}] & \text{if } k = n, \\ 0 & \text{if } k < n. \end{cases}$$

Problem 17.38.

(a) Show that $\Sigma : \pi_1(S^1) \to \pi_2(S^2)$ is an isomorphism.

(b) Prove Theorem 17.37.

HINT. $S^1 \subseteq \mathbb{C}$ is a topological group under multiplication.

Corollary 17.39. *For any $n > 1$ and any indexing set $\mathcal{J}$,*

$$\pi_n(\textstyle\bigvee_{\mathcal{J}} S^n) \cong \bigoplus_{\mathcal{J}} \mathbb{Z} \cdot [\mathrm{in}_j].$$

Problem 17.40.

(a) Show that any map $f : S^n \to \bigvee_{\mathcal{J}} S^n$ must factor through a finite subwedge.

(b) Prove Corollary 17.39.

Hint. For a finite collection $\{X_j\}$, $\pi_n(\prod X_j) \cong \prod \pi_n(X_j) \cong \bigoplus \pi_n(X_j)$.

We have finally found homotopy functors that can distinguish between spheres and disks, and this is good news because it shows that our intuition that $S^n \not\simeq *$ is accurate.

Exercise 17.41. Generalize and prove Problem 1.13.

Higher Homotopy Groups. Algebraic topology can be said to have begun when Poincaré introduced the 'homology groups' of a space in the 1890s. These are functors $\widetilde{H}_n : \mathcal{T}_* \to \mathrm{Ab}\,\mathcal{G}$, and they have the property that $\widetilde{H}_k(X^n) = 0$ for $k \neq n$, and $\widetilde{H}_n(S^n) \cong \mathbb{Z}$. Thus when the homotopy groups $\pi_*(X)$ were first introduced, and it was proved that $\pi_k(S^n) = 0$ for $k < n$ and $\pi_n(S^n) \cong \mathbb{Z} \cdot [\mathrm{id}_{S^n}]$, most topologists thought that it would ultimately turn out that π_n and $\widetilde{H}_n$ were really the same functor.

This conventional wisdom was shattered when Hopf showed that in fact the 'higher homotopy groups' of spheres are not all trivial!

Problem 17.42 (Hopf)**.** Show that $\pi_3(S^2) \cong \mathbb{Z}$, and exhibit a generator.

Hint. This is an instance where projective spaces offered deep insight into homotopy theory.

Now it is known that the homotopy groups of every simply-connected finite complex are nonzero in infinitely many dimensions; and there is no simply-connected finite complex all of whose homotopy groups are known. The structure of $\pi_*(S^n)$ is extremely complicated and messy, but there is structure. For example, the Whitehead product and composition of maps between spheres endow these groups with useful algebraic operations.

17.6. Eilenberg-Mac Lane Spaces

Let G be a group, and let $n \in \mathbb{N}$. A CW complex L is called an **Eilenberg-Mac Lane space of type** (G, n) if its homotopy groups are given by

$$\pi_k(L) \cong \begin{cases} G & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that we used $\cong$ instead of $=$ in this definition. This is an important and somewhat subtle point. The group G is a certain set with a multiplication rule. For example, it might be a set of permutations (which are bijective functions from a set to itself) or it might be a set of cosets, etc. The group $\pi_n(L)$ is a set of homotopy classes of maps from a sphere into L. Unless the stars align perfectly, it is essentially impossible for $\pi_n(L)$ to be *equal* to G. Instead, when we say that L is an Eilenberg-Mac Lane space of type (G, n), we mean that the only nonzero homotopy group of L is $\pi_n(L)$ and we have in mind a *particular choice* of a group isomorphism $\theta : \pi_n(L) \to G$. Deciding on a choice of isomorphism is sometimes referred to as giving L the **structure** of an Eilenberg-Mac Lane space; different isomorphisms constitute different structures. Most of the time, these technical considerations can be safely ignored, but it is occasionally crucial to keep complete control of the situation.

Exercise 17.43.

(a) Show that S^1 is an Eilenberg-Mac Lane space of type $(\mathbb{Z}, 1)$. How many different such structures does it have?

(b) Show that $\mathbb{R}\mathrm{P}^\infty = K(\mathbb{Z}/2, 1)$ and $\mathbb{C}\mathrm{P}^\infty = K(\mathbb{Z}, 2)$. Which Eilenberg-Mac Lane space is $\mathbb{H}\mathrm{P}^\infty$?

(c) Show that for any discrete group G, the classifying space BG is an Eilenberg-Mac Lane space of type $(G, 1)$. How many ways can BG be structured as an Eilenberg-Mac Lane space of type $(G, 1)$?

17.6.1. Maps into Eilenberg-Mac Lane Spaces. Since the homotopy groups of Eilenberg-Mac Lane spaces are so simple, maps into them are especially amenable to construction and analysis by obstruction theory. In fact, maps from an $(n-1)$-connected domain into an Eilenberg-Mac Lane space are entirely determined by the algebra of their homotopy groups.

Theorem 17.44. *Let G be a group, and let Y be an Eilenberg-Mac Lane space of type (G, n). Then for any $(n-1)$-connected CW complex X, the map*

$$\phi : [X, Y] \longrightarrow \mathrm{Hom}(\pi_n(X), \pi_n(Y)) \qquad \textit{given by} \qquad \phi(f) = f_*$$

is bijective.

Problem 17.45.

(a) Explain why it is enough to prove Theorem 17.44 in the special case that X has dimension at most $n + 1$.

(b) Show that an $(n-1)$-connected and $(n+1)$-dimensional CW complex X sits in a cofiber sequence $\bigvee_{i \in \mathcal{I}} S^n \xrightarrow{\alpha} \bigvee_{j \in \mathcal{J}} S^n \xrightarrow{i} X$.

(c) Show that the induced map $i_* : \pi_n(\bigvee_{\mathcal{J}} S^n) \to \pi_n(X)$ is surjective.

(d) Use Theorem 16.27 to show that ϕ is injective.

(e) Let $\pi_n(X) = H$ and let $h : H \to G$ be any group homomorphism. Show that there is a map $\beta : \bigvee_{j\in\mathcal{J}} S^n \to K(G,n)$ whose induced map $\pi_n(\bigvee_{\mathcal{J}} S^n) \to G$ is $h \circ i_*$.

(f) Show that ϕ is surjective.

Suppose X and Y are Eilenberg-Mac Lane spaces of type (G,n). This means that we have in mind particular isomorphisms

$$\phi : \pi_n(X) \xrightarrow{\cong} G \qquad \text{and} \qquad \theta : \pi_n(Y) \xrightarrow{\cong} G.$$

According to Theorem 17.44, the composite $\theta^{-1} \circ \phi : \pi_n(X) \xrightarrow{\cong} \pi_n(Y)$ is induced by a unique homotopy class $X \to Y$, which is manifestly a weak homotopy equivalence and hence a homotopy equivalence, since X and Y are CW complexes. Thus we can unambiguously write $K(G,n)$ to denote any Eilenberg-Mac Lane space of type (G,n).

We have mentioned Milnor's theorem (Theorem 4.83), which has as a special case that if X is a CW complex, then ΩX is homotopy equivalent to a CW complex. This implies the following.

Corollary 17.46. *If $n > 0$, then $\Omega K(G,n) \simeq K(G,n-1)$.*

Problem 17.47. Prove Corollary 17.46.

Problem 17.48. Show that the rule $G \mapsto K(G,n)$ is the object part of a functor $K(\,?\,,n) : \mathrm{AB}\,\mathcal{G} \to \mathrm{H}\mathcal{T}_*$.

17.6.2. Existence of Eilenberg-Mac Lane Spaces. We have now shown that the Eilenberg-Mac Lane spaces that exist are unique up to canonical equivalence in $\mathrm{H}\mathcal{T}_*$. We have also seen that there are Eilenberg-Mac Lane spaces of the form $K(G,1)$ for every group G. Since $\pi_n(X)$ is an abelian group for all $n > 1$, it remains to show that $K(G,n)$ exists for all abelian groups.

Theorem 17.49. *For every abelian group G and every $n \geq 1$, there is an Eilenberg-Mac Lane space $K(G,n)$ which is a topological monoid.*

Problem 17.50.

(a) Show that if there is an $(n-1)$-connected space X with $\pi_n(X) \cong G$, then there is an Eilenberg-Mac Lane space of type (G,n).

(b) Show that if F is a free abelian group, then $K(F,n)$ exists.

Problem 17.51. Let $0 \to F_1 \xrightarrow{d} F_0 \to G \to 0$ be a free resolution of an abelian group G (see Section A.1).

(a) Show that $K(F_1, n)$ and $K(F_0, n)$ exist and that there is a map $\delta : K(F_1, n) \to K(F_0, n)$ such that the square

$$\begin{tikzcd}
\pi_n(K(F_1,n)) \arrow[r, "\delta_*"] \arrow[d, "\cong"'] & \pi_n(K(F_0,n)) \arrow[d, "\cong"] \\
F_1 \arrow[r, "d"] & F_0
\end{tikzcd}$$

commutes, where the vertical isomorphisms are the structure maps for the Eilenberg-Mac Lane spaces.

(b) Determine the fiber of δ and prove Theorem 17.49.

Proposition 17.52. *If $0 \to A \to B \to C \to 0$ is an exact sequence of abelian groups, then the corresponding sequence*

$$K(A, n) \longrightarrow K(B, n) \longrightarrow K(C, n)$$

of Eilenberg-Mac Lane spaces is a fibration sequence.

Problem 17.53. Prove Proposition 17.52.

Exercise 17.54. Is it possible for $K(G, 1)$ to be an H-space if G is a nonabelian group?

Exercise 17.55. Criticize the following argument:

> *We know that for any space X, ΩX is an H-space that is H-equivalent to a topological monoid $\Omega_M X$; and we know that $\Omega^2 X$ is an abelian H-space. Thus $\Omega^2 X$ is H-equivalent to an abelian topological monoid.*

It can be shown that if G is an abelian group, then there is an Eilenberg-Mac Lane space $K(G, n)$ which is a strictly associative abelian topological monoid (see Problem 20.64).

Problem 17.56. Show that if G is an abelian group and $f : X \to K(G, n)$ is nontrivial, then $\Sigma f : \Sigma X \to \Sigma K(G, n)$ is also nontrivial.

17.7. Suspension in Dimension 1

We were able to show that the suspension $\Sigma : \pi_1(S^1) \to \pi_2(S^2)$ is an isomorphism, but it was not a direct consequence of the Freudenthal Suspension Theorem. The Freudenthal Suspension Theorem only guarantees that the suspension $\Sigma : \pi_1(X) \to \pi_2(X)$ is *surjective* (if X is path-connected). To complete our understanding of the suspension map, we need to determine its kernel.

Exercise 17.57.

(a) Is $\Sigma : \pi_1(X) \to \pi_2(X)$ surjective when X is not path-connected?

(b) Show that if $\pi_1(X)$ is nonabelian, then Σ has a kernel.

The **commutator** of two elements $x, y \in G$ is the element $[x, y] = x^{-1}y^{-1}xy \in G$. The **commutator subgroup** of a group G is the smallest subgroup of G that contains all the commutators in G; this is frequently denoted G', but we'll use the more suggestive (and also common) notation $[G, G]$. A surjective homomorphism $\phi : G \to H$ is called an **abelianization** of G if $\ker(\phi) = G'$. The quotient map $G \to G/G'$ may be called the 'standard abelianization' of G. A *nontrivial* group G is called **perfect** if $G' = G$; perfect groups are precisely those that have trivial abelianizations.

We'll frequently need to impose conditions on fundamental groups such as: $\pi_1(X)$ is not a nontrivial perfect group; or $\pi_1(X)$ has no nontrivial perfect subgroups. A group with the latter property is known as a **hypoabelian group**. If $N \lhd G$, then G is hypoabelian if and only if both N and G/N are hypoabelian. We'll generally use the phrase 'has no nontrivial perfect subgroups' instead of the more obscure term 'hypoabelian' in our statements.

Proposition 17.58. *If X is path-connected, then the suspension map*

$$\Sigma : \pi_1(X) \longrightarrow \pi_2(\Sigma X)$$

is an abelianization.

Problem 17.59. Prove Proposition 17.58.

HINT. Let A be an abelianization of $\pi_1(X)$ and consider maps $X \to K(A, 1)$.

We deduce that, even for spaces that are not simply-connected, suspension usually, *but not always*, increases connectivity by one.

Theorem 17.60. *The following are equivalent:*

(1) $\pi_1(X)$ *is not a nontrivial perfect group,*

(2) $\mathrm{conn}(\Sigma X) = \mathrm{conn}(X) + 1$.

Problem 17.61. Prove Theorem 17.60.

It is often useful to know that Theorem 17.60 applies to the homotopy fiber F_f of a map f.

Problem 17.62. Let $f : X \to Y$, where $\pi_1(X)$ has no nontrivial perfect subgroups. Show that $\pi_1(F_f)$ has no nontrivial perfect subgroups and, in particular, it is not a nontrivial perfect group.

Problem 17.63. Suppose $\pi_1(X)$ has no nontrivial perfect quotients, and suppose $f : X \to Y$ with $\pi_1(f)$ surjective and nontrivial. Show that $\Sigma f \not\simeq *$.

17.8. Additional Topics and Problems

17.8.1. Stable Phenomena. We have seen several situations in which suspension makes a significant change in a topological problem. For example we have seen that $\Sigma(X \times Y) \simeq \Sigma(X \vee Y \vee (X \wedge Y))$, though we strongly suspect that this is not true without the suspension; and later (in Problem 19.38) we'll build a nontrivial space X such that $\Sigma X \simeq *$.

The Freudenthal Suspension Theorem implies that if we are studying CW complexes that are very highly connected in comparison to their dimension, many properties are *unchanged* by suspension. Such properties are called **stable phenomena**, and maps and spaces that satisfy the required dimension and connectivity conditions are said to be **in the stable range**.

Problem 17.64.

(a) Show that for any space X the suspension maps

$$\Sigma : \pi_{n+t}(\Sigma^t X) \longrightarrow \pi_{n+t+1}(\Sigma^{t+1} X)$$

are isomorphisms for all sufficiently large t.

(b) More generally, show that if X is a finite-dimensional CW complex, then the suspension maps

$$\Sigma : [\Sigma^t X, \Sigma^t Y] \longrightarrow [\Sigma^{t+1} X, \Sigma^{t+1} Y]$$

are bijections for sufficiently large t.

How large is 'sufficiently large'?

The group $\pi_{n+t}(\Sigma^t X)$ (for t sufficiently large) is called the n^{th} **stable homotopy group** of X and is denoted $\pi_n^S(X)$. Analogously, the stable group (it *is* a group) of homotopy classes $[X, Y]^S$ is the limiting value of $[\Sigma^t X, \Sigma^t Y]$.

17.8.2. The James Splitting. Using the James construction, we can give an explicit formula for $\Sigma\Omega\Sigma X$ in terms of the space X and using only basic operations. This formula is called the **James splitting**.

Problem 17.65.

(a) Show that the suspension of the inclusion $J^{n-1}(X) \hookrightarrow J^n(X)$ can be identified as in the diagram

$$\begin{tikzcd}
\Sigma J^{n-1}(X) \arrow[r] \arrow[d, equal] & \Sigma J^n(X) \arrow[d, "\simeq"] \\
\Sigma J^{n-1}(X) \arrow[r, hook, "\mathrm{in}_1"] & \Sigma J^{n-1}(X) \vee \Sigma X^{\wedge n}.
\end{tikzcd}$$

(b) Show that $\Sigma\Omega\Sigma X \simeq \Sigma\left(\bigvee_{n\geq 1} X^{\wedge n}\right)$.

17.8.3. The Hilton-Milnor Theorem. The wedge of two (or more) spaces is fundamentally a domain-type object, and so computing the homotopy groups of a wedge is a fundamentally difficult and interesting problem.

According to Problem 15.86, there is a decomposition

$$\Omega(X \vee Y) \simeq \Omega X \times \Omega(X \rtimes \Omega Y)$$

for any well-pointed X and Y. This implies the Hilton-Milnor theorem, which concerns the loop space on a wedge of two suspensions. This topological approach to the Hilton-Milnor theorem is due to Brayton Gray [**73**].

Theorem 17.66 (Hilton-Milnor-Gray). *If A and B are well-pointed spaces, then*

$$\Omega(\Sigma A \vee \Sigma B) \simeq \Omega(\Sigma A) \times \Omega\Sigma\left(B \vee \left(\bigvee_{n=1}^{\infty} B \wedge A^{\wedge n}\right)\right).$$

Problem 17.67. Prove Theorem 17.66.

HINT. Use Problem 5.152 and the James splitting.

It is instructive to apply Theorem 17.66 to a wedge of spheres.

Problem 17.68. Let $m \leq n$

(a) Apply Theorem 17.66 to $S^m \vee S^n$.
(b) Repeat, always collapsing to a sphere of smallest dimension. What happens to the connectivity of the confusing loop space piece as you repeat the process?
(c) Argue that $\Omega(S^m \vee S^n)$ is homotopy equivalent to a big product of loop spaces of spheres of various dimensions.

Determining the list of spheres that show up in the splitting of $\Omega(S^m \vee S^n)$ is a complicated combinatorial problem, which has been worked out semi-explicitly by Peter Hilton. They may be put in bijective correspondence with a basis for a free graded Lie algebra.

Project 17.69. Work out the spheres that appear in the Hilton-Milnor splitting of $\Omega(S^n \vee S^m)$.

17.8.4. Problems.

Problem 17.70. Show that $\Sigma : \pi_*(S^3) \to \pi_*(S^4)$ is injective.

Problem 17.71. Let $f : A \to B$ be a homomorphism of abelian groups. Show that the homotopy fiber of the corresponding map $K(A, n) \to K(B, n)$ is $K(\ker(f), n) \times K(\operatorname{coker}(f), n-1)$.

Problem 17.72. Write out the cellular structure of $J(S^n) \simeq \Omega S^{n+1}$. Is it possible that $\Omega S^3 \cong \mathbb{C}\mathrm{P}^\infty$?

Exercise 17.73. Work out $J(X)$ and $\Omega\Sigma X$ for $X \in \mathcal{T}_*$ discrete and for $X = Y_+$ where $Y \in \mathcal{T}_\circ$. (The identification $J(X) \sim \Omega\Sigma X$ fails for these spaces.)

If $K(G,n)$ exists, then $K(G,n-1) \simeq \Omega K(G,n)$ is an associative H-space. But what about $K(G,n)$? Must it be an H-space? Here's a direct proof.

Problem 17.74. Let G be a group with multiplication $\mu : G \times G \to G$, and suppose $K(G,n)$ exists.

(a) Show that G is abelian if and only if μ is a homomorphism.

(b) Show that if G is abelian, then the homomorphism μ induces a map $m : K(G,n) \times K(G,n) \to K(G,n)$ which is an H-space multiplication.

(c) Show that in the situation of part (b), the multiplication m is homotopy associative.

(d) Show that if G is not abelian, then $K(G,1)$ is *not* an H-space.

Problem 17.75.

(a) Show that if X is an n-fold loop space, the map $X \to \Omega^n\Sigma^n X$ has a left homotopy inverse. Compare with Problem 17.38.

Hint. This is an instance of an abstract result: if $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ are adjoint, then $RX \to RLRX$ has a left inverse.

(b) Show that $\Sigma^n : [X, K(G,m)] \to [\Sigma^n X, \Sigma^n K(G,m)]$ is injective.

Problem 17.76. Show that if X is $(n-1)$-connected and $(2n-1)$-dimensional, then there is a space Y such that $X \simeq \Sigma Y$.

Problem 17.77. Establish a bijection $\phi : [\Omega S^2, \Omega S^2] \xrightarrow{\cong} \prod_{n=1}^\infty \pi_n(S^2)$. The domain and target are groups; is ϕ a homomorphism?

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Chapter 18

Comparing Pushouts and Pullbacks

In this chapter, we resolve the two basic questions: how close is a homotopy pullback square to being a homotopy pushout square and how close is a homotopy pushout square to being a homotopy pullback square?

To make these questions precise, we begin by observing that in any strictly commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

there are canonical comparison maps $\xi : Q \to D$ from the homotopy pushout Q of $C \leftarrow A \to B$ and $\zeta : A \to P$ to the homotopy pullback P of $C \to D \leftarrow B$. The square is a strong homotopy pushout square if and only if ξ is a homotopy equivalence, and it is a strong homotopy pullback if and only if ζ is a homotopy equivalence. We quantify the extent to which the square is 'close to being a homotopy pushout' by measuring the connectivity of the comparison map ξ; dually, the connectivity of ζ measures the extent to which the square is 'close to being a homotopy pullback'.

18.1. Pullbacks and Pushouts

In this section, we start with a strong homotopy pullback square and determine the homotopy type of the fiber of the comparison map $\xi : Q \to D$. Once we know this fiber, we can estimate its connectivity and thereby quantify the extent to which the square is 'like a homotopy pushout'.

18.1.1. The Fiber of $\xi : Q \to D$. Suppose our given square is a strong homotopy pullback square, and form the comparison map $\xi : Q \to D$ using the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & \overline{B} \arrow[r, "\simeq"] \arrow[d] & B \arrow[dd] \\
\overline{C} \arrow[r] \arrow[d, "\simeq"'] & Q \arrow[dr, dotted, "\xi"] & \\
C \arrow[rr] & & D.
\end{tikzcd}$$

We will explicitly determine the homotopy type of the fiber of $\xi : Q \to D$ in terms of the fibers of the maps in the square.

Theorem 18.1. *If the strictly commutative square*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d, "\beta"] \\
C \arrow[r, "\gamma"] & D
\end{tikzcd}$$

is a strong homotopy pullback square, then the comparison map ξ fits into a fiber sequence $F_\beta * F_\gamma \to Q \xrightarrow{\xi} D$.

Since we can easily estimate the connectivity of a join, we derive an estimate of the connectivity of the comparison map.

Corollary 18.2. *If $B \to D$ is a $(b-1)$-equivalence and $C \to D$ is a $(c-1)$-equivalence, then $\xi : Q \to D$ is a $(b+c-1)$-equivalence.*

To obtain the fiber of ξ, introduce the map $\mathcal{P}(D) \to D$ (which you should imagine perpendicular to the page), and form pullbacks to obtain a cubical diagram whose base is the strong homotopy pushout square in the upper left corner of the diagram above.

Problem 18.3.

(a) Determine the spaces and maps in the top face of the cube.

Hint. Use Problem 7.44.

(b) Prove Theorem 18.1.

(c) Derive Corollary 18.2.

Exercise 18.4. Explain the sense, if any, in which the result of Theorem 18.1 is functorial.

18.1.2. Ganea's Fiber-Cofiber Construction. We return, as promised, to the Ganea construction that we introduced in Section 9.7.3. The construction starts with a fibration sequence $F \to E \xrightarrow{p} B$ with a well-pointed base and constructs the diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & * \arrow[d] \arrow[ddr, bend left, "*"] & \\
E \arrow[r] \arrow[drr, bend right, "p"'] & E/F \arrow[dr, dotted, "g"] & \\
 & & B
\end{tikzcd}$$

in which the square is a strong homotopy pushout. The map $g : E/F \to B$ constructed in this way is known as Ganea's **fiber-cofiber construction** on $p : E \to B$. It is frequently useful to convert g to a fibration $G(p) : G(E) \to B$ whose fiber is denoted $G(F)$. The procedure that replaces $p : E \to B$ with $G(p) : G(E) \to B$ is called the **Ganea construction**.

We are of course interested in determining the homotopy fiber of g. For this, we pull back the path fibration $\mathcal{P}(B) \to B$ over the square, resulting in the Mather cube

$$\begin{tikzcd}
F \times \Omega B \arrow[rr, "\mathrm{pr}_2"] \arrow[dr, "a"] \arrow[dd, "\mathrm{pr}_1"'] & & \Omega B \arrow[dr] \arrow[dd, "\mathrm{pr}_1"] & \\
 & \widetilde{F} \arrow[rr] \arrow[dd] & & F_g \arrow[dd, "j"] \\
F \arrow[rr, "\mathrm{in}_0"] \arrow[dr, "i"] & & * \arrow[dr] & \\
 & E \arrow[rr] & & G(E),
\end{tikzcd}$$

where the map a is the one we studied in Section 8.3.3. The Second Cube Theorem guarantees that the fiber F_g is the homotopy pushout of the upper square. To determine that homotopy type, we make a well-chosen cofibrant replacement.

Problem 18.5. Let $\psi : F \times \Omega B \to F \times \Omega B$ be the map defined by the rule $\psi(x, \omega) = (\omega^{-1} \cdot x, \omega)$, where the dot represents the action of ΩX on F.

(a) Show that the diagram

$$\begin{tikzcd}
F \arrow[d, "e"'] & F \times \Omega B \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\psi"'] & \Omega B \arrow[d, "\mathrm{in}_2"] \\
\widetilde{F} \arrow[d] & F \times \Omega B \arrow[l, "a"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \Omega B \arrow[d, "\mathrm{pr}_1"] \\
E & F \arrow[l, "i"'] \arrow[r, "\mathrm{in}_0"] & *
\end{tikzcd}$$

commutes up to homotopy.

(b) Show that the homotopy fiber of $g : G(E) \to B$ is $F * \Omega B$.

Since the Ganea construction converts a fibration to another fibration, it can be iterated. We use the notation $G_n(p) : G_n(E) \to B$ for the n^{th} iterate, so that

$$G_1(p) = G(p) \qquad \text{and} \qquad G_{n+1}(p) = G(G_n(p)).$$

In the special case $p = @_0 : \mathcal{P}(X) \to X$, the path fibration over X, it is customary to write $G_n(X)$ instead of $G_n(\mathcal{P}(X))$ and p_n instead of $G_n(@_0)$.

Problem 18.6. Let $p : E \to B$ be a fibration with fiber F.

(a) Show that the iterated Ganea construction gives a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & G_n(E) \arrow[rr] \arrow[dr, "G_n(p)"'] & & G_{n+1}(E) \arrow[r] \arrow[dl, "G_{n+1}(p)"] & \cdots \\
& & B & &
\end{tikzcd}$$

in which the horizontal maps are cofibrations.

(b) Show that induced map $G_\infty(E) \to B$ from the (homotopy) colimit of the telescope diagram on the top row is a weak homotopy equivalence.

Exercise 18.7. What do you need to know about $p : E \to B$ to be able to conclude that $G_\infty(E) \to B$ is a genuine homotopy equivalence?

18.2. Comparing the Fiber of f to Its Cofiber

We can use Ganea's construction to compare the cofiber of a map f with the suspension of its fiber. Under fairly mild conditions on fundamental groups, we have

$$\mathrm{conn}(C_f) = \mathrm{conn}(F) + 1 = \mathrm{conn}(\Sigma F_f),$$

so that the connectivity of the cofiber determines that of f.

Problem 18.8. Convert $f : X \to Y$ to the fibration $p : E_f \to Y$; this comes with a standard deformation retraction $E_f \to X$. Let $g : E_f \cup CF \to Y$ be the result of applying Ganea's fiber-cofiber construction on p.

(a) Show that the square

$$\begin{tikzcd} E_f \arrow[r] \arrow[d, "\simeq"'] & E_f \cup CF \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is homotopy commutative.

(b) Show that there is a map $q \simeq g$ making the diagram

$$\begin{tikzcd} E_f \arrow[r] \arrow[d, "\simeq"'] & E_f \cup CF \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

strictly commutative.

(c) Use the diagram

$$\begin{tikzcd} CE_f \arrow[d] & E_f \arrow[l] \arrow[r] \arrow[d] & E_f \cup CF \arrow[d, "q"] \\ CX & X \arrow[l] \arrow[r] & Y \end{tikzcd}$$

to construct a comparison map $\phi : \Sigma F \to C$.

(d) Show that if Y is $(n-1)$-connected and f is an $(m-1)$-equivalence, then ϕ is an $(n+m-1)$-equivalence.

Theorem 18.9. *Let $f : X \to Y$, where $\pi_1(X)$ has no perfect subgroups and Y is simply-connected. Then*

$$\mathrm{conn}(C_f) = \mathrm{conn}(\Sigma F_f) = \mathrm{conn}(F_f) + 1.$$

Problem 18.10. Prove Theorem 18.9.

We now have criteria guaranteeing that $\mathrm{conn}(C_f) = \mathrm{conn}(F_f) + 1$ and that $\mathrm{conn}(\Sigma X) = \mathrm{conn}(X)+1$. Putting these together, we obtain conditions that ensure $\mathrm{conn}(\Sigma f) = \mathrm{conn}(f) + 1$.

Corollary 18.11. *Let $f : X \to Y$ where $\pi_1(X)$ has no perfect subgroups and Y is simply-connected. For $n \geq 1$, the following are equivalent:*

(1) f is an n-equivalence,

(2) Σf is an $(n+1)$-equivalence.

Problem 18.12. Prove Corollary 18.11.

18.3. The Blakers-Massey Theorem

Now we quantify the extent to which a strong homotopy pushout square is 'like' a homotopy pullback square. Start with a strong homotopy pushout square and convert the maps $B \to D$ and $C \to D$ to fibrations to build the diagram

$$\begin{tikzcd} A \arrow[rr] \arrow[dd] \arrow[dr, dotted, "\zeta"] & & B \arrow[d, "\simeq"] \\ & Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & \overline{B} \arrow[d] \\ C \arrow[r, "\simeq"] & \overline{C} \arrow[r] & D. \end{tikzcd}$$

This defines the comparison map $\zeta : A \to Q$ from A to the homotopy pullback Q of $C \to D \leftarrow B$; we will estimate the connectivity of ζ.

Theorem 18.13 (Blakers-Massey)**.** *Suppose that, in the strong homotopy pushout square*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] \arrow[dr, phantom, "\text{HPO}"] & B \arrow[d, "\beta"] \\ C \arrow[r, "\gamma"] & D, \end{tikzcd}$$

i is a $(b-1)$-equivalence and j is a $(c-1)$-equivalence, with $b, c \geq 3$. Then the comparison map $\zeta : A \to Q$ to the homotopy pullback is a $(b+c-1)$-equivalence.

Exercise 18.14. Show by example that the estimate $\operatorname{conn}(\zeta) \geq b+c-1$ cannot be improved in general. Can you find an example in which the connectivity of ζ is *strictly* larger than $b+c-1$?

We follow a pretty proof due to M. Mather [**115**]. Using the Second Cube Theorem, we reduce the problem to the special case in which $D \simeq *$ and all spaces are simply-connected. Pulling back from the path-loop fibration

$\mathcal{P}(D) \to D$ produces the diagram

$$
\begin{tikzcd}
Z \arrow[rrrr] \arrow[drr, dotted, "\theta"] \arrow[ddr] \arrow[ddd] & & & & F_\beta \arrow[dr] \arrow[ddd] & & \\
& & P \arrow[rrr] \arrow[dr] \arrow[ddd] & & & \overline{F}_\beta \arrow[dr] \arrow[ddd] & \\
& F_\gamma \arrow[rr] \arrow[ddd] & & \overline{F}_\gamma \arrow[rrr] \arrow[ddd] & & & \mathcal{P}(D) \arrow[ddd] \\
A \arrow[rrrr] \arrow[drr, dotted, "\zeta"] \arrow[ddr] & & & & B \arrow[dr] & & \\
& & Q \arrow[rrr] \arrow[dr] & & & \overline{B} \arrow[dr] & \\
& C \arrow[rr] & & \overline{C} \arrow[rrr] & & & D.
\end{tikzcd}
$$

Your job is to show that it suffices to prove Theorem 18.13 for the top of this diagram.

Problem 18.15.

(a) Show that $P \simeq \overline{F}_\beta \times \overline{F}_\gamma$.

(b) Show that the square

$$
\begin{tikzcd}
Z \arrow[r, "\theta"] \arrow[d] & P \arrow[d] \\
A \arrow[r, "\zeta"] & Q
\end{tikzcd}
$$

is a strong homotopy pullback square.

(c) Show that the square

$$
\begin{tikzcd}
Z \arrow[r] \arrow[d] & F_\beta \arrow[d] \\
F_\gamma \arrow[r] & \mathcal{P}(D)
\end{tikzcd}
$$

is a strong homotopy pushout square.

(d) Show that
- $\mathrm{conn}(\theta) = \mathrm{conn}(\zeta)$,
- $\mathrm{conn}(Z \to F_\beta) = \mathrm{conn}(A \to B)$, and
- $\mathrm{conn}(Z \to F_\gamma) = \mathrm{conn}(A \to C)$.

Conclude that it suffices to prove Theorem 18.13 for diagrams of the form

$$\begin{tikzcd}
A \arrow[rr, "\alpha"] \arrow[dr, dotted, "\zeta"] \arrow[dd, "\beta"'] & & B \arrow[d, equal] \\
& B \times C \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & B \arrow[d] \\
C \arrow[r, equal] & C \arrow[r] & *
\end{tikzcd}$$

in which the outer square is a strong homotopy pushout and B and C are simply-connected.

Let's get a feel for homotopy pushout squares in which the homotopy pushout is contractible.

Exercise 18.16.

(a) Show that the homotopy pushout of $C \xleftarrow{q_2} B \vee C \xrightarrow{q_1} B$, where the maps are the canonical quotient maps, is contractible.

(b) Show that if

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & *
\end{tikzcd}$$

is a homotopy pushout square, then its suspension may be identified with

$$\begin{tikzcd}
\Sigma B \vee \Sigma C \arrow[r, "q_1"] \arrow[d, "q_2"'] & \Sigma B \arrow[d] \\
\Sigma C \arrow[r] & *.
\end{tikzcd}$$

(c) Show that any such square is a *strong* homotopy pushout square.

The examples given in Exercise 18.16(a) by no means exhaust all possible prepushout diagrams with contractible pushouts (for another kind of example, see Problem 19.38). But they give an important clue about the nature of such squares: they suggest that the comparison map ζ is related to the inclusion $B \vee C \to B \times C$; and this turns out to be the case.

Problem 18.17.

(a) Using the diagram

$$\begin{tikzcd}
C \arrow[d, equal] & A \arrow[l] \arrow[r] \arrow[d, "\zeta"] & B \arrow[d, equal] \\
C & B \times C \arrow[l, "\mathrm{pr}_2"'] \arrow[r, "\mathrm{pr}_1"] & B,
\end{tikzcd}$$

determine the homotopy type of ΣC_ζ.

(b) Suppose B is $(b-1)$-connected and C is $(c-1)$-connected, with $b, c \geq 2$. What is the connectivity of C_ζ?

(c) Deduce Theorem 18.13.

Comparing a Cofiber Sequence to a Fiber Sequence. Let $A \hookrightarrow B$ be a cofibration, and consider the strong homotopy pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "HPO"] & B \arrow[d, "f"] \\ * \arrow[r] & C. \end{tikzcd}$$

In this case, the homotopy pullback of the diagram $* \to C \leftarrow B$ is the homotopy fiber F of f. Our work above reveals the connectivity of the comparison map $\zeta : A \to F$, but we can do more in this case: we'll identify the map $\Sigma\zeta$ explicitly.

Theorem 18.18. *With the setup above, the map $\Sigma\zeta$ can be naturally identified as in the strictly commutative diagram*

$$\begin{tikzcd} \Sigma A \arrow[r, "\mathrm{in}_1"] \arrow[d, equal] & \Sigma A \rtimes \Omega C \arrow[d, "\simeq"] \\ \Sigma A \arrow[r, "\Sigma\zeta"] & \Sigma F. \end{tikzcd}$$

Problem 18.19. Build a Mather cube over the square by pulling back from the path fibration $\mathcal{P}(D) \to D$.

(a) Show that there is a strong homotopy pushout square of the form

$$\begin{tikzcd} A \times \Omega C \arrow[r, "\mu"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "HPO"] & F \arrow[d] \\ \Omega C \arrow[r] & *. \end{tikzcd}$$

(b) Show that $\Sigma F \simeq \Sigma A \rtimes \Omega C$.

(c) Prove Theorem 18.18.

Pairs and Triads. We have systematically avoided using pairs of spaces whenever possible; but it is important to understand how the theorems we have proved are formulated in terms of pairs and triads.

In many, if not most, expositions of algebraic topology and homotopy theory, one defines the homotopy groups of a pair (X, A) as the set

$$\pi_n(X, A) = [(D^n, S^{n-1}), (X, A)]$$

of pointed homotopy classes of maps of pairs and proves the exactness of the sequence

$$\cdots \to \pi_n(A) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, A) \longrightarrow \pi_{n-1}(A) \to \cdots .$$

If you uncoil the definitions, you'll find that the exponential law leads to an isomorphism

$$\pi_n(X, A) \cong \pi_{n-1}(\mathrm{Fib}(A \hookrightarrow X)).$$

Thus the exact sequence of a pair is just oddball notation for something well understood: the long exact sequence of a fiber sequence. In other contexts, 'relative' functors of pairs (X, A) may be understood to be applying the corresponding 'absolute' functor applied to the cofiber X/A; the two interpretations are coordinated via the comparison map $\Sigma F \to C$ from the suspension of the fiber to the cofiber.

The Blakers-Massey theorem was originally expressed [**25**, **26**] in terms of a related notion: a **triad** is a triple $(X; A, B)$, where $X = A \cup B$. A triad gives rise to a commutative square

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

and hence to a map of pairs $(A, A \cap B) \to (X, B)$. If the triad is 'excisive' (which means, essentially, that the square above is a strong homotopy pushout square) the induced map on homotopy groups (of pairs) fits into a long exact sequence of homotopy groups in which every third entry is a 'homotopy group of a triad' $\pi_n(X; A, B)$. Unwinding the definitions will reveal that these groups are isomorphic to (dimension-shifted) homotopy groups of the iterated fiber of the square.

Exercise 18.20. Carry out the suggested unravelings.

18.4. The Delooping of Maps

The duality-minded reader will have been unsatisfied with Chapter 17, since we made no mention there of the dual question concerning the behavior of the looping operation $\Omega : [X, Y] \to [\Omega X, \Omega Y]$. In this section we prove a result that is roughly dual to the Freudenthal Suspension Theorem, showing that in a certain range of 'coconnectivity', the looping operation is bijective. We get more detailed information by applying Theorem 18.1 to the homotopy pullback square established in Section 9.6.

To prove the results of this section, we need to know that the loop space of a CW complex X is homotopy equivalent to a CW complex. This is a consequence of Milnor's theorem [**132**, Thm. 3], stated in this text as Theorem 4.83.

18.4.1. The Connectivity of Looping. We saw in Section 9.6 that the diagram

$$\begin{tikzcd}
[X, Y] \arrow[r, "\Omega"] \arrow[d, equal] & [\Omega X, \Omega Y] \arrow[d, "\cong"] \\
[X, Y] \arrow[r, "\lambda^*"] & [\Sigma\Omega X, Y]
\end{tikzcd}$$

commutes, where $\lambda : \Sigma\Omega X \to X$ is adjoint to $\mathrm{id}_{\Omega X}$. It follows that the study of the operation Ω is equivalent to the study of the map λ. Since we are interested in when Ω is bijective (or surjective), we will determine the connectivity of the map λ.

Problem 18.21.

(a) Show that there is a strictly commutative triangle

$$\begin{tikzcd}
\Sigma\Omega X \arrow[rr, "\simeq"] \arrow[dr, "\lambda"'] & & G_1(X) \arrow[dl] \\
& X, &
\end{tikzcd}$$

where $G_1(X) \to X$ is the result of applying the Ganea construction to the path-loop fibration $\mathcal{P}X \to X$.

HINT. You can embed $C\Omega X$ into $\mathcal{P}X$ using the map $(\omega, t) \mapsto \omega_t$, where

$$\omega_t(s) = \begin{cases} \omega(t) & \text{if } s \leq t \\ \omega(s) & \text{if } s \geq t. \end{cases}$$

(b) Estimate the connectivity of λ.

You have proved our first theorem: the 'dual' to the Freudenthal Suspension Theorem.

Theorem 18.22. *Suppose X is an $(n-1)$-connected CW complex and $\pi_k(Y) = 0$ for $k \geq 2n-1$. Then the looping map*

$$\Omega : [X, Y] \longrightarrow [\Omega X, \Omega Y]$$

is bijective. If $\pi_k(Y) = 0$ for $k \geq 2n$, then Ω is injective.

According to Problem 18.5, the homotopy fiber of λ is the join $\Omega X * \Omega X$. Later on, we'll want to know the fiber inclusion $j : \Omega X * \Omega X \to \Sigma\Omega X$.

Problem 18.23.

(a) Show that j is an induced map of homotopy pushouts as in the diagram

$$\begin{tikzcd}
\Omega X \times \Omega X \arrow[rr, "\mathrm{pr}_2"] \arrow[dr, "\mathrm{pr}_1"] \arrow[dd, "m"'] & & \Omega X \arrow[dr] \arrow[dd] & \\
& \Omega X \arrow[rr] \arrow[dd] & & \Omega X * \Omega X \arrow[dd, dotted, "j"] \\
\Omega X \arrow[rr] \arrow[dr] & & * \arrow[dr] & \\
& * \arrow[rr] & & \Sigma\Omega X,
\end{tikzcd}$$

where $m(\omega, \tau) = \tau^{-1}\omega$.

(b) Show that there is a commutative square

$$\begin{tikzcd}
\Omega X * \Omega X \arrow[r, "q"] \arrow[d, "j"'] & \Sigma(\Omega X \times \Omega X) \arrow[d, "\Sigma m"] \\
\Sigma\Omega X \arrow[r, "\mathrm{id}"] & \Sigma\Omega X
\end{tikzcd}$$

in which the map q comes from the Mayer-Vietoris sequence you studied in Problem 9.2.

18.4.2. The Kernel and Cokernel of Looping. G. W. Whitehead [**182**] realized that the homotopy pushout square of Section 9.6 makes it possible (for certain targets) to insert the loop operation Ω into an exact sequence involving the reduced diagonal. Since the square is a homotopy pullback, it is approximately a homotopy pushout, and so we can estimate the connectivity of the induced map z between the cofibers in the diagram

$$\begin{tikzcd}
\Sigma\Omega X \arrow[r, "\lambda"] \arrow[d] & X \arrow[r] \arrow[d, "\Delta"] & C_\lambda \arrow[d, "z"] \\
X \vee X \arrow[r] & X \times X \arrow[r] & X \wedge X.
\end{tikzcd}$$

Proposition 18.24. *If X is $(n-1)$-connected and $\pi_1(X)$ has no nontrivial perfect subgroups, then the map $z : C_\lambda \to X \wedge X$ is a $(3n-1)$-equivalence.*

Corollary 18.25. *If $\pi_k(Y) = 0$ for $k \geq 3n-2$, then the sequence*

$$[\Omega X, \Omega Y] \xleftarrow{\ \Omega\ } [X, Y] \xleftarrow{\ \overline{\Delta}^*\ } [X \wedge X, Y]$$

is exact.

Problem 18.26. Prove Proposition 18.24 and derive Corollary 18.25.

If we can understand $\mathrm{Im}(\overline{\Delta}^*)$, we can get information about $\ker(\Omega)$.

Problem 18.27. Show that if X is an $(n-1)$-connected co-H-space and $\pi_k(Y) = 0$ for $k \geq 3n-2$, then $\Omega : [X, Y] \to [\Omega X, \Omega Y]$ is injective.

18.5. The n-Dimensional Blakers-Massey Theorem

In Blakers-Massey type theorems, we study squares (or, more generally, n-cubes) that are approximately homotopy pushout squares, as measured by the connectivity of certain comparison maps. As we argued in Section 15.5, in this kind of context it is almost a matter of intellectual honesty to change the focus from homotopy equivalences to weak homotopy equivalences. Thus we can talk about a **Serre homotopy pushout square**, which is a commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in which the comparison map $\xi : P \to D$ from the homotopy pushout is a weak homotopy equivalence; the standard terminology (which we'll use from now on) is that this is a **homotopy cocartesian square**. There are easily formulated dual notions.

In this section we'll give a brief summary of a generalization of the Blakers-Massey theorem to higher-dimensional cubes. The result is not used in the main part of this text, and the proof is very complicated, so we will simply discuss the broad outlines of the argument and leave it as a project to work through a detailed proof.

The n-dimensional Blakers-Massey theorem implies a very interesting result: it is possible to reconstruct a simply-connected space X using homotopy limits involving only wedges of copies of ΣX.

18.5.1. Blakers-Massey Theorem for n-Cubes. Let Δ^n be the category whose objects are finite (ordered, if you like) subsets of $\mathbf{n} = \{1, 2, \ldots, n\}$ and whose morphisms are the inclusion maps. This is the 'n-skeleton' of the simplicial category Δ; it can be drawn as a commutative n-dimensional cube. We call a functor $F : \Delta^n \to \mathcal{T}$ a **cube diagram**. For $n \geq 1$, the **punctured n-cube** is the full subcategory $\widehat{\Delta}^n \subseteq \Delta^n$ containing all the objects except the initial $\varnothing$.

A cube diagram $F : \Delta^n \to \mathcal{T}$ is called a **homotopy cocartesian cube** if the natural comparison map

$$\xi : \operatorname{colim} F|_{\widehat{\Delta}^n} \longrightarrow F(\mathbf{n})$$

is a weak homotopy equivalence; these are the n-dimensional analogs of strong homotopy pushout squares. The cube F is called **strongly homotopy cocartesian** if each 2-dimensional subcube is a homotopy cocartesian square.

Problem 18.28. Show that a strongly cocartesian cube is determined up to pointwise weak equivalence by the maps $F(\varnothing \to \{k\})$.

Given an n-cube diagram $F : \Delta^n \to \mathcal{T}$, let F_k^0 and F_k^1 be the restrictions to the $(n-1)$-dimensional subcubes with constant k^{th} coordinate. Then F can be viewed as a morphism $\Phi_k : F_k^0 \to F_k^1$ of $(n-1)$-cubes. Then we may form homotopy fibers in each coordinate, yielding a new diagram

$$\mathrm{Fib}_k(F) : \Delta^{n-1} \longrightarrow \mathcal{T}.$$

The **iterated fiber** of F is $\mathrm{Fib}(F) = \mathrm{Fib}_1 \circ \mathrm{Fib}_2 \circ \cdots \circ \mathrm{Fib}_n(F)$.

Problem 18.29.

(a) Show that the iterated fiber is the fiber of the natural map from $F(\varnothing)$ to $\mathrm{holim}_* F|_{\widehat{\Delta}^n}$, the homotopy limit of the punctured cube.

(b) Conclude that the homotopy type of the iterated (co)fiber is independent of the order of fiber-takings.

(c) The diagram F, viewed as a diagram morphism Φ_k, induces a map from the iterated fiber of the diagram F_k^0 to that of F_k^1. Show that the fiber of this map is the iterated fiber of F.

Because of Problem 18.29(b), it makes sense to use the connectivity of the iterated fiber to measure the extent to which an n-cube is approximately cartesian. A **k-cartesian** cube is one whose iterated fiber is k-connected.

Theorem 18.30. *Let $F : \Delta^n \to \mathcal{T}_*$ be a strongly cocartesian n-cube, and say that $F(\varnothing \to \{k\})$ is a c_k-equivalence, where $c_k \geq 2$. Then the cube F is $((\sum c_i) - (n-1))$-cartesian.*

Because an n-cube can be thought of as a diagram morphism from one $(n-1)$-cube to another, we can interpret this n-dimensional Blakers-Massey theorem as an estimate of the connectivity of an induced map between iterated fibers.

Problem 18.31. Show that Theorem 18.30 is equivalent to the following statement: if $\Phi : F \to G$ is a map of $(n-1)$-cubes satisfying the connectivity conditions of Theorem 18.30 and Φ is a pointwise c_n-equivalence, then the induced map of iterated fibers is a $((\sum c_i) - (n-1))$-equivalence.

Theorem 18.30 can be proved by induction on n, starting with the base case $n = 2$, which is Theorem 18.13.

Exercise 18.32. Criticize this argument:

> *We already know the case $n = 2$. For $n > 2$, we can view the given cube F as a square of $(n-2)$-cubes by restricting the first and second coordinates. Forming homotopy colimits gives a square*
>
> $$\begin{tikzcd} X_{0,0} \arrow[r] \arrow[d] & X_{0,1} \arrow[d] \\ X_{1,0} \arrow[r] & X_{1,1}, \end{tikzcd}$$
>
> *in which the horizontal maps induced maps of iterated fibers, and so are $((\sum_{k=2}^{n} c_i) - (n-2))$-connected, and the vertical maps are c_1-connected. Since the iterated fiber of the original cube is the iterated fiber of this square, the base case (Theorem 18.13) gives the result.*

Project 18.33. Work through a proof of Theorem 18.30. Formulate the dual theory, including finding an estimate for the extent to which a strongly cartesian cube is approximately a cocartesian cube.

The Blakers-Massey theorem in case $n = 2$ was originally proved by Blakers and Massey in [**25**], using the language of pairs and triads. Barratt and Whitehead [**21**] and (independently) Toda [**169**] generalized this to n-ads, at least for simply-connected spaces; the proof for n-ads of not necessarily simply-connected spaces had to wait for Ellis and Steiner [**60**], some thirty years later. Our formulation in terms of diagrams is heavily influenced by Goodwillie's exposition [**69**], which is a good place to look for a detailed proof.

18.5.2. Recovering X from ΣX.

The Ganea construction can be viewed as a recipe for recovering a space X from its loop space ΩX using only products and homotopy colimits.

Problem 18.34. Prove the assertion made above.

What about the dual? If we are given ΣX, can we use wedges and homotopy limits to recover X? Certainly, there is no hope for spaces which lose lots of information when they are suspended, which can happen if X is not simply-connected (e.g., we know that suspension can kill substantial portions of π_1). So let's focus on simply-connected spaces.

Fix a simply-connected space X, and for each n, let F_n be the strongly cocartesian n-cube F_n based on the maps $f_k : X \to *$ for $k = 1, \ldots, n$.

Explicitly, we replace each f_k with $\mathrm{in}_0 : X \hookrightarrow CX$; then we have

$$F(I) = X \cup \bigcup_{i \in I} (CX)_i$$

and all the maps are the evident inclusions. Write $\widehat{F}_n = F_n|_{\widehat{\Delta}^n}$ for the corresponding punctured n-cube and $X_{(n)} = \mathrm{holim}_* \widehat{F}_n$ for its homotopy limit.

Problem 18.35.

(a) Show that $F_n(I)$ is homotopy equivalent to a wedge of copies of ΣX for all $I \neq \varnothing$.

(b) Show that F_n is a strongly cocartesian cube.

(c) Construct maps $X_{(n)} \to X_{(n-1)}$.

(d) Let E_n be the punctured n-cube that is constant at X, and show that there is a natural map $E_n \to F_n$. Use it to construct the commutative ladder

$$\begin{tikzcd} \cdots \arrow[r] & X \arrow[r, equal] \arrow[d] & X \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & X_{(n+1)} \arrow[r] & X_{(n)} \arrow[r] & \cdots . \end{tikzcd}$$

(e) What can you say about the connectivity of $X \to X_{(n)}$?

(f) The ladder gives rise to a comparison map $\xi : X \to X_{(\infty)}$ from X to $X_{(\infty)}$, the homotopy limit of the bottom row. Show that ξ is a weak homotopy equivalence.

What Have We Done? We have clearly achieved something interesting, and now we are faced with the challenge of saying exactly what it was. We will articulate two different statements that follow from and describe the work we have done.

The first, and more standard, approach is to observe that an order-preserving map $I \hookrightarrow J$ of ordered finite subsets of $\mathbb{N}$ induces a map

$$X \cup \bigcup_{i \in I} (CX)_i \longrightarrow X \cup \bigcup_{j \in J} (CX)_j,$$

and so defines a (covariant) functor $F_\infty : \Delta \to \mathcal{T}_*$, where Δ is the simplicial category defined in Section 15.6.2.

We have not given a construction for homotopy limits of diagrams F with shape Δ, but such homotopy limits do exist, and they can be formed in two steps as the homotopy limit of the telescope of homotopy limits of the restrictions of F to Δ^n. Thus we have our first interpretation.

Theorem 18.36. *If X is simply-connected, then X is weakly homotopy equivalent to* $\operatorname{holim}_* F_\infty$.

This theorem can be traced back to Barratt and was stated without proof by Hopkins [**93**]. A proof that works for non-simply-connected spaces with well-behaved fundamental groups was given by Bousfield [**33**].

Another way to view our construction is to say that if you are given the spaces $\bigvee \Sigma X$ and the power to form homotopy limits, then you can recover the original space X. Let's say a **resolving class** is a full subcategory $\mathcal{R}$ of $\mathcal{T}_*$ that is closed under weak homotopy equivalence and under homotopy limits: if $X \sim Y$ and $Y \in \mathcal{R}$, then $X \in \mathcal{R}$; and if $F : \mathcal{I} \to \mathcal{R}$ is any diagram, then $\operatorname{holim}_* F \in \mathcal{R}$. If $\mathcal{Y}$ is a collection of spaces, we write $\mathcal{R}(\mathcal{Y})$ for the smallest resolving class containing $\mathcal{Y}$.

Theorem 18.37. *If X is simply-connected, then $X \in \mathcal{R}(\{\bigvee_1^n \Sigma X \mid n \in \mathbb{N}\})$.*

Repeated application of Theorem 18.37 allows us to reconstruct X from the N-fold suspension of X (and its wedges).

Corollary 18.38. *If X is simply-connected and $N \in \mathbb{N}$, then*

$$X \in \mathcal{R}\big(\big\{\textstyle\bigvee_1^n \Sigma^N X \mid n \in \mathbb{N}\big\}\big).$$

Problem 18.39. Derive Corollary 18.38 from Theorem 18.37.

This has another interesting corollary, which we will come back to later in Chapters 36 and 37.

Corollary 18.40. *If* $\operatorname{map}_*\big(Q, \bigvee_1^n \Sigma^N X\big) \sim *$ *for all n, then* $\operatorname{map}_*(Q, X) \sim *$.

Problem 18.41.

(a) Show that $\mathcal{R} = \{Z \mid \operatorname{map}_*(Q, Z) \sim *\}$ is a resolving class.

(b) Prove Corollary 18.40.

18.6. Additional Topics, Problems and Projects

18.6.1. Blakers-Massey Exact Sequence of a Cofibration. The Blakers-Massey theorem gives us a kind-of-lengthy exact sequence for the homotopy groups of a *cofiber sequence.*

Problem 18.42. Let $A \to B \to C$ be a cofiber sequence in which A is $(a-1)$-connected and C is $(c-1)$-connected. Show that there is an exact sequence

$$\pi_{a+c-2}(A) \longrightarrow \pi_{a+c-2}(B) \longrightarrow \pi_{a+c-2}(C) \longrightarrow \pi_{a+c-3}(A) \to \cdots$$

$$\cdots \to \pi_1(C) \longrightarrow \pi_0(A) \longrightarrow \pi_0(b) \longrightarrow \pi_0(C).$$

What can you say about the exactness of the sequence

$$\cdots \to [X, \Sigma^n A] \longrightarrow [X, \Sigma^n B] \longrightarrow [X, \Sigma^n C] \longrightarrow [X, \Sigma^{n+1} A] \to \cdots \quad ?$$

18.6.2. Exact Sequences of Stable Homotopy Groups. The Blakers-Massey theorem implies that, in the stable range, there is precious little difference between cofiber sequences and fiber sequences.

Problem 18.43. Let $A \to B \to C$ be a cofiber sequence of CW complexes.

(a) Show that if X is a finite-dimensional CW complex, the sequence

$$[X, A]^S \longrightarrow [X, B]^S \longrightarrow [X, C]^S$$

is exact.

(b) Show that if A, B and C are finite-dimensional, then the sequence

$$[A, X]^S \longleftarrow [B, X]^S \longleftarrow [C, X]^S$$

is exact.

Problem 18.44. Show that a cofiber sequence $A \to B \to C$ gives rise to a long exact sequence

$$\cdots \to \pi_n^S(A) \longrightarrow \pi_n^S(B) \longrightarrow \pi_n^S(C) \longrightarrow \pi_{n-1}^S(A) \to \cdots$$

of stable homotopy groups.

18.6.3. Simultaneously Cofiber and Fiber Sequences. It is good practice to always be mindful of which spaces, maps and so on are domain-type and which are target-type. In this problem, you will investigate sequences that fit both descriptions equally well.

Problem 18.45.

(a) Suppose $A \to B \to C$ is a cofiber sequence. Give conditions on the spaces A, B and C that guarantee that the sequence is also a fiber sequence.

(b) Now suppose $X \to Y \to Z$ is a fiber sequence. What do you need to know about X, Y and Z in order to conclude that the sequence is a cofiber sequence?

18.6.4. The Zabrodsky Lemma. The condition $\mathrm{map}_*(X, Y) \sim *$ plays a very important role in homotopy theory. The Zabrodsky lemma is frequently the key to making use of trivial mapping spaces (see Section 20.3, for instance).

Problem 18.46. Suppose $\mathrm{map}_*(A, Q) \sim *$. Show that in any cofiber sequence $A \to B \to C$, the induced map $\mathrm{map}_*(C, Q) \to \mathrm{map}_*(B, Q)$ is a weak homotopy equivalence.

The Zabrodsky lemma, which follows easily from the Ganea construction, establishes a similar implication for fiber sequences. This is a powerful and surprising result because it gives very good information about what happens when a (target-type) fibration is used in the domain.

Theorem 18.47 (Zabrodsky lemma)**.** *If $F \to E \to B$ is a fibration sequence, and if* $\operatorname{map}_*(F, Y) \simeq *$, *then* $\operatorname{map}_*(E, Y) \leftarrow \operatorname{map}_*(B, Y)$ *is a (weak) homotopy equivalence.*

Start with the fibration $p : E \to B$, and apply the Ganea construction iteratively, resulting in a ladder

$$\begin{tikzcd} \cdots \arrow[r] & G_n(p) \arrow[r] \arrow[d, "p_n"] & G_{n+1}(p) \arrow[r] \arrow[d, "p_{n+1}"] & \cdots \\ \cdots \arrow[r, equal] & B \arrow[r, equal] & B \arrow[r, equal] & \cdots . \end{tikzcd}$$

Problem 18.48.

(a) Show that if $\operatorname{map}_*(F, Y) \sim *$, then $\operatorname{map}_*(F \wedge Q, Y) \sim *$ for every space $Y \in \mathcal{T}_*$.

(b) Show that $\operatorname{map}_*(G_{n+1}(p), Y) \to \operatorname{map}_*(G_n(p), Y)$ is a weak homotopy equivalence.

(c) Show that the homotopy colimit of the top row in the ladder is (weakly) homotopy equivalent to B.

(d) Prove Theorem 18.47.

18.6.5. Problems and Projects.

Problem 18.49. Show that in a homotopy pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "HPO"] & B \arrow[d, "g"] \\ C \arrow[r] & D \end{tikzcd}$$

the connectivity of f is equal to the connectivity of g. Under what conditions can you prove the same thing for a homotopy pushout square?

Project 18.50. Construct a comparison map $\Sigma^2 F \to C$ from the double suspension of the iterated fiber of a square to the iterated cofiber. How much of an equivalence is it? Can you generalize your result to n-cubes? (Consult Mather [**115**] for guidance.)

Problem 18.51.

(a) Show that if $\operatorname{map}_*(X, Q) \sim * \sim \operatorname{map}_*(Y, Q)$, then $\operatorname{map}_*(X \times Y, Q) \sim *$.

(b) Show that if $\operatorname{map}_*(\Omega X, Y) \sim *$, then $\operatorname{map}_*(X, Y) \sim *$.

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Chapter 19

Some Computations in Homotopy Theory

Having now set up the fundamentals of homotopy theory, we now begin to apply it. In this chapter we do some explicit computations involving homotopy sets $[X, Y]$ and apply those calculations to prove topological theorems.

We begin by studying the degree of a reflection and then of the antipodal map. We use this to prove results about fixed points of self-maps of spheres. We also determine the degree of the twisting homeomorphism $S^n \wedge S^m \to S^m \wedge S^n$, which appears ubiquitously in algebraic topology in the form of the **Milnor sign convention**.

We show that maps from one wedge of n-spheres to another are fully described by certain integer matrices. This is used to define and study Moore spaces and to produce an example of a noncontractible CW complex X whose suspension is trivial.

In the final two sections, we define the smash product pairing of homotopy groups and use it to express the smallest nontrivial homotopy group of a smash product in terms of the homotopy groups of the factors. This is applied to construct certain maps between Eilenberg-Mac Lane spaces that will be crucial in our development of cohomology algebras in the next part of the book.

19.1. The Degree of a Map $S^n \to S^n$

The degree of a map $S^n \to S^n$ was first defined by L. E. J. Brouwer in 1910. A pointed map $f : S^n \to S^n$ induces a homomorphism

$$\begin{tikzcd} \pi_n(S^n) \arrow[r, "f_*"] \arrow[d, equal] & \pi_n(S^n) \arrow[d, equal] \\ \mathbb{Z} \cdot [\mathrm{id}_{S^n}] \arrow[r] & \mathbb{Z} \cdot [\mathrm{id}_{S^n}] \end{tikzcd}$$

from an infinite cyclic group *to itself*, which is necessarily multiplication by some integer d, which is called the **degree** of f and is denoted $\deg(f)$.

Exercise 19.1. Determine the degree of id_{S^n} and of $*$. Suppose $f : S^n \to S^n$ is a homotopy equivalence; what can you say about $\deg(f)$?

On the face of it, this definition applies only to pointed maps, but it can be easily extended to unpointed maps.

Problem 19.2.

(a) Show that the forgetful map $[S^n, S^n] \to \langle S^n, S^n \rangle$ is bijective for $n \geq 1$, so degree makes sense for unpointed maps, too.

(b) Show that $f \simeq g$ in $\mathcal{T}_\circ$ if and only if $\deg(f) = \deg(g)$.

Exercise 19.3. Let $f : S^n \to S^n$. Since $\pi_n(S^n) \simeq \mathbb{Z} \cdot [\mathrm{id}_{S^n}]$, if $f : S^n \to S^n$, we have $[f] = d \cdot [\mathrm{id}_{S^n}]$ for some $d \in \mathbb{Z}$. Show that $d = \deg(f)$.

The set $[S^n, S^n]$ has a second binary operation given by composition of functions.

Problem 19.4.

(a) Show that composition defines a product making $[S^n, S^n]$ into a ring.

(b) Show that $\deg : [S^n, S^n] \to \mathbb{Z}$ is a ring isomorphism.

(c) Determine the degree of the co-H inverse map $\nu : S^n \to S^n$.

19.1.1. The Degree of a Reflection and the Antipodal Map. Now we come to our first interesting computation: the degree of the antipodal map, which is based on the degree of a reflection.

Reflections. Any two reflections are (freely) homotopic to one another, so they all have the same degree. To determine this degree, it suffices to work with the most convenient reflection.

Problem 19.5.

(a) Let $S^n \subseteq \mathbb{R}^{n+1}$, and let $r : S^n \to S^n$ be the restriction of the reflection about the hyperplane $\mathbb{R}^n \times 0 \subseteq \mathbb{R}^{n+1}$. Show that $r = \nu : S^n \to S^n$.

(b) Let $P_0, P_1 \subseteq \mathbb{R}^{n+1}$ be hyperplanes through the origin, and let $R_0, R_1 : S^n \to S^n$ be the reflections through P_1 and P_2, respectively. Show that R_0 and R_1 are freely homotopic (i.e., they are homotopic as unpointed maps).

Hint. Hyperplanes are determined by their normal vectors.

(c) What is the degree of a reflection?

The Antipodal Map. The **antipodal map** on the n-sphere is the map $\alpha_n : S^n \to S^n$ given by $\alpha(x) = -x$. You will determine the degree of α_n and derive some interesting consequences.

Problem 19.6.

(a) What is the degree of $\alpha_1 : S^1 \to S^1$?

(b) Show that $\alpha_{n+1} + \Sigma\alpha_n = 0$ in $\pi_{n+1}(S^{n+1})$.

(c) Determine the degree of α_n as a function of n.

(d) For which n is $\alpha \simeq \mathrm{id}_{S^n}$?

Another approach to determining the degree of α_n is to express it in terms of reflections.

19.1.2. Computation of Degree. We appropriate (and mangle) an idea from differential topology: call a point $y \in S^n$ a **regular value** for $f : S^n \to S^n$ if $f^{-1}(y) = \{x_1, \ldots, x_m\}$ is a finite set and y has an open neighborhood V such that

- the inclusion $\partial\overline{V} \hookrightarrow \overline{V}$ is pointwise homeomorphic to $S^{n-1} \hookrightarrow D^n$,
- $f^{-1}(V)$ is the disjoint union $\coprod_{i=1}^m U_i$ with $x_i \in U_i$, and
- the closures $\overline{U}_i$ are pairwise disjoint and f restricts to homeomorphisms $\overline{U}_i \to \overline{V}$ for each i.

We'll refer to n as the **multiplicity** of the regular value y.

We can use a regular value for f to compute its degree. If $y \in S^n$ is a regular value for $f : S^n \to S^n$, then we may construct the diagram

$$\begin{tikzcd}
S^n - U_i \arrow[d] & \partial\overline{U}_i \arrow[l] \arrow[r, "\cong"] \arrow[d] & \partial\overline{V} \arrow[r] \arrow[d] & S^n - V \arrow[d] \\
S^n & \overline{U}_i \arrow[l] \arrow[r] & \overline{V} \arrow[r] & S^n.
\end{tikzcd}$$

Taking cofibers of the vertical maps produces the diagram

$$\begin{tikzcd}
S^n \arrow[rrr, "f_i"] \arrow[d, "\simeq"'] & & & S^n \arrow[d, "\simeq"] \\
S^n & \overline{U}_i \arrow[ul] \arrow[r] \arrow[d] & \overline{V} \arrow[ur] \arrow[d] & S^n \\
& S^n \arrow[ul, "\cong"] \arrow[r] & S^n, \arrow[ur, "\cong"'] &
\end{tikzcd}$$

which we use to define the map $f_i : S^n \to S^n$, at least up to homotopy.

Exercise 19.7. Show that $\deg(f_i) = \pm 1$.

Theorem 19.8. *Let $f : S^n \to S^n$ with $n \geq 2$. With the notation from the discussion above, $f = f_1 + f_2 + \cdots + f_n \in \pi_n(S^n)$, and so*

$$\deg(f) = \sum_{i=1}^{n} \deg(f_i).$$

Corollary 19.9. *Let $f : S^n \to S^n$.*

(a) If $y_1, y_2 \in S^n$ are regular values for f, then their multiplicities are congruent to each other, and to $\deg(f)$, modulo 2.

(b) If y is a regular value for f with odd multiplicity, then $f \not\simeq *$.

(c) If f has degree d, then every regular value for f must have multiplicity at least d.

Problem 19.10.

(a) Show that $S^n/(S^n - \coprod U_i) \cong \bigvee_{1=1}^m S^n$.

(b) Show that the diagram

$$\begin{tikzcd}
S^n \arrow[r, "\phi_m"] \arrow[d, equal] & \bigvee_{1=1}^m S^n \arrow[d, "\cong"] \\
S^n \arrow[r, "q"] & S^n/(S^n - \coprod U_i)
\end{tikzcd}$$

commutes up to homotopy, where q is the quotient map and ϕ_m is the m-fold pinch map.

(c) Prove Theorem 19.8.

(d) Derive Corollary 19.9.

Exercise 19.11. Find examples of maps $f : S^n \to S^n$ with degree d and a point $x \in S^n$ such that $f^{-1}(x)$ is a singleton.

The remaining question is: how do we determine the degree of f_i? For this, we quote a simple result from differential topology.

Proposition 19.12. *If $f : S^n \to S^n$ is a homeomorphism, and if the derivative $Df(x)$ is an invertible linear transformation, then*

$$\deg(f) = \operatorname{sign}(\det(Df(x))).$$

We'll take this for granted. This approach to degree makes sense more generally for maps $f : M \to N$ from one n-manifold to another.

Project 19.13. Prove Proposition 19.12.

19.2. Some Applications of Degree

Our knowledge of the degree of the antipodal map, together with some simple topology, leads to some important conclusions about self-maps of spheres.

19.2.1. Fixed Points and Fixed Point Free Maps. A **fixed point** for a self-map $f : X \to X$ is a point $x \in X$ such that $f(x) = x$. Using degree, we can discover quite a lot about fixed points of self-maps of spheres.

Problem 19.14.

(a) Let $x, y \in S^n$ and assume that $x \neq -y$. Write down an explicit parametrization for the shortest arc on the sphere joining x to y. What happens when $x = -y$?

(b) Suppose $f : S^n \to S^n$ is a map so that $f(x) \neq -x$ for every $x \in S^n$. Show that $f \simeq \mathrm{id}_{S^n}$.

(c) Suppose $f : S^n \to S^n$ is a map with no fixed points. Show that f is homotopic to the antipodal map α_n.

  Hint. Show $\alpha_n \circ f \simeq \mathrm{id}_{S^n}$.

Now you can derive a nice and surprising result about maps of spheres without fixed points.

Theorem 19.15. *If $f : S^{2n} \to S^{2n}$ has no fixed points, then there is an $x \in S^{2n}$ such that $f(x) = -x$.*

Problem 19.16.

(a) Prove Theorem 19.15.

(b) Find examples of maps $S^{2n+1} \to S^{2n+1}$ without fixed points and with $f(x) \neq -x$ for all x.

19.2.2. Vector Fields on Spheres. Let $S^n \subseteq \mathbb{R}^{n+1}$ be the standard n-sphere. A vector $v \in \mathbb{R}^{n+1}$ is tangent to S^n at x if and only if $x \perp v$. A **vector field** on $S^n \subseteq \mathbb{R}^{n+1}$ is a function $v : S^n \to \mathbb{R}^{n+1}$ such that $v(x)$ is tangent to S^n at the point x. A **nonzero** vector field is a vector field $v : S^n \to \mathbb{R}^{n+1}$ such that $v(x) \neq 0$ for all $x \in S^n$.

Theorem 19.17. *There is a nonzero vector field on $S^n \subseteq \mathbb{R}^{n+1}$ if and only if n is odd.*

Problem 19.18.

(a) Show that S^n has a nonzero vector field if and only if there is a function $f : S^n \to S^n$ such that $f(x) \perp x$ for all $x \in S^n$.

(b) Show that if S^n has a nonzero vector field, then n must be odd.

(c) If $n = 2k - 1$ is odd, then $S^n \subseteq \mathbb{R}^{n+1} \cong \mathbb{C}^k$. Use this to construct a nonzero vector field on S^n.

19.2.3. The Milnor Sign Convention. Let $T : S^n \wedge S^m \to S^m \wedge S^n$ be the **twist map** that interchanges smash factors. We identify $S^n \wedge S^m$ with S^{n+m} by iterating the homeomorphism $\Sigma S^n = S^1 \wedge S^n \cong S^{n+1}$ of Problem 3.3. Thus we obtain standard homeomorphisms

$$S^n \wedge S^m \cong (S^1)^{\wedge n} \wedge S^m \cong S^{n+m}.$$

Using the commutative square

$$\begin{tikzcd}
S^n \wedge S^m \arrow[r, "T"] \arrow[d, "\cong"'] & S^m \wedge S^n \arrow[d, "\cong"] \\
S^{n+m} \arrow[r, "\widetilde{T}"] & S^{n+m},
\end{tikzcd}$$

we view T as a map $\widetilde{T}$ from S^{n+m} to itself; when we talk about the degree of T, we really mean the degree of the corresponding map $\widetilde{T}$.

Theorem 19.19. *The degree of $T : S^n \wedge S^m \to S^m \wedge S^n$ is $(-1)^{nm}$.*

The full statement follows easily from the special case $n = m = 1$.

Problem 19.20.

(a) Think of S^1 as the quotient of I by identifying the endpoints. Then $S^1 \wedge S^1$ is a quotient of $I \times I$ by a certain equivalence relation. Describe the map T in terms of the square $I \times I$.

(b) Show that, under the identification of part (a), the map $T : S^2 \to S^2$ is the reflection about a certain plane.

(c) Prove Theorem 19.19 in the case $n = m = 1$, and derive the full statement.

Problem 19.21. Use Theorem 19.8 to give a different proof of Theorem 19.19.

Theorem 19.19 is the reason that algebraic operations derived from the smash product introduce signs. The general convention, called the **Milnor sign convention**, is that when two things x and y with dimensions $|x|$ and

$|y|$ are moved past one another, a sign $(-1)^{|x|\cdot|y|}$ must be introduced, as we do, for example, when forming the tensor product of graded modules (see Section A.3). This commutativity formula is called **graded commutativity**; but when all the algebra in sight is graded (i.e., elements x have dimensions $|x|$ associated to them), the rule $xy = (-1)^{|x|\cdot|y|}yx$ is referred to simply as *commutativity.*

19.2.4. Fundamental Theorems of Algebra. Next we show that every polynomial with coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$: the Fundamental Theorem or Algebra. We also show that (certain) quaternionic polynomials have quaternionic zeros.

Complex Polynomials. Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial. We hope to show that there is a $z \in \mathbb{C}$ such that $p(z) = 0$. If we assume that p has no zeros, then we can view it as a function $p : \mathbb{C} \to \mathbb{C} - \{0\}$.

Theorem 19.22 (Fundamental Theorem of Algebra)**.** *If $p : \mathbb{C} \to \mathbb{C}$ is a polynomial function, then there is $z \in \mathbb{C}$ such that $p(z) = 0$.*

Problem 19.23. Suppose $p : \mathbb{C} \to \mathbb{C}$ has no zeros. Write $p_n(z) = z^n$; we may assume that $p(z) = p_n(z) + u(z)$, where $\deg(u) < n$.

(a) Show that the function $q : \mathbb{C} - \{0\} \to S^1$ given by $z \mapsto \frac{z}{|z|}$ is a homotopy equivalence.

(b) For $r > 0$, define $\mathrm{in}_r : S^1 \to \mathbb{C}$ by the rule $\mathrm{in}_r(z) = rz$. Determine the degree of the composite map $q \circ p_n \circ \mathrm{in}_r : S^1 \to S^1$.

(c) Show that for sufficiently large r, $|u(rz)| < |rz|^n$ for all $z \in S^1$. Conclude that $p \circ \mathrm{in}_r \simeq p_n \circ \mathrm{in}_r : S^1 \to \mathbb{C} - \{0\}$ for sufficiently large r.

(d) Prove Theorem 19.22.

Quaternionic Polynomials. We can ask a similar question for quaternions, and indeed the same argument can be used to prove the corresponding result. For variety, though, we'll use a different approach.

Because $\mathbb{H}$ is not associative, the algebra is trickier, and our statement about roots of polynomials must be scaled back. A typical quaternionic monomial of 'degree' n has the form

$$q_0(zq_1)(zq_2)\cdots(zq_n)$$

in which z is the variable and the quaternions q_i are 'coefficients'. It is important to realize that it may not be possible to simplify a monomial into the form qz^n, and so we cannot necessarily collect together all of the monomials of the same degree into a single monomial. Thus it is possible—or at least conceivable—to have a polynomial which can only be written using several degree n terms, but which grows at a rate slower than $|z|^n$.

To avoid this possibility, we will study quaternionic polynomials with a single monomial of highest degree: thus we write

$$p(z) = \overbrace{q_0 z q_1 z q_2 \cdots q_{n-1} z q_n}^{\bar{p}(z)} + u(z)$$

in which u is a sum of monomials of degree less than n.

Theorem 19.24 (Eilenberg-Niven)**.** *If p is a quaternionic polynomial with a single monomial of highest degree, then p has a zero in $\mathbb{H}$.*

This does not mean that quaternionic algebra is comparable to that of the complex numbers.

Exercise 19.25. Show that the polynomial $p(z) = z^2 + 1$ has infinitely many zeros. In fact, q is a zero of p if and only if $|q| = 1$ and the real part of q is zero.

A quaternion whose real part is zero is called a **pure quaternion**.

Problem 19.26. Let $a \in S^3 \subseteq \mathbb{H}$ be a pure quaternion.

(a) Show that the rule $1 \mapsto 1$ and $i \mapsto a$ defines an injective ring homomorphism $\mathbb{C} \hookrightarrow \mathbb{H}$.
(b) Show that if $z \in \operatorname{Span}\{1, a\}$, then $z^n \in \operatorname{Span}\{1, a\}$ for all $n \geq 1$.
(c) Show that if $q \notin \mathbb{R}$, then the equation $z^n = q$ has exactly n solutions in $\mathbb{H}$.

Exercise 19.27. Verify that for $x, y \in \mathbb{H}$, $|xy| = |x| \cdot |y|$.

We may interpret a quaternionic polynomial p as a function $p : \mathbb{H} \to \mathbb{H}$. Since $|p(z)|$ increases to ∞ as $|z|$ increases, p induces a map $p_\infty : S^4 \to S^4$ from one-point compactification of $\mathbb{H}$ to itself.

Problem 19.28. Suppose $p = \bar{p}(z) + u(z)$ is a quaternionic polynomial with a single term of highest degree n.

(a) Show that $p_\infty \simeq \bar{p}_\infty : S^4 \to S^4$.
(b) Let $f(z) = z^n$, and show that $\bar{p}_\infty \simeq f_\infty : S^4 \to S^4$.
HINT. $\mathbb{H} - \{0\}$ is path-connected.

We proceed to compute the degree of f_∞.

Problem 19.29.

(a) Use Proposition 19.12 to show that $i \in \mathbb{C} \subseteq \mathbb{H}$ is a regular value for f_∞.
(b) By computing the Jacobian matrices for f at the points in $f^{-1}(\mathbf{i})$, determine the degree of f_∞.
(c) Show that $p_\infty \not\simeq *$, and prove Theorem 19.24.

Compare these proofs to the Intermediate Value Theorem proof that every real polynomial with odd degree has a root: if such a polynomial p had no root, then we would be forced to conclude $S^0 \simeq *$.

19.3. Maps Between Wedges of Spheres

In this section, we completely determine the set of all homotopy classes of maps $\bigvee_{\mathcal{J}} S^n \to \bigvee_{\mathcal{I}} S^n$ for $n \geq 2$ and explicitly describe their compositions. Wedges of 1-spheres are trickier, because the groups involved are nonabelian, and our conclusions are not as strong.

We will identify the sets of homotopy classes of maps between simply-connected wedges of n-spheres with certain groups of matrices. Given indexing sets $\mathcal{I}$ and $\mathcal{J}$, write $M_{\mathcal{I}\times\mathcal{J}}(R)$ for the set of all matrices indexed on the product set[1] $\mathcal{I} \times \mathcal{J}$, with entries in the ring R. When $\mathcal{I} = \{1, 2, , \ldots, n\}$ and $\mathcal{J} = \{1, 2, \ldots, m\}$, this is usually written $M_{n\times m}(R)$.

For any $f : \bigvee_{i\in\mathcal{I}} S^n \to \bigvee_{j\in\mathcal{J}} S^n$, the $(i, j)^{\text{th}}$ **coordinate map** is the map $f_{ij} : S^n \to S^n$ defined by the diagram

$$\begin{tikzcd}
S^n \arrow[r, "f_{ij}"] \arrow[d, "\mathrm{in}_j"'] & S^n \\
\bigvee_{j\in\mathcal{J}} S^n \arrow[r, "f"] & \bigvee_{i\in\mathcal{I}} S^n, \arrow[u, "q_i"']
\end{tikzcd}$$

where q_i is the collapse to the i^{th} sphere. The map f_{ij} represents an element of $\pi_n(S^n)$, which has a degree. We define a matrix $A(f) : \mathcal{I} \times \mathcal{J} \to \mathbb{Z}$ by setting $A(i, j) = \deg(f_{ij})$.

Problem 19.30. Show that the matrix $A(f) \in M_{\mathcal{I}\times\mathcal{J}}(\mathbb{Z})$ has only finitely many nonzero entries in each column.

We write $\overline{M}_{\mathcal{I}\times\mathcal{J}}(R)$ for the (additive) subgroup of $M_{\mathcal{I}\times\mathcal{J}}(R)$ consisting of those matrices with only finitely many nonzero entries in each column.

Theorem 19.31. *The function*

$$A : \left[\bigvee_{\mathcal{J}} S^n, \bigvee_{\mathcal{I}} S^n\right] \longrightarrow \overline{M}_{\mathcal{I}\times\mathcal{J}}(\mathbb{Z}) \qquad \text{given by} \qquad A : f \mapsto A(f)$$

is an isomorphism of (additive) abelian groups.

Problem 19.32.

(a) Show that the inclusion in : $\bigvee_{\mathcal{I}} S^n \hookrightarrow \prod_{\mathcal{I}} S^n$ of the wedge into the **weak product**[2] is a $(2n-1)$-equivalence.

[1] This might give you pause when $\mathcal{I}$ or $\mathcal{J}$ (or both!) are infinite, but such a matrix is simply a function $A : \mathcal{I} \times \mathcal{J} \to R$.

[2] The weak product is the colimit of the finite products.

(b) Prove Theorem 19.31.

Theorem 19.31 implies that if $f, g : \bigvee_{\mathcal{J}} S^n \to \bigvee_{\mathcal{I}} S^n$, then $f \simeq g$ if and only if $f_* = g_*$ on $\pi_n(\,?\,)$.

Problem 19.33. Let $\mathcal{I} = \mathcal{J}$, so that we can compose elements of the homotopy set $[\bigvee_{\mathcal{I}} S^n, \bigvee_{\mathcal{I}} S^n]$ with each other.

(a) Show that composition gives $[\bigvee_{\mathcal{I}} S^n, \bigvee_{\mathcal{I}} S^n]$ the structure of a ring.

(b) Show that $\overline{M}_{\mathcal{I}\times\mathcal{I}}(\mathbb{Z})$ is a ring under matrix multiplication.

(c) Verify that $\alpha : [\bigvee_{\mathcal{I}} S^n, \bigvee_{\mathcal{I}} S^n] \to \overline{M}_{\mathcal{I}\times\mathcal{I}}(\mathbb{Z})$ is a ring isomorphism.

Induced Maps and Matrices. We have seen how to encode the homotopy class of a map $f : \bigvee_{\mathcal{J}} S^n \to \bigvee_{\mathcal{I}} S^n$ from one wedge of n-spheres to another in the matrix $A(f)$. Suppose now that Y is some other space, and consider the induced map

$$f^* : [\bigvee_{\mathcal{I}} S^n, Y] \longrightarrow \left[\bigvee_{\mathcal{J}} S^n, Y\right].$$

What can we say about this map? We have

$$\left[\bigvee_{\mathcal{J}} S^n, Y\right] \cong \prod_{\mathcal{J}} \pi_n(Y)$$

and similarly $[\bigvee_{\mathcal{I}} S^n, Y] \cong \prod_{\mathcal{I}} \pi_n(Y)$. Problem 1.50 shows that we can describe maps from a sum to a product as matrices of maps, but here we have a map from a *product* to a product, a categorically unpleasant situation. We might be tempted to abandon the general question and retreat to the safety of finite indexing sets, so that the product is isomorphic to the sum, but the compactness of spheres offers another approach.

Problem 19.34. Show that f^* is a map of finite type (see Section A.8).

Because of Problem 19.34 and the theory developed in Section A.8, f^* is the unique finite-type map extending the map ϕ in the diagram

$$\begin{tikzcd}
\bigoplus_i \pi_n(Y) \arrow[d] \arrow[dr, bend left, "\phi"] & \\
\prod_i \pi_n(Y) \arrow[r, "f^*"] & \prod_j \pi_n(Y),
\end{tikzcd}$$

and ϕ is determined by a matrix Φ each of whose entries is a homomorphism from $\pi_n(Y)$ to $\pi_n(Y)$. Among the self-maps of $\pi_n(Y)$, of course, are those that are simply multiplication by some integer.

Theorem 19.35. *If $f : \bigvee_{\mathcal{J}} S^n \to \bigvee_{\mathcal{I}} S^n$ has matrix $A = A(f)$, then the induced map $f^* : [\bigvee_{\mathcal{I}} S^n, Y] \to [\bigvee_{\mathcal{J}} S^n, Y]$ may be identified with the unique finite-type map $\prod_{\mathcal{J}} \pi_n(Y) \to \prod_{\mathcal{I}} \pi_n(Y)$ whose matrix is the transpose A^T.*

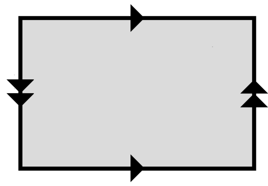

Figure 19.1. Gluing up a Klein bottle

Problem 19.36.

(a) Show that if $g : S^n \to S^n$ has degree d, then $g^* : \pi_n(Y) \to \pi_n(Y)$ is multiplication by d.

(b) Prove Theorem 19.35.

The Non-Simply-Connected Case. Now we want to study maps $f : \bigvee_{\mathcal{J}} S^1 \to \bigvee_{\mathcal{I}} S^1$. Since the domain of f is a sum, f is determined by the maps $f_j = f \circ \mathrm{in}_j : S^1 \to \bigvee_{\mathcal{I}} S^1$. But now, in contrast with the simply-connected case, the target cannot be replaced with a product, so our analysis stops here: f is determined by the $\mathcal{J}$-tuple

$$(f \circ \mathrm{in}_1, f \circ \mathrm{in}_2, \ldots) \in \prod_{\mathcal{J}} \pi_1\left(\bigvee_{\mathcal{I}} S^1\right).$$

In this formula, each map $f \circ \mathrm{in}_j$ is an element of $\pi_1(\bigvee_{\mathcal{I}} S^1)$. Since this is a free group on generators $\{\mathrm{in}_i \mid i \in \mathcal{I}\}$, the element $f \circ \mathrm{in}_j$ has a unique expression as a reduced word in the symbols $\{\mathrm{in}_i, \mathrm{in}_i^{-1} \mid i \in \mathcal{I}\}$. This is about as far as we can push the argument in this case.

If we suspend f, the factors in our words suddenly commute, so we can collect terms and add the exponents.

Problem 19.37. The **Klein bottle** K is the space obtained from the square $I \times I$ by gluing the edges as indicated in Figure 19.1.

(a) Show that K is the cofiber of a map $\kappa : S^1 \to S^1 \vee S^1$. Describe κ explicitly as an element of the free group $\pi_1(S^1 \vee S^1) = F(\mathrm{in}_1, \mathrm{in}_2)$.

(b) Give a presentation for $\pi_1(K)$. Is this group abelian?

(c) Determine the homotopy type of ΣK. What is $\pi_2(\Sigma K)$?

Strange Behavior of Suspension. There are monsters in homotopy theory: nontrivial spaces whose suspensions are contractible; spaces which are not simply-connected and whose suspension has connectivity exactly n; and other oddities.

You'll have to take for granted that the group with the presentation

$$G = \langle a, b, c, d \mid bab^{-1}a^{-2},\ cbc^{-1}b^{-2},\ dcd^{-1}c^{-2},\ ada^{-1}d^{-2} \rangle$$

is not a trivial group (see [**84**]).

Problem 19.38. Write a, b, c and d, respectively, for the summand inclusions $\mathrm{in}_1, \mathrm{in}_2, \mathrm{in}_3$ and $\mathrm{in}_4 \in \pi_1(\bigvee_1^4 S^1)$, and set

$$r_1 = bab^{-1}a^{-2}, \quad r_2 = cbc^{-1}b^{-2}, \quad r_3 = dcd^{-1}c^{-2} \quad \text{and} \quad r_4 = ada^{-1}d^{-2}.$$

Let $f : \bigvee_1^4 S^1 \to \bigvee_1^4 S^1$ be the map given by the 4-tuple (r_1, r_2, r_3, r_4), and let X be its cofiber.

(a) Show that $X \not\simeq *$.

(b) Show that $\Sigma X \simeq *$.

Hint. What is the matrix $A(\Sigma f)$?

Problem 19.39. Show that for each n, there is a path-connected CW complex Y_n such that Y_n is not simply-connected but ΣY_n is n-connected but not $(n+1)$-connected.

Problem 19.40. Show that if $\Sigma X \simeq *$, then for any connected CW complex Y, $X \wedge Y \simeq *$.

Problem 19.41. Give an example of a homotopy pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "HPO"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

in which $B \to D$ is a homotopy equivalence and $A \to C$ is not a homotopy equivalence.

19.4. Moore Spaces

In this section we introduce and study Moore spaces, which occupy a position in homotopy theory that is roughly dual to that of Eilenberg-Mac Lane spaces.

We write $F(X)$ for the free abelian group on the generating set X.

Problem 19.42. Let $n \geq 2$.

(a) Show that the rule $i \mapsto \mathrm{in}_i$ induces an isomorphism $F(\mathcal{I}) \xrightarrow{\cong} \pi_n\left(\bigvee_{\mathcal{I}} S^n\right)$.

(b) Let $d : F(\mathcal{J}) \to F(\mathcal{I})$ be a map from one free abelian group to another. Show that there is a unique homotopy class of maps $\delta : \bigvee_{\mathcal{J}} S^n \to \bigvee_{\mathcal{I}} S^n$ making the diagram

$$\begin{tikzcd} F(\mathcal{J}) \arrow[r, "d"] \arrow[d, "\cong"'] & F(\mathcal{I}) \arrow[d, "\cong"] \\ \pi_n(\bigvee_{\mathcal{J}} S^n) \arrow[r, "\delta_*"] & \pi_n(\bigvee_{\mathcal{I}} S^n) \end{tikzcd}$$

commute. We say that δ *induces* d *on homotopy groups.*

Now if G is an abelian group, it has a **free resolution**, which is a short exact sequence

$$0 \to F_1 \xrightarrow{d} F_0 \longrightarrow G \to 0,$$

where $F_0 = F(\mathcal{I})$ and $F_1 = F(\mathcal{J})$ for some indexing sets $\mathcal{I}$ and $\mathcal{J}$. Given $n \geq 2$, let $\delta : \bigvee_{\mathcal{J}} S^n \to \bigvee_{\mathcal{I}} S^n$ be a map inducing d on homotopy groups, and write $M(G,n)$ for the cofiber C_δ; this is the **Moore space** for the group G in dimension n.

Problem 19.43. Show that the construction of $M(G,n)$ gives rise to a canonical isomorphism $G \xrightarrow{\cong} \pi_n(M(G,n))$.

Thus any two Moore spaces share the same low-dimensional homotopy groups. Since $M(G,n)$ is constructed using cofiber sequences, it is hard to get information about the homotopy groups in dimensions greater than $2n-1$. Nevertheless, any two Moore spaces have the same homotopy groups in *all* dimensions, because they are homotopy equivalent to each other.

Theorem 19.44. *The homotopy type of $M(G,n)$ depends only on the number n and the group G, and not on the choice of free resolution of G.*

The proof of Theorem 19.44 will make use of some computation of homotopy sets involving Moore spaces.

Problem 19.45. Show that if $n \geq 2$, then $M(G,n)$ is a suspension. In fact, $\Sigma M(G,n) \simeq M(G,n+1)$.

Homotopy Sets Involving Moore Spaces. We can compute the stable homotopy groups of a Moore space in terms of the homotopy groups of spheres using the algebraic functors $\mathrm{Tor}(\,?\,,\,?\,)$ and $?\otimes ?$ (see Section A.1 for definitions and basic properties).

Proposition 19.46. *Let $G \in \mathrm{AB}\mathcal{G}$ and $n \geq 2$. There are exact sequences*

$$0 \to G \otimes \pi_{n+k}(S^n) \longrightarrow \pi_{n+k}(M(G,n)) \longrightarrow \mathrm{Tor}(G, \pi_{n+k}(S^{n+1})) \to 0$$

for $k < n-1$.

The cases $k=0$ and $k=1$ are so frequently useful that we'll state them separately.

Corollary 19.47. *If $G \in \mathrm{AB}\mathcal{G}$ and $n \geq 2$, then*

$$\pi_n(M(G,n)) \cong G \qquad \text{and} \qquad \pi_{n+2}(M(G,n+1)) \cong G \otimes \pi_{n+2}(S^{n+1}).$$

Problem 19.48. Suppose $M(G,n)$ has been constructed using the free resolution $0 \to F_1 \xrightarrow{d} F_0 \to G \to 0$.

(a) Show that for each $k < n-1$ there is an exact sequence of the form

$$F_1 \xrightarrow{d} F_0 \longrightarrow \pi_n(M(G,n)) \longrightarrow F_1 \xrightarrow{d} F_1.$$

(b) Prove Proposition 19.46 and derive Corollary 19.47.

Now we are prepared to prove the uniqueness of Moore spaces up to homotopy type.

Problem 19.49. Suppose $0 \to H_1 \xrightarrow{\epsilon} H_0 \to G \to 0$ is another free resolution of the group G.

(a) Show that there is a commutative square

$$\begin{tikzcd} 0 \arrow[r] & F_1 \arrow[r, "\delta"] \arrow[d, "\beta"] & F_0 \arrow[r] \arrow[d, "\alpha"] & G \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & H_1 \arrow[r, "\epsilon"] & H_0 \arrow[r] & G \arrow[r] & 0. \end{tikzcd}$$

(b) Prove Theorem 19.44 by realizing the maps α, β, δ and ϵ as maps between wedges of spheres.

Exercise 19.50. Discuss the issues involved in defining $M(G,1)$.

Since they are defined as cofibers, Moore spaces are much more comfortable as domains, and we can describe the maps out of them without restricting to the stable range.

Proposition 19.51. *Show that there are natural exact sequences*

$$0 \to \mathrm{Ext}(G, \pi_{n+1}(X)) \longrightarrow [M(G,n), X] \longrightarrow \mathrm{Hom}(G, \pi_n(X)) \to 0.$$

Problem 19.52. Prove Proposition 19.51 by applying the functor $[\,?\,, X]$ to a cofiber sequence that defines $M(G,n)$.

Problem 19.53. Show that if X is $(n-1)$-connected, then X has an $(n+1)$-skeleton of the form $X_{n+1} = M(\pi_n(X), n) \vee \bigvee S^{n+1}$.

Homotopy classes of maps $f : M(G,n) \to K(G,n)$ can be identified with homomorphisms $\phi : G \to G$ using the diagram

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "\cong"'] & G \arrow[d, "\cong"] \\ \pi_n(M(G,n)) \arrow[r, "f_*"] & \pi_n(K(G,n)), \end{tikzcd}$$

in which the vertical maps are the canonical isomorphisms that come with the structure of a Moore space or an Eilenberg-Mac Lane space. We write $i : M(G,n) \to K(G,n)$ for the map corresponding to id_G.

Problem 19.54. Let $n \geq 2$ and $G \in \text{AB}\,\mathcal{G}$.

(a) Show that $i : M(G, n) \to K(G, n)$ is an $(n+1)$-skeleton.

(b) Show C_i is $(n+1)$-connected and $\pi_{n+2}(C_i) \cong G \otimes \pi_{n+2}(S^{n+1})$.

The groups $[M(G, n), X]$ are sometimes called the homotopy groups of X with **coefficients** in G and are denoted

$$\pi_{n+1}(X; G) = [M(G, n), X],$$

at least for G finite. Since, for fixed X, the groups in this sequence are contravariant functors of G, P. Hilton used the notation $\pi_{n+1}(G; X)$ for these groups and called G the **contraficients**; this didn't really catch on. The index shifting here has led a number of authors—including the eponymous Moore—to use the notation $P^{n+1}(G)$ for what we call $M(G, n)$ (at least for G finite); with this notation, $\pi_n(X; G) = [P^n(G), X]$.

19.5. Homotopy Groups of a Smash Product

Let X and Y be any two spaces. We can take any two elements $\alpha \in \pi_n(X)$ and $\beta \in \pi_m(Y)$ (so $\alpha : S^n \to X$ and $\beta : S^m \to Y$) and smash them together to give us a map

$$\alpha \wedge \beta : S^n \wedge S^m \longrightarrow X \wedge Y.$$

Using the standard homeomorphisms $S^n \wedge S^m \cong S^{n+m}$ that we established in Section 19.2.3, we view $\alpha \wedge \beta$ as an element of $\pi_{n+m}(X \wedge Y)$. We have defined a function

$$\pi_n(X) \times \pi_m(Y) \longrightarrow \pi_{n+m}(X \wedge Y),$$

which we call the **smash product pairing** of homotopy groups. This is a rule for multiplying an element of a group by an element from another group to get a product in a third group; such operations are called **pairings** or **external products**.

Problem 19.55. Show that the homotopy class of $\alpha \wedge \beta$ only depends on the homotopy classes of α and β and that the smash product is functorial in both variables.

Since we have not yet made any study of the algebraic properties of this map, it is—as far as we know—simply a map of pointed sets.

19.5.1. Algebraic Properties of the Smash Product. But the situation is quite a bit nicer than that: we will show that the exterior product is bilinear, associative and (graded) commutative. The bilinearity ultimately boils down to the distributivity law for smashes of maps over wedges of maps.

Problem 19.56.

(a) Let $\alpha : A \to X$, $\beta : B \to Y$ and $\gamma : C \to Z$. Then the diagram

$$\begin{tikzcd} A \wedge (B \vee C) \arrow[r, "\alpha\wedge(\beta\vee\gamma)"] \arrow[d, "\cong"'] & X \wedge (Y \vee Z) \arrow[d, "\cong"] \\ (A \wedge B) \vee (A \wedge C) \arrow[r, "(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)"] & (X \wedge Y) \vee (X \wedge Z) \end{tikzcd}$$

is commutative.

(b) Write down the analogous diagram for smashing on the right, and argue that it also commutes.

(c) Show that the exterior product is bilinear.

Since the tensor product is the universal object for bilinear maps, we have the following immediate consequence.

Proposition 19.57. *The smash product defines a natural group homomorphism*

$$\wedge : \pi_n(X) \otimes \pi_m(Y) \longrightarrow \pi_{n+m}(X \wedge Y).$$

The smash product operation is also commutative, in the graded sense.

Proposition 19.58. *For any X and Y, the diagram*

$$\begin{tikzcd} \pi_n(X) \otimes \pi_m(Y) \arrow[r, "\wedge"] \arrow[d, "(-1)^{nm}\tau"'] & \pi_{n+m}(X \wedge Y) \arrow[d, "T_*"] \\ \pi_m(Y) \otimes \pi_n(X) \arrow[r, "\wedge"] & \pi_{n+m}(Y \wedge X) \end{tikzcd}$$

commutes, where τ is the algebraic twist map given by $\tau(\alpha \otimes \beta) = \beta \otimes \alpha$.

Problem 19.59. Prove Proposition 19.58.

19.5.2. Nondegeneracy. So far, nothing we have said guarantees that this smash product pairing is nontrivial—it might be that $\alpha \wedge \beta \simeq *$ for every α and β as far as we know! But this is actually far from true—indeed, the tensor product models the smash product perfectly in low dimensions.

Proposition 19.60. *If X is $(n-1)$-connected and Y is $(m-1)$-connected, where $m, n \geq 1$,*[3] *then the smash product map*

$$\wedge : \pi_n(X) \otimes \pi_m(Y) \longrightarrow \pi_{n+m}(X \wedge Y)$$

is a natural isomorphism.

[3] If m or n is 1, we must assume that $\pi_n(X)$ and $\pi_m(Y)$ are abelian groups.

Problem 19.61.

(a) Show that if $X = S^n$, then $\wedge : \pi_n(X) \otimes \pi_m(Y) \to \pi_{m+n}(S^n \wedge Y)$ is simply the suspension homomorphism, and verify Proposition 19.60 in this case.

(b) Verify the proposition for $X = M(G, n)$.

HINT. Use the cofiber sequence that defines $M(G, n)$.

(c) Prove Proposition 19.60.

19.6. Smash Products of Eilenberg-Mac Lane Spaces

We use our study of smash products to build some maps involving Eilenberg-Mac Lane spaces. These maps will be used later to define a natural algebra structure on the cohomology groups of a space. Proposition 19.60 gives an isomorphism

$$\pi_{n+m}(K(G,n) \wedge K(H,m)) \cong G \otimes H$$

which we use to relate Eilenberg-Mac Lane spaces for G, H and $G \otimes H$.

Let $n, m \geq 1$ and let G and H be abelian groups. According to Problem 16.20, $K(G,n) \wedge K(H,m)$ is $(n+m-1)$-connected, and according to Proposition 19.60, $\pi_{n+m}(K(G,n) \wedge K(H,m)) \cong G \otimes H$. Theorem 17.44 ensures that there is a unique homotopy class

$$c : K(G,n) \wedge K(H,m) \longrightarrow K(G \otimes H, n+m)$$

such that the diagram

$$\begin{tikzcd}
\pi_{n+m}(K(G,n) \wedge K(H,m)) \arrow[r] & \pi_{n+m}K(G \otimes H, n+m) \arrow[d, "\cong"] \\
\pi_n(K(G,n)) \otimes \pi_m(K(H,m)) \arrow[u, "\wedge", "\cong"'] \arrow[r, "\cong"] & G \otimes H,
\end{tikzcd}$$

built by combining Proposition 19.60 with the isomorphisms $\pi_n(K(G,n)) \cong G$, $\pi_m(K(H,m)) \cong H$ and $\pi_{n+m}(K(G \otimes H, n+m)) \cong G \otimes H$ that come as part of the structure of Eilenberg-Mac Lane spaces, is commutative. Now let

$$T : K(G,n) \wedge K(H,m) \longrightarrow K(H,m) \wedge K(G,n)$$

be the twist map, and consider the diagram

$$\begin{tikzcd}
K(G,n) \wedge K(H,m) \arrow[r] \arrow[d, "T"'] & K(G \otimes H, n+m) \arrow[d, dashed, "t"] \\
K(H,m) \wedge K(G,n) \arrow[r] & K(H \otimes G, n+m).
\end{tikzcd}$$

Problem 19.62.

(a) Show that there is a unique map $t : K(G\otimes H, n+m) \to K(G\otimes H, n+m)$ making the diagram commute.

(b) According to Theorem 17.44, the map t corresponds to a homomorphism $\phi : G \otimes H \to H \otimes G$. Show that $\phi = (-1)^{nm}\tau$, where τ is the unsigned twist map $\tau(\alpha \otimes \beta) = \beta \otimes \alpha$.

Problem 19.63. Show that the maps c are associative, in the sense that the diagrams

$$\begin{tikzcd} K(G,n) \wedge K(H,m) \wedge K(J,p) \arrow[r] \arrow[d] & K(G \otimes H, n+m) \wedge K(J,p) \arrow[d] \\ K(G,n) \wedge L(H \otimes J, m+p) \arrow[r] & K(G \otimes H \otimes J, n+m+p) \end{tikzcd}$$

commute up to homotopy.

Disconnected Eilenberg-Mac Lane Spaces. First of all, if G is an abelian group, then we define $K(G, 0)$ to be the set G, with the discrete topology. Then $K(G, 0)$ is automatically a CW complex, and it is automatically a group object in $\mathrm{H}\mathcal{T}_*$, because it is a group!

Since $K(G, 0)$ is just a big wedge of copies of S^0, one for each nonidentity element of G, when we form the smash product $K(G, 0) \wedge K(H, n)$ we get

$$K(G,0) \wedge K(H,n) = \left(\bigvee_{g\neq 1\in G} S^0\right) \wedge K(H,m) = \bigvee_{g\neq 1\in G} K(H,m).$$

Applying π_m to this space yields $\bigoplus H$ and not $G \otimes H$.

Nevertheless, we would like to have a map

$$c : K(G,0) \wedge K(H,m) \longrightarrow K(G \otimes H, m).$$

Since we have $K(G, 0) \wedge K(H, m)$ written as a wedge, we just need to define c on each wedge summand—in other words, we need to define maps $K(H, m) \to K(G \otimes H, m)$, one for each nonidentity element $g \in G$. But the answer is right in front of us! If $g \in G$, then we can define

$$\phi_g : H \longrightarrow G \otimes H \quad \text{by the formula} \quad \phi_g(h) = g \otimes h.$$

Since

$$[K(G,n), K(G \otimes H, n)] \cong \mathrm{Hom}(G, G \otimes H),$$

there is a unique homotopy class $c_g \in [K(G, n), K(G \otimes H, n)]$ such that $(c_g)_* = \phi_g$. We define

$$c = (c_g \mid g \in G - \{1\}) : K(G,0) \wedge K(H,m) \longrightarrow K(G \otimes H, m).$$

This is the map we want.

Exercise 19.64. Show that the same construction works equally well to define a nice map $K(G,n) \wedge K(H,0) \to K(G \otimes H, n)$.

Ring Structures and Eilenberg-Mac Lane Spaces. Now let R be a ring. Since the multiplication $R \times R \to R$ is bilinear, it defines a new map from the tensor product, $R \otimes R \to R$, which corresponds to a homotopy class

$$m : K(R \otimes R, n) \longrightarrow K(R, n).$$

Using the map m, we define a family of maps

$$\mu_{m,n} : K(R,m) \wedge K(R,n) \longrightarrow K(R, n+m).$$

Problem 19.65. Show that the maps $\mu_{m,n}$ are associative and commutative, in the sense that the diagrams

$$\begin{tikzcd}
K(R,m) \wedge K(R,n) \wedge K(R,p) \arrow[r, "\mu_{m,n}\wedge \mathrm{id}"] \arrow[d, "\mathrm{id}\wedge\mu_{n,p}"'] & K(R,n+m) \wedge K(R,p) \arrow[d, "\mu_{m+n,p}"] \\
K(R,m) \wedge K(R,n+p) \arrow[r, "\mu_{m,n+p}"] & K(R,n+m+p)
\end{tikzcd}$$

and

$$\begin{tikzcd}
K(R,m) \wedge K(R,n) \arrow[r, "T"] \arrow[d, "\mu_{m,n}"'] & K(R,n) \wedge K(R,m) \arrow[d, "\mu_{n,m}"] \\
K(R,n+m) \arrow[r, "(-1)^{nm}"] & K(R,n+m)
\end{tikzcd}$$

commute up to homotopy.

The dimension zero deserves a bit of careful study. Write

$$K(R,0) \wedge K(R,n) \xrightarrow{c} K(R \otimes R, n) \xrightarrow{m} K(R,n).$$

Since $K(R,0) \wedge K(R,n) = \bigvee_{r \neq 0 \in R} K(R,n)$, we just need to determine the restriction of μ to the r^{th} summand.

Problem 19.66. If $r \in R$, then multiplication by r defines a homomorphism $\phi_r : R \to R$, and hence a map $f_r : K(R,n) \to K(R,n)$. Show that the composite

$$K(R,n) \xrightarrow{\mathrm{in}_r} K(R,0) \wedge K(R,n) \xrightarrow{c} K(R \otimes R, n) \xrightarrow{m} K(R,n)$$

is homotopic to f_r.

Finally, we mention that if R has a multiplicative identity element, then $\phi_1 = \mathrm{id}$ and so $f_1 \simeq \mathrm{id}$, which means that the diagram

$$\begin{tikzcd} K(R,n) \arrow[r, "\mathrm{in}_1"] \arrow[dr, "\mathrm{id}"'] & K(R,0) \wedge K(R,n) \arrow[d, "m \circ c"] \\ & K(R,n) \end{tikzcd}$$

is homotopy commutative.

19.7. An Additional Topic and Some Problems

19.7.1. Smashing Moore Spaces. In Problem 19.38 you built a noncontractible space X with $S^1 \wedge X \simeq *$. The construction depended on strange nonabelian features of the fundamental group $\pi_1(X)$. Since things seem to behave much more nicely with simply-connected spaces (the homotopy groups are abelian, for instance), it might seem reasonable to guess that if X and Y are simply-connected and not contractible, then $X \wedge Y$ is not contractible either; but simply-connected spaces *can* have a contractible smash product.

Problem 19.67. Let $N = M(\mathbb{Z}/p, n)$ and $M = M(\mathbb{Z}/q, m)$, where p and q are two distinct prime numbers.

(a) Since M is a suspension, $[M, M]$ has a natural group structure. Show that $(p \cdot \mathrm{id}_{S^n}) \wedge \mathrm{id}_M$ is homotopic to $p \cdot \mathrm{id}_{\Sigma^n M}$.

(b) Show that if $k \leq 2(n+m) - 1$, then the map $\pi_k(\Sigma^n M) \to \pi_k(\Sigma^n M)$ induced by $p \cdot \mathrm{id}_{\Sigma^n M}$ is given by multiplication by p. How much of an equivalence is it?

(c) Show that $N \wedge M \simeq *$, so that

$$\mathrm{conn}(N \wedge M) > \mathrm{conn}(N) + \mathrm{conn}(M) + 1.$$

Later we will see that, for simply-connected spaces, this kind of algebraic incompatibility is the *only* way to have a contractible smash product.

Problem 19.68. Is $M(G, n) \wedge M(H, m) \simeq M(G \otimes H, n + m)$?

Problem 19.69. Give an example of a sequence $A \to B \to C$ which is simultaneously a fiber sequence and a cofiber sequence.

19.7.2. Problems.

Problem 19.70. Let G be finitely generated. Show that $M(G, n)$ splits as a wedge.

Problem 19.71. Suppose two of the three spaces in the cofiber sequence $A \to B \to C$ are n-connected. What can you say about the connectivity of the third?

Problem 19.72. Identify $S^n = S^1 \wedge S^1 \wedge \cdots \wedge S^1$, and let $\sigma \in \mathrm{Sym}(n)$ be a permutation. Let $f_\sigma : S^n \to S^n$ be the map that applies the permutation σ to the smash coordinates of S^n. Show that $\deg(f_\sigma) = \mathrm{sign}(\sigma)$.

Problem 19.73. For this problem, take for granted that $\pi_{n+1}(S^n) = \mathbb{Z}/2$ for all $n \geq 3$ (this is proved in Problem 25.40). Let $M = M(\mathbb{Z}/p^r, n)$ where p is an odd prime; write m for the order of the element id_M in the group $[M, M]$.

(a) Show that m must be a power of p.

(b) Show that $\mathrm{order}(\alpha) \leq m$ for all $\alpha \in \pi_n(M)$.

(c) Show that $p^r \mathrm{id}_M = 0$ factors through S^{n+1} and conclude $p^r \mathrm{id}_M = 0$.

(d) What is m?

Problem 19.74. Show that the abelian group $\left[\bigvee_{\mathcal{I}} S^n, \bigvee_{\mathcal{J}} S^n\right]$ is a module over each of the rings $\left[\bigvee_{\mathcal{I}} S^n, \bigvee_{\mathcal{I}} S^n\right]$ and $\left[\bigvee_{\mathcal{J}} S^n, \bigvee_{\mathcal{J}} S^n\right]$. Interpret this in terms of matrices.

Problem 19.75. Evaluate $[M(\mathbb{Z}/a, n), K(G, n+1)]$.

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Chapter 20

Further Topics

To begin this chapter we give the long-promised example of a prepushout diagram which has no pushout in the homotopy category. Then we study certain very tightly controlled (infinite) cone decompositions with respect to Moore spaces; these are roughly dual to Postnikov decompositions. We also give a brief overview of the theory of infinite symmetric products.

But the biggest surprise is that the dimension n in which a (simply-connected) space X first has nontrivial p-torsion (or elements of infinite order) is a *stable* invariant of the space. Thus ΣX has its first p-torsion in dimension $n+1$, even if these groups are far outside the range in which the Freudenthal Suspension Theorem applies. Even better, the suspension $\Sigma : \pi_n(X) \to \pi_{n+1}(\Sigma X)$ induces an isomorphism of the p-torsion.

20.1. The Homotopy Category Is Not Complete

In explaining the need for the homotopy colimit, we argued that we could not fall back on the categorical colimit in the homotopy category, because there are diagrams in $\mathrm{H}\mathcal{T}_*$ without colimits. In this section we explicitly construct a prepushout diagram in $\mathrm{H}\mathcal{T}_*$ with no pushout.

Problem 20.1. A prepushout diagram $C \leftarrow A \to B$ in $\mathrm{H}\mathcal{T}_*$ is the image of many prepushout diagrams in $C \leftarrow A \to B$ in $\mathcal{T}_*$. Show that all of these preimage diagrams have homotopy equivalent homotopy pushouts.

Because of Problem 20.1, we can meaningfully talk about the **homotopy pushout** of a prepushout diagram in $\mathrm{H}\mathcal{T}_*$.

Problem 20.2. Let $C \leftarrow A \to B$ be a prepushout diagram in $\mathrm{H}\mathcal{T}_*$ with homotopy pushout D, and suppose it has a categorical pushout P.

(a) Show there is a comparison map $\xi : P \to D$ in $\mathrm{H}\mathcal{T}_*$.

(b) Show that i has a left homotopy inverse $r : D \to P$, so that P is a retract of D in $\mathrm{H}\mathcal{T}_*$.

Now we use our (very limited) knowledge of homotopy groups to get some control over retracts of spheres.

Problem 20.3. Suppose P is a homotopy retract of S^n, so that there are maps $r : S^n \to P$ and $j : P \to S^n$ with $r \circ j \simeq \mathrm{id}_P$.

(a) Show that $e = j \circ r$ is idempotent (i.e., $e \circ e \simeq e$).

(b) What can you say about the degree of $j \circ r$?

(c) Show that either $P \simeq *$ or $P \simeq S^n$.

Our example involves Moore spaces.

Problem 20.4. Let $d \in \mathbb{N}$ with $d \geq 2$, and consider the long cofiber sequence

$$\cdots \to S^n \xrightarrow{\mathbf{d}} S^n \xrightarrow{j} M(\mathbb{Z}/d, n) \xrightarrow{\partial} S^{n+1} \to \cdots$$

in which $\mathbf{d}$ is of degree d.

(a) Show $M(\mathbb{Z}/d, n) \not\simeq *$.

(b) Show that the induced map $\partial^* : [S^{n+1}, S^{n+1}] \to [M(\mathbb{Z}/d, n), S^{n+1}]$ is nontrivial but not injective.

(c) Show that the prepushout diagram $* \leftarrow S^n \xrightarrow{j} M(\mathbb{Z}/d, n)$ has no pushout in $\mathrm{H}\mathcal{T}_*$.

20.2. Cone Decompositions with Respect to Moore Spaces

We think of CW decompositions and Postnikov decompositions as being roughly dual to one another, but there is a significant difference: the Postnikov analysis of a space attaches (or removes) one group at a time, dimension-by-dimension. But in our CW constructions, we construct each group in two steps: first we have a map which is surjective on the homotopy (or homology) groups, and then we attach more disks to render it bijective. In this section we'll describe a domain-type construction that works one group at a time, provided X is simply-connected. The trade-off is that the groups in question are not the homotopy groups of X.

Write $G_2 = \pi_2(X)$ and $X_{(2)} = M(G_2, 2)$. If X is simply-connected, then we can find a map $X_{(2)} \to X$ which induces an isomorphism on π_k for $k \leq 2$.

Theorem 20.5. *If X is a simply-connected CW complex, then the map $X_{(2)} \to X$ has an infinite cone decomposition*

$$\begin{tikzcd} M(G_3,2) \arrow[d] & M(G_4,3) \arrow[d] & & M(G_{n+1},n) \arrow[d] & & \\ X_{(2)} \arrow[r] & X_{(3)} \arrow[r] & \cdots \arrow[r] & X_{(n)} \arrow[r] & \cdots \arrow[r] & X \end{tikzcd}$$

with respect to the collection of Moore spaces.

Suppose we have constructed the n-equivalence $X_{(n)} \to X$; then let F be its homotopy fiber and write $G_{n+1} = \pi_n(F)$.

Problem 20.6.

(a) Show there is a map $M(G_{n+1}, n) \to F$ inducing an isomorphism on $\pi_n(\,?\,)$.

(b) Let $X_{(n+1)}$ be the cofiber of the composite $M(G_{n+1}, n) \to F \to X_{(n)}$. Extend the square

$$\begin{tikzcd} M(G_{n+1},n) \arrow[r, equal] \arrow[d] & M(G_{n+1},n) \arrow[d] \\ F \arrow[r] & X_{(n)}, \end{tikzcd}$$

by taking cofibers, and use the resulting diagram to show that $X_{(n)} \to X$ extends to an $(n+1)$-equivalence $X_{(n+1)} \to X$.

(c) Finish the proof of Theorem 20.5, and discuss the naturality of the construction.

A Moore Space Decomposition of $B\mathbb{Z}/m$. Because they are not simply connected, the classifying spaces $B\mathbb{Z}/m$ cannot have Moore space cone decompositions as we understand them, but they are very closely related to Moore spaces.

Problem 20.7. Show that there is a cofiber sequence of the form $S^1 \cup_n D^2 \to B\mathbb{Z}/m \to X$ in which X has a cone decomposition

$$\begin{tikzcd} M(\mathbb{Z}/n,2) \arrow[d] & M(\mathbb{Z}/n,4) \arrow[d] & & M(\mathbb{Z}/n,2n) \arrow[d] & & \\ * \arrow[r] & X_4 \arrow[r] & \cdots \arrow[r] & X_{2n} \arrow[r] & \cdots & X. \end{tikzcd}$$

Hint. Refer to the construction of Section 16.5.3.

20.3. First p-Torsion Is a Stable Invariant

Let $\mathcal{P}$ be a set of prime numbers and suppose (for simplicity) that X is a space each of whose homotopy groups is finite. Then we can ask: what is the first dimension n for which $\pi_n(X)$ has nontrivial $\mathcal{P}$-torsion? If this dimension is c, then we say that the **$\mathcal{P}$-connectivity** of X is $c-1$ and we write $\mathrm{conn}_{\mathcal{P}}(X) = c-1$. If we were only able to detect $\mathcal{P}$-torsion, this would be our best estimate of the actual connectivity of X. If $\mathrm{conn}_{\mathcal{P}}(X)$ happens to be in the stable range for X, the Freudenthal Suspension Theorem guarantees that $\mathrm{conn}_{\P}(\Sigma X) = \mathrm{conn}_{\P}(X) + 1$.

In a stunning turn of events, this relationship holds for all simply-connected spaces! Not only that, $\mathcal{P}$-connectivity behaves just like ordinary connectivity with respect to smash products, and hence the same is true for joins. It follows easily that the main elements of the apparatus of homotopy theory—such as the Blakers-Massey theorem and its dual, the Freudenthal Suspension Theorem, the comparison between fiber and cofiber and so on—have extremely faithful 'mod $\mathcal{P}$' counterparts.

These amazing facts are usually proved in the way they were originally discovered: using the extremely powerful and intricate machinery of spectral sequences and Serre classes of abelian groups. We'll prove them here using much simpler ideas; later we'll develop the Serre approach to prove other qualitative results about homotopy groups.

For the whole section, we work entirely with abelian groups and with a fixed partition of the prime numbers into disjoint subsets $\mathcal{P}$ and $\mathcal{Q}$.

20.3.1. Setting Up. We call an abelian group G a **$\mathcal{P}$-group** if every element $g \in G$ has finite order divisible only by the primes in $\mathcal{P}$.

We need a fact from pure algebra.

Lemma 20.8. *Let p be a prime number. If G is an abelian group with an element $g \in G$ of infinite order, then there is a nontrivial map $G \to P$ for some p-group P.*

You may take this for granted for now, but you are asked to prove it in Problem A.45 in Appendix A.

Exercise 20.9. Prove Lemma 20.8.

The connections between homotopy properties and the algebraic conditions depend on the following lemma.

Lemma 20.10. *If H is a finite $\mathcal{Q}$-group and P is a $\mathcal{P}$-group, then*

$$\mathrm{map}_*(K(H,m), K(P,n)) \sim *$$

for all m and n.

Problem 20.11.

(a) Show that it suffices to prove Lemma 20.10 in the special case $m = 1$ and $H = \mathbb{Z}/q^r$ with $q \in \mathcal{Q}$.

(b) Show that $\mathrm{map}_*(S^1 \cup_{q^r} D^2, K(P, n)) \sim *$ for all n.

(c) Let X be the cofiber of the map $S^1 \cup_{q^r} D^2 \to B\mathbb{Z}/q^r$ defined in Problem 20.7. Show that $\mathrm{map}_*(X, K(P, n)) \sim *$, and complete the proof of Lemma 20.10.

We replace 'having no $\mathcal{P}$-torsion' with 'maps trivially to all $\mathcal{P}$-groups'. We begin with a long list of equivalent conditions.

Proposition 20.12. *The following are equivalent:*

(1) $\mathrm{Hom}(H, P) = 0$ *for all $\mathcal{P}$-groups P,*

(2) H is a $\mathcal{Q}$-group,

(3) $\mathrm{map}_*(BK, K(P, n)) \sim *$ *for all n, all finite subgroups $K \subseteq H$ and all $\mathcal{P}$-groups P,*

(4) $\mathrm{map}_*(BH, K(P, n)) \sim *$ *for all n and all abelian $\mathcal{P}$-groups P,*

(5) $\mathrm{map}_*(K(H, m), K(P, n)) \sim *$ *for all n, m and all $\mathcal{P}$-groups P.*

Problem 20.13. Prove Proposition 20.12.

HINT. Use the results of Section 16.5.4 to prove that (3) implies (4) and the Zabrodsky lemma to prove that (4) implies (5).

20.3.2. Connectivity with Respect to $\mathcal{P}$. We say that a path-connected space X has **$\mathcal{P}$-connectivity** $\mathrm{conn}_{\mathcal{P}}(X) \geq n$ if and only if $\pi_k(X)$ is a $\mathcal{Q}$-group for all $k \leq n$.

Exercise 20.14. What does $\mathcal{P}$-connectivity mean in the extreme cases $\mathcal{P} = \varnothing$ and $\mathcal{P} = \{\text{all primes}\}$?

We have made much use of the fact that for $(n-1)$-connected spaces X, homotopy classes of maps $X \to K(G, n)$ are perfectly modeled by their induced maps on homotopy groups. Now we show that if $\mathrm{conn}_{\mathcal{P}}(X) = n-1$, the homotopy groups perfectly describe the homotopy classes of maps $X \to K(P, n)$, where P is a $\mathcal{P}$-group.

Theorem 20.15. *If X is simply-connected and* $\mathrm{conn}_{\mathcal{P}}(X) \geq n - 1$, *then for every $\mathcal{P}$-group P, the map*

$$\phi : [X, K(P, n)] \longrightarrow \mathrm{Hom}(\pi_n(X), P) \qquad \textit{given by } f \mapsto f_*$$

is a group isomorphism.

Thus the essentially algebraic notion of $\mathcal{P}$-connectivity can be understood in homotopy-theoretical terms.

Corollary 20.16. *If* $\operatorname{conn}_{\mathcal{P}}(X) \geq n-1$*, then the following are equivalent:*

(1) $\operatorname{conn}_{\mathcal{P}}(X) = n-1$,

(2) $[X, K(P,n)] \neq *$ *for some* $\mathcal{P}$*-group* P.

Our proof of Theorem 20.15 uses the Postnikov system of X, which gives us a fibration sequence $\Omega P_{n-1}X \to X\langle n-1\rangle \to X \to P_{n-1}(X)$.

Problem 20.17. Suppose $\operatorname{conn}_{\mathcal{P}}(X) \geq n-1$.

(a) Show that if Z has finitely many nonzero homotopy groups, all of which are $\mathcal{Q}$-groups, then $\operatorname{map}_*(Z, K(P,m)) \simeq *$ for all $\mathcal{P}$-groups P and all $m \geq 1$.

(b) Show that if $\operatorname{conn}_{\mathcal{P}}(X) \geq n-1$, then the induced map

$$\operatorname{map}_*(X\langle n-1\rangle, K(P,m)) \longrightarrow \operatorname{map}_*(X, K(P,m))$$

is a weak homotopy equivalence for any $\mathcal{P}$-group P and any $m \geq 1$.

(c) Prove Theorem 20.15.

(d) Derive Corollary 20.16.

Hint. Use the Zabrodsky lemma.

Corollary 20.16 implies the stability of $\mathcal{P}$-connectivity.

Theorem 20.18. *If* X *is simply-connected, then*

$$\operatorname{conn}_{\mathcal{P}}(\Sigma X) = \operatorname{conn}_{\mathcal{P}}(X) + 1.$$

Problem 20.19.

(a) Show that $[X, K(G,n)] \cong [\Sigma X, K(G,n+1)]$ for any abelian group G.

(b) Prove Theorem 20.18.

The stability of the first $\mathcal{P}$-torsion implies that the $\mathcal{P}$-connectivity of the spaces in a cofiber sequence are related in the same way as the ordinary connectivities.

Problem 20.20. Let $A \to B \to C$ be a cofiber sequence of simply-connected CW complexes.

(a) Suppose you know the $\mathcal{P}$-connectivities of two of the three spaces in the sequence. What can you say about the third?

(b) Show that if $\operatorname{conn}_{\mathcal{P}}(A) = \infty$, then $\operatorname{conn}_{\mathcal{P}}(B) = \operatorname{conn}_{\mathcal{P}}(C)$.

Finally we see that infinite $\mathcal{P}$-connectivity implies the weak contractibility of a great many mapping spaces.

Corollary 20.21. *The following are equivalent:*

(1) $\operatorname{conn}_{\mathcal{P}}(X) = \infty$,

(2) $\mathrm{map}_*(X, K(P, n)) \sim *$ *for all* $P \in \mathcal{P}$ *and all* $n \geq 1$.

Problem 20.22.

(a) Write down $\pi_k(\mathrm{map}_*(X, K(P, n)))$ as a homotopy set $[\,?\,,\,?\,]$.

(b) Prove Corollary 20.21.

20.3.3. $\mathcal{P}$-Connectivity and Moore Spaces. In this section, we study the $\mathcal{P}$-connectivity of Moore spaces and apply our results to characterize $\mathrm{conn}_{\mathcal{P}}(X)$ in terms of the groups G_n that appear in the cone decompositions of Section 20.2.

Theorem 20.23. *Let* $n \geq 2$. *The following are equivalent:*

(1) G *is a* $\mathcal{Q}$*-group,*

(2) $\mathrm{conn}_{\mathcal{P}}(M(G, n)) = \infty$.

Since (2) obviously implies (1), we focus our attention on the converse. You'll prove that (1) implies (2) by relating the Moore space to an Eilenberg-Mac Lane space using the map $i_n : M(G, n) \to K(G, n)$ that you studied in Problem 19.54.

Problem 20.24. Assume G is a $\mathcal{Q}$-group.

(a) Show that $\mathrm{conn}_{\mathcal{P}}(M(G, n+1)) = \mathrm{conn}_{\mathcal{P}}(M(G, n)) + 1$. Conclude that it suffices to prove Theorem 20.23 for n large.

(b) Show that $\mathrm{conn}_{\mathcal{P}}(M(G, n))$ must be either n, $n + 1$ or ∞.

(c) Show that $\mathrm{conn}_{\mathcal{P}}(M(G, n)) \neq n$.

(d) Show that $[M(G, n), K(P, n + 1)] \cong [C_{i_n}, K(P, n + 2)]$.

(e) Prove Theorem 20.23.

Hint. If Q is a $\mathcal{Q}$-group, then so is $Q \otimes G$ for an abelian group G.

Problem 20.25. Show that if G is a torsion group, then each homotopy group $\pi_k(M(G, n))$ is also a torsion group.

Now we relate the $\mathcal{P}$-connectivity of X to the mysterious groups G_n that appear in the cone decomposition

$$\begin{tikzcd}
M(G_3,2) \arrow[d] & M(G_3,2) \arrow[d] & & M(G_n,n-1) \arrow[d] & & \\
X(2) \arrow[r] & X(3) \arrow[r] & \cdots \arrow[r] & X(n) \arrow[r] & \cdots \arrow[r] & X
\end{tikzcd}$$

of Section 20.2.[1] Strangely, the first n for which $\pi_n(X)$ is not a $\mathcal{Q}$-group is the same as the first n for which G_n is not a $\mathcal{Q}$-group.

[1]Recall that $X(2) = M(G_2, 2)$.

Theorem 20.26. *If X is simply-connected, then*

$$\mathrm{conn}_{\mathcal{P}}(X) = \min\{n \mid G_n \notin \mathcal{C}_{\mathcal{P}}\} - 1.$$

To prove this, you'll show by induction that $\mathrm{conn}_{\mathcal{P}}(X) \geq n$ if and only if G_k is a $\mathcal{Q}$-group for all $k \leq n$.

Problem 20.27. Show that $\mathrm{conn}_{\mathcal{P}}(X) \geq 2$ if and only if G_2 is a $\mathcal{Q}$-group.

The inductive step is the meat of the proof.

Problem 20.28. Suppose $\mathrm{conn}_{\mathcal{P}}(X) \geq n-1$ and that G_k is a $\mathcal{Q}$-group for all $k < n$.

(a) Determine $\mathrm{conn}_{\mathcal{P}}(X(n-1))$.

(b) Let C be the cofiber of $X(n-1) \to X$; show that $\mathrm{conn}_{\mathcal{P}}(X) = \mathrm{conn}_{\mathcal{P}}(C)$.

(c) Show that $\mathrm{conn}_{\mathcal{P}}(X) = n-1$ if and only if G_n is not a $\mathcal{Q}$-group.

(d) Prove Theorem 20.26.

Problem 20.29.

(a) Show that if X is a homotopy retract of Y, then $\mathrm{conn}_{\mathcal{P}}(X) \leq \mathrm{conn}_{\mathcal{P}}(Y)$.

(b) Show that if $\mathrm{conn}_{\mathcal{P}}(X) > \dim(X)$, then $\mathrm{conn}_{\mathcal{P}}(X) = \infty$.

20.3.4. The First $\mathcal{P}$-Torsion of a Smash Product. Next we determine the effect of the smash product on $\mathcal{P}$-connectivity. There is a wrinkle here: if there were two primes in $\mathcal{P}$, then we could have one space X whose homotopy groups were p_1-torsion, and another space Y with p_2-torsion homotopy groups. Then $X \wedge Y \simeq *$ so $\mathrm{conn}_{\mathcal{P}}(X \wedge Y) = \infty$. Thus $\mathcal{P}$-connectivity of a smash product is governed by an inequality.

Problem 20.30. Show that if $\mathrm{conn}_{\mathcal{P}}(X) = \infty$, then for any other space Y,

$$\mathrm{conn}_{\mathcal{P}}(X \wedge Y) = \infty$$

as well.

That is, if $\pi_*(X)$ has no $\mathcal{P}$-torsion and no elements of infinite order, then neither does $\pi_*(X \wedge Y)$.

Theorem 20.31. *If X and Y are simply-connected, then*

$$\mathrm{conn}_{\mathcal{P}}(X \wedge Y) \geq \mathrm{conn}_{\mathcal{P}}(X) + \mathrm{conn}_{\mathcal{P}}(Y) + 1.$$

Corollary 20.32. *If X and Y are simply-connected, then*

$$\mathrm{conn}_{\mathcal{P}}(X * Y) \geq \mathrm{conn}_{\mathcal{P}}(X) + \mathrm{conn}_{\mathcal{P}}(Y) + 2.$$

Exercise 20.33. Show by example that the inequality in Theorem 20.31 cannot be improved to equality in general. Can you find conditions on the homotopy groups of X and Y that would ensure equality?

Start with Moore space cone decompositions of X and Y, and smash them together to give a decomposition of $Z = X \wedge Y$ as in Section 9.4.2.

Problem 20.34. Suppose $\mathrm{conn}_{\mathcal{P}}(X) = n - 1$ and $\mathrm{conn}_{\mathcal{P}}(Y) = m - 1$.

(a) What is the first k for which $\mathrm{conn}_{\mathcal{P}}(Z(k)) \neq \infty$?

(b) Prove Theorem 20.31 and derive Corollary 20.32.

20.3.5. $\mathcal{P}$-Local Homotopy Theory. Some of the most powerful tools in homotopy theory are the theorems that compare domain-type objects to target-type objects, usually by estimating the connectivity of some kind of comparison map. Their proofs usually came in two steps: first, finding a geometric approximation of the fiber of the comparison map, often by some kind of join; and then an estimate of the connectivity of the fiber, generally following from the known behavior of connectivity with respect to join. Since the $\mathcal{P}$-connectivity of $X * Y$ is related to the $\mathcal{P}$-connectivities of X and Y in the same way as ordinary connectivity, most of these results have '$\mathcal{P}$-local' counterparts which can be proved by slight modifications of existing proofs.

To articulate these theorems, though, we need to establish a notion of n-equivalences modulo $\mathcal{Q}$. A homomorphism $f : G \to H$ of abelian groups is **$\mathcal{P}$-surjective** if its cokernel is a $\mathcal{Q}$-group; f is **$\mathcal{P}$-injective** if its kernel is a $\mathcal{Q}$-group; and it is a $\mathcal{P}$-isomorphism if it is both. A map $f : X \to Y$ of simply-connected spaces is an **n-equivalence modulo $\mathcal{Q}$** (or **at** $\mathcal{P}$) if the induced map

$$f_* : \pi_k(X) \longrightarrow \pi_k(Y)$$

is a $\mathcal{P}$-isomorphism if $k < n$ and a $\mathcal{P}$-surjection if $k = n$.

Problem 20.35. Let X and Y be simply-connected. Show that $f : X \to Y$ is an n-equivalence modulo $\mathcal{Q}$ if and only if the homotopy fiber F satisfies $\mathrm{conn}_{\mathcal{P}}(F) \geq n$.

The rest of this section is devoted to a brief overview of the $\mathcal{P}$-local analogs of the results in Chapter 18.

Homotopy Pushouts and Pullbacks Modulo $\mathcal{Q}$. In any strictly commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "\beta"] \\ C \arrow[r, "\gamma"] & D, \end{tikzcd}$$

we may construct a comparison map $\xi : Q \to D$ from the homotopy pushout Q of $C \leftarrow A \to B$. We estimate the connectivity of ξ modulo $\mathcal{Q}$.

Theorem 20.36. *Assume the square is a strong homotopy pullback square and that β is a $(b-1)$-equivalence modulo $\mathcal{Q}$ and γ is a $(c-1)$-equivalence modulo $\mathcal{Q}$. Then $\xi : Q \to D$ is a $(b+c-1)$-equivalence modulo $\mathcal{Q}$.*

Problem 20.37. Prove Theorem 20.36.

Comparing Fibers and Cofibers Modulo $\mathcal{Q}$. One of the triumphs of our work in Chapter 18 was to show that the connectivity of a map of simply-connected spaces could be determined by the connectivity of the cofiber. This remains true modulo $\mathcal{P}$.

We can detect the $\mathcal{P}$-connectivity of a map by studying either its fiber or its cofiber.

Problem 20.38. Let $A \to B \to C$ be a cofiber sequence of simply-connected spaces, and let F be the homotopy fiber of $B \to C$.

(a) Show that F is simply-connected.
(b) Show that $\mathrm{conn}_{\mathcal{P}}(F) = \mathrm{conn}_{\mathcal{P}}(A)$.

Hint. Use Theorem 18.18.

Theorem 20.39. *Let $f : X \to Y$ be a map of simply-connected spaces with fiber F and cofiber C. If F is also simply-connected, then*

$$\mathrm{conn}_{\mathcal{P}}(C) = \mathrm{conn}_{\mathcal{P}}(F) + 1.$$

Problem 20.40.

(a) Show that Theorem 20.39 is true if f is a principal cofibration: there is a map $A \to X$ such that $A \to X \xrightarrow{f} Y$ is a cofiber sequence.
(b) Show that if f has a finite length cone decomposition $\{A_k \to X_k \to X_{k+1}\}$ with $\mathrm{conn}_{\mathcal{P}}(A_k) < \mathrm{conn}_{\mathcal{P}}(A_{k+1})$ for all k, then the same is true.
(c) Use the Ganea construction to show that any map $f : X \to Y$ has a factorization $X \to G \to Y$ in which the first map may be handled by (b) and the second map is as highly connected as you like.
(d) Prove Theorem 20.39 in full generality.

Corollary 20.41. *If $f : X \to Y$ is a map of simply-connected spaces, then* $\mathrm{conn}_{\mathcal{P}}(\Sigma f) = \mathrm{conn}_{\mathcal{P}}(f) + 1$.

The Blakers-Massey Theorem Modulo $\mathcal{Q}$. Now we start with a strong homotopy pushout square

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\gamma"] & D \end{tikzcd}$$

and construct a comparison map $\zeta : A \to P$, where P is the homotopy pullback of $C \to D \leftarrow B$.

Theorem 20.42. *If i is a $(b-1)$ equivalence modulo $\mathcal{Q}$ and j is a $(c-1)$-equivalence modulo $\mathcal{Q}$, then ζ is a $(b+c-1)$-equivalence modulo $\mathcal{Q}$.*

Problem 20.43. Prove Theorem 20.42 by adapting the method of proof of Theorem 18.13.

Freudenthal Suspension Modulo $\mathcal{Q}$. We know that if $\pi_n(X)$ is the lowest homotopy group that is not a $\mathcal{Q}$-group, then $\pi_{n+1}(\Sigma X)$ is the lowest homotopy group of ΣX that is not a $\mathcal{Q}$-group. But we have not yet established any relationship between these two groups. Now we'll show that the suspension map $\Sigma : \pi_n(X) \to \pi_{n+1}(\Sigma X)$ is an isomorphism modulo $\mathcal{Q}$.

Theorem 20.44. *If X is simply-connected and $\mathrm{conn}_{\mathcal{P}}(X) = n-1$, then the suspension map $\Sigma : X \to \Omega\Sigma X$ is a $(2n-1)$-equivalence modulo $\mathcal{Q}$.*

We'll prove this by estimating the connectivity of the cofiber of Σ. The key to our proof, not surprisingly, is the James construction.

Problem 20.45. Write $J^k = J^k(X)$ for the k^{th} stage in the construction of $J = J(X)$.

(a) Estimate the $\mathcal{P}$-connectivity of the cofiber of the inclusion $J^k \hookrightarrow J^{k+1}$.

(b) Show that if k is large enough, then the composite $J^k \to J \to \Omega\Sigma X$ is an ordinary $2n$-equivalence, and hence a $2n$-equivalence modulo $\mathcal{Q}$.

(c) Prove Theorem 20.44.

20.4. Hopf Invariants and Lusternik-Schnirelmann Category

In Section 9.7.3 we showed that the Lusternik-Schnirelmann category of a space X is the least n for which the n^{th} Ganea fibration $p_n : G_n(X) \to X$ has a section. We also showed in Problem 9.121 that certain maps h involving the fibers of the Ganea fibrations could give insight to the relation between the category of X and the category of the cofiber C_α of a map $\alpha : A \to X$. In this section we define a specific construction of such maps.

20.4.1. Berstein-Hilton Hopf Invariants. First of all, we have shown that the first Ganea fibration $p_1 : G_1(X) \to X$ may be identified with the evaluation map $@ : \Sigma\Omega X \to X$. Since every suspension has category 1, this map has a section if $X = \Sigma A$. Among the collection of all possible sections, there is one preferred choice, namely the map

$$\sigma_{\Sigma A} : \Sigma A \longrightarrow \Sigma\Omega\Sigma A \qquad \text{given by} \qquad [a,t] \mapsto [\omega_a, t],$$

where $\omega_a : I \to \Sigma A$ is given by $t \mapsto [a,t]$.

Now let $\alpha : \Sigma A \to X$ and consider the cofiber C_α. If $\mathrm{cat}(X) = n$, then $\mathrm{cat}(C_\alpha) \leq n + 1$, and we want to know how to tell whether or not $\mathrm{cat}(C_\alpha) \leq n$. Form the diagram

$$\begin{tikzcd}
G_1(\Sigma A) \arrow[r, "G_1(\alpha)"] \arrow[dd] & G_1(X) \arrow[rr, "G_1(j)"] \arrow[dd] \arrow[dr, "i_n"] & & G_1(C_\alpha) \arrow[dd] \arrow[dr, "i_n"] & \\
& & G_n(X) \arrow[rr, "G_n(j)"] \arrow[dd] & & G_n(C_\alpha) \arrow[dd] \\
A \arrow[r, "\alpha"] \arrow[uu, bend left, "\sigma_{\Sigma A}"] & X \arrow[rr, "j"] \arrow[dr, equal] & & C_\alpha \arrow[dr, equal] & \\
& & X \arrow[rr, "j"] \arrow[uu, bend left, "\sigma"] & & C_\alpha .
\end{tikzcd}$$

There can be many choices for σ; each one gives rise to a map

$$\delta_\sigma(\alpha) = (\sigma \circ \alpha) - (i_n \circ G_1(\alpha) \circ \sigma_{\Sigma A}).$$

Problem 20.46.

(a) Show that for any space Z and any $n \geq 1$, the map $p_n : G_n(Z) \to Z$ splits after looping; that is, Ωp_n has a section.

(b) Show that there is a unique map $H_\sigma(\alpha) : \Sigma A \to F_n(X)$ so that the composite $\Sigma A \xrightarrow{H_\sigma(\alpha)} F_n(X) \to G_n(X)$ is $\delta_\sigma(\alpha)$.

The map $H_\sigma(\alpha)$ is called a Berstein-Hilton **Hopf invariant** of α. It is important to remember that the map α has a large collection of Hopf invariants and to consider the fundamental gadget to be the **Hopf set** of α, which we denote $\mathcal{H}(\alpha) = \{H_\sigma(\alpha) \mid \sigma \text{ is a section of } p_n\}$. This set gives insight into the category of C_α.

Proposition 20.47. *Let $\alpha : \Sigma A \to X$. If $* \in \mathcal{H}(\alpha)$, then $\mathrm{cat}(C_\alpha) \leq n$.*

Problem 20.48.

(a) Show that for each section $\sigma : X \to G_n(X)$, the diagram

$$\begin{tikzcd}
\Sigma A \arrow[r, "\alpha"] \arrow[d, "H_\sigma(\alpha)"'] & X \arrow[r, "j"] \arrow[d, "G_n(j)\circ\sigma"'] & C_\alpha \arrow[d, equal] \\
F_n(C_\alpha) \arrow[r] & G_n(C_\alpha) \arrow[r, "p_n"] & C_\alpha
\end{tikzcd}$$

commutes up to homotopy.

(b) Prove Proposition 20.47.

In the terminology of Section 9.7.4, where we first saw Hopf invariants, Problem 20.48(a) asserts that $\mathcal{H}(\alpha) \subseteq \mathscr{H}(\alpha)$.

We end this section with a basic formula involving these Hopf invariants.

Problem 20.49. Consider the maps $\Sigma A \xrightarrow{\alpha} \Sigma B \xrightarrow{\beta} X$ with $H_{\sigma_{\Sigma B}}(\alpha) = 0$.

(a) Show that α is a co-H-map.

(b) Show that $\Sigma\Omega(\alpha) \circ \sigma_{\Sigma A} = \sigma_{\Sigma B} \circ \alpha$.

(c) Show that $H_\sigma(\gamma \circ \beta) = H_\sigma(\gamma) \circ \beta$ for any section $\sigma : X \to G_n(X)$.

This is particularly easy to apply in the case of cell attachments.

Problem 20.50. Show that if $\alpha : S^n \to S^m$, then $\mathcal{H}(\alpha)$ is a singleton set.

20.4.2. Stanley's Theorems on Compatible Sections. We illustrate the power of these Hopf invariants by proving two theorems of D. Stanley. First we show that the Lusternik-Schnirelmann category of the skeleta of a noncontractible space is weakly increasing with dimension. Then we derive an extremely useful consequence: in certain common situations, the converse of Proposition 20.47 holds.

Theorem 20.51 (Stanley). *Let X be a CW complex with $\mathrm{cat}(X) \le n$. Then for each section $\sigma : X \to G_n(X)$ and each $r \in \mathbb{N}$ there is a section $\sigma_r : X_r \to G_n(X_r)$ which is compatible with σ in the sense that the square*

$$\begin{tikzcd} G_n(X_r) \arrow[r, "G_n(j)"] \arrow[d, shift left, "q_n"] & G_n(X) \arrow[d, shift right, "p_n"'] \\ X_r \arrow[r, "j"] \arrow[u, bend left, "\sigma_r"] & X \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

commutes both ways: $p_n \circ G_n(j) = j \circ q_n$ and $G_n(j) \circ \sigma_r = \sigma \circ j$.

We'll prove this for the special case in which the CW decomposition of X is simply-connected and the inclusion $j : X_r \hookrightarrow X$ is an $(r+1)$-equivalence. Both of these can be removed entirely, but the proof becomes substantially more complicated, and this is the case we will need.

Problem 20.52. Assume that X and X_r are simply-connected and that $j : X_r \to X$ is an $(r+1)$-equivalence, and consider the square

$$\begin{tikzcd} G_n(X_r) \arrow[r, "G_n(j)"] \arrow[d, shift left, "q_n"] & G_n(X) \arrow[d, shift right, "p_n"'] \\ X_r \arrow[r, "j"] \arrow[u, dotted, bend left, "\sigma_r"] & X. \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

(a) Estimate the connectivity of $F_n(X_r) \to F_n(X)$ and $G_n(X_r) \to G_n(X)$.

(b) Show that there is a map σ_r such that $j = p_n \circ G_n(j) \circ \sigma_r$.

(c) Show that σ_r is a section of q_n.

We end the section by showing that when high-dimensional cells are attached to a finite-dimensional complex, the change in category is entirely detected by the Hopf invariant.

Theorem 20.53. *Let $W \xrightarrow{\alpha} X \xrightarrow{j} C_\alpha$ be a cofiber sequence in which X is n-dimensional and $W = \bigvee S^m$ with $m \geq n$. Then the following are equivalent:*

(1) $* \in \mathcal{H}(\alpha)$,

(2) $\operatorname{cat}(C_\alpha) = \operatorname{cat}(X)$.

The implication (1) implies (2) is simply Proposition 20.47. For the converse, we begin with the commutative diagram

$$\begin{tikzcd} F_n(X) \arrow[r, "F_n(j)"] \arrow[d, "u"'] & F_n(C_\alpha) \arrow[d, "v"] \\ G_n(X) \arrow[r, "G_n(j)"] \arrow[d, shift left, "q_n"] & G_n(C_\alpha) \arrow[d, shift right, "p_n"'] \\ X \arrow[r, "j"] \arrow[u, shift left, bend left, "\tau"] & C_\alpha \arrow[u, shift right, bend right, "\sigma"'] \end{tikzcd}$$

with compatible sections guaranteed by Theorem 20.51.

Problem 20.54.

(a) Show that $v \circ F_n(j) \circ H_\tau(\alpha) = 0$.

(b) Prove Theorem 20.53.

The proof relies on Theorem 20.51, so we can consider it completely proved in the special case X is simply-connected and $m > n$.

20.5. Infinite Symmetric Products

The James construction produces the free topological monoid on a space X. The usefulness of the construction suggests that we attempt to build other algebraic gadgets from spaces. The **infinite symmetric product** of a space X is the free *abelian* monoid generated by X.

This construction will not be used in the main flow of the text, so we will be content to sketch the main ideas and leave it to you to look up or invent the lengthy or technical arguments needed to fill in the gaps. The basic reference is [**55**], but the textbook [**12**] also has an exposition of the core ideas.

20.5.1. The Free Abelian Monoid on a Space. The construction is most conveniently done by a colimit procedure. If $X \in \mathcal{T}_*$, then the symmetric group $\mathrm{Sym}(n)$ acts on the n-fold product X^n by permuting the factors. We define $SP^n(X)$ to be the quotient by the equivalence relation generated by the $\mathrm{Sym}(n)$-action. The inclusions

$$X^n = X^n \times * \hookrightarrow X^{n+1} \quad \text{and} \quad \mathrm{Sym}(\{1, \ldots, n\}) \hookrightarrow \mathrm{Sym}(\{1, \ldots, n, n+1\})$$

induce inclusion maps $SP^n(X) \hookrightarrow SP^{n+1}(X)$. We define $SP^\infty(X)$ to be the (homotopy) colimit of the cofibrant telescope diagram

$$SP^0(X) \longrightarrow SP^1(X) \to \cdots \to SP^n(X) \longrightarrow SP^{n+1}(X) \to \cdots .$$

Problem 20.55. Let $X \in \mathcal{T}_*$ be a CW complex.

(a) Show that $SP^n(X)$ inherits a CW structure from X^n and that the maps in the telescope are inclusions of subcomplexes, and hence cofibrations.

(b) Deduce that $SP^\infty(X)$ is a CW complex.

Write $[x_1, x_2, \ldots, x_n]$ for the equivalence class in $SP^\infty(X)$ of the point $(x_1, x_2, \ldots, x_n, *, *, \ldots) \in X^\infty$ and define the multiplication map

$$\mu : SP^\infty(X) \times SP^\infty(X) \longrightarrow SP^\infty(X)$$

by $\mu([x_1, x_2, \ldots, x_n], [y_1, y_2, \ldots, y_m]) = [x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m]$.

Problem 20.56.

(a) Show that $SP^\infty(X)$ is an abelian topological monoid.

(b) Show that every map $f : X \to M$ from X to an abelian topological monoid M extends to a unique homomorphism $\phi : SP^\infty(X) \to M$, which is continuous if f is continuous.

Problem 20.57. Suppose X is a CW complex, and let $\mathcal{K}$ denote the category of finite subcomplexes of X and their inclusions. Show that the natural maps

$$\mathrm{colim}_{K \in \mathcal{K}}\, SP^n(K) \longrightarrow SP^n(X)$$

and

$$\mathrm{colim}_{K \in \mathcal{K}}\, SP^\infty(K) \longrightarrow SP^\infty(X)$$

are homeomorphisms.

Project 20.58. Investigate the extent to which $SP^\infty(\,?\,)$ respects homotopy.

20.5.2. Symmetric Products of Cofiber Sequences. What happens when we apply the functor $SP^\infty(\,?\,)$ to a cofiber sequence?

Problem 20.59. Let $i : A \to X$ be a cofibration with $B = X/A$. Show that for each $b = [b_1, \ldots, b_n] \in SP^\infty(B)$, there is a pullback square

$$\begin{tikzcd} SP^\infty(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & SP^\infty(X) \arrow[d] \\ \{b\} \arrow[r] & SP^\infty(B). \end{tikzcd}$$

This is very suggestive! Essentially every map we have seen in which all the points in the target have the same preimage has been a fibration, so it is entirely reasonable to ask whether $SP^\infty(X) \to SP^\infty(X/A)$ must be a fibration. The main theorem about the homotopy theory of symmetric products is that SP^∞ transforms cofiber sequences into *quasifibration* sequences.

Theorem 20.60 (Dold-Thom)**.** *Let $A \hookrightarrow X$ be a cofibration and write $B = X/A$. Then the induced map $SP^\infty(X) \to SP^\infty(X/A)$ is a quasifibration with fiber $SP^\infty(A)$.*

Project 20.61. Work through a proof of Theorem 20.60.

Problem 20.62. Show that $SP^\infty(X) \simeq \Omega SP^\infty(\Sigma X)$ if X is a CW complex.

20.5.3. Some Examples. Because of Theorem 20.60 and Problem 20.57, the behavior of $SP^\infty(\,?\,)$ on CW complexes is determined by its effect on spheres; and because of Problem 20.62 we can focus on S^2.

Problem 20.63. Consider S^2 as $\mathbb{C} \cup \{\infty\}$ with ∞ as basepoint, and let $P_n(\mathbb{C})$ be the space of nonzero polynomials over $\mathbb{C}$ with degree at most n.

(a) Note that $P_n(\mathbb{C}) \cong \mathbb{C}^{n+1} - \{0\}$; let $q : P_n(\mathbb{C}) \to \mathbb{C}\mathrm{P}^n$ be the quotient map. Show that $q(f) = q(g)$ if and only if f and g have the same roots (counted with multiplicity).

(b) Define $\phi_n : SP^n(S^2) \to P_n(\mathbb{C})$ by the rules

$$[z_1, z_2, \ldots, z_n] \longmapsto \prod_{i=1}^{n} (z - z_i) \qquad \text{and} \qquad [\infty] \mapsto 1.$$

Show that $q \circ \phi_n : SP^n(S^2) \to \mathbb{C}\mathrm{P}^n$ is a homeomorphism.

(c) Show that $SP^\infty(S^2) \cong \mathbb{C}\mathrm{P}^\infty$.

(d) Show that $SP^\infty(S^n) = K(\mathbb{Z}, n)$ for $n \geq 2$.

Now let's work out the infinite symmetric products of Moore spaces.

Problem 20.64.

(a) Show that if $f : S^n \to S^n$ is of degree d, then the induced map $SP^\infty(f)$ may be identified with the map $K(\mathbb{Z}, n) \to K(\mathbb{Z}, n)$ corresponding to the 'multiplication by d' map $d : \mathbb{Z} \to \mathbb{Z}$.

(b) Determine the homotopy type of $SP^\infty(M(G, n))$, where $M(G, n)$ is the Moore space for the (abelian) group G in dimension n.

(c) Show that the natural inclusion $M(G, n) \hookrightarrow SP^\infty(M(G, n))$ is an $(n+1)$-equivalence.

20.5.4. Symmetric Products and Eilenberg-Mac Lane Spaces. We have seen that the infinite symmetric products of spheres and Moore spaces are Eilenberg-Mac Lane spaces. In fact, the symmetric products of all CW complexes turn out to be (possibly infinite) products of Eilenberg-Mac Lane spaces.[2]

Problem 20.65. Suppose we are given abelian topological monoids X_i, indexed by $i \in \mathcal{I}$.

(a) Show that if X and Y are abelian topological monoids, then so is $X \times Y$.

(b) Show that the infinite *weak* product $\overline{\prod}_{i\in\mathcal{I}} X_i$ is also an abelian topological monoid.

(c) Show that if for each $i \in \mathcal{I}$ the map $f_i : X_i \to Y$ is a homomorphism of abelian topological monoids, then in the diagram

$$\begin{tikzcd}
\bigvee_{i\in\mathcal{I}} X_i \arrow[r, "(f_n)"] \arrow[d, "\mathrm{in}"'] & Y \\
\overline{\prod}_{i\in\mathcal{I}} X_i \arrow[ur, dotted, "f"'] &
\end{tikzcd}$$

there is a unique homomorphism f making the diagram commute.

Problem 20.66. Let X be an abelian topological monoid and write $G_n = \pi_n(X)$ for each n.

(a) Show that there is a *homomorphism* $K(G_n, n) \to X$ which induces the identity on $\pi_n(\,?\,)$.

(b) Show that there is a homomorphism $\overline{\prod}_n K(G_n, n) \to X$ which is a weak homotopy equivalence.

Problem 20.66 implies that for CW complex X, there is a homotopy equivalence $\overline{\prod}_n K(G_n, n) \to SP^\infty(X)$ where $G_n = \pi_n(SP^\infty(X))$.

[2] Such products are called **generalized Eilenberg-Mac Lane spaces**, or **GEMs**.

Problem 20.67.

(a) Show that the groups G_n defined in Section 20.2 are the same as the groups $G_n = \pi_n(SP^\infty(X))$.

(b) Conclude that if $f : X \to Y$ is a weak homotopy equivalence, then so is the induced map $SP^\infty(f)$.

20.6. Additional Topics, Problems and Projects

20.6.1. Self-Maps of Projective Spaces. There is a large body of research concerning the sets $[X, X]$ of homotopy classes of self-maps of spaces. In this section we'll get a taste of the theory by investigating the homotopy sets $[\mathbb{F}\mathrm{P}^n, \mathbb{F}\mathrm{P}^n]$.

Problem 20.68. Let $f : \mathbb{F}\mathrm{P}^m \to \mathbb{F}\mathrm{P}^n$. Show that for each $k \le m$ there is a map $f_k : \mathbb{F}\mathrm{P}^k \to \mathbb{F}\mathrm{P}^k$ making the diagram

$$\begin{tikzcd}
\mathbb{F}\mathrm{P}^k \arrow[r, "f_k"] \arrow[d] & \mathbb{F}\mathrm{P}^k \arrow[d] \\
\mathbb{F}\mathrm{P}^m \arrow[r, "f"] & \mathbb{F}\mathrm{P}^n
\end{tikzcd}$$

commute up to homotopy.

If $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$ and $f : \mathbb{F}\mathrm{P}^n \to \mathbb{F}\mathrm{P}^n$, $f_* : \pi_d(\mathbb{F}\mathrm{P}^n) \to \pi_d(\mathbb{F}\mathrm{P}^n)$ is a map $\mathbb{Z} \to \mathbb{Z}$, so it is multiplication by some integer, called the **degree** of f.

Problem 20.69.

(a) Show that if $m \le n$, then the restriction $[\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n] \to [\mathbb{C}\mathrm{P}^{m-1}, \mathbb{C}\mathrm{P}^n]$ is bijective.

(b) Show that the degree map $[\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n] \to \mathbb{Z}$ is bijective.

In the case $\mathbb{F} = \mathbb{R}$, the map $f_* : \pi_1(\mathbb{R}\mathrm{P}^n) \to \pi_1(\mathbb{R}\mathrm{P}^n)$ is a map $\mathbb{Z}/2 \to \mathbb{Z}/2$, and we can define the degree of f, which is an element of $\mathbb{Z}/2$. Thus we have a partition

$$[\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^n] = \mathcal{D}_0(\mathbb{R}\mathrm{P}^n) \sqcup \mathcal{D}_1(\mathbb{R}\mathrm{P}^n)$$

of $[\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^n]$ into the sets of maps with degree 0 or 1 modulo 2.

Problem 20.70. The cofiber sequence $S^1 \xrightarrow{2} S^1 \to \mathbb{R}\mathrm{P}^2 \to S^2 \xrightarrow{2} S^2$ gives rise to an action $[S^2, \mathbb{R}\mathrm{P}^2] \times [\mathbb{R}\mathrm{P}^2, \mathbb{R}\mathrm{P}^2] \to [\mathbb{R}\mathrm{P}^2, \mathbb{R}\mathrm{P}^2]$. Show that

$$\mathcal{D}_0(\mathbb{R}\mathrm{P}^2) = \mathrm{Orbit}(*) \qquad \text{and} \qquad \mathcal{D}_1(\mathbb{R}\mathrm{P}^2) = \mathrm{Orbit}(\mathrm{id}_{\mathbb{R}\mathrm{P}^2}).$$

We'll completely determine the set $\mathcal{D}_0(\mathbb{R}\mathrm{P}^n)$ as a function of n.

Problem 20.71. Let $n \geq 2$.

(a) Show that if $m < n$, then maps $f : \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$ are determined up to homotopy by their degree.

(b) Show that if $f : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ has degree zero, then f has a lift in the diagram

$$\begin{tikzcd} & S^n \arrow[d, "p"] \\ \mathbb{R}\mathrm{P}^n \arrow[r, "f"] \arrow[ur, dotted, "\phi"] & \mathbb{R}\mathrm{P}^n, \end{tikzcd}$$

where $p : S^n \to \mathbb{R}\mathrm{P}^n$ is the universal cover.

(c) Show that $p_* : [\mathbb{R}\mathrm{P}^n, S^n] \to \mathcal{D}_0(\mathbb{R}\mathrm{P}^n)$ is bijective.

(d) Determine the homotopy type of $\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{n-2}$.

Hint. Use the cone decomposition of Problem 20.7.

(e) Determine $[\mathbb{R}\mathrm{P}^n, S^n]$ and therefore $\mathcal{D}_0(\mathbb{R}\mathrm{P}^n)$.

20.6.2. Fiber of Suspension and Suspension of Fiber. Let $f : A \to B$, let F_f be its homotopy fiber and let C be its cofiber. Also suspend f and let $F_{\Sigma f}$ be its homotopy fiber. Then we can ask: how do the spaces ΣF_f and $F_{\Sigma f}$ compare? To answer the question, we need to construct a map from one to the other; then we can attempt to estimate the connectivity of the map.

The construction of the map is a straightforward combination of the comparison maps we have been studying: it is the vertical composite in the middle of the diagram

$$\begin{tikzcd} F_f \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & \Sigma F_f \arrow[d, "\xi"] & & \\ A \arrow[r, "f"] & B \arrow[r] & C \arrow[r] \arrow[d, "\zeta"'] & \Sigma A \arrow[r, "\Sigma f"] \arrow[d, equal] & \Sigma B \arrow[d, equal] \\ & & F_{\Sigma f} \arrow[r] & \Sigma A \arrow[r, "\Sigma f"] & \Sigma B. \end{tikzcd}$$

Proposition 20.72. *If B is $(b-1)$-connected and f is a $(c-1)$-equivalence, then the comparison map $\zeta \circ \xi : \Sigma F_f \to F_{\Sigma f}$ is a $(b+c-2)$-equivalence.*

Problem 20.73. Prove Proposition 20.72.

20.6.3. Complexes of Reduced Product Type. The James reduced product construction produces from a CW complex X the free topological monoid generated by X. By imposing commutativity relations on $J(X)$, Dold and Thom built $SP^\infty(X)$, the free abelian monoid on X. In this section, we look at another collection of ideas inspired by the James construction.

The idea is to build topological monoids by imposing 'homotopy relations' instead of genuine algebraic relations. Instead of forcing elements to be trivial by identifying them with $*$, we make them homotopically trivial by attaching cones; just as a new relation must be propagated throughout a monoid by identifying nontrivial words with one another, so the introduction of a homotopy relation gives rise to a host of new homotopies that must be imposed on the original topological monoid.

The construction is done in such a way as to preserve a crucial feature of the multiplication in James construction: not only is the product cellular, but the restriction of the product to any open product cell $\text{int}(D^n)\times\text{int}(D^m)$ in $J\times J$ is a *homeomorphism* onto its image in J. Topological monoids with this special property are said to be of **reduced product type** and are called **RPT complexes** for short. These spaces can be used to give fairly concrete cellular models for the loop spaces of CW complexes.

This topic, being tangential to the main flow of the book, will be covered very superficially. The problems offer less guidance, and the hard work is left as projects for the interested and highly motivated reader. Good references are [**97**], [**94**] and [**96**].

Building RPT Complexes. Let X be a CW complex with a cellular multiplication $\mu : X \times X \to X$. Then for each n and m we may form the space $X[n,m]$ as the (homotopy) pushout in the square

$$\begin{tikzcd}
(X_{n-1}\times X_m)\cup(X_n\times X_{m-1}) \arrow[rr, "\mu"] \arrow[d] \arrow[drr, phantom, "\text{pushout}"] & & X_{n+m-1} \arrow[d] \\
X_n\times X_m \arrow[rr] & & X[n,m].
\end{tikzcd}$$

The pushout property guarantees a canonical map $X[n,m]\to X$, and X is said to be of **reduced product type** if for each n and m, the map $X[n,m]\to X$ is the inclusion of a subcomplex; we generally say that X is an **RPT complex**. A **homomorphism** of RPT complexes is simply a cellular map that is a homomorphism of monoids.

Exercise 20.74. Show that $*$ and $J(X)$ are RPT complexes. Is $SP^\infty(X)$ an RPT complex?

The basic theorem for the construction of RPT complexes shows how the attachment of cells to an RPT complex can be extended to an inclusion of RPT complexes.

Theorem 20.75. *Let M be an RPT complex which is a subcomplex of another CW complex Y. Then there is an RPT complex N containing Y such that*

(a) the inclusion $i : M \to N$ is a homomorphism and

(b) i satisfies the following universal property: if G is a topological monoid and $f : Y \to G$ such that $f \circ i : M \to G$ is a homomorphism, then there is a unique homomorphism $g : N \to G$ making the diagram

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[dr, bend right] & Y \arrow[r] \arrow[d, "f"] & N. \arrow[dl, dotted, "\exists!\, g"] \\ & G & \end{tikzcd}$$

commute.

The idea of this proof is very simple. We know how to multiply elements of X, and we need to attach cells to allow us to multiply elements of X by elements of Y not in X. The very strict RPT condition essentially forces the construction on us. We form the pushout

$$\begin{tikzcd} (X \times X \times X) \cup (* \times Y \times *) \arrow[r] \arrow[d, "\mu"'] \arrow[dr, phantom, "\text{pushout}"] & X \times Y \times X \arrow[d] \\ Y \arrow[r] & Y_1. \end{tikzcd}$$

Note that Y_1 can be thought of as the space of all products of at most one element of Y with arbitrarily many elements of X (on either side).

Problem 20.76.

(a) Show that the inclusion $Y \to J(Y_1)$ satisfies all of the requirements of Theorem 20.75 *except* the uniqueness of the map g.

(b) Construct $\overline{Y}$ from $J(Y_1)$ by quotienting out by an appropriate equivalence relation, and thereby prove Theorem 20.75.

Modeling Loop Spaces. We finish our brief discussion of RPT complexes by indicating how they can be used to give cellular models for loop spaces of CW complexes. Let's say that an **RPT model** for ΩX is a homomorphism $M \to \Omega_M(X)$ that is a weak homotopy equivalence.

Theorem 20.77. *If X is a simply-connected CW complex, then there is an RPT model $M \to \Omega_M(X)$.*

The proof of this theorem is by CW induction, starting with the observation that if X is a suspension, then the James construction provides an RPT model for ΩX. For the inductive step, we suppose we have an RPT model $M \to \Omega_M X$, and we hope to find one for $\Omega_M(X \cup_\alpha D^{n+1})$. The diagram

$$S^{n-1} \longrightarrow \Omega_M S^n \xrightarrow{\Omega\alpha} \Omega_M X \xleftarrow{\sim} M$$

gives a well-defined homotopy class $\beta : S^{n-1} \to M$. Applying Theorem 20.75 to the inclusion $M \hookrightarrow M \cup_\beta D^n$ yields an RPT complex and a map $N \to \Omega_M(X \cup_\alpha D^{n+1})$. To show that N is an RPT model for $\Omega_M(X \cup_\alpha D^{n+1})$, construct a contractible space P with an 'RPT action' of N, such that the 'space of maximal orbits' $P//N$ is $X \cup_\alpha C^{n+1}$. This does the job because it is true in general that the quotient $P \to P//N$ of an RPT action is a quasifibration.

Project 20.78. Prove Theorem 20.77.

20.6.4. Problems and Projects.

Problem 20.79. Theorem 20.23 was one of the keys to our study of $\mathcal{P}$-connectivity in Section 20.3. In this problem, you'll rederive it, for the Moore spaces $M(\mathbb{Z}/p^r, n)$, in a different way. Let $p^r : S^3 \to S^3$, where $p \in \mathcal{P}$, and let F be its fiber.

(a) Show that $M(\mathbb{Z}/p^r, 3)$ is a retract of ΣF.

(b) Show that $\mathrm{conn}_{\mathcal{Q}}(F) = \infty$.

(c) Prove that $\mathrm{conn}_{\mathcal{Q}}(M(\mathbb{Z}/p^r, n)) = \infty$ for all $n \geq 2$.

Problem 20.80. Let p and q be different primes. Show that if $\pi_*(X)$ are finite p-groups and $\pi_*(Y)$ are finite q-groups, then $X \wedge Y \simeq *$.

Problem 20.81. Let $\mathcal{P} \sqcup \mathcal{Q}$ be a partition of the prime numbers and let Y be a 2-connected CW complex with $\mathrm{conn}_{\mathcal{Q}}(X) = \infty$. Show that the $\mathcal{Q}$-subgroup (that is, the subgroup consisting of all elements whose order is a product of primes in $\mathcal{Q}$) of $[X, \Omega^2 Y]$ is trivial.

Project 20.82. Investigate the $\mathcal{P}$-connectivity of smash products of *maps*.

Part 5

Cohomology and Homology

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Chapter 21

Cohomology

We have seen in the last few chapters that very useful information can be obtained about a space X by studying the homotopy classes of maps from X into Eilenberg-Mac Lane spaces. In this chapter we codify the technique by defining the n^{th} cohomology of X with coefficients in the abelian group G to be $\widetilde{H}^n(X;G) = [X, K(G,n)]$. Thus each group G gives rise to a collection of functors indexed by the nonnegative integers. Taken together, these functors are a cohomology theory: a collection of homotopy functors that are related to one another via the suspension operation and which carry cofiber sequences to exact sequences.

We construct a multiplicative structure involving cohomology theories which, when we use coefficients in a ring R, yields an R-algebra structure on the cohomology of spaces. We find relations between cohomology theories with different coefficients and prove a simple result about the cohomology of a product (smash or cartesian) of two spaces.

Consult Appendix A for basic results and terminology about graded algebra, exact sequences and other algebraic prerequisites.

21.1. Cohomology

Eilenberg-Mac Lane spaces are good spaces to map *into* (in contrast to spheres, Moore spaces, etc.) so the homotopy functors $[\,?\,, K(G,n)]$ are fairly well-behaved. We have already seen that these functors can be used to determine the connectivity of spaces (at least simply-connected spaces). These functors, and the relations between them, form a general framework called a **cohomology theory**, and our aim in this section is to study the basic properties of cohomology theories.

21.1.1. Represented Ordinary Cohomology. The n^{th} **cohomology** group of X with coefficients in the abelian group G is the abelian group

$$\widetilde{H}^n(X;G) = [X, K(G,n)].$$

It is convenient to define $\widetilde{H}^n(X;G) = 0$ for all $n < 0$. It is frequently useful to work with the entire collection

$$\widetilde{H}^*(X;G) = \{\widetilde{H}^n(X;G) \mid n \in \mathbb{Z}\}$$

at once, rather than one group at a time. A collection $A^* = \{A^n \mid n \in \mathbb{Z}\}$ of abelian groups is called a **graded abelian group**. A homomorphism $f^* : A^* \to B^*$ of graded abelian groups is a collection of homomorphisms $\{f^n : A^n \to B^n \mid n \in \mathbb{Z}\}$. These are the morphisms in the category $\mathrm{AB}\,\mathcal{G}^*$ of graded abelian groups. Thus cohomology with coefficients in G defines a contravariant functor

$$\widetilde{H}^*(\,?\,;G) : \mathrm{H}\mathcal{T}_* \longrightarrow \mathrm{AB}\,\mathcal{G}^*.$$

We call this **represented cohomology** because the functors $\widetilde{H}^*(\,?\,;G)$ are represented in the sense of Proposition 1.21.

An element $u \in \widetilde{H}^n(X;G)$ is called an n**-dimensional cohomology class**. The dimension of $u \in \widetilde{H}^*(X;G)$ is denoted $|u|$. Sometimes it is useful, though we do not do it in this book, to think of a graded abelian group as the direct sum (or product) of its constituent groups. With this definition, it makes sense to work with elements of the form $x + y$ with $|x| \neq |y|$. When this possibility is allowed, it is frequently useful to distinguish the **homogeneous** elements, which are elements that lie in A^n for some n.

Since $\pi_k(K(G,n)) = 0$ for $k > n$, adding k-cells to X has absolutely no effect on $\widetilde{H}^n(\,?\,;G)$ for $n < k-1$, and likewise for collapsing k-cells of X for $k < n-1$. This suggests that $\widetilde{H}^n(X;G)$ measures the n-dimensional features of X and, in particular, that $\widetilde{H}^0(X;G)$ depends only on the path components of X.

Exercise 21.1. Show that $\widetilde{H}^0(X;G) \cong G \times G \times \cdots \times G$, where the number of factors is one less than the number of path components of X.

We will return to this intuition in much more detail later.

The fundamental properties of cohomology were laid down by Eilenberg and Steenrod in their seminal work [**59**]. They are known as the **Eilenberg-Steenrod axioms**.

Theorem 21.2. *For any abelian group G,*

(a) $\widetilde{H}^n(\,?\,;G)$ is a contravariant homotopy functor $\mathcal{T}_* \to \mathrm{AB}\,\mathcal{G}$,

(b) if $A \to B \to C$ is a cofiber sequence, then

$$\widetilde{H}^n(A;G) \longleftarrow \widetilde{H}^n(B;G) \longleftarrow \widetilde{H}^n(C;G)$$

is an exact sequence, and

(c) there are natural isomorphisms $\widetilde{H}^n(\,?\,;G) \xrightarrow{\cong} \widetilde{H}^{n+1}(\Sigma\,?\,;G)$.

The suspension ΣA^* of a graded abelian group A^* is defined by setting $(\Sigma A)^n = A^{n-1}$ for each n. In this notation, Theorem 21.2(c) asserts that there is a natural isomorphism $\Sigma \circ \widetilde{H}^*(\,?\,;G) \xrightarrow{\cong} \widetilde{H}^*(\,?\,;G) \circ \Sigma$.

Corollary 21.3. *Let $A \to B \to C$ be a cofiber sequence. Then there is a natural long exact sequence*

$$\cdots \to \widetilde{H}^n(C) \longrightarrow \widetilde{H}^n(B) \longrightarrow \widetilde{H}^n(A) \longrightarrow \widetilde{H}^{n+1}(C) \to \cdots .$$ [1]

Problem 21.4. Prove Theorem 21.2 and derive Corollary 21.3.

HINT. Use Corollary 17.46.

More About Eilenberg and Steenrod. The list of properties given by Eilenberg and Steenrod is significantly longer than that given in Theorem 21.2. Because the language of categories and functors was new at the time of their formulation, there were many axioms explicitly laying out the functoriality and naturality.

Also, they used the language of pairs, so that where we have the cohomology of the cofiber of $A \to B$, they used the cohomology of the pair (B, A). Our suspension isomorphism was encoded in the **Excision Axiom**, which says that if $V \subseteq \operatorname{int}(A) \subseteq X$, then $(X - V, A - V) \hookrightarrow (X, A)$ must induce an isomorphism on cohomology groups.

Finally, Eilenberg and Steenrod imposed a **Weak Equivalence Axiom**: a weak homotopy equivalence must induce isomorphisms on cohomology. In Problem 21.12 you are asked to show that represented cohomology does not satisfy this axiom.

21.1.2. Cohomology Theories.

Any contravariant homotopy functor $\widetilde{h}^* : \mathcal{T}_* \to \mathrm{AB}\,\mathcal{G}^*$ satisfying the Eilenberg-Steenrod axioms

- if $A \to B \to C$ is a cofiber sequence, then $\widetilde{h}^*(A) \leftarrow \widetilde{h}^*(B) \leftarrow \widetilde{h}^*(C)$ is an exact sequence and
- there is a natural isomorphism $\Sigma \circ \widetilde{h}^* \xrightarrow{\cong} \widetilde{h}^* \circ \Sigma$

of Theorem 21.2 is called a **cohomology theory**. The graded abelian group $\widetilde{h}^*(S^0)$ is known as the **coefficients** of $\widetilde{h}^*$.

[1] Here we're beginning to use a very common convention: don't write down the coefficient group G unless you need to; also, the maps are the only reasonable ones, so we omit their labels.

If $\mathcal{C} \subseteq \mathcal{T}_*$ is a full subcategory that is closed under homotopy equivalence and the formation of cofibers, then it makes sense to ask whether a functor $\widetilde{h}^* : \mathcal{C} \to \text{AB}\,\mathcal{G}^*$ satisfies the Eilenberg-Steenrod axioms. If it does, then we call $\widetilde{h}^*$ a *cohomology theory defined on* $\mathcal{C}$. We'll use this idea several times, typically taking $\mathcal{C} = \mathbf{CW}_*$ to be the category of all spaces homotopy equivalent to a CW complex, or $\mathcal{F}_*$, the category of *finite* CW complexes.

A graded group A^* is said to be **concentrated** in degree n if $A^k = 0$ for $k \neq n$. For each n there is a functor $\text{in}_n : \text{AB}\,\mathcal{G} \to \text{AB}\,\mathcal{G}^*$ which takes a group A and produces the graded group A^* given by $A^k = 0$ for $k \neq n$ and $A^n = A$. It is customary to use in_0 to identify $\text{AB}\,\mathcal{G}$ as a subcategory of $\text{AB}\,\mathcal{G}^*$, often with no mention made of the distinction.

If the graded abelian group $\widetilde{h}^*(S^0)$ is concentrated in degree 0, then $\widetilde{h}^*$ is called an **ordinary cohomology theory** with **coefficient group** $\widetilde{h}^0(S^0)$. It is customary to use a capital H for ordinary cohomology theories. We will show later that *ordinary* cohomology theories are largely determined by their coefficients. If $\widetilde{h}^*(S^0)$ is more complicated, then $\widetilde{h}^*$ is called an **extraordinary cohomology theory**.

Problem 21.5. Show that $\widetilde{H}^*(\,?\,; G)$ is an ordinary cohomology theory, and conclude

$$\widetilde{H}^n(S^k; G) = \begin{cases} G & \text{if } n = k, \\ 0 & \text{otherwise.} \end{cases}$$

Transformations of Cohomology Theories. Let $\widetilde{h}^*$ and $\widetilde{k}^*$ be cohomology theories, and let $T : \widetilde{h}^* \to \widetilde{k}^*$ be a natural transformation of *functors* $\mathcal{T}_* \to \text{AB}\,\mathcal{G}^*$. If T commutes with the suspension isomorphisms, in the sense that the diagram

$$\begin{tikzcd} \Sigma\widetilde{h}^*(X) \arrow[r, "\Sigma T_X"] \arrow[d, "\cong"'] & \Sigma\widetilde{k}^*(X) \arrow[d, "\cong"] \\ \widetilde{h}^*(\Sigma X) \arrow[r, "T_X"] & \widetilde{k}^*(\Sigma X) \end{tikzcd}$$

commutes for every space X, then we call T a **natural transformation of cohomology theories**.

Problem 21.6. Show that if $T : \widetilde{h}^* \to \widetilde{k}^*$ is a transformation of cohomology theories, then for every cofiber sequence $A \to B \to C$, the diagram

$$\begin{tikzcd} \cdots \arrow[r] & \widetilde{h}^n(C) \arrow[r] \arrow[d, "T_n"] & \widetilde{h}^n(B) \arrow[r] \arrow[d, "T_n"] & \widetilde{h}^n(A) \arrow[r] \arrow[d, "T_n"] & \widetilde{h}^{n+1}(C) \arrow[r] \arrow[d, "T_{n+1}"] & \widetilde{h}^{n+1}(B) \arrow[r] \arrow[d, "T_{n+1}"] & \cdots \\ \cdots \arrow[r] & \widetilde{k}^n(C) \arrow[r] & \widetilde{k}^n(B) \arrow[r] & \widetilde{k}^n(A) \arrow[r] & \widetilde{k}^{n+1}(C) \arrow[r] & \widetilde{h}^{n+1}(B) \arrow[r] & \cdots \end{tikzcd}$$

commutes.

Proposition 21.7. *If $T : \widetilde{h}^* \to \widetilde{k}^*$ is a natural transformation of cohomology theories, then the following are equivalent:*

(1) $T : \widetilde{h}^*(S^n) \to \widetilde{k}^*(S^n)$ is an isomorphism for some n,

(2) $T : \widetilde{h}^*(S^n) \to \widetilde{k}^*(S^n)$ is an isomorphism for all n,

(3) T is an isomorphism for all finite CW complexes.

Problem 21.8. Prove Proposition 21.7.

Proposition 21.7 cannot be extended to guarantee that T is an isomorphism on spaces that are not homotopy equivalent to finite complexes unless the cohomology theories in question satisfy additional properties. There are two additional properties that appear naturally in many examples.

The first of these concerns the cohomology of a wedge. The inclusions $\mathrm{in}_j : X_j \hookrightarrow \bigvee_{\mathcal{J}} X_j$ of the summands into a wedge induce maps $\widetilde{h}^*(\bigvee_{\mathcal{J}} X_j) \to \widetilde{h}^*(X_j)$ and hence a comparison map

$$w : \widetilde{h}^*\left(\bigvee_{\mathcal{J}} X_j\right) \longrightarrow \prod_{\mathcal{J}} \widetilde{h}^*(X_j).$$

Exercise 21.9. Explain how to interpret w as a natural transformation.

Problem 21.10.

(a) Show that if $\mathcal{J}$ is finite, the map w is an isomorphism, no matter what cohomology theory $\widetilde{h}^*$ is used.

(b) Show that in the examples $\widetilde{H}^n(X; G) = [X, K(G, n)]$ the map w is an isomorphism for *all* wedges, finite or infinite.

If $\widetilde{h}^*$ is a cohomology theory with the property that the comparison map $w : \widetilde{h}^*\left(\bigvee_{\mathcal{J}} X_j\right) \to \prod_{\mathcal{J}} \widetilde{h}^*(X_j)$ is an isomorphism for all wedges $\bigvee_{\mathcal{J}} X_j$ (finite or infinite), then we say that $\widetilde{h}^*$ satisfies the **Wedge Axiom**.

The Wedge Axiom gives us the information we need to extend Proposition 21.7 to finite-dimensional CW complexes.

Problem 21.11. Show that if, in Proposition 21.7, the theories $\widetilde{h}^*$ and $\widetilde{k}^*$ satisfy the Wedge Axiom, then $T : \widetilde{h}^*(X) \to \widetilde{k}^*(X)$ is an isomorphism for all finite-dimensional CW complexes.

The Wedge Axiom is actually powerful enough to force T to be an isomorphism for all CW complexes, as we will see when we study the cohomology of the homotopy colimits of telescope diagrams.

We can make the jump from CW complexes to all spaces if $\widetilde{h}^*$ satisfies the **Weak Equivalence Axiom**, which requires that every weak homotopy equivalence $X \to Y$ should induce an isomorphism $\widetilde{h}^*(Y) \to \widetilde{h}^*(X)$. This is essentially the same thing as extending the theory $\widetilde{h}^*$ from CW complexes to all spaces using the singular approach that we will discuss at the end of this chapter.

Problem 21.12.

(a) Show that the discrete space $N = \{0, 1, 2, \ldots\}$ and the subspace $L = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$ are weakly equivalent.

(b) Show that the cohomology theory $\widetilde{H}^*(\,?\,; G) = [\,?\,, K(G, n)]$ does not satisfy the Weak Equivalence Axiom.

21.1.3. Cohomology and Connectivity. The J. H. C. Whitehead theorem tells us that a map $f : X \to Y$ between path-connected CW complexes is a homotopy equivalence if and only if the induced maps $f_* : \pi_k(X) \to \pi_k(Y)$ are isomorphisms for all k. For simply-connected spaces, we can make a similar conclusion by studying the induced maps on cohomology.

Proposition 21.13. *If X is a path-connected CW complex and $\pi_1(X)$ is not a nontrivial perfect group, then the following are equivalent:*

(1) $\mathrm{conn}(X) = n$,

(2) $\widetilde{H}^k(X; G) = 0$ *for all $k < n$ and all abelian groups G, and there is a group G for which* $\widetilde{H}^n(X; G) \neq 0$.

Problem 21.14. Prove Proposition 21.13.

We can determine the connectivity of a map by applying Proposition 21.13 to its cofiber.

Theorem 21.15. *Let X and Y be simply-connected CW complexes, and let $f : X \to Y$. The following are equivalent:*

(1) $f^* : \widetilde{H}^k(Y; G) \to \widetilde{H}^k(X; G)$ *is an isomorphism for all $k < n$ and is injective for $k = n$, no matter what coefficient group G is used,*

(2) f *is an n-equivalence.*

Problem 21.16. Prove Theorem 21.15.

Exercise 21.17. Find examples to show that the simply-connected hypothesis in these results cannot be removed. Can it be weakend?

21.1.4. Cohomology of Homotopy Colimits. Cohomology theories are well suited to studying domain-type spaces. In particular, the cohomology of a homotopy colimit is comparatively easy to compute. More precisely, these computations are usually quite difficult, but at least there are methods of attack ready to be tried.

Homotopy Pushouts. The cohomology groups of the spaces in a homotopy pushout square fit into a long exact sequence known as the **Mayer-Vietoris exact sequence**.

Problem 21.18 (Mayer-Vietoris sequence)**.** Let $\widetilde{h}^*$ be a cohomology theory and let

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] \arrow[dr, phantom, "HPO"] & B \arrow[d, "k"] \\ C \arrow[r, "l"] & D \end{tikzcd}$$

be a homotopy pushout square. Show that there is a natural long exact sequence

$$\cdots \to \widetilde{h}^n(D) \xrightarrow{(l^*,k^*)} \widetilde{h}^n(B) \oplus \widetilde{h}^n(C) \xrightarrow{(-i^*,j^*)} \widetilde{h}^n(A) \longrightarrow \widetilde{h}^{n+1}(D) \to \cdots .$$

Telescope Diagrams. Let $\widetilde{h}^*$ be a cohomology theory that satisfies the Wedge Axiom, and consider the homotopy colimit of the telescope diagram

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} \cdots .$$

Assume that all the maps in the diagram are cofibrations, and write X for its (homotopy) colimit. Applying $\widetilde{h}^k$ to this telescope yields the map

$$\begin{tikzcd} & & & \widetilde{h}^k(X) \arrow[dll] \arrow[dl] & \\ \cdots & \widetilde{h}^k(X_n) \arrow[l] & \widetilde{h}^k(X_{n+1}) \arrow[l, "(f_n)_*"] & \cdots \arrow[l] & \end{tikzcd}$$

from $\widetilde{h}^k(X)$ to the tower. Ideally, the cohomology of the (homotopy) colimit of the telescope would be the (categorical) limit of the tower. This is frequently true, and even when it is not, its deviation from truth can be quantified.

The **shift map** for the telescope diagram of spaces is the map

$$\mathrm{sh} : \bigvee X_n \longrightarrow \bigvee X_n$$

whose restriction to X_n is the composite $X_n \xrightarrow{f_n} X_{n+1} \xhookrightarrow{\mathrm{in}_{n+1}} \bigvee X_n$. There is an entirely analogous shift map for the tower of groups.

Problem 21.19. Let $G_0 \leftarrow G_1 \leftarrow \cdots \leftarrow G_n \leftarrow \cdots$ be a tower of abelian groups, and let

$$\mathrm{sh} : \prod G_n \longrightarrow \prod G_n$$

be the shift map. Show that $\ker(\mathrm{id} - \mathrm{sh}) = \lim G_n$.

We have studied limits of towers of groups before, in Section 9.5.2. There we introduced the standard notation for the cokernel of $\sigma = \mathrm{id} - \mathrm{sh}$: it is written $\lim^1 G_n = \mathrm{coker}(\mathrm{id} - \mathrm{sh})$ and read 'lim-one'. With this notation, we have an exact sequence

$$0 \to \lim G_n \longrightarrow \prod G_n \xrightarrow{\sigma} \prod G_n \longrightarrow \lim{}^1 G_n \to 0.$$

Proposition 21.20. *If X is the homotopy colimit of the telescope diagram $\cdots \to X_n \xrightarrow{f_n} X_{n+1} \to \cdots$, then there are exact sequences*

$$0 \to \lim{}^1 \widetilde{h}^k(X_n) \longrightarrow \widetilde{h}^k(X) \longrightarrow \lim \widetilde{h}^k(X_n) \to 0.$$

Problem 21.21. Let X be as in Proposition 21.20.

(a) Show that X is the homotopy pushout in the square

$$\begin{tikzcd} (\bigvee X_n) \vee (\bigvee X_n) \arrow[r, "(\mathrm{id},\mathrm{sh})"] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\mathrm{HPO}"] & \bigvee X_n \arrow[d, "j"] \\ \bigvee X_n \arrow[r] & X, \end{tikzcd}$$

where ∇ denotes the fold map.

HINT. Convert ∇ to a cofibration by replacing $\bigvee X_n$ with $(\bigvee X_n) \rtimes I$.

(b) Show that there is a cofiber sequence

$$\bigvee \Sigma X_n \xrightarrow{\mathrm{in}_1 - \mathrm{in}_2} \Big(\bigvee \Sigma X_n\Big) \vee \Big(\bigvee \Sigma X_n\Big) \xrightarrow{\nabla} \bigvee \Sigma X_n.$$

(c) Show that there is a cofiber sequence

$$\bigvee \Sigma X_n \xrightarrow{\mathrm{id} - \Sigma\,\mathrm{sh}} \bigvee \Sigma X_n \xrightarrow{j} \Sigma X.$$

(d) Write $\sigma_k : \prod_n \widetilde{h}^k(X_n) \to \prod_n \widetilde{h}^k(X_n)$ for $\mathrm{id} - \mathrm{sh}_k$, where sh_k is the algebraic shift map for the tower of k-dimensional cohomology groups. Show that there are exact sequences

$$\prod \widetilde{h}^{k-1}(\Sigma X_n) \xrightarrow{\sigma_{k-1}} \prod \widetilde{h}^{k-1}(\Sigma X_n) \to \widetilde{h}^k(X) \to \prod \widetilde{h}^k(\Sigma X_n) \xrightarrow{\sigma_k} \prod \widetilde{h}^k(\Sigma X_n).$$

(e) Prove Proposition 21.20.

Exercise 21.22. Explain the sense, if any, in which the exact sequences of Proposition 21.20 are functorial.

We know that a CW complex X is the homotopy colimit of its skeleta. In ordinary represented cohomology, the $\lim^1$ 'error term' for the cohomology of this homotopy colimit is trivial.

Problem 21.23.

(a) Show that if a tower $\cdots \leftarrow G_n \leftarrow G_{n+1} \leftarrow \cdots$ is eventually constant, then $\lim^1 G_n = 0$.

(b) Show that when we express a CW complex X as the homotopy colimit of its skeleta $X_0 \to X_1 \to \cdots$, we have $\widetilde{H}^*(X) \cong \lim \widetilde{H}^*(X_n)$.

Problem 21.23 allows us to extend Proposition 21.7 (via Problem 21.11) from finite CW complexes to all CW complexes.

Problem 21.24. Suppose $\widetilde{h}^*$ and $\widetilde{k}^*$ are cohomology theories that satisfy the Wedge Axiom. Show that if the natural transformation of cohomology theories $T : \widetilde{h}^* \to \widetilde{k}^*$ is an isomorphism on spheres, then it is an isomorphism on all CW complexes.

21.1.5. Cohomology for Unpointed Spaces. We have defined cohomology theories as functors $\widetilde{h}^* : \mathcal{T}_* \to \mathrm{AB}\,\mathcal{G}^*$; these are sometimes called **reduced cohomology**. For unpointed spaces $X \in \mathcal{T}_\circ$, we use the **unreduced cohomology** of X, defined by the formula

$$h^*(X; G) = \widetilde{h}^*(X_+; G).$$

In the other direction, it is sometimes useful to apply the unreduced theory to a pointed space, so we write $h^*(X)$ for $h^*(X_-)$ for $X \in \mathcal{T}_*$.

Problem 21.25.

(a) Show that there is a canonical isomorphism $H^0(*; G) \cong G$.

(b) The map $X \to *$ induces a map $G = H^0(*; G) \to H^0(X; G)$. Show that this is an isomorphism if X is path-connected.

Problem 21.26. Show that a cofibration $A \hookrightarrow X$ in $\mathcal{T}_\circ$ gives rise to a long exact sequence of the form

$$\cdots \to h^{k-1}(A) \longrightarrow \widetilde{h}^k(X/A) \longrightarrow h^k(X) \longrightarrow h^k(A) \to \cdots .$$

Hint. Use Problem 5.143.

The distinction between the reduced and unreduced cohomology groups of a pointed space is easily understood and frequently negligible.

Problem 21.27. Let X be a pointed space, let $\widetilde{h}^*$ be a (reduced) cohomology theory, and let h^* be the corresponding unreduced theory.

(a) Show that $h^*(X) \cong \widetilde{h}^*(X) \oplus \widetilde{h}^*(S^0)$.

(b) Show that if $n \geq 1$, then $H^n(X; G) \cong \widetilde{H}^n(X; G)$.

21.2. Basic Computations

We have introduced a great deal of machinery and terminology, but how do we actually compute the cohomology of a space? In this section, we describe a few simple approaches that are frequently useful, and provide many examples for you to hone your skills.[2]

21.2.1. Cohomology and Dimension. If X is a CW complex, then the group $\widetilde{H}^n(X;G)$ depends only on the cells in dimensions $n-1, n$ and $n+1$.

Proposition 21.28. *If X is a CW complex, then the inclusion $X_{n+1} \hookrightarrow X$ and the quotient map $X_{n+1} \to X_{n+1}/X_{n-2}$ induce isomorphisms*

$$\widetilde{H}^n(X;G) \xrightarrow{\cong} \widetilde{H}^n(X_{n+1};G) \xleftarrow{\cong} \widetilde{H}^n(X_{n+1}/X_{n-2};G).$$

Proposition 21.28 implies some very easily applied criteria for the vanishing of ordinary cohomology groups.

Corollary 21.29. *If X has no cells in dimension n, then $\widetilde{H}^n(X;G) = 0$.*

Corollary 21.30. *If X is a $(c-1)$-connected and d-dimensional CW complex, then $\widetilde{H}^n(X;G) = 0$ for $n < c$ and $n > d$.*

The proofs of these results are essentially an exercise in reinterpreting what we already know about maps into Eilenberg-Mac Lane spaces.

Problem 21.31. Prove Proposition 21.28 and derive Corollaries 21.29 and 21.30.

Now you are ready to work out some examples.

Problem 21.32.

(a) Determine $\widetilde{H}^*(\mathbb{C}\mathrm{P}^n;G)$ and $\widetilde{H}^*(\mathbb{H}\mathrm{P}^n;G)$ for $1 \le n \le \infty$. What can you say about $\mathbb{R}\mathrm{P}^n$?

(b) Determine $H^*(\Omega S^{n+1};G)$ for $n \ge 2$. What can you say about the cohomology of ΩS^2?

(c) Determine $\widetilde{H}^*(\prod_1^r S^n;G)$ for $n \ge 2$. What can you say about $\prod_1^r S^1$?

(d) Determine $\widetilde{H}^*(S^2 \times \mathbb{C}\mathrm{P}^n)$. What can you say about $S^3 \times \mathbb{C}\mathrm{P}^n$?

(e) Determine $\widetilde{H}^*(\mathbb{C}\mathrm{P}^n \wedge \mathbb{H}\mathrm{P}^m;G)$.

21.2.2. Suspension Invariance. The cohomology of X is the same—up to a dimension shift—as that of ΣX. This suggests a strategy: if the cohomology of ΣX can be computed, then we can first evaluate the cohomology of ΣX, then shift dimensions to obtain $\widetilde{h}^*(X)$.

Problem 21.33. Compute the cohomology of the space constructed in Problem 19.38.

Problem 21.34. Determine $\widetilde{H}^*(\Omega\Sigma\mathbb{C}\mathrm{P}^n;G)$.

[2]One reason for putting so many computations *here* is to emphasize the fact that they can be done straight from the definition, with little or no extra machinery.

Problem 21.35.

(a) Determine $\widetilde{H}^*(S^3 \times \mathbb{C}\mathrm{P}^n; G)$.

(b) More generally, determine $\widetilde{H}^*(S^n \times X)$ in terms of the (unknown) $\widetilde{H}^*(X)$.

In the next few problems, you are asked to determine the cohomology of spaces using a general cohomology theory. The answer you give should be in terms of the (unspecified) coefficients $h^*(S^0)$. For example, you could write $\widetilde{h}^*(S^n) = \Sigma^n \widetilde{h}^*(S^0)$.

It will greatly clarify things to set up some notation. Write $P = S^{n_1} \times S^{n_2} \times \cdots \times S^{n_r}$; for a subset $S \subseteq \{1, 2, \ldots, r\}$, we set $P_S = \prod_{i \in S} S^{n_i}$ and define $n_S = \sum_{i \in S} n_i = \dim(P_S)$. Write

$$\mathrm{pr}_S : P \longrightarrow P_S \qquad \text{and} \qquad q_S : P_S \longrightarrow S^{n_S}$$

for the evident projection and quotient maps.

Problem 21.36. Show that $\widetilde{h}^*(S^{n_1} \times S^{n_2} \times \cdots \times S^{n_r}) \cong \bigoplus_S \Sigma^{n_S} \widetilde{h}^*(S^0)$.

Problem 21.37. Determine the cohomology $\widetilde{h}^*$ of the spaces ΩS^2 and $\Omega \mathbb{C}\mathrm{P}^n$.

Problem 21.38. Determine $\widetilde{h}^*(K)$, where K is the Klein bottle (defined in Problem 19.37).

21.2.3. Exact Sequences. The most powerful, and most subtle, of the basic computational tools is the exact sequence of a cofiber sequence. The subtlety is that the use of these sequences requires us to know the maps between cohomology groups, not just the groups. This can seem quite daunting at first, so we begin with some results that help to identify the cohomology of maps.

Problem 21.39. Let X be a co-H-space with comultiplication $\phi : X \to X \vee X$. Since the quotient maps $q_1, q_2 : X \vee X \to X$ give an isomorphism $\widetilde{h}^*(X \vee X) \to \widetilde{h}^*(X) \times \widetilde{h}^*(X)$, the map induced by ϕ can be identified with a map

$$\phi^* : \widetilde{h}^*(X) \times \widetilde{h}^*(X) \longrightarrow \widetilde{h}^*(X).$$

(a) Show that, under this identification, $\phi^*(u, v) = u + v$.

(b) Determine the effect on cohomology of the inverse map $\nu : X \to X$.

We apply this to understand the effect on cohomology of maps between wedges of spheres.

Problem 21.40.

(a) Show that if $f : S^n \to S^n$ is a map of degree d, then the induced map $f^* : \widetilde{h}^*(S^n; G) \to \widetilde{h}^*(S^n; G)$ is multiplication by d.

(b) Show that if $f : S^n \to S^n$ is a map of degree d, then the map $f \wedge \mathrm{id}_X$ induces multiplication by d on $\widetilde{h}^*(\Sigma^n X)$.

(c) Let $f : \bigvee_{\mathcal{J}} S^n \to \bigvee_{\mathcal{I}} S^n$ be any map. According to Theorem 19.31, f may be identified with a matrix $A(f) \in \overline{M}_{\mathcal{I}\times\mathcal{J}}(\mathbb{Z})$. Show that the induced map

$$f^* : \prod_{\mathcal{I}} \widetilde{h}^*(S^n) \longrightarrow \prod_{\mathcal{J}} \widetilde{h}^*(S^n)$$

is of finite type and is given by the matrix transpose $A(f)^T$.

Now you are prepared to work out some more examples.

Problem 21.41. Let X be the homotopy pushout of $S^n \xleftarrow{a} S^n \xrightarrow{b} S^n$, where the arrows are labelled with their degrees. Determine $\widetilde{H}^*(X;G)$.

Problem 21.42.

(a) Compute $\widetilde{H}^*(M(\mathbb{Z}/n,k);\mathbb{Z})$.

(b) Compute $\widetilde{H}^*(M(\mathbb{Z}/n,k);G)$ for a general abelian group G.

Problem 21.43. Determine $\widetilde{H}^*(M(\mathbb{Z}/n,k)\wedge M(\mathbb{Z}/m,l);G)$.

HINT. Moore spaces are suspensions.

Problem 21.44. Determine $\widetilde{H}^*(\Omega M(\mathbb{Z}/n,k);\mathbb{Z})$ and $\widetilde{H}^*(\Omega M(\mathbb{Z}/n,k);\mathbb{Z}/p)$, where p is prime.

21.2.4. Cohomology of Projective Spaces. Sometimes there are no shortcuts, and the only way to compute the cohomology of a space is to do the hard work of really understanding how it is built. We end this section with an extremely important example of this kind of computation.

To calculate the cohomology of real projective spaces, we will work by induction on the cellular cone decomposition

$$\begin{tikzcd}
S^0 \arrow[d, "q_0"'] & S^1 \arrow[d, "q_1"'] & S^2 \arrow[d, "q_2"'] & & S^{n-1} \arrow[d, "q_{n-1}"'] & \\
\mathbb{R}\mathrm{P}^0 \arrow[r] & \mathbb{R}\mathrm{P}^1 \arrow[r] & \mathbb{R}\mathrm{P}^2 \arrow[r] & \cdots \arrow[r] & \mathbb{R}\mathrm{P}^{n-1} \arrow[r] & \mathbb{R}\mathrm{P}^n.
\end{tikzcd}$$

The cofiber sequences $S^n \to \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^{n+1}$ give us exact sequences, but they don't simply fall apart for us, so we'll need to determine some of the maps in the sequence. Start by writing our cone decomposition of $\mathbb{R}\mathrm{P}^n$ vertically in the left column of the following diagram. Then extend the 'L-shaped' cofiber sequences to long zig-zag cofiber sequences, and weave them together, resulting in the diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d, dashed] & & \\
S^{n-1} \arrow[r, dashed] & \mathbb{R}\mathrm{P}^{n-1} \arrow[r] \arrow[d, dashed] & S^{n-1} \arrow[r] & \Sigma\mathbb{R}\mathrm{P}^{n-2} \arrow[r, dashed] \arrow[d] & S^{n-1} \arrow[r, dashed] & \cdots \\
S^{n} \arrow[r, "q_n"'] \arrow[rr, dotted, bend left, "\boxed{\delta_n}"] & \mathbb{R}\mathrm{P}^{n} \arrow[r, dashed, "j_n"'] \arrow[d] & S^{n} \arrow[r, dashed, "\Sigma q_{n-1}"'] \arrow[rr, dotted, bend left, "\boxed{\Sigma\delta_{n-1}}"] & \Sigma\mathbb{R}\mathrm{P}^{n-1} \arrow[r, "\Sigma j_{n-1}"'] \arrow[d, dashed] & S^{n} \arrow[r] & \cdots \\
S^{n+1} \arrow[r, dashed] & \mathbb{R}\mathrm{P}^{n+1} \arrow[r] \arrow[d, dashed] & S^{n+1} \arrow[r] & \Sigma\mathbb{R}\mathrm{P}^{n} \arrow[r, dashed] \arrow[d] & S^{n+1} \arrow[r, dashed] & \cdots \\
 & \vdots & & \vdots & &
\end{tikzcd}$$

in which the dotted zig-zag is the cofiber sequence mentioned above, and the other zig-zags are the cofiber sequences for other values of n.

The key to our computation is to understand the maps δ_n. Since they are maps between spheres, this amounts to computing their degree.

Lemma 21.45. *The map $\delta_n = j_n \circ q_n : S^n \to S^n$ has degree $1 + (-1)^{n+1}$. That is, $\deg(\delta_n) = 0$ if n is even, and $\deg(\delta_n) = 2$ if n is odd.*[3]

Problem 21.46. Prove Lemma 21.45 as follows.

(a) Show that there is a map $\beta : S^n \vee S^n \to S^n$ so that the solid arrows in

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] & \mathbb{R}\mathrm{P}^{n-1} \arrow[d] \\
S^{n} \arrow[r, "q_n"] \arrow[d, "\phi"'] \arrow[dr, dotted, "\delta_n" description] & \mathbb{R}\mathrm{P}^{n} \arrow[d, "j_n"] \\
S^n \vee S^n \arrow[r, "\beta"'] & S^n
\end{tikzcd}$$

form a strictly commutative diagram, where the vertical maps are cofiber sequences and ϕ is the pinch map that we used to define the group structure on $\pi_n(X)$.

(b) Since $\beta \in [S^n \vee S^n, S^n] \cong [S^n, S^n] \times [S^n, S^n]$, we can write $\beta = (\beta_1, \beta_2)$. Show that $\delta_n = [\beta_1] + [\beta_2]$.

(c) Show that β_1 and β_2 are homeomorphisms (it is enough to show that they are bijective). Conclude that $|\deg(\beta_1)| = |\deg(\beta_2)| = 1$.

[3]If we choose our generator for $H^n(S^n; \mathbb{Z})$ differently, then the map will have degree 0 or -2, respectively, but this has no practical effect on our applications.

(d) Prove Lemma 21.45 by showing that $\beta_2 = \alpha_n \circ \beta_1$, where $\alpha_n : S^n \to S^n$ is the antipodal map.

Now choose a coefficient group G, and apply the functor $H^n(\,?\,; G)$ to the big diagram of spaces we constructed above to obtain

$$\begin{tikzcd}
 & \vdots & & \vdots & \\
0 & 0 \arrow[l, dashed] \arrow[u] & 0 \arrow[l] & 0 \arrow[l] \arrow[u, dashed] & 0 \arrow[l, dashed] \\
H^n(S^n) & H^n(\mathbb{R}\mathrm{P}^n) \arrow[l, "q_n^*"'] \arrow[u, dashed] & H^n(S^n) \arrow[l, dashed, "j_n^*"'] \arrow[ll, dotted, bend right, "\delta_n^* = 0 \text{ or } 2"'] & H^n(\Sigma\mathbb{R}\mathrm{P}^n) \arrow[l, dashed, "q_{n-1}^*"'] \arrow[u] & H^n(\Sigma S^{n-1}) \arrow[l, "j_{n-1}^*"'] \arrow[ll, dotted, bend right, "\delta_{n-1}^* = 2 \text{ or } 0"'] \\
H^n(S^{n+1}) & H^n(\mathbb{R}\mathrm{P}^{n+1}) \arrow[l, dashed] \arrow[u] & 0 \arrow[l] & H^n(\Sigma\mathbb{R}\mathrm{P}^n) \arrow[l] \arrow[u, dashed] & 0. \arrow[l, dashed] \\
 & \vdots \arrow[u, dashed] & & \vdots \arrow[u] &
\end{tikzcd}$$

(Why are we justified in labeling the second curved arrow δ^*_{n-1} instead of $(\Sigma\delta_{n-1})^*$?) There is a lot of information hidden in this diagram. Your next problem is to tease out some of it.

Problem 21.47.

(a) Show that $j_n^* : H^n(S^n; G) \to H^n(\mathbb{R}\mathrm{P}^n; G)$ is surjective. Conclude that $H^n(\mathbb{R}\mathrm{P}^n) \cong \frac{H^n(S^n)}{\ker(j_n^*)}$.

(b) Show that $\mathrm{Im}(q_n^*) = \mathrm{Im}(\delta_n^*)$ and $\mathrm{Im}(q_{n-1}^*) = \mathrm{Im}(\delta_{n-1}^*)$.

(c) Determine $H^n(\mathbb{R}\mathrm{P}^n; G)$ in terms of G; your answer will depend on whether n is even or odd.

Here are some more useful bits of information about this diagram.

Problem 21.48.

(a) Show that $H^n(\mathbb{R}\mathrm{P}^{n+1}; G) \to H^n(\mathbb{R}\mathrm{P}^n; G)$ is injective.

(b) Show that q_n^* is either the zero map or multiplication by 2.

(c) If $G = \mathbb{Z}/2$, show that j_n^* is injective, no matter what n is.

Here is the promised calculation, which we state in terms of the algebraic functors Tor and $\otimes$ (see Section A.5). What you really need to know for now is that multiplication by any $n \in \mathbb{Z}$ defines a homomorphism $\mathbf{n} : G \to G$, and there is an exact sequence

$$0 \to \mathrm{Tor}(G, \mathbb{Z}/n) \longrightarrow G \xrightarrow{\mathbf{n}} G \longrightarrow G \otimes \mathbb{Z}/n \to 0.$$

That is, $G \otimes \mathbb{Z}/n \cong G/nG$ and $\mathrm{Tor}(G, \mathbb{Z}/n) = \{g \in G \mid n \cdot g = 0\}$.

Proposition 21.49. *The cohomology of* $\mathbb{R}\mathrm{P}^n$ *is given by*

$$H^k(\mathbb{R}\mathrm{P}^n; G) = \begin{cases} G & \text{if } k = 0, \\ G \otimes \mathbb{Z}/2 & \text{if } 1 \le k < n,\ k \text{ even}, \\ \mathrm{Tor}(G, \mathbb{Z}/2) & \text{if } 1 \le k < n,\ k \text{ odd}, \end{cases}$$

$$H^n(\mathbb{R}\mathrm{P}^n; G) = \begin{cases} G & \text{if } n \text{ is odd}, \\ G \otimes \mathbb{Z}/2 & \text{if } n \text{ is even}. \end{cases}$$

Problem 21.50. Prove Proposition 21.49.

HINT. Use induction.

Exercise 21.51. Work out $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z})$ and $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2)$.

We have done this important and nontrivial computation by studying the web of interlocking cofiber sequences that come from the CW decomposition of $\mathbb{R}\mathrm{P}^n$. This is a very powerful technique, and we'll revisit this approach to calculation in much greater generality several times in later chapters. The following exercise hints at the basic algebraic construction underlying this kind of computation.

Exercise 21.52. Referring to the large diagram of cohomology groups we constructed,

(a) show that $\delta_k^* \circ \delta_{k-1}^* = 0$, so that $\mathrm{Im}(\delta_{k-1}^*) \subseteq \ker(\delta_k^*)$;

(b) show that $H^k(\mathbb{R}\mathrm{P}^{n+1}) \cong \frac{\ker(\delta_k^*)}{\mathrm{Im}(\delta_{k-1}^*)}$.

21.3. The External Cohomology Product

The smash product is the basis for a very useful multiplicative structure in cohomology. Cohomology classes $u \in \widetilde{H}^n(X; G)$ and $v \in \widetilde{H}^m(Y; H)$ are homotopy classes of maps

$$u : X \longrightarrow K(G, n) \qquad \text{and} \qquad v : Y \longrightarrow K(H, m).$$

Smashing these maps together gives a map

$$u \wedge v : X \wedge Y \longrightarrow K(G, n) \wedge K(H, m).$$

This map can be made into a cohomology class by composing with the map $c : K(G, n) \wedge K(H, m) \to K(G \otimes H, n + m)$ that we constructed and studied in Section 19.6. The **external product** of u and v is the cohomology class

$$u \bullet v \in \widetilde{H}^{n+m}(X \wedge Y; G \otimes H) = [X \wedge Y, K(G \otimes H, n + m)]$$

defined by the diagram

$$
\begin{tikzcd}
X \wedge Y \arrow[rr, "u \bullet v"] \arrow[dr, "u \wedge v"'] & & K(G \otimes H, n+m) \\
& K(G,n) \wedge K(H,m). \arrow[ur, "c"'] &
\end{tikzcd}
$$

The external product defines maps (of pointed sets)

$$\widetilde{H}^n(X;G) \times \widetilde{H}^m(Y,H) \longrightarrow \widetilde{H}^{n+m}(X \wedge Y; G \otimes H)$$

for every n and m. Notice that $|u \bullet v| = |u| + |v|$.

The first thing to do is to establish the basic algebraic properties of this product.

Proposition 21.53. *The external product is bilinear, associative and commutative; that is, the identities*

(a) $(u+v) \bullet w = u \bullet w + v \bullet w$ *and* $u \bullet (v+w) = u \bullet v + u \bullet w$,

(b) $(u \bullet v) \bullet w = u \bullet (v \bullet w)$,

(c) $u \bullet v = (-1)^{|u||v|} v \bullet u$

hold for all $u \in \widetilde{H}^*(X;G)$, $v \in \widetilde{H}^*(Y;H)$ *and* $w \in \widetilde{H}^*(Z,;J)$.[4]

The bilinearity of the external product means that it defines natural group homomorphisms

$$\kappa_{X,Y} : \widetilde{H}^n(X;G) \otimes \widetilde{H}^m(Y;H) \longrightarrow \widetilde{H}^{n+m}(X \wedge Y; G \otimes H),$$

which we call **Kunneth map@Künneth map**.

Problem 21.54. Prove Proposition 21.53.

Tensor Products of Graded Groups. We can simplify our notation using tensor products of graded abelian groups. If A^* and B^* are two graded abelian groups, then we define a new graded abelian group $A^* \otimes B^*$ by setting

$$(A^* \otimes B^*)^n = \bigoplus_{i+j=n} A^i \otimes B^j.$$

This should remind you of the way the i-cells in X and the j-cells in Y give rise to $(i+j)$-cells in $X \wedge Y$. With this notation, our external Künneth map takes the form

$$\kappa_{X,Y} : \widetilde{H}^*(X;G) \otimes \widetilde{H}^*(Y;H) \longrightarrow \widetilde{H}^*(X \wedge Y; G \otimes H).$$

[4]Strictly speaking, $u \bullet v \in \widetilde{H}^*(X \wedge Y; G \otimes H) = [X \wedge Y, K(G \otimes H, n+m)]$, and $v \bullet u \in [Y \wedge X, K(H \otimes G, n+m)]$. These different, but isomorphic, homotopy sets are related by the homotopy equivalences t and T discussed in Section 19.6. Part (c) of Proposition 21.53 *really* means that $t_*(u \bullet v) = (-1)^{nm} T^*(v \bullet u)$.

Unpointed Künneth Maps. Although the external cohomology product is defined in terms of smash products, it is also useful in studying the cohomology of ordinary cartesian products of spaces. In fact, since

$$X_+ \wedge Y_+ = (X \times Y)_+,$$

we have the commutative square

$$\begin{tikzcd} H^*(X;G) \otimes H^*(Y;H) \arrow[r, "\kappa"] \arrow[d, equal] & H^*(X \times Y; G \otimes H) \arrow[d, equal] \\ \widetilde{H}^*(X_+;G) \otimes \widetilde{H}^*(Y_+H) \arrow[r, "\kappa"] & \widetilde{H}^*(X_+ \wedge Y_+; G \otimes H). \end{tikzcd}$$

21.4. Cohomology Rings

In this section we use the diagonal map to convert the external product to an *internal* product. When we use ring coefficients, this construction allows us to give $H^*(X;R)$ the structure of a graded R-algebra.

21.4.1. Graded R-Algebras. Let's start with one more bit of algebraic terminology. A **graded R-algebra** is a graded R-module A^* equipped with a multiplication map

$$\mu : A^* \otimes_R A^* \longrightarrow A^*.$$

The graded algebra A^* is **unital** if there is a unit map $\eta : R \to A^*$ such that the diagram

$$\begin{tikzcd} R \otimes A^* \arrow[r] \arrow[dr, bend right] & A^* \otimes A^* \arrow[d, "\mu"] & A^* \otimes R \arrow[l] \arrow[dl, bend left] \\ & A^* & \end{tikzcd}$$

commutes; it is **associative** or (graded) **commutative** if the evident diagrams commute.

We'll write $\mathbf{Mod}_R$ for the category of R-modules and $\mathbf{Alg}_R$ for the category of R-algebras; then $\mathbf{Mod}_R^*$ and $\mathbf{Alg}_R^*$ will denote the corresponding graded categories.

Exercise 21.55. Write down the 'evident diagrams'; then formulate the notion of a graded module over a graded R-algebra.

Problem 21.56. Let A^* be a commutative graded ring, and let $x \in A^*$. Show that if $|x|$ is odd, then $2x^2 = 0$.

We say that a cohomology theory $h^* : \mathcal{T}_\circ \to \text{AB}\,\mathcal{G}^*$ is **multiplicative** if there is a lift in the diagram

$$\begin{tikzcd} & \mathbf{Alg}^*_R \arrow[d] \\ \mathcal{T}_\circ \arrow[ur, dotted] \arrow[r, "h^*"] & \mathbf{Mod}^*_R \end{tikzcd}$$

of functors and natural transformations. It is certainly conceivable that a given theory might have several lifts; a choice of lift amounts to giving h^* the structure of a multiplicative cohomology theory. In this section, we'll show how to give ordinary represented cohomology with coefficients in a ring R the structure of a multiplicative cohomology theory.

21.4.2. Internalizing the Exterior Product. If we take $X = Y$, the external product gives us a map

$$\widetilde{H}^*(X;G) \otimes \widetilde{H}^*(X;H) \longrightarrow \widetilde{H}^*(X \wedge X; G \otimes H).$$

Let's go one step further: from $\widetilde{H}^*(X \wedge X; G \otimes H)$ to $H^*(X; G \otimes H)$. For this we need a map $X \to X \wedge X$. Luckily, we have already seen just such a map. The diagonal map is $\Delta : X \to X \times X$ given by $x \mapsto (x, x)$. The **reduced diagonal map** is the composite map $\bar{\Delta}$ in the diagram

$$\begin{tikzcd} X \arrow[rr, "\bar{\Delta}"] \arrow[dr, bend right, "\Delta"'] & & X \wedge X \\ & X \times X \arrow[ur, bend right, "\wedge"'] & \end{tikzcd}$$

where the symbol $\wedge : X \times X \to X \wedge X$ denotes the standard quotient map. Then $\bar{\Delta}^* : \widetilde{H}^*(X \wedge X) \to \widetilde{H}^*(X)$, and we define the (external) **cup product** of $u \in \widetilde{H}^n(X;G)$ and $v \in \widetilde{H}^m(X;H)$ to be

$$u \cdot v = \bar{\Delta}^*(u \bullet v) \in \widetilde{H}^{n+m}(X; G \otimes H).$$

Getting back to spaces and maps, the cup product of u and v is the composition in the diagram

$$\begin{tikzcd} X \arrow[r, "u \cdot v"] \arrow[d, "\bar{\Delta}"'] & K(G \otimes H, n+m) \\ X \wedge X \arrow[r, "u \wedge v"] & K(G,n) \wedge K(H,m). \arrow[u, "c"'] \end{tikzcd}$$

Problem 21.57. Show that the cup product is a homomorphism

$$\widetilde{H}^*(X;G) \otimes \widetilde{H}^*(X;H) \longrightarrow \widetilde{H}^*(X; G \otimes H).$$

In order to internalize this still-external product, we first need to restrict to the case $G = H$; then we need to relate $G \otimes G$ to G. This forces us to use coefficients in a ring R. If R is a ring, the multiplication is a homomorphism $R \otimes R \to R$, and so it induces a map $m : K(R \otimes R, n) \to K(R, n)$. Throwing this map into the mix, we get an internal product

$$\begin{tikzcd}
\widetilde{H}^n(X;R) \otimes \widetilde{H}^m(X;R) \arrow[r] \arrow[d, "\kappa"'] & \widetilde{H}^*(X;R) \\
\widetilde{H}^{n+m}(X \wedge X; R \otimes R) \arrow[r, "m_*"] & \widetilde{H}^{n+m}(X \wedge X; R). \arrow[u, "\bar{\Delta}^*"']
\end{tikzcd}$$

This internal cup product is given by the formula

$$u \cdot v = \bar{\Delta}^* \, m_*(u \bullet v) \in \widetilde{H}^*(X;R),$$

or, in terms of spaces again, the cup product of u and v is the composition in the diagram

$$\begin{tikzcd}
X \arrow[rr, "u \cdot v"] \arrow[d, "\bar{\Delta}"'] & & K(R, n+m) \\
X \wedge X \arrow[r, "u \wedge v"] & K(R,n) \wedge K(R,m) \arrow[r, "c"] & K(R \otimes R, n+m). \arrow[u, "m"']
\end{tikzcd}$$

The Künneth Map with Ring Coefficients. It is worth considering the composition $H^*(X;R) \otimes H^*(Y;R) \longrightarrow H^*(X \times Y;R)$ of the unpointed Künneth map with the map on Eilenberg-Mac Lane spaces induced by the multiplication $R \otimes R \to R$. This map makes it easy to understand the maps induced by the inclusions

$$\mathrm{in}_X : X \longrightarrow X \times Y \qquad \text{and} \qquad \mathrm{in}_Y : Y \longrightarrow X \times Y$$

and the projections

$$\mathrm{pr}_X : X \times Y \longrightarrow X \qquad \text{and} \qquad \mathrm{pr}_Y : X \times Y \longrightarrow Y.$$

Problem 21.58.

(a) Show that $\mathrm{pr}_X^*(u) = u \bullet 1$.

(b) Show that $\mathrm{in}_X^*(u \bullet v) = uv$ if $v \in R \subseteq H^0(Y;R)$ and is zero otherwise.

(c) Show that $\mathrm{pr}_Y^*(v) = 1 \bullet v$ and that $\mathrm{in}_Y^*(u \bullet v) = uv$ if $u \in R \subseteq H^0(X;R)$ and is zero otherwise.

21.4.3. R-Algebra Structure. Let R be a ring and let X be a pointed space. Then the map $X_+ \to *_+$ which collapses X to a point induces a homomorphism $R = H^0(*;R) \hookrightarrow H^0(X;R)$, and the cup product defines a homomorphism

$$R \otimes H^n(X;R) \longrightarrow H^0(X;R) \otimes H^n(X;R) \longrightarrow H^n(X;R),$$

which we will denote by $r \otimes u \mapsto r \cdot u$. Thus the cup product gives $H^*(X;R)$ the structure of a graded R-module.

Problem 21.59. Show that $1_R \cdot x = x$ for all $x \in H^*(X;R)$.

Problem 21.60. Multiplication by $r \in R$ defines a homomorphism $R \to R$, and hence a map $r : K(R,n) \to K(R,n)$. Show that $r_*(u) = r \cdot u$ for every $u \in H^*(X;R)$, where on the right-hand side, the $\cdot$ refers to the R-module structure on $H^*(X;R)$.

The reduced cohomology $\widetilde{H}^*(X;R)$ also has an R-algebra structure and the canonical map $\widetilde{H}^*(X;R) \to H^*(X;R)$ is an algebra map. However, the algebra $\widetilde{H}^*(X;R)$ is not as nice as $H^*(X;R)$ because it has no multiplicative identity. On the other hand, when studying the Künneth map, we usually work with reduced cohomology because the notation is nicer: we study $X \wedge Y$ instead of $(X \times Y)_+$.

The internal product inherits the basic algebraic properties of the external product. Thus we have the following structure theorem.

Theorem 21.61. *The cup product is bilinear, so it induces a homomorphism of graded abelian groups*

$$H^*(X;R) \otimes_R H^*(X;R) \longrightarrow H^*(X;R),$$

giving $H^*(X;R)$ the structure of an associative and graded commutative R-algebra.

Problem 21.62.

(a) Derive Theorem 21.61 from Proposition 21.53.

(b) Show that in $H^*(X;\mathbb{Z}/p)$, with p odd, if $|x|$ is odd, then $x^2 = 0$.

Multiplicative Properties of the Künneth Map. We know that the Künneth map is a homomorphism of graded R-modules

$$\kappa : H^*(X;R) \otimes_R H^*(Y;R) \longrightarrow H^*(X \times Y;R),$$

but the situation is actually even nicer. The tensor product $H^* \otimes K^*$ of two graded R-algebras may also be given the structure of a graded R-algebra by defining

$$(h_1 \otimes k_1) \cdot (h_2 \otimes k_2) = (-1)^{|k_1||h_2|}(h_1 h_2 \otimes k_1 k_2).$$

Thus $H^*(X;R) \otimes_R H^*(Y;R)$ and $H^*(X \times Y;R)$ both have R-algebra structures, and it makes sense to ask whether κ is an R-algebra homomorphism.

Proposition 21.63. *The map $\kappa : H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R)$ is a homomorphism of graded R-algebras.*

Problem 21.64. Prove Proposition 21.63.

HINT. Use the formula $(\mathrm{pr}_X \times \mathrm{pr}_Y) \circ \Delta_{X \times Y} = \mathrm{id}_{X \times Y}$.

21.5. Computing Algebra Structures

Once again, we pause to illustrate methods and approaches for the computation of the algebra structures; again, we hope to emphasize that we are making computations with a minimum of machinery.

21.5.1. Products of Spheres. You determined the cohomology groups of a product of spheres in Problem 21.36. In the following problems, you will determine the algebra structure of the cohomology of such a product.

We will use the notation established in the discussion leading up to Problem 21.36. In addition, for $S \subseteq \{1, 2, \ldots, r\}$ we let $u_S : S^{n_S} \to K(R, n_S)$ be the map corresponding to $1 \in R \cong \pi_{n_S}(K(R, n_S))$. Finally, we define cohomology classes $x_S \in H^{n_S}(P; R)$ by the compositions

$$\begin{tikzcd} P \arrow[r, "x_S"] \arrow[d, "\mathrm{pr}_S"'] & K(R, n_S) \\ P_S \arrow[r, "q_S"] & S^{n_S}. \arrow[u, "u_S"'] \end{tikzcd}$$

Problem 21.65. Show that $H^*(P; R) \cong \bigoplus_S \Sigma^{n_S} R \cdot x_S$ as R-modules.

HINT. Try the case $r = 2$ to get your bearings.

We'll determine the algebra structure in terms of the generators x_S.

Problem 21.66.

(a) Show that $x_S^2 = 0$ for every S.

(b) Suppose $\max(S) < \min(T)$. Show that the diagram

$$\begin{tikzcd} P \arrow[r, "\Delta"] \arrow[d, "\mathrm{pr}_{S\cup T}"'] & P \times P \arrow[r, "\mathrm{pr}_S \times \mathrm{pr}_T"] & P_S \times P_T \arrow[r, "q_S \times q_T"] & S^{n_S} \times S^{n_T} \arrow[d, "q"] \\ P_{S\cup T} \arrow[rrr, "q_{S\cup T}"] & & & S^{n_S + n_T} \end{tikzcd}$$

commutes, where q is the obvious quotient map.

(c) Show $x_S = \prod_{i\in S} x_i$ and conclude that $H^*(P; R) = \Lambda_R(x_1, \ldots, x_r)$.

Problem 21.67. Let $P = \overbrace{S^n \times S^n \times \cdots \times S^n}^{r \text{ factors}}$, and let $x_1, x_2, \ldots, x_r \in H^n(P; R)$ be the generators we defined above. Evaluate $(x_1 + x_2 + \cdots + x_r)^r$.

HINT. The answer should depend on the parity of n.

21.5.2. Bootstrapping from Known Cohomology. Whenever we build a new space from known spaces, the new space comes equipped with maps from (or to) the known ones. These maps induce algebra homomorphisms that we can use to derive information about the cohomology of the new space.

Cohomology of the Loop Space of a Sphere. In Problem 21.32 you used the James splitting to compute the cohomology groups of ΩS^{n+1}. Choose generators $y_k \in H^{nk}(\Omega S^{n+1}; R) \cong R$. Then $y_k \cdot y_l = c_{k,l}\, y_{k+l}$ for some coefficient $c_{k,l} \in R$. Since the James construction relates ΩS^{n+1} to products of spheres, whose cohomology algebras we fully understand, we can use what we know about products of spheres to determine the coefficients $c_{k,l}$ and hence the algebra structure of $H^*(\Omega S^{n+1})$.

Problem 21.68.

(a) Show that there is a (homotopy) pushout square

$$\begin{tikzcd} T^{k-1}(S^n) \arrow[r] \arrow[d, "\mathrm{in}"'] & J^{k-1}(S^n) \arrow[d] \\ S^n \times \cdots \times S^n \arrow[r, "q_k"] & J^k(S^n), \end{tikzcd}$$

where $T^{k-1}(S^n)$ is the fat wedge.

(b) Show that $q_k^* : H^{nk}(J^k(S^n); R) \to H^{nk}(S^n \times \cdots \times S^n; R)$ is an isomorphism.

(c) Show that $q_1^*(y_1) = x_1 + \cdots + x_k$, where $x_i \in H^n(S^n \times \cdots \times S^n; R)$ are the standard generators we defined above.

Problem 21.69. We want to determine any and all algebraic relations among the various generators $y_i \in H^{2ni}(\Omega S^{2n+1}; \mathbb{Z}) \cong \mathbb{Z}$.

(a) Because $H^{nk}(\Omega S^{2n+1}; \mathbb{Z}) \cong \mathbb{Z} \cdot y_k$, we know that $y_1^k = c_k\, y_k$, for some coefficient $c_k \in \mathbb{Z}$. Determine the integer c_k.[5]

(b) Use (a) to determine the coefficient $c_{k,l}$ in the equation $y_k \cdot y_l = c_{k,l}\, y_{k+l}$.

(c) Summarize what you have learned by giving a complete description of the ring structure on $H^*(\Omega S^{2n+1}; \mathbb{Z})$.

The computation of the cohomology of $H^*(\Omega S^{2n}; \mathbb{Z})$ is similar, but a bit more involved, so it is left as a project (Project 25.139).

21.5.3. Cohomology Algebras for Projective Spaces. Our next goal is to determine the cohomology ring structure for $\mathbb{F}\mathrm{P}^n$.

[5] Up to a sign—you can, if you like, go back and redefine the generators y_k so that each of the coefficients c_k is ≥ 0.

Generators for $H^{kd}(\mathbb{F}\mathrm{P}^n; R)$. In Section 21.2 we computed the cohomology *groups* of the projective spaces. To detail the multiplicative structure, though, we need to have explicitly defined generators, so that is where we begin.

As usual, the situation is simplest when $\mathbb{F} \neq \mathbb{R}$.

Problem 21.70. Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$, and let R be a ring with multiplicative identity 1_R.

(a) Show that $\pi_{kd}(\mathbb{F}\mathrm{P}^n/\mathbb{F}\mathrm{P}^{k-1}) \cong \mathbb{Z}$, generated by the map u_k given by the composition

$$\begin{tikzcd} S^{kd} \arrow[rr, "u_k"] \arrow[dr, "\cong"'] & & \mathbb{F}\mathrm{P}^n/\mathbb{F}\mathrm{P}^{k-1} \\ & \mathbb{F}\mathrm{P}^k/\mathbb{F}\mathrm{P}^{k-1}. \arrow[ur] & \end{tikzcd}$$

(b) Show that there is a unique homotopy class $\bar{x}_k : \mathbb{F}\mathrm{P}^n/\mathbb{F}\mathrm{P}^{k-1} \to K(R, k)$ carrying u_k to $1_R \in R \cong \pi_{kd}(K(R, kd))$.

(c) Define $x_k \in H^{kd}(\mathbb{F}\mathrm{P}^n; R)$ to be the composite

$$\mathbb{F}\mathrm{P}^n \xrightarrow{q} \mathbb{F}\mathrm{P}^n/\mathbb{F}\mathrm{P}^{k-1} \xrightarrow{\bar{x}_k} K(R, kd).$$

Show that $H^{kd}(\mathbb{F}\mathrm{P}^n; R) = R \cdot x_k$.

The cells in the real projective spaces appear in consecutive dimensions, so the computation of the group $\pi_k(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{k-1})$ is a bit more delicate.

Problem 21.71.

(a) Show that the inclusion $\mathbb{R}\mathrm{P}^{k+1}/\mathbb{R}\mathrm{P}^{k-1} \hookrightarrow \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{k-1}$ induces an isomorphism on $\pi_k(\ ?\ )$.

(b) Show that there is a diagram

$$\begin{tikzcd} S^k \arrow[r] \arrow[d, equal] & \mathbb{R}\mathrm{P}^k \arrow[r] \arrow[d] & \mathbb{R}\mathrm{P}^{k+1} \arrow[d] \\ S^k \arrow[r, "\delta_k"] & S^k \arrow[r, "u_k"] & \mathbb{R}\mathrm{P}^{k+1}/\mathbb{R}\mathrm{P}^{k-1} \end{tikzcd}$$

in which the rows are cofiber sequences.

(c) Because of part (a), we'll feel free to write u_k for the image of u_k in $\pi_k(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{k-1})$. Show that

$$\pi_k(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{k-1}) = \begin{cases} \mathbb{Z} \cdot u_k & \text{if } k \text{ is even,} \\ \mathbb{Z}/2 \cdot u_k & \text{if } k \text{ is odd.} \end{cases}$$

This computation shows that we must treat the even dimensions differently from the odd dimensions.

Problem 21.72. Let R be a ring with multiplicative identity 1_R.

(a) Show that there is a unique map $\bar{x}_{2k} : \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2k-1} \to K(R, 2k)$ such that $(\bar{x}_{2k})_*(u_{2k}) = 1_R \in R \cong \pi_{2k}(K(R, 2k))$.

(b) Now suppose that $2 \cdot 1_R = 0 \in R$. Show that there is a unique cohomology class $\bar{x}_{2k+1} : \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{2k} \to K(R, 2k+1)$ such that

$$(\bar{x}_{2k+1})_*(u_{2k+1}) = 1_R \in R \cong \pi_{2k+1}(K(R, 2k+1)).$$

(c) Assuming $\bar{x}_k \in H^k(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{k-1}; R)$ has been defined for the ring R, define $x_k \in H^k(\mathbb{R}\mathrm{P}^n; R)$ to be the composite

$$\mathbb{R}\mathrm{P}^n \xrightarrow{q} \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{k-1} \xrightarrow{\bar{x}_k} K(R, k)$$

and show that $H^{kd}(\mathbb{F}\mathrm{P}^n; R) = R \cdot x_k$.

Exercise 21.73. Express $\mathbb{R}\mathrm{P}^{k+1}/\mathbb{R}\mathrm{P}^{k-1}$ as a wedge of Moore spaces.

The Multiplicative Structure of the Projective Spaces. Now that we have clearly defined generators, we are prepared to establish the algebra structure of the cohomology of the projective spaces.

Theorem 21.74. *Let R be a ring with multiplicative identity 1_R, and let $n \leq \infty$.*

(a) For $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$, $H^*(\mathbb{F}\mathrm{P}^n; R) \cong R[x_d]/(x_d^{n+1})$ where $x_d \in H^d(\mathbb{F}\mathrm{P}^n; R)$.

(b) If $2 \cdot 1_R = 0$, then $H^*(\mathbb{R}\mathrm{P}^n; R) \cong R[x_1]/(x_1^{n+1})$ where $x_1 \in H^1(\mathbb{R}\mathrm{P}^n; R)$.

(We interpret x_1^∞ as 0.)

Since cup products in $H^*(X)$ are defined in terms of the diagonal map $\Delta : X \to X \times X$, we begin with a detailed study of the topology of the diagonal map for $\mathbb{F}\mathrm{P}^n$. We approach the diagonal for $\mathbb{F}\mathrm{P}^n$ via the diagonal for D^{nd}.

It will simplify our work considerably if, instead of using the standard round disk D^{nd}, we can instead work with the homeomorphic n-fold product $(D^d)^n = D^d \times \cdots \times D^n$. Our first task is to establish a very nice homeomorphism between these spaces, so that our work with the product will transfer automatically to give conclusions about the ordinary disks.

We consider the ordinary disk D^{nd} as a subspace of $\mathbb{F}^n$, so that points in D^{nd} are n-tuples $(x_1, x_2, \ldots, x_n)$ with $x_k \in \mathbb{F}$. Write

$$V_k = \{(x_1, x_2, \ldots, x_n) \mid |x_i| \leq |x_k| \text{ for all } i\} \qquad \text{for } k = 1, \ldots, n,$$

and define a map $\sigma : D^{nd} \to (D^d)^n$ by setting

$$\sigma(x_1, x_2, \ldots, x_n) = \frac{1}{|x_k|}(x_1, x_2, \ldots, x_n)$$

for $(x_1, x_2, \ldots, x_n) \in V_k$.

Problem 21.75.

(a) Show that σ is well-defined and is a homeomorphism.

(b) Show that if $z \in S^{d-1}$ and $x \in D^{nd}$, then $\sigma(z \cdot x) = z \cdot \sigma(x)$.

(c) Show that the diagram

$$\begin{tikzcd}
S^{nd-1} \arrow[r, "\cong"] \arrow[d] & \partial(D^d)^n \arrow[d] \\
D^{nd} \arrow[r, "\cong"] & (D^d)^n
\end{tikzcd}$$

commutes, where $\partial(D^d)^n$ denotes the boundary of $(D^d)^n$, which is the set of all points x such that $|x_k| = 1$ for at least one k.

Fix $a, b \in \mathbb{N}$ such that $a + b = n$. Then $(D^d)^n = (D^d)^a \times (D^d)^b$, and we can write elements of $(D^d)^n$ in the form (x, y) with $x \in (D^d)^a$ and $y \in (D^d)^b$. For clarity later, we'll let $0^a \in \mathbb{F}^s$ and $0^b \in \mathbb{F}^t$ denote the origins in the two factors. Now consider the diagram of pairs

$$\begin{tikzcd}
((D^d)^n, *) \arrow[r, "\Delta"] \arrow[d, equal] & ((D^d)^n, *) \boxplus ((D^d)^n, *) \arrow[d, "u"] \\
((D^d)^n, *) \arrow[r] \arrow[d, "r"'] & ((D^d)^n, S^{ad-1} \times 0^b) \boxplus ((D^d)^n, 0^a \times S^{bd-1}) \\
((D^d)^n, S^{nd-1}) \arrow[r, dotted, "\xi"] & ((D^d)^a \times 0^b, S^{ad-1} \times 0^b) \boxplus (0^a \times (D^d)^b, 0^a \times S^{bd-1}), \arrow[u, "i"']
\end{tikzcd}$$

where $(D^d)^a$ is identified with the first a coordinates, and $(D^d)^b$ with the last b. We want to find a homeomorphism ξ which renders the diagram commutative *up to a homotopy of pairs*. In fact, it is easy to define this map and homotopy:

$$\xi(x, y) = ((x, 0), (0, y)) \quad \text{and} \quad H((x, y), t) = ((x, ty), (tx, y)).$$

Problem 21.76.

(a) Show that $H : u \circ \Delta \simeq i \circ \xi \circ r$.

(b) Show that H is a homotopy of pairs.

(c) Show that H respects the action of S^{d-1}; that is,

$$H(z \cdot (x, y), t) = z \cdot H((x, y), t)$$

for all $z \in S^{d-1}$, $(x, y) \in (D^d)^a \times (D^d)^b$ and $t \in I$.

Now we can use our homeomorphism σ to make the corresponding conclusions about the ordinary round disks.

Lemma 21.77. *There is a homeomorphism of pairs $\zeta : (D^{nd}, S^{nd-1}) \to (D^{ad}, S^{ad-1}) \times (D^{bd}, S^{bd-1})$ that makes the diagram*

$$\begin{tikzcd}
(D^{nd},*) \arrow[rr, "\Delta"] \arrow[d, equal] & & (D^{nd},*) \boxplus (D^{nd},*) \arrow[d] \\
(D^{nd},*) \arrow[r, "d"] \arrow[d, "r"'] & (D^{nd}, S^{ad-1}) \boxplus (D^{nd}, S^{bd-1}) & \\
(D^{nd}, S^{n-1}) \arrow[r, dotted, "\zeta"] & (D^{ad}, S^{ad-1}) \boxplus (D^{bd}, S^{bd-1}) \arrow[u, "i"'] &
\end{tikzcd}$$

commute up to a homotopy of pairs $G : i \circ \zeta \circ r \simeq d$ such that $G(z \cdot x, t) = z \cdot G(x,t)$ for each $z \in S^{d-1}$ and each $t \in I$.

Exercise 21.78. Prove Lemma 21.77 in full detail.

Now consider the characteristic maps $D^{nd} \to \mathbb{F}\mathrm{P}^n$ that we discussed in Section 3.3 when we defined CW decompositions for projective spaces. If we apply them to each disk in the diagram of Lemma 21.77, we obtain the diagram of pairs

$$\begin{tikzcd}
(\mathbb{F}\mathrm{P}^n,*) \arrow[r, "\Delta"] \arrow[d] & (\mathbb{F}\mathrm{P}^n,*) \boxplus (\mathbb{F}\mathrm{P}^n,*) \arrow[d] \\
(\mathbb{F}\mathrm{P}^n,*) \arrow[r] \arrow[d] & (\mathbb{F}\mathrm{P}^n, \mathbb{F}\mathrm{P}^{a-1}) \boxplus (\mathbb{F}\mathrm{P}^n, \widehat{\mathbb{F}\mathrm{P}}^{b-1}) \\
(\mathbb{F}\mathrm{P}^n, \mathbb{F}\mathrm{P}^{n-1}) \arrow[r] & (\mathbb{F}\mathrm{P}^a, \mathbb{F}\mathrm{P}^{a-1}) \boxplus (\widehat{\mathbb{F}\mathrm{P}}^{b}, \widehat{\mathbb{F}\mathrm{P}}^{b-1}) \arrow[u]
\end{tikzcd}$$

which is commutative up to a homotopy of pairs, because the homotopy of Lemma 21.77 respects the S^{d-1} action.[6] Collapsing the pairs yields the diagram

$$\begin{tikzcd}
\mathbb{F}\mathrm{P}^n \arrow[r, "\bar{\Delta}"] \arrow[d, equal] & \mathbb{F}\mathrm{P}^n \wedge \mathbb{F}\mathrm{P}^n \arrow[d] \\
\mathbb{F}\mathrm{P}^n \arrow[r] \arrow[d, "j_n"'] & (\mathbb{F}\mathrm{P}^n/\mathbb{F}\mathrm{P}^{a-1}) \wedge (\mathbb{F}\mathrm{P}^n/\widehat{\mathbb{F}\mathrm{P}}^{b-1}) \\
S^{nd} \arrow[r, "\cong"] & S^{ad} \wedge S^{bd}. \arrow[u]
\end{tikzcd}$$

[6] Remember that $\mathbb{F}\mathrm{P}^{t-1} \subseteq \mathbb{F}\mathrm{P}^n$ is the set of points (in homogeneous coordinates) of the form $[x_1, x_2, \ldots, x_t, 0, 0, \ldots, 0]$; the notation $\widehat{\mathbb{F}\mathrm{P}}^{t-1}$ is supposed to indicate that typical elements have the form $[0, 0, \ldots, 0, x_1, x_2, \ldots, x_t]$.

The commutativity of this diagram is the key to our understanding of the cup product in $H^*(\mathbb{F}\mathrm{P}^n)$.

Problem 21.79. Let R be a ring whose multiplicative identity 1_R satisfies $2 \cdot 1_R = 0 \in R$. Show that

(a) $j_n^* : H^n(S^n) \to H^n(\mathbb{R}\mathrm{P}^n)$ is an isomorphism.

(b) Let $n \leq m$. Show that the inclusion $\mathbb{R}\mathrm{P}^n \hookrightarrow \mathbb{R}\mathrm{P}^m$ induces isomorphisms $H^k(\mathbb{R}\mathrm{P}^m) \to H^k(\mathbb{R}\mathrm{P}^n)$ for $k \leq n$.

To prove Theorem 21.74, we just have to show that $x_a x_b = x_{a+b}$. Because of Problem 21.79, it suffices to verify this in the cohomology of $\mathbb{F}\mathrm{P}^n$, where $n = a + b$.

Problem 21.80. Prove Theorem 21.74 by studying the diagram

$$\begin{tikzcd}
\mathbb{F}\mathrm{P}^n \arrow[r, "\bar{\Delta}"] \arrow[d, equal] & \mathbb{F}\mathrm{P}^n \wedge \mathbb{F}\mathrm{P}^n \arrow[r, "x_a \wedge x_b"] \arrow[d] \arrow[rr, bend left, "x_a \bullet x_b"] & K_{ad} \wedge K_{bd} \arrow[r] \arrow[d, equal] & K_{nd} \arrow[d, equal] \\
\mathbb{F}\mathrm{P}^n \arrow[r] \arrow[d, "j_n"'] & (\mathbb{F}\mathrm{P}^n/\mathbb{F}\mathrm{P}^{a-1}) \wedge (\mathbb{F}\mathrm{P}^n/\mathbb{F}\mathrm{P}^{b-1}) \arrow[r, "\bar{x}_a \wedge \bar{x}_b"] & K_{ad} \wedge K_{bd} \arrow[r] & K_{nd} \\
S^{nd} \arrow[r, "\cong"] & S^{ad} \wedge S^{bd}. \arrow[u] \arrow[urr, bend right, "w"'] & &
\end{tikzcd}$$

We imposed the condition $2 \cdot 1_R = 0 \in R$ for the real projective spaces so that we could make use of our odd-dimensional generators. But the even-dimensional generators are perfectly well-defined for all rings R. Let's define an 'evenly-graded' abelian group

$$H^{\mathrm{even}}(\mathbb{R}\mathrm{P}^n; R) = \{H^{2k}(\mathbb{R}\mathrm{P}^n; R) \mid k = 0, 1, 2, \ldots\}$$

and notice that the multiplication in $H^*(\mathbb{R}\mathrm{P}^n; R)$ restricts to an algebra structure in $H^{\mathrm{even}}(\mathbb{R}\mathrm{P}^n; R)$.

Problem 21.81. Let $m = \lfloor \frac{n}{2} \rfloor$, and show that $H^{\mathrm{even}}(\mathbb{R}\mathrm{P}^n; R) = R[x_2]/(x_2^m)$.

21.6. Variation of Coefficients

In this section we study the relationships between the cohomology of X with different coefficient groups (or rings). The underlying topological phenomenon is Proposition 17.52, which says that short exact sequences of abelian groups give rise to fibration sequences of Eilenberg-Mac Lane spaces.

Two Useful Lemmas. The fibration sequence of Eilenberg-Mac Lane spaces that comes from a short exact sequence of abelian groups gives rise to a long exact sequence in cohomology.

Problem 21.82. Show that if $0 \to A \to B \to C \to 0$ is a short exact sequence of abelian groups, then there is a long exact sequence

$$\cdots \to \widetilde{H}^n(X;A) \longrightarrow \widetilde{H}^n(X;B) \longrightarrow \widetilde{H}^n(X;C) \longrightarrow \widetilde{H}^{n+1}(X;A) \to \cdots$$

which is natural in both X and the sequence.

The second of our useful lemmas shows how to cut a long exact sequence into many short exact sequences.

Problem 21.83. Show that if $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A' \xrightarrow{f'} B'$ is exact, then the maps g and h induce an exact sequence

$$0 \to \operatorname{coker}(f) \longrightarrow C \longrightarrow \ker(f') \to 0.$$

21.6.1. Universal Coefficients. The Universal Coefficients Theorem allows us to compute the cohomology with respect to G, given the cohomology with respect to $\mathbb{Z}$. It can be generalized to compute the cohomology with respect to the R-module M given the cohomology with respect to R.

Theorem 21.84 (Universal Coefficients Theorem)**.** *Suppose either*

(1) G is a finitely generated abelian group or

(2) X is of finite type.[7]

Then there are natural exact sequences

$$0 \to \widetilde{H}^n(X;\mathbb{Z}) \otimes G \longrightarrow \widetilde{H}^n(X;G) \longrightarrow \operatorname{Tor}(\widetilde{H}^{n-1}(X;\mathbb{Z}), G) \to 0$$

for each n.

Problem 21.85.

(a) Let $0 \to F_1 \to F_0 \to G \to 0$ be a free resolution of the group G, and use Problem 21.82 to produce a long exact sequence of cohomology groups in various coefficients. Cut the long exact sequence into a system of interlocking short exact sequences as in Problem 21.83.

(b) Let F be a free abelian group. Show that if either
   (1) F is finitely generated or
   (2) X_{n+1} is a finite complex,
   then there is a natural isomorphism $\widetilde{H}^n(X;F) \cong \widetilde{H}^n(X;\mathbb{Z}) \otimes F$.

(c) Use part (b) to identify the groups in the short sequences of part (a) and so prove the Universal Coefficients Theorem.

[7] I.e., X has a CW decomposition with only finitely many cells in each dimension.

Problem 21.86. Use Theorem 21.84 to determine $\widetilde{H}^*(\mathbb{R}\mathrm{P}^n; G)$ in terms of $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z})$.

Theorem 21.84 makes it possible to compute cohomology with coefficients in an abelian group from cohomology with integer coefficients. Since abelian groups are $\mathbb{Z}$-modules, this suggests a generalization: a universal coefficients theorem that allows us to compute $\widetilde{H}^*(X; M)$ in terms of $\widetilde{H}^*(X; R)$, for an R-module M. The key point is Problem A.38, which shows that if R is a **principal ideal domain** (generally abbreviated **PID**), then every R-module M has a free resolution $0 \to F_1 \to F_0 \to M \to 0$.

Problem 21.87. Show that there are universal coefficients sequences with $\mathbb{Z}$ and G replaced with a PID R and an R-module M, respectively.

Problem 21.88.

(a) Show that if $\widetilde{H}^*(X; R) = 0$, then $\widetilde{H}^*(X; M) = 0$ for all R-modules M.

(b) Show that if X is a simply-connected CW complex and $\widetilde{H}^*(X; \mathbb{Z}) = 0$, then $X \simeq *$.

(c) Show that for any CW complex X, if $\widetilde{H}^*(X; \mathbb{Z}) \simeq *$, then $\Sigma X \simeq *$.

One of the most important consequences of Theorem 21.84 is the special case in which $\widetilde{H}^*(X; R)$ is a free R-module.

Problem 21.89. Show that if $\widetilde{H}^*(X; R)$ is a free R-module, then, for any R-module M, $\widetilde{H}^*(X; M) \cong \widetilde{H}^*(X; R) \otimes_R M$.

21.7. A Simple Künneth Theorem

In algebraic topology, any theorem that gives a formula for the cohomology (of whatever kind) of a product of spaces is referred to as a *Künneth theorem*. We'll show how to compute the cohomology of $X \wedge Y$ in terms of $\widetilde{H}^*(X; R)$ and $H^*(Y; R)$, under the assumption that $\widetilde{H}^*(Y; R)$ is a free R-module.[8] This is certainly a restricted statement, but it is nevertheless enormously useful.

In Chapter 24 we will prove a more general result based on a detailed examination of the attaching maps of the cells in a product.

We start with the simplest possible case: the cohomology of $S^1 \wedge X \cong \Sigma X$. We know that $\widetilde{H}^*(S^1; R)$ is a free graded R-module, so we have a composite isomorphism

$$\widetilde{H}^k(X; R) \xrightarrow{\cong} R \otimes_R \widetilde{H}^k(X; R) \xrightarrow{\cong} \widetilde{H}^*(S^1; R) \otimes_R H^k(X; R) \xrightarrow{\cong} H^{k+1}(\Sigma X; R)$$

[8] Which is automatically true if R is a field.

and a suspension isomorphism

$$\Sigma : \widetilde{H}^k(X;R) \xrightarrow{\cong} \widetilde{H}^{k+1}(\Sigma X;R)$$

as part of the structure of a cohomology theory. The key to our analysis is that these are in fact the same map.

Problem 21.90. Show that the diagram

$$\begin{tikzcd} \widetilde{H}^*(X;R) \arrow[r, "\cong"] \arrow[dr, "\Sigma"'] & R \otimes_R \widetilde{H}^k(X;R) \arrow[r, "\cong"] & \widetilde{H}^1(S^1;R) \otimes_R \widetilde{H}^k(X;R) \arrow[dl, "\kappa"] \\ & \widetilde{H}^{k+1}(\Sigma X;R) & \end{tikzcd}$$

is commutative. Conclude that in general, the external product

$$\kappa : \widetilde{H}^*(S^n;R) \otimes_R \widetilde{H}^*(X;R) \longrightarrow \widetilde{H}^*(\Sigma^n X;R)$$

can be identified with the n-fold suspension isomorphism.

We know from Proposition 21.63 that the external product is a homomorphism of graded R-algebras

$$\widetilde{H}^*(X;R) \otimes_R \widetilde{H}^*(Y;R) \longrightarrow \widetilde{H}^*(X \wedge Y;R).$$

Our goal is to show that this map is actually an **isomorphism** in many cases, including the case where R is a field.

Theorem 21.91 (Künneth)**.** *Let X be a CW complex and suppose that $\widetilde{H}^*(Y;R)$ is a free R-module. Then the Künneth map*

$$\kappa : \widetilde{H}^*(X;R) \otimes_R \widetilde{H}^*(Y;R) \longrightarrow \widetilde{H}^*(X \wedge Y;R)$$

is an isomorphism of graded R-algebras.

The corresponding statement for unreduced cohomology tells us about the cohomology algebra of a cartesian product.

Corollary 21.92. *Let X be a CW complex and suppose that $H^*(Y;R)$ is a free R-module. Then the Künneth map*

$$\kappa : H^*(X;R) \otimes_R H^*(Y;R) \longrightarrow H^*(X \times Y;R)$$

is an isomorphism of graded R-algebras.

Since κ is already known to be an algebra homomorphism, we really only need to show that κ is bijective.

Problem 21.93.

(a) Let Y be any space, and define functors $\widetilde{J}^n(\,?\,) = \widetilde{H}^n(\,?\, \wedge Y;R)$. Show that the functors $\widetilde{J}^n$ constitute a cohomology theory.

(b) Suppose that $\widetilde{H}^*(Y;R)$ is a free R-module, and define

$$\widetilde{L}^*(\,?\,) = \widetilde{H}^*(\,?\,;R) \otimes_R \widetilde{H}^*(Y;R).$$

Show that the functors L^n constitute a cohomology theory.

(c) Show that the external cohomology product defines a natural transformation of cohomology theories $\Phi : \widetilde{L}^* \to \widetilde{J}^*$.

(d) Derive Theorem 21.91 and Corollary 21.92.

Problem 21.94. Use Theorem 21.91 to compute $H^*(S^{n_1} \times \cdots \times S^{n_r};R)$. Compare your results with Problem 21.66(c).

21.8. The Brown Representability Theorem

Our definition of cohomology began, secretly, back in Chapter 17 with the difficult construction of the Eilenberg-Mac Lane spaces $K(G,n)$. With these spaces in hand, we defined $\widetilde{H}^n(X;G) = [X, K(G,n)]$ and the basic properties of these functors flowed easily from the general theory. For example, the relation $\Omega K(G,n+1) \simeq K(G,n)$ gives rise to the suspension isomorphism of Theorem 21.2(b); this isomorphism yields the long exact sequence relating the cohomology groups of the spaces in a cofiber sequence, and so on.

When faced with an abstract cohomology theory $\widetilde{h}^*$, wouldn't it be nice to know there were spaces $L(n)$ and natural isomorphisms $[\,?\,, L(n)] \xrightarrow{\cong} \widetilde{h}^n$? Such an isomorphism would allow us to apply all that we have learned about computing and analyzing homotopy sets to understand the otherwise utterly abstract functors $\widetilde{h}^n$. The Brown Representability Theorem tells us that there are such spaces, though the natural transformations are only guaranteed to be isomorphisms for CW complexes.

One very nice corollary of Brown's theorem is that all natural transformations of cohomology theories (on the category of pointed CW complexes, anyway) are represented by maps between the representing spaces. Another important corollary is the uniqueness of ordinary cohomology on the category of CW complexes: for each abelian group G, there is a unique ordinary cohomology theory with coefficients G.

21.8.1. Representing Homotopy Functors.

21.8.1. Representing Homotopy Functors. Here are two properties that a contravariant homotopy functor $F : \mathcal{T}_* \to \mathbf{Sets}_*$ might satisfy:

(1) *Wedge Axiom.* If $\{X_\alpha\}$ is any set of spaces, the natural comparison map $F(\bigvee X_\alpha) \to \prod F(X_\alpha)$ is an isomorphism.

(2) *Mayer-Vietoris Axiom.* Apply F to a homotopy pushout square like so:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "j"] \\ C \arrow[r, "g"] & D, \end{tikzcd} \qquad \begin{tikzcd} F(A) & F(B) \arrow[l, "i^*"'] \\ F(C) \arrow[u, "f^*"] & F(D). \arrow[l, "g^*"] \arrow[u, "j^*"'] \end{tikzcd}$$

For any pair of elements $u \in F(B)$ and $v \in F(C)$ such that $f^*(u) = i^*(v)$, there is an element $w \in F(D)$ such that $g^*(w) = v$ and $j^*(w) = u$. Pictorially:

$$\begin{tikzcd} t & u \arrow[l, mapsto] \\ v \arrow[u, mapsto] & w. \arrow[l, mapsto, dotted] \arrow[u, mapsto, dotted] \end{tikzcd}$$

These two properties are enough to guarantee representability of the functor F, at least for CW complexes.

Problem 21.95. Show that if F is representable, then F satisfies the Wedge and Mayer-Vietoris Axioms.

The Brown Representability Theorem is the converse of Problem 21.95.

Theorem 21.96 (Brown Representability)**.** *If $F : \mathcal{T}_* \to \mathrm{AB}\,\mathcal{G}$ is a contravariant homotopy functor that satisfies the Wedge and Mayer-Vietoris axioms, then*

(a) there is a CW complex E with an element $e \in F(E)$ such that the natural transformation

$$@_e : [X, E] \longrightarrow F(X) \qquad \textit{given by} \quad f \mapsto f^*(e)$$

is an isomorphism for every CW complex X, and

(b) the space E and the element $e \in F(E)$ are unique up to homotopy in the sense that if E' and $e' \in F(E')$ are another CW complex and element which represent F, then there is a homotopy equivalence, unique up to homotopy, $g : E \to E'$ such that $g^*(e') = e$.

The element $e \in F(E)$ is called a **fundamental class** for the functor F. Because of the Yoneda lemma, natural transformations of representable functors are also representable.

Corollary 21.97. *Let F and G be two functors satisfying the conditions of Theorem 21.96, so that they are represented by $e_F \in F(E_F)$ and $e_G \in G(E_G)$. If $\Phi : F \to G$ is a natural transformation, then there is a unique* *homotopy class* $\phi : E_F \to E_G$ *such that the diagram*

$$\begin{tikzcd} [\,?\,, E_F] \arrow[r, "\phi_*"] \arrow[d, "\cong"'] & [\,?\,, E_G] \arrow[d, "\cong"] \\ F \arrow[r, "\Phi"] & G \end{tikzcd}$$

commutes.

The first step in our proof of Theorem 21.96 is to adapt the proof of Proposition 21.7 to reduce the proof of Theorem 21.96 to the problem of representing F on spheres.

Problem 21.98. Let $F : \mathcal{T}_* \to \text{AB}\,\mathcal{G}$ be a homotopy functor satisfying the Wedge and Mayer-Vietoris properties.

(a) Show that if E is a space as in Theorem 21.96, then E is an abelian H-space.

(b) Show that if $A \to B \to C$ is a cofiber sequence, then the sequence

$$F(A) \longleftarrow F(B) \longleftarrow F(C) \longleftarrow F(\Sigma A) \longleftarrow F(\Sigma B)$$

is exact.

(c) Show that if E and $e \in F(E)$ are such that $@_e : [S^n, E] \to F(S^n)$ is an isomorphism for all n, then $@_e$ is an isomorphism for all CW complexes.

Our plan is to construct E skeleton-by-skeleton, with elements $e_n \in F(E_n)$ that represent F on spheres of dimension at most n. We start by setting $E_{-1} = *$ and $e_{-1} = 0 \in F(E_{-1})$.

Problem 21.99. Suppose we are given E_n and $e_n \in F(E_n)$ such that the transformation $@_{e_n} : [S^k, E_n] \to F(S^k)$ is an isomorphism for $k < n$ and surjective for $k = n$.

(a) Let $\ker(@_n) = \{g_\beta \mid \beta \in \mathcal{J}\}$ and define $\widetilde{E}_{n+1}$ by the (homotopy) pushout square

$$\begin{tikzcd} \bigvee_{\mathcal{J}} S^n \arrow[r, "(g_\beta)"] \arrow[d, "\mathrm{in}"'] \arrow[dr, phantom, "\text{HPO}"] & E_n \arrow[d] \\ \bigvee_{\mathcal{J}} D^{n+1} \arrow[r] & \widetilde{E}_{n+1}. \end{tikzcd}$$

Show that there is an element $\widetilde{e}_{n+1} \in F(\widetilde{E}_{n+1})$ such that the evaluation map $@_{\widetilde{e}_{n+1}} : [S^k, \widetilde{E}_{n+1}] \to F(S^k)$ is an isomorphism for all $k \leq n$.

(b) Construct the space E_{n+1} and the element $e_{n+1} \in F(E_{n+1})$ so that, in addition, $[S^{n+1}, E_{n+1}] \to F(S^{n+1})$ is surjective.

(c) Now we set $E = \operatorname{colim} E_n$, which is a CW complex by construction. Show that there is an element $e \in F(E)$ such that the inclusion $E_n \hookrightarrow E$ induces $e \mapsto e_n$.

HINT. Express E as a (homotopy) pushout involving the spaces E_n.

(d) Prove Theorem 21.96(a).

The proof of Theorem 21.96(b) follows the typical pattern of proving uniqueness in category theory.

Problem 21.100. Suppose $e \in F(E)$ and $e' \in F(E')$ both represent F for all CW complexes X.

(a) Find maps $f : E \to E'$ and $g : E' \to E$ such that $g^*(e) = e'$ and $f^*(e') = e$.

(b) Show that f and g induce inverse isomorphisms on $[X, ?\,]$ for CW complexes X, and conclude that E and E' are weakly homotopy equivalent.

(c) Prove Theorem 21.96(b).

21.8.2. Representation of Cohomology Theories. Theorem 21.96 applies to a very general class of functors, but we are primarily interested in the functors that constitute a cohomology theory.

Problem 21.101. Let $\widetilde{h}^*$ be a cohomology theory satisfying the Wedge Axiom.

(a) Show that each functor $\widetilde{h}^n$ satisfies the Mayer-Vietoris property and hence is representable by a space $E(n)$.

(b) Show that $\Omega E(n+1)$ and $E(n)$ are weakly homotopy equivalent.

An extremely important consequence of this theorem is that ordinary cohomology theories are completely determined by their coefficient groups, at least as long as you plug in CW complexes.

Corollary 21.102. *Two reduced ordinary cohomology theories $\widetilde{H}^*$ and $\widetilde{K}^*$ satisfying the Wedge Axiom and having the same coefficient group G are naturally equivalent on the category of CW complexes.*

Problem 21.103. Prove Corollary 21.102.

Exercise 21.104. Criticize the following argument:

> *If $\widetilde{h}^*$ and $\widetilde{k}^*$ have the same coefficient groups, then their representing spaces have the same homotopy groups and hence are weakly homotopy equivalent. Therefore all cohomology theories—ordinary or not—are determined by their coefficients, at least for CW complexes.*

Problem 21.101 implies that the representing spaces $E(n)$ are *infinite loop spaces*. A space X is called an **infinite loop space** if there are spaces $X(0), X(1), X(2), \ldots, X(n), \ldots$ such that $X = X(0)$ and $X(n) \simeq \Omega X(n+1)$ for all $n \geq 0$. Giving X the *structure* of an infinite loop space amounts to providing a list of homotopy equivalences $\{f_n : X(n) \xrightarrow{\simeq} \Omega X(n+1)\}$.

Conversely, every infinite loop space defines a cohomology theory. We extend the list of spaces in the infinite loop space $\mathbf{E} = \{E(0), E(1), \ldots\}$ to negative indices by setting $E(-n) = \Omega^n E(0)$.

Problem 21.105. Show that if $E(0)$ has the structure of an infinite loop space, then the functors $\widetilde{h}^n(X) = [X, E(n)]$ define a cohomology theory.

21.8.3. Representing a Functor on Finite Complexes. Sometimes we have functors whose behavior is known on finite CW complexes, but not necessarily all CW complexes: for example, a cohomology theory that does not—or is not known to—satisfy the Wedge Axiom. If we impose some mild finiteness conditions on F, we can still find a representing space.[9]

Theorem 21.106. *Let $F : \mathrm{H}\mathcal{T}_* \to \mathrm{AB}\,\mathcal{G}$ be a contravariant homotopy functor that satisfies the finite Wedge Axiom[10] and the Mayer-Vietoris axiom. Assume further that $F(S^n)$ is a finitely generated abelian group for each n. Then*

(a) there is a CW complex E of finite type with an element $e \in F(E)$ such that the natural transformation

$$@_e : [X, E] \longrightarrow F(X) \quad \textit{given by } f \mapsto f^*(e)$$

is an isomorphism for every finite CW complex X, and

(b) the space E and the element $e \in F(E)$ are unique up to homotopy in the sense that if E' and $e' \in F(E)$ are another CW complex and element which represent F, then there is a homotopy equivalence, unique up to homotopy, $g : E \to E'$ such that $g^*(e') = e$.

The proof is exactly the same as the one we have already given for Theorem 21.96, with the exception that at each stage of the construction we have to be sure that the required finiteness conditions hold.

Problem 21.107.

(a) Show that it suffices to represent F on spheres.

(b) Let $F : \mathcal{T}_* \to \mathrm{AB}\,\mathcal{G}$ be a functor such that $F(S^n)$ is finitely generated for each n. Show that $F(K)$ is finitely generated for each finite CW complex K.

[9] This is due to Adams [**3**].

[10] That is, the wedge axiom for finite wedges.

(c) Show that in the inductive construction we used in the proof of Theorem 21.96, each group $F(E_n)$ is finitely generated. Explain how to modify the construction of E_{n+1} so that we only have to attach finitely many disks in each dimension.

(d) Complete the proof of Theorem 21.106.

21.9. The Singular Extension of Cohomology

Because of the Brown Representability Theorem we feel that cohomology theories are more or less well-defined on CW complexes, or at least on finite CW complexes. But because we must study spaces, such as mapping spaces, that are not necessarily of the homotopy type of CW complexes, we do need cohomology theories defined on all of $\mathcal{T}$.

The most useful approach is one we will call the **singular extension**. The idea is actually quite simple: since we know how to evaluate $\widetilde{h}^*$ on CW complexes, we take a general space X, replace it with a weakly equivalent CW complex $\overline{X}$, and compute the cohomology of $\overline{X}$. Here are the details.

Problem 21.108. Let $\widetilde{h}^*$ be a cohomology theory defined, at least, on all CW complexes.

(a) Show that if $\overline{X} \to X$ and $\widetilde{X} \to X$ are two CW replacements for X, then there is a canonical isomorphism $\widetilde{h}^*(\widetilde{X}) \xrightarrow{\cong} \widetilde{h}^*(\overline{X})$.

(b) Show that if $\overline{f}$ and $\widetilde{f}$ are two lifts of $f : X \to Y$, then the isomorphisms of part (a) establish an isomorphism between $\overline{f}^*$ and $\widetilde{f}^*$.

In view of Problem 21.108, we can choose our favorite functorial CW replacement functor, which I'll call CW, and define

$$\widetilde{h}^*_{\mathrm{sing}}(\,?\,) = \widetilde{h}^*(\mathrm{CW}(\,?\,)).$$

This is called the **singular extension** of the cohomology theory $\widetilde{h}^*$.

Problem 21.109.

(a) Show that if $A \xrightarrow{f} B \to C$ is a cofiber sequence and if $\bar{f} : \overline{A} \to \overline{B}$ is a cellular replacement for f, then the induced map of cofibers $C_{\bar{f}} \to C$ is also a cellular replacement.

(b) Show that if $A \xrightarrow{f} B \to C$ is a cofiber sequence, then its cellular replacement $\overline{A} \to \overline{B} \to \overline{C}$ is also a cofiber sequence.

(c) Show that if $\widetilde{h}^*$ is a cohomology theory defined for CW complexes, then $\widetilde{h}^*_{\mathrm{sing}}$ is a cohomology theory satisfying the Weak Equivalence Axiom.

Problem 21.110.

(a) Show that there is a canonical comparison functor $\xi : \widetilde{h}^*(\,?\,) \to \widetilde{h}^*_{\text{sing}}$, and that ξ is an equivalence for all CW complexes.

(b) Show that $\widetilde{h}^*$ satisfies the Weak Equivalence Axiom if and only if ξ is an isomorphism for all spaces.

Another way to extend a cohomology theory $\widetilde{h}^*$ defined on CW complexes is to use Theorem 21.96 to find representations $\widetilde{h}^n \cong [\,?\,, L(n)]$ and define $\widetilde{h}^n(X) = [X, L(n)]$ for all spaces, not just CW complexes. Such a theory does not necessarily satisfy the Weak Equivalence Axiom.

Problem 21.111.

(a) Show that if we do this with a cohomology theory, we get a cohomology theory.

(b) Show by example that the resulting theory need not satisfy the Weak Equivalence Axiom.

Project 21.112. What must be true of a space X to conclude that every weak homotopy equivalence $f : A \to B$ induces a bijection $f^* : [B, X] \xrightarrow{\cong} [A, X]$?

21.10. An Additional Topic and Some Problems and Projects

21.10.1. Cohomology of $B\mathbb{Z}/n$. We need to understand the cohomology of $B\mathbb{Z}/m$ for $m > 1$, and we have already done this for $m = 2$, since $B\mathbb{Z}/2 = \mathbb{R}\mathrm{P}^\infty$.

We begin with a simple derivation of the additive structure of $H^*(B\mathbb{Z}/m)$. For this section, let's fix n and write $B = B\mathbb{Z}/m$.

Problem 21.113. Use the cofiber sequence $S^1 \cup_n D^2 \to B\mathbb{Z}/m \to X$ and the decomposition of Problem 20.7 to determine the cohomology groups of $B\mathbb{Z}/m$ for $m \geq 2$. Compare your answer to the computations in Proposition 21.49.

Our approach to the multiplicative structure is an adaptation of the one we used in Section 21.5.3. Recall that most of the technical work involved there was in proving Lemma 21.77. The multiplicative structure followed by applying certain characteristic maps to the cells in the lemma.

In Problem 16.53 we described a CW structure on B with a single cell in each dimension. Furthermore, B_{2n-1} was constructed as a quotient of S^{2n-1}, and B_{2n} was the mapping cone of the quotient map $q_{2n-1} : S^{2n-1} \to B_{2n-1}$. Let $\chi_{2n} : D^{2n} \to B_{2n}$ be the characteristic map of the $2n$-cell.

Theorem 21.114. *Let R be a ring with $m \cdot 1 = 0$. Then*

$$H^*(B\mathbb{Z}/m; R) \cong R[x_2] \otimes E(x_1).$$

Problem 21.115. Let R be a ring with $n \cdot 1_R = 0$.

(a) Compute the homotopy groups of B_k/B_{k-1}.

(b) Construct generators $x_k \in H^k(B; R)$.

(c) Construct a diagram analogous to the one in Problem 21.80 and use it to analyze the product $x_a \cdot x_b \in H^{a+b}(B\mathbb{Z}/p; \mathbb{Z}/p)$.

(d) Show that if $a \neq b$, then $x_a \cdot x_b \neq 0$.

(e) Prove Theorem 21.114.

For other rings, the situation can be more complicated, but the structure for the even dimensions remains quite easy to describe.

Problem 21.116. Show that $H^{\mathrm{even}}(B\mathbb{Z}/m; R) \cong (R/mR)[x_2]$.

21.10.2. Problems and Projects.

Problem 21.117.

(a) Show that if $m \leq n$, then $[\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n] \cong \widetilde{H}^2(\mathbb{C}\mathrm{P}^m; \mathbb{Z}) \cong \mathbb{Z}$. Compare with Problem 20.69.

(b) Show that if $m > n$, then every map $f : \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ induces the zero map $\widetilde{H}^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \to \widetilde{H}^*(\mathbb{C}\mathrm{P}^m; \mathbb{Z})$.

Problem 21.118. Let T be a transformation of cohomology theories, and assume $T : \widetilde{h}^*(S^0) \to \widetilde{k}^*(S^0)$ is zero. Show that for each finite CW complex X, there is an n such that $T^n : \widetilde{h}^*(X) \to \widetilde{h}^*(X)$ is zero. How is the number n related to the space X? What can you say about the behavior of T on infinite complexes?

Problem 21.119. The suspension transformations $\sigma_Z : Z \to \Omega\Sigma Z$ can be used to define maps $\Omega^n(\sigma_{\Sigma^n X}) : \Omega^n\Sigma^n X \to \Omega^{n+1}\Sigma^{n+1}X$ which yield a telescope diagram

$$\cdots \to \Omega^n\Sigma^n X \longrightarrow \Omega^{n+1}\Sigma^{n+1}X \longrightarrow \Omega^{n+2}\Sigma^{n+2}X \to \cdots .$$

Show that the homotopy colimit $\Omega^\infty\Sigma^\infty X$ of the telescope diagram is an infinite loop space (at least in the Serre sense: it is weakly homotopy equivalent to a loop space, which is itself weakly homotopy equivalent to a loop space, etc.).

Problem 21.120. Show that

$$X \mapsto \operatorname{colim}\left(\cdots \to [\Sigma^{n+t}X, \Sigma^n E] \to [\Sigma^{n+t+1}X, \Sigma^{n+1}E] \to \cdots\right)$$

is a cohomology theory defined on the category of finite complexes. Can it be bootstrapped to a genuine cohomology theory?

Project 21.121. A compact space M is called a compact n**-dimensional manifold** if every point $x \in M$ has a neighborhood U homeomorphic to $\mathbb{R}^n$. It is known that a compact n-manifold has the structure of an n-dimensional CW complex with a single n-dimensional cell.

(a) What can you say about the n^{th} cohomology of an n-dimensional manifold?

(b) Define **degree** for a map $f : M \to N$ from one n-manifold to another.

(c) Define **regular value** for a map $f : M \to N$ and explain how to use a regular value to determine the degree of f.

Problem 21.122. Develop a Künneth formula for half-smash products.

Project 21.123. Prove Theorem 21.96 for functors $F : \mathrm{H}\mathcal{T}_* \to \mathbf{Sets}_*$.

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Chapter 22

Homology

Homology theories are defined by simply replacing the word 'contravariant' in our definition of a cohomology theory with the word 'covariant' and reversing arrows where appropriate. Of course, it is quite easy to define something, only to discover that no such thing exists, and the concept of homology should instantly raise your suspicions. Unlike any other functor we have studied so far, homology is a *covariant* functor that behaves well with respect to *domains*. Since they are so much more counterintuitive, it should come as no surprise that the construction of homology theories is much more involved than the construction of cohomology theories. Strangely, and historically, homology predated cohomology by several decades.

In the first section, we develop the fundamental properties of homology theories, closely following the development of the corresponding ideas for cohomology theories. The development is purely from the definition, since we do not produce any examples until the second section. After exhibiting ordinary and extraordinary homology theories, we turn to multiplicative structures in homology, including a simple Künneth theorem. The relationship between (ordinary) homology and cohomology is discussed, and we end with an investigation of the homology and cohomology of H-spaces.

22.1. Homology Theories

In this section we give the abstract definition of homology theories and briefly outline the basic formal consequences, all of which closely parallels the exposition in Section 21.1.

22.1.1. Homology Theories. A (reduced) **homology theory** $\widetilde{h}_*$ is a covariant homotopy functor $\widetilde{h}_* : \mathcal{T}_* \to \text{AB}\,\mathcal{G}^*$ such that

- there is a natural isomorphism $\widetilde{h}_* \circ \Sigma \xrightarrow{\cong} \Sigma \circ \widetilde{h}_*$ and
- if $A \to B \to C$ is a cofiber sequence, then the sequence
$$\widetilde{h}_*(A) \longrightarrow \widetilde{h}_*(B) \longrightarrow \widetilde{h}_*(C)$$
is exact.

The graded abelian group $\widetilde{h}_*(S^0)$ is called the **coefficient group** for the homology theory $\widetilde{h}_*$. If the coefficient group is simply the abelian group G concentrated in dimension 0, then $\widetilde{h}_*$ is called an **ordinary homology theory** with coefficients in G.

As for cohomology, it is customary to use a capital H for ordinary theories and a small h for generic theories. Specific extraordinary theories will of course have their own notations.

Problem 22.1. Show that if $\widetilde{h}_*$ is a homology theory, then a cofiber sequence $A \to B \to C$ gives rise naturally to a long exact sequence of the form
$$\cdots \to \widetilde{h}_n(A) \longrightarrow \widetilde{h}_n(B) \longrightarrow \widetilde{h}_n(C) \longrightarrow \widetilde{h}_{n-1}(A) \to \cdots .$$

Exercise 22.2. Do the homotopy groups $\pi_n(\,?\,)$ define a homology theory?

Problem 22.3. Give a definition for transformations $T : \widetilde{h}_* \to \widetilde{k}_*$ of homology theories, and prove the analog for homology of Proposition 21.7: if T is an isomorphism on spheres, then it is an isomorphism for all finite CW complexes.

Unreduced Homology Theories. For each reduced homology theory $\widetilde{h}_* : \mathcal{T}_* \to \text{AB}\,\mathcal{G}^*$, there is a corresponding **unreduced homology theory** $h_* : \text{H}\mathcal{T}_\circ \to \text{AB}\,\mathcal{G}^*$ defined by the rule $h_*(X) = \widetilde{h}_*(X_+)$. The exactness property for the unreduced homology of a cofiber sequence $A \to B \to C$ is that the sequence
$$h_n(A) \longrightarrow h_n(B) \longrightarrow \widetilde{h}_n(C)$$
should be exact for each n.[1]

Wedges and Weak Equivalences. The inclusions $\text{in}_j : X_j \hookrightarrow \bigvee_{\mathcal{J}} X_j$ of the summands of a wedge induce maps $\widetilde{h}_*(X_j) \to \widetilde{h}_*(\bigvee_{\mathcal{J}} X_j)$ and hence a natural transformation
$$w : \bigoplus_{\mathcal{J}} \widetilde{h}_*(X_j) \longrightarrow \widetilde{h}_*\left(\bigvee_{\mathcal{J}} X_j\right).$$

[1]Recall that if $A \to B$ is a cofibration in $\mathcal{T}_\circ$, then $B/A \in \mathcal{T}_*$.

For finite wedges, w is an isomorphism, but just as for cohomology, a homology theory does not automatically behave well with respect to infinite wedges. We say the homology theory $\widetilde{h}_*$ satisfies the **Wedge Axiom** if w is an isomorphism for *every* wedge, infinite or finite.

Homology theories, like cohomology theories, are well-behaved on CW complexes, but their behavior on more general spaces is mysterious. We can dispel the mystery by imposing a **Weak Equivalence Axiom**: if $f : X \to Y$ is a weak homotopy equivalence, then $f_* : \widetilde{h}_*(X; G) \to \widetilde{h}_*(Y; G)$ must be an isomorphism.

There is a unique homology theory $\widetilde{h}_*^{\text{sing}}$ which is defined on all spaces, satisfies the Weak Equivalence Axiom, and is equipped with a natural transformation $\widetilde{h}_*^{\text{sing}} \to \widetilde{h}^*$ that is an isomorphism on CW complexes. We call it the **singular extension** of $\widetilde{h}_*$, defined by the rule

$$\widetilde{h}_*^{\text{sing}}(\,?\,) = \widetilde{h}_*(\mathrm{CW}(\,?\,)),$$

where $\mathrm{CW} : \mathcal{T}_* \to \mathcal{T}_*$ is a natural CW replacement. The weak equivalences $\mathrm{CW}(X) \to X$ induce the natural comparison $\xi : \widetilde{h}_*^{\text{sing}} \to \widetilde{h}_*$.

Problem 22.4. Let $\widetilde{h}_*$ be a homology theory.

(a) Show that $\widetilde{h}_*^{\text{sing}}$ is a homology theory.

(b) Show that ξ is a transformation of homology theories.

(c) Show that $\widetilde{h}_*$ satisfies the Weak Equivalence Axiom if and only if the comparison map $\xi : \widetilde{h}_*^{\text{sing}} \to \widetilde{h}_*$ is an isomorphism.

22.1.2. Homology and Homotopy Colimits. Homology, like cohomology, works well with domain-type constructions like homotopy colimits. Thus we have a Mayer-Vietoris sequence for homotopy pushout squares, and, in the presence of the Wedge Axiom, a formula for the homology of the homotopy colimit of a telescope.

Problem 22.5. Establish a Mayer-Vietoris exact sequence relating the homology of the spaces in a homotopy pushout square.

The **shift map** for a telescope $A_0 \xrightarrow{d^0} A_1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} A_n \xrightarrow{d^n} \cdots$ of abelian groups is the map $\mathrm{sh} : \bigoplus A_n \to \bigoplus A_n$ given by the formula $\mathrm{sh}(x_0, x_1, x_2, \ldots) = (0, d^0(x_0), d^1(x_1), \ldots)$.

Problem 22.6.

(a) Show that $\operatorname{colim} A_n = \operatorname{coker}(\mathrm{id} - \mathrm{sh})$.

(b) Show that $\mathrm{id} - \mathrm{sh}$ is injective.

Problem 22.7. Let $\widetilde{h}^*$ be a homology theory satisfying the Wedge Axiom. Show that if $X_0 \to X_1 \to \cdots \to X_n \to \cdots$ is a telescope in $\mathcal{T}_*$ having homotopy colimit X, the natural maps $X_n \to X$ induce an isomorphism

$$\operatorname{colim} \widetilde{h}_*(X_n) \xrightarrow{\cong} \widetilde{h}_*(X).$$

There is a uniqueness theorem for ordinary homology.

Theorem 22.8. *Let $T : \widetilde{h}^* \to \widetilde{k}^*$ be a natural transformation from one ordinary homology theory to another, and assume that T is an isomorphism on spheres.*

(a) If $\widetilde{h}^*$ and $\widetilde{k}^*$ satisfy the Wedge Axiom, then T is an isomorphism for all CW complexes.

(b) If $\widetilde{h}^*$ and $\widetilde{k}^*$ also satisfy the Weak Equivalence Axiom, then T is an isomorphism for all spaces.

Problem 22.9. Prove Theorem 22.8.

Theorem 22.8 leaves open the possibility that there are two ordinary homology theories with the same coefficients, but there are no transformations that induce an isomorphism between them. If this were the case, then we couldn't be sure that the two theories would agree, even on finite complexes. However, this cannot happen: we will show in Chapter 24 that there is an ordinary homology theory $\widetilde{H}_*(\,?\,;G)$ (defined on all CW complexes) that maps to any other ordinary homology theory with coefficients in G and is an isomorphism on spheres.

22.1.3. The Hurewicz Theorem. Homology has an advantage over cohomology when it comes to the detection of connectivity. According to Proposition 21.13, we can detect connectivity with ordinary cohomology, but we need to use infinitely many different coefficient groups. By contrast, the connectivity of a (simply-connected) space is determined by just its integral homology. In fact, more is true: there is a natural transformation $\mathrm{H} : \pi_* \to \widetilde{H}_*(\,?\,;\mathbb{Z})$, and if X is $(n-1)$-connected, H is an isomorphism in dimensions up to and including n.

By definition, $\widetilde{H}_*(S^0;\mathbb{Z}) \cong \mathbb{Z}$, and we fix a generator s_0 for this group.[2] The suspension isomorphisms

$$\widetilde{H}_0(S^0;\mathbb{Z}) \xrightarrow{\cong} \widetilde{H}_1(S^1;\mathbb{Z}) \to \cdots \to \widetilde{H}_n(S^n;\mathbb{Z}) \xrightarrow{\cong} \widetilde{H}_{n+1}(S^{n+1};\mathbb{Z}) \to \cdots$$

carry s_0 to generators $s_n \in \widetilde{H}_n(S^n;\mathbb{Z})$ for each n. The **Hurewicz map** is the map

$$\mathrm{H} : \pi_n(X) \longrightarrow \widetilde{H}_n(X;\mathbb{Z}) \qquad \text{given by } \mathrm{H} : \alpha \mapsto \alpha_*(s_n).$$

[2] Later when we introduce a multiplicative structure, we'll be able to distinguish between $+1$ and -1, and we will retroactively define s_0 to be the positive generator.

Problem 22.10.

(a) Show that H is a natural transformation $\pi_* \to \widetilde{H}_*(\,?\,;\mathbb{Z})$.

(b) Show that H is a homomorphism.

(c) Show that the diagram

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{\quad \mathrm{H} \quad} & \widetilde{H}_n(X;\mathbb{Z}) \\ {\scriptstyle \Sigma}\downarrow & & \downarrow{\scriptstyle \cong} \\ \pi_{n+1}(\Sigma X) & \xrightarrow{\quad \mathrm{H} \quad} & \widetilde{H}_{n+1}(\Sigma X;\mathbb{Z}) \end{array}$$

commutes for all spaces and all n.

Now we are prepared to state and prove the **Hurewicz theorem**.

Theorem 22.11 (Hurewicz)**.** *Let $\widetilde{H}_*(\,?\,;\mathbb{Z})$ be the ordinary homology theory with integer coefficients and satisfying the Wedge and Weak Equivalence Axioms. If X is $(n-1)$-connected, then $\mathrm{H} : \pi_k(X) \to H_k(X;\mathbb{Z})$ is*

- *an isomorphism for $k \le n$ if $n > 1$,*
- *abelianization for $k = n = 1$.*

We begin by reducing the proof of the theorem for all X to the special case in which X is a highly-connected Moore space.[3]

Problem 22.12.

(a) Show that it suffices to prove Theorem 22.11 for simply-connected spaces.

(b) Show that it suffices to prove Theorem 22.11 for CW complexes.

(c) Show that it suffices to prove Theorem 22.11 for highly-connected Moore spaces.

It is a simple matter to prove Theorem 22.11 for wedges of spheres. Since a Moore space is constructed as the cofiber in the sequence

$$W_1 \longrightarrow W_0 \longrightarrow M(G,n),$$

where W_0 and W_1 are wedges of n-spheres, and the induced map $\pi_n(W_1) \to \pi_n(W_0)$ is a free resolution of G, it is reasonable to hope that we will be able to bootstrap from spheres to Moore spaces.

Problem 22.13. Let $M = M(G,n)$.

(a) Prove Theorem 22.11 for $X = \bigvee S^n$.

[3] This approach was suggested by to M. Arkowitz.

(b) Show that there is a commutative diagram

$$\begin{tikzcd}
\pi_n(W_1) \arrow[r] \arrow[d, "\mathrm{H}"'] & \pi_n(W_0) \arrow[r] \arrow[d, "\mathrm{H}"'] & \pi_n(M) \arrow[r] \arrow[d, "\mathrm{H}"] & \pi_n(\Sigma W_1) \arrow[r] \arrow[d, "\mathrm{H}"] & \pi_n(\Sigma W_0) \arrow[d, "\mathrm{H}"] \\
\widetilde{H}_n(W_1;\mathbb{Z}) \arrow[r] & \widetilde{H}_n(W_0;\mathbb{Z}) \arrow[r] & \widetilde{H}_n(M;\mathbb{Z}) \arrow[r] & \widetilde{H}_n(\Sigma W_1;\mathbb{Z}) \arrow[r] & \widetilde{H}_n(\Sigma W_0;\mathbb{Z})
\end{tikzcd}$$

with exact rows.

(c) Prove Theorem 22.11.

Note that the proof given here makes no use whatsoever of any particular construction of an ordinary homology theory. In fact it cannot, because we have yet to construct one!

22.1.4. Computation. It is high time to do some computation. The basic tools and techniques for computing homology are the same as for cohomology: exploit the cellular structure, suspend the space, use the exact sequences associated to cofiber sequences, and, if all else fails, make a careful study of the topology of the space in question. We assume that our homology theories satisfy the Wedge and Weak Equivalence axioms.

First, we establish the impact of the cellular structure of a CW complex on its ordinary homology.

Proposition 22.14. *For a CW complex X, there are isomprhisms*

$$\widetilde{H}_n(X_{n+1}/X_{n-2};G) \xleftarrow{\cong} \widetilde{H}_n(X_{n+1};G) \xrightarrow{\cong} \widetilde{H}_n(X;G).$$

Corollary 22.15.

(a) If X is a CW complex with no n-cells, then $\widetilde{H}_n(X;G) = 0$.

(b) If X is $(c-1)$-connected and d-dimensional, then $\widetilde{H}_n(X : G) = 0$ for $n < c$ and $n > d$.

Problem 22.16. Prove Proposition 22.14 and Corollary 22.15.

Next we consider the homomorphisms induced by maps between wedges of spheres.

Proposition 22.17. *Let $f : \bigvee_{\mathcal{J}} S^n \to \bigvee_{\mathcal{I}} S^n$ be determined by the matrix $A(f)$. Then the induced map may be identified via the diagram*

$$\begin{tikzcd}
\widetilde{h}_*(\bigvee_{\mathcal{J}} S^n) \arrow[r, "f_*"] \arrow[d, "\cong"'] & \widetilde{h}_*(\bigvee_{\mathcal{I}} S^n) \arrow[d, "\cong"'] \\
\bigoplus_{\mathcal{J}} \widetilde{h}_*(S^n) \arrow[r, "A(f)"] & \bigoplus_{\mathcal{I}} \widetilde{h}_*(S^n)
\end{tikzcd}$$

with the map of direct sums given by the same matrix $A(f)$.

Problem 22.18. Prove Proposition 22.17.

Now you have all the tools you need to do most homology calculations.

Problem 22.19. Compute the homology of the spaces in Section 21.2.

22.2. Examples of Homology Theories

We have plenty of experience with cohomology theories, but we have not yet seen a homology theory. Indeed, it should seem very strange for a *covariant* functor to behave well with respect to *domains*. Could it be that we have a definition without examples?

22.2.1. Stabilization of Maps. The Freudenthal Suspension Theorem implies that repeating the suspension operation can change a given homotopy group only finitely many times (see Problem 17.64). In other words, the sequence of groups

$$\pi_n(X) \longrightarrow \pi_{n+1}(\Sigma X) \to \cdots \to \pi_{n+t}(\Sigma^t X) \longrightarrow \pi_{n+t+1}(\Sigma^{t+1} X) \to \cdots$$

stabilizes for large t. Hence the colimit of the sequence is just the group to which the sequence eventually stabilizes. This group is called the n^{th} **stable homotopy group** of X, and it is denoted $\pi_n^S(X)$, so that

$$\pi_n^S(X) = \operatorname{colim}\big(\cdots \to \pi_{n+t}(\Sigma^t X) \to \pi_{n+t+1}(\Sigma^{t+1} X) \to \cdots\big).$$

There is a canonical map $\Sigma^\infty : \pi_n(X) \to \pi_n^S(X)$ which just takes α to the element represented by $\Sigma^t \alpha$, where t is large enough, or larger.

Theorem 22.20. *The stable homotopy groups $\pi_*^S(\,?\,)$ define a homology theory satisfying the Wedge and Weak Equivalence Axioms.*

Since π_n^S is clearly a homotopy functor, you can prove the theorem by checking that π_*^S satisfies the two defining properties of a homology theory.

Problem 22.21.

(a) Construct a natural isomorphism $s_X : \pi_{n+1}^S(\Sigma X) \to \pi_n^S(X)$.

(b) Show that, if $A \to B \to C$ is a cofiber sequence, then

$$\pi_n^S(A) \longrightarrow \pi_n^S(B) \longrightarrow \pi_n^S(C)$$

is exact.

Hint. Since you are trying to get an exact sequence using a cofiber sequence in the target, you should use Theorem 18.13.

Stable homotopy groups are the simplest special case of a vast family of homology theories.

Problem 22.22. Show that for any finite-dimensional CW complex A, the functors

$$\widetilde{h}_n(X) = \operatorname{colim}\left(\cdots \to [\Sigma^t(\Sigma^n A), \Sigma^t X] \to [\Sigma^{t+1}(\Sigma^n A), \Sigma^{t+1} X] \to \cdots\right)$$

constitute a homology theory.

Notice that it makes perfect sense to plug in **negative** values for n.

Exercise 22.23. Is $h_n(X) = \operatorname{colim}[\Sigma^{n+t} A, \Sigma^t X]$ a cohomology theory if A is not finite-dimensional?

22.2.2. Ordinary Homology. Stable homotopy groups are very nice, but they are also very complicated. It is known that the coefficient groups $\pi_k^S(S^0)$ are nonzero for infinitely many values of k, so π_*^S is certainly not an ordinary homology theory. In this section, we give the more complicated construction of the vastly more simple ordinary homology theories.

Fix an abelian coefficient group G. Let $\lambda_t : \Sigma K(G,t) \to K(G,t+1)$ be the adjoint of the identity map $K(G,t) \to K(G,t) \simeq \Omega K(G,t+1)$; then define $\tau_t = (\mathrm{id}_X \wedge \lambda_t)_* \circ \Sigma$, as in the diagram

$$\begin{tikzcd}[column sep=small]
\pi_{n+t}(X \wedge K(G,t)) \arrow[rr, "\tau_t"] \arrow[dr, bend right, "\Sigma"'] & & \pi_{n+t+1}(X \wedge K(G,t+1)) \\
& \pi_{n+t+1}(X \wedge \Sigma K(G,t)). \arrow[ur, bend right, "(\mathrm{id}_X \wedge \lambda_t)_*"'] &
\end{tikzcd}$$

Problem 22.24. Show that, for fixed n and large enough t, the map

$$\tau_t : \pi_{n+t}(X \wedge K(G,t)) \longrightarrow \pi_{n+t+1}(X \wedge K(G,t+1))$$

is an isomorphism.

Problem 22.24 ensures that the sequence of groups

$$\cdots \to \pi_{n+t}(X \wedge K(G,t)) \xrightarrow{\tau_t} \pi_{n+t+1}(X \wedge K(G,t+1)) \to \cdots$$

is eventually constant, and this constant value is the colimit of the telescope diagram. We define

$$\widetilde{H}_n(X;G) = \operatorname{colim}\left(\cdots \to \pi_{n+t}(X \wedge K(G,t)) \xrightarrow{\tau_t} \pi_{n+t+1}(X \wedge K(G,t+1)) \to \cdots\right);$$

to justify this notation, we must show that this rule defines a reduced ordinary homology theory with coefficients in the group G.

Problem 22.25.

(a) Show that $\widetilde{H}^n(\,?\,;G)$ is a functor.

(b) Construct natural isomorphisms $\widetilde{H}_n(\Sigma X;G) \xrightarrow{\cong} \widetilde{H}_{n-1}(X;G)$.

(c) Show that, if $A \to B \to C$ is a cofiber sequence, then
$$\widetilde{H}_n(A;G) \longrightarrow \widetilde{H}_n(B;G) \longrightarrow \widetilde{H}_n(C;G)$$
is exact.

HINT. Let t be large enough that all three groups are represented by homotopy sets.

(d) Determine $\widetilde{H}_k(S^n;G)$ for all k.

(e) Show that $\widetilde{H}_*(\,?\,;G)$ satisfies the Wedge Axiom.

(f) Show that $\widetilde{H}_*(\,?\,;G)$ satisfies the Weak Equivalence Axiom.

You have proved the existence of ordinary homology theories.

Theorem 22.26. *The functors $\widetilde{H}_*$ define a reduced ordinary homology theory with coefficients in the group G and satisfying the Wedge and Weak Equivalence Axioms.*

We will see later (in Chapter 24) that, up to isomorphism, there is a unique ordinary theory defined on CW complexes that satisfies the Wedge Axiom. This implies that the theory constructed here is the *unique* ordinary theory defined on $\mathcal{T}_*$ satisfying the Wedge and Weak Equivalence Axioms.[4]

Problem 22.27. Give another proof of the Hurewicz theorem, using the construction above and the fact that $S^n \hookrightarrow K(\mathbb{Z},n)$ is an $(n+1)$-equivalence.

22.2.3. Infinite Loop Spaces and Homology. In Section 21.8 we saw that, if the spaces E_n represent the cohomology theory $\widetilde{h}^*$, then the suspension isomorphism forces homotopy equivalences $\Omega E_{n+1} \simeq E_n$. That is, each space E_n is an **infinite loop space**.

Now let $\mathbf{E} = \{E_0, E_1, \ldots\}$ be an infinite loop space. Then we can write $\lambda_t : \Sigma E_t \to E_{t+1}$ for the adjoint of the homotopy equivalence $E_t \xrightarrow{\simeq} \Omega E_t$, and define τ_t to be the composite
$$\pi_{n+t}(X \wedge E_t) \xrightarrow{\Sigma} \pi_{n+t+1}(X \wedge \Sigma E_t) \xrightarrow{(\mathrm{id}_X \wedge \lambda_t)_*} \pi_{n+t+1}(X \wedge E_{t+1}).$$
Now we define $\widetilde{E}_n(X)$ to be the colimit of the telescope diagram
$$\cdots \to \pi_{n+t}(X \wedge E_t) \xrightarrow{\tau_t} \pi_{n+t+1}(X \wedge E_{t+1}) \to \cdots .$$

Problem 22.28.

(a) Show that the functors $\widetilde{E}_n$ define a homology theory.

(b) Suppose the infinite loop space $\mathbf{E}$ represents the cohomology theory $\widetilde{E}^*$. Express the coefficients of $\widetilde{E}^*$ in terms of the coefficients of $\widetilde{E}_*$.

[4]Interestingly, represented homology is singular homology, while represented cohomology differs from singular cohomology.

This problem shows that every cohomology has a corresponding homology theory.

22.3. Exterior Products and the Künneth Theorem for Homology

Next we develop the exterior product for ordinary homology groups and prove a simple Künneth theorem.

22.3.1. The Exterior Product in Homology. Let $\alpha \in \widetilde{H}_n(X; A)$ and $\beta \in \widetilde{H}_m(Y; B)$. These homology classes are represented by homotopy classes of maps

$$\alpha : S^{n+t} \longrightarrow X \wedge K(A, t) \qquad \text{and} \qquad \beta : S^{m+s} \longrightarrow Y \wedge K(B, s).$$

Smashing these together and twisting smash factors, we obtain the diagram

$$\begin{tikzcd} S^{(m+n)+(t+s)} \arrow[r, "\alpha\bullet\beta"] \arrow[d, "\alpha\wedge\beta"] & (X \wedge Y) \wedge K(A \otimes B, t+s) \\ (X \wedge K(A,t)) \wedge (Y \wedge K(B,s)) \arrow[r, "\mathrm{id}\wedge T\wedge \mathrm{id}"] & (X \wedge Y) \wedge (K(A,t) \wedge K(B,s)), \arrow[u, "\mathrm{id}\wedge c"'] \end{tikzcd}$$

which defines the **exterior product** $\alpha \bullet \beta$ of the homology classes α and β.

Problem 22.29. Show that the exterior product is well-defined, natural, graded commutative, and associative.

Problem 22.29 implies that the exterior product defines a natural transformation

$$\kappa_{X,Y} : \widetilde{H}_*(X; A) \otimes \widetilde{H}_*(Y; B) \longrightarrow \widetilde{H}_*(X \wedge Y; A \otimes B).$$

The maps $\kappa_{X,Y}$ are called **Künneth maps**. In the unpointed context, these maps take the form

$$\kappa : H_*(X; A) \otimes H_*(Y; B) \longrightarrow H_*(X \times Y; A \otimes B).$$

Ideally, the Künneth maps would be isomorphisms. This is not always the case but, just as with cohomology, it is the case when at least one of $\widetilde{H}^*(X; R)$ or $\widetilde{H}^*(Y; R)$ is a free R-module.

Theorem 22.30 (Künneth)**.** *If either $\widetilde{H}^*(X; R)$ or $\widetilde{H}^*(Y; R)$ is a free graded R-module, then*

$$\kappa : \widetilde{H}_*(X; A) \otimes \widetilde{H}_*(Y; B) \longrightarrow \widetilde{H}_*(X \wedge Y; A \otimes B)$$

is an isomorphism.

The proof of Theorem 22.30 is entirely analogous to that of Theorem 21.91.

Problem 22.31. Prove Theorem 22.30.

22.4. Coalgebra Structure for Homology

At this point in our development of cohomology, we internalized the external product using the diagonal map, which gave cohomology (with ring coefficients) a natural ring structure. For homology we expect a dual structure: homology with ring coefficients should carry a natural *coalgebra* structure. But when we apply homology to the diagonal map, we obtain the diagram

$$\begin{tikzcd}
H_*(X) \arrow[r, dotted, dash] \arrow[dr, "\Delta_*"'] & \text{no map!} \arrow[r, dotted] & H_*(X) \otimes H_*(X) \arrow[dl, "\kappa_{X,X}"] \\
& H_*(X \times X) &
\end{tikzcd}$$

and there is no composite map to study, because the Künneth map points in the wrong direction.

The way to proceed is to restrict our attention to those spaces X for which $\kappa : H_*(X;R) \otimes H_*(X;R) \to H_*(X \times X;R)$ is an isomorphism. For such spaces, we can establish a coalgebra structure on $H_*(X;R)$, with coproduct

$$\delta = (\kappa_{X,X})^{-1} \circ \Delta_* : H_*(X) \longrightarrow H_*(X) \otimes H_*(X).$$

The counit $\epsilon : H_*(X) \to R$ is the map induced by the collapse map $X \to *$.

Theorem 22.32. *Suppose $H_*(X;R)$ and $H_*(Y;R)$ are free graded R-modules, and let $f : X \to Y$. Then*

(a) the maps δ and ϵ give $H_*(X;R)$ and $H_*(Y;R)$ the structure of associative and (graded) commutative coalgebras,

(b) the induced map $f_* : H_*(X;R) \to H_*(Y;R)$ is a map of coalgebras, and

(c) the Künneth map $\kappa : H_*(X;R) \otimes H_*(X;R) \to H_*(X \times Y;R)$ is an isomorphism of coalgebras.

Problem 22.33. Prove Theorem 22.32.

Problem 22.34. Let A_* be a (graded) coalgebra over the ring R. Show that the diagonal $\delta : A_* \to A_* \otimes A_*$ satisfies

$$\delta(x) = 1 \otimes x + \left(\sum a_i \otimes b_i\right) + x \otimes 1,$$

where for each i, the elements a_i, b_i are both in the kernel of the augmentation $A_* \to R$.

The elements of A_* such that $\delta(x) = 1 \otimes x + x \otimes 1$ play an important role in theory of coalgebras (and Hopf algebras); such elements are called **primitive**.

22.5. Relating Homology to Cohomology

We have seen in Section 22.2.3 that a cohomology theory gives rise to a corresponding homology theory. To what extent is it possible to compute $\widetilde{h}_*(X)$ from $\widetilde{h}^*(X)$?

In this section, we see that ordinary cohomology classes can be used to define functions defined on ordinary homology, which gives us maps $\widetilde{H}^*(\,?\,) \to \mathrm{Hom}(\widetilde{H}_*(\,?\,), R)$. We end with a universal coefficient theorem relating cohomology to homology. This is the result underlying the algebraic duality between homology and cohomology.

22.5.1. Pairing Cohomology with Homology. Let $u \in \widetilde{H}^k(X; A)$ and $\alpha \in \widetilde{H}_n(X; B)$. Thus u is a homotopy class of maps $X \to K(A, k)$ and α is represented by a map

$$\alpha : S^{n+t} \longrightarrow X \wedge K(B, t)$$

for sufficiently large t. We define $\langle u, \alpha\rangle \in \widetilde{H}_{n-k}(X; A \otimes B)$ to be the class represented by the composition

$$\begin{tikzcd}
S^{n+t} \arrow[rrr, "\langle u,\alpha\rangle"] \arrow[d, "\alpha"'] & & & X \wedge K(A \otimes B, k+t) \\
X \wedge K(B,t) \arrow[r, "\bar{\Delta}\wedge\mathrm{id}"] & (X \wedge X) \wedge K(B,t) \arrow[rr, "\mathrm{id}\wedge u\wedge\mathrm{id}"] & & X \wedge (K(A,k) \wedge K(B,t)). \arrow[u, "\mathrm{id}\wedge c"']
\end{tikzcd}$$

The function $(u, \alpha) \mapsto \langle u, \alpha\rangle$ has some simple and useful algebraic properties.

Problem 22.35. Show that $\langle\,?\,,\,?\,\rangle$ is natural in both variables. That is, suppose $f : X \to Y$, $u \in \widetilde{H}^*(Y)$, $\alpha \in \widetilde{H}_*(X)$. Then we can form

$$\langle u, f_*(\alpha)\rangle \in \widetilde{H}_{n-k}(Y) \qquad \text{and} \qquad \langle f^*(u), \alpha\rangle \in \widetilde{H}_{n-k}(X).$$

Show that $f_*(\langle f^*(u), \alpha\rangle) = \langle u, f_*(\alpha)\rangle$.

Problem 22.36. Show that ϕ is bilinear, so that it induces maps

$$\langle\,?\,,\,?\,\rangle : \widetilde{H}^k(X; A) \otimes \widetilde{H}_n(X; B) \longrightarrow \widetilde{H}_{n-k}(X; A \otimes B).$$

Clearly we can plug in X_+ for X, and so we get maps

$$\langle\,?\,,\,?\,\rangle : H^n(X; A) \otimes H_n(X; B) \longrightarrow H_{n-k}(X; A \otimes B)$$

defined and natural for unpointed spaces.

22.5.2. Nondegeneracy. We work now in the unpointed context and specialize to the case $n = k$ (and X is path-connected). In this case our map has the form

$$\langle\, ?\,,\, ?\,\rangle : H^n(X;R) \otimes H_n(X;R) \longrightarrow H_0(X;R\otimes R) \cong R.$$

Furthermore, if $f : X \to Y$, then the induced map on H_0 is simply the identity on R, so that naturality reduces to

$$\langle f^*(u), \alpha\rangle = \langle u, f_*(\alpha)\rangle.$$

Thus, the rules $u \mapsto \langle u, ?\,\rangle$ and $\alpha \mapsto \langle\, ?\,, \alpha\rangle$ define natural homomorphisms $H^n(X;R) \to \mathrm{Hom}_R(H_n(X;R), R)$ and $H_n(X;R) \to \mathrm{Hom}_R(H^n(X;R), R)$.

Theorem 22.37. *If X is a CW complex of finite type, and either $\widetilde{H}_*(X;R)$ or $\widetilde{H}^*(X;R)$ is a free graded R-module, then the natural transformations*

$$H^n(X;R) \longrightarrow \mathrm{Hom}_R(H_n(X;R), R)$$

and

$$H_n(X;R) \longrightarrow \mathrm{Hom}_R(H^n(X;R), R)$$

are isomorphisms.

Corollary 22.38. *If R is a field, then for CW complexes X of finite type there are natural isomorphisms*

$$H^*(X;R) \cong \mathrm{Hom}_R(H_*(X;R), R).$$

The argument is very similar to our proof of the simple Künneth theorem in Section 21.7.

Problem 22.39. Suppose R is a field.

(a) Show that $h^n(\,?\,) = \mathrm{Hom}_R(H_n(\,?\,;R), R)$ is a cohomology theory defined on (at least) the category of finite CW complexes.

(b) Show that u is a natural transformation of cohomology theories.

(c) Prove Theorem 22.37.

Exercise 22.40. Criticize the following argument:

> *Since* $\mathrm{Hom}_R(F, R) \cong F$ *for any finitely generated free R-module, if X is a finite complex and either $H_n(X;R)$ or $H^n(X;R)$ is a free R-module, then so is the other, and they are isomorphic.*

22.6. H-Spaces and Hopf Algebras

The multiplication map for an H-space X provides the homology and cohomology of X with an algebra structure that is natural with respect to H-maps. This observation is due to Pontrjagin, and so the algebra $H_*(X;R)$ is generally called the **Pontrjagin algebra** of X. If $H_*(X;R)$ is a free R-module, the diagonal map gives $H_*(X;R)$ a coalgebra structure. These two structures are related to each other, making $H_*(X;R)$ into a **Hopf algebra**. This structure is very rigid, and so it is very useful, both theoretically and computationally.

You should consult Section A.6 for information about the basic theory of coalgebras and Hopf algebras.

22.6.1. The Pontrjagin Algebra of an H-Space. If X is an H-space then we can form the composite map

$$H_*(X) \otimes H_*(X) \xrightarrow{\kappa} H_*(X \times X) \xrightarrow{\mu_*} H_*(X),$$

which we will also call μ_*. Note also that the inclusion of the basepoint $* \hookrightarrow X$ induces a map $\eta : R \to H_*(X)$.

Theorem 22.41. *Let X and Y be H-spaces, and let $f : X \to Y$ be an H-map.*

(a) The maps μ_* and η give $H_*(X;R)$ the structure of graded R-algebra.

(b) If X is an associative H-space, then $H_*(X;R)$ is an associative R-algebra; if X is commutative, then $H_*(X;R)$ is graded commutative.

(c) $f_* : H_*(X;R) \to H_*(Y;R)$ is a homomorphism of R-algebras.

Problem 22.42. Prove Theorem 22.41.

An H-space structure on X likewise endows its cohomology with extra structure.

Problem 22.43. Let X and Y be H-spaces whose cohomology groups are finitely generated free R-modules; let $f : X \to Y$ be an H-map.

(a) Show that the multiplication μ gives $H^*(X;R)$ the structure of a coalgebra over R.

(b) Show that f^* is a coalgebra map.

22.6.2. Pontrjagin and Künneth. If X and Y are H-spaces, then $X \times Y$ is an H-space with multiplication defined by the diagram

$$\begin{tikzcd}
(X \times Y) \times (X \times Y) \arrow[rr, "\mu_{X\times Y}"] \arrow[dr, bend right, "\mathrm{id}\times T\times \mathrm{id}"'] & & X \times Y \\
& (X \times X) \times (Y \times Y). \arrow[ur, bend right, "\mu_X \times \mu_Y"'] &
\end{tikzcd}$$

The domain and target of the Künneth map

$$\kappa : H_*(X \times Y) \xrightarrow{\cong} H_*(X) \otimes H_*(Y)$$

are R-algebras because X, Y and $X \times Y$ are H-spaces, so it makes sense to inquire whether the Künneth map κ is an algebra map.

Problem 22.44. Show that κ is a homomorphism of R-algebras.

It follows that if one or the other of $H_*(X)$ and $H_*(Y)$ is a free graded R-module, κ is an R-algebra isomorphism. The Künneth map also behaves well with respect to the coalgebra structure of the cohomology of an H-space.

Problem 22.45. Let X and Y be H-spaces whose cohomology groups are free R-modules. Show that the Künneth map

$$\kappa : H^*(X; R) \otimes_R H^*(Y; R) \longrightarrow H^*(X \times Y; R)$$

is an isomorphism of coalgebras.

22.6.3. The Homology and Cohomology of an H-Space. Here's the situation we have. For any space X, $H^*(X; R)$ has an algebra structure, but $H_*(X; R)$ has an algebra structure only if X is an H-space. If the homology $H_*(X; R)$ is a free and finitely generated R-module, then it has a coalgebra structure; but even if it is free and finitely generated, $H^*(X; R)$ only has a coalgebra structure if X is an H-space.

So let's specialize to those H-spaces X for which $H_*(X; R)$ and $H^*(X; R)$ are free R-modules of finite type. Now both $H^*(X; R)$ and $H_*(X; R)$ have two structures each: they are both coalgebras and algebras over R. The key observation is that these two structures are compatible with each other, in the sense that the coproduct map is a map of algebras and the product map is a map of coalgebras. This is a lot easier to understand in diagram form: if we write $A = H_*(X)$ with coproduct δ_A and product μ_A, then this simply means that the diagram

$$\begin{tikzcd} A \otimes A \arrow[r, "\delta_{A\otimes A}"] \arrow[d, "\mu_A"'] & (A \otimes A) \otimes (A \otimes A) \arrow[d, "\mu_{A\otimes A}"] \\ A \arrow[r, "\delta_A"] & A \otimes A \end{tikzcd}$$

is commutative, where the maps $\mu_{A\otimes A}$ and $\delta_{A\otimes A}$ are defined as in Section A.6. An algebra with compatible coalgebra structure is known as a **Hopf algebra**.

The relation between the homology and cohomology algebras of a finite-type H-space is that of dual Hopf algebras.

Theorem 22.46. *If X is an H-space of finite type and $H_*(X; R)$ is a free graded R-module, then $H^*(X)$ and $H_*(X)$ are dual Hopf algebras.*

Problem 22.47. Prove Theorem 22.46.

Hopf algebras have so much structure that a comparatively small amount of information about them suffices to determine them up to isomorphism. Thus Theorem 22.46 is an invaluable tool for calculation when, for example, the cohomology algebra of an H-space is known and its homology is needed.

Problem 22.48. You know from Section 21.5.2 that $H^*(\Omega S^{2n+1};\mathbb{Z})$ is a divided polynomial algebra. Use this information to determine the structure of $H_*(\Omega S^{2n+1};\mathbb{Z})$ as a Hopf algebra.

Problem 22.49. In the same way, determine the algebra $H_*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$.

HINT. Show that every $x \in H^2(\mathbb{C}\mathrm{P}^\infty)$ must be primitive.

Direct computations of the homology of Lie groups in the early 20th century led topologists to conjecture that the rational homology of a compact H-space must be the same as the homology of a product of rational spheres. H. Hopf showed that the rational homology must be a Hopf algebra and verified the conjecture by proving Theorem A.49.

Problem 22.50. Using Theorem A.49, show that if X is a finite-dimensional H-space, then there are odd integers $n_1, n_2, \ldots, n_m$ such that

$$H^*(X;\mathbb{Q}) \cong H^*(S^{n_1} \times \cdots \times S^{n_m};\mathbb{Q}).$$

It is reasonable to wonder about the topological *meaning* of this theorem. There is a hidden connection between an arbitrary H-spaces and products of spheres, which will be explained in Section 34.5.

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Chapter 23

Cohomology Operations

Since cohomology groups are defined as sets of homotopy classes, they are—on the face of it—nothing more than pointed sets. But since Eilenberg-Mac Lane spaces are H-spaces, we were able to factor the cohomology functors through the category of abelian groups, thus imposing additional structure on them. When the coefficients are taken in a ring, a multiplicative structure becomes available. Each additional structure gives us more leverage on the cohomology groups and the maps between them. For instance, a perfectly nice map of sets might be incompatible with the group structures; and a perfectly good homomorphism of graded abelian groups might not respect the multiplication. Thus the more structure we can impose, the better off we are, because then less information takes us further.

Cohomology operations are the answer to: what is the upper limit on how much structure can be put on cohomology? Natural algebraic structures are essentially natural transformations from one cohomology group to another, so why not take the bull by the horns and study cohomology together with *all* of its natural transformations? A natural transformation of cohomology is called a cohomology operation. We study them abstractly for a bit, then turn to constructing the most important examples in ordinary mod p cohomology.

We work with cohomology theories satsifying the Wedge and Weak Equivalence Axioms, so that we may always assume, without loss of generality, that our spaces are CW complexes and our cohomology classes are homotopy classes of maps into Eilenberg-Mac Lane spaces.

23.1. Cohomology Operations

A **cohomology operation** from one cohomology theory $\widetilde{h}^*$ to another $\widetilde{k}^*$ is a natural transformation of functors

$$\Phi : \widetilde{h}^n \longrightarrow \widetilde{k}^{n+d},$$

where $\widetilde{h}^n$ and $\widetilde{k}^{n+d}$ are considered—by composition with the forgetful functor $\mathrm{AB}\,\mathcal{G}^* \to \mathbf{Sets}_*$—as functors $\mathcal{T}_* \to \mathbf{Sets}_*$. The **degree** of Φ is d.

It is not hard to find simple examples of cohomology operations that illustrate the need for the seemingly artificial intrusion of the forgetful functor into our definition. If $\widetilde{h}^*$ is a multiplicative cohomology theory and $p \in \mathbb{N}$, then we may define $(\,?\,)^p : \widetilde{h}^n \to \widetilde{h}^{np}$ by $(\,?\,)^p : u \to u^p$ for $u \in \widetilde{h}^n(X)$. Unfortunately, this rule is unlikely to be a homomorphism, which means that the power operation $(\,?\,)^p$ is not a natural transformation of functors with values in the category of abelian groups. But after we forget the group structures, then the p-power operation is a natural transformation.

Because cohomology theories are representable on CW complexes, the Yoneda lemma applies to a general cohomology operation $\Phi : \widetilde{h}^n \to \widetilde{k}^{n+d}$.

Theorem 23.1. *Suppose $\widetilde{h}^n$ is represented by the space E_n and $\widetilde{k}^{n+d}$ is represented by the space F_{n+d}. Then there are bijections*

$$\{\textit{cohomology operations } \widetilde{h}^n \to \widetilde{k}^{n+d}\} \cong [E_n, F_{n+d}] \cong \widetilde{k}^{n+d}(E_n).$$

Problem 23.2. Prove Theorem 23.1.

We will be particularly interested in cohomology operations in ordinary cohomology with $\mathbb{Z}/p$ coefficients, where p is prime. Because of Theorem 23.1, we can compute these cohomology operations by determining the cohomology groups

$$[K(\mathbb{Z}/p, n), K(\mathbb{Z}/p, n+d)] = \widetilde{H}^{n+d}(K(\mathbb{Z}/p, n); \mathbb{Z}/p).$$

These calculations are far from easy, but later (in Chapter 33), we will completely determine these groups.

Problem 23.3. Show that in ordinary cohomology (with whatever coefficients you like) there are no nontrivial cohomology operations of negative degree.

Some Simple Examples. Maps $f : K(A, n) \to K(B, n)$ are in bijective correspondence with $\mathrm{Hom}(A, B)$, so each homomorphism $f : A \to B$ gives rise to a family of **coefficient transformations**

$$\widetilde{H}^n(X; A) \longrightarrow \widetilde{H}^n(X; B),$$

one for each n.

There is another kind of cohomology operation that derives from the algebra of coefficient groups. A short exact sequence $0 \to A \to B \to C \to 0$ gives rise to a long fibration sequence

$$\cdots \to \Omega K(C, n+1) \xrightarrow{\beta} K(A, n+1) \longrightarrow K(B, n+1) \longrightarrow K(C, n+1).$$

All the maps here induce coefficient transformations except for

$$\beta : K(C, n) \longrightarrow K(A, n+1).$$

This map is called the **Bockstein map** associated to the given short exact sequence. The induced cohomology operation $\widetilde{H}^n(\,?\,; C) \to \widetilde{H}^{n+1}(\,?\,; A)$ is known as the **Bockstein operation** associated to the sequence.

Problem 23.4.

(a) Show that the Bockstein associated to $0 \to A \to B \to C \to 0$ is trivial if and only if the sequence is a split exact sequence.

(b) Show that the only cohomology operations $\widetilde{H}^n(\,?\,; C) \to \widetilde{H}^{n+1}(\,?\,; A)$ are Bocksteins.

(c) Show that the Bocksteins associated to the exact sequences

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \qquad \text{and} \qquad 0 \to A \xrightarrow{-f} B \xrightarrow{-g} C \to 0$$

are equal operations.

Problem 23.5. Let $\beta_n : \widetilde{H}^n(\,?\,; \mathbb{Z}/p) \to \widetilde{H}^{n+1}(\,?\,; \mathbb{Z}/p)$ be the Bockstein operation corresponding to the sequence $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$. Show that

$$\beta_n : \widetilde{H}^n(M(\mathbb{Z}/p, n); \mathbb{Z}/p) \longrightarrow \widetilde{H}^{n+1}(M(\mathbb{Z}/p, n); \mathbb{Z}/p)$$

is nontrivial.

Cohomology operations act diagonally on the cohomology of a wedge. This is enough to show, sometimes, that a map is nontrivial.

Problem 23.6.

(a) Show that, under the isomorphism $\widetilde{h}^*(X \vee Y) \cong \widetilde{h}^*(X) \times \widetilde{h}^*(Y)$, every cohomology operation Φ acts by the rule $\Phi(u, v) = (\Phi(u), \Phi(v))$.

(b) Let $f : S^n \to S^m$, where $n \geq m$. Show that if there is a nonzero cohomology operation $\Phi : \widetilde{H}^m(C_f) \to \widetilde{H}^{n+1}(C_f)$, then $f \not\simeq *$.

Cohomology Operations in Unreduced Cohomology. Since we define unreduced cohomology in terms of reduced cohomology, a cohomology operation, every 'reduced' operation has a corresponding 'unreduced' operation, and vice versa.

Problem 23.7. You have shown that $h^*(X) \cong \widetilde{h}^*(S^0) \times \widetilde{h}^*(X)$. Show that every cohomology operation respects this splitting.

23.2. Stable Cohomology Operations

Let's consider all the cohomology operations from $\widetilde{h}^n$ to $\widetilde{k}^m$ for various values of m and n. If ϕ and θ have the same dimension of definition and the same degree, then we can add them in the usual way: $(\phi+\theta)(x) = \phi(x)+\theta(x)$. If the target dimension of ϕ is equal to the domain dimension of θ, then we can define the composite $\theta \circ \phi$. But without dimensional restrictions, neither sums nor composites of cohomology operations make sense. To get around this problem, we will focus on special *families* of cohomology operations called stable operations.

23.2.1. The Same Operation in All Dimensions. A **stable cohomology operation** $\Phi : \widetilde{h}^* \to \widetilde{k}^*$ of **degree** d is a collection $\Phi = \{\Phi_n \mid n \in \mathbb{Z}\}$ of cohomology operations, where

- Φ_n is a cohomology operation $\widetilde{h}^n \to \widetilde{k}^{n+d}$ and
- Φ_n and Φ_{n-1} are related to each other by the suspension isomorphism, meaning that the diagram

$$\begin{tikzcd} \widetilde{h}^{n-1} \arrow[r, "\Phi_{n-1}"] \arrow[d, "\Sigma", "\cong"'] & \widetilde{k}^{(n-1)+d} \arrow[d, "\Sigma"', "\cong"] \\ \widetilde{h}^n \circ \Sigma \arrow[r, "\Phi_n"] & \widetilde{k}^{n+d} \circ \Sigma \end{tikzcd}$$

commutes for each n.

Problem 23.8. Show that a stable cohomology operation Φ of degree d is precisely a transformation $\Phi : \Sigma^d \widetilde{h}^* \to \widetilde{k}^*$ of cohomology theories.

Now if Φ and Θ are stable cohomology operations, where Θ has degree d, then we can define the sum $\Phi + \Theta = \{\Phi_n + \Theta_n\}$ and composite $\Phi \circ \Theta = \{\Phi_{n+d} \circ \Theta_n\}$.

Exercise 23.9. Check that $\Phi + \Theta$ and $\Phi \circ \Theta$ are stable operations.

Let $\Phi : \widetilde{h}^* \to \widetilde{k}^*$ be a stable cohomology operation of degree d. Suppose that $\widetilde{h}^*$ is represented by the spaces $\{E_n\}$ and that $\widetilde{k}^*$ is represented by spaces $\{F_n\}$. Then Φ_n is the map induced by some function $\phi_n : E_n \to F_{n+d}$ and Φ_{n-1} is represented by $\phi_{n-1} : E_{n-1} \to F_{(n-1)+d}$. How are the maps ϕ_n and ϕ_{n-1} related?

Proposition 23.10. *Let $\Phi_n : \widetilde{h}^n \to \widetilde{k}^{n+d}$ and $\Phi_{n+1} : \widetilde{h}^{n+1} \to \widetilde{k}^{n+1+d}$ be cohomology operations represented by the maps $\phi_n : E_n \to F_{n+d}$ and $\phi_{n+1} : E_{n+1} \to F_{n+1+d}$. Then the following are equivalent:*

(1) $\Sigma \circ \Phi_n = \Phi_{n+1} \circ \Sigma$,

(2) ϕ_{n-1} *is pointwise equivalent in* $\mathrm{H}\mathcal{T}_*$ *to* $\Omega\phi_n$*; that is, the diagram*

$$\begin{tikzcd} E_{n-1} \arrow[r, "\phi_{n-1}"] \arrow[d, "\simeq"'] & F_{(n-1)+d} \arrow[d, "\simeq"] \\ \Omega E_n \arrow[r, "\Omega\phi_n"] & \Omega F_{n+d} \end{tikzcd}$$

is homotopy commutative,

(3) *(in the case* $\widetilde{h}^* = \widetilde{k}^* = \widetilde{H}^*(\,?\,; R)$*)*

$$\Phi_{n+1}(i_1 \bullet u) = i_1 \bullet \Phi_n(u)$$

where $i_1 \in \widetilde{H}^1(S^1; R) \cong R$ *corresponds to* 1_R*.*

Problem 23.11. Prove Proposition 23.10.

Proposition 23.10 implies that the map representing a stable operation must be an infinite loop map. This implies that all stable operations on representable cohomology theories are additive. In fact, stable operations are additive in all cohomology theories, not just the representable ones.

Problem 23.12.

(a) Show that every operation is additive on $\widetilde{h}^*(\Sigma X)$.

(b) Show that every stable operation is additive.

The set of stable operations from a cohomology theory $\widetilde{h}^*$ to itself has an algebraic structure. For a graded ring $\mathcal{A}$, write $\mathbf{Mod}_{\mathcal{A}}$ for the category of graded $\mathcal{A}$-modules.

Corollary 23.13. *If* $\widetilde{h}^*$ *is a cohomology theory, then the set* $\mathcal{A} = \mathcal{A}(\widetilde{h}^*)$ *of all stable cohomology operations* $\Phi : \widetilde{h}^* \to \widetilde{h}^*$ *is a graded ring under addition and composition, and the functor* $\widetilde{h}^*$ *lifts through the forgetful map* $\mathbf{Mod}_{\mathcal{A}} \to \mathrm{AB}\,\mathcal{G}^*$ *as in the diagram*

$$\begin{tikzcd} & \mathbf{Mod}_{\mathcal{A}} \arrow[d, "\text{forget}"] \\ \mathcal{T}_* \arrow[ur, bend left, "\widetilde{h}^*"] \arrow[r, "\widetilde{h}^*"] & \mathrm{AB}\,\mathcal{G}^*. \end{tikzcd}$$

It is virtually unheard of to distinguish notationally between the two functors labelled $\widetilde{h}^*$ in the diagram of Corollary 23.13.

Problem 23.14. Prove Corollary 23.13.

The ring of $\mathcal{A}(\widetilde{h}^*)$ is known as the **Steenrod algebra** of the cohomology theory $\widetilde{h}^*$. It is named for Norman Steenrod, who was among the first to study cohomology operations; he constructed all the cohomology operations from $\widetilde{H}^*(\,?\,;\mathbb{Z}/p)$ to itself [**158**] (though the proof that his list was complete came later).

Steenrod algebras are usually (graded) commutative. The multiplicative structure of the Steenrod algebras for $\widetilde{H}^*(\,?\,;\mathbb{Z}/p)$ is encoded in the **Ádem relations**, which we will derive (for $p = 2$) later in this chapter. They are named for José Ádem, who was the first to find them [**9–11**].

23.2.2. Extending an Operation to a Stable Operation. Suppose we have a cohomology operation $\Phi_n : \widetilde{h}^n \to \widetilde{k}^{n+d}$ (represented by a map which we call ϕ_n), and we wonder if it is part of a stable cohomology operation Φ. In dimensions below n, it is easy to define: Φ_{n-k} is forced to be the operation represented by the map $\phi_{n-k} = \Omega^k \phi_n$. But in dimensions $n+1, n+2, \ldots$ we have a much more difficult problem: we need to deloop ϕ_n to obtain a map ϕ_{n+1} such that $\phi_n \simeq \Omega\phi_{n+1}$; then we have to deloop ϕ_{n+1}; and so on.

Problem 23.15.

(a) Show that Φ_n is part of a stable cohomology operation if and only if, for each $k \geq 0$, the representing map ϕ_{n+k} can be delooped to a map ϕ_{n+k+1}. Can a given operation Φ_n be a part of more than one stable operation?

(b) Show that every cohomology operation $\widetilde{H}^n(\,?\,;G) \to \widetilde{H}^{n+d}(\,?\,;G)$ with $n \geq 1$ and $d < n$ is part of a stable cohomology operation.

Problem 23.16.

(a) Let $\beta_n : \widetilde{H}^n(\,?\,;C) \to \widetilde{H}^{n+1}(\,?\,;A)$ be the Bockstein associated to the short exact sequence $0 \to A \to B \to C \to 0$. Show that $\beta = \{(-1)^n\beta_n\}$ is a stable cohomology operation.

(b) Show that the power operation $(\,?\,)^2 : \widetilde{H}^1(\,?\,;\mathbb{Z}/2) \to \widetilde{H}^2(\,?\,;\mathbb{Z}/2)$ is part of a unique stable cohomology operation $\mathrm{Sq}^1 : \widetilde{H}^n(\,?\,;\mathbb{Z}/2) \to \widetilde{H}^{n+1}(\,?\,;\mathbb{Z}/2)$.

23.2.3. Cohomology of $B\mathbb{Z}/p$. Let's determine the action of the Bockstein in the groups $H^*(B\mathbb{Z}/p;\mathbb{Z}/p) \cong \mathbb{Z}/p$. This will lead us to define canonical generators in each dimension. In dimension 1, we have

$$H^1(B\mathbb{Z}/p;\mathbb{Z}/p) = \mathbb{Z}/p \cdot x,$$

where x is the homotopy class of the identity map $\mathrm{id} : B\mathbb{Z}/p \to B\mathbb{Z}/p$.

Problem 23.17. Show that the Bockstein β associated to the exact sequence $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$ is an isomorphism

$$\beta : H^1(B\mathbb{Z}/p;\mathbb{Z}/p) \longrightarrow H^2(B\mathbb{Z}/p;\mathbb{Z}/p).$$

Therefore, if we set $y = \beta(x)$, Theorem 21.114 implies that

$$H^*(B\mathbb{Z}/p; \mathbb{Z}/p) \cong \Lambda(x) \otimes \mathbb{Z}/p[y].$$

Thus $H^{2n+\epsilon}(B\mathbb{Z}/p; \mathbb{Z}/p) = \mathbb{Z}/p \cdot x^\epsilon y^n$, where $\epsilon \in \{0, 1\}$.

23.3. Using the Diagonal Map to Construct Cohomology Operations

In this section, we study a technique for constructing cohomology operations based on the reduced diagonal maps $\bar{\Delta} : X \to X^{\wedge p}$. The technique requires as input a certain kind of natural transformation, and we will construct such a transformation.

23.3.1. Overview. The basic plan is both simple and ingenious. The group $\mathbb{Z}/p$ acts on the smash product $X^{\wedge p}$ by cycling the smash factors; giving X the trivial $\mathbb{Z}/p$-action, we find that the reduced diagonal map $\bar{\Delta} : X \to X^{\wedge p}$ is $\mathbb{Z}/p$-equivariant.[1] A pointed space with a $\mathbb{Z}/p$-action may be interpreted as a diagram $\mathbb{Z}/p \to \mathcal{T}_*$, and in this context the equivariance of $\bar{\Delta}$ simply means that it is a map of diagrams.

Since $\bar{\Delta}$ is a diagram map, we can apply the homotopy colimit functor, which, by Problem 8.82, produces a map

$$\widetilde{\Delta} : B\mathbb{Z}/p \ltimes X \longrightarrow \Lambda(X)$$

between the homotopy colimit of the constant diagram X to the homotopy colimit (which we denote $\Lambda(X)$) of the more interesting diagram $X^{\wedge p}$. Using the Künneth formula, we identify the target of the induced map

$$\widetilde{\Delta}^* : \widetilde{H}^*(\Lambda(X); \mathbb{Z}/p) \longrightarrow \widetilde{H}^*(B\mathbb{Z}/p \ltimes X; \mathbb{Z}/p)$$

with the tensor product

$$\widetilde{H}^*((B\mathbb{Z}/p)_+ \wedge X; \mathbb{Z}/p) \cong H^*(B\mathbb{Z}/p; \mathbb{Z}/p) \otimes \widetilde{H}^*(X; \mathbb{Z}/p).$$

In Theorems 21.74 and 21.114 you showed that

$$H^*(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[x] \qquad \text{and} \qquad H^*(B\mathbb{Z}/p; \mathbb{Z}/p) \cong \Lambda(x) \otimes \mathbb{Z}/p[y]$$

where $|x| = 1$ and $y = \beta(x)$. Thus for each prime $p \geq 2$, $H^k(B\mathbb{Z}/p; \mathbb{Z}/p) \cong \mathbb{Z}/p$ is one-dimensional with a canonical generator which we'll refer to as γ_k. Now each element $z \in \widetilde{H}^*(B\mathbb{Z}/p \ltimes X; \mathbb{Z}/p)$ has a unique expression of the form

$$z = \sum_{k \geq 0} \gamma_k \otimes y_k \qquad \text{with} \qquad y_k \in H^{|z|-k}(X; \mathbb{Z}/p).$$

Now suppose we have a natural transformation

$$\ell : \widetilde{H}^n(\,?\,; \mathbb{Z}/p) \longrightarrow \widetilde{H}^m(\Lambda(\,?\,); \mathbb{Z}/p).$$

[1] That is, $\bar{\Delta}(a \cdot x) = a \cdot \bar{\Delta}(x)$ for $x \in X$ and $a \in \mathbb{Z}/p$.

Then for $\alpha \in \widetilde{H}^n(X;\mathbb{Z}/p)$ we have a canonical expression

$$\widetilde{\Delta}^*(\ell(\alpha)) = \sum_d \gamma_{(m-n)-d} \otimes \phi^d(\alpha),$$

which defines functions $\phi^d : \widetilde{H}^n(X;\mathbb{Z}/p) \to \widetilde{H}^{n+d}(X;\mathbb{Z}/p)$.

Problem 23.18. Show that the functions $\phi^d : \widetilde{H}^n(\,?\,;\mathbb{Z}/p) \to \widetilde{H}^{n+d}(\,?\,;\mathbb{Z}/p)$ define cohomology operations.

The only mystery in this construction is the natural transformation ℓ. We will show that there is a natural transformation

$$\lambda : \widetilde{H}^n(\,?\,;\mathbb{Z}/p) \longrightarrow \widetilde{H}^{np}(\Lambda(\,?\,);\mathbb{Z}/p)$$

which is nicely related to the p-power operation, and in the next chapter we'll use the resulting operations to define the Steenrod reduced powers.

23.3.2. The Transformation λ. In this section, we establish the existence of the natural transformation that we will use to produce cohomology operations. We begin by explicitly laying out our construction of homotopy colimits of $\mathbb{Z}/p$-shaped diagrams. In Problem 16.53 we gave the infinite-dimensional sphere S^∞ the structure of a CW complex for which the $\mathbb{Z}/p$-action is free and cellular; we'll use that structure throughout the following discussion.

Problem 23.19. Show that the diagonal actions of $\mathbb{Z}/p$ on $S^\infty \ltimes X$ and $S^\infty \ltimes X^{\wedge p}$ are cofibrant $\mathbb{Z}/p$-shaped diagrams in $\mathcal{T}_*$. Conclude that

$$\begin{tikzcd}
S^\infty \ltimes X \arrow[r, "\mathrm{id}_{S^\infty} \ltimes \overline{\Delta}"] \arrow[d, "\mathrm{pr}_2"'] & S^\infty \ltimes X^{\wedge p} \arrow[d, "\mathrm{pr}_2"] \\
X \arrow[r, "\bar{\Delta}"] & X^{\wedge p}
\end{tikzcd}$$

is a cofibrant replacement for the diagram map $\bar{\Delta}$.

The map $\widetilde{\Delta}$ of pointed homotopy colimits induced by the reduced diagonal is the induced map of categorical colimits on the top row of the square in Problem 23.19. The inclusions $X \hookrightarrow S^\infty \ltimes X$ and $X^{\wedge p} \hookrightarrow S^\infty \ltimes X^{\wedge p}$ give rise to commutative squares

$$\begin{tikzcd}
X \arrow[r, "\overline{\Delta}_X"] \arrow[d, "\mathrm{in}_2"'] & X^{\wedge p} \arrow[d, "j"] \\
B\mathbb{Z}/p \ltimes X \arrow[r, "\widetilde{\Delta}_X"] & \Lambda(X).
\end{tikzcd}$$

Problem 23.20.

(a) Show that Λ is a functor and that each of the four maps in the square above defines a natural transformation.

(b) Show that if $f \simeq g$, then $\Lambda(f) \simeq \Lambda(g)$.
HINT. Find an equivariant homotopy $f^{\wedge p} \simeq g^{\wedge p}$.

Now we are prepared to state and prove the existence of the transformation λ.

Proposition 23.21. *There is a unique natural transformation*

$$\lambda : \widetilde{H}^n(\,?\,;\mathbb{Z}/p) \longrightarrow \widetilde{H}^{np}(\Lambda(\,?\,);\mathbb{Z}/p)$$

such that $j^*(\lambda(\alpha)) = \alpha \wedge \cdots \wedge \alpha \in \widetilde{H}^{np}(X^{\wedge p};\mathbb{Z}/p)$ *for all* $\alpha \in \widetilde{H}^n(X;\mathbb{Z}/p)$.

We prove Proposition 23.21 by the method of the universal example. That is, we show that the element $\lambda(\iota_n)$ exists (and is unique) for the fundamental class $\iota_n = [\mathrm{id}_{K(\mathbb{Z}/p,n)}] \in \widetilde{H}^n(K(\mathbb{Z}/p,n);\mathbb{Z}/p)$ and derive the transformation.

Problem 23.22.

(a) Show that transformations $\ell : \widetilde{H}^n(\,?\,;\mathbb{Z}/p) \to \widetilde{H}^m(\Lambda(\,?\,);\mathbb{Z}/p)$ are in bijective correspondence with elements $\ell(\iota_n) \in \widetilde{H}^m(\Lambda(K(\mathbb{Z}/p,n));\mathbb{Z}/p)$.

(b) Show that $j^*(\ell(\iota_n)) = \iota_n \wedge \cdots \wedge \iota_n$ if and only if $j^*(\ell(\alpha)) = \alpha \wedge \cdots \wedge \alpha$ for all $\alpha \in \widetilde{H}^n(X;\mathbb{Z}/p)$ and all spaces X.

Now we simply have to find the element $\lambda(\iota_n) \in \widetilde{H}^{np}(\Lambda(K(\mathbb{Z}/p,n));\mathbb{Z}/p)$, which we interpret as a homotopy class of maps

$$\Lambda(K(\mathbb{Z}/p,n)) \longrightarrow K(\mathbb{Z}/p,np).$$

Such homotopy classes are determined, according to Corollary 16.29, by their restrictions to the $(np+1)$-skeleton of $\Lambda(K(\mathbb{Z}/p,n))$.

The Subcomplex $\Lambda_1(X)$. To simplify the construction of $\lambda(\iota_n)$, we introduce a subcomplex $\Lambda_1(X) \subseteq \Lambda(X)$ which, for $(n-1)$-connected spaces X, contains the $(np+1)$-skeleton of $\Lambda(X)$, and we show how to construct maps $\Lambda_1(X) \to Y$ from a map $f : X \to Y$ and a homotopy $f \circ T \simeq f$.

Let Z be any CW complex with a cellular action of $\mathbb{Z}/p$; write $T : Z \to Z$ for the map corresponding to your favorite generator of $\mathbb{Z}/p$. The inclusion $S^1 \hookrightarrow S^\infty$ induces an equivariant inclusion $S^1 \ltimes Z \hookrightarrow S^\infty \ltimes Z$ which leads to an inclusion

$$S^1 \ltimes_{\mathbb{Z}/p} Z \longrightarrow S^\infty \ltimes_{\mathbb{Z}/p} Z$$

of orbit spaces, where $S^1 \ltimes_{\mathbb{Z}/p} Z$ stands for $(S^1 \ltimes Z)/\mathbb{Z}/p$, in a straightforward extension of the notation for the Borel construction introduced in Section 15.4.3.

Problem 23.23. Let Z be $(k-1)$-connected, and suppose that $\mathbb{Z}/p$ acts cellularly on Z. Show that the map $S^1 \ltimes_{\mathbb{Z}/p} Z \to S^\infty \ltimes_{\mathbb{Z}/p} Z$ is a $(k+1)$-equivalence.

Problem 23.24. Let J denote the interval $[0, \frac{2\pi}{p}]$, and continue to assume that Z is $(k-1)$-connected.

(a) Show that $S^1 \ltimes_{\mathbb{Z}/p} Z$ is the pushout in the strong homotopy pushout square

$$\begin{tikzcd} Z \vee Z \arrow[r] \arrow[d, "(\mathrm{id},T)"'] \arrow[dr, phantom, "\text{pushout}"] & J \ltimes Z \arrow[d] \\ Z \arrow[r, "j_1"] & S^1 \ltimes_{\mathbb{Z}/p} Z. \end{tikzcd}$$

(b) Show that maps $S^1 \ltimes_{\mathbb{Z}/p} Z \to Y$ are in bijective correspondence with the set of pairs (f, H), where $f : Z \to Y$ and H is a pointed homotopy $f \simeq T \circ f$.

(c) Show that if $H, K : f \simeq T \circ f$ are homotopic homotopies, then the corresponding maps $S^1 \ltimes_{\mathbb{Z}/p} Z \to Y$ are homotopic.

(d) Show that $j_1 : Z \to S^1 \ltimes_{\mathbb{Z}/p} Z$ is a k-equivalence.

Problem 23.25. Let X be an $(n-1)$-connected CW complex and define $\Lambda_1(X) = S^1 \ltimes_{\mathbb{Z}/p} X^{\wedge p}$. Show that

(a) $j_1 : X^{\wedge p} \to \Lambda_1(X)$ is an np-equivalence,

(b) the inclusion $\Lambda_1(X) \hookrightarrow \Lambda(X)$ is an $(np+1)$-equivalence,

(c) the inclusions $* \ltimes X^{\wedge p} \hookrightarrow S^1 \ltimes X^{\wedge p} \hookrightarrow S^\infty \ltimes X^{\wedge p}$ give rise to a commutative diagram

$$\begin{tikzcd} & X^{\wedge p} \arrow[dl, "j_1"'] \arrow[dr, "j"] & \\ \Lambda_1(X) \arrow[rr] & & \Lambda(X). \end{tikzcd}$$

Proof of Proposition 23.21. We will prove Proposition 23.21 by constructing a class $\lambda_1 \in \widetilde{H}^{np}(\Lambda_1(X); \mathbb{Z}/p)$ that restricts to $(\iota_n)^{\wedge p}$ and then extending λ_1 to the required class $\lambda(\iota_n) \in \widetilde{H}^{np}(\Lambda(X); \mathbb{Z}/p)$.

Problem 23.26.

(a) Show that the diagram

$$\begin{tikzcd} K(\mathbb{Z}/p, n)^{\wedge p} \arrow[rr, "T"] \arrow[dr, "(\iota_n)^{\wedge p}"'] & & K(\mathbb{Z}/p, n)^{\wedge p} \arrow[dl, "(\iota_n)^{\wedge p}"] \\ & K(\mathbb{Z}/p, np) & \end{tikzcd}$$

commutes up to homotopy.

(b) Show that there is a homotopy class $\lambda_1 : \Lambda_1(K(\mathbb{Z}/p, n)) \to K(\mathbb{Z}/p, np)$ whose restriction to $K(\mathbb{Z}/p, n)^{\wedge p}$ is $(\iota_n)^{\wedge p}$.

(c) Prove Proposition 23.21.

23.4. The Steenrod Reduced Powers

The fundamental cohomology operations in ordinary cohomology with $\mathbb{Z}/p$ coefficients are the Bockstein and the Steenrod reduced powers. This is because, as we will see in Section 33.5, all stable operations are sums of compositions of these, and all operations are polynomial in stable ones.[2] Using the transformation λ of Proposition 23.21, in the construction of Section 23.3.1 we have operations

$$\theta_n^d : \widetilde{H}^n(\,?\,;\mathbb{Z}/p) \longrightarrow \widetilde{H}^{n+d}(\,?\,;\mathbb{Z}/p)$$

defined by the formula

$$\widetilde{\Delta}^*(\lambda(\alpha)) = \sum_d \gamma_{(np-n)-d} \otimes \theta_n^d(\alpha).$$

Using these operations, we will prove that the p^{th} power operation $\alpha \mapsto \alpha^p$ in $\widetilde{H}^*(\,?\,;\mathbb{Z}/p)$ is part of a stable cohomology operation; these stable operations are known as the **Steenrod reduced powers**.

23.4.1. Unstable Relations. The **unstable relations** among the operations θ_n^d describe the influence of the dimension n on their behavior.

Problem 23.27.

(a) Show that $\theta_n^{n(p-1)}(\alpha) = \alpha^p$.

(b) Show that if $d > n(p-1)$, then $\theta_n^d = 0$.

The big job is to establish the **Cartan formula**, which governs the behavior of the operations θ_n^k on products.

Theorem 23.28. *For $\alpha \in \widetilde{H}^n(X;\mathbb{Z}/2)$ and $\beta \in \widetilde{H}^m(X;\mathbb{Z}/2)$,*

$$\theta_{n+m}^k(\alpha \cdot \beta) = \sum_{i+j=k} \theta_n^i(\alpha) \cdot \theta_m^j(\beta).$$

For $\alpha \in \widetilde{H}^n(X;\mathbb{Z}/p)$ and $\beta \in \widetilde{H}^m(X;\mathbb{Z}/p)$, there is a very similar formula of the form

$$\theta_{n+m}^k(\alpha \cdot \beta) = \sum_{i+j=k} a_{n,m}(i,j)\, \theta_n^i(\alpha) \cdot \theta_m^j(\beta).$$

[2]This is an amazing feature of ordinary cohomology: it is not true that in general every operation can be expressed in terms of unstable ones.

The coefficients in the sum vary according to intricate formulas. Since we are only planning to sketch the odd prime case here, you will only determine the one coefficient that we need; the others will be be left as an exercise.

To prove Theorem 23.28, we need to know how the natural transformation λ of Proposition 23.21 behaves with respect to products.

Lemma 23.29. *The transformation* $\lambda : \widetilde{H}^n(\,?\,;\mathbb{Z}/p) \to \widetilde{H}^{np}(\Lambda(\,?\,);\mathbb{Z}/p)$ *satisfies*

$$\lambda(\alpha \cdot \beta) = (-1)^{\binom{p}{2}|\alpha||\beta|}\lambda(\alpha)\cdot\lambda(\beta)$$

for all $\alpha, \beta \in \widetilde{H}^*(X;\mathbb{Z}/p)$.

Problem 23.30.

(a) Show that for $\alpha, \beta \in \widetilde{H}^*(X;\mathbb{Z}/p)$,

$$(\alpha\cdot\beta)\wedge\cdots\wedge(\alpha\cdot\beta) = (-1)^{\binom{p}{2}|\alpha||\beta|}(\alpha\wedge\cdots\wedge\alpha)\cdot(\beta\wedge\cdots\wedge\beta)$$

in $\widetilde{H}^*(X^{\wedge p};\mathbb{Z}/p)$.

(b) Prove Lemma 23.29.

Problem 23.31.

(a) Prove Theorem 23.28 for $p = 2$ by evaluating $\widetilde{\Delta}^*(\lambda(\alpha\cdot\beta)) = \widetilde{\Delta}^*(\lambda(\alpha)\cdot\lambda(\beta))$.

(b) Show that $\theta^k_{n+1}(i_1 \bullet u) = (-1)^k i_1 \bullet \theta^k_n(u)$ for all p (even or odd), where $u \in \widetilde{H}^n(X;\mathbb{Z}/p)$ and $i_1 \in \widetilde{H}^1(S^1;\mathbb{Z}/p)$ is the canonical generator.

Problem 23.32. Establish Cartan formulas for unreduced cohomology and for exterior products.

23.4.2. Extending the p^{th} Power to a Stable Operation.

The Cartan formula gives us leverage on the suspension operation, because (as we showed in Section 21.7) the suspension isomorphism may be identified with the external product with the canonical generator $i_1 \in \widetilde{H}^1(S^1;\mathbb{Z}/p)$. To use the Cartan formula to study the suspension operation, we need a single piece of actual knowledge.

Lemma 23.33. *The operation* $\theta^0_1 : \widetilde{H}^1(S^1;\mathbb{Z}/p) \to \widetilde{H}^1(S^1;\mathbb{Z}/p)$ *is an isomorphism.*

Since the target and domain of the operation in Lemma 23.33 are both $\mathbb{Z}/p$, the operation is either zero or an isomorphism, and we have to show that it is not zero.

We'll prove the case $p = 2$ and leave the general case as a project. This computation amounts to coming to a clear understanding of the map $\widetilde{\Delta}$, which is a map of quotient spaces induced by $\mathrm{id}_I \times \Delta : I \times I \to I \times I^2$.

Problem 23.34.

(a) Express the map $\widetilde{\Delta}$ as the induced map of pushouts of the form

$$\begin{tikzcd}
D^2 \arrow[d] & S^1 \arrow[l] \arrow[r, "\boxed{?}"] \arrow[d] & \boxed{??} \arrow[d] \\
D^3 & S^2 \arrow[l] \arrow[r, "\boxed{???}"] & \boxed{????}.
\end{tikzcd}$$

(b) Show that there is a cofiber sequence $S^1 \vee S^2 \xrightarrow{\widetilde{\Delta}} \Lambda_1(S^1) \to S^2 \vee S^3$.

(c) Show that $H^1(\Lambda_1(S^1); \mathbb{Z}/2) = 0$ and prove Lemma 23.33 in the case $p = 2$.

Exercise 23.35. Determine the homotopy type of $\Lambda_1(S^1)$ when $p = 2$.

We'll use Lemma 23.33 (which we assume for all p) to prove the main theorem of this section: the p^{th} power operation in $\widetilde{H}^*(\,?\,;\mathbb{Z}/p)$ is part of a stable cohomology operation.

Theorem 23.36.

(a) Each operation $(\,?\,)^2 : \widetilde{H}^d(?;\mathbb{Z}/2) \to H^{2d}(\,?\,;\mathbb{Z}/2)$ *is part of a stable operation* Sq^d *of degree* d*.*

(b) For odd p*, each operation* $(\,?\,)^p : \widetilde{H}^{2d}(?;\mathbb{Z}/p) \to H^{2dp}(\,?\,;\mathbb{Z}/p)$ *with even-dimensional domain is part of a stable operation* P^d *of degree* $2d(p-1)$*.*

The operations of Theorem 23.36(a) are called the **Steenrod squares** and are denoted Sq^d. The operations of part (b) are called the **reduced** p **powers** and are denoted P^d, or P_p^d when the prime p might be in doubt.

Problem 23.37.

(a) Show that $\theta^d_{n+1}(\Sigma u) = \pm \Sigma \theta^d_n(u)$.

(b) Prove Theorem 23.36.

Generically, the Steenrod reduced powers and the Steenrod squares are referred to as **Steenrod operations** and written $\mathrm{St}^d = P^d$ or $\mathrm{St}^d = \mathrm{Sq}^{2d}$; when handling $p = 2$ together with the odd primes, we use the β notation instead of Sq^1.

Problem 23.38. Show that $\mathrm{Sq}^1 = \beta$.

Exercise 23.39. Is $(\,?\,)^p : \widetilde{H}^{2d+1}(?;\mathbb{Z}/p) \to H^{(2d+1)p}(\,?\,;\mathbb{Z}/p)$ part of a stable operation when p is odd?

23.5. The Ádem Relations

The Steenrod algebra $\mathcal{A}_p$ of stable cohomology operations from $\widetilde{H}^*(\,?\,;\mathbb{Z}/p)$ to itself contains the operations β and St^d for $d \geq 1$. It also contains all sums and composites of those basic operations. As it turns out (and as we will show later), these give a complete list of all the stable mod p cohomology operations. To fully determine the structure of $\mathcal{A}_p$, then, we must determine any and all relations among the various stable operations.[3] In this section, we prove the Ádem relations which, as we will see in Section 33.5, generate all the relations and therefore completely determine the algebraic structure of $\mathcal{A}_p$.

Composition of Steenrod operations is necessarily related to the iterated reduced diagonal map

$$X \longrightarrow X^{\wedge p} \longrightarrow X^{\wedge(p\times p)},$$

and the Ádem relations result from an extra topological symmetry that appears in this map. To explain this, we write elements of $X^{\wedge p}$ as equivalence classes of *horizontal* p-tuples $[x_1, x_2, \ldots, x_p]$ and elements of $X^{\wedge(p\times p)} = (X^{\wedge p})^{\wedge p}$ as equivalence classes of *vertical* p-tuples of elements of $X^{\wedge p}$. Thus an element of $X^{\wedge(p\times p)}$ is an equivalence class of $p \times p$ arrays of elements of X. With this notation, the second reduced diagonal map $X^{\wedge p} \to X^{\wedge(p\times p)}$ carries a p-tuple to the 'matrix' having every row equal to the given p-tuple. Now we have a 'transpose operation' $\tau : X^{\wedge(p\times p)} \to X^{\wedge(p\times p)}$ and the triangle

$$\begin{tikzcd} & X \arrow[dl, "\bar{\Delta}"'] \arrow[dr, "\bar{\Delta}"] & \\ X^{\wedge(p\times p)} \arrow[rr, "\tau"] & & X^{\wedge(p\times p)} \end{tikzcd}$$

is strictly commutative in the category of $\mathbb{Z}/p \times \mathbb{Z}/p$-shaped diagrams. The map τ forces the equality of two different expressions involving the Steenrod operations; a bunch of algebraic manipulation reduces this equation to the more easily applied Ádem relations.

To emphasize the generality of the derivation, we establish the basic symmetry formula for general primes p. But the algebraic technicalities required to derive the Ádem relations from the symmetry formula are significantly more complex for odd primes, so we will only work out the details for $p = 2$.[4]

[3] Algebraically, we have a map $\mathbb{Z}/p[\beta, \mathrm{St}^1, \mathrm{St}^2, \ldots] \to \mathcal{A}_p$ (which we'll later show is a surjection), and we want to determine the kernel.

[4] See [**81**] for the details of the derivation for p odd.

23.5.1. Steenrod Operations on Polynomial Rings. The Cartan formula completely determines the action of the Steenrod operations on polynomial algebras. These formulas feature in many computations, and in particular they are crucial in our derivation of the Ádem relations.

Problem 23.40. Let X be any space such that $H^*(X;\mathbb{Z}/2) \cong \mathbb{Z}/2[x]$, a polynomial algebra on a single generator x in dimension d. To determine the action of the Steenrod algebra on $H^*(X;\mathbb{Z}/2)$, we must evaluate $\mathrm{Sq}^n(x^k)$ for all n and k.

(a) Show that $\mathrm{Sq}^m(x^k) = 0$ unless $m = nd$ for some n.

(b) Define $a_{k,n} \in \mathbb{Z}/2$ by the formula $\mathrm{Sq}^{nd}(x^k) = a_{k,n}x^{n+k}$. Working by induction using the Cartan formula, determine the coefficients $a_{k,1}$.

(c) Using the Cartan formula, show $a_{k,n} = a_{k-1,n} + a_{k-1,n-1}$.

(d) Determine the coefficients $a_{k,n}$.

Of course, the projective spaces are the primary examples of spaces with polynomial cohomology.

Problem 23.41. If $x \in \widetilde{H}^*(\mathbb{F}\mathrm{P}^\infty;\mathbb{Z}/2)$, and $N \in \mathbb{N}$, then there is a stable cohomology operation Φ such that $\Phi(x) \neq 0$ and $|\Phi(x)| > N$.

23.5.2. The Fundamental Symmetry Relation. When we write points of $X^{\wedge(p\times p)}$ as $p\times p$ arrays of points from X and consider the iterated diagonal

$$X \longrightarrow X^{\wedge p} \longrightarrow X^{\wedge(p\times p)},$$

each of these spaces comes with an action of $\mathbb{Z}/p \times \mathbb{Z}/p$. Specifically,

- X has a trivial action;
- for $X^{\wedge p}$, the first coordinate of $\mathbb{Z}/p \times \mathbb{Z}/p$ acts trivially, and the second coordinate acts by cycling through the (horizontal) smash coordinates; and
- the first coordinate of $\mathbb{Z}/p \times \mathbb{Z}/p$ acts on $X^{\wedge(p\times p)}$ by cycling the rows while the second coordinate cycles the columns.

We will study the homotopy colimit of the p^2-fold reduced diagonal. Since $S^\infty \times S^\infty$ is a free $(\mathbb{Z}/p\times\mathbb{Z}/p)$-CW complex, we have the cofibrant replacements

$$(S^\infty \times S^\infty) \ltimes X \longrightarrow (S^\infty \times S^\infty) \ltimes X^{\wedge p} \longrightarrow (S^\infty \times S^\infty) \ltimes X^{\wedge(p\times p)}.$$

Write $\Lambda_2(X)$ for the homotopy colimit of the $\mathbb{Z}/p\times\mathbb{Z}/p$-action on $X^{\wedge(p\times p)}$, which we construct as the categorical colimit of the diagonal action of $\mathbb{Z}/p\times\mathbb{Z}/p$ on $(S^\infty \times S^\infty) \ltimes X^{\wedge(p\times p)}$, and write $\widehat{\Delta} : (B\mathbb{Z}/p \times B\mathbb{Z}/p) \ltimes X \to \Lambda_2(X)$ for the induced map of homotopy orbit spaces.

Problem 23.42.

(a) Show that there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\bar{\Delta}"] \arrow[d] & X^{\wedge p} \arrow[r, "\bar{\Delta}"] \arrow[d] & X^{\wedge (p\times p)} \arrow[d, "j_2"] \\ (B\mathbb{Z}/p \times B\mathbb{Z}/p) \ltimes X \arrow[r] & B\mathbb{Z}/p \ltimes \Lambda(X) \arrow[r] & \Lambda_2(X). \end{tikzcd}$$

(b) Show that if X is $(n-1)$-connected, then $j_2 : X^{\wedge(p\times p)} \to \Lambda_2(X)$ is an np^2-equivalence.

Now we come to our extra symmetry. The group $\mathbb{Z}/2$ acts on $X^{\wedge(p\times p)}$ by the transpose action t. If we let $\mathbb{Z}/2$ act on $S^\infty \times S^\infty$ by the switch map T, then the square

$$\begin{tikzcd} (S^\infty \times S^\infty) \ltimes X \arrow[r] \arrow[d, "T\times \mathrm{id}_X"'] & (S^\infty \times S^\infty) \ltimes X^{\wedge(p\times p)} \arrow[d, "T\times t"] \\ (S^\infty \times S^\infty) \ltimes X \arrow[r] & (S^\infty \times S^\infty) \ltimes X^{\wedge(p\times p)} \end{tikzcd}$$

commutes.

Exercise 23.43. Show that the vertical maps are *not* $\mathbb{Z}/p\times\mathbb{Z}/p$-equivariant.

Problem 23.44.

(a) For $g = (g_1, g_2) \in \mathbb{Z}/p \times \mathbb{Z}/p$, write $g^T = (g_2, g_1)$. Show that the $\mathbb{Z}/2$-action

$$T \ltimes t : (S^\infty \times S^\infty) \ltimes X^{\wedge(p\times p)} \longrightarrow (S^\infty \times S^\infty) \ltimes X^{\wedge(p\times p)}$$

satisfies $(T\times t)(g\cdot z) = g^T \cdot (T\times t)(z)$ for $g \in \mathbb{Z}/p\times\mathbb{Z}/p$.

(b) Conclude that $T \times t$ induces a natural map $\tau : \Lambda_2(X) \to \Lambda_2(X)$ making the square

$$\begin{tikzcd} (B\mathbb{Z}/p \times B\mathbb{Z}/p) \ltimes X \arrow[r, "\widehat{\Delta}_X"] \arrow[d, "T\ltimes \mathrm{id}_X"'] & \Lambda_2(X) \arrow[d, "\tau"] \\ (B\mathbb{Z}/p \times B\mathbb{Z}/p) \ltimes X \arrow[r, "\widehat{\Delta}_X"] & \Lambda_2(X) \end{tikzcd}$$

commute.

Problem 23.45.

(a) Show that $\Lambda(\Lambda(X))$ is the quotient of $S^\infty \times S^\infty \times X^{p\times p}$ by the equivalence relation generated by
   - independent cyclic permutations in each of the p columns, and
   - cyclic permutation of the columns.

(This relation is given by the action of a group known as the **wreath product** of $\mathbb{Z}/p$ with itself.[5])

(b) Construct a natural transformation $\Lambda_2(X) \to \Lambda(\Lambda(X))$ such that

$$\begin{tikzcd} B\mathbb{Z}/p \ltimes \Lambda(X) \arrow[rr, "\widetilde{\Delta}_{\Lambda(X)}"] \arrow[dr, bend right] & & \Lambda(\Lambda(X)) \\ & \Lambda_2(X) \arrow[ur, bend right] & \end{tikzcd}$$

commutes.

Now that we have a pretty good understanding of the homotopy colimit of the iterated diagonal, we need to use our knowledge to determine its effect on the class $\lambda(\lambda(\iota_n))$. Write $\lambda = \lambda(\iota_n)$ and λ_2 for the composite $\Lambda_2(K_n) \to \Lambda(\Lambda(K_n)) \xrightarrow{\lambda(\lambda)} K_{p^2n}$.

Problem 23.46. Show that λ_2 is the unique element of $\widetilde{H}^{np}(\Lambda_2(K_n))$ such that $j_2^*(\lambda_2) = (\iota_n)^{\wedge p^2}$.

Problem 23.47. Specialize to the universal example $X = K_n = K(\mathbb{Z}/p, n)$ and $\alpha = \iota_n \in H^n(K_n; \mathbb{Z}/p)$.

(a) Show that there is a commutative diagram

$$\begin{tikzcd} (B\mathbb{Z}/p \times B\mathbb{Z}/p) \ltimes K_n \arrow[r, "\widehat{\Delta}_{K_n}"] \arrow[d, "T\ltimes \mathrm{id}"'] & \Lambda_2(K_n) \arrow[d, "\tau"'] \arrow[dr, "\lambda_2"] & \\ (B\mathbb{Z}/p \times B\mathbb{Z}/p) \ltimes K_n \arrow[r, "\widehat{\Delta}_{K_n}"] \arrow[d, "\mathrm{id}\times\widetilde{\Delta}_{K_n}"'] & \Lambda_2(K_n) \arrow[r, "\lambda_2"] \arrow[d] & K_{p^2n} \\ B\mathbb{Z}/p \ltimes \Lambda(K_n) \arrow[r, "\widetilde{\Delta}_{\Lambda(K_n)}"] & \Lambda(\Lambda(K_n)). \arrow[ur, "\lambda(\lambda)"'] & \end{tikzcd}$$

(b) Derive the formula

$$(T \times \mathrm{id})^*(\mathrm{id} \times \widetilde{\Delta})^*\widetilde{\Delta}^*(\lambda(\lambda)) = (\mathrm{id} \times \widetilde{\Delta})^*\widetilde{\Delta}^*(\lambda(\lambda)).$$

Algebraic Form of the Fundamental Symmetry Relation. Using our knowledge of the operation of $\mathcal{A}_2$ on $\widetilde{H}^*(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2)$, we will translate the fundamental symmetry relation

$$(T \times \mathrm{id})^*\left(\boxed{(\mathrm{id} \times \widetilde{\Delta})^*\widetilde{\Delta}^*(\lambda(\lambda))}\right) = \boxed{(\mathrm{id} \times \widetilde{\Delta})^*\widetilde{\Delta}^*(\lambda(\lambda))}$$

[5] The wreath product is denoted $\mathbb{Z}/p \wr \mathbb{Z}/p$.

of Problem 23.47(b) into an algebraic equation that will lead, after a bit of manipulation of binomial coefficients, to the Ádem relations. At this point we abandon the odd primes and focus our attention on $p = 2$.

Problem 23.48. Using the Cartan formula, show that

$$(\mathrm{id} \times \widetilde{\Delta})^* \widetilde{\Delta}^*(\lambda(\lambda)) = \sum_{i,j,k} \binom{n-j}{k} \; x^{2n-i} \otimes y^{n-j+k} \otimes \mathrm{Sq}^{i-k}\mathrm{Sq}^{j}(\iota_n),$$

where $H^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2) \cong \mathbb{Z}/2[x] \otimes \mathbb{Z}/2[y]$.

Next we want to apply the function $(\tau \times \mathrm{id})^*$. This is facilitated by making a substitution which reduces the operation of τ^* to a simple interchange of indices.

Problem 23.49.

(a) Define l by the equation $k = n+j-l$ and eliminate k from the expression of Problem 23.48.

(b) Show that the fundamental symmetry relation is equivalent to the equations

$$\sum_{j} \binom{n-j}{n+j-l} \mathrm{Sq}^{i+l-n-j}\mathrm{Sq}^{j}(\iota_n) = \sum_{j} \binom{n-j}{n+j-i} \mathrm{Sq}^{i+l-n-j}\mathrm{Sq}^{j}(\iota_n)$$

for each i and l.

The equation of Problem 23.49(b) essentially encodes all the relations among the Steenrod squares. For convenience, though, it is much more useful to have a formula that tells us how to rewrite an expression of the form $\mathrm{Sq}^a\mathrm{Sq}^b$.

Messing Around with Binomial Coefficients. The derivation of the most commonly used form of the Ádem relations from the equations in Problem 23.49(b) involves manipulation of binomial coefficients modulo 2, using three basic facts:

(1) $\binom{a}{b} = \binom{a}{a-b}$ for all $0 \le b \le a$,

(2) $\binom{a}{b} \equiv \binom{2^r+a}{b} \equiv \binom{2^r+a}{2^r+b}$ mod 2 for $1 < b \le a < 2^r$, and

(3) $\binom{2^r-1-b}{b} \equiv 0$ mod 2 for $0 < b < 2^{r-1}$.

These can all be proved by simple algebraic or combinatorial arguments.

Exercise 23.50. Prove these three formulas.

The Ádem relations follow from this by choosing n, i, l in such a way that the left-hand side of the expression in Problem 23.49(b) has only a single nonzero term.

Problem 23.51. Let $n = 2^r - 1 + s$ and let $l = n + s$ with $r \gg 0$.

(a) Show that $\binom{n-j}{l-2j} = 0$ except when $j = s$.

(b) Show that $\binom{n-j}{i-2j} = \binom{s-j-1}{i-2j}$.

(c) Prove that

$$\mathrm{Sq}^i \mathrm{Sq}^s(\alpha) = \sum_j \binom{s-j-1}{i-2j} \mathrm{Sq}^{(i+s)-j}\mathrm{Sq}^j(\alpha)$$

for all spaces X and all $\alpha \in \widetilde{H}^*(X;\mathbb{Z}/2)$, and so complete the proof of the Ádem relations.

23.6. The Algebra of the Steenrod Algebra

First we collect what we have learned, and then we study modules over the Steenrod algebra.

23.6.1. Fundamental Properties of Steenrod Operations. Finally, we collect the principal properties of the Steenrod reduced powers together in two omnibus theorems: one for $p = 2$ and one for odd primes.

Theorem 23.52. *The Steenrod squares are stable operations satisfying the following properties:*

(a) Sq^0 *is the identity,*

(b) Sq^1 *is the Bockstein associated to* $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$,

(c) $\mathrm{Sq}^d : \widetilde{H}^n(\,?\,;\mathbb{Z}/2) \to \widetilde{H}^{n+d}(\,?\,;\mathbb{Z}/2)$ *is a natural group homomorphism,*

(d) the ***Ádem relations***[6]

$$\mathrm{Sq}^a\mathrm{Sq}^b = \sum_j \binom{b-j-1}{a-2j} \mathrm{Sq}^{a+b-j}\mathrm{Sq}^j,$$

(e) the ***Cartan formula***

$$\mathrm{Sq}^d(u \cdot v) = \sum_{i+j=d} \mathrm{Sq}^i(u)\mathrm{Sq}^j(v),$$

(f) the ***unstable conditions***

(i) $\mathrm{Sq}^d(u) = u^2$ *if* $|u| = d$,

(ii) $\mathrm{Sq}^d(u) = 0$ *if* $|u| < d$.

Here is the version for operations in $\widetilde{H}^*(\,?\,;\mathbb{Z}/p)$ with p an odd prime.

[6] In fact they completely determine the algebraic structure of the Steenrod algebra, because these relations generate all the relations. The sum as written is over all i, but most terms are zero because we define $\binom{m}{n} = 0$ if $n > m$ or if $n < 0$.

Theorem 23.53. *The Steenrod powers P^k and the Bockstein β are stable operations satisfying the following properties:*

(a) P^0 *is the identity,*

(b) $P^k : \widetilde{H}^n(\,?\,;\mathbb{Z}/p) \to \widetilde{H}^{n+2k(p-1)}(\,?\,;\mathbb{Z}/p)$ *is a natural group homomorphism,*

(c) there are two collections of mod p ***Ádem relations***

$$P^a P^b = \sum_j (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j} P^j$$

and

$$\begin{aligned} P^a \beta P^b \quad = \quad & \sum_j (-1)^{a+j} \binom{(p-1)(b-j)}{a-pj} \beta P^{a+b-j} P^j \\ & - \sum_j (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj-1} P^{a+b-j} \beta P^j, \end{aligned}$$

(d) the ***Cartan formulas***

$$P^k(u \cdot v) = \sum_{i+j=k} P^i(u) P^j(v) \qquad \text{and} \qquad \beta(u \cdot v) = \beta(u) \cdot v + (-1)^{|v|} u \cdot \beta(v),$$

(e) the ***unstable conditions***

(i) $P^k(u) = u^p$ *if* $|u| = 2k$,

(ii) $P^k(u) = 0$ *if* $|u| < 2k$ *and*

(iii) $\beta P^k(u) = 0$ *if* $|u| < 2k+1$.

We have already proved Theorem 23.52, and we have proved a good deal of Theorem 23.53.

Exercise 23.54. Make a careful accounting of which parts of Theorem 23.53 have been proved and which have not. Also, check that all of Theorem 23.52 has been proved.

23.6.2. Modules and Algebras over $\mathcal{A}$. We'll focus our attention on the prime 2. Since $\mathcal{A}_2$ is a graded algebra, we can talk about (graded) modules over $\mathcal{A}_2$. We say that a graded $\mathcal{A}_2$-module M is an **unstable module** if it satisfies the **unstable condition**:

$$\mathrm{Sq}^n(x) = 0 \qquad \text{if } |x| < n.$$

An **unstable algebra** over $\mathcal{A}$ is a graded $\mathbb{Z}/2$-algebra A that is an unstable module over $\mathcal{A}$ satisfying

(1) the Cartan formula $\mathrm{Sq}^n(x \cdot y) = \sum_{i+j=n} \mathrm{Sq}^i(x)\mathrm{Sq}^j(y)$ and

(2) $\mathrm{Sq}^n(x) = x^2$ if $|x| = n$.

We write $\mathcal{U}$ for the category of unstable modules over $\mathcal{A}$ and $\mathcal{K}$ for the category of unstable algebras over $\mathcal{A}$.

Much of the work done in this chapter can be summarized in the statement that there is a lift in the diagram

$$\begin{tikzcd}
 & \mathcal{K} \arrow[d, "\text{forget}"] \\
\mathcal{T}_\circ \arrow[r, "H^*(\,?\,;\mathbb{Z}/2)"] \arrow[ur, dotted, bend left] & \mathrm{AB}\,\mathcal{G}^*.
\end{tikzcd}$$

The Steenrod Algebra and Symmetric Polynomials. The algebras of symmetric polynomials are an important example of an unstable algebra over $\mathcal{A}_2$ that is not (or, at least, not obviously) the cohomology of a space.

For any ring R, the symmetric group $\mathrm{Sym}(n)$ acts on the polynomial algebra $R[x_1, \ldots, x_n]$ according to the rule

$$(\sigma \cdot f)(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

We say that $f \in R[x_1, \ldots, x_n]$ is called a **symmetric polynomial** if $\sigma \cdot f = f$ for all $\sigma \in \mathrm{Sym}(n)$. The set of all symmetric polynomials is a subalgebra of $R[x_1, \ldots, x_n]$, denoted

$$R[x_1, \ldots, x_n]^{\mathrm{Sym}(n)}.$$

Write $\mathcal{P}_k(\mathbf{n})$ for the set of k-element subsets of $\mathbf{n} = \{1, 2, \ldots, n\}$; and for $I \in \mathcal{P}_k(\mathbf{n})$, write $x_I = \prod_{i \in I} x_i$. Then the k^{th} **elementary symmetric function** is given by

$$\sigma_k = \sum_{I \in \mathcal{P}_k(\mathbf{n})} x_I.$$

For example, $\sigma_1 = x_1 + x_2 + \cdots + x_n$ and $\sigma_n = x_1 x_2 \cdots x_n$.

Problem 23.55. Show that $\prod_{i=1}^n (1 + x_i) = \sum_{j=0}^n \sigma_j$.

Theorem 23.56. *The algebra of symmetric polynomials is polynomial on the elementary symmetric functions:*

$$R[x_1, \ldots, x_n]^{\mathrm{Sym}(n)} = R[\sigma_1, \sigma_2, \ldots, \sigma_n].$$

You should take this for granted, since it is a standard theorem in pure algebra.[7] The first thing to do is to verify that the symmetric polynomials lie in $\mathcal{K}$.

Problem 23.57.

(a) Show that $H^*((\mathbb{R}\mathrm{P}^\infty)^n; \mathbb{Z}/2) = \mathbb{Z}/2[x_1, \ldots, x_n]$.

(b) Show that if $\sigma \in \mathrm{Sym}(n)$, there is a map $T_\sigma : (\mathbb{R}\mathrm{P}^\infty)^n \to (\mathbb{R}\mathrm{P}^\infty)^n$ whose induced map is given by the formula $(T_\sigma)^*(f) = \sigma \cdot f$.

[7] PROJECT. Prove it!

(c) Show that $\mathbb{Z}/2[x_1, \ldots, x_n]^{\mathrm{Sym}(n)}$ is an unstable $\mathcal{A}_2$-algebra.

Our investigation of the structure of $\mathbb{Z}/2[x_1, \ldots, x_n]^{\mathrm{Sym}(n)}$ as an $\mathcal{A}$-module will be facilitated by making a brief foray into the world of non-homogenous elements. We define the **total square** to be the operation

$$\mathrm{Sq} = \mathrm{Sq}^0 + \mathrm{Sq}^1 + \cdots + \mathrm{Sq}^n + \cdots .$$

This is an operator that takes an element to a sum of elements in different degrees. The power of the total square is that it is a multiplicative homomorphism.

Problem 23.58.

(a) Show that the total square $\mathrm{Sq}(x)$ is actually a finite sum.

(b) Show that $\mathrm{Sq}(xy) = \mathrm{Sq}(x)\mathrm{Sq}(y)$ for any $x, y \in H^*(X; \mathbb{Z}/2)$.

We also order the monomials in the elementary symmetric polynomials $\sigma_1, \sigma_2, \ldots, \sigma_n$ by setting

$$\sigma_1^{a_1}\sigma_2^{a_2}\cdots\sigma_n^{a_n} < \sigma_1^{b_1}\sigma_2^{b_2}\cdots\sigma_n^{b_n}$$

if $a_i = b_i$ for $i < k$ and $a_k < b_k$.

Problem 23.59.

(a) Determine $\mathrm{Sq}(x) \in H^*(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]$.

(b) Determine $\mathrm{Sq}(\sigma_n)$ in $H^*((\mathbb{R}\mathrm{P}^\infty)^n; \mathbb{Z}/2)$.

(c) Show that $\mathrm{Sq}^i(\sigma_n) = \sigma_i\sigma_n$.

The action of $\mathcal{A}_2$ on the symmetric polynomials can be used to verify the Ádem relations. If $R \in \mathcal{A}$ has degree n or less, then $R = 0$ if and only if $R(\sigma_n) = 0 \in H^*((\mathbb{R}\mathrm{P}^\infty)^n; \mathbb{Z}/2)$.

Project 23.60. Write the Ádem relation for $\mathrm{Sq}^a\mathrm{Sq}^b$ in the form $R_{a,b} = 0$. Prove the Ádem relations hold in $\mathcal{A}_2$ by verifying $R_{a,b}(\sigma_n) = 0$.

23.6.3. Indecomposables and Bases. Now we'll make a brief but informative study of the algebraic properties of the Steenrod algebra.

Proposition 23.61. *The cohomology operation* Sq^n *is a sum of compositions of squares of lower index if and only if* n *is not a power of* 2.

Problem 23.62.

(a) Suppose n is not a power of 2 and write $n = 2^a(2b+1)$ with $b > 0$. Expand the Ádem relation for $\mathrm{Sq}^{2^a}\mathrm{Sq}^{2^a(2b+1)}$. Show that the coefficient of $\mathrm{Sq}^n\mathrm{Sq}^0$ is nonzero.

(b) Evaluate $\mathrm{Sq}^m(x^{2^k})$ where $x \in H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2)$ and $m \leq 2^k$.

(c) Prove Proposition 23.61.

It follows from Proposition 23.61 that the composites of Sq^{2^a} for various a span $\mathcal{A}$, but these composites are not linearly independent.

Exercise 23.63. Find a sum of composites which is zero.

So we turn our attention to finding a basis for $\mathcal{A}_2$. We begin by introducing some notation and terminology for compositions of Steenrod operations. Given a sequence $I = (i_k, i_{k-1}, \ldots, i_1)$, write

$$\mathrm{Sq}^I = \mathrm{Sq}^{i_k} \circ \mathrm{Sq}^{i_{k-1}} \circ \cdots \circ \mathrm{Sq}^{i_1}.$$

The **degree** of I is $|I| = i_1 + i_2 + \cdots + i_k$. Finally, I is an **admissible** sequence if $i_j \geq 2i_{j-1}$ for all j; the **excess** of an admissible sequence I is

$$\begin{aligned} e(I) &= (i_k - 2i_{k-1}) + (i_{k-1} - 2i_{k-2}) + \cdots + (i_2 - 2i_1) + i_1 \\ &= i_k - (i_{k-1} + \cdots + i_1) \end{aligned}$$

(we assign the values $i_j = 0$ for $j \leq 0$).

Lemma 23.64. *The following are equivalent for an admissible sequence:*

(1) $e(I) \geq n$,
(2) $\mathrm{Sq}^I(x) = 0$ *for all* x *with* $|x| < n$,
(3) if $x \in H^n(\Sigma X; \mathbb{Z}/2)$, *then* $\mathrm{Sq}^I(x) \neq 0$.

Problem 23.65. Prove Lemma 23.64.

Theorem 23.66. *Let I be any index set. Then Sq^I is a linear combination of operations Sq^{J_α} with J_α admissible.*

Define the **moment** of I to be $m(I) = \sum j i_j$.

Problem 23.67.

(a) Apply the Ádem relations to show by induction on $m(I)$ that every Sq^I is a linear combination of admissible monomials.

Hint. If $I = (i_1, \ldots, i_k)$ is not admissible, then it can be written as a concatenation of three sequences $I = (I_1, (i,j), I_2)$ where (i,j) is not admissible.

(b) Use Problem 23.59 to show that if $I = (i_1, i_2, \ldots, i_k)$, then

$$\mathrm{Sq}^I(\sigma_n) = \sigma_n \left(\sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_k} + \boxed{\text{monomials of lower order}} \right).$$

(c) Show that the set $\{\mathrm{Sq}^I(\sigma_n) \mid I \text{ admissible and } |I| \leq n\}$ is linearly independent in $H^*((\mathbb{R}\mathrm{P}^\infty)^n; \mathbb{Z}/2)$.

(d) Finish the proof of Theorem 23.66.

The Same Story for Odd Primes. Since $\beta^2 = 0$, we may put any composition of Steenrod operations into the form

$$P^I = \beta^{\epsilon_{k+1}} \circ P^{i_k} \circ \beta^{\epsilon_k} \circ P^{i_{k-1}} \circ \ldots \circ \beta^{\epsilon_3} \circ P^{i_2} \circ \beta^{\epsilon_2} \circ P^{i_1} \circ \beta^{\epsilon_1}$$

in which each ϵ_i is either 0 or 1. We say that the sequence

$$I = (\epsilon_{k+1}, i_k, \epsilon_k, i_{k-1}, \epsilon_{k-1}, \ldots, \epsilon_3, i_2, \epsilon_2, i_1, \epsilon_1)$$

is **admissible** if $i_j \geq p i_{j-1} + \epsilon_{j-1}$ for each j. If I is admissible, then its **excess** is

$$e(I) = \sum 2(i_j - p i_{j-1}) - \epsilon_{j-1}.$$

Theorem 23.68. *Every operation P^I is a linear combination of operations P^{J_α} with J_α admissible.*

Project 23.69. Prove Theorem 23.68.

23.7. Wrap-Up

23.7.1. Delooping the Squaring Operation. Theorem 23.36 implies that the squaring operation deloops. The reason is that it explicitly provides an element whose loop is squaring, which is a bit unsatisfying. In this section you'll prove from basic principles that the square can be delooped.

Let's rephrase this in terms of the homotopy theory of spaces. The squaring operation $x \mapsto x^2$ is represented by a map

$$\mathrm{SQ}^n_n : K(\mathbb{Z}/2, n) \longrightarrow K(\mathbb{Z}/2, 2n).$$

If this were part of a stable cohomology operation, then there would be a map

$$\mathrm{SQ}^n_{n+1} : K(\mathbb{Z}/2, n+1) \longrightarrow K(\mathbb{Z}/2, 2n+1)$$

such that $\Omega\mathrm{SQ}^n_{n+1} \simeq \mathrm{SQ}^n_n$; also there would be a map SQ^n_{n+2} such that $\Omega\mathrm{SQ}^n_{n+2} = \mathrm{SQ}^n_{n+1}$, and so on. If all of these maps exist, then the sequence $\mathrm{Sq}^n = \{\mathrm{SQ}^n_k\}$ will constitute a stable cohomology operation of degree n extending the squaring operation.

Problem 23.70. Use the abbreviation $K_n = K(\mathbb{Z}/2, n)$; assume that $n \geq 1$.

(a) Show that $\bar{\Delta}^* : \widetilde{H}^{2n}(K_n \wedge K_n; \mathbb{Z}/2) \to \widetilde{H}^{2n}(K_n; \mathbb{Z}/2)$ is injective.

HINT. Find a space X and a class $u \in \widetilde{H}^n(X; \mathbb{Z}/2)$ with $u^2 \neq 0$.

(b) Use Proposition 18.24 (also see Problem 25.124) to show that if $m > n$, then there is a unique map $\mathrm{SQ}^n_m : K(\mathbb{Z}/2, m) \to K(\mathbb{Z}/2, m+n)$ such that $\Omega\mathrm{SQ}^n_{m+1} = \mathrm{SQ}^n_m$.

(c) Prove Theorem 23.36(a) in a different way.

23.7.2. Additional Problems and Projects.

Project 23.71. Develop a theory of homology operations.

Project 23.72. Prove Lemma 23.33 and derive the Ádem relations for p odd.

Exercise 23.73. Determine the signs in the Cartan formula for p odd.

Project 23.74. Write $P_p^n(x^k) = c_{n,k}x^{k+n(p-1)}$ for $x \in H^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/p)$. Determine the coefficients $c_{n,k}$.

Problem 23.75. Determine the action of the mod p Steenrod algebra on the algebra $H^*(B\mathbb{Z}/p; \mathbb{Z}/p)$.

Problem 23.76. For a prime p write $I^k(n) = (p^{k-1}n, \cdots, p^2n, pn, n)$ for $k \geq 1$ and $I^0(n) = (0)$.

(a) Show that if $|x| = n$, then $\mathrm{Sq}^{I^k(n)}(x) = x^{2^k}$.

(b) Show that if $|x| = 2n$, then $P^{I^k(n)}(x) = x^{p^k}$.

Project 23.77. Suppose X is an abelian topological group. Show how to define homology operations, based on the multiplication $\mu_p : X^p \to X$. Can you get away with a less restrictive hypothesis on X?

Problem 23.78. Suppose d is not a power of 2, and let $f : S^{n+d} \to S^n$.

(a) Show that the action of the Steenrod algebra on $\widetilde{H}^*(C_f; \mathbb{Z}/2)$ is trivial.

(b) Determine the algebra structure of $H^*(C_f; \mathbb{Z}/2)$.

(c) What can you say about the algebra structure in $H^*(C_f; \mathbb{Z})$?

Problem 23.79. Is $\mathrm{Sq}^{2^{n+1}} = \mathrm{Sq}^{2^n} \circ \mathrm{Sq}^{2^n}$?

Problem 23.80. Show that all stable operations are zero in $\widetilde{H}^*(\Omega S^n)$.

Problem 23.81. Determine the structure of $\widetilde{H}^*(\Omega\Sigma\mathbb{C}\mathrm{P}^2; \mathbb{Z}/2)$ as an $\mathcal{A}_2$-module.

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Chapter 24

Chain Complexes

In Section 21.2.4 we computed the cohomology of projective spaces by analyzing the web of long cofiber sequences that resulted naturally from their CW decompositions. In this chapter we apply the same technique to general CW complexes and discover that cohomology (and homology) can be computed by a two-step process: first compute a chain complex (which is functorial for cellular maps of CW complexes), and then form the (algebraic) homology groups of the chain complex. The fact that this process computes the topologically defined cohomology groups of CW complexes constitutes a proof of the uniqueness of ordinary cohomology.

From our point of view, chain complexes emerge as a method of computation and are certainly not central to the definition of the ordinary cohomology or homology of spaces. However, the fact that cohomology can be computed via chain complexes has a number of very useful consequences, because it allows us to apply the purely algebraic theory of chain complexes to the topological cohomology of spaces. Thus we obtain extensions of the Universal Coefficients Theorem and the Künneth theorem.

We show that the multiplicative structure in cohomology can be obtained from a chain algebra structure on cellular chain complexes. Interestingly, the chain algebra structure is not natural unless we choose our CW replacements with great care, for example, by using the *singular chain complex*.

Singular cohomology is the ordinary cohomology of a functorial CW replacement functor $\mathrm{CW}(\,?\,)$. Any choice leads to chain complexes that compute singular cohomology, but these complexes are not natural unless $\mathrm{CW}(f)$ is a cellular map for each f. The geometric realization of the singular simplicial set is a CW replacement that converts ordinary maps of spaces to cellular maps of CW complexes. The chain complex computed by this choice is called the *singular chain complex*; it is natural on the entire category $\mathcal{T}_*$. Moreover, the construction has a rich enough structure that it is possible to find a functorial homotopy from the diagonal map to a cellular map, which is why we are able to give the singular chain complex a functorial chain algebra structure.

24.1. The Cellular Complex

In this section we will see how a natural approach computing the cohomology of a CW complex leads to the concept of a cochain complex, and to the conclusion that the (algebraic) cohomology of the cellular cochain complex of X is naturally isomorphic to the (topological) cohomology of X.

24.1.1. The Cellular Cochain Complex of a Space. When we say that a path-connected space X is a CW complex, we are implicitly (or explicitly) asserting that we have in mind a cone decomposition of the form

$$\begin{tikzcd}
\bigvee S^0 \arrow[d] & \bigvee S^1 \arrow[d] & & \bigvee S^n \arrow[d] & \bigvee S^{n+1} \arrow[d] & \\
X_0 \arrow[r] & X_1 \arrow[r] & \cdots \arrow[r] & X_n \arrow[r] & X_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

for X. That is, X is the homotopy colimit of the telescope diagram, and each 'L-shaped' sequence $\bigvee S^n \to X_n \to X^{n+1}$ is a cofiber sequence. Just as we did in our study of $\mathbb{R}\mathrm{P}^n$ in Section 21.2.4, we extend these sequences to obtain a diagram of long interwoven cofiber sequences

$$\begin{tikzcd}
& \vdots \arrow[d] & & \vdots \arrow[d] & & \\
\bigvee S^{n-1} \arrow[r, dotted] & X_{n-1} \arrow[r] \arrow[d, dotted] & X_{n-1}/X_{n-2} \arrow[r] & \Sigma X_{n-2} \arrow[r] \arrow[d] & \Sigma(X_{n-2}/X_{n-3}) \arrow[r] & \cdots \\
\bigvee S^{n} \arrow[r] \arrow[rr, bend left, "\delta_{n-1}"] & X_{n} \arrow[r, dotted] \arrow[d] & X_{n}/X_{n-1} \arrow[r, dotted] \arrow[rr, bend right, "\delta_n"] & \Sigma X_{n-1} \arrow[r] \arrow[d, dotted] & \Sigma(X_{n-1}/X_{n-2}) \arrow[r] & \cdots \\
\bigvee S^{n+1} \arrow[r] & X_{n+1} \arrow[r] \arrow[d] & X_{n+1}/X_{n} \arrow[r] & \Sigma X_{n} \arrow[r, dotted] \arrow[d] & \Sigma(X_{n}/X_{n-1}) \arrow[r, dotted] & \cdots \\
& \vdots & & \ddots & &
\end{tikzcd}$$

Exercise 24.1. Show that the big diagram above is functorial on the category of CW complexes and cellular maps.

We write $C^n(X;G) = \widetilde{H}^n(X_n/X_{n-1};G)$. The quotient X_n/X_{n-1} is a wedge of n-spheres, in natural bijective correspondence with the set $\mathrm{Cell}_n(X)$

of n-cells of X; similarly, $\Sigma(X_{n-1}/X_{n-2})$ is a wedge of n-spheres in bijective correspondence with $\mathrm{Cell}_{n-1}(X)$. The collapse maps $q_i : X_n/X_{n-1} \to S^n_i$ to the i^{th} summand define isomorphisms

$$C^n(X;G) \cong \prod_{\mathrm{Cell}_n(X)} G$$

and the maps δ_n induce homomorphisms $d^n : C^n(X;G) \to C^{n+1}(X;G)$ defined by the diagram

$$\begin{tikzcd}
C^{n+1}(X;G) \arrow[d, equal] & & C^n(X;G) \arrow[ll, "d^n"'] \arrow[d, equal] \\
\widetilde{H}^{n+1}(X_{n+1}/X_n;G) & \widetilde{H}^{n+1}(\Sigma(X_n/X_{n-1});G) \arrow[l, "\delta_n^*"'] & \widetilde{H}^n(X_n/X_{n-1};G). \arrow[l, "\cong"']
\end{tikzcd}$$

We write $\mathcal{C}^*(X;G)$ for the sequence of groups $C^n(X;G)$ and the homomorphisms d^n between them; this is the **cellular cochain complex** of X.

We'll generally leave the coefficient group G out of the notation and simply write $\mathcal{C}^*(X)$.

Problem 24.2. Show that $d^n \circ d^{n-1} = 0$ for each n.

24.1.2. Chain Complexes and Algebraic Homology. Our attempt to compute the cohomology of a CW complex by studying its skeleta has led us, inexorably,[1] to a certain kind of algebraic gadget: a sequence $\mathcal{C}^* = \{C^n\}$ of abelian groups together with maps $d_n : C^n \to C^{n+1}$ satisfying the relation $d^n \circ d^{n-1} = 0$ for each n. These objects are known as **cochain complexes**. There is an entirely analogous notion of a **chain complex**, which is a sequence of spaces $\mathcal{C}_* = \{C_n\}$ together with maps $d_n : C_n \to C_{n-1}$ that *decrease* index and satisfy $d_n \circ d_{n+1} = 0$.

A standard convention holds that upper subscripts and lower subscripts are related by $C^n = C_{-n}$. If we adopt this convention, the only difference between a chain complex and a cochain complex is the notation. Therefore, we will frequently use the term **chain complex** generically for either concept. The structure maps d^n in a chain complex are called **differentials** or **boundary maps** (or **coboundary maps** if you insist on distinguishing between chain and cochain complexes).

[1] And against the will of the author.

There is a category **Chain** whose objects are chain complexes and whose morphisms are sequences $f = \{f^n : C^n \to D^n\}$ commuting with the differentials, which means that for each n, the diagram

$$\begin{tikzcd} C^{n+1} \arrow[r, "f^{n+1}"] & D^{n+1} \\ C^n \arrow[u, "d_C^n"] \arrow[r, "f_n"] & D_n \arrow[u, "d_D^n"'] \end{tikzcd}$$

commutes—such a sequence is called a **chain map**. For any ring R, there is a category $\mathbf{Chain}_R$ of chain complexes of R-modules, which are simply chain complexes in which the groups are R-modules and the maps are R-linear. The forgetful functor $\mathbf{Mod}_R \to \mathbf{Mod}_{\mathbb{Z}}$ extends to a functor $\mathbf{Chain}_R \to \mathbf{Chain}$ (since $\mathbf{Chain} = \mathbf{Chain}_{\mathbb{Z}}$).

There is a very important functor $H^* : \mathbf{Chain} \to \mathrm{AB}\,\mathcal{G}^*$ from chain complexes to graded abelian groups, known as **homology**.[2] Since $d^n \circ d^{n-1} = 0$, we have

$$\mathrm{Im}(d^{n-1}) = B^n(\mathcal{C}^*) \subseteq Z^n(C^*) = \ker(d^n).$$

The elements of $B^n(\mathcal{C}^*)$ are called the **boundaries** in $\mathcal{C}^n$ and the elements of $Z^n(\mathcal{C}^*)$ are called the **cycles** (or **cocycles**) in $\mathcal{C}^n$. Since $B^n(\mathcal{C}^*) \subseteq Z^n(C^*)$, we can form the quotient group

$$H^n(\mathcal{C}^*) = Z^n(\mathcal{C}^*)/B^n(\mathcal{C}^*),$$

which is known as the n^{th} **homology group** of $\mathcal{C}^*$. This is often called the n^{th} (algebraic) **cohomology group** of $\mathcal{C}^*$ when upper indexing is used.

Problem 24.3. Show that a chain map $f : \mathcal{C}^* \to \mathcal{D}^*$ induces maps $H^n(\mathcal{C}^*) \to H^n(\mathcal{D}^*)$ for each n, so that H^* is a functor.

You can read more about the pure algebra of chain complexes and homology in Section A.4.

24.1.3. Computing the Cohomology of Spaces via Chain Complexes. We have stumbled upon a construction that takes CW complexes and produces chain complexes. How can we resist looking at the algebraic homology of those chain complexes?

Problem 24.4. Show that the construction of Section 24.1 defines a functor $\mathcal{C}^* : \mathbf{CW}_* \to \mathbf{Chain}$.

Take a look at some examples.

[2] To distinguish it from the homology of spaces, we will sometimes refer to this functor as *algebraic* homology.

Exercise 24.5.

(a) Use two different cellular decompositions for S^1 to construct two different chain complexes for S^1. Then compute the cohomology groups of these chain complexes.

(b) Define several cellular maps from one of your decompositions to the other: one that is a homeomorphism, one that is of degree 2, and one that is not a homeomorphism but is a homotopy equivalence. Determine their effect on chain complexes and on the (algebraic) homology of the chain complexes.

The amazing fact[3] is that algebraic homology groups of the cochain complex $\mathcal{C}^*(X)$ are naturally isomorphic to the topologically defined cohomology groups $\widetilde{H}^*(X)$.

Theorem 24.6. *For any reduced ordinary cohomology theory $\widetilde{H}^*$, there is a natural isomorphism $\Phi : \widetilde{H}^*(\,?\,) \to H^*(\mathcal{C}^*(\,?\,))$ of functors defined on the category of CW complexes and cellular maps.*

The proof involves a careful investigation of the diagram

$$
\begin{tikzcd}
 & & & \widetilde{H}^n(X) \arrow[d, "\theta"] & \\
 & \widetilde{H}^{n-1}(X_n) \arrow[r] \arrow[d, dotted] & \widetilde{H}^n\left(X_{n+1/n}\right) \arrow[r] & \widetilde{H}^n(X_{n+1}) \arrow[d, "\phi"] & \\
\widetilde{H}^{n-1}\left(X_{n-1/n-2}\right) \arrow[r, "\delta"] \arrow[rr, bend left, "\boxed{d^{n-1}}"] & \widetilde{H}^{n-1}(X_{n-1}) \arrow[r, dotted, "\gamma"] \arrow[d] & \widetilde{H}^n\left(X_{n/n-1}\right) \arrow[r, dotted, "\beta"] \arrow[rr, bend right, "\boxed{d^n}"'] & \widetilde{H}^n(X_n) \arrow[r, "\alpha"] \arrow[d, dotted] & \widetilde{H}^{n+1}\left(X_{n+1/n}\right) \\
 & \widetilde{H}^{n-1}(X_{n-2}) \arrow[r] & \widetilde{H}^{n-1}(X_{n-1/n-2}) \arrow[r] & \widetilde{H}^n(X_{n-1}) &
\end{tikzcd}
$$

which is obtained by applying $\widetilde{H}^n(\,?\,; G)$ to the large diagram of cofiber sequences that we constructed in Section 24.1.1, and in which I have used the space-saving notation $X_{n/m}$ to stand for X_n/X_m. Before launching into the proof, you should simplify this diagram as much as possible.

Exercise 24.7. Identify all the groups in this diagram which are necessarily zero. Which maps are necessarily surjective? Which must be injective?

Let $Z_n = \ker(d^n)$ and $B_n = \operatorname{Im}(d^{n-1})$, so that $B_n \subseteq Z_n \subseteq C^n(X)$ and $H^n(\mathcal{C}^*(X)) = Z_n/B_n$.

Problem 24.8.

(a) Show that $B_n = \ker(\beta)$. Conclude that $\widetilde{H}^n(\mathcal{C}^*(X)) \cong \beta(Z_n)$.

(b) Show that $\beta(Z_n) = \ker(\alpha)$.

[3]First observed by Emmy Noether.

(c) Show that $\ker(\alpha) \cong \widetilde{H}^n(X)$.

(d) Prove Theorem 24.6.

Problem 24.9. Use Theorem 24.6 to give a different proof of Corollary 21.102.

24.1.4. Chain Complexes for Homology Theories. An entirely parallel development leads to the conclusion that the homology of a CW complex X can be computed using chain complexes.

Problem 24.10. Define a cellular chain complex functor $\mathcal{C}_* : \mathbf{CW}_* \to \mathbf{Chain}$.

Theorem 24.11. *Let $\widetilde{H}_*$ be a reduced ordinary homology theory, and let $\mathcal{C}_*(X)$ be the corresponding cellular chain complex. Then there is a natural isomorphism*

$$\Phi : H_*(\mathcal{C}_*(\,?\,)) \longrightarrow \widetilde{H}_*(\,?\,)$$

of functors defined on the category CW complexes and cellular maps.

Problem 24.12. Prove Theorem 24.11.

Problem 24.13. Write down the cellular chain complex for $B\mathbb{Z}/p$ and use it to compute $H_*(B\mathbb{Z}/p; G)$ and $H^*(B\mathbb{Z}/p; G)$.

24.1.5. Uniqueness of Cohomology and Homology. It is conceivable that different ordinary cohomology theories with the same coefficients could somehow give rise to different cellular chain complexes. But actually the coefficient group G specifies the chain complex $\mathcal{C}^*(X; G)$ up to isomorphism for finite complexes, so Theorem 24.6 implies the uniqueness of ordinary cohomology, at least for finite CW complexes.

We have already seen that the *groups* $C^n(X; G)$ and $C_n(X; G)$ depend only on the group G and the cellular structure of X—specifically, the number of cells that X has in dimension n. It remains to study the boundary maps d^n and d_n which are induced by the topological boundary maps

$$\delta_n : \bigvee_{\mathrm{Cell}_{n+1}(X)} S^{n+1} \longrightarrow \bigvee_{\mathrm{Cell}_n(X)} S^{n+1}.$$

These are maps from one wedge of n-spheres to another, and in Section 19.3 we showed that maps f of this kind are fully described by the integer matrix $A(f)$ which records the degrees of its various coordinate functions.

Problem 24.14. Show that the boundary maps $(\delta_n)_*$ and δ_n^* of the chain complex $\mathcal{C}^*(X)$ are given by the matrices $A(\delta_n)$ and $A(\delta_n)^T$ for all CW complexes X and Y.

Corollary 24.15. *Any two ordinary cohomology theories (or homology theories) with the same group of coefficients are naturally isomorphic on the category of finite CW complexes and cellular maps.*

Problem 24.16. Prove Corollary 24.15.

24.2. Applying Algebraic Universal Coefficients Theorems

We establish certain basic algebraic properties of the cellular chain and cochain complexes. We also show how these complexes, for various coefficients, are related to one another. This information is fed into purely algebraic theorems about the homology of (co)chain complexes, resulting in very general universal coefficients theorems.

24.2.1. Constructing New Chain Complexes. Our abstract algebraic results apply only when the given chain complexes are complexes of free abelian groups (or R-modules), so we begin by noting when the cellular chain complexes of a space are free.

Problem 24.17. Let X be a CW complex.

(a) Show that $\mathcal{C}_*(X;\mathbb{Z})$ is a chain complex of free abelian groups.

(b) Show that if X is of finite type, then $\mathcal{C}^*(X;\mathbb{Z})$ is a chain complex of finitely generated free abelian groups.

Here is the general result that we use to construct new chain complexes from old ones.

Problem 24.18. Let $F : \mathcal{AB}\mathcal{G} \to \mathcal{AB}\mathcal{G}$ be a functor such that $F(0) = 0$. Then F extends to a functor F : **Chain** $\to$ **Chain** (which switches upper indices and lower indices if F is contravariant).

In particular, given a chain complex $\mathcal{C}^*$, we may construct the chain complexes $\mathcal{C}^* \otimes G$ and $\mathrm{Hom}(\mathcal{C}^*, G)$ for any abelian group G.

Problem 24.19. Let X be a CW complex and let G be an abelian group. Show that

(a) $\mathcal{C}_*(X;G) \cong \mathcal{C}_*(X;\mathbb{Z}) \otimes G$ and

(b) $\mathcal{C}^*(X;G) \cong \mathrm{Hom}_{\mathbb{Z}}(C_*(X;\mathbb{Z}), G)$.

If X is of finite type, then the relationship between $\mathcal{C}^*(X)$ and $\mathcal{C}_*(X)$ is symmetrical for spaces, because the complexes $\mathcal{C}^*(X;\mathbb{Z})$ and $\mathcal{C}_*(X;\mathbb{Z})$ are both free.

Problem 24.20. Suppose X is a CW complex of finite type and G is an abelian group.

(a) Show that $\mathcal{C}^*(X;G) \cong \mathcal{C}^*(X;\mathbb{Z}) \otimes G$.

(b) Show that $\mathcal{C}_*(X;G) \cong \mathrm{Hom}(\mathcal{C}^*(X;\mathbb{Z}), G)$.

Exercise 24.21. Show that Problems 24.19 and 24.20 remain true when $\mathbb{Z}$ is replaced by a PID R and G is replaced by an R-module M.

24.2.2. Universal Coefficients Theorems. The relationships between topological homology and cohomology of spaces with various coefficients can be addressed by studying the algebraic homology and cohomology of the chain complexes $\mathcal{C}^* \otimes G$ and $\mathrm{Hom}(\mathcal{C}_*, G)$. The homology groups of these complexes are determined up to isomorphism, according to a theorem of pure homological algebra that we quote here.

Theorem 24.22. *If $\mathcal{C}_*$ is a free abelian chain complex, then there are short exact sequences*

(a) $0 \to H_*(\mathcal{C}_*) \otimes G \to H_*(\mathcal{C}_* \otimes G) \to \Sigma\,\mathrm{Tor}(H_*(\mathcal{C}_*), G) \to 0$ *and*

(b) $0 \to \Sigma\,\mathrm{Ext}(H_*(\mathcal{C}_*), G) \to H^*(\mathrm{Hom}(\mathcal{C}_*, G)) \to \mathrm{Hom}(H_*(\mathcal{C}_*), G) \to 0$

of graded abelian groups. The sequences split for algebraic reasons, but not naturally.

Project 24.23. Prove Theorem 24.22.

We immediately harvest sequences describing the homology and cohomology of a space.

Theorem 24.24. *For any space X, there are exact sequences*

(a) $0 \to \widetilde{H}_*(X;\mathbb{Z}) \otimes G \to \widetilde{H}_*(X;G) \to \Sigma\,\mathrm{Tor}(\widetilde{H}_*(X;\mathbb{Z}), G) \to 0$ *and*

(b) $0 \to \Sigma\,\mathrm{Ext}(\widetilde{H}_*(X;\mathbb{Z}), G) \to \widetilde{H}^*(X;G) \to \mathrm{Hom}(\widetilde{H}_*(\mathcal{X};\mathbb{Z}), G) \to 0$

of graded abelian groups. The sequences split, but not naturally.

Problem 24.25. Use Theorem 24.22 to prove Theorem 24.24.

Problem 24.26. Suppose X is of finite type. Derive formulas for $\widetilde{H}^*(X;G)$ in terms of $\widetilde{H}^*(X;\mathbb{Z})$ and for $\widetilde{H}_*(X;G)$ in terms of $\widetilde{H}^*(X;\mathbb{Z})$.

24.3. The General Künneth Theorem

We proved a Künneth theorem in Chapter 21 which allowed us to determine the cohomology of a product, provided one of the factors had free cohomology. In this section we apply a purely algebraic Künneth theorem given in Section A.5.3 to the cellular chain complex and derive much more general Künneth formulas.

24.3.1. The Cellular Complexes of a Product. In Section 9.4.2 we studied the boundary maps arising from the natural cone decomposition of a product of two spaces having cone decompositions. When we restrict this construction to CW decompositions, the formula we established relates the boundary map in the cellular chain complex of $X \times Y$ to those in $\mathcal{C}_*(X)$ and in $\mathcal{C}_*(Y)$.

Problem 24.27.

(a) Use the smash product to define an external product

$$H^n(X_n/X_{n-1}) \otimes H^m(Y_m/Y_{m-1}) \to H^{n+m}((X_n \times Y_m)_{n+m}/(X_n \times Y_m)_{n+m-1}).$$

(b) Show that $(X_n \times Y_m)_{n+m}/(X_n \times Y_m)_{n+m-1}$ is naturally a wedge summand of $(X \times Y)_{n+m}/(X \times Y)_{n+m-1}$.

(c) Define an exterior product $\kappa : \mathcal{C}^*(X) \otimes \mathcal{C}^*(Y) \to \mathcal{C}^*(X \times Y)$.

Theorem 24.28. *Let X and Y be CW complexes. Then*

(a) $\mathcal{C}_*(X) \otimes \mathcal{C}_*(Y) \to \mathcal{C}_*(X \times Y)$ *is an isomorphism, and*

(b) $\mathcal{C}^*(X) \otimes \mathcal{C}^*(Y) \to \mathcal{C}^*(X \times Y)$ *is an isomorphism provided at least one of X and Y is of finite type.*

Furthermore, under these isomorphisms, the algebraic exterior product of chain complexes coincides with the topological exterior product in the sense that the diagram

$$\begin{tikzcd}
H^*(\mathcal{C}^*(X)) \otimes H^*(\mathcal{C}^*(Y)) \arrow[r, "\kappa"] & H^*(\mathcal{C}^*(X) \otimes \mathcal{C}^*(Y)) \\
H^*(X) \otimes H^*(Y) \arrow[r, "\kappa"] \arrow[u, "\cong"] & H^*(X \times Y) \arrow[u, "\cong"']
\end{tikzcd}$$

commutes (and so does the analogous one for homology).

Problem 24.29. Prove Theorem 24.28.

Hint. Use Theorem 9.81.

24.3.2. Künneth Theorems for Spaces. Theorem 24.28 reduces the computation of the homology of $X \times Y$ to the computation of the homology of a tensor product of chain complexes, and Theorem A.47 shows us how to find the homology groups of a tensor product.

Theorem 24.30 (Topological Künneth Theorem)**.** *There are exact sequences*

$$0 \to H_*(X;G) \otimes H_*(Y;H) \to H_*(X \times Y; G \otimes H) \to \Sigma\,\mathrm{Tor}(H^*(X;G), H^*(Y;H)) \to 0$$

and

$$0 \to H^*(X;G) \otimes H^*(Y;H) \to H^*(X \times Y; G \otimes H) \to \Sigma\,\mathrm{Tor}(H^*(X;G), H^*(Y;H)) \to 0,$$

where the first maps are the topologically defined exterior products. The sequences are natural, and they split, but the splitting is not natural.

Problem 24.31. Prove Theorem 24.30.

24.4. Algebra Structures on $C^*(X)$ and $C_*(X)$

We have a natural algebra structure on $H^*(X;R)$. In this section, we will show that we can define algebra structures on the chain complexes $\mathcal{C}^*(X;R)$ that induce the standard multiplicative structure on $\widetilde{H}^*(X;R)$.

A **chain algebra** is a chain complex $\mathcal{C}^*$ with chain map $\mu : \mathcal{C}^* \otimes \mathcal{C}^* \to \mathcal{C}^*$.

Problem 24.32. Show that if $\mathcal{C}^*$ is a chain algebra, then the rule $[x]\otimes[y] \mapsto [xy]$ gives $H^*(\mathcal{C}^*)$ the structure of a graded ring.

Let X be a CW complex, and let $\widetilde{\Delta}$ be a cellular approximation for the diagonal map $\Delta : X \to X \times X$. Then we have the composition

$$\begin{tikzcd} \mathcal{C}^*(X) \otimes \mathcal{C}^*(X) \arrow[rr, "\mu"] \arrow[dr, "\kappa"'] & & \mathcal{C}^*(X) \\ & \mathcal{C}^*(X \times X). \arrow[ur, "\widetilde{\Delta}^*"'] & \end{tikzcd}$$

Problem 24.33. Show that μ gives $\mathcal{C}^*(X)$ the structure of a chain algebra and that the induced product on $H^*(X)$ is the ordinary cup product.

Different choices of $\widetilde{\Delta}$ give different algebra structures on $\mathcal{C}^*(X)$, but according to our work, they must all induce the same algebra structure in homology. A map $f : X \to Y$ induces a map $\mathcal{C}^*(f) : \mathcal{C}^*(Y) \to \mathcal{C}^*(X)$ of chain complexes, but because there is no canonical choice of $\widetilde{\Delta}$, we cannot expect $\mathcal{C}^*(f)$ to respect the multiplication.

Exercise 24.34. Is μ (graded) commutative? Associative?

Chain Algebras for Homology. A similar construction can be carried out for the chain complex that computes the homology $H_*(X;R)$ when X is an H-space.

Problem 24.35. Let X be an H-space with multiplication $\mu : X \times X \to X$.

(a) Show that the composite
$$\mathcal{C}_*(X) \otimes \mathcal{C}_*(X) \longrightarrow \mathcal{C}_*(X \times X) \xrightarrow{\mu_*} \mathcal{C}_*(X)$$
of the external product with the map induced by μ gives $\mathcal{C}_*(X)$ the structure of a chain algebra.

(b) Show that the induced map on $H_*(X;R)$ is the Pontrjagin product.

24.5. The Singular Chain Complex

If $\widetilde{H}^*$ is an ordinary cohomology theory satisfying the Wedge and Weak Equivalence Axioms, then we know that $\widetilde{H}^*$ is naturally isomorphic to the singular extension of its restriction to CW complexes. So let's study the functor $\widetilde{H}^*(\mathrm{CW}(\,?\,))$.

As we know, different choices of CW will yield isomorphic theories. In this section we will specialize to the functor $\mathrm{CW}(\,?\,) = |\Delta_\bullet(\,?\,)|$ (see Section 15.6.2). With this choice, we call $\mathcal{S}_* = \mathcal{C}_*(\mathrm{CW}(\,?\,)) : \mathcal{T}_* \to$ **Chain** the **singular chain complex** functor.

Theorem 24.36. *There is a natural isomorphism $\widetilde{H}^*(X) \xrightarrow{\cong} H^*(\mathcal{S}^*(X))$ for all spaces X and likewise for $\widetilde{H}_*$.*

Problem 24.37. Prove Theorem 24.36.

The group $\mathcal{S}_n(X)$ has canonical generators which are in bijective correspondence with the n-cells of $\mathrm{CW}(X)$. Because of our choice of cellular replacement functor, these cells correspond in turn to maps $\sigma : \Delta^n \to X$. That is, the generator σ corresponds to a chain map $\sigma_* : \mathcal{C}_n(\Delta^n) \to \mathcal{C}_n(\mathrm{CW}(X))$ with $\sigma_*(\mathrm{id}) = \sigma$. Thus the structure of the chain complex $\mathcal{S}_*(X)$ is determined by the value of the differential on $\mathrm{id} \in \mathcal{C}_n(\Delta^n)$.

We determine this differential using the method of the universal example: it suffices to determine the value of the differential on the single simplex $\mathrm{id} : \Delta^n \to \Delta^n$. For this, we need to recall that Δ^n has a canonical CW decomposition, in which the k-cells are the convex hulls of the $(k+1)$-element subsets of the set $\{e_0, e_1, \ldots, e_n\}$ of its vertices.

Problem 24.38. Let Δ^n be the n-simplex with its canonical simplicial structure, and let $\sigma : \Delta^n \to X$.

(a) Let $d : \mathcal{C}_n(\Delta^n) \to \mathcal{C}_{n-1}(\Delta^n)$ be the boundary map, and determine $d(\mathrm{id})$.

(b) Show that σ gives rise to a simplicial map $\bar{\sigma} : \Delta^n \to \mathrm{CW}(X)$ such that $\bar{\sigma}_*(\mathrm{id}) = \sigma \in \mathcal{S}_*(X)$.

(c) Describe the boundary operator $\mathcal{S}_n(X) \to \mathcal{S}_{n-1}(X)$ in terms of the maps $\Delta^{n-1} \to X$ that index the canonical generators of $\mathcal{S}_{n-1}(X)$.

Project 24.39. What would it take to use this as the definition of homology and prove that it satisfies the Eilenberg-Steenrod axioms?

If we use the geometric realization, it is possible to give a functorial map $\widetilde{\Delta} : |\Delta_\bullet(X)| \to |\Delta_\bullet(X \times X)|$ that is homotopic to the diagonal. This boils down to defining a simplicial structure on $\Delta^n \times \Delta^n$ and then choosing a homotopy from $\Delta^n \to \Delta^n \times \Delta^n$ to a simplicial map. This was done by Alexander and Whitney. Thus the singular cochain complex functor factors through the category of cochain algebras.

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Chapter 25

Topics, Problems and Projects

This chapter contains a large collection of applications of homology and cohomology; they are generally unrelated to one another.

25.1. Algebra Structures on $\mathbb{R}^n$ and $\mathbb{C}^n$

A normed **$\mathbb{F}$-algebra** is an $\mathbb{F}$-algebra A equipped with a **norm** $|?| : A \to \mathbb{R}$ with the basic properties (1) $|x| \geq 0$ for all $x \in A$ and $|x| = 0$ if and only if $x = 0$, (2) $|xy| = |x||y|$ for all $x, y \in A$ and (3) $|cx| = |c||x|$ for $c \in \mathbb{F}$ and $x \in A$. We know some examples of normed $\mathbb{R}$-algebras: $\mathbb{R}$, $\mathbb{R}^2 = \mathbb{C}$ and $\mathbb{R}^4 = \mathbb{H}$. Are there other values of n for which $\mathbb{R}^n$ can be given the structure of a normed $\mathbb{R}$-algebra?

Since vector spaces are contractible, it would be understandable if you thought that homotopy theory could have no application to the study of such things. But, as we have seen, if A is an n-dimensional normed $\mathbb{F}$-algebra, then S^{nd-1} can be given the structure of an H-space. In fact, more is true: $\mathbb{F}\mathrm{P}^{n-1}$ must also be an H-space. Also, since there are very few values of n for which there is a comultiplication $H^*(\mathbb{F}\mathrm{P}^{n-1}) \to H^*(\mathbb{F}\mathrm{P}^{n-1}) \otimes H^*(\mathbb{F}\mathrm{P}^{n-1})$, we find very restrictive conditions on the dimension of A.

Problem 25.1. Suppose A is an n-dimensional normed $\mathbb{F}$-algebra, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

(a) Show that $\mathbb{F}\mathrm{P}^{n-1}$ must be an H-space.

(b) By studying $\mu_*(x_1)^n$, where x_1 generates $H^1(\mathbb{F}\mathrm{P}^{n-1})$, determine numerical conditions on n under which such a diagram cannot exist. Conclude that there is no multiplication $\mu : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n$ unless n is ... what?[1]

Your result can be used to give another proof (due to R. Bruner) of the Fundamental Theorem of Algebra.

Theorem 25.2 (Fundamental Theorem of Algebra)**.** *The field* $\mathbb{C}$ *is algebraically closed.*

Problem 25.3. Prove Theorem 25.2 by showing that if there were a nonconstant polynomial $f \in \mathbb{C}[x]$ without a root in $\mathbb{C}$, then there would have to be a normed $\mathbb{C}$-algebra A with $1 < \dim_{\mathbb{C}}(A) < \infty$.

Exercise 25.4. Why does this proof not apply to $\mathbb{R}$? Does it apply to $\mathbb{H}$?

25.2. Relative Cup Products

Let $\widetilde{h}^*$ be any multiplicative cohomology theory, with unreduced counterpart h^*. If $A \hookrightarrow X$ is a cofibration, then there is an induced map $\widetilde{h}^*(X/A) \to h^*(X)$. The image of this map is an ideal I_A, and we'll see how these ideals interact for $A, B \subseteq X$. As an application, we derive a relationship between the multiplicative structure in $\widetilde{h}^*(X)$ and the Lusternik-Schnirelmann category of X.

25.2.1. A New Exterior Cup Product. Here we introduce a variant of the reduced diagonal map which is useful for studying the cup products of cohomology classes $u \in \widetilde{h}^*(X)$ which are known to vanish on certain subsets of X.

Problem 25.5. Let $f : A \to X$ and let I_f denote the image of induced map $\widetilde{h}^*(C_f) \to h^*(X)$.

(a) Show that $I_f \subseteq h^*(X)$ is a two-sided ideal.

(b) Show that when $f : * \to X$, $I_f \cong \widetilde{h}^*(X)$.

Since we can replace any map with a cofibration, we now focus on cofibrations $f : A \hookrightarrow X$. In this case, we have $I_A = \mathrm{Im}(\widetilde{h}^*(X/A) \to h^*(X))$ is a two-sided ideal. If $A, B \subseteq X$ are subcomplexes, then we may form the ideals I_A and I_B. What can we say about their product $I_A \cdot I_B$?

To answer this question, we introduce a modified reduced diagonal map.

Problem 25.6. Let $A \hookrightarrow X$ and $B \hookrightarrow X$ be cofibrations and let $\widetilde{h}^*$ be a multiplicative cohomology theory.

[1] The conditions will be different for different $\mathbb{F}$.

(a) Show that the dotted arrow in the diagram

$$\begin{tikzcd} X \arrow[rr, "\overline{\Delta}"] \arrow[d, "q_{A\cup B}"'] & & X \wedge X \arrow[d, "q_A \wedge q_B"] \\ X/(A\cup B) \arrow[rr, dotted] & & (X/A)\wedge(X/B) \end{tikzcd}$$

can be filled in—in exactly one way—to make the diagram strictly commutative.

(b) Prove that $I_A \cdot I_B \subseteq I_{A\cup B}$.

(c) Show that if $A \cup B \hookrightarrow X$ is a cofibration, then there is an exterior product

$$\widetilde{h}^*(X/A) \otimes \widetilde{h}^*(X/B) \longrightarrow \widetilde{h}^*(X/(A\cup B)).$$

(d) Show that if X is a CW complex and $A, B \subseteq X$ are subcomplexes, then the diagram

$$\begin{tikzcd} H^*(X)\otimes H^*(X) \arrow[rr] & & H^*(X) \\ \widetilde{H}^*(X/A)\otimes H^*(X/B) \arrow[rr] \arrow[u, "q\otimes \mathrm{id}"] & & \widetilde{H}^*(X/(A\cup B)) \arrow[u, "q"'] \end{tikzcd}$$

commutes.

Exercise 25.7. Explain the naturality of the relative cup product.

Problem 25.8. Show that there is an external product of the form

$$\widetilde{H}^*(X/A;G)\otimes \widetilde{H}^*(Y/B;H) \longrightarrow \widetilde{H}^*((X\times Y)/(A\times Y\cup X\times B))$$

and that $\widetilde{H}^*((X\times Y)/(A\times Y\cup X\times B)) = \widetilde{H}^*((X/A)\wedge(Y/B))$.

25.2.2. Lusternik-Schnirelmann Category and Products. You showed in Theorem 9.109 that the **Lusternik-Schnirelmann category** of a CW complex X is the least integer n for which X has a cover $X = A_0 \cup A_1 \cup \cdots \cup A_n$ where each A_k is a subcomplex of X and each inclusion map $A_k \hookrightarrow X$ is nullhomotopic (and $\mathrm{cat}(X) = \infty$ if there X has no finite cover of this kind).

The **nilpotency** of an ideal $I \subseteq R$ is the greatest n such that $I^n \neq 0$; we'll denote it by $\mathrm{nil}(I)$. When $R = h^*(X)$ and $I = \widetilde{h}^*(X)$, then $\mathrm{nil}(\widetilde{h}^*(X))$ is known as the **cup length** of X with respect to the theory h^*. It is the length of the longest 'nontrivial' product in the ring $h^*(X)$.

Theorem 25.9. *If X is a CW complex, then* $\mathrm{nil}(\widetilde{h}^*(X)) \leq \mathrm{cat}(X)$.

Problem 25.10. Prove Theorem 25.9.

The relationship between of Lusternik-Schnirelmann category and cup length is one of the most powerful basic tools in the theory.

Problem 25.11.

(a) Determine the Lusternik-Schnirelmann category of $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_r}$.

(b) Determine the Lusternik-Schnirelmann category of $\mathbb{F}\mathrm{P}^n$.

(c) Determine $\mathrm{cat}(\Omega S^n)$ and $\mathrm{cat}(\Lambda(S^n))$.

Problem 25.12. Is it possible that $X \times Y$ is a co-H-space?

It can happen that $\mathrm{nil}(\widetilde{h}^*(X))$ is strictly less than $\mathrm{cat}(X)$, but certain very restrictive conditions on the dimension and connectivity of X guarantee equality.

Theorem 25.13. *If X is $(n-1)$-connected and nk-dimensional, then the following are equivalent:*

(1) $\mathrm{cat}(X) = k$,

(2) $\mathrm{wcat}(X) = k$,

(3) there are coefficients G such that the product

$$H^n(X;G)^{\otimes k} \longrightarrow H^{nk}(X;G^{\otimes k})$$

is nonzero.

Problem 25.14. Prove Theorem 25.13.

25.3. Hopf Invariants and Hopf Maps

The **Hopf map** is the map $\eta : S^3 \to S^2$ which we defined in Section 15.1.2 when we proved that $S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ is a fiber bundle. In Problem 17.42 you used the long exact sequence of the fibration $S^3 \to \mathbb{C}\mathrm{P}^1$ to show that $\pi_3(S^2) = \mathbb{Z} \cdot \eta$.

When Hopf showed—in 1935, without the benefit of the long exact sequence of a fibration—that η is nontrivial, it became the first known nontrivial map from a sphere to a sphere of a different dimension. Before this, many topologists thought that the functors $H_n(\,?\,;\mathbb{Z})$ and $\pi_n(\,?\,)$ might actually be naturally equivalent.

Hopf observed that for each $x \in S^2$, the preimage $\eta^{-1}(x)$ is a circle; and furthermore, if $x \neq y \in S^2$, then the circles $\eta^{-1}(x)$ and $\eta^{-1}(y)$ are *linked*. So he defined a function on maps $\alpha : S^3 \to S^2$ like this:

(a) if α is not differentiable, replace α with a differentiable map which is homotopic to α; then the preimage of almost every point $x \in S^2$ will be a circle or a union of circles;

(b) the linking number of any two such preimages is called the *Hopf invariant* of α, and it is denoted $H(\alpha)$.

He proved that $H(\alpha)$ is a well-defined *homomorphism* $\pi_3(S^2) \to \mathbb{Z}$, so that if $\alpha \simeq \beta$, then $H(\alpha) = H(\beta)$ and, of course, $H(*) = 0$. Then, since $H(\eta) = 1$, it must be that $\eta \not\simeq *$.

About 15 years later, Steenrod [**157**] recast the Hopf invariant in terms of cohomology algebras. Let $\alpha : S^{2n-1} \to S^n$, and let C_α be the cofiber of α. Then $\widetilde{H}^k(C_\alpha; \mathbb{Z})$ is nontrivial only in dimensions n and $2n$; assign generators x_n and y_{2n} to these groups. Then

$$x_n^2 = a \cdot y_{2n} \qquad \text{for some integer } a \in \mathbb{Z}.$$

Steenrod showed that $a = \pm H(\alpha)$, with the sign depending only on the choice of generator y. We take this cup product formula as our definition of the **Hopf invariant**.

Problem 25.15. Show that $H(\eta) = 1$.

25.3.1. The Hopf Invariant Is a Homomorphism. It may seem strange that a function defined in terms of multiplication should be an additive homomorphism, but that is the case. We will prove this using an argument due to James [**104**].

Theorem 25.16. *The Hopf invariant is a homomorphism $\pi_{2n-1}(S^n) \to \mathbb{Z}$.*

We can focus only on the case $n > 1$.

Exercise 25.17. Show that if $n = 1$, $H(\alpha) = 0$ for all f.

It will be convenient to use a nonstandard formulation of the condition that a map be a homomorphism.

Problem 25.18.

(a) Show that $H(0) = 0$.

(b) Show that the following are equivalent:
   (1) $H : \pi_{2n-1}(S^n) \to \mathbb{Z}$ is a homomorphism,
   (2) if $\alpha_1 + \alpha_2 + \alpha_3 = 0 \in \pi_{2n-1}(S^n)$, then

$$H(\alpha_1) + H(\alpha_2) + H(\alpha_3) = 0 \in \mathbb{Z}.$$

You will show that H is a homomorphism by verifying the second condition in Problem 25.18(b). So let $\alpha_1 + \alpha_2 + \alpha_3 = 0 \in \pi_{2n-1}(S^n)$ and form the three spaces

$$X_1 = S^n \cup_{\alpha_1} D^{2n}, \qquad X_2 = S^n \cup_{\alpha_2} D^{2n} \qquad \text{and} \qquad X_3 = S^n \cup_{\alpha_3} D^{2n}$$

and a fourth space Y obtained by attaching all three disks to S^n. There are natural inclusions $X_i \hookrightarrow Y$ for $i = 1, 2, 3$.

Exercise 25.19. Express Y in terms of X_1, X_2 and X_3 using colimits.

Problem 25.20. Write $a_1 = H(\alpha_1)$, $a_2 = H(\alpha_2)$ and $a_3 = H(\alpha_3)$.

(a) Determine the rings $H^*(X_i;\mathbb{Z})$ for $i = 1, 2, 3$.

(b) Determine the ring $H^*(Y;\mathbb{Z})$ and the maps $H^*(Y;\mathbb{Z}) \to H^*(X_i;\mathbb{Z})$ for $i = 1, 2, 3$.

(c) Consider the cofiber sequence

$$\bigvee_1^3 S^{2n-1} \xrightarrow{(\alpha_1,\alpha_2,\alpha_3)} S^n \hookrightarrow Y \xrightarrow{q} \bigvee_1^3 S^{2n},$$

and show that there is a map $f : S^{2n} \to Y$ such that $q_*(f)$ is the three-fold folding map $\nabla_3 \in \pi_{2n}(\bigvee_1^3 S^{2n})$.

(d) Prove Theorem 25.16 by studying $f^* : H^*(Y;\mathbb{Z}) \to H^*(S^{2n};\mathbb{Z})$.

The Hopf invariant has other nice algebraic properties.

Problem 25.21. Let $f : S^n \to S^n$, $g : S^{2n-1} \to S^{2n-1}$ and let $\alpha \in \pi_{2n-1}(S^n)$.

(a) Show that $H(f \circ \alpha) = (\deg(f))^2 \cdot H(\alpha)$.

(b) Show that $H(\alpha \circ g) = \deg(g) \cdot H(\alpha)$.

25.3.2. The Hopf Construction. Determining the complete list of values of n for which there is a map with Hopf invariant equal to one was a major problem throughout the 1950s. This was not just idle navel-gazing: the answer to this problem resolves several other important problems. The connection is made using the **Hopf construction**.

Given a map $\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$, set up a morphism of categorical (and homotopy) pushout squares

$$\begin{tikzcd}
S_1^{n-1} \times S_2^{n-1} \arrow[rr] \arrow[dr] \arrow[dd, "\mu"'] & & D_1^n \times S_2^{n-1} \arrow[dr] \arrow[dd, "g"' near start] & \\
& S_1^{n-1} \times D_2^n \arrow[rr] \arrow[dd, "f" near start] & & S_1^{n-1} * S_2^{n-1} \arrow[dd, "h"] \\
S^{n-1} \arrow[rr] \arrow[dr] & & D_+^n \arrow[dr] & \\
& D_-^n \arrow[rr] & & S^n.
\end{tikzcd}$$

To define the maps f and g, we write elements of D^n in the form tz where $t \in I$ and $z \in S^{n-1}$ and set $f(x, ty) = [\mu(x,y), t]$; similarly $g(tx, y) = [\mu(x,y), t]$. The resulting map of pushouts $h : S^{2n-1} \to S^n$ is known as the **Hopf construction** on μ.

One way to distinguish between maps $\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$ is to consider the composition $S^{n-1} \vee S^{n-1} \to S^{n-1} \times S^{n-1} \xrightarrow{\mu} S^{n-1}$; since this composite is a map from one wedge of spheres to another, it may be identified with a matrix $A = [d_1, d_2]$, which we call the **bidegree** of μ.

Proposition 25.22. *Let $\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$ and let $h : S^{2n-1} \to S^n$ be the map produced from μ by the Hopf construction. If μ has bidegree $[d_1, d_2]$, then $H(h) = d_1 d_2$.*

The verification of Proposition 25.22 will make use of the relative cup products introduced in Section 25.2. Write K for the cofiber of h. Since $S_1^{n-1} * S_2^{n-1}$ is the boundary of $\partial D_1^n \times D_2^n$, K is the categorical (and homotopy) pushout in the square

$$\begin{tikzcd}
S_1^{n-1} * S_2^{n-1} \arrow[r] \arrow[d, "h"'] \arrow[dr, phantom, "\text{pushout}"] & D_1^n \times D_2^n \arrow[d] \\
S^n \arrow[r] & K.
\end{tikzcd}$$

Problem 25.23.

(a) Construct and explain the commutative diagram

$$\begin{tikzcd}
 & H^n(K) \otimes H^n(K) \arrow[r] & H^{2n}(K) \\
H^n(D_+^n/S^{n-1}) \otimes H^n(D_-^n/S^{n-1}) \arrow[d, "u \otimes v"'] & H^n(K/D_-^n) \otimes H^n(K/D_+^n) \arrow[l, "\cong"'] \arrow[u, "\cong"'] \arrow[r] \arrow[d] & H^{2n}(K/S^n) \arrow[u, leftrightarrow, "\cong"'] \arrow[d, "\cong"'] \\
H^n(D_1^n/S_1^{n-1}) \otimes H^n(D_2^n/S_2^{n-1}) & H^n(D_1^n/S_1^{n-1}) \otimes H^n(D_2^n/S_2^{n-1}) \arrow[l, "\cong"'] \arrow[r, "\cong"] & H^n\left(D_1^n \times D_2^n/S_1^{n-1} * S_2^{n-1}\right).
\end{tikzcd}$$

(b) Prove Proposition 25.22 by determining the maps u and v.

25.3.3. Hopf Invariant One. In this section we ask what numbers can be the Hopf invariant of a map $S^{2n-1} \to S^n$. The answer depends on the dimension n.

Problem 25.24.

(a) Show that $H(\pi_{2n-1}(S^n)) = 0$ if n is odd.

(b) We know that $J^2(S^n) = S^n \cup_\alpha D^{2n}$ for some map $\alpha : S^{2n-1} \to S^n$. Show that if n is even, $H(\alpha) = \pm 2$.

(c) Show that $\pi_{2n-1}(S^n) \cong \mathbb{Z} \oplus G_n$ where G_n is some other abelian group.

Problem 25.24 implies that if n is even, $H(\pi_{2n-1}(S^n))$ is either all of $\mathbb{Z}$ or else the subgroup $2\mathbb{Z} \subseteq \mathbb{Z}$. The image of the Hopf invariant will be all of $\mathbb{Z}$ if and only if there is an element $\alpha \in \pi_{2n-1}(S^n)$ with $H(\alpha) = 1$.

Problem 25.25. Show that if S^n is an H-space, then there is an element $\alpha \in \pi_{2n-1}(S^n)$ with $H(\alpha) = 1$.

We have shown that $\mathbb{R}^{n+1}$ cannot be a normed $\mathbb{R}$-algebra except possibly in dimensions $n+1 = 2^k$; and we know that $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^4$ and $\mathbb{R}^8$ do have such structures. It can be shown that this question is equivalent to the Hopf invariant one question, since the following are equivalent:

(1) there is an element $\alpha \in \pi_{2n-1}(S^n)$ with Hopf invariant one,
(2) S^{n-1} can be given the structure of an H-space,
(3) $\mathbb{R}^n$ can be given the structure of normed $\mathbb{R}$-algebra.

The existence of elements with Hopf invariant one can be rephrased in terms of cohomology operations.

Problem 25.26. Show that the following are equivalent:

(1) $\pi_{2n-1}(S^n)$ has no element with Hopf invariant one,
(2) For any $X = S^m \cup D^{m+n}$, Sq^n acts trivially in $\widetilde{H}^*(X; \mathbb{Z}/2)$.

Problem 25.27. Using the algebra of $\mathcal{A}_2$, show that if n is not a power of 2, then $\pi_{2n-1}(S^n)$ does not contain an element with Hopf invariant one.

The Hopf Invariant One problem was finally solved by J. F. Adams in a monumental paper in which he undertook a deep study of the Steenrod squares, ultimately showing:

> *Except for $2^n = 1, 2, 4$ or 8, Sq^{2^n} may be expressed as a sum of compositions of secondary cohomology operations*[2] *of degree strictly less than 2^n.*

Problem 25.28. Use Adams' conclusion to show that there are no maps $\alpha \in \pi_{2n-1}(S^n)$ with $H(\alpha) = 1$ for $n \notin \{1, 2, 4, 8\}$.

Later we will sketch a second proof of Adams' 'Hopf invariant one' theorem, using cohomology operations in K-theory (see Section 32.7).

25.3.4. Generalization. The Hopf invariant has a vast array of generalizations, and the whole story is much too complicated to cover in detail. We will take a brief look at three of these notions.

Factoring the Reduced Diagonal. Since the Hopf invariant is defind in terms of cup product, it should not come as a surprise that the Hopf invariant can be described, without cohomology, in terms of the homotopy properties of the reduced diagonal map $\bar{\Delta} : X \to X^{\wedge k+1}$.

Problem 25.29. Consider the cofiber sequence

$$S^{2n-1} \xrightarrow{f} S^n \xrightarrow{i} Y \xrightarrow{q} S^{2n} \to \cdots .$$

[2] A secondary cohomology operation is an operation which is defined only on certain *subsets* of the cohomology of X and which takes values in certain *quotients* of the cohomology of X.

(a) Show that in the diagram

$$\begin{tikzcd} Y \arrow[r, "\bar{\Delta}"] \arrow[d, "q"'] & Y \wedge Y \\ S^{2n} \arrow[r, dotted, "\phi"] & S^n \wedge S^n \arrow[u, "i\wedge i"'] \end{tikzcd}$$

there is a unique homotopy class of maps ϕ making the diagram commute in $\mathrm{H}\mathcal{T}_*$.

(b) Show that $\deg(\phi) = H(f)$.

Problem 25.29 gives us a new definition of the Hopf invariant that can be applied equally well to *all* maps $f : S^m \to S^n$. If $f : S^m \to S^n$, then we form the cofiber sequence $S^m \xrightarrow{f} S^n \xrightarrow{i} Y \xrightarrow{q} S^{m+1} \to \cdots$ and construct the diagram

$$\begin{tikzcd} Y \arrow[r, "\bar{\Delta}"] \arrow[d, "q"'] & Y \wedge Y \\ S^{m+1} \arrow[r, dotted, "\phi"] & S^n \wedge S^n. \arrow[u, "i\wedge i"'] \end{tikzcd}$$

We define the **Hopf invariant** of a map $f : S^m \to S^n$ to be the unique homotopy class $\phi : S^m \to S^n \wedge S^n$ making the diagram commute up to homotopy.

Problem 25.30. Verify that there is a unique homotopy class ϕ making the diagram commute in $\mathrm{H}\mathcal{T}_*$.

If we are willing to accept a *set-valued* Hopf invariant, then we can generalize further. Given a map $S^m \to X$ where $\mathrm{wcat}(X) = n$,[3] we may set up the cofiber sequence $S^m \xrightarrow{f} X \xrightarrow{i} Y \xrightarrow{q} S^{m+1} \to \cdots$ and the diagram

$$\begin{tikzcd} Y \arrow[r, "\bar{\Delta}"] \arrow[d, "q"'] & Y^{\wedge(n+1)} \\ S^m \arrow[r, dotted, "\phi"] & X^{\wedge(n+1)}. \arrow[u, "i^{\wedge(n+1)}"'] \end{tikzcd}$$

Problem 25.31. Show that there is a map ϕ making the diagram commute.

We define the **Hopf set** of f to be the set $\overline{\mathcal{H}}(f)$ of all $\phi \in \pi_{n+1}(X^{\wedge(n+1)})$ making the diagram commute up to homotopy. This notion (in a slightly different form) was introduced by Berstein and Hilton [**24**] for the study of (weak) category, because it captures the relationship between the weak category of X and that of Y.

[3]The **weak category** of X is the least integer n such that the reduced diagonal map $\bar{\Delta} : X \to X^{\wedge(n+1)}$ is nullhomotopic.

Proposition 25.32. *If $Y = X \cup_f D^{n+1}$, then* $\mathrm{wcat}(Y) \le \mathrm{wcat}(X)$ *if and only if* $0 \in \overline{\mathcal{H}}(f)$.

Problem 25.33. Prove Proposition 25.32.

The James Hopf Invariant. A different generalization of the Hopf invariant may be derived from the James splitting. In fact, this construction fits the original Hopf invariant into an infinite family of Hopf-type invariants.

The James splitting is the canonical homotopy equivalence $\Sigma\Omega S^{n+1} \simeq \Sigma \bigvee_{k=1}^{\infty} S^{nk}$. Composition of $\Sigma\widehat{f}$ with the collapse to the k^{th} summand is a map

$$\Sigma\Omega S^{n+1} \longrightarrow S^{nk+1}$$

whose adjoint $H_k : \Omega S^{n+1} \to \Omega S^{nk+1}$ is called the k^{th} **James Hopf invariant** map. We are justified in calling these maps (generalized) *Hopf* invariants, because the induced map

$$(H_2)_* : \pi_{4m-1}(S^{2m}) \longrightarrow \pi_{4m-1}(S^{4m-1}) \cong \mathbb{Z}$$

can be identified with the ordinary Hopf invariant defined in terms of cup products. In fact, H_2 agrees—in a range of dimensions—with the Hopf invariant defined by factoring the reduced diagonal.

Project 25.34. Show that if $f : S^m \to S^n$ with $m < 3n - 3$, then $\Sigma H_2(f)$ is the same as the Hopf invariant H defined following Problem 25.29.

Hint. Show that the diagram

$$\begin{tikzcd}
S^m \arrow[rr, "H_2(\alpha)"] \arrow[d, equal] & & S^{2n-1} \arrow[d, "{[\mathrm{id},\mathrm{id}]}"] \\
S^{2n-1} \arrow[rr] \arrow[d, "\alpha"'] & & S^n \vee S^n \arrow[d, hook] \\
S^n \arrow[r] & S^n \vee S^n \arrow[r, hook] & C_f \vee C_f
\end{tikzcd}$$

commutes up to homotopy.

The deeper theory of James Hopf invariants is founded on the following basic observation.

Lemma 25.35. *The k^{th} Hopf invariant induces an isomorphism*

$$(H_k)_* : H_{2nk}(\Omega S^{2n+1}; \mathbb{Z}) \xrightarrow{\cong} H_{2nk}(\Omega S^{2nk+1}; \mathbb{Z}).$$

Problem 25.36.

(a) Replacing ΩS^{2n+1} with $J = J(S^{2n})$, show that there is a map h making the diagram

$$\begin{tikzcd} J^{k-1} \arrow[r] & J \arrow[r] \arrow[d, "H_p"'] & J/J^{k-1} \arrow[dl, dotted, "h"] \\ & \Omega S^{2nk+1} & \end{tikzcd}$$

commute, where the top row is a cofiber sequence.

(b) Prove Lemma 25.35.

We will see later that the invariants H_p for p prime can shed considerable light on the homotopy groups of spheres. Many of these results are based on the homotopy type of the fiber of H_p, which is of course intimately bound up with the map induced by H_p on cohomology.

Problem 25.37. Determine the map

$$(H_p)^* : H^*(\Omega S^{2np+1}; \mathbb{Z}/p) \longrightarrow H^*(\Omega S^{2n+1}; \mathbb{Z}/p).$$

For later use, it will be nice to know how the James Hopf invariant is related to the Berstein-Hilton Hopf invariant of Section 20.4.

Problem 25.38. Let $\alpha : S^n \to S^{m+1}$.

(a) Show that $H_k(\alpha)$ is adjoint to $S^n \xrightarrow{\delta(\alpha)} \Sigma\Omega S^{m+1} \xrightarrow{q} S^{mk+1}$.

(b) Deduce that if $H_k(\alpha) \neq 0$ for some k, then $\mathcal{H}(\alpha) \neq \{*\}$.

Problem 25.39. Consider the Hopf invariants of maps $f : S^{4n-1} \to S^{2n}$.

(a) Show that the fiber $F_1(S^{2n})$ has the homotopy type $S^{4n-1} \vee Z$, where Z is $(6n-3)$-connected.

(b) Show that the Berstein-Hilton Hopf invariant of f is $H(f)$ times the inclusion $S^{4n-1} \hookrightarrow F_1(S^{2n})$.

(c) Determine $H_k(f)$ for all k.

25.4. Some Homotopy Groups of Spheres

Using the map η and our knowledge of its Hopf invariant, we can compute some of the 'higher' homotopy groups of spheres.

25.4.1. The Group $\pi_{n+1}(S^n)$. Using the James construction, we can develop a good enough understanding of the suspension homomorphism to determine $\pi_{n+1}(S^n)$ for all n.

Problem 25.40.

(a) Show that $\pi_{n+1}(S^n) \neq 0$ for $n \geq 2$.

(b) Determine the kernel of the suspension map $\Sigma : \pi_3(S^2) \to \pi_4(S^3)$.
HINT. Study the ring $H^*(J(S^2); \mathbb{Z})$.

(c) Determine the groups $\pi_{n+1}(S^n)$ for $n \geq 3$.

Since $\pi_{n+1}(S^n)$ has only two elements, there are only two homotopy types for spaces with cellular decompositions of the form $X = S^n \cup D^{n+2}$.

Problem 25.41. Show that if $X \simeq S^n \cup D^{n+2}$, then

$$X \simeq \begin{cases} S^n \vee S^{n+2} & \text{if } \mathrm{Sq}^2 : \widetilde{H}^n(X; \mathbb{Z}/2) \to \widetilde{H}^{n+2}(X; \mathbb{Z}/2) \text{ is zero,} \\ \Sigma^{n-2}\mathbb{C}\mathrm{P}^2 & \text{if } \mathrm{Sq}^2 : \widetilde{H}^n(X; \mathbb{Z}/2) \to \widetilde{H}^{n+2}(X; \mathbb{Z}/2) \text{ is nonzero.} \end{cases}$$

Problem 25.42. Determine the homotopy type of $\mathbb{C}\mathrm{P}^{n+1}/\mathbb{C}\mathrm{P}^{n-1}$.

Exercise 25.43. Criticize the following argument:

> *In $\widetilde{H}^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z}/2)$, the operation Sq^2 is nontrivial in every other even dimension. Therefore, depending on the parity of n, the quotient $\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-3}$ has the homotopy type of either $\Sigma^{n-6}\mathbb{C}\mathrm{P}^2 \vee S^{2n}$ or $S^{2(n-2)} \vee \Sigma^{n-4}\mathbb{C}\mathrm{P}^2$.*

Problem 25.44. Determine the homotopy type of $\mathbb{R}\mathrm{P}^{n+1}/\mathbb{R}\mathrm{P}^{n-1}$.

25.4.2. Composition of Hopf Maps. We have seen that the cohomology operation Sq^2 can be used to show that for every n, $\Sigma^n \eta \not\simeq *$. Therefore, we have, for each n, maps

$$\eta : S^{n+2} \longrightarrow S^{n+1} \qquad \text{and} \qquad \eta : S^{n+1} \longrightarrow S^n.$$

This raises an interesting question: is the composite map $\eta^2 = \eta \circ \eta$ trivial or not?

Theorem 25.45 (G. W. Whitehead)**.** *For $n \geq 2$, $\pi_{n+2}(S^n) \cong \mathbb{Z}/2 \cdot \eta^2$.*

The case $n = 2$ is easy.

Problem 25.46. Show that $\pi_4(S^2) = \mathbb{Z}/2 \cdot \eta^2$.

Next we show that $\eta^2 : S^{n+2} \to S^n$ is nonzero for all $n \geq 2$. Suppose, for a contradiction, that $\eta^2 = 0$, and consider the resulting homotopy commutative diagram

$$\begin{tikzcd} S^{n+4} \arrow[r] \arrow[d, "\eta"'] & * \arrow[d] \\ S^{n+3} \arrow[r, "\eta"] & S^{n+2}. \end{tikzcd}$$

Problem 25.47.

(a) Form cofibers of the rows and columns to create a 3×3 diagram with iterated cofiber Q in the bottom right corner. Show that Q has a CW decomposition of the form $Q = S^{n+2} \cup D^{n+4} \cup D^{n+6}$.

(b) Show that the operations

$$\widetilde{H}^{n+2}(Q;\mathbb{Z}/2) \xrightarrow{\mathrm{Sq}^2} \widetilde{H}^{n+4}(Q;\mathbb{Z}/2) \xrightarrow{\mathrm{Sq}^2} \widetilde{H}^{n+6}(Q;\mathbb{Z}/2)$$

are both nontrivial.

(c) Use the Ádem relations to show $\eta^2 \neq 0$.

(d) Show that the following are equivalent:
   (1) Theorem 25.45,
   (2) the suspension maps $\Sigma^n : \pi_4(S^2) \to \pi_{n+4}(S^{n+2})$ are surjective for all $n \geq 0$,
   (3) the suspension map $\Sigma : \pi_4(S^2) \to \pi_5(S^3)$ is surjective.

We will prove Theorem 25.45 by establishing the third condition. This is a fairly complicated undertaking.

Problem 25.48.

(a) By studying the square

$$\begin{tikzcd} S^3 \arrow[r, "\Sigma"] \arrow[d, "\eta"'] & \Omega S^4 \arrow[d, "\Omega\Sigma\eta"] \\ S^2 \arrow[r, "\Sigma"] & \Omega S^3, \end{tikzcd}$$

show that $\Sigma : \pi_4(S^2) \to \pi_5(S^3)$ is onto if and only if $(\Sigma\eta)_* : \pi_5(S^4) \to \pi_5(S^3)$ is onto.

(b) Show that $(\Sigma\eta)_* : \pi_5(S^4) \to \pi_5(S^3)$ is onto if and only if $q_* : \pi_5(\Sigma\mathbb{C}\mathrm{P}^2) \to \pi_5(S^5)$ is injective, where q is the map defined by the cofiber sequence

$$S^4 \xrightarrow{\Sigma\eta} S^3 \xrightarrow{\partial} \Sigma\mathbb{C}\mathrm{P}^2 \xrightarrow{q} S^5 \xrightarrow{\Sigma^2\eta} S^4.$$

(c) By studying the James construction for $\mathbb{C}\mathrm{P}^2$, show that there is a 5-equivalence $\mathbb{C}\mathrm{P}^2 \vee S^4 \to \Omega\Sigma\mathbb{C}\mathrm{P}^2$. Conclude that $\pi_5(\Sigma\mathbb{C}\mathrm{P}^2) \cong \mathbb{Z}$, and finish the proof of Theorem 25.45.

25.5. The Borsuk-Ulam Theorem

A map $f : S^m \to S^n$ is an **odd map** if $f(-x) = -x$ for all $x \in S^m$. The algebra $H^*(\mathbb{R}\mathrm{P}^n;\mathbb{Z}/2)$ constrains the dimensions n and m for which an odd map can exist.

Theorem 25.49. *Let $f : S^m \to S^n$ be an odd map. Then $m \leq n$.*

Problem 25.50. Let $f : S^m \to S^n$ be an odd map.

(a) Show that the solid arrow part of the diagram

$$\begin{tikzcd} S^m \arrow[r, "f"] \arrow[d, "q_n"'] & S^n \arrow[d, "q_n"] \\ \mathbb{RP}^m \arrow[r, "\phi"'] \arrow[ur, dotted, "g"] & \mathbb{RP}^n \end{tikzcd}$$

commutes.

(b) Let $u_1 \in H^1(\mathbb{RP}^n; \mathbb{Z}/2)$ be the unique nonzero element. Show that if $\phi^*(u_1) = 0$, then there is a map g making the bottom triangle in the diagram commute.

(c) Show that $g \circ q_n = \pm f$.

Hint. q_n is a covering.

(d) Show that the formula $g \circ q_n = \pm f$ is incompatible with the oddness of f, and conclude that $\phi^*(u_1) \neq 0$.

(e) Prove Theorem 25.49.

Now we use Theorem 25.49 to prove the **Borsuk-Ulam theorem**.

Theorem 25.51 (Borsuk-Ulam)**.** *If $f : S^n \to \mathbb{R}^n$, then there is a point $x \in S^n$ such that $f(x) = f(-x)$.*

Problem 25.52. Prove Theorem 25.51 by studying the function $g : S^n \to S^{n-1}$ given by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}.$$

The Ham Sandwich Theorem. Suppose you have a ham sandwich—that is, three sets, B_1, B_2 and $H \subseteq \mathbb{R}^3$, which represent the bread (top and bottom slices) and the ham, and you want to cut it neatly in half. That is, you want to divide each of the regions B_1, B_2 and H into two pieces of equal volume. The question is this: *can you cut the ham and the bread with a single knife slice?* More precisely, is there a single plane which divides each of B_1, B_2 and H into two pieces with equal volume?

Theorem 25.53 (Ham Sandwich Theorem)**.** *Let $A_1, \ldots, A_n$ be Lebesgue measurable subsets of $\mathbb{R}^n$. Then there is an $(n-1)$-dimensional plane through the origin which bisects each of the sets A_i into two pieces of equal measure.*

For each $x \in S^n$, let $V_x \subseteq \mathbb{R}^n$ denote the $(n-1)$-dimensional subspace perpendicular to x. Then V_x divides $\mathbb{R}^n$ into two half-spaces: H_x^+, which contains x, and H_x^-, which contains $-x$. For each $i = 1, \ldots, n$, define a continuous function $f_i : S^n \to \mathbb{R}$ by setting

$$f_i(x) = \mu(A_i \cap H_x^+).$$

Problem 25.54. Use the functions f_i to prove Theorem 25.53.

25.6. Moore Spaces and Homology Decompositions

In Section 19.4, we defined Moore spaces $M(G, n)$ in terms of free resolutions of the (abelian) group G; here we show that Moore spaces are characterized by their homology. Finally, we revisit the cone decompositions of Section 20.2 and see that the mysterious groups G_n are actually the integral homology groups $\widetilde{H}_n(X; \mathbb{Z})$.

25.6.1. Homology of Moore Spaces. Just as Eilenberg-Mac Lane spaces are determined by their homotopy groups, Moore spaces are characterized by their integral homology.

Proposition 25.55. *If N is a simply-connected CW complex such that $H_*(N; \mathbb{Z}) \cong H_*(M(G, n); \mathbb{Z})$, then $N \simeq M(G, n)$.*

Problem 25.56.

(a) Determine $H_*(M(G, n); \mathbb{Z})$.

(b) Show that if X is $(n-1)$-connected, then the map

$$[M(G, n), X] \longrightarrow \operatorname{Hom}(G, H_n(X; \mathbb{Z}))$$

given by $f \mapsto f_*$ is surjective.

(c) Prove Proposition 25.55.

This constitutes another proof of Theorem 19.44.

25.6.2. Cohomology Operations in Moore Spaces. Since $M(G, n)$ is homotopy equivalent to a CW complex with nontrivial cells in dimensions n and $n+1$ only, the only cohomology operations in $\widetilde{H}^*(M(G, n); H)$ that could possibly be nontrivial are the Bocksteins.

Problem 25.57. Compute the cohomology $\widetilde{H}^*(M(G, n); H)$.

Let's focus now on Moore spaces for the groups $\mathbb{Z}/p^r$, which is enough for us to understand Moore spaces for finitely generated abelian groups—see Problem 25.134. Write $\beta_{a,b}$ for the Bockstein derived from the short exact sequence

$$0 \to \mathbb{Z}/p^a \longrightarrow \mathbb{Z}/p^{a+b} \longrightarrow \mathbb{Z}/p^b \to 0.$$

Problem 25.58. Fix a and b; for which values of r is the operation

$$\beta_{a,b} : \widetilde{H}^n(M(\mathbb{Z}/p^r, n); \mathbb{Z}/p^b) \longrightarrow \widetilde{H}^{n+1}(M(\mathbb{Z}/p^r, n); \mathbb{Z}/p^a)$$

nonzero? For which values is it injective? Surjective?

25.6.3. Maps Between Moore Spaces. Let's take a closer look at the maps from $M(G,n) \to M(H,n)$ with $n \geq 3$. For such n, the homotopy group $\pi_{n+1}(S^n) = \mathbb{Z}/2 \cdot \eta$ is stable and the suspension maps

$$\Sigma : [M(G,n), M(H,n)] \longrightarrow [M(G,n+1), M(H,n+1)]$$

are isomorphisms, so we can determine all groups of homotopy classes at once.

Problem 25.59. Let $n \geq 3$ and let G and H be abelian groups.

(a) Show that there are exact sequences

$$0 \to \operatorname{Ext}(G, H \otimes \mathbb{Z}/2) \longrightarrow [M(G,n), M(H,n)] \longrightarrow \operatorname{Hom}(G,H) \to 0.$$

(b) Show that if multiplication by 2 is an isomorphism in either G or H, then $[M(G,n), M(H,n)] \cong \operatorname{Hom}(G,H)$.

In particular, $[M(\mathbb{Z}/p,n), M(\mathbb{Z}/p,n)] \cong \mathbb{Z}/p$ if $n \geq 3$ and p is odd. The situation when G or H has two-torsion can be more interesting. In the simplest case $G = H = \mathbb{Z}/2$ our exact sequence takes the form

$$0 \to \mathbb{Z}/2 \longrightarrow [M(\mathbb{Z}/2,n), M(\mathbb{Z}/2,n)] \longrightarrow \mathbb{Z}/2 \to 0,$$

and we see that $[M(\mathbb{Z}/2,n), M(\mathbb{Z}/2,n)]$ is isomorphic to either $\mathbb{Z}/4$ or the Klein four group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Theorem 25.60. $[M(\mathbb{Z}/2,n), M(\mathbb{Z}/2,n)] \cong \mathbb{Z}/4 \cdot \mathrm{id}_{M(\mathbb{Z}/2,n)}$.

To prove this, it suffices to produce a map $f : M(\mathbb{Z}/2,n) \to M(\mathbb{Z}/2,n)$ whose order is not 2. You'll show that twice the identity map is nontrivial.

Problem 25.61.

(a) Show that if $2 \cdot \mathrm{id}_{M(\mathbb{Z}/2,n)} \simeq *$, then $M(\mathbb{Z}/2,n) \wedge M(\mathbb{Z}/2,n)$ would be homotopy equivalent to $M(\mathbb{Z}/2,2n) \vee M(\mathbb{Z}/2,2n+1)$. Show that, if this were the case, Sq^2 would act trivially in $\widetilde{H}^*(M(\mathbb{Z}/2,n) \wedge M(\mathbb{Z}/2,n); \mathbb{Z}/2)$.

(b) Use the Cartan formula to prove Theorem 25.60.

Significance of Theorem 25.60. The fact that the $\mathrm{id}_{M(\mathbb{Z}/2,n)}$ has order 4 instead of order 2 is not a mere curiosity. In [**146**], S. Schwede uses this fact to prove a theorem completely characterizing the stable category. The stable *homotopy* category has been known since the 1960s (at least), but there was no known corresponding stable model category which it was the homotopy category *of*. In the 1990s, various categories were found to fill this gap, and they were shown to be equivalent to one another. But there remained the possibility that there were 'exotic' stable categories having the same stable homotopy category. Schwede (and Shipley) [**146**, **148**] proved that this is not the case, and the fact that the order of $\mathrm{id}_{M(\mathbb{Z}/2,n)}$ is 4—together with loosely analogous results for the odd primes—is actually key to making the proof work.

25.6.4. Homology Decompositions. In Section 20.2 we constructed cone decompositions of the form

$$\begin{tikzcd} M_1 \arrow[d] & M_2 \arrow[d] & & M_n \arrow[d] & M_{n+1} \arrow[d] & \\ * \arrow[r] & X_{(1)} \arrow[r] & \cdots \arrow[r] & X_{(n-1)} \arrow[r] & X_{(n)} \arrow[r] & \cdots \end{tikzcd}$$

in which each space M_n is a Moore space of the form $M(G_n, n-1)$, where G_n is a group that depends (mysteriously) on X.

Problem 25.62.

(a) Show that the maps $M_n \to X(n-1)$ induce zero on $\widetilde{H}_*(\,?\,;\mathbb{Z})$.

(b) Determine the induced map $\widetilde{H}_*(X(n);\mathbb{Z}) \to \widetilde{H}_*(X;\mathbb{Z})$.

(c) Show that $G_n = H_n(X;\mathbb{Z})$.

Problem 25.62 shows that the filtration $\cdots \to X_{(n)} \to X_{(n+1)} \to \cdots$ of X can be thought of as building X one homology group at a time. For this reason, these cone decompositions are known as a **homology decompositions**. This implies that integral homology suffices to determine the $\mathcal{P}$-connectivity of a space.

Problem 25.63. Let $\mathcal{P} \sqcup \mathcal{Q}$ be a partition of the prime numbers, and let X be a simply-connected space. Show that

$$\mathrm{conn}_{\mathcal{P}}(X) + 1 = \min\{n \mid \widetilde{H}_n(X;\mathbb{Z}) \text{ is not a } \mathcal{Q}\text{-group}\}.$$

Finally, we derive a $\mathcal{P}$-local version of the Hurewicz theorem.

Theorem 25.64. *Let X be simply-connected with $\mathrm{conn}_{\mathcal{P}}(X) = n-1$. Then the Hurewicz map*

$$\mathrm{H} : \pi_n(X) \longrightarrow H_n(X;\mathbb{Z})$$

is a $\mathcal{P}$-isomorphism.

Recall that in our explicit construction of ordinary homology, the Hurewicz map can be identified via the diagram

$$\begin{tikzcd} \pi_n(X) \arrow[r, "\mathrm{H}"] \arrow[d, "\Sigma^t"'] & \widetilde{H}_n(X;\mathbb{Z}) \arrow[d, "\cong"] \\ \pi_{n+t}(X \wedge S^t) \arrow[r, "\mathrm{id}_X \wedge i"] & \pi_{n+t}(X \wedge K(\mathbb{Z}, t)) \end{tikzcd}$$

in which t is large and $i : S^t \hookrightarrow K(\mathbb{Z}, t)$ is the t-skeleton.

Problem 25.65.

(a) Show that it suffices to prove Theorem 25.64 for highly-connected X.

(b) Show that the cofiber C of $i : S^t \to K(\mathbb{Z}, t)$ is $(t + 1)$-connected.

(c) Show that $\mathrm{conn}_{\mathcal{P}}(X \wedge C) \geq \mathrm{conn}_{\mathcal{P}}(X) + (t + 2)$.

(d) Prove Theorem 25.64.

25.7. Finite Generation of $\pi_*(X)$ and $H_*(X)$

In our work with $\mathcal{P}$-connectivity, we have needed to admit the possibility that the homotopy groups or homology groups of spaces could be unfathomably large and complicated $\mathcal{P}$-groups. But how large can these groups get in practice?

If X is a finite complex, or even a CW complex of finite type, then the cellular chain complex approach to homology and cohomology shows that the homology groups of X must be finitely generated abelian groups. In this section we'll see that, if X is simply-connected, the same is true for homotopy groups.

Theorem 25.66. *If X is simply-connected, then the following are equivalent:*

(1) $\pi_k(X)$ is finitely generated for $k \leq n$,

(2) $\widetilde{H}_k(X;\mathbb{Z})$ is finitely generated for $k \leq n$,

(3) $\widetilde{H}^k(X;\mathbb{Z})$ is finitely generated for $k \leq n$.

This has a number of very important 'qualitative' consequences about how the topology of a CW complex impacts its homotopy groups.

Corollary 25.67. *Let X be a simply-connected CW complex.*

(a) If X has finite type (i.e., finitely many cells in each dimension), then $\pi_k(X)$ is finitely generated for all k.

(b) If $\pi_k(X)$ is finitely generated for all k, then X is homotopy equivalent to a CW complex of finite type.

Corollary 25.68. *If X is simply-connected, then the following are equivalent:*

(a) $\pi_k(X)$ is finite for all $k \leq n$, and

(b) $\widetilde{H}_k(X;\mathbb{Z})$ is finite for all $k \leq n$.

We begin the proof of Theorem 25.66 by showing that we can disregard cohomology and concentrate just on homology.

Problem 25.69. Show that $\widetilde{H}^k(X;\mathbb{Z})$ is finitely generated for $k \leq n$ if and only if $\widetilde{H}_k(X;\mathbb{Z})$ is finitely generated for $k \leq n$.

Problem 25.70. Suppose that $A \to B \to C$ is a cofiber sequence. Show that it cannot be that only one of the spaces has a nonfinitely generated homology group.

Now we take a look at the presence of nonfinitely generated groups in the homology of spaces in a fibration sequence.

Problem 25.71. Let $F \to E \to B$ be a fibration sequence of path-connected spaces. Suppose $\widetilde{H}_k(F;\mathbb{Z})$ is finitely generated for $k \leq n$. Show that either $\widetilde{H}_k(B;\mathbb{Z})$ and $\widetilde{H}_k(E;\mathbb{Z})$ are both finitely generated for all $k \leq n$ or both are *not* finitely generated for some $k \leq n$.

Hint. Let $G_m(p) : G_m(E) \to B$ be the result of m applications of the Ganea construction to the fibration p; show $\widetilde{H}_k(G_m(E);\mathbb{Z})$ is finitely generated for each $k \leq n$.

This implies that the homology groups of an Eilenberg-Mac Lane space for a finitely generated abelian group must be finitely generated.

Problem 25.72.

(a) Show that if G is a finitely generated abelian group, then $\widetilde{H}_k(K(G,n);\mathbb{Z})$ is finitely generated for each $n \geq 1$ and each k.

(b) Show that if X has finitely many nonzero homotopy groups, all of which are finitely generated, then $\widetilde{H}_k(X;\mathbb{Z})$ is finitely generated for all k.

Hint. Work by induction on the Postnikov decomposition of X.

When $n > 1$, Problem 25.72(a) is a special case of Theorem 25.66. It is crucial to our proof that the result for Eilenberg-Mac Lane spaces holds even in the non-simply-connected case.

Problem 25.73.

(a) Let X be a CW complex, and let $X \to P_n(X)$ be the n^{th} Postnikov section. Show that $X \to P_n(X)$ is an $(n+1)$-skeleton.

(b) Prove that if $\pi_k(X)$ is finitely generated for all $k \le n$, then $\widetilde{H}_k(X;\mathbb{Z})$ is finitely generated for all $k \le n$.

You proved that (1) implies (2) in Theorem 25.66. To prove the converse, suppose X is simply-connected and $\widetilde{H}_k(X;\mathbb{Z})$ is finitely generated for $k \le n$ and that $\pi_k(X)$ is finitely generated for $k < n$. We have to show that $\pi_n(X)$ is finitely generated.

Problem 25.74. Prove Theorem 25.66 by showing that the sequence

$$\widetilde{H}_{n+1}(P_{n-1}(X);\mathbb{Z}) \longrightarrow \widetilde{H}_n(K(\pi_n(X),n);\mathbb{Z}) \longrightarrow \widetilde{H}_n(P_n(X);\mathbb{Z})$$

is exact.

Problem 25.75. Derive Corollaries 25.67 and 25.68.

Problem 25.76.

(a) Show that (1) implies (2) (and (3)) in Theorem 25.66 even for non-simply-connected spaces.

(b) Show that the simply-connected hypothesis in Theorem 25.66 is necessary to prove that (2) implies (1) by studying $\pi_2(S^2 \vee S^1)$.

Hint. What is the universal cover of $S^1 \vee X$ if X is simply-connected?

25.8. Surfaces

A **surface** is a 2-dimensional manifold. There is a complete classification of the compact surfaces, expressed in terms of an operation called the **connected sum**. If M and N are 2-dimensional manifolds, then we may cut small disks out of each, leaving compact manifolds $\overline{M}$ and $\overline{N}$, each having boundary S^1. Now form the (homotopy) pushout diagram

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & \overline{M} \arrow[d] \\ \overline{N} \arrow[r] & M\#N. \end{tikzcd}$$

The space $M\#N$ is a compact 2-manifold without boundary, and it is called the **connected sum** of M and N.

Theorem 25.77. *Every orientable surface is a connected sum of S^2 with finitely many tori $S^1 \times S^1$, and every nonorientable surface is a connected sum of an orientable surface with $\mathbb{R}\mathrm{P}^2$.*

You should take this result for granted for the purposes of these problems (a nice exposition of Conway's ZIP proof can be found in [**176**]). Note that the classification, as stated, does not assert that the manifolds on this list are pairwise nonhomeomorphic to each other.

Problem 25.78. Let $T = S^1 \times S^1$ be the torus, and let $\overline{T}$ be the space obtained by deleting the interior of a small disk from T. Let $j : S^1 \to \overline{T}$ be the inclusion of the boundary of the disk.

(a) Show that j is not nullhomotopic, but its suspension is trivial.[4]

(b) Show that the maps $j_* : H_*(S^1) \to H_*(\overline{T})$ and $j^* : H^*(\overline{T}) \to H^*(S^1)$ are trivial for all coefficients.

Problem 25.79. Determine the cohomology ring $H^*(M;\mathbb{Z})$, where M is an orientable compact 2-manifold. Show that if M and N are connected sums of different numbers of tori, then they are not homotopy equivalent, so there are no repetitions in the list of orientable 2-manifolds.

Problem 25.80. Determine the cohomology ring $H^*(M;\mathbb{Z}/2)$, where M is a nonorientable compact 2-manifold. Show that if M and N are connected sums of different numbers of tori, then they are not homotopy equivalent, so there are no repetitions in the list of nonorientable 2-manifolds.

Problem 25.81. The Klein bottle K (defined in Problem 19.37) is one of the spaces on the list; which one?

25.9. Euler Characteristic

The **Euler characteristic** of X of a finite CW complex X is the integer

$$\chi(X) = \sum_{k=0}^{\infty} (-1)^k \dim_F(H_k(X;F))$$

where F is any field you like. It is often convenient to work with the **reduced Euler characteristic** $\widetilde{\chi}(X)$ defined by replacing $H_k(X;\mathbb{F})$ with $\widetilde{H}_k(X;\mathbb{F})$.

Exercise 25.82.

(a) Show that homotopy equivalent spaces have equal Euler characteristic.

(b) Express $\widetilde{\chi}(X)$ in terms of $\chi(X)$.

(c) Show that the Euler characteristic can be defined (for finite CW complexes) equally well with cohomology.

25.9.1. Independence of the Field. It is a bit funny that we have not attempted to assert any control over the field F used to compute the Euler characteristic. Why do we not have an array of Euler characteristics, one for each field F?

[4] You may take for granted that, because of the homogeneity of a manifold, if j_1 and j_2 are obtained by the deletion of different disks, then they are (freely) homotopic to one another.

To see that the choice is immaterial, we take a brief algebraic detour and define the Euler characteristic of a graded F-vector space A_* by setting

$$\chi(A_*) = \sum_{k=0}^{\infty}(-1)^k \dim_F(A_k).$$

Problem 25.83. Show that if C_* is a chain complex over a field F, then $\chi(C_*) = \chi(H_*(C_*))$.

Problem 25.84.

(a) Show that if X is a finite CW complex, then

$$\chi(X) = \sum_{k=0}^{\infty}(-1)^k \left(\text{number of } k\text{-cells in } X\right).$$

(b) Conclude that $\chi(X)$ is independent of the choice of field F.

25.9.2. Axiomatic Characterization of Euler Characteristic. Since the Euler characteristic is defined in terms of homology, it is a homotopy invariant of finite CW complexes; but since it can be computed in terms of the cellular structure of X, it does not actually depend on all the machinery we have developed so far. In this section, we'll see that the Euler characteristic is determined by a few simple properties.

Theorem 25.85 (Watts)**.** *The reduced Euler characteristic is the unique function*

$$\gamma : \{\textit{finite CW complexes}\} \longrightarrow \mathbb{Z}$$

satisfying the properties

(a) $\gamma(S^0) = 1$ *and*

(b) if $A \to X \to B$ *is a cofiber sequence, then* $\gamma(X) = \gamma(A) + \gamma(B)$.

Problem 25.86. Suppose γ satisfies the conditions of Theorem 25.85.

(a) Show that $\gamma(X \vee Y) = \gamma(X) + \gamma(Y)$.

(b) Show that $\gamma(D^n) = 0$ for all n.

Hint. Use part (a).

(c) Show that $\gamma(S^n) = (-1)^n$.

(d) Prove Theorem 25.85.

25.9.3. Poincaré Series. The Euler characteristic throws away a lot of information by taking an alternating sum. We can retain this information by studying the **Poincaré series**

$$p_X(t) = \sum_{n=0}^{\infty} \dim_{\mathbb{Q}}(H_n(X;\mathbb{Q})) \cdot t^n$$

(this is the **generating function** for the sequence $\{H_n(X;\mathbb{Q})\}$). This definition makes sense for any space—even infinite-dimensional ones—X whose homology is finitely generated in all degrees. If X is a finite CW complex, though, the Poincaré series is just a polynomial, and it is possible to recover the Euler characteristic of X from p_X.

Exercise 25.87.

(a) Explain how to find $\chi(X)$ given $p_X(t)$.

(b) Is $p_X(t)$ independent of the field? That is, do we get the same series if we replace $\mathbb{Q}$ with some other field F?

The Poincaré series is well behaved with respect to products.

Problem 25.88. Suppose X and Y have finitely generated homology in all degrees.

(a) Show that $p_{X\times Y}(t) = p_X(t)\cdot p_Y(t)$.

(b) Show that $\chi(X\times Y) = \chi(X)\times\chi(Y)$ if X and Y are finite CW complexes.

It can be useful to have a reduced version of the Poincaré series, defined in terms of the reduced homology $\widetilde{H}_*(X;\mathbb{Q})$, available.

Problem 25.89.

(a) Determine $\widetilde{p}_{X\wedge Y}$ in terms of $\widetilde{p}_X$ and $\widetilde{p}_Y$.

(b) Show that $\widetilde{p}_{X\vee Y} = \widetilde{p}_X + \widetilde{p}_Y$.

(c) Show that $\widetilde{p}_{\Sigma X} = t\cdot\widetilde{p}_X$.

(d) Find formulas for $p_{X\wedge Y}$, $p_{X\vee Y}$ and $p_{\Sigma X}$ in terms of p_X and p_Y.

25.9.4. More Examples. We finish our discussion of the Euler characteristic and Poincaré polynomials with some computations.

Problem 25.90.

(a) Determine the Euler characteristics of S^n, $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$.

(b) What is the Euler characteristic of a Moore space?

Problem 25.91. Show that surfaces are classified by their Euler characteristic. How can you tell from the Euler characteristic whether a surface is orientable?

Problem 25.92. Express the Poincaré series for $\mathbb{R}\mathrm{P}^\infty$, $\mathbb{C}\mathrm{P}^\infty$, $\mathbb{H}\mathrm{P}^\infty$ and ΩS^{n+1} as rational functions (not infinite sums).

Problem 25.93. Express $p_{\Omega\Sigma X}$ as a rational function of p_X.

Serre asked whether the Poincaré polynomial of the loop space of any simply-connected finite complex must be a rational function. In [**13**] Anick constructed examples of spaces X such that $p_{\Omega X}$ is not a rational function.

25.10. The Künneth Theorem via Symmetric Products

In Section 20.5 we introduced the infinite symmetric product $SP^\infty(X)$ on a space X and sketched a proof that $SP^\infty(\,?\,)$ converts cofiber sequences $A \to B \to C$ to quasifibration sequences $SP^\infty(A) \to SP^\infty(B) \to SP^\infty(C)$. In this section, you'll use symmetric products to give a new proof of the Künneth formula for homology.

Amazingly, the homotopy groups of $SP^\infty(X)$ are easily understood as a functor of X.

Theorem 25.94. *For any connected CW complex X, $\pi_n(SP^\infty(X)) \cong H_n(X;\mathbb{Z})$.*

Problem 25.95. Prove Theorem 25.94.

We will argue that there are exact sequences of the form given in the statement of Theorem 24.30 (for homology), but we will not attempt to show that the first map is the Künneth map as we have defined it.

Problem 25.96.

(a) Let Y be an abelian topological monoid, and let X be another space. Show that $\mathrm{map}(X,Y)$ is also an abelian topological monoid.

(b) Explicitly describe the homotopy type of $\mathrm{map}_*(Y, K(G,n))$ in terms of the cohomology groups of Y.

(c) Explicitly describe the space $\mathrm{map}_*(X, \mathrm{map}_*(Y, K(G,n)))$.
Hint. Use a universal coefficients theorem.

(d) Prove Theorem 24.30.

Project 25.97. Is the exterior product given here the same as the one defined in Section 21.3.

25.11. The Homology Algebra of $\Omega\Sigma X$

Let's begin by establishing a universal property for tensor algebras (see Section A.1.3 for the definition).

Problem 25.98.

(a) Show that if M is an R-module and A is an R-algebra, then each R-module homomorphism $M \to A$ has a unique extension to an R-algebra map $T(M) \to A$ making the triangle

$$\begin{tikzcd} M \arrow[r] \arrow[d] & A \\ T(M) \arrow[ur, dotted] & \end{tikzcd}$$

commute.

(b) Show that the tensor algebra functor $T(\,?\,)$ is left adjoint to the forgetful functor $\mathbf{Alg}_R \to \mathbf{Mod}_R$.

The universal property of the tensor algebra functor $T(\,?\,)$ produces a unique algebra homomorphism j making the diagram

$$\begin{tikzcd} \widetilde{H}_*(X) \arrow[r, "\sigma_*"] \arrow[d] & H_*(\Omega\Sigma X) \\ T(\widetilde{H}_*(X)) \arrow[ur, dotted, "\exists!\, j"'] & \end{tikzcd}$$

commutative, where $\sigma : X \to \Omega\Sigma X$ is the adjoint of $\mathrm{id}_{\Sigma X}$. The **Bott-Samelson theorem** shows that very often j is an isomorphism.

Theorem 25.99 (Bott-Samelson)**.** *Let R be a PID, and assume $H_*(X;R)$ is a free graded R-module. Then the map*

$$j : T(\widetilde{H}_*(X;R)) \longrightarrow H_*(\Omega\Sigma X;R)$$

is an isomorphism of R-algebras.

Problem 25.100.

(a) Show that for each n there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T_{n-1}(\widetilde{H}_*(X)) \arrow[r] \arrow[d, "j_{n-1}"] & T_n(\widetilde{H}_*(X)) \arrow[r] \arrow[d, "j_n"] & \widetilde{H}^*(X)^{\otimes n} \arrow[r] \arrow[d, "\kappa"] & 0 \\ 0 \arrow[r] & H_*(J^{n-1}(X)) \arrow[r] & H_*(J^n(X)) \arrow[r] & \widetilde{H}_*(X^{\wedge n}) \arrow[r] & 0, \end{tikzcd}$$

where κ is the (iterated) Künneth map.

(b) Prove Theorem 25.99.

Problem 25.101. Determine the structure of the algebra $H_*(\Omega S^n;\mathbb{Z})$.

25.12. The Adjoint λ_X of $\mathrm{id}_{\Omega X}$

The adjoint of the identity map $\mathrm{id}_{\Omega X} : \Omega X \to \Omega X$ is a map $\lambda_X : \Sigma\Omega X \to X$. In this section we'll study the homology (and cohomology) of this map.

The Horribly Named Homology Suspension. The suspension operation on homotopy groups is the composition

$$\pi_k(X) \xrightarrow{\sigma_*} \pi_{k+1}(\Omega\Sigma X) \xrightarrow{\Omega^{-1}} \pi_{k+1}(\Sigma X),$$

where $\sigma : X \to \Omega\Sigma X$ is adjoint to the identity on ΣX and Ω is the standard isomorphism $\pi_{k+1}(X) \xrightarrow{\cong} \pi_k(\Omega X)$. The composition literally takes a homotopy class $\alpha : S^k \to X$ and returns its suspension. Strictly dualizing leads to a composition

$$H^{k-1}(\Omega X) \xleftarrow{\Sigma^{-1}} H^k(\Sigma\Omega X) \xleftarrow{\lambda^*} H^k(X)$$

which deserves to be called the **loop operation**. We'll denote it by Ω.

Problem 25.102. Show that the map Ω is precisely the function that takes a homotopy class $u : X \to K(G,n)$ and returns Ωu.

Because homology is an unnatural monster, we get a confusing situation when we apply it to the map λ_X. The map we get is

$$H_k(\Omega X) \xrightarrow{\Sigma} H_{k+1}(\Sigma\Omega X) \xrightarrow{\lambda_*} H_{k+1}(X)$$

which is 'dual' to the other one. But the duality is *algebraic*, not categorical; nevertheless, some misguided souls[5] decided to call this composite the **homology suspension**. We will not call it this, but you should know what is meant by homology suspension. Even though some authors use σ_* for this operation, we use Ω^*, indicating that (for field coefficients, anyway) it is the vector space dual of the operation Ω in cohomology.

Induced Maps. The real work of this section is to determine the effect of the properties of the map $\Omega^* : H_k(\Omega X) \to H_{k+1}(X)$. In Section 18.4 you identified λ_X with the Ganea construction on the path-loop fibration $@_0 : \mathcal{P}(X) \to X$ and discovered that its homotopy fiber has the homotopy type of the join $\Omega X * \Omega X$. You also showed that the fiber inclusion $j : \Omega X * \Omega X \to \Sigma\Omega X$ fits into a commutative square

$$\begin{tikzcd}
\Omega X * \Omega X \arrow[r, "q"] \arrow[d, "j"'] & \Sigma(\Omega X \times \Omega X) \arrow[d, "\Sigma m"] \\
\Sigma\Omega X \arrow[r, "\mathrm{id}"] & \Sigma\Omega X
\end{tikzcd}$$

where $m : \Omega X \times \Omega X \to \Omega X$ is given by $m : (\omega, \tau) \mapsto \tau^{-1}\omega$.

Problem 25.103.

(a) Show that the image of q is $\Sigma(\widetilde{H}_*(\Omega X) \otimes \widetilde{H}^*(\Omega X))$.

(b) Show that the image of Ω^* is precisely suspension of the module of decomposables in the Pontrjagin algebra $H_*(\Omega X)$.

(c) Conclude that the map Ω^* kills decomposables.

[5]Not to mention excellent mathematicians!

25.13. Some Algebraic Topology of Fibrations

Theorem 25.99 suggests an intuition for the homology of the loop space. The homology of $\Omega\Sigma X$ is a free graded algebra having multiplicative generators in bijective correspondence with a basis for the free graded R-module $\widetilde{H}_*(X;R) = \Sigma^{-1}\widetilde{H}^*(\Sigma X;R)$. Unfortunately, simple examples show that this is too much to ask for when it comes to the loop spaces of nonsuspensions.

Exercise 25.104. Find an example where $H_*(\Omega X;R)$ is not a free graded algebra.

In 1955 J. F. Adams and P. J. Hilton [**8**] found that a construction of this kind—but applied at the pre-homology chain complex level—does give a comparatively simple model for the chain algebra of ΩX when X is a simply-connected CW complex with $X_1 = *$.

Define the graded tensor algebra

$$\mathrm{AH}_*(X;R) = T(\Sigma^{-1}\widetilde{C}_*(X;R)),$$

where $C_*(X;R)$ is the cellular chain complex of X, giving a functor from the category of CW complexes and cellular maps to the category of graded R-algebras. We hope to construct a differential d on $\mathrm{AH}_*(X;R)$ and a chain equivalence $\mathrm{AH}_*(X;R) \to \mathcal{S}_*(\Omega X;R)$.

For $X = *$, there is nothing to do. Suppose now that $Y = X \cup_\alpha D^{n+1}$ and we have constructed a chain equivalence

$$\theta_X : \mathrm{AH}_*(X;R) \longrightarrow C_*(\Omega X;R).$$

By construction, $\mathrm{AH}_*(Y;R)$ is obtained from $\mathrm{AH}_*(X;R)$ by adding a single n-dimensional generator $[D^{n+1}]$ corresponding to the attached cell; and because of the product rule, we will have defined the differential in $\mathrm{AH}_*(Y;R)$ once we specify $d([D^{n+1}])$.

Let $\beta : S^{n-1} \to \Omega X$ be the adjoint of α, and let $s_{n-1} \in H_{n-1}(S^{n-1})$ be the canonical generator.

Problem 25.105. Show $\beta_*([s_{n-1}]) = [\theta_X(z)]$ for some $z \in H_*(\mathrm{AH}_*(X;R))$.

Now we define $d([D^{n+1}]) = z$ and extend the differential to the whole algebra $\mathrm{AH}_*(Y;R)$ using the product rule.

Problem 25.106. Check that $d^2 = 0$ in $\mathrm{AH}_*(Y;R)$.

The last step in the inductive construction is to define the comparison map $\theta_Y : \mathrm{AH}_*(Y;R) \to \mathcal{S}_*(\Omega Y;R)$. But this is a bit involved and nitpicky, so we'll simply assert that it can be done and leave the verification to the interested reader.

Theorem 25.107 (Adams-Hilton). *It is possible to extend the chain equivalence θ_X to another one $\theta_Y : \mathrm{AH}_*(Y; R) \to \mathcal{S}_*(\Omega Y; R)$.*

It is possible to lift the Adams-Hilton theorem out of the realm of pure algebra and give it true geometric meaning.

Project 25.108. Prove Theorem 25.107 by showing that the Adams-Hilton model for ΩX is precisely the cellular chain algebra of the RPT model of ΩX.

Perhaps inspired by the Adams-Hilton model, several authors sought out computable models for the chain complexes for the spaces in a fibration sequence. Results of this kind can be derived from cellular models of fibration sequences [**118**].

25.14. A Glimpse of Spectra

We have shown that every cohomology theory satisfying the Wedge Axiom is represented (at least on the category of CW complexes) by a sequence of spaces $E(n)$ together with homotopy equivalences $E(n) \xrightarrow{\simeq} \Omega E(n+1)$. In our construction of ordinary homology, we used the representing spaces for ordinary cohomology (the Eilenberg-Mac Lane spaces), but the maps we used were the adjoints $\Sigma K(G, n) \to K(G, n+1)$, which are not homotopy equivalences. In the late 1950s Lima (and G. W. Whitehead) made the big conceptual leap: the homotopy equivalence is utterly immaterial to the construction!

If we omit the condition of homotopy equivalence, we are led to the concept of a *spectrum*. A **spectrum**[6] is a collection $\mathbf{E} = \{E(n), \epsilon_n\}$, where $\epsilon_n : \Sigma E(n) \to E(n+1)$ is an arbitrary map. Given a spectrum $\mathbf{E}$, we can define, for each space X, maps

$$\begin{tikzcd}
{[\Sigma^t X, E(n+t)]} \arrow[rr] \arrow[dr, "\Sigma"'] & & {[\Sigma^{t+1} X, E(n+t+1)]} \\
& {[\Sigma^{t+1} X, \Sigma E(n+t)]} \arrow[ur, "(\epsilon_{n+t})_*"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
{[S^{n+t}, E(t) \wedge X]} \arrow[rr] \arrow[dr, "\Sigma"'] & & {[\Sigma^{n+t+1} X, E(t+1)]} \\
& {[S^{n+t+1} X, \Sigma E(t) \wedge X].} \arrow[ur, "(\epsilon_t \wedge \mathrm{id}_X)_*"'] &
\end{tikzcd}$$

[6]These are also called **prespectra** in deference to more highly structured objects that have assumed the title of **spectra**.

Thus we obtain two telescope diagrams in the category $\mathrm{AB}\,\mathcal{G}$, and we use the notation

$$\begin{aligned}\widetilde{\mathbf{E}}^n(X) &= \operatorname{colim}\big(\cdots \to [\Sigma^t X, E(n+t)] \to [\Sigma^{t+1}X, \Sigma E(n+t)] \to \cdots\big),\\ \widetilde{\mathbf{E}}_n(X) &= \operatorname{colim}\big(\cdots \to [S^{n+t}, E(t)\wedge X] \to [S^{n+t+1}X, \Sigma E(t)\wedge X] \to \cdots\big)\end{aligned}$$

for their colimits. The collections $\widetilde{\mathbf{E}}_* = \{\widetilde{\mathbf{E}}_n\}$ and $\widetilde{\mathbf{E}}^* = \{\widetilde{\mathbf{E}}^n\}$ are functors $\mathcal{T}_* \to \mathrm{AB}\,\mathcal{G}^*$ from the category of pointed topological spaces to the category of graded abelian groups.

Theorem 25.109. *The functors $\widetilde{\mathbf{E}}_n$ constitute a homology theory, and similarly $\widetilde{\mathbf{E}}^*$ is a cohomology theory.*

Problem 25.110. Prove Theorem 25.109.

There is a version of the Brown Representability Theorem for homology which says that every homology theory can be constructed from a spectrum as above. Thus we have correspondences

$$\{\text{cohomology theories}\} \longleftrightarrow \{\text{spectra}\} \longleftrightarrow \{\text{homology theories}\}.$$

Exercise 25.111. Are homology theories in bijective correspondence with cohomology theories?

25.15. A Variety of Topics

25.15.1. Contractible Smash Products. Let's investigate the algebraic relations between the cohomology of X and Y if $X \wedge Y \simeq *$.

Problem 25.112.

(a) Show that if X is not path-connected, then $X \wedge Y \simeq *$ if and only if $Y \simeq *$.

(b) Show that if X and Y are connected CW complexes, then $X \wedge Y$ is simply-connected.

(c) Suppose that for every prime $p \geq 0$, at least one of $H^*(X;\mathbb{Z}/p)$ and $H^*(Y;\mathbb{Z}/p)$ is trivial, and show that $X \wedge Y \simeq *$.

(d) On the other hand, show that if there is even one prime p for which $H^*(X;\mathbb{Z}/p)$ and $H^*(Y;\mathbb{Z}/p)$ are both nontrivial, then $X \wedge Y \not\simeq *$.

Problem 25.113.

(a) Find spaces X and Y for which the inclusion $X \vee Y \hookrightarrow X \times Y$ is a homotopy equivalence.

(b) Find a sequence $A \to B \to C$ which is both a fibration sequence and a cofibration sequence.

Problem 25.114. Characterize those cofiber sequences $A \to B \to C$ that are also fiber sequences; and characterize the fiber sequences $F \to E \to B$ that are also cofiber sequences. Must such a sequence be trivial?

Problem 25.115. Let X be a noncontractible CW complex such that $\Sigma X \simeq *$. Suppose $\mathrm{map}_*(X, Y) \not\simeq *$. Show that Y cannot be expressed as $\mathrm{map}_*(A, B)$ for any path-connected CW complex A and any space B.

25.15.2. Phantom Maps. In Section 9.5 we introduced the notion of **phantom maps**, which are maps $f : X \to Y$ in which X is a CW complex and $f|_{X_n} \simeq *$ for all n. There you found that there is a universal example for phantom maps out of a given space X, and it has the form

$$\Theta_X : X \longrightarrow \bigvee_n \Sigma X_n.$$

Now we have some powerful functors that we can bring to bear on this question and prove the existence of nontrivial phantom maps.

Phantom Maps and Postnikov Sections. Any space Y is homotopy equivalent to the limit of the tower $\cdots \to P_n(Y) \to P_{n-1}(Y) \to \cdots$ of its Postnikov approximations. Since these spaces have only finitely many nontrivial homotopy groups, the maps into them from CW complexes only depend on their restrictions to finite-dimensional skeleta.

Problem 25.116. Let $f : X \to Y$. Show that f is a phantom map if and only if each composite $X \xrightarrow{f} Y \to P_n(Y)$ is trivial.

Thus we can apply the theory we developed in Section 9.5.3 to study phantom maps.

Problem 25.117. Show that $\mathrm{Ph}(X, Y) \cong \lim^1[X, \Omega P_n(Y)]$.

Now we can prove the existence or nonexistence of some phantom maps.

Problem 25.118. Let $\mathbb{F}$ be $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and let $n \geq 0$.

(a) Show that $\Sigma^n \mathbb{F}\mathrm{P}^\infty$ does not have a nontrivial finite-dimensional CW complex as a homotopy retract.
(b) Show that there are nontrivial phantom maps $\mathbb{F}\mathrm{P}^\infty \to Y$ for some Y.

Problem 25.119. Show that there are nontrivial phantom maps $B\mathbb{Z}/p \to Y$ for well-chosen spaces Y.

We complement this result by giving a homological criterion guaranteeing the vanishing of all phantom maps.

Proposition 25.120.

(a) If $H^*(X;\mathbb{Z})$ is finite in each dimension, then $\mathrm{Ph}(X,Y) = *$ *for any finite-type CW complex Y.*

(b) If $\pi_*(Y)$ is finite in each dimension, then $\mathrm{Ph}(X,Y) = *$ *for any finite type X.*

Problem 25.121. Prove Proposition 25.120.

Problem 25.122. Show that there are no phantom maps from $B\mathbb{Z}/p$ to any simply-connected CW complex of finite type.

25.15.3. The Serre Exact Sequence. The Serre exact sequence is the homological reflection of the Blakers-Massey exact sequence which results from the symmetrical fact that, in low dimensions, there is no distinction between cofiber and fiber sequences.

Problem 25.123. Let $F \to E \to B$ be a fibration sequence where F is $(f-1)$-connected and B is $(b-1)$-connected. Show that there is an exact sequence

$$\widetilde{H}^0(B) \longrightarrow \widetilde{H}^0(E) \longrightarrow \widetilde{H}^0(F) \longrightarrow \widetilde{H}^1(F) \to \cdots$$

$$\cdots \to \widetilde{H}^{a+c-3}(F) \longrightarrow \widetilde{H}^{a+c-2}(B) \longrightarrow \widetilde{H}^{a+c-2}(E) \longrightarrow \widetilde{H}^{a+c-2}(F).$$

Also formulate and prove a Serre exact sequence for homology groups.

25.15.4. The G. W. Whitehead Exact Sequences. The results of Section 18.4 can be used to insert the looping of cohomology classes into an exact sequence.

Problem 25.124.

(a) Using Problem 18.26, show that if X is $(n-1)$-connected, then there is an exact sequence of the form

$$\widetilde{H}^1(X \wedge X) \xrightarrow{\bar{\Delta}^*} \widetilde{H}^1(X) \xrightarrow{\lambda^*} \widetilde{H}^1(\Sigma\Omega X) \longrightarrow \widetilde{H}^2(X \wedge X) \to \cdots$$

$$\cdots \to \widetilde{H}^{3n-2}(X \wedge X) \xrightarrow{\bar{\Delta}^*} \widetilde{H}^{3n-2}(X) \xrightarrow{\lambda^*} \widetilde{H}^{3n-2}(\Sigma\Omega X).$$

(b) Formulate the corresponding result for homology.

Problem 25.125. Let F be a field. Show that if $k \le 3n-4$ and $u \in H^k(X;F)$ with $\lambda^*(u) = 0$, then u must be decomposable in the graded F-algebra $H^*(X;F)$.

Whitehead showed in [**184**] that his exact sequence gives enough information to construct the Steenrod squares.

Problem 25.126. Use Whitehead's exact sequence to show that the squaring operation SQUARE : $\widetilde{H}^n(X;\mathbb{Z}/2) \to \widetilde{H}^{2n}(X;\mathbb{Z}/2)$ is part of a stable cohomology operation. Can you prove it is unique?

HINT. Use the method of the universal example.

Unfortunately, there is no obvious adaptation of this approach that deloops the p-power operations, because when $p > 2$, the cohomology classes involved are outside the range in which the sequence is exact.

25.15.5. Hopf Algebra Structure on the Steenrod Algebra. The Cartan formula can be interpreted as giving the Steenrod algebra $\mathcal{A}_p$ the structure of a Hopf algebra.

Problem 25.127.

(a) Show that the rule $\delta : \mathrm{Sq}^n \mapsto \sum_{i+j=n} \mathrm{Sq}^i \otimes \mathrm{Sq}^j$ extends to a unique algebra map $\delta : \mathcal{A}_2 \to \mathcal{A}_2 \otimes \mathcal{A}_2$.

(b) Show that, for p an odd prime, the rule $\delta : P^n \mapsto \sum_{i+j=n} P^i \otimes P^j$ and $\delta : \beta \mapsto 1\otimes\beta + \beta\otimes 1$ extends to a unique algebra map $\delta : \mathcal{A}_p \to \mathcal{A}_p \otimes \mathcal{A}_p$.

(c) Show that these maps give $\mathcal{A}_p$ the structure of a Hopf algebra.

If M and N are $\mathcal{A}_p$-modules, then $M\otimes N$ is naturally an $\mathcal{A}_p\otimes\mathcal{A}_p$ module, with the action given by

$$(\theta \otimes \phi) \cdot (m \otimes n) = (-1)^{|\phi|\cdot|m|}\theta(m) \otimes \phi(n).$$

The diagonal δ gives a way for $\mathcal{A}_p$ to act on $M \otimes N$, namely

$$\theta \cdot (m \otimes n) = \delta(\theta) \cdot (m \otimes n),$$

where the right-hand side is the already-defined action of $\mathcal{A}_p\otimes\mathcal{A}_p$ on $M\otimes N$.

Problem 25.128. Show that the following are equivalent:

(1) the Cartan formula,

(2) the cup product $H^*(X)\otimes H^*(X) \to H^*(X)$ is a map of $\mathcal{A}_p$-modules.

The Steenrod algebra is highly noncommutative, but its coproduct is commutative. This suggests that the dual Hopf algebra $\mathcal{A}_p^*$ might be easier to work with. The structure of the dual algebra was worked out by Milnor in [**131**].

Theorem 25.129 (Milnor)**.**

(a) The Hopf algebra $\mathcal{A}_2^*$ is a polynomial algebra on generators ξ_n with $|\xi_n| = 2^n - 1$ and the diagonal is given by

$$\xi_n \longmapsto \sum_i \xi_{n-i}^{2^i} \otimes \xi_i.$$

(b) For odd primes p, the Hopf algebra $\mathcal{A}_p^*$ is the tensor product of an exterior algebra on generators τ_i with $|\tau_n| = 2p^n - 1$ and a polynomial algebra on generators ξ_n with $|\xi_n| = 2(p^n - 1)$ and the diagonal is given by

$$\xi_n \longmapsto \sum_i \xi_{n-i}^{p^i} \otimes \xi_i \qquad \textit{and} \qquad \tau_n \mapsto \tau_n \otimes 1 + \sum_i \xi_{n-i}^{p^i} \otimes \tau_i.$$

We will not use this result in this book, so we leave it to the interested reader to work through the proof.

25.16. Additional Problems and Projects

We end this long chapter with a motley assortment of conceptual and computational challenges.

Problem 25.130. Show that if the groups $H^k(X;\mathbb{Z})$ are finitely generated, then so are the groups $H^k(\Omega X;\mathbb{Z})$. What can you say if $H^k(X;\mathbb{Z})$ is only known to be finitely generated for $k \leq n$?

Problem 25.131. Show that if the homotopy groups $\pi_*(X)$ are finitely generated, then $\mathrm{conn}_{\mathcal{P}}(X) \geq n$ if and only if $\widetilde{H}^k(X;\mathbb{Z}/p) = 0$ for all $k \leq n$ and all $p \in \mathcal{P}$.

Project 25.132. Investigate the relationship between the Universal Coefficients Theorem and the Bockstein operation.

Problem 25.133. Show that if X is simply-connected and $\widetilde{H}_n(X;G) = 0$ for all G, then X has a CW replacement $\overline{X} \to X$ with no n-cells; that is, $\overline{X}_n = \overline{X}_{n-1}$.

Problem 25.134. Suppose G and H are finite abelian groups and $|G|$ and $|H|$ are relatively prime. Show that

$$M(G \oplus H, n) \simeq M(G,n) \times M(H,n) \quad \text{and} \quad K(G \oplus H, n) \simeq K(G,n) \vee K(H,n).$$

Problem 25.135. Let P be a finite abelian $\mathcal{P}$-group.

(a) Show that $\pi_k(M(P,n))$ is a finite $\mathcal{P}$-group for all $k \geq n > 1$.

(b) Show that $\widetilde{H}_k(K(P,n);\mathbb{Z})$ is a finite $\mathcal{P}$-group for all $k \geq n \geq 1$.

Problem 25.136.

(a) Let $\nu : S^7 \to S^4$ be the attaching map of the 8-cell in $\mathbb{HP}^\infty$. Show that $\Sigma^n \nu \not\simeq *$ for all n and that $\nu^2 \not\simeq *$.

(b) Prove similar results for the map $\sigma : S^{15} \to S^8$ that derives from the Cayley numbers.

(c) What do your results tell you about the homotopy groups of spheres?

(d) Investigate compositions like $\eta \circ \nu$, etc.

Problem 25.137.

(a) Let $u \in \widetilde{H}^*(K(\mathbb{Z}/p, n); \mathbb{Z}/p)$. Show that for every $N > 0$, there is a cohomology operation θ with degree $|\theta| \geq N$ such that $\theta(u) \neq 0$.

(b) Show that if X is a finite complex and $f : K(\mathbb{Z}/p, n) \to X$, then the induced map $f^* : \widetilde{H}^*(X; \mathbb{Z}/p) \to \widetilde{H}^*(K(\mathbb{Z}/p, n); \mathbb{Z}/p)$ is trivial.

This result has a vast generalization, which is the subject of Chapter 37.

Project 25.138. Generalize Problems 21.42, 21.43 and 21.44 to general Moore spaces $M(G, n)$ and general cohomology theories $\widetilde{h}^*$.

Project 25.139. Determine the cohomology algebra $H^*(\Omega S^{2n}; \mathbb{Z})$.

Problem 25.140. Write down a Cartan formula for the operation βP^k.

Project 25.141. Define $\mathcal{E}(X) = \{\mathrm{conn}(f) \mid f : X \to X\}$. What can you say about the sets $\mathcal{E}(X)$? Consider the implications of the connectivity and dimension of X. What if X is infinite-dimensional? Can you find a space X for which $\mathcal{E}(X) = \mathbb{N}$? Can $\mathcal{E}(X) = 2\mathbb{N}$?

Problem 25.142. Let $n \geq 3$.

(a) Show that there are $(n+2)$-equivalences $\Sigma^{n-2}\mathbb{C}\mathrm{P}^2 \to K(\mathbb{Z}, n)$.

(b) Show that there are $(n+3)$-equivalences

$$M(\mathbb{Z}/2, n+2) \vee S^{n+2} \longrightarrow K(\mathbb{Z}, n)/S^n.$$

(c) What can you say about the homology and cohomology of $K(G, n)$ in low dimensions?

Problem 25.143. Determine the cohomology ring of $\mathbb{F}\mathrm{P}^n/\mathbb{F}\mathrm{P}^{k-1}$. What is the Lusternik-Schnirelmann category of $\mathbb{F}\mathrm{P}^n/\mathbb{F}\mathrm{P}^{k-1}$?

Problem 25.144. If $\widetilde{h}^*$ is a multiplicative cohomology theory, show that $\widetilde{h}^*(X)$ is an $h^*(S^0)$-module. Suppose $\widetilde{h}^*(Y)$ is a free $\widetilde{h}^*(S^0)$-module; show that

$$\widetilde{h}^*(X \wedge Y) \cong \widetilde{h}^*(X) \otimes_{\widetilde{h}^*(S^0)} \widetilde{h}^*(Y).$$

Problem 25.145. Let $\widetilde{h}^* = \{\widetilde{h}^n\}$ be a collection of functors $\widetilde{h}^n : \mathrm{H}\mathcal{T}_* \to$ **Sets**$_*$, equipped with natural isomorphisms $\widetilde{h}^n \to \widetilde{h}^{n+1} \circ \Sigma$. Show that each $\widetilde{h}^n$ factors through $\mathrm{AB}\,\mathcal{G}$ so that $\widetilde{h}^*$ is a cohomology theory in the standard definition.

Project 25.146. Generalize Problem 21.40 to maps between wedges of spheres that are not all in the same dimension. Can you similarly generalize Theorem 19.35?

Problem 25.147. Show that every element $u \in H^*(X; \mathbb{Z}/n)$ has the property that $nu = 0$.

Problem 25.148. Let $\mathcal{P} \sqcup \mathcal{Q}$ be a partition of the prime numbers and let X be a simply-connected CW complex with $\mathrm{conn}_{\mathcal{Q}}(X) = \infty$.

(a) Show that for each $q \in \mathcal{Q}$, the map $q \cdot \mathrm{id}_{\Sigma^2 X} : \Sigma^2 X \to \Sigma^2 X$ induces an isomorphism on $H_*(X; \mathbb{Z})$. Conclude that $q \cdot \mathrm{id}_{\Sigma^2 X}$ is a homotopy equivalence.

(b) Show that the abelian group $[\Sigma^2 X, Y]$ is a $\mathcal{P}$-group for every space Y.

Problem 25.149. Assume that $\dim(X) \leq n$. Show that there is an isomorphism $[X, S^n] \xrightarrow{\cong} \widetilde{H}^n(X, \mathbb{Z})$. This is known as **Hopf's theorem**.

Problem 25.150. Show that if X is $(n-1)$-connected, the Hurewicz map $h : \pi_{n+1}(X) \to H_{n+1}(X)$ is surjective. What can you say about the kernel?

Problem 25.151. Show that if M is an R-module, then $H^*(X; M)$ is an $H^*(X; R)$-module.

Project 25.152. The conditions of Theorem 21.96 make sense for homotopy functors $F : \mathcal{T}_* \to \mathbf{Sets}_*$. Prove the Brown Representability Theorem for set-valued functors.

Project 25.153. The additivity of the Hopf invariant[7] can be generalized quite substantially. Let $u \in H^p(X; \mathbb{Z})$ and $v \in H^q(X; \mathbb{Z})$ with $p, q > 1$ and $u \cdot v = 0$. For each $\alpha \in \pi_{p+q-1}(X)$, we may form the space $Y = X \cup_\alpha D^{p+q}$; there will be unique classes $\bar{u}, \bar{v} \in H^*(Y; \mathbb{Z})$ whose restrictions to X are u and v, and the map $Y \to Y/X \hookrightarrow K(\mathbb{Z}, p+q)$ defines a class $w \in H^{p+q}(Y)$. Show that $\bar{u} \cdot \bar{v}$ is a multiple of w. Define a function k on $\pi_{p+q-1}(X)$ by the formula $\bar{u} \cdot \bar{v} = k(\alpha) \cdot w$. Under what conditions on X is k well-defined? What is the target of k? Assuming it is well-defined, under what conditions is it a homomorphism?

Problem 25.154. A map $f : \Sigma\mathbb{C}\mathrm{P}^2 \to \Sigma\mathbb{C}\mathrm{P}^2$ induces maps

$$H^3(\Sigma\mathbb{C}\mathrm{P}^2; \mathbb{Z}) \longrightarrow H^3(\Sigma\mathbb{C}\mathrm{P}^2; \mathbb{Z}) \qquad \text{and} \qquad H^5(\Sigma\mathbb{C}\mathrm{P}^2; \mathbb{Z}) \longrightarrow H^5(\Sigma\mathbb{C}\mathrm{P}^4; \mathbb{Z})$$

which are multiplication by integers d_3 and d_5, respectively; we call the ordered pair (d_3, d_5) the **bidegree** of f.

(a) Show that every $f : \Sigma\mathbb{C}\mathrm{P}^2 \to \Sigma\mathbb{C}\mathrm{P}^2$ fits into a ladder of cofiber sequences

$$\begin{tikzcd} S^3 \arrow[r] \arrow[d, "d_3"'] & \Sigma\mathbb{C}\mathrm{P}^2 \arrow[r] \arrow[d, "f"] & S^5 \arrow[r] \arrow[d, "d_5"] & \cdots \\ S^3 \arrow[r] & \Sigma\mathbb{C}\mathrm{P}^2 \arrow[r] & S^5 \arrow[r] & \cdots . \end{tikzcd}$$

(b) Show that there is a map $f : \Sigma\mathbb{C}\mathrm{P}^2 \to \Sigma\mathbb{C}\mathrm{P}^2$ with bidegree (d_3, d_5) if and only if $d_3 \equiv d_5 \bmod 2$.

[7]Proved in Theorem 25.16.

(c) Show that if f and g have the same bidegree, then $f - g$ has bidegree $(0, 0)$.

(d) Show that if h has bidegree $(0, 0)$, then it has a factorization

$$\begin{tikzcd} \Sigma\mathbb{C}\mathrm{P}^2 \arrow[r, "f"] \arrow[d, "q"'] & \Sigma\mathbb{C}\mathrm{P}^2 \\ S^5 \arrow[r, "\phi"] & S^3 \arrow[u, "i"'] \end{tikzcd}$$

for some map $\phi : S^5 \to S^3$.

(e) Conclude that the bidegree map $[\Sigma\mathbb{C}\mathrm{P}^2, \Sigma\mathbb{C}\mathrm{P}^2] \to \mathbb{Z} \times \mathbb{Z}$ is injective.

(f) Determine all bidegrees of suspensions of maps $\mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$; conclude that suspension $[\mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^2] \to [\Sigma\mathbb{C}\mathrm{P}^2, \Sigma\mathbb{C}\mathrm{P}^2]$ is not surjective.

Project 25.155 (M. G. Barratt [**66**])**.** In Problem 18.5 you showed that if $p : E \to B$ is a fibration with fiber F, then the fiber of induced map $E/F \to B$ is homotopy equivalent to $F * \Omega B$; thus the homotopy fiber is entirely determined by the homotopy types of F and ΩB. It is natural to ask whether the dual statement is true. Let $i : A \to X$ with cofiber C; convert $X \to C$ to a fibration $E \to C$, and let Q be the cofiber of $A \to E$. Does the homotopy type of Q depend only on the homotopy types of ΣA and C? Show that if $A = S^p \times S^q$ and $Z = S^p \vee S^q \vee S^{p+q}$, then $\Sigma A \simeq \Sigma Z$, which is the cofiber of $A \to *$ and $Z \to *$. Thus if the answer were yes, then $\Omega\Sigma A/A$ and $\Omega\Sigma B/B$ would be homotopy equivalent.

Show that their cohomology algebras $H^*(Q; \mathbb{Z})$ are not isomorphic.

Project 25.156. Moore spaces in dimension 2 are exceptional because the group $\pi_3(S^2) = \mathbb{Z} \cdot \eta$ is not in the stable range. Determine the structure of $[M(\mathbb{Z}/n, 2), M(\mathbb{Z}/m, 2)]$ (see [**14**]).

Problem 25.157. Generalize the Hopf construction so that it applies to maps $\mu : X \times Y \to Z$.

Part 6

Cohomology, Homology and Fibrations

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Chapter 26

The Wang Sequence

Some of the most powerful tools of homotopy theory are techniques for calculating the cohomology groups of the spaces in a fibration sequence. These techniques use the second cube theorem together with cone decompositions of the base to obtain information about the cohomology of the total space.

Since suspensions have one-step cone decompositions, the case where the base is a suspension is particularly easy, and our aim in this chapter is to explore this case in some detail. We finish with a bit of preliminary exploration of the cohomology of fibrations over bases with longer cone decompositions.

26.1. Trivialization of Fibrations

A homotopy equivalence $f : F \times B \to E$ making the diagram

$$\begin{tikzcd} F \times B \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{pr}_2"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

strictly commutative is called a **trivialization** of the fibration p; a fibration is called a **trivial fibration** if it has a trivialization.

Exercise 26.1. Show that every fibration over a contractible base is trivial.

Since CW complexes are constructed from disks, which are contractible, fibrations over CW complexes are built by gluing together many trivial fibrations. The complexity of fibrations over CW complexes stems from the fact that the individual trivializations are generally incompatible with one another, so they cannot be pieced together to trivialize the whole fibration.

In this section we develop results that use admissible maps (defined in Section 8.2.1) to gain some control over these trivializations.

Proposition 26.2. *Let $p : E \to B$ be a fibration, and let $f : X \to B$ be homotopic to the constant map $X \to a \in B$. Then there is a strong homotopy pullback square*

$$\begin{array}{ccc} X \times F_a & \xrightarrow{\ \phi\ } & E \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{\ f\ } & B \end{array}$$

in which each map $\phi|_{x \times F_a} : F_a \to F_{f(x)}$ is admissible for p.

Problem 26.3. Let $H : * \simeq f$, and prove Proposition 26.2 by studying the diagram

$$\begin{array}{ccc} X \times F & \xrightarrow{\ i \circ \mathrm{pr}_2\ } & E \\ {\scriptstyle \mathrm{in}_0}\downarrow & & \downarrow{\scriptstyle p} \\ X \times F \times I & \xrightarrow{\ H \circ \mathrm{pr}_{1,3}\ } & B. \end{array}$$

In the situation of Proposition 26.2, let $q : P \to X$ be the pullback of p by f. Then the map ϕ in Proposition 26.2 gives a trivialization

$$\begin{array}{ccccc} X \times F_a & \xrightarrow{\ \simeq\ } & P & \longrightarrow & E \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & {\scriptstyle q}\downarrow & \text{pullback} & \downarrow{\scriptstyle p} \\ X & = & X & \longrightarrow & B \end{array}$$

which we will call an **admissible trivialization** of the pullback.

26.2. Orientable Fibrations

Our goal in this and the next several chapters is to study fibrations using cohomology theories, and we will have to relate the cohomology $h^*(F_a)$ to the (isomorphic) cohomology $h^*(F_b)$. Call a homomorphism $\theta : h^*(F_b) \to h^*(F_a)$ **admissible** for h^* if $\theta = h^*(\phi_\omega)$ for some path ω from a to b in B. Our theorems will apply to fibrations which satisfy the condition

($\star$) *there is a unique admissible homomorphism $h^*(F_b) \to h^*(F_a)$*

for any $a, b \in B$. If the fibration $p : E \to B$ satisfies this condition, then we say that p is **orientable** for the cohomology theory h^*. If p is orientable for h^*, we write θ_{ab} for the unique admissible isomorphism $h^*(F_b) \to h^*(F_a)$.

The orientability of a fibration depends heavily on the cohomology theory we are using. It is easy to find fibrations p which are orientable for $H^*(\,?\,;\mathbb{Z}/2)$, but not for $H^*(\,?\,;\mathbb{Z})$. Some fibrations are not orientable at all!

Problem 26.4. Let $h : \mathrm{H}\mathcal{T}_\circ \to \mathcal{C}$ be any homotopy functor.

(a) Define what it means for a fibration $p : E \to B$ to be h-orientable.

(b) Let $p : E \to B$ be a fibration with fiber F. Show that the following are equivalent:
  (1) p is nonorientable for some homotopy functor h,
  (2) p is nonorientable for $\langle\, ?\,, F\rangle$.

(c) Show that if B is simply-connected, then $p : E \to B$ is orientable with respect to every homotopy functor h.

Problem 26.5.

(a) Show that the double cover $S^0 \to S^1 \to \mathbb{R}\mathrm{P}^1$ is nonorientable for all nontrivial representable functors h.

(b) Completely characterize the functors h for which the double cover $S^0 \to S^1 \to \mathbb{R}\mathrm{P}^1$ is h-orientable. Can you find an example of such a functor?

(c) Show that every fibration with fiber S^n (with $n \geq 1$) is $H^*(\,?\,; \mathbb{Z}/2)$-orientable.

(d) Show that the Klein bottle K is the total space of a fibration $p : K \to S^1$ which is orientable over $H^*(\,?\,; G)$ if and only if $2G = 0$.

Problem 26.6. Show that a fibration $p : E \to B$ with a path-connected base is orientable with respect to a homotopy functor h if and only if $h(\phi) = \mathrm{id}$ for every admissible map $\phi : F_a \to F_a$.

26.3. The Wang Cofiber Sequence

Let $p : E \to \Sigma A$ be a fibration. Since the base is a suspension, it is possible to use the second cube theorem to identify the homotopy type of the cofiber E/F and thereby place $h^*(E)$ into a useful exact sequence. When the base is a sphere, this cohomology exact sequence is called the **Wang sequence**.

26.3.1. Fibrations over a Suspension. Let $p : E \to \Sigma A$ be a fibration with fiber F. Then we have a strong homotopy pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathrm{in}_0"'] & * \arrow[d] \\ CA \arrow[r, "j"] & \Sigma A \end{tikzcd}$$

and by pulling back the fibration p, we arrive at the Mather cube

$$\begin{tikzcd}
A \times F \arrow[rr] \arrow[dr, "J"] \arrow[dd] & & Q \arrow[dr, "\alpha"] \arrow[dd] & \\
& F \arrow[rr, crossing over, "i"] & & E \arrow[dd, "p"] \\
A \arrow[rr] \arrow[dr] & & CA \arrow[dr, "q"] & \\
& * \arrow[rr] \arrow[from=uu, crossing over] & & \Sigma A.
\end{tikzcd}$$

We take $i : F \to E$ to be the inclusion of the fiber over $*$ and proceed to identify the map $A \times F \to Q$.

Problem 26.7. Construct a trivialization of $Q \to CA$, and construct a strong homotopy pushout square

$$\begin{tikzcd}
A \times F \arrow[r, "\mathrm{in}\times\mathrm{id}_F"] \arrow[d, "J"'] & CA \times F \arrow[d] \\
F \arrow[r, "i"] & E.
\end{tikzcd}$$

Conclude that there is a homotopy equivalence $e : \Sigma A \wedge F_+ \to E/F$.

Problem 26.7 shows that the homotopy type of the cofiber of the inclusion $i : F \hookrightarrow E$ depends only on the homotopy types of ΣA and F, and not on the particular fibration p. This leads to the **Wang cofiber sequence** associated to the fibration sequence $F \to E \to \Sigma A$.

Theorem 26.8. *Let $F \xrightarrow{i} E \xrightarrow{p} \Sigma A$ be a fibration sequence. Then there is a cofiber sequence of the form*

$$F \xrightarrow{i} E \longrightarrow \Sigma A \rtimes F.$$

Problem 26.9. Prove Theorem 26.8.

26.3.2. The Wang Exact Sequence. When $A = S^{n-1}$, the Wang cofiber sequence only involves the spaces F and E (and their suspensions), so it is particularly easy to work with. Its long exact sequence in cohomology is called the **Wang exact sequence**. Here are the key points about the Wang exact sequence of a fibration over a sphere.

Theorem 26.10. *Let $F \xrightarrow{i} E \xrightarrow{p} S^n$ be a fibration sequence. Then there is an exact sequence*

$$\cdots \to \widetilde{H}^{k-1}(F) \xrightarrow{\delta} H^{k-n}(F) \xrightarrow{\theta} \widetilde{H}^k(E) \xrightarrow{i^*} \widetilde{H}^k(F) \to \cdots$$

in any coefficients. If $\widetilde{H}^*$ has ring coefficients, so that $\widetilde{H}^*$ is a multiplicative cohomology theory, then

(a) if $u \in H^*(F)$ and $v \in H^*(E)$, then $\theta(u \cdot i^*(v)) = \theta(u) \cdot v$, and

(b) δ is a (graded) ***derivation****: for any $u, v \in H^*(F)$,*

$$\delta(u \cdot v) = \delta(u) \cdot v + (-1)^{|v|} u \cdot \delta(v).$$

On the face of it, part (a) it just a formula which could conceivably be useful in certain computations. But the formula has an important algebraic interpretation. Using the map $i^* : H^*(E) \to H^*(F)$, we can define an $H^*(E)$-module structure on $H^*(F)$, using the rule

$$H^*(F) \otimes H^*(E) \longrightarrow H^*(F) \qquad \text{given by} \qquad u \otimes v \mapsto u \cdot i^*(v).$$

Of course, $H^*(E)$ is also a right $H^*(E)$-module in the obvious way, and Theorem 26.10(a) tells us that θ is an $H^*(E)$-module homomorphism.

Problem 26.11. Establish the exact sequence of Theorem 26.10, being very explicit about the definitions of the maps δ and θ. Also be sure to justify the use of reduced cohomology $\widetilde{H}^*$ and unreduced homology H^* in the various terms of the sequence.

The proofs of parts (a) and (b) of Theorem 26.10 are outlined in the following section.

Exercise 26.12.

(a) Explain clearly the sense in which the Wang sequence is natural.

(b) Investigate the extension of Theorem 26.10 to more general cohomology theories.

(c) Write down the Wang sequence for (generalized) homology.

26.3.3. Proof of Theorem 26.10(a). We'll make use of the 'relative cup product'

$$\widetilde{H}^*(X/A) \otimes H^*(X) \longrightarrow \widetilde{H}^*(X/A)$$

which you studied in Section 25.2. Continue to write $e : S^n \wedge F_+ \to E/F$ for the map you found in Problem 26.7.

Problem 26.13. Let $u \in H^*(F)$ and $v \in H^*(E)$, and let $w \in \widetilde{H}^*(E/F)$ be the unique class such that $e^*(w) = \Sigma^n(u)$, so that $\theta(u) = q^*(w)$. We can multiply $w \in \widetilde{H}^*(E/F)$ by $v \in \widetilde{H}^*(E)$, yielding an element $w \cdot v \in \widetilde{H}^*(E/F)$.

(a) Show that $q^*(w \cdot v) = q^*(w) \cdot v$.

(b) Show that $e^*(w \cdot v) = \Sigma^n(u \cdot i^*(v))$.

HINT. Recall that $\Sigma^n(u) = i_n \bullet u$; use associativity.

(c) Now evaluate $\theta(u \cdot i^*(v))$ and prove Theorem 26.10(a).

26.3.4. Proof of Theorem 26.10(b). We begin with a careful analysis of the cofibration $S^{n-1} \hookrightarrow D^n$. We let $s_n \in \widetilde{H}^n(S^n; R)$ denote the standard generator corresponding to the multiplicative identity element $1 \in R \cong \pi_n(K(R,n))$.

Problem 26.14. Study the cofiber sequence $S^{n-1}_+ \to D^n_+ \to S^n \xrightarrow{\partial} \Sigma(S^{n-1}_+)$.

(a) Show that $\Sigma(S^{n-1}_+) \simeq S^n \vee S^1$, and that the map $\partial : S^n \to S^n \vee S^1$ is simply the inclusion of the first summand.

(b) Show that the induced map $\partial^* : H^{k-1}(S^{n-1}) \to \widetilde{H}^k(S^n)$ is determined by $\partial^*(1) = 0$ and $\partial^*(s_{n-1}) = s_n$.

Now get the fiber involved by smashing the cofiber sequence above with F_+, yielding

$$\begin{tikzcd}
S^{n-1}_+ \wedge F_+ \arrow[r] \arrow[d, equal] & D^n_+ \wedge F_+ \arrow[r] \arrow[d, equal] & S^n \wedge F_+ \arrow[r, "\partial\wedge\mathrm{id}"] \arrow[d, equal] & \Sigma(S^{n-1}_+ \wedge F_+) \arrow[r] \arrow[d, equal] & \cdots \\
(S^{n-1} \times F)_+ \arrow[r] & (D^n \times F)_+ \arrow[r] & S^n \rtimes F \arrow[r, "\partial\wedge\mathrm{id}"] & \Sigma((S^{n-1} \times F)_+) \arrow[r] & \cdots .
\end{tikzcd}$$

Problem 26.15.

(a) Show that $(\partial \wedge \mathrm{id})^* : H^{k-1}(S^{n-1} \times F) \to \widetilde{H}^k(S^n \wedge F_+)$ is given by the formulas $(\partial \wedge \mathrm{id})^*(1 \bullet u) = 0$ and $(\partial \wedge \mathrm{id})^*(s_{n-1} \bullet v) = s_n \bullet v$.

(b) Show that the map $\mathrm{in}_2 : F \to S^{n-1} \times F$ satisfies $J \circ \mathrm{in}_2 = \mathrm{id}_F$.

(c) Conclude that $J^* : H^k(F) \to H^k(S^{n-1} \times F)$ is given by the formula
$$J^*(u) = 1 \bullet u + s_{n-1} \bullet (\text{something}).$$

Now we are prepared to prove part (b) of Theorem 26.10.

Problem 26.16. Let $s_n \in \widetilde{H}^n(S^n)$ be our standard generator.

(a) Show that there is a commutative diagram

$$\begin{tikzcd}
 & & H^{k-n}(F) & \\
\widetilde{H}^k(S^{n-1} \times F) & \widetilde{H}^k(D^n \times F) \arrow[l] & \widetilde{H}^k(S^n \rtimes F) \arrow[l] \arrow[u, "\Sigma^{-n}", "\cong"'] & \widetilde{H}^{k-1}(S^{n-1} \times F) \arrow[l, dashed, "\partial^*"'] \\
\widetilde{H}^k(F) \arrow[u, "J^*"] & \widetilde{H}^k(E) \arrow[l, "i^*"'] \arrow[u] & \widetilde{H}^k(E/F) \arrow[l] \arrow[u, "e^*", "\cong"'] & \widetilde{H}^{k-1}(F) \arrow[l] \arrow[u, "J^*"'] \arrow[uul, dotted, "\delta"]
\end{tikzcd}$$

with exact rows.

(b) Show that $J^*(u) = 1 \bullet u + s_{n-1} \bullet \delta(u)$.

Hint. Write $J^*(u) = 1 \bullet u + s_{n-1} \bullet z$ and evaluate $\delta(u)$.

(c) Finish the proof of Theorem 26.10(b) by evaluating $J^*(x) \cdot J^*(y)$ and $J^*(xy)$, making use of the fact that J^* is a ring homomorphism.

26.4. Some Algebraic Topology of Unitary Groups

Having built some machinery for computation, we now take a break from the purely theoretical and study some examples. In this section, we apply the Wang sequence to determine the ordinary cohomology rings of the unitary groups.

26.4.1. The Cohomology of the Unitary Groups. According to Theorem 15.36, the map $A \mapsto A \cdot e_n$ defines a fiber bundle $p : U(n) \to S^{2n-1}$ whose fiber is the stabilizer of e_n, which is simply $U(n-1)$. We will apply the Wang sequence to the fibration sequence

$$U(n-1) \xrightarrow{i} U(n) \xrightarrow{p} S^{2n-1}$$

and work out the cohomology of unitary groups $U(n)$ by induction.

Define $x_n = p^*(s_{2n-1}) \in H^{2n-1}(U(n); R)$, where R is any ring of coefficients you like.

Problem 26.17. Show that $U(n-1) \to U(n)$ is a $(2n-2)$-equivalence. Conclude that if $k < n$, there is a unique element in $H^{2k-1}(U(n); R)$ that restricts to $x_k \in H^{2k-1}(U(k); R)$.

We write $x_k \in H^{2k-1}(U(n); R)$ for the element guaranteed by Problem 26.17, which is a bit ambiguous. To be absolutely precise (read: pedantic) about it, we could write $x_k^{(k)} = p^*(s_{2k-1})$ and $x_k^{(n)} \in H^*(U(n); R)$ for the unique element that restricts to $x_k^{(k)}$. But this is way too much notation, and we will follow the usual practice.

Theorem 26.18. *The cohomology of $U(n)$ is the exterior algebra*

$$H^*(U(n); R) \cong \Lambda_R(x_1, x_2, \ldots, x_n)$$

and the map $i^* : H^*(U(n); R) \to H^*(U(n-1); R)$ is determined by the formulas $i^*(x_k) = x_k$ for $k < n$ and $i^*(x_n) = 0$.

Problem 26.19.

(a) Show that in the Wang exact sequence $x_n = \theta(1)$. Show that $\theta(u) = u \cdot x_n$ for $u \in H^*(U(n-1); \mathbb{Z})$.

(b) Show that δ is the zero homomorphism. Conclude that $H^*(U(n); \mathbb{Z})$ and $\Lambda_{\mathbb{Z}}(x_1, x_2, \ldots, x_n)$ are isomorphic as graded abelian groups, but not necessarily as algebras (yet).

(c) Show that $x_k^2 = 0 \in H^*(U(n); \mathbb{Z})$ for all k. Explain why there is a ring homomorphism $\phi : \Lambda_{\mathbb{Z}}(x_1, \ldots, x_n) \to H^*(U(n); \mathbb{Z})$.

(d) Show that ϕ is an isomorphism.

Hint. First show it is surjective.

(e) Use a Universal Coefficients Theorem to finish the proof of Theorem 26.18.

Problem 26.20. Show that $H_*(U(n);\mathbb{Z})$ is a free finitely generated graded abelian group.

26.4.2. The Homology Algebra of the Unitary Groups. Because $U(n)$ is a topological group and because its homology is free and finitely generated, there is a Pontrjagin multiplication giving $H_*(U(n);\mathbb{Z})$ the structure of a graded $\mathbb{Z}$-algebra.

Theorem 26.21. *The homology algebra of $U(n)$ is*

$$H_*(U(n);\mathbb{Z}) \cong \Lambda(y_1, y_2, \ldots, y_n)$$

where $y_k \in H_{2k-1}(U(n);\mathbb{Z})$ is dual[1] *to $x_k \in H^{2k-1}(U(n);\mathbb{Z})$.*

Since $H^*(U(n);\mathbb{Z})$ is a free abelian group, the map induced by the multiplication $\mu : U(n) \times U(n) \to U(n)$ gives $H^*(U(n);\mathbb{Z})$ the structure of a Hopf algebra, and $H_*(U(n);\mathbb{Z})$ is the dual Hopf algebra. It follows that if we can determine the Hopf algebra structure of $H^*(U(n);\mathbb{Z})$, then we can determine the structure of $H_*(U(n);\mathbb{Z})$ by the purely algebraic process of dualization. Since we already know the algebra structure, it remains to determine the diagonal, which is the map induced by the H-space multiplication μ.

Problem 26.22.

(a) Show that $\mu^*(x_n) = 1 \otimes x_n + (\sum u_i \otimes v_i) + x_n \otimes 1$ where $1 \leq |u_i|, |v_i| \leq 2n$ for each i.

(b) Using the diagram

$$\begin{tikzcd} U(n-1) \times U(n-1) \arrow[r, "\mu"] \arrow[d] & U(n-1) \arrow[d] \\ U(n) \times U(n) \arrow[r, "\mu"] & U(n), \end{tikzcd}$$

show that $\sum u_i \otimes v_i = 0$, so that x_n is primitive in $H^*(U(n);\mathbb{Z})$.

(c) Use Theorem A.54 to prove Theorem 26.21.

26.4.3. Cohomology of the Special Unitary Groups. The **special unitary groups** are the subgroups $SU(n) \subseteq U(n)$ consisting of all those matrices (or transformations) with determinant 1.

[1]That is, algebraically dual, via Theorem 22.37.

Problem 26.23.

(a) Show that $H^*(SU(n); R) \cong \Lambda_R(x_2, x_3, \ldots, x_n)$.

(b) Determine the homology algebra of $SU(n)$.

The computation of Problem 26.23 has a very nice topological counterpart. In Section 9.3.3 we constructed maps $\gamma : \Sigma\mathbb{C}\mathrm{P}^{n-1} \to SU(n)$ for $n \geq 1$ making the diagrams

$$\begin{tikzcd} \Sigma\mathbb{C}\mathrm{P}^{n-1} \arrow[r, "\gamma"] \arrow[d, "q"'] & SU(n) \arrow[d, "p"] \\ \Sigma S^{2n-2} \arrow[r, "\cong"] & S^{2n-1} \end{tikzcd}$$

commute. The induced maps $H_*(\Sigma\mathbb{C}\mathrm{P}^{n-1}) \to H_*(SU(n))$ can be identified with the inclusion of a generating set.

For any exterior algebra $\Lambda_R(V)$, there is a quotient map $\Lambda_R(V) \to V$, given by sending all nontrivial products to zero.

Problem 26.24.

(a) Let V be a free graded R-module of finite type and let $f : V \to \Lambda_R(V)$. Show that if the composite $V \to \Lambda_R(V) \to V$ is an isomorphism, then f extends to an isomorphism $\bar{f} : \Lambda_R(V) \to \Lambda_R(V)$.

Hint. Show that, modulo $(k+1)$-fold products, the image of $\bar{f}$ contains all k-fold products.

(b) Show that the map $\gamma_* : \widetilde{H}_*(\Sigma\mathbb{C}\mathrm{P}^{n-1}; R) \to H_*(SU(n); R)$ extends to an algebra isomorphism

$$\Lambda_R(\widetilde{H}_*(\Sigma\mathbb{C}\mathrm{P}^{n-1}; R)) \longrightarrow H_*(SU(n); R).$$

The map γ was first introduced to define cells in $SU(n)$; and the products of these basic cells gave us a cellular decomposition for $SU(n)$. Problem 26.24(b) shows that this topology is reflected in the fact that the image of γ_* generates homology algebra $H_*(SU(n); R)$. For these reasons, $\Sigma\mathbb{C}\mathrm{P}^{n-1}$ is sometimes referred to as a **generating complex** for $SU(n)$.

26.4.4. Cohomology of the Stiefel Manifolds. Recall that the unitary **Stiefel manifold** $V_k(\mathbb{C}^n)$ of k-planes in $\mathbb{C}^n$ is topologized by the orbit-stablizer bijection $U(n)/U(n-k) \to V_k(\mathbb{C}^n)$.

Problem 26.25. Let $k \leq n$.

(a) Show there are fiber sequences $V_k(\mathbb{C}^{n-1}) \to V_k(\mathbb{C}^n) \to S^{2n-1}$ for $k \leq n$.

(b) Compute the cohomology algebras $H^*(V_k(\mathbb{C}^n); R)$.

Exercise 26.26. Criticize the following argument:

> *Since the Stiefel manifolds are related to one another by the fiber sequences $V_k(\mathbb{C}^{n-1}) \to V_k(\mathbb{C}^n) \to S^{2n-1}$ which are entirely analogous to the ones used in the calculation for $U(n)$ and $SU(n)$, the homology algebras of Stiefel manifolds are also exterior algebras.*

26.5. The Serre Filtration

The Wang cofiber sequence for a fibration $p : E \to \Sigma A$ derives from the Mather cube that is constructed by pulling p back over the standard homotopy pushout diagram expressing ΣA as a supension. These cubes are particularly easy to analyze, since two of the spaces in the base square are contractible.

In this section, we relax our hypotheses about the base and study fibrations over a mapping cone $X = B \cup_f CA$. This leads to a cofiber sequence involving E, F and E_B, the total space of the pullback of p over B. Generalizing further, we start with a fibration over a space B with a cone decomposition and use pullbacks to define a filtration of the total space. When the cone decomposition is a CW decomposition of B, this is called the **Serre filtration** of E.

26.5.1. The Fundamental Cofiber Sequence. We begin by establishing the basic homotopy-theoretic situation.

Problem 26.27. Let $X = B \cup_f CA$, and let $p : E \to X$ be a fibration; write $E_B \to B$ for the pullback of p over the inclusion $B \hookrightarrow X$. Show that there is a diagram

$$\begin{tikzcd}
A \times F \arrow[r] \arrow[d, "J"'] \arrow[dr, phantom, "\text{HPO}"] & CA \times F \arrow[r] \arrow[d] & \Sigma A \wedge F_+ \arrow[d, "e", "\simeq"'] \\
E_B \arrow[r, "i"] & E \arrow[r, "q"] & E/E_B
\end{tikzcd}$$

where the rows are cofiber sequences and e is an induced map of cofibers.

The cofiber sequence of Problem 26.27 gives rise to a long exact sequence in cohomology having algebraic properties similar to those in the ordinary Wang sequence.

Proposition 26.28. *Let $p : E \to B$ where $X = B \cup_f CA$, and write $E_B \to B$ for the pullback of p over B. Then*

(a) there is a long exact sequence

$$\cdots \to \widetilde{H}^{k+1}(E_B) \xrightarrow{\delta} H^k(\Sigma A \wedge F_+) \xrightarrow{\theta} \widetilde{H}^k(E) \xrightarrow{i^*} \widetilde{H}^k(E_B) \to \cdots ,$$

and

(b) *if the coefficients are a ring R, then the map θ is a homomorphism of $H^*(E;R)$-modules.*

Thus if we understand E_B, A and F, then we have a reasonable hope of being able to determine the cohomology of E.

Problem 26.29.

(a) Prove Proposition 26.28, being sure to define the maps δ and θ carefully.

(b) Define the $H^*(E;R)$-module structures on $H^*(E;R)$ and $H^*(\Sigma A \wedge F_+;R)$ and prove Proposition 26.28(b).

We will show that δ is a derivation in the special case $A = \bigvee_{\mathcal{I}} S^{n-1}$, which is only slightly more general than the situation that gives rise to the Wang sequence. But this case has the virtue that, by iteration, it gives information about the cohomology of the spaces in any fibration whose base is a CW complex. Write w_i for the standard generator of the i^{th} copy of S^n in ΣA, so that

$$\widetilde{H}^k(\Sigma A \rtimes F) = \prod_{\mathcal{I}} w_i \bullet H^{k-n}(F).$$

Thus if $u \in \widetilde{H}^{k-1}(F)$, then we can write $\delta(u)$ in the form $\prod_{\mathcal{I}} \delta_i(u)$.

Problem 26.30.

(a) Show that $J_*(u) = 1 \bullet u + \prod_{\mathcal{I}} w_i \bullet \delta_i(u)$.

(b) Show that δ is a derivation.

Hint. Use the fact that $w_i \cdot w_j = 0$ for all i, j.

Project 26.31. Do we need this to be a wedge of spheres? How general can A be and still get δ to be a derivation?

26.5.2. Pullbacks over a Cone Decomposition of the Base. Suppose now that we have a fibration $p : E \to B$ with fiber F, and a cone decomposition

$$\begin{tikzcd}
A_0 \arrow[d] & A_1 \arrow[d] & & A_s \arrow[d] & A_{s+1} \arrow[d] & & \\
* \arrow[r] & B_{(1)} \arrow[r] & \cdots \arrow[r] & B_{(s)} \arrow[r] & B_{(s+1)} \arrow[r] & \cdots \arrow[r] & B
\end{tikzcd}$$

of B (meaning that the sequences $A_s \to B_{(s)} \to B_{(s+1)}$ are cofiber sequences and B is the homotopy colimit of the telescope diagram). Let $E_{(s)} \to B_{(s)}$ be the pullback of p by the inclusion $B_{(s)} \hookrightarrow B$, so that we have the telescope diagram

$$F = E_{(0)} \longrightarrow E_{(1)} \to \cdots \to E_{(s)} \longrightarrow E_{(s+1)} \to \cdots$$

whose categorical colimit is E; this is called the **Serre filtration** of E.

Problem 26.32.

(a) Show that the telescope diagram is cofibrant, so E is also its homotopy colimit.

(b) Show that $E_{(j+1)}/E_{(j)} \simeq \Sigma A_j \rtimes F$.

Exercise 26.33. Is the Serre filtration a cone decomposition?

The long cofiber sequences can be arranged into a large commutative diagram which is entirely analogous to the one we constructed in Section 24.1 when we developed the cellular chain complex. There is quite a lot of information in this diagram, but it can be hard to even begin to tease it out. In later chapters we will use the formalism of the *spectral sequence* to do the necessary bookkeeping.

A Simple Special Case. The next step up in complexity from suspensions is spaces X with two-step cone decompositions, so that $X = \Sigma A \cup CB$ (such spaces are sometimes called **two-cones**). Consider a fibration $p : E \to X$, where $X = \Sigma A \cup CB$. We write $E_{\Sigma A}$ for the pullback of E over ΣA, and let's use the Wang technique to analyze p. In this case our big diagram simplifies to

$$\begin{tikzcd}
F \arrow[d] & & & & & \\
E_{\Sigma A} \arrow[r] \arrow[d, dotted] & \Sigma A \rtimes F \arrow[r] & \Sigma F \arrow[d] & & & \\
E \arrow[r, dotted] & \Sigma B \rtimes F \arrow[r, dotted] & \Sigma E_{\Sigma A} \arrow[r] \arrow[d, dotted] & \Sigma^2 A \rtimes F \arrow[r] & \Sigma^2 F \arrow[d] & \\
& & \Sigma E \arrow[r, dotted] & \Sigma B \rtimes F \arrow[r, dotted] & \Sigma^2 E_{\Sigma A} \arrow[r] \arrow[d, dotted] & \cdots \\
& & & & \vdots &
\end{tikzcd}$$

in which the sequences of arrows of the same kind (solid, dotted) are cofiber sequences.

Problem 26.34. Show that the path-loop fibration

$$\Omega(\Sigma A \cup CB) \longrightarrow \mathcal{P}(\Sigma A \cup CB) \longrightarrow \Sigma A \cup CB$$

gives rise to a cofiber sequence

$$\Sigma A \rtimes \Omega X \longrightarrow \Sigma \Omega X \longrightarrow \Sigma B \rtimes \Omega X.$$

Exercise 26.35. What does Problem 26.34 tell you about the loop space of $\mathbb{C}\mathrm{P}^2$? Is it something you could have learned some other way?

Project 26.36. Use this technique to study the loop space of a 3-cone.

26.6. Additional Topics, Problems and Projects

26.6.1. Clutching. In Problem 26.7, or more generally, in Problem 26.27, you were able to put three of the maps into easily understandable standard forms, but you made no attempt to identify the map $J : A \times F \to E_B$. It is the map J that completely determines the fibration p.

The map J is the recipe by which the trivial fibration $CA \times F \to CA$ is glued onto the known fibration $E_B \to B$; this is known as a **clutching** construction, and J is sometimes called a **clutching map**

Problem 26.37. Make the identifications of Problem 26.7.

(a) Show that if $J = \mathrm{pr}_2 : A \times F \to F$, then the fibration p is the trivial fibration $\mathrm{pr}_2 : \Sigma A \times F \to \Sigma A$.

(b) Compare $J : A \times F \to F$ with the holonomy action $\Omega\Sigma A \times F \to F$.

Problem 26.38. Show that every clutching map gives rise to a fibration.

26.6.2. Orthogonal and Symplectic Groups. The same general setup using the field $\mathbb{R}$ gives the **orthogonal groups** $O(n)$ and the special orthogonal groups $SO(n)$; but they have a bit of trickiness to them, very much analogous to the trickiness we had to deal with when we studied the cohomology of $\mathbb{R}\mathrm{P}^n$. When we use quaternions, we get the **symplectic groups** $Sp(n)$, which have many formal similarities to the $U(n)$ groups. There is no 'special' analog of the symplectic groups: because the quaternions are not associative, there is no reasonable notion of determinant.

Project 26.39.

(a) Compute the cohomology algebras $H^*(V_k(\mathbb{R}^n); \mathbb{Z}/2)$.

(b) Compute the cohomology algebras $H^*(V_k(\mathbb{H}^n); R)$.

(c) Determine the homology algebras of $SO(n)$ and $Sp(n)$.

(d) Find generating complexes for $SO(n)$ and $Sp(n)$.

26.6.3. The Homotopy Groups of S^3. In this section, we'll learn more than we have any right to expect about the homotopy groups of S^3.

Theorem 26.40.

(a) For $k \neq 3$, $\pi_k(S^3)$ is a finite group.

(b) For each prime p, $\pi_k(S^3)$ has no p-torsion in dimensions $k < 2p$, and the p-torsion in $\pi_{2p}(S^3)$ is isomorphic to $\mathbb{Z}/p$.

The proof of Theorem 26.40 is founded on the computation of the homology of the 3-connected cover $S^3\langle 3\rangle$.

Problem 26.41.

(a) Show that there is a fiber sequence $\mathbb{C}\mathrm{P}^\infty \to S^3\langle 3\rangle \to S^3$.

(b) Determine the cohomology groups $\widetilde{H}^*(S^3\langle 3\rangle; \mathbb{Z})$.

Hint. δ is a derivation.

(c) Determine the homology groups $\widetilde{H}_*(S^3\langle 3\rangle; \mathbb{Z})$.

Hint. Use a Universal Coefficients Theorem.

Now that we have the basic computation at hand, we can use our knowledge of $\mathcal{P}$-connectivity to prove the theorem.

Problem 26.42. Prove Theorem 26.40.

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Chapter 27

Cohomology of Filtered Spaces

At the end of Chapter 26, we outlined a way to analyze the total space of a fibration $p : E \to B$. If B is the homotopy colimit of a cofibrant telescope diagram $B_{(0)} \to B_{(1)} \to \cdots \to B_{(s)} \to B_{(s+1)} \to \cdots \to B$, then we define $E_{(s)} \to B_{(s)}$ to be the pullback of p over $B_{(s)} \to B$, and E is the (homotopy) colimit of the cofibrant telescope

$$E_{(0)} \longrightarrow E_{(1)} \to \cdots \to E_{(s)} \longrightarrow E_{(s+1)} \to \cdots \to E.$$

Such a description is called a **filtration** of E.

If the original telescope was a cone decomposition, then we can use the Second Cube Theorem to explicitly determine the homotopy types of the quotients $E_{(s)}/E_{(s-1)}$ in the cofiber sequences $E_{(s-1)} \to E_{(s)} \to E_{(s)}/E_{(s-1)}$. Thus we are led to ask: if we are given a filtration of a space X with known quotients $X_{(s)}/X_{(s-1)}$, what can we learn about the cohomology of X? In this chapter we take the first steps in the systematic study of this question.

What kind of answer can we reasonably hope for? A topological filtration of X gives rise to an algebraic filtration of $H^*(X)$, given by $\mathcal{F}^s H^*(X) = \ker(H^*(X) \to H^*(X_{(s)}))$. We will use the information encoded in our diagram to approximate the **filtration quotients**

$$\mathrm{Gr}^s H^*(X) = \mathcal{F}^{s-1}(H^*(X))/\mathcal{F}^s(H^*(X)).$$

How does one 'approximate' a group? We begin by constructing groups $E_1^{s,n}$ with naturally defined subgroups $\mathcal{B}^{s,n} \subseteq \mathcal{Z}^{s,n} \subseteq E_1^{s,n}$ such that

$$\mathrm{Gr}^s(H^n(X)) \cong \mathcal{Z}^{s,n}/\mathcal{B}^{s,n}.$$

We say that $E_1^{s,n}$ approximates $\mathrm{Gr}^s(H^n(X))$ because information about the group $E_2^{s,n}$ gives insight into our real interest, $\mathrm{Gr}^s(H^n(X))$. For example, $|E_1^{s,n}|$ is an upper bound for $|\mathrm{Gr}^s(H^n(X))|$; and if $E_1^{s,n}$ is finitely generated, then so is $\mathrm{Gr}^s(H^n(X))$.

The main theorem of this chapter shows that there are subgroups $B_1^{s,n}$ and $Z_1^{s,n}$ that fit into a chain $0 \subseteq B_1^{s,n} \subseteq \mathcal{B}^{s,n} \subseteq \mathcal{Z}^{s,n} \subseteq Z_1^{s,n} \subseteq E_1^{s,n}$. We think of the group $E_2^{s,n} = Z_1^{s,n}/B_1^{s,n}$ as an improved approximation because it is formed using less of the extraneous portions of the group $E_1^{s,n}$.

27.1. Filtered Spaces and Filtered Groups

We'll begin by establishing some of the basic theory of filtered spaces, filtered groups and their associated graded objects.

27.1.1. Subquotients and Correspondence. If G is an abelian group with two subgroups $B, Z \subseteq G$ such that $0 \subseteq B \subseteq Z \subseteq G$, then we can form the quotient group Z/B, which is quotient-of-a-subgroup. We refer to groups formed from G in this way as **subquotients** of the group G.

The study of subquotients is greatly facilitated by the Correspondence Theorem and some related notation. Let $f : G \to H$ be a morphism of algebraic gadgets, such as (abelian) groups, or R-modules. Then each subgroup $A \subseteq G$ gives rise to a subgroup $f(A) \subseteq H$, and each $Z \subseteq H$ corresponds to its inverse image subgroup $f^{-1}(Z) \subseteq G$.

The following is a basic theorem of elementary algebra, which you may have seen in the form of a collection of numbered 'Isomorphism Theorems'.

Theorem 27.1 (Correspondence Theorem). *Let $f : G \to H$ be a group homomorphism.*

(a) The functions $A \mapsto f(A)$ and $Z \mapsto f^{-1}(Z)$ are inverse bijections between the sets

$$\{A \mid \ker(f) \subseteq A \subseteq G\} \qquad \textit{and} \qquad \{Z \mid 0 \subseteq Z \subseteq \mathrm{Im}(f)\}.$$

(b) If $A \subseteq B$ correspond to $Y \subseteq Z$, then
  (1) f induces a bijection $B/A \to Z/Y$ and
  (2) A is a normal subgroup of B if and only if Y is a normal subgroup of Z; and in this case the bijection $B/A \to Z/Y$ is group isomorphism.

We will take the Correspondence Theorem for granted, as it is basic algebra. If you have not seen this particular statement, it will be an excellent exercise for you to prove it.[1]

[1]You can start by showing that f and f^{-1} establish a Galois connection between the subgroups of G and the subgroups of H.

There is a very useful diagrammatic notation for keeping track of this correspondence. We use vertical (or diagonal) lines to represent subgroup containments and horizontal lines to represent corresponding subgroups. Thus the diagram

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[dd, dotted, no head] & H \arrow[d, dotted, no head] \\
& H \arrow[d, no head] \\
G \arrow[r, no head] \arrow[d, no head] & \mathrm{Im}(f) \arrow[d, no head] \\
B \arrow[r, no head] \arrow[d, no head] & Z \arrow[d, no head] \\
A \arrow[r, no head] \arrow[d, no head] & Y \arrow[d, no head] \\
\ker(f) \arrow[r, no head] \arrow[d, no head] & 0 \\
0 &
\end{tikzcd}$$

shows some of the relationships established by the homomorphism f; the last sentence of Theorem 27.1 asserts that the rule $x \cdot A \mapsto f(x) \cdot Y$ defines an isomorphism $B/A \xrightarrow{\cong} Z/Y$ of subquotients.

27.1.2. Filtered Spaces. A **filtration** of a space X is a sequence of nested subspace like so:

$$\cdots \subseteq X_{(s)} \subseteq X_{(s+1)} \subseteq \cdots \subseteq X.$$

Since we can use mapping cylinders to replace any map with a cofibration, it is essentially the same thing to think of a filtration as a sequence of cofibrations

$$\cdots \to X_{(s)} \longrightarrow X_{(s+1)} \to \cdots \to X.$$

This is clearly a domain-type notion. There is a dual kind of filtration which can be thought of as a sequence of fibrations

$$X \to \cdots \to X_{(s+1)} \longrightarrow X_{(s)} \to \cdots .$$

We will be focusing on the cohomology (and homology) of filtered spaces, so we will work exclusively with the first kind of filtration, which we call an **ascending filtration**. From now on, we will refer to a space with an ascending filtration simply as a **filtered space**.

Very often, filtrations are presented in the form $X_{(0)} \subseteq X_{(1)} \subseteq \cdots \subseteq X_{(s)} \subseteq \cdots \subseteq X$, with the smaller-indexed subspaces not specified. In this case we simply assign the values $X_{(-s)} = \varnothing$ or $X_{(-s)} = *$ for $s \in \mathbb{N}$, according to whether we are working in $\mathcal{T}_\circ$ or $\mathcal{T}_*$.

A map $f : X \to Y$ from one filtered space to another is called a **filtered map** if $f(X_{(s)}) \subseteq Y_{(s)}$ for each s. This guarantees that there are maps $f_{(s)} : X_{(s)} \to Y_{(s)}$ making the ladder

$$\begin{tikzcd}
X_{(0)} \arrow[r] \arrow[d, "f_{(0)}"'] & X_{(1)} \arrow[r] \arrow[d, "f_{(1)}"'] & \cdots \arrow[r] & X_{(s)} \arrow[r] \arrow[d, "f_{(s)}"'] & X_{(s+1)} \arrow[r] \arrow[d, "f_{(s+1)}"] & \cdots \arrow[r] & X \arrow[d, "f"] \\
Y_{(0)} \arrow[r] & Y_{(1)} \arrow[r] & \cdots \arrow[r] & Y_{(s)} \arrow[r] & Y_{(s+1)} \arrow[r] & \cdots \arrow[r] & Y
\end{tikzcd}$$

strictly commutative. We write $\text{FILT}(\mathcal{T})$ for the category whose objects are filtered spaces in $\mathcal{T}$ and whose morphisms are filtered maps.

When you decide on a cone decomposition of a space X, you have defined, in passing, a filtration on X. We'll call such a filtration a **cone filtration**.

Exercise 27.2.

(a) Is $\text{FILT}(\mathcal{T})$ a diagram category?

(b) A CW decomposition of a space is a special kind of cone decomposition. What is a filtered morphism from one CW complex to another?

(c) Define a filtered homotopy of filtered maps.

The product of two filtered spaces X and Y is given the filtration

$$(X \times Y)_{(s)} = \bigcup_{i+j=s} X_{(i)} \times Y_{(j)}.$$

This is called the **standard filtration** on the product of two filtered spaces. In our category $\mathcal{T}_\circ$, the standard filtration on a product $X \times Y$ of CW complexes (with the CW filtrations) is also a CW decomposition.

Problem 27.3. Consider the filtrations on spaces X and Y given by CW decompositions. Show that the diagonal map $\Delta : X \to X \times X$ is not a map of filtered spaces but that it is homotopic to a filtered map $\widetilde{\Delta} : X \to X \times X$. Is it possible to choose the homotopy $\Delta \simeq \widetilde{\Delta}$ to be a filtered homotopy?

We say that a filtration of X is **multiplicative** if the diagonal map

$$\Delta : X \longrightarrow X \times X$$

is homotopic in $\mathcal{T}$ to a filtered map, where we use the standard filtration on $X \times X$.

Exercise 27.4. Is $X \times Y$ with the canonical filtration a categorical product in the category of filtered spaces? What about in the homotopy category of filtered spaces?

Filtering Smash Products. Let X and Y be filtered spaces, and give $X \times Y$ the standard filtration. Then $X \wedge Y$ is filtered by the spaces

$$(X \wedge Y)_{(s)} = (X \times Y)_{(s)}/X_{(s)} \vee Y_{(s)}.$$

We call this the **standard filtration** of $X \wedge Y$.

It is frequently useful to work with smash products instead of cartesian products.

Problem 27.5. If we attach disjoint basepoints to the spaces in the filtration of X we obtain a new filtration $\cdots \to (X_{(s)})_+ \to (X_{(s+1)})_+ \to \cdots \to X_+$. Show that the standard filtration of $X_+ \wedge Y_+$ is the standard filtration of $(X \times Y)_+$.

27.1.3. Filtered Algebraic Gadgets. An ascending filtration of an algebraic gadget (i.e., a group or an R-module) H is defined analogously: it is a sequence of subgadgets

$$\cdots \subseteq H_s \subseteq H_{s+1} \subseteq \cdots \subseteq H.$$

A decreasing filtration has the form

$$H \supseteq H_1 \supseteq \cdots \supseteq H_s \supseteq H_{s+1} \supseteq \cdots .$$

We will need both kinds of algebraic filtrations.

Exercise 27.6. Find some examples of increasing and decreasing filtrations in pure algebra.

If A and B are filtered R-modules, then $A \otimes B$ has a standard filtration, given by

$$(A \otimes B)_n = \sum_{i+j=n} A_i \otimes B_j.$$

An ascending **filtered R-algebra** is an R-algebra A with a filtration

$$\cdots \subseteq A_{(s+1)} \subseteq A_{(s)} \subseteq A_{(s-1)} \subseteq \cdots \subseteq A,$$

in which $A_{(s)} \cdot A_{(t)} \subseteq A_{(s+t)}$ for all s and t.

Problem 27.7. Show that if A is a filtered R-algebra and $A \otimes A$ is given the standard filtration, then the multiplication $A \otimes A \to A$ is a filtered map.

Project 27.8. Formulate the corresponding definitions and results for descending filtrations on R-algebras.

27.1.4. Linking Topological and Algebraic Filtrations. We can use a filtration on a pointed[2] space X to construct a descending algebraic filtration on the cohomology groups $H^*(X)$, by setting

$$\mathcal{F}^s\left(H^n(X)\right) = \ker\left(\widetilde{H}^n(X) \to \widetilde{H}^n(X_{(s)})\right).$$

When there is no (or little) risk of confusion, we will write $\mathcal{F}^s$ rather than $\mathcal{F}^s\widetilde{H}^n(X)$, etc. Thus we have a descending filtration

$$\widetilde{H}^n(X) \supseteq \cdots \supseteq \mathcal{F}^s \supseteq \mathcal{F}^{s+1} \supseteq \cdots.$$

The homology of X inherits an ascending filtration, defined by

$$\mathcal{F}_s\widetilde{H}_n(X) = \mathrm{Im}(\widetilde{H}_n(X_{(s)}) \to \widetilde{H}_n(X)).$$

Problem 27.9. Let X and Y be filtered pointed spaces, and give $\widetilde{H}^*(X)$ and $\widetilde{H}^*(Y)$ the filtrations as described above.

(a) Show that the exterior product $\widetilde{H}^*(X) \otimes \widetilde{H}^*(Y) \to \widetilde{H}^*(X \times Y)$ is a filtered map.

HINT. Refer to Section 25.2.

(b) Show that if X has a multiplicative filtration, then the cup product map $\widetilde{H}^*(X) \otimes \widetilde{H}^*(X) \to \widetilde{H}^*(X)$ is a filtered map.

Exercise 27.10. Let X be a pointed CW complex. Explicitly describe the corresponding filtrations of $\widetilde{H}^*(X)$ and $\widetilde{H}_*(X)$.

27.1.5. The Functors Gr^* and Gr_*. It is a standard procedure in algebra to try to understand a filtered object H by studying the **filtration quotients**, which we denote as follows. For a descending filtration (such as the filtration that $H^*(X)$ inherits from a filtered space X), the s^{th} filtration quotient is given by

$$\mathrm{Gr}^s H = \mathcal{F}^{s-1}H/\mathcal{F}^s H.$$

For an ascending filtration, we define

$$\mathrm{Gr}_s H = \mathcal{F}_s H/\mathcal{F}_{s-1}H.$$

We will refer to the complete list of filtration quotients as $\mathrm{Gr}^* H$ or $\mathrm{Gr}_* H$; these are graded gadgets and are generally called the **associated graded** gadget.

Problem 27.11. Show that Gr^* and Gr_* are functors.

Problem 27.12. Let A be a filtered R-algebra, and let $[x] \in \mathrm{Gr}^s(A)$ and $[y] \in \mathrm{Gr}^t(A)$.

(a) Show that for any $x \in [x]$ and any $y \in [y]$, the product $xy \in A_{(s+t-1)}$.

[2] There is no loss in generality in focusing on pointed spaces, since we can study an unpointed space X by applying our techniques to X_+.

(b) Show that the class $[xy] \in \mathrm{Gr}^{s+t}(A)$ is well-defined.

(c) Show that the rule $[x]\otimes[y] \mapsto [xy]$ gives $\mathrm{Gr}^*(A)$ the structure of a graded R-algebra.

Exercise 27.13. Let X be a CW complex, with the CW filtration. Determine the associated graded group $\mathrm{Gr}_* \widetilde{H}_*(X;R)$ and the graded algebra $\mathrm{Gr}^* \widetilde{H}^*(X;R)$.

Topological Subquotients. If X is a filtered space, then it makes sense to study the cofibers

$$\mathrm{Gr}^s(X) = X_{(s)}/X_{(s-1)}.$$

If X has a descending filtration (in which all the maps are fibrations), then the filtration quotients are the fibers

$$\mathrm{Gr}_s(X) = \mathrm{Fiber}(X_{(s)} \to X_{(s-1)}).$$

Our goal in this chapter is to begin to understand the algebraic filtration quotients of the cohomology $H^*(X)$ in terms of the cohomology $H^*(X_{(s)}/X_{(s-1)})$ of the topological filtration quotients.

We have temporarily used the notation Gr^s and Gr_s for topological subquotients to make the analogy with the algebraic situation clear. But since it is not at all standard notation, we abandon it now. However, since these subquotients frequently appear in rather large diagrams, we will sometimes use the space-saving notation

$$X_{(s)/(t)} = X_{(s)}/X_{(t)}.$$

27.1.6. Convergence. Philosophically, a filtered group H is 'built' from the pieces $\mathrm{Gr}^s H$ for the various values of s, and we expect that understanding the pieces will give us insight into our real interest, H.

Problem 27.14. Let $0 = H_{-1} \subseteq H_0 \subseteq \cdots \subseteq H_k = H$ be a finite length filtration of an abelian group G.

(a) Show that if $\mathrm{Gr}^s H$ is a finite group for all s, then H is also a finite group.

(b) Show that if $\mathrm{Gr}^s H$ is a $\mathcal{P}$-group (for some set $\mathcal{P}$ of prime numbers) for all s, then H is also a $\mathcal{P}$-group.

(c) Show that if $\mathrm{Gr}^s H$ is a finitely generated group for all s, then H is also a finitely generated group.

Here's a slightly more subtle situation in which information about $\mathrm{Gr}^*(f)$ gives information about f.

Problem 27.15. Let $f : G \to H$ be a map of filtered groups, with induced map $\mathrm{Gr}^*(f) : \mathrm{Gr}^* G \to \mathrm{Gr}^* H$. Assume $H_{(s)} = 0$ for sufficiently small s and that $H = \bigcup H_{(s)}$. Show that if $\mathrm{Gr}^*(f)$ is surjective, then f is surjective.

Exercise 27.16. Give an example showing that the converse of Problem 27.15 need not be true. Can you find an example in which f is injective? Can you find extra hypotheses on f that allow you to prove the converse?

Problem 27.17. Let M be an R-module with a finite filtration. Show that if $\mathrm{Gr}^* M$ is free, then M is free.

In most of the algebraic examples we have seen, the filtrations have been **finite**—i.e., there are only finitely many values of s for which $H_s \neq H_{s+1}$. Furthermore, they have been **exhaustive**, meaning that $H = \bigcup_s H_s$ (when the filtration is finite, this is the same as saying $H = H_s$ for large s).

27.1.7. Indexing of Associated Graded Objects. When we apply the functor Gr^* to $H^*(X)$, the result is an array of groups $\mathrm{Gr}^s H^n(X)$ that are indexed by two integers; such groups are called **bigraded**. We refer to s as the **filtration degree** and n as the **total dimension** and we call the grading by s and n as the **natural grading**.

It sometimes simplifies things—either conceptually or notationally—to reindex a bigraded group by setting $t = n - s$, so that $\mathrm{Gr}^s H^n(X) = \mathrm{Gr}^s H^{s+t}(X)$. We'll refer to this as the **diagonal indexing**.

A map $\phi : X \to Y$ of bigraded gadgets whose effect on degrees is that $\phi : X^{s,t} \to Y^{s+a,t+b}$ for all s and t is said to have **bidegree** (a, b) and **total degree** $a + b$.

Exercise 27.18. Suppose $f : X \to Y$ has bidegree (a, b) in the (s, n)-indexing. What is the bidegree of f in the diagonal indexing?

27.2. Cohomology and Cone Filtrations

In this section we use a filtration of a pointed space X to obtain information about the bigraded object $\mathrm{Gr}^* \widetilde{H}^*(X)$. Later this whole story will be repackaged in its proper algebraic formalism. Our goal in this chapter is to develop an understanding of the topology underlying the algebra and to take a useful first step along the way.

For clarity, in this section we concentrate on filtrations that arise from cone decompositions; we will see in Section 27.3 that—because of the stability of cohomology—the same conclusions hold equally well in the general case.

27.2.1. Studying Cohomology Using Filtrations. Suppose X has a cone filtration

$$\begin{tikzcd}
 & A_s \arrow[d, "\alpha_s"'] \arrow[dd, bend left, dotted, "\partial_s"] & A_{s+1} \arrow[d, "\alpha_{s+1}"'] \arrow[dd, bend left, dotted, "\partial_{s+1}"] & A_{s+2} \arrow[d, "\alpha_{s+2}"] & & \\
\cdots \arrow[r] & X_{(s)} \arrow[r] \arrow[d, "q_s"'] & X_{(s+1)} \arrow[r] \arrow[d, "q_{s+1}"'] & X_{(s+2)} \arrow[r] \arrow[d, "q_{s+2}"] & \cdots \arrow[r] & X \\
 & \Sigma A_{s-1} & \Sigma A_s & \Sigma A_{s+1}. & &
\end{tikzcd}$$

It is most natural to work in the category $\mathcal{T}_*$ of pointed spaces, since we will want to view cohomology classes as pointed homotopy classes of maps into Eilenberg-Mac Lane spaces. If we were given such a diagram in $\mathcal{T}_\circ$, we could replace each $X_{(s)}$ with $(X_{(s)})_+$ and have a pointed filtration.

The topological filtration of X gives rise (as in Section 27.1.4) to a filtration of the cohomology $\widetilde{H}^n(X; G)$, defined by

$$\mathcal{F}^s \widetilde{H}^n(X) = \ker(\widetilde{H}^n(X; G) \to \widetilde{H}^n(X_{(s)}; G)).$$

We'll say that an element $x \in \mathcal{F}^{s-1} - \mathcal{F}^s$ has **filtration** s. If u has filtration s, then it represents a nonzero class $[u] \in \mathrm{Gr}^s \widetilde{H}^n(X) = \mathcal{F}^{s-1}/\mathcal{F}^s$.

In terms of spaces and maps, an element $u \in \widetilde{H}^n(X; G)$ is a homotopy class of maps $u : X \to K(G, n)$ and the filtration of u is the smallest s for which the composite $X_{(s)} \to X \xrightarrow{u} K_n$ is nontrivial (we'll write K_n for $K(G, n)$ in this section).

Problem 27.19. Let $u : X \to K_n$ have filtration s in $\widetilde{H}^n(X; G) = [X, K_n]$.

(a) Show that there is a pointed map $\bar{u} : \Sigma A_{s-1} \to K_n$ making the square

$$\begin{tikzcd}
X_{(s)} \arrow[r] \arrow[d, "q_s"'] & X \arrow[d, "u"] \\
\Sigma A_{s-1} \arrow[r, "\bar{u}"] & K_n
\end{tikzcd}$$

commute up to homotopy.

(b) Show that $\bar{u}$ determines the class $[u] \in \mathrm{Gr}^s \widetilde{H}^n(X)$.

This shows that every element $[u] \in \mathrm{Gr}^* \widetilde{H}^n(X)$ is determined by some element $\bar{u} \in \widetilde{H}^n(\Sigma A_{s-1})$. Be careful, though: it is not necessarily true that every element of $\widetilde{H}^n(\Sigma A_{s-1})$ determines an element of $\mathrm{Gr}^* \widetilde{H}^n(X)$.

We can give this a more algebraic formulation.

Problem 27.20.

(a) Show that the restriction $\widetilde{H}^n(X) \to \widetilde{H}^n(X_{(s)})$ induces an injective map $\operatorname{Gr}^s \widetilde{H}^n(X) \hookrightarrow \widetilde{H}^n(X_{(s)})$.

(b) Show that $\operatorname{Gr}^s \widetilde{H}^n(X)$ is a subquotient of $\widetilde{H}^n(\Sigma A_{s-1})$.

Explicitly, Problem 27.20(b) tells us that $\widetilde{H}^n(\Sigma A_{s-1})$ contains subgroups $\mathcal{B}^{s,n} \subseteq \mathcal{Z}^{s,n} \subseteq \widetilde{H}^n(\Sigma A_{s-1})$ and that the map $\widetilde{H}^n(\Sigma A_{s-1}) \to \widetilde{H}^n(X_{(s)})$ induces an isomorphism

$$g : \mathcal{Z}^{s,n}/\mathcal{B}^{s,n} \xrightarrow{\cong} \operatorname{Gr}^s \widetilde{H}^n(X).$$

The groups $\mathcal{B}^{s,n}$ and $\mathcal{Z}^{s,n}$ are actually not very hard to describe. First of all, the Correspondence Theorem tells us that $\mathcal{B}^{s,n} = \ker(g)$. For $\mathcal{Z}^{s,n}$, let's begin by saying that $v \in \widetilde{H}^n(\Sigma A_{s-1})$ **represents** $[u] \in \operatorname{Gr}^s \widetilde{H}^n(X)$ if $q_s^*(v) = [u]$; then $\mathcal{Z}^{s,n} = (q_s^*)^{-1}(\operatorname{Gr}^s \widetilde{H}^n(X))$ is the set of all $v : \Sigma A_{s-1} \to K_n$ which represent elements of $\operatorname{Gr}^s \widetilde{H}^n(X)$. Topologically, this means that $v : \Sigma A_{s-1} \to K_n$ is in $\mathcal{Z}^{s,n}$ if and only if there is a map $u : X \to K_n$ making the square

$$\begin{tikzcd}
X_{(s)} \arrow[r] \arrow[d, "q_s"'] & X \arrow[d, dotted, "u"] \\
\Sigma A_{s-1} \arrow[r, "v"] & K_n
\end{tikzcd}$$

commute up to homotopy.

These descriptions are pretty concrete and would be fine if we had loads of information about the spaces $X_{(s)}$ in our filtration. Unfortunately, the situation is not usually so nice: we can generally expect to have a pretty good understanding of the spaces A_s and the boundary maps $\partial_s : A_s \to \Sigma A_{s-1}$; but information about X and $X_{(s)}$ is hard to come by. For this reason, we hope to approximate the groups $\mathcal{Z}^{s,n}$ and $\mathcal{B}^{s,n}$ by subgroups of $\widetilde{H}^n(\Sigma A_{s-1})$ which can be described in terms of the spaces A_s and the boundary maps ∂_s. We interpret Problem 27.20(b) as a first approximation of $\operatorname{Gr}^s \widetilde{H}^n(X)$.

First Approximation: $\operatorname{Gr}^s \widetilde{H}^n(X)$ *is a subquotient of* $\widetilde{H}^n(\Sigma A_{n-1})$.

Ultimately we'll construct an infinite sequence of progressively better approximations and show that, under reasonable hypotheses, the approximations fully determine $\operatorname{Gr}^s \widetilde{H}^n(X)$.

27.2.2. Approximating $\mathcal{Z}^{s,n}$ and $\mathcal{B}^{s,n}$. For now, though, we'll focus on improving our first approximation to a second approximation. We'll tackle the general problem in Chapter 30 with the aid of a great deal of notation.

Write $v_{(s)} : X_{(s)} \to K_n$ for the composite $X_{(s)} \to \Sigma A_{s-1} \xrightarrow{v} K_n$, so that $\mathcal{Z}^{s,n}$ is precisely the set of all $v : \Sigma A_{s-1} \to K_n$ for which $v_{(s)}$ extends to a map $v_{(\infty)} : X \to K_n$. We'll try to construct such an extension by working one step at a time up the filtration of X. The first step is to find what we need to be able to extend $v_{(s)}$ to a map $v_{(s+1)} : X_{(s+1)} \to K_n$.

Problem 27.21.

(a) Show that $v_{(s)}$ can be extended to a map $v_{(s+1)} : X_{(s+1)} \to K_n$ if and only if $\partial_s \circ v \simeq *$.

(b) Show that if $v : \Sigma A_{s-1} \to K_n$ has a factorization of the form

$$\Sigma A_{s-1} \xrightarrow{\Sigma\partial_{s-1}} \Sigma^2 A_{s-2} \xrightarrow{w} K_n$$

for some w, then $v \in \mathcal{B}^{s,n}$.

Now is the time to introduce some standard and useful notation. We write $d_1^{s,n}$ for the map defined by the square

$$\begin{tikzcd} \widetilde{H}^n(\Sigma A_{s-1}) \arrow[r, "d_1^{s,n}"] \arrow[d, equal] & \widetilde{H}^{n+1}(\Sigma A_s) \\ \widetilde{H}^n(\Sigma A_{s-1}) \arrow[r, "\partial_s^*"] & \widetilde{H}^n(A_s). \arrow[u, "\cong"', "\Sigma"] \end{tikzcd}$$

Then we set

$$Z_1^{s,n} = \ker(d_1^{s,n}) \qquad \text{and} \qquad B_1^{s,n} = \mathrm{Im}(d_1^{s-1,n-1}).$$

According to Problem 27.21(a), $Z_1^{s,n}$ is precisely the set of all $v \in \widetilde{H}^n(\Sigma A_{s-1})$ that can be extended to a map $v_{(s+1)} : X_{(s+1)} \to K_n$. Together with Problem 27.21(b), we see that

$$0 \subseteq B_1^{s,n} \subseteq \mathcal{B}^{s,n} \subseteq \mathcal{Z}^{s,n} \subseteq Z_1^{s,n} \subseteq \widetilde{H}^n(\Sigma A_{s-1})$$

which means that the first approximation of $\mathrm{Gr}^s\, \widetilde{H}^n(X)$ is improved by our second approximation.

Second Approximation: $\mathrm{Gr}^s\, \widetilde{H}^n(X)$ *is a subquotient of* $Z_1^{s,n}/B_1^{s,n}$.

27.3. Approximations for General Filtered Spaces

Now we go through the exact same constructions as in Section 27.2, but this time we work with cohomology groups instead of maps and spaces. This is as straightforward as an empty exercise, but it actually has some content because, after applying cohomology, the distinction between generic filtrations and those arising from cone decompositions vanishes.

27.3.1. Algebraic Repackaging. Given a filtered space X, we form the groups

$$E_1^{s,n} = \widetilde{H}^n(X_{(s)}/X_{(s-1)}),$$

which we imagine we can compute. Each map $X_{(s-1)} \to X_{(s)}$ gives rise to a long cofiber sequence

$$X_{(s-1)} \xrightarrow{j_{s-1}} X_{(s)} \xrightarrow{q_s} X_{(s)}/X_{(s-1)} \xrightarrow{\alpha_{s-1}} \Sigma X_{(s-1)} \to \cdots .$$

The groups $E_1^{*,*}$ are related to one another by boundary maps $d_1^{s,n}$ defined by the compositions

$$\begin{tikzcd}
\widetilde{H}^n\left(X_{(s)}/X_{(s-1)}\right) \arrow[rr, "d_1^{s,n}"] \arrow[dr, "q_s^*"'] & & \widetilde{H}^{n+1}(X_{(s+1)}/X_{(s)}) \\
& \widetilde{H}^n(X_{(s)}). \arrow[ur, "\alpha_s^*"'] &
\end{tikzcd}$$

We denote the whole collection of groups $E_1^{s,n}$ together with the maps d_1 (we omit the superscripts unless there is some fear of confusion) by $E_1^{*,*}$; we also collect together the groups $D_1^{s,n} = \widetilde{H}^n(X_{(s)})$ into a bigraded object $D_1^{*,*}$. The maps j_s^*, α_s^* and q_s^* for various s define maps

$$j : D_1^{*,*} \longrightarrow D_1^{*,*}, \quad \alpha : D_1^{*,*} \longrightarrow E_1^{*,*} \qquad \text{and} \qquad q : E_1^{*,*} \longrightarrow D_1^{*,*}$$

of bigraded objects. With this notation, we can repackage the whole structure in the form of a triangle

$$\begin{tikzcd}
& E_1^{*,*} \arrow[dr, "q"] & \\
D_1^{*,*} \arrow[ur, "\alpha"] & & D_1^{*,*} \arrow[ll, "j"']
\end{tikzcd}$$

which is exact at each vertex. Such a structure is called an **exact couple**.

Problem 27.22. Show that this construction of the exact couple of a filtered space X is functorial.

Exercise 27.23. What are the bidegrees of the maps α, q and j?

27.3.2. Algebraic Homology and Exact Couples. In any exact couple, the group $E_1^{*,*}$ has the structure of a (bigraded) cochain complex, and our approximations to $\mathrm{Gr}^s\, \widetilde{H}^n(X)$ can be phrased in terms of the homology of this complex.

Problem 27.24. Show that the map $d_1 = \alpha \circ q : E_1^{*,*} \to E_1^{*,*}$ is a differential, so that $E_1^{*,*}$ is a cochain complex.[3] Show also that this construction is functorial on the category of filtered spaces.

[3]That is, show that $d_1 \circ d_1 = 0$.

The algebraic cohomology groups of the cochain complex $E_1^{*,*}$ have two indices, n and s. We'll write

$$E_2^{s,n} = H^{s,n}(E_1^{*,*})$$

for the cohomology of this complex in position (s, n).

If the filtration on X is a cone decomposition, then the work of Section 27.2 establishes our first theorem about the approximation of $\mathrm{Gr}^s H^n(X)$.

Theorem 27.25. *If X is a filtered pointed space, then the filtration quotient*

$$\mathrm{Gr}^s \widetilde{H}^n(X) = \mathcal{F}^{s-1}\widetilde{H}^n(X)/\mathcal{F}^s\widetilde{H}^n(X)$$

is, in a natural way, a subquotient of $E_2^{s,n}(X)$.[4]

The naturality statement in Theorem 27.25 is actually entirely straightforward; the tricky part is to carefully express what it means. First of all, if $f : X \to Y$ is a map of filtered spaces, then the induced map $f^* : \widetilde{H}^*(Y) \to \widetilde{H}^*(X)$ is a map of filtered groups, so it induces maps $\phi^{s,n} : \mathrm{Gr}^s \widetilde{H}^n(Y) \to \mathrm{Gr}^s \widetilde{H}^n(X)$. On the other hand f also induces a map of exact couples, hence a chain map $f_1^{*,*} : E_1^{*,*}(Y) \to E_1^{*,*}(X)$. Theorem 27.25 asserts that

$$f_1^{*,*}(\mathcal{Z}^{s,n}(Y)) \subseteq \mathcal{Z}^{s,n}(X) \qquad \text{and} \qquad f_1^{*,*}(\mathcal{B}^{s,n}(X)) \subseteq \mathcal{B}^{s,n}(Y)$$

and that the induced map $\theta^{s,n} : \mathrm{Gr}^s \widetilde{H}^n(Y) \to \mathrm{Gr}^s \widetilde{H}^n(X)$ is precisely $\phi^{s,n}$.

Problem 27.26. Prove Theorem 27.25.

27.3.3. Topological Boundary Maps for a Filtration. When we studied (in Section 27.2.2) filtrations coming from cone decomposition, the boundary maps $d_1^{s,n}$ were induced by the topological boundary maps of the cone decomposition. This approach was not available to us in the study of a more general filtered space, because the inclusions $X_{(s)} \hookrightarrow X_{(s+1)}$ were not part of cofiber sequences $A_s \to X_{(s)} \to X_{(s+1)}$. Nevertheless, in Section 27.3 we were able to construct *algebraic* boundary maps after applying cohomology.

Problem 27.27. Let $\cdots \to X_{(s)} \to X_{(s+1)} \to \cdots \to X$ be a filtered space.

(a) Show that ΣX has a cone decomposition

$$\begin{tikzcd} & X_{(s)}/X_{(s-1)} \arrow[d] & X_{(s+1)}/X_{(s)} \arrow[d] & X_{(s+2)}/X_{(s+1)} \arrow[d] & \\ \cdots \arrow[r] & \Sigma X_{(s-1)} \arrow[r] & \Sigma X_{(s)} \arrow[r] & \Sigma X_{(s+1)} \arrow[r] & \cdots . \end{tikzcd}$$

[4]In the diagonal indexing, the statement is that $\mathrm{Gr}^s \widetilde{H}^{s+t}(X)$ is a subquotient of $E_2^{s,t}$.

(b) Show that boundary maps $\partial_s : X_{(s+1)}/X_{(s)} \to \Sigma(X_{(s)}/X_{(s-1)})$ for this cone decomposition induce the algebraic ones in the sense that the diagram

$$\begin{tikzcd} \widetilde{H}^{n+1}(\Sigma(X_{(s)}/X_{(s-1)})) \arrow[r, "\partial_s^*"] \arrow[d, "\Sigma^{-1}"', "\cong"] & \widetilde{H}^{n+1}(X_{(s+1)}/X_{(s)}) \arrow[d, equal] \\ \widetilde{H}^{n}(X_{(s)}/X_{(s-1)}) \arrow[r, "d_1^{s,n}"] & \widetilde{H}^{n+1}(X_{(s+1)}/X_{(s)}) \end{tikzcd}$$

commutes.

Thus we see that, even for generic filtrations, the algebraic boundary maps can be viewed as having been induced by underlying topological maps.

27.4. Products in $E_1^{*,*}(X)$

In this section, we show that the functor $E_1^{*,*}(\,?\,)$ has an external product which is compatible with, and approximates, the external product $\widetilde{H}^*(X) \otimes \widetilde{H}^*(Y) \to \widetilde{H}^*(X \wedge Y)$. This implies that for spaces X with multiplicative filtrations, $E_1^{*,*}(X)$ and $E_2^{*,*}(X)$ have algebra structures that approximate the algebra structure that $\mathrm{Gr}^*\,\widetilde{H}^*(X)$ inherits from $\widetilde{H}^*(X)$.

27.4.1. The Exterior Product for $Z_1^{*,*}$. Let X and Y be filtered pointed spaces. For each s and t, the relative cup product defined in Problem 25.8 is a map

$$\begin{tikzcd} \widetilde{H}^{n}(X_{(s)}/X_{(s-1)}) \otimes \widetilde{H}^{m}(Y_{(t)}/Y_{(t-1)}) \arrow[r, "?\bullet ?"] \arrow[d, equal] & \widetilde{H}^{n+m}((X_{(s)} \wedge Y_{(t)})/(X_{(s)} \wedge Y_{(t)})_{(s+t-1)}) \arrow[d, dotted, "\ell"] \\ Z_1^{s,n}(X;G) \otimes Z_1^{s,n}(Y,H) \arrow[r] & Z_1^{s,n}(X \wedge Y; G \otimes H). \end{tikzcd}$$

To define our exterior product

$$Z_1^{*,*}(X;G) \otimes Z_1^{*,*}(Y,H) \to Z_1^{*,*}(X \wedge Y, G \otimes H),$$

it suffices to construct the map ℓ.

Problem 27.28. Let $T = (X_{(s+1)} \wedge Y_{(t-1)}) \cup (X_{(s-1)} \wedge Y_{(t+1)})$.

(a) Show that

$$(X \wedge Y)_{(s+t+1)}/T \simeq A \vee B \vee (X_{(s+1)} \wedge Y_{(t+1)})_{s+t+1}/T$$

for two spaces A and B that you can easily write down.

(b) Define, in a canonical way, an extension of $u_{(s+1)} \bullet v_{(t+1)}$ to a map $u \star v : (X \wedge Y)_{s+t+1} \to K_{n+m}$ that is trivial on $(X \wedge Y)_{s+t-1}$.

(c) Show that the rule $(u, v) \mapsto u \star v$ is bilinear, so that it defines an exterior product

$$E_1^{s,n}(X) \otimes E_1^{t,m}(Y) \longrightarrow E_1^{s+t,n+m}(X \wedge Y).$$

(d) Show that if $u \in E_1^{s,n}(X)$ represents $[x] \in \operatorname{Gr}^s H^n(X)$ and $v \in E_1^{t,m}$ represents $[y] \in \operatorname{Gr}^t H^m(Y)$, then $u \star v \in E_1^{s+t,n+m}(X \wedge Y)$ represents $[x \bullet y]$ in $\operatorname{Gr}^{s+t} H^{n+m}(X \wedge Y)$.

27.4.2. Boundary Maps for a Smash of Filtered Spaces. The bigraded group $E_1^{*,*}(X) \otimes E_1^{*,*}(Y)$ inherits a differential from its factors:

$$d_1^{X \wedge Y}(x \otimes y) = d_1^X(x) \otimes y + (-1)^{|y|} x \otimes d_1^Y(y).$$

We want to know that the exterior product is a chain map with respect to this differential, and for this we have to analyze the boundary maps in the filtration for $X \wedge Y$.

You showed in Section 24.3 that a very similar exterior product was a chain map, and the approach here is analogous. The only wrinkle is that we need to maneuver around the fact that we do not have a cone filtration in general.

Problem 27.29. Let X and Y be filtered pointed spaces.

(a) Show that the boundary maps for the standard filtration of $\Sigma X \wedge \Sigma Y$ are the two-fold suspensions of the boundary maps for $X \wedge Y$.

(b) Use Theorem 9.81 to determine the boundary maps for $\Sigma X \wedge \Sigma Y$.

(c) Show that the exterior product is a chain map.

27.4.3. Internalizing the External Product. Now suppose X has a multiplicative filtration, which means that the diagonal $\overline{\Delta} : X \to X \wedge X$ is homotopic to a filtered map $\widetilde{\Delta} : X \to X \wedge X$. Then the composition

$$E_1^{*,*}(X) \otimes E_1^{*,*}(X) \xrightarrow{\ ?\bullet ?\ } E_1^{*,*}(X \wedge X) \xrightarrow{E_1^{*,*}(\widetilde{\Delta})} E_1^{*,*}(X)$$

gives $E_1^{*,*}(X)$ an internal product.

Theorem 27.30. *Let X be a filtered pointed space.*

(a) The differential $d_1^{*,*} : E_1^{*,*}(X) \to E_1^{*,*}(X)$ is a derivation with respect to the internal product.

(b) The induced product on $\operatorname{Gr}^* \widetilde{H}^*(X) \otimes \operatorname{Gr}^* \widetilde{H}^*(X) \to \operatorname{Gr}^* \widetilde{H}^*(X)$ is the same as that induced by the cup product in $\widetilde{H}^*(X)$.

Corollary 27.31. *If X is a pointed filtered space, then $E_2^{*,*}(X)$ inherits a product from $E_1^{*,*}(X)$ and this product induces the multiplication that $\operatorname{Gr}^* \widetilde{H}^*(X)$ inherits from the cup product in $\widetilde{H}^*(X)$.*

Problem 27.32. Prove Theorem 27.30 and derive Corollary 27.31.

27.5. Pointed and Unpointed Filtered Spaces

Problem 27.5 shows us that all of the constructions that we have done in the pointed category $\mathcal{T}_*$ have counterparts in the unpointed category $\mathcal{T}_\circ$.

Problem 27.33. Let X be a filtered unpointed space, and let X_+ be the corresponding filtered pointed space. Show that the boundary maps for X are *the same as* the boundary maps for X_+.

In the same way, if we plug in X_+ and Y_+ for X and Y in the pointed exterior product $E_1^{*,*}(X) \otimes E_1^{*,*}(Y) \to E_1^{*,*}(X \wedge Y)$, we obtain the unpointed exterior product

$$E_1^{*,*}(X) \otimes E_1^{*,*}(Y) \longrightarrow E_1^{*,*}(X \times Y),$$

and it enjoys all of the same nice algebraic properties.

It follows, in particular, that if X has a multiplicative filtration, then $E_1^{*,*}(X)$ is a chain algebra.[5] Thus $E_2^{*,*}(X)$ inherits the structure of a bigraded algebra, and this structure induces the same multiplication in $\mathrm{Gr}^* H^*(X)$ as the cup product in $H^*(X)$ induces.

27.6. The Homology of Filtered Spaces

We can go through very similar arguments for homology. A filtered space X gives rise to the large diagram of interlocking cofiber sequences, and on applying homology, we obtain a big algebraic diagram, or, if you prefer, an exact couple

$$\begin{tikzcd} & E^1_{*,*} \arrow[dl] & \\ D^1_{*,*} \arrow[rr] & & D^1_{*,*} \arrow[ul] \end{tikzcd}$$

where

$$E^1_{s,n} = \widetilde{H}_n(X_{(s)}/X_{(s-1)}) \qquad \text{and} \qquad E^1_{s,n} = H_n(X_{(s)}).$$

Exercise 27.34. Write out the big algebraic diagram that results from applying homology to the big topological diagram. Then define $E^1_{s,n}$ and the differentials $d^1_{s,n}$.

Reasoning very much like what we did for cohomology leads to the following theorem concerning $E^2_{*,*} = H_{*,*}(E^1_{*,*})$.

Theorem 27.35. *If X is a filtered space, then the filtration quotient*

$$\mathrm{Gr}_s H_n(X) = \mathcal{F}_s\widetilde{H}_n(X)/\mathcal{F}_{s-1}\widetilde{H}_n(X)$$

is, in a natural way, a subquotient of $E^2_{s,n} = H_{s,n}(E^1_{*,*}, d^1)$.

[5] That is, it is an algebra and a chain complex, and the differential is a derivation.

Problem 27.36. Prove Theorem 27.35.

Project 27.37. Suppose X is an H-space filtered by sub-H-spaces. Show that $E^1_{*,*}$ inherits an algebra structure and that $d^1_{*,*}$ is a derivation.

27.7. Additional Projects

Project 27.38. Suppose X has a decreasing filtration. Set up and prove a theorem analogous to Theorem 27.25 for getting information about $\pi_*(X)$. Can you generalize from $\pi_n(X) = [S^n, X]$ to $[A, X]$ for other spaces A?

Project 27.39. Topologize the mapping space $\mathrm{map}_{\mathrm{FILT}}(X, Y)$ of filtered maps from X to Y. Is there an exponential law

$$\mathrm{map}_{\mathrm{FILT}}(X \times Y, Z) \cong \mathrm{map}_{\mathrm{FILT}}(X, \mathrm{map}_{\mathrm{FILT}}(Y, Z)) \ ?$$

Project 27.40. How much of the analysis of $[X, K(G, n)]$ in Section 27.2 can be applied to $[X, Y]$ for more general spaces Y?

Hint. You can use the coaction of ΣA_{s-1} on $X_{(s)}$ to make sense of expressions like $v_{(s+1)} - \bar{v}_{(s+1)}$.

Project 27.41. Try to characterize the maps $v : \Sigma A_{s-1} \to K_n$ such that $v_{(s)}$ extends to $v_{(s+2)} : X_{(s+2)} \to K_n$. How far can you go?

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Chapter 28

The Serre Filtration of a Fibration

If $p : E \to B$ is a fibration in which the base B is a CW complex, then by taking pullbacks over the skeleta $\cdots \subseteq B_{s-1} \subseteq B_s \subseteq \cdots \subseteq B$ we obtain the **Serre filtration** $\cdots \to E_{(s-1)} \to E_{(s)} \to \cdots \to E$ of the total space. This topological filtration gives rise to algebraic filtrations in cohomology and homology whose filtration quotients $\mathrm{Gr}^s H^{s+t}(E)$ can be estimated, using Theorem 27.25, by groups which we denote[1] $E_2^{s,t}(p)$. In this chapter we'll derive explicit formulas for $E_2^{s,t}(p)$ in terms of the cohomology of the spaces F and B and determine the algebraic structure of the bigraded object $E_2^{*,*}(p)$.

28.1. Identification of E_2 for the Serre Filtration

We begin this section by determining $E_1^{s,t}(p)$. The resulting formula suggests a pleasant and plausible conjecture about the differential d_1, and we find that this conjecture is true, at least as long as the fibration p is orientable for our cohomology theory. Once we have identified d_1, we give an explicit and useful formula for $E_2^{s,t}(p)$ in terms of $H^*(B)$ and $H^*(F)$.

28.1.1. Cohomology with Coefficients in Cohomology. First of all, we know from Problem 26.32 that $E_{(s)}/E_{(s-1)} \simeq B_s/B_{s-1} \rtimes F$. Since the

[1] We always use the diagonal indexing in the context of the Serre filtration—see Section 27.1.7.

cohomology of $B_s/B_{s-1} \simeq \bigvee S^s$ is free abelian, we have

$$\begin{aligned} E_1^{s,t}(p) &= \widetilde{H}^{s+t}(E_{(s)}/E_{(s-1)}) \\ &\cong \widetilde{H}^{s+t}((B_s/B_{s-1}) \rtimes F) \\ &\cong \widetilde{H}^{s}(B_s/B_{s-1}) \otimes H^t(F) \\ &\cong C^s(B; H^t(F)), \end{aligned}$$

the s^{th} group in the cellular cochain complex for B with coefficients in $H^t(F)$. In other words, there is a canonical isomorphism of bigraded abelian groups

$$E_1^{*,*}(p) \longrightarrow C^*(B; H^*(F)).$$

An optimistic person with lots of time on his or her hands might hope that this is not just an isomorphism of bigraded groups; he or she would hope that it was a chain map. If this were true, we would be able to determine $E_2^{s,t}(p)$ by the simple computation

$$\begin{aligned} E_2^{s,t}(p) &= H^{s,t}(E_1^{*,*}(p)) \\ &\cong H^s(C^*(B; H^t(F))) \\ &\cong H^s(B; H^t(F)). \end{aligned}$$

That this is actually the case is the content of our main theorem on the Serre filtration.

Theorem 28.1. *Let $p : E \to B$ be a fibration which is orientable for the ordinary cohomology theory H^*, whose coefficients we suppress. Then there is a natural*[2] *isomorphism*

$$E_2^{s,t}(p) \cong H^s(B; H^t(F)).$$

The proof, which you will work through presently, makes absolutely no use of the fact that we have applied an *ordinary* cohomology theory to the Serre filtration. In fact, the vast majority of the proof consists of analyzing the topological boundary maps that induce the differentials. If we apply a general cohomology theory h^*, the resulting formula is

$$E_2^{s,t}(p) = H^s(B; h^t(F)).$$

When applied to the (orientable) fibration $\mathrm{id}_X : X \to X$ with fiber $F = *$, this shows that $\mathrm{Gr}^s\, h^{s+t}(X)$ is a subquotient of $H^s(X; h^t(*))$, so the ordinary cohomology of X together with the coefficient groups of h^* provide an upper bound, so to speak, for $h^*(X)$.

Before delving into the proof, let's look at a very simple example illustrating the power of Theorem 28.1.

[2]We will discuss the meaning of naturality in detail in Section 28.2.4.

Problem 28.2. Let $\mathbb{F}$ be a field, and let $p : E \to B$ be an $H^*(\,?\,;\mathbb{F})$-orientable fibration with fiber F. Suppose $H^*(B;\mathbb{F})$ and $H^*(F;\mathbb{F})$ are concentrated in even degrees (i.e., $H^k(F;\mathbb{F}) = 0$ and $H^k(B;\mathbb{F}) = 0$ for k odd). Show that $H^*(E;\mathbb{F})$ is also concentrated in even degrees.

Exercise 28.3. Does Problem 28.2 generalize to spaces whose cohomology is concentrated in odd dimensions? What if the cohomology is concentrated in dimensions divisible by 3? What if nonfield coefficients are used?

More about Orientability. If we omit all orientability hypotheses, the group $E_2^{s,t}(p)$ can still be thought of as being the s^{th} cohomology of B, but with coefficients $H^t(F_x)$ that 'vary from place to place'. More precisely, the orientability condition allows us to identify the groups $H^t(F_x)$ and $H^t(F_y)$ (for $x, y \in B$) in a canonical way, and this is crucial to identifying the differential d_1. In the general case, the groups $H^t(F_x)$ and $H^t(F_y)$ are isomorphic, but not canonically. Instead, we have a structure $H^t(\mathcal{F})$ which assigns to each point $x \in B$ the group $H^t(F_x)$, and to each path from x to y in B an isomorphism $H^t(F_y) \to H^t(F_x)$. These isomorphisms satisfy various rules[3] and are called a **bundle of groups** or a **system of local coefficients**. There is a way to define cohomology with coefficients in any bundle of groups $\mathcal{G}$ on B (called **cohomology with local coefficients**), and with this definition,

$$E_2^{s,t}(p) \cong H^s(B; H^t(\mathcal{F})).$$

If the bundle $\mathcal{G}$ is *constant* at the group G, then $H^*(X;\mathcal{G}) \cong H^*(X;G)$, so our result is a special case of this more general statement.

28.2. Proof of Theorem 28.1

The differentials $d_1^{s,*}$ are induced (modulo a dimension shift) by the topological boundary map ∂_s^E in the diagram

$$\begin{tikzcd}
E_{(s-1)} \arrow[r] \arrow[d] & E_{(s)} \arrow[d] & \\
E_{(s-1)} \cup CE_{(s-2)} \arrow[r] \arrow[d, "q"'] & E_{(s)} \cup CE_{(s-1)} \arrow[d, "q"'] \arrow[dd, dotted, bend left, "\partial_s^E"] & \bigvee_{\mathcal{J}} S^s \rtimes F \arrow[l, "\alpha_s"', "\simeq"] \arrow[dd, dotted, "\delta"] \\
\Sigma E_{(s-2)} \arrow[r] & \Sigma E_{(s-1)} \arrow[d] & \\
& \Sigma\left(E_{(s-1)}/E_{(s-2)}\right) \arrow[r, "\Sigma\beta_{s-1}", "\simeq"'] & \Sigma\left(\bigvee_{\mathcal{I}} S^{s-1} \rtimes F\right).
\end{tikzcd}$$

[3]Which are best expressed by saying that they constitute a functor $\Pi(X) \to \textsc{ab}\,\mathcal{G}$, where $\Pi(X)$ is the fundamental groupoid of X.

We know from Problem 26.32 that there are homotopy equivalences α_s identifying the subquotients $E_{(s)}/E_{(s-1)}$ with wedges of copies of $S^s \rtimes F$, one for each s-cell of B (and likewise with β_{s-1}), so ∂_s^E can be identified with the other dotted arrow δ. Our proof amounts to constructing the maps α_s and β_{s-1} carefully enough that the resulting map δ has two properties:

- there is a commutative square

$$\begin{tikzcd}
\bigvee_{\mathcal{J}} S^s \rtimes F \arrow[r, "\delta"] \arrow[d, "\mathrm{pr}_1"'] & \Sigma\left(\bigvee_{\mathcal{I}} S^{s-1} \rtimes F\right) \arrow[d, "\mathrm{pr}_1"] \\
B_s/B_{s-1} \arrow[r, "d^B"] & \Sigma\left(B_{s-1}/B_{s-2}\right),
\end{tikzcd}$$

  where d^B is the boundary map for the CW filtration of B (which implies that δ carries fibers to fibers), and
- the restriction of δ to each fiber is admissible for the fibration p.

Taken together with Lemma 4.32, these properties imply that the coordinate map δ_{ij} defined by the square

$$\begin{tikzcd}
S^s \rtimes F \arrow[r, "\delta_{ij}"] \arrow[d, "\mathrm{in}_j"'] & S^s \rtimes F \\
\bigvee_{\mathcal{J}} S^s \rtimes F \arrow[r, "\delta"] & \bigvee_{\mathcal{I}} S^s \rtimes F \arrow[u, "q_i"']
\end{tikzcd}$$

is homotopic to $d_{ij}^B \rtimes \phi_{ij}$ where d_{ij}^B is the (i,j)-coordinate of the boundary map for B and ϕ_{ij} is admissible for the fibration p. If p is orientable for the cohomology theory h^*, then $h^*(\delta)$ may be identified with the map whose (i,j)-coordinate function is $d_{ij} \otimes \mathrm{id}_{h^*(F)}$, and this implies Theorem 28.1.

28.2.1. Setting Up. We begin by building the diagram

$$\begin{tikzcd}
Q_s \arrow[rr] \arrow[dr] \arrow[dd] & & E_{(s-1)} \arrow[rr] \arrow[dr] \arrow[dd] & & * \arrow[dr] \arrow[dd] & \\
& P_s \arrow[rr] \arrow[dd] & & E_{(s)} \arrow[rr] \arrow[dd] & & E_{(s)}/E_{(s-1)} \arrow[dd] \\
\coprod_J S^{s-1} \arrow[rr] \arrow[dr] & & B_{s-1} \arrow[rr] \arrow[dr] & & * \arrow[dr] & \\
& \coprod_J D^s \arrow[rr] & & B_s \arrow[rr] & & B_s/B_{s-1},
\end{tikzcd}$$

in which the left cube is the Mather cube that results from pulling back the fibration $E_{(s)} \to B_s$ and the right cube is built by forming the induced map $E_{(s)}/E_{(s-1)} \to B_s/B_{s-1}$ of categorical (and homotopy) pushouts.

Problem 28.4.

(a) Show that all three squares on the top of the diagram are strong homotopy pushout squares.

(b) Show that the mapping cone of $\coprod_J S^s \to \coprod_J D^s$ is $\bigvee_{\mathcal{J}} S^s$.

(c) Show that in the squares

$$\begin{tikzcd}
P_s \cup CQ_s \arrow[r, "\simeq"] \arrow[d] & E_{(s)} \cup CE_{(s-1)} \arrow[d] \\
\bigvee_{\mathcal{J}} S^s \arrow[r, "\simeq"] & B_s \cup CB_{s-1}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
P_s/Q_s \arrow[r, "\cong"] \arrow[d] & E_{(s)}/E_{(s-1)} \arrow[d] \\
\bigvee_{\mathcal{J}} S^s \arrow[r, "\cong"] & B_s/B_{s-1}
\end{tikzcd}$$

the maps on top restrict to the identity on each fiber.

Now we identify the spaces Q_s and P_s.

Problem 28.5. Show that there is a strictly commutative diagram

$$\begin{tikzcd}
Q_s \arrow[rr, "\simeq"] \arrow[dr] \arrow[dd] & & \coprod_{\mathcal{J}} S^{s-1} \times F \arrow[rr, "\simeq"] \arrow[dr] \arrow[dd] & & Q_s \arrow[dr] \arrow[dd] & \\
& P_s \arrow[rr, "\simeq"] \arrow[dd] & & \coprod_{\mathcal{J}} D^s \times F \arrow[rr, "\simeq"] \arrow[dd] & & P_s \arrow[dd] \\
\coprod_{\mathcal{J}} S^{s-1} \arrow[rr, equal] \arrow[dr] & & \coprod_{\mathcal{J}} S^{s-1} \arrow[rr, equal] \arrow[dr] & & \coprod_{\mathcal{J}} S^{s-1} \arrow[dr] & \\
& \coprod_{\mathcal{J}} D^s \arrow[rr, equal] & & \coprod_{\mathcal{J}} D^s \arrow[rr, equal] & & \coprod_{\mathcal{J}} D^s
\end{tikzcd}$$

in which the maps in the right cube are admissible trivializations and the maps in the left cube are homotopy inverses over $\coprod_{\mathcal{J}} S^{s-1}$ and $\coprod_{\mathcal{J}} D^s$, respectively.

The maps α_s and β_s that are the core of our analysis of the boundary map are composites of the homotopy equivalences of Problem 28.5 with the ones you will construct in the next problem.

Problem 28.6. Show that there are commutative diagrams

$$\begin{tikzcd}
\bigvee_{\mathcal{J}} S^s \rtimes F \arrow[r, "\bar{\alpha}_s", "\simeq"'] \arrow[d] & P_s \cup CQ_s \arrow[d] \\
\bigvee_{\mathcal{J}} S^s \arrow[r, equal] & \bigvee_{\mathcal{J}} S^s
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
P_s/Q_s \arrow[r, "\bar{\beta}_s", "\simeq"'] \arrow[d] & \bigvee_{\mathcal{J}} S^s \rtimes F \arrow[d] \\
\bigvee_{\mathcal{J}} S^s \arrow[r, equal] & \bigvee_{\mathcal{J}} S^s
\end{tikzcd}$$

in which the restrictions of $\bar{\alpha}_s$ and $\bar{\beta}_s$ to each fiber are admissible for p.

28.2.2. The Topological Boundary Map. Now we are prepared to identify the topological boundary map (up to suspension) in the Serre filtration. Consider the diagram

$$\begin{tikzcd}
E_{(s-1)} \arrow[r] \arrow[d] & E_{(s)} \arrow[d] & & \\
E_{(s-1)} \cup CE_{(s-2)} \arrow[r] \arrow[d, "q"'] & E_{(s)} \cup CE_{(s-1)} \arrow[d, "q"'] \arrow[ddd, dotted, bend left, "\partial_s^E"] & P_s \cup CQ_s \arrow[l] & \bigvee_{\mathcal{J}} S^s \rtimes F \arrow[l, "\bar{\alpha}_s", "\simeq"'] \arrow[ll, dotted, bend right, "\alpha_s"'] \arrow[ddd, dotted, "\delta"] \\
\Sigma E_{(s-2)} \arrow[r] & \Sigma E_{(s-1)} \arrow[dd] & & \\
& & & \\
& \Sigma E_{(s-1)} \cup CE_{(s-2)} \arrow[r, "\cong"] \arrow[rr, dotted, bend left, "\Sigma\beta_{s-1}"] & \Sigma(P_{s-1}/Q_{s-1}) \arrow[r, "\Sigma\bar{\beta}_{s-1}", "\simeq"'] & \Sigma\left(\bigvee_{\mathcal{I}} S^{s-1} \rtimes F\right)
\end{tikzcd}$$

(the maps q are collapse maps).

Problem 28.7.

(a) Show that all the maps in the diagram restrict to the identity on fibers except (possibly) $\bar{\alpha}_s$, δ and $\Sigma\bar{\beta}_{s-1}$.

(b) Show that all the maps in the diagram restrict to admissible maps on fibers.

(c) Show that the diagram

$$\begin{tikzcd}
\bigvee_{\mathcal{J}} S^s \rtimes F \arrow[r, "\delta"] \arrow[d] & \bigvee_{\mathcal{I}} S^s \rtimes F \arrow[d] \\
\bigvee_{\mathcal{J}} S^s \arrow[r, "\partial_s^B"] & \bigvee_{\mathcal{I}} S^s
\end{tikzcd}$$

commutes on the nose.

(d) Show that $\delta_{ij} \simeq d_{ij} \rtimes \phi_{ij}$, where the maps d_{ij} are the coordinate maps for the space B and ϕ_{ij} is admissible for the fibration.

Hint. Use Lemma 4.32.

28.2.3. Identifying the Differential. Now it's all over but the shouting: it only remains to apply our cohomology theory to the topological boundary maps and deduce Theorem 28.1.

Problem 28.8.

(a) Show that $\phi_{ij}^* : H^*(F) \to H^*(F)$ is the identity.

(b) Show that the diagram

$$\begin{tikzcd}
H^m(S^s \rtimes F) \arrow[r, "\delta^*_{ij}"] \arrow[d, "\cong"'] & H^m(S^s \rtimes F) \arrow[d, "\cong"] \\
H^{m-s}(F) \arrow[r, "d_{ij}"] & H^{m-s}(F)
\end{tikzcd}$$

commutes, where, as usual, we identify the map $d_{ij} : S^s \to S^s$ with its numerical degree $\deg(d_{ij}) \in \mathbb{Z} \cong \pi_s(S^s)$.

(c) Show that δ is a map of finite type,[4] so that its induced map δ^* in cohomology is determined by its coordinate maps δ^*_{ij}.

(d) Prove Theorem 28.1 by showing that the diagram

$$\begin{tikzcd}
E_1^{s,t} \arrow[r, "d_1"] \arrow[d, equal] & E_1^{s+1,t} \arrow[d, equal] & \\
H^{s+t}(E_{(s-1)/(s-2)}) & H^{s+t+1}(E_{(s)/(s-1)}) \arrow[r, "\cong"] \arrow[d, "\cong"] & H^{s+t+1}(E_{(s)} \cup CE_{(s-1)}) \arrow[dl, bend left, "\bar{\alpha}^*_{s+1}"] \\
H^{s+t}(\bigvee_{\mathcal{I}} S^s \rtimes F) \arrow[r, "\delta^*"] \arrow[u, "\beta^*_{s-1}", "\cong"'] \arrow[d, equal] & H^{s+t}(\bigvee_{\mathcal{J}} S^s \rtimes F) \arrow[d, equal] & \\
\prod_{\mathcal{I}} H^{s+t}(S^s \rtimes F) \arrow[r, "(\delta^*_{ij})"] \arrow[d, "\cong"'] & \prod_{\mathcal{J}} H^{s+t}(S^s \rtimes F) \arrow[d, "\cong"] & \\
\prod_i H^t(F) \arrow[r, "(d_{ij})"] \arrow[d, "\cong"'] & \prod_j H^t(F) \arrow[d, "\cong"] & \\
C^s(B; H^t(F)) \arrow[r, "d"] & C^{s+1}(B; H^t(F)) &
\end{tikzcd}$$

commutes.

28.2.4. Naturality of $E_2^{*,*}$. We have defined a construction that takes a fibration $p : E \to B$ with fiber F and returns the bigraded abelian group

$$E_2^{*,*}(p) = H^*(B; H^*(F))$$

and identifies two bigraded subgroups $B_2^{*,*} \subseteq Z_2^{*,*}$ and an isomorphism $Z_2^{s,t}/B_2^{s,t} \cong \mathrm{Gr}^s(H^{s+t}(E))$.

---

[4] See Section A.8.

To discuss the naturality of the construction, we introduce the category **Fib** whose objects are fibrations, and whose morphisms are strictly commutative diagrams

$$\begin{tikzcd} E_1 \arrow[r, "g"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "h"] & B_2. \end{tikzcd}$$

The strongest naturality statement we could hope for would be that g and h induce maps

$$E_1^{*,*}(p_2) \longrightarrow E_1^{*,*}(p_1) \qquad \text{and} \qquad E_2^{*,*}(p_2) \longrightarrow E_2^{*,*}(p_1)$$

and that under the identification of $E_2^{*,*}$ given by Theorem 28.1, this agrees with the map

$$H^s(E_2; H^t(F_2)) \xrightarrow{(f^*)_*} H^s(E_2; H^t(F_1)) \xrightarrow{h^*} H^s(E_1; H^t(F_1)),$$

where $f : F_1 \to F_2$ is the restriction of g to the fibers.

As is frequently the case with intricate constructions, it can be done as naturally as one could hope for, but only at the expense of giving up hands-on control of the spaces involved. Specifically, the construction as given cannot possibly satisfy these ideal naturality conditions, because the whole thing is founded on the cellular structure of the base. They do hold, however, provided B_1 and B_2 are CW complexes and f is cellular.

Problem 28.9. Show that $E_2^{*,*}$ satisfies the desired naturality properties for maps of fibrations in which the map $h : B_1 \to B_2$ is a cellular map of CW complexes.

If we are given a map of fibrations in which the map $h : B_1 \to B_2$ of the bases is not—or is not known to be—cellular (for instance, if one or both of B_1 and B_2 is not a CW complex), then we appeal to the Weak Equivalence Axiom and cellular replacement. The key is that there is a functorial CW replacement functor, which we'll refer to as CW.[5]

Given a fibration $p : E \to B$, we form the pullback of p over the cellular replacement, resulting in the square

$$\begin{tikzcd} \widetilde{E} \arrow[r, "\sim"] \arrow[d, "\widetilde{p}"'] \arrow[dr, phantom, "\text{pullback}"] & E \arrow[d, "p"] \\ \mathrm{CW}(B) \arrow[r, "\sim"] & B. \end{tikzcd}$$

[5]Such a functor was constructed, for example, in Problem 16.14.

Now the maps $\mathrm{CW}(B) \to B$, $\widetilde{E} \to E$ and the restrictions $\widetilde{F} \to F$ to fibers are weak homotopy equivalences, so they induce isomorphisms on (singular) cohomology. We define $E_1^{*,*}(p) = E_1^{*,*}(\widetilde{p})$; then

$$E_2^{s,t}(p) = H^{s,t}(E_1^{*,*}(p)) \cong H^s(\mathrm{CW}(B); H^t(\widetilde{F})) \cong H^s(B; H^t(F)).$$

Now we have a construction that works for all fibrations and we can ask about its naturality. Applying our natural CW approximation, we obtain the diagram

$$\begin{tikzcd}
\widetilde{E}_1 \arrow[rr, "\widetilde{g}"] \arrow[dr] \arrow[dd, "\widetilde{p}_1"'] & & \widetilde{E}_2 \arrow[dr] \arrow[dd, "\widetilde{p}_2"] & \\
& E_1 \arrow[rr, "g"] \arrow[dd, "p_1"'] & & E_2 \arrow[dd, "p_2"] \\
\mathrm{CW}(B_1) \arrow[rr, "\bar{h}"] \arrow[dr, "\sim"'] & & \mathrm{CW}(B_2) \arrow[dr, "\sim"] & \\
& B_1 \arrow[rr, "h"] & & B_2
\end{tikzcd}$$

in which $\bar{h} = \mathrm{CW}(h)$ is cellular.

Problem 28.10. Prove the naturality part of Theorem 28.1.

28.3. External and Internal Products

In Section 27.4 we defined external products in $E_1^{*,*}$ and $E_2^{*,*}$ for any filtered spaces, and internal products for spaces with multiplicative filtrations. In this section we show that for the Serre filtration, the external product may be identified via Theorem 28.1 with composition of the ordinary external product with the coefficient transformation induced by the exterior product $H^*(F_1) \otimes H^*(F_2) \to H^*(F_1 \times F_2)$. The Serre filtration is multiplicative, and it follows that the internal product in $E_2^{*,*}(p)$ coincides with the ordinary (external) cup product in $H^*(B; H^*(F))$.

28.3.1. External Products for $E_2^{*,*}(p)$. We want to show that the external product is compatible with the identifications of Theorem 28.1.

Theorem 28.11. *Let $p : X \to A$ and $q : Y \to B$ be fibrations with fibers F and G, respectively, orientable with respect to H^*. The isomorphisms of*

Theorem 28.1 identify the exterior product in $E_2^{*,*}$ like so:

$$\begin{tikzcd}
E_2^{*,*}(p) \otimes E_2^{*,*}(q) \arrow[rr, "?\bullet?"] \arrow[d, "\cong"'] & & E_2^{*,*}(p \times q) \arrow[d, "\cong"] \\
H^*(A; H^*(F)) \otimes H^*(B; H^*(G)) \arrow[rr] \arrow[drr, bend right=10, "?\bullet?"'] & & H^*(A \times B; H^*(F \times G)) \\
& & H^*(A \times B; H^*(F) \otimes H^*(G)). \arrow[u, "(?\bullet?)_*"']
\end{tikzcd}$$

The horizontal map in the middle line is often called the exterior product *with respect to the pairing $H^*(F) \otimes H^*(G) \to H^*(F \times G)$.*

The topological filtration quotient $(E_1 \times E_2)_{(s)}/(E_1 \times E_2)_{(s-1)}$ splits as a wedge with summands

$$(S^i \wedge F_+) \wedge (S^j \wedge G_+) = S^i \wedge S^j \rtimes F_+ \wedge G_+ = S^i \wedge S^j \rtimes (F \times G).$$

The proof of Theorem 28.11 amounts to proving a corresponding identification for the external product on $E_1^{*,*}(p \times q)$.

Problem 28.12.

(a) Show that the diagram

$$\begin{tikzcd}
H^{s+t}(X_{(s)/(s-1)}) \otimes H^{u+v}(Y_{(t)/(t-1)}) \arrow[r, equal] \arrow[d, "?\bullet?"'] & C^s(A; H^t(F)) \otimes C^u(B; H^v(G)) \arrow[d, "?\bullet?"] \\
H^{s+t+u+v}(X_{(s)/(s-1)} \wedge (Y_{(t)/(t-1)})) \arrow[d, "\text{summand}"'] & C^{s+u}(A \times B; H^t(F) \otimes H^v(G)) \arrow[d, "(?\bullet?)_*"] \\
H^{s+t+u+v}((X \times Y)_{(s+t)/(s+t-1)}) \arrow[r, equal] & C^{s+u}(A \times B; H^{t+v}(F \times G))
\end{tikzcd}$$

commutes.

(b) Prove Theorem 28.11.

28.3.2. Internalizing Using the Diagonal. We are more interested in producing an internal product. For this, we simply need to show that the Serre filtration is multiplicative.

Problem 28.13. Show that the Serre filtration on E is multiplicative.

It follows from our work in Section 27.4 that $E_2^{*,*}(p)$ inherits an algebra structure.

Theorem 28.14. *Let $p : E \to B$ be a fibration with fiber F, orientable with respect to H^*. The isomorphisms of Theorem 28.1 identify the external cup* *product in $E_2^{*,*}$ like so:*

$$\begin{tikzcd}
E_2^{*,*}(p) \otimes E_2^{*,*}(p) \arrow[r] \arrow[d, "\cong"'] & E_2^{*,*}(p) \arrow[d, "\cong"] \\
H^*(B; H^*(F)) \otimes H^*(B; H^*(F)) \arrow[r] \arrow[dr, "\text{cup}"'] & H^*(B; H^*(F)) \\
& H^*(B; H^*(F) \otimes H^*(F)). \arrow[u, "(\text{cup})_*"']
\end{tikzcd}$$

If coefficients are in a ring, so that H^* is a multiplicative cohomology theory, then the cup product $H^*(F) \otimes H^*(F) \to H^*(F \times F)$ can be replaced with the internal cup product and $E_2^{*,*}(p)$ becomes a bigraded R-algebra.

Problem 28.15. Prove Theorem 28.14.

28.4. Homology and the Serre Filtration

The Serre filtration can also be used to study the homology of the total space of a fibration. Since the reasoning is entirely analogous to what we have already done, we simply state the results.

Theorem 28.16. *Let $p : E \to B$ be a fibration with fiber F which is orientable over H_*. Then there is a natural isomorphism*

$$E^2_{s,t}(p) \cong H_s(B; H_t(F)).$$

Problem 28.17. Prove Theorem 28.16.

There are external products and edge homomorphisms for $E^2_{*,*}$.

Problem 28.18.

(a) Identify the external product in $E_2^{*,*}$ with the corresponding external homology products.

(b) Define the horizontal and edge homomorphisms and show that they can be identified with the maps induced by $p : E \to B$ and $i : F \to E$.

28.5. Additional Problems

Problem 28.19. Let $p : E \to B$ be a fibration with fiber F, and consider $E^2_{*,*}(\Omega p)$. Then all spaces involved are H-spaces and all the maps are H-maps. Define internal products

$$E^2_{*,*}(p) \otimes E^2_{*,*}(p) \longrightarrow E^2_{*,*}(p)$$

and identify them in terms of the Pontrjagin products in $H_*(\Omega E)$, $H_*(\Omega B)$ and $H_*(\Omega F)$.

Problem 28.20. Consider the pullback square

$$\begin{tikzcd} E_A \arrow[r] \arrow[d, "p_A"'] & E \arrow[d, "p"] \\ A \arrow[r] & B \end{tikzcd}$$

where p and p_A are fibrations with fiber F and $A \hookrightarrow B$ is a relative CW complex. Then B is obtained from A by repeated cell attachments, giving a filtration

$$A = B_0 \longrightarrow B_1 \to \cdots \to B_{s-1} \longrightarrow B_s \to \cdots \to B.$$

Pulling back this filtration gives a Serre-like filtration of E, and we can apply our apparatus to it. Show that $\mathrm{Gr}^s H^{s+t}(E/E_A)$ is a subquotient of

$$E_2^{s,t} \cong H^s(B/A; H^*(F)).$$

Can you define products?

Problem 28.21. Consider a map of one fibration to another:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd}$$

with induced map $F \to G$ of fibers. Show that there is a filtration of Y/X whose filtration quotients are $(B_s/B_{s-1}) \rtimes (G/F)$. Show that $\mathrm{Gr}^s H^{s+t}(Y/X)$ is a subquotient of

$$E_2^{s,t} \cong H^s(B; H^t(G/F)).$$

Can $E_2^{*,*}$ be given an algebra structure?

Problem 28.22.

(a) Show that $E_2^{*,*}(p)$ is naturally dependent on the groups $E_2^{*,0}(p)$ and $E_2^{0,*}(p)$, in the sense that if $p \to q$ is a map of fibrations, then there is a ladder of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & E_2^{s,0}(p) \otimes E_2^{0,t}(p) \arrow[r] \arrow[d, "f_X \otimes f_Y"] & E_2^{s,t}(p) \arrow[r] \arrow[d, "f"] & \mathrm{Tor}(E_2^{s+1,0}(p), E_2^{0,t}(p)) \arrow[r] \arrow[d, "\mathrm{Tor}(f_X, f_Y)"] & 0 \\ 0 \arrow[r] & E_2^{s,0}(q) \otimes E_2^{0,t}(q) \arrow[r] & E_2^{s,t}(q) \arrow[r] & \mathrm{Tor}(E_2^{s+1,0}(q), E_2^{0,t}(q)) \arrow[r] & 0. \end{tikzcd}$$

(b) If coefficients are in a field F, then $E_2^{s,t}(p) = E_2^{0,s}(p) \otimes E_2^{0,t}(p)$.

We will say that $E_2^{*,*}(p)$ admits a **universal coefficients decomposition**.

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Chapter 29

Application: Incompressibility

Before building even more layers of machinery for analyzing the Serre filtration, we take a break and use the tools we already have to prove a substantial and interesting theorem with many surprising corollaries.

A map $f : X \to Y$ is **compressible** if it factors, up to homotopy, through a finite-dimensional CW complex and f is called **incompressible** otherwise. In this chapter you will prove the following theorem [**177**].

Theorem 29.1 (Weingram). *Let $f : X \to Y$ where X is homotopy equivalent to a strictly associative H-space. If there is an n for which $H_{2n}(Y;\mathbb{Z})$ is finitely generated and the composite*

$$\pi_{2n}(X) \xrightarrow{\mathrm{H}} \widetilde{H}_{2n}(X;\mathbb{Z}) \xrightarrow{f_*} \widetilde{H}_{2n}(Y;\mathbb{Z})$$

is nonzero, then f is incompressible.

The statement of Theorem 29.1 is very interesting because H-spaces, which have target-type structure, are used in the domain and CW complexes, which have domain-type structure, are used in the target, and we are able to understand something important and useful about these mixed-up maps.

Theorem 29.1 makes it easy to derive surprising results about H-spaces and fibrations. For example, every fibration $p : E \to S^3$ whose fiber is a finite complex must have a section, and every finite complex which is an H-space must have even connectivity. You'll work through these consequences in Section 29.4. You will derive Theorem 29.1 from the following very special case.

Theorem 29.2 (Weingram)**.** *If G is a finitely generated abelian group and $n \geq 1$, then every nontrivial map $\Omega S^{2n+1} \to K(G, 2n)$ is incompressible.*

29.1. Homology of Eilenberg-Mac Lane Spaces

To prove Theorem 29.2, you will need to make use of a lot of the tools you have developed in the past few chapters. You will also need two bits of information about the homology of Eilenberg-Mac Lane spaces.

29.1.1. Exponents for $\widetilde{H}_*(K(\mathbb{Z}/p^r); G)$. An abelian p-group P has **exponent** p^e if $p^e P = 0$. Our proof of Theorem 29.2 will rely on the fact that the homology groups $\widetilde{H}_s(K(\mathbb{Z}/p^r); G)$ have exponents which are independent of the group G (though they may vary with s).

Problem 29.3. Show that for any $r, s > 0$, there is an integer ε_s such that $p^{\varepsilon_s} \cdot H_s(K(\mathbb{Z}/p^r, 2n); G) = 0$ for any coefficient group G.

Hint. Use Problem 25.135.

29.1.2. The Homology Algebra $H_*(K(\mathbb{Z}, 2n); \mathbb{Z})$. If G is an abelian group, then $K(G, 2n) = \Omega^2 K(G, 2n+2)$, which is a commutative group-like space. Hence $H_*(K(G, 2n); \mathbb{Z})$ has a Pontrjagin product giving its homology the structure of a commutative algebra. From the Hurewicz theorem, we know that $H_{2n}(K(\mathbb{Z}, 2n); \mathbb{Z}) \cong \mathbb{Z}$, so we may find generators

$$H_{2n}(K(\mathbb{Z}, 2n); \mathbb{Z}) \cong \mathbb{Z} \cdot \alpha \qquad \text{and} \qquad H^{2n}(K(\mathbb{Z}, 2n); \mathbb{Z}) \cong \mathbb{Z} \cdot \beta$$

that are (algebraically) dual to each other in the sense that $\langle \beta, \alpha \rangle = 1$.

Problem 29.4. Let α and β be as above.

(a) Show that β generates a polynomial subalgebra $\mathbb{Z}[\beta] \subseteq H^*(K(\mathbb{Z}, 2n); \mathbb{Z})$.
Hint. For each m, find a space X and a class $u \in H^{2n}(X; \mathbb{Z})$ such that u^m has infinite order.

(b) Show that $\langle \beta^m, \alpha^m \rangle = m!$ for each $m \geq 1$.
Hint. Use the naturality of the pairing $\langle\, ?\, ,\, ?\, \rangle$.

29.2. Reduction to Theorem 29.1

We begin our proof of Theorem 29.1 in this section by showing that it follows from the special case Theorem 29.2.

29.2.1. Compressible Maps. First, let's develop a little intuition for the notion of compressibility.

Exercise 29.5.

(a) Give an example of an incompressible map.

(b) Show that if either X or Y is a finite-dimensional CW complex, then every map from X to Y is compressible.

(c) Show that if Y is a CW complex, then $f : X \to Y$ is compressible if and only if $f \simeq \phi$, where $\phi(X) \subseteq Y_N$ for some $N \in \mathbb{N}$.

(d) Show that if f is compressible, then $f_* : H_*(X) \to H_*(Y)$ is trivial in all sufficiently high dimensions, and similarly for H^*.

(e) Show that if $f : X \to Y$ is compressible, then $\mathrm{CW}(f) : \mathrm{CW}(X) \to \mathrm{CW}(Y)$ is also compressible (where $\mathrm{CW}(\,?\,)$ is any CW replacement functor).

(f) Show that if $f : X \to Y$ and $g : Y \to Z$, then $f \circ g$ is compressible if either f or g is compressible.

(g) Give an example of incompressible maps $f : X \to Y$ and $g : Y \to Z$ whose composite is compressible.

(h) Suppose $f : X \to Y$ is compressible. What, if anything, does this tell you about $f_* : \pi_*(X) \to \pi_*(Y)$?

The result of Exercise 29.5(d) is the key to many simple proofs that maps are not compressible.

Problem 29.6.

(a) Let $f : K(\mathbb{Z}/p, n) \to X$ be any map, and assume that there is a class $u \in H^n(X; \mathbb{Z}/p)$ such that $f^*(u) \neq 0$. Show that f is not compressible.

(b) Let X be an H-space and let $f : X \to \Omega S^{2n+1}$ be an H-map. Show that if f_* is nonzero on $H_{2n}(\,?\,; \mathbb{Z})$, then f is not compressible.

29.2.2. The Reduction. In order to apply Theorem 29.2, we need to have maps $\Omega S^{2n+1} \to K(G, 2n)$. The James construction shows that ΩS^{2n+1} may be regarded as the free topological monoid on S^{2n}, so it is really enough to find maps $S^{2n} \to K(G, 2n)$. We produce such maps by composing homotopy element $\alpha \in \pi_{2n}(X)$ with a cohomology class $u \in \widetilde{H}^{2n}(X; G)$.

Lemma 29.7. *Let X be a CW complex of finite type. Let $\alpha \in \pi_n(X)$, and assume that its image $\mathrm{H}(\alpha) \in \widetilde{H}_*(X; \mathbb{Z})$ under the Hurewicz map is nonzero. Then there is a cohomology class $u \in \widetilde{H}^n(X; G)$, where G is finitely generated, such that $u \circ \alpha \in H^n(S^n; G)$ is nonzero.*

Problem 29.8. Let X be a CW complex of finite type and let $\alpha \in \pi_n(X)$ as in the statement of Lemma 29.7.

(a) Show that $\pi_n(X/X_{n-1})$ is a finitely generated (abelian) group.

(b) Show that $q \circ \alpha \neq 0 \in \pi_n(X/X_{n-1})$.

(c) Let $p : X/X_{n-1} \to K(\pi_n(X/X_{n-1}), n)$ be the n^{th} Postnikov section of X/X_{n-1}. Define $u = p \circ q$, and use it to prove Lemma 29.7.

Now we have the tools to reduce Theorem 29.1 to Theorem 29.2.

Problem 29.9. Derive Theorem 29.1 from Theorem 29.2.

Exercise 29.10. Show that the conclusion of Theorem 29.1 holds as long as one or the other (or both) of X and Y is homotopy equivalent to a CW complex.

29.2.3. Maps from ΩS^{2n+1} to $K(G, 2n)$. We'll get a start on the proof of Theorem 29.2 by investigating what special properties a map $\Omega S^{2n+1} \to K(G, 2n)$ must have.

Problem 29.11.

(a) Show that every map $S^{2n} \to K(G, 2n)$ extends to a unique H-map $\Omega S^{2n+1} \to K(G, 2n)$.

(b) Show that every map $\Omega S^{2n+1} \to K(G, 2n)$ is homotopic to an H-map.

(c) Show that if $f : \Omega S^{2n+1} \to K(G, 2n)$ is nontrivial, then it induces a nontrivial map on $\widetilde{H}_{2n}(\,?\,; \mathbb{Z})$.

Problem 29.12. Prove Theorem 29.2 in the case $G = \mathbb{Z}$.

29.3. Proof of Theorem 29.2

The first step in the proof of Theorem 29.2 is to reduce it to the special case $G = \mathbb{Z}/p^r$ for some r. To handle this case, consider the coefficient map $K(\mathbb{Z}, 2n) \to K(\mathbb{Z}/p^r, 2n)$, which we can take to be a fibration. You'll show that any map $f : \Omega S^{2n+1} \to K(\mathbb{Z}/p^r, 2n)$ can be lifted to a map $\phi : \Omega S^{2n+1} \to K(\mathbb{Z}, 2n)$ and that if f is compressible, then the image of ϕ_* must be contained in some finite stage of the Serre filtration on $H_*(K(\mathbb{Z}, 2n); \mathbb{Z})$.

Then we will use our knowledge of the algebras $H_*(K(\mathbb{Z}, 2n); \mathbb{Z})$ and $H_*(K(\mathbb{Z}/p^r, 2n); \mathbb{Z})$ to show that the image of ϕ_*, if it is nontrivial, cannot be contained in any finite stage of the Serre filtration. This contradiction will establish the theorem.

29.3.1. Reduction to the Case $G = \mathbb{Z}/p^r$. If G is a finitely generated abelian group, then $G \cong \prod C_k$ where each C_k is either $\mathbb{Z}$ or a cyclic group with prime power order. Thus $K(G, n) \simeq \prod K(C_k, n)$, which leads to the following lemma.

Lemma 29.13. *The following are equivalent:*

(1) there is a compressible map $\Omega S^{2n+1} \to K(G, 2n)$ where G is finitely generated,

(2) there is a compressible map $\Omega S^{2n+1} \to K(\mathbb{Z}/p^r, 2n)$ for some prime p and some $r \geq 1$.

Problem 29.14. Prove Lemma 29.13.

Lemma 29.13 implies that Theorem 29.2 will follow once we show that every nontrivial map $\Omega S^{2n+1} \to K(\mathbb{Z}/p^r, 2n)$ is incompressible.

29.3.2. Compressibility and the Serre Filtration. The key to the proof is to relate the notion of compressiblity to the Serre filtration.

Problem 29.15. Let $p : E \to B$ be a fibration, and let $f : X \to B$. Assume that there is a map $\phi : X \to E$ lifting f as in the diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, bend left, "\phi"] \arrow[r, "f"] & B. \end{tikzcd}$$

Let $\cdots \to E_{(s)} \to E_{(s+1)} \to \cdots \to E$ be the Serre filtration corresponding to the CW filtration of B. Show that f is compressible if and only if for some N, there is a map $\phi_{(N)} : X \to E_{(N)}$ making the diagram

$$\begin{tikzcd} & E_{(N)} \arrow[r] & E \\ X \arrow[ur, bend left, "\phi_{(N)}"] \arrow[urr, "\phi"'] & & \end{tikzcd}$$

commute up to homotopy. Conclude that if f is compressible, then

$$\phi_*(H_*(X)) \subseteq \mathrm{Im}\left(H_*(E_{(N)}) \to H_*(E)\right) = \mathcal{F}_N(H_*(E))$$

for some N.

We want to apply this result to the map $f : \Omega S^{2n+1} \to K(\mathbb{Z}/p^r, 2n)$, which means that we need a fibration with base $K(\mathbb{Z}/p^r, 2n)$. The short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p^r \to 0$ gives rise to a fibration sequence

$$K(\mathbb{Z}, 2n) \xrightarrow{j} K(\mathbb{Z}, 2n) \xrightarrow{q} K(\mathbb{Z}/p^r, 2n)$$

where j is the map induced by multiplication by p^r. This is the fibration we want. As written, the notation here is confusing, because the fiber and total space are denote by the same symbol, $K(\mathbb{Z}, 2n)$. For clarity, for the rest of this chapter we'll use K to denote the total space and F for the fiber.

Exercise 29.16. Let $x \in H_{2n}(K; R)$, where R is any ring at all.

(a) Show that $x \in \mathrm{Im}(j_*)$ if and only if x is divisible by p^r.

(b) Show that $q_*(x) \neq 0$ if and only if x is not divisible by p^r.

If we fix a CW decomposition of $K(\mathbb{Z}/p^r, 2n)$, then we get a corresponding Serre filtration $\cdots \to K_{(s)} \to K_{(s+1)} \to \cdots \to K$. We will assume that our CW decomposition of $K(\mathbb{Z}/p^r, 2n)$ has only a single 0-cell, so that $K_{(0)} = F$, the fiber of q. Thus, we want to study the diagram

$$\begin{tikzcd} & K \arrow[d, "q"] \\ \Omega S^{2n+1} \arrow[ur, dotted, "\phi"] \arrow[r, "f"] & K(\mathbb{Z}/p^r, 2n). \end{tikzcd}$$

Problem 29.17.

(a) Show that the map ϕ exists and that it induces a nonzero map on H_{2n}.

(b) Show that ϕ is an H-map.

The topological Serre filtration of K is used to define an algebraic filtration of $H_*(K) = H_*(K;\mathbb{Z})$ given by $\mathcal{F}_s H_*(K) = \mathrm{Im}(H_*(K_{(s)}) \to H_*(K))$. From now on, we'll write $\mathcal{F}_s$ for $\mathcal{F}_s(H_*(K))$.

Lemma 29.18. *For each N, there is an integer $\kappa = \kappa(N)$ such that*

$$p^\kappa \mathcal{F}_N(H_*(K)) \subseteq F_0(H_*(K)).$$

The crucial content of Lemma 29.18 is that the integer κ is *independent of the dimension* $*$.

Problem 29.19. Use homology with coefficients in any ring at all.

(a) Show that for each s, there is an integer κ_s such that $p^{\kappa_s}\,\mathrm{Gr}_s(H_*(K)) = 0$ for some e.

(b) Show that $p^{\kappa_s}\mathcal{F}_s \subseteq \mathcal{F}_{s-1}$.

(c) Finish the proof of Lemma 29.18.

29.3.3. Consequences of Membership in F_0. Now we turn to the problem of characterizing the zeroth filtration $\mathcal{F}_0 = \mathrm{Im}\,(j_* : H_*(F) \to H_*(K))$. For $m \in \mathbb{N}$, write $e_p(m)$ for the largest power of p which divides $m!$. A standard theorem of elementary number theory, known as *de Polignac's formula* [**140**], states that

$$e_p(m) = \sum_{k\geq 1} \left\lfloor \frac{m}{p^k} \right\rfloor .$$

Problem 29.20. Show that the fibration sequence $F \xrightarrow{j} K \xrightarrow{q} K(\mathbb{Z}/p^r, 2n)$ is the loops of another fibration sequence.

Lemma 29.21. *Suppose $u \in \mathbb{Z}$ is not divisible by p. Then if $u \cdot p^t\,\alpha^m \in \mathcal{F}_0$, then $t \geq rm + e_p(m)$.*

Problem 29.22. Let $\alpha \in H_{2n}(F)$ and $\beta \in H^{2n}(K)$ be the generators of Section 29.1.2, and suppose there is a z such that $j_*(z) = u \cdot p^t\,\alpha^m$.

(a) What are $j^*(\beta^m)$ and $j_*(\alpha^m)$?

(b) Prove Lemma 29.21 by comparing $\langle \beta^k, j_*(z)\rangle$ and $\langle j^*(\beta^k), z\rangle$.

29.3.4. Completing the Proof. Now you are ready to finish the proof of Theorem 29.2. We of course continue to use the notation and setup of the previous problems.

Problem 29.23. If $f : \Omega S^{2n+1} \to K(\mathbb{Z}/p^r, 2n)$ is a compressible map, then f has a factorization (up to homotopy) of the form

$$\Omega S^{2n+1} \longrightarrow K_{(N)} \longrightarrow K \longrightarrow K(\mathbb{Z}/p^r, 2n)$$

for some N. We write ϕ for the composite $\Omega S^{2n+1} \to K$ and $\kappa = \kappa(N)$ for the number guaranteed by Lemma 29.18.

(a) Show that $\mathcal{F}_0(H_*(K))$ is a subalgebra of $H_*(K)$.

(b) Show that if f is nontrivial, then $e_p(m) \geq m - \kappa$ for all $m \in \mathbb{N}$.

HINT. If $x \in H_{2n}(\Omega S^{2n+1}; \mathbb{Z})$, then $\phi_*(x) = p^s u\alpha$ for some $s < r$ and some $u \in \mathbb{Z}$ not divisible by p.

(c) Prove Theorem 29.2.

HINT. Consider $m = p^z - 1$.

29.4. Consequences of Theorem 29.1

In this section, you will use Theorem 29.1 to derive a number of consequences. First, you will find a condition on the connectivity of a finite-dimensional associative H-space. Then you will answer the question that originally motivated the theorem: must every fibration $p : E \to S^3$ have a section?

29.4.1. The Connectivity of a Finite H-Spaces. Since the Hurewicz map is guaranteed to be nontrivial in dimension $\mathrm{conn}_{\mathcal{P}}(X) + 1$, we have some leverage for the production of maps in these dimensions.

Theorem 29.24. *Let X be a finite CW complex which is an associative H-space. Then the image of the Hurewicz map is zero in all even dimensions.*

Corollary 29.25. *If X is a finite-dimensional associative H-space and $\mathcal{P}$ is a collection of primes, then the $\mathcal{P}$-connectivity of X is even.*

Corollary 29.25 includes the corresponding statement for ordinary connectivity, which is $\mathcal{P}$-connectivity with $\mathcal{P} = \{\text{all primes}\}$.

Problem 29.26. Prove Theorem 29.24 and derive Corollary 29.25.

HINT. Show that a map $S^{2n} \to X$ extends to a map $\Omega S^{2n+1} \to X$.

29.4.2. Sections of Fibrations over Spheres. D. Gottlieb [**71**] asked whether every fibration $p : E \to S^3$ with a finite-dimensional fiber F must have a section. Weingram proved Theorem 29.2 as a tool for answering this question, so it is fitting that we end our discussion of compressibility by resolving it.

Corollary 29.27. *Let $F \to E \xrightarrow{p} S^3$ be a fibration sequence, where F is a finite, simply-connected CW complex. Then p has a section.*

The more general case, in which F is not necessarily simply-connected, is also true, but the argument is more complicated (see Project 29.30). This corollary leaves open the possibility of the existence of fibrations over S^3 with infinite fibers and no section.

Before proving Corollary 29.27, we study the obstruction to the existence of a section of a fibration over a sphere.

Problem 29.28. Let $F \to E \xrightarrow{p} S^3$ be a fibration sequence, with $n \geq 2$.

(a) Show that p has a section if and only if $p_* : \pi_*(E) \to \pi_*(S^n)$ is onto.

(b) Extend the fibration to obtain the fiber sequence

$$\cdots \to \Omega S^n \xrightarrow{\partial} F \longrightarrow E \xrightarrow{p} S^n.$$

Let $\alpha \in \pi_{n-1}(\Omega S^n)$ be the homotopy class of the canonical map $S^{n-1} \to \Omega S^n$. Show that p has a section if and only if $\partial_*(\alpha) = 0$ (the class $\partial_*(\alpha)$ is called the **obstruction** to the existence of a section).

Problem 29.29. Prove Corollary 29.27.

29.5. Additional Problems and Projects

Project 29.30. Prove Corollary 29.27 with the 'simply-connected' hypothesis replaced with simply 'connected'.

HINT. Form a new fibration in which each fiber is the universal cover of the given fiber; then apply the result as stated to the new fibration.

Project 29.31. What can you say about fibrations over S^3 which have no section? What can you say about their fibers? Can they be simply-connected?

Problem 29.32. Let X be $2n$-connected, and let $f : \Omega X \to Y$, where Y is finite-dimensional. Show that $f_* : \pi_{2n}(\Omega X) \to \pi_{2n}(Y)$ must be the zero homomorphism.

Problem 29.33. Let $f : \Omega S^{2n+1} \to X$ be a map, where $H_{2n}(X)$ is finitely generated, and suppose $f_* : H_{2n}(\Omega S^{2n+1}) \to H_{2n}(X)$ is nontrivial. Show that X is not finite-dimensional.

Problem 29.34. Generalize Corollary 29.27 to obtain conditions under which a fibration $p : E \to S^{2n-1}$ with fiber F must have a section.

Problem 29.35. Show that $\beta^m \in H^{2nm}(K(\mathbb{Z}, 2n); \mathbb{Z})$ is not divisible by any nonunit in $\mathbb{Z}$.

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Chapter 30

The Spectral Sequence of a Filtered Space

In this chapter we resume our study of the cohomology of a generic filtered space X. We have shown that the topological filtration on X gives rise to an algebraic filtration of $\widetilde{H}^*(X)$ and that the filtration quotients $\mathrm{Gr}^s\,\widetilde{H}^n(X)$ can be approximated by groups $E_1^{s,n}(X)$. These groups constitute a bigraded chain complex, and their algebraic homology $E_2^{s,n}(X) = H^{s,n}(E_1^{*,*})$ is an even better approximation to $\mathrm{Gr}^s\,\widetilde{H}^n(X)$.

Now we improve this second approximation by constructing differentials $d_2 : E_2^{s,n} \to E_2^{s+2,n+1}$ that make $E_2^{*,*}(X)$ into a bigraded chain complex and showing that $\mathrm{Gr}^s\,\widetilde{H}^n(X)$ is a subquotient of the algebraic cohomology group $E_3^{s,n}(X) = H^{s,n}(E_2^{*,*})$. More generally, we construct, for each $r \geq 1$, approximations $E_r^{*,*}(X)$ for $\mathrm{Gr}^*\,\widetilde{H}^*(X)$ and differentials d_r in such a way that $E_{r+1}^{*,*}(X) = H^{*,*}(E_r^{*,*})$ is a better (but not necessarily *strictly* better) estimate for $\mathrm{Gr}^*\,\widetilde{H}^*(X)$ than $E_r^{*,*}$.

Thus our program to understand the cohomology of a filtered space has led us to a new kind of algebraic gadget: an infinite sequence of bigraded chain complexes $E_r^{*,*}(X)$ for $r \geq 1$, related to one another by $E_{r+1}^{*,*}(X) = H^{*,*}(E_r^{*,*})$. Such an apparatus is called a **spectral sequence**. We take a brief look at the algebra of generic spectral sequences, including defining the limit of a spectral sequence, and setting down the definition of a spectral sequence of algebras.

We end the chapter with an omnibus theorem collecting together everything we have learned about the spectral sequences of filtered spaces, including their multiplicative structure and convergence.

30.1. Approximating $\operatorname{Gr}^s \widetilde{H}^n(X)$ by $E_r^{s,n}(X)$

For notational simplicity, in this section we will suppress the coefficients G. Thus we will write $\widetilde{H}^*(X)$ instead of $\widetilde{H}^*(X;G)$ and $u : X \to K_n$ instead of $u : X \to K(G,n)$.

In Section 27.2.1 you showed that if X is a filtered pointed space, then $\widetilde{H}^n(X_{(s)})$ contains a subgroup isomorphic to $\operatorname{Gr}^s \widetilde{H}^n(X)$ and that this subgroup is contained in the image of $q_s^* : \widetilde{H}^n(X_{(s)}/X_{(s-1)}) \to \widetilde{H}^n(X_{(s)})$. Using the Correspondence Theorem, you found a sequence of subgroups $\mathcal{B}^{s,n} \subseteq \mathcal{Z}^{s,n} \subseteq \widetilde{H}^n(X_{(s)}/X_{(s-1)})$ such that $\mathcal{Z}^{s,n}/\mathcal{B}^{s,n} \cong \operatorname{Gr}^s \widetilde{H}^n(X)$.

Our goal is to approximate $\operatorname{Gr}^s \widetilde{H}^n(X) \cong \mathcal{Z}^{s,n}/\mathcal{B}^{s,n}$ by estimating $\mathcal{Z}^{s,n}$ from above and $\mathcal{B}^{s,n}$ from below. Since

$$\mathcal{Z}^{s,n} \subseteq \left\{ v : \Sigma A_{s-1} \to K_n \;\middle|\; \begin{array}{l} v_{(s)} : X_{(s)} \to K_n \text{ has extensions} \\ v_{(s+m)} : X_{(s+m)} \to K_n \text{ for each } 0 \le m < \infty \end{array} \right\},$$

we have chosen to approximate $\mathcal{Z}^{s,n}$ by the subgroups

$$Z_r^{s,n} = \left\{ v : \Sigma A_{s-1} \to K_n \;\middle|\; \begin{array}{l} v_{(s)} : X_{(s)} \to K_n \text{ has extensions} \\ v_{(s+m)} : X_{(s+m)} \to K_n \text{ for each } 0 \le m \le r \end{array} \right\}.$$

On the other side, we approximate

$$\mathcal{B}^{s,n} = \{(\Sigma\alpha_s)^*(\Sigma u) \mid u : X_{s-1} \to K_{n-1}\}$$

(where Σu denotes the image of u under the suspension isomorphism in cohomology) by

$$\widehat{B}_r^{s,n} = \{(\Sigma\alpha_s)^*(\Sigma u) \mid u : X_{s-1} \to K_{n-1} \text{ and } u|_{X_{(s-r-1)}} = 0\}.$$

Thus there is a chain of subgroups

$$\widehat{B}_1^{s,n} \subseteq \widehat{B}_2^{s,n} \subseteq \cdots \subseteq \widehat{B}_r^{s,n} \subseteq \mathcal{B}^{s,n} \subseteq \mathcal{Z}^{s,n} \subseteq \widehat{Z}_r^{s,n} \subseteq \cdots \subseteq \widehat{Z}_2^{s,n} \subseteq \widehat{Z}_1^{s,n},$$

and $\operatorname{Gr}^s H^n(X) \cong \mathcal{Z}^{s,n}/\mathcal{B}^{s,n}$ is a subquotient of $E_r^{s,n} = Z_{r-1}^{s,n}/B_{r-1}^{s,n}$ for each r, and these approximations improve as r increases.

The challenge is to describe the groups $\widehat{Z}_r^{s,n}$ and $\widehat{B}_r^{s,n}$ in useful ways. Our first approximation ($r = 0$) was given by $0 \subseteq \mathcal{B}^{s,n}$ and $\mathcal{Z}^{s,n} \subseteq E_1^{s,n}$. Then we introduced the differential $d_1 = \alpha \circ q$ and showed that

$$\widehat{B}_1^{s,n} = \operatorname{Im}(d_1) \subseteq \mathcal{B}^{s,n} \qquad \text{and} \qquad \mathcal{Z}^{s,n} \subseteq \widehat{Z}_1^{s,n} = \ker(d_1),$$

so that $E_2^{s,n} = H^{s,n}(E_1^{*,*})$ is a better approximation for $\operatorname{Gr}^s \widetilde{H}^n(X)$. In this section we define differentials d_r in $E_r^{*,*}$ such that $Z_r^{*,*} = \ker(d_r)$ and $B_r^{*,*} = \operatorname{Im}(d_r)$ correspond, via the Correspondence Theorem, to $\widehat{Z}_r^{*,*}$ and $\widehat{B}_r^{*,*}$, respectively. The situation is expressed clearly by the Correspondence

Theorem diagram

$$\begin{tikzcd}
E_1 \arrow[d, dash] & & \\
\widehat{Z}_{r-1} \arrow[r, dash] \arrow[d, dash] & E_r \arrow[d, dash] & \\
\widehat{Z}_r \arrow[r, dash] \arrow[d, dash] & Z_r \arrow[r, dash] \arrow[d, dash] & E_{r+1} \arrow[d, dash] \\
\widehat{B}_r \arrow[r, dash] \arrow[d, dash] & B_r \arrow[r, dash] \arrow[d, dash] & 0 \\
\widehat{B}_{r-1} \arrow[r, dash] \arrow[d, dash] & 0 & \\
0. & &
\end{tikzcd}$$

In this way we build an infinite sequence of increasingly accurate approximations $E_r^{*,*}(X)$ to $\mathrm{Gr}^s\,\widetilde{H}^n(X)$, each of which is the (algebraic) homology of the previous one: $E_{r+1}^{*,*} = H^{*,*}(E_r^{*,*})$. Such an algebraic gadget is called a **spectral sequence**.

30.1.1. Topological Description of d_r. We begin by giving a topological construction of the differentials d_r on $E_r^{*,*}(X)$. Just as in Section 27.2, we work with a cone filtration for clarity. Thus we have a space X with a cone decomposition

$$\begin{tikzcd}
 & A_s \arrow[d, "\alpha_s"'] & A_{s+1} \arrow[d, "\alpha_{s+1}"'] & & A_{s+r-1} \arrow[d, "\alpha_{s+r-1}"] & A_{s+r} \arrow[d, "\alpha_{s+r}"] & \\
\cdots \arrow[r, "j_{s-1}"] & X_{(s)} \arrow[r, "j_s"] \arrow[d, "q_s"'] & X_{(s+1)} \arrow[r, "j_{s+1}"] \arrow[d, "q_{s+1}"'] & \cdots \arrow[r] & X_{(s+r-1)} \arrow[r, "j_{s+r-1}"] \arrow[d, "q_{s+r-1}"] & X_{(s+r)} \arrow[r, "j_{s+r}"] \arrow[d, "q_{s+r}"] & \cdots \\
 & \Sigma A_{s-1} & \Sigma A_s & & \Sigma A_{s+r-2} & \Sigma A_{s+r-1}. &
\end{tikzcd}$$

A map $v : \Sigma A_{s-1} \to K_n$ gives rise by composition to a map $v_{(s)} : X_{(s)} \to K_n$, and we are attempting to extend $v_{(s)}$ to maps $v_{(s+r)} : X_{(s+r)} \to K_n$.

By definition, $v \in \widehat{Z}_{r-1}^{s,n}(X)$ if and only if there is at least one extension $v_{(s+r-1)} : X_{(s+r-1)} \to K_n$. The differential d_r is induced by the highly indexed function

$$\delta_r : \widehat{Z}_{r-1}^{s,n} \longrightarrow \widehat{Z}_{r-1}^{s+r-1,n+1} / \widehat{B}_{r-1}^{s+r-1,n+1}$$

given by the simple rule

$$\delta_r : v \longmapsto \left[\Sigma\left(v_{(s+r-1)} \circ \alpha_{s+r-1}\right)\right],$$

where Σ denotes the isomorphism $\Sigma : \widetilde{H}^n(A_{s+r}) \xrightarrow{\cong} \widetilde{H}^{n+1}(\Sigma A_{s+r})$ and brackets denote cosets of the subgroup $B_{r-1}^{*,*}$.

A Well-Defined Differential. Since there is ambiguity in the choice of $v_{(s+r)}$, our first task is to show that δ_r is well-defined. Once this is done, we show that, as promised, δ_r induces a map $d_r : E_r^{*,*} \to E_r^{*,*}$ and finally verify that $d_r \circ d_r = 0$.

Problem 30.1. Let $v_{(s+r)}, \bar{v}_{(s+r)} : X_{(s+r-1)} \to K_n$ be two extensions of $v_{(s)}$.

(a) Show that $(v_{(s+r)} - \bar{v}_{(s+r)})|_{X_{(s)}} = 0$.

(b) Show that there is a map $w : \Sigma A_s \to K_n$ such that

$$\begin{tikzcd} X_{(s+1)} \arrow[r] \arrow[d] & X_{(s+r)} \arrow[d, "v_{(s+r)}-\bar{v}_{(s+r-1)}"] \\ \Sigma A_s \arrow[r, "w"] & K_n \end{tikzcd}$$

commutes up to homotopy.

(c) Conclude that δ_r is well-defined.

Now we show δ_r induces a differential $d_r : E_r^{s,n}(X) \to E_r^{s+r-1,n+1}(X)$.

Problem 30.2.

(a) Show that if $v \in B_{r-1}^{s,n}(X)$, then $\delta_r(v) = 0$.

(b) Show that $d_r \circ d_r = 0$.

To finish our construction, we need to show that the kernel and image of d_r correspond to the subgroups $\widehat{Z}_r^{*,*}(X)$ and $\widehat{B}_r^{*,*}(X)$ of $E_1^{*,*}$.

Identifying the Kernel. We begin by studying $Z_r^{*,*}(X) = \ker(d_r)$ and the corresponding subgroup in $E_1^{*,*}$.

Problem 30.3. Show that $\ker(\delta_r)$ is the subgroup of $\widehat{Z}_{r-1}^{s,n}(X)$ corresponding to $\ker(d_r)$.

Now it remains to show that $\widehat{Z}_r^{*,*} = \ker(d_r)$.

Problem 30.4. Assume that $v \in \widehat{Z}_{r-1}^{s,n}$, so that $v_{(s)}$ has an extension $v_{(s+r-1)}$.

(a) Suppose $v_{(s)}$ extends to $v_{(s+r)}$, and let $\bar{v}_{(s+r-1)} = v_{(s+r)}|_{X_{(s+r-1)}}$. Show that $(v_{(s+r)} - \bar{v}_{(s+r)})_{X_{(s)}} = 0$, and conclude that there is an element $w \in \widehat{Z}_{r-2}^{s+1,n}$ such that $\delta_r(v) = \delta_{r-1}(w)$.

(b) Show that if $\delta_r(v) = \delta_{r-1}(w)$ for some $w \in \widehat{Z}_{r-2}^{s+1,n}(X)$, then $v_{(s)}$ can be extended to a map $v_{(s+r)}$.

(c) Show that $\ker(\delta_r) = \widehat{Z}_r^{s,n}(X)$.

Identifying the Image. We conclude our construction by showing that $B_r^{s,n} = \mathrm{Im}(d_r) \subseteq E_r^{s,n}$ corresponds to $\widehat{B}_r^{s,n} \subseteq \widehat{Z}_{r-1}^{s,n}$. For this, we introduce the notation $\chi : \widehat{Z}_{r-1}^{s,n} \to E_r^{s,n}$ for the canonical quotient map. Now we can clearly express our goal: we have to show that

$$\widehat{B}_r^{s,n}(X) = \chi^{-1}(B_r^{s,n}).$$

We'll handle the two containments separately.

Problem 30.5. Show that $v : \Sigma A_{s-1} \to K_n$ is an element of $\widehat{B}_r^{s,n}$ if and only if there are maps $u : X_{(s-1)} \to K_{n-1}$ and $w : \Sigma A_{s-r-1} \to K_{n-1}$ making the diagram

$$\begin{tikzcd}
 & A_{s-1} \arrow[d, "\alpha_{s-1}"'] \arrow[dd, bend left=60, "\Sigma^{-1}v"] \\
X_{(s-r)} \arrow[r] \arrow[d, "q_{s-r}"'] & X_{(s-1)} \arrow[d, "u"'] \\
\Sigma A_{s-r-1} \arrow[r, "w"] & K_{n-1}
\end{tikzcd}$$

commute up to homotopy.

Problem 30.6. Let $v \in \widehat{B}_r^{s,n}(X)$, and choose maps $u : X_{(s-1)} \to K_{n-1}$ and $w : \Sigma A_{s-r-1} \to K_{n-1}$ as in Problem 30.5.

(a) Show that $\chi(v) = \delta_r(w)$.

(b) Conclude that $\widehat{B}_r^{s,n} \subseteq \chi^{-1}(B_r^{s,n})$.

Problem 30.7. Suppose $\chi(v) = d_r(w)$ for some $w : \Sigma A_{s-r-1} \to K_{n-1}$.

(a) Show that there are elements $z \in \widehat{B}_{r-1}^{s,n}$ and $u : X_{s-1} \to K_{n-1}$ making the diagram

$$\begin{tikzcd}
 & A_{s-1} \arrow[d, "\alpha_{s-1}"'] \arrow[dd, bend left=60, "\Sigma^{-1}(v+z)"] \\
X_{(s-r)} \arrow[r] \arrow[d, "q_{s-r}"'] & X_{(s-1)} \arrow[d, "u"'] \\
\Sigma A_{s-r-1} \arrow[r, "w"] & K_{n-1}
\end{tikzcd}$$

commute up to homotopy.

(b) Show that $\widehat{B}_{r-1}^{s,n} \subseteq \chi^{-1}(B_r^{s,n})$, and conclude that $v \in \widehat{B}_r^{s,n}$.

30.1.2. The Algebraic Approach. Now we rederive the results of Section 30.1.1, this time making use of the algebraic language and notation of exact couples. While this makes the topological content a little more obscure, it both streamlines the derivation and makes it plain that it applies to all filtered spaces, not just the cone filtrations.

In Section 27.3.1 we saw that the vast web of exact sequences that result from a filtration can be collapsed into an **exact couple**: a triangle

$$\begin{tikzcd} & E_1^{*,*} \arrow[dr, "q"] & \\ D_1^{*,*} \arrow[ur, "\alpha"] & & D_1^{*,*} \arrow[ll, "j"'] \end{tikzcd}$$

which is exact at each vertex. In this diagram

$$D_1^{s,n} = \widetilde{H}^n(X_{(s)}) \qquad \text{and} \qquad E_1^{s,n} = \widetilde{H}^n(X_{(s)}/X_{(s-1)}),$$

and the maps are induced by $j_s : X_{(s)} \to X_{(s+1)}$, $q_s : X_{(s)} \to X_{(s)}/X_{(s-1)}$ and $\alpha_s : X_{(s+1)}/X_{(s)} \to \Sigma X_{(s)}$.

For the rest of this section we abandon all references to topology, and work with the purely algebraic gadget. We want to approximate to the groups

$$\mathcal{Z}^{s,n} = q^{-1}\left(\bigcap\nolimits_{m\geq 0} \mathrm{Im}(j^m)\right) \qquad \text{and} \qquad \mathcal{B}^{s,n} = \mathrm{Im}(\alpha),$$

with an eye toward estimating the quotients $\mathcal{Z}^{s,n}/\mathcal{B}^{s,n}$. Just as in Section 30.1.1, the approximation is entirely straightforward: we estimate $\mathcal{Z}^{s,n}$ by $\widehat{Z}_r^{s,n} = q^{-1}(\mathrm{Im}(j^{r-1}))$ and $\mathcal{B}^{s,n}$ by $\widehat{B}_r^{s,n} = \alpha(\ker(j^r))$. These groups fit into a chain

$$\widehat{B}_1^{s,n} \subseteq \widehat{B}_2^{s,n} \subseteq \cdots \subseteq \widehat{B}_r^{s,n} \subseteq \mathcal{B}^{s,n} \subseteq \mathcal{Z}^{s,n} \subseteq \widehat{Z}_r^{s,n} \subseteq \cdots \subseteq \widehat{Z}_2^{s,n} \subseteq \widehat{Z}_1^{s,n},$$

and we see immediately that the groups $E_r^{s,n} = Z_{r-1}^{s,n}/B_{r-1}^{s,n}$ are a sequence of increasingly accurate approximations of $\mathcal{Z}^{s,n}/\mathcal{B}^{s,n}$.

The challenge is to describe these groups in a way that makes them computable. We do this by introducing differentials $d_r : E_r^{*,*} \to E_r^{*,*}$ given by the fomula

$$d_r([a]) = [\alpha \circ j^{-(r-1)} \circ q(a)].$$

Before anything else, we have to show that d_r is a well-defined differential.

Problem 30.8. Let $[a] \in E_r^{s,n} = H^{s,n}(E_{r-1}^{*,*})$.

(a) Show that $q(a) \in \mathrm{Im}(j^{r-1})$.

(b) Show that if $[a] = [\bar{a}]$, then $q(a) = q(\bar{a})$.

(c) Show that if $b \in j^{-(r-1)}(q(a))$, then $\alpha(b) \in Z_{r-1}^{s+r,n+1}$.

(d) Show that if $b, \bar{b} \in j^{-(r-1)}(q(a))$, then $j^{r-2}(b - \bar{b}) = q(z)$ for some $z \in Z_{r-2}^{s+1,n}$, and conclude that $\alpha(b) - \alpha(\bar{b}) \in B_{r-1}^{s+r,n+1}$.

(e) Deduce that d_r is a well-defined map $E_r^{*,*} \to E_r^{*,*}$.

(f) Show that $d_r \circ d_r = 0$.

Next we have to show that the kernel and image of d_r behave properly. Write $Z_r^{s,n} = \ker(d_r) \subseteq E_r^{s,n}$ and $B_r^{s,n} = \mathrm{Im}(d_r) \subseteq E_r^{s,n}$.

Problem 30.9. Show that

(a) $\widehat{Z}_r^{s,n} \subseteq \widehat{Z}_{r-1}^{s,n}$ corresponds to $Z_r^{s,n} \subseteq E_r^{s,n}$ and

(b) $\widehat{B}_r^{s,n} \subseteq \widehat{Z}_{r-1}^{s,n}$ corresponds to $B_r^{s,n} \subseteq E_r^{s,n}$

under the composite homomorphism $\widehat{Z}_{r-1}^{s,n} \to Z_{r-1}^{s,n} \to E_r^{s,n}$.

30.2. Some Algebra of Spectral Sequences

In this section we make a brief study of spectral sequences in their own right. We discuss morphisms of spectral sequences, multiplicative structures and convergence.

30.2.1. The Category of Spectral Sequences. There is a category $\mathbf{SS}_R$ of spectral sequences of R-modules. The objects, of course, are spectral sequences. The morphisms are a bit complicated to describe.

A chain map $f_r : E_r^{*,*} \to F_r^{*,*}$ (of bidegree $(0,0)$) induces a map $f_{r+1} = (f_r)_* : E_{r+1}^{*,*} \to F_{r+1}^{*,*}$. The map f_{r+1} need not commute with the differential d_{r+1}, but if it does, then we can define f_{r+2}, and so on. A **homomorphism** $f : E \to F$ **of spectral sequences** is a sequence of chain maps $f_r : E_r^{*,*} \to F_r^{*,*}$ such that, for all $r \geq 1$, $f_{r+1} = (f_r)_*$ under the isomorphisms $E_{r+1}^{*,*} \cong H^{*,*}(E_r^{*,*})$ and $F_{r+1}^{*,*} \cong H^{*,*}(F_r^{*,*})$.

Problem 30.10. Let $f : E \to F$ be a homomorphism of spectral sequences. Show that if f_r is an isomorphism (or zero) for some r, then f_s is an isomorphism (or zero) for all $s \geq r$.

30.2.2. Exact Couples and Filtered Modules. There are two important algebraic situations in which spectral sequences arise.

Exact Couples. We have already seen that every exact couple

$$\mathcal{C}: \quad \begin{tikzcd} & E \arrow[dr, "q"] & \\ D \arrow[ur, "\alpha"] & & D \arrow[ll, "j"] \end{tikzcd}$$

gives rise, in a natural way, to a spectral sequence. We'll give another, much slicker, derivation of the same spectral sequence using the concept of a derived exact couple.

The exactness of the triangle implies that $d = \alpha \circ q : E \to E$ is a differential. The **derived** exact couple is the exact couple

$$\mathcal{C}' : \quad \begin{tikzcd} & E' \arrow[dr, "q'"] & \\ D' \arrow[ur, "\alpha'"] & & D' \arrow[ll, "j'"] \end{tikzcd}$$

defined by setting

$$E' = H(E) \qquad \text{and} \qquad D' = \mathrm{Im}(j).$$

Problem 30.11. Define maps q', j' and α' making $\mathcal{C}'$ into an exact couple.

HINT. D' is both the subgroup $\mathrm{Im}(\beta) \subseteq D$ and isomorphic to $D/\,\mathrm{Im}(q)$.

Since $\mathcal{C}'$ is an exact couple, we can iterate the procedure, producing exact couples $\mathcal{C}''$, $\mathcal{C}'''$, etc.; now it is clearly time to improve our notation. Starting with the exact couple $(\mathcal{C})$, we obtain an infinite sequence $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \ldots,$ of exact couples with $\mathcal{C}^{(r+1)} = (\mathcal{C}^{(r)})'$ for each r. Writing

$$\mathcal{C}^{(r)} : \quad \begin{tikzcd} & E_r \arrow[dr, "q_r"] & \\ D_r \arrow[ur, "\alpha_r"] & & D_r \arrow[ll, "j_r"] \end{tikzcd}$$

for the r^{th} exact couple in the sequence, we have see that we have constructed a spectral sequence $E_* = \{(E_r, \alpha_r \circ q_r) \mid r \geq 1\}$.

Problem 30.12.

(a) Set up the category $\mathbf{EC}_R$ of exact couples of R-modules, and show that this construction defines a functor $\mathbf{EC}_R \to \mathbf{SS}_R$.

(b) Show that the spectral sequence constructed in this way is naturally isomorphic to the spectral sequence built with the construction of Section 30.1.2.

In this exposition of spectral sequences we have made no mention whatsoever of grading, let alone bigrading. It is almost universally true that the manipulation of spectral sequences involves keeping careful track of multiple indices, but the indices should be thought of as helpful tools, rather than fundamental to the concept of a spectral sequence.

Filtered Differential Modules. In abstract algebra, spectral sequences are often constructed from filtered chain complexes. We'll show that a filtered chain complex gives rise to an exact couple, and then we're off and running, using the construction just outlined above. This approach has the drawback that it pulls you deeper into the mucky details, but this is offset by the greater ability to manipulate and understand the resulting sequences, or by an affection for mucky details.

Let M be a differential R module with a descending filtration (an analogous construction works for ascending filtrations)

$$M \supseteq \cdots \supseteq M_{(s)} \supseteq M_{(s-1)} \supseteq \cdots$$

by differential submodules. Then we have short exact sequences

$$0 \to M_{(s-1)} \longrightarrow M_{(s)} \longrightarrow M_{(s)}/M_{(s-1)} \to 0$$

which lead—by virtue of Theorem A.27—to long exact sequences in homology. These sequences can be woven together to form an exact couple with

$$E_1^{s,n} = H^n(M_{(s)}/M_{(s-1)}) \qquad \text{and} \qquad D_1^{s,n} = H^n(M_{(s)}).$$

By Problem 30.12, this exact couple leads naturally to a spectral sequence.

30.2.3. Multiplicative Structure. We can get a great deal of computational power by working with spectral sequences that have additional algebraic structure. The most common additional structure is a product that is compatible with the differentials.

A **chain algebra** is a chain complex C^* with the structure of a graded algebra such that the differential satisfies the **product rule** (also known as the **Leibniz rule**)

$$d(xy) = d(x) \cdot y + (-1)^{|x|} x \cdot d(y).$$

Problem 30.13. Show that if C^* is a chain algebra, then the multiplication in C^* induces one on $H^*(C^*)$.

If E is a spectral sequence with the natural grading (in terms of filtration s and dimension n), then we say that $x \in E_r^{s,n}$ has **dimension** or **degree** n; if E has the diagonal indexing, then $x \in E_r^{s,t}$ is assigned the **bidegree** $\|x\| = (s,t)$ and the **total degree** $|x| = s + t$. In either case, the (total) degree gives meaning to the Leibniz rule in $E_r^{*,*}$, so we can ask whether the chain complexes $E_r^{*,*}$ are chain algebras or not. If each term $E_r^{*,*}$ has been given the structure of a bigraded chain algebra, then we can compare the given multiplication on $E_{r+1}^{*,*}$ with the one induced from $E_r^{*,*}$. If for every r these products agree, then we say that E is a **spectral sequence of algebras**.

30.2.4. Convergence of Spectral Sequences. For a spectral sequence E, we write

$$\widehat{B}_\infty^{s,n} = \bigcup_r \widehat{B}_r^{s,n} \qquad \text{and} \qquad \widehat{Z}_\infty^{s,n} = \bigcap_r \widehat{Z}_r^{s,n}.$$

We say that E **converges** to the limiting value $E_\infty^{s,n} = \widehat{Z}_\infty^{s,n} / \widehat{B}_\infty^{s,n}$.

It happens frequently that the 'geometry' of a spectral sequence ensures that it achieves its limit in only finitely many steps.

Problem 30.14. Show that if $d_i = 0$ for $i \geq r$, then $E_\infty^{*,*} = E_r^{*,*}$.

In the situation of Problem 30.14, the spectral sequence is said to **collapse** at the r^{th} term. A slightly more subtle, and much more common, situation occurs when there is an $R = R(s,n)$ such that for all $r \geq R$, all differentials d_r with either target or domain $E_r^{s,n}$ are zero.

Problem 30.15.

(a) Show that if there is such an R, then $E_\infty^{s,n} = E_R^{s,n}$.

(b) Show that for a first quadrant spectral sequence with diagonal indexing we can take $R(s,t) = \max\{s+1, t+2\}$.

Project 30.16. Express the limit of the spectral sequence of an exact couple in terms of more standard algebraic or categorical constructions.

30.3. The Spectral Sequences of Filtered Spaces

Now we return our attention to our only real interest in spectral sequences: to gain information about filtered spaces.

30.3.1. Multiplicative Structures. We have defined an exterior product $E_1^{*,*}(X) \otimes E_1^{*,*}(Y) \to E_1^{*,*}(X \wedge Y)$ and shown that d_1 satisfies the Leibniz rule, so that there is an exterior product $E_2^{*,*}(X) \otimes E_2^{*,*}(Y) \to E_2^{*,*}(X \wedge Y)$. We'll show that this generalizes to all r. Specifically, we'll show that the exterior product as we have defined it restricts to exterior products

$$\widehat{Z}_{r-1}^{*,*}(X) \otimes \widehat{Z}_{r-1}^{*,*}(Y) \longrightarrow \widehat{Z}_{r-1}^{*,*}(X \wedge Y)$$

that induce exterior products in $E_r^{*,*}$, and we'll show that d_r is a derivation with respect to it.

Problem 30.17. Let $u \in \widehat{Z}_{r-1}^{s,n}(X;G)$ and $v \in \widehat{Z}_{r-1}^{t,m}(X;H)$.

(a) Show that the diagram

$$\begin{tikzcd}
(X_{(s-1)} \wedge Y_{(t+r-1)}) \cup (X_{(s+r-1)} \wedge Y_{(t-1)}) \arrow[d] \arrow[dr, "*"] & \\
X_{(s+r-1)} \wedge Y_{(t+r-1)} \arrow[r, "u_{(s+r-1)} \wedge v_{(t+r-1)}"] & K(G,n) \wedge K(H,m)
\end{tikzcd}$$

commutes.

(b) Write $T = (X_{(s-1)} \wedge Y) \cup (X \wedge Y_{(t-1)})$. Show that $u_{(s+r-1)} \wedge v_{(t+r-1)}$ extends to a unique map

$$w : (X_{(s+r-1)} \wedge Y_{(t+r-1)}) \cup T \longrightarrow K(G, n) \wedge K(H, m)$$

that is trivial on T.

(c) Show $u \bullet v \in E_1^{s+t,n+m}(X \wedge Y; G \otimes H)$ extends to a map

$$(u \bullet v)_{((s+t)+(r-1))} : (X \times Y)_{((s+t)+(r-1))} \longrightarrow K(G \otimes H, n+m).$$

(d) Conclude that $u \bullet v \in \widehat{Z}_{r-1}^{s+t,n+m}(X \wedge Y; G \otimes H)$.

We have not yet made any attempt to relate the product structure in $E_r^{*,*}$ to the differential $d_r^{*,*}$, so we take up that question next. Recall that a tensor product $C \otimes D$ of chain complexes is given the differential determined by the formula

$$d_{C \otimes D}(c \otimes d) = d_C(c) \otimes d + (-1)^{|c|} c \otimes d_D(d).$$

Proposition 30.18. *Let X and Y be pointed filtered spaces. Then for each $r \geq 1$, the exterior product*

$$E_r^{*,*}(X) \otimes E_r^{*,*}(Y) \longrightarrow E_r^{*,*}(X \wedge Y)$$

is a chain map. Explicitly, if $u \bullet v \in E_r^{*,*}(X \wedge Y)$ is the exterior product of $u \in E_r^{*,*}(X)$ and $v \in E_r^{*,*}(Y)$, then

$$d_r(u \cdot v) = d_r(u) \bullet v + (-1)^{|u|} u \bullet d_r(v).$$

Let $u : X_{(s)}/X_{(s-1)} \to K_n$ and $v : Y_{(t)}/Y_{(t-1)} \to K_m$ be maps which extend to $u_{(s+r-1)} : X_{(s+r-1)} \to X$ and $v_{(t+r-1)} : Y_{(t+t-1)}$. According to the definition given in Section 30.1.1, $d_r(u \bullet v)$ is the cohomology class represented by the composite

$$\bigvee_{i+j=s+t+r} A_i * B_j \longrightarrow (X \wedge Y)_{s+t+r-1} \xrightarrow{w} K_{n+m}$$

where w is any extension of $X_s \wedge Y_t \to K_n \wedge K_m \to K_{n+m}$ (there is such an extension because of Problem 30.17). Since this is a map out of a wedge, it suffices to determine the summands. Write $\theta_{i,j}$ for the restriction to the summand $A_i * B_j$.

Problem 30.19.

(a) Using Problem 30.17, show that $\theta_{i,j} = 0$ except if $i = s$ or $j = t$.

(b) Show that $\theta_{s,t+r}$ is defined by the diagram

$$\begin{tikzcd}
(CA_{s-1} \times B_{t+r-1}) \cup (A_{s-1} \times CB_{t+r-1}) \arrow[r, "\theta_{s,t+r}"] \arrow[d, "q_1"'] & K_{n+m} \\
\Sigma A_{s-1} \rtimes B_{t+r-1} \arrow[r, "\mathrm{id}\wedge\beta_{t+r-1}"] & (X_{(s)}/X_{(s-1)}) \wedge Y_{(t+r-1)} \arrow[u, "u\wedge v_{(t+r-1)}"']
\end{tikzcd}$$

where q_1 is the cofiber of the inclusion of the second term in the union.

(c) Show that $\theta_{s+r,t}$ is defined by the diagram

$$\begin{tikzcd}
(CA_{s+r-1} \times B_{t-1}) \cup (A_{s+r-1} \times CB_{t-1}) \arrow[r, "\theta_{s+r,t}"] \arrow[d, "q_2"'] & K_{n+m} \\
A_{s+r-1} \ltimes \Sigma B_{t-1} \arrow[r, "\alpha_{s+r-1}\wedge\mathrm{id}"] & X_{(s+r-1)} \wedge (Y_{(t)}/Y_{(t-1)}) \arrow[u, "u_{(s+r-1)}\wedge v"']
\end{tikzcd}$$

where q_2 is the cofiber of the inclusion of the first term in the union.

(d) Show that the coproduct on $(CA_i \times B_j) \cup (A_i \times CB_j)$ is naturally

$$(CA_i \times B_j) \cup (A_i \times CB_j) \longrightarrow (\Sigma A_i \rtimes B_j) \vee (A_i \ltimes \Sigma B_j).$$

(e) Prove Proposition 30.18.

Internalizing the External Product. When we apply our spectral sequence construction to space with a multiplicative filtration, we can internalize the exterior product and get a spectral sequence of algebras.

Proposition 30.20. *Show that if X has a multiplicative filtration and R is a ring, then $E(X;R)$ is a spectral sequence of R-algebras.*

Problem 30.21. Prove Proposition 30.20.

30.3.2. Convergence. Next we establish conditions on a filtered pointed space X that ensure that $E_\infty^{s,n} \cong \mathrm{Gr}^s\, \widetilde{H}^n(X)$.

Problem 30.22. Show that in the spectral sequence of a filtered space,

$$B_\infty^{s,n} \subseteq \mathcal{B}_\infty^{s,n} \subseteq \mathcal{Z}_\infty^{s,n} \subseteq Z_\infty^{s,n}.$$

Conclude that $\mathrm{Gr}^s\, \widetilde{H}^n(X)$ is a subquotient of $E_\infty^{s,n}(X)$.

If either of the containments $B_\infty^{s,n} \subseteq \mathcal{B}_\infty^{s,n}$ or $\mathcal{Z}_\infty^{s,n} \subseteq Z_\infty^{s,n}$ is strict, then $\mathrm{Gr}^s\, \widetilde{H}^n(X)$ is not entirely determined by the spectral sequence. But if both containments are equalities, complete knowledge of the spectral sequence will include complete knowledge of the filtration quotients.

Proposition 30.23. *Let X be a filtered space such that, for each n,*

(C1) $\widetilde{H}^n(X) \to \widetilde{H}^n(X_{(s)})$ is an isomorphism when s is large enough and

(C2) if $n > 0$, the group $\widetilde{H}^n(X_{(s)}) = 0$ when s is small enough.

Then the spectral sequence $\{E_r^{*,*}(X), d_r\}$ *converges to* $\mathrm{Gr}^s \widetilde{H}^n(X)$.

Problem 30.24. Prove Proposition 30.23.

Problem 30.25. Show that $(C1)$ and $(C2)$ can be weakened to

(D1) $H^n(X) \to \lim_s \widetilde{H}^n(X_{(s)})$ is an isomorphism and

(D2) $\operatorname{colim}_s \widetilde{H}^n(X_{(s)}) = 0$

for each n.

30.3.3. The Grand Conclusion. We conclude this chapter by collecting together everything we have discovered about the spectral sequences of filtered spaces in one theorem.

Cohomology Spectral Sequence of a Filtered Space.

Theorem 30.26. *For each coefficient group G, there is a functor*

$$E(\,?\,; G) : \mathrm{FILT}(\mathcal{T}_*) \longrightarrow \mathbf{SS}$$

such that

(a) $\mathrm{Gr}^s \widetilde{H}^n(X; G)$ *is a subquotient of* $E_r^{s,n}(X; G)$ *for all* r*, and if the filtration on* X *satisfies properties (C1) and (C2), then the sequence converges to* $\mathrm{Gr}^s \widetilde{H}^n(X)$,

(b) there are exterior products $E(X; G) \otimes E(Y; H) \to E(X \wedge Y; G \otimes H)$, *and*

(c) if the filtration of X *is multiplicative and* R *is a ring, then* $E(X; R)$ *is a spectral sequence of algebras.*

If we want to study an unpointed filtered space X, we apply Theorem 30.26 to the associated pointed space X_+. The conclusions are exactly the same, except that reduced cohomology $\widetilde{H}^*$ must be replaced with unreduced cohomology H^* and the unpointed exterior product has the form

$$E(X; G) \otimes E(Y; H) \longrightarrow E(X \times Y; G \otimes H).$$

Homology Spectral Sequence of a Filtered Space. Entirely analogous results hold for the homology of filtered spaces. To state them, we need the conditions

$(\mathrm{C1})_*$ $H_n(X_{(s)}) \to H_n(X)$ is an isomorphism for s large enough and

$(\mathrm{C2})_*$ $H_n(X_{(s)}) = 0$ if s is small enough

(dual to (C1) and (C2)) to guarantee convergence.

Theorem 30.27. *For each coefficient group G, there is a functor*

$$E(\,?\,; G) : \mathrm{FILT}(\mathcal{T}_*) \longrightarrow \mathbf{SS}$$

such that

(a) $\mathrm{Gr}_s\,\widetilde{H}_n(X;G)$ *is a subquotient of* $E^r_{s,n}(X;G)$ *for all* r, *and if the filtration on* X *satisfies properties (C1)*$_*$ *and (C2)*$_*$, *then the sequence converges to* $\mathrm{Gr}_s\,\widetilde{H}_n(X)$ *and*

(b) *there are exterior products* $E(X;G)\otimes E(Y;H)\to E(X\wedge Y;G\otimes H)$.

In the unpointed case, we replace reduced homology $\widetilde{H}_*$ with unreduced homology and use the unreduced exterior product

$$E(X;G)\otimes E(Y;H)\longrightarrow E(X\times Y;G\otimes H).$$

If X is an H-space with a nice enough filtration, its spectral sequence is a spectral sequence of algebras.

Theorem 30.28. *If* X *is an H-space whose multiplication* $\mu: X\times X\to X$ *is homotopic to a filtered map and* R *is a ring, then* $E(X;R)$ *is a spectral sequence of algebras.*

Project 30.29. Prove Theorem 30.28.

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Chapter 31

The Leray-Serre Spectral Sequence

Now we resume our study of the Serre filtration. Theorem 30.26 tells us that the Serre filtration leads to cohomology spectral sequences, which are called *Leray-Serre spectral sequences*.

We begin by collecting together all the main properties of the Leray-Serre spectral sequences. We determine the topological significance of the groups on the bottom and left edges of the spectral sequence and of the differentials joining them.

The rest of the chapter is devoted to developing the skill of working with spectral sequences. This begins with some very simple computations, followed by some general situations in which a spectral sequence can be reduced to an ordinary exact sequence. Thus we give a new derivation of the Wang sequence and establish some other exact sequences for fibrations satisfying various special hypotheses.

31.1. The Leray-Serre Spectral Sequence

The Leray-Serre spectral sequence is the spectral sequence that emerges when we feed the Serre filtration into the spectral sequence of a filtered space. In this section, we carefully write down the main properties of these spectral sequences.

As we have discussed before, this construction can be applied to any fibration by replacing $p : E \to B$ by its pullback $\bar{p} : \overline{E} \to \overline{B}$ over a CW replacement $\overline{B} \to B$.

31.1.1. The Spectral Sequences Associated to the Serre Filtration. We begin by stating the basic existence theorem for the Leray-Serre spectral sequence.

Theorem 31.1. *Let $p : E \to B$ be a fibration with fiber F, and assume that it is orientable with respect to H^*. Then there is a natural spectral sequence $E(p) = \{E_r^{*,*}, d_r\}$ which converges to $\mathrm{Gr}^* H^*(E)$ and whose second term is*

$$E_2^{s,t}(p) \cong H^s(B; H^t(F)).$$

If the coefficients are in a ring, then this is a spectral sequence of algebras, and the identifications of $E_2^{*,*}$ and of $E_\infty^{*,*}$ are algebra isomorphisms.

The naturality here means that the Leray-Serre spectral sequence is the object part of a functor $\mathbf{Fib} \to \mathbf{SS}_R$, which follows from the naturality of Theorem 30.26 and the naturality statement of Theorem 28.1. What remains is to verify the conditions (C1) and (C2) of Proposition 30.23 and to check that the Serre filtration is multiplicative.

Problem 31.2. Show that the Serre filtration for any fibration $p : E \to B$ satisfies conditions (C1) and (C2).

Problem 31.3. Let $p : E \to B$ be a fibration over a CW complex B.

(a) Show that the CW filtration of B is multiplicative.

(b) Use the homotopy lifting property to show that the Serre filtration of E is multiplicative.

Drawing the Leray-Serre Spectral Sequence. When we work with spectral sequences, we usually draw the groups as an array of dots in the plane, with the nonzero groups $E_r^{s,t}$ represented by the dot at position (s, t). For the Leray-Serre spectral sequence, the groups $E_r^{s,t}$ are zero if either s or t is negative, so these dots are concentrated in the first quadrant (such spectral sequences are often referred to as **first quadrant spectral sequences**). The differentials d_r go from dot to dot as indicated in Figure 31.1.

31.1.2. Nondegeneracy of the Algebra Structure. Let $p : E \to B$ be a fibration with fiber F, and consider the algebra $E_2(p; R)$, where R is a ring. For any R-module M, the Universal Coefficients Theorem gives injective maps $H^s(B; R) \otimes_R M \to H^s(B; M)$. Taking $M = H^t(F; R)$, we obtain an injection

$$H^s(B; R) \otimes_R H^t(F; R) \xrightarrow{\mu} H^s(B; H^t(F; R)) = E_2^{s,t}(p; R),$$

which we'll denote $u \otimes v \mapsto u \cdot v$. The tensor product $H^*(B; R) \otimes_R H^*(F; R)$ has a canonical algebra structure, determined by the rule

$$(a \otimes b) \cdot (x \otimes y) = (-1)^{|b||x|} ax \otimes by.$$

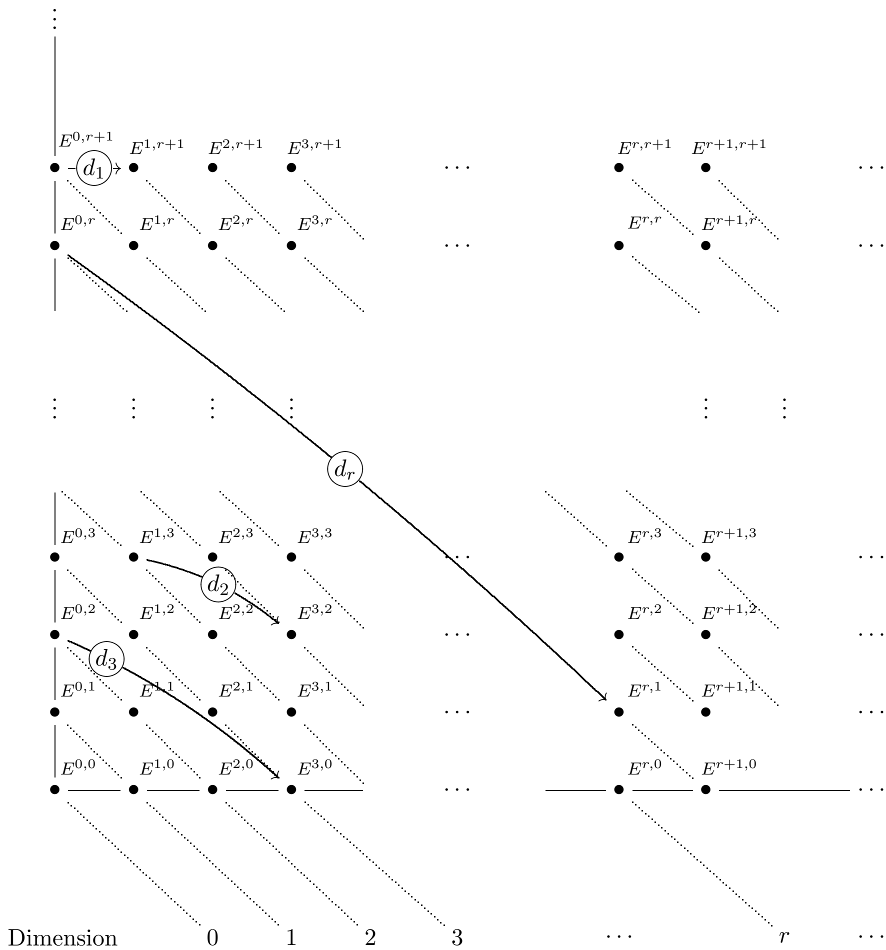

Figure 31.1. A spectral sequence with diagonal indexing

Proposition 31.4. *The map* $\mu : H^*(B;R) \otimes_R H^*(F;R) \to E_2^{*,*}(p;R)$ *is an algebra homomorphism.*

Problem 31.5. Prove Proposition 31.4.

Hint. Ultimately, the products in the two sides boil down to smashing together cohomology classes.

Now let's consider the product structure in $E_r(p;R)$ for $r > 2$. If $f : R \to S$ is a homomorphism of rings, then we may give S the structure of an R-module by setting $r * s = f(r) \cdot s$. Thus a map $f : X \to Y$ gives $H^*(X)$ the structure of an $H^*(Y)$-module.

Problem 31.6. Let $f : X \to Y$, and let R be a ring. Show that the multiplication $H^*(Y;R) \otimes H^*(X;R) \to H^*(X;R)$ is the composite of the exterior product with the map induced by $(f, \mathrm{id}_X) : X \to Y \times X$.

Applying this construction to $p : E \to B$, we find that p endows $H^*(E;R)$ with the structure of an $H^*(B;R)$-module. This module structure can be approximated by the Leray-Serre spectral sequence.

Problem 31.7. Let $p : E \to B$ be a fibration.

(a) Show that there is a map of fibrations

$$\begin{tikzcd} E \arrow[r, "p\times \mathrm{id}_E"] \arrow[d, "p"'] & B \times E \arrow[d, "\mathrm{id}_B \times p"] \\ B \arrow[r, "\Delta"] & B \times B. \end{tikzcd}$$

(b) Show that $p \times \mathrm{id}_E$ is homotopic to a filtered map with respect to the Serre filtrations.

Proposition 31.8. *The cohomology Leray-Serre spectral sequence with coefficients in the ring R is a module over $H^*(B;R)$. Explicitly, there are multiplications*

$$\mu_r : H^n(B;R) \otimes_R E_r^{s,t}(p;R) \longrightarrow E_r^{s+n,t}(p;R)$$

such that

(a) $d_r(x \cdot y) = (-1)^{|x|} x \cdot d_r(y)$ for $x \in H^n(B;R)$ and $y \in E_r^{s,t}(p;R)$,

(b) when $r = 2$, $E_2^{s,t}(p;R) = H^s(B; H^t(F;R))$, and the multiplication is the ordinary cup product with respect to $R \otimes_R H^*(F;R) \xrightarrow{\cong} H^*(F;R)$, and

(c) in general the module structure is given by $\mu_r(x \otimes y) = [x] \cdot y$, where $[x]$ denotes the coset of x in the group $E_r^{n,0}(p;R) \cong H^n(B;R)/B_{r-1}^{n,0}(p;R)$.

Problem 31.9. Prove Proposition 31.8.

There is a corresponding product $H^*(E) \otimes H^*(B) \to H^*(E)$ on the other side. Everything is exactly the same in this case, except that the rule for the differential is simpler: $d_r(x \cdot y) = d_r(y) \cdot x$.

31.1.3. Two Relative Variants. Let $p : E \to B$ be a fibration with fiber F. Assume that $A \subseteq B$ is a relative CW complex, so that B is obtained from A by attaching cells, and consider the pullback square

$$\begin{tikzcd} E_0 \arrow[r] \arrow[d, "p_0"'] \arrow[dr, phantom, "\text{pullback}"] & E_1 \arrow[d, "p_1"] \\ A \arrow[r] & B. \end{tikzcd}$$

Pulling back over the relative skeleta of B yields a Serre-like filtration of the quotient E/E_A, and hence to a spectral sequence.

Problem 31.10. Show that the spectral sequence converges to $H^*(E_1/E_0)$ and has $E_2^{s,t} \cong H^s(B/A; H^t(F))$.

There is a second kind of relative Serre filtration, in which the fibers are relativized. Thus we consider squares

$$\begin{tikzcd} E_0 \arrow[r] \arrow[d, "p_0"'] & E_1 \arrow[d, "p_1"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

with induced map $F_0 \to F_1$ of fibers. This square induces a map from the Serre filtration of E_0 to that of E_1; when we form cofibers of the maps $(E_0)_{(s)} \to (E_1)_{(s)}$, we obtain a Serre-inspired filtration of E_1/E_0, and from there a spectral sequence.

Problem 31.11. Show this spectral sequence converges to $\mathrm{Gr}^* H^*(E_1/E_0)$ and that $E_2^{s,t} \cong H^s(B; H^t(F_1/F_0))$.

Project 31.12. Investigate the possibility of defining algebra structures in these spectral sequences.

31.1.4. The Homology Leray-Serre Spectral Sequence. We could also apply homology to the Serre filtration associated to the fibration $p : E \to B$, and the result is a spectral sequence that converges to $\mathrm{Gr}_* H_*(E)$.

Theorem 31.13. *Let $p : E \to B$ be a fibration with fiber F, and assume that it is orientable with respect to H_*. Then there is a natural spectral sequence $\{E^r_{*,*}, d^r\}$ which converges to $\mathrm{Gr}_* H_*(E)$ and whose second term is*

$$E^2_{s,t} \cong H_s(B; H_t(F)).$$

Project 31.14. Prove Theorem 31.13 in detail.

Problem 31.15. Show that if $p : E \to B$ is an H-map, then the homology Leray-Serre spectral sequence is a spectral sequence of algebras.

31.2. Edge Phenomena

If $p : E \to B$ is a fibration with fiber $i : F \to E$, then $H^*(E)$ inherits a Serre filtration, and the associated graded object $\mathrm{Gr}^*(H^*(E))$ may be obtained as subquotients from the 2-dimensional array of groups $E_2^{s,t}(p) = H^s(B; H^t(F))$. What was not entirely obvious at the outset—but *is* entirely obvious in view of Theorem 28.1—is that these groups are zero whenever one or the other (or both) of the indices is negative. Thus our 2-dimensional array of groups is really indexed by the first quadrant of the (s,t)-plane.

We show in this section that the subquotients that appear on the left and bottom edges of our bigraded group $E_2^{*,*}(p)$ can be identified with the induced maps p^* and i^*.

31.2.1. Edge Filtration Quotients. Let $p : E \to B$ be a fibration with fiber F. The Serre filtration $\cdots \to E_{(s)} \to E_{(s+1)} \to \cdots \to E$ on E gives us the algebraic Serre filtration defined by

$$\mathcal{F}^s H^*(E) = \ker(H^*(E) \to H^*(E_{(s)})).$$

For fixed cohomological dimension $*$, this filtration is actually finite, in the sense that the sequence actually only has finitely many proper inclusions.

Problem 31.16.

(a) Determine the connectivities of the maps in the diagram

$$\begin{tikzcd} E_{(s-1)} \arrow[r, "j"] \arrow[d] & E \arrow[r, "q"] \arrow[d, "p"] & E/E_{(s-1)} \arrow[d, "\xi"] \\ B_{s-1} \arrow[r] & B \arrow[r, "b"] & B/B_{s-1}. \end{tikzcd}$$

(b) Show that $\mathcal{F}^s H^s(E) = 0$ and $\mathcal{F}^{-1} H^s(E) = H^s(E)$.

It follows that $\mathrm{Gr}^s\, H^s(E) = \mathcal{F}^{s-1} H^s(E)$, which is a subgroup of $H^s(E)$, and that $\mathrm{Gr}^0\, H^s(E) = H^s(E)/\mathcal{F}^0$, which is a quotient of $H^s(E)$.

Proposition 31.17. *Let $p : E \to B$ be an H^*-orientable fibration in which B is path-connected, and write $i : F \to E$ for the inclusion of the fiber. Then in the Leray-Serre spectral sequence for p,*

$$\begin{aligned} E_{r+1}^{r,0}(p) &\cong \mathrm{Gr}^r\, H^r(E) \\ &= \mathrm{Im}(p^* : H^r(B) \to H^r(E)) \end{aligned}$$

and

$$\begin{aligned} E_{r+1}^{0,r-1}(p) &\cong \mathrm{Gr}^0\, H^{r-1}(E) \\ &= \mathrm{Im}(i^* : H^{r-1}(E) \to H^{r-1}(F)). \end{aligned}$$

The path-connected hypothesis is not much of a hardship, because the cohomology of the total space of fibration may be studied over each path component separately.

Problem 31.18. Prove Proposition 31.17.

31.2.2. One Step Back. We can actually push this analysis back one step and determine the groups $E_r^{r,0}$ and $E_r^{0,r-1}$. For this, we need to introduce the Ganea construction $g : E/F \to B$ on the fibration p, which figures in the diagram

$$\begin{tikzcd} E \arrow[r, "q"] \arrow[d, "p"'] & E/F \arrow[d, "g"] \\ B \arrow[r, equal] & B. \end{tikzcd}$$

Problem 31.19. Show that the Serre filtration on E induces a filtration of E/F and that all the maps in the square are filtered maps.

We'll refer to the spectral sequence of the filtered space E/F as $E(g)$.

Problem 31.20. Show that if p is H^*-orientable, then

$$E_r^{r,0}(g) = E_\infty^{r,0}(g) \cong \mathrm{Im}(g^* : H^*(B) \to H^*(E/F)).$$

Problem 31.21.

(a) Show that $E_r^{s,0}(g) \to E_r^{s,0}(p)$ is an isomorphism for $r \leq s$.

(b) Determine $E_r^{r,0}(p)$.

31.2.3. Edge Homomorphisms. We have a tantalizing situation: we know that $\mathrm{Gr}^r H^r(E) = \mathrm{Im}(p^* : H^r(B) \to H^r(E))$ is a subquotient of $E_2^{s,0} \cong H^s(B)$ and that $\mathrm{Gr}^0 H^s(E) = \mathrm{Im}(i^* : H^s(E) \to H^s(F))$ is a subquotient of $E_2^{0,s} \cong H^s(F)$. There are painfully obvious ways in which to realize these associated graded groups as subquotients; is it possible that our construction has found a natural way to obtain $\mathrm{Im}(p^*)$ and $\mathrm{Im}(i^*)$ as subquotients of $H^s(B)$ and $H^s(F)$ that is different from the obvious one?

To answer this question, we introduce the **edge homomorphisms**.

Bottom Edge Homomorphism. We begin by studying the homomorphism determined by the bottom edge of our graded gadget $E_2^{*,*}$.

Problem 31.22. Show that $Z_1^{s,0} = \mathcal{Z}^{s,0}$ (see Sections 27.2 and 27.3 for the definition of $\mathcal{Z}^{s,t}$), so that for each $r \geq 1$

$$E_r^{s,0} = Z_1^{s,0}/B_{r-1}^{s,0},$$

which is a quotient (not just a subquotient) of $E_2^{s,0} \cong H^s(B)$.

It follows, in particular, that $E_{s+1}^{s,0} = \mathrm{Gr}^s H^s(E)$ is naturally a quotient of $H^s(B)$. The bottom **edge homomorphism** is the map **edge** defined by the composition

$$\begin{tikzcd} H^s(B) \arrow[r, "\mathbf{edge}"] \arrow[d, "\cong"] & H^s(E) \\ E_2^{s,0} \arrow[r] & \mathrm{Gr}^s\, H^s(E), \arrow[u, hook] \end{tikzcd}$$

where q is the quotient map.

Problem 31.23. Show that the edge homomorphism is natural, in the sense that if we have a map (g, f) from p_1 to p_2, then the diagram

$$\begin{tikzcd} H^s(B_2) \arrow[r, "\cong"] \arrow[d, "f^*"'] & E_2^{s,0}(p_2) \arrow[r] \arrow[d, "(f,g)^{*,*}"'] & \mathrm{Gr}^s\, H^s(E_2) \arrow[r, hook] \arrow[d] & H^s(E_2). \arrow[d, "g^*"] \\ H^s(B_1) \arrow[r, "\cong"] & E_2^{s,0}(p_1) \arrow[r] & \mathrm{Gr}^s\, H^s(E_1) \arrow[r, hook] & H^s(E_1) \end{tikzcd}$$

commutes.

Problem 31.24. By applying naturality to the map $(p, \mathrm{id}_B) : p \to \mathrm{id}_B$ of fibrations, show that there is a commutative square

$$\begin{tikzcd} H^s(B) \arrow[d, "p^*"'] & E_2^{s,0}(\mathrm{id}_B) \arrow[l, "\mathbf{edge}"', "\cong"] \arrow[d, "\cong"', "(p,\mathrm{id}_B)^{*,*}"] \\ H^s(E) & E_2^{s,0}(p). \arrow[l, "\mathbf{edge}"'] \end{tikzcd}$$

In view of this last result, we won't use the notation **edge** for this map any more; but we will occasionally refer to p^* as an 'edge homomorphism'.

Bottom Edge Homomorphism for the Ganea Construction. Nearly identical reasoning leads to an understanding of the edge homomorphisms in the spectral sequence of the filtered space E/F.

Problem 31.25. Let $p : E \to B$ be a fibration with fiber F, and let $g : E/F \to B$ be the Ganea construction on p. Define the bottom edge homomorphism in the spectral sequence $E(g)$, and identify it with $g^* : H^*(B) \to H^*(E/F)$.

Left Edge Homomorphism. Now we take a look at the left edge. The reasoning is very similar to what we did on the bottom, so I'll leave most of the work to you.

Problem 31.26.

(a) Define the natural left edge homomorphism $\mathbf{edge} : H^t(E) \to H^t(F)$.

(b) Using the square

$$\begin{tikzcd} F \arrow[r, "j"] \arrow[d] & E \arrow[d, "p"] \\ * \arrow[r] & B, \end{tikzcd}$$

identify **edge** with the induced map $j^* : H^t(E) \to H^t(F)$.

Exercise 31.27. What can you say about the edge homomorphisms when F or B is not path-connected?

31.2.4. The Transgression. The transgression is a useful correspondence between cohomology classes of the fiber and the base of a fibration. It is typically defined by tracing through various commutative diagrams of cohomology groups. More diagram chasing reveals that the transgression can be identified with certain differentials in the Leray-Serre spectral sequence. We take a different route, defining the transgression topologically and establishing the relation with the spectral sequence by the method of the universal example.

Transgessive Pairs. Let $p : E \to B$ be a fibration with fiber F. We call a pair of cohomology classes

$$u : B \longrightarrow K(G, n) \qquad \text{and} \qquad v : F \longrightarrow K(G, n-1)$$

a **transgressive pair** if the right square in the diagram

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "p"'] & E \cup CF \arrow[r, "\delta"] \arrow[d, "g"'] & \Sigma F \arrow[d, "\Sigma v"] \\ B \arrow[r, equal] & B \arrow[r, "u"] & K(G, n) \end{tikzcd}$$

commutes up to homotopy, where $g : E \cup CF$ is the Ganea construction on p and $\Sigma v : \Sigma F \to K_n$ is the image of v under the suspension isomorphism $\Sigma : \widetilde{H}^{n-1}(F; G) \to \widetilde{H}^n(\Sigma F; G)$; in other words, $\Sigma v \in [\Sigma F, K(G, n)]$ is the adjoint of $v \in [F, \Omega K(G, n)]$.

We say that $v : F \to K(G, n-1)$ is **transgressive** if it is part of a transgressive pair.

Problem 31.28. Let $p : E \to B$ be a fibration with fiber inclusion $i : F \to E$, and suppose $v : F \to K(G, n-1)$ is transgressive.

(a) Show that $\{u \in H^n(B; G) \mid (u, v) \text{ is a transgressive pair}\}$ is a coset of the subgroup $(g^*)^{-1}(\mathrm{Im}(\partial^*))$.

(b) Write $U = H^n(B)/(g^*)^{-1}(\mathrm{Im}(\partial^*))$ and $V = (\partial^*)^{-1}(\mathrm{Im}(g^*))$. Show that V is precisely the set of transgressive elements of $H^{n-1}(F; G)$ and that each element $v \in V$ gives rise to a well-defined element $\tau(v) = [u] \in U$.

(c) Show that the function τ fits into an exact sequence

$$0 \to \mathrm{Im}(i^*) \longrightarrow V \xrightarrow{\tau} U \longrightarrow \mathrm{Im}(p^*) \to 0.$$

The function τ is called the **transgression**.

Cohomology Operations and the Transgression. Since the transgression is defined in terms of maps of spaces, it commutes with cohomology operations.

Problem 31.29. Let $p : E \to B$ be a fibration with fiber F.

(a) If $u : X \to K(G, n)$ and $v : F \to K(G, n-1)$ is a transgressive pair, then for any stable cohomology operation $\theta : H^*(\,?\,; G) \to H^*(\,?\,; H)$, $\theta(u)$ and $\theta(v)$ are also a transgressive pair.

(b) Write $V(G)$ and $U(G)$ for the groups U and V for cohomology with coefficients in the group G. Show that a stable cohomology operation θ induces maps $\theta : U(G) \to U(H)$ and $\theta : V(G) \to V(H)$ making the square

$$\begin{tikzcd} U(G) \arrow[r, "\tau"] \arrow[d, "\theta"'] & V(G) \arrow[d, "\theta"] \\ U(H) \arrow[r, "\tau"] & V(H) \end{tikzcd}$$

commute.

Exercise 31.30. What if θ is not a stable operation?

The Transgression in the Path-Loop Fibration. In the very important path-loop fibrations, the groups U and V simplify a bit, and we can give a genuine homotopy-theoretical interpretation of the transgression τ.

Problem 31.31. Let $p = @_0 : \mathcal{P}(X) \to X$.

(a) Show that $u : X \to K(G, n)$ and $v : \Omega X \to K(G, n-1)$ are a transgressive pair for the fibration $@_0 : \mathcal{P}(X) \to X$ if and only if $v = \Omega u$.

(b) Conclude that the transgression

$$\tau : \widetilde{H}^{n-1}(\Omega X; G) \longrightarrow \widetilde{H}^n(X; G)/\ker(\lambda^*)$$

is given by $v \mapsto [\Omega^{-1}(v)]$.

The Transgression in the Leray-Serre Spectral Sequence. Now we show that the differentials that stretch from edge to edge in the Leray-Serre spectral sequence for $p : E \to B$ can be described in terms of the transgression.

Problem 31.32. Write $\widetilde{\Sigma F}$ for the space $(E \cup CF) \cup CE$.

(a) Show that the canonical inclusion $\phi : \Sigma F \hookrightarrow \widetilde{\Sigma F}$ is a homotopy equivalence.

(b) Show that if (u, v) is a transgressive pair, then Σv has a representative $\widetilde{\Sigma v}$ making the diagram

$$\begin{tikzcd}
E \arrow[r, "j"] \arrow[d, "p"'] & E \cup CF \arrow[r, "\widetilde{\delta}"] \arrow[d, "g"'] & \widetilde{\Sigma F} \arrow[d, "\widetilde{\Sigma v}"] \\
B \arrow[r, equal] & B \arrow[r, "u"] & K(G,n)
\end{tikzcd}$$

strictly commutative.

Lemma 31.33. *If $p : E \to B$ is a fibration with fiber F, then the following are equivalent:*

(1) (u, v) is a transgressive pair for p,

(2) there is a strictly commutative cube

$$\begin{tikzcd}
E \arrow[rr, "\mathrm{in}_0"] \arrow[dd, "j"'] \arrow[dr, equal] & & CE \arrow[dr, "H"] \arrow[dd] & \\
& E \arrow[rr] \arrow[dd, "p"] & & \mathcal{P}(K(G,n)) \arrow[dd, "@_0"] \\
E \cup CF \arrow[rr] \arrow[dr, "g"'] & & \widetilde{\Sigma F} \arrow[dr, "\widetilde{\Sigma v}"] & \\
& B \arrow[rr, "u"] & & K(G,n)
\end{tikzcd}$$

in which $\widetilde{\Sigma v} \circ \phi : \Sigma F \to K(G,n)$ is homotopic to Σv,

(3) there is a strictly commutative diagram

$$\begin{tikzcd}
F \arrow[r, "w"] \arrow[d, "i"'] & K(G,n-1) \arrow[d] \\
E \arrow[r] \arrow[d, "p"'] & \mathcal{P}(K(G,n)) \arrow[d, "@_0"] \\
B \arrow[r, "u"] & K(G,n)
\end{tikzcd}$$

in which the map w is homotopic to v.

Problem 31.34. Prove Lemma 31.33.

Now we are in an excellent position, because we have established a connection between the transgressive pair (u, v) and the much simpler transgressive pair $(\mathrm{id}_{K(G,n)}, \mathrm{id}_{K(G,n-1)})$.

Problem 31.35. Show that in the Leray-Serre spectral sequence for the path-loop fibration $\mathcal{P}(K(G,n)) \to K(G,n)$, the differential $d_r : E_r^{0,n} \to E_r^{r,n-r+1}$ is trivial for $r < n$ and d_n is the identity.

Theorem 31.36. *Let $p : E \to B$ be an H^*-orientable fibration with fiber F. Then under the identification of $E_2^{*,*}(p)$, the differential d_n can be identified using the diagram*

$$\begin{tikzcd} V \arrow[r, "\cong"] \arrow[d, "\tau"'] & E_n^{0,n-1}(p) \arrow[d, "d_n"] \\ U \arrow[r, "\cong"] & E_n^{n,0}(p). \end{tikzcd}$$

Theorem 31.36 gives information about the effect of cohomology operations in the Leray-Serre spectral sequence.

Corollary 31.37. *Let θ be a stable cohomology operation of degree k. Then the differentials*

$$d_n : E_n^{0,n-1}(p) \longrightarrow E_n^{n,0}(p) \qquad \text{and} \qquad d_{n+k} : E_n^{0,n+k-1}(p) \longrightarrow E_{n+k}^{n+k,0}(p)$$

satisfy

$$d_{n+k}(\theta(v)) = \theta(d_n(v)).$$

The proof naturally divides into two parts. First we show that every transgressive element of $H^{n-1}(F; G)$ actually lives in $E_n^{0,n-1}(p)$ and that the differential behaves as required on all such elements.

Problem 31.38. Let (u, v) be a transgressive pair for the fibration $p : E \to B$. Show that $d_r(v) = 0$ for $r < n$ and $d_n(v) = [u]$.

Hint. Find a map of fibrations as in Lemma 31.33(3).

To finish the proof, we study the effect of d_n on elements of $E_n^{0,n-1}(p)$ that are not known (at the outset) to be transgressive.

Problem 31.39. Let $v \in E_n^{0,n-1}(p)$, and let $u \in d_n(v)$.

(a) Show that there is a commutative square

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "p"'] & \mathcal{P}(K(G,n)) \arrow[d, "@_0"] \\ B \arrow[r, "u"] & K(G,n). \end{tikzcd}$$

(b) Consider the map of spectral sequences induced by the square in part (a). Show that the induced map $h_2^{*,*} : E_2^{*,*}(@_0) \longrightarrow E_2^{*,*}(p)$ carries $\mathrm{id}_{K(G,n)} \in E_2^{n,0}(@_0)$ to u and $\mathrm{id}_{K(G,n-1)} \in E_2^{0,n-1}(@_0)$ to an element $z \in E_2^{0,n-1}(p)$ such that $d_r(z) = 0$ for $r < n$ and $d_n(z) = [u]$.

(c) Show that in $E_n^{*,*}(p)$, the differential $d_n : E_2^{0,n-1}(p) \longrightarrow E_2^{n,0}(@_0)$ is injective. Conclude that $h^{*,*}(\mathrm{id}_{K(G,n-1)}) = v$.

(d) Show that v is transgressive and $d_n(v) = [\tau(v)]$.

(e) Derive Corollary 31.37.

31.3. Simple Computations

The Leray-Serre spectral sequence is a large and complicated gadget, and it takes some getting used to. In this section, you will work through several examples of varying complexity. The intent is that these problems will illustrate some of the basic ways in which the spectral sequence is frequently used. We begin with a number of computations and then use the spectral sequence to rederive several theorems that we have already seen.

31.3.1. Fibration Sequences of Spheres. We found in Section 15.1.2 that since $\mathbb{F}\mathrm{P}^1 \cong S^d$, there are fibration sequences

$$S^0 \longrightarrow S^1 \longrightarrow S^1, \qquad S^1 \longrightarrow S^3 \longrightarrow S^2 \qquad \text{and} \qquad S^3 \longrightarrow S^7 \longrightarrow S^4.$$

These sequences have provided valuable footholds in the study of the homotopy groups of spheres and other applications. Now we consider the general question: suppose $S^e \to S^b$ is a fibration with fiber S^f; then what can be said about the dimensions e, b and f?

Problem 31.40. Let $S^f \to S^e \xrightarrow{p} S^b$ is a fibration sequence, and let $E(p;\mathbb{Z})$ be the Leray-Serre spectral sequence for p.

(a) Draw $E_2^{*,*}(p)$, identifying all nonzero groups.

(b) Draw $E_\infty^{*,*}(p)$, identifying all nonzero groups.

(c) Show that the sequence does not collapse at $E_2^{*,*}(p)$.

(d) Show that the sequence collapses unless … *something* is true about the numbers e, b and f.

31.3.2. Cohomology of Projective Spaces. Now we use the Leray-Serre spectral sequence to compute the cohomology rings of projective spaces.

Problem 31.41. Let $E(p_n;\mathbb{Z})$ be the spectral sequence, with $\mathbb{Z}$ coefficients, for the fibration sequence $S^1 \to S^{2n+1} \xrightarrow{p_n} \mathbb{C}\mathrm{P}^n$. Let $\mathbf{s}_1 \in H^1(S^1;\mathbb{Z})$ be the canonical generator.

(a) Show that the spectral sequence collapses at $E_3^{*,*}$.

(b) Show that $d_2 : E_2^{0,1} \to E_2^{2,0}$ is an isomorphism and, more generally, that $d_2 : E_2^{2s,1} \to E_2^{2s+2,0}$ is an isomorphism for all $s < n$.

(c) Show that $H^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) = \mathbb{Z} \cdot x$.

(d) Show that if $b \neq 0 \in H^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$, then $b \cdot \mathbf{s}_n \neq 0$.

(e) Use the fact that d_2 is a derivation to show that $H^{2s}(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \cong \mathbb{Z} \cdot x^s$ for $s \leq n$.

(f) Argue that $H^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ must be concentrated in even degrees, and deduce the full ring structure of $H^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$.

Problem 31.42. Use the same technique to deduce the cohomology algebras of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$.

31.3.3. Cohomology of the Loop Space of a Sphere. Here is yet another simple but instructive recomputation.

Problem 31.43. Consider the Leray-Serre spectral sequence for the fibration sequence $\Omega(S^{2n+1}) \to \mathcal{P}(S^{2n+1}) \xrightarrow{@_0} S^{2n+1}$.

(a) Write out $E_\infty^{*,*}(@_0)$.

(b) Show that d_r is nonzero for exactly one r and that the nonzero differentials must be isomorphisms.

(c) Show that $E_2^{0,2nk}(@_0) = \mathbb{Z} \cdot y_{(k)}$ for all k.

(d) Use the fact that the differentials are derivations to determine the coefficient in the equation $y_{(1)}^k = c_k \cdot y_{(k)}$, where $c_k \in \mathbb{Z}$.

Project 31.44. Use the Leray-Serre spectral sequence to determine the cohomology algebra $H^*(\Omega S^{2n}; \mathbb{Z})$.

31.3.4. Rational Exterior and Polynomial Algebras. When we use cohomology with coefficients in $\mathbb{Q}$, the denominators that showed up in Problem 31.43 lose their significance, and the algebra of the spectral sequence is considerably more straightforward.

Problem 31.45. Let X be a simply-connected space.

(a) Show $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x]$ with $|x| = 2n$ if and only if $H^*(\Omega X; \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(y)$ with $|y| = 2n - 1$ and $\tau(y) = x$.

(b) Show $H^*(X; \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(x)$ with $|x| = 2n + 1$ if and only if $H^*(\Omega X; \mathbb{Q}) \cong \mathbb{Q}[y]$ with $|y| = 2n$ and $\tau(y) = x$.

Problem 31.46. Determine the cohomology algebra $H^*(K(\mathbb{Z}, n); \mathbb{Q})$.

The simple insight of Problem 31.45 is a clue to some much more powerful and generally applicable connections between the algebra of a space and it loop space that are expressed in theorems of Borel and Postnikov (which are Theorem 33.21 and Theorem 33.40 in this book).

31.3.5. Construction of Steenrod Squares. Our construction (in Chapter 23) of the Steenrod reduced powers was a long process, and one of the main consequences was the existence theorem: the p^{th} power operation on classes in dimension $H^n(\,?\,;\mathbb{Z}/p)$ is part of a stable cohomology operation. We did this initially by direct construction (for all primes p), and later (in Section 23.7.1) you proved it for $p = 2$ using the G. W. Whitehead pullback square. Here you will use what you now know about the transgression to derive the existence and uniqueness of Sq^n.

Problem 31.47. Show that the squaring operation

$$\text{SQUARE} : H^n(\,?\,;\mathbb{Z}/2) \longrightarrow H^{2n}(\,?\,;\mathbb{Z}/2)$$

is part of a unique stable cohomology operation (which is of course Sq^n).

In the next chapter we see that a similar argument can be used for odd primes too.

Exercise 31.48. Compare your solutions to Problems 31.47 and 23.70.

31.4. Simplifying the Leray-Serre Spectral Sequence

In this section you'll derive (or rederive) some theoretical results for the computation of the cohomology of the total space of fibrations for which the Leray-Serre spectral sequence degenerates in one way or another.

31.4.1. Two Simplifying Propositions. We begin with two propositions which allow us to greatly simplify spectral sequences, either by guaranteeing their collapse or by reducing them to ordinary exact sequences.

Proposition 31.49. *Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a fibration sequence in which either $H^*(F;R)$ or $H^*(B;R)$ is free over the ring R. Then the following are equivalent:*

(1) $j^* : H^*(E;R) \to H^*(F;R)$ is surjective,

(2) the Leray-Serre spectral sequence $E(p;R)$ collapses at E_2.

Problem 31.50. Prove Proposition 31.49.

Proposition 31.49 tells us when all of the differentials in a spectral sequence are zero. The next simplest possibility is that for each group $E_2^{s,t}$, there is at most one nontrivial differential. This is the case when all but one of the groups that could possibly be the target of a differential are zero.

Proposition 31.51. *Let X be a filtered space with (diagonally indexed) spectral sequence $E(X)$ that converges to $\mathrm{Gr}^* H^*(X)$.*

(a) *Suppose that all of the groups* $E_2^{s,t}(X)$ *with* $s + t = n$ *are zero, except perhaps* $E_2^{s_1,t_1}$ *and* $E_2^{s_2,t_2}$*, with* $s_1 < s_2$*. Show that there is an exact sequence*

$$0 \to \operatorname{Gr}^{s_2} H^n(X) \longrightarrow H^n(X) \longrightarrow \operatorname{Gr}^{s_1} H^n(X) \to 0.$$

(b) *Assume in addition that at most two of the groups* $E_2^{s,t}(X)$ *with* $s+t = n+1$ *are nonzero, say* $E_2^{u_1,v_1}$ *and* $E_2^{u_2,v_2}$*, with*

$$u_1 < u_2, \quad u_1 \leq s_1 + 1 \quad \text{and} \quad u_2 \leq s_2 + 1.$$

Finally, assume that $E_2^{s,t}(X) = 0$ *for* $s + t = n - 1$ *and* $s < s_1 - 1$ *and for* $s + t = n + 2$ *and* $s > u_2 + 1$*. Show that there is an exact sequence*

$$0 \to \operatorname{Gr}^{s_1} H^n(X) \longrightarrow E_2^{s_1,t_1}(X) \xrightarrow{d_{u_2-s_1}} E_2^{u_2,v_2}(X) \longrightarrow \operatorname{Gr}^{u_2} H^{n+1}(X) \to 0$$

(interpret d_1 *as zero in this problem).*

Problem 31.52. Prove Proposition 31.51.

Hint. The hypotheses are atrocious to write down, so draw a picture to see what they really mean.

Proposition 31.51 is often used together with Lemma A.16 (Section A.2).

31.4.2. The Leray-Hirsch Theorem. We apply Proposition 31.49 to derive the Leray-Hirsch theorem.

Suppose $F \xrightarrow{j} E \xrightarrow{p} B$ is a fibration sequence in which $H^*(F)$ is a free R-module and $j^* : H^*(E) \to H^*(F)$ is surjective. Then there exists a map $\theta : H^*(F) \to H^*(E)$ of R modules such that $j^* \circ \theta = \mathrm{id}_{H^*(F)}$, and we define a map of $H^*(B)$-modules

$$\Phi : H^*(B) \otimes_R H^*(F) \longrightarrow H^*(E)$$

by the rule $\Phi(x \otimes y) = p^*(x) \cdot \theta(y)$. Now we are prepared to state the **Leray-Hirsch theorem**.

Theorem 31.53 (Leray-Hirsch theorem)**.** *Let* $p : E \to B$ *be an* H^**-orientable fibration with fiber* F*. If* $H^*(B)$ *and* $H^*(F)$ *are finitely generated* R*-modules, then the map* $\Phi : H^*(B) \otimes_R H^*(F) \to H^*(E)$ *is an isomorphism of* $H^*(B)$*-modules.*

Note that Theorem 31.53 does not claim that Φ is a homomorphism of algebras. The surjectivity of j^* guarantees that the Leray-Serre spectral sequence collapses, so that $\operatorname{Gr}^* H^*(E) \cong_R H^*(B) \otimes H^*(F)$ as bigraded R-algebras. The question is whether the map Φ is an isomorphism.

Problem 31.54. Filter $H^*(B) \otimes_R H^*(F)$ by setting $\mathcal{F}^s = \Phi^{-1}(\mathcal{F}^s H^*(E))$.

(a) Show that the $H^*(B)$-module structure on $H^*(E)$ gives $\operatorname{Gr}^* H^*(E)$ the structure of an $H^*(B)$-module.

(b) Show that $\mathrm{Gr}^*(\Phi)$ is a homomorphism of $H^*(B)$-modules.

(c) Show that $\theta(H^s(F)) \cap \mathcal{F}^0 H^*(E) = 0$; conclude that $H^*(F) \otimes H^0(B) \subseteq \mathrm{Im}(\mathrm{Gr}^*(\Phi))$.

(d) Show that $H^*(E)$ is a free graded R-module, finitely generated in each dimension.

(e) Complete the proof of the Leray-Hirsch theorem.

Exercise 31.55.

(a) Show that if θ is induced by a map $E \to F$ of spaces, then Φ is an algebra homomorphism.

(b) Construct an example of a fibration satisfying the hypotheses of Theorem 31.53 for which Φ is not an algebra homomorphism.

Project 31.56. Let $F \xrightarrow{j} E \xrightarrow{p} B$ be an H^*-orientable fiber sequence with either $H^*(B)$ or $H^*(F)$ free. Proposition 31.49 implies that if j^* is surjective, then p^* is injective. Is the converse true?

31.4.3. Exact Sequences for Fibrations Involving Spheres. Next we consider fibrations in which either the base or the fiber is a sphere. Since the cohomology of the sphere is so simple, $E_2^{*,*}(p)$ is comparatively easy to understand, and the Leray-Serre spectral sequences degenerate to exact sequences.

Problem 31.57 (Wang sequence)**.** Let $p : E \to S^n$ with fiber F. Use Proposition 31.51 to derive the Wang sequence, including the algebraic properties of its differential.

Now we consider **spherical fibrations**: fibrations whose fiber is a sphere. Just as with fibrations over spheres, the Leray-Serre spectral sequence of a spherical fibration is so simple that it can be reduced to an exact sequence, called the **Gysin sequence**.

Proposition 31.58 (Gysin sequence)**.** *Let $p : E \to B$ be an H^*-orientable fibration with fiber S^n. Show that there is an exact sequence*

$$\cdots \to H^k(B) \xrightarrow{p^*} H^k(E) \xrightarrow{\gamma} H^{k-n}(B) \xrightarrow{d} H^{k+1}(B) \to \cdots .$$

Problem 31.59. Prove Proposition 31.58.

If $p : E \to B$ is a spherical fibration with fiber S^n, then the class $e = d_{n+1}(\mathbf{s}_n) \in H^{n+1}(B)$ is called the **Euler class** of p.

Problem 31.60. Show that the homomorphism d in the Gysin sequence is given by $d(x) = e \cdot x$.

If we are willing to do the hard work of setting up the Gysin sequence, we can easily compute the cohomology rings of projective spaces.

Problem 31.61.

(a) Compute the cohomology rings of $\mathbb{C}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$ by studying the Gysin sequence for $S^{d-1} \to S^{nd+(d-1)} \to \mathbb{F}\mathrm{P}^n$.

(b) Similarly, compute $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2)$. Be careful! $\mathbb{R}\mathrm{P}^n$ is *not* simply-connected.

Project 31.62. Somehow the existence of the fibration sequence $S^{d-1} \to S^{nd+(d-1)} \to \mathbb{F}\mathrm{P}^n$ puts very strong restrictions on the effect of the diagonal map $\mathbb{F}\mathrm{P}^n \to \mathbb{F}\mathrm{P}^n \wedge \mathbb{F}\mathrm{P}^n$ in cohomology. Try to explain the connection in terms of spaces and maps as opposed to algebra.

The analysis of fibrations whose fibers are spheres becomes particularly easy if we impose some special algebraic conditions—conditions which are actually present in some important examples.

Theorem 31.63. *Let $p : E \to B$ be an H^*-orientable fibration with fiber S^{2n-1}. Assume that there is an algebra homomorphism $\sigma : H^*(E) \to H^*(B)$ such that $\sigma \circ p^* = \mathrm{id} : H^*(B; R) \to H^*(B; R)$. Then there is an algebra isomorphsim*

$$H^*(B; R) \cong H^*(E; R) \otimes_R R[e]$$

where $e \in H^{2n}(B; R)$ is the Euler class of the fibration. Furthermore, the map $p^* : H^*(B; R) \to H^*(E; R)$ corresponds to the unique ring homomorphism which is the identity on $H^*(B)$, sends $1 \mapsto 1$ and sends $e \mapsto 0$.

Problem 31.64.

(a) Show that the rule $\Phi(u \otimes e^k) = \sigma(u) \cdot e^k$ defines an algebra homomorphism $\Phi : H^*(E; R) \otimes R[e] \to H^*(B; R)$.

(b) Show that if $\Phi\left(\sum_i u_i \otimes e^{k_i}\right) = 0$, then all $k_i \geq 1$. Conclude that Φ is injective.

 Hint. Try to find an element of $\ker(\Phi)$ of minimum degree.

(c) Show that $H^*(B; R) = \sigma(H^*(E; R)) + e \cdot H^*(B; R)$. Conclude that Φ is surjective.

You have proved Theorem 31.63.

31.4.4. The Thom Isomorphism Theorem. In his work on cobordism, René Thom introduced the **Thom space** of a vector bundle and proved the Thom Isomorphism Theorem to determine its cohomology (and homology).

The Thom Space of a Spherical Fibration. We begin by giving a slightly nonstandard account of Thom spaces. Let $p : E \to B$ be a fibration with fiber F. The mapping cylinder of p fits into the homotopy commutative diagram

$$\begin{tikzcd} E \arrow[r, dashed] \arrow[d, "p"'] & M_p \arrow[r] \arrow[d] & C_p \\ B \arrow[r, equal] \arrow[ur, bend left] & B \end{tikzcd}$$

in which the solid arrow part is strictly commutative. Thus each fibration $p : E \to B$ gives rise, in a natural way, to a map $B \to C_p$. For each $x \in B$, we may construct the morphism

$$\begin{tikzcd} F_x \arrow[r] \arrow[d] & E \arrow[d] \\ \{x\} \arrow[r] & B \end{tikzcd}$$

of fibrations and thereby obtain natural maps $\xi_x : \Sigma F_x \to C_p$ for each $x \in X$.

When we apply this construction to a fibration in which the fiber is a sphere S^n, then we call C_p the **Thom space** of the fibration and denote it $T(p)$.

Problem 31.65. Determine $T(p)$ when $p = \mathrm{pr}_1 : B \times S^n \to B$.

Our construction is equivalent to the standard approach, but looks slightly different. The usual approach is to start with an $(n+1)$-dimensional vector bundle $p : E \to B$ in which it is possible to introduce a **Riemannian metric**, which is a continuous assignment of inner products to the fibers of p. The vectors of unit length (or less), and constitute subbundles $S(E) \subseteq D(E) \subseteq E$, known as the **sphere bundle** and the **disk bundle** associated to E. The **Thom space** of E is the quotient space $T(p) = D(E)/S(E)$.

Problem 31.66.

(a) Show that $S(E) \hookrightarrow D(E)$ is a cofibration.

(b) Show that there is a commutative square

$$\begin{tikzcd} S(E) \arrow[r] \arrow[d, equal] & M_{S(p)} \arrow[d, "\cong"] \\ S(E) \arrow[r] & D(E) \end{tikzcd}$$

and conclude that the two definitions of $T(p)$ are canonically homeomorphic.

There is yet another common description of the Thom space, but it is only valid for vector bundles with compact bases.

Problem 31.67. Show that if $p : E \to B$ is a vector bundle and B is compact, then $T(p)$ is also homeomorphic to the one-point compactification E_∞ of E.

The Thom Class of a Spherical Fibration. A **Thom class** for a spherical fibration $p : E \to B$ with fiber S^n is an element $U \in H^{n+1}(T(p))$ such that for each $x \in B$, the map $\xi_x : \widetilde{H}^{n+1}(T(p); R) \to \widetilde{H}^{n+1}(\Sigma S^n; R)$ carries U to a generator of $\widetilde{H}^{n+1}(\Sigma S^n; R) \cong R$.

If p has a Thom class U, then we define $\Psi : H^*(B) \to \widetilde{H}^*(T(p))$ by the rule

$$\Psi(x) = p^*(x) \cdot U,$$

where the product is the relative product $H^*(E) \otimes H^*(T(p)) \to H^*(T(P))$ defined in Section 25.2. The Thom Isomorphism Theorem shows that the map Ψ must be an isomorphism.

Theorem 31.68 (Thom Isomorphism Theorem)**.** *If $p : E \to B$ is a spherical fibration with Thom class U, then the map $\Psi : H^*(B) \to \widetilde{H}^*(T(p))$ is an isomorphism.*

Problem 31.69. Use Problem 31.11 to prove Theorem 31.68.

Project 31.70. Find a proof of Theorem 31.68 using a local-to-global approach instead of spectral sequences.

31.4.5. The Serre Exact Sequence. Finally, we derive the Blakers-Massey and Serre sequences (Theorems 18.13 and Problem 25.123). The basic content of these theorems is that, in a certain range of dimensions depending on the connectivity of the spaces involved, fibration sequences are indistinguishable from cofibration sequences.

Problem 31.71 (Serre exact sequence)**.** Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a fibration sequence, in which F is $(f-1)$-connected and B is $(b-1)$-connected.

(a) Write out the E_2 term of the Leray-Serre spectral sequence for p.

(b) For which dimensions k do the hypotheses of Proposition 31.51 hold?

(c) Show that there is an exact sequence

$$\cdots \to H^{k-1}(B) \longrightarrow H^k(B) \longrightarrow H^k(E) \longrightarrow H^k(F) \to \cdots$$

valid for $k \le b + f - 2$.

Problem 31.72. Derive the Blakers-Massey exact sequence involving the homotopy groups of the spaces in a cofiber sequence from the Serre exact sequence.

31.5. Additional Problems and Projects

Problem 31.73. Let $p : E \to B$ be an H^*-orientable fibration with fiber F. Show that if either $\dim(F) < r$ or $\dim(B) \leq r$, then the Leray-Serre spectral sequence collapses at $E_{r+1}^{*,*}(p)$.

Problem 31.74. Show that if u and v are a transgressive pair for the fibration $p : E \to B$, then the diagram

$$\begin{tikzcd} \Omega B \arrow[rr, "\delta"] \arrow[dr, "\Omega u"'] & & F \arrow[dl, "v"] \\ & K(G, n-1) & \end{tikzcd}$$

commutes up to homotopy, where δ is the connecting map in the long fiber sequence of p.

Project 31.75. Let $p : E \to B$ be a fibration with fiber F. Define what it means for homology classes $a \in H_n(B; G)$ and $b \in H_{n-1}(F; G)$ to be a transgressive pair, and construct a homology transgression map τ. Show that if p is H_*-orientable, then the differential $d_n : E_{n,0}^n(p) \to E_{0,n-1}^n$ may be identified with τ.

Problem 31.76. Let $A \xrightarrow{i} B \xrightarrow{q} C$ be a cofiber sequence, let $p = p_C : E \to C$ be an H^*-orientable fibration, and form pullbacks resulting in the diagram

$$\begin{tikzcd} E_A \arrow[r] \arrow[d, "p_A"'] & E_B \arrow[r] \arrow[d, "p_B"'] & E_C \arrow[d, "p_C"] \\ A \arrow[r, "i"] & B \arrow[r, "q"] & C. \end{tikzcd}$$

(a) Show that the induced sequence

$$E_2^{*,*}(p_A) \xleftarrow{i^{*,*}} E_2^{*,*}(p_B) \xleftarrow{q^{*,*}} E_2^{*,*}(p_C)$$

is exact.

(b) Show that if $x \in E_2^{*,*}(p_B)$ is not in the image of $q^{*,*}$, then x is not a boundary in $E_r^{*,*}(p_B)$ for any r.

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Chapter 32

Application: Bott Periodicity

We begin this chapter by determining the cohomology algebra of the unitary groups $BU(n)$. Because of the fibration sequences $S^{2n-1} \to BU(n-1) \to BU(n)$, this computation is a simple application of Theorem 31.63.

Once this is done, we undertake a much more intricate computation. Write $D_n \subseteq U(n)$ for the subspace of diagonal matrices, which is easily seen to be isomorphic to $(S^1)^n$. Then the inclusion $i : D_n \hookrightarrow U(n)$ induces a map $i^* : H^*(BU(n); \mathbb{Z}) \to H^*(BD_n; \mathbb{Z})$. We show that restricting the target of i^* yields an isomorphism of $H^*(BU(n); \mathbb{Z})$ with the algebra of symmetric polynomials in $t_1, t_2, \ldots, t_n$. Since $H^*(BD_n; \mathbb{Z}) = \mathbb{Z}[t_1, \ldots, t_n]$ with $|t_i| = 2$ for each i, it is easy to work with the algebra and Steenrod operations there. Thus this isomorphism is a powerful tool when it comes to doing algebraic-topological calculations in $H^*(BU(n))$.

We give $BU = \operatorname{colim} BU(n)$ the structure of an H-space, so that its homology and cohomology are dual Hopf algebras. Using symmetric functions, we determine the diagonal in $H^*(BU; \mathbb{Z})$ and deduce that $H_*(BU; \mathbb{Z})$ is the polynomial algebra $\mathbb{Z}[a_1, a_2, \ldots]$ with $|a_n| = 2n$.

Then we turn our attention to the homology algebras $H_*(\Omega SU(n))$ and discover that when $n = \infty$, we have an abstract isomorphism $H_*(\Omega SU) \cong H_*(BU)$. Thus we are led to conjecture the Bott Periodicity Theorem: ΩSU and BU are homotopy equivalent. We define the Bott map $\beta : BU \to \Omega SU$ and conclude the chapter by proving that it is a homotopy equivalence.

32.1. The Cohomology Algebra of $BU(n)$

In this section, we make our simple computation of the cohomology algebra $H^*(BU(n); R)$. Since $U(n)/U(n-1) \cong S^{2n-1}$, Problem 16.52 gives us the fibration sequences

$$\cdots \to U(n-1) \longrightarrow U(n) \longrightarrow S^{2n-1} \longrightarrow BU(n-1) \longrightarrow BU(n).$$

We will focus on the last three terms $S^{2n-1} \to BU(n-1) \to BU(n)$ because they enable us to apply the Gysin sequence to do an inductive computation of the cohomology of $BU(n)$.

Problem 32.1.

(a) Show that $BU(1) = \mathbb{C}\mathrm{P}^\infty$.

(b) Show that $H^*(BU(1); R) = R[c_1]$ with $|c_1| = 2$.

Problem 32.1 is the first step in an inductive computation that results in the following theorem.

Theorem 32.2. *The cohomology ring of $BU(n)$ is given by*

$$H^*(BU(n); R) \cong R[c_1, c_2, \ldots, c_n]$$

where $|c_k| = 2k$. Furthermore, the map $H^*(BU(n); R) \to H^*(BU(n-1); R)$ induced by the inclusion $BU(n-1) \to BU(n)$ is determined by $c_k \mapsto c_k$ for $k < n$ and $c_n \mapsto 0$.

Problem 32.3. We have already established Theorem 32.2 in the case $n = 1$; assume it is correct for $n - 1 \geq 1$.

(a) Show that $BU(n-1) \to BU(n)$ is a $(2n-1)$-equivalence. Conclude that p^* is surjective. For $k < n$, define $c_k \in H^*(BU(n))$ to be the unique element which maps to $c_k \in H^*(BU(n-1))$.

(b) Define the class c_n and use Theorem 31.63 to prove Theorem 32.2.

32.2. The Torus and the Symmetric Group

Inside of $U(n)$ is the subgroup D_n of diagonal matrices. The inclusion $D_n \hookrightarrow U(n)$ induces a map $BD_n \to BU(n)$. We will identify its induced map on cohomology algebras.

32.2.1. The Action of the Symmetric Group. In this section we define and study an action of symmetric group $\mathrm{Sym}(n)$ on the space D_n and its induced action on BD_n.

Problem 32.4. Show that D_n is isomorphic to $S^1 \times \cdots \times S^1$, and conclude that $BD_n \cong \mathbb{C}\mathrm{P}^\infty \times \cdots \times \mathbb{C}\mathrm{P}^\infty$.

It follows that the map in cohomology induced by $BD_n \to BU(n)$ takes the form

$$R[c_1, c_2, \ldots, c_n] \longrightarrow R[t_1, t_2, \ldots, t_n],$$

where $|c_k| = 2k$ and $|t_k| = 2$ for $k = 1, 2, \ldots, n$.

Next we look at the maps induced by conjugation.

Problem 32.5. Let G be a path-connected topological group, and let $a \in G$. Show that the function $\chi_a : G \to G$ given by $g \mapsto a^{-1}ga$ is homotopic, through homomorphisms, to id_G. Conclude that $B\chi_a \simeq \mathrm{id}_{BG}$.

Let $\mathrm{Sym}(n)$ denote the symmetric group of permutations of the set $\{1, 2, \ldots, n\}$. For $\sigma \in \mathrm{Sym}(n)$, define $T_\sigma : \mathbb{C}^n \to \mathbb{C}^n$ to be the unique linear transformation such that $T_\sigma(e_i) = e_{\sigma(i)}$ for each i. This defines an injective map $\mathrm{Sym}(n) \hookrightarrow U(\mathbb{C}^n)$, so we will consider $\mathrm{Sym}(n)$ as a subgroup of $U(\mathbb{C}^n)$; we'll often blur the distinction between σ and T_σ thus.

Problem 32.6.

(a) Let $\sigma \in \mathrm{Sym}(n) \subseteq U(\mathbb{C}^n)$. Show that χ_σ restricts to the homomorphism $\chi_\sigma : D_n \to D_n$ which permutes the factors of $D_n \cong S^1 \times \cdots \times S^n$ by the permutation σ.

(b) Show that the induced map $B\chi_\sigma : BD_n \to BD_n$ is again permutation of the factors by the permutation σ.

(c) Determine the map $\mathbb{Z}[t_1, \ldots, t_n] \to \mathbb{Z}[t_1, \ldots, t_n]$ induced by $B\chi_\sigma$.

32.2.2. Identifying $H^*(BU(n))$ with Symmetric Polynomials. Now we show that the effect of the action of $\mathrm{Sym}(n)$ on elements in the image of $H^*(BU(n)) \to H^*(BD_n)$ is trivial. It follows that this map factors through an algebra homomorphism

$$H^*(BU(n)) \longrightarrow H^*(BD_n)^{\mathrm{Sym}(n)},$$

where $H^*(BD_n)^{\mathrm{Sym}(n)}$ denotes the subalgebra of $H^*(BD_n)$ of those polynomials that are unchanged by permutation of the variables.

This algebra is known as the ring of **symmetric polynomial** on the indeterminates $t_1, t_2, \ldots, t_n$; an account of the basic algebraic theory of symmetric polynomials can be found in Section A.7.

Problem 32.7. Use the commutative diagram

$$\begin{tikzcd} BD_n \arrow[r, "\phi_n"] \arrow[d, "B\chi_\sigma"'] & BU(n) \arrow[d, "B\chi_\sigma"] \\ BD_n \arrow[r, "\phi_n"] & BU(n) \end{tikzcd}$$

to show that the image of $H^*(BU(n);R) \to H^*(BD_n;R) \cong R[t_1,\ldots,t_n]$ is contained in the ring $R[t_1,\ldots,t_n]^{\mathrm{Sym}(n)}$.

Hint. Use Problem 32.6.

It follows that there is a map ρ making the diagram

$$\begin{tikzcd} & H^*(BD_n)^{\mathrm{Sym}(n)} \arrow[d] \\ H^*(BU(n)) \arrow[ur, bend left, "\rho_n"] \arrow[r, "\phi_n^*"] & H^*(BD_n) \end{tikzcd}$$

commutative. Let's focus now on cohomology with coefficients in an integral domain R, so that we can use Theorem A.57 to identify $H^*(BD_n;R)^{\mathrm{Sym}(n)}$ with the ring

$$R[t_1,\ldots,t_n]^{\mathrm{Sym}(n)} = R[\sigma_1,\sigma_2,\ldots,\sigma_n],$$

where σ_k denotes the k^{th} elementary symmetric function on the variables $t_1,t_2,\ldots,t_n$.

Exercise 32.8. Determine the degree of $\sigma_k(t_1,\ldots,t_n)$ in $H^*(BD_n;R)$.

First we constructed an algebra map $H^*(BU(n)) \to H^*(BD_n;R)^{\mathrm{Sym}(n)}$; and now we have shown that $H^*(BU(n);R)$ and $H^*(BD_n;R)^{\mathrm{Sym}(n)}$ are isomorphic graded algebras. There is only one thing to hope for.

32.2.3. The Main Theorem. The amazing, beautiful, important and useful thing is that ρ_n is an *isomorphism* of $H^*(BU(n);R)$ with the ring of symmetric polynomials.

Theorem 32.9. *Let R be an integral domain. Then the map*

$$\rho_n : H^*(BU(n);R) \longrightarrow H^*(BD_n;R)^{\mathrm{Sym}(n)}$$

is given by $c_k \mapsto \sigma_k$ and hence is an isomorphism of algebras.

We will prove Theorem 32.9 by induction on n.

Problem 32.10.

(a) Show that Theorem 32.9 holds in the case $n=1$.

(b) Show that it suffices to prove Theorem 32.9 in the case $R=\mathbb{Z}$.

(c) Show that ρ_n is an isomorphism if and only if it is surjective.

As it turns out, Problem 32.10 is a red herring, since we need to know that ρ is injective before we can prove that it is surjective.

Problem 32.11.

(a) Show there is a fibration sequence of the form

$$\mathbb{C}\mathrm{P}^{n-1} \longrightarrow \mathbb{C}\mathrm{P}^\infty \times BU(n-1) \xrightarrow{\widetilde{\phi}_n} BU(n).$$

(b) Show that the cohomology spectral sequence—with coefficients in any ring R—for this fibration collapses at the $E_2^{*,*}$-term. Conclude that $\widetilde{\phi}_n^*$ is injective.

(c) Show that there are commutative triangles

$$\begin{tikzcd} \mathbb{C}\mathrm{P}^\infty \times BD_{n-1} \arrow[r, "\mathrm{id}\times\phi_{n-1}"] \arrow[dr, "\phi_n"'] & \mathbb{C}\mathrm{P}^\infty \times BU(n-1) \arrow[d, "\widetilde{\phi}_n"] \\ & BU(n). \end{tikzcd}$$

(d) Show by induction that $\phi_n^* : H^*(BU(n)) \to H^*(BD_n)$ is injective.

It remains to show that ρ_n is surjective, and we know that it is in the case $n = 1$. So we assume that ρ_k is an isomorphism for $k < n$ and show that ρ_n is surjective. For this it suffices to show that $\sigma_k \in \mathrm{Im}(\rho)$ for $k = 1, 2, \ldots, n$.

Problem 32.12. Consider the commutative square

$$\begin{tikzcd} BD_{n-1} \arrow[r] \arrow[d, "\phi_{n-1}"'] & BD_n \arrow[d, "\phi_n"] \\ BU(n-1) \arrow[r] & BU(n). \end{tikzcd}$$

(a) Show that $\sigma_k \in \mathrm{Im}(\rho)$ for $k < n$.

(b) Show that $\phi_n^*(c_n) = a \cdot \sigma_n + f(\sigma_1, \ldots, \sigma_{n-1}) \in \mathbb{Z}[t_1, \ldots, t_n]$ for some integer a and some polynomial f in $n-1$ variables.

(c) Show that $f(\sigma_1, \ldots, \sigma_{n-1}) = 0$, so that $\phi_n^*(c_n) = a \cdot \sigma_n \in H^*(BD_n; \mathbb{Z})$.

(d) Show that $\phi_n^*(c_n) = [a] \cdot \sigma_n(t_1, \ldots, t_n) \neq 0 \in H^*(BD_n; \mathbb{Z}/m)$ for any integer $m > 1$ (where $[a]$ denote the congruence class of a mod m).

(e) Show that $a = \pm 1$ and complete the proof of Theorem 32.9.

We rename our polynomial algebra generators, if necessary, so that $\rho(c_k) = \sigma_k$ in $H^{2k}(BU(n); R)$ rather than $-\sigma_k$.

Project 32.13. Decide whether or not it is necessary.

32.3. The Homology Algebra of BU

Now we turn our attention to the homology of the classifying space BU of the infinite unitary group U, which we may build as the colimit of the telescope

$$\cdots \to BU(n) \longrightarrow BU(n+1) \to \cdots \to BU.$$

The spaces $BU(n)$ are not H-spaces, but their union BU is an H-space, and its homology $H_*(BU;\mathbb{Z})$ has a Pontrjagin product making it into a Hopf algebra dual to its cohomology. Our goal in this section is to determine the structure of $H_*(BU;\mathbb{Z})$ as a graded algebra.

32.3.1. H-Structure for BU. The standard inclusions $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ given by $(z_1, z_2, \ldots, z_n) \mapsto (z_1, z_2, \ldots, z_n, 0)$ induce maps $U(n) \hookrightarrow U(n+1)$ given on matrices by the formula

$$A \longmapsto \left[\begin{array}{c|c} A & \\ \hline & 1 \end{array}\right].$$

The colimit of the sequence $\cdots \to U(n) \to U(n+1) \to \cdots$ is the **infinite unitary group** U. These maps induce a telescope diagram of classifying spaces $\cdots \to BU(n) \to BU(n+1) \to \cdots$, and its colimit is the space BU.

Problem 32.14.

(a) Show that U is the group of inner-product preserving T automorphisms of $\mathbb{C}^\infty$ such that $\ker(T - \mathrm{id})$ is finite-dimensional.

(b) Show that there is a fibration sequence $U \to EU \to BU$ in which EU is contractible, so that BU is a classifying space for the group U.

We need to talk about unitary groups for other complex inner product spaces and Problem 32.14(a) gives us the key. We define $U(V)$ to be the group of all inner product preserving transformations T of V such that $\ker(T - \mathrm{id})$ is finite-dimensional. If $V \subseteq W$, then there is an induced inclusion $U(V) \hookrightarrow U(W)$, which you explored in Problem 9.36.

Problem 32.15. Let U and V be two complex inner product spaces.

(a) Show that $U(V) \times U(W)$ is naturally a subgroup of $U(V \oplus W)$.

(b) Show that there is a canonical extension in the diagram

$$\begin{tikzcd} BU(V) \vee BU(W) \arrow[r, "{(BU(\mathrm{in}_V), BU(\mathrm{in}_W))}"] \arrow[d, "\mathrm{in}"'] & BU(V \oplus W) \\ BU(V) \times BU(W). \arrow[ur, "\mathrm{juxt}"'] & \end{tikzcd}$$

For $V = \mathbb{C}^n$ and $W = \mathbb{C}^m$, the map juxt is represented on matrices by the juxtaposition rule

$$(A, B) \longmapsto \left[\begin{array}{c|c} A & \\ \hline & B \end{array}\right].$$

We are hoping to define an H-structure on BU, and the fundamental idea is that because $\mathbb{C}^\infty$ is infinite-dimensional, we can shuffle the coordinates of $\mathbb{C}^\infty$ with one another. To be more precise about this, we give $\mathbb{C}^\infty$ the standard basis $\{e_1, e_2, \ldots, e_n, \ldots\}$ and define

$$\mathbb{C}^{\mathrm{odd}} = \mathrm{span}\{e_k \mid k \text{ is odd}\} \qquad \text{and} \qquad \mathbb{C}^{\mathrm{even}} = \mathrm{span}\{e_k \mid k \text{ is even}\}.$$

The maps $k \mapsto 2k-1$ and $k \mapsto 2k$ induce isomorphisms $\mathbb{C}^\infty \to \mathbb{C}^{\mathrm{odd}}$ and $\mathbb{C}^\infty \to \mathbb{C}^{\mathrm{even}}$ and ultimately homeomorphisms

$$\mathrm{pr}_{\mathrm{odd}} : BU \xrightarrow{\cong} BU(\mathbb{C}^{\mathrm{odd}}) \qquad \text{and} \qquad \mathrm{pr}_{\mathrm{even}} : BU \xrightarrow{\cong} BU(\mathbb{C}^{\mathrm{even}}).$$

Lemma 32.16. *The diagram*

$$\begin{tikzcd} BU \arrow[r, "\mathrm{pr}_{\mathrm{odd}}"] \arrow[d, "\mathrm{pr}_{\mathrm{even}}"'] \arrow[dr, "\mathrm{id}_{BU}"] & BU(\mathbb{C}^{\mathrm{odd}}) \arrow[d, "\mathrm{in}_{\mathrm{odd}}"] \\ BU(\mathbb{C}^{\mathrm{even}}) \arrow[r, "\mathrm{in}_{\mathrm{even}}"'] & BU \end{tikzcd}$$

is homotopy commutative.

Problem 32.17. Let W be a complex inner-product space.

(a) Establish a bijection between the set of n-frames $F \in V_n(W)$ and the collection of inner-product preserving embeddings $T : \mathbb{C}^n \to W$.

(b) Explain how each n-frame induces a map $f : BU(\mathbb{C}^n) \to BU(W)$.

(c) Show that a path $\alpha : I \to V_n(W)$ from F_0 to F_1 induces a homotopy between the corresponding maps $f_0, f_1 : BU(n) \to BU(W)$.

The maps we are interested in are maps between classifying spaces of unitary groups induced by injective maps between their standard bases. For any strictly increasing function $\alpha : \mathbb{N} \to \mathbb{N}$, we define $T_\alpha : \mathbb{C}^\infty \to \mathbb{C}^\infty$ to be the unique linear embedding given by $T_\alpha(e_i) = e_{\alpha(i)}$ for $i \in \mathbb{N}$. Such functions α are determined by their images, which could be any infinite subset $S \subseteq \mathbb{N}$.

Problem 32.18. Consider the ∞-frames F_0 and F_1 in $\mathbb{C}^\infty$ corresponding to the subsets

$$S_0 = \{1, 2, \ldots, n, \alpha(n+1), \alpha(n+2), \ldots\}$$

and

$$S_1 = \{1, 2, \ldots, n, n+1, \alpha(n+2), \ldots\}.$$

Write $f_0, f_1 : BU \to BU$ for the induced maps.

(a) Find a path from F_0 to F_1 in $V_\infty(\mathbb{C}^\infty)$ that is constant away from the $(n+1)^{\mathrm{st}}$ vector.

(b) Show that the induced maps $f_0, f_1 : BU \to BU$ are homotopic by a homotopy that is constant on $BU(n)$.

(c) Use an infinite concatenation of homotopies to prove that if $S \subseteq \mathbb{N}$ is an infinite subset, then the corresponding $f : BU \to BU$ is homotopic to id_{BU}.

(d) Prove Lemma 32.16.

Now we can define our H-structure on BU. The **shuffle product** on BU is the map $BU \times BU \to BU$ given by

$$\begin{tikzcd} BU \times BU \arrow[rr, "\text{shuffle}"] \arrow[d, "\mathrm{pr}_{\mathrm{odd}} \times \mathrm{pr}_{\mathrm{even}}"'] & & BU \arrow[d, equal] \\ BU(\mathbb{C}^{\mathrm{odd}}) \times BU(\mathbb{C}^{\mathrm{even}}) \arrow[rr, "\mathrm{juxt}"'] & & BU(\mathbb{C}^{\mathrm{odd}} \oplus \mathbb{C}^{\mathrm{even}}) \end{tikzcd}$$

using the fact that $\mathbb{C}^\infty = \mathbb{C}^{\mathrm{odd}} \oplus \mathbb{C}^{\mathrm{even}}$. We'll write $\mathrm{sh} : BU \times BU \to BU$ for the shuffle product from now on.

Problem 32.19. Show that $\mathrm{sh} : BU \times BU \to BU$ gives BU the structure of an H-space.

32.3.2. The Diagonal of $H^*(BU;\mathbb{Z})$. In Section 32.1 we determined the cohomology algebra of $BU(n)$. Now we'll extend that computation to find $H^*(BU;\mathbb{Z})$ and then determine the map sh^*.

Problem 32.20.

(a) Determine the connectivity of the inclusion $BU(n) \hookrightarrow BU$.

(b) Show that $H^*(BU;\mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \ldots]$.

(c) Show that $H^*(BU;\mathbb{Z})$ has the structure of a Hopf algebra.

To determine the dual Hopf algebra $H_*(BU;\mathbb{Z})$, we need to determine the diagonal, which is the map $\mathrm{sh}^* : H^*(BU;\mathbb{Z}) \to H^*(BU;\mathbb{Z}) \otimes H^*(BU;\mathbb{Z})$ induced by the H-structure $\mathrm{sh} : BU \times BU \to BU$.

Proposition 32.21. *The map* $\mathrm{sh}^* : H^*(BU;\mathbb{Z}) \to H^*(BU;\mathbb{Z}) \otimes H^*(BU;\mathbb{Z})$ *is given by*

$$\mathrm{sh}^*(c_n) = \sum_{i+j=n} c_i \otimes c_j.$$

The proposition is clearly true for c_1.

Problem 32.22.

(a) Argue that $\mathrm{sh}^*(c_n)$ must have the form $\sum_{i+j=n} a_i \cdot (c_i \otimes c_j)$ for some coefficients $a_i \in \mathbb{Z}$, and show that

$$(\phi_i \times \phi_j)^*(\mu^*(c_n)) = a_i \cdot \sigma_i(t_1, \ldots, t_i) \otimes \sigma_j(t_{i+1}, \ldots, t_n).$$

(b) Show that for each $i+j=n$, the diagram

$$\begin{tikzcd} BD_i \times BD_j \arrow[r, "\mathrm{id}"] \arrow[d, "\phi_i \times \phi_j"'] & BD_n \arrow[d, "\phi_n"] \\ BU \times BU \arrow[r, "\mu"] & BU \end{tikzcd}$$

commutes up to homotopy.

(c) Prove Proposition 32.21.

32.3.3. The Pontrjagin Algebra $H_*(BU;\mathbb{Z})$. Now that we have determined the Hopf algebra structure of $H^*(BU;\mathbb{Z})$, we can deduce the structure of $H_*(BU;\mathbb{Z})$ by pure algebra.

Theorem 32.23. $H_*(BU;\mathbb{Z}) \cong \mathbb{Z}[a_1, a_2, \ldots]$.

Problem 32.24. Prove Theorem 32.23.

32.4. The Homology Algebra of $\Omega SU(n)$

Now we turn our attention to the homology of the loop spaces $\Omega SU(n)$, for $n \leq \infty$. The fibration sequence that we used in Section 32.1 to determine the cohomology algebras $H^*(BU(n);\mathbb{Z})$ can be extended to the left, yielding the sequence

$$\Omega SU(n) \longrightarrow \Omega SU(n+1) \longrightarrow \Omega S^{2n+1},$$

and this is the key to our inductive computation.

Theorem 32.25. *For each $n \leq \infty$, the homology algebra of $\Omega SU(n+1)$ is*

$$H_*(\Omega SU(n+1);\mathbb{Z}) \cong \mathbb{Z}[b_1, b_2, \ldots, b_n] \qquad \text{with} \qquad |b_k| = 2k,$$

and the maps $i_* : H_*(\Omega SU(m)) \to H_*(\Omega SU(n+1))$ (induced by the standard inclusions, with $m \leq n+1$) are given by $i_*(b_k) = b_k$.[1]

You will prove Theorem 32.25 inductively by applying the Leray-Serre spectral sequence to the fibration sequences $\Omega SU(n) \to \Omega SU(n+1) \to \Omega S^{2n+1}$.

Problem 32.26.

(a) Show that the homology Leray-Serre spectral sequences for the fibration sequences $\Omega SU(n) \to \Omega SU(n+1) \xrightarrow{p} \Omega S^{2n+1}$ collapse at E_2. Conclude that there is $H_*(\Omega SU(n+1)) \cong \mathbb{Z}[b_1, b_2, \ldots, b_n]$ as graded abelian groups, anyway.

(b) Write (as usual) $\sigma : S^{2n} \to \Omega S^{2n+1}$ for the map adjoint to $\mathrm{id}_{S^{2n+1}}$. Show that $\sigma_* : H_{2n}(S^{2n}) \to H_{2n}(\Omega S^{2n+1})$ is an isomorphism.

[1] This justifies our not introducing notation like $b_k^{(n+1)}$ for the generator of $H_*(\Omega SU(n+1))$ in degree $2k$.

(c) Show that $p_* : H_{2n}(\Omega SU(n+1)) \to H_{2n}(\Omega S^{2n+1})$ is surjective.

(d) Construct an algebra homomorphism
$$\Phi : H_*(\Omega SU(n); \mathbb{Z}) \otimes H_*(\Omega S^{2n+1}; \mathbb{Z}) \longrightarrow H_*(\Omega SU(n+1); \mathbb{Z}).$$

Write $A = H_*(\Omega S^{2n+1})$ and $A_{\leq s}$ for the graded subgroup which is A in dimensions $\leq s$ and zero for dimensions $> s$. Then we define a filtration by setting
$$F_s(A \otimes H_*(\Omega S^{2n+1})) = A_{\leq s} \otimes H_*(\Omega S^{2n+1}; \mathbb{Z}).$$

Problem 32.27.

(a) Show that this filtration is multiplicative and that Φ is a filtered map.

(b) Show that the algebra map $\mathrm{Gr}^* \Phi$ is surjective on the left and bottom edges.[2] Conclude that $\mathrm{Gr}^* \Phi$ is surjective.

(c) Complete the proof of Theorem 32.25 by showing Φ itself is surjective.

Multiple Multiplications. In our computation of the homology algebra $H_*(\Omega SU)$, we used the fact that, because $\Omega SU(n) \to \Omega SU(n+1) \to \Omega S^{2n+1}$ is a fibration sequence of H-maps, the corresponding Serre filtration of $H_*(\Omega SU(n))$ is multiplicative. Thus we implicitly used the multiplication on $\Omega SU(n)$ given by concatenation of loops. But $\Omega SU(n)$ also has a multiplication $\Omega\mu$, where μ is the standard product in the group SU.

Problem 32.28. Show that the Pontrjagin product in $H_*(\Omega SU)$ induced by $\Omega\mu$ is the same as the one determined in Theorem 32.25.

32.5. Generating Complexes for ΩSU and BU

The homology algebras $H_*(BU)$ and $H_*(\Omega SU)$ are polynomial algebras on submodules isomorphic to $\mathbb{Z}$ in each even dimension and zero otherwise. In this section we'll show that these generating submodules may be realized topologically by maps $\phi : \mathbb{C}\mathrm{P}^\infty \to BU$ and $\gamma : \mathbb{C}\mathrm{P}^\infty \to \Omega SU$ whose induced maps on homology are precisely the inclusions of the generating submodules. For this reason, we call $\phi : \mathbb{C}\mathrm{P}^\infty \to BU$ and $\gamma : \mathbb{C}\mathrm{P}^\infty \to \Omega SU$ **generating complexes** for BU and ΩSU, respectively.

32.5.1. Generating Complex for BU. We know that $BU(1) = \mathbb{C}\mathrm{P}^\infty$, so we have a natural inclusion $\phi : \mathbb{C}\mathrm{P}^\infty \hookrightarrow BU$. Furthermore, under this homotopy equivalence, $H^*(\mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}[c_1]$. Since these cohomology groups are free and finitely-generated abelian groups, each power c_1^n of c_1 has a well-defined dual class $(c_1^n)^*$ which generates $H_{2n}(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}) \cong \mathbb{Z}$.

[2] Use diagonal indexing for *both* algebras.

Problem 32.29.

(a) Show that $a_k \in H_*(BU;\mathbb{Z})$ may be chosen to be $a_k = \phi_*((c_1^k)^*)$.

HINT. Trace through the proof of Theorem 32.23.

(b) Show that if A is a graded algebra and $f : H_*(\mathbb{C}\mathrm{P}^\infty) \to A$ is a homomorphism of graded abelian *groups*, then in the diagram

$$\begin{tikzcd} H_*(\mathbb{C}\mathrm{P}^\infty) \arrow[d, "\phi_*"'] \arrow[dr, bend left, "f"] & \\ H_*(BU) \arrow[r, dotted, "g"] & A \end{tikzcd}$$

there is a unique algebra map g extending f.

32.5.2. Generating Complexes for $\Omega SU(n)$. For $V \subseteq \mathbb{C}^\infty$ and $z \in S^1$, define $T_{z,V}$ to be the transformation that is multiplication by z on the subspace V and which is the identity on $V^\perp$ (defined in Section 9.3.3).

We have already constructed, in Section 9.3.3, maps $\Sigma\mathbb{C}\mathrm{P}^{n-1} \to SU(n)$ whose images generate the cellular structure of $SU(n)$. Now we consider the adjoint maps $\gamma_n : \mathbb{C}\mathrm{P}^{n-1} \to \Omega SU(n)$, which are given by the formula

$$\gamma_n : \ell \longmapsto \boxed{z \mapsto T_{\bar{z},\mathbb{C}} \cdot T_{z,\ell}}\,,$$

where $z \in S^1 \subseteq \mathbb{C}$.

Problem 32.30.

(a) Show that the diagram

$$\begin{tikzcd} \mathbb{C}\mathrm{P}^{n-1} \arrow[r] \arrow[d, "\gamma_n"'] & \mathbb{C}\mathrm{P}^n \arrow[r] \arrow[d, "\gamma_{n+1}"'] & S^{2n} \arrow[d, "g"] \\ \Omega SU(n) \arrow[r] & \Omega SU(n+1) \arrow[r] & \Omega S^{2n+1} \end{tikzcd}$$

commutes, where the top row is a cofiber sequence and the bottom row is a fiber sequence.

(b) Show that the map g induces an isomorphism on H_{2n}.

(c) Show that in Problem 32.26(c), the element $b_n \in H_{2n}(\Omega SU(n+1))$ can be chosen to be in the image of $(\gamma_n)_*$.

(d) Conclude that $\gamma_{n+1} : \mathbb{C}\mathrm{P}^n \to \Omega SU(n+1)$ is a generating complex for $n \leq \infty$.

Exercise 32.31. Write down the map $g : S^{2n} \to \Omega S^{2n+1}$ explicitly.

32.6. The Bott Periodicity Theorem

You have shown that the H-spaces BU and ΩSU have isomorphic homology algebras and that they have very similar generating complexes. These results suggest that it is not so crazy to suspect that ΩSU and BU are homotopy equivalent; and this is the content of the Bott Periodicity Theorem.

The plan of the proof is actually quite simple, given the work we have already done. We define the **Bott map** $\beta : BU \to \Omega SU$ by the formula

$$\beta(V) : z \longmapsto T_{\bar{z},\mathbb{C}^{\dim(V)}} \circ T_{z,V}.$$

We'll show that β is an H-map and that the triangle

$$\begin{tikzcd} & \mathbb{CP}^\infty \arrow[dl, "\phi"'] \arrow[dr, "\gamma"] & \\ BU \arrow[rr, "\beta"] & & \Omega SU \end{tikzcd}$$

commutes. This is sufficient to show that the map induced by β in homology is an algebra isomorphism, and hence a homotopy equivalence.

32.6.1. Shuffling Special Unitary Groups. It is easy to see that β does not carry shuffle products in BU to either loop concatenations or to $\Omega\mu$ products in ΩSU. Showing that β is an H-map, then, will require us to show that the diagram

$$\begin{tikzcd} BU \times BU \arrow[r, "\mathrm{sh}"] \arrow[d, "\beta\times\beta"'] & BU \arrow[d, "\beta"] \\ \Omega SU \times \Omega SU \arrow[r, "\Omega\mu"] & \Omega SU \end{tikzcd}$$

is homotopy commutative. We'll do this by constructing a shuffle product on ΩSU and showing: (1) β carries shuffles in BU to shuffles in ΩSU and (2) the shuffle product in ΩSU is homotopic to the standard multiplications.

We use the ideas we developed in Section 32.3.1. Thus the isomorphisms $\mathbb{C}^\infty \to \mathbb{C}^{\text{odd}}$ and $\mathbb{C}^\infty \to \mathbb{C}^{\text{even}}$ induce isomorphisms

$$\mathrm{pr}_{\text{even}} : SU \longrightarrow SU(\mathbb{C}^{\text{even}}) \qquad \text{and} \qquad \mathrm{pr}_{\text{odd}} : SU \longrightarrow SU(\mathbb{C}^{\text{odd}})$$

and the inclusions $\mathbb{C}^{\text{odd}} \hookrightarrow \mathbb{C}^\infty$ and $\mathbb{C}^{\text{even}} \hookrightarrow \mathbb{C}^\infty$ induce maps

$$\mathrm{in}_{\text{even}} : SU(\mathbb{C}^{\text{even}}) \longrightarrow SU \qquad \text{and} \qquad \mathrm{in}_{\text{odd}} : SU(\mathbb{C}^{\text{odd}}) \longrightarrow SU.$$

Now we define a shuffle product by the diagram

$$\begin{tikzcd} SU(\mathbb{C}^\infty) \times SU(\mathbb{C}^\infty) \arrow[r, "\text{shuffle}"] \arrow[d, "\mathrm{pr}_{\text{odd}}\times\mathrm{pr}_{\text{even}}"'] & SU(\mathbb{C}^\infty) \arrow[d, equal] \\ SU(\mathbb{C}^{\text{odd}}) \times SU(\mathbb{C}^{\text{even}}) \arrow[r] & SU(\mathbb{C}^{\text{odd}} \oplus \mathbb{C}^{\text{even}}). \end{tikzcd}$$

We want to know that the shuffle product is homotopic to the ordinary one, and it suffices to show that the composite from upper left to lower right in its defining square is homotopic to the identity.

Problem 32.32.

(a) Show that the diagram

$$\begin{tikzcd} SU \arrow[r, "\mathrm{pr}_{\mathrm{odd}}"] \arrow[d, "\mathrm{pr}_{\mathrm{even}}"'] \arrow[dr, "\mathrm{id}_{SU}" description] & SU(\mathbb{C}^{\mathrm{odd}}) \arrow[d, "\mathrm{in}_{\mathrm{odd}}"] \\ SU(\mathbb{C}^{\mathrm{even}}) \arrow[r, "\mathrm{in}_{\mathrm{even}}"'] & SU \end{tikzcd}$$

commutes up to homotopy.

(b) Show that the shuffle product in ΩSU is homotopic to the product $\Omega\mu$.

32.6.2. Properties of the Bott Map. As we mentioned in the introduction to this section, the proof of the Bott Periodicity Theorem is founded on two basic properties of the map β: it is an H-map, and it carries the generating complex for its domain to the generating complex for its target.

Problem 32.33. Write $e_{\mathrm{odd}} = \mathrm{in}_{\mathrm{odd}} \circ \mathrm{pr}_{\mathrm{odd}}$ for both BU and ΩSU, and define e_{even} analogously.

(a) Show that the square

$$\begin{tikzcd} BU \times BU \arrow[r, "e_{\mathrm{odd}} \times e_{\mathrm{even}}"] \arrow[d, "\beta"'] & BU \times BU \arrow[d, "\beta"] \\ \Omega SU \times \Omega SU \arrow[r, "e_{\mathrm{odd}} \times e_{\mathrm{even}}"] & \Omega SU \times \Omega SU \end{tikzcd}$$

commutes on the nose.

(b) Show that β commutes (on the nose) with the shuffle products in BU and ΩSU.

Problem 32.34. Show that the diagram

$$\begin{tikzcd} & \mathbb{CP}^\infty \arrow[dl, "\phi"'] \arrow[dr, "\gamma"] & \\ BU \arrow[rr, "\beta"] & & \Omega SU \end{tikzcd}$$

is strictly commutative.

32.6.3. Bott Periodicity. We are ready to prove the Bott Periodicity Theorem.

Theorem 32.35 (Bott)**.** *The Bott map $\beta : BU \to \Omega SU$ is a homotopy equivalence.*

Problem 32.36. Prove Theorem 32.35.

In the next problem, you will see why Theorem 32.35 called the Bott *Periodicity* Theorem.

Problem 32.37.

(a) Show that $\Omega U \simeq \mathbb{Z} \times \Omega SU$.

(b) Determine the homotopy type $\Omega^n BU$.

(c) Determine the homotopy groups $\pi_k(BU)$ for $k \leq 2$.

(d) Determine all the homotopy groups of U and BU.

(e) For what values of k do you know the homotopy groups of $\pi_k(BU(n))$? What about $\pi_k(U(n))$?

Bott Periodicity for Other Fields. There are Bott periodicity theorems for the infinite orthogonal group O and the infinite symplectic group Sp. The statement is much more complicated, since the period is 8 rather than 2. The list of spaces is

X	ΩX	$\Omega^2 X$	$\Omega^3 X$	$\Omega^4 X$	$\Omega^5 X$	$\Omega^6 X$	$\Omega^7 X$
BO	O	O/U	U/Sp	$\mathbb{Z} \times BSp$	Sp	Sp/U	U/O

so that the real and symplectic cases are actually the same, but shifted. It follows that the homotopy groups of these spaces are 8-periodic; in fact, the first 8 homotopy groups of BO are

$$\begin{aligned} \pi_1(BO) &= \mathbb{Z}/2, & \pi_5(BO) &= 0, \\ \pi_2(BO) &= \mathbb{Z}/2, & \pi_6(BO) &= 0, \\ \pi_3(BO) &= 0, & \pi_7(BO) &= 0, \\ \pi_4(BO) &= \mathbb{Z}, & \pi_8(BO) &= \mathbb{Z}. \end{aligned}$$

Proofs of Bott Periodicity. Bott periodicity is one of the most-proved theorems in modern mathematics. Bott's original proof made ingenious use of Morse theory. A short time later, Toda and Moore devised homotopy-theoretical proofs similar to the one we've just worked through. There are proofs that amount to detailed study of how vector bundles over $X \times S^n$ are related to vector bundles over X. Still others are based on complex analysis, or algebra.

There is an extraordinarily direct proof that originated in a paper of MacDuff [**108**]. For an infinite-dimensional $\mathbb{C}$-inner product space V, we say that an automorphism $T : V \to V$ has finite type if $\ker(T - \mathrm{id})$ is finite-dimensional; we'll only study automorphisms of finite type. Write $H(V \oplus V)$ for the space of Hermitian linear transformations of V to itself (i.e., those for which $T^* = T$) and let $E(V)$ be the subspace consisting of those matrices all of whose eigenvalues (which are necessarily real) lie in the interval $[0, 1]$. Then $E(V) \simeq *$, and the function

$$p : E(V) \longrightarrow U(V \oplus V) \qquad \text{given by} \qquad p(A) = \exp(2\pi i A)$$

can be shown to be a quasifibration, and it is nearly tautological that the quasifibers are given by

$$p^{-1}(X) = \mathbb{Z} \times BU(\ker(X - \mathrm{id})).$$

Thus $\Omega U \simeq \mathbb{Z} \times BU$, which is equivalent to Theorem 32.35. A proof using this method has appeared in the recent book [**12**] by Aguilar, Gitler and Prieto. M. Behrens [**22, 23**] also gave a proof along the same lines, but streamlined by the use of coordinate-free linear algebra.

32.7. K-Theory

The Bott Periodicity Theorem immediately implies that the space $\mathbb{Z} \times BU$ is an infinite loop space, and since we have seen that infinite loop spaces represent cohomology theories, there is a cohomology theory $\widetilde{K}^*$ with

$$\widetilde{K}^0(X) = [X, \mathbb{Z} \times BU].$$

This was the first nontrivial example of an extraordinary cohomology theory, and within a few years of its discovery, it had been used to resolve several long-standing conjectures and to prove unexpected and important theorems. The K-theory of compact spaces has a natural interpretation in terms of complex vector bundles, and since vector bundles play important roles throughout mathematics, this theory established a powerful connection between homotopy theory and other areas of mathematics. In this section we briefly outline some of the basic features of K-theory; see Atiyah's book [**19**] for details.

32.7.1. K-Theory and Vector Bundles. First of all, let's write $\mathrm{Vect}(X)$ for the set of equivalence classes of complex vector bundles. If X is compact, then pullback of the universal bundle over $BU(n)$ defines a bijection

$$\langle X, BU(n)\rangle \xrightarrow{\cong} \mathrm{Vect}_n(X).$$

Write ε^k for the trivial bundle $\mathrm{pr}_1 : X \times \mathbb{C}^k \to X$.

All of the standard constructions on vector spaces, such as direct sum, tensor product, exterior power, and so on, can be extended to vector bundles over X. Thus if we have vector bundles ξ and ζ over X, we also have the **Whitney sum** bundle $\xi \oplus \zeta$, the tensor product bundle $\xi \otimes \zeta$, and the k^{th} exterior power bundle $\Lambda^k(\xi)$.

The rule $? \oplus \varepsilon^k$ defines a natural transformation $\mathrm{Vect}_n(X) \to \mathrm{Vect}_{n+k}(X)$ which is represented by the standard inclusion $BU(n) \hookrightarrow BU(n+k)$. Now we say that two bundles are **stably equivalent** if it is possible to add trivial bundles to each of them so that the sums are equivalent vector bundles.

Problem 32.38. Let X be compact.

(a) Suppose two complex vector bundles are classified by $f : X \to BU(n)$ and $g : X \to B(m)$. Show that they are stably equivalent if and only if there is an M such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & BU(n) \arrow[d] \\ BU(m) \arrow[r] & BU(M) \end{tikzcd}$$

commutes up to unpointed homotopy.

(b) Show that there is a bijection from $\langle X, BU\rangle$ to the set of stable equivalence classes of complex vector bundles over X.

The set $\mathrm{Vect}(X) = \bigcup_n \mathrm{Vect}_n(X)$ is a semigroup under sum $\oplus$ of vector bundles. Grothendieck introduced a procedure for turning an abelian semigroup S with sum $\oplus$ into a group $G(S)$: start with the free abelian group $F(S)$ and form the quotient by the subgroup of all elements of the form $x + y - x \oplus y$.

Problem 32.39. Show that $G(\,?\,)$ is left adjoint to the forgetful functor from the category of abelian groups to that of abelian semigroups.

If X is compact, then define $K(X) = G(\mathrm{Vect}(X))$. Assigning to each bundle over X its dimension defines a function $\mathrm{Vect}(X) \to \mathbb{N}$ and induces an augmentation

$$\dim : K(X) = G(\mathrm{Vect}(X)) \longrightarrow \mathbb{Z} = G(\mathbb{N}),$$

and we write $\widetilde{K}(X) = \ker(\dim)$. Now, viewing $\widetilde{K}^0(X)$ as the set of stable equivalence classes of vector bundles, we can define a map

$$\nu : \widetilde{K}^0(X) \longrightarrow K(X)$$

by setting $\nu([\xi]) = \xi - \varepsilon^{\dim(\xi)}$.

Problem 32.40. Show that ν is well-defined.

Now we have a second interpretation of K-theory in terms of bundles.

Theorem 32.41. *If X is compact, then ν induces is a natural isomorphism*

$$\widetilde{K}^0(X) \xrightarrow{\cong} \widetilde{K}(X).$$

Because of Theorem 32.41, it is frequently useful to dispense with the graded theory $\widetilde{K}^*$ and study the functor $\widetilde{K}$.

Problem 32.42. Determine $\widetilde{K}(S^n)$.

Problem 32.43. Show that the tensor product of vector bundles induces a natural ring structure on $\widetilde{K}(X)$.

32.7.2. Cohomology Operations in K-Theory. K-theory is known to have lots of cohomology operations, but very few of them are known. In this section we define the **Adams operations** and set down their basic properties.

The exterior power functor $\Lambda^k(\,?\,)$ on vector spaces induces an exterior power functor $\lambda^k : \widetilde{K}(\,?\,) \to \widetilde{K}(\,?\,)$. Adams defined

$$\psi^k(\xi) = Q_k(\lambda^1(\xi), \lambda^2(\xi), \ldots, \lambda^k(\xi)),$$

where Q_k is the k^{th} **Newton polynomial** (see Section A.7).

Theorem 32.44 (Adams)**.** *The operations ψ^k satisfy the following conditions.*

(a) $\psi^1 = \text{id}$.

(b) $\psi^k \circ \psi^l = \psi^{kl} = \psi^l \circ \psi^k$ *for any* $k, l \neq 0$.

(c) If p is prime, then for any $x \in K(X)$, $\psi^k(x) \equiv x^p \bmod p$.

(d) If $x \in K(X)$ is represented by a line bundle, then $\psi^k(x) = x^k$, the k^{th} tensor power of x.

(e) $\psi^k(x) = kx$ *if* $x \in \widetilde{K}(S^2)$.

(f) The diagram

$$\begin{tikzcd} \widetilde{K}(X) \arrow[r, "\psi^k"] \arrow[d, "\beta"'] & \widetilde{K}(X) \arrow[d, "\beta"] \\ \widetilde{K}(\Sigma^2 X) \arrow[r, "k\psi^k"] & \widetilde{K}(\Sigma^2 X) \end{tikzcd}$$

commutes.

We will not prove this.

There are Adams operations ψ^k for negative k, defined by setting ψ^{-1} to be complex conjugation of bundles and extending to other negative values of k by forcing condition Theorem 32.44(b) to hold.

Problem 32.45. Determine the effect of ψ^k on $\widetilde{K}(S^{2n})$.

These operations were used in [**4**] to determine, for each n, the maximum number of linearly independent vector fields that can be found on S^n. Shortly thereafter, Adams and Atiyah used them to give a short proof of the Hopf Invariant One Theorem.

Theorem 32.46. *If there is a map $f : S^{2n-1} \to S^n$ with Hopf invariant one, then $n = 1, 2, 4$ and 8.*

We end our discussion of K-theory with this pretty argument. We know that for $n > 1$, there cannot be such a map unless n is even.

Problem 32.47. Let $f : S^{4n-1} \to S^{2n}$.

(a) Show that $\widetilde{K}(C_f) \cong \mathbb{Z} \cdot \alpha \oplus \mathbb{Z} \cdot \beta$ where α restricts to a generator of $\widetilde{K}(S^{2n})$ and β is the image of a generator of $\widetilde{K}(S^{4n})$.

(b) Show that there is an integer $h(f)$ such that $\alpha^2 = h(f) \cdot \beta$.

Proposition 32.48. *The integer $h(f)$ is the Hopf invariant $H(f)$ (up to sign).*

You should take this for granted and use it to derive Theorem 32.46.

Problem 32.49. Let $f : S^{4n-1} \to S^{2n}$ be a map with Hopf invariant one, and identify $\widetilde{K}(C_f)$ as in Problem 32.47.

(a) Show that $\psi^k(\beta) = k^n\beta$.

(b) Show that $\psi^k(\alpha) = k^n\alpha + u(k)\beta$ for some $u(k) \in \mathbb{Z}$.

(c) Show that $u(2)$ must be odd.

(d) Show that the relation $\psi^2 \circ \psi^3 = \psi^3 \circ \psi^2$ implies 2^n divides $3^n - 1$.

(e) Prove Theorem 32.46.

32.8. Additional Problems and Projects

Problem 32.50.

(a) Explain how the map $\phi_n : BD_n \to BU(n)$ determines the $\mathcal{A}_2$-action on $H^*(BU(n))$.

(b) Determine $\mathrm{Sq}^k(c_j)$ for $j \leq 5$ and all k.

(c) Determine $\mathrm{Sq}^4(c_1^2 c_2 c_4^3)$.

Project 32.51. Work out the cohomology of $BO(n)$ and BO; relate them to the cohomology of the corresponding diagonal matrices.

Problem 32.52. Determine $H^*(SO(n); \mathbb{Z}/2)$.

Problem 32.53.

(a) Show that

$$\widetilde{K}^n(X) = \begin{cases} [X, \mathbb{Z} \times BU] & \text{if } n \text{ is even,} \\ [X, U] & \text{if } n \text{ is odd.} \end{cases}$$

(b) What are the coefficients of $\widetilde{K}^*$?

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Chapter 33

Using the Leray-Serre Spectral Sequence

In this chapter we work out the cohomology of Eilenberg-Mac Lane spaces with coefficients in $\mathbb{Z}/p$, p prime. This computation was one of the very first triumphs of the spectral sequence technique, and the results allow us to completely determine the Steenrod algebra for ordinary cohomology with coefficients in $\mathbb{Z}/p$.

The result follows fairly easily from a characterization of certain special kinds of abstract spectral sequences due, for $p = 2$, to Borel [**27**], and, for odd primes, to Postnikov [**141**]. We will derive these results from the Zeeman Comparison Theorem, which we prove in the first section of the chapter.

We end with a brief study of certain global properties of the homotopy groups of spheres, including the first few nonzero p-torsion groups. In the next chapter, this information will be used to produce counterexamples to Ganea's conjecture on the Lusternik-Schnirelmann category of $X \times S^k$.

33.1. The Zeeman Comparison Theorem

When we use the the Leray-Serre spectral sequence for a fibration to compute the cohomology of the fiber or the base, we are essentially making a model for the spectral sequence and arguing that the actual spectral sequence must be isomorphic to the model. In this section we will establish a general principle, first articulated by Zeeman, which streamlines and conceptualizes many arguments of this kind.

A map of spectral sequences $f : \widetilde{E} \to E$ restricts to three maps

$$f_Y : \widetilde{E}_2^{0,*} \longrightarrow E_2^{0,*}, \qquad f_X : \widetilde{E}_2^{*,0} \longrightarrow E_2^{*,0} \qquad \text{and} \qquad f_\infty : \widetilde{E}_\infty^{*,*} \longrightarrow E_\infty^{*,*},$$

which, in the Leray-Serre spectral sequence, are directly induced by the topological maps on the fiber, base and total space, respectively. Furthermore, if f is induced by a morphism from one fibration to another, then it has a **universal coefficients decomposition**: for each s and t, there is a ladder

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{E}_2^{s,0} \otimes \widetilde{E}_2^{0,t} \arrow[r] \arrow[d, "f\otimes f"] & \widetilde{E}_2^{s,t} \arrow[r] \arrow[d, "f"] & \mathrm{Tor}(\widetilde{E}_2^{s+1,0}, \widetilde{E}_2^{0,t}) \arrow[r] \arrow[d, "\mathrm{Tor}(f,f)"] & 0 \\ 0 \arrow[r] & E_2^{s,0} \otimes E_2^{0,t} \arrow[r] & E_2^{s,t} \arrow[r] & \mathrm{Tor}(E_2^{s+1,0}, E_2^{0,t}) \arrow[r] & 0 \end{tikzcd}$$

of short exact sequences.

Theorem 33.1 (Zeeman Comparison Theorem)**.** *Let $f : \widetilde{E} \to E$ be a map of first quadrant spectral sequences that admits a universal coefficients decomposition. The following are equivalent:*

(1) two of the maps f_X, f_Y and f_∞ are isomorphisms,

(2) all three of the maps f_X, f_Y and f_∞ are isomorphisms,

(3) f is an isomorphism of spectral sequences.

Theorem 33.1 applies equally well to spectral sequences like the cohomology Leray-Serre spectral sequence in which the differential d_r has degree $(r, 1-r)$ and to those like the homology Leray-Serre spectral sequence with differentials d^r of degree $(-r, r-1)$.

A spectral sequence E is **acyclic** if it converges to $E_\infty = 0$. Theorem 33.1 has a very useful reduction when the spectral sequences involved are acyclic.

Corollary 33.2. *Let $f : \widetilde{E} \to E$ be a map of first quadrant spectral sequences that admits a universal coefficients decomposition. If E and $\widetilde{E}$ are acyclic, then f_X is an isomorphism if and only if f_Y is an isomorphism.*

Some of Theorem 33.1 is easily proved.

Problem 33.3.

(a) Show that (3) implies (2) and that (2) implies (1).

(b) Show that if f_X and f_Y are isomorphisms, then f is an isomorphism of spectral sequences (i.e., (3) holds).

Thus the real content of Theorem 33.1 is that if f_∞ is an isomorphism, then f_X is an isomorphism if and only if f_Y is an isomorphism.

Let's assume that f_X and f_∞ are isomorphisms and show that f_Y is an isomorphism. The idea is to show by induction on T that $f_Y = f_2^{0,t}$ is an isomorphism for $t \leq T$. The key to the induction is to show that the inductive hypothesis—which describes the behavior of f on the t-axis of the spectral sequence—implies rather a lot about the map f in the 'interior' of the spectral sequence.

The actual algebra underlying this is quite simple, but it can be obscured by the constellations of indices that surround everything connected with a spectral sequence. So we start with a friendly lemma about abelian groups.

Lemma 33.4. *Consider the commutative diagram*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] \arrow[d, "g"'] & C \arrow[d, "\cong"', "h"] \\ D \arrow[r, "\delta"] & E \arrow[r, "\epsilon"] & F \end{tikzcd}$$

of abelian groups in which the composites on top and bottom are zero, and write

$$\Gamma : \ker(\beta)/\operatorname{Im}(\alpha) \longrightarrow \ker(\epsilon)/\operatorname{Im}(\delta)$$

for the map induced by g. Then

(a) if g is surjective, then Γ is surjective, and

(b) if g is bijective and f is surjective, then Γ is bijective.

Problem 33.5. Prove Lemma 33.4.

Now we apply these simple algebraic facts to our inductive proof of Theorem 33.1.

Lemma 33.6. *Let f be as in Theorem 33.1, and assume f_X is an isomorphism. If $f_Y = f_2^{0,t}$ is an isomorphism for $t \leq T$, then for every $s \geq 0$*

(a) $f_r^{s,t}$ is surjective for all r and all $t \leq T$ and

(b) $f_r^{s,t}$ is bijective for all r and t satisfying $t \leq T - r + 2$.

Problem 33.7. Use Lemma 33.4 to prove the two parts of Lemma 33.6 together by induction on r.

Since our spectral sequences are first quadrant spectral sequences, $E_r^{s,t} = E_\infty^{s,t}$ for large r; thus $f_r^{s,t}$ is an isomorphism when r is large enough. We will argue that if $f_2^{0,t}$ is an isomorphism for all $t \leq T$ and if $f_{r+1}^{0,t}$ is an isomorphism for all $t \leq T+1$, then $f_r^{0,t}$ is also an isomorphism for all $t \leq T+1$.

Problem 33.8. Assume $f_2^{0,t}$ is an isomorphism for $t \le T$ and that $f_{r+1}^{0,t}$ is an isomorphism for $t \le T+1$.

(a) Show that there is a map of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{E}_{r+1}^{0,t} \arrow[r] \arrow[d, "f_{r+1}^{0,t}"] & \widetilde{E}_r^{0,t} \arrow[r, "\widetilde{d}_r^{0,t}"] \arrow[d, "f_r^{0,t}"] & \widetilde{Z}_r^{r,t-r+1} \arrow[r] \arrow[d, "f_r^{r,t-r+1}"] & 0 \\ 0 \arrow[r] & E_{r+1}^{0,t} \arrow[r] & E_r^{0,t} \arrow[r, "d_r^{0,t}"] & Z_r^{r,t-r+1} \arrow[r] & 0. \end{tikzcd}$$

(b) Show that $f_r^{r,t-r+1}$ is an isomorphism for $t \le T+1$.

(c) Prove that $f_r^{0,t}$ is an isomorphism for $t \le T+1$.

(d) Show that f_Y is an isomorphism.

The proof of the remaining implication is entirely analogous.

Problem 33.9. Prove that if f_Y and f_∞ are isomorphisms, then so is f_X.

More subtle statements can be derived from the same argument.

Project 33.10. Let $f : \widetilde{E} \to E$ be a map of first quadrant spectral sequences having a universal coefficients decomposition.

(a) Suppose first that E and $\widetilde{E}$ are acyclic. What can you say about f_X if f_Y is an isomorphism in dimensions $\le n$? What can you say about f_Y if f_X is an isomorphism in dimensions $\le n$?

(b) What can you say if E and $\widetilde{E}$ are not necessarily acyclic but f_∞ is an isomorphism in some range of dimensions?

33.2. A Rational Borel-Type Theorem

If we are content to work with coefficients in a field $\mathbb{F}$ of characteristic zero, we can easily prove a very powerful Borel-type theorem relating the cohomology of a space to that of its loop space.

In addition to the odd-dimensional monogenic acyclic spectral sequences, we now define $E[2n]$ to be $\mathbb{Q}[b] \otimes E(a)$, where $\|b\| = (0, 2n)$ and $\tau(b) = a$. The differentials are forced by the transgression: $d_{2n+1}(b^k) = k \cdot (a \otimes b^{k-1})$.

Problem 33.11.

(a) Show that $E[2n]$ is an acyclic spectral sequence.

(b) Show that if E is an acyclic spectral sequence over $\mathbb{F}$ and $y \in E_2^{0,2n}$ is transgressive, with $\tau(y) = [x]$, then there is a unique map of spectral sequences of algebras $E[2n] \to E$ carrying b to y and a to x.

Theorem 33.12. *Let X be simply-connected. Then the following are equivalent:*

(1) $H^*(\Omega B;\mathbb{Q}) = \Lambda_{\mathbb{Q}}(V)$,

(2) $H^*(B;\mathbb{Q}) = \Lambda_{\mathbb{Q}}(\Sigma V)$.

Problem 33.13. Prove Theorem 33.12 by constructing an isomorphism of spectral sequences.

33.3. Mod 2 Cohomology of $K(G,n)$

In this section we work out the algebras $H^*(K(G,n);\mathbb{Z}/2)$ for all finitely generated abelian groups.

Problem 33.14. Show that to determine $H^*(K(G,n);\mathbb{Z}/2)$ for all finitely generated abelian groups G, it suffices to determine

$$H^*(K(\mathbb{Z},n);\mathbb{Z}/2) \qquad \text{and} \qquad H^*(K(\mathbb{Z}/2^r,n);\mathbb{Z}/2)$$

for all $n, r \geq 1$.

33.3.1. The Transgression.

We saw in Section 31.2.4 that the behavior of the transgression in the Leray-Serre spectral sequences for $H^*(\,?\,;G)$ can be deduced from the universal example $K(G,n) \to * \to K(G,n+1)$. So let's consider the case $G = \mathbb{Z}/2$.

Problem 33.15. Let $\iota_n \in H^n(K(\mathbb{Z}/2,n);\mathbb{Z}/2)$ be the cohomology class corresponding to the identity map.

(a) Show that ι_n is transgressive and that $d_{n+1}(\iota_n) = \iota_{n+1}$.[1]

(b) Let I be a sequence of indices. Show that $d_{n+|I|+1}(\mathrm{Sq}^I \iota_n) = [\mathrm{Sq}^I \iota_{n+1}]$.

(c) Show that if $y \in H^m(K(\mathbb{Z}/2,n);\mathbb{Z}/2)$ is transgressive, with $d_{m+1}(y) = [x]$, then y^2 is also transgressive and $d_{2m+1}(y^2) = [\mathrm{Sq}^n(x)]$.

The **free monogenic acyclic spectral sequence** with generator b in dimension n is the spectral sequence of $\mathbb{Z}/2$-algebras

$$E[n] = \mathbb{Z}/2[a] \otimes \Lambda(b)$$

where $\|a\| = (n+1,0)$, $\|b\| = (0,n)$ and $d_n(b) = a$.

Exercise 33.16. Draw the spectral sequences $E[n]$. Show that the conditions put on the differentials determine the spectral sequences up to isomorphism. For each n, verify that $E[n]$ is an acyclic spectral sequence of algebras and that it has a Künneth decomposition.

[1] Why can we write ι_{n+1} and not $[\iota_{n+1}]$?

We should justify calling these spectral sequences *free*. There is an interesting wrinkle here: we will find a map from one spectral sequence of algebras to another, but the map need not be an algebra map. However, the restriction of the map to the bottom edge *is* a map of algebras.

Problem 33.17. Let $p : E \to B$ be a fibration with fiber F, and suppose $y \in H^n(F;\mathbb{Z}/2)$ transgresses to $[x]$, where $x \in H^{n+1}(B;\mathbb{Z}/2)$. Show that there is a unique map of spectral sequences $f : E[n] \to E(p)$ such that

- $f(b) = y$ and $f(a) = x$,
- f is compatible with the universal coefficients decompositions, and
- $f_X : \mathbb{F}[a] \to E_2^{*,0}(p)$ is an algebra map.

Hint. First prove it for $@_0 : \mathcal{P}(K(\mathbb{Z}/2, n+1)) \to K(\mathbb{Z}/2, n+1)$.

Exercise 33.18. Find an example in which $f^{0,*}$ is not an algebra map.

33.3.2. Simple Systems of Generators. A subset $\{x_i \mid i \in \mathcal{I}\}$ of a graded $\mathbb{Z}/2$-algebra A is a **simple system** of generators for A if there are only finitely many elements x_i in any one dimension and the set of monomials $\{x_{i_1} x_{i_2} \cdots x_{i_r} \mid i_1, i_2, \cdots, i_r \in \mathcal{I}\}$ is a basis for A as a $\mathbb{Z}/2$-vector space.

Problem 33.19.

(a) Show that $\mathbb{F}[x]/(x^m)$ has a simple system of generators if and only if m is a power of 2 or if $m = \infty$.

(b) Show that if A and B have simple systems of generators, then so does $A \otimes B$.

It follows that all polynomial algebras have simple systems of generators.

Problem 33.20. Show that if $A = \mathbb{Z}/2[\{x_i \mid i \in I\}]$, then the set

$$\{x_i^{2^k} \mid i \in \mathcal{I} \text{ and } k \geq 1\}$$

is a simple system of generators for A.

33.3.3. Borel's Theorem. Now we show that if the algebra $H^*(\Omega X;\mathbb{Z}/2)$ is nice enough, then it determines $H^*(X;\mathbb{Z}/2)$. We state the theorem without reference to spectral sequences, but it is proved by studying the Leray-Serre spectral sequence. Indeed, it is frequently stated as a theorem about abstract acyclic spectral sequences having the same general form as the Leray-Serre spectral sequence.

Theorem 33.21 (Borel). *Suppose the path-loop fibration $@_0 : \mathcal{P}(X) \to X$ is $H^*(\,?\,;\mathbb{Z}/2)$-orientable and that $H^*(\Omega X;\mathbb{Z}/2)$ has a simple system of transgressive generators $\{y_i \mid i \in \mathcal{I}\}$. If $\tau(y_i) = [x_i]$, then*

$$H^*(X;\mathbb{Z}/2) = \mathbb{Z}/2\left[\{x_i \mid i \in \mathcal{I}\}\right].$$

Problem 33.22. Write $n_i = |y_i|$ and let $\widetilde{E} = \bigotimes_{\mathcal{I}} E[n_i]$. Prove Theorem 33.21 by constructing an isomorphism $f : \widetilde{E} \to E(@_0)$ of spectral sequences.

Theorem 33.21 is very convenient for inductive arguments, because the restriction that the fiber have a simple system of generators is conveyed effortlessly to the base.

In Section 23.6.3 we established the notation $\mathrm{Sq}^I = \mathrm{Sq}^{i_m} \circ \mathrm{Sq}^{i_{m-1}} \circ \cdots \circ \mathrm{Sq}^{i_1}$ for a sequence $I = (i_m, i_{m-1}, \ldots, i_1)$, and in Problem 23.76 we introduced the special sequences

$$I^k(n) = (2^{k-1}n, \ldots, 2^2 n, 2n, n)$$

for $k \geq 1$ and $I^0(n) = (0)$.

Problem 33.23. Suppose $H^*(\Omega X; \mathbb{Z}/2)$ has a simple system of transgressive generators $\{y_i \mid i \in \mathcal{I}\}$. If $\tau(y_i) = [x_i]$ and $|x_i| = m_i$, then

$$\{\mathrm{Sq}^{I^k(m_i)}(x_i) \mid i \in \mathcal{I} \text{ and } k \geq 0\}$$

is a simple system of generators for $H^*(X; \mathbb{Z}/2)$.

This result already gives us insight into the cohomology of Eilenberg-Mac Lane spaces.

Problem 33.24. Show that $H^*(K(\mathbb{Z}/2^r, n); \mathbb{Z}/2)$ and $H^*(K(\mathbb{Z}, n); \mathbb{Z}/2)$ are polynomial algebras for all $r \geq 1$ and all $n > 1$.

33.3.4. Mod 2 Cohomology of Eilenberg-Mac Lane Spaces. At this point, determining the cohomology of Eilenberg-Mac Lane spaces essentially boils down to formulating the correct inductive hypothesis and then applying Problem 33.23.

Excessive Notation. In Theorem 23.66 we showed that every operation Sq^J is a linear combination of the Sq^I with I an **admissible** sequence. Recall that $I = (i_m, i_{m-1}, \ldots, i_1)$ is admissible if $i_j \geq 2i_{j-1}$ for each j, and the **excess** of an admissible sequence I is

$$\begin{aligned} e(I) &= (i_m - 2i_{m-1}) + (i_{m-1} - 2i_{m-2}) + \cdots + (i_2 - 2i_1) + i_1 \\ &= i_m - (i_{m-1} + \cdots + i_1). \end{aligned}$$

We may extend any admissible sequence I by concatenating it with the sequence $I^k(n + |I|)$:

$$\begin{aligned} J_k(I) &= (I^k(n + |I|), I) \\ &= (2^{k-1}(n + |I|), \ldots, 2(n + |I|), n + |I|, i_m, \ldots, i_1). \end{aligned}$$

We need to keep track of what happens to excess when sequences are extended according to this rule.

Lemma 33.25. *If J is an admissible sequence, then the following are equivalent:*

(1) $e(J) = n$,

(2) $J = J_k(I)$ for some admissible I with $e(I) < n$ and some $k \geq 0$.

Problem 33.26. Prove Lemma 33.25.

Cohomology of $K(\mathbb{Z}/2, n)$. Finally we can determine the cohomology of $K(\mathbb{Z}/2, n)$.

Theorem 33.27 (Cartan-Serre)**.** *For $n \geq 1$,*

$$H^*(K(\mathbb{Z}/2, n); \mathbb{Z}/2) = \mathbb{Z}/2\left[\left\{\mathrm{Sq}^I(\iota_n) \mid e(I) < n\right\}\right].$$

Problem 33.28. Prove Theorem 33.27 by induction on n.

Cohomology of $K(\mathbb{Z}, n)$. We have developed a powerful inductive argument that will certainly prove the correct statement about the cohomology of $K(\mathbb{Z}, n)$. The challenge, then, is to settle on the correct statement.

Problem 33.29. Determine $H^*(K(\mathbb{Z}, n); \mathbb{Z})$ for $n \geq 1$.

Cohomology of $K(\mathbb{Z}/2^r, n)$. The result for $\mathbb{Z}/p^r$ is essentially the same; the only difference is that the operation Sq^1, which we have shown is the Bockstein operation associated to the exact sequence

$$0 \to \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \to 0,$$

must be replaced with the operation β_r associated to

$$0 \to \mathbb{Z}/2^{r-1} \longrightarrow \mathbb{Z}/2^r \longrightarrow \mathbb{Z}/2 \to 0.$$

Project 33.30. Carry out the proof.

33.4. Mod p Cohomology of $K(G, n)$

Now we work out $H^*(K(G, n); \mathbb{Z}/p)$ for all finitely generated abelian groups G. Just as for the mod 2 case, it suffices to compute $H^*(K(\mathbb{Z}, n); \mathbb{Z}/p)$ and $H^*(K(\mathbb{Z}/p^r, n); \mathbb{Z}/p)$ for $r \geq 1$; and just as in the mod 2 case, we will do the case $r = 1$ in detail and leave it to you to adapt the argument to the other cases.

In its broad outlines, the computation is no different from that of the previous section. We study the behavior of the transgression in the universal example, namely the path-loop fibration for $K(\mathbb{Z}/p^r, n + 1)$, and use it to analyze the transgression in general path-loop fibrations. This leads to a Borel-type theorem (due to Postnikov) which fuels a simple inductive argument that requires some combinatorial analysis of admissible sequences.

33.4.1. The mod p Path-Loop Transgression. We need to define simple spectral sequences that we can assemble into a map into the Leray-Serre spectral sequence of the path-loop fibration on X. To learn what form these basic spectral sequences should have, we study

$$H^*(K(\mathbb{Z}/p, n); \mathbb{Z}/p) \otimes H^*(K(\mathbb{Z}/p, n+1); \mathbb{Z}/p),$$

the E_2-term of the Leray-Serre spectral sequence for the fibration $@_0 : \mathcal{P}(K(\mathbb{Z}/p, n+1)) \to K(\mathbb{Z}/p, n+1)$.

Let $u \in H^*(K(\mathbb{Z}/p, n+1); \mathbb{Z})$ and $v \in H^*(K(\mathbb{Z}/p, n); \mathbb{Z})$ be the cohomology classes corresponding to the identity maps. Since $\Omega u = v$, we know that $\tau(v) = u$ (modulo the trivial subgroup of $H^{n+1}(K(\mathbb{Z}/p, n+1); \mathbb{Z}/p)$).

Problem 33.31. Suppose n is even.

(a) Determine $d_{n+1}(v^k)$ for $k = 1, 2, \ldots, p-1$.

(b) Show that v^p is transgressive, and determine $\tau(v^p)$.

Now we have a puzzle: what happens to $u \otimes v^{p-1}$? The answer is given by the **Kudo Transgression Theorem**.

Lemma 33.32. *If n is even, then in the Leray-Serre spectral sequence for the fibration $\mathcal{P}(K(\mathbb{Z}/p, n+1)) \to K(\mathbb{Z}/p, n+1)$,*

(a) $u \otimes v^{p-1}$ is not a boundary in E_r for any r,

(b) $d_r(u \otimes v^{p-1}) = 0$ for $r \leq n(p-1)$, and

(c) $d_{n(p-1)+1}(u \otimes v^{p-1}) = [-\beta P^n(u)]$.

Parts (a) and (b) were verified in Problem 33.31. We'll prove a slightly weaker version of (c).

Problem 33.33.

(a) Suppose X is a space such that $p \cdot \widetilde{H}^*(X; \mathbb{Z}) = 0$. Show that if $x \in H^*(X; \mathbb{Z}/p)$, then either $\beta(x) \neq 0$ or $x = \beta(y)$ for some y.

(b) Show that in Lemma 33.32, $d_{n(p-1)+1}(u \otimes v^{p-1}) = a \cdot [\beta P^n(u)]$ for some unit $a \in \mathbb{Z}/p$.

Free Monogenic Acyclic Spectral Sequences. In characteristic p, the square of an odd-dimensional class is necessarily zero, so there are two families of free spectral sequences, depending on the parity of $|a|$. First, we have

$$E[2n-1] = \mathbb{Z}/p[a] \otimes \Lambda(b)$$

where $\|a\| = (2n, 0)$, $\|b\| = (0, 2n-1)$ and $d_{2n}(b) = a$. Next, we define

$$E[2n] = (\Lambda(a) \otimes \mathbb{Z}/p[z]) \otimes (\mathbb{Z}/p[b]/(b^p))$$

where $\|a\| = (2n+1, 0)$, $\|b\| = (0, 2n)$, $\|z\| = (2np+2, 0)$, and the differential is determined by the rules

$$d_{2n+1}(z^k \otimes b^j) = j \cdot az^k \otimes b^{j-1} \qquad \text{and} \qquad d_{2(p-1)n+1}(az^k \otimes b^{p-1}) = z^{k+1}.$$

Both of these are spectral sequences of algebras.

Exercise 33.34. Draw the spectral sequences $E[n]$. Show that the conditions put on the differentials are consistent and determine the spectral sequences up to isomorphism. For each n, verify that $E[n]$ is an acyclic spectral sequence of algebras with a Künneth decomposition.

The spectral sequences $E[2n-1]$ actually do satisfy the natural freeness property. But the sense in which these spectral sequences $E[2n]$ are free is similar to the characteristic 2 case.

Proposition 33.35. *Let E be the Leray-Serre spectral sequence of an orientable path-loop fibration $@_0 : \mathcal{P}(X) \to X$. Let $y \in E_2^{n,0}$ be transgressive, and let $\tau(y) = [x]$.*

(a) If n is odd, then there is a unique map $f : E[n] \to E$ of spectral sequences of algebras such that $f(b) = y$ and $f(a) = x$.

(b) If n is even, then among the spectral sequence maps compatible with the Künneth decompositions, there is a unique $f : E[n] \to E$ such that $f(b) = y$ and $f(a) = x$.

In either case, restriction $f_X : E[n]_2^{*,0} \to E_2^{*,0}$ of f to the bottom edge is an algebra map.

Problem 33.36. Prove Proposition 33.35.

33.4.2. Postnikov's Theorem. In the paper [**141**], M. M. Postnikov proved a theorem that plays the same role in characteristic p as Borel's theorem plays in characteristic 2. To state it, we need a new definition: a subset $\{x_i \mid i \in \mathcal{I}\}$ of a graded algebra A over $\mathbb{Z}/p$ is a **simple system** of generators for A if the set of monomials

$$\left\{ x_{i_1}^{e_1} \cdot x_{i_2}^{e_2} \cdots x_{i_r}^{e_r} \;\middle|\; \begin{array}{l} e_j = 1 \text{ if } |x_{i_j}| \text{ is odd, and} \\ 1 \le e_j < p \text{ if } |x_{i_j}| \text{ is even} \end{array} \right\}$$

is a basis for A as a $\mathbb{Z}/p$-vector space. Some authors distinguish the odd characteristic case by calling such a set a **p-simple system of generators**.

Problem 33.37.

(a) Show that if $|x|$ is even, then $\mathbb{Z}/p[x]/(x^m)$ has a simple system of generators if and only if m is a power of p or if $m = \infty$.

(b) Show that if $|x|$ is odd, then the exterior algebra $E(x)$ has a simple system of generators.

(c) Show that if A and B have simple systems of generators, then so does $A\otimes B$.

Recall that the free graded algebra $\Lambda(V)$ on a graded $\mathbb{Z}/p$-vector space V is the quotient of the tensor algebra $T(V)$ by the ideal imposing the commutativity relation $xy=(-1)^{|x|\cdot|y|}yx$.

If V is concentrated in even dimensions, then $\Lambda(V)=\mathbb{Z}/p[V]$, the polynomial algebra on V. On the other hand, if V is concentrated in odd dimensions, then $\Lambda(V)$ is the exterior algebra on V.

Problem 33.38.

(a) Show that the functor $V\mapsto\Lambda(V)$ satisfies the following universal property: there is a natural inclusion $V\hookrightarrow\Lambda(V)$, and any map of graded $\mathbb{Z}/p$-vector spaces $V\to A$, where A is a graded-commutative $\mathbb{Z}/p$-algebra, has a unique extension in the diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d] & A \\ \Lambda(V). \arrow[ur, dotted, bend right, "\phi"'] & \end{tikzcd}$$

(b) Show that $\Lambda(V\oplus W)\cong\Lambda(V)\otimes\Lambda(W)$.

(c) Every graded vector space has a decomposition $V=V^{\text{odd}}\oplus V^{\text{even}}$ as the sum of its even- and odd-dimensional subspaces. Conclude that $\Lambda(V)$ is a tensor product of an exterior algebra and a polynomial algebra.

Problem 33.39. Show that $\Lambda(V)$ has a simple system of generators

$$\{x_j\mid |x_j|=n_j\text{ is odd}\}\cup\{P^{I^k(n_i)}(x_i)\mid |x_i|=n_i\text{ is even}\}.$$

It will be convenient to divide a list $\mathcal{I}$ of generators of a graded algebra A into two pieces:

$$\begin{aligned}\mathcal{I} &= \mathcal{I}^{\text{odd}}\cup\mathcal{I}^{\text{even}}\\ &= \{x_i\mid i\in\mathcal{I}\text{ and }|x_i|\text{ is odd}\}\cup\{x_i\mid i\in\mathcal{I}\text{ and }|x_i|\text{ is even}\}.\end{aligned}$$

Theorem 33.40 (Postnikov)**.** *Assume that the path-loop fibration over X is $H^*(\,?\,;\mathbb{Z}/p)$-orientable. If $H^*(\Omega X;\mathbb{Z}/p)$ has a p-simple system of generators $\{y_i\mid i\in\mathcal{I}\}$, then $H^*(X;\mathbb{Z}/p)$ is the free graded-commutative algebra*

$$\Lambda\left(\{x_i\mid i\in\mathcal{I}^{\text{even}}\}\cup\left\{P^{I^k(|y_i|/2)}x_i, P^{I^k(|y_i|+1)}\beta P^{|y_i|/2}(x_i)\,\middle|\, i,j\in\mathcal{I}^{\text{even}}\right\}\right),$$

where $x_i\in\tau(y_i)$.

Problem 33.41. Construct an isomorphism $f:\bigotimes_{\mathcal{I}}E[r_i]\to E$ of spectral sequences, where $r_i=|x_i|$, and prove Theorem 33.40.

33.4.3. Mod p Cohomology of Eilenberg-Mac Lane Spaces. We are ready to execute our computation.

Theorem 33.42 (Cartan)**.** *The algebra $H^*(K(\mathbb{Z}/p,n);\mathbb{Z}/p)$ is the free graded-commutative algebra on the generators $\{P^I(\iota_n) \mid e(I) < n\}$.*

Project 33.43. Prove Theorem 33.42.

Hint. Characterize the admissible sequences with excess $\leq n$.

Essentially the same method of proof can be used to determine the cohomology algebras of the other cyclic groups.

Project 33.44. Determine $H^*(K(\mathbb{Z},n);\mathbb{Z}/p)$ and $H^*(K(\mathbb{Z}/p^r,n);\mathbb{Z}/p)$.

33.5. Steenrod Operations Generate $\mathcal{A}_p$

We defined the Steenrod operations and determined their algebraic structure in Chapter 23. The Steenrod operations generate many others: stably by composition and summation; and unstably by composition, multiplication and summation. The calculations of Sections 33.4 and 33.3 imply that the operations constructed in these ways exhaust all the cohomology operations.

On the other hand, when we established the Adém relations, we made no attempt to show that they were a complete set of relations. Now we have enough information to deduce that no more relations are necessary to determine the Steenrod algebra.

First we show that we have found all of the cohomology operations for ordinary cohomology with coefficients in $\mathbb{Z}/p$.

Theorem 33.45.

(a) The operations P^I for admissible sequences I form a basis for $\mathcal{A}_p$.

(b) Every operation $H^n(\,?\,;\mathbb{Z}/p) \to H^*(\,?\,;\mathbb{Z}/p)$ is a polynomial in the operations P^I with I admissible and $e(I) < n$.

Problem 33.46.

(a) Show that $(\mathcal{A}_p)^k \cong H^{n+k}(K(\mathbb{Z}/p,n);\mathbb{Z}/p)$ for $k < n$.

(b) Prove Theorem 33.45.

Next we establish the structure of the mod p Steenrod algebra $\mathcal{A}_p$. Write $\mathcal{Z}$ for the two-sided ideal of $\mathbb{Z}/p[\beta, P^1, P^2, \ldots, P^n, \ldots]$ generated by the Adém relations.

Theorem 33.47. *The map $\alpha : \mathbb{Z}/p[\beta, P^1, P^2, \ldots, P^n, \ldots] \to \mathcal{A}_p$ carrying β to β and P^n to β is surjective with kernel $\mathcal{Z}$.*

This follows easily from Theorem 33.45. The connection is made by the $\mathcal{A}_p$-linear maps $e_n : \mathcal{A}_p \to \widetilde{H}^*(K(\mathbb{Z}/p,n);\mathbb{Z}/p)$ given by $1 \mapsto \iota_n$.

Problem 33.48.

(a) Show that $e_n \circ \alpha$ is surjective in the stable range.

(b) Show that the kernel of $e_n \circ \alpha$ is $\mathscr{Z}$ in the stable range.

(c) Prove Theorem 33.47.

33.6. Homotopy Groups of Spheres

The computation of homotopy groups is quite difficult, so results that apply to *all* the homotopy groups of a space are surprising and wonderful. In this section you will establish a variety of 'qualitative' global results about the homotopy groups of spheres. Specifically, you will determine exactly which groups $\pi_k(S^n)$ are finite and which are infinite, and you will find the first few nontrivial p-torsion groups for each prime p.

We'll make use of the following simple computation.

Problem 33.49. Determine $H^*(K(\mathbb{Z}, n); \mathbb{Q})$ for all n.

33.6.1. Finiteness for Homotopy Groups of Spheres. In this section, we show that, with the exception of the groups $\pi_n(S^n)$ and $\pi_{4n-1}(S^{2n})$, which we know contain a copy of $\mathbb{Z}$, all of the homotopy groups of spheres are finite.

Theorem 33.50. *For $n \geq 1$,*

$$\pi_k(S^{2n-1}) \quad \textit{is} \quad \begin{cases} \mathbb{Z}, & \textit{if } k = 2n-1, \\ \textit{finite}, & \textit{otherwise}, \end{cases}$$

$$\pi_k(S^{2n}) \quad \textit{is} \quad \begin{cases} \mathbb{Z}, & \textit{if } k = 2n, \\ \mathbb{Z} \oplus F, & \textit{where } F \textit{ is finite, if } k = 4n-1, \\ \textit{finite}, & \textit{otherwise.} \end{cases}$$

We begin with the simpler odd-dimensional spheres.

Problem 33.51.

(a) Show that the groups $\widetilde{H}^*(S^{2n+1}\langle 2n+1\rangle; \mathbb{Z})$ are finitely generated.

(b) Show that $\widetilde{H}^*(S^{2n+1}\langle 2n+1\rangle; \mathbb{Q}) = \mathbb{Q}$.

(c) Show that $\pi_{n+k}(S^{2n+1})$ is a finite group for $k > 0$.

Using the Stiefel manifold $V_2(\mathbb{R}^{2n+1})$, we reduce the statement for even spheres to that for odd spheres.

Problem 33.52.

(a) Show that $V_2(\mathbb{R}^{2n+1}) = M(\mathbb{Z}/2, 2n-1) \cup D^{4n-1}$.

(b) Let $q : V_2(\mathbb{R}^{2n+1}) \to S^{2n-1}$ be the quotient map which collapses the Moore space in the decomposition of part (a). Show that for every k, the kernel and cokernel of $q_* : \pi_k(V_2(\mathbb{R}^{2n+1})) \to \pi_k(S^{4n-1})$ are finite 2-groups.

(c) Finish the proof of Theorem 33.50.

Hint. The Stiefel manifold sits in a useful fiber sequence.

33.6.2. Low-Dimensional p-Torsion. In this section, we determine the smallest dimension k in which $\pi_k(S^n)$ contains p-torsion, and we determine that first p-torsion group.

Theorem 33.53. *If $n \geq 1$ and p is an odd prime, then*

$$\left(\pi_{(2n+1)+k}(S^{2n+1})\right)_{(p)} = \begin{cases} \mathbb{Z}/p \cdot \alpha_i & \text{if } k = 2i(p-1)-1,\ 1 \leq i \leq p-1, \\ \mathbb{Z}/p \cdot \beta_{i-1} & \text{if } k = 2i(p-1)-2,\ n < i \leq p-1, \\ 0 & \text{otherwise for } k \leq 2p(p-1)-2. \end{cases}$$

We'll only prove this for $i = 1$ and $i = 2$ (and these are the only groups we will make use of in this text).

Just as for the finiteness theorem, a similar result for even spheres follows directly from that for odd spheres; the formulation and proof of the situation for even spheres is left to you.

Problem 33.54. Determine the low-dimensional p-torsion in $\pi_*(S^{2n})$.

For them, the inductive proof boils down to a study of the double suspension map $\sigma^2 : S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$ adjoint to $\mathrm{id}_{S^{2n+1}}$, which turns out to be a $\mathcal{P}$-isomorphism for a ridiculously large range of dimensions.

Problem 33.55.

(a) Show that there is an algebra map $\mathbb{Z}/p[y_{2n}] \to H^*(\Omega S^{2n+1}; \mathbb{Z}/p)$, which is an isomorphism in dimensions less than $2np$.

(b) Show that the map $(\sigma^2)^* : H^*(\Omega^2 S^{2n+1}; \mathbb{Z}/p) \to H^*(S^{2n-1}; \mathbb{Z}/p)$ induced by the double suspension is an isomorphism in dimensions less than $2np - 2$.

(c) Show that the double suspension $\Sigma^2 : \pi_k(S^{2n-1}) \to \pi_{k+2}(S^{2n+1})$ is a p-isomorphism for $k < 2np - 3$.

(d) Show that it suffices to prove Theorem 33.53 for S^3.

Problem 33.56. Use Theorem 26.40 to prove part of Theorem 33.53.

Corollary 33.57. *Show that for $p < 2n - 1$, the homotopy groups of $M(\mathbb{Z}/p, 2p)$ are given by*

$$\pi_k(M(\mathbb{Z}/p, 2p)) = \begin{cases} \mathbb{Z}/p & \text{for } k = 2p, 4p-2 \text{ and } 4p-3, \\ 0 & \text{otherwise for } k \leq 4p-2. \end{cases}$$

Problem 33.58. Use Problem 33.56 to prove Corollary 33.57.

Hint. Use the exact sequences of Proposition 19.46.

We can use Moore spaces to get information about the p-part of $\pi_{n+k}(S^n)$ for k up to slightly less than $4p$.

Problem 33.59.

(a) Show that there is a map $f : M(\mathbb{Z}/p, 2p) \to S^3\langle 3\rangle$ which induces an isomorphism on $\pi_{2p}(\,?\,)$.

(b) Show that f induces an isomorphism of the p-torsion in $\pi_k(M(\mathbb{Z}/p, 2p))$ with the p-torsion in $\pi_k(S^3\langle 3\rangle) \cong \pi_k(S^3)$ for all $k < 4p - 1$.

Hint. Compute what you can of the mod p cohomology of $\operatorname{coker}(f)$.

(c) Prove the rest of Theorem 33.53.

33.7. Spaces Not Satisfying the Ganea Condition

Recall that a space X satisfies the **Ganea condition** if $\operatorname{cat}(X \times S^k) = \operatorname{cat}(X) + 1$ for all $k \geq 1$. T. Ganea asked in [**68**] whether there are spaces that do not satisfy this condition, and this remains one of the single most influential problems in the theory of Lusternik-Schnirelmann category. Examples of spaces that do not satisfy the condition were first found by N. Iwase [**98**] in the mid-1990s, and we will construct them here.

We'll look for spaces of the form $S^n \cup_\alpha D^{m+1}$, where $\alpha : S^m \to S^n$. Iwase's insight was that the crucial property for α is that it should have a nontrivial Hopf invariant that vanishes after suspension.[2]

Problem 33.60. Let $\mathcal{H}(\alpha) = \{h\}$.

(a) Show that if $m > n$, then $\operatorname{cat}(C_\alpha) = 2$ if and only if $h \not\simeq *$.

(b) Show that $\operatorname{cat}(C_\alpha \times S^r) = 2$ if $\Sigma^r h = *$.

The work we have already done makes it easy to find an attaching map satisfying the required conditions.

Problem 33.61. Let β generate the p-torsion subgroup $\mathbb{Z}/p \subseteq \pi_{4p-3}(S^3)$.

(a) Show that $\Sigma^2\beta = 0$.

(b) Show that $H(\eta\beta) \neq 0$.

(c) Show that $\operatorname{cat}(C_\alpha) = 2$.

(d) Show that $\operatorname{cat}(C_\alpha \times S^k) = 2$ for $k \geq 2$.

[2]It seems that this idea was in the air at the time; several other researchers were thinking along these lines, but it was Iwase who figured out how to translate the vague feeling into actual examples.

33.8. Spectral Sequences and Serre Classes

The results we have established connecting the torsion in the homotopy groups of a space with that in the homology and cohomology groups were first proved by J.-P. Serre using the Leray-Serre spectral sequence. To prove them, he introduced the notion of what is now called a **Serre class** of abelian groups. Though it is logically redundant for us, it is a nice theory, and it is good for your cultural well-roundedness.

33.8.1. Serre Classes. A nonempty class $\mathcal{C}$ of abelian groups is called a **Serre class** if it satisfies the following properties:

(C1) $0 \in \mathcal{C}$.

(C2) If $A \in \mathcal{C}$ and $A \cong B$, then $B \in \mathcal{C}$.

(C3) If $A \to B \to C$ is an exact sequence and $A, C \in \mathcal{C}$, then $B \in C$.

Some theorems about Serre classes will need one or both of the following extra properties enjoyed by some Serre classes:

(XC1) If $A \in \mathcal{C}$, then $H_n(K(A,1);\mathbb{Z}) \in \mathcal{C}$ for all n.

(XC2) If $A_i \in \mathcal{C}$ for all $i \in \mathcal{I}$, then $\bigoplus_{\mathcal{I}} A_i \in \mathcal{C}$.

Problem 33.62. Let $\mathcal{C}$ be a Serre class.

(a) Show that $\mathcal{C}$ is closed under the formation of subquotients.

(b) Show that (XC2) is equivalent to $A \otimes B$ and $\mathrm{Tor}(A, B) \in \mathcal{C}$ for all B (note that B is not necessarily in $\mathcal{C}$).

Exercise 33.63. A collection $\mathcal{C}$ of abelian groups containing 0 is a Serre class if and only if in any short exact sequence $0 \to A \to B \to C \to 0$, we have $A, C \in \mathcal{C}$ if and only if $B \in \mathcal{C}$.

Exercise 33.64. Verify that the following are Serre classes:

(a) the class of *all* abelian groups,

(b) the class of all trivial groups,

(c) the class of all finite abelian groups,

(d) the class of all *torsion* abelian groups—that is, groups in which every element has finite order,

(e) the class of all finitely generated abelian groups,

(f) the class of all finite $\mathcal{P}$-groups,

(g) the class of all finite abelian groups whose order is *not* divisible by the prime number p.

33.8.2. Some Algebra of Serre Classes. Serre classes are generally used to make precise ideas like 'ignoring torsion prime to p'. To *be* precise, let $\mathcal{C}$ be a Serre class, and let $\phi : G \to H$ be a homomorphism of abelian groups. Then we say

- ϕ is $\mathcal{C}$-**injective** if $\ker(\phi) \in \mathcal{C}$,
- ϕ is $\mathcal{C}$-**surjective** if $\operatorname{coker}(\phi) \in \mathcal{C}$,
- ϕ is a $\mathcal{C}$-**isomorphism** if it is both $\mathcal{C}$-injective and $\mathcal{C}$-surjective.

Finally, two abelian groups X and Y are $\mathcal{C}$-**equivalent** if there is a third group Q and $\mathcal{C}$-isomorphisms $X \leftarrow Q \to Y$. We write $X \cong_{\mathcal{C}} Y$ to denote that X and Y are $\mathcal{C}$-equivalent.

Exercise 33.65. Let $\mathcal{C}$ be a Serre class.

(a) Show that $\cong_{\mathcal{C}}$ is an equivalence relation.

(b) Show that $X \cong_{\mathcal{C}} Y$ if and only if there is a group Q and $\mathcal{C}$-isomorphisms $X \to Q \leftarrow Y$.

The following very simple proposition contains key properties that we will use over and over in applying the theory of Serre classes to the study of the Leray-Serre spectral sequence.

Proposition 33.66. *Let $f : A \to B$.*

(a) If $B \in \mathcal{C}$, then $A \cong_{\mathcal{C}} \ker(f)$.

(b) If $A \in \mathcal{C}$, then $B \cong_{\mathcal{C}} \operatorname{coker}(f)$.

(c) If $0 \to A \to B \to C \to D \to 0$ is exact, and $A, D \in \mathcal{C}$, then $B \cong_{\mathcal{C}} C$.

(d) If $0 = F_{-1} \subseteq F_0 \subseteq \cdots \subseteq F_n = A$ and the associated graded group $\mathrm{Gr}_s(A) = F_s/F_{s-1}$ is in $\mathcal{C}$ for $s \neq k$, then $A \cong_{\mathcal{C}} \mathrm{Gr}_k(A) \cong_{\mathcal{C}} F_k$.

Project 33.67. Formulate and prove the Five Lemma mod $\mathcal{C}$.

33.8.3. Serre Classes and Topology. Serre classes are tailor-made for navigating the repeated formation of subquotients that occurs in a spectral sequence. When applied in the Leray-Serre spectral sequence, they provide different proofs of our qualitative results on homotopy and homology groups.

Our first topological result using Serre classes is a mod $\mathcal{C}$ version of the Hurewicz theorem.

Theorem 33.68. *The following are equivalent for simply-connected X.*

(1) $\pi_k(X) \in \mathcal{C}$ for $k < n$,

(2) $H_k(X) \in \mathcal{C}$ for $k < n$.

In this case, the $\pi_n(X) \cong_{\mathcal{C}} H_n(X)$.

Problem 33.69. Use the Leray-Serre spectral sequence for the path-loop fibration $\mathcal{P}(X) \to X$ to prove Theorem 33.68 by induction on n.

We say X is $(n-1)$**-connected mod** $\mathcal{C}$ if $\pi_k(X) \in \mathcal{C}$ for $k < n$. One of our biggest basic theorems was that ordinary connectivity can also be determined using homology, and we prove a mod $\mathcal{C}$ version here. Write $\mathcal{D}_H(\mathcal{C}) = \{X \mid H_*(X) \in \mathcal{C}\}$.

Theorem 33.70. *Let $\mathcal{C}$ be a Serre class satisfying (XC1) and (XC2) and let $F \to E \to B$ be an orientable fibration sequence. If two of F, E and B are in $\mathcal{D}_H$, then so is the third.*

Problem 33.71. Prove Theorem 33.70.

Theorem 33.72. *Let $f : X \to Y$ be a map between simply-connected spaces. Then the following are equivalent:*

(1) C_f is n-connected mod $\mathcal{C}$,

(2) F_f is $(n-1)$-connected mod $\mathcal{C}$.

If these conditions hold, then the comparison map $\xi : \Sigma F_f \to C_f$ induces a $\mathcal{C}$-isomorphism $\xi_* : \pi_{n+1}(C_f) \xrightarrow{\cong_{\mathcal{C}}} \pi_n(F_f)$.

Problem 33.73. Prove Theorem 33.72.

33.9. Additional Problems and Projects

Problem 33.74. Define the product $\mathcal{C} \star \mathcal{D}$ of two Serre classes to be the smallest Serre class containing $A \otimes B$ and $\mathrm{Tor}(A, B)$ for all $A \in \mathcal{C}$ and $B \in \mathcal{D}$. Let $p : E \to B$ be a fibration with fiber F. Show that if $H^k(B) \in \mathcal{B}$ and $H^k(F) \in \mathcal{F}$ for all $k > 0$, then $H^k(E) \in \mathcal{B} \star \mathcal{F}$ for all $k > 0$.

Project 33.75. Problem 33.74 is nice, but it seems wasteful to worry about p-groups in low dimensions if the p-torsion does not occur until later. Define a **graded Serre class** $\mathcal{C}_*$ to be a sequence $\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_n \subseteq \cdots$ of Serre classes and say that a graded abelian group A is in $\mathcal{C}_*$ if and only if $A_n \in \mathcal{C}_n$ for each n. Define a product $\star$ for graded Serre classes in such a way that for an orientable fibration $p : E \to B$ with fiber F, if $H^*(B) \in \mathcal{B}_*$ and $H^*(F) \in \mathcal{F}_*$, then $H^*(E) \in \mathcal{B}_* \star \mathcal{F}_*$.

Problem 33.76. Use the theorems of Borel and Postnikov to determine the cohomology rings of $BU(n)$ and $BSp(n)$ over $\mathbb{Z}/p$ for p an odd prime and also over $\mathbb{Q}$.

Project 33.77. We have a fibration $\mathbb{C}\mathrm{P}^\infty \to * \to K(\mathbb{Z}, 3)$. Apply the Leray-Serre spectral sequence to determine as much as you can of the cohomology ring of $H^*(K(\mathbb{Z}, 3); \mathbb{Z}/2)$.

Problem 33.78. Show that if X is a finite complex, then $H^*(\Omega X; \mathbb{Q})$ is not a nontrivial exterior algebra.

Problem 33.79. Determine the cohomology algebra $H^*(\Omega^2 S^n; \mathbb{Z}/p)$.

Part 7

Vistas

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Chapter 34

Localization and Completion

Localization is a general term for constructions or procedures that invert elements of algebraic gadgets or morphisms in categories. Indeed, there is a useful point of view that holds that the objects of ordinary algebra are simply special kinds of categories, so the distinction between algebraic gadgets and categories is perhaps illusory.

In algebra, the most familiar example of localization is probably the formation of the field of fractions of an integral domain. This may be easily generalized: if $P \subseteq R$ is a prime ideal, then we may introduce multiplicative inverses for all the elements of $Q = R - P$, resulting in the ring denoted $R_{(P)}$ (also sometimes denoted $Q^{-1}R$ to indicate that it is permissible to divide by elements of Q in this new ring), and referred to as R localized at P (or localized away from Q).

Now suppose we have a category $\mathcal{C}$ and a class of morphisms $\mathcal{W}$ in $\mathcal{C}$. We wish to find a functor $\mathcal{C} \to \mathcal{W}^{-1}\mathcal{C}$ which is initial among all functors that render the morphisms in $\mathcal{W}$ as isomorphisms, which is to say that it solves the problem

$$\begin{tikzcd}
\mathcal{W} \arrow[r] \arrow[d] & \mathcal{C} \arrow[r] \arrow[d] & \mathcal{W}^{-1}\mathcal{C} \arrow[dl, dotted, "\exists !"] \\
\mathcal{D}^{\times} \arrow[r] & \mathcal{D}, &
\end{tikzcd}$$

in which $\mathcal{D}^{\times}$ denotes the category whose objects are the same as in $\mathcal{D}$ and whose morphisms are the isomorphisms of $\mathcal{D}$. It is easy to construct a candidate for $\mathcal{W}^{-1}\mathcal{C}$ by just inserting inverses willy-nilly, but this can lead to serious set-theoretical problems.[1] The functor $\mathcal{C} \to \mathcal{W}^{-1}\mathcal{C}$ (if it exists) is the localization of $\mathcal{C}$ away from $\mathcal{W}$.

We cannot invert just any class of morphisms, and in practice some extra structure is usually required. For example, if $\mathcal{W}$ is the class of weak equivalences of a model category $\mathcal{C}$, then the functor $\mathcal{C} \to \mathrm{HO}\,\mathcal{C}$ to the homotopy category is localization with away from $\mathcal{W}$. Thus if there are classes of cofibrations and fibrations that, together with the given $\mathcal{W}$ as weak equivalences, give $\mathcal{C}$ the structure of a model category, then $\mathcal{W}$ can be inverted.

In this chapter we will study (homotopy) localization of spaces, first in great generality and then focusing on algebraic localization of spaces. We give a brief account of rational homotopy theory and conclude with a short treatment of some applications of localization to H-spaces.

34.1. Localization and Idempotent Functors

Let $\mathcal{C}$ be a category, and let $\mathcal{W}$ be some fixed class of morphisms in $\mathcal{C}$. An object $X \in \mathcal{C}$ is called **$\mathcal{W}$-local** if the induced maps

$$w^* : \mathrm{mor}_{\mathcal{C}}(B, X) \longrightarrow \mathrm{mor}_{\mathcal{C}}(A, X)$$

are bijective for every $w : A \to B$ in $\mathcal{W}$. It frequently happens that, when $\mathcal{W}^{-1}\mathcal{C}$ exists, it is isomorphic to the full subcategory of $\mathcal{C}$ whose objects are the $\mathcal{W}$-local objects of $\mathcal{C}$. Then the localization can be thought of as a functor $E : \mathcal{C} \to \mathcal{C}$ which turns out to be idempotent. In this section we explore the interplay between idempotent functors and localization.

34.1.1. Idempotent Functors. An **idempotent functor** is a functor $E : \mathcal{C} \to \mathcal{C}$ equipped with a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to E$ (called a **coagumentation**) such that the morphisms

$$\eta_{EX} : EX \longrightarrow E^2X \qquad \text{and} \qquad E(\eta_X) : EX \longrightarrow E^2X$$

are the same and are both equivalences for every $X \in \mathcal{C}$. We say that an object X is **E-local** if $\eta_X : X \to E(X)$ is an equivalence; and a morphism $f : X \to Y$ is an **E-equivalence** if $E(f)$ is an isomorphism. Write $\mathcal{W}_E$ for the class of all E-equivalences.

Problem 34.1.

(a) Show that X is E-local if and only if $X \cong E(Y)$ for some Y.

[1] Exercise. Concoct an example where the morphisms $\mathrm{mor}(X, Y)$ are not a set.

(b) Show that if X and Y are E-local and $f : Y \to Z$ is an E-equivalence, then in any diagram of the form

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ X \arrow[r] \arrow[ur, dotted, "\phi"] & Z \end{tikzcd}$$

the lift ϕ exists and is unique.

(c) Show that if f is an E-equivalence and Y and Z are E-local, then in any diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Z \\ Y \arrow[ur, dotted, "\phi"'] & \end{tikzcd}$$

the map ϕ exists and is unique.

Let $\mathcal{D} \subseteq \mathcal{C}$ be the full subcategory whose objects are the E-local objects of $\mathcal{C}$. Then E has a factorization of the form

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "E"] \arrow[dr, bend right, "\lambda"'] & & \mathcal{C} \\ & \mathcal{D}. \arrow[ur, bend right, "\mathrm{in}"'] & \end{tikzcd}$$

Problem 34.2.

(a) Show that the functors λ and in are an adjoint pair.

(b) Show that λ is localization of $\mathcal{C}$ with respect to $\mathcal{W}_E$.

Algebraic Localization. Let $\mathcal{P} \sqcup \mathcal{Q}$ be a partition of the prime numbers. It is fairly standard to write $\mathbb{Z}_{(\mathcal{P})}$ for the subring of all elements of $\mathbb{Q}$ whose denominators are $\mathcal{Q}$-numbers; this is known as $\mathbb{Z}$ **localized at** $\mathcal{P}$, or **localized away** from $\mathcal{Q}$.[2] If $\mathcal{P}$ is the singleton set $\{p\}$, we write $\mathbb{Z}_{(p)}$ instead. Some authors write $\mathbb{Z}_p$ for this, which conflicts with the other common usage $\mathbb{Z}_p = \mathbb{Z}/p$ for the same notation; we've chosen to avoid confusion by using $\mathbb{Z}_p$ for absolutely nothing.

Exercise 34.3.

(a) Show that every ring R such that $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$ is equal to $\mathbb{Z}_{(\mathcal{P})}$ for some prime set $\mathcal{P}$.

(b) Show that an abelian group G is a $\mathbb{Z}_{(\mathcal{P})}$-module if and only if for each $q \in \mathcal{Q}$ and each $x \in G$, there is exactly one $y \in G$ such that $x = qy$.

[2]Another notation, more common in algebra than topology, is $\mathcal{Q}^{-1}\mathbb{Z}$.

In the category $\text{AB}\,\mathcal{G}$ of abelian groups, each prime $q \in \mathcal{Q}$ may be considered as a map $q : \mathbb{Z} \to \mathbb{Z}$, so we can attempt to localize $\text{AB}\,\mathcal{G}$ with respect to $\mathcal{Q}$.

Problem 34.4.

(a) Show that the map $L_{\mathcal{P}} : G \to G \otimes \mathbb{Z}_{(\mathcal{P})}$ induced by $\mathbb{Z} \hookrightarrow \mathbb{Z}_{(\mathcal{P})}$ (and the canonical isomorphsm $G \cong G \otimes \mathbb{Z}$) is $\mathcal{P}$-localization.

(b) Show that $L_{\mathcal{P}}$ is an exact functor.

The functor $L_{\mathcal{P}}$ is usually called **$\mathcal{P}$-localization**, and its effect is to eliminate all q-torsion for $q \in \mathcal{Q}$. It can be extended from abelian groups to nilpotent groups in a canonical way [**86**].

34.1.2. Homotopy Idempotent Functors. More closely aligned with our interests are the **homotopy idempotent functors**, which are homotopy functors $E : \mathcal{T}_* \to \mathcal{T}_*$ such that $E(\eta_X)$ is a homotopy equivalence and $E(\eta_X) \simeq \eta_{EX}$ for all X. Thus a homotopy functor $E : \mathcal{T}_* \to \mathcal{T}_*$ is homotopy idempotent if and only if the induced functor $\text{H}\mathcal{T}_* \to \text{H}\mathcal{T}_*$ is idempotent in the sense of the previous section.

Fix a map $f : A \to B$, where A and B are cofibrant. A space X is **f-local** if $f^* : \text{map}_*(B, X) \to \text{map}_*(A, X)$ is a weak homotopy equivalence, and a map $g : Y \to Z$ is an **f-equivalence** if $g^* : \text{map}_*(Z, X) \to \text{map}_*(Y, X)$ is a homotopy equivalence for all f-local spaces X. There is a homotopy localization for the class $\mathcal{W}_f$ of all f-equivalences.

Theorem 34.5. *There is a homotopy idempotent functor $L_f : \mathcal{T}_* \to \mathcal{T}_*$ such that for every space X, $L_f(X)$ is an f-local space and $X \to L_f(X)$ is an f-equivalence.*

The terminology given here is both standard and in conflict with the algebraic terminology. That is, a p-local group is one in which p is the only prime that is not invertible, while an f-local space is one for which f *is* invertible.

Problem 34.6. Explain how to use Theorem 34.5 to invert any set of maps.

Now we have two ways to localize: first, if $\mathcal{W}$ is the class of weak equivalences in some model structure; and second, if $\mathcal{W} = \mathcal{W}_f$ for some map f. It turns out that the second case is a special case of the first.

Theorem 34.7 (Hirschhorn). *For any f, there is a model structure whose weak equivalences are the f-equivalences; and in this structure f-localization is fibrant replacement.*

It would be nice to be able to prove Theorem 34.7 first and then derive Theorem 34.5 as a consequence. But this is counter to the existing argument: Hirschhorn uses Theorem 34.5 as a key element in his proof.

Now we can hardly avoid asking if we have found all of the homotopy idempotent functors. Do they all come from model category structures? The best answer to this question is quite subtle and ultimately rests on what axioms for set theory you accept.

Theorem 34.8 (Casacuberta, Scevenels, Smith)**.** *In ordinary ZFC, there are indempotent functors that are not of the form L_f; but if one assumes Vopěnka's Principle, then every idempotent functor is of the form L_f for some f.*[3]

34.1.3. Simple Explorations. Before beginning the proof of Theorem 34.5, we explore the idea of f-localization to try to develop some intuition for it.

Simple Example: Nullification. Historically, the first homotopy-theoretical localizations were the Postnikov sections.

Problem 34.9. Let $f : * \to S^{n+1}$ be the inclusion of the basepoint. Show that the functors $P_n(?)$ and $L_f(\,?\,)$ are naturally homotopy equivalent.

Because of Theorem 34.5, these functors have a vast generalization: we can consider localization with respect to the map $* \to X$ for any space X. Since these localization functors depend only on the pointed space X (which determines the map f), they are generally written $P_X(\,?\,)$.

The assignment $X \mapsto P_X$ may be used to define a partial order on the spaces in homotopy category $\mathrm{H}\mathcal{T}_*$: we say $X < Y$ if $P_X(Y) \sim *$. If $X < Y$ and $Y < X$, then we say that X and Y are **Bousfield equivalent**. The equivalence class of X is denoted $\langle X \rangle$. These classes inherit a partial order: $\langle X \rangle < \langle Y \rangle$ if and only if $X < Y$. Thus we are led to an intriguing new combinatorial structure that encodes certain global properties of the homotopy category $\mathrm{H}\mathcal{T}_*$.

Problem 34.10. Let $A \xrightarrow{f} B \to C$ be a cofiber sequence.

(a) Show that if X is f-local, then X is C-null.

(b) Show that if X is C-null, then X is Σf-local.

(c) Show that if X is C-null and if $[A, X] = *$, then X is f-local.

[3]ZFC stands for Zermelo-Frankel-Choice, which is the go-to axiomatization for people who want axioms for their set theory; Vopěnka's Principle is an extra axiom for sets which is known to not follow from ZFC and which is thought—but not known—to be consistent with ZFC.

Localization with Respect to Self-Maps. Suppose $f : A \to A$. Then we can compose f with itself many times, leading to maps $f^{\circ n} = f \circ f \circ \cdots \circ f : A \to A$, and even infinitely many times, yielding

$$f^{\circ\infty} : A \longrightarrow \operatorname{hocolim}(A \xrightarrow{f} A \xrightarrow{f} \cdots).$$

How are the localizations with respect to these maps related to one another?

Problem 34.11. Show that if X is f-local, then f is $f^{\circ n}$-local for all $n \leq \infty$. Is the converse true?

34.2. Proof of Theorem 34.5

To prove Theorem 34.5, we use a small object argument, in which the word ‘small’ is code for ‘extremely large’. The construction requires a little theory of ordinal numbers, which is described in Section A.9. In particular, we need to talk about transfinite iteration of functors F equipped with coaugmentations $X \to F(X)$. This is done by transfinite induction: $F^\beta(X) = F(F^{\beta-1}(X))$ if β is a successor ordinal; and if β is a limit ordinal, $F^\beta(X)$ is the homotopy colimit of the diagram $\beta \to \mathcal{T}_*$ that takes $\alpha \to \alpha+1$ to $F^\alpha(X) \to F^{\alpha+1}(X)$.

34.2.1. The Shape of a Small Object Argument. Here is how small object arguments typically work. The goal is to construct, for a space X, a map $X \to \Phi(X)$, where $\Phi(X)$ is required to satisfy some property that is expressed in terms of certain test diagrams of maps into $\Phi(X)$. We find a construction $X \to F(X)$ such that $F(X)$ satisfies the required condition on the test diagrams which happen to factor through $X \to F(X)$.

Now we repeat the construction some ordinal number α of times large enough that *every test diagram* into

$$\Phi(X) = F^\alpha(X) = \operatorname{hocolim}_{\beta<\alpha} F^\beta(X)$$

factors through the composition $F^\beta X \to F^{\beta+1} X \to \Phi(X)$. How large must α be? The most simple-minded thing to do is to count up the cardinality $\mathfrak{a}$ of all the points involved in the test diagrams. Each of these points must be mapped into $F^\beta(X)$ for some $\beta < \alpha$. If the cofinality of α is larger than $\mathfrak{a}$, the entire test diagram will map into $F^\beta(X)$ for some $\beta < \alpha$, and this will imply that $\Phi(X)$ has the required property.[4]

[4]If we assume that A and B are CW complexes, then we can get away with cofinality greater than the number of cells in $T_f(n)$.

34.2.2. The Property to Be Tested. The first thing to do is to replace $f : A \to B$ with a weakly equivalent cofibration between cofibrant spaces. We want $Y = L_f(X)$ to be f-local, which means that $f^* : \mathrm{map}_*(B, Y) \to \mathrm{map}_*(A, Y)$ should be a weak homotopy equivalence.

Problem 34.12.

(a) Show that different replacements of f with cofibrations yield equivalent concepts of f-equivalence.

(b) Show that Y is f-local if in any strictly commutative diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "u"] \arrow[d] & \mathrm{map}_*(B, Y) \arrow[d, "f^*"] \\ D^n \arrow[r, "v"] \arrow[ur, dotted] & \mathrm{map}_*(A, Y) \end{tikzcd}$$

of solid arrows, the dotted arrow can be filled in to make the whole diagram strictly commutative.

Now we have a criterion for Y to be f-local, but it involves spaces of maps into Y, rather than Y itself. To get this into a more useful form, we introduce two new spaces. First, $T_f(n)$ is the (homotopy) pushout in the square

$$\begin{tikzcd} A \times S^{n-1} \arrow[r, "f \times \mathrm{id}"] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & B \times S^{n-1} \arrow[d] \\ A \times D^n \arrow[r] & T_f(n) \end{tikzcd}$$

and note that $T_f(n)$ comes naturally with an inclusion into $B \times D^n$. Next, $P_f(X)$ is the (homotopy) pushout in

$$\begin{tikzcd} A \times \mathrm{map}_*(B, X) \arrow[r, "f \times \mathrm{id}"] \arrow[d, "\mathrm{id}_A \times f^*"'] \arrow[dr, phantom, "\text{(H)PO}"] & B \times \mathrm{map}_*(B, X) \arrow[d] \\ A \times \mathrm{map}_*(A, X) \arrow[r] & P_f(X). \end{tikzcd}$$

The space $P_f(X)$ is equipped with an 'inclusion' $j : P_f(X) \hookrightarrow B \times \mathrm{map}_*(A, X)$ and an evaluation map $@ : P_f(X) \to X$.

Problem 34.13. Maps $u : S^{n-1} \to \mathrm{map}_*(B, Y)$ and $v : D^n \to \mathrm{map}_*(A, Y)$ are adjoint to maps $\hat{u} : S^{n-1} \times B \to Y$ and $\hat{v} : D^n \times A \to Y$.

(a) Show that the solid arrow part of the diagram in Problem 34.12(b) commutes if and only if $\hat{u}$ and $\hat{v}$ define a map $(\hat{u}, \hat{v}) : T_f(n) \to Y$.

(b) Show that if u and v make the square in Problem 34.12(b) commute, then the solid arrow part of the cube

$$\begin{tikzcd}
T_f(n) \arrow[rr] \arrow[dr, "{(u,v)}"] \arrow[dd, "{(\hat{u},\hat{v})}"'] & & B \times D^n \arrow[dr, "\mathrm{id}\times v"] \arrow[dd, dotted, "\hat{\delta}" near start] & \\
& P_f(X) \arrow[rr, "j" near start] \arrow[dd, "@" near start] & & B \times \mathrm{map}_*(A, Y) \arrow[dd, dotted, "\varepsilon"] \\
Y \arrow[rr, equal] \arrow[dr, equal] & & Y \arrow[dr, equal] & \\
& Y \arrow[rr, equal] & & Y
\end{tikzcd}$$

commutes.

(c) Show that the dotted arrow in Problem 34.12 can be filled in if and only if there is a map $\widehat{\delta} : B \times D^n$ making the back square in the cube commute.

(d) Conclude that if there is a map ε making the front face commute up to homotopy, then Y is f-local.

34.2.3. The Construction. Inspired by Problem 34.13, we define the functor ℓ and the natural transformation $\eta : \mathrm{id} \to \ell$ by forming the standard homotopy pushout in the square

$$\begin{tikzcd}
P_f(X) \arrow[r, "j"] \arrow[d, "@"'] \arrow[dr, phantom, "\mathrm{HPO}"] & B \times \mathrm{map}_*(A, X) \arrow[d] \\
X \arrow[r, "\eta"] & \ell X.
\end{tikzcd}$$

Finally, we define $L_f(X) = \ell^\alpha(X)$, where the ordinal number α is chosen so that its cofinality is greater than the cardinality of $T_f(n)$ for all n.

Problem 34.14. Show that for any space X, $L_f(X)$ is f-local.

It remains to show that $X \to L_f(X)$ is an f-equivalence.

Problem 34.15. Suppose Z is an f-local space.

(a) Show that all of the maps in the square defining $P_f(X)$ become weak equivalences after applying $\mathrm{map}_*(\,?\,, Z)$.

(b) Show that $j^* : \mathrm{map}_*(B \times \mathrm{map}_*(A, X), Z) \to \mathrm{map}_*(P_f(X), Z)$ is a weak homotopy equivalence.

(c) Show that $\mathrm{map}_*(\ell(X), Z) \to \mathrm{map}_*(X, Z)$ is a weak homotopy equivalence.

(d) Finish the proof of Theorem 34.5.

34.2.4. Connectivity of $L_f(X)$. The construction of L_f involves both mapping spaces and formation of colimits over ridiculous diagrams; what can we say about the space $L_f(X)$ apart from its defining property? Most importantly, since most of the homotopy theory we have developed works most smoothly for simply-connected spaces, under what conditions can we guarantee that $L_f(X)$ is simply-connected?

Proposition 34.16. *If X is simply-connected, then $L_f(X)$ is also simply-connected for any map $f : A \to B$ with A path-connected and B simply-connected.*

The key to the proof is to study the localization of covering spaces.

Problem 34.17.

(a) Show that if X is path-connected, then $L_f(X)$ is also path-connected.
HINT. Use Problem 34.15(c).

(b) Show that if X is simply-connected and the universal cover $\overline{L_f(X)}$ is f-local, then $L_f(X)$ is simply-connected.

Let X be path-connected and let $p : \overline{X} \to X$ be a covering map, and consider the commutative square

$$\begin{array}{ccc} \mathrm{map}_*(B,\overline{X}) & \xrightarrow{\overline{f^*}} & \mathrm{map}_*(A,\overline{X}) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ \mathrm{map}_*(B,X) & \xrightarrow{f^*} & \mathrm{map}_*(A,X). \end{array}$$

Problem 34.18. Let $f : A \to B$ with A path-connected and B simply-connected.

(a) Show that $p_* : \mathrm{map}_*(A,\overline{X}) \to \mathrm{map}_*(A,X)$ is the inclusion of some collection of path components and $p_* : \mathrm{map}_*(B,\overline{X}) \to \mathrm{map}_*(B,X)$ is a homeomorphism.

(b) Show that if f^* is a weak equivalence, then $\overline{f^*}$ is a weak equivalence.
HINT. Restrict your attention to a single path component of $\mathrm{map}_*(A,X)$.

(c) Prove Proposition 34.16.

34.3. Homotopy Theory of $\mathcal{P}$-Local Spaces

In Section 20.3 we showed that the entire apparatus of elementary homotopy theory works just fine (at least for simply-connected spaces) when we choose to ignore a certain set $\mathcal{Q}$ of primes. Thus we proved the Blakers-Massey, Freduenthal, and other theorems as seen through '$\mathcal{Q}$-colored glasses' (ensuring that we can only see the $\mathcal{P}$-local phenomena). There is a very good overarching topological reason for this: (simply-connected) *spaces* can be localized at $\mathcal{P}$.

This kind of localization is by far the most frequently used kind—indeed it is frequently used in the literature tacitly and with little or no warning or explanation. We will show that it exists by exhibiting the correct map f and appealing to Theorem 34.5; but $\mathcal{P}$-localizations can be easily constructed using the basic techniques of homotopy theory, and we present the simpler construction in detail here. The problem with this construction is that it applies only to simply-connected spaces, while our earlier construction applies to all spaces (though it is only guaranteed to be $\mathcal{P}$-localization on the simply-connected ones).

After we work out our hands-on construction, we study the interplay of basic homotopy-theoretical operations and localization of simply-connected spaces with respect to $\mathcal{P}$. Finally, we investigate the reconstruction of X from its localizations.

34.3.1. $\mathcal{P}$-Localization of Spaces. Let $f : \bigvee_{q\in\mathcal{Q}} S^1 \to \bigvee_{q\in\mathcal{Q}} S^1$ be the wedge of the degree q maps, one for each prime $q \in \mathcal{Q}$. We write $L_{\mathcal{P}}$ for the functor L_f and refer to an f-local space as being **$\mathcal{P}$-local**.

It is easy to recognize f-local spaces and $\mathcal{P}$-localization maps.

Theorem 34.19. *Let $h : X \to Y$, where X and Y are simply-connected. Then the following are equivalent:*

(1) h is $\mathcal{P}$-localization,

(2) $h_* : \pi_*(X) \to \pi_*(Y)$ is algebraic $\mathcal{P}$-localization,

(3) $h_* : H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$ is algebraic $\mathcal{P}$-localization.

Corollary 34.20. *If X is simply-connected, the following are equivalent:*

(1) X is $\mathcal{P}$-local,

(2) $\pi_*(X)$ is a graded $\mathbb{Z}_{(\mathcal{P})}$-module,

(3) $H_*(X;\mathbb{Z})$ is a graded $\mathbb{Z}_{(\mathcal{P})}$-module.

Project 34.21. Prove Theorem 34.19.

Problem 34.22. Use Theorem 34.19 to prove Corollary 34.20.

The connection between this theory and the $\mathcal{P}$-local version of homotopy theory is already apparent.

Problem 34.23. Show that $\mathrm{conn}_{\mathcal{P}}(X) = \mathrm{conn}(X_{(\mathcal{P})})$ if X is simply-connected.

One very convenient thing about simply-connected p-local spaces is that the very nicely behaved cohomology theory $\widetilde{H}^*(\,?\,;\mathbb{Z}/p)$ gives very good information about them.

Proposition 34.24. *Let $f : X \to Y$ be a map of p-local simply-connected CW complexes of finite type. Then the following are equivalent:*

(1) f is a homotopy equivalence,

(2) $f^* : \widetilde{H}^*(Y;\mathbb{Z}/p) \to \widetilde{H}^*(X;\mathbb{Z}/p)$, is an isomorphism

(3) $f_* : \widetilde{H}_*(X;\mathbb{Z}/p) \to \widetilde{H}_*(Y;\mathbb{Z}/p)$ is an isomorphism.

Problem 34.25.

(a) Show that if X is p-local, simply-connected, and of finite type, then $X \simeq *$ if and only if $\widetilde{H}^*(X;\mathbb{Z}/p) = 0$.

(b) Prove Proposition 34.24.

34.3.2. Hands-On Localization of Simply-Connected Spaces. Now we temporarily forget the functor L_f of the previous section and instead take the theorems of that section as definitions. That is, we say that a simply-connected space is $\mathcal{P}$-local if $\pi_*(X)$ is a graded $\mathbb{Z}_{(\mathcal{P})}$-module and that a map $X \to Y$ of simply-connected spaces is $\mathcal{P}$-localization if its induced maps in π_* and $H_*(\,?\,;\mathbb{Z})$ are algebraic $\mathcal{P}$-localizations. Our goal is to construct the $\mathcal{P}$-localization of simply-connected spaces by induction on their homology decompositions. The first step, of course, is to define it for Moore spaces.

Problem 34.26. Show that $M(G,n) \to M(G_{(\mathcal{P})},n)$ is $\mathcal{P}$-localization.

Write $G_n = H_n(X;\mathbb{Z})$, $X_{(2)} = M(G_2,2)$ and $M_n = M(G_{n+1},n)$; then we construct a $\mathcal{P}$-localization for X by induction on its homology decomposition

$$\begin{tikzcd} M_2 \arrow[d] & M_3 \arrow[d] & & M_n \arrow[d] & \\ X(2) \arrow[r] & X(3) \arrow[r] & \cdots \arrow[r] & X(n) \arrow[r] & \cdots . \end{tikzcd}$$

Problem 34.26 provides a localization for $X(2)$, so let us assume that we have constructed a localization $X(n) \to (X(n))_{(\mathcal{P})}$.

Problem 34.27.

(a) Show that the dotted arrow in the diagram

$$\begin{tikzcd} M_n \arrow[r] \arrow[d] & (M_n)_{(\mathcal{P})} \arrow[d, dotted] \\ X(n) \arrow[r] & (X(n))_{(\mathcal{P})} \end{tikzcd}$$

can be filled in to make the square commute.

(b) Show that the induced map ϕ of cofibers in

$$\begin{tikzcd}
M_n \arrow[rr] \arrow[dr] \arrow[dd] & & CM_n \arrow[dr] \arrow[dd] & \\
& X(n) \arrow[rr] \arrow[dd] & & X(n+1) \arrow[dd, dotted, "\phi"] \\
(M_n)_{(\mathcal{P})} \arrow[rr] \arrow[dr] & & C(M_n)_{(\mathcal{P})} \arrow[dr] & \\
& (X(n))_{(\mathcal{P})} \arrow[rr] & & Z
\end{tikzcd}$$

is $\mathcal{P}$-localization; thus we write $(X(n+1))_{(\mathcal{P})}$ for the space Z.

(c) Show that the induced map of $X \to \operatorname{colim}(X(n))_{(\mathcal{P})}$ is $\mathcal{P}$-localization.

34.3.3. Localization of Homotopy-Theoretic Constructions. The exactness of algebraic localization together with the algebraic characterization of localization of simply-connected spaces implies that $\mathcal{P}$-localization preserves homotopy pushout and pullback squares. This is the foothold we need to show that $\mathcal{P}$-localization commutes with the basic constructions of homotopy theory.

Theorem 34.28. *Let $\mathcal{P}$ be a set of primes, and consider the squares*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A_{(\mathcal{P})} \arrow[r] \arrow[d] & B_{(\mathcal{P})} \arrow[d] \\
C_{(\mathcal{P})} \arrow[r] & D_{(\mathcal{P})}.
\end{tikzcd}$$

(a) If the first is a homotopy pushout square of simply-connected spaces, then so is the second.

(b) If the first is a homotopy pullback square of simply-connected spaces, then so is the second.

Problem 34.29. Prove Theorem 34.28.

Problem 34.30.

(a) Show that if $A \to X \to Y$ is a cofiber sequence of simply-connected spaces, then so is its localization $A_{(\mathcal{P})} \to X_{(\mathcal{P})} \to Y_{(\mathcal{P})}$, and similarly for fiber sequences.

(b) Show that if X and Y are simply-connected, then

$$\begin{array}{ll}
(X \times Y)_{(\mathcal{P})} \simeq X_{(\mathcal{P})} \times Y_{(\mathcal{P})}, & (\Sigma X)_{(\mathcal{P})} \simeq \Sigma X_{(\mathcal{P})}, \\
(X \vee Y)_{(\mathcal{P})} \simeq X_{(\mathcal{P})} \vee Y_{(\mathcal{P})}, & (\Omega X)_{(\mathcal{P})} \simeq \Omega(X_{(\mathcal{P})}), \\
(X \wedge Y)_{(\mathcal{P})} \simeq X_{(\mathcal{P})} \wedge Y_{(\mathcal{P})}, & (X * Y)_{(\mathcal{P})} \simeq X_{(\mathcal{P})} * Y_{(\mathcal{P})}.
\end{array}$$

Now we can give new and more directly topological proofs of the results of Section 20.3.

Problem 34.31. By applying our earlier results to the localizations of the spaces involved, give topological proofs of $\mathcal{P}$-local versions of the Blakers-Massey theorem, the Freudenthal Suspension Theorem, and so on.

Problem 34.32. Show that $\mathrm{cat}(X_{(\mathcal{P})}) \leq \mathrm{cat}(X)$ if X is simply-connected.

34.3.4. Recovering a Space from Its Localizations. If we are given $X_{(\mathcal{P})}$ and $X_{(\mathcal{Q})}$, then there is no prime at which we have lost information about X. In fact, a simply-connected space can be reconstructed from any such 'complementary pair' of localizations.

Problem 34.33. If $\mathcal{P}_1 \subseteq \mathcal{P}_2$, we get natural maps $X_{(\mathcal{P}_2)} \to X_{(\mathcal{P}_1)}$.

In the end, we reach the important special case $\mathcal{P} = \varnothing$. The localization $X_{(\varnothing)}$ is often written X_0, or $X_{\mathbb{Q}}$, and is called the **rationalization** of X. We'll use $X_{\mathbb{Q}}$, since it is utterly unambiguous.

Problem 34.34. Let $\mathcal{P} \sqcup \mathcal{Q}$ be a partition of the prime numbers. Then for any simply-connected space X, the square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_{(\mathcal{P})} \arrow[d] \\ X_{(\mathcal{Q})} \arrow[r] & X_{\mathbb{Q}} \end{tikzcd}$$

is a homotopy pullback square.

This result brings up an interesting possibility. Suppose you have a $\mathcal{P}$-local space P and a $\mathcal{Q}$-local space Q whose rationalizations are homotopy equivalent. Then you can form the homotopy pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "HPB"] & P \arrow[d] \\ Q \arrow[r] & P_{\mathbb{Q}} \end{tikzcd}$$

and build a new space X having the given spaces P and Q as its localizations. Next, consider two different spaces X and Y whose rationalizations are homotopy equivalent, and apply the above construction to $X_{(\mathcal{P})}$ and $Y_{(\mathcal{Q})}$. The result is a new space Z which looks like X from the point of view of $\mathcal{P}$ and like Y from the point of view of $\mathcal{Q}$. This technique is called **Zabrodsky mixing**.

Mislin Genus. Two simply-connected CW complexes X and Y are said to have the same **Mislin genus** if $X_{(p)} \simeq Y_{(p)}$ for all primes p. It is possible for nonhomotopy equivalent spaces—even finite complexes—to have the same Mislin genus, though they must appear to be very similar in many respects.

Problem 34.35. Show that if X and Y are simply-connected CW complexes with the same Mislin genus, then $H^*(X) \cong H^*(Y)$ and $\pi_*(X) \cong \pi_*(Y)$.

Properties of spaces that are shared by all spaces of the same genus (such as their cohomology and homotopy groups) are called **generic** properties.

Inspired at least in part by the result of Problem 34.32, C. McGibbon asked whether Lusternik-Schnirelmann category was a generic property. Roitberg showed that the answer is no, using cofibers of nontrivial phantom maps that become trivial on localization at every prime. The question remains open for finite CW complexes.

34.4. Localization with Respect to Homology

Let h_* be a homology theory. We'll call a map $f : A \to B$ an **h_*-equivalence** if it induces an isomorphism on h_*. If A and B are simply-connected and h_* is ordinary homology with coefficients in $\mathbb{Z}$, then every h_*-equivalence is a weak equivalence, but this is not the case for other homology theories. A space X is **h_*-local** if the map $f^* : \mathrm{map}_*(B, X) \to \mathrm{map}_*(A, X)$ is a weak equivalence for each h_*-equivalence f.

The canonical references for most of these results are [**30, 32**].

34.4.1. Construction of h_*-Localization. We would like to have a localization functor L_{h_*} that inverts all the h_*-equivalences and replaces each space X with an h_*-local one, but there is a serious problem: there are too many maps to invert! Since the class of all h_*-equivalences is not a set, we cannot use the technique of Problem 34.6. Nevertheless, the required localization does exist.

Theorem 34.36. *Let $\mathfrak{a}$ be the cardinality of the coefficient group $h_*(*)$, and let*

$$\mathcal{W} = \left\{ f : A \to B \;\middle|\; \begin{array}{l} h_*(f) \text{ is an isomorphism, and} \\ A \cup B \text{ has fewer than } \mathfrak{a} \text{ cells} \end{array} \right\}.$$

Then X is h_*-local if and only if X is $\mathcal{W}$-local.

We'll take this for granted.

Project 34.37. Prove Theorem 34.36.

Corollary 34.38. *Every homology theory h_* has a localization functor L_{h_*}.*

Problem 34.39. Use Theorem 34.36 to prove Corollary 34.38.

Problem 34.40. Let h_* be a homology theory. Show that if $\widetilde{h}_*(X) = 0$ and Y is h_*-local, then $\mathrm{map}_*(X, Y) \sim *$.

34.4.2. Ordinary Cohomology Theories. Consider localization with respect to ordinary homology theories. Some of these localizations are becoming familiar.

Problem 34.41. Show that if $h_* = H_*(\,?\,; \mathbb{Z}_{(\mathcal{P})})$, then h_*-localization of simply-connected spaces with respect to h_* is the same as $\mathcal{P}$-localization.

On the other hand, other coefficients give us genuinely new functors.

Completion Functors. Localization with respect to $H_*(\,?\,; \mathbb{Z}/p)$ is called p**-completion** and is generally denoted $X \to X_p^\wedge$. Localization with respect to $H_*(\,?\,; \mathbb{Z})$ is simply called **completion** and is denoted $X^\wedge$.

Problem 34.42.

(a) Show that if X is simply-connected and $\widetilde{H}_*(X; \mathbb{Z}/p) = 0$, then $X_p^\wedge \simeq *$.

(b) Show that if Y is p-complete, then the map $X \to X_p^\wedge$ induces a weak equivalence $\mathrm{map}_*(X_p^\wedge, Y) \to \mathrm{map}_*(X, Y)$.

Algebraic Characterization of p-Completion. If we apply the p-completion functor to an Eilenberg-Mac Lane space for an abelian group, we get a map

$$G = \pi_n(K(G, n)) \longrightarrow \pi_n(K(G, n)_p^\wedge)$$

which is independent of the choice of $n \geq 2$, so we introduce the notation

$$G_p^\wedge = \pi_n(K(G, n)_p^\wedge)$$

and call the map $G \to G_p^\wedge$ the p**-completion** of G. There is an algebraic characterization of the p-completions of abelian groups. A group G is called **Ext-p-complete** if

$$\mathrm{Hom}(\mathbb{Z}[\tfrac{1}{p}], G) = 0 = \mathrm{Ext}(\mathbb{Z}[\tfrac{1}{p}], G).$$

Theorem 34.43.

(a) An abelian group is the p-completion of another abelian group if and only if it is Ext-p-complete.

(b) The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}_p^\wedge$ of $\mathbb{Z}$ into p***-adic integers*** *is p-completion of $\mathbb{Z}$.*[5]

We'll take this for granted and derive some practical consequences.

[5] $\mathbb{Z}_p^\wedge$ is the limit of the tower of surjections $\cdots \to \mathbb{Z}/p^r \to \mathbb{Z}/p^{r-1} \to \cdots$.

Corollary 34.44. *If G is finitely generated, then the composite map*

$$G \xrightarrow{\cong} G \otimes \mathbb{Z} \longrightarrow G \otimes \mathbb{Z}_p^\wedge$$

is p-completion.

Problem 34.45.

(a) Show that the p-completion of $\mathbb{Z}/p^r$ is $\mathbb{Z}/p^r$.

(b) Show that $(G \times H)_p^\wedge \cong G_p^\wedge \times H_p^\wedge$.

(c) Prove Corollary 34.44.

Now we are prepared to state our algebraic recognition principle for p-completions.

Theorem 34.46. *If X is a simply-connected CW complex of finite type, then $f : X \to Y$ is p-completion if and only if the induced map $f_* : \pi_*(X) \to \pi_*(Y)$ can be identified with the algebraic p-completion*

$$\pi_*(X) \longrightarrow \pi_*(X)_p^\wedge.$$

Finally, we show that for simply-connected spaces, p-completion can be accomplished by nullification.

Proposition 34.47. *Let $M = M(\mathbb{Z}[\frac{1}{p}], 1)$. Then for simply-connected spaces X, the nullification $X \to P_M(X)$ is p-completion.*

Problem 34.48. Let X be simply-connected.

(a) Show that for each $n \geq 2$ there is an exact sequence

$$0 \leftarrow \mathrm{Hom}(\mathbb{Z}[\tfrac{1}{p}], \pi_{n-1}(X)) \longleftarrow [\Sigma^{n-1} M, X] \longleftarrow \mathrm{Ext}(\mathbb{Z}[\tfrac{1}{p}], \pi_n(X)) \leftarrow 0.$$

(b) Prove Proposition 34.47.

Homotopy Properties of Completion. The p-completion functors are not quite as nice as the p-localization functors, but they do behave well with respect to homotopy pullback.

Theorem 34.49. *If D is simply-connected and the left square of*

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "HPB"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A_p^\wedge \arrow[r] \arrow[d] \arrow[dr, phantom, "HPB"] & B_p^\wedge \arrow[d] \\ C_p^\wedge \arrow[r] & D_p^\wedge \end{tikzcd}$$

is a homotopy pullback square, then the right square is also a homotopy pullback square.

You should take this for granted.

Corollary 34.50. *If $F \to E \to B$ is a fibration sequence with B simply-connected, then $F_p^\wedge \to E_p^\wedge \to B_p^\wedge$ is also a fibration sequence.*

Rebuilding a Space from Its Completions. Just as we were able to recover a space X from its localizations, we can recover X from its completions.

Theorem 34.51. *Let $\mathcal{P}$ be a set of prime numbers. Then the square*

$$\begin{tikzcd}
X_{(\mathcal{P})} \arrow[r] \arrow[d] \arrow[dr, phantom, "HPB"] & \prod_{p\in\mathcal{P}} X_p^{\wedge} \arrow[d] \\
X_{\mathbb{Q}} \arrow[r] & \left(\prod_{p\in\mathcal{P}} X_p^{\wedge}\right)_{\mathbb{Q}}
\end{tikzcd}$$

is a homotopy pullback square.

We use Theorem 34.51 to prove the equivalence of certain mapping spaces.

Corollary 34.52. *If $\widetilde{H}_*(B;\mathbb{Z}[\frac{1}{p}]) = 0$, then the induced map*

$$\mathrm{map}_*(B,X) \longrightarrow \mathrm{map}_*(B,X_p^{\wedge})$$

is weak equivalence.

Problem 34.53. Let B be as in Corollary 34.52.

(a) Show that $\widetilde{H}_*(B;\mathbb{Q}) = 0$ and $\widetilde{H}_*(B;\mathbb{Z}/q) = 0$ for all $q \neq p$.

(b) Prove Corollary 34.52.

34.4.3. Other Connective Homology Theories. A homology theory h_* is called **connective** if $h_k(*) = 0$ for all sufficiently small k. Localization with respect to connective theories is the same as localization with respect to the coefficients.

Theorem 34.54 (Bousfield)**.** *If h_* is connective and satisfies the Wedge Axiom, then localization with respect to h_* is either $\mathcal{P}$-localization or $\mathcal{P}$-completion, where $\mathcal{P}$ is some set of primes.*

We will not prove this, or use it, in this text.

34.5. Rational Homotopy Theory

The homotopy theory of rational spaces is *almost* trivial: after suspension, all spaces and maps are entirely determined by their homology, which means that it is equivalent to the theory of graded $\mathbb{Q}$ vector spaces. Dually, after looping, spaces and maps are determined by their effect on homotopy groups, again reducing the theory to $\mathbb{Q}$-vector spaces.

Though the degeneracy of loops and suspensions makes it much simpler than ordinary homotopy theory, rational homotopy theory is decidedly *not* trivial. Interesting and important theorems about ordinary spaces can be proved by studying their rationalizations.

One reason for the great success of the theory is that the homotopy theory of rational spaces (even those that are not suspensions or loop spaces) is isomorphic to the homotopy theory of two different categories of algebras. The subject of rational homotopy is vast, and we can only give the smallest hint of its flavor here. If you find your curiosity piqued, you should move on to the books [**65**, **79**].

34.5.1. Suspensions and Loop Spaces. In this section we show that the suspensions of simply-connected rational space split as wedges of spheres and that the loop spaces of simply-connected rational spaces split as products of Eilenberg-Mac Lane spaces.

The two dual results are fundamentally based on the following simple observation about rational spheres and Eilenberg-Mac Lane spaces.

Problem 34.55. Show that $S^{2n+1}_{\mathbb{Q}} = K(\mathbb{Q}, 2n+1)$.

From now on, we will tacitly assume that all spaces are rational spaces; thus we'll write S^n for $S^n_{\mathbb{Q}}$.

Rational Suspensions. The splitting of suspensions results, ultimately, from the combination of Problem 34.55 and the James splitting.

Theorem 34.56. *If X is a simply-connected rational space of finite type, then the following are equivalent:*

(1) X is a suspension,

(2) there is a map $X \to \bigvee S^{n_\alpha}$ inducing a surjection on cohomology,

(3) X is homotopy equivalent to a wedge of spheres.

We begin with some simple but useful results which produce maps of spaces from algebraic phenomena.

Problem 34.57.

(a) Let W_1 and W_2 be finite-type wedges of rational spheres. Show that the map
$$[W_1, W_2] \longrightarrow \mathrm{Hom}_{\mathrm{AB}\,\mathcal{G}_*}(H^*(W_2), H^*(W_1))$$
given by $g \mapsto g^*$ is surjective.

(b) Show that if $f : X \to \bigvee S^{n_\alpha}$, then there is $g : \bigvee S^{n_\alpha} \to \bigvee S^{m_\beta}$ such that $\widetilde{H}^*(\bigvee S^{n_\alpha}; \mathbb{Q}) = \mathrm{Im}(g^*) \oplus \ker(f^*)$.

Problem 34.58.

(a) Show that conditions Theorem 34.56(2) and (3) hold for $X = \Sigma K(\mathbb{Q}, n)$.

(b) Prove Theorem 34.56.

Rational Loop Spaces. The dual statement depends on a splitting of ΩX instead of the James splitting.

Theorem 34.59. *If X is a simply-connected rational space of finite type, then the following are equivalent:*

(1) X is a loop space,

(2) there is a map $\prod_\alpha K(\mathbb{Q}, n_\alpha) \to X$ which induces a surjection on homotopy groups,

(3) X is homotopy equivalent to a product of Eilenberg-Mac Lane spaces.

We continue to tacitly assume that all spaces have been rationalized.

Problem 34.60.

(a) If P and Q are finite-type products of rational Eilenberg-Mac Lane spaces, then the map $[P, Q] \to \mathrm{Hom}_{\mathrm{AB}\,\mathcal{G}_*}(\pi_*(P), \pi_*(Q))$ is surjective.

(b) Show that parts (2) and (3) of Theorem 34.59 hold if $X = S^n$.

(c) Prove Theorem 34.59.

34.5.2. Sullivan Models. Now we give a brief account of the first of the algebraic models for rational spaces.

Commutative Differential Graded Algebras. If a commutative graded $\mathbb{Q}$-algebra A has a differential d which satisfies the standard product rule, then A is called a **commutative differential graded algebra**, commonly referred to as a CDGA.

A CDGA A is **simply-connected** if $A^0 \cong \mathbb{Q}$ and $A^1 = 0$. A simply-connected[6] CDGA is a **Sullivan algebra** if it has the form $\Lambda(V)$, where V is a graded $\mathbb{Q}$-vector space with $V_0 = V_1 = 0$. A Sullivan algebra is called **minimal** if the image of the differential lies in the subalgebra $\Lambda^{\geq 2}(V)$ generated by all nontrivial products.

Exercise 34.61. Why does it make sense to use the terminology *minimal*?

There is a generalization of this notion: if A is any (graded) commutative differential graded algebra, then we may consider differential graded algebra structures on $A \otimes \Lambda V$ that extend the given one on A. Thus we allow the differential of $v \in V$ to involve elements of A, but we require $d(A) \subseteq A$. Such a CDGA, or really the map $A \to A \otimes \Lambda V$, is called a **relative Sullivan algebra**.

Theorem 34.62. *There is a model category structure on the category of simply-connected (graded) commutative differential graded algebras in which*

[6] This definition can be extended to non-simply-connected algebras, but the increase in technicalities is considerable.

- *the weak equivalences are the quasi-isomoprhisms: maps which induce isomorphisms on homology,*
- *the fibrations are the surjections, and*
- *the cofibrations are the retracts of relative Sullivan algebras.*

We will not prove it. You should consult [**65**] for details.

Models for Rational Spaces. The best theorem explaining the algebraic modeling of rational spaces shows that the model category of simply-connected CDGAs is Quillen equivalent to a certain model category of simplicial sets, which in turn agrees with the homotopy theory of simply-connected rational spaces. We will not go into all of these technicalities but will content ourselves with a more 'practical' account of the representation of spaces by algebras.

The algebra of singular cochains $S^*(X;\mathbb{Q})$ is not (graded) commutative. It was shown (by Sullivan) that there is a natural quasi-isomorphism

$$A_{PL}(X) \longrightarrow \mathcal{S}^*(X;\mathbb{Q}),$$

where $A_{PL}(X)$ is commutative, and hence an object of our model category. Now we can find a cofibrant replacement $\Lambda V \to A_{PL}(X)$, which is unique up to the appropriate notion of homotopy in the category of CDGAs. But minimal Sullivan algebras are such rigid gadgets that a quasi-isomorphism between minimal Sullivan algebras is actually an isomorphism. Thus we see that each space X gives rise to a well-defined minimal Sullivan algebra, called the **minimal model** of X.

Theorem 34.63.

(a) *The minimal model establishes a bijection between homotopy types of finite type simply-connected spaces and isomorphism classes of minimal Sullivan algebras.*

(b) *Homotopy classes of maps between simply-connected spaces of finite type correspond bijectively to homotopy classes of maps between the corresponding Sullivan algebras.*

Problem 34.64. Determine minimal models for S^n and $\mathbb{C}\mathrm{P}^n$.

The minimal model $(\Lambda(V), d)$ is of course intimately bound up with the homotopy theory of X. The differential can be split into 'homogeneous' pieces $d = d_1 + \cdots + d_k + \cdots$, where d_k increases wordlength by exactly k. The first term d_1 is known as the **quadratic part** of d.

Theorem 34.65. *Let $(\Lambda(V), d)$ be the minimal model for the space X.*

(a) *There is a natural isomorphism $V \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Q}}(\pi^*(X), \mathbb{Q})$.*

(b) *The isomorphism of (a) identifies d_1, the quadratic part of d, with the algebraic dual of the Whitehead product.*

To clarify, part (b) says that under the isomorphism of part (a), the map $d_1 : V \to V \otimes V$ may be identified with the composition

$$\mathrm{Hom}(\pi_*(X), \mathbb{Q}) \otimes \mathrm{Hom}(\pi_*(\mathbb{Q})) \to \mathrm{Hom}(\pi_*(X) \otimes \pi_*(X), \mathbb{Q}) \to \mathrm{Hom}(\pi^*(X)),$$

in which the second map is the algebraic dual of the Whitehead product.

34.5.3. The Lie Model. There is a dual algebraic model, which is a differential graded Lie algebra, built from the homology of X and computing the homotopy groups of X. The differential in a differential graded Lie algebra (often abbreviated DGLA) also breaks up as a sum $d = d_0 + d_1 + \cdots + d_k + \cdots$ where d_k increases product length by k. We say that $\mathbb{L}$ is **minimal** if $d_0 = 0$.

Theorem 34.66.

(a) *There is a bijection between homotopy types of simply-connected finite type rational spaces and minimal graded differential Lie algebras.*

(b) *The homology of the DGLA $\mathbb{L}$ corresponding to X is $H_*(\mathbb{L}) \cong \pi_*(\Omega X)$.*

(c) *After looping, the composite isomorphism $H_*(\mathbb{L}) \cong \pi_*(\Omega X) \cong \pi_{*+1}(X)$ identifies the bracket in $\mathbb{L}$ with the Whitehead product in $\pi_*(X)$.*

34.5.4. Elliptic and Hyperbolic. One of the big surprises in rational homotopy theory is the emergence of a division of the collection of finite complexes into two radically different classes. For any rational space X, we can consider the growth of its homotopy groups $\pi_{\leq n}(X) = \bigoplus_{k=1}^{n} \pi_k(X)$ as n increases.

Theorem 34.67. *Let X be a nontrivial rational finite complex. Then*

- *either* $\dim(\pi_*(X)) < \infty$ *(i.e., $\pi_k(X) = 0$ for large k)*
- *or* $\dim_{\mathbb{Q}}(\pi_{\leq n}(X))$ *grows exponentially with n.*

More precisely, the exponential growth here means that there is a $c > 0$ and an $N \in \mathbb{N}$ such that $\dim_{\mathbb{Q}}(\pi_{\leq n}(X)) \geq e^{cn}$ for all $n > N$. Spaces of the first kind are called **rationally elliptic** and those of the second kind are called **rationally hyperbolic**. For example, spheres and projective spaces are elliptic; and wedges of spheres are hyperbolic.

34.5.5. Lusternik-Schnirelmann Category of Rational Spaces. We say that a space X satisfies the **Ganea condition** if

$$\mathrm{cat}(X \times S^k) = \mathrm{cat}(X) + 1 \qquad \text{for all} \quad k \geq 1.$$

Ganea [**68**] asked whether there were any spaces which do not satisfy this condition, a question which has had enormous influence on the study of

Lusternik-Schnirelmann category. You constructed spaces which do not satisfy the Ganea condition in Section 33.7; but for a long time, it seemed as though perhaps there were no such examples. This feeling was strengthened in the early 1990s by the proof—a combination of the work of B. Jessup [**106**] and K. Hess [**82**]—that all simply-connected rational spaces do satisfy the Ganea condition. Actually more is true: it was shown in [**64**] that $\mathrm{cat}(X \times Y) = \mathrm{cat}(X) + \mathrm{cat}(Y)$ for any two simply-connected rational spaces.

34.6. Further Topics

We begin by developing an exact sequence, called the **EHP sequence**, which fits the James Hopf invariant H_2 into an exact sequence involving the suspension map. Then we investigate the general principle that the commutativity of the rationalization of a finite diagram of finite complexes does not actually require the inversion of all the primes. Rather, there should be a finite set $\mathcal{Q}$ of primes which account for all of the pre-rationalization obstructions to commutativity, and after $\mathcal{Q}$ has been inverted, the diagram will commute.

This implies, for example, that odd-dimensional spheres are H-spaces when localized away from the prime 2 and that, more generally, H-spaces split into products of odd-dimensional spheres after a certain finite collection of primes are inverted.

34.6.1. The EHP Sequence. Now we come back to Hopf invariants. Using Lemma 25.35 as input into the Leray-Serre spectral sequence, we can determine the p-local homotopy type of the fiber of the p^{th} James Hopf invariant.

Proposition 34.68. *The p-localization of the sequence $J^{p-1} \hookrightarrow J \xrightarrow{H_p} \Omega S^{2np+1}$ is a fiber sequence.*

Problem 34.69. Let F_p be the fiber of H_p.

(a) Construct a map $\xi : J^{p-1} \to F_p$.

(b) Show that ξ induces an isomorphism on mod p homology.

(c) Prove Proposition 34.68.

Hint. Show that the homology Leray-Serre spectral sequence for the sequence $F_p \to \Omega S^{2n+1} \xrightarrow{H_p} \Omega S^{2np+1}$ collapses at E_2.

The Hopf invariant maps and the corresponding fiber sequences can be used to study the p-localizations of the homotopy groups of spheres. This is especially simple for $p = 2$

Problem 34.70. Show that the 2-localizations of the homotopy groups of spheres fit into a long exact sequence

$$\cdots \to \pi_k(S^n) \xrightarrow{E} \pi_{k+1}(S^{n+1}) \xrightarrow{H} \pi_{k+1}(S^{2n+1}) \xrightarrow{P} \pi_{k-1}(S^n) \to \cdots .$$

The sequence of Problem 34.70 is known as the **EHP sequence**. The 'E' stands for 'Einhängung', the German word for suspension; 'H' is for 'Hopf invariant'; and 'P' indicates that the map can be described in terms of Whitehead products.

34.6.2. Spheres Localized at $\mathcal{P}$. We know that an odd-dimensional rational sphere is an H-space (Problem 34.55). Thus the general principal enunciated above suggests that, except for finitely many primes, $S^{2n+1}_{(p)}$ should be an H-space. This was proved by Adams in [**2**] before $\mathcal{P}$-localization functors were constructed.

Problem 34.71. Show that if X is an H-space, then for any collection of primes $\mathcal{P}$, $X_{(\mathcal{P})}$ is also an H-space.

Theorem 34.72. *Let $\mathcal{P} = \{$all odd primes$\}$. Then for every $n \geq 1$, $S^{2n+1}_{(\mathcal{P})}$ is an H-space.*

Problem 34.73. Prove Theorem 34.72.

Hint. Use Problem 9.133.

Project 34.74. Investigate the possibility that $S^{2n-1}_{(p)}$ could be homotopy associative, commutative, and so on.

34.6.3. Regular Primes. If X is an H-space, then X is a retract of $\Omega\Sigma X$, and so, rationally, X is a product of Eilenberg-Mac Lane spaces. If X is finite-dimensional, this forces the factors to be odd-dimensional spheres, so we can write

$$X \simeq S^{n_1} \times \cdots \times S^{n_r},$$

in which each n_i is an odd number, and we may as well index them so that $n_1 \leq n_2 \leq \cdots \leq n_r$. In this case, we say that X has **type** $(n_1, n_2, \ldots, n_r)$.

Theorem 34.75. *If X is an H-space of type $(n_1, n_2, \ldots, n_r)$ and p is large enough, then $X_{(p)} \simeq S^{n_1} \times \cdots \times S^{n_r}$.*

A prime number for which the splitting occurs is called a **regular prime** for the H-space X.

Problem 34.76. Prove Theorem 34.75.

Even if p is not regular for X, it can happen that $X_{(p)}$ splits as a product of simpler spaces. If each factor in such a splitting is either a sphere or a sphere bundle over a sphere, then the prime p is called **quasiregular** for the H-space X.

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Chapter 35

Exponents for Homotopy Groups

Determining the homotopy groups of simply-connected finite complexes is a notoriously difficult problem, and a great deal of work has been devoted to computations, especially for spheres, Moore spaces, and various topological groups. In the stable range the homotopy groups are a homology theory, so their computation ultimately depends on the homotopy groups of spheres. Even outside the stable range, knowledge of $\pi_*(S^n)$ can provide useful information about the homotopy groups of other spaces.

This chapter is not about specific computations of homotopy groups; rather it is concerned with global properties of the entire graded group $\pi_*(S^n)$. Specifically, we will study the p-exponent of $\pi_*(S^n)$. If G is a group such that $x^n = 1$ for every $x \in G$, then G is said to have **exponent** n. For nilpotent groups, whose torsion splits as a product of its Sylow subgroups, we can talk about the p-**exponent**, which is the ordinary exponent of the Sylow p-subgroup. We are of course interested in homotopy groups, which are customarily written additively; thus the p-exponent of $\pi_k(X)$ is the least p^n so that $p^n x = 0$ for every p-torsion element of $\pi_k(X)$.

So now we ask: if X is a finite complex, does $\pi_*(X)$ have a p-exponent? What is that exponent? In [**170**], Toda showed that for odd primes the homotopy groups $\pi_*(S^{2n+1})$ have p-exponent dividing p^{2n}. Barratt conjectured that the p-exponent is actually equal to p^n. Later Gray [**72**] showed that there are elements in $\pi_*(S^{2n+1})$ having order p^n, and finally in [**43**], the conjecture was proved.

Theorem 35.1 (Cohen, Moore, Neisendorfer). *If p is an odd prime, then homotopy groups $\pi_*(S^{2n+1})$ have p-exponent p^n.*

The proof of Theorem 35.1 is too involved for us to tackle here. Instead we will indicate, in very broad strokes, the basic thrust of the proof. The proof by induction on n is accomplished by finding, for each $n \geq 2$, factorizations

$$\begin{tikzcd}
\Omega^2 S^{2n+1}_{(p)} \arrow[rr, "p"] \arrow[dr] & & \Omega^2 S^{2n+1}_{(p)} \\
& S^{2n-1}_{(p)} \arrow[ur] &
\end{tikzcd}$$

of the degree p map on the H-space $\Omega^2 S^{2n+1}_{(p)}$. The construction given in [**43**] depends heavily on algebraic computation; but a nice topological construction was found recently by Gray and Theriault [**77**].

Problem 35.2. Show that this kind of factorization implies that the exponent of $\pi_*(S^{2n+1})$ is at most p times that of $\pi_*(S^{2n-1})$.

Because of Problem 35.2, the proof of Theorem 35.1 reduces to showing that $\pi_*(S^3)$ has exponent p. This is due to Selick [**149**].

Theorem 35.3 (Selick). *If p is odd, then $\pi_*(S^3)$ has p-exponent p.*

Our goal in this chapter is to prove Theorem 35.3. This proof is also based on a clever factorization. We will define maps α and $\Omega^2(\gamma)$ that fit into a commutative triangle (in which all spaces have been silently localized at p)

$$\begin{tikzcd}
\Omega^2(S^3\langle 3\rangle) \arrow[rr, "\simeq"] \arrow[dr, "\alpha"'] & & \Omega^2(S^3\langle 3\rangle) \\
& \Omega^2 S^{2p+1}\{p\}, \arrow[ur, "\Omega^2(\gamma)"'] &
\end{tikzcd}$$

where $\Omega^2 S^{2p+1}\{p\}$ denotes the fiber of the degree p map $\Omega^2 S^{2p+1} \xrightarrow{p} \Omega^2 S^{2p+1}$.

Problem 35.4. Let p be an odd prime.

(a) Show that $\Omega^2 S^{2p+1}\{p\} \simeq \mathrm{map}_*(M(\mathbb{Z}/p, 2), S^{2p+1})$.

(b) Show that the commutativity of the triangle implies Theorem 35.3.

In the final section of this chapter, we apply Theorem 35.1 to produce some interesting examples: maps $f : X \to Y$ of finite complexes such that $\Sigma^2 f$ induces zero on homotopy groups.

Throughout this chapter we will work in the category of p-local spaces, except when otherwise noted.

35.1. Construction of α

In this section we will construct the map α. We will make use of a construction, called the deviation, which takes a map f between H-spaces and returns another map which measures how far f is from being an H-map. Thus we begin with the definition of the deviation and an exploration of its basic properties.

35.1.1. Deviation. The construction of α makes repeated and essential use of the concept of deviation. Suppose $f : X \to Y$, where X is an H-space, and Y is a group-like space. We assume that X is well-pointed, and so its multiplication μ may be replaced with a homotopic one which has $*$ as strict multiplicative identity. Now we may ask whether or not f is an H-map.

The answer can be given in terms of a new map $D(f) : X \times X \to Y$ defined by the rule

$$D(f)(x, y) = f(y)^{-1} f(x)^{-1} f(xy)$$

and called the **deviation** of f.

Problem 35.5. Let $f : X \to Y$, where Y is group-like and X is an H-space.

(a) Show that f is an H-map if and only if $D(f) \simeq *$.

(b) Show that if $g : Y \to Z$ is a homomorphism of topological groups, then $D(g \circ f) = g \circ D(f)$.

Thus $D(f)$ may be considered the obstruction to f being an H-map.

The deviation can be iterated to give higher-order deviation maps $D_n(f) : X^n \to Y$. Fix an $(n-1)$-tuple $\bar{x} \in X^{n-1}$ and write $\mathrm{in}_{\bar{x}} : X \to X^n$ for the inclusion map $\mathrm{in}_{\bar{x}} : y \mapsto (\bar{x}, y)$. The inductive definition of $D_n(f)$ begins by setting $D_1(f) = f$ and then defining

$$\begin{aligned} D_{n+1}(f)(\bar{x}, y, z) &= D(D_n(f) \circ \mathrm{in}_{\bar{x}})(y, z) \\ &= D_n(f)(\bar{x}, z)^{-1} D_n(f)(\bar{x}, y)^{-1} D_n(f)(\bar{x}, yz). \end{aligned}$$

If X is an H-space and Y is group-like, then every function $f : X \to Y$ gives rise to a sequence of functions $f_n : X \to Y$ defined by the composition

$$\begin{tikzcd} X \arrow[rr, "f_n"] \arrow[dr, "\Delta_n"'] & & Y \\ & X^n. \arrow[ur, "D_n(f)"'] & \end{tikzcd}$$

If Y is homotopy-commutative and X is homotopy-associative, then we can write down a more explicit formula for the homotopy class of the n^{th}

deviation. It will help to introduce an 'f-ed up' version of the classical symmetric functions:

$$\sigma_k^f(x_1, \ldots, x_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} f(x_{i_1} \cdots x_{i_k}).$$

Problem 35.6. Let Y be a homotopy-commutative group-like space, and let X be a homotopy-associative H-space.

(a) Show that for $\bar{x} \in X^{n-1}$ and $y, z \in X$,

$$\sigma_k^f(\bar{x}, y, z) = \left(\sigma_k^f(\bar{x}, y) + \sigma_k^f(\bar{x}, z) + \sigma_{k-1}^f(\bar{x}, yz)\right) - \left(\sigma_k^f(\bar{x}) + \sigma_{k-1}^f(\bar{x})\right).$$

(b) Show that $D_n(f) = \sum_{k=1}^n (-1)^{n-k} \sigma_k^f$ in the group $[X^n, Y]$.

(c) Let $m \in \mathbb{N}$ and also use $m : X \to X$ to denote the map $x \mapsto x^m$. Then

$$f \circ m = \sum_{j=1}^{m} \binom{m}{j} f_j.$$

Hint. Use part (b) and an identity for binomial coefficients.

The deviation is just one of a whole family of obstructions to various algebraic properties. Probably the most important of these is the commutator, which measures the noncommutativity of an algebraic gadget.

35.1.2. Deviation and Lusternik-Schnirelmann Category. Obstructions such as deviation tend to vanish on spaces with small Lusternik-Schnirelmann category. The first instance of this kind of argument—applied to commutators—is probably G. W. Whitehead's proof of the graded Jacobi identity for the Whitehead product [**181**]. Steenrod [**159**] used the deviation to demonstrate that certain cohomology operations vanish on spaces with 'small' category, and we'll prove a similar result here.

The n^{th} deviation of a map f kills the fat wedge.

Proposition 35.7. *If $f : X \to Y$ where X is an H-space and Y is group-like, then for each $n \ge 1$, the composition*

$$T^n X \xrightarrow{j} X^n \xrightarrow{D_n(f)} Y$$

is trivial.

Corollary 35.8. *If $\mathrm{cat}(A) < n$ and $g : A \to X$, then $f_n \circ g \simeq *$.*

Problem 35.9. Let $f : X \to Y$ where X is an H-space and Y is group-like.

(a) Suppose Proposition 35.7 is true for $n - 1$. Show that $D_{n-1}(f)$ is homotopic to a map $\widetilde{D} : X^{n-1} \to Y$ such that $\widetilde{D}(T^{n-1}(X)) = *$.

(b) Show that $D_n(f)(\bar{x}, y, z) \simeq D(\widetilde{D} \circ \mathrm{in}_{\bar{x}})(y, z)$.

(c) Prove Proposition 35.7 and derive Corollary 35.8.

35.1.3. Deviation and Ganea Fibrations. Consider the Ganea fibrations over BG, where G is a topological group. We obtain the split[1] fibration sequence

$$\begin{tikzcd} \cdots \arrow[r] & \Omega G_1(BG) \arrow[r, "\Omega p_1"'] & \Omega BG \arrow[l, bend right, "s"'] \arrow[r] & G * G \arrow[r] & G_1(BG) \arrow[r, "p_1"] & BG. \end{tikzcd}$$

Proposition 35.10. *If G is an H-space, then for any $n \geq 2$, the composite*

$$G \xrightarrow{s} \Omega G_1(BG) \xrightarrow{i} \Omega G_n(BG)$$

is an H-map.

Problem 35.11.

(a) Show that the lift exists in the diagram

$$\begin{tikzcd} & \Omega F_1(BG) \arrow[r, "\Omega j"] \arrow[d] & \Omega F_n(BG) \arrow[d] \\ G \times G \arrow[ur, dotted] \arrow[r, "D(s)"] & \Omega G_1(BG) \arrow[r, "\Omega i"] \arrow[d, "\Omega p_1"] & \Omega G_n(BG) \arrow[d] \\ & G \arrow[r, equal] & G. \end{tikzcd}$$

(b) Show that the map $j : F_1(BG) \to F_n(BG)$ is trivial.

(c) Prove Proposition 35.10.

35.1.4. Compositions of Order p. To construct a map into $S^{2n+1}\{p\}$, we first build one with target S^{2n+1} and show that it has order p. The following lemma is the perfect tool for this purpose.

Lemma 35.12. *Let $f : X \to Y$, where X is a homotopy-associative H-space and let Y be a p-local homotopy-commutative group-like space. Suppose that for all $q \in \mathbb{N}$, the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \arrow[d, "q^p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is homotopy commutative. If $g : A \to X$ with $\operatorname{cat}(A) < p$, then the order of the element $f \circ g \in [A, Y]$ divides p.

The proof boils down to establishing some entertaining combinatorial formulas involving the deviation.

[1] You proved that p_n splits after looping in Problem 20.46.

Problem 35.13. Let $f : X \to Y$ be a map as in Lemma 35.12, and let $m > 1$.

(a) Show that $f_m = (m^p - m) \cdot f - \sum_{j=2}^{m-1} \binom{m}{j} \cdot f_j$.

(b) Show that $f_m = \lambda_m \cdot f$, where $p | \lambda_m$.

(c) Show that $f_p = (p \cdot u) \cdot f$ where $u \in \mathbb{Z}_{(p)}$ is a unit.

(d) Prove Lemma 35.12.

The bizarre hypotheses of Lemma 35.12 were not invented according to some childish whim or momentary impulse: they were formulated precisely to apply to the p^{th} Hopf invariant H_p.

Problem 35.14.

(a) Determine the suspension Σq of the degree q map $q : \Omega S^{m+1} \to \Omega S^{m+1}$.

(b) Show that for any $q \in \mathbb{Z}$ and any $m \geq 1$, the diagram

$$\begin{tikzcd}
\Omega S^{m+1} \arrow[r, "H_p"] \arrow[d, "q"'] & \Omega S^{mp} \arrow[d, "q^p"] \\
\Omega S^{m+1} \arrow[r, "H_p"] & \Omega S^{mp}
\end{tikzcd}$$

commutes up to homotopy, where H_p is the p^{th} James Hopf invariant and q is the degree q map.

HINT. Theorem 34.72 implies that the degree q^p map is a loop map; study the adjoint square.

35.1.5. Definition of α. Now all the pieces are in place and we can define our map α. We begin by constructing the diagram

$$\begin{tikzcd}
 & & & \Omega S^{2p+1}\{p\} \arrow[d, "\Omega i"] \\
G_2(\Omega(S^3\langle 3\rangle)) \arrow[r, "p_2"] \arrow[urrr, dotted, "\ell"] & \Omega(S^3\langle 3\rangle) \arrow[r] \arrow[rr, bend right, "H'"'] & \Omega S^3_{(p)} \arrow[r, "H_p"] & \Omega S^{2p+1}_{(p)} \arrow[d, "p"] \\
 & & & \Omega S^3_{(p)}
\end{tikzcd}$$

and asking whether there is a map ℓ making it commute up to homotopy.

Problem 35.15. Show that the map ℓ exists.

HINT. Use Problem 35.14.

We are finally able to define $\alpha : \Omega^2(S^3\langle 3\rangle) \to \Omega^2 S^{2p+1}\{p\}$ to be the composite in the triangle

$$\begin{tikzcd}
\Omega^2(S^3\langle 3\rangle) \arrow[rr, "\alpha"] \arrow[dr, bend right, "s"'] & & \Omega^2 S^{2p+1}\{p\} \\
& \Omega G_2(S^3\langle 3\rangle) \arrow[ur, bend right, "\Omega(\ell)"'] &
\end{tikzcd}$$

where s is the section of Ωp_2 we studied in Section 35.1.3.

Problem 35.16. Show that α is an H-map.

35.2. Spectral Sequence Computations

We'll study the maps α and $\Omega(\gamma)$ by explicitly determining their induced maps on homology. This requires us to undertake several computations with the Leray-Serre spectral sequence.

35.2.1. The Dual of the Bockstein. It will simplify our computations if we can use the dual Bockstein operations, which we denote

$$\beta_* : H_{n+1}(\,?\,;\mathbb{Z}/p) \longrightarrow H_n(\,?\,;\mathbb{Z}/p).$$

If $\beta : K(\mathbb{Z}/p, n) \to K(\mathbb{Z}/p, n+1)$ is the map which induces the more familiar cohomology Bockstein, then β_* is simply the map defined by the square

$$\begin{tikzcd}
\pi_{(n+1)+t}(X \wedge K(\mathbb{Z}/p, t)) \arrow[r, "(\mathrm{id}_X \wedge \beta)_*"] \arrow[d, "\cong"'] & \pi_{n+(t+1)}(X \wedge K(\mathbb{Z}/p, t+1)) \arrow[d, "\cong"] \\
\widetilde{H}_{n+1}(X;\mathbb{Z}/p) \arrow[r, "\beta_*"] & \widetilde{H}_n(X;\mathbb{Z}/p)
\end{tikzcd}$$

where the vertical maps are isomorphisms for t large enough.

Problem 35.17.

(a) Show that β_* is a natural transformation.

(b) Show that if X is of finite type, then the homology operation β_* in $\widetilde{H}_*(X;\mathbb{Z}/p)$ is the vector space dual of the cohomology operation β in $\widetilde{H}^*(X;\mathbb{Z}/p)$.

The operation β_* can be understood in terms of Moore spaces and homology decompositions.

Problem 35.18. Show that if $\widetilde{H}_n(X;\mathbb{Z}) = \mathbb{Z}/p \oplus G$, then there are classes $x \in \widetilde{H}_{n+1}(X;\mathbb{Z}/p)$ and $y \in \widetilde{H}_n(X;\mathbb{Z}/p)$ such that $\beta_*(x) = y$.

35.2.2. The Homology Algebra of $\Omega(S^3\langle 3\rangle)$. Let $u : \Omega S^3 \to \mathbb{C}\mathrm{P}^\infty$ generate $H^2(\Omega S^3; \mathbb{Z})$; then the fiber of u is $\Omega c : \Omega(S^3\langle 3\rangle) \to \Omega S^3$, the loops on the 3-connected cover of S^3. We will determine the homology algebra $H_*(\Omega(S^3\langle 3\rangle); \mathbb{Z}/p)$ by studying the Leray-Serre spectral sequence for the fibration sequence

$$\Omega(S^3\langle 3\rangle) \xrightarrow{\Omega c} \Omega S^3 \xrightarrow{u} \mathbb{C}\mathrm{P}^\infty.$$

Problem 35.19.

(a) Show that u is an H-map.

(b) Determine the homology algebra $H_*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/p)$.

(c) Determine the induced map $u_* : H_*(\Omega S^3; \mathbb{Z}/p) \to H_*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/p)$.

We'll use the information gained in this problem to analyze the Leray-Serre spectral sequence for $u : \Omega S^3 \to \mathbb{C}\mathrm{P}^\infty$.

Problem 35.20.

(a) Determine the bottom edge homomorphism in the spectral sequence.

(b) Show that there is a generator $x_{2p-1} \in H^{2p-1}(\Omega(S^3\langle 3\rangle))$ which is in the image of the transgression d^{2p}. Show that there is another generator $y_{2p} \in H^{2p}(\Omega(S^3\langle 3\rangle))$ such that $\beta_*(y_{2p}) = x_{2p-1}$.

(c) Show that $y_{2p}^n \neq 0$ for all n.

(d) Determine $E_\infty^{*,*}(u; \mathbb{Z}/p)$.

(e) Show that $H_*(\Omega(S^3\langle 3\rangle); \mathbb{Z}/p) \cong (\mathbb{Z}/p)[y_{2p}] \otimes \Lambda(x_{2p-1})$.

We can easily deduce the map induced by the loops on the 3-connected cover $c : S^3\langle 3\rangle \to S^3$ from the work done in Problem 35.20.

Problem 35.21. Show that $(\Omega c)_*$ factors as in the diagram

$$\begin{tikzcd}
H_*(\Omega(S^3\langle 3\rangle)) \arrow[rr, "(\Omega c)_*"] \arrow[dr, bend right, "q"'] & & H_*(\Omega(S^3)) \\
& \mathbb{Z}/p[y_{2p}], \arrow[ur, bend right, "\theta"'] &
\end{tikzcd}$$

where $\theta(y_{2p}) = x_2^p$. Conclude that θ is an isomorphism in dimensions divisible by $2p$.

35.2.3. The Homology Algebra of $\Omega^2(S^3\langle 3\rangle)$. Now we compute the homology algebra $H_*(\Omega^2(S^3\langle 3\rangle); \mathbb{Z}/p)$ by determining the spectral sequence for the path-loop fibration $\Omega^2(S^3\langle 3\rangle) \to \mathcal{P}(\Omega(S^3\langle 3\rangle)) \to \Omega(S^3\langle 3\rangle)$. The job is simplified because the spectral sequence is acyclic and the base algebra has a simple system of generators.

Problem 35.22.

(a) Show $H_*(\Omega(S^3\langle 3\rangle))$ has a p-simple system of transgressive generators.

(b) Map a model spectral sequence into the one in question and deduce

$$H_*(\Omega^2(S^3\langle 3\rangle);\mathbb{Z}/p) \cong \bigotimes_{k=1}^{\infty} \Lambda[a_{2p^k-1}] \otimes \mathbb{Z}/p[\beta_* a_{2p^k-1}],$$

where β_* is the dual of the Bockstein operation.

For the path-loop fibration, the transgression is easily defined: for elements $u \in H_n(X;\mathbb{Z}/p)$ and $v \in H_{n-1}(\Omega X;\mathbb{Z}/p)$, we have $d^n(u) = [v]$ if and only if $(\lambda_X)_*(v) = u$. Thus d^n is a sort of inverse to $(\lambda_X)_*$, defined on a subgroup of $H_*(X)$ and taking its values in a quotient of $H_*(\Omega X)$.

We interpret λ_* as a map $\Omega^* : H_k(\Omega X) \to H_{k+1}(X)$.

Problem 35.23.

(a) Let $f : X \to Y$. Show that $\lambda_Y \circ \Sigma\Omega(f) \simeq f \circ \lambda_X$.

(b) Show that if $f : X \to Y$, then $\Omega^* \circ (\Omega f)_* = f_* \circ \Omega^*$.

(c) Show that $\Omega^*(a_{2p^k-1}) = (x_{2p})^{p^{k-1}}$.

(d) Show that Ω^* is zero on decomposables.

35.2.4. The Homology Algebra $H_*(\Omega S^{2p+1}\{p\})$. Our final computation is based on the fibration sequence $S^{2p+1}\{p\} \xrightarrow{\Omega i} \Omega S^{2n+1} \xrightarrow{p} \Omega S^{2n+1}$.

Problem 35.24.

(a) Show that $(\Omega p)_*$ is the trivial homomorphism of algebras.

(b) Show that

$$H_*(\Omega S^{2p+1}\{p\}) \cong \mathbb{Z}/p[y_{2p}] \otimes E(\beta_*(y_{2p})) \otimes \bigotimes_{k=2}^{\infty} \left(E(z_{2p^k-1}) \otimes \mathbb{Z}/p[\beta_*(z_{2p^k-1})] \right).$$

The edge homomorphism in your spectral sequence determines the map induced by Ωi on homology.

Problem 35.25.

(a) Show that $(\Omega i)_* : H_*(\Omega S^{2p+1}\{p\}) \to H_*(S^{2p+1})$ is an algebra map that factors

$$\begin{tikzcd}
H_*(\Omega S^{2p+1}\{p\}) \arrow[rr, "(\Omega i)_*"] \arrow[dr, bend right, "q"'] & & H_*(S^{2p+1}) \\
& \mathbb{Z}/p[y_{2p}]. \arrow[ur, bend right, "\phi"'] &
\end{tikzcd}$$

(b) Show that ϕ is an isomorphism.

35.3. The Map γ

The definition of the map γ is quite simple. The real work in this section is in determining the map induced by $\Omega\gamma$ in homology.

Problem 35.26. Show that there is a fiber sequence

$$S^3\langle 3\rangle \longrightarrow S^3 \longrightarrow K(\mathbb{Z}/3) \longrightarrow Z \longrightarrow \mathbb{HP}^\infty \longrightarrow K(\mathbb{Z}/4).$$

Since $S^3\langle 3\rangle \simeq \Omega Z$, we see that $S^3\langle 3\rangle$ is an H-space, and we are entitled to refer to Z as $B(S^3\langle 3\rangle)$. Let $\alpha : S^{2p+1} \to B(S^3\langle 3\rangle)$ be a nonzero map in $\pi_{2p+1}(B(S^3\langle 3\rangle)) \cong \pi_{2p}(S^3) \cong \mathbb{Z}/p$ and define γ to be the induced map of fibers in the diagram

$$\begin{tikzcd} S^{2p+1}\{p\} \arrow[r] \arrow[d, dotted, "\gamma"'] & S^{2p+1} \arrow[r, "p"] \arrow[d] & S^{2p+1} \arrow[d, "\alpha"] \\ S^3\langle 3\rangle \arrow[r] & \mathcal{P}(B(S^3\langle 3\rangle)) \arrow[r] & B(S^3\langle 3\rangle). \end{tikzcd}$$

Now we need to determine the map $\Omega\gamma$ induces on homology. This is one of the few times in this book where we will use the naturality of the Leray-Serre spectral sequence.

Problem 35.27.

(a) Show $\alpha : S^{2p+1} \to B(S^3\langle 3\rangle)$ induces an isomorphism on $H_{2p+1}(\,?\,;\mathbb{Z}/p)$.

(b) Show that $(\Omega\gamma)_*(\beta_*(y_{2p})) = u \cdot \beta_*(x_{2p})$ where u is a unit.

Hint. Use naturality in the Leray-Serre spectral sequence.

(c) Show that $(\Omega\gamma)_*$ factors

$$\begin{tikzcd} H_*(\Omega S^{2p+1}\{p\}) \arrow[rr, "(\Omega\gamma)_*"] \arrow[dr, bend right, "q"'] & & H_*(S^3\langle 3\rangle) \\ & \mathbb{Z}/p[y_{2p}] \arrow[ur, bend right, "\psi"'] & \end{tikzcd}$$

where ψ is an isomorphism.

35.4. Proof of Theorem 35.3

Problem 34.25 implies that we only have to check that the composite $\beta \circ \alpha$ induces an isomorphism on $H_*(\,?\,;\mathbb{Z}/p)$. Since both α and $\Omega^2(\gamma)$ are H-maps, the induced map on homology is a homomorphism of Pontrjagin algebras, which gives us a powerful tool for simplifying the problem.

Problem 35.28. Let $\phi : A \to A$ be an automorphism of a graded $\mathbb{Z}/p$-algebra of finite type. Show that the following are equivalent:

(1) ϕ is injective,
(2) ϕ is surjective,
(3) the induced map $Q\phi : QA \to QA$ of indecomposables[2] is surjective,
(4) the induced map $Q\phi : QA \to QA$ of indecomposables is injective.

All of these, of course, are then equivalent to ϕ being an isomorphism.

The rest of the proof of Theorem 35.3 comes down to determining the homology algebra $H_*(\Omega^2(S^3\langle 3\rangle); \mathbb{Z}/p)$ and verifying that $(\Omega^2(\gamma)\circ\alpha)_*$ induces an isomorphism of indecomposables.

35.4.1. The Map Induced by the Hopf Invariant. The Hopf invariant $H : \Omega S^3 \to \Omega S^{2p+1}$ is constructed in such a way that it induces an isomorphism $H_* : H_{2p}(\Omega S^3; \mathbb{Z}) \to H_{2p}(\Omega S^{2p+1})$. Unfortunately, H is not an H-map, so we cannot immediately deduce the higher-dimensional effect of H from this.

The key is to work out the cohomology and then dualize. Recall that $H^*(\Omega S^{2n+1}; \mathbb{Z}_{(p)})$ is the divided polynomial algebra $\Delta_{\mathbb{Z}_{(p)}}(x_{2n})$. Thus the Hopf invariant induces an algebra map $H^* : \Delta_{\mathbb{Z}_{(p)}}(y_{2p}) \to \Delta_{\mathbb{Z}_{(p)}}(x_2)$.

Problem 35.29.

(a) Show that $H^*(y_{2p}) = \frac{1}{p!}x_2^p$.
(b) Use de Polignac's formula to show that $H^*(\frac{1}{n!}y_{2p}^n) = u \cdot \frac{1}{(np)!}x_2^{np}$, where u is a unit in $\mathbb{Z}_{(p)}$.
(c) Conclude that $H_* : H_*(\Omega S^3; \mathbb{Z}/p) \to H_*(\Omega S^{2p+1})$ is surjective (hence an isomorphism in all dimensions divisible by $2p$).

35.4.2. Finishing the Argument. It suffices to show that $H'_* \circ \Omega^*$ is injective on $Q(H_*(\Omega^2(S^3\langle 3\rangle)))$. But even more is true: because of the action of the dual of the Bockstein, we can concentrate our efforts on the odd-dimensional indecomposables.

Problem 35.30. Show that it suffices to prove $(\Omega^2\gamma)_*(\Omega\ell)_*s_*(a_{2p^k-1})$ is indecomposable for each k.

The next step is to show that the study of $(\Omega^2\gamma)_*(\Omega\ell)_*s_*$ can be replaced with the study of an entirely different map. This replacement is based on the following consequence of our computations: the maps $(H' \circ \Omega\gamma)_*$ and $(\Omega i)_*$ agree up to an automorphism of the range.

[2] See Section A.3.1.

Problem 35.31.

(a) Show that there is an isomorphism $\phi : H_*(\Omega S^{2np+1}) \to H_*(\Omega S^{2np+1})$ making the triangle

$$\begin{tikzcd} & H_*(\Omega S^{2np+1}\{p\}) \arrow[dl, "(H'\circ\Omega\gamma)_*"'] \arrow[dr, "\Omega(i)_*"] & \\ H_*(\Omega S^{2np+1}) \arrow[rr, "\phi"] & & H_*(\Omega S^{2np+1}) \end{tikzcd}$$

commute.

(b) Show that $(\phi \circ (H')_* \circ \Omega^*) \circ \big((\Omega^2\gamma)_* \circ (\Omega\ell)_* \circ s_*\big) = H'_* \circ \Omega^*$.

Hint. You will have to use the naturality of λ_* three times.

(c) Show that $(\Omega^2\gamma)_* \circ (\Omega\ell)_* \circ s_*(a_{2p^k-1})$ is indecomposable for each k, and derive Theorem 35.3.

35.5. Nearly Trivial Maps

If $\Sigma f : \Sigma X \to \Sigma Y$ is nontrivial, then $\Omega\Sigma f$ is also nontrivial, because Σf is a retract of $\Sigma\Omega\Sigma f$. This suggests that Σf must induce a nontrivial map on π_*. Using the exponent theorem of Cohen, Moore and Neisendorfer, we shatter that reasonable expectation.

Theorem 35.32. *Let X be a finite complex with $\pi_{2n+1}(X)$ finite, and suppose $g : X \to S^{2n+1}$ such that $\Sigma^{2k+2}g$ has finite order divisible by p^{n+k+1}. Then there is an s large enough that for each $m \le k$, the composite*

$$\Sigma^{2m}X \xrightarrow{p^{n+k}\Sigma^{2m}g} S^{2(n+m)+1} \xrightarrow{i} M(\mathbb{Z}/p^s, 2(n+m)+1)$$

is nontrivial and induces the zero map on homotopy groups.

Before proceeding to the proof of Theorem 35.32, we show that there actually are spaces and maps satisfying its hypotheses.

Corollary 35.33. *For each $k, n \ge 1$ there is a map*

$$f : M(\mathbb{Z}/p^{n+k+1}, 2n) \longrightarrow M(\mathbb{Z}/p^s, 2n+1)$$

whose $2k$-fold suspension is nontrivial and induces the zero map on homotopy groups.

Problem 35.34. Use Theorem 35.32 to prove Corollary 35.33.

As mentioned above, the proof depends heavily on Theorem 35.1. We will also make essential use of the fact that an odd-dimensional sphere becomes an H-space when localized at an odd prime.

Problem 35.35. Let $d : S^{2n+1} \to S^{2n+1}$ be a map of degree d, and let p be an odd prime.

(a) Show that the induced map $d : S^{2n+1}_{(p)} \to S^{2n+1}_{(p)}$ of p-localizations is homotopic to d times the identity.

(b) Show that the induced map $d_* : \pi_*(S^{2n+1}_{(p)}) \to \pi_*(S^{2n+1}_{(p)})$ is multiplication by d.

The basic idea underlying the construction is quite simple.

Problem 35.36. Let X be any space with $\pi_{2n+1}(X)$ a finite group, and let $g : X \to S^{2n+1}$ be any map. Show that the map f defined by the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & M(\mathbb{Z}/p^s, 2n+1) \\ S^{2n+1} \arrow[r, "p^{n+k+1}"] & S^{2n+1} \arrow[u, "i"'] \end{tikzcd}$$

has the property that $\Sigma^{2m} f$ induces the zero map on homotopy groups for $m \leq k+1$.

The tricky part is in choosing the map g and the integer s so that $\Sigma^{2i} f$ is nontrivial. For this, we consider the diagram

$$\begin{tikzcd} & S^{2n+1} \arrow[r, "\eta"] \arrow[d, "p^s"'] & S^{2n+1}_{(p)} \arrow[d, "p^s"] \\ X \arrow[r, "h"] \arrow[dr, bend right, "i\circ h"'] & S^{2n+1} \arrow[r, "\eta"] \arrow[d, "i"'] & S^{2n+1}_{(p)} \arrow[d, "j"] \\ & M(\mathbb{Z}/p^s, 2n+1) \arrow[r, equal] & M(\mathbb{Z}/p^s, 2n+1). \end{tikzcd}$$

Problem 35.37. Assume that $\Sigma^{2k}(\eta \circ h) \not\simeq *$.

(a) Show that if $i \circ h \simeq *$, then there is a lift in the diagram

$$\begin{tikzcd} & S^{2(n+k)+1}_{(p)} \arrow[d, "p^s"] \\ \Sigma^{2k} X \arrow[r, "\eta\circ\Sigma^{2k} h"] \arrow[ur, dotted] & S^{2(n+k)+1}_{(p)}. \end{tikzcd}$$

(b) Show that if X is a finite complex, then the group $[X, S^{2(n+k)+1}_{(p)}]$ has an exponent p^e and that if $s \geq e$, then the lift cannot exist.

(c) Prove Theorem 35.32.

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Chapter 36

Classes of Spaces

Using the fundamental 'orthogonality relation' $\mathrm{map}_*(X, Y) \sim *$, we define a Galois connection involving classes of pointed spaces. The 'closed' classes in this connection are, on the domain side, called **strongly closed classes** and, on the target side, **strong resolving classes**. Working with these classes rather than with individual spaces can greatly simplify arguments involving the homotopy theory of mapping spaces.

After defining the basic notions and getting a feel for the formal theory of these classes, we turn to a detailed study of strong resolving classes. Since they are firmly on the target side of things, we focus on the extent to which such classes are closed under domain-type operations such as wedges, cofibers and the like.

Strongly closed classes have a weaker cousin: a **closed class** is a class of pointed spaces that is closed under weak homotopy equivalence and pointed homotopy colimits. The smallest closed class containing the space A is denoted $\mathcal{C}(A)$; it is convenient and suggestive to express the relation $X \in \mathcal{C}(A)$ in the form of a **cellular inequality** $A \ll X$. We sketch a proof of the fundamental theorem of E. Dror Farjoun which establishes a cellular inequality involving the homotopy fiber of an induced map of homotopy pushouts. This is used in the final section to develop a calculus of closed classes, resulting in cellular inequalities shedding light on the basic theorems of homotopy theory, such as the Blakers-Massey theorem.

In this chapter we consistently use the Serre model structure.

36.1. A Galois Correspondence in Homotopy Theory

Let us consider the collection of classes $\mathcal{X}$ or $\mathcal{Y}$ of pointed topological spaces and denote them by $\mathfrak{X}$ and $\mathfrak{Y}$. We define functions

$$\Phi : \mathfrak{X} \longrightarrow \mathfrak{Y} \qquad \text{and} \qquad \Theta : \mathfrak{Y} \longrightarrow \mathfrak{X}$$

by the rules

$$\Phi(\mathcal{X}) = \{Y \mid \operatorname{map}_*(X, Y) \sim * \text{ for all } X \in \mathcal{X}\}$$

and

$$\Phi(\mathcal{Y}) = \{X \mid \operatorname{map}_*(X, Y) \sim * \text{ for all } Y \in \mathcal{Y}\}.$$

We should justify the idea that this is a Galois correspondence.

Problem 36.1.

(a) Show that $\mathcal{X} \subseteq \Theta(\Phi(\mathcal{X}))$ and $\mathcal{Y} \subseteq \Phi(\Theta(\mathcal{Y}))$.

(b) Show that $\Phi(\mathcal{X}) = \Phi(\Theta(\Phi(\mathcal{X})))$ and $\Theta(\mathcal{Y}) = \Theta(\Phi(\Theta(\mathcal{Y})))$.

(c) Conclude that Θ and Φ establish a bijection between the image of Φ and the image of Θ.

The subcollection $\overline{\mathfrak{C}} = \operatorname{Im}(\Theta) \subseteq \mathfrak{X}$ is called the collection of **strongly closed classes** of spaces, and the subcollection $\overline{\mathfrak{R}} = \operatorname{Im}(\Phi) \subseteq \mathfrak{Y}$ is called the collection of **strong resolving classes** of spaces. We will generally use $\overline{\mathcal{C}}$ to denote a strong closed class and $\overline{\mathcal{R}}$ to denote a strong resolving class.

Problem 36.2. Suppose $\overline{\mathcal{C}} \in \overline{\mathfrak{C}}$ and $\overline{\mathcal{R}} \in \overline{\mathfrak{R}}$ correspond under the bijection given by Θ and Φ.

(a) Show that $\overline{\mathcal{C}}$ and $\overline{\mathcal{R}}$ are closed under weak homotopy equivalences.

(b) Show that $\overline{\mathcal{C}}$ is closed under homotopy colimits: if $F : \mathcal{I} \to \mathcal{T}_*$ is a diagram with $F(i) \in \overline{\mathcal{C}}$ for each $i \in \mathcal{I}$, then $\operatorname{hocolim}_* F \in \overline{\mathcal{C}}$.

(c) Show that $\overline{\mathcal{R}}$ is closed under homotopy limits.

(d) Show that $\overline{\mathcal{C}}$ is closed under extensions by cofibrations: if $A \to X \to B$ is a cofiber sequence and $A, B \in \overline{\mathcal{C}}$, then $X \in \overline{\mathcal{C}}$ also.

(e) Show that $\overline{\mathcal{R}}$ is closed under extensions by fibrations.

(f) Show that $\mathcal{C} \cap \mathcal{R} = \{\text{weakly contractible spaces}\}$.

Each collection of spaces generates a strongly closed class and a strong resolving class, given explicitly by

$$\overline{\mathcal{C}}(\mathcal{X}) = \Theta \circ \Phi(\mathcal{X}) \qquad \text{and} \qquad \overline{\mathcal{R}}(\mathcal{Y}) = \Phi \circ \Theta(\mathcal{Y}).$$

Of particular importance are the cases $\overline{\mathcal{C}}(X) = \overline{\mathcal{C}}(\{X\})$ and $\overline{\mathcal{R}}(Y) = \overline{\mathcal{R}}(\{Y\})$.

Problem 36.3.

(a) Show that $\{X \mid X \text{ is } n\text{-connected}\}$ is a strongly closed class.

(b) Show that if X is n-connected, then so is every space in $\overline{\mathcal{C}}(X)$.

(c) What are the corresponding statements for strong resolving classes?

Problem 36.4. Determine $\overline{\mathcal{C}}(S^n)$ and $\overline{\mathcal{R}}(K(\mathbb{Z}, n))$.

HINT. The first is easy; the second is tricky.

Problem 36.5. Let $\overline{\mathcal{C}} \in \overline{\mathfrak{C}}$ and $\overline{\mathcal{R}} \in \overline{\mathfrak{R}}$.

(a) Suppose A is a retract of X, so that we have maps $i : A \to X$ and $r : X \to A$ whose composition is the identity id_A. Determine the categorical limit and colimit of the doubly infinite sequence

$$\cdots \xrightarrow{i} X \xrightarrow{r} A \xrightarrow{i} X \xrightarrow{r} A \xrightarrow{i} \cdots .$$

(b) Now suppose A is only a homotopy retract of X, so that $r \circ i \simeq \mathrm{id}_A$. Show that if $X \in \overline{\mathcal{C}}$, then $A \in \overline{\mathcal{C}}$ and similarly if $X \in \overline{\mathcal{R}}$, then $A \in \overline{\mathcal{R}}$.

(c) Show that if $\overline{\mathcal{C}} \neq \varnothing$, then $* \in \mathcal{C}$, and similarly for $\overline{\mathcal{R}}$.

(d) Show that $\overline{\mathcal{C}}$ is closed under the formation of cofibers and that $\overline{\mathcal{R}}$ is closed under the formation of homotopy fibers.

Here are some simple examples.

Problem 36.6.

(a) Show that the class of all f-local spaces is a resolving class.

(b) Show that the class of X-null spaces is a strong resolving class.

36.2. Strong Resolving Classes

We say that a space Z is **$\mathcal{Y}$-resolvable** if $X \in \overline{\mathcal{R}}(\mathcal{Y})$. If $\mathcal{Y} = \{Y\}$, then we call a $\mathcal{Y}$-resolvable space Y-resolvable. If $Z \in \overline{\mathcal{R}}(\{S^n \mid n \geq 1\})$, then we say that Z is **spherically resolvable**.

In this section we give criteria for deciding if a space is $\mathcal{Y}$-resolvable, with a strong interest in the special case of spherical resolvability. This boils down to several statements about closure of strong resolving classes under various domain-type operations, such as the formation of wedges and cofibers. It is too much to ask that these be true generally throughout a resolving class. Our results tend to have the form 'if an array of spaces related to X is contained in $\overline{\mathcal{R}}$, then X is also in $\overline{\mathcal{R}}$'.

Problem 36.7.

(a) Is there a space Y such that every CW complex is Y-resolvable?

(b) Show that the Lie groups $SO(n), U(n)$, and $Sp(n)$ are spherically resolvable.

36.2.1. Manipulating Classes of Spaces. We begin with some notation and terminology. For a collection $\mathcal{A}$ of spaces (which we assume is closed under weak equivalence), we write

$$\Sigma\mathcal{A} = \{\Sigma A \mid A \in \mathcal{A}\} \qquad \text{and} \qquad \Omega\mathcal{A} = \{\Omega A \mid A \in \mathcal{A}\}.$$

We say that $\mathcal{A}$ is **closed under suspension** if $\Sigma\mathcal{A} \subseteq \mathcal{A}$ and that $\mathcal{A}$ is **closed under wedges** if $A \vee B \in \mathcal{A}$ whenever $A, B \in \mathcal{A}$; likewise $\mathcal{A}$ is **closed under smash products** if $A \wedge B \in \mathcal{A}$ whenever $A, B \in \mathcal{A}$.

We also need to work with certain kinds of infinite wedges. Fix a collection $\mathcal{A}$ of spaces. A space W is a **finite-type wedge** of spaces in $\mathcal{A}$ if it is possible to express W as a wedge $W = \bigvee_{i\in\mathcal{I}} A_i$ in which each $A_i \in \mathcal{A}$, and for each n, $\operatorname{conn}(A_i) \leq n$ for only finitely many $i \in \mathcal{I}$. For any collection $\mathcal{A}$, we write $\mathcal{A}^\vee$ for the collection of all finite-type wedges of spaces in $\mathcal{A}$.

Proposition 36.8. *If $\mathcal{A}$ is closed under smash products and suspension, then*

$$\Sigma\Omega\left(\Sigma^2\mathcal{A}^\vee\right) \subseteq \Sigma^2\mathcal{A}^\vee.$$

Problem 36.9. Let $A = \bigvee_{i\in\mathcal{I}} \Sigma^2 A_i \in \Sigma^2\mathcal{A}^\vee$.

(a) Show that ΩA is homotopy equivalent to a product of loop spaces on spaces in $\Sigma^2\mathcal{A}$.

(b) Show that $\Sigma\Omega A$ is the suspension of a wedge of smash products of loop spaces on spaces in $\Sigma^2\mathcal{A}$.

(c) Use the James splitting to complete the proof of Proposition 36.8.

36.2.2. Closure under Finite-Type Wedges. It seems very unlikely that classes of spaces defined in such a thoroughly target-type fashion as resolving classes could possibly be closed under the formation of wedges. But some resolving classes contain smaller collections all of whose wedges lie in the larger resolving class.

Proposition 36.10. *Let $\overline{\mathcal{R}}$ be a strong resolving class. If $\mathcal{A} \subseteq \overline{\mathcal{R}}$ is closed under suspension and smash, then $\mathcal{A}^\vee \subseteq \overline{\mathcal{R}}$.*

The finite-type hypothesis on the wedges is imposed to facilitate an inductive argument. Connectivity imposes a partial order on spaces: $X < Y$ if $\operatorname{conn}(X) > \operatorname{conn}(Y)$. For our proof of Proposition 36.10 we will use a refinement of this partial order. Suppose W is a finite-type wedge of spaces in $\mathcal{A}$ and that W is $(n-1)$-connected but not n-connected. Then we set $r(W)$ to be the least number of $\mathcal{A}$-summands that are $(n-1)$-connected but not n-connected in any expression of W as a wedge of spaces in $\mathcal{A}$. Now if W_1 and W_2 are both finite-type wedges of spaces in $\mathcal{A}$, we say that $W_1 < W_2$ if and only if

(1) W_1 is more highly connected than W_2 or

(2) W_1 and W_2 have the same connectivity, and $r(W_1) < r(W_2)$.

Problem 36.11. Let W be a finite-type wedge of spaces in $\Sigma\mathcal{A}$, and write

$$W = A_1 \vee A_2 \vee \cdots \vee A_k \vee \widetilde{W}$$

where W and A_i have the same connectivity for each i and $\widetilde{W}$ is a wedge of spaces in $\Sigma\mathcal{A}$ of strictly larger connectivity. Let F be the fiber of the collapse map $q : W \to A_k$.

(a) Determine the homotopy type of F.

(b) Show that $F \in \Sigma\mathcal{A}^\vee$ and $W < F$.

Now we prove that $W \in \overline{\mathcal{R}}$. Set $V_0 = W$; given V_n, define V_{n+1} to be the fiber of the map guaranteed by Problem 36.11(a), and so build a tower

$$V_0 \longleftarrow V_1 \leftarrow \cdots \leftarrow V_n \longleftarrow V_{n+1} \leftarrow \cdots .$$

Problem 36.12.

(a) Show that the homotopy limit of the tower is weakly contractible.

(b) Let $\overline{\mathcal{R}} = \Theta(\mathcal{X})$, and let $X \in \mathcal{X}$. Show that

$$\mathrm{map}_*(X, V_{n+1}) \longrightarrow \mathrm{map}_*(X, V_n)$$

is a weak homotopy equivalence.

(c) Show that $W \in \overline{\mathcal{R}}$ and so complete the proof of Proposition 36.10.

Corollary 36.13. *If $S^n \in \overline{\mathcal{R}}$ for all $n \geq 1$, then every finite type wedge of spheres is also in $\overline{\mathcal{R}}$.*

36.2.3. Desuspension in Resolving Classes. You showed in Section 18.5.2 that a simply-connected space can be recovered as a homotopy limit involving the wedges of its suspensions. We even went so far as to express this result in Theorem 18.37 in the language of resolving classes. Since every strong resolving class is a resolving class, we have the following corollary.

Corollary 36.14. *Let X be simply-connected. If $\bigvee_{i=1}^k \Sigma^n X \in \overline{\mathcal{R}}$ for some n and all $k \in \mathbb{N}$, then $X \in \overline{\mathcal{R}}$.*

Closure under Formation of Cofibers. It is hard to imagine that a resolving class could be closed under the formation of cofibers. But we can get a result of this kind if we restrict our attention to smaller subcollections within the resolving class.

Theorem 36.15. *Let $\overline{\mathcal{R}}$ be a strong resolving class. Suppose $\mathcal{A} \subseteq \overline{\mathcal{R}}$ is closed under suspension and smash product. Then every space X with finite $\mathcal{A}^\vee$-cone length is in $\overline{\mathcal{R}}$.*

We begin by establishing some basic formulas involving cone length.

Problem 36.16.

(a) Show that for any collection $\mathcal{A}$, $\mathrm{cl}_{\Sigma\mathcal{A}}(X) \leq \mathrm{cl}_{\mathcal{A}}(X)$.

(b) Show that $\mathrm{cl}_{\mathcal{A}}(A \wedge X) \leq \mathrm{cl}_{\mathcal{A}}(X)$ if $\mathcal{A}$ is closed under smash products and $A \in \mathcal{A}$.

(c) Show that $\mathrm{cl}_{\mathcal{A}}(X \vee Y) \leq \max\{\mathrm{cl}_{\mathcal{A}}(X), \mathrm{cl}_{\mathcal{A}}(Y)\}$ if $\mathcal{A}$ is closed under wedges.

(d) Show that it suffices to prove Theorem 36.15 for the collection $\Sigma^2\mathcal{A}^\vee$.

You'll prove this by induction on $\mathrm{cl}_{\mathcal{A}^\vee}$ using the following lemma.

Lemma 36.17. *Let $\mathcal{A}$ be a collection of spaces that is closed under smash products and suspension. Let $L \to V \to W$ be a cofiber sequence with $W \in \Sigma^2\mathcal{A}^\vee$, and let F be the fiber of $V \to W$. Then $\mathrm{cl}_{\Sigma^2\mathcal{A}^\vee}(\Sigma F) \leq \mathrm{cl}_{\mathcal{A}^\vee}(L)$.*

Problem 36.18. Prove Lemma 36.17 and Theorem 36.15.

Hint. Use Theorem 18.18.

36.2.4. Spherical Resolvability of Finite Complexes. We conclude this section with a surprising theorem showing that finite complexes, which are constructed from spheres by the fundamentally domain-type operations of wedge and cofiber, can also be built from spheres in some mysterious target-type fashion.

Corollary 36.19. *Every simply-connected finite complex is spherically resolvable.*

Problem 36.20. Prove Corollary 36.19.

36.3. Closed Classes and Fibrations

Now we turn our attention to the domain side. Just as strong resolving classes had their weaker cousin, i.e., the resolving class, the notion of strongly closed classes can be weakened to closed classes. A class $\mathcal{C}$ of pointed spaces is a **closed class** if it is closed under weak homotopy equivalence and the formation of homotopy colimits.

In this section we study how the domain-type closed classes behave with respect to (target-type) fibration sequences. These results are expressed as cellular inequalities.

36.3.1. Cellular Inequalities. For any collection $\mathcal{X}$ of spaces, we write $\mathcal{C}(\mathcal{X})$ for the smallest closed class containing $\mathcal{X}$ and say that Z is an **$\mathcal{X}$-cellular** space if $Z \in \mathcal{C}(\mathcal{X})$. When $\mathcal{X} = \{X\}$, we write $\mathcal{C}(X)$ for $\mathcal{C}(\mathcal{X})$; and we write $X \ll Z$ if $Z \in \mathcal{C}(X)$. This is known as a **cellular inequality**.

Problem 36.21.

(a) Show that if X is not path-connected, then every space is X-cellular.

(b) Show that if $X \ll Y$, then $\mathrm{conn}(X) \leq \mathrm{conn}(Y)$.

(c) Let $\widetilde{h}_*$ be a homology theory satisfying the Wedge Axiom. Show that if $\widetilde{h}^*(X) = 0$ and $X \ll Y$, then $\widetilde{h}_*(Y) = 0$.

Next we establish some simple closure properties of cellular inequalities.

Problem 36.22.

(a) If $A \ll B$ and $X \ll Y$, then

$$A \vee X \ll B \vee Y \qquad \text{and} \qquad A \wedge X \ll B \wedge Y.$$

(b) Show that $X \vee Y \ll X \times Y$ for any X and Y.

Hint. Replace X and Y with CW complexes, and show by induction that $X \vee Y \ll (X \vee Y) \cup (X \times Y_n)$.

Strongly closed classes give rise to a related notion of cellular inequality: $X < Y$ if and only if $X \in \overline{\mathcal{C}}(Y)$. Except for Problem 36.24(b), we will not use these inequalities in this book.

Cellular Inequalities for Unpointed Spaces. On the face of it, the expression $X \ll Y$ is meaningless if X and Y are not both pointed spaces, but we actually can make sense of such an expression when the spaces involved are CW complexes. If X and Y are unpointed CW complexes, we define $X \ll Y$ if and only if the cellular inequality $(X, x) \ll (Y, y)$ of pointed spaces holds for some choice of basepoints $x \in X$ and $y \in Y$.

Problem 36.23.

(a) Let X be an unpointed CW complex. Show that if x is any point in X, then the inclusion $x \hookrightarrow X$ is an unpointed cofibration.

Hint. Let $x \in \mathrm{int}(D^n) \subseteq X$, and show that $\{x\} \hookrightarrow X_n$ is a cofibration.

(b) Show that if X is connected, then the pointed homotopy type of X is independent of the choice of basepoint.

(c) Show that $X \ll Y$ is well-defined and does not depend on the choice of basepoints x and y, even if X is not connected.

36.3.2. Closed Classes and Fibration Sequences. Theorem 18.47, the Zabrodsky lemma, has an interpretation in terms of cellular inequalities.

Problem 36.24. Let $F \to E \to B$ be a fibration sequence.

(a) Show that $F \vee E \ll B$.

(b) Show that $F \vee B < E$.

Problem 36.24(a) implies a useful relation between the homotopy fiber of a composition and those of the factors.

Problem 36.25. Show that if $f : X \to Y$ and $g : Y \to Z$, then

$$F_f \vee \Omega F_g \ll F_{g\circ f}.$$

36.3.3. E. Dror Farjoun's Theorem. The deeper theory of closed classes rests on a fundamental observation of E. Dror Farjoun, which gives information about the homotopy fiber of an induced map of homotopy colimits. The theorem as stated in [**63**] applies to homotopy colimits over very general shape categories $\mathcal{I}$, but we will only need the result for prepushout diagrams.

Theorem 36.26 (Farjoun)**.** *Let $\delta : D \to Z$ be a map of homotopy pushouts induced by the diagram morphism*

$$\begin{tikzcd} C \arrow[d, "\gamma"'] & A \arrow[l] \arrow[r] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ Y & W \arrow[l] \arrow[r] & X. \end{tikzcd}$$

Then the homotopy fibers are related by the cellular inequality

$$F_\alpha \vee F_\beta \vee F_\gamma \ll F_\delta.$$

To set up for the proof of this theorem, we first replace the spaces involved with weakly equivalent simplicial complexes and replace the maps α, β and γ with weakly equivalent fibrations. Thus the preimages by α, β and γ of the simplices in the bottom row of the diagram are homotopy equivalent to the homotopy fibers.

Thus Theorem 36.26 will follow from the following more general theorem.

Theorem 36.27. *Let $f : K \to L$ be a surjective simplicial map of simplicial complexes, and let F be its homotopy fiber. Then*

$$\bigvee_{\sigma\subseteq L} f^{-1}(\sigma) \ll F.$$

The standard construction of the homotopy fiber of f is to convert f to a fibration $\bar{f}$ and then form the pullback of $\bar{f}$ over $* \hookrightarrow L$. The first great idea in the proof of Theorem 36.26 is to instead leave f alone and construct F as the pullback

$$\begin{tikzcd} F \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\text{pullback}"] & K \arrow[d, "f"] \\ P \arrow[r, "p"'] & L \end{tikzcd}$$

over the simplicial replacement $p : P \to L$ for the fibration $@_0 : \mathcal{P}(L) \to L$.

Problem 36.28. Show that if $\sigma \subseteq P$ is a simplex of P, then $q^{-1}(\sigma) \simeq f^{-1}(\tau)$ for some simplex $\tau \subseteq L$.

This allows us to construct the homotopy fiber F as a homotopy colimit. Recall from Project 15.90 the simplex category $\mathcal{S}_Y$ whose objects are the simplices of Y and whose morphisms are the inclusions of simplices. Also recall from Theorem 15.91 that a simplicial map $f : X \to Y$ gives rise to a functor $\Theta_f : \mathcal{S}_Y \to \mathcal{T}_\circ$ given by $\Theta_f : \sigma \mapsto f^{-1}(\sigma)$ and that $X = \mathrm{hocolim}_\circ\, \Theta_f$.

Thus F is an unpointed homotopy colimit of a diagram involving only the spaces $f^{-1}(\sigma)$. This is a good first step, but we need to express F as a *pointed* homotopy colimit of the spaces $f^{-1}(\sigma)$. This is where we need to use the second great idea in the proof: the following theorem of Amit (unpublished master's thesis).

Theorem 36.29. *If $\Phi : \mathcal{I} \to \mathcal{T}_\circ$ is an unpointed diagram of simplicial complexes. If the classifying space $B\mathcal{I}$ is contractible, then*

$$\bigvee_{i\in\mathcal{I}} \Phi(i) \ll \mathrm{hocolim}_\circ\, \Phi.$$

(Note that this cellular inequality makes sense because of Problem 36.23.)

We will not work through a proof of Amit's theorem. A proof can be found in [**63**].

Problem 36.30. Use Theorem 36.29 to prove Theorems 36.27 and 36.26.

Exercise 36.31. Why must f be surjective in Theorem 36.27?

36.4. The Calculus of Closed Classes

Theorems 36.26 is extremely powerful, and we use it now to derive a great many interesting and illuminating results about closed classes. Most of these results were given in a series of paper by W. Chachólski [**39**–**41**].

36.4.1. Fibers and Cofibers. The induced map of cofibers in a homotopy pushout square is homotopy equivalence. This implies that the fibers have the same connectivity; and even more is true by virtue of the Blakers-Massey theorem. But there is a very straightforward cellular relation between these fibers.

Proposition 36.32. *In the homotopy pushout square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\mathrm{HPO}"] & B \arrow[d] \\ C \arrow[r, "g"] & D, \end{tikzcd}$$

the homotopy fibers of f and g satisfy the cellular inequality $F_f \ll F_g$.

Problem 36.33. Prove Proposition 36.32.

Hint. Express g as an induced map of homotopy pushouts.

This proposition packages the result of Theorem 36.26 in a form that is often more convenient to use than the original statement.

Problem 36.34. Suppose $A \xrightarrow{i} X \xrightarrow{q} B$ is a cofiber sequence. Show that

$$F_i \ll \Omega B \qquad \text{and} \qquad A \ll F_q.$$

What do these results tell you in the special case of the cofiber sequence $A \to * \to \Sigma A$?

Proposition 36.32 can be generalized to relate the fibers of parallel arrows in an commutative square.

Problem 36.35. Consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & B \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r, "u"] \arrow[drr, bend right, "g"'] & P \arrow[dr, dotted, "\xi"] & \\
& & D.
\end{tikzcd}$$

Show that $F_f \vee \Omega F_\xi \ll F_g$.

36.4.2. Loops and Suspensions. Next we study the interplay of cellular inequalities and the basic operations Σ and Ω. A strong result for suspension should be expected, and the equally strong result for loop spaces should be shocking.

Proposition 36.36. *If $A \ll B$, then*

(a) $\Sigma A \ll \Sigma B$ and

(b) if A is path-connected, then $\Omega A \ll \Omega B$.

Problem 36.37. Let $\mathcal{C}$ be a closed class.

(a) Show that $\Sigma^{-1}\mathcal{C} = \{X \mid \Sigma X \in \mathcal{C}\}$ is also a closed class.

(b) Show that $\Omega^{-1}\mathcal{C} = \{X \mid X \text{ is simply-connected and } \Omega X \in \mathcal{C}\}$ is a closed class.

(c) Prove Proposition 36.36.

Exercise 36.38. Give an example to show that Proposition 36.36(b) is false without the path-connected hypothesis.

36.4.3. Adjunctions. The next important element in the calculus of cellular inequalities is a rule that allows us to interchange loops on the right with suspensions on the left.

Proposition 36.39. *For B path-connected, $\Sigma A \ll B$ if and only if $A \ll \Omega B$.*

Problem 36.40.

(a) Show that if X is path-connected, then $\Sigma\Omega X \ll X$.

Hint. Show that $\Sigma\Omega X \ll \Omega X * \Omega X$.

(b) Show that $X \ll \Omega\Sigma X$ for any space X.

(c) Prove Proposition 36.39.

Proposition 36.39 implies some equalities among closed classes.

Problem 36.41.

(a) Show that if X is path-connected, then $\mathcal{C}(\Omega\Sigma\Omega X) = \mathcal{C}(\Omega X)$.

(b) Show that $\mathcal{C}(\Sigma\Omega\Sigma X) = \mathcal{C}(\Sigma X)$.

In Section 20.6.2 we studied the relation between the fiber of the suspension of a map f and the suspension of its fiber F_f. We are now equipped to extend that result to a cellular inequality.

Problem 36.42. Let $f : X \to Y$.

(a) Show that $\Sigma F_f \ll C_f$.

(b) Show that $\Sigma F_f \ll F_{\Sigma f}$.

Project 36.43. Investigate cellular relations between the fibers of a pair of adjoint maps.

Finally, suppose you know $\Sigma X \ll \Sigma Y$ or $\Omega X \ll \Omega Y$. It is sometimes possible to conclude from such inequalities that $X \ll Y$.

Problem 36.44. Show that the following are equivalent:

(1) $\Sigma A \ll \Sigma B$ if and only if $A \ll B$ for every space A,

(2) $\mathcal{C}(B) = \mathcal{C}(\Omega Z)$ for some path-connected space Z.

Also state and prove the dual result.

36.4.4. A Cellular Blakers-Massey Theorem. In this section we revisit the proof of the Blakers-Massey theorem that we gave in Section 18.3, with an eye to finding a cellular generalization.

Theorem 36.45. *Suppose the outer square in the diagram*

$$\begin{tikzcd}
A \arrow[drr, bend left, "j"] \arrow[ddr, bend right, "i"'] \arrow[dr, dotted, "\zeta"] & & \\
& Q \arrow[r] \arrow[d] \arrow[dr, phantom, "HPB"] & B \arrow[d, "\beta"] \\
& C \arrow[r, "\gamma"] & D
\end{tikzcd}$$

is a strong homotopy pushout square, and let $\zeta : A \to Q$ be the comparison map from A to the homotopy pullback Q. Then $F_\beta \wedge F_\gamma \ll F_{\Sigma\zeta}$.

Before embarking on the proof, you should verify that we have actually generalized the Blakers-Massey theorem.

Problem 36.46. Show that Theorem 36.45 implies Theorem 18.13.

In Section 18.3 we were interested in the connectivity of the map ζ. We established a strong homotopy pullback square

$$\begin{tikzcd}
Z \arrow[r, "\theta"] \arrow[d] \arrow[dr, phantom, "HPB"] & P \arrow[d] \\
A \arrow[r, "\zeta"] & Q
\end{tikzcd}$$

and deduced that we could just as well determine the connectivity of θ. Now we are interested in finding a *cellular* estimate for the fiber of $\Sigma\zeta$, so we need to investigate what kind of cellular relations, if any, exist between the fibers $F_{\Sigma\zeta}$ and $F_{\Sigma\theta}$.

Problem 36.47. Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & B \arrow[d, "v"] \arrow[ddr, bend left] & \\
C \arrow[r, "u"] \arrow[drr, bend right, "g"'] & P \arrow[dr, dotted, "\xi"] & \\
& & D
\end{tikzcd}$$

in which the outer square is a homotopy pullback and the inner square is a homotopy pushout.

(a) Show that $F_{\Sigma f} \vee \Sigma(F_u * F_v) \ll F_{\Sigma\xi}$.

(b) Show that it suffices to prove Theorem 36.45 in the case $D = *$.

Now we attack the simpler special case.

Problem 36.48. Consider the diagram

$$\begin{tikzcd}
A \arrow[drr, bend left] \arrow[dr, dotted, "\theta"] \arrow[ddr, bend right] & & \\
& B \times C \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HPB}"] & B \arrow[d] \\
& C \arrow[r, "u"] & *
\end{tikzcd}$$

in which the outer rectangle is a homotopy pushout square.

(a) Show that there is a ladder of Mayer-Vietoris cofiber sequences

$$\begin{tikzcd}
B \vee C \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & \Sigma A \arrow[d, "\Sigma\theta"] \\
B \vee C \arrow[r] & B * C \arrow[r] & \Sigma(B \times C).
\end{tikzcd}$$

(b) Show that $B \wedge C \ll F_{\Sigma\theta}$.

(c) Prove Theorem 36.45.

Many of the other core theorems of homotopy theory have cellular versions.

Problem 36.49. Derive a cellular version of the Freudenthal Suspension Theorem from Theorem 36.45.

Project 36.50. Show that the class of simply-connected $\mathcal{P}$-local spaces is a closed class, and derive the main results of $\mathcal{P}$-local homotopy theory.

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Chapter 37

Miller's Theorem

In this chapter we will prove an extension of a very important theorem of Haynes Miller. To state it, we write $\mathcal{S}$ for the collection of all wedges of spheres.

Theorem 37.1 (Miller)**.** *If K is a CW complex with cone length $\mathrm{cl}_{\mathcal{S}}(K) < \infty$ (e.g., a finite-dimensional CW complex), then $\mathrm{map}_*(B\mathbb{Z}/p, K) \sim *$.*

Since its proof in the early 1980s, this theorem has inspired a tremendous amount of research and played a fundamental role in the proofs of a great many important or interesting (or both) theorems. Miller's proof of Theorem 37.1 is based on a spectral sequence due to Bousfield and Kan [**30**], which begins with some nonabelian homological algebra involving the cohomology algebras of X and Y and provides information about $\mathrm{map}_*(X, Y)$. According to [**127**], the proof amounts to 'constructing a long chain of spectral sequences and then showing that, miraculously, the initial term of the initial spectral sequence is trivial.'

In keeping with the point of view of this book, our proof is much more focused on spaces than on algebra, though some algebra is of course necessary. We begin by reducing the proof of Theorem 37.1 to showing that the space of maps from $B\mathbb{Z}/p$ to an odd-dimensional sphere is trivial.

Theorem 37.2. *Let X be a CW complex with finite homotopy groups. Then the following are equivalent:*

(1) $\mathrm{map}_*(X, S^{2n+1}) \sim *$ *for all sufficiently large n,*

(2) $\mathrm{map}_*(X, K) \sim *$ *for all simply-connected CW complexes K with* $\mathrm{cl}_{\mathcal{S}}(K) < \infty$.

The plan of the proof of Theorem 37.2 is actually quite sensible. We study the strong resolving class

$$\overline{\mathcal{R}} = \{K \mid \mathrm{map}_*(X, K) \sim *\},$$

hoping to show that if $\overline{\mathcal{R}}$ contains all highly-connected odd-dimensional spheres, then it must contain all simply-connected CW complexes with finite $\mathcal{S}$-cone length. Using the Hilton-Milnor theorem, we show that if $\overline{\mathcal{R}}$ contains the odd-dimensional spheres, then it must also contain all simply-connected wedges of spheres. The proof is completed by appealing to Theorem 36.15.

To retain control over the mapping space for non-simply-connected targets, we must impose a condition on $\pi_1(X)$.

Proposition 37.3. *If $\pi_1(X)$ has no nontrivial perfect quotients (which is the case if and only if $\pi_1(X)$ is hypoabelian), then the simply-connected hypothesis in Theorem 37.2(2) may be removed.*

Thus the proof of Theorem 37.1 boils down to studying the space of maps from $B\mathbb{Z}/p$ into odd-dimensional spheres. We introduce (without going into full detail) the Massey-Peterson tower of a space with particularly simple cohomology and use it to reduce the whole problem to a question involving the homological algebra of modules over the Steenrod algebra. Then (taking a theorem about injective modules for granted) we establish the required algebraic vanishing result and derive Theorem 37.1.

We finish the chapter by demonstrating the vanishing of mapping spaces for certain more general domains and targets. Finally, we use the power of Miller's theorem to derive some interesting results.

In this chapter, we work with a fixed prime p, which may be 2 or not. All unspecified cohomology is taken with $\mathbb{Z}/p$ coefficients. We write $\mathcal{A}$ for the mod p Steenrod algebra $\mathcal{A}_p$ and $\mathcal{U} = \mathcal{U}_p$ and $\mathcal{K} = \mathcal{K}_p$ for the categories of unstable $\mathcal{A}$-modules and $\mathcal{A}$-algebras, respectively.

We will sometimes use the following interpretation of the triviality of a mapping space.

Problem 37.4. Show that $\mathrm{map}_*(X, K) \sim *$ if and only if $[\Sigma^t X, K] = *$ for all $t \geq 0$.

37.1. Reduction to Odd Spheres

In this section we prove Theorem 37.2 and Proposition 37.3, which, taken together, reduce the proof of Theorem 37.1 to showing that the space of maps from $B\mathbb{Z}/p$ to each odd-dimensional sphere is weakly contractible.

37.1.1. From Odd Spheres to Wedges of Spheres. We begin by showing that if the strong resolving class $\overline{\mathcal{R}}$ contains highly-connected odd-dimensional spheres, then it must contain all finite wedges of spheres.

Problem 37.5. Let $X \in \mathcal{T}_*$, and write $\overline{\mathcal{R}} = \{K \mid \mathrm{map}_*(X, K) \sim *\}$.

(a) Suppose A and B are connected CW complexes which have cells only in even dimensions. Show that the homotopy fiber of the collapse map

$$\Sigma A \vee \Sigma B \longrightarrow \Sigma B$$

has a CW decomposition with nontrivial cells only in odd dimensions.

(b) Show that if $\Sigma^{2N}\mathcal{S}^{\mathrm{odd}} \subseteq \overline{\mathcal{R}}$, then $(\Sigma^{2N}\mathcal{S}^{\mathrm{odd}})^{\vee} \subseteq \overline{\mathcal{R}}$.

(c) If $\mathcal{R}$ contains S^{2n+1} for all sufficiently large n, then $\overline{\mathcal{R}}$ contains all simply-connected finite-type wedges of spheres.

Hint. Adapt the proof of Proposition 36.10.

Our goal is to show that there are no nontrivial maps $\Sigma^t X \to W$. In the next problem we get a good start by showing that any such maps must at least be phantom maps (see Section 9.5).

Problem 37.6. Let X be a CW complex of finite type, let W be a wedge of spheres, and let $f : \Sigma^t X \to W$.

(a) Show that $f(\Sigma^t(X_n))$ is contained in a finite subwedge of W.

(b) Show that $f(\Sigma^t X)$ is contained in a countable subwedge of W.

(c) Suppose now that $\mathrm{map}_*(X, V) \sim *$ for all simply-connected finite wedges of spheres V, and show that

$$[\Sigma^t X, W] = \mathrm{Ph}(\Sigma^t X, W)$$

for any simply-connected wedge of spheres W.

(d) Conclude that to prove Theorem 37.2 it suffices to show $\mathrm{Ph}(\Sigma^t X, W) = *$ for all countable wedges of spheres W and all $t \geq 0$.

37.1.2. Vanishing Phantoms. Recall that the phantom set $\mathrm{Ph}(X, Y)$ can be identified with $\lim^1 [\Sigma X_n, Y]$. There is a very powerful condition which can be used to show that a tower $\cdots \leftarrow G_n \leftarrow G_{n+1} \leftarrow \cdots$ has trivial $\lim^1$. We say that the tower is **Mittag-Leffler** if for each n the sequence of images

$$G_n \supseteq \cdots \supseteq \mathrm{Im}(G_{n+k} \to G_n) \supseteq \mathrm{Im}(G_{n+k+1} \to G_n) \supseteq \cdots$$

eventually stabilizes.

Theorem 37.7. *If the tower $\cdots \leftarrow G_n \leftarrow G_{n+1} \leftarrow \cdots$ is Mittag-Leffler, then $\lim^1 G_n = *$. If the groups G_n are countable, then the converse holds.*

Project 37.8. Prove Theorem 37.7.[1]

[1] The first statement is standard; the second is a theorem of McGibbon and Steiner [**60**].

Interestingly, the Mittag-Leffler condition is blind to the algebraic structures in the groups G_n. This allows us to relate the phantom sets for maps into different targets with homotopy equivalent loop spaces.

Problem 37.9.

(a) Given a map of towers of countable groups

$$\begin{tikzcd} \cdots \arrow[r] & G_{n+1} \arrow[r] \arrow[d, "f_{n+1}"'] & G_n \arrow[r] \arrow[d, "f_n"] & \cdots \\ \cdots \arrow[r] & H_{n+1} \arrow[r] & H_n \arrow[r] & \cdots \end{tikzcd}$$

in which the maps f_n are bijections, but not necessarily homomorphisms, show that $\lim^1 G_n = *$ if and only if $\lim^1 H_n = *$.

(b) Let X be a CW complex with finite homotopy groups and let K and L be simply-connected countable CW complexes with $\Omega K \sim \Omega L$. Show that $\mathrm{Ph}(X, K) = *$ if and only if $\mathrm{Ph}(X, L) = *$.

Now we use an extension of the argument you used to prove the Hilton-Milnor theorem to finish our proof.

Problem 37.10. Let $W = \bigvee_{i=1}^{\infty} S^{n_i}$ with $n_i \leq n_{i+1}$ for all i and write W_k for the subwedge $\bigvee_{i=1}^{k} S^{n_i}$. Let $j_k : W_k \to W_{k+1}$ be the inclusion.

(a) Show that $\Omega(j_k) : \Omega W_k \hookrightarrow \Omega W_{k+1}$ may be identified with the inclusion $\mathrm{in}_1 : \Omega W_k \hookrightarrow \Omega W_k \times F_k$, where F_k is the homotopy fiber of j_k.

(b) Show that $F_k \simeq \Omega P_k$, where P_k is a (weak) product of spheres.[2]

(c) Show that $\Omega W \sim \Omega P$, where P is a weak product of spheres.

Problem 37.11.

(a) Let X be a CW complex of finite type, and let $\mathcal{Y}$ be a collection of spaces such that for each n, only finitely many of the spaces in $\mathcal{Y}$ are not n-connected. Show that if each tower $\{[\Sigma X_n, Y_i]\}$ is Mittag-Leffler, and if Y is a weak product of spaces in $\mathcal{Y}$, then the tower $\{[\Sigma X_n, Y]\}$ is also Mittag-Leffler.

(b) Prove Theorem 37.2.

37.1.3. Non-Simply-Connected Targets. To finally complete the reduction of Miller's theorem to the special case in which the targets are odd-dimensional spheres, we consider maps from X into targets K that are not simply-connected.

[2] Recall that a weak product is the (homotopy) colimit of the finite subproducts.

Problem 37.12. Show that if $\mathrm{cl}_{\mathcal{S}}(K) < \infty$ and $q : L \to K$ is any path-connected covering space, then $\mathrm{cl}_{\mathcal{S}}(L) < \infty$.

Hint. Use a Cube Theorem.

Problem 37.13. Assume that $\pi_1(X)$ has no nontrivial perfect quotients. Let $f : \Sigma^t X \to K$, and let $q : L \to K$ be the unique path-connected covering such that $\mathrm{Im}(f_*) = \mathrm{Im}(q_*) \cong \pi_1(L)$.

(a) Show that there is a lift in the diagram

$$\begin{tikzcd} & L \arrow[d, "q"] \\ \Sigma^t X \arrow[ur, dotted, bend left, "\phi"] \arrow[r, "f"] & K \end{tikzcd}$$

and ϕ induces a surjection on fundamental groups.

(b) Show that if f_* is nontrivial, then there is an abelian group Q and a map $L \to K(Q, 1)$ so that the composite $X \to L \to K(Q, 1)$ is nontrivial.

(c) Show that if $\phi \not\simeq *$, then $\Sigma^t \phi \not\simeq *$ for all $t \geq 0$.

(d) Prove Proposition 37.3.

37.2. Modules over the Steenrod Algebra

In this section we establish the vanishing of a certain Ext group involving unstable modules over the Steenrod algebra.

37.2.1. Projective $\mathcal{A}$-Modules. In order to do homological algebra with $\mathcal{A}$-modules, we need 'enough projectives', so we begin by exhibiting free $\mathcal{A}$-modules. You are probably used to thinking of a free module over R as a sum of modules of the form $R \cdot x$, which is isomorphic to R as a module. For unstable modules, the situation is a bit more subtle. The unstable relations imply that if $|x_i| = n$, then the module $\mathcal{A} \cdot x_i$ is not isomorphic to $\Sigma^n \mathcal{A}$; rather, it is isomorphic to

$$F(n) = \Sigma^n(\mathcal{A}/I(n)),$$

where $I(n)$ is the submodule of $\mathcal{A}$ generated by all Steenrod operations P^I with $e(I) > n$.

Exercise 37.14. Determine the structure of $F(0)$ and $F(1)$.

Problem 37.15. Let $\iota_n \in F(n)$ be the coset of $\mathrm{id} \in \mathcal{A}$.

(a) Show that $F(n)$ is a free module on ι_n, in the sense that for any unstable $\mathcal{A}$-module M and any $x \in M$, there is a unique map $f : F(n) \to M$ in $\mathcal{U}$ such that $f(\iota_n) = x$.

(b) Show that every sum of copies of the modules $F(n)$ is projective.[3]

(c) Show that $\mathcal{U}$ has enough projectives: every module M is the target of a surjection $P \to M$, with P a projective unstable $\mathcal{A}$-module.

(d) Show that if M is of finite type, then P can be chosen to be of finite type.

Since $\mathcal{U}$ has enough projectives, it is possible to build projective resolutions and so carry out the basic constructions of homological algebra.

37.2.2. Homological Algebra. Recall that a projective resolution of a module M is an exact sequence $0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$ where each P_n is projective. These resolutions play a role in homological algebra similar to that played by cofibrant replacement in homotopy theory.

Problem 37.16.

(a) Show that every module $M \in \mathcal{U}$ has a projective resolution in which P_n is $(n-1)$-connected.

(b) Show that if M is a module of finite type, then it can be arranged that in addition each P_n is of finite type.

Now we can construct the functors $\mathrm{Ext}^s_{\mathcal{U}}(\,?\,,\,?\,)$. Given modules M and N, apply $\mathrm{Hom}_{\mathcal{U}}(\,?\,, N)$ to your favorite projective resolution $P_* \to M \to 0$, yielding the chain complex $\mathrm{Hom}_{\mathcal{U}}(P_*, N)$. Then

$$\mathrm{Ext}^s_{\mathcal{U}}(M, N) = H^s(\mathrm{Hom}_{\mathcal{U}}(P_*, N)).$$

It is standard procedure to show that these groups are independent of the choice of resolution.

37.2.3. The Functor $\overline{T}$. In this section we introduce the functor $\overline{T}$ which is left adjoint to $\widetilde{H}^*(B\mathbb{Z}/p) \otimes\ ?$ and use it to establish the vanishing of the required Ext groups.

Some Injectives. Let $d_n : \mathcal{U} \to \mathrm{Vect}_{\mathbb{Z}/p}$ be the functor which takes a module M and returns the vector space dual $(M^n)^*$ of the dimension n group M^n. It can be shown that, because it is exact and carries sums to products, d_n is representable: there is a module $J(n)$ and a natural isomorphism

$$\mathrm{Hom}_{\mathcal{U}}(\,?\,, J(n)) \xrightarrow{\cong} d_n(\,?\,).$$

Since the functor d_n is exact, so is $\mathrm{Hom}_{\mathcal{U}}(\,?\,, J(n))$, which means that $J(n)$ is an injective object in $\mathcal{U}$.

Now we can state the crucial theorem, due to Lannes and Zarati, on injective modules in $\mathcal{U}$.

[3] See Section A.5 for the definition.

Theorem 37.17. *For each n, $\widetilde{H}^*(B\mathbb{Z}/p) \otimes J(n)$ is injective in $\mathcal{U}$.*

The proof of Theorem 37.17 requires quite a bit of preliminary work, so we won't prove it here. A very nice exposition of it can be found in L. Schwartz's book [**145**].

The Functors T and $\overline{T}$. Given a module $M \in \mathcal{U}$, it is not obvious on the face of it whether or not the functor $M \otimes\ ?$ has a left adjoint. But once you decide to prove it, you find that it is not too difficult to show that these functors do have left adjoints. Taking $M = H^*(B\mathbb{Z}/p)$, we obtain a functor which has been heavily studied and which is always denoted T; we will use the reduced form $\overline{T}$, which is adjoint to $\widetilde{H}^*(B\mathbb{Z}/p) \otimes\ ?$. These two functors are related by the formula $T = \mathrm{id} \oplus \overline{T}$.

Theorem 37.18. *The functor $\widetilde{H}^*(B\mathbb{Z}/p) \otimes\ ?$ has a left adjoint $\overline{T} : \mathcal{U} \to \mathcal{U}$, and it preserves exact sequences.*

For the usual T-functor proof of Miller's theorem, it must be shown that the functor $\overline{T}$ is also left adjoint to $\widetilde{H}^*(B\mathbb{Z}/p) \otimes\ ?$ in the category $\mathcal{K}$ of unstable *algebras* over the Steenrod algebra and that it preserves tensor products of algebras; we do not need to work with unstable algebras at all.

Let $0 \to A \to B \to C \to 0$ be an exact sequence of unstable $\mathcal{A}$-modules; we have to show that the sequence

$$0 \to \overline{T}(A) \longrightarrow \overline{T}(B) \longrightarrow \overline{T}(C) \to 0$$

is exact. This condition must be checked at each dimension separately; that is, we need to show $0 \to \overline{T}(A)^n \to \overline{T}(B)^n \to \overline{T}(C)^n \to 0$ is exact for each n.

Problem 37.19.

(a) Show that for each n, the functor $\mathrm{Hom}_{\mathcal{U}}(\,?\,, \widetilde{H}^*(B\mathbb{Z}/p) \otimes J(n))$ is exact.

(b) Prove Theorem 37.18.

The functor $\overline{T}$ enjoys many other good commutativity properties. For example, since $\overline{T}$ is a left adjoint, it commutes with coproducts, such as direct sums, telescopes, and so on.

Proposition 37.20. *The functor $\overline{T}$ commutes with suspension of modules.*

Taking Proposition 37.20 for granted, we use it to derive some simple consequences.

Problem 37.21.

(a) Show that $\overline{T}(F(n)) = \bigoplus_{i=0}^{n-1} F(i)$.
Hint. Show that $\overline{T}(F(n))$ represents the same functor as $\bigoplus_{k<n} F(k)$.

(b) Show that $\overline{T}(\mathbb{Z}/p) = 0$.

(c) Show that $\overline{T}(M) = 0$ if M has trivial $\mathcal{A}$-action.

Vanishing Results. Next we establish a vanishing theorem for $\overline{T}$ which will allow us to prove the required vanishing result for $\mathrm{Ext}_{\mathcal{U}}$.

Lemma 37.22. *If M is finite, then $\overline{T}(M) = 0$.*

Problem 37.23. Let $M \in \mathcal{U}$ be finite.

(a) Show that $\overline{T}(M) = 0$ if M has a filtration whose filtration quotients are trivial modules.

(b) Prove Lemma 37.22.

Now we are equipped to prove our basic algebraic vanishing theorem.

Theorem 37.24 (Miller)**.** *If M is a finite module, then*

$$\mathrm{Ext}^s(M, \Sigma^t \widetilde{H}^*(B\mathbb{Z}/p)) = 0$$

for all s and t.

Problem 37.25. Let $M \in \mathcal{U}$ be locally finite, and let $P_* \to M \to 0$ be a free resolution.

(a) Show that if P is free and of finite type, then $\overline{T}(P)$ is also free and of finite type.

(b) Show that $\overline{T}(P_*)$ is a free resolution of the trivial module 0.

(c) Prove Theorem 37.24.

37.3. Massey-Peterson Towers

In this section we show that spaces with very nice cohomology can be given filtrations that are intimately related to the homological algebra, in the category $\mathcal{U}$, of their cohomology.

37.3.1. Relating Algebras and Modules. The unreduced cohomology of a pointed space is an unstable algebra over the Steenrod algebra, and the inclusion $* \hookrightarrow X$ induces a map $H^*(X) \to \mathbb{Z}/p$ known as an **augmentation**. The kernel of the augmentation is known as the **augmentation ideal**.

Let $\mathcal{K}^+$ denote the category of augmented unstable algebras over $\mathcal{A}$. By forming the augmentation ideal $I(A)$ (which is the kernel of the augmentation), we get a functor $I : \mathcal{K}^+ \to \mathcal{U}$. The functor I has a left adjoint U which takes an unstable left $\mathcal{A}$-module and returns the 'free' unstable algebra it generates. Thus

$$\mathrm{Hom}_{\mathcal{K}}(U(M), A) \cong \mathrm{Hom}_{\mathcal{U}}(M, I(A)).$$

A space X is said to have **very nice** cohomology if $H^*(X) \cong U(M)$ for some finite-type module $M \in \mathcal{U}$.

Problem 37.26. Show that if $P = \bigoplus_{i\in\mathcal{I}} F(n_i)$ is a projective module of finite type, then there are natural isomorphisms

$$U(P) \cong H^*\left(\prod_{i\in\mathcal{I}} K(\mathbb{Z}/p, n_i)\right) \cong \bigotimes_{i\in\mathcal{I}} H^*(K(\mathbb{Z}/p, n_i)).$$

Problem 37.27. Show that $H^*(S^n) = U(\Sigma^n \mathbb{Z}/p)$ for all n if $p = 2$, and only for odd n if p is odd.

37.3.2. Topologizing Modules and Resolutions. In this section we show how maps and resolutions of $\mathcal{A}$-modules may be realized as maps induced in cohomology from maps of spaces.

Topological Realization of Free Modules. We have actually defined our single-generator free modules as the cohomology of spaces. For a free module $P = \bigoplus_{i\in\mathcal{I}} F(n_i)\cdot x_i$ of finite type we define

$$K(P) = \prod_{i\in\mathcal{I}} K(\mathbb{Z}/p, n_i),$$

so that $H^*(K(P)) = U(P)$. A homomorphism $f : P \to Q$ induces a well-defined map $K(f) : K(Q) \to K(P)$, defined by requiring the i^{th} coordinate

$$K(Q) \xrightarrow{K(f)} K(P) \xrightarrow{\text{pr}_i} K(\mathbb{Z}/p, n_i)$$

to be the cohomology class $f(x_i) \in Q \subseteq U(Q) = H^*(K(Q))$. Thus we have the commutative diagram

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d] & Q \arrow[d] \\
H^*(K(P)) \arrow[r, "K(f)^*"] & H^*(K(Q)).
\end{tikzcd}$$

Problem 37.28. Show that if $f : P \to Q$ is a map of projective $\mathcal{A}$-modules, the map $K(f)^* : [Y, K(Q)] \to [Y, K(P)]$ may be naturally identified using the diagram

$$\begin{tikzcd}
{[Y, K(Q)]} \arrow[r, "K(f)^*"] \arrow[d, "\cong"'] & {[Y, K(P)]} \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathcal{U}}(Q, \widetilde{H}^*(Y)) \arrow[r, "f^*"] & \mathrm{Hom}_{\mathcal{U}}(P, \widetilde{H}^*(Y)).
\end{tikzcd}$$

Topologizing Resolutions. Let Y be simply-connected with very nice cohomology (that is, $H^*(Y) \cong U(M)$ for some module $M \in \mathcal{U}$ is of finite type). Let

$$0 \leftarrow M \xleftarrow{\epsilon} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} \cdots \xleftarrow{d_s} P_s \xleftarrow{d_{s+1}} P_{s+1} \xleftarrow{d_{s+2}} \cdots$$

be a projective resolution of M in $\mathcal{U}$ in which each P_s is of finite type. We begin the construction of the Massey-Peterson tower by setting $X_0 = K(P_0)$, defining $j_0 : Y \to Y_0$ to be the map which induces ϵ on cohomology, and letting $k_0 = K(d_1)$, the map that induces $U(d_1)$ on cohomology. Inductively, suppose we have constructed the part of the tower of fibrations

$$\begin{tikzcd}
Y \arrow[drr, dotted] \arrow[ddrr, dotted] \arrow[dddrr, dotted] & & \vdots \arrow[d, "q_3"] & \\
& \Omega^2 K(P_2) \arrow[r, "i_2"] & Y_2 \arrow[r, "k_2"] \arrow[d, "q_2"] & \Omega^2 K(P_3) \\
& \Omega K(P_1) \arrow[r, "i_1"] & Y_1 \arrow[r, "k_1"] \arrow[d, "q_1"] & \Omega K(P_2) \\
& K(P_0) \arrow[r, "i_0"] & K(P_0) \arrow[r, "k_0"] & K(P_1)
\end{tikzcd}$$

so that for $r < s$ the composite $\Omega^r K(P_r) \to Y_r \to \Omega^r K(P_{r+1})$ is the map $\Omega^r K(d_{r+1})$. The inductive step involves the study of the cube

$$\begin{tikzcd}
Y_s \arrow[rr, dotted, "k_s"] \arrow[dr] \arrow[dd] & & \Omega^s K(P_{s+1}) \arrow[dr] \arrow[dd] & \\
& \mathcal{P}(\Omega^{s-1} K(P_s)) \arrow[rr] \arrow[dd] & & \mathcal{P}(\Omega^{s-1} K(P_{s+1})) \arrow[dd] \\
Y_{s-1} \arrow[rr, dotted, "h"] \arrow[dr, "k_{s-1}"'] & & \mathcal{P}(\Omega^{s-1} K(P_{s+1})) \arrow[dr] & \\
& \Omega^{s-1} K(P_s) \arrow[rr, "\Omega^{s-1} K(d_s)"] & & \Omega^{s-1} K(P_{s+1}).
\end{tikzcd}$$

We will not prove it here, but it is true that at each stage, a homotopy $h : Y_{s-1} \to \mathcal{P}(\Omega^{s-1} K(P_{s+1}))$ can be found such that the induced map k_s of homotopy fibers makes the diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[dr, bend right, "*"'] & Y_s \arrow[d, "k_s"] \arrow[dr, bend left, "*"] & \\
& \Omega^s K(P_{s+1}) \arrow[r, "\Omega^s K(d_{s+2})"] & \Omega^s K(P_{s+2})
\end{tikzcd}$$

commute up to homotopy. Let's summarize the main features of the construction.

Theorem 37.29. *Let Y be a simply-connected space of finite type, with cohomology $H^*(Y) = U(M)$ for some $M \in \mathcal{U}$, and let $P_* \to M \to 0$ be a projective resolution of M. Then, in the Massey-Peterson tower,*

(a) the boundary map $\Omega^s K(P_s) \to Y_s \to \Omega^s K(P_{s+1})$ is $\Omega^s K(d_s)$,

(b) the natural map from Y to the limit of $\cdots \to Y_s \to Y_{s-1} \to \cdots$ is the p-completion $Y \to Y_p^\wedge$.

Problem 37.30. Prove Theorem 37.29(a).

Project 37.31. Assume that the construction of the discussion above can be carried out in a natural way.

(a) Show that $\lim Y_s$ is p-complete.

(b) Show that the induced map $H^*(\lim Y_s; \mathbb{Z}/p) \to H^*(Y; \mathbb{Z}/p)$ is an isomorphism, and so prove Theorem 37.29(b).

37.3.3. The Groups $E_2^{s,t}(X, Y)$. Massey-Peterson towers are built very carefully so as to ensure that the resulting E_2-term is easy to describe. If we apply the functors $[\Sigma^{t-s}X, ?\,]$ to the Massey-Peterson tower for the space X, we obtain something very much like an exact couple with

$$E_1^{s,t}(X, Y) = [\Sigma^{t-s}Y, \Omega^s K(P_s)] \cong \mathrm{Hom}_{\mathcal{U}}(P_s, \widetilde{H}^*(\Sigma^t(X)).$$

The thing that makes it not quite an exact couple is that the objects in the $D_1^{s,s}$ positions are pointed sets, not groups. Nevertheless, we can identify the boundary maps and determine $E_2^{s,t}(X, Y)$.

Proposition 37.32. $E_2^{s,t}(X, Y) = \mathrm{Ext}^s_{\mathcal{U}}(M, \widetilde{H}^*(\Sigma^t X))$.

Problem 37.33. Prove it.

37.3.4. A Condition for the Omniscience of Cohomology. For any two spaces X and Y, evaluation of cohomology gives a map

$$\mathcal{H} : [X, Y] \longrightarrow \mathrm{Hom}_{\mathcal{K}}(H^*(Y), H^*(X))$$

which encodes the extent to which cohomology—together with the algebra structure and the cohomology operations—determines the maps from X to Y. If $H^*(Y) = U(M)$, then we can express the evaluation in the form

$$\mathcal{H} : [X, Y] \longrightarrow \mathrm{Hom}_{\mathcal{U}}(M, \widetilde{H}^*(X)),$$

where $\mathrm{Hom}_{\mathcal{U}}$ is the morphisms in the category $\mathcal{U}$ of modules over $\mathcal{A}$, rather than the category $\mathcal{K}$ of $\mathcal{A}$-algebras.

Theorem 37.34. *Let X be a finite-type CW complex which is the homotopy colimit of a telescope diagram*

$$\cdots \to X_{(k)} \longrightarrow X_{(k+1)} \to \cdots \to X$$

in which each $X_{(k)}$ is a finite complex whose integral homology groups are finite p-groups, and let Y be simply-connected with $H^*(Y) \cong U(M)$ for some finite-type module $M \in \mathcal{U}$. If $\operatorname{Ext}^s(M, \widetilde{H}^*(\Sigma^s X)) = 0$ for $s > 0$, then the following are equivalent:

(1) $f : X \to Y$ is nullhomotopic,

(2) $f^* : \widetilde{H}^*(Y) \to \widetilde{H}^*(X)$ is zero.

Actually, quite a lot more is true: the conditions of Theorem 37.34 imply that $\mathcal{H}$ is injective; and if $\operatorname{Ext}^s(M, \widetilde{H}^*(\Sigma^{s+1} X)) = 0$ for $s > 0$, then it is surjective as well. We will only prove the given statement, since it is all we need.

Part (1) trivially implies (2). You will prove the converse, beginning by showing by induction that the composites f_s defined by the triangles

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f_s"'] & Y \arrow[d] \\ & Y_s \end{tikzcd}$$

are all trivial.

Problem 37.35. Show that $f_0 : X \to Y_0$ is trivial.

For the inductive step, it will be helpful to refer to the diagram

$$\begin{tikzcd} K(\Omega^s P_{s-1}) \arrow[r] \arrow[rr, bend left, "K(\Omega^s d_s)"] & \Omega Y_{s-1} \arrow[r] \arrow[rr, bend right, "*"'] & K(\Omega^s P_s) \arrow[r, "i"] \arrow[rr, bend left, "K(\Omega^s d_{s+1})"] & Y_s \arrow[r] \arrow[d] & K(\Omega^s P_{s+1}) \\ & & & Y_{s-1}. & \end{tikzcd}$$

Problem 37.36. Now suppose that $f_{s-1} : X \to Y_s$ is trivial.

(a) Show that $f_s = i \circ h$ for some map $h \in E_1^{s,s}(X, Y)$.

(b) Show that h is a cycle, representing a class in $[h] \in E_2^{s,s}(X, Y)$.

(c) Show that h is a boundary, and conclude that $f_s = *$.

Now we have a map $f : X \to Y$ such that each composite $f_s : X \to Y_s$ is trivial, and we have to show that f is trivial. We prove it first in the special case X is a finite complex.

Problem 37.37. Suppose X is a finite complex whose integral homology groups are finite p-groups, and let $f : X \to Y$ with $f_s \simeq *$ for all s.

(a) Show that $[\Sigma X, Y_s]$ is a finite group for each s.

(b) Show that the natural map $[X, \lim Y_s] \to \lim[X, Y_s]$ has trivial kernel.

(c) Show that the composite $X \xrightarrow{f} Y \to \lim Y_s$ is trivial.

(d) Show that f is trivial.

Hint. Use Corollary 34.52.

Finally, we bootstrap from finite complexes to complexes of finite type.

Problem 37.38. Let $f : X \to Y$ where X and Y satisfy the conditions of Theorem 37.34 and $f^* : \widetilde{H}^*(Y) \to \widetilde{H}^*(X)$ is trivial.

(a) Show that if $\mathrm{Ext}^s(M, \widetilde{H}^*(\Sigma^s X)) = 0$ for all $s > 0$, then $f \in \mathrm{Ph}(X, Y)$.

(b) Prove Theorem 37.34.

At last we have all of the tools necessary to prove Miller's theorem.

Problem 37.39. Prove Theorem 37.1.

37.4. Extensions and Consequences of Miller's Theorem

37.4.1. The Sullivan Conjecture. Theorem 37.1 is frequently referred to as the **Sullivan conjecture**, since it resolves a special case of a conjecture due to Sullivan [**166**]. The conjecture (extended from $G = \mathbb{Z}/2$) concerns $\mathbb{Z}/p$-actions. If $\mathbb{Z}/p$ acts on the space Y, collapse map $EG \to *$ induces the comparison map

$$\mathrm{map}_G(*, Y) \longrightarrow \mathrm{map}_G(EG, Y)$$

from the genuine fixed point set to the homotopy fixed point set. This map is generally not a weak homotopy equivalence, but Sullivan conjectured that it would become a weak homotopy equivalence after p-completion.

Problem 37.40. Show that Miller's theorem resolves the special case of Sullivan's conjecture concerned with trivial actions.

After the initial breakthrough, the full resolution of the conjecture was not long in coming.

37.4.2. $B\mathbb{Z}/p$-Nullification. Recall that a space Y is X**-null** if the mapping space $\mathrm{map}_*(X, Y)$ is weakly contractible, or, equivalently, if Y is local with respect to the map $* \to X$. Since Miller's theorem is concerned with contractible mapping spaces, it can be expressed in this language.

Problem 37.41.

(a) Show that every space K with $\mathrm{cl}_{\mathcal{S}}(K) < \infty$ is $B\mathbb{Z}/p$-null.

(b) Show that if Y is simply-connected, then Y is $B\mathbb{Z}/p$-null if and only if the p-completion $Y_p^\wedge$ is $B\mathbb{Z}/p$-null.

Problem 37.42. Show that if $P_{B\mathbb{Z}/p}(X) \sim *$, then $\mathrm{map}_*(X, K) \sim *$ for all K with $\mathrm{cl}_{\mathcal{S}}(K) < \infty$.

To date, no one has ever produced an example of a space X with $\mathrm{map}_*(X, K) \sim *$ for all finite-dimensional CW complexes K without appealing to Miller's theorem in one form or another. This suggests the following conjecture.

Conjecture 37.43. If X is a p-local space and $\mathrm{map}_*(X, K) \sim *$ for all finite-dimensional spaces K, then $B\mathbb{Z}/p < X$.

37.4.3. Neisendorfer Localization. The preceding results have been interesting to one degree or another, but now we have come to an amazing and totally unexpected result that is a fairly straightforward consequence of Miller's theorem in the form we have proved it. Write $M = M(\mathbb{Z}[\frac{1}{p}], 1)$, and define the **Neisendorfer localization** functor

$$\mathcal{L}(X) = P_M\left(P_{B\mathbb{Z}/p}(X)\right).$$

The amazing thing about this functor is that it enables us to recover (up to p-completion) a simply-connected finite complex X from any one of its connected covers $X\langle n\rangle$.

Problem 37.44. Show that $\mathcal{L}(X) = \left(P_{B\mathbb{Z}/p}(X)\right)^\wedge_p$ if X is simply-connected.

Theorem 37.45 (Neisendorfer)**.** *Let X be a simply-connected finite complex, and assume that $\pi_2(X)$ is finite. Then*

$$\mathcal{L}\left(X\langle n\rangle\right) = \mathcal{L}\left(X\langle n\rangle^\wedge_p\right) = X^\wedge_p$$

for each $n \geq 1$.

We begin the proof of the theorem by characterizing the map $X \to \mathcal{L}(X)$ for simply-connected spaces.

Problem 37.46. Let X be simply-connected.

(a) Show that $\mathcal{L}(X)$ is p-complete and $B\mathbb{Z}/p$-null.

(b) Conclude that the natural map $\mathcal{L}(X) \to \mathcal{L}(\mathcal{L}(X))$ is a homotopy equivalence.

(c) Show that $X \to \mathcal{L}(X)$ is determined up to pointwise equivalence in $\mathrm{H}\mathcal{T}_*$ by the universal property:

If $f : X \to Z$ and Z is p-complete and $B\mathbb{Z}/p$-null, then f factors uniquely up to homotopy through $X \to \mathcal{L}(X)$.

(d) Show that if X satisfies the conditions of Theorem 37.45 and Z is both p-complete and $B\mathbb{Z}/p$-null, then

$$\begin{tikzcd}
\operatorname{map}_*(X\langle m-1\rangle, Z) \arrow[r, "\sim"] \arrow[d, "\sim"'] & \operatorname{map}_*(X\langle m\rangle, Z) \arrow[d, "\sim"] \\
\operatorname{map}_*(X\langle m-1\rangle_p^\wedge, Z) \arrow[r, "\sim"] & \operatorname{map}_*(X\langle m\rangle_p^\wedge, Z)
\end{tikzcd}$$

commutes up to homotopy.

(e) Prove Theorem 37.45.

37.4.4. Serre's Conjecture. In [**153**], J.-P. Serre conjectured that if a finite complex X has $\widetilde{H}_*(X;\mathbb{Z}/p) \neq 0$, then infinitely many of the homotopy groups $\pi_*(X)$ must contain p-torsion. This was proved by McGibbon and Neisendorfer in [**123**] using Miller's theorem. We'll give a very pretty alternate proof, due to Neisendorfer [**139**] (and also depending on Miller's theorem), that makes use of the functor $\mathcal{L}$.

Theorem 37.47 (McGibbon and Neisendorfer)**.** *If X is a simply-connected finite complex and $\pi_n(X) \otimes \mathbb{Z}/p = 0$ for all large n, then $\widetilde{H}_*(X;\mathbb{Z}/p) = 0$.*

We prove Theorem 37.47 in two steps: first in the special case where $\pi_2(X)$ is finite; then we use the special case to show that the special case is actually the general case.

Problem 37.48. Suppose X is as in Theorem 37.47 and $\pi_2(X)$ is finite. Show that $X_p^\wedge \simeq *$ and conclude that $\widetilde{H}_*(X;\mathbb{Z}/p) = 0$.

Now we consider the possibility that $\pi_2(X)$ might be infinite.

Problem 37.49.

(a) Show that there is a fiber sequence $\prod S^1 \to Y \to X \to \prod \mathbb{C}\mathrm{P}^\infty$ where Y satisfies the conditions of Theorem 37.47 and $\pi_2(Y)$ is finite.

(b) Complete the proof of Theorem 37.47 by showing that $X \to \prod \mathbb{C}\mathrm{P}^\infty$ is an $H_*(\,?\,;\mathbb{Z}/p)$-equivalence.

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Appendix A

Some Algebra

The purpose of this chapter is to summarize the definitions, theorems and techniques of pure algebra that are essential tools in algebraic topology, but which are not standard topics in introductory algebra courses.

A.1. Modules, Algebras and Tensor Products

In this section we briefly review basic terminology and results concerning modules over a commutative ring R. We define tensor products so that we can talk about algebras, and we end with a list of some frequently used algebras.

A.1.1. Modules. Let R be a commutative ring (with unit). An R-**module** is an abelian group M together with a rule $R \times M \to M$ (usually denoted $(r, m) \mapsto r \cdot m$, or simply rm) for multiplying module elements by ring elements. This multiplication is required to satisfy the following rules:

- $r(m + n) = rm + rn$,
- $(r + s)m = rm + sm$,
- $r(sm) = (rs)m$,
- $1 \cdot m = m$

for all $m, n \in M$ and all $r, s \in R$. If M and N are R-modules, then a group homomorphism $f : M \to N$ is a **module homomorphism** if $f(rm) = rf(m)$ for all $m \in M$ and $r \in R$.

The theory of R-modules, as it appears in basic algebraic topology, is primarily a straightforward generalization of the theory of abelian groups.

Exercise A.1. Show that every abelian group is a $\mathbb{Z}$-module and that every group homomorphism is a $\mathbb{Z}$-module homomorphism.

Problem A.2. Show that if $f : M \to N$ is an R-module homomorphism, then $\ker(f)$, $\mathrm{Im}(f)$ and $\mathrm{coker}(f)$ are all R-modules.

We write $\mathbf{Mod}_R$ for the category whose objects are R-modules and whose morphisms $\mathrm{Hom}_R(M, N)$ are the R-module homomorphisms from M to N. Thus we may define the (categorical) sum and the product of a collection of R-modules, and these may be constructed in the usual way.

Problem A.3. Give explicit constructions for $\prod_{\mathcal{I}} M_i$ and $\bigoplus_{\mathcal{I}} M_i$.

Free Modules. The ring R itself is an R-module, using the ring multiplication. An R-module F is called a **free module** if $F \cong \mathrm{Free}(\mathcal{I}) = \bigoplus_{\mathcal{I}} R$ for some indexing set $\mathcal{I}$.

Problem A.4.

(a) Show that the rule $\mathcal{I} \mapsto \mathrm{Free}(\mathcal{I})$ defines a functor $\mathrm{Free} : \mathbf{Sets} \to \mathbf{Mod}_R$.

(b) Show that Free is left adjoint to the forgetful functor $\mathbf{Mod}_R \to \mathbf{Sets}$.

(c) Show that if F is a free module and the vertical sequence in the diagram

$$\begin{tikzcd} & A \arrow[d] \\ F \arrow[ur, dotted, "\ell"] \arrow[r] & B \arrow[d] \\ & 0 \end{tikzcd}$$

is exact, then the lift ℓ exists.

A.1.2. Bilinear Maps and Tensor Products. Let M, N and Z be R-modules; a function $f : M \times N \to Z$ gives rise to functions

$$f_m : N \longrightarrow Z \qquad \text{and} \qquad f^n : M \longrightarrow Z$$

given by $f_m(n) = f(m, n)$ and $f^n(m) = f(m, n)$. We say that f is **bilinear** if the functions f_m and f^n are R-homomorophisms for each $m \in M$ and $n \in N$.

The **tensor product** of R-modules M and N is the R-module $M \otimes_R N$ which serves as a universal target for bilinear maps. Thus $M \otimes_R N$ comes with a bilinear map $b : M \times N \to M \otimes_R N$, and for any bilinear map $f : M \times N \to Z$, there is a unique R-module map $\bar{f} : M \otimes_R N \to Z$ making

the diagram

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "b"'] & Z \\ M \otimes_R N \arrow[ur, dotted, "\exists!\, \bar{f}"'] & \end{tikzcd}$$

commute. Since it is defined by a universal property, the tensor product is well-defined up to isomorphism, provided it exists.

In fact, tensor products do exist, because we can construct them explicitly. First form the free module

$$F = \mathrm{Free}\left(\{m \otimes n \mid m \in M, n \in N\}\right),$$

where (for the moment) we view $m \otimes n$ as an abstract symbol; we'll impose algebraic meaning on these symbols presently. Let $X \subseteq F$ be the smallest submodule that contains all elements of the forms

- $m \otimes (rn_1 + sn_2) - (r(m \otimes n_1) + s(m \otimes n_2))$,
- $(rm_1 + sm_2) \otimes n - (r(m_1 \otimes n) + s(m_2 \otimes n))$,
- $(rm) \otimes n - m \otimes (rn)$

with $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r, s \in R$. Then the tensor product can be defined to be $M \otimes_R N = F/X$, with bilinear map $M \times N \to M \otimes_R N$ given by $(m, n) \mapsto m \otimes n$.

Problem A.5.

(a) Show that the rule $r \cdot (m \otimes n) = (rm) \otimes n$ gives F/X the structure of an R-module.

(b) Prove that F/X has the universal property of the tensor product.

Problem A.6.

(a) Show that the map $R \otimes_R M \to M$ given by $r \otimes m \mapsto rm$ is an R-module isomorphism, and likewise $M \cong M \otimes_R M$.

(b) Show that $? \otimes_R N$ and $M \otimes_R ?$ are functors.

(c) Show that there are natural isomorphisms

$$\mathrm{Hom}_R(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C)),$$

an algebraic exponential law.

(d) Show that there are natural distributivity isomorphisms

$$A \otimes_R (B \oplus C) \cong (A \otimes_R B) \oplus (A \otimes_R C).$$

A.1.3. Algebras. An R**-algebra** is an R-module A which is also a ring with unit in such a way that the map $R \to A$ given by $r \mapsto r \cdot 1_A$ is a ring homomorphism. We write $\mathbf{Alg}_R$ for the category of R-algebras and algebra maps.

Problem A.7. Let A be an R-module that is a ring. Show that A is an R-algebra if and only if the product $A \times A \to A$ is R-bilinear. That is, an R-algebra is an R-module A together with R-module homomorphisms

$$\mu : A \otimes_R A \longrightarrow A \quad \text{(product)} \qquad \text{and} \qquad \epsilon : R \longrightarrow A \quad \text{(unit)}$$

satisfying the ring properties.

In terms of diagrams, 'satisfying the ring properties' means that the diagrams

$$\begin{tikzcd}
A \arrow[r, "\cong"] \arrow[drr, bend right, "\mathrm{id}"'] & A \otimes R \arrow[r, "\mathrm{id}\otimes\epsilon"] & A \otimes A \arrow[d, "\mu"] & R \otimes A \arrow[l, "\epsilon\otimes\mathrm{id}"'] & A \arrow[l, "\cong"'] \arrow[dll, bend left, "\mathrm{id}"] \\
 & & A & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes\mu"] \arrow[d, "\mu\otimes\mathrm{id}"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"] & A
\end{tikzcd}$$

commute. The algebra A is **commutative** if the triangle

$$\begin{tikzcd}
A \otimes A \arrow[rr, "T"] \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl] \\
 & A &
\end{tikzcd}$$

commutes.

If A and B are algebras, then the R-module $A \otimes B$ can also be given the structure of an algebra, using the product

$$(A \otimes B) \otimes (A \otimes B) \xrightarrow{\mathrm{id}\otimes T\otimes\mathrm{id}} (A \otimes A) \otimes (B \otimes B) \xrightarrow{\mu_A\otimes\mu_B} A \otimes B$$

(where $T : B \otimes_R A \to A \otimes_R B$ is the **twist map** given by $T(b \otimes a) = a \otimes b$) and the unit $R \xrightarrow{\cong} R \otimes R \xrightarrow{\epsilon_A\otimes\epsilon_B} A \otimes B$.

We conclude this section with a small bestiary of simple R-algebras that frequently arise in algebraic topology.

Tensor Algebra. Let R be a ring, and let M be an R-module. The **tensor algebra** $T(M)$ of M is the R-module

$$T(M) = \bigoplus_{n=0}^{\infty} M^{\otimes n} \qquad \text{where} \qquad M^{\otimes n} = \overbrace{M \otimes_R \cdots \otimes_R M}^{n \text{ factors}}$$

(we adopt the convention that $M^{\otimes 0} = R$). This is clearly an R-module, but we want to give it the structure of an R-algebra.

Because $\otimes$ distributes over $\oplus$, we have a natural isomorphism

$$T(M) \otimes_R T(M) \cong \bigoplus_{i,j} M^{\otimes i} \otimes_R M^{\otimes j},$$

so to define a map $\mu : T(M) \otimes_R T(M) \to T(M)$, it suffices to specify its restiction to each summand. On the (i,j)-summand, we define μ to be the composite

$$M^{\otimes i} \otimes_R M^{\otimes j} \xrightarrow{\cong} M^{\otimes(i+j)} \hookrightarrow T(M).$$

Problem A.8.

(a) Show that the rule $T : M \to T(M)$ is a functor $T : \mathbf{Mod}_R \to \mathbf{Alg}_R$.

(b) Show that T is left adjoint to the forgetful functor $\mathbf{Alg}_R \to \mathbf{Mod}_R$.

Polynomial Algebra. The **polynomial algebra** on M is the module obtained from $T(M)$ by imposing the commutativity relation $x \otimes y = y \otimes x$. Explicitly, define C to be the smallest ideal of $T(M)$ containing all elements of the form $x \otimes y - y \otimes x$; then $R[M] = T(M)/C$.

Problem A.9. Let $\widetilde{\mathbf{Alg}}_R$ denote the category of commutative R-algebras (this is temporary notation). Show that the rule $M \mapsto R[M]$ is a functor that is left adjoint to the forgetful functor $\widetilde{\mathbf{Alg}}_R \to \mathbf{Mod}_R$.

Problem A.10. Show that $R[M \oplus N] \cong R[M] \otimes R[N]$.

We'll also use **truncated polynomial algebras**, typified by the quotient $R[x]/(x^{n+1})$.

Exterior Algebra. The **exterior algebra** on M is the module obtained from $T(M)$ by imposing the *anticommutativity* relation $x \otimes y = -y \otimes x$. Thus we let Z be the smallest ideal of $T(M)$ containing all elements of the form $x \otimes y + y \otimes x$ and define $\Lambda(M) = T(M)/Z$.

Problem A.11. Show that $\Lambda(\,?\,)$ is a left adjoint.

Problem A.12. Compare $\Lambda(M)$ to the truncated polynomial algebra on M in which every element squares to zero.

Divided Polynomial Algebra. We'll be less categorical about divided polynomial algebras and begin by defining the **divided polynomial algebra** $\Delta_R(x)$ on one 'generator'. As an R-module, this is simply a sum

$$\Gamma_R(x) = \bigoplus_{n=0}^{\infty} R \cdot x_{(n)},$$

subject to the relations $x_{(0)} = 1$, $x_{(1)} = x$ and in general $x_{(1)}^n = n! x_{(n)}$. For a finite set of generators, we define

$$\Gamma_R(x_1, x_2, \ldots, x_m) = \bigotimes_{k=1}^{m} \Delta_R(x_k).$$

Problem A.13. In a divided polynomial algebra, $x_{(k)} \cdot x_{(l)} = c_{k,l} x_{(k+l)}$ for some $c_{k,l} \in \mathbb{Z}$. Find a formula for $c_{k,l}$.

When the ring R is $\mathbb{Z}/p$ (with p prime), divided polynomial algebras split as tensor products of trunctated polynomial algebras.

Problem A.14. Show that

$$\Gamma_{\mathbb{Z}/p}(x) \cong \bigotimes_{k=0}^{\infty} \mathbb{Z}/p[x_{(p^k)}]/(x_{(p^k)}^p).$$

A.2. Exact Sequences

A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of R-modules is said to be an **exact sequence** (or **exact at** B) if $\mathrm{Im}(f) = \ker(g)$. A longer sequence is called exact if it is exact at each module in the sequence. An exact sequence of the form $0 \to A \to B \to C \to 0$ is called a **short exact sequence**.

Problem A.15. What can you say about $f : A \to B$ in the following situations?

(a) $A \to B \to 0$ is exact.

(b) $0 \to A \to B$ is exact.

(c) $0 \to A \to B \to 0$ is exact.

It is frequently useful to cut long exact sequences into short ones or to splice short ones to obtain longer ones.

Lemma A.16.

(a) *If the sequences* $A \to B \xrightarrow{f} C \to 0$ *and* $0 \to C \xrightarrow{g} D \to E$ *are exact, then the sequence*

$$A \longrightarrow B \xrightarrow{g \circ f} D \longrightarrow E$$

is also exact.

(b) If $A \xrightarrow{f} B \to C \to D \xrightarrow{g} E$ *is exact, then there is a short exact sequence*

$$0 \to \operatorname{coker}(f) \longrightarrow C \longrightarrow \ker(g) \to 0.$$

Problem A.17. Prove Lemma A.16.

A functor $F : \mathbf{Mod}_R \to \mathbf{Mod}_R$ is called an **exact functor** if the image $F(A) \to F(B) \to F(C)$ of any exact sequence $A \to B \to C$ is also exact.

Problem A.18. Let $F : \mathbf{Mod}_R \to \mathbf{Mod}_R$. Show that F is exact if and only if F carries every short exact sequence to a short exact sequence.

Problem A.19. Let $0 \to A \to B \to C \to 0$ be a short exact sequence. Show that the following are equivalent:

(1) the map $A \to B$ has a left inverse,
(2) the map $B \to C$ has a right inverse,
(3) the sequence is pointwise isomorphic to the standard trivial extension $0 \to A \to A \oplus C \to C \to 0$ given by projection and inclusion.

If one (i.e., all) of these conditions holds, then we say that the sequence is a **split exact sequence**, or that the sequence **splits**.

Easily one of the most important theorems for analyzing exact sequences is the **Five Lemma**.

Theorem A.20 (Five Lemma)**.** *In the diagram*

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "\phi"] & A_4 \arrow[r] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

if the rows are exact and f_1, f_2, f_4 *and* f_5 *are isomorphisms, then* ϕ *is also an isomorphism.*

Problem A.21. Prove Theorem A.20. Examine your proof carefully: what is the bare minimum you need to know to conclude ϕ is injective? What do you need to know to conclude ϕ is surjective?

A.3. Graded Algebra

Very often in this text, the algebraic gadgets we use will come with a notion of dimension—they will be **graded**. A **graded R-module** is a list

$$M^* = \{\ldots, M^n, M^{n+1}, \ldots\}$$

of R-modules. The module M^n is the **component** of M in **degree** n, or **dimension** n. Very often, but not always, we will work with graded modules which are **bounded below**: $M^n = 0$ for all sufficiently small n.

If $M^n = 0$ for all n except one (say $M^k \neq 0$), then M^* is said to be **concentrated in degree** k. If $M^k = G$, then we say that M is G, concentrated in degree k. We consider the category of ungraded modules as a subcategory of the category of graded modules by identifying $G \in \mathbf{Mod}_R$ with the graded module which is G, concentrated in dimension 0. The **suspension** of a graded R-module M is the module ΣM given by $(\Sigma M)^n = M^{n-1}$. Thus if $G \in \mathbf{Mod}_R$, then $\Sigma^n G$ is the graded module with G concentrated in degree n.

It is common in algebraic topology to adopt the following convention relating upper and lower indices: $M^n = M_{-n}$. When we form tensor products, the degree satisfies $|x \otimes y| = |x| + |y|$. Thus $M^k \otimes N_k$ is of degree zero.

Two graded R-modules have a direct sum which is constructed dimensionwise: $(M^* \oplus N^*)^n = M^n \oplus N^n$. It is more complicated to construct the tensor product of graded modules:

$$(M^* \otimes_R N^*)^n = \bigoplus_{i+j=n} M^i \otimes N^j.$$

If M^* and N^* are both bounded below (or above), then this is a finite sum; otherwise, the sum could be infinite.

Now we can define a **graded R-algebra** to be a graded R-module A equipped with a multiplication $\mu : A \otimes_R A \to A$ satisfying the properties of Section A.1.3. A graded R-module M is an **A module@A-module** if there is a multiplication $A \otimes_R M \to M$ satisfying the conditions of A.1.1.

Now we can define the graded tensor algebra, polynomial algebra, exterior algebra and divided polynomial algebra just as in Section A.1.3.

Graded Commutativity. In homotopy theory, the degree $(-1)^{nm}$ of the twist map $S^n \wedge S^m \to S^m \wedge S^n$ propagates throughout all the algebraic functors that we define, so that the vast majority of the algebras we encounter are **graded commutative algebras**. The **graded twist map** is the map

$$T : M^* \otimes N^* \longrightarrow N^* \otimes M^*$$

given by $T(m \otimes n) = (-1)^{|m|\cdot|n|} n \otimes m$, and we say that a graded algebra A is called **graded commutative** if the diagram

$$\begin{tikzcd} A \otimes A \arrow[rr, "T"] \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\ & A & \end{tikzcd}$$

commutes.

If $1 = -1$ in the ring R, then graded commutativity is indistinguishable from ordinary commutativity. This is the case, for example, if $R = \mathbb{Z}/2$.

Free Graded Commutative Algebras. Let M be a graded R-module, and impose the graded commutativity rule $xy = (-1)^{|x|\cdot|y|}yx$ on the tensor algebra $T_R(M)$ to obtain the **free graded commutative algebra** on M.

There is notational trouble here, because there does not seem to be a standard notation for these algebras, except in rational homotopy theory, where—in direct conflict with the long-established notation for exterior algebras—they are universally denoted $\Lambda_{\mathbb{Q}}(M)$. Fortunately, we won't need to use these algebras more that a few times, so we'll introduce the (necessarily nonstandard) notation $\Phi_R(M)$ for the free graded commutative R-algebra on M.

Problem A.22.

(a) Show that the inclusion $M \hookrightarrow \Phi_R(M)$ is the universal example for R-module maps from M to graded-commutative R-algebras.

(b) Show that if $1 = -1$ in R, then $\Phi_R(M)$ is the polynomial algebra on M.

When $1 \neq -1$ in R, the situation is more interesting, and we obtain a genuinely new algebra.

Problem A.23. Write $M = M^{\text{even}} \oplus M^{\text{odd}}$, and show that

$$\Phi_R(M) \cong R[M^{\text{even}}] \otimes_R \Lambda_R(M^{\text{odd}}).$$

A.3.1. Decomposables and Indecomposables. An **augmentation** of a graded module M is a map $M \to R$, where R is considered as the graded module concentrated in dimension 0. In the cohomology of spaces, the inclusion of the basepoint $* \hookrightarrow X$ gives $H^*(X; R)$ the structure of an **augmented algebra**.

If A is an augmented graded R-algebra, then we can form the **augmentation ideal** $\overline{A} = \ker(A \to R)$; this can be thought of as the ideal of nonunits in A. The image of the composition $\overline{A} \otimes_R \overline{A} \hookrightarrow A \otimes_R A \xrightarrow{\mu} A$ is called the module of **decomposables** in A, and its cokernel

$$Q(A) = A/(\text{decomposables})$$

is called the module of **indecomposables** in A.

An augmented R-module M is called **connected** if $M^n = 0$ for $n < 0$ and the augmentation is an isomorphism $M^0 \xrightarrow{\cong} R$.

Problem A.24. Let A be a connected augmented graded $\mathbb{F}$-algebra, where $\mathbb{F}$ is a field. Show that A is generated by QA, in the sense that if $B \subseteq A$ is an augmented subalgebra and the composite $B \to A \to QA$ is surjective, then $B = A$.

Hint. Study the smallest dimension n in which $B^n \neq A^n$.

A.4. Chain Complexes and Algebraic Homology

A **chain complex** is a sequence of R-modules and homomorphisms

$$\cdots \to A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \to \cdots$$

in which each composite $d^{n+1} \circ d^n$ is trivial.[1] The maps d^n are called the **differentials** of the chain complex; they are also called **boundary maps**. We can assemble the groups A^n into one graded object A^*, so that the differentials become a single map $d : A^* \to A^*$ with $d^2 = d \circ d = 0$.

Problem A.25. Show that $\operatorname{Im}(d) \subseteq \ker(d)$.

A.4.1. Homology of Chain Complexes.

The image of d is known as the module of **boundaries** of A^*, and the kernel of d is known as the module of **cycles** of A^*. In view of the problem above, we can define the n^{th} (algebraic) **homology module**[2] of a chain complex is the graded R-module given by

$$H^n(A^*) = \ker(d^n)/\operatorname{Im}(d^{n-1}).$$

A **chain map**[3] is a map $f : A^* \to B^*$ from one chain complex to another that is compatible with the differentials in the sense that $f \circ d_A = d_B \circ f$.

Problem A.26.

(a) Show that there is a category whose objects are chain complexes and whose morphisms are chain maps.

(b) Show that homology is a functor from the category of chain complexes to the category of graded R-modules.

Theorem A.27. *If $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is a short exact sequence of chain complexes, then there is a long exact sequence*

$$\cdots \to H^n(A) \xrightarrow{i^*} H^n(B) \xrightarrow{j^*} H^n(C) \xrightarrow{\partial} H^{n+1}(A) \to \cdots .$$

Problem A.28.

(a) Show that the rule $\partial([c]) = [(i^{-1} \circ d_B \circ j^{-1})(c)]$ (brackets denote homology classes) is a well-defined homomorphism $H^n(C) \to H^{n+1}(A)$.

(b) Prove Theorem A.27.

[1] Some authors use the term **cochain complex** for a chain complex as we have defined it and reserve **chain complex** for complexes with lower indices and maps which decrease index. Since we have the convention $A^n = A_{-n}$, these two kinds of complexes are really just different notations for the same thing, so we only use one terminology.

[2] Authors who distinguish between chain complexes and cochain complexes also distinguish between *homology* and *cohomology*.

[3] Or *cochain map*, if you insist on thinking of a chain complex as different from a cochain complex.

A chain complex A^* is called **acyclic** if $H^*(A^*) = 0$. A chain map $f : A^* \to B^*$ is called a **quasi-isomorphism** if the induced map $f^* : H^*(A^*) \to H^*(B^*)$ is an isomorphism.

Problem A.29. Show that if $f : A^* \hookrightarrow B^*$, then f is a quasi-isomorphism if and only if B^*/A^* is acyclic.

Tensor Product of Chain Complexes. We can define a tensor product of chain complexes. The tensor product of graded R-modules is already defined, so we just need to define the differential on it. This is given by

$$d_{\mathcal{C}\otimes\mathcal{D}}(c \otimes d) = d_{\mathcal{C}}(c) \otimes d + (-1)^{|c|} c \otimes d_{\mathcal{D}}(d).$$

Problem A.30. Show that $\mathcal{C}_* \otimes \mathcal{D}_*$ is a chain complex.

A.5. Some Homological Algebra

In this section we introduce the functors Tor and Ext, which measure the failure of $\otimes$ and Hom to preserve exactness.

A.5.1. Projective Resolutions and Tor_R**.** An R-module P is **projective** if in any diagram of the form

$$\begin{tikzcd} & A \arrow[d] \\ P \arrow[ur, dotted, "\phi"] \arrow[r, "f"] & B \arrow[d] \\ & 0 \end{tikzcd}$$

where the column is exact the map ϕ can be found so that the triangle commutes.

Problem A.31.

(a) Show that every free module is projective.

(b) Show that every R-module is the target of a surjection $P \to M$ where P is a projective R-module.

(c) Show that P is projective if and only if the functor $F(?) = \mathrm{Hom}_R(P, ?\,)$ is an exact functor—i.e., if $A \to B \to C$ is exact, then so is $F(A) \to F(B) \to F(C)$.

The condition of Problem A.31(b) is often referred to by saying that the category of R-modules has **enough projectives**. This is the property that ensures the existence of projective resolutions. We can consider any R-module as a chain complex concentrated in degree 0. A **projective resolution** for M is a quasi-isomorphism

$$\begin{tikzcd} \cdots \arrow[r] & P_n \arrow[r] \arrow[d] & P_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[d] \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & M \end{tikzcd}$$

in which the chain complex P_* is acyclic and each P_n is projective.

Theorem A.32. *Every R-module has a projective resolution. If $f : M \to N$ and if $P_* \to M$ and $Q_* \to N$ are projective resolutions, then there is a chain map $\phi : P_* \to Q_*$ such that the diagram*

$$\begin{tikzcd} P_* \arrow[r, "\phi"] \arrow[d] & Q_* \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

commutes. Furthermore, ϕ is unique up to chain homotopy.

Problem A.33. Prove Theorem A.32.

Now if we are given two modules M and N and a projective resolution P_* of M, we define

$$\mathrm{Tor}_n^R(M, N) = H_n(P_* \otimes N).$$

Problem A.34.

(a) Show that $\mathrm{Tor}_n^R(\,?\,,\,?\,)$ is well-defined and functorial.

(b) Show that if $Q_* \to N$ is a projective resolution, then for any R-module M, $\mathrm{Tor}_n^R(M, N) \cong H_n(M \otimes Q_*)$.

(c) Show that there are natural isomorphisms $\mathrm{Tor}_0^R(M, N) \xrightarrow{\cong} M \otimes_R N$.

Theorem A.35. *Suppose $0 \to A \to B \to C$ is a short exact sequence of R-modules, and let N be another R-module. Then there is a long exact sequence*

$$\cdots \to \mathrm{Tor}_n^R(A, N) \to \mathrm{Tor}_n^R(B, N) \to \mathrm{Tor}_n^R(C, N) \to \mathrm{Tor}_{n-1}^R(A, N) \to \cdots$$

$$\cdots \to \mathrm{Tor}_1(C, N) \to A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0.$$

Problem A.36. Prove Theorem A.35.

Problem A.37.

(a) Show that if $0 \to A \to B \to C \to 0$ is exact and $\mathrm{Tor}_1^R(C, N) = 0$, then $0 \to A \otimes N \to B \otimes N \to C \otimes N \to 0$ is exact as well. Is the converse true?

(b) Show that the functor $? \otimes N$ is exact if and only if $\mathrm{Tor}_1^R(M, N) = 0$ for all M.

(c) Show that if $\mathrm{Tor}_1^R(M, N) = 0$ for all M, then $\mathrm{Tor}_n^R(M, N) = 0$ for all M and all $n \geq 1$.

Finally we observe that if R is a **principal ideal domain** (almost always abbreviated to **PID**), the situation with Tor is substantially simpler because every module has a short projective resolution.

Problem A.38. Let R be a PID.

(a) Show that if M is an R-module, then there is a surjection $F \to M$ where F is free.

(b) Show that every submodule of a free module is free.

(c) Show that every R-module M has a projective resolution of the form $0 \to F_1 \to F_0 \to M \to 0$ with F_0 and F_1 free.

(d) Show that $\mathrm{Tor}_n^R(M, N) = 0$ for $n \geq 2$.

Since the 'higher-dimensional' Tor groups are all trivial when R is a PID, it is common practice to abbreviate the notation $\mathrm{Tor}_1^R(M, N)$ to $\mathrm{Tor}^R(M, N)$ or even just $\mathrm{Tor}(M, N)$ in this case.

A.5.2. Injective Resolutions and $\mathrm{Ext}_R^n(\,?\,,\,?\,)$. Next we make an analogous study based on the adjoint functor $\mathrm{Hom}_R(\,?\,,\,?\,)$. We say that an R-module J is **injective** if in any diagram of the form

$$\begin{tikzcd} & A \arrow[dl, dotted, "\theta"'] \\ J & B \arrow[l, "g"] \arrow[u] \\ & 0 \arrow[u] \end{tikzcd}$$

where the column is exact, the map θ can be found to make the triangle commute. An **injective resolution** is a quasi-isomorphism

$$\begin{tikzcd} N \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ J^0 \arrow[r] & J^1 \arrow[r] & \cdots \arrow[r] & J^{n-1} \arrow[r] & J^n \arrow[r] & \cdots \end{tikzcd}$$

in which each J^n is injective.

Theorem A.39.

(a) *The category $\mathbf{Mod}_R$ of R-modules has enough injectives: if $M \in \mathbf{Mod}_R$, then there is an injective J and an injective map $M \hookrightarrow J$.*

(b) *Every R-module M has an injective resolution.*

(c) *If $M \to I^*$ and $N \to J^*$ are injective resolutions and $f : M \to N$, then there is a chain map (which is unique up to chain homotopy) $\phi : I^* \to J^*$ making*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \arrow[d] \\ I^* \arrow[r, "\phi"] & J^* \end{tikzcd}$$

commute.

Project A.40. Prove Theorem A.39.

Then for any modules M and N we find an injective resolution $N \to J^*$ and form the chain complex $\mathrm{Hom}_R(M, J^*)$ and define

$$\mathrm{Ext}^n_R(M, n) = H^n(\mathrm{Hom}_R(M, J^*)).$$

Problem A.41. Show that $\mathrm{Ext}^n_R(\,?\,,\,?\,)$ is functorial, and that $\mathrm{Ext}^0_R(\,?\,,\,?\,)$ is naturally equivalent to $\mathrm{Hom}_R(\,?\,,\,?\,)$. What does Ext^n_R tell you about exactness?

Using Projectives to Compute Ext^n_R. Interestingly, while the tensor product $\,?\otimes_R ?\,$ is naturally a commutative operation, the functor $\mathrm{Hom}_R(\,?\,,\,?\,)$ depends very differently on the two variables. In particular, (target-type) injective resolutions are utterly inappropriate for the domain variable.

Problem A.42. Show that if $P_* \to M$ is a projective resolution, then there are canonical isomorphisms

$$H^n(\mathrm{Hom}_R(P_*, N)) \xrightarrow{\cong} \mathrm{Ext}^n_R(M, N).$$

Injective Abelian Groups. It is possible to give a complete list of the injective abelian groups (which are also known as **divisible groups**).

Theorem A.43. *If $J \in \mathcal{AB}\mathcal{G}$ is injective, then J is a sum of copies of $\mathbb{Q}$ and groups of the form $\mathbb{Z}/p^\infty$ for p prime.*

Project A.44. Prove Theorem A.43.

We'll use Theorem A.43 to prove Lemma 20.8 from Section 20.3.

Problem A.45. Let p be a prime and let G be an abelian group with $g \in G$ of infinite order.

(a) Show that there is a homomorphism $f : G \to \mathbb{Q}$ such that $f(g) \neq 0$.

(b) Let $q \in \mathbb{Q}$. Show that there is a homomorphism $h : \mathbb{Q} \to \mathbb{Z}/p$ such that $h(q) \neq 0$.

(c) Deduce Lemma 20.8.

A.5.3. Algebraic Künneth and Universal Coefficients Theorems. In this section we use the functors $\otimes_R$ and Tor_1^R to relate $H_*(\mathcal{C}_* \otimes \mathcal{D}_*)$ to $H_*(\mathcal{C}_*)$ and $H_*(\mathcal{D}_*)$. We begin by defining an algebraic version of the external product. If $a \in \mathcal{C}_*$ and $d(a) = 0$, then the coset of a defines a homology class, which we denote $[a] \in H_*(\mathcal{C}_*)$.

Problem A.46. Show that the rule $[c] \otimes [d] \mapsto [c \otimes d]$ gives a well-defined **algebraic Künneth map** $\kappa : H_*(\mathcal{C}_*) \otimes H_*(\mathcal{D}_*) \to H_*(\mathcal{C}_* \otimes \mathcal{D}_*)$.

The purely algebraic Künneth formula for the homology of a tensor product of chain complexes involves the derived functor Tor_1^R, which is defined for graded R-modules by the formula

$$\left(\mathrm{Tor}_1^R(A_*, B_*)\right)_n = \bigoplus_{i+j=n} \mathrm{Tor}_1^R(A_i, B_j).$$

Theorem A.47. *Let $\mathcal{C}_*$ and $\mathcal{D}_*$ be chain complexes of R-modules, where R is a PID. Assume that at least one of $\mathcal{C}_*$ and $\mathcal{D}_*$ is a chain complex of free R-modules. Then there is a short exact sequence*

$$0 \to H_*(\mathcal{C}_*) \otimes H_*(\mathcal{D}_*) \xrightarrow{\kappa} H_*(\mathcal{C}_* \otimes \mathcal{D}_*) \longrightarrow \Sigma\,\mathrm{Tor}(H^*(\mathcal{C}_*), H^*(\mathcal{D}_*)) \to 0$$

in which the first map is the algebraic external product. The sequence is natural in both variables. If both complexes are free, then the sequence splits, but splitting is not natural.

A.6. Hopf Algebras

In this section we develop the basic theory of Hopf algebras. Hopf algebras are, depending on your point of view, cogroup objects in the category of algebras, or group objects in the category of coalgebras. They occur naturally as the homology (or cohomology) of an H-space, and it was in the study of H-spaces that H. Hopf first introduced them.

A.6.1. Coalgebras. We begin by establishing the necessary definitions and constructions for coalgebras. A **coalgebra** can be defined abstractly as the dual of an algebra. Explicitly, a coalgebra A over a ring R is an R-module together with maps

$$\delta : A \longrightarrow A \otimes A \quad \text{(diagonal)} \qquad \text{and} \qquad \eta : A \longrightarrow R \quad \text{(counit)}$$

making the diagrams

$$\begin{tikzcd} & & A \arrow[lld, bend right, "\mathrm{id}"'] \arrow[d, "\delta"] \arrow[rrd, bend left, "\mathrm{id}"] & & \\ A & A \otimes R \arrow[l, "\cong"'] & A \otimes A \arrow[l, "\eta\otimes\mathrm{id}"'] \arrow[r, "\mathrm{id}\otimes\eta"] & R \otimes A \arrow[r, "\cong"'] & A \end{tikzcd}$$

and

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "\delta"'] & A \otimes A \arrow[d, "\mathrm{id}\otimes\delta"] \\ A \otimes A \arrow[r, "\delta\otimes\mathrm{id}"] & A \otimes A \otimes A \end{tikzcd}$$

commute. The diagonal δ is also called a **comultiplication**. A coalgebra A is **cocommutative** if the diagram

$$\begin{tikzcd} & A \arrow[dl, "\delta"'] \arrow[dr, "\delta"] & \\ A \otimes A \arrow[rr, "T"] & & A \otimes A \end{tikzcd}$$

commutes, where T is the twist map. A **coalgebra homomorphism** is a map $f : A \to B$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\delta_A"'] & B \arrow[d, "\delta_B"] \\ A \otimes A \arrow[r, "f\otimes f"] & B \otimes B \end{tikzcd}$$

commutes. All of these definitions can be applied in the graded context without essential change.

Problem A.48. Show that if A is a coalgebra, then for any $x \in X$, $\delta(x)$ has the form

$$\delta(x) = x \otimes 1 + (\textstyle\sum_i a_i \otimes b_i) + 1 \otimes x.$$

An element $x \in A$ is called **primitive** if $\delta(x) = x \otimes 1 + 1 \otimes x$. The primitive elements are closed under addition and scalar multiplication, so they constitute a submodule of **primitives** $P(C) \subseteq C$, which is the dual notion to the module of decomposables of an algebra.

Tensor Products of Coalgebras. If A and B are coalgebras, then we can form their tensor product $A \otimes B$, which is automatically an R-module. But we can give it the structure of a coalgebra, too. The diagonal is given by the composite

$$A \otimes B \xrightarrow{\delta_A \otimes \delta_B} (A \otimes A) \otimes (B \otimes B) \xrightarrow{\mathrm{id}_A \otimes T \otimes \mathrm{id}_B} (A \otimes B) \otimes (A \otimes B)$$

and counit

$$A \otimes B \xrightarrow{\eta_A \otimes \eta_B} R \otimes R \xrightarrow{\cong} R.$$

A.6.2. Hopf Algebras. A **Hopf algebra** is a (graded) R-module which is simultaneously an algebra and a coalgebra; furthermore, the multiplication $A \otimes A \to A$ must be a coalgebra map and the comultiplication $A \to A \otimes A$ must be an algebra map. Explicitly, let A be an R-algebra with multiplication μ and an R-coalgebra with comultiplication δ. Then $A \otimes A$ inherits a multiplication and a comultiplication, as detailed earlier. For A to be a Hopf algebra, we require

$$\mu : A \otimes A \longrightarrow A \qquad \text{and} \qquad \delta : A \longrightarrow A \otimes A$$

to be a coalgebra map and an algebra map, respectively.

Classification. Hopf algebras have so much structure that there are comparatively few of them. Thus they can be classified for many rings.

Theorem A.49 (Hopf)**.** *Let A^* be a finite-dimensional, graded-commutative, graded-cocommutative Hopf algebra over a field of characteristic* 0*. Then A (as an algebra) is a free exterior algebra with generators of odd degree.*

Exercise A.50. Find examples of Hopf algebras of the kind in Theorem A.49 having isomorphic algebra structures but nonisomorphic coalgebra structures.

There are also characterizations for characteristic p, but they are more complicated, and we won't make use of them.

A.6.3. Dualization of Hopf Algebras. The *description* of Hopf algebras is self-dual, so it should come as no surprise that the (algebraic) dual of a Hopf algebra A is again a Hopf algebra, provided, of course, that A is of finite type.

Proposition A.51. *If A is a finite-type Hopf algebra over a field $\mathbb{F}$, then its dual $A^* = \mathrm{Hom}_{\mathbb{F}}(A, \mathbb{F})$ is also a Hopf algebra.*

Project A.52. Investigate the indecomposables and the primitives in Hopf algebras and their duals.

Important Dualizations. Some of our computations of homology or cohomology algebras proceed by first computing the dual Hopf algebra, and then deriving the required Hopf algebra by purely algebraic dualization. Here are two dualization rules that are frequently useful.

First we consider the dual of a polynomial algebra.

Theorem A.53. *Let A be the commutative Hopf algebra $A = R[x_1, x_2, \ldots]$ with diagonal*

$$\delta(x_k) = \textstyle\sum_{i+j=k} x_i \otimes x_j.$$

Then the dual A^* is the polynomial algebra $R[y_1, y_2, \ldots]$ where

- $y_i^* = (x_1^i)^*$ *and*
- $\mu^*(y_k) = \sum_{i+j} y_i \otimes y_j$ *(where we interpret $y_0 = 1$).*

Now we determine the duals of exterior algebras.

Theorem A.54. *Let $A = \Lambda(x_1, x_2 \ldots)$ be a Hopf algebra that is free and of finite type, with a simple system of primitive generators. Then the dual A^* is the exterior algebra $\Lambda(x_1^*, x_2^*, \ldots)$ where x_k^* is dual to x_k with respect to the basis $\{x_{i_1} \cdots x_{i_k} \mid i_1 < \cdots < i_k\}$.*

Project A.55. Prove Theorems A.53 and A.54.

A.7. Symmetric Polynomials

The symmetric group $\mathrm{Sym}(n)$ acts on the ring $R[t_1, \ldots, t_n]$ by the rule

$$\sigma \cdot f(t_1, \ldots, t_n) = f(t_{\sigma(1)}, \ldots, t_{\sigma(n)}).$$

Inside of $R[t_1, \ldots, t_n]$ is the subring consisting of all those polynomials $f(t_1, \ldots, t_n)$ which are not changed when the variables are permuted, i.e., those polynomials for which $\sigma \cdot f = f$ for all $\sigma \in \mathrm{Sym}(n)$. The standard notation for this subring is

$$R[t_1, \ldots, t_n]^{\mathrm{Sym}(n)},$$

and it is known as the ring of **symmetric polynomials**.

Here are some examples of symmetric polynomials:

$$\begin{aligned} \sigma_1(t_1, t_2, t_3) &= t_1 + t_2 + t_3, \\ \sigma_2(t_1, t_2, t_3) &= t_1 t_2 + t_2 t_3 + t_1 t_3, \\ \sigma_3(t_1, t_2, t_3) &= t_1 t_2 t_3. \end{aligned}$$

These examples are part of a general pattern: the k^{th} **elementary symmetric polynomial** on variables $t_1, t_2, \ldots, t_n$ is

$$\sigma_k(t_1, t_2, \ldots, t_n) = \sum_{i_1 < \cdots < i_k} t_{i_1} t_{i_2} \cdots t_{i_k}.$$

Problem A.56.

(a) What is the degree of $\sigma_k(t_1, \ldots, t_n) \in H^*(BD(n); R) \cong R[t_1, \ldots, t_n]$?

(b) Show that the map $R[t_1, \ldots, t_n] \to R[t_1, \ldots, t_{n-1}]$ given by $t_k \mapsto t_k$ for $k < n$ and $t_n \mapsto 0$ carries $\sigma_k(t_1, \ldots, t_n)$ to $\sigma_k(t_1, \ldots, t_{n-1})$ for $k < n$ and kills σ_n.

We'll make use of the following theorem of pure algebra.

Theorem A.57. *If R is an integral domain, then*

$$R[t_1, \ldots, t_n]^{\mathrm{Sym}(n)} = R[\sigma_1, \ldots, \sigma_n],$$

the polynomial ring generated by $\sigma_1, \sigma_2, \ldots, \sigma_n$.

Project A.58. Prove Theorem A.57.

The polynomial $s_k(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^n$ is symmetric, and so Theorem A.57 implies that s_k is a polynomial combination of the elementary symmetric functions:

$$s_k = Q_k(\sigma_1, \ldots, \sigma_n).$$

The polynomial Q_k is known as the k^{th} **Newton polynomial**.

Problem A.59. Write out the first few Newton polynomials.

A.8. Sums, Products and Maps of Finite Type

The category of R-modules has a strange and wonderful property: for any two R-modules the comparison map $M \oplus N \to M \times N$ from the sum to the product is an isomorphism. This makes the matrix representation of morphisms described in Problem 1.50 especially powerful, since compositions are accurately modeled by matrix multiplications. In this section we study a condition that will enable us to find and use matrix representatives for maps between infinite products.

The difference between infinite sums and infinite products of R-modules is in how many nontrivial entries are allowed in a long tuple: in a sum, only finitely many nonzero entries are allowed, but there is no restriction for products. We are accustomed to thinking of $(x, 0)$ and $(0, y) \in M \oplus N$ as $x \in M \subseteq M \oplus N$ and $y \in N \subseteq M \oplus N$, so that $(x, y) = x + y$. Thus we see that in an infinite product, certain 'infinite sums' are permitted; but the value of a homomorphism on an infinite sum is not determined by the values on the summands. The composition of $f : \prod A_j \to \prod B_i$ with the comparison map $\bigoplus A_j \to \prod A_j$ is a map from a sum to a product, so it can be represented by a matrix $A(f)$ whose (i, j)-entry is the **coordinate map** f_{ij} defined by the diagram

$$\begin{tikzcd}
A_j \arrow[r, "f_{ij}"] \arrow[d, "\mathrm{in}_j"'] & B_i \\
\prod_{\mathcal{J}} A_j \arrow[r, "f"] & \prod_{\mathcal{I}} B_i. \arrow[u, "\mathrm{pr}_i"']
\end{tikzcd}$$

Unfortunately, it can happen that different maps f have the same matrix.

Let $\mathcal{C}$ be a category (such as $\mathbf{Mod}_R$) in which the comparison maps from finite sums to finite products are isomorphisms. We say that a morphism $f : \prod X_i \to \prod Y_j$ in $\mathcal{C}$ is of **finite type** if each $\mathrm{pr}_j \circ f$ depends only on finitely many entries in the domain. More precisely, f is of finite type if for each $i \in \mathcal{I}$ there is a finite subset $\mathcal{Q}_i \subseteq \mathcal{J}$ such that the diagram

$$\begin{tikzcd} \prod_{\mathcal{J}} A_j \arrow[r, "f"] \arrow[d, "q"'] & \prod_{\mathcal{I}} B_i \arrow[d, "\mathrm{pr}_i"] \\ \prod_{\mathcal{Q}_i} A_j \arrow[r] & B_i \end{tikzcd}$$

commutes.[4]

Problem A.60. Let $f : \prod_{\mathcal{J}} A_j \to \prod_{\mathcal{I}} B_i$ be a morphism of finite type.

(a) Show that each row of the matrix $A(f)$ has only finitely many nonzero entries.

(b) Show that there is a unique morphism ϕ making the square

$$\begin{tikzcd} \bigoplus X_i \arrow[r, "\phi"] \arrow[d] & \bigoplus Y_j \arrow[d] \\ \prod X_i \arrow[r, "f"] & \prod Y_j \end{tikzcd}$$

commutative.

Getting back to R-modules, we see that the finite-type condition gives *a way* (not the only way) to make sense of the image of a special infinite sum: since the infinite sum of the images is actually a finite sum in each coordinate, we can simply add them up! Thus each finite-type matrix is the matrix of a map (more than one, probably) between products.

Now we come to our main result on finite-type maps.

Theorem A.61. *Let $\mathcal{C}$ be a category in which the comparison maps from finite sums to finite products are isomorphisms.*

(a) If $f, g : \prod_{\mathcal{J}} A_j \to \prod_{\mathcal{I}} B_i$ are both of finite type, then $f = g$ if and only if $A(f) = A(g)$.

(b) If f and g are composable and of finite type, then $A(f \circ g) = A(f) \cdot A(g)$, where the product on the right is matrix multiplication.

Problem A.62. Prove Theorem A.61.

A.9. Ordinal Numbers

Ordinal numbers are equivalence classes of linearly ordered sets, and they can be constructed inductively by the rule $\alpha = \bigcup_{\beta<\alpha} \beta$, starting with $0 = \varnothing$. Thus we have

$$0 = \varnothing, \quad 1 = \{\varnothing\}, \quad 2 = \{\varnothing, \{\varnothing\}\}, \quad \text{and so on.}$$

[4]Note that if $\mathcal{Q}$ does the job, then so does any finite set $\mathcal{R}$ containing $\mathcal{Q}$.

The first infinite ordinal is $\omega = \{0, 1, \ldots, n, n+1, \ldots\}$ and its successor is $\omega + 1 = \{0, 1, \ldots, n, n+1, \ldots, \omega\}$. Each ordinal is either a **successor ordinal** or a **limit ordinal**. If we forget the order in an ordinal, we are left with a bare set which has a cardinality. Thus the cardinality of the ordinal n is the number n, and the cardinality of ω is $\aleph_0$.

Let's consider maps of (not necessarily ordered) sets S into a limit ordinal α. For example, if S is finite, then every map $S \to \omega$ must factor through some finite ordinal $\beta < \omega$; but we cannot limit the (finite) cardinality of β. More interesting is the first uncountable ordinal Ω. If S is a countable set, then every map $S \to \Omega$ must also factor through some countable ordinal $\beta < \Omega$. In general, each limit ordinal α has a **cofinality**, which is a cardinal $\mathfrak{a}$ such that every map $S \to \alpha$ with cardinality at most $\mathfrak{a}$ must factor through an ordinal $\beta < \alpha$.

Theorem A.63. *For each cardinal $\mathfrak{a}$ there is an ordinal α whose cofinality exceeds $\mathfrak{a}$.*

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This text develops classical homotopy theory from a modern point of view, meaning that the exposition is informed by the theory of model categories and that homotopy limits and colimits play central roles. The exposition is guided by the principle that it is generally preferable to prove topological results using topology (rather than algebra). The language and basic theory of homotopy limits and colimits make it possible to penetrate deep into the subject with just the rudiments of algebra. The text does reach advanced territory, including the Steenrod algebra, Bott periodicity, localization, the Exponent Theorem of Cohen, Moore, and Neisendorfer, and Miller's Theorem on the Sullivan Conjecture. Thus the reader is given the tools needed to understand and participate in research at (part of) the current frontier of homotopy theory. Proofs are not provided outright. Rather, they are presented in the form of directed problem sets. To the expert, these read as terse proofs; to novices they are challenges that draw them in and help them to thoroughly understand the arguments.

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