All of the manifolds that we explicitly discussed admit CW-structures:

- S^n, $\mathbb{RP}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$ explicitly mentioned
- $V_{k,n}$, $Gr_{k,n}$: CW-structures exist.

In all these cases: manifold dimension $=$ CW-dimension

This is no coincidence: suppose that a compact manifold M admits a CW-structure; let $x \in M$ be an interior point of a cell of top dimension (in the CW-structure). Then the open cell is an open neighborhood homeomorphic to $\mathbb{R}^n$ with $n =$ CW-dimension of X. Since the manifold dimension is intrinsic, $n =$ manifold dimension.

Cor: Let M be a compact n-manifold that admits a CW-structure. Then $H_i(M;A) \cong H_i^{cell}(M;A) = 0$ for $i > n$.

Warning: compact manifolds do not in general admit CW-structures.
but: every smooth compact manifold admits a triangulation, and hence a CW-structure $\Big\}$ difficult facts

Thm: Let M be an n-manifold, A an abelian group and K a compact subset of M. Then

i) $H_i(M, M\setminus K; A) = 0$ for $i > n$.

ii) A class in $H_n(M, M\setminus K; A)$ is zero if and only if its restriction to $H_n(M, M\setminus\{x\}; A)$ is zero for all $x \in K$.

Note: M need not be compact. But if M is compact, then $K = M$ is allowed, and then both statements refer to the absolute homology of M.

Proof: Proceed in 6 steps.
(The proof follows the proof of Lemma A.7 in Appendix A of Milnor-Stasheff`s book "Characteristic Classes")

Step 1: $M = \mathbb{R}^n$, K is a compact convex nonempty subset. For every $x \in K$, K can be linearly contracted onto x.
Let $R > 0$ be large enough so that $K \subseteq S^{n-1}_R = \{y \in \mathbb{R}^n : |x| = R\}$. Then the inclusions
$$S^{n-1}_{2R} \subseteq M\setminus K \subseteq M\setminus\{x\}$$
are homotopy equivalences.

So they induce isomorphisms
$$\begin{array}{ccc} H_i(M|K) & \longrightarrow & H_i(M|x) \\ \| & & \| \\ H_i(M, M\setminus K, A) & & H_i(M, M\setminus\{x\}; A) \end{array}$$
is an isomorphism.

In particular, $H_i(M|K) \cong H_i(M|x) \cong 0$ for $i > n$.

Step 2: M any n-manifold, $K = K_1 \cup K_2$ for K_1, K_2 compact; suppose the statements are true for K_1, K_2 and $K_1 \cap K_2$.
Then the statements also hold for K.
We a long exact Mayer-Vietoris sequence for the local homology groups.

Construction:
$$M\setminus(K_1\cap K_2) = \underbrace{(M\setminus K_1)}_{\text{open}} \cup \underbrace{(M\setminus K_2)}_{\text{open}} \quad \text{and} \quad (M\setminus K_1)\cap(M\setminus K_2) = M\setminus(K_1\cup K_2) = M\setminus K$$

The theorem of small simplices shows that the map
$$\begin{array}{ccc} \dfrac{C_*(M\setminus K_1)\oplus C_*(M\setminus K_2)}{C_*(M\setminus K)} & \xrightarrow{\simeq} & C_*(M\setminus(K_1\cap K_2)) \\ & & \downarrow \\ & & C_*(M) \end{array}$$
is an isomorphism of all homology groups.

So also the chain map
$$D := \frac{C_*(M)}{\left(\dfrac{C_*(M\setminus K_1)\oplus C_*(M\setminus K_2)}{C_*(M\setminus K)}\right)} \xrightarrow{\simeq} \frac{C_*(M)}{C_*(M\setminus(K_1\cap K_2))}$$
is also a quasi-isomorphism.

The source is a short exact sequence of chain complexes:
$$0 \to \frac{C_*(M)}{C_*(M\setminus K)} \longrightarrow \frac{C_*(M)}{C_*(M\setminus K_1)} \oplus \frac{C_*(M)}{C_*(M\setminus K_2)} \longrightarrow D \longrightarrow 0$$

This yields a long exact sequence of homology groups:
$$\cdots \to \underset{\overset{\|}{0}}{H_{i+1}(M|K_1\cap K_2)} \xrightarrow{\partial} H_i(M|K) \longrightarrow \underset{\overset{\|}{0}}{H_i(M|K_1)} \oplus \underset{\overset{\|}{0}}{H_i(M|K_2)} \longrightarrow \underset{\overset{\|}{0}}{H_i(M|K_1\cap K_2)} \to \cdots$$
For $i > n$

So $H_i(M|K) = 0$ for $i > n$.

For $i = n$: exact sequence
$$0 \longrightarrow H_n(M|K) \xrightarrow{(r^K_{K_1}, r^K_{K_2})} H_n(M|K_1) \oplus H_n(M|K_2)$$

So if $\alpha \in H_n(M|K)$ satisfies $r^K_x(\alpha)=0$ in $H_n(M|x)$ for all $x \in K$.

Then $r^{K_1}_x(r^K_{K_1}(\alpha)) = r^K_x(\alpha) = 0$ for all $x \in K_1$.

Since we assumed that K_1 satisfies the theorem, this means that $r^K_{K_1}(\alpha)=0$. Similarly, $r^K_{K_2}(\alpha)=0$.

So $\alpha = 0$ by the injectivity of $(r^K_{K_1}, r^K_{K_2})$.

Step 3: $M=\mathbb{R}^n$, $K = K_1 \cup \dots \cup K_m$ for $K_1,\dots,K_m$ convex compact subsets.

We argue by induction on m. The case $m=1$ was step 1. Suppose that $m>1$.

Then $K = (K_1 \cup \dots \cup K_{m-1}) \cup K_m$, $\quad (K_1\cup\dots\cup K_{m-1}) \cap K_m = (K_1 \cap K_m) \cup \dots \cup (K_{m-1}\cap K_m)$

each is convex compact

Statement is true by induction

So the claim is true for $K = (K_1 \cup\dots\cup K_{m-1}) \cup K_m$ by step 2.

Step 4: $M = \mathbb{R}^n$, K any compact subset of $\mathbb{R}^n$. Let $\alpha \in H_i(\mathbb{R}^n|K)$ be any class.

Claim: There is a compact neighborhood N of K and a class $\alpha' \in H_i(\mathbb{R}^n|N)$ such that $r^N_K(\alpha')=\alpha$.

Proof: We represent $\alpha = [x + C_i(\mathbb{R}^n\setminus K)]$ for some chain $x \in C_i(\mathbb{R}^n)$ with $d_i(x) \in C_{i-1}(\mathbb{R}^n \setminus K)$.

Then $d_i(x) = \sum_{\text{finite}} a_j \cdot (f_j : \Delta^{i-1} \to \mathbb{R}^n\setminus K)$ for some $a_j \in A$, f_j continuous.

Then $L = \text{support}(d_i(x)) = \bigcup f_j(\Delta^{i-1})$ a compact subset of $\mathbb{R}^n \setminus K$.

Since L and K are disjoint compact subsets of $\mathbb{R}^n$, there is a compact neighborhood N of K in $\mathbb{R}^n$ that is still disjoint from L. Then $d_i(x) \in C_{i-1}(L) \subseteq C_{i-1}(\mathbb{R}^n\setminus N)$.

So the class $\alpha' = [x + C_i(\mathbb{R}^n \setminus N)] \in H_i(\mathbb{R}^n|N)$ satisfies $r^N_K(\alpha') = \alpha$.

Proof of Step 4: Since N is a neighborhood of K, each $x\in K$ has a metric open ball around it that is contained in N. Since K is compact, finitely many of these balls cover K. Since N is closed, these closed balls will belong to N.

So $K \subseteq B_1 \cup\dots\cup B_m \subseteq N$, where each B_i is a convex compact subset of $\mathbb{R}^n$.

Now suppose $i>n$ and $\alpha \in H_i(\mathbb{R}^n|K)$. Let $\alpha' \in H_i(\mathbb{R}^n|N)$ satisfy $r^N_K(\alpha')=\alpha$.

By step 3 $r^N_{B_1\cup\dots\cup B_m}(\alpha')=0$, hence also $\alpha = r^{B_1\cup\dots\cup B_m}_K(r^N_{B_1\cup\dots\cup B_m}(\alpha')) = 0$.

Now let $i=n$ and let $\alpha \in H_n(\mathbb{R}^n|K)$ be such that $r^K_x(\alpha)=0$ for all $x\in K$.

Then $\beta = r^N_{B_1\cup\dots\cup B_m}(\alpha')$ has the property that it restricts to 0 in $H_n(\mathbb{R}^n|B_j)$ for all $j=1,\dots,m$, because each B_j contains at least one point in K. So $\beta=0$ by step 3, and hence also

$$\alpha = r^{B_1\cup\dots\cup B_m}_K(\beta) = 0.$$

Step 5: There is an open neighborhood U of K in M and a homeomorphism $\varphi: U \xrightarrow{\cong} \mathbb{R}^n$.

Then we contemplate the commutative diagram

$$\begin{tikzcd}
H_i(M|K) \arrow[ddrr, bend right, "r^K_x"'] & & H_i(U|K) \arrow[ll, "\cong"', "\text{excision}"] \arrow[rr, "\cong", "\varphi_*"'] \arrow[dd, "r^K_x"] & & H_i(\mathbb{R}^n|\varphi(K)) \arrow[dd, "r^{\varphi(K)}_{\varphi(x)}"] \arrow[r, phantom, "\cong"] & \begin{cases} 0 & \text{for } i>n \end{cases} \\
& & & & & \\
& & H_i(U|x) \arrow[rr, "\varphi_*", "\cong"'] & & H_i(\mathbb{R}^n|\varphi(x)) &
\end{tikzcd}$$

$i=n$ these maps detect as $\varphi(x)$ varies in $\varphi(K)$.

$\Rightarrow$ $H_i(M|K)=0$ for $i>n$

and r^K_x detect classes in $H_n(M|K)$ for all $x\in K$.

Step 6: M any n-manifold, K any compact subset of M.

Claim: We can write $K = K_1 \cup\dots\cup K_m$ such that each K_i is compact and has an open neighborhood homeomorphic to $\mathbb{R}^n$.

Proof: Each $x\in K$ has an open nbh homeomorphic to $\mathbb{R}^n$. So inside this there is a compact neighborhood N_x of x.

$K \subseteq \bigcup_{x\in K} N_x$, since K is compact, $K \subseteq N_{x_1}\cup\dots\cup N_{x_m}$ for some $x_1,\dots,x_m \in K$.

So $K = (K\cap N_{x_1}) \cup\dots\cup (K\cap N_{x_m})$, each $K\cap N_{x_i}$ is compact and contained in an open subset [illegible] $_{x \in K}$ [illegible] $_{x_1} \cup \dots \cup$ [illegible] $_{x_m}$ [illegible]

So $K = (K \cap N_{x_1}) \cup \dots \cup (K \cap N_{x_m})$, each $K \cap N_{x_i}$ is compact and contained in an open subset homeomorphic to $\mathbb{R}^n$.

Proof of Step 6: induction on m. For $m = 1$ we appeal to Step 5.

The inductive step for $m \geq 2$ is taken care of by Step 2. $\square$