Cup product

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Let X be a simplicial set and R a ring. Reminder: $C^n(X;R) = \text{map}(X_n, R)$ under pointwise addition

$d^n : C^n(X;R) \longrightarrow C^{n+1}(X;R)$ is given by

$$(d^n f)(y) = \sum_{i=0}^{n+1} (-1)^i \cdot f(d_i^* (y)) \qquad f: X_n \to R, \; y \in X_{n+1}$$

__Construction__ Let X be a simplicial set and R a ring.

The __cup product__ or __Alexander-Whitney map__ is the family of map

$$\cup : C^n(X;R) \times C^m(X;R) \longrightarrow C^{n+m}(X;R) \qquad \text{for } n,m \geq 0$$

is defined by

$$(f \cup g)(x) = f(d_{front}^*(x)) \cdot g(d_{back}^*(x)) \qquad f: X_n \to R, \; g: X_m \to R, \; x \in X_{n+m}$$

where

$$d_{front} : [n] \longrightarrow [n+m], \quad d_{front}(i) = i$$

$$d_{back} : [m] \longrightarrow [n+m], \quad d_{back}(i) = n+i$$

__Thm__: (i) The Alexander-Whitney map is biadditive and satisfies the relation

$$d(f \cup g) = d(f) \cup g + (-1)^n \cdot f \cup d(g) \qquad \text{for } f \in C^n(X,R), \; g \in C^m(X,R)$$

(ii) For every morphism $\alpha : Y \to X$ of simplicial sets, $f \in C^n(X;R)$, $g \in C^m(X;R)$, we have
(Naturality)

$$\alpha^*(f \cup g) = \alpha^*(f) \cup \alpha^*(g) \quad \text{in } C^{n+m}(Y,R)$$

(iii) (Associativity) If $h \in C^k(X,R)$ is another cochain, then $(f \cup g) \cup h = f \cup (g \cup h)$ in $C^{n+m+k}(X;R)$

(Unitality) and $1 \cup f = f = f \cup 1$, where $1 \in C^0(X;R)$ is the constant function with value $1 \in R$.

__Proof__: (i) Let $d_{front} : [n] \longrightarrow [n+m]$ and $d_{back} : [m] \longrightarrow [n+m]$ be as in the definition of $\cup$.

Then

$$(\times) \qquad d_i \circ d_{front} = \begin{cases} d_{front} \circ d_i & 0 \leq i \leq n+1 \\ d_{front} & n+1 \leq i \leq n+m+1 \end{cases} \qquad [n] \longrightarrow [n+m+1]$$

$$(*) \qquad d_i \circ d_{back} = \begin{cases} d_{back} & 0 \leq i \leq n \\ d_{back} \circ d_{i-n} & n \leq i \leq n+m+1 \end{cases} \qquad [m] \longrightarrow [n+m+1]$$

So for $x \in X_{n+m+1}$, we get

$$d(f \cup g)(x) = \sum_{i=0}^{n+m+1} (-1)^i (f \cup g)(d_i^*(x)) = \sum_{i=0}^{n+m+1} (-1)^i f(d_{front}^*(d_i^*(x))) \cdot g(d_{back}^*(d_i^*(x)))$$

$(i = j+n)$

$$= \sum_{i=0}^{n} (-1)^i f(d_{front}^*(x)) \cdot g(d_{back}^*(x)) + \sum_{j=1}^{m+1} (-1)^{n+j} f(d_{front}^*(d_{j+n}^*(x))) \cdot g(d_{back}^*(d_{j+n}^*(x)))$$

terms $i = n+1$ and $j = 0$ cancel!

$$\overset{(\times)}{=} \sum_{i=0}^{n+1} (-1)^i f(d_i^*(d_{front}^*(x))) \cdot g(d_{back}^*(x)) + \sum_{j=0}^{m+1} (-1)^{n+j} f(d_{front}^*(x)) \cdot g(d_j^*(d_{back}^*(x)))$$

$$= d(f)(d_{front}^*(x)) \cdot g(d_{back}^*(x)) + (-1)^n \cdot f(d_{front}^*(x)) \cdot d(g)(d_{back}^*(x))$$

$$= \left((d(f) \cup g) + (-1)^n \cdot (f \cup d(g)) \right)(x)$$

So $d(f \cup g) = (df \cup g) + (-1)^n \cdot (f \cup dg)$

(ii)

$\alpha : Y \to X$ morphism of simplicial sets, $f \in C^n(X;R)$, $g \in C^m(X;R)$, $y \in Y_{n+m}$

$$(\alpha^*(f \cup g))(y) = (f \cup g)(\alpha_{n+m}(y))$$

$$= f(d_{front}^*(\alpha_{n+m}(y))) \cdot g(d_{back}^*(\alpha_{n+m}(y)))$$

α morphism of simplicial sets

$$= f(\alpha_n(d_{front}^*(y))) \cdot g(\alpha_m(d_{back}^*(y)))$$

$$= \alpha^*(f)(d_{front}^*(y)) \cdot \alpha^*(g)(d_{back}^*(y)) = (\alpha^*(f) \cup \alpha^*(g))(y)$$

(iii)

$f \in C^n(X;R)$, $g \in C^m(X;R)$, $h \in C^k(X;R)$, $x \in X_{n+m+k}$

$$((f \cup g) \cup h)(x) = (f \cup g)(d_{front}^*(x)) \cdot h(d_{back}^*(x))$$

$$= f(d_{front}^*(d_{front}^*(x))) \cdot g(d_{back}^*(d_{front}^*(x))) \cdot h(d_{back}^*(x))$$

$$= f(d_{front}^*(x)) \cdot g(d_{middle}^*(x)) \cdot h(d_{back}^*(x)) \quad \text{where } d_{front} : [n] \longrightarrow [n+m+k], \; i \mapsto i$$

$$= \dots = (f \cup (g \cup h))(x)$$

$d_{middle} : [m] \longrightarrow [n+m+k]$, $i \longmapsto n+i$

$d_{back} : [k] \longrightarrow [n+m+k]$, $n+m+i$

Also, $(f \cup 1)(x) = f(d_{front}^*(x)) \cdot 1(d_{back}^*(x)) = f(x) \cdot 1 = f(x)$ because $d_{front} = Id : [n] \longrightarrow [n+0]$.

So $f \cup 1 = f$. Similarly, $1 \cup f = f$. $\square$

Def: A differential graded ring is a cochain complex $A = (A^n, d^n)_{n \in \mathbb{Z}}$ equipped with biadditive maps
$$\cdot : A^n \times A^m \longrightarrow A^{n+m} \quad \text{for all } n, m \in \mathbb{Z}$$
and an element $1 \in A^0$ such that:

- the product $\cdot$ is associative and has 1 as two sided unit.
- the Leibniz rule holds: $\quad d(a \cdot b) = (da) \cdot b + (-1)^n \cdot a \cdot (db)$ for $a \in A^n$, $b \in A^m$.

Example: For every simplicial set X and ring R, the cochain complex $C^*(X;R)$ is a differential graded ring under cup product.

Example: The de Rham complex of a smooth manifold is a differential graded ring under exterior product ($\wedge$ product) of differential forms.

Construction: Let $A = (A^n, d, \cdot)$ be a differential graded ring. The multiplication induces a well-defined product on cohomology
$$\cdot : H^n(A) \times H^m(A) \longrightarrow H^{n+m}(A), \quad [a] \cdot [b] = [a \cdot b]$$

This is well-defined: if $a \in A^n$, $b \in A^m$ are cocycles, then $d(a \cdot b) = \underbrace{(da)}_{=0} \cdot b + (-1)^n \cdot a \cdot \underbrace{(db)}_{=0} = 0$

so $a \cdot b$ is also a cocycle.

Suppose $x \in A^{n-1}$
$$(a + dx) \cdot b = a \cdot b + (dx) \cdot b = a \cdot b + d(x \cdot b)$$
so $[a \cdot b] = [(a+dx) \cdot b]$, so $[a \cdot b]$ only depends on the cohomology $[a]$ and not the representing cocycle; similarly in the other argument.

The associativity and unitality is inherited; note $\quad d(1) = d(1 \cdot 1) = (d1) \cdot 1 + 1 \cdot (d1) = 2 \cdot d(1)$

$\Rightarrow$ $0 = d(1)$, i.e. the multiplicative unit $1 \in A^0$ is a cocycle, and $[1] \in H^0 A$ is a multiplicative unit again.

Def: The cup product in the cohomology of a simplicial set X with coefficients in a ring R is defined by
$$\cup : H^n(X;R) \times H^m(X;R) \longrightarrow H^{n+m}(X;R), \quad [x] \cup [y] = [x \cup y].$$

Thm: Let X be a simplicial set and R a ring.

(i) The cup product in R-cohomology of X is associative and unital.

(ii) For a morphism of simplicial sets $\alpha : Y \longrightarrow X$, the relation
$$\alpha^*([x] \cup [y]) = \alpha^*[x] \cup \alpha^*[y].$$

Remark: There is also a generalization of the cup product to relative cohomology. Let $A, B \subseteq X$ be two simplicial subsets. Then the cup product restricts to a biadditive pairing
$$C^n(X, A; R) \times C^m(X, B; R) \longrightarrow C^{n+m}(X, A \cup B; R)$$

Recall: $C^n(X, A; R) = \{ f \in C^n(X;R) = map(X_n, R) \text{ such that } f(A_n) = \{0\} \}$

$f \in C^n(X, A; R)$, $g \in C^m(X, B; R)$, $\quad x \in (A \cup B)_{n+m} = A_{n+m} \cup B_{n+m}$, then
$$(f \cup g)(x) = f(d_{front}^*(x)) \cdot g(d_{back}^*(x)) = 0 \quad \text{because one of the two factors is } 0.$$

Taking $A = B$ gives a pairing $\quad C^n(X, A; R) \times C^m(X, A; R) \longrightarrow C^{n+m}(X, A; R)$
that is associative and satisfies the Leibniz rule, and it passes to cohomology.