Poincare duality versus cup product

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Def: Let A and B abelian groups. A biadditive map $\langle\ ,\ \rangle : A \times B \longrightarrow \mathbb{Z}$ is non-singular (or non-degenerate), if the "adjoint" maps

$$A \longrightarrow \operatorname{Hom}(B,\mathbb{Z}) \quad \text{and} \quad B \longrightarrow \operatorname{Hom}(A,\mathbb{Z}) \quad \text{are isomorphisms.}$$
$$a \longmapsto \langle a,-\rangle \qquad\qquad b \longmapsto \langle -,b\rangle$$

Remark: Because $\mathbb{Z}$ is torsion free, so are $\operatorname{Hom}(B,\mathbb{Z})$ and $\operatorname{Hom}(A,\mathbb{Z})$; so if there is such a non-singular pairing $A\times B \longrightarrow \mathbb{Z}$, then A and B are in particular torsion free.

We recall the evaluation homomorphism $\Phi : H^i(X;\mathbb{Z}) \longrightarrow \operatorname{Hom}(H_i(X,\mathbb{Z}),\mathbb{Z})$ from the universal coefficient theorem. This satisfies the relation

$$\Phi(\alpha)(\xi) = \varepsilon(\xi \cap \alpha)$$

for $\alpha \in H^i(X;\mathbb{Z})$, $\xi \in H_i(X;\mathbb{Z})$, where $\varepsilon : H_0(X;\mathbb{Z}) \longrightarrow \mathbb{Z}$ is the augmentation $\varepsilon[\Sigma a_i x_i] = \Sigma a_i$.

Let M be a compact oriented n-manifold. The biadditive pairing

$$H^i(M,\mathbb{Z}) \times H^{n-i}(M,\mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$(\alpha,\beta) \longmapsto \varepsilon(\underset{\in H_n(M;\mathbb{Z})}{[M]} \cap \underbrace{(\alpha\cup\beta)}_{H^n(M;\mathbb{Z})}) = \varepsilon(D_M(\alpha\cup\beta))$$

factors over a biadditive map

$$\langle\ ,\ \rangle : \frac{H^i(M,\mathbb{Z})}{\text{torsion}} \times \frac{H^{n-i}(M,\mathbb{Z})}{\text{torsion}} \longrightarrow \mathbb{Z}, \quad \text{the } \underline{\text{Poincaré duality}}\ \underline{\text{pairing}}. \qquad (*)$$

Theorem: Let M be a compact oriented n-manifold such that $H_i(M;\mathbb{Z})$ is finitely generated for all $i \geq 0$. Then the Poincaré duality pairing $(*)$ is non-singular.

Proof: The UCT gives an exact sequence

$$0 \to \underbrace{\operatorname{Ext}(H_{n-i-1}(M,\mathbb{Z}),\mathbb{Z})}_{\text{torsion}} \longrightarrow H^{n-i}(M,\mathbb{Z}) \xrightarrow{\ \Phi\ } \underbrace{\operatorname{Hom}(H_{n-i}(M;\mathbb{Z}),\mathbb{Z})}_{\text{torsion free}} \longrightarrow 0$$

So Φ factors over an isomorphism

$$\begin{tikzcd}
\frac{H^{n-i}(M,\mathbb{Z})}{\text{torsion}} \arrow[r, "\cong"] \arrow[ddr, bend right, "\beta \mapsto \langle -,\beta\rangle"'] & \operatorname{Hom}(H_{n-i}(M,\mathbb{Z}),\mathbb{Z}) \arrow[d, "\operatorname{Hom}(D_M,\mathbb{Z})", "\cong"'] \\
 & \operatorname{Hom}(H^i(M,\mathbb{Z}),\mathbb{Z}) \\
 & \operatorname{Hom}(\frac{H^i(M,\mathbb{Z})}{\text{torsion}},\mathbb{Z}) \arrow[u, "\cong"]
\end{tikzcd}$$

So the map $\beta \longmapsto \langle -,\beta\rangle$ is an isomorphism.

Because $\alpha\cup\beta = \pm\, \beta\cup\alpha$, the other pairing $\alpha \longmapsto \langle \alpha,-\rangle$ is also an isomorphism. □

Thm: Let $x \in H^2(\mathbb{C}P^n,\mathbb{Z})$ be a generator, $n \geq 1$. Then $H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$.

Proof: We argue by induction on n. For $n=1$, $\mathbb{C}P^1 \cong S^2$ the statement holds. Now suppose that $n \geq 2$.
The inclusion $\mathbb{C}P^{n-1} \longrightarrow \mathbb{C}P^n$ induces isomorphisms $H^k(\mathbb{C}P^n,\mathbb{Z}) \longrightarrow H^k(\mathbb{C}P^{n-1},\mathbb{Z})$ for $0 \leq k \leq 2n-2$.
These maps are multiplicative and send $x \in H^2(\mathbb{C}P^n,\mathbb{Z})$ to a generator in $H^2(\mathbb{C}P^{n-1},\mathbb{Z})$.
So the classes $1, x, x^2, \dots, x^{n-1}$ are generators of their respective cohomology groups because their restrictions to $\mathbb{C}P^{n-1}$ have this property. It remains to show that x^n generates $H^{2n}(\mathbb{C}P^n,\mathbb{Z})$.
Since the cohomology groups of $\mathbb{C}P^n$ are torsion free, the map (Poincaré duality pairing)

$$H^2(\mathbb{C}P^n,\mathbb{Z}) \xrightarrow{\ \cong\ } \operatorname{Hom}(H^{2n-2}(\mathbb{C}P^n;\mathbb{Z}),\mathbb{Z})$$
$$\alpha \longmapsto \{\beta \longmapsto \varepsilon([\mathbb{C}P^n] \cap (\alpha\cup\beta))\}$$

is an isomorphism.

The LHS is generated by x; the RHS is generated by the homomorphism $f : H^{2n-2}(\mathbb{C}P^n,\mathbb{Z}) \longrightarrow \mathbb{Z}$ with $f(x^{n-1}) = 1$.

So x is sent to $\pm f$, i.e.

$$\varepsilon([\mathbb{C}P^n] \cap x^n) = \pm 1$$

So $x^n \in H^{2n}(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}$ cannot be divisible by any $l \geq 2$, and hence x^n is a generator. □

Corollary: The cohomology ring of $\mathbb{CP}^\infty$ is $H^*(\mathbb{CP}^\infty;\mathbb{Z}) = \mathbb{Z}[x]$,

where $x \in H^2(\mathbb{CP}^\infty;\mathbb{Z})$ is a generator.

Proof: We must show that x^k generates $H^{2k}(\mathbb{CP}^\infty;\mathbb{Z})$ for all $k>0$. The inclusion $\mathbb{CP}^k \hookrightarrow \mathbb{CP}^\infty$ induces an isomorphism of cohomology rings up to dimension $2k$. So the claim follows from the previous calculation of $H^*(\mathbb{CP}^k;\mathbb{Z})$. □

Theorem Let M be a compact n-manifold. Then the mod-2 Poincaré duality pairing

$$H^i(M;\mathbb{F}_2) \times H^{n-i}(M;\mathbb{F}_2) \longrightarrow \mathbb{F}_2$$
$$(\alpha,\beta) \longmapsto \Phi(\alpha\cup\beta)(\nu_M) = \varepsilon(\nu_M \cap (\alpha\cup\beta))$$

is non-singular, i.e. the adjoint homomorphism

$\nu_M \in H_n(M;\mathbb{F}_2)$ mod-2 fundamental class.

$$H^i(M,\mathbb{F}_2) \longrightarrow H^{n-i}(M,\mathbb{F}_2)^*$$
$$H^{n-i}(M;\mathbb{F}_2) \longrightarrow (H^i(M,\mathbb{F}_2))^*$$ are isomorphisms of $\mathbb{F}_2$-vector spaces.

Proof: Similar as in $\mathbb{Z}$-version for oriented manifold. Using that

$$\Phi: H^i(M;\mathbb{F}_2) \longrightarrow \mathrm{Hom}_{\mathbb{F}_2}(H_i(M;\mathbb{F}_2);\mathbb{F}_2) = H_i(M;\mathbb{F}_2)^*$$

is always an isomorphism because taking dual vector spaces is exact. □

Theorem: Let $x \in H^1(\mathbb{RP}^n;\mathbb{F}_2)$ be the generator, $n\geq 1$. Then $H^*(\mathbb{RP}^n;\mathbb{F}_2) = \mathbb{F}_2[x]/(x^{n+1})$.

Also $H^*(\mathbb{RP}^\infty,\mathbb{F}_2) = \mathbb{F}_2[x]$, for $x\in H^1(\mathbb{RP}^\infty,\mathbb{F}_2)$ the generator.

Corollary: $H^*(\mathbb{RP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[y]/(2y)$, where $y \in H^2(\mathbb{RP}^\infty,\mathbb{Z})$ is the generator.

Proof: By comparing the cellular cochain complexes we see that reduction of coefficients $\mathbb{Z}\longrightarrow\mathbb{Z}/2$ induces isomorphisms $H^{2k}(\mathbb{RP}^\infty;\mathbb{Z}) \longrightarrow H^{2k}(\mathbb{RP}^\infty;\mathbb{F}_2)$. Because $\mathbb{Z}\longrightarrow\mathbb{Z}/2$ is a ring homomorphism, the coefficient reduction map is multiplicative. Since it sends $y\in H^2(\mathbb{RP}^\infty,\mathbb{Z})$ to $x^2\in H^2(\mathbb{RP}^\infty,\mathbb{F}_2)$ all of whose powers are non-zero, all powers of y are non-zero, too. □