WORKSHOP SHEET 1

ALGEBRAIC GEOMETRY [MATH11120]

Try to do some exercises before tutorial.

Each exercise has $(\star)$-part. They are for self-study only.

Exercise 1. Let Σ be a subset in $\mathbb{P}^2_{\mathbb{C}}$ such that Σ is not contained in one line in $\mathbb{P}^2_{\mathbb{C}}$.

(a) Let $[a_{11} : a_{12} : a_{13}]$, $[a_{21} : a_{22} : a_{23}]$, and $[a_{31} : a_{32} : a_{33}]$ be points in $\mathbb{P}^2_{\mathbb{C}}$. Prove that these three points are contained in one line in $\mathbb{P}^2_{\mathbb{C}}$ if and only if the determinant of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is zero. Similarly, show the determinant of this matrix is zero if and only if the lines $a_{11}x + a_{12}y + a_{13}z = 0$, $a_{21}x + a_{22}y + a_{23}z = 0$ and $a_{31}x + a_{32}y + a_{33}z = 0$ all pass through one point in $\mathbb{P}^2_{\mathbb{C}}$.

(b) Suppose that $|\Sigma| \leqslant 6$. Prove that there exists a line $L \subset \mathbb{P}^2_{\mathbb{C}}$ that contains exactly two points of the set Σ.

$(\star)$ Suppose that $|\Sigma| = 7$. Prove that there exists a line $L \subset \mathbb{P}^2_{\mathbb{C}}$ that contains exactly two points of the set Σ.

Exercise 2. Observe that no three points among the four points $[1 : 2 : 3]$, $[1 : 0 : -1]$, $[2 : 5 : 1]$ and $[-1 : 7 : 1]$ in $\mathbb{P}^2_{\mathbb{C}}$ are collinear.

(a) Find the projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi([1 : 2 : 3]) = [1 : 0 : 0]$, $\phi([1 : 0 : -1]) = [0 : 1 : 0]$, $\phi([2 : 5 : 1]) = [0 : 0 : 1]$ and $\phi([-1 : 7 : 1]) = [1 : 1 : 1]$.

(b) Let $\mathcal{C}$ be the conic in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$-xy + 2y^2 - 3xz + 7yz + 3z^2 = 0.$$

Find a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi(\mathcal{C})$ is given by $xy = 0$.

$(\star)$ Let $\mathcal{C}$ be the conic in $\mathbb{P}^2$ that is given by

$$x^2 + xy - 2y^2 + 3xz + 3yz + z^2 = 0.$$

Then $\mathcal{C}$ contains the point $[-2 : 1 : 3]$. Find a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi([-2 : 1 : 3]) = [0 : 0 : 1]$ and $\phi(\mathcal{C})$ is given by $xz + y^2 = 0$.

Workshop 1 will be on Friday 25th January 2019 at 12:10pm at room 4325C at JCMB.

Exercise 3. Let λ be a complex number. Put

$$f(x,y,z) = x^3 + y^3 + z^3 + \lambda xyz.$$

Let C be a subset in $\mathbb{P}^2_{\mathbb{C}}$ given by $f(x,y,z)=0$. Let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, so that $\omega^3 = 1$. Denote by Σ the subset in $\mathbb{P}^2_{\mathbb{C}}$ consisting of the following 9 points:

$$[1:-1:0], [1:-\omega:0], [1:-\omega^2:0],$$
$$[1:0:-1], [1:0:-\omega], [1:0:-\omega^2],$$
$$[0:1:-1], [0:1:-\omega], [0:1:-\omega^2].$$

(a) Check that C contains Σ. Show that the set Σ is not contained in any line in $\mathbb{P}^2_{\mathbb{C}}$. Going through all pairs of points in Σ, one can see that every line $L \subset \mathbb{P}^2_{\mathbb{C}}$ that passes through two points in Σ contains another point in Σ. Check this in some cases.

(b) Suppose that $\lambda^3 \neq -27$. Show that there is no point $[a:b:c] \in \mathbb{P}^2_{\mathbb{C}}$ such that

$$\frac{\partial f(a,b,c)}{\partial x} = \frac{\partial f(a,b,c)}{\partial y} = \frac{\partial f(a,b,c)}{\partial z} = 0.$$

Use Bezout theorem to show that the homogeneous polynomial $f(x,y,z)$ is irreducible. Conclude that C is a smooth irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree 3. Pick a point $P \in \Sigma$. Find the equation of the line $L_P \subset \mathbb{P}^2_{\mathbb{C}}$ that is tangent to the curve C at the point P. Show that $L_P \cap C = P$.

($\star$) Suppose that $\lambda^3 = -27$. Show that there are 3 points $[a:b:c] \in \mathbb{P}^2_{\mathbb{C}}$ such that

$$\frac{\partial f(a,b,c)}{\partial x} = \frac{\partial f(a,b,c)}{\partial y} = \frac{\partial f(a,b,c)}{\partial z} = 0.$$

Use Bezout theorem to deduce that the curve C is a union of 3 different lines in $\mathbb{P}^2_{\mathbb{C}}$. Conclude that $f(x,y,z)$ is a product of 3 different polynomials in $\mathbb{C}[x,y,z]$ of degree 1. Find these 3 polynomials explicitly.

Exercise 4. Denote by V_n the vector space (over $\mathbb{C}$) consisting of all homogeneous polynomials in $\mathbb{C}[x,y,z]$ of degree n. Let C_d be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree $d \geqslant 3$.

(a) Show that the dimension of V_3 is 10. Use this to prove that for every 9 points in $\mathbb{P}^2_{\mathbb{C}}$, there is a non-zero polynomial $f(x,y,z) \in V_3$ that vanishes at these 9 points.

(b) Suppose that $d=5$. Use Bezout theorem and part (a) to show that the curve C_d has at most 6 singular points.

($\star$) Show that V_n is of dimension $\frac{(n+1)(n+2)}{2}$. Use this and Bezout theorem to show that the curve C_d has at most $\frac{(d-1)(d-2)}{2}$ singular points.

Exercise 5. Denote by V_n the vector space (over $\mathbb{C}$) consisting of all homogeneous polynomials in $\mathbb{C}[x,y,z]$ of degree n. Let Σ be a finite subset in $\mathbb{P}^2_{\mathbb{C}}$, and let $V_n(\Sigma)$ be the vector subspace of V_n consisting of all polynomials in V_n that vanish at each point of the subset Σ. We say that Σ imposes independent linear conditions (ILC) on V_n in the case when

$$\dim_{\mathbb{C}}\Big(V_n(\Sigma)\Big) = \dim_{\mathbb{C}}\big(V_n\big) - |\Sigma|.$$

Otherwise we say that Σ imposes dependent linear conditions (DLC) on V_n.

(a) Prove that the following two conditions are equivalent:

- the subset Σ imposes ILC on V_n,
- for every point $P \in \Sigma$, there is a polynomial

$$f(x, y, z) \in V_n$$

such that $f(P) \neq 0$ and $f(Q) = 0$ for every point $Q \in \Sigma \setminus P$.

(b) Use part (a) to show that Σ imposes ILC on V_n in each of the following cases:

(i) when Σ imposes ILC on V_k for some $k \leqslant n$,

(ii) $|\Sigma| \leqslant n + 1$,

(iii) $n = 1$, $|\Sigma| = 3$, and Σ is not contained in one line in $\mathbb{P}^2_{\mathbb{C}}$,

(iv) $n = 2$, $|\Sigma| = 5$, and every line in $\mathbb{P}^2_{\mathbb{C}}$ contains at most 3 points of Σ.

($\star$) Show that Σ imposes DLC on V_n in each of the following cases:

(i) $dn + 2$ points of $|\Sigma|$ are contained in an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree d,

(ii) $n = 3$, $|\Sigma| = 9$, and $\Sigma = C \cap Z$, where C and Z are irreducible curves of degree 3.