Stiefel manifolds

Friday, 28 May 2021 18:03

$0 \le k \le n$. The <u>Stiefel manifold</u> is $V_{k,n} = \{ (v_1, \dots, v_k) \in (\mathbb{R}^n)^k \; : \; \langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \}$

= space of orthonormal k-frames in $\mathbb{R}^n$

= k-frames in $\mathbb{R}^n$

$V_{k,n}$ carries the subspace topology of $(\mathbb{R}^n)^k$; since $V_{k,n} \subseteq (S^{n-1})^k$ is a closed subset, $V_{k,n}$ is compact.

<u>Example</u>: $V_{0,n} = \{\emptyset\}$ is a one-point space.

$V_{1,n} = S^{n-1}$

$$V_{n,n} \xleftrightarrow{\cong} O(n)$$

$(v_1, \dots, v_n) \longmapsto$ matrix with columns $(v_1, \dots, v_n)$

$(Ae_1, \dots, Ae_n) \longleftarrow\!\shortmid A$

$e_i = (0, \dots, 0, 1, 0, \dots, 0)$ i-th vector in the coordinate basis of $\mathbb{R}^n$.
$\uparrow$ i-th spot

$$SO(n) \longrightarrow V_{n-1,n}$$
$$A \longmapsto (Ae_1, \dots, A \cdot e_{n-1})$$

is a continuous bijection between compact Hausdorff spaces, hence a homeomorphism.

Bijective: let $(v_1, \dots, v_{n-1})$ be an $(n-1)$-frame in $\mathbb{R}^n$, then the orthogonal complement of the span of $v_1, \dots, v_{n-1}$ is 1-dimensional. So there are exactly 2 unit vectors in this complement. Exactly one of these results in an ONB $(v_1, \dots, v_{n-1}, v_n)$ of determinant $+1$.

<u>Prop</u>: The space $V_{k,n}$ is a manifold of dimension

$$(n-1) + (n-2) + \dots + (n-k) = nk - \frac{k(k+1)}{2}.$$

<u>Proof</u>: By induction on k. $k=0$: $V_{0,n} = *$ $\rightsquigarrow$ 0-dimensional manifold

$k=1$: $V_{1,n} = S^{n-1}$ $\rightsquigarrow$ $(n-1)$-dimensional manifold.

Now suppose $k \geq 2$. We consider the map $f: S^{n-1}_+ = \{ w \in S^{n-1} : w_1 > 0 \} =$ "northern hemisphere" $\longrightarrow O(n)$

as the composite

$$S^{n-1}_+ \longrightarrow Gl_n(\mathbb{R}) \xrightarrow[\text{orthonormalization}]{\text{Gram-Schmidt}} O(n)$$

$$w \longmapsto (w, e_2, e_3, \dots, e_n) = \begin{pmatrix} w_1 & & & 0 \\ w_2 & 1 & & \\ \vdots & & \ddots & \\ w_n & 0 & & 1 \end{pmatrix}$$

Properties of f:

- f is continuous
- $f(e_1) = f(1, 0, \dots, 0) = E_n =$ identity matrix
- $f(w) \cdot e_1 = w$ for all $w \in S^{n-1}_+$

<u>Warning</u>: There is no continuous map $f: S^{n-1} \longrightarrow O(n)$ such that $f(w) \cdot e_1 = w$ for all $w \in S^{n-1}$.

We define $U = \{ (v_1, \dots, v_k) \in V_{k,n} : v_1 \in S^{n-1}_+ \}$, this is an open neighborhood of $(e_1, \dots, e_k) \in V_{k,n}$.

The map

$$U \longrightarrow S^{n-1}_+ \times V_{k-1,n-1}$$

is a homeomorphism.

$$(v_1, \dots, v_k) \longmapsto (v_1, \; f(v_1)^{-1}(v_2), \dots, f(v_1)^{-1}(v_k))$$

- this is well-defined: $f(v_1)^{-1}$ is an orthogonal matrix such that $f(v_1)^{-1}(v_1) = e_1$

since $f(v_1)^{-1}$ is orthogonal matrix and $v_2, \dots, v_k$ are a k-frame in $(v_1)^\perp$

$\Longrightarrow$ $f(v_1)^{-1}(v_2), \dots, f(v_1)^{-1}(v_k) \longmapsto (e_1)^\perp = 0 \oplus \mathbb{R}^{n-1}$

- continuous
- and has a continuous inverse:

$$S^{n-1}_+ \times V_{k-1,n-1} \longrightarrow U$$

$$(v, w_1, \dots, w_{k-1}) \longmapsto (v, \; f(v)\underbrace{(0, w_1)}_{\mathbb{R}^n}, \dots, f(v)(0, w_{k-1}))$$

<u>Conclusion</u>: the point $(e_1, \dots, e_k) \in V_{k,n}$ has an open neighborhood homeomorphic to $S^{n-1}_+ \times V_{k-1,n-1}$, which is a manifold of dimension

Conclusion: The point $(e_1,\dots,e_k)\in V_{k,n}$ has an open neighborhood homeomorphic to $S^{n-1}_+ \times V_{k-1,n-1}$, which is a manifold of dimension

$$(n-1)+(n-2)+(n-3)+\dots+((n-1)-(k-1)) \quad \text{by induction.}$$

$\Rightarrow$ so $(e_1,\dots,e_k)$ has an open neighborhood homeomorphic to $\mathbb{R}^d$, $d=(n-1)+(n-2)+\dots+(n-k)$.

Now let $(v_1,\dots,v_k)\in V_{k,n}$ be any point. Complete to an orthonormal basis

$$A=(v_1,\dots,v_k,v_{k+1},\dots,v_n)\in O(n)$$

Then

$$A: V_{k,n}\longrightarrow V_{k,n}, \quad (w_1,\dots,w_k)\longmapsto (Aw_1,\dots,Aw_k)$$

is a self-homeomorphism of $V_{k,n}$ that sends $(e_1,\dots,e_k)$ to $(v_1,\dots,v_k)$. So also $(v_1,\dots,v_k)$ has an open neighbourhood homeomorphic to $\mathbb{R}^d$. □

Remark: What we really showed is that the map $V_{k,n}\longrightarrow S^{n-1}$, $(v_1,\dots,v_k)\longmapsto v_1$ is a "locally trivial fiber bundle" with fiber $V_{k-1,n-1}$.

Complex Stiefel manifolds:

$$V^{\mathbb{C}}_{k,n}=\left\{(v_1,\dots,v_k)\in(\mathbb{C}^n)^k : \langle v_i,v_j\rangle=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}\right\}$$

= space of (complex) k-frames in $\mathbb{C}^n$

As in the real case, one shows that $V^{\mathbb{C}}_{k,n}$ is a compact manifold of dimension

$$(2n-1)+(2n-3)+\dots+(2n-2k+1) = 2nk-k^2$$

Special case:

$$V^{\mathbb{C}}_{1,n} = \text{unit sphere in } \mathbb{C}^n = S^{2n-1}$$

$$V^{\mathbb{C}}_{n-1,n}\cong SU(n), \qquad V^{\mathbb{C}}_{n,n}\cong U(n)$$

[Same inductive proof, with Gram-Schmidt orthonormalization for hermitian inner product spaces; in inductive step, you work over $S^{2n-1}_+=\{(v_1,\dots,v_n)\in S^{n-1} : \mathrm{Re}(v_1)>0\}$

Quaternion Stiefel manifolds: compact manifolds $V^{\mathbb{H}}_{k,n}$ of dimension

$$(4n-1)+(4n-5)+\dots+(4n-4k+3) = 4nk-k(2k-1).$$

$$V^{\mathbb{H}}_{1,n} = \text{unit sphere in } \mathbb{H}^n = S^{4n-1}$$

$$V^{\mathbb{H}}_{n,n} = Sp(n) = \{A\in M(n\times n,\mathbb{H}) : A\cdot\bar{A}^t=\bar{A}^t\cdot A=E_n\}$$