Commutativity of the ∪-product, revisited

We give another application of the theorem of acyclic models: we explain a proof of the graded-commutativity of the cup product that is very different in flavor from the formula-based proof involving the $\cup_1$-product. Given two chain complexes C and D, we define the *symmetry isomorphism*

$$\tau_{C,D} \ : \ C \otimes D \ \longrightarrow \ D \otimes C$$

in dimension n as the map

$$(\tau_{C,D})_n \ : \ \bigoplus_{p+q=n} C_p \otimes D_q \ \longrightarrow \ \bigoplus_{p+q=n} D_q \otimes C_p$$
$$(\tau_{C,D})_n(c \otimes d) \ = \ (-1)^{pq} \cdot d \otimes c \ .$$

I omit the verification that for varying n, these maps indeed form a chain map. But I encourage you to perform this verification yourself; it will also explain why the sign is chosen the way it is.

At the end of the proof of the boundary formula for the shuffle map we also showed that the Eilenberg-Zilber map is symmetric in the sense of the relation $x\nabla y = (-1)^{pq}\cdot\tau(y\nabla x)$, where x and y have degrees p and q, respectively. With our present notation, this relation can be equivalently formulated as the commutativity of the following square of chain complexes and chain maps:

$$\begin{tikzcd}
C_*(X)\otimes C_*(Y) \arrow[r, "\mathrm{EZ}"] \arrow[d, "\tau_{C_*(X),C_*(Y)}"'] & C_*(X\times Y) \arrow[d, "C_*(\mathrm{flip})"] \\
C_*(Y)\otimes C_*(X) \arrow[r, "\mathrm{EZ}"'] & C_*(Y\times X)
\end{tikzcd}$$

Here $\mathrm{flip} : X \times Y \longrightarrow Y \times X$ is the isomorphism of simplicial sets that interchanges the two factors.

You should beware that the Alexander-Whitney map *does not* have the analogous symmetry property: the square of chain complexes and chain maps

$$\begin{tikzcd}
C_*(X\times Y) \arrow[r, "\mathrm{AW}"] \arrow[d, "C_*(\mathrm{flip})"'] & C_*(X)\otimes C_*(Y) \arrow[d, "\tau_{C_*(X),C_*(Y)}"] \\
C_*(Y\times X) \arrow[r, "\mathrm{AW}"'] & C_*(Y)\otimes C_*(X)
\end{tikzcd}$$

does *not* commute. So in this regard, the Eilenberg-Zilber and Alexander-Whitney maps behave very differently. However, the next proposition shows that this square at least commutes up to natural chain homotopy.

Proposition 1. *The two natural chain maps*

$$\tau_{C_*(X),C_*(Y)} \circ \mathrm{AW} \ , \ \mathrm{AW} \circ C_*(\mathrm{flip}) \ : \ C_*(X\times Y) \ \longrightarrow \ C_*(Y)\otimes C_*(X)$$

are naturally chain homotopic.

Proof. We will show that the natural chain map

$$\kappa \ = \ \tau_{C_*(X),C_*(Y)} \circ \mathrm{AW} \ - \ \mathrm{AW} \circ C_*(\mathrm{flip}) \ : \ C_*(X\times Y) \ \longrightarrow \ C_*(Y)\otimes C_*(X) \ .$$

satisfies the hypotheses of the theorem of acyclic models. In chain dimension 0, the two natural chain maps $\tau_{C_*(X),C_*(Y)} \circ \mathrm{AW}$ and $\mathrm{AW} \circ C_*(\mathrm{flip})$ coincide, so the transformation κ_0 is trivial. For $n \geq 0$, the functor

$$C_n(X\times Y) \ = \ \mathbb{Z}[X_n \times Y_n]$$

is representable by (Δ^n, Δ^n); and we showed in the previous video that the complex $C_*(\Delta^n) \otimes C_*(\Delta^n)$ is acyclic in positive dimensions. So the theorem of acyclic models applies, and it provides the desired natural chain homotopy. $\square$

We use Proposition 1 to give another proof of the graded-commutativity of the $\cup$-product. We specialize Proposition 1 to $X = Y$, and we write $\Delta : X \longrightarrow X \times X$ for the diagonal morphism. The flip morphism of $X \times X$ fixes the diagonal, i.e., composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{\text{flip}} X \times X$$

is also the diagonal. So if we precompose the chain homotopy of Proposition 1 with the chain morphism $C_*(\Delta) : C_*(X) \longrightarrow C_*(X \times X)$, we conclude that the triangle of chain complexes and chain maps

$$\begin{tikzcd} & C_*(X) \otimes C_*(X) \arrow[d, "\tau_{C_*(X),C_*(X)}"] \\ C_*(X) \arrow[ur, bend left, "\mathrm{AW} \circ C_*(\Delta)"] \arrow[dr, bend right, "\mathrm{AW} \circ C_*(\Delta)"'] & \\ & C_*(X) \otimes C_*(X) \end{tikzcd}$$

commutes up to a natural chain homotopy.

Now we let R be a commutative ring. We apply the functor $\mathrm{Hom}(-, R)$ to the previous triangle of chain complexes and the chain homotopy. The result is a triangle of cochain complexes and cochain maps

$$\begin{tikzcd} \mathrm{Hom}(C_*(X) \otimes C_*(X), R) \arrow[dr, bend left, "\mathrm{Hom}(\mathrm{AW} \circ C_*(\Delta), R)"] \arrow[dd, "\mathrm{Hom}(\tau_{C_*(X),C_*(X)}, R)"'] & \\ & \mathrm{Hom}(C_*(X), R) \; = \; C^*(X; R) \\ \mathrm{Hom}(C_*(X) \otimes C_*(X), R) \arrow[ur, bend right, "\mathrm{Hom}(\mathrm{AW} \circ C_*(\Delta), R)"'] & \end{tikzcd}$$

that commutes up to a natural cochain homotopy. Homotopic cochain maps induce the same map on cohomology, so the triangle of cohomology groups

$$\begin{tikzcd} H^n(\mathrm{Hom}(C_*(X) \otimes C_*(X), R)) \arrow[dr, bend left, "H^n(\mathrm{Hom}(\mathrm{AW} \circ C_*(\Delta), R))"] \arrow[dd, "H^n(\mathrm{Hom}(\tau_{C_*(X),C_*(X)}, R))"'] & \\ & H^n(X; R) \\ H^n(\mathrm{Hom}(C_*(X) \otimes C_*(X), R)) \arrow[ur, bend right, "H^n(\mathrm{Hom}(\mathrm{AW} \circ C_*(\Delta), R))"'] & \end{tikzcd} \tag{2}$$

commutes for all $n \geq 0$.

We let $f \in C^p(X; R) = \mathrm{Hom}(C_p(X), R)$ and $g \in C^q(X; R) = \mathrm{Hom}(C_q(X), R)$ be two cocycles of the simplicial set X with coefficients in the commutative ring R. Then the map

$$f \otimes g \;:\; (C_*(X) \otimes C_*(X))_{p+q} \xrightarrow{\text{project}} C_p(X) \otimes C_q(X) \xrightarrow{f \otimes g} R \otimes R \xrightarrow{\text{multiplication}} R$$

is a $(p+q)$-cocycle of the complex $\mathrm{Hom}(C_*(X) \otimes C_*(X), R)$, where the first map is the projection from

$$(C_*(X) \otimes C_*(X))_{p+q} \;=\; \bigoplus_{j=0}^{p+q} C_j(X) \otimes C_{p+q-j}(X)$$

to the summand indexed by $j = p$. Moreover,

$$[f] \cup [g] \;=\; [f \cup g] \;=\; H^n(\mathrm{Hom}(\mathrm{AW} \circ C_*(\Delta), R))(f \otimes g)$$

in the group $H^{p+q}(X;R)$. Indeed,

$$H^n(\mathrm{Hom}(\mathrm{AW}\circ C_*(\Delta),R))(f\otimes g) \;=\; [(f\otimes g)\circ \mathrm{AW}_{p+q}\circ C_{p+q}(\Delta)]\ ,$$

and

$$\begin{aligned}((f\otimes g)\circ \mathrm{AW}_{p+q}\circ C_{p+q}(\Delta))(x) &= (f\otimes g)(\mathrm{AW}_{p+q}(x,x))\\ &= (f\otimes g)\left(\sum_{j=0}^{p+q} d^*_{\mathrm{front}}(x)\otimes d^*_{\mathrm{back}}(x)\right)\\ &= f(d^*_{\mathrm{front}}(x))\cdot g(d^*_{\mathrm{back}}(x)) \;=\; (f\cup g)(x)\end{aligned}$$

for all $x\in X_{p+q}$. Given this relation, the commutativity of the triangle (2) yields

$$\begin{aligned}[f]\cup[g] &= H^n(\mathrm{Hom}(\mathrm{AW}\circ C_*(\Delta),R))(f\otimes g)\\ &= H^n(\mathrm{Hom}(\mathrm{AW}\circ C_*(\Delta),R))(H^n(\mathrm{Hom}(\tau_{C_*(X),C_*(X)},R))(f\otimes g))\\ &= (-1)^{pq}\cdot H^n(\mathrm{Hom}(\mathrm{AW}\circ C_*(\Delta),R))(g\otimes f)\\ &= (-1)^{pq}\cdot [g]\cup[f]\ .\end{aligned}$$