1. Quivers and their representations: Basic definitions and examples.

1.1. Quivers.

A *quiver* Q (sometimes also called a directed graph) consists of vertices and oriented edges (arrows): loops and multiple arrows are allowed. An arrow goes from some vertex (its tail) to some vertex (its head), if we denote the tail of the arrow α by $t(\alpha)$, the head by $h(\alpha)$, we see that we deal with two set-theoretical maps

$$t, h \colon Q_1 \to Q_0,$$

where Q_0 denotes the set of vertices, Q_1 the set of arrows. Here is the formal definition of a *quiver* $Q = (Q_0, Q_1, t, h)$: there are given two sets Q_0, Q_1 and two maps $h, t \colon Q_1 \to Q_0$, the elements of Q_0 are called *vertices,* the elements of Q_1 are called *arrows,* and for every arrow $\alpha \in Q_1$, there is defined its *tail* $t(\alpha)$ and its *head* $h(\alpha)$. One depicts this in the usual way:

$$t(\alpha) \xrightarrow{\alpha} h(\alpha). \quad \text{or also} \quad \alpha \colon t(\alpha) \to h(\alpha)$$

(actually, often we will draw arrows from right to left, or also in any possible direction). Arrows α with $h(\alpha) = t(\alpha)$ are called *loops.*

Given a quiver Q, one may delete the orientation of the arrows and obtains in this way the *underlying graph* $\overline{Q}$, this is the triple consisting of the two sets Q_0, Q_1 and the functions which attaches to $\alpha \in Q_1$ the set $\{t(\alpha), h(\alpha)\}$ (this means that one does no longer distinguish which one of the vertices is the head and which one is the tail. The reverse process will be called *choosing an orientation.*

> The wording was chosen by Gabriel (1972): “quiver” means literally a box for holding arrows. Before Gabriel, quivers were called “diagram schemes” by Grothendieck.

Here is a collection of typical quivers, with the names which are now usually attached, often these names refer just to the underlying graph.

$\circ \longrightarrow \circ$ $\quad\mathbb{A}_2$ $\qquad\qquad$ $\circ \longrightarrow \circ \longrightarrow \circ$ $\quad\mathbb{A}_3$ $\qquad\qquad$ $\circ \longrightarrow \circ \longleftarrow \circ$ $\quad\mathbb{A}_3$

$$\begin{tikzcd} \circ \arrow[dr] & \circ \arrow[d] & \circ \arrow[dl] \\ & \circ & \end{tikzcd} \quad \mathbb{D}_4$$

$$\begin{tikzcd} & & \circ \arrow[d] & & \\ \circ \arrow[r] & \circ \arrow[r] & \circ & \circ \arrow[l] & \circ \arrow[l] \end{tikzcd} \quad \mathbb{E}_6$$

$$\begin{tikzcd} \circ \arrow[loop above] \end{tikzcd} \quad \mathbb{L} \text{ (loop)} \qquad\qquad \circ \rightrightarrows \circ \quad \text{Kronecker quiver}$$

Of course, one may consider much more complicated quivers, say with 1000 vertices and 7000 arrows, but the representation theory already of quite small quivers usually turns out to be quite complicated. There are quivers with many edges which we will deal with, for example

$$\circ \;\text{——}\; \circ \;\text{——}\; \circ \;-\; \cdots \;-\; \circ \;\text{——}\; \circ \qquad \mathbb{A}_n$$

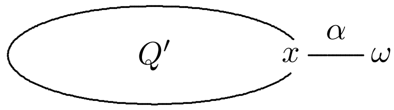
with n vertices, usually labeled $1, 2, \dots, n$, and with $n-1$ arrows α_i with $\{t(\alpha_i), h(\alpha_i)\} = \{i, i+1\}$, but usually one is interested in rather small quivers, for example the Dynkin quivers $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, or the corresponding Euclidean quivers $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$.

If Q is a quiver, a *subquiver* Q' of Q is of the form $Q' = (Q'_0, Q'_1, t', h')$, with subsets $Q'_0 \subseteq Q_0$, $Q'_1 \subseteq Q_1$, such that $t(Q'_1) \subseteq Q'_0$ and $h(Q'_1) \subseteq Q'_0$, and such that t', h' are the restrictions of t, h, respectively.

For example, a quiver of type $\mathbb{A}_3$ has two subquivers of type $\mathbb{A}_2$.

A quiver Q is said to be *connected*, provided for any decomposition $Q_0 = Q'_0 \cup Q''_0$ of the set of vertices of Q with non-empty subsets Q'_0, Q''_0, there is an arrow α such that $h(\alpha) \in Q'_0, t(\alpha) \in Q''_0$ or $h(\alpha) \in Q''_0, t(\alpha) \in Q'_0$.

A connected quiver with n vertices and $n-1$ arrows is called a *tree quiver* (this just means that the underlying graph is a tree in the sense of graph theory). The tree quivers can be constructed inductive as follows: first of all, the quiver $\mathbb{A}_1$ (it consists of a single vertex and there is no arrow) is a tree quiver, and a quiver Q with $n \geq 2$ vertices is a tree quiver provided it is obtained from a tree quiver Q' with $n-1$ vertices by *attaching an arm of the form* $\mathbb{A}_2$ *at the vertex* x (this means that x is a vertex of Q' and one obtains Q_0 by adding to Q'_0 a vertex, say labeled ω, and that one obtains Q_1 by adding to Q'_1 an arrow α such that $\{h(\alpha), t(\alpha)\} = \{x, \omega\}$)



1.2. Representations of a quiver.

Let k be some field. Al the vector spaces to be considered are assumed to be k-spaces. For most considerations, the structure of k itself will not play a role, but we should stress that we always work with a **fixed** (commutative) field k.

A *representation* of the quiver Q is of the form $M = (M_x, M_\alpha)_{x,\alpha}$, where M_x is a vector space, for every vertex $x \in Q_0$, and $M_\alpha \colon M_{t(\alpha)} \to M_{h(\alpha)}$ is a linear map, for every $\alpha \in Q_1$; instead of M_α one often writes just α. Thus, representations of quivers are nothing else than collections of vector spaces and linear maps between these vector spaces. *We usually will assume that the vector spaces which we consider are finite-dimensional* (however most of the considerations carry over to the general case of dealing with vector spaces of arbitrary dimension).

Given a representation $M = (M_x, M_\alpha)_{x,\alpha}$, we call the sum of the dimensions of the vector spaces M_x with $x \in Q_0$ the *dimension* of M and denote it by $\dim M$. Later it will be convenient to denote the maps M_α with $\alpha \in Q_1$ just by α (clearly an abuse of notation, but quite convenient).

> Why do we use the letter M for a representation of a quiver? The representations of a quiver M may be considered as the "**m**odules" over the "path algebra" of Q, see section 4.

Of course, for any quiver there is defined the corresponding *zero representation* (or "trivial" representation) with all the vector spaces being zero (and all the maps being zero maps). The zero representation is usually just denoted by 0.

Representations M with all vector spaces M_x of dimension at most 1 are said to be *thin.*

> We will deal with thin representations in 1.6. Here is a typical representation which is not thin (and not isomorphic to a direct sum of thin modules). We deal with a quiver of type $\mathbb{D}_4$:
>
> $$\begin{tikzcd} \circ \arrow[dr] & \circ \arrow[d] & \circ \arrow[dl] \\ & \circ & \end{tikzcd} \qquad\qquad \begin{tikzcd} k0 \arrow[dr] & 0k \arrow[d] & \Delta \arrow[dl] \\ & k^2 & \end{tikzcd}$$
>
> with $\Delta = \{(x, x) \mid x \in k\}$ and all the maps being the corresponding inclusion maps. In section 3 we will see that this is an "indecomposable representation" (but we did not yet define what means "indecomposable").
>
> Also we may be interested in a vector space V with 4 subspaces $U_1,\ U_2,\ U_3,\ U$ such that $U_1 \subseteq U_2 \subseteq U_3$. Such a system can be considered as a representation of the following quiver of type $\mathbb{A}_5$
>
> $$\circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longleftarrow \circ \qquad \mathbb{A}_5$$
>
> namely as
>
> $$U_1 \to U_2 \to U_3 \to V \leftarrow U$$
>
> where again all the maps are the inclusion maps.

When looking at representations of quivers, we often will replace a given representation by an "isomorphic" one, whenever this is suitable. Given representations M, M' of a quiver Q, an *isomorphism* $f = (f_x)_x \colon M \to M'$ is given by vector space isomorphisms $f_x \colon M_x \to M'_x$ such that for any arrow $\alpha \colon t(\alpha) \to h(\alpha)$ the following diagram commutes:

$$\begin{tikzcd} M_{t(\alpha)} \arrow[r, "f_{t(\alpha)}"] \arrow[d, "M_\alpha"'] & M'_{t(\alpha)} \arrow[d, "M'_\alpha"] \\ M_{h(\alpha)} \arrow[r, "f_{h(\alpha)}"] & M'_{h(\alpha)}. \end{tikzcd}$$

(We often will have to consider such diagrams, they are given by an arrow say $\alpha \colon x \to y$; the usual convention will be to draw the data concerning M vertically on the left, those concerning M' vertically on the right, and the maps f_x horizontally.)

Of course, the commuativity of the diagram above implies that also the diagram

$$\begin{array}{ccc} M'_{t(\alpha)} & \xrightarrow{f^{-1}_{t(\alpha)}} & M_{t(\alpha)} \\ {\scriptstyle M'_\alpha}\downarrow & & \downarrow{\scriptstyle M_\alpha} \\ M'_{h(\alpha)} & \xrightarrow{f^{-1}_{h(\alpha)}} & M_{h(\alpha)}. \end{array}$$

commutes; thus if $f = (f_x)_x \colon M \to M'$ is an isomorphism, also $f^{-1} = (f_x^{-1})_x \colon M' \to M$ is an isomorphism.

> **Slogan:** *Representation theory studies properties of representations which are invariant under isomorphisms.*
>
> In section 3, we will introduce the notion of a homomorphism $f \colon M \to M'$; isomorphisms are special homomorphisms.

For example:

Polishing. *Let Q be a quiver, $\alpha \colon x \to y$ an arrow of Q, but not a loop, and M a representation such that M_α is injective. Then M is isomorphic to a representation M' such that M'_α is the inclusion of a subspace* (namely the inclusion of the image of M_α into M_y).

Proof: Let M' be defined as follows: Let $M'_x = \mathrm{Im}(M_\alpha)$, and $M'_a = M_a$ for all vertices $a \neq x$ of Q. If $\beta \colon b \to x$ and $b \neq x$, let $M'_\beta = M_\alpha M_\beta \colon M_b \to M'_a$. If $\gamma \colon x \to c$ and $c \neq x$, let $M'_\gamma = M_\gamma (M_\alpha)^{-1} \colon M'_x \to M_c$. Finally, if $\delta \colon x \to x$, let $M'_\delta = M_\alpha M_\delta (M_\alpha)^{-1} \colon M'_x \to M'x$. Always note that $(M_\alpha)^{-1}$ is defined on M'_a and the definition for $\gamma = \alpha$ shows that $M'_\alpha = M_\gamma (M_\alpha)^{-1} \colon M'_x \to M_y$ is just the inclusion map. The representations M and M' are isomorphic, with an isomorphism $f : M \to M'$ given by $f_a = 1$ for $a \neq x$ and $f_x = M_\alpha \colon M_x \to M'_x$; in order to see that f is a homomorphism, let us exhibit two typical squares: on the left we consider an arrow $\beta \colon b \to x$, on the right an arrow $\gamma \colon x \to c$.

$$\begin{array}{ccc} M_b & = & M'_b \\ {\scriptstyle M_\beta}\downarrow & & \downarrow{\scriptstyle M_\alpha M_\beta} \\ M_x & \xrightarrow{M_\alpha} & M'_x \end{array} \qquad\qquad \begin{array}{ccc} M_x & \xrightarrow{M_\alpha} & M'_x \\ {\scriptstyle M_\gamma}\downarrow & & \downarrow{\scriptstyle M_\gamma (M_\alpha)^{-1}} \\ M_c & = & M'_c \end{array}$$

Warning: If M is a representation with several of the maps M_α being injective, one may try to replace successively all these maps by inclusion maps, but in general this will not be possible. For example, consider a cycle with all the maps being invertible. If there are n arrows, we may replace $n-1$ of them by corresponding identity maps, but trying to replace also the remaining map by an identity map may destroy a previous identity map.

Similarly, *if M is a representation of a quiver Q and $\alpha \colon x \to y$ is an arrow which is not a loop, such that M_α is surjective, then M is isomorphic to a representation M' such that M'_α is the canonical projection $M_y \to M_y / \mathrm{Ker}(M_\alpha)$.*

1.3. Direct decomposition.

Given a representation M of a quiver Q, a *direct sum decomposition* of M is of the following form: for every $x \in Q_0$, there is given a direct sum $M_x = M'_x \oplus M''_x$ and for every $\alpha\colon x \to y$, one has $M_\alpha(M'_x) \subseteq M'_y$ and $M_\alpha(M''_x) \subseteq M''_y$. One may denote the restriction of M_α to M'_x by $M'_\alpha\colon M'_x \to M'_y$, and similarly, the restriction of M_α to M''_x by $M''_\alpha\colon M''_x \to M''_y$. One obtains in this way representations $M' = (M'_x, M'_\alpha)_{x,\alpha}$ and $M'' = (M''_x, M''_\alpha)_{x,\alpha}$ and one writes $M = M' \oplus M''$.

The representation theory of quivers is concerned with the following question: given a representation M of some quiver Q, is it possible to decompose the representation? If there is no non-trivial decomposition and M is non-zero, then M is said to be indecomposable: To repeat: M is *indecomposable* if and only if $M \neq 0$ and for any decomposition $M = M' \oplus M''$, either $M' = 0$ or $M'' = 0$.

There is the following question: describe all the indecomposable representations of a given quiver. For some (quite small) quivers, this will be possible (and indeed for all the examples exhibited above), but in general it seems to be impossible (there is a notion of “wildness”: nearly all the large quiver are wild and one does not expect that there is a decent way to classify all the indecomposable representations of any wild quiver).

> **Slogan:** *Representation theory studies the isomorphism classes of indecomposable representations.*

Let us consider the quiver $\mathbb{A}_2$, we label the vertices 1 and 2 so that the unique arrow is $\alpha\colon 2 \to 1$. The representations of Q are of the form $M = (M_2, M_1, M_\alpha)$, where M_1, M_2 are vector spaces and $M_\alpha\colon M_2 \to M_1$ is a linear map, we will denote M just by writing $M = (M_\alpha\colon M_2 \to M_1)$. There are three indecomposable representations of V which are easy to describe:

$$(0 \to k), \quad (k \to 0), \quad (1_k\colon k \to k).$$

(and later it will turn out that these are the only indecomposable representations up to isomorphism. Why are these representations indecomposable? This should be clear for the first two representations, thus let us look at the third one: write it as $M = (M_\alpha\colon M_2 \to M_1)$ with $M_1 = M_2 = k$ and M_α the identity map. What is important is only that $M_\alpha \neq 0$. Assume we have given a direct decomposition $M = M' \oplus M''$, thus $M_2 = M'_2 \oplus M''_2$, $M_1 = M'_1 \oplus M''_1$, such that $M_\alpha(M'_2) \subseteq M'_1$ and $M_\alpha(M''_2) \subseteq M''_1$. Since $M_2 = k$ is one-dimensional, we must have $M'_2 = 0$ or $M''_2 = 0$. Without loss of generality, we can assume that $M''_2 = 0$, thus $M'_2 = M_2$. Now M_α is non-zero and maps M'_2 into M'_1, therefore also $M'_1 \neq 0$. Since $M_1 = M'_1 \oplus M''_1$ is one-dimensional and $M'_1 \neq 0$, it follows that $M''_1 = 0$. Thus $M'' = 0$.

Given a representation M and for every $x \in Q_0$ a subspace M'_x of M_x with $M_\alpha(M'_x) \subseteq M'_y$, for every arrow $\alpha\colon x \to y$, then we may denote the restriction of M_α to M'_x by M'_α, for $\alpha\colon x \to y$, and we obtain in this way a representation $M' = (M'_x, M'_\alpha)$ of Q which is called a *subrepresentation* of M.

Direct decomposition $M = M' \oplus M''$ are given by subrepresentations M', M'' of M such that $M_x = M'_x \oplus M''_x$ for all x.

Definition. We say that a representation N of a quiver Q with a vertex y is *y-sincere,* provided for any direct decomposition $N = N' \oplus N''$ with $N''_y = 0$ we have $N'' = 0$.

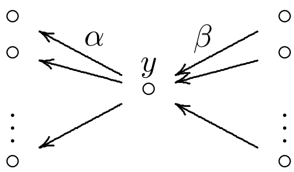
Proposition. *Let Q be obtained from a quiver Q' by attaching an arm of type $\mathbb{A}_2$ at the vertex x. Let M be an indecomposable representation of Q with support not contained in Q'. Then the restriction M' of M to Q' is x-sincere.*

Prof: Let $N = N' \oplus N''$ with $N''_x = 0$. Then we obtain a direct decomposition of $M = M' \oplus M''$ by taking $M'_\omega = M_\omega$, $M''_\omega = 0$ and such that the restriction of M' to Q' is N', the restriction of M'' to Q' is N''. Since $M' \neq 0$, and M is indecomposable, we conclude that $M'' = 0$, thus $N'' = 0$.

1.4. The simple representations $S(x)$.

Let x be a vertex of Q. The representation $S(x)$ of Q is defined by $S(x)_x = k$, $S(x)_y = 0$ for $y \neq 0$, and $S(x)_\alpha = 0$ for all arrows α (note that the latter condition concerns only loops $\alpha\colon x \to x$).

Proposition. *If y is a vertex of the quiver Q, and M a representation of Q, define subspaces K_y, I_y of M_y as follows: K_y is the intersection of the kernels of the maps M_α, where α is an arrow with tail $t(\alpha) = y$ and I_y is the sum of the images of the maps M_β where β is an arrow with head $h(\beta) = y$. Then $S(y)$ is a direct summand of M if and only if $K_y \not\subseteq I_x$.*



Better:

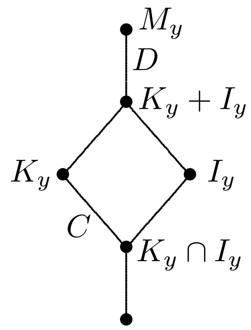
Splitting off copies of $S(y)$. *Let C, D be subspaces of M_y such that*

$$(K_y \cap I_y) \oplus C = K_y \quad \text{and} \quad (K_y + I_y) \oplus D = M_y.$$

Let $M'_y = I_y \oplus D$ and $M'_x = M_x$ for all $x \neq y$. Let $M''_y = C$ and $M''_x = 0$ for all $x \neq y$. Then M', M'' are subrepresentations of M, and $M = M' \oplus M''$. The representation M' has no direct summand of the form $S(x)$, whereas M'' is a direct sum of copies of $S(x)$.

If x is a vertex, a representation without a direct summands $S(x)$ may be called *y-reduced*, thus here we deal with the *y-reduction.*

Proof: Here is the lattice of the relevant subspace of M_y:



We will need that $(K_y \cap I_y) \oplus C = K_y$ implies that $I_y \oplus C = K_y + I_y$.

In order to see that M' is a subrepresentation of M, we need to look only at arrows β ending in y, since $M'_x = M_x$ for $x \neq y$. But by construction M_y contains I_y, thus the image of any map $M\beta$ with $h(\beta) = y$. In order to see that M'' is a subrepresentation of M, we only have to note that C is contained in the kernel of any map M_α with $t(\alpha) = y$. Actually, this also shows that M'' is a direct sum of copies of $S(y)$. Namely, take a basis $\mathcal{B}$ of C and observe that any $b \in \mathcal{B}$ yields a copy of $S(y)$.

Since $I_y \oplus C = K_y + I_y$, it follows that $M_y = I_y \oplus C \oplus D = M'_y \oplus M''_y$, and therefore $M = M' \oplus M''$. Looking at M', we see that the sum of the images of the maps M'_β with $h(\beta) = y$ is precisely I_y, whereas the intersection of the kernels of the maps M'_α with $t(\alpha) = y$ is $K_y \cap I_y$ and thus a subset of I_y.

Of course, in general, if M has a direct summand isomorphic to $S(y)$, there is an element $b \in M_y$ which belongs to K_x and not to I_y, thus $K_y \not\subseteq I_y$. Conversely, the splitting-off assertion shows: If $K_y \not\subseteq I_y$, then $K_y \cap I_y$ is a proper subspace of K_y and therefore $C \neq 0$. The splitting-off assertion shows that we split off the the direct sum of c copies of $S(x)$, where c is the dimension of C, thus the dimension of $K_y/(K_y \cap I_y)$.

Corollaries. *Let y be a vertex of Q and M an indecomposable representation of Q which is not isomorphic to $S(y)$.*

(a) *Always, $K_y \subseteq I_y$.*

(b) *If y is a source, then $K_y = 0$.*

(c) *If y is a sink, then $I_y = M_y$.*

Namely, if y is a source, then $I_y = 0$, and $K_y \subseteq I_y = 0$. And if y is a sink, then $K_y = M_y$ and then $M_y = K_y \subseteq I_y$.

Let us consider again the quiver Q of type $\mathbb{A}_2$ with the arrow $\alpha\colon 2 \to 1$. Let M be an indecomposable representation. If M is not isomorphic to $S(1)$, then M_α has to be surjective, according to (c). If M is not isomorphic to $S(2)$, then M_α has to be injective, according to (b). Thus if M is neither isomorphic to $S(1)$ not to $S(2)$, then M_α is both injective and surjective, thus a vector space isomorphism. It follows that M is isomorphic to a direct sum of say n copies of $(1_k\colon k \to k)$ (here n may be a non-negative integer or some cardinality. Namely, choose a basis $\mathcal{B}$ of M_2, this yields a vector space isomorphism

$\Phi\colon k^n \to M_2$ and a commutative diagram

$$\begin{array}{ccc} k^n & \xrightarrow{\Phi} & M_2 \\ {\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle M_\alpha} \\ k^n & \xrightarrow{M_\alpha\Phi} & M_1 \end{array}$$

which is an isomorphism of representations of Q. Note that the left vertical map is the direct sum of n copies of $(1_k\colon k \to k)$. We see:

Let Q be the quiver of type $\mathbb{A}_2$. Any representation of Q is a direct sum of copies of $S(1), S(2)$ and $(1_k\colon k \to k)$, thus of thin representations.

1.5. The indecomposable representations of quivers of type $\mathbb{A}$.

Let us first consider the quivers of type $\mathbb{A}_3$.

(1) The quiver Q of type $\mathbb{A}_3$ with linear orientation. This is the following quiver

$$\underset{\circ}{1} \xleftarrow{\ \alpha\ } \underset{\circ}{2} \xleftarrow{\ \beta\ } \underset{\circ}{3}$$

Splitting off copies of $S(1)$ we can assume that we deal with a representation M with M_α surjective; splitting off copies of $S(3)$ we can assume that M_β is injective. After polishing. we can assume that M_β is the inclusion of a subspace U of $V = M_2$, and that there is a subspace U' of V such that M_α is the canonical projection $V \to V/U'$. Thus we deal with a vector space V with two subspaces U, U' and consider the corresponding representation of Q:

$$V/U \longleftarrow V \longleftarrow U'$$

One knows that there is a basis $\mathcal{B}$ of V such that both subspaces U, U' are generated by subsets of $\mathcal{B}$. But this means that we can decompose M into a direct sum of copies of the following representations

$$\begin{array}{cccc} 0 \longleftarrow k \xleftarrow{\ 1\ } k & 0 \longleftarrow k \longleftarrow 0 & k \xleftarrow{\ 1\ } k \xleftarrow{\ 1\ } k & k \xleftarrow{\ 1\ } k \longleftarrow 0 \\ b \in U \cap U' & b \in U \setminus U' & b \in U' \setminus U & b \notin U \cup U' \end{array}$$

(always, we specify which elements $b \in \mathcal{B}$ give rise to the representation in question). In particular, we see: *Any indecomposable representation of Q is thin.*

(2) The 2-subspace quiver. This is the following quiver

$$\underset{\circ}{1} \xrightarrow{\ \alpha\ } \underset{\circ}{2} \xleftarrow{\ \beta\ } \underset{\circ}{3}$$

Splitting off copies of $S(1)$ and of $S(3)$, we can assume that we deal with a representation M with both M_α, M_β injective. After polishing. we can assume that M_α and M_β are the inclusions subspaces U, U' of $V = M_2$, respectively. Thus we deal with a vector space V with two subspaces U, U'.

$$U \longrightarrow V \longleftarrow U'$$

One knows that there is a basis $\mathcal{B}$ of V such that both subspaces U, U' are generated by subsets of $\mathcal{B}$. But this means that we can decompose M into a direct sum of copies of the following representations:

$$\underset{b \in U \cap U'}{k \xrightarrow{1} k \xleftarrow{1} k} \qquad \underset{b \in U \setminus U'}{k \xrightarrow{1} k \longleftarrow 0} \qquad \underset{b \in U' \setminus U}{0 \longrightarrow k \xleftarrow{1} k} \qquad \underset{b \notin U \cup U'}{0 \longrightarrow k \longleftarrow 0}$$

(again, we specify which elements $b \in \mathcal{B}$ give rise to the representation in question). Also here, we see: *Any indecomposable representation of Q is thin.*

(3) The 2-factor-space quiver. This is the quiver

$$\overset{1}{\circ} \xleftarrow{\alpha} \overset{2}{\circ} \xrightarrow{\beta} \overset{3}{\circ}$$

One uses vector space duality in order to relate the representations of the 2-factor-space quiver Q and the representations of the 2-subspace quiver Q' (at least when dealing with finite-dimensional representations):

representation of Q	representation of Q'
$M_1 \xleftarrow{M_\alpha} M_2 \xrightarrow{M_\beta} M_3$	$M_1^* \xrightarrow{M_\alpha^*} M_2^* \xleftarrow{M_\beta^*} M_3^*$

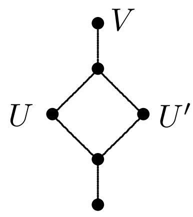
Or else, one shows that any polished representation of Q without direct summands $S(1), S(3)$ is of the form

$$V/U \longleftarrow V \longrightarrow V/U'$$

where U, U' are subspaces of a vector space V. Thus, again, we see: *any indecomposable representation is thin.*

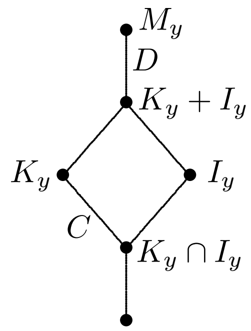
Altogether we have shown: *If Q is a quiver of type $\mathbb{A}_3$, then any indecomposable representation is thin.*

> Let us stress that the indecomposable non-simple representations of all quivers of type $\mathbb{A}_3$ have been obtained by looking at a vector space V with two subspaces U, U', thus by looking at the following subspace lattice of a vector space V:



Slogan: *Representation theory of quivers is just (a higher form of) linear algebra.*

Recall: The splitting-off of copies of $S(y)$ as considered in section 1.4 also relies on the same subspace lattice, namely we were dealing with:



Proposition. *Let Q be a quiver of type $\mathbb{A}_n$. Then any indecomposable representation of Q is thin.*

Proof: As we know already, the assertion is true for $n \leq 3$. Thus, consider now some $n \geq 4$. By induction, we may assume that any indecomposable representation of a quiver of type $\mathbb{A}_{n-1}$ is thin.

Let M be an indecomposable representation of Q with underlying graph

$$\overset{1}{\circ} \text{———} \overset{2}{\circ} \text{———} \cdots \qquad \text{———} \overset{n-1}{\circ} \text{———} \overset{n}{\circ}$$

and assume that both $M_1 \neq 0$ and $M_n \neq 0$. The restriction M' of M to the full subquiver Q' with vertices $1, \ldots, n-1$ is a direct sum of thin representations which are $(n-1)$-sincere, according to Proposition 1.3. For example, if $n = 6$, then M' is the direct sum of copies of the following 5 representations of Q' (where the edges have to be replaced by corresponding arrows, and all the maps $k \to k$ are identity maps):

$$\begin{array}{ccccccccc}
k & — & k & — & k & — & k & — & k \\
0 & — & k & — & k & — & k & — & k \\
0 & — & 0 & — & k & — & k & — & k \\
0 & — & 0 & — & 0 & — & k & — & k \\
0 & — & 0 & — & 0 & — & 0 & — & k
\end{array}$$

We claim that M' is *increasing from left to right*: this should mean that for any arrow α with $t(\alpha), h(\alpha) \in [1, n-1]$, the map $M'_\alpha (= M_\alpha)$ is a monomorphism provided $t(\alpha) < h(\alpha)$, and an epimorphism otherwise. Similarly, consider the restriction M'' of M to the full subquiver with vertices $2, \ldots, n$. The representation M'' is a direct sum of thin representations which are 2-sincere, according to Proposition 1.3. Thus M'' is *decreasing from left to right*: for any arrow α with $t(\alpha), h(\alpha) \in [2, n]$, the map $M''_\alpha (= M_\alpha)$ is an epimorphism, if $t(\alpha) < h(\alpha)$, otherwise a monomorphism.

It follows that all the maps M_α with $t(\alpha), h(\alpha) \in [2, n-1]$ are bijective, thus up to isomorphism, we can assume that these maps are identity maps. But then it is sufficient to look at the representation

$$M_1 \;\text{———}\; \begin{matrix} M_2 \\ = M_{n-1} \end{matrix} \;\text{———}\; M_n$$

(where the edges have to be replaced by the appropriate arrows), thus at a representation of a quiver of type $\mathbb{A}_3$. We know that this representation is a direct sum of thin representations, thus also M itself is a direct sum of thin representations. But since by assumption M is indecomposable, we conclude that M is thin.

Remark. The proof provides a normal form for all the indecomposable representations of Q. Namely, *any indecomposable representation of the quiver Q of type $\mathbb{A}$ is isomorphic to a representation M using as vector spaces only* 0 *and* k *and as non-zero maps only the identity map* $1\colon k \to k$. Thus M is determined by the pair of numbers $i \le j$, such that the support quiver $Q(M)$ consists of the vertices x with $i \le x \le j$ and all the arrows in-between. In particular, the classification of the indecomposable representations uses only combinatorial data.

> Actually, the assertion concerning the normal form is an easy consequence, once we have established that any indecomposable representation is thin. We just have to use the process of polishing inductively, starting at one end. It is easy to see that all thin representations of tree quivers can be polished in this way, as we will outline in the next section.

Corollary. *Let V be a vector space with two filtrations*

$$\begin{aligned} U_1 \subseteq U_2 \subseteq \cdots \subseteq U_p \subseteq V, \\ U'_1 \subseteq U'_2 \subseteq \cdots \subseteq U'_q \subseteq V. \end{aligned}$$

Then there is a basis $\mathcal{B}$ of V such that any of the subspaces U_i, U'_j is generated by a subset of $\mathcal{B}$.

Proof: Consider the quiver Q of type $\mathbb{A}_n$ with $n = p+q+1$ with vertices labeled $1, \dots, p, 1', \dots, q'$ and 0 and the following orientation

$$\overset{1}{\circ} \longrightarrow \overset{2}{\circ} \;—\; \cdots \;\rightarrow \overset{p}{\circ} \longrightarrow \overset{0}{\circ} \longleftarrow \overset{q'}{\circ} \leftarrow \;\cdots\; — \overset{2'}{\circ} \longleftarrow \overset{1'}{\circ}$$

The two filtrations yield the following representation of Q (all maps are the inclusion maps):

$$U_1 \longrightarrow U_2 \;—\; \cdots \;\rightarrow U_p \longrightarrow V \longleftarrow U'_q \leftarrow \;\cdots\; — U'_2 \longleftarrow U'_1$$

If we write this representation as a direct sum of indecomposable representations, thus of thin representations, and choose in any direct summand N a non-zero element $b \in N_0$, we obtain the required basis.

> **Slogan:** *One can use the representation theory of quivers in order to solve vector space problems.*

Theorem 1. *Let Q be a finite connected quiver. Then all indecomposable modules are thin if and only if Q is of type $\mathbb{A}_n$.*

Proof: We have seen above that the indecomposable representations of a quiver of type $\mathbb{A}_n$ are thin. Conversely, assume now Q is a connected quiver and all its indecomposable representations are thin. We look at some special cases of Q.

The Kronecker quiver Q. It has two vertices, labeled $1, 2$ and two arrows $\alpha, \beta \colon 2 \to 1$. Define M as follows: $M_1 = M_2 = k^2$, M_α the identity matrix, $M_\beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. One easily checks that M is indecomposable.

More general: Cycles. Say assume there are pairwise different vertices $x(1), \dots, x(n)$ with arrows $\alpha(i)$ such that $\{h(\alpha(i)), t(\alpha(i))\} = \{x(i), x(i+1)\}$ for all $1 \le i \le n$ (and $x(n+1) = x(1)$; in the Kronecker case, one also requires $\alpha(1) \neq \alpha(2)$). As in the Kronecker case, take $M_{x(i)} = k^2$, and take for all but one arrows $\alpha(i)$ the identity matrix, and $M_{\alpha(n)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Again, we get an indecomposable representation.

The quivers of type D_4. We have mentioned already the case of the subspace orientation. In general, we have to distinguish the 4 different orientations. The construction is quite similar in all cases. Namely, we start with the subspaces $k0, 0k, \Delta = \{(x, x) \mid x \in k\}$ of k^2 and take the following representations:

$$\begin{tikzcd}[column sep=small] k0 \arrow[dr] & 0k \arrow[d] & \Delta \arrow[dl] \\ & k^2 & \end{tikzcd} \quad \begin{tikzcd}[column sep=small] k0 \arrow[dr] & 0k \arrow[d] & k^2/\Delta \\ & k^2 \arrow[ur] & \end{tikzcd} \quad \begin{tikzcd}[column sep=small] k0 \arrow[dr] & k^2/0k & k^2/\Delta \\ & k^2 \arrow[u] \arrow[ur] & \end{tikzcd} \quad \begin{tikzcd}[column sep=small] k^2/k0 & k^2/0k & k^2/\Delta \\ & k^2 \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

The maps which we use are either the inclusion maps or the canonical projections. One may check directly that these representations are indecomposable. In section 1.7 we will see that the corresponding endomorphism rings are all equal to k.

If Q is not of type $\mathbb{A}_n$, then Q has a subquiver which is either a cycle or of type $\mathbb{D}_4$.

A quiver is said to be *representation-finite* (or to be of *finite representation type,* provided the number of isomorphism classes of indecomposable representations is finite. We have shown above that *any quiver of type $\mathbb{A}$ is representation-finite.*

> The connected quivers of finite representation type have been determined by Gabriel, they are the quivers whose underlying graph is a "Dynkin diagram". A typical example of a quiver which is not representation finite is the loop quiver $\mathbb{L}$; as we will point out in the next section, this is a consequence of the Jordan normal form of linear endomorphisms which usually is established in a Linear Algebra course.

2. Simple representations, thin representations.

2.1. Paths.

Given a representation M of the quiver Q, one defines the support quiver $Q(M)$ of M as follows: its vertices are the vertices x of Q with $M_x \neq 0$, the arrows of $Q(M)$ are the arrows α of Q such that $M_\alpha \neq 0$.

If Q is a quiver, a *path* $w = \alpha_1 \cdots \alpha_n$ in Q of length $n \geq 1$ is a sequence of arrows $\alpha_1, \ldots, \alpha_n$ such that $t(\alpha_i) = h(\alpha_{i+1})$ for $1 \leq i \leq n-1$. One calls $h(w) = t(\alpha_1)$ the *head* of w and $t(w) = t(\alpha_n)$ the *tail* of w; we also say that this is *a path from* $t(w)$ *to* $h(w)$. Such a path w should be visualized as follows:

$$\underset{h(w)}{\circ} \xleftarrow{\alpha_1} \circ \leftarrow \;\cdots\; - \circ \xleftarrow{\alpha_n} \underset{t(w)}{\circ}$$

In addition, any vertex x of Q is considered as a *path of length* 0 with head x and tail x, and then denoted by e_x. Actually, in order to have available a common notation for all the paths, we write also $(h(\alpha_1)|\alpha_1, \cdots, \alpha_n|t(\alpha_n))$ for the paths of length at least 1 and $e_x = (x||x)$ for those of length 0.

If two paths $w = (x|\alpha_1, \ldots, \alpha_n|y)$ and $w' = (x'|\alpha'_1, \ldots, \alpha'_{n'}|y')$ are given, the *concatenation* ww' is defined provided $y = x'$ and then $ww' = (x|\alpha_1, \ldots, \alpha_n, \alpha'_1, \ldots, \alpha'_{n'}|y')$; otherwise (if $y \neq x'$), we write $ww' = 0$.

> Here, 0 is a new element called the *zero path.* We later will consider the so-called path algebra kQ of the quiver; it is the k-vector space with basis all the non-zero paths, and 0 will be the zero vector in kQ; using the product of paths which we just have introduced, kQ becomes an associative k-algebra.

A quiver Q is called *strongly connected* provided for every pair x, y of vertices of Q there is a path w of length at least 1 with $h(w) = x, t(w) = y$. (In the definition, the vertices x, y are **not** assumed to be different; in particular, a strongly connected quiver has for every vertex x a path w of length at least 1 with $h(w) = x = t(w)$.)

An *oriented cycle* is a path w of length at least 1 such that $h(w) = t(w)$. Such an oriented cycle $w = \alpha_1 \cdots \alpha_n$ is said to be *elementary* provided the vertices $t(\alpha_i)$ with $1 \leq i \leq n$ are pairwise different.

If $w = (x|\alpha_1, \ldots, \alpha_n|y)$ is a path, then we write wM or $w(M)$ for the image $\alpha_1 \cdots \alpha_n(M_y)$ in M_x. In particular, if $e_x = (x||x)$ is a path of length 0, then $e_xM = M_x$.

> This explains why we draw arrows as pointing from right to left when we deal with paths: in this way we follow the convention of writing maps on the left of argument so that the composition of first a map f and second a map g has to be denoted as gf.

2.2. Simple representations.

A representation S of Q is said to be *simple* (or *irreducible* provided S is non-zero and any non-zero subrepresentation of S is equal to S. The representations $S(x)$ with $x \in Q_0$ are obviously simple.

The loop quiver. Let $\mathbb{L}$ be the loop quiver, it has just one vertex, say x and just one arrow, the loop $x \to x$. The representations of $\mathbb{L}$ are pairs (V, ϕ), where V is a vector space and $\phi\colon V \to V$ a linear map (a vector space endomorphism). If $(V, \phi), (V', \phi')$ are representations of $\mathbb{L}$, then an isomorphism $f\colon (V, \phi) \to (V', \phi')$ is an invertible linear map $f\colon V \to V'$ such that $\phi' = f\phi f^{-1}$. This shows that $(V, \phi), (V', \phi')$ are isomorphic if and only if the endomorphisms ϕ and ϕ' are similar.

> Similarity of vector space endomorphisms (and of square matrices) is a basic concept in Linear Algebra. Recall that two $(n{\times}n)$-matrices Φ, Φ' with coefficients in the field k are said to be similar provided there is an invertible $(n \times n)$-matrix F such that $\Phi' = F\Phi F^{-1}$. In case the two linear maps $\phi\colon V \to V$ and $\phi'\colon V' \to V'$ are similar, the vector spaces V, V' are isomorphic, thus (in the finite-dimensional case) isomorphic to the vector space k^n for some natural number n. Such isomorphisms are obtained by choosing in V a basis $\mathcal{B}$ and in V' a basis $\mathcal{B}'$. With respect to these bases, we can write ϕ and ϕ' as $(n{\times}n)$-matrices Φ, Φ', respectively, and obviously ϕ, ϕ' are similar linear maps if and only if Φ, Φ' are similar matrices.
>
> Here we deal with a situation where the structure of the field k does play a role: In Linear Algebra, the classification problem for square matrices up to similarity is usually only discussed in case k is algebraically closed (say if $k = \mathbb{C}$ is the field of complex numbers). In that case, the classification is given by the Jordan normal form which we will recall below. For the moment, we are only interested in the simple representations of $\mathbb{L}$, they correspond to the "irreducible" square matrices.

Clearly, one-dimensional representations of any quiver have to be simple. A one-dimensional representation of $\mathbb{L}$ is up to isomorphism of the form (k, λ) where λ denotes the multiplication map $\lambda\cdot\colon k \to k$ (with $a \in k$ being sent to λa). Namely, if (V, ϕ) is a one-dimensional representation of $\mathbb{V}$, choose a non-zero vector $b \in V$. It generates V, since V is one-dimensional, thus we have $\phi(b) = \lambda b$ for some $\lambda \in k$. Note that we have $\phi(a) = \lambda a$ for all $a \in V$; namely write $a = \mu b$ with $\mu \in k$, then

$$\phi(a) = \phi(\mu b) = \mu\phi(b) = \mu\lambda b = \lambda(\mu b) = \lambda a.$$

This shows, that λ is an **invariant** of (V, ϕ) and that different λ's yield non-isomorphic representations.

In case k is an infinite field, we obtain in this way infinitely many isomorphism classes of simple representations of $\mathbb{L}$. In case k is algebraically closed we get in this way all isomorphism classes of simple representations, otherwise not.

> Recall that k is said to be *algebraically closed* provided any polynomial of degree at least 1 has a zero, or alternatively, provided every

endomorphism of a non-zero vector space over k has an eigenvector. It is the latter condition which we need here: If $\phi\colon V \to V$ is an endomorphism, and $v \in V$ is an eigenvector of ϕ, say with eigenvalue λ, then the subspace $\langle v\rangle$ of V generated by v yields a one-dimensional subrepresentation of (V, ϕ) (since $\phi(\langle v\rangle) \subseteq \langle v\rangle$), and this subrepresentation is isomorphic to (k, λ).

Also in case k is a finite field, there are infinitely many isomorphism classes of simple representations of $\mathbb{L}$, they can be constructed as follows: Take a finite field extension $k \subseteq K$. One knows (see any algebra course dealing with finite fields) that there is a so-called "primitive" element $t \in K$, namely an element which generates K as a k-algebra. Then (K, t) *is a simple representation of* $\mathbb{L}$, here t again denotes the multiplication map $t\colon K \to K$ which maps $a \in K$ to ta.

Altogether we see: *For any field k, there are infinitely many isomorphism classes of simple representations of* $\mathbb{L}$. Let us add: *If (V, ϕ) is a simple representation of $\mathbb{L}$, then ϕ is bijective unless (V, ϕ) is isomorphic to $(k, 0) = S(*)$*, where $*$ is the unique vertex of $\mathbb{L}$. Namely, if ϕ is not bijective, then it has an eigenvector with eigenvalue 0, thus $(k, 0)$ is a subrepresentation of (V, ϕ).

If k is algebraically closed, the classification of the similarity classes of linear endomorphisms of vector spaces or, equivalently, of square matrices with coefficients in k, is given by the Jordan normal form: any square matrix is similar to the direct sum of Jordan blocks, a Jordan block is of the form

$$\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}$$

this is called the Jordan block $J(\lambda, n)$ of size n with eigenvalue λ. Two Jordan normal forms are similar provided they consist (up to possible permutations of the block) of the same Jordan blocks. The diagonal sum of square matrices $A_1, \dots, A_m$ is the matrix

$$\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_m \end{bmatrix}.$$

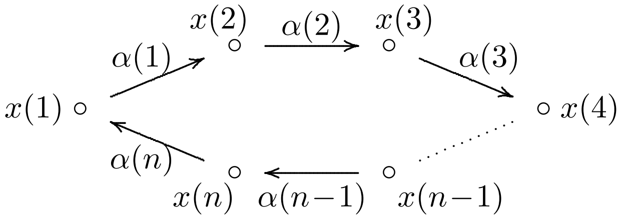
Let us reformulate these considerations in terms of the representation theory of quivers. The diagonal sum of square matrices corresponds to the direct sum of representations of $\mathbb{L}$. The Jordan blocks $J(\lambda, n)$ yield indecomposable representations of the loop quiver $\mathbb{L}$, namely

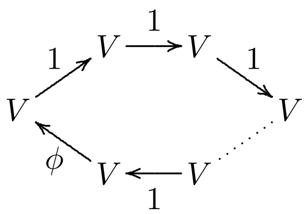
$(k^n, J(\lambda, n))$; different Jordan blocks yield non-isomorphic representations, and, if k is algebraically closed, any indecomposable representation of $\mathbb{L}$ is isomorphic to some $(k^n, J(\lambda, n))$. In case we deal with an arbitrary (not necessarily algebraically closed) field, then, as we have mentioned, there are additional simple representations S of $\mathbb{L}$; and there are the corresponding indecomposable representations with a “filtration” of arbitrary length with all factors isomorphic to a fixed simple representation S.

Elementary oriented cycles. Recall that an elementary oriented cycle is a path

$$w = \alpha(1) \cdots \alpha(n)$$

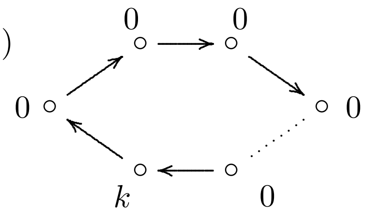
of length at least 1 such that $h(w) = t(w)$ and such that the vertices $x(i) = t(\alpha(i))$ with $1 \leq i \leq n$ are pairwise different. Given such an elementary oriented cycle w in the quiver Q, and a representation (V, ϕ) of the loop quiver $\mathbb{L}$, we define a representation $M = M(w, V, \phi)$ of Q as follows: Let $M_{x(i)} = V$ for $1 \leq i \leq n$ and $M_y = 0$ for the remaining vertices; let $M_{\alpha(i)}$ be the identity map for $1 \leq i < n$, let $M_{\alpha(n)} = \phi$, and $M_\beta = 0$ for the remaining arrows β. Thus, the essential part of M looks as follows:



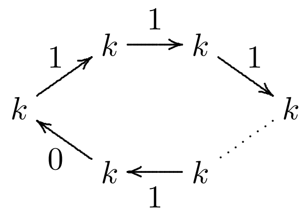


Note that if $(V, \phi) = (k, 0) = S(*)$, and $n \geq 2$, then $M(w, V, \phi) = M(w, k, 0)$ has the simple module $S(x_n)$ as submodule:

$S(x(n))$



$M(w, k, 0)$



Thus, $M(x, k, 0)$ is not simple (for $n \geq 2$). Let us now consider the representations $M(w, V, \phi)$, where (V, ϕ) is a simple representation of V which is not isomorphic to $(k, 0) = S(*)$.

Claim: *The construction $(V, \phi) \mapsto M(w, V, \phi)$ furnishes an injection from the set of isomorphism classes of the simple representations of $\mathbb{L}$ different from $S(*)$ into the set of isomorphism classes of the simple representations of Q.*

Proof: Let (V, ϕ) be a simple representation of $\mathbb{L}$ which is not isomorphic to $S(*)$ and let $M = M(w, V, \phi)$. Let M' be a non-zero submodule of M. We claim that $M_{x(1)} \neq 0$. In general, if M' is a submodule of M and $\alpha \colon x \to y$ is an arrow, such that M_α is injective, then with $M'_x \neq 0$ also $M'_y \neq 0$, since $0 \neq M_\alpha(M'_x) \subseteq M'_y$. Thus, in our case where all the maps $M_{\alpha(i)}$ are invertible, we see that $M'_{x(i)} \neq 0$ implies that also $M'_{x(i+1)} \neq 0$ (as usual we let $x(n+1) = x(1)$).

Let $U = M'_{x(1)} (\neq 0)$. Since the maps $M_{\alpha(i)}$ are the identity map for $1 \leq i < n$, we see inductively that $U = M_{\alpha(i)}(U) \subseteq M_{x(i+1)}$, thus $U \subseteq M'_{x(n)}$. Finally, $\phi(U) = M_{\alpha(n)}(U) \subseteq M_{x(1)} = U$ shows that $(U, \phi|U)$ is a non-zero submodule of (V, ϕ). Since (V, ϕ) is simple, we must have $U = V$ and therefore $M'_{x(i)} = M_{x(i)}$ for all i, thus $M' = M$. This shows that $M(w, V, \phi)$ is a simple representation of Q.

Now assume that (V, ϕ) and (V', ϕ') are simple representations of $\mathbb{L}$ and that there is an isomorphism $f = f_{x(i)} \colon M = M(w, V, \phi) \to M(w, V', \phi') = M'$. There are the commutative squares

$$\begin{tikzcd} M_{x(i)} \arrow[r, "f_{x(i)}"] \arrow[d, "M_{\alpha(i)}"'] & M'_{x(i)} \arrow[d, "M'_{\alpha(i)}"] \\ M_{x(i+1)} \arrow[r, "f_{x(i+1)}"] & M'_{x(i+1)} \end{tikzcd}$$

which are of the form

$$\text{for } 1 \leq i < n \quad \begin{tikzcd} V \arrow[r, "f_{x(i)}"] \arrow[d, "1"'] & V' \arrow[d, "1"] \\ V \arrow[r, "f_{x(i+1)}"] & V' \end{tikzcd} \qquad \text{and for } i = n \quad \begin{tikzcd} V \arrow[r, "f_{x(n)}"] \arrow[d, "\phi"'] & V' \arrow[d, "\phi'"] \\ V \arrow[r, "f_{x(1)}"] & V' \end{tikzcd}$$

It follows first that $f_{x(i)} = \psi$ for some fixed linear map ψ and all then $1 \leq i \leq n$, and that $\phi'\psi = \psi\phi$. Since ψ is invertible, we see that ϕ, ϕ' are similar linear maps, thus (V, ϕ) and (V', ϕ') are isomorphic representations of $\mathbb{L}$.

Thus, we see: *If Q has an orientec cycle, then there are infinitely many isomorphism classes of simple representations.*

Lemma. *The support quiver of a simple representation which is not of the form $S(x)$ for any vertex x is strongly connected.*

Proof. Let S be a simple representation which is not isomorphic to a representation of the form $S(x)$. If the support quiver $Q(S)$ of S has only one vertex, say x, there must be a loop $x \to x$, since otherwise $Q(S)$ is the quiver $\mathbb{A}_1$ and the representations of this quiver are just the direct sums of copies of $S(x)$. On the other hand, the quiver with one vertex and at least one loop is strongly connected.

Now assume that the support quiver of S has at least two vertices. Let $x \neq y$ be vertices of $Q(S)$ and assume that there is no path from x to y. For any vertex z, let $W(z, x)$ be the set of paths from x to z and let N_z be the sum of the subspaces $w(S_x)$. where $w \in W(z, x)$. Clearly, N is a subrepresentation of S. We have $N_x = S_x$, thus $N \neq 0$. And we have $N_y = 0$, whereas $S_y \neq 0$, thus $N \neq S$. This shows that N is a non-zero proper subrepresentation of S, thus S is not simple. This contradiction shows that there is a path from x to y, for any pair $x \neq y$ of vertices. But then there is also a proper path from x to itself, namely the concatenation of a path from x to some vertex $y \neq x$ with a path from y to x.

Theorem. *Let Q be a finite quiver. The following conditions are equivalent:*
(i) *There is an oriented cycle in Q.*
(ii) *There are infinitely many isomorphism classes of simple representations.*
(iii) *There is at least one simple representation which is not isomorphic to a representation of the form $S(x)$ with $x \in Q_0$.*

Proof: (i) $\implies$ (ii): Let w be an elementary oriented cycle and consider the representations M with support quiver being given by (the vertices and arrows of) w.

(ii) $\implies$ (iii): The number of representations of the form $S(x)$ is $|Q_0|$, thus finite.

(iii) $\implies$ (i). Let S be simple, not isomorphic to $S(x)$. It has been shown in the Lemma that the support quiver $Q(S)$ of S is strongly simply connected, thus $Q(S)$, and therefore Q, has an oriented cylce.

2.3. Factor representations and filtrations.

Let Q be a quiver and M a representation of Q. If M' is a subrepresentations of M, thus for every vertex x of Q, there is given a subspace M'_x of M_x, and if $\alpha\colon x \to y$ is an arrow, then $M_\alpha(M'_x) \subseteq M'_y$. We can form the factor spaces $M''_x = M_x/M'_x$, and since for the arrow $\alpha\colon x \to y$, we have $M_\alpha(M'_x) \subseteq M'_y$, one knows that M_α induces a linear map $M''_x = M_x/M'_x \to M_y/M'_y = M''_y$ which we denote by M''_α (the elements of M''_x are residue classes of the form $v + M'_x$ with $v \in M_x$ and M''_α is defined by $M''_\alpha(v+M'_x) = M_\alpha(v) + M'_y$; it is basic knowledge in Linear Algebra (and also easy to check) that this definition of M''_α is well-defined and yields again a linear map. We see that we obtain a representation $M'' = (M''_x, M''_\alpha)_{x,\alpha}$ which is called a *factor representation* (or a *factor module*) of M.

Sometimes we will consider filtrations of a representation M. A *filtration* of M is given by a chain of submodules of M, say

$$0 = M^{(0)} \subseteq M^{(1)} \subseteq M^{(2)} \subseteq \cdots \subseteq M^{(m)} = M,$$

and one calls the factor representations $M^{(i)}/M^{(i-1)}$ with $1 \leq i \leq m$ the *factors* of the filtration.

2.4. Nilpotent representations.

We say that a representation M is *nilpotent* provided it has a filtration with all factors being of the form $S(x)$ (with x vertices of the quiver).

Proposition. *Let Q be a quiver and M a representation of Q of dimension m. Then the following conditions are equivalent:*
(a) *M is nilpotent.*

(b) $w(M) = 0$ *for every path w of length m,*
(c) *There is a natural number t such that $w(M) = 0$ for any path of length t.*

Proof. (a) $\implies$ (b). We use induction on m. If $m = 1$, then $M = S(x)$ for some vertex x and then all maps M_α are zero-maps. Now if w is a path of length 1, then $w = \alpha$ for some arrow α and $w(M) = \alpha(M_{t(\alpha)}) = 0$.

Now assume that the implication has been shown for some $m \geq 1$, let M be of dimension $m + 1$. Since M is nilpotent, it has a filtration such that all factors are (1-dimensional and) of the form $S(x)$ for vertices x. This shows that there is a submodule M' of M of dimension m which is nilpotent and such that M/M' is isomorphic to some $S(x)$. By induction, we know that $w(M') = 0$ for all paths of length m. Let α be any arrow. We claim that $\alpha(M) \subseteq M'$. Let $\alpha\colon y \to z$, thus $\alpha(M) = M_\alpha(M_y)$. Now $M''_\alpha\colon M_y/M'_y \to M_z/M'_z$ is induced by M_α and is the zero map, since $M'' = S(x)$, thus $M_\alpha(M_y) \subseteq M'_z$.

If w is a path of length $m+1$, it is of the form $w = w'\alpha$, where w' is a path of length m and α is an arrow (thus a path of length 1). It follows that

$$w(M) = (w'\alpha)(M) = w'(\alpha(M)) \subseteq w(M') = 0.$$

This concludes the proof.

For (b) $\implies$ (c), nothing has to be shown.

(c) $\implies$ (a). Let M be non-zero and assume that there is a natural number t such that $w(M) = 0$ for any path of length t. For any vertex x and any natural number i, let $I(x, i)$ be the sum of the images $w(M)$, where w is a path of length i with head z, thus $I(x, i) \subseteq M_x$. We show the following: If $I(x, i) = M_x$ for all x and some $i \geq 1$, then also $I(x, i + 1) = M_x$ for all x. Namely, let $\alpha_j\colon x(j) \to y$ be the arrows with head y. Then $M_{x(j)} = I(x(j), i)$ and therefore

$$M_y = I(y, 1) = \sum_j \alpha_j(M_{x(j)}) = \sum \alpha_j(I(x(j), i) = \sum_w w(M)$$

where the last sum is indexed by all paths of length $j + 1$ with head y (observe that any path of length $j + 1$ with head y is of the form $\alpha_j w'$ with w' a path of length j with head $x(j)$.

Since by assumption, we have $I(x, t) = 0$ for all x, it follows that there has to be a vertex y such that there has to exist some z such that $I(z, 1)$ is a proper subspace of M_z. Define M' as follows: let M'_z be a maximal subspace of M_z which includes $I(z, 1)$, and let $M'_x = M_x$ for $x \neq z$. Since $I(z, 1) \subseteq M'_z$, it follows that M' is a subrepresentation of M and also that M/M' is isomorphic to $S(z)$.

Of course, with M also M' satisfies the condition (c), thus M' is nilpotent, and therefore also M is nilpotent.

In order to show the implication (c) $\implies$ (a), one also may start to construct a subrepresentation M' of M which is isomorphic to some $S(x)$ as follows: Let t be minimal such that $w(M) = 0$ for all paths of length t. If $t = 0$, then M itself is a direct sum of representations of the form $S(x)$. If $t \geq 0$, there is a path w with $w(M) \neq 0$. Any non-zero element in $w(M)$ generates a subrepresentation which is of the form $S(h(w))$. Now one may look at M/M'. By induction, M/M' is nilpotent, thus it remains to be seen that a representation M with a subrepresentation M' such that both M' and M/M' are nilpotent, is nilpotent.

The following is easy to verify: Let M be a nilpotent representation, say with a filtrations

$$0 = M^{(0)} \subseteq M^{(1)} \subseteq M^{(2)} \subseteq \cdots \subseteq M^{(m)} = M,$$

such that the factors $M^{(i)}/M^{(i-1)}$ with $1 \leq i \leq m$ are of the form $S(x)$. Then *m is the dimension of M and for any vertex x, the number of factors $M^{(i)}/M^{(i-1)}$ isomorphic to $S(x)$ is equal to the k-dimension of M_x.*

These numbers are called the Jordan-Hölder multiplicities. We will discuss them later in detail.

2.5. Thin representations.

Recall that a representation $M = (M_x, M_\alpha)_{x,\alpha}$ is thin provided all the vector spaces M_x are of dimension at most 1. Also we recall that the support quiver $Q(M)$ is given by the vertices x with $M_x \neq 0$ and the arrows α with $M_\alpha \neq 0$.

Let us start with two quite trivial general assertions.

Lemma.
(a) *If M is indecomposable, $Q(M)$ is connected.*
(b) *If M, M' are isomorphic representations, then $Q(M) = Q(M')$.*

Proof: (a) Assume that we can write $Q(M)$ as the disjoint union of two (non-empty) quivers Q', Q''. Let $M'_x = M_x$ if x is a vertex of Q' and $M'_x = 0$ otherwise. Similarly, let $M''_x = M_x$ if x is a vertex of Q'' and $M''_x = 0$ otherwise. Then one easily sees that both M', M'' are non-zero subrepresentations of M and that $M = M' \oplus M''$, thus M is decomposable.

(b) If $f = (f_x)_x \colon M \to M'$ is an isomorphism, then $f_x \colon M_x \to M'_x$ is an isomorphism for all vertices x, thus $M_x \neq 0$ if and only if $M'_x \neq 0$. Also, if $\alpha \colon x \to y$ is an arrow, then we have the following commutative diagram

$$\begin{tikzcd} M_x \arrow[r, "f_x"] \arrow[d, "M_\alpha"'] & M'_x \arrow[d, "M'_\alpha"] \\ M_y \arrow[r, "f_y"] & M'_y \end{tikzcd}$$

thus $M'_\alpha = f_x^{-1} M_\alpha f_y$ shows that $M_\alpha \neq 0$ if and only if $M'_\alpha \neq 0$.

Proposition 1. *Let M be a thin representation of Q. Then M is indecomposable if and only if the support quiver of M is connected.*

Proof: One direction is true in general, as the lemma above shows. Now let T be a tree, M a thin representation and $Q(M)$ connected. Now since $Q(M)$ is a connected subquiver of a tree quiver, it is again a tree quiver, thus, without loss of generality, we can assume that $Q = Q(M)$. If Q has only one vertex x, then $M = S(x)$. Thus we can assume that Q has at least two vertices. Since Q is a tree it is obtained from a quiver Q' by attaching an arm of the form $\mathbb{A}_2$ at a vertex x, say adding an arrow α with $\{h(\alpha), t(\alpha)\} = \{x, \omega\}$. By induction, the restriction M' of M to Q' is indecomposable. Since M_α is non-zero, it follows that with M' also M is indecomposable.

Proposition 2. *Let Q be a tree and M, M' indecomposable thin representations of Q. Then M, M' are isomorphic if and only if M, M' have the same support quiver.*

Proof: One direction is true in general. We have to consider the reverse implication, thus assume that M, N are indecomposable thin representations of a tree quiver Q such that $Q(M) = Q(N)$. As in the previous proof, we can assume that $Q = Q(M) = Q(N)$ and that Q has at least 2 vertices and is obtained from a quiver Q' by attaching an arm of the form $\mathbb{A}_2$ at a vertex x, say adding an arrow α with $\{h(\alpha), t(\alpha)\} = \{x, \omega\}$. Now by induction, the restrictions M' of M and N' of N to Q' are isomorphic, say there is an isomorphism $f = (f_x)_x \colon M' \to N'$. We need to define $f_\omega \colon M_\omega \to N_\omega$ so that we obtain an isomorphism $M \to N$. But this just means that we have to choose f_ω so that one of the following diagrams commutes, the left one in case $\alpha \colon x \to \omega$, the right one in case $\alpha \colon \omega \to x$.

$$\begin{array}{ccc} M_x & \xrightarrow{f_x} & N_x \\ {\scriptstyle M_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle N_\alpha} \\ M_\omega & \xrightarrow{f_\omega} & N_\omega \end{array} \qquad \begin{array}{ccc} M_x & \xrightarrow{f_x} & N_x \\ {\scriptstyle M_\alpha}\big\uparrow & & \big\uparrow{\scriptstyle N_\alpha} \\ M_\omega & \xrightarrow{f_\omega} & N_\omega \end{array}$$

thus, in case $\alpha \colon x \to \omega$, take $f_\omega = N_\alpha f_x M_\alpha^{-1}$, in case $\alpha \colon \omega \to x$, take $f_\omega = N_\alpha^{-1} f_x M_\alpha$.

As an immediate consequence of Propositions 1 and 2 we see: *If Q is a tree and M is an indecomposable thin representation with support quiver Q, then M is isomorphic to the representation M' with $M'_x = k$ for all vertices x and M'_α the identity map, for all arrows α;* we may call M' the *normal form*).

Proposition 3. *Let Q be a tree and M an indecomposable thin representation of Q. If Y is a representation with a subrepresentation X such that both X and Y/X are isomorphic to M, then X is a direct summand of Y* (thus, there is a subrepresentation Z of Y with $Y = X \oplus Z$ and Z is necessarily also isomorphic to M.)

> We later may reformulate this assertion by saying that for a tree quiver Q any indecomposable thin representation is "exceptional".

Proof: Again, we assume that $Q = Q(M)$ and use induction. In case Q has only one vertex $*$, the representation Y has to be the direct sum of two copies of $S(*)$.

Now consider the case that Q has at least 2 vertices. Before we continue, let us note that for α an arrow in $Q = Q(M)$, the map M_α is bijective, thus X_α and $(Y/X)_\alpha$

are bijective and therefore also Y_α is bijective. We assume that Q is obtained from a quiver Q' by attaching an arm of the form $\mathbb{A}_2$ at a vertex x, say adding an arrow α with $\{h(\alpha), t(\alpha)\} = \{x, \omega\}$. We consider the restrictions X' of X and Y' of Y to Q'. By induction, there is a subrepresentation Z' of Y' such that $Y' = X' \oplus Z'$. If $\alpha\colon x \to \omega$, let $Z_\omega = Y_\alpha(Z_x)$. If $\alpha\colon \omega \to x$, let $Z_\omega = Y_\alpha^{-1}(Z_x)$. Then $Y = X \oplus Z$.

Elementary cycle. We consider now a quiver of type $\widetilde{\mathbb{A}}_n$, such a quiver has $n+1$ vertices, say labeled $0, 1, \ldots, n$ and $n+1$ arrows $\alpha(0), \ldots, \alpha(n)$ with $\{t(\alpha(i)), h(\alpha(i)\} = \{i, i+1\}$ (modulo $n+1$). *The isomorphism classes of thin indecomposable representations with support quiver Q are indexed by the non-zero element of the base field k.* Thus, if k is an infinite field, there are infinitely many isomorphism classes of indecomposable thin representations. If k is a finite field, say with q elements, then the number of isomorphism classes of indecomposable thin representations with support quiver Q is equal to $q-1$.

Proof: If M is an indecomposable thin representation with support quiver Q, it is isomorphic to a representation M' with $M'_x = k$ for all $x \in Q_0$ and $M'_{\alpha(i)}$ the identity map for all the arrows $\alpha(i)$ with $1 \le i \le n$. Namely, let Q' be obtained from Q by deleting one arrow, say $\alpha(0)$, but keeping all the vertices. Then Q' is a tree, and we may write $M'|Q'$ in normal form. Now $M'_{\alpha(0)}$ is an arbitary non-zero element of k and this element is uniquely determined by the isomorphism class of M.

3. Homomorphisms.

3.1. Definition, some properties.

If M, M' are representations of the quiver Q, a *homomorphism* $f\colon M \to M'$ is of the form $f = (f_x)_x$ with linear maps $f_x\colon M_x \to M'_x$ for all $x \in Q_0$ such that the following diagrams for every arrow $\alpha\colon x \to y$ commute

$$\begin{array}{ccc} M_x & \xrightarrow{f_x} & M'_x \\ {\scriptstyle M_\alpha}\downarrow & & \downarrow{\scriptstyle M'_\alpha} \\ M_y & \xrightarrow{f_y} & M'_y \end{array}$$

To repeat: one has such a diagram for every arrow α of the quiver; the vertical data on the left are part of M, those on the right are part of M', the horizontal maps are those which combine to form f.

Of course, given a representation M, there is always the identity homomorphism $1_M\colon M \to M$ with $(1_M)_x$ the identity map of M_x. Also, for any pair M, M' of representations, there is the zero homomorphism $0\colon M \to M'$ (with $0_x\colon M_x \to M'_x$ being the zero map).

Examples. Consider the three representations

$$(0 \to k), \quad (k \to 0), \quad (1_k\colon k \to k)$$

of the quiver Q of type $\mathbb{A}_2$, and let us determine whether there are non-zero homomorphisms $M \to M'$ or not. Of course, If $M = (0 \to k)$ and $M' = (k \to 0)$, there cannot be a non-zero homomorphism $f\colon M \to M'$, since $f = (f_1, f_2)$ and for $f_1\colon M_1 \to M'_1$ and for $f_2\colon M_2 \to M'_2$ there only exist the zero maps. Now let $M = (0 \to k)$ and $M' = (1\colon k \to k)$, and look for pairs $f = (f_1, f_2)$ with $f_1\colon M_1 \to M'_1$ and $f_2\colon M_2 \to M'_2$. For f_1 the only possibility is the zero map, whereas for $f_2\colon k \to k$ we may try to take any scalar multiplication, say take the multiplication by $c \in k$ (as a map $k \to k$). But of course, we have to check whether the following diagram is commutative:

$$\begin{array}{ccc} 0 & \longrightarrow & k \\ \downarrow & & {\scriptstyle 1}\downarrow \\ k & \xrightarrow{c} & k \end{array}$$

it always is, thus there are non-zero homomorphisms $(0 \to k) \longrightarrow (1\colon k \to k)$. (Note that when drawing this square, as well as the following ones, we follow the convention mentioned above: the vertical maps are those of the form M_α, M'_α, whereas the horizontal ones are those of the form f_1 and f_2.) On the other hand, if we are looking for homomorphisms $(1\colon k \to k) \longrightarrow (0 \to k)$, we have to deal with the diagram

$$\begin{tikzcd} k \arrow[r] \arrow[d, "1"'] & 0 \arrow[d] \\ k \arrow[r, "c"] & k \end{tikzcd}$$

and here it turns out that the diagram commutes only in case $c = 0$, thus there is no non-zero homomorphism $(1\colon k \to k) \longrightarrow (0 \to k)$.

In a similar way, one deals with homomorphisms between $(k \to 0)$ and $(1\colon k \to k)$. The only homomorphism $(k \to 0) \longrightarrow (1\colon k \to k)$ is the zero homomorphism, since the following diagram on the left commutes only for $c = 0$.

$$\begin{tikzcd} k \arrow[r, "c"] \arrow[d] & k \arrow[d, "1"] \\ 0 \arrow[r] & k \end{tikzcd} \qquad \begin{tikzcd} k \arrow[r, "c"] \arrow[d, "1"'] & k \arrow[d] \\ k \arrow[r] & 0 \end{tikzcd}$$

On the other hand, the above diagram on the right commutes for all c, thus any $c \in k$ defines a homomorphism $(1\colon k \to k) \longrightarrow (k \to 0)$.

Summarizing these considerations, we see that we can order the indecomposable representations of Q

$$(0 \to k), \quad (1_k\colon k \to k), \quad (k \to 0)$$

so that non-invertible homomorphisms go from left to right.

If M, M', M'' are representations of the quiver Q, and $f\colon M \to M'$, $g\colon M' \to M''$ are homomorphisms, then the definition $(gf)_x = g_x f_x$ yields a homomorphism $gf = ((gf)_x)_x\colon M \to M''$, the *composition* of these homomorphisms. Note that the composition is both associative and bilinear.

Let M, N be representations of the quiver Q. Let $\mathrm{Hom}(M, N)$ be the set of homomorphisms $f\colon M \to N$. This set $\mathrm{Hom}(M, N)$ *is a k-space* with respect to the following addition and scalar multiplication: Let $f = (f_x)_x$ and $f' = (f'_x)_x$ be homomorphisms $M \to N$ and $c \in k$, we define $f + f'$, $cf\colon M \to N$ by $(f + f')_x = f_x + f'_x$ and $(cf)_x = cf_x$. (Here, one has to check that $f + f'$ as well as cf are again homomorphisms; also one has to check that with this definition of addition and scalar multiplication, the vector space axioms are satisfied.) In particular, the zero homomorphism $M \to N$ is the zero element of the vector space $\mathrm{Hom}(M, N)$. It should be stressed that for finite-dimensional representations M, N, also $\mathrm{Hom}(M, N)$ is a finite-dimensional k-space.

If M, M', N, N' are representations of the quiver Q, and $f\colon M \to M'$ is a homomorphism, then the composition yields a k-linear map

$$\mathrm{Hom}(f, N)\colon \mathrm{Hom}(M', N) \to \mathrm{Hom}(M, N),$$

it is defined by $\mathrm{Hom}(f,N)(h) = hf$ for $h \in \mathrm{Hom}(M',N)$. Similarly, if $g\colon N \to N'$ is a homomorphism, then the composition yields a k-linear map

$$\mathrm{Hom}(M,g)\colon\ \mathrm{Hom}(M,N) \to \mathrm{Hom}(M,N'),$$

it is defined by $\mathrm{Hom}(f,N)(h) = gh$ for $h \in \mathrm{Hom}(M,N)$.

Let $f\colon M \to M'$ be a homomorphism. We say that f is a *monomorphism,* or an *epimorphism,* or an *isomorphism,* provided all the maps f_x are injective, or surjective, or bijective, respectively. Note that *if $f\colon M \to M'$ is an isomorphism, then $f^{-1}\colon M' \to M$* defined by $(f^{-1})_x = (f_x)^{-1}$ *is again a homomorphism,* and of course also an isomorphism. (Proof: Let $\alpha\colon x \to y$ be an arrow of Q. It follows from $M'_\alpha f_x = f_y M_\alpha$ that $M_\alpha(f_x)^{-1} = (f_y)^{-1}M'_\alpha$. This is what is needed in order that $((f_x)^{-1})_x$ is a homomorphism.) If an isomorphism $f\colon M \to M'$ exists, then M, M' are said to be *isomorphic.*

Note that the composition of two monomorphisms, epimorphisms, isomorphisms is again a monomorphism, epimorphism, isomorphism, respectively. The following is quite easy to check: *A homomorphism $f\colon M \to M'$ is an isomorphism if and only if there is a homomorphism $g\colon M' \to M$ such that $gf = 1_M$ and $fg = 1_{M'}$.*

Let us also record the following observations: If $f\colon M \to M'$ is a homomorphism such that all the maps f_x for $x \in Q_0$ are inclusion maps, then M' is a subrepresentation of M and f is called the corresponding *inclusion map.* Of course, an inclusion map is a monomorphism. Also recall that given a subrepresentation M' of a representation M, then we form the factor representation $M'' = M/M'$ and there are the canonical projection maps $q_x\colon M_x \to (M/M')_x = M''_x$, they combine to a homomorphism $q\colon M \to M'' = M/M'$. Of course, q is an epimorphism.

If $f\colon M \to M'$ is a homomorphism of representations of Q, then its *kernel* $\mathrm{Ker}(f)$ is the subrepresentation of M with $(\mathrm{Ker}(f))_x = \mathrm{Ker}(f_x)$, and the *image* $\mathrm{Im}(f)$ is the subrepresentation of M' with $(\mathrm{Im}(f))_x = \mathrm{Im}(f_x)$. Finally, define $M'/\mathrm{Im}(f)$ to be the *cokernel* of f. Also note: Given a monomorphism $u\colon M \to M'$, then M is isomorphic to the image of u. If $q\colon M \to M'$ is an epimorphism, then M' is isomorphic to $M/\mathrm{Ker}(q)$.

If M, M', N, N' are representations of Q, then there are canonical identifications:

$$\begin{aligned}\mathrm{Hom}(M, N \oplus N') &= \mathrm{Hom}(M,N) \oplus \mathrm{Hom}(M,N'),\\ \mathrm{Hom}(M \oplus M', N) &= \mathrm{Hom}(M,N) \oplus \mathrm{Hom}(M',N).\end{aligned}$$

3.2. Endomorphism rings.

Let Q be a quiver and M a representation of Q. A homomorphism $f\colon M \to M$ is called an *endomorphism* of M. In case f is invertible, one calls it an *automorphism.* The set $\mathrm{End}(M)$ of all endomorphisms of M is a ring, even a k-algebra, it is called the *endomorphism ring* of M. (Since $\mathrm{End}(M) = \mathrm{Hom}(M,M)$, it is a k-space, the composition of endomorphisms yields an associative multiplication which is bilinear, thus satisfies the distributivity laws. The identity map $1 = 1_M$ is the unit element of the ring $\mathrm{End}(M)$. If $c \in k$, the scalar multiple $c \cdot 1$ is the scalar multiplication on M (sending $a \in M$ to ca); these scalar multiples $c \cdot 1$ commute with all endomorphisms, thus the map $k \to \mathrm{End}(M)$ which sends c to $c \cdot 1$ is a ring homomorphism from k into the center of $\mathrm{End}(M)$, in this way, $\mathrm{End}(M)$ is a k-algebra.) In case $M \neq 0$, the ring homomorphism $k \to \mathrm{End}(M)$ defined by $c \mapsto c \cdot 1$ is injective; we may consider this as an embedding of k into $\mathrm{End}(M)$. Of course, if M is a finite-dimensional representation, then $\mathrm{End}(M)$ is a finite-dimensional k-algebra.

Let us stress that the endomorphism ring $\mathrm{End}(M)$ of a representation M is usually non-commutative (as one knows already from the case of the quiver $\mathbb{A}_1$; the representations of this quiver are just vector spaces, and the endomorphism ring of a vector space V is commutative only in case the dimension of V is at most 1).

Of special interest are the idempotents in $\mathrm{End}(M)$. Recall that an element e of a ring is called an *idempotent* provided $e^2 = e$; the elements 0 and 1 of $\mathrm{End}(M)$ are always idempotents, and it is interesting to know whether there are additional idempotents.

Lemma. *Given any representation M of a quiver Q, there is a bijection between the set of idempotents in* $\mathrm{End}(M)$ *and the direct decompositions $M = M' \oplus M''$, where the idempotent e corresponds to the direct decomposition $M = \mathrm{Im}(e) \oplus \mathrm{Ker}(e)$, and conversely, the direct decomposition $M = M' \oplus M''$ corresponds to the canonical projection of M onto M' (with kernel M'').*

The canonical projection of $M = M' \oplus M''$ onto M' is given by the map which sends $a' + a''$ (where $a' \in M', a'' \in M''$) onto a'.

Proof: Many things have to be verified.

Let us start with e an idempotent. We know already that both $\mathrm{Im}(e)$ and $\mathrm{Ker}(e)$ are subrepresentations of M, thus we only have to verify we obtain in this way a direct decomposition. First, $\mathrm{Im}(e) \cap \mathrm{Ker}(e) = 0$; namely, if $a' \in \mathrm{Im}(e)$, then a' is of the form $a' = e(a)$ for some $a \in M$; if a' also belongs to $\mathrm{Ker}(e)$, then $0 = e(a') = e(ea) = (e^2)a = ea = a'$. Second, $\mathrm{Im}(e) + \mathrm{Ker}(e) = M$; namely, if $a \in M$, then $a = e(a) + (1-e)(a)$ and $e(a)$ belongs to $\mathrm{Im}(e)$, whereas $(1-e)(a)$ obviously belongs to $\mathrm{Ker}(e)$.

Next, start with a direct decomposition $M = M' \oplus M''$ and let e be the canonical projection onto M'. Here one has to verify that this is indeed a homomorphism (it is the composition of the projection $M \to M/M''$, the identification $M/M'' \to M'$ and the embedding $M' \to M$). In addition, we need to know that $e^2 = e$, but this is obvious from the definition.

If we start with the idempotent e, and consider the direct decomposition $M = \mathrm{Im}(e) \oplus \mathrm{Ker}(e)$, then it turns out that the canonical projection of M onto $\mathrm{Im}(e)$ is precisely e. Namely, consider an element $a = a' + a''$ with $a' \in \mathrm{Im}(e)$ and $a'' \in \mathrm{Ker}(e)$. Let $a' = e(b)$ for some $b \in M$. If we apply e to $a = a' + a''$, we obtain $e(a) = e(a') + e(a'') = e(e(b)) = e(b) = a'$ (using that $e^2 = e$ and that $e(a'') = 0$).

Conversely, if we start with the direct decomposition $M = M' \oplus M''$ and consider the canonical projection e of M onto M', then clearly M' is the image of e, whereas M'' is contained in the kernel of e. It only remains to observe that M'' has to be the kernel of e: if an element $a \in M$ belongs to the kernel of a, then write $a = a' + a''$ with $a' \in M', a'' \in M''$;

as we know, $a' \in \operatorname{Im}(e), a'' \in \operatorname{Ker}(e)$. Since both a, a'' belong to $\operatorname{Ker}(e)$, also $a' = a - a''$ belongs to $\operatorname{Ker}(e)$. Thus $a' \in \operatorname{Im}(e) \cap \operatorname{Ker}(e) = 0$ (this we have shown for any idempotent e), therefore $a = a'' \in M''$.

Corollary. *Let M be a non-zero representation. Then M is indecomposable if and only if the only idempotents in* $\operatorname{End}(M)$ *are* 0 *and* 1.

As an example, let us calculate the endomorphism ring in one example. We deal with the 3-subspace quiver with vertices labeled $0, 1, 2, 3$ as shown below on the left, and we consider the representation M shown on the right.

$$\begin{tikzcd} \overset{1}{\circ} \arrow[dr] & \overset{2}{\circ} \arrow[d] & \overset{3}{\circ} \arrow[dl] \\ & \underset{0}{\circ} & \end{tikzcd} \qquad\qquad \begin{tikzcd} k0 \arrow[dr] & 0k \arrow[d] & \Delta \arrow[dl] \\ & k^2 & \end{tikzcd}$$

with $\Delta = \{(c, c) \mid c \in k\}$, or better $\Delta = \{\left[\begin{smallmatrix} c \\ c \end{smallmatrix}\right] \mid c \in k\}$.

Let $f = (f_0, f_1, f_2, f_3)$ be an endomorphism, thus $f_0 \colon k^2 \to k^2$ is given by a (2×2)-matrix F with coefficients in k. The commutativity of the diagram

$$\begin{tikzcd} k0 \arrow[r, "f_1"] \arrow[d, "u"'] & k0 \arrow[d, "u"] \\ k^2 \arrow[r, "f_0"] & k^2 \end{tikzcd}$$

(here, u denotes the inclusion map) implies that F is an upper triangular matrix, Similarly, the commutativity of the diagram

$$\begin{tikzcd} 0k \arrow[r, "f_2"] \arrow[d, "u"'] & 0k \arrow[d, "u"] \\ k^2 \arrow[r, "f_0"] & k^2 \end{tikzcd}$$

shows that F is a lower triangular matrix. Thus F is a diagonal matrix, say $F = \begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix}$. But there is a third commutativity condition:

$$\begin{tikzcd} \Delta \arrow[r, "f_3"] \arrow[d, "u"'] & \Delta \arrow[d, "u"] \\ k^2 \arrow[r, "f_0"] & k^2 \end{tikzcd}$$

it asserts that F maps Δ into Δ. But $F \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$. It follows that $d_1 = d_2$, thus F is a scalar matrix, say the multiplication by $c \in k$. But then also f_1, f_2, f_3 (being restrictions of f_0) are the multiplication by c, therefore $\mathrm{End}(M) = k$. In particular, we see that M *is indecomposable.*

Proposition. *Let M, M' be representations of the quiver Q such that the support quivers $Q(M)$ and $Q(M')$ are disjoint. Then*

$$\mathrm{End}(M \oplus M') = \mathrm{End}(M) \times \mathrm{End}(M').$$

Given two rings R, R' we denote by $R \times R'$ the product; it is defined on the set $R \times R'$ using component wise addition and multiplication.

Proof: In general, we have

$$\mathrm{End}(M \oplus M') = \begin{bmatrix} \mathrm{End}(M) & \mathrm{Hom}(M', M) \\ \mathrm{Hom}(M, M') & \mathrm{End}(M') \end{bmatrix}.$$

Since $Q(M) \cap Q(M') = \emptyset$, we have $\mathrm{Hom}(M', M) = 0 = \mathrm{Hom}(M, M')$.

3.3. Recollection of general results.

Here we should insert some general results from ring and module theory.

The rings which we will consider here are (associative, and not necessarily commutative) rings with 1. Note that we allow that a ring consists just of one element, this is the zero-ring (and there it holds that $0 = 1$). The zero ring arises naturally as the endomorphism ring of the zero representation of a quiver (and similarly as the endomorphism ring of the zero modules in module theory).

Recall that a ring R is said to be *local,* provided it is not the zero ring and has a unique maximal left ideal I. This maximal left ideal is necessarily a two-sided ideal, and contains every left ideal, it is called the *radical* of R. Also, R/I is a division ring. Note that a ring R is local if and only if the set of non-invertible elements is closed under addition, thus if R is local, also the opposite ring is local (thus R is local if and only if R has a unique maximal right ideal).

The only idempotents of a local ring are 0 and 1 (but there are many non-local rings which have only these two idempotents, for example the ring $\mathbb{Z}$ of the integers).

Fitting Lemma. *An endomorphism of a finite-dimensional indecomposable module is either bijective or nilpotent.*

Corollary. *A finite-dimensional algebra which has only* $0, 1$ *as idempotents, is a local ring with nilpotent radical.*

Proof: Just consider the algebra as a module over itself.

Corollary. *Let Q be a quiver and M a finite-dimensional indecomposable representation of Q. Then* $\mathrm{End}(M)$ *is a local ring with nilpotent radical.*

The locality of endomorphism rings of indecomposable objects has strong consequences, the most important one is the uniqueness of direct decompositions, as formulated in the theorem of Krull-Remak-Schmidt. Since this is usually formulated for module categories, we will discuss this result when we have identified the category of representations of a finite quiver Q with the category of finite-dimensional modules over the path algebra kQ.

3.4. Homomorphisms between thin indecomposable representations.

Proposition. *Let M, M' be thin indecomposable representations of a tree quiver Q. Then* $\operatorname{Hom}(M, M')$ *is at most one-dimensional.*

Proof. Let $f = (f_x)_x \colon M \to M'$ be a homomorphism. Clearly $f_x \neq 0$ implies that x belongs to $Q(M) \cap Q(M')$ (this holds true for general quivers). We assume that Q is a tree, and that M, M' are indecomposable representations. Thus $Q(M), Q(M')$ are again trees and if $Q(M) \cap Q(M') \neq \emptyset$, then $Q(M) \cap Q(M')$ is a tree. If there is an arrow $\alpha \colon x \to y$ in $Q(M) \cap Q(M')$, then we see that $f_x = f_y$:

$$\begin{array}{ccc} M_x & \xrightarrow{f_x} & M'_x \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ M_y & \xrightarrow{f_y} & M'_y \end{array} \qquad\qquad \begin{array}{ccc} k & \xrightarrow{f_x} & k \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ k & \xrightarrow{f_y} & k \end{array}$$

Note that the proof shows: *If M, M' are thin indecomposable representations of a tree quiver Q and $f \colon M \to M'$ is a non-zero homomorphism, then $Q(M) \cap Q(M')$ is a connected subquiver of Q and the image of f is the thin representation with support quiver $Q(M) \cap Q(M')$.*

In general, it is not difficult to decide whether $\operatorname{Hom}(M, M')$ is zero or 1-dimensional, Let us write down the rule in a special case:

The case of a linearly ordered $\mathbb{A}_n$-quiver.

We consider a linearly ordered quiver Q of type $\mathbb{A}_n$, say with the following vertices:

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow \cdots \longleftarrow n{-}1 \longleftarrow n$$

For every pair of integers i, j with $1 \leq i \leq j \leq n$, we define a representation $[i, j]$ with $[i, j]_x = k$ if $i \leq x \leq j$ and $[i, j]_x = 0$ otherwise, and such that $[i, j]_\alpha$ is the identity map whenever possible. We know that we obtain in this way all the indecomposable representations of Q, one from each isomorphism class.

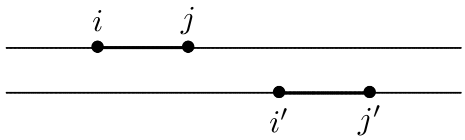
Proposition.

$$\operatorname{Hom}([i, j], [i', j']) = \begin{cases} k & \textit{if} \quad i \leq i' \leq j \leq j' \\ 0 & \qquad \textit{otherwise} \end{cases} ,$$

and the image of any non-zero homomorphism $[i,j] \to [i',j']$ *is just* $[i',j]$.
If $i = i' \le j \le j'$, *then any non-zero homomorphism* $[i,j] \to [i',j']$ *is a monomorphism.*
If $i \le i' \le j = j'$, *then any non-zero homomorphism* $[i,j] \to [i',j']$ *is an epimorphism.*

Proof: Note that the considerations to be done will be the same as those in the special case $\mathbb{A}_2$ discussed in 3.1. Let $f\colon [i,j] \to [i',j']$.

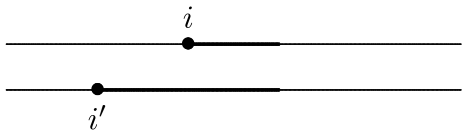
We distinguish several cases. First, assume that $j < i'$.



In this case $Q([i,j]) \cap Q([i',j']) = \emptyset$. Similarly, if $j' < i$, then $Q([i,j]) \cap Q([i',j']) = \emptyset$. In both cases we have $\mathrm{Hom}([i,j],[i',j']) = 0$.

From now on, we assume that $i' \le j$ and $i \le j'$.

Let $i' < i$, thus we deal with

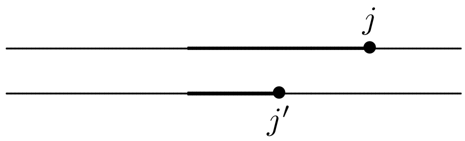


and both $i \le j, j'$. In particular, $i \in Q([i,j]) \cap Q([i',j'])$. Let us consider the arrow $i{-}1 \leftarrow i$:

$$\begin{tikzcd} {[i,j]_i} \arrow[r, "f_i"] \arrow[d] & {[i',j']_i} \arrow[d] \\ {[i,j]_{i-1}} \arrow[r] & {[i',j']_{i-1}} \end{tikzcd} \qquad \begin{tikzcd} k \arrow[r, "f_i"] \arrow[d] & k \arrow[d, "1"] \\ 0 \arrow[r] & k \end{tikzcd}$$

we see that we must have $f_i = 0$. But if $f_x = 0$ for some x in the intersection of the support quivers, then $f = 0$.

Let $j' < j$, thus we deal with

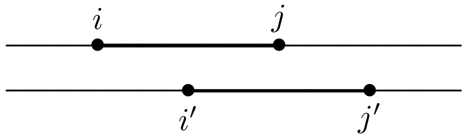


and both $i, i' \le j'$. In particular, $j' \in Q([i,j]) \cap Q([i',j'])$. Let us consider the arrow $j' \leftarrow j'{+}1$:

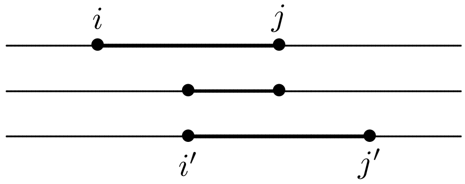
$$\begin{tikzcd} {[i,j]_{j'+1}} \arrow[r] \arrow[d] & {[i',j']_{j'+1}} \arrow[d] \\ {[i,j]_{j'}} \arrow[r, "f_{j'}"'] & {[i',j']_{j'}} \end{tikzcd} \qquad \begin{tikzcd} k \arrow[r] \arrow[d, "1"'] & 0 \arrow[d] \\ k \arrow[r, "f_{j'}"'] & k \end{tikzcd}$$

we see that we must have $f_{j'} = 0$, and again it follows that $f = 0$. Thus, in all cases discussed so far, $\mathrm{Hom}([i,j],[i',j']) = 0$.

It remains to consider the case $i' \leq i \leq j' \leq j$.



In this case we claim that there is a non-zero map $f\colon [i,j] \to [i',j']$, or even better that there is an epimorphism $g\colon [i,j] \to [i',j]$ and a monomorphism $h\colon [i',j] \to [i',j']$



so that we can take for f the composition $f = hg$. In order to define such maps f, g, h, just take $f_x = g_x = h_x = 1_k$ for $i' \leq x \leq j$ and zero otherwise (of course, one has to check that the diagrams in question commute). It follows that in this last case, $\mathrm{Hom}([i,j],[i',j'])$ is non-zero (and one-dimensional).

Example: $n = 4$. We consider the quiver Q of type $\mathbb{A}_4$

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4$$

with the indecomposable representations $[i,j]$ with $1 \leq i \leq j \leq 4$. The following picture arranges these representations in a triangle.

$$\begin{tikzcd}
{[1,1]} \arrow[dr] \arrow[rr, dotted, no head] & & {[2,2]} \arrow[dr] \arrow[rr, dotted, no head] & & {[3,3]} \arrow[dr] \arrow[rr, dotted, no head] & & {[4,4]} \\
& {[1,2]} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & {[2,3]} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & {[3,4]} \arrow[ur] & \\
& & {[1,3]} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & {[2,4]} \arrow[ur] & & \\
& & & {[1,4]} \arrow[ur] & & &
\end{tikzcd}$$

The arrows $[i,j] \to [i,j+1]$ (drawn in southeast direction) indicate the existence of a so-called "irreducible" monomorphism, the arrows $[i,j] \to [i+1,j]$ (drawn in northeast direction) indicate the existence of a so-called "irreducible" epimorphism. Note that all the paths in southeast direction indicate the existence of corresponding monomorphisms, the paths in northeast direction indicate the existence of corresponding epimorphisms. As we know, any non-zero homomorphism $[i,j] \to [i',j']$ has as image an indecomposable representation, namely $[i',j]$, and there is a corresponding concatenation of a northeast path followed by a southeast path (for example, any non-zero homomorphism $[1,3] \to [3,4]$ has a factorization $[1,3] \to [2,3] \to [3,3] \to [3,4]$).

Thus, we deal with a visualization of the category $\mathcal{A}$ of indecomposable representations of Q, it is called the "Auslander-Reiten quiver" of the category of representations of Q. We also should mention the meaning of the dotted lines: they indicate "relations": In the upper line, they indicate that the composition of maps corresponding to a southeast arrow and the next northeast arrow is zero. The lower dotted lines mark the commutativity of the corresponding squares. Actually, this Auslander-Reiten quiver (with its vertices, arrows and dotted lines) provides a presentation of the category $\mathcal{A}$. by generators and relation.
Such an Auslander-Reiten quiver is not just a quiver, but a so-called translation quiver; the translation is indicated by the dotted lines. Any translation quiver may be considered as a 2-dimensional simplicial complex. In our case, the triangles can be seen quite well, all are bounded by a northeast arrow, a southeast arrow and a dotted line. In our example, these (small) triangles fit together to form a large triangle.

Again we see that we can order the indecomposable representations in such a way, that non-invertible homomorphisms go in one direction (here from left to right.

4. The path algebra of a quiver.

4.1. Paths.

For definitions see section 2.1 (In particular: path; head, tail, length of a path; concatenation; oriented cycle).

Lemma. *Let Q be a quiver. If there is a path of length at least $|Q_0|$, then there are cyclic paths, and thus infinitely many paths.*

Proof: Assume that there exists a path of length greater or equal to $|Q_0|$. Then there exists a path of length $|Q_0|$, say $\alpha_n \cdots \alpha_1$. Consider the vertices $x_i = t(\alpha_i)$ for $1 \le i \le n$ and $x_{n+1} = h(\alpha_n)$. Then these are $n+1$ vertices, thus there has to exists $i < j$ with $x_i = x_j$. Let $w = \alpha_{j-1} \cdots \alpha_i$, this is a path with head and tail $x_i = x_j$, thus a cyclic path. But then w^m is a path for any natural number m. The path w has length $j - i \ge 1$, thus w^m has length $m(j-i)$. This shows that these paths are pairwise different.

Corollary. *Let Q be a quiver. The number of paths is finite if and only if Q is finite and there are no oriented cycles.*

Proof: The number of paths of length at most 1 is $|Q_0| + |Q_1|$, thus an infinite quiver has infinitely many paths. Also, any oriented cycle w gives rise to infinitely many paths, namely the paths w^m with m a natural number.

Conversely, assume that Q is a finite quiver. The number of paths of length 0 is $|Q_0|$, the number of paths of length s is at most $|Q_1|^s$. Thus, if there are infinitely many paths, there has to exist paths of arbitrarily large length. According to the lemma, this implies that there are oriented cycles.

4.1. The path algebra of a quiver.

Definition: Let kQ be the vector space with basis the set of all paths in Q, and with the following *multiplication:* if w, w' are paths, let ww' be the concatenation of w and w' provided the tail of w is the head of w', and the zero vector otherwise, and extend this multiplication bilinearly to kQ.

Note that kQ is an associative k-algebra. Proof of the associativity: Let w, w', w'' be paths. Then both $(ww')w''$ and $w(w'w'')$ are the concatenation of w on the left, w' in the middle and w'' on the right, in case both conditions $t(w) = h(w')$ and $t(w') = h(w'')$ are satisfied, and otherwise the zero element (since $(ww')0 = 0$, $0(w'w'') = 0$, according to bilinearity.

Since the multiplication is defined on a basis and extended bilinearly, we clearly deal with a k-algebra.

The elements e_x with $x \in Q_0$ are pairwise orthogonal idempotents.

Below we also will see that any e_x is a primitive idempotent.

If Q_0 is finite, then kQ has a unit element, namely $\sum_{x \in Q_0} e_x$. Proof: Let $e = \sum_{x \in Q_0} e_x$. We have to show that $ew = w = we$ for any path w (then we also have $er = r = re$ for any linear combination r of paths, thus for any element $r \in kQ$). Let

w be a path with tail x and head y, then $e_y w = w$ and $e_z w = 0$ for all $z \neq y$, thus $ew = e_y w + \sum_{z \neq y} e_z w = w$. Similarly, $we_x = w$ and $we_z = 0$ for $z \neq x$.

> More generally, we can say that for an arbitrary quiver Q the path algebra always has sufficiently many idempotents. Recall that a ring R is said to have *sufficiently many idempotents* provided there is a set of pairwise orthogonal idempotents e_i in R indexed by a set I such that for any element $r \in R$, there is a finite subset $I' \subseteq I$ such that $\left(\sum_{i \in I'} e_i\right) r = r = r \left(\sum_{i \in I'} e_i\right)$. In our case $R = kQ$, we take $I = Q_0$.

> **Warning.** *A path algebra has usually many additional idempotents.* Example: Let $\alpha \colon x \to y$ be an arrow which is not a loop. Then $e_x + \alpha$ is an idempotent. Namely:
>
> $$(e_x + \alpha)^2 = e_x^2 + e_x\alpha + \alpha e_x + \alpha^2 = e_x + 0 + \alpha + 0.$$

Finite-dimensionality. *The algebra kQ is finite-dimensional if and only if there are only finitely many paths in Q, thus if and only if Q is a finite quiver without oriented cycles.*

The ideal kQ_+. Let kQ_+ be the subspace of kQ with basis all paths of length at least 1. This is clearly an ideal of kQ.

Also, let kQ_0 be the subspace of kQ with basis the paths of length 0. This is a subalgebra, it is a direct sum of copies of k (one for each vertex x), with component wise multiplication (or, we may reformulate this by saying that kQ_0 is the path algebra of the quiver $(Q_0, \emptyset)$ with the same vertices as Q, but no arrows.

Now $kQ = kQ_0 \oplus kQ_+$, or better $kQ_0 \ltimes kQ_+$, since this is a semi-direct product (kQ_0 is a subalgebra, kQ_+ an ideal).

The powers of kQ_+ can be described easily: $(kQ_+)^m$ *is the subspace with basis the set of paths of length at least m,* for all natural numbers m.

There are the following consequences:

(a) *If there is no path of length m, then $(kQ_+)^m = 0$.*

(b) *If Q is a finite quiver without oriented cycles, say with n vertices, then kQ_+ is a nilpotent ideal: $(kQ_+)^n = 0$.*

It follows that if Q is a finite quiver without oriented cycles, then kQ_+ is the radical of kQ (it is a nilpotent ideal, with semisimple factor ring).

> **Warning.** In general, kQ_+ is not the (Jacobson or nil) radical of kQ. For example, in case $Q = \mathbb{L}$, the algebra $k\mathbb{L}$ is the polynomial ring in one variable: its radical is 0, whereas kQ_+ is a maximal ideal.

If Q is a quiver, one calls a vertex x a *source* provided no arrow ends in x, and a *sink* provided no arrow starts in x. *All vertices of Q are sinks or sources, if and only if there* *are no paths of length* 2 *if and only if* $(kQ_+)^2 = 0$. Typical examples of quivers with all the vertices sinks or sources are the subspace quivers $\mathbb{S}_n$.

Description of kQ by generators and relations. Looking at the construction, we see that kQ is generated as a k-algebra by the paths of length at most 1 in Q. Also, we see that the following relations are satisfied:

- If x is a vertex, then $e_x^2 = e_x$,
- If $x \neq y$ are vertices, then $e_x e_y = 0$,
- If $\alpha\colon x \to y$ is an arrow, then $e_y \alpha = \alpha = \alpha e_x$.

Actually, it is not difficult so see that these are all the relations needed in order to define kQ by generators and relations.

4.3. Examples of path algebras.

(a) **The loop quiver $\mathbb{L}$.** We have $k\mathbb{L} = k[T]$, the polynomial ring in one variable with coefficients in k.

(b) **The n-loops quiver.** Let Q be the quiver with one vertex and $n \geq 2$ loops. Then kQ is the free (non-commutative!) algebra in n generators.

(c) **The linearly oriented quiver Q of type $\mathbb{A}_n$.** Here, $kQ = T_n(k)$, the ring of upper triangular $(n \times n)$-matrices.

$$\underset{\circ}{1} \xleftarrow{\ \alpha_2\ } \underset{\circ}{2} \xleftarrow{\ \alpha_3\ } \underset{\circ}{3} \longleftarrow \cdots\cdots \longleftarrow \underset{\circ}{n-1} \xleftarrow{\ \alpha_n\ } \underset{\circ}{n}$$

An isomorphism $\eta\colon kQ \to T_n(k)$ is defined as follows:

$$\eta(e_i) = E_{ii}, \quad \eta(\alpha_i) = E_{i-1,i}.$$

(By definition, $E_{i,j}$ is the $(n \times n)$-matrix with one coefficient 1, namely in the intersection of the i-th row and the j-th column, all other coefficients being zero.) Note: paths of length at least 1 are of the form $\alpha_i \alpha_{i+1} \cdots \alpha_j$, with $2 \leq i \leq j \leq n$, and

$$\eta(\alpha_i \alpha_{i+1} \cdots \alpha_j) = E_{i-1,i} E_{i,i+1} \cdots E_{j-1,j} = E_{i-1,j}$$

for the longest path (there is such a path) we see:

$$\eta(\alpha_2 \alpha_3 \cdots \alpha_j) = E_{1,n}.$$

(d) **The n-subspace quiver $\mathbb{S}_n$.** The path algebra $k\mathbb{S}_n$ of the n-subspace quiver

$$\begin{tikzcd}
\overset{1}{\circ} \arrow[drr, "\gamma_1"'] & \overset{2}{\circ} \arrow[dr, "\gamma_2"] & \cdots\cdots & \overset{n}{\circ} \arrow[dl, "\gamma_n"] \\
 & & \underset{0}{\circ} &
\end{tikzcd}$$

is the subalgebra of $T_{n+1}(k)$ of matrices with non-zero coefficients only on the diagonal and in the first row:

$$\begin{bmatrix} * & * & \cdots & * \\ & * & & \\ & & \ddots & \\ & & & * \end{bmatrix}$$

An isomorphism is defined as follows:

$$e_0 \mapsto E_{11}, \quad e_i \mapsto E_{i+1,i+1}, \quad \alpha_i \mapsto E_{1,i+1}.$$

for $1 \leq i \leq n$.

(e) **The Kronecker quiver $\mathbb{K}$.** This is the quiver:

$$\begin{tikzcd} \overset{1}{\circ} & \overset{2}{\circ} \arrow[l, bend right, "\alpha"'] \arrow[l, bend left, "\beta"] \end{tikzcd}$$

It is of interest, since the representations of $\mathbb{K}$ are pairs of linear maps $\alpha, \beta \colon M_2 \to M_1$, in matrix language, one deals with *matrix pencils.* The path algebra can be written as the (2×2)-matrices

$$\begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix}$$

Note that in general, given two rings R, S and a bimodule ${}_RM_S$, the set of matrices of the form

$$\begin{bmatrix} r & m \\ 0 & s \end{bmatrix}$$

with $r \in R,\ m \in M,\ s \in S$, with the usual matrix addition and matrix multiplication, a ring: for such upper triangular (2×2)-matrices, we need the addition in R, in M and in S separately, the multiplication in R and in S, as well as multiplications $R \times M \to M$ and $M \times S \to M$, and the bimodule axioms are just the correct axioms in order to obtain a ring. This ring is denoted by

$$\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}.$$

There is a fancy way to realize $k\mathbb{K}$, namely to consider the subspace R of $M_2(k[t])$ with k-basis

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix},$$

this obvious is a subring and of course of the form $\begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix}$.
But this is fancy only on first sight. Namely it turns out that the inclusion of $R \to M_2(k[t])$ is a categorical epimorphism of rings (be aware that categorical epimorphisms of rings do not have to be surjective!) and provides a full embedding of the category of $M_2(k[t])$-modules into the category of $k\mathbb{K}$-modules. This kind of embeddings are of great interest.

4.4. Representations of quivers, modules over the path algebra.

Reminder: Given a ring R with identity $1 = 1_R$, an *R-module* M is by definition an abelian group M with a given biadditive map $R \times M \to M$, called the scalar multiplication, the image of (r, m) under this map is usually denoted just by rm, such that the following two rules are satisfied:

- $r(r'm) = (rr')m$ for all $r, r' \in R$, and all $m \in M$.
- $1_R m = m$ for all $m \in M$.

One can show that the last condition is equivalent to the condition $RM = M$; here RM denotes the abelian subgroup of M generated by the set of elements of M of the form rm with $r \in R, m \in M$.

Theorem. *Let Q be a quiver with finitely many vertices and k a field. The category of representations of Q over k is equivalent to the category of kQ-modules.*

The following functors are equivalences which are inverse to each other:

Given a representation $(M_x, M_\alpha)_{x,\alpha}$ of the quiver Q, let $M = \bigoplus_{x \in Q_0} M_x$ be the corresponding kQ module, with operation by the paths when ever possible: thus the path $(y|\alpha_1, \ldots, \alpha_m|x)$ sends $a \in M_x$ to $\alpha_1 \cdots \alpha_m(a) \in M_y$, and the elements in M_z with $z \neq x$ to zero.

Conversely, given a kQ-module M, let $M_x = e_x M$ and for $\alpha\colon x \to y$ let $M_\alpha\colon M_x \to M_y$ be the multiplication with α (note that $\alpha = e_y \alpha e_x$).

It is straightforward (but tedious to verify) that this works well. (See for example the text books by Auslander-Reiten-Smalø (Theorem III.1.5, p.57) or Assem-Simson-Skowronski.)

What about morphisms? Of course, if we start with a homomorphism

$$(f_x)_x\colon (M_x, M_\alpha)_{x,\alpha} \to (M'_x, M'_\alpha)_{x,\alpha},$$

we just form $f = \bigoplus_x f_x\colon \bigoplus_x M_x \to \bigoplus_x M'_x$.

Conversely, assume that there are given two kQ-modules M, M' and a module homomorphism $f\colon M \to M'$. The important fact is that $f(e_x M) \subseteq e_x M'$ for any $x \in Q_0$ (this is due to the fact that f commutes with scalar multiplication, here with the multiplication with the scalar $e_x \in kQ$. Thus, denote by f_x the restriction of f to M_x (with values in M'_x), then we really have $f = \bigoplus_x f_x\colon \bigoplus_x M_x \to \bigoplus_x M'_x$.

SLOGAN: The representations of a quiver Q are just the kQ-modules.

If $M = (M_x, M_\alpha)_{x,\alpha}$ is a representation of Q, then the information provided by the M_x and the M_α is quite different: the vector spaces M_x are subspaces "of the module M", we may think of M as $M = \bigoplus_x M_x$, whereas the maps M_α provide the action of kQ on M.

If we denote the category of representations of Q over k by $\operatorname{Rep}(Q,k)$, and the module category of a ring R by $\operatorname{Mod} R$, then the Theorem can be noted as follows:

$$\operatorname{Rep}(Q,k) \simeq \operatorname{Mod} kQ$$

Also, if we denote the category of finite-dimensional representations of Q over k by $\operatorname{rep}(Q,k)$, and the category of finite-dimensional kQ-modules by $\operatorname{mod} kQ$, then we similarly have:

$$\operatorname{rep}(Q,k) \simeq \operatorname{mod} kQ$$

The categories $\operatorname{Rep}(Q,k)$ and $\operatorname{Mod} kQ$ are not only equivalent, but (nearly) isomorphic. Recall that an *equivalence* of categories $\mathcal{C}$ and $\mathcal{D}$ requires the existence of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ such that the composition GF is naturally equivalent to the identity of $\mathcal{C}$, and the composition FG is naturally equivalent to the identity of $\mathcal{D}$, whereas for an *isomorphism* one requires that GF and $\mathcal{C}$ GF **are** the respective identity functors. Let us look at our functors. The functor $F\colon \operatorname{Rep}(Q,k) \to \operatorname{Mod} kQ$ attaches to the given vector spaces M_x indexed by Q_0 the direct sum $M = \bigoplus_x M_x$, this is an **external** direct sum, the functor G sends a module M to the set of spaces e_xM indexed by Q_0, note that the spaces M_x are subspaces of M and $M = \bigoplus_x M_x$, but this is now an **internal** direct sum. The composition FG of applying first G, then F would be the identity, if we would use the internal direct sum, not the external direct sum when applying the functor F. Also, when we look at the composition GF, we are faced with the question whether $e_y(\bigoplus_x M_x)$ can be considered as being equal to M_y, or only (canonically) isomorphic to M_y.

4.5. Finite-dimensional k-algebras in general.

This is a report (essentially without proofs) which outlines in which way the representation theory of quivers can be used in order to study the module category of a finite-dimensional k-algebra.

Any finite-dimensional k-algebras Λ (associative, with 1) has a maximal nilpotent ideal J (called its *radical*) and Λ/J is a semisimple k-algebra: it is the product of finitely many matrix rings over division k-algebras.

Proposition 1. *Let Λ be a finite-dimensional k-algebra with radical J such that $\Lambda/J = k \times \cdots \times k$ (n copies of k) and such that $J^r = 0$* (such an r exists, since J is nilpotent). *Then Λ is isomorphic as a k-algebra to kQ/I, where Q is a quiver with n vertices, and I is an ideal with $(kQ_+)^r \subseteq I \subseteq (kQ_+)^2$. The quiver Q is uniquely determined by Λ* (and called the *quiver* of Λ).

Conversely, if Q is a quiver with n vertices, and I is an ideal with $(kQ_+)^r \subseteq I \subseteq (kQ_+)^2$, then $\Lambda = kQ/I$ is a finite-dimensional k-algebra with radical kQ_+/I and Λ modulo its radical is of the form $k \times \cdots \times k$ with n copies of k.

Idea of proof: Start with a finite-dimensional k-algebra Λ. We need to find the quiver of Q. The theorem mentions already how many vertices we need. We want to construct an algebra homomorphism $\eta\colon kQ \to \Lambda$, and we want to have the elements $\eta(e_x)$ from the start. These elements have to be orthogonal idempotents in Λ. Now $\Lambda/J = k \times \cdots \times k$ (with n copies of k) has precisely n primitive idempotents, namely the elements $\overline{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the i-th position. It is well-known that a complete set of primitive pairwise orthogonal idempotents can be lifted modulo any nilpotent ideal, thus there is a complete set of primitive pairwise orthogonal idempotents $e_1, \dots, e_n$, with $e_i + J = \overline{e}_i$. These are the elements we are looking for. Thus we take as vertices of Q the numbers $1, 2, \dots, n$, and we will start to define η by setting $\eta(e_i) = e_i$ for $1 \le i \le n$ (the first e_i is the path of length 0 corresponding to the vertex i, the second e_i is an idempotent in Λ).

Next, consider J/J^2 and multiply this bimodule from the left by e_i, from the right by e_j, we obtain a k-vector space $e_i(J/J^2)e_j$, its dimension yields the number of arrows $j \to i$. Actually, let us choose elements $a_1, \dots, a_t$ in e_iJe_j which form modulo J^2 a basis of $e_i(J/J^2)e_j$. By definition, there are precisely t arrows $j \to i$ in Q, label them $\alpha_1, \dots, \alpha_t$. We continue to define η by setting $\eta(\alpha_1) = a_1,\ \dots,\ \eta(\alpha_t) = a_t + J^2$.

We have defined Q, thus there is the corresponding path algebra kQ. We have described in which way we want to define $\eta(w)$ for all the paths of length at most 1 and we extend the definition to all of kQ, so that η is multiplicative and k-linear. Since the paths of length at most 1 are generators of the algebra kQ, we have to verify that the relations which define kQ are satisfied for the elements $e_1, \dots, e_n$ and the chosen elements in J/J^2. However, this is clear: the elements $e_1, \dots, e_n$ are orthogonal idempotents, and all the elements $a \in e_iJe_j$ satisfy $e_iae_j = a$. This shows that we obtain a k-algebra homomorphism

$$\eta\colon kQ \to \Lambda.$$

It remains to be shown that η is surjective (this means, we have to show that the chosen elements in Λ generate Λ). And we have to see that the kernel I of η satisfies

$$(kQ_+)^r \subseteq I \subseteq (kQ_+)^2.$$

Application. *Let Λ be a finite-dimensional k-algebra with radical J such that $\Lambda/J = k \times \cdots \times k$. Let Q be the quiver of Λ. Then* $\operatorname{Mod}\Lambda$ *is a full exact subcategory of* $\operatorname{Mod} kQ$ *and* $\operatorname{mod}\Lambda$ *is a full exact subcategory of* $\operatorname{mod} kQ$.

Proof: This is just a special case of the following general result: *If I is an ideal of the ring R, then* $\operatorname{Mod} R/I$ *is a full exact subcategory of* $\operatorname{Mod} R$, it consists just of those

R-modules M which are annihilated by I (this means that $ra = 0$ for all $r \in I$ and all $a \in M$).

In our case, dealing with the algebra Λ, we write $\Lambda = kQ/I$ where I is an ideal von kQ. [Actually, since we know that we can assume that $(kQ_+)^r \subseteq I \subseteq (kQ_+)^2$, we have more information about the embedding $\operatorname{Mod}\Lambda \subseteq \operatorname{Mod} kQ$: for example, the simple modules $S(x)$ with $x \in Q_0$ are annihilated by I, thus they are in the subcategory.]

Proposition 2. *If Λ is a finite-dimensional k algebra, then there exists a finite-dimensional k-algebra Λ' (unique up the algebra isomorphisms) such that the module categories of Λ and Λ' are equivalent and all simple factor algebras of Λ' are division ring.*

The algebra Λ' can be constructed as follows: Let $e_1, \dots, e_n$ be a complete set of pairwise inequivalent, but pairwise orthogonal primitive idempotents, and let $e = \sum e_i$. Then take $\Lambda' = e\Lambda e$. The algebra Λ is called a *basic* algebra, the algebras Λ and Λ' are said to be *Morita equivalent.*

Summery, in case k is algebraically closed. Let Λ be a finite-dimensional k-algebra, where k is an algebraically closed field. According to Proposition 2, there is a basic k-algebra Λ' which is Morita-equivalent to Λ. Since k is algebraically closed, the only finite-dimensional k-algebra which is a division ring, is k itself. Let J' be the radical of Λ', let $(J')^r = 0$ It follows that $\Lambda'/J' = k \times \cdots \times k$, thus there is a quiver Q and an ideal I with $(kQ_+)^r \subseteq I \subseteq (kQ_+)^2$ such that Λ' and kQ/I are isomorphic. Altogether, we see:

- The categories $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda'$ are equivalent (this is a Morita equivalence),
- the categories $\operatorname{mod}\Lambda'$ and $\operatorname{mod} kQ/I$ are equivalent (or even isomorphic; this is trivial, since the algebras Λ' and kQ/I are isomorphic,
- the category $\operatorname{mod} kQ/I$ is a full exact subcategory of $\operatorname{mod} kQ$,

thus there is a full exact embedding of $\operatorname{mod}\Lambda$ into $\operatorname{mod} kQ$.

4.6. The indecomposable projective kQ-modules $P(x)$.

Let x be a vertex of the quiver Q. Let $P(x)$ be the vector space with basis the set of all paths w with tail x. By definition, $P(x)$ is a subspace of kQ, but it is even a submodule, thus a left ideal. And we have:

$$kQ = \bigoplus_x P(x).$$

Proposition. *The evaluation map $f \mapsto f_x(e_x)$ yields a natural isomorphism*

$$\eta_M \colon \operatorname{Hom}(P(x), M) \to M_x$$

for all kQ-modules M.

Proof: Let $f \colon P(x) \to M$ be a homomorphism, then $f_x(e_x) = f_x(e_x^2) = e_x f_x(e_x)$, thus $f_x(e_x)$ is an element of $M_x = e_x M$, thus we really get a (set-theoretical) map $\eta = \eta_M \colon \operatorname{Hom}(P(x), M) \to M_x$. And clearly η is k-linear. We have to show that η is surjective and that its kernel is zero.

In order to show that the map η is surjective, let $a \in M_x$. For every path w with tail x and head y, the path w lies in $P(x)_y$, we have to define $f_y(w) \in M_y$. Thus, let $w = \alpha_1 \cdots \alpha_n$; we take (and have to take)

$$f_y(w) = f_y(\alpha_1 \cdots \alpha_n) = \alpha_1 \cdots \alpha_n(a).$$

In this way, f is defined on all paths in $P(x)$ and we extend it k-linearly in order to obtain $f\colon P(x) \to M$. Actually, it is easy to verify that we obtain not just a map, but a homomorphism $f\colon P(x) \to M$, and by definition, $f_x(e_x) = a$.

Now let us consider the kernel, thus let $f\colon P(x) \to M$ be a homomorphism such that $f_x(e_x) = 0$. But then for any path w with tail x and head y, we have $f_y(w) = f_y(we_x) = wf_x(e_x) = 0$, thus $f = 0$.

What means the naturality? If there is given a homomorphism $g\colon M \to M'$ of quiver representations, then the following square must commute:

$$\begin{array}{ccc}
\mathrm{Hom}(P(x), M) & \xrightarrow{\eta_M} & M_x \\
{\scriptstyle \mathrm{Hom}(P(x),g)}\big\downarrow & & \big\downarrow{\scriptstyle g_x} \\
\mathrm{Hom}(P(x), M') & \xrightarrow{\eta_{M'}} & M'_x
\end{array}$$

Start with $f \in \mathrm{Hom}(P(x), M)$, to the right we get $\eta_N(f) = f_x(e_x)$, under g_x we get $g_x f_x(e_x)$. On the other hand, $\mathrm{Hom}(P(x), g)(f) = gf$, and $\eta_{M'}(gf) = (gf)(e_x) = g_x f_x(e_x)$.

Corollary. *If $p\colon M' \to M$ is a surjective homomorphism of quiver representations, then, for every homomorphism $f\colon P(x) \to M$, there is a homomorphism $f'\colon P(x) \to M'$ such that $pf' = f$.* Thus $P(x)$ is a projective module.

Proof: Since p is surjective, $p_x\colon M'_x \to M_x$ is a surjective linear map. Now assume there is given $f\colon P(x) \to M$. Then $f_x(e_x) \in M_x$, thus there is $a \in M'_x$ such that $p_x(a) = f_x(e_x)$. According to the Proposition, there is $f'\colon P(x) \to M'$ with $f'(e_x) = a$ (the surjectivity of $\eta_{M'}$). But then

$$\eta_M(f) = f_x(e_x) = p_x(a) = p_x f'_x(a) = (pf')_x(a = \eta_M(pf').$$

The injectivity of η_M asserts that $f = pf'$.

Of course, if Q has only finitely many vertices, then $R = kQ$ is a ring with 1, and it is well-known, that the module ${}_RR$ (the ring considered as a left module over itself) is projective, as well as that direct summands of projective modules are projective. Thus, since $kQ = \bigoplus_x P(x)$ is a direct sum of left ideals, thus left modules, we see that all the modules $P(x)$ are projective left modules.

5. Extensions.

Given representations M, N of a quiver, we want to introduce a vector space $\operatorname{Ext}^1(M, N)$ which measures the possible extensions. Here, by an *extension* of R-modules (where R is a ring, for example the path algebra of a quiver) one means a short exact sequence

$$0 \to N \xrightarrow{f} Y \xrightarrow{g} M \to 0.$$

Note that such an exact sequence just means that $f\colon N \to Y$ is an injective homomorphism and g a cokernel of f, thus g is up to isomorphism uniquely determined by f, but the information given by f itself is (up to the isomorphism $f\colon N \to f(N)$) just the inclusion $f(N) \subseteq Y$. Let me repeat this as a slogan:

> SLOGAN: To consider extensions means nothing else then to study submodules of modules (to be precise: we do not mean the study of a submodule as a module in its own right, but the study of the **embedding** of the submodule into the given module).
>
> A typical question is the following: Given a submodule N of Y, is it a direct summand? Formulated in the language of "extensions", this is the question whether the sequence $0 \to N \to Y \to M \to 0$ splits or not.

The sections 5.1 - 5.3 deal with modules in general. Here, we start with a ring R, all modules are R-modules.

5.1. Split extension.

If N, M are modules and $\sigma\colon N \to M$, and $\rho\colon M \to N$ are maps with $\rho\sigma = 1_N$, then σ is said to be a *split monomorphism* (with *retraction* ρ), and ρ is said to be a *split epimorphism* (with *section* σ).

Lemma. *Let*

$$0 \to N \xrightarrow{f} Y \xrightarrow{g} M \to 0.$$

be an exact sequence. The following conditions are equivalent:
(1) *f is a split monomorphism.*
(1′) *g is a split epimorphism.*
(1″) *There is a submodule Y' such that $f(N) \oplus Y' = Y$.*

Proof should be well-known. For example, if (1) holds, thus there is $\rho\colon Y \to N$ with $\rho f = 1_N$. Let $Y' = \operatorname{Ker}(\rho)$. In order to see $Y = f(N) \oplus Y'$, take $y \in Y$ and write is as $f = f\rho(y) + (y - f\rho(y))$; here $f\rho(y) \in f(N)$ and $(y - f\rho(y)) \in Y'$. Also, If $y \in f(N) \cap Y'$, then $y = f(x)$ for some $x \in N$, thus $0 = \rho(y) = \rho f(x) = x$, and therefore $y = f(x) = 0$. Thus (1‴) holds. If (1‴) holds, then $g|Y'\colon Y' \to M$ is an isomorphism, thus take for σ the composition of $(g|Y')^{-1}$ with the inclusion map $Y' \to Y$; this yields (1″).

Proposition. *Let R be a k-algebra, and let*

$$0 \to N \xrightarrow{f} Y \xrightarrow{g} M \to 0.$$

be an exact sequence, where N, M (thus also Y) are finite-dimensional k-modules. The following conditions are equivalent:

(1) *f is a split monomorphism.*
(2) *Y is isomorphic to $N \oplus M$.*
(3) $\dim_k \operatorname{End}(Y) = \dim_k \operatorname{End}(N \oplus M)$.
(3′) $\dim_k \operatorname{End}(Y) \geq \dim_k \operatorname{End}(N \oplus M)$.

Proof: Trivially, (1) $\implies$ (2) $\implies$ (3) $\implies$ (3′). Thus, let us assume (3′). We may assume that f is an inclusion map, thus $N \subseteq Y$ and that $M = Y/N$ with g the projection map. Let $\eta\colon \operatorname{End}(M) \to \operatorname{Hom}(N, M)$ be defined by $\eta(\phi) = g\phi f$ and let E be the kernel of η, thus

$$\dim_k \operatorname{End}(Y) \leq \dim_k E + \dim_k \operatorname{Hom}(N, M).$$

Note that

$$E = \{\phi \in \operatorname{End}(Y) \mid \phi(N) \subseteq N.\}.$$

Define $\eta'\colon E \to \operatorname{End}(N) \oplus \operatorname{End}(M)$ by $\eta'(\phi) = (\phi|N, \overline{\phi})$, where $\overline{\phi}$ is the endomorphism of $M = Y/N$ induced by ϕ. Let E' be the kernel of η'. Then

$$E' = \{\phi \in \operatorname{End}(Y) \mid \phi(N) = 0, \phi(M) \subseteq N\},$$

thus E' is isomorphic as the vector space to $\operatorname{Hom}(M, N)$ (here, $\phi\colon M \to N$ corresponds to $f\psi g$). We see:

$$\begin{aligned} \dim_k E &\leq \dim_k E' + \dim_k \operatorname{End}(N) + \dim_k \operatorname{End}(M) \\ &= \dim_k \operatorname{Hom}(M, N) + \dim_k \operatorname{End}(N) + \dim_k \operatorname{End}(M). \end{aligned}$$

Here is a picture which shows the filtration of $\operatorname{End}(Y)$ we are dealing with, as well as the information on the corresponding factors which we have obtained:

$$\begin{array}{r|l} \operatorname{End}(Y) & \\ & \subseteq\ \operatorname{Hom}(M, N) \\ E & \\ & \subseteq\ \operatorname{End}(N) \oplus \operatorname{End}(M) \\ E' & \\ & \operatorname{Hom}(N, M) \\ 0 & \end{array}$$

Altogether we see that

$$\begin{aligned} \dim_k \operatorname{End}(Y) &\leq \dim_k E + \dim_k \operatorname{Hom}(N, M) \\ &\leq \dim_k \operatorname{Hom}(M, N) + \dim_k \operatorname{End}(N) + \dim_k \operatorname{End}(M) + \dim_k \operatorname{Hom}(N, M) \\ &= \dim_k \operatorname{End}(N \oplus M) \leq \dim_k \operatorname{End}(Y). \end{aligned}$$

This shows that all the inequality signs have to be equality signs, in particular, the map $\eta'\colon E \to \operatorname{End}(N) \oplus \operatorname{End}(M)$ has to be surjective, thus $(1_N, 0_N) = \eta'(\phi)$ for some $\phi \in E$.

But $\eta'(\phi) = (1_N, 0_N)$ means that $\phi|N = 1_N$ and $\phi(Y) \subseteq N$. Since $\phi(Y) \subseteq N$, we can write $\phi = f\rho$ for some $\rho\colon Y \to N$. Then $f = \phi f = f\rho f$, thus, since f is injective, $1_N = \rho f$. This shows that f is a split monomorphism.

> Of special interest seems to be the implication (2) $\implies$ (1). Whereas the converse implication is trivial, this one is not. In this context, it seems worthwhile to draw the attention to the weaker conditions that N is a direct summand of Y, or that M is a direct summand of Y. Such sequences $0 \to N \to N \oplus Y' \to M \to 0$ and $0 \to N \to M \oplus Y'' \to M \to 0$ are sometimes called *Riedtmann-Schofield sequences* and if such a sequence exists, one says that M is a *degeneration* of Y' (or that N is a degeneration of Y'', respectively), see for example Ringel: The ladder construction of Prüfer modules.

As mentioned above, given representations M, N of a quiver, we are going to introduce a vector space $\operatorname{Ext}^1(M,N)$ which measures the possible extensions. Actually, we are mainly interested to know whether $\operatorname{Ext}^1(M,N) = 0$ or not. This can be reformulated quite easily: The formulation $\operatorname{Ext}^1(M,N) = 0$ means just the following: Given a module Y with submodule N such that Y/N is isomorphic to M, then the embedding $N \to Y$ is *splits:* There is a submodule Y' of Y with $N \oplus Y' = Y$.

> It was Kaplansky who stressed that it sometimes may be sufficient to work with the condition $\operatorname{Ext}^1(M,N) = 0$ without introducing the groups Ext^1 .

If M is a module with $\operatorname{Ext}^1(M,M) = 0$, then M is said *to have no self-extensions.* An indecomposable module without self-extensions is called an *exceptional* module. The kQ-modules which we are interested in, are mainly the exceptional kQ-modules.

> Warning. This terminology is in some sense irritating. For example, for a Dynkin quiver, all the indecomposables are exceptional, thus to be exceptional is nothing special! One of the reasons for the naming comes from commutative ring theory, where it is very unusual to deal with a module without self-extensions. This has influenced people dealing with vector bundles, since they usually coming from commutative ring theory. Thus, the first appearance of the word "exceptional" in the sense as mentioned here, was in the realm of vector bundles, here we should mention the school of Rudakov. It was shifted to quiver representation by Crawley-Boevey, since it turned out that there were several parallel results.

5.2. Equivalence classes of short exact sequences.

Definition: Let $\epsilon = (0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0)$ and $\epsilon' = (0 \to X \xrightarrow{f'} Y' \xrightarrow{g'} Z \to 0)$ be exact sequences (with identical first and last modules). These extensions are called *equivalent* provided there is a commutative diagram of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle h} & & \| & & \\
0 & \longrightarrow & X & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z & \longrightarrow & 0.
\end{array}$$

Note that, *if such a diagram exists, then the map h is necessarily an isomorphism.*

Proof: First, let us show that h is injective. Thus, take $y \in Y$ with $h(y) = 0$. Then $g(y) = g'h(y) = 0$, thus there is $x \in X$ with $f(x) = y$. Then $f'(x) = hf(x) = h(y) = 0$, and, since f' is injective, $x = 0$, thus $y = f(x) = 0$.

Second, in order to see that h is surjective, start with $y' \in Y'$. There is $y \in Y$ with $g(y) = g'(y')$, since g is surjective. Now

$$g'(y' - h(y)) = g'(y) - g'h(y) = g'(y) - g(y) = 0,$$

thus $y' - h(y) = f'(x)$ for some $x' \in X'$. Then $y' = h(y) + f'(x) = h(y) + hf(x) = h(y + f(x))$ shows that y is in the image of h.

As a consequence, it is obvious that the relation for short exact sequences to be equivalent is really an equivalence relation: if ϵ, ϵ' (in this order) are equivalent, say using the map h, then also ϵ', ϵ are equivalent, use h^{-1}; if in addition also ϵ', ϵ'' are equivalent, say using the map h', then ϵ, ϵ'' are equivalent: use $h'h$.

The set of equivalence classes of exact sequences $\epsilon = (0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0)$ with X, Z being fixed will be denoted by $\operatorname{Ext}^1(Z, X)$.

> We have introduced here $\operatorname{Ext}^1(Z, X)$ just as a set (or as a set with a distinguished element, namely the equivalence class of split exact sequences). Usually, one defines on $\operatorname{Ext}^1(Z, X)$ an addition, the so-called Baer addition, so that $\operatorname{Ext}^1(Z, X)$ becomes an abelian group. In case one deals with a k-algebra R, the set $\operatorname{Ext}^1(Z, X)$ should be endowed even with the structure of a k-space. We avoid this at the moment, but later we will identify the set Ext^1 (in the case where R is the path algebra of a quiver) with a k-space, and this k-space structure of $\operatorname{Ext}^1(Z, X)$ is the usual one.
>
> When we speak about the **set** of equivalence classes, we have to worry whether there may be set-theoretical difficulties. Fortunately, in the usual categories we are working with, say the category of modules over a ring R, the class of modules which are isomorphic to a fixed one may not be a set (but just a class), however the class of isomorphism classes of modules with fixed cardinality is a set, as is the class of equivalence classes of exact sequences of the form $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ with X, Z both being fixed.

We have seen that if the sequences $\epsilon = (0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0)$ and $\epsilon' = (0 \to X \xrightarrow{f'} Y' \xrightarrow{g'} Z \to 0)$ are equivalent, then Y, Y' are isomorphic, but not every isomorphism $h\colon Y \to Y'$ will not provide a commutative diagram as required for the equivalence — the easiest example to have in mind is the following (here we assume that R is a k-algebra): assume that $h\colon Y \to Y'$ is an isomorphism which provides a a commutative diagram as required, and let $c \neq 0$ be an element of k, then also $ch\colon Y \to Y'$ is an isomorphism, but it will **not** provide such a commutative diagram unless $X = 0 = Z$.

5.3. Construction of extensions using projective modules.

Proposition. *Assume that there is given a surjective map $p\colon P \to Z$ with P projective, let ΩZ be the kernel of p, thus we deal with the exact sequence*

$$\epsilon: \qquad 0 \to \Omega Z \xrightarrow{u} P \to Z \to 0.$$

Then any short exact sequence ending in Z is induced from the sequence ϵ.

This means that given a short exact sequence $0 \to X \to Y \to Z \to 0$, there is a map $\phi\colon \Omega Z \to X$ and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega Z & \xrightarrow{u} & P & \longrightarrow & Z & \longrightarrow & 0 \\ & & {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi'} & & \Big\| & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array} .$$

Note that up to equivalence of short exact sequences, the upper sequence and the map ϕ together determine the lower exact sequence uniquely: Namely for such a commutative diagram, the left square

$$\begin{array}{ccc} \Omega Z & \xrightarrow{u} & P \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi'} \\ X & \longrightarrow & Y, \end{array}$$

is a pushout diagram, thus up to isomorphism Y is of the form

$$Y = P \oplus X/\{(-u(a), \phi(a) \mid a \in \Omega Z\}.$$

The proposition asserts that there is a surjective map

$$\delta\colon \operatorname{Hom}(\Omega Z, X) \to \operatorname{Ext}^1(Z, X),$$

and the kernel of this map are the morphisms $\Omega Z \to X$ which factor through $u\colon \Omega Z \to P$.

These assertions are usually formulated in terms of the long exact sequence which one obtains when we apply the functor $\mathrm{Hom}(-, X)$ to the exact sequence

$$0 \to \Omega Z \xrightarrow{u} P \to Z \to 0 \qquad (\epsilon).$$

Namely, we obtain the exact sequence

$$0 \to \mathrm{Hom}(Z, X) \to \mathrm{Hom}(P, X) \xrightarrow{\mathrm{Hom}(u,X)} \mathrm{Hom}(\Omega Z, X) \xrightarrow{\delta} \mathrm{Ext}^1(Z, X) \to 0$$

The map δ is called the *connecting homomorphism,* it attaches to $\phi\colon \Omega Z \to X$ the exact sequence induced from ϵ by ϕ.

Let us return to quivers and their representations.

5.4. Realization of extensions of quiver representations by quiver data.

Let Q be a quiver. Given representations M, N of Q, let

$$D(M, N) = \bigoplus_{\alpha \in Q_1} \mathrm{Hom}_k(M_{t(\alpha)}, N_{h(\alpha)}).$$

This is the set whose elements we will use in order to construct short exact sequences starting with N and ending in M. For any $e = (e_\alpha)_\alpha$ in $D(M, N)$, we may consider the representation $W(M, N, e)$ as follows:

$$W(M, N, e)_x = N_x \oplus M_x, \quad W(M, N, e)_\alpha = \begin{bmatrix} N_\alpha & e_\alpha \\ 0 & M_\alpha \end{bmatrix}$$

Note that N is a submodule of $Y = W(M, N, e)$ and the corresponding factor module Y/N can be identified with M; thus, there is the following short exact sequence

$$\epsilon(M, N, e) = \left(0 \to N \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} W(M, N, e) \xrightarrow{[0\ 1]} M \to 0 \right).$$

Lemma. *If Y is a representation of Q with a subrepresentation N and $M = Y/N$, with inclusion map $u\colon N \to P$ and projection map $p\colon Y \to M$, then there is a commutative diagram*

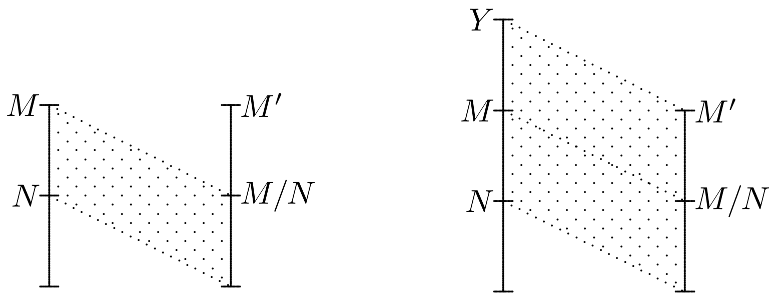
$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}"] \arrow[d, equal] & W(M,N,e) \arrow[r, "{[0\ 1]}"] \arrow[d, "\phi"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "u"] & Y \arrow[r, "p"] & M \arrow[r] & 0
\end{tikzcd}$$

Proof: For any vertex x, choose a submodule C_x such that $N_x \oplus C_x = Y_x$. Using the map p, we actually may identify C_x with M_x, thus we assume $N_x \oplus M_x = Y_x$. Let $\alpha\colon x \to y$ be an arrow. Note that $Y_\alpha(M_x) \subseteq N_y$, thus we may consider the restriction of Y_α to M_x and denote it by $e_\alpha\colon M_x \to N_y$. Using these identifications, we see that the identity map

$$\phi_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} : W(M,N,e)_x = N_x \oplus M_x \longrightarrow N_x \oplus M_x = Y_x$$

yields an isomorphism (even an identification) $\phi = (\phi_x)_x\colon W(M,N,e) \to Y$. Of course, this is the required isomorphism which we need in the Lemma.

We say that an abelian category such as a module category is *hereditary* provided the following condition is satisfied: If M is a module with a submodule N and there is an embedding $M/N \to M'$ for some module M', then there exists a module Y with submodule M, such that there is an isomorphism $Y/N \to M'$ which is the identity on M/N (by assumption, M/N is both a submodule of Y/N as well as of M').



Let us stress, for those familiar with Ext^2 or at least with Ext^1, that this definition of heredity coincides with the usual one, namely with the condition that globally $\mathrm{Ext}^2 = 0$, or, equivalently, that for any monomorphism u and any object N, the induced map $\mathrm{Ext}^1(N,u)$ is surjective. What we did, is that we have reformulated the surjectivity assertion, by saying that for any exact sequence ϵ, which we can assume to be of the form $0 \to N \to M \to M/N \to 0$ (with an inclusion map $N \to M$), and for any embedding $u\colon M/N \to M'$, there exists an exact sequence $\epsilon' = (0 \to N \to Y \to M' \to 0)$ which induces ϵ, thus $\epsilon = \mathrm{Ext}^1(N,u)(\epsilon')$.

Theorem. *The category* $\mathrm{Rep}(Q,k)$ *is hereditary.*

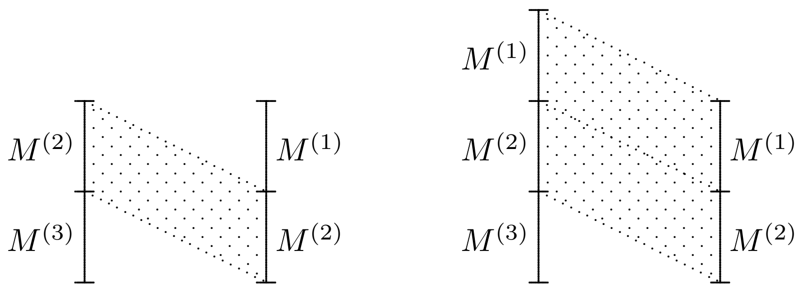
Proof: We assume that we have given three representations, say $M^{(1)}, M^{(2)}, M^{(3)}$, and extensions

$$W(M^{(1)}, M^{(2)}, e) \qquad \text{and} \qquad W(M^{(2)}, M^{(3)}, e').$$

Then the required representations are those of the form

$$\left(M_x^{(1)} \oplus M_x^{(2)} \oplus M_x^{(3)}, \begin{bmatrix} M^{(1)} & e_x & * \\ & M^{(2)} & e'_x \\ & & M^{(3)} \end{bmatrix} \right),$$

where $*$ is arbitrary.



Remark. Any Serre subcategory of a hereditary category is hereditary, thus the category of nilpotent representations is also hereditary.

Reformulation. *If $e\colon X \to I$ is an epimorphism, $m\colon I \to Y$ is a monomorphism, then there is a monomorphism $m'\colon X \to J$ and an epimorphism $e'\colon J \to Y$ such that the sequence*

$$0 \to X \xrightarrow{\begin{bmatrix} m' \\ e \end{bmatrix}} I \oplus J \xrightarrow{\begin{bmatrix} -e' & m \end{bmatrix}} Y \to 0$$

is exact.

Proof: Choose J with submodule X such that $J/m(I) = Y$, let $m'\colon X \to J$ the inclusion map and $e'\colon J \to Y$ the projection.

Lemma. *If X, Y are indecomposable objects of finite length in a hereditary category, and $\operatorname{Ext}^1(Y, X) = 0$, then any non-zero morphism $X \to Y$ is a monomorphism or an epimorphism.*

Proof. Let $f\colon X \to Y$ be a non-zero morphism which is neither a monomorphism nor an epimorphism, let I be the image of f. Then there is an exact sequence

$$0 \to X \to I \oplus J \to Y \to 0.$$

But this sequence cannot split, since otherwise $X \oplus Y$ is isomorphic to $I \oplus Y$, but $I \neq 0$ and an indecomposable direct summand I' of I has length smaller than the length of X or Y, thus cannot be isomorphic to X of Y, contrary to the Krull-Remak-Schmidt theorem.

Corollary. *If M is an exceptional object of finite length in a hereditary category, then $\operatorname{End}(M)$ is a division ring.*

Proof: If $f\colon M \to M$ is non-invertible, then it is neither a monomorphism nor an epimorphism. The previous result shows that $f = 0$.

5.5. Modules without self-extensions.

We are interested in the exceptional modules or, more generally, in modules without self-extensions.

Proposition 1. *Let M, N be representations of the quiver Q. Let $\alpha\colon x \to y$ be an arrow of the quiver. Assume that M_α has a non-trivial kernel, and that N_α has a non-trivial cokernel. Then* $\operatorname{Ext}^1(M,N) \neq 0$.

Proof. Write $M_x = \operatorname{Ker}(M_\alpha) \oplus C$ for some subspace C, and take a non-zero element $b \in M_y$ which does not belong to $\operatorname{Im}(N_\alpha)$. Let $e_\alpha\colon M_x \to N_y$ be defined as follows: it shall be zero on C and it shall map $\operatorname{Ker}(M_\alpha)$ surjectively onto $\langle b\rangle$ (such a linear map exists, sine we assume that $\operatorname{Ker}(M_\alpha)$ is non-zero. For the remaining arrows β of the quiver, let $e_\beta = 0$. We consider $W = W(M,N,e)$, in particular

$$W_\alpha = \begin{bmatrix} M_\alpha & e_\alpha \\ & N_\alpha \end{bmatrix} : M_x \oplus N_x \longrightarrow M_y \oplus N_y.$$

Clearly, the image of W_α is $\operatorname{Im}(M_\alpha) \oplus (\operatorname{Im} N_\alpha) + \langle b\rangle$, and the latter plus sign concerns also a direct sum inside N_y. Therefore W_α has rank equal to $\operatorname{rank} M_\alpha + \operatorname{rank} N_\alpha + 1$. But this shows that $W = W(M,N,e)$ cannot be isomorphic to $M \oplus N$, thus $\operatorname{Ext}^1(M,N) \neq 0$.

Corollary. *Let M be a representation of a quiver Q without self-extensions. Then, for any arrow α, the map M_α has maximal rank.*

(We recall that a vector space map $V \to V'$ is said to have maximal rank, provided its rank is as large as possible, namely $\min\{\dim_k V, \dim_k V'\}$, or, equivalently, provided the map is a monomorphism or an epimorphism.)

Proposition 2. *Let M be a module without self-extensions. Let w be a path with $wM = 0$. Let α be an arrow with $t(\alpha = h(w)$ and $\alpha w M = 0$. Then $M_{h(\alpha)} = 0$.*

Proof: Write $M_x = wM \oplus C$ for some subspace C, and choose some non-zero element $b \in M_y$. Let $e_\alpha\colon M_x \to N_y$ be defined as follows: it shall be zero on C and it shall map wM surjectively onto $\langle b\rangle$. For the remaining arrows β of the quiver, let $e_\beta = 0$. We consider $W = W(M,M,e)$ and the extension

$$0 \to M \to W(M,M,e) \to M \to 0.$$

This sequence does not split. Now assume that $\alpha w M = 0$, then also $\alpha w(M \oplus M) = 0$. However, by construction, $\alpha w W(M,M,e) = \langle b\rangle \neq 0$ and therefore $W(M,M,e)$ is not isomorphic to $M \oplus M$. But this implies $\operatorname{Ext}^1(M,M) \neq 0$, contrary to the assumption. This contradiction shows that we must have $\alpha w M \neq 0$.

Corollary. *Let M be a nilpotent module without self-extensions. Then the support quiver $Q(M)$ does not have cyclic paths.*

Proof: Assume that there is a cyclic path v in the support quiver. Since M is nilpotent, there is some t with $v^tM = 0$. Let $v^t = \alpha_s \cdots \alpha_1$ Choose $m \geq 0$ maximal with $wM \neq 0$, where $w = \alpha_m \cdots \alpha_1$. Then $m < s$ and $\alpha w M = 0$ for $\alpha = \alpha_{m+1}$. According to the Proposition, $M_{h(\alpha)} = 0$. But then α does not belong to the support quiver of Q, a contradiction.

It is not too difficult to show that the support quiver of no module without self-extensions has cyclic paths.

5.6. The standard guide.

Given representations M, N of the quiver Q and an element $e \in D(M,N)$, we have constructed a representation $W(M,N,e)$, or better even, an extension

$$\epsilon(M,N,e) = \left(0 \to N \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} W(M,N,e) \xrightarrow{[0\ 1]} M \to 0 \right)$$

and we know that we obtain in this way all extensions. One may ask when are two such extensions equivalent.

> For example, if we start with the quiver of tape $\mathbb{A}_2$ and consider $M = N$ the two-dimensional indecomposable representation, then obviously all the extensions $\epsilon(M,N,e)$ are equivalent.

If M, N are representations, we consider the following linear map which we call the *standard guide* Ξ_{MN} for M and N:

$$\Xi_{MN} \colon \bigoplus_x \mathrm{Hom}_k(M_x, N_x) \longrightarrow \bigoplus_a \mathrm{Hom}_k(M_{t(\alpha)}, N_{h(\alpha)}) = D(M,N),$$

defined by

$$(\Xi_{MN}(f))_\alpha = N_\alpha f_{t(\alpha)} - f_{h(\alpha)} M_\alpha.$$

where $f = (f_x)_x$ with k-linear maps $f_x \colon M_x \to N_x$. First, let us note:

Proposition. *The kernel of* Ξ_{MN} *is* $\mathrm{Hom}(M,N)$.

Now let us look at the cokernel. Note that if $e = (e_\alpha)_\alpha$ is an element of the target $D(M,N)$ of Ξ_{MN}, then there is defined the representation $W(M,N,e)$ and the extension $\epsilon(M,N,e)$.

Theorem. *The map* $e \mapsto \epsilon(e)$ *yields a bijection*

$$\mathrm{Cok}(\Xi_{MN}) \longrightarrow \mathrm{Ext}^1(M,N).$$

Under this bijection, the zero element $e = 0$ *is sent to the split exact sequence.*

Proof: The last sentence is trivial. Thus, let us prove the first sentence. We have to understand what it means that the exact sequences $\epsilon(M,N,e)$ and $\epsilon(M,N,e')$ are equivalent: there has to exist a map $h \colon W(M,N,e) \to W(M,N,e')$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\begin{bmatrix}1\\0\end{bmatrix}"] \arrow[d, equal] & W(M,N,e) \arrow[r, "{[0\ 1]}"] \arrow[d, "h"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "\begin{bmatrix}1\\0\end{bmatrix}"'] & W(M,N,e') \arrow[r, "{[0\ 1]}"'] & M \arrow[r] & 0
\end{tikzcd}$$

For such a map $h = (h_x)_x$, the maps $h_x \colon W(M, N, e)_x \to W(M, N, e')_x$ have to be of the form

$$h_x = \begin{bmatrix} 1 & f_x \\ 0 & 1 \end{bmatrix} : h_x \colon W(M, N, e)_x = N_x \oplus M_x \to N_x \oplus M_x = W(M, N, e')_x,$$

with $f_x \colon M_x \to N_x$. Now for every arrow $\alpha \colon x \to y$, we must have $h_y W_\alpha = W'_\alpha h_x$, or, written in matrices:

$$\begin{bmatrix} 1 & f_y \\ & 1 \end{bmatrix} \begin{bmatrix} N_\alpha & e_\alpha \\ & M_\alpha \end{bmatrix} = \begin{bmatrix} N_\alpha & e'_\alpha \\ & M_\alpha \end{bmatrix} \begin{bmatrix} 1 & f_x \\ & M_\alpha \end{bmatrix},$$

thus

$$e'_\alpha - e'_\alpha = N_\alpha f_x - f_y M_\alpha = \Xi(f)_\alpha.$$

Of course, also conversely, if $e'_\alpha - e'_\alpha = \Xi(f)_\alpha$, then the extensions $\epsilon(M, N, e)$ and $\epsilon(M, N, e')$ are equivalent.

Remark. Since the cokernel is a vector space, we may (and will) consider also $\mathrm{Ext}^1(M, N)$ as a vector space.

> As we have mentioned, there is a direct way to define an addition (the Baer addition) and scalar multiplication an the set of equivalence classes of extensions. If one uses the Baer addition on $\mathrm{Ext}^1(M, N)$, one has to show that the bijection established in the Theorem is in fact a vector space isomorphism. Below we will see that the vector space operations on $\mathrm{Ext}^1(M, N)$ as defined here coincide with those which we obtain when we calculate $\mathrm{Ext}^1(M, N)$ using a projective presentation of M, thus with the standard definition.

5.7. The standard resolution of a quiver representation.

The standard guide Ξ_{MN} can be obtained from a certain projective presentation of M, namely the standard presentation, by applying the functor $\mathrm{Hom}(-, N)$. In order to define the standard presentation of M, we define the following two projective modules:

$$P^s(M) = \bigoplus_{x \in Q_0} P(x) \otimes_k M_x,$$

$$\Omega^s(M) = \bigoplus_{\alpha \in Q_1} P(h(\alpha)) \otimes_k M_{t(\alpha)}.$$

The tensor product $\otimes_k$ which we use here, means just the following: if V is a vector space of dimension v, then $P(x) \otimes_k V$ is the direct sum of v copies of $P(x)$; if we choose a basis of V, we may think of the copies being indexed by the elements of the basis. Of course, with $P(x)$ also $P(x) \otimes_k V$ is projective, for any vector space V.

The *standard resolution* of M is given as follows:

$$0 \to \Omega^s(M) \xrightarrow{d} P^s(M) \xrightarrow{p} M \to 0,$$

where the maps are defined as follows:

$$\begin{aligned} p(w \otimes a) &= wa && \text{for} \quad w \in P(x),\ a \in M_x, \\ d(w \otimes a) &= w\alpha \otimes a - w \otimes \alpha a && \text{for} \quad w \in P(h(\alpha)),\ a \in M_{t(\alpha)}. \end{aligned}$$

Proposition 1. *The standard resolution of any representation M is an exact sequence.*

The proof is just a direct calculation, fiddling around with linear combinations of paths, see the Lecture Notes by Crawley-Boevey.

The use of the tensor product $\otimes_k$ has the following advantage: We have seen in section 4.6 that the evaluation map $f \mapsto f_x(e_x)$ yields an isomorphism $\mathrm{Hom}(P(x), N) \to N_x$. Of course, there is also a corresponding isomorphism $\mathrm{Hom}_k(k, N_x) \to N_x$ which sends $\phi\colon k \to N_x$ to $\phi(1)$, thus we may combine these isomorphisms (or better, the first isomorphism which the inverse of the second) in order to obtain a canonical isomorphism

$$\mathrm{Hom}(P(x), N) \to \mathrm{Hom}_k(k, N).$$

Using the tensor product $\otimes_k$, this yields an isomorphism

$$\eta\colon \mathrm{Hom}(P(x) \otimes_k V, N) \to \mathrm{Hom}_k(V, N)$$

for any vector space V, we call it the *evaluation map.*

Consider now representations M, N of the quiver Q and take the standard resolution of M. If we apply the functor $\mathrm{Hom}(-, N)$ to the standard resolution of M, we obtain the following exact sequence:

$$0 \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(P^s(M), N) \xrightarrow{\mathrm{Hom}(d,N)} \mathrm{Hom}(\Omega^s(M), N) \to \mathrm{Ext}^1(M, N) \to 0.$$

Let us consider the evaluation maps

$$\begin{aligned} \eta_0\colon \mathrm{Hom}(P^s(M), N) &\longrightarrow \bigoplus_x \mathrm{Hom}_k(M_x, N_x) \\ \eta_1\colon \mathrm{Hom}(\Omega^s(M), N) &\longrightarrow \bigoplus_a \mathrm{Hom}_k(M_{t(\alpha)}, N_{h(\alpha)}) \end{aligned}$$

Here, the index x runs through the set Q_0 of the vertices, the index α through the set Q_1 of arrows; also, we wrote $\otimes$ instead of $\otimes_k$.

Proposition 2. *Let M, N be representations of the quiver Q. The standard guide Ξ_{MN} is just* $\operatorname{Hom}(d, N)$*, where u is the standard presentation of M,* namely the following diagram commutes:

$$\begin{tikzcd}
\operatorname{Hom}(\bigoplus_x P(x) \otimes M_x, N) \arrow[r, "\operatorname{Hom}(d,N)"] \arrow[d, "\eta_0"'] & \operatorname{Hom}(\bigoplus_\alpha P(h(\alpha)) \otimes M_{t(\alpha)}, N) \arrow[d, "\eta_1"] \\
\bigoplus_x \operatorname{Hom}_k(M_x, N_x) \arrow[r, "\Xi_{MN}"] & \bigoplus_a \operatorname{Hom}_k(M_{t(\alpha)}, N_{h(\alpha)})
\end{tikzcd}$$

Proof. Let $f = (f_x)_x \in \bigoplus_x \operatorname{Hom}_k(M_x, N_x)$. Under the inverse of the middle map η, we obtain the homomorphism defined by the maps $P(x) \otimes_k M_x \to N_x$ with $w \otimes a \mapsto w f_x(a)$ for $w \in P(x)$ and $a \in M_x$, let us call it f'. Under $\operatorname{Hom}(d, N)$, we get the homomorphism $\operatorname{Hom}(d, N)(f'))$, let us look at its restriction to $P(y) \otimes M_x$ (where $t(\alpha) = x$, and $h(\alpha) = y$, thus $\alpha\colon x \to y$), it maps $e_y \otimes a$ first (under d) to $\alpha \otimes a - e_y \otimes \alpha a$ and then under f' to $\alpha f_x(a) - f_y(\alpha a)$, thus to $N_\alpha f_x(a) - f_y M_\alpha(a)$.

But Ξ_{MN} also sends $f = (f_x)_x$ to the element of $D(M, N)$ whose component indexed by $\alpha\colon x \to y$ is $N_\alpha f_x - f_y M_\alpha$.

Corollary. *The maps* $\operatorname{Hom}(d, N)$ *and* Ξ_{MN} *have the same kernel, namely* $\operatorname{Hom}(M, N)$ *and the same cokernel, namely* $\operatorname{Ext}^1(M, N)$*.*

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}(M, N) \arrow[r] \arrow[d, equal] & \operatorname{Hom}(P^s(M), N) \arrow[r, "\operatorname{Hom}(d,N)"] \arrow[d, "\eta_0"'] & \operatorname{Hom}(\Omega^s(M), N) \arrow[r] \arrow[d, "\eta_1"] & \operatorname{Ext}^1(M, N) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Hom}(M, N) \arrow[r] & \bigoplus_x \operatorname{Hom}_k(M_x, N_x) \arrow[r, "\Xi_{MN}"] & \bigoplus_a \operatorname{Hom}_k(M_{t(\alpha)}, N_{h(\alpha)}) \arrow[r] & \operatorname{Ext}^1(M, N) \arrow[r] & 0
\end{tikzcd}$$

It seems to be of interest to compare the cokernel maps: we either may start (in the upper row) with an element ϕ in $\operatorname{Hom}(\Omega^s(M), N)$ and form the induced exact sequence with respect to ϕ, or else (in the lower row) we may take an element $e \in D(M, N) = \bigoplus_\alpha (M_{t(\alpha)}, N_{h(\alpha)}$ and form the extension $\epsilon(M, N, e)$. What we obtain, for $\eta_1(\phi) = e$, are exact sequences which are equivalent.

6. Dynkin quivers, Euclidean quivers, wild quivers.

This last section is more sketchy, its aim is, on the one hand, to provide a short survey concerning the difference between the Dynkin quivers, the Euclidean quivers and the remaining ones, but also, on the other hand, to draw the attention to some important techniques not covered in the lectures (but note that some of the definitions are not given and several proofs are missing).

6.1. The theorems of Gabriel and Kac.

A finite dimensional algebra is said to be *representation-finite* provided there are only finitely many isomorphism classes of indecomposable representations. The starting result for the representation theory of quivers was Gabriel's theorem:

Theorem (Gabriel). (a) *A connected quiver is representation finite if and only if it is a Dynkin quiver.*

The number of the indecomposable representations for the different Dynkin types is as follows:

$\mathbb{A}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
$\frac{1}{2}n(n+1)$	$n(n-1)$	36	69	120

note that the numbers do not depend on the orientation! Actually, as observed by Tits, there is a bijection between the indecomposable representations and the positive roots of the corresponding simple complex Lie algebra $\mathbf{g}$. This bijection is furnished by the dimension vector $\mathbf{dim}$ (it will be introduced in section 6.2). Recall that the (finite-dimensional) simple complex Lie algebras have been classified by Cartan, they are labeled by the Dynkin diagrams (including also the types $\mathbb{B}_n, \mathbb{C}_n, \mathbb{F}_4, \mathbb{G}_2$, which do not play a role when dealing with representations of quivers).

(b) *If Q is a Dynkin quiver and* $\mathbf{g}$ *is the corresponding simple complex Lie algebra, then* $\mathbf{dim}$ *yields a bijection between the set of isomorphism classes of indecomposable representations of Q and the set of positive roots of* $\mathbf{g}$.

Again, this is an assertion which shows that some invariants for quiver representations do not depend on the orientation of the quiver, namely here the dimension vectors of the indecomposable representations. One special representation of the Dynkin quiver Q should be mentioned: there is a unique indecomposable representation of maximal dimension, it corresponds to the unique maximal root. For example, for type $\mathbb{E}_8$, the dimension vector of the maximal indecomposable representation is

$$\begin{matrix} & & 3 & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 \end{matrix}$$

In our list of the Dynkin diagrams we have added on the right side the corresponding maximal root. It plays an important role (not only in Lie theory, but also) in the representation theory of quivers.

Gabriel's theorem was extended to arbitrary finite quivers by Kac. Given any quiver Q without loops (or better just its underlying graph $\overline{Q}$), there is a corresponding (usually infinite-dimensional) complex Lie algebra $\mathbf{g}$, the Kac-Moody Lie algebra of type $\overline{Q}$, as well as a corresponding root system; here the roots are divided into two classes: the real roots and the imaginary roots (in the special case of dealing with a Dynkin quiver, the Kac-Moody Lie algebra of type $\overline{Q}$ is just the finite-dimensional simple Lie algebra of type $\overline{Q}$, and there are no imaginary roots). Kac has shown:

Theorem (Kac). *If Q is a finite quiver without loops and $\mathbf{g}$ the corresponding Kac-Moody Lie algebra, then* **dim** *yields a surjective map from the set of isomorphism classes of indecomposable representations of Q onto the set of positive roots of $\mathbf{g}$.*

If r is a positive real root of $\mathbf{g}$, then $\mathbf{dim}^{-1}(r)$ is a single isomorphism class. If r is a positive imaginary root, and k is infinite, then $\mathbf{dim}^{-1}(r)$ consists if infinitely many isomorphism classes.

Actually, there is a corresponding result also for quivers with loops, but one needs to define the corresponding Lie algebras, or, at least, the corresponding root systems.

6.2. The Euler form.

The exact sequence

$$0 \to \operatorname{Hom}(M,N) \to \bigoplus_x \operatorname{Hom}_k(M_x,N_x) \xrightarrow{\Xi} \bigoplus_a \operatorname{Hom}_k(M_{t(\alpha)},N_{h(\alpha)}) \to \operatorname{Ext}^1(M,N) \to 0$$

shows that the dimension difference

$$\dim_k \operatorname{Hom}(M,N) - \dim_k \operatorname{Ext}^1(M,N)$$

only depends on the dimensions of the various vector spaces M_x, N_x.

In order to formulate this properly, let us consider the free abelian group $\mathbb{Z}Q_0$ with basis Q_0, its elements will be written in the form $d = (d_x)_x$ with integers d_x for all $x \in Q_0$. If M is a representation of Q, then we may consider the element $\mathbf{dim}\, M = (\dim_k M_x)_x$ as such an element, it is called the *dimension vector* of M

We define on $\mathbb{Z}Q_0$ a bilinear form depending on the quiver Q as follows: If $d, d' \in \mathbb{Z}Q_0$, let

$$\langle d, d'\rangle = \sum_{x\in Q_0} d_x d'_x - \sum_{\alpha\in Q_1} d_{t(\alpha)} d'_{h(\alpha)};$$

we are also interested in the corresponding quadratic form

$$q(d) = \langle d, d\rangle.$$

Proposition. *If M, M' are representations of Q, then*

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, M'\rangle = \dim \operatorname{Hom}(M,M') - \dim \operatorname{Ext}^1(M,M').$$

Corollary 1. *If M is an exceptional representation of Q with* $\mathrm{End}(M) = k$, *then* $q(\mathbf{dim}\, M) = 1$.

Remark: The condition $\mathrm{End}(M) = k$ is actually always satisfied. We know already that $\mathrm{End}(M)$ is a division ring, see section 5.4. Thus, in case k is algebraically closed, it follows directly that $\mathrm{End}(M) = k$. However, also in general one can show that $\mathrm{End}(M) = k$ for any exceptional representation of a quiver.

Corollary 2. *If M is a representation with* $\mathrm{End}(M)$ *a division ring and* $\mathrm{Ext}^1(M, M) \neq 0$, *then* $q(\mathbf{dim}\, M) \leq 0$.

Proof. Let $D = \mathrm{End}(M)^{\mathrm{op}}$. Since $\mathrm{Ext}^1(M, M)$ is a non-zero D-D-bimodule, the k-dimension of $\mathrm{Ext}^1(M, M)$ is at least $\dim_k D$, thus

$$q(\mathbf{dim}\, M) = \dim_k D - \dim_k \mathrm{Ext}^1(M, M) \leq 0.$$

In case we deal with a quiver without cyclic paths, the group $\mathbb{Z}Q_0$ can be identified with the Grothendieck group $K_0(\mathrm{mod}\, kQ)$ of finite-dimension representations of Q modulo all exact sequences. Namely, according to the Jordan-Hölder theorem, the Grothendieck group $K_0(\mathrm{mod}\, kQ)$ is the free abelian group with basis the set of isomorphism classes of simple kQ-modules. But if Q has no cyclic paths, then we know that the simple representations are of the form $S(x)$, with $x \in Q_0$, thus we may identify the basis vector of $\mathbb{Z}Q_0$ with index $x \in Q_0$ with the isomorphism class of $S(x)$. If we do so, then for every representation M of the quiver, its dimension vector $\mathbf{dim}\, M$ has as coordinate with index x just the Jordan-Hölder multiplicity of $S(x)$ in M.

6.3. The quadratic form of a quiver.

We have introduced in 6.2 a quadratic form $q = q_Q$ on the free abelian group $\mathbb{Z}Q_0$. By definition,

$$q(d) = \sum_{x \in Q_0} d_x^2 - \sum_{\alpha \in Q_1} d_{t(\alpha)} d_{h(\alpha)}.$$

Note that in contrast to the bilinear form $\langle -, - \rangle$, this quadratic form only depends on the underlying graph $\overline{Q}$ of Q, and not on the orientation of the edges.

Proposition. *Let Q be a finite connected quiver and q the corresponding quadratic form.*

(a) *If Q is a Dynkin quiver, then q is positive definite,*

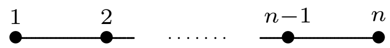
(b) *If Q is a Euclidean quiver, then q is positive semi-definite with radical of rank 1.semi*

(c) *If Q is neither a Dynkin quiver nor a Euclidean quiver, then q is indefinite.*

Proof. This is standard knowledge, say in Lie theory: the first assertions are used in order to classify the finite-dimensional semi-simple Lie algebras, see any such book. An elementary (and very nice) reference is the Bernstein-Gelfand-Ponomarev paper.

Here is an outline of the main steps: In the Dynkin case, one may consider the quadratic forms case by case. A good procedure seems to be to consider first the cases $\mathbb{A}_n$, and then trees with a unique branching vertex c such that c has precisely three neighbors (we may call such a graph a star with 3 arms).

Thus, let us start with the case $\mathbb{A}_n$, with $n \geq 1$:

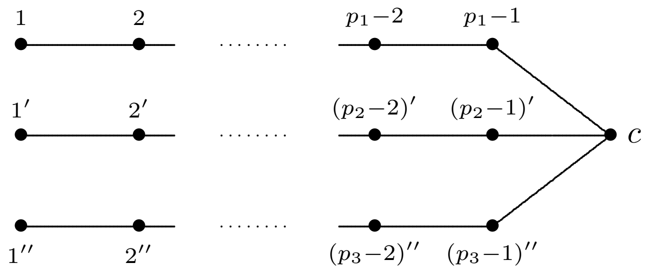


We may rewrite the quadratic form as follows:

$$q(d) = \quad \sum\nolimits_{i=1}^{n-1} \tfrac{i}{2(i+1)}\Big(\tfrac{i+1}{i}d_i - d_{i+1}\Big)^2 + \Big(1 - \tfrac{n-1}{2n}\Big)d_n^2.$$

Since $q(d)$ is written as a linear combination of squares with positive coefficients, it follows that q is positive semi-definite. But q is even positive definite, since the n linear forms $\frac{i+1}{i}d_i - d_{i+1}$ (with $1 \leq 1 \leq n-1$) and d_n are linearly independent.

Now we look at a star with three arms; such a graph may be obtained by starting with three graphs of type $\mathbb{A}_n$ where $n = p_1, p_2, p_3$ and identifying the vertices say on the right to get one vertex $c = p_1 = (p_2)' = (p_3)''$, here is a picture:



Using our knowledge about the graphs of type $\mathbb{A}_n$, we may rewrite the quadratic form for our star as

$$\begin{aligned} q(d) = \quad & \sum\nolimits_{i=1}^{p_1-1} \tfrac{i}{2(i+1)}\Big(\tfrac{i+1}{i}d_i - d_{i+1}\Big)^2 \\ & + \sum\nolimits_{i=1}^{p_2-1} \tfrac{i}{2(i+1)}\Big(\tfrac{i+1}{i}d_{i'} - d_{(i+1)'}\Big)^2 \\ & + \sum\nolimits_{i=1}^{p_3-1} \tfrac{i}{2(i+1)}\Big(\tfrac{i+1}{i}d_{i''} - d_{(i+1)''}\Big)^2 \\ & + \Big(1 - \frac{p_1-1}{2p_1} - \frac{p_2-1}{2p_2} - \frac{p_3-1}{2p_3}\Big)d_c^2. \end{aligned}$$

We see that we deal with a linear combination of squares, and the decisive coefficient is the coefficient

$$\lambda \;=\; 1 - \frac{p_1-1}{2p_1} - \frac{p_2-1}{2p_2} - \frac{p_3-1}{2p_3}$$

of d_c^2, which can be positive, zero or negative (depending on the numbers p_1, p_2, p_3), whereas all the other coefficients are of the form $\frac{i}{2(i+1)}$, thus positive. Now

$$\begin{aligned}\lambda &= 1 - \frac{p_1 - 1}{2p_1} - \frac{p_2 - 1}{2p_2} - \frac{p_3 - 1}{2p_3} \\ &= \frac{2p_1p_2p_3 - (p_1 - 1)p_2p_3 - (p_2 - 1)p_1p_3 - (p_3 - 1)p_1p_2}{2p_1p_2p_3} \\ &= \frac{-p_1p_2p_3 + p_2p_3 + p_1p_3 + p_1p_2}{2p_1p_2p_3} \\ &= \tfrac{1}{2}\Big(-1 + \tfrac{1}{p_1} + \tfrac{1}{p_2} + \tfrac{1}{p_3}\Big)\end{aligned}$$

We see that

$$\begin{aligned}\lambda > 0 &\quad\Longleftrightarrow\quad \tfrac{1}{p_1} + \tfrac{1}{p_2} + \tfrac{1}{p_3} > 1 \\ \lambda = 0 &\quad\Longleftrightarrow\quad \tfrac{1}{p_1} + \tfrac{1}{p_2} + \tfrac{1}{p_3} = 1 \\ \lambda < 0 &\quad\Longleftrightarrow\quad \tfrac{1}{p_1} + \tfrac{1}{p_2} + \tfrac{1}{p_3} < 1\end{aligned}$$

The following is easy to see:

Lemma. *The triples $p_1 \le p_2 \le p_3$ with $\sum_i \frac{1}{p_i} > 1$ are the following:*

$$(1, p_2, p_3), \quad (2, 2, p_3), \quad (2, 3, 3), \quad (2, 3, 4), \quad (2, 3, 5).$$

(the corresponding graphs are the Dynkin diagrams $\mathbb{A}_{p_2+p_3-1}, \mathbb{D}_{p_3+2}, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$).

There are precisely three triples $p_1 \le p_2 \le p_3$ with $\sum_i \frac{1}{p_i} = 1$, namely the triples

$$(3, 3, 3), \quad (2, 4, 4), \quad (2, 3, 6).$$

(the corresponding graphs are the Euclidean diagrams $\widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8$).

The reader may wonder whether the convention which we use here (and which seems to be widely accepted) is reasonable: For example, looking at the graph $\mathbb{E}_7$, we say that it has an $\mathbb{A}_2$-arm, an $\mathbb{A}_3$-arm and an $\mathbb{A}_4$-arm, thus we draw the attention to the triple of numbers $(2, 3, 4)$ and not to $(1, 2, 3)$ which would correspond to the optical impression of having arms of length 1, 2, and 3. The formulae presented above, as well as many other ones which express properties of stars with 3 arms seem to be sufficient justification.

The importance of these triples of numbers was stressed already by Felix Klein in his book *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade* (1884).

Let us turn the attention to the radical of the quadratic form q. By definition, the *radical* of a quadratic form q is the set (indeed subgroup) of all elements r with $q(d + r) =$

$q(d)$ for all vectors d. In particular, any vector r in the radical of q satisfies $q(r) = 0$ (but we stress that the converse is not true). Now in our case

$$q(d + r) = q(d) + q(r) + \langle d, r\rangle + \langle r, d\rangle.$$

It follows that r belongs to the radical if and only if

$$\langle e(x), r\rangle + \langle r, e(x)\rangle = 0,$$

for all vertices $x \in Q_0$ (here, $e(x)$ denotes the canonical basis vector in $\mathbb{Z}Q_0$, with coefficients $(e(x))_x = 1$ and $(e(x))_y = 0$ for $y \neq x$). But clearly:

$$\langle e(x), r\rangle + \langle r, e(x)\rangle = 2r_x - \sum_{t(\alpha)=x} r_{h(\alpha)} - \sum_{h(\alpha)=x} r_{t(\alpha)}.$$

Thus we see that r *belongs to the radical of* q *provided* $2r_x$ *is equal to the sum of the neighboring values* r_y. For example, if $Q = \widetilde{E}_8$, then there is the following radical vector:

$$\begin{array}{cccccccc} & & 3 & & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

and it generates the radical (as a subgroup of $\mathbb{Z}Q_0$). In our list of the Euclidean diagrams we have added on the right side a vector r which is positive and turns out to generate the radical. Note that in all cases we see: if we delete a vertex x with $r_x = 1$ (one such vertex is encircled, but usually there are several such vertices), then the graph which we obtain is the corresponding Dynkin diagram and the restriction of r is precisely the maximal root for the Dynkin diagram. This shows quite clearly, that for the study of the Euclidean diagrams, the maximal root of the corresponding Dynkin diagram plays a decisive role.

6.4. Dynkin quivers.

Proposition. *Let* Q *be a Dynkin quiver and* M *a representation of* Q. *If* $\mathrm{End}(M)$ *is a division ring, then* $\mathrm{Ext}^1(M, M) = 0$.

Proof: Assume that $\mathrm{End}(M)$ is a division ring and that $\mathrm{Ext}^1(M, M) \neq 0$. Then, according to section 6.2, we have $q(\mathbf{dim}\, M) \leq 0$. However, q is positive definite, by 6.3.

Proposition. *Let* Q *be a Dynkin quiver and* M *an indecomposable representation of* Q. *Then* $\mathrm{End}(M)$ *is a division ring.*

Proof: See for example the Lectures by Crawley-Boevey, section 2.

Corollary. *If* k *is an algebraically closed field and* M *is an indecomposable representation of a Dynkin quiver, then* $q(\mathbf{dim}\, M) = 1$.

Note that for a positive definite quadratic form on a finitely generated free abelian group, there are only finitely many vectors d with $q(d) = 1$. Thus it follows that *a Dynkin* *quiver is* of bounded representation type (this means that the indecomposable representations are of bounded length), and therefore *representation-finite* (according to Rojter who proved the first Brauer-Thrall conjecture).

We have not yet shown that actually **dim** provides (for a Dynkin quiver) a bijection between the isomorphism classes of the indecomposable representations and the positive vectors d with $q(d) = 1$ (for the Dynkin graphs, the vectors d with $q(d) = 1$ are precisely the positive roots of the corresponding Lie algebra). It still remains to show: For any positive root r, there is an indecomposable, and there is up to isomorphism only one.

Definition: We say that a representation M of Q is *in general position* provided $\dim_k \mathrm{End}(M) \le \dim_k \mathrm{End}(M')$ for all representations M' with $\mathbf{dim}\, M = \mathbf{dim}\, M'$.

Lemma. *Assume that M is a representation in general position and let $M = M' \oplus M''$ be a direct decomposition. Then* $\mathrm{Ext}^1(M', M'') = 0$.

Proof: This is a direct consequence of Proposition 5.1. Namely, if $0 \to M' \to Y \to M'' \to 0$ is a non-split exact sequence, then

$$\dim_k \mathrm{End}(Y) < \dim_k \mathrm{End}(M' \oplus M'') = \dim_k \mathrm{End}(M),$$

but of course $\mathbf{dim}\, Y = \mathbf{dim}\, M$.

Thus, if M is in general position and $M = \bigoplus M_i$ with indecomposable representations M_i, then $\mathrm{Ext}^1(M_i, M_j) = 0$ for all $i \neq j$.

Corollary. *Let Q be a Dynkin quiver. Let $r \in \mathbb{Z}Q_0$ with $q(r) = 1$. Then any representation M of Q with* $\mathbf{dim}\, M = r$ *which is in general position is indecomposable and has endomorphism ring k.*

Proof: Let M be a representation of Q with $\mathbf{dim}\, M = r$ which is in general position and write it as $M = \bigoplus M_i$ with indecomposable representations M_i. As we just have seen, $\mathrm{Ext}^1(M_i, M_j) = 0$ for all $i \neq j$. But we know that we also have $\mathrm{Ext}^1(M_i, M_i) = 0$ for all i, thus $\mathrm{Ext}^1(M, M) = 0$. But then

$$1 = q(\mathbf{dim}\, M) = \dim_k \mathrm{End}(M)$$

shows that $\mathrm{End}(M) = k$, thus M is indecomposable.

In particular, there exists an indecomposable representation M with $\mathbf{dim}\, M$ the maximal root!

6.5. More about the Dynkin quivers.

As is the last section, let us consider again a Dynkin quiver Q. We have seen that in case k is algebraically closed, then $\mathrm{End}(M) = k$ for any indecomposable representation. This is true for k an arbitrary field, but this needs some further considerations. There are several possible proofs available, anyone provides further information.

(a) Knitting the **Auslander-Reiten quiver** (case by case). If one could show from the beginning that we deal with a quiver of finite representation type, then it would be sufficient to know that we deal with a preprojective component (because for M in a preprojective component, $\mathrm{End}(M) \simeq \mathrm{End}(P)$ for some indecomposable projective module, and if P is an indecomposable projective module and Q has no cyclic paths, then $\mathrm{End}(P) = k$).

(b) Use of the **Coxeter transformation.** Here, one uses only knowledge which concerns the quadratic form. This method also shows directly that all the indecomposable modules are determined by the dimension vector.

(c) Use of **reflection functors.** Here one relates the representations of quivers with the same underlying graph but may-be different orientation to each other.

(d) **Schofield induction.** We know that all indecomposables are exceptional, thus are obtained by Schofield induction from the simple kQ-modules $S(x)$. Inductively, we see that $\mathrm{End}(M) = k$ for all exceptional modules.

6.6. Euclidean quivers.

As for the Dynkin quivers, also for the Euclidean quivers the full classification of all the indecomposable representations is known and is quite easy to overlook.

The special case of the Kronecker quiver

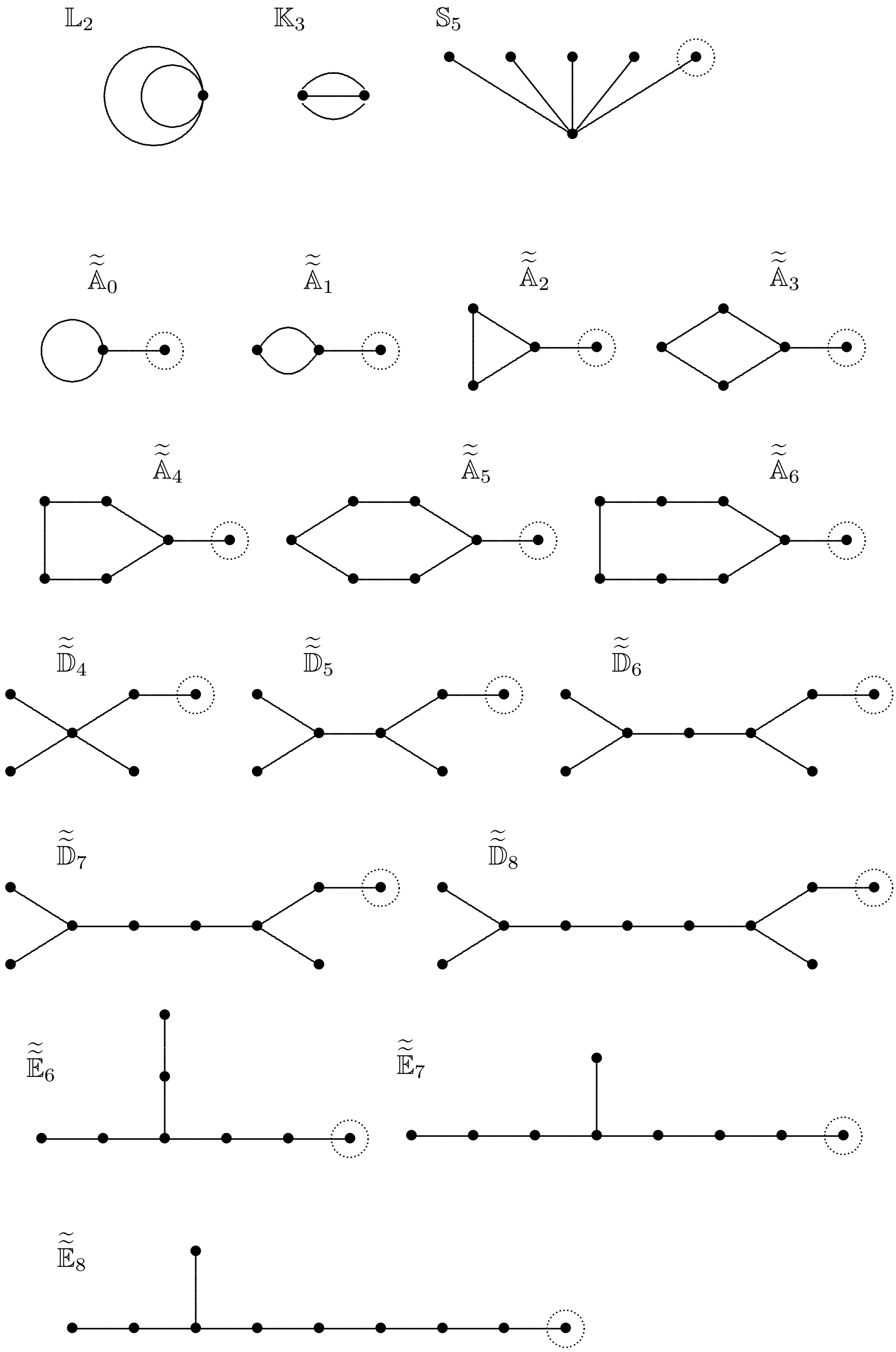
$$\circ \rightrightarrows \circ$$

was investigated already by Weierstrass and then solved by Kronecker in 1890.

The next case which was studied was the 4-subspace quiver $\widetilde{\mathbb{D}}_4$, see Gelfand-Ponomarev, 1970. The solution for arbitrary Euclidean quivers is due to Donovan-Freislich and Nazarova (1973).

6.7. Wild quivers.

Let us deal with the following list of graphs.



Note that any of these graphs has at most 10 vertices.

Proposition. *These are the minimal graphs with indefinite quadratic form* q*. Always, there exists a vector* d *with positive integer coefficients such that* $q(d) = -1$.

Proof. First, let us show the existence of the vector d. For the graph $\mathbb{L}_2$, we take $d = (1)$, of course $q(1) = 1^2 - 2 = -1$. Similarly, for $\mathbb{K}_3$, take $d = (1, 1)$, we have $q(d) = 1^2 + 1^2 - 3 = -1$. The remaining graphs are obtained from a Euclidean graph E by adding a vertex ω (see the encircled vertex) and an edge connecting ω to say x. For E there exists a vector d' with positive integer coefficients such that $d'_x = 2$. Namely, in the case $\mathbb{S}_5$ take for d' the positive radical generator, whereas in all the other cases take for d' twice the positive radical generator. Let d be defined by $d_y = d'_y$ for the vertices y of E and $d_\omega = 1$. Then $q(d) = q(d') + d_\omega^2 - d_x d_\omega = 0 + 1 - 2 = -1$. Thus, always we have found d such that $q(d) = -1$, in particular we see that q is indefinite.

It remains to show that any graph $\overline{Q}$ with indefinite quadratic form contains a subgraph in the list. Of course, as a minimal graph with indefinite quadratic form, $\overline{Q}$ has to be connected.

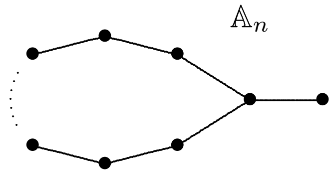
If there is a vertex with at least two loops, then $\mathbb{L}_2$ is a subgraph. Thus, we can assume that there is no vertex with more than one loop.

If there is a vertex x with one loop, then there have to be additional vertices, thus there is vertex ω connected to x, thus $\widetilde{\widetilde{\mathbb{A}}}_0$ is a subgraph. Now we can assume that there are no loops.

If there are multiple edges, then $\mathbb{K}_3$ or $\widetilde{\widetilde{\mathbb{A}}}_0$ has to be a subgraph. Thus we can assume that there are no multiple edges.

If there is a cycle, then there is an elementary cycle, as well as a vertex x on this cycle with a neighbor say ω outside the cycle, let us denote the corresponding subgraph by $\widetilde{\widetilde{\mathbb{A}}}_n$ provided the cycle consists of $n + 1$ vertices (here, ≥ 2):

$\widetilde{\widetilde{\mathbb{A}}}_n$

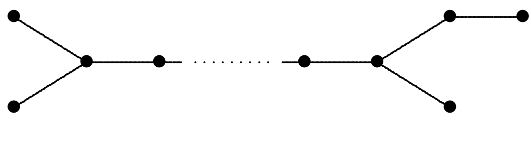


For $n \geq 7$, the graph $\widetilde{\widetilde{\mathbb{A}}}_n$ contains $\widetilde{\widetilde{\mathbb{E}}}_7$ as a subgraph, the remaining graphs $\widetilde{\widetilde{\mathbb{A}}}_n$ with $2 \leq n \leq 6$ occur in the list. Thus we can assume that $\overline{Q}$ is a tree.

If there is a vertex with at least 4 neighbors, then $\mathbb{S}_5$ or $\widetilde{\widetilde{\mathbb{D}}}_4$ is a subgraph. Thus we can assume that any vertex has at most three neighbors.

If there are two vertices both having three neighbors, there has to be a subgraph of the form $\widetilde{\widetilde{\mathbb{D}}}_n$ (with $n + 2$ vertices):

$\widetilde{\widetilde{\mathbb{D}}}_n$



If $n \geq 9$, then $\widetilde{\widetilde{\mathbb{D}}}_n$ contains $\widetilde{\widetilde{\mathbb{E}}}_7$ as a subgraph. The remaining graphs $\widetilde{\widetilde{\mathbb{D}}}_n$ with $5 \leq n \leq 8$ are in the list.

It remains to deal with the stars with 3 arms, say with arms $\mathbb{A}_{p_1}, \mathbb{A}_{p_2}, \mathbb{A}_{p_3}$ as considered in section 6.2, where $2 \leq p_1 \leq p_2 \leq p_3$. If $p_1 \geq 4$, or if $p_1 = 3$ and $p_2 = 3$, then $\widetilde{\widetilde{\mathbb{E}}}_6$ has to be a subgraph. Thus $p_1 = 2$. If $p_2 \geq 4$, then $\widetilde{\widetilde{\mathbb{E}}}_7$ has to be a subgraph. On the other hand, if $p_1 = 2$ and $p_2 = 2$, then we deal with a Dynkin diagram of type $\mathbb{D}$, impossible. The cases $p_1 = 2, p_2 = 3$ remain: since the quadratic form is indefinite, we must have $p_3 \geq 7$, thus $\widetilde{\widetilde{\mathbb{E}}}_8$ is a subgraph. This completes the proof.

A finite-dimensional k-algebra Λ is called *strictly wild,* provided there is a full exact embedding of the category of finite-dimensional representations of the quiver $\mathbb{L}_2$ into the category $\operatorname{mod}\Lambda$. A quiver is said to be strictly wild provided its path algebra is strictly wild.

Theorem. *If Q is a connected quiver which is neither a Dynkin quiver nor a Euclidean quiver, then Q is strictly wild.*

Sketch of proof. We use the **process of simplification** as outlined in Ringel, *Representations of K-species and bimodules,* J.Algebra 1976. This amounts to the following inductive procedure: given any quiver Q in the list, we have to find a finite set $\mathcal{N} = \{N_1, \dots, N_t\}$ of representations of Q such that $\operatorname{End}(N_i) = k$, $\operatorname{Hom}(N_i, N_j) = 0$ for all $i \neq j$ in $\{1, \dots, t\}$ (such a set may be called a *set of orthogonal bricks*) such that the Ext-quiver $\Delta(\mathcal{N})$ of $\mathcal{N}$ is already known to be strictly wild (by definition, the Ext-*quiver* $\Delta(\mathcal{N})$ has t vertices labeled $[N_1], \dots, [N_t]$ and the number of arrows $[N_i] \to [N_j]$ is given by $-\dim_k \operatorname{Ext}^1(N_i, N_j)$). We may use the Euler form in order to determine this number: Since $\operatorname{End}(N_i) = k$, we have

$$\begin{aligned} \dim_k \operatorname{Ext}^1(N_i, N_i) &= -q(\mathbf{dim}\, N_i) + 1, \\ \dim_k \operatorname{Ext}^1(N_i, N_j) &= -\langle \mathbf{dim}\, N_i, \mathbf{dim}\, N_j \rangle \qquad \text{for} \quad i \neq j. \end{aligned}$$

We distinguish five cases.

(1) $\mathbb{K}_3$. Let Q be a quiver of type $\mathbb{K}_3$, let N be any two-dimensional indecomposable representation of Q and $\mathcal{N} = \{N\}$. Since $q(\mathbf{dim}\, N) = -1$, we see that $\Delta(\mathcal{N}) = \mathbb{L}_2$.

(2) $\widetilde{\widetilde{\mathbb{A}}}_0$, say with subspace arm, thus we consider the quiver

$$Q \qquad \begin{tikzcd} \circ \arrow[loop left] & \circ \arrow[l] \end{tikzcd}$$

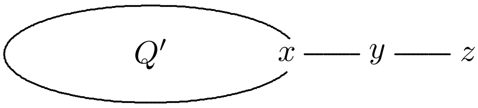
Let N be the following representation of Q

$$\begin{tikzcd} k^2 \arrow[loop left, "{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}"] & & k \arrow[ll, "{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}"'] \end{tikzcd}$$

One easily checks that $\operatorname{End}(N) = k$. Also here, let $\mathcal{N} = \{N\}$. Then $\dim_k \operatorname{Ext}^1(N, N) = 2$, thus again $\Delta(\mathcal{N}) = \mathbb{L}_2$. In case the arm has factor space orientation, we proceed similarly.

(3) $\widetilde{\widetilde{\mathbb{A}}}_n$ with $n \geq 1$. Let Q be such a quiver, thus Q is obtained form a quiver Q' of type $\widetilde{A}_n$ by adding an $\mathbb{A}_2$-arm. Let ω be the vertex outside Q'. Let N be any indecomposable thin representation of Q' with $N_y = k$ for all vertices y of Q', and $\mathcal{N} = \{N, S(\omega)\}$. Then $\Delta(\mathcal{N})$ is a quiver of type $\widetilde{\widetilde{\mathbb{A}}}_0$.

(4) The cases $\widetilde{\widetilde{\mathbb{D}}}_n$ and $\widetilde{\widetilde{\mathbb{E}}}_m$ with $4 \leq n \leq 8$ and $6 \leq m \leq 8$. These quivers are obtained from a Dynkin quiver Q' by adding an $\mathbb{A}_3$-arm in a vertex x of Q', say

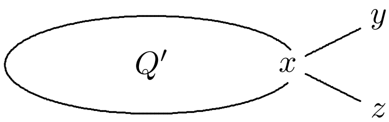


Let N be the maximal indecomposable representation of Q' and note that in all cases $\dim_k N_x = 2$. It follows that $\Delta(\mathcal{N})$ is of the following form

$$\begin{tikzcd}
{[N]} \arrow[r, bend left, no head] \arrow[r, bend right, no head] & {[S(y)]} \arrow[r, no head] & {[S(z)]}
\end{tikzcd}$$

thus of type $\widetilde{\widetilde{\mathbb{A}}}_1$.

(5) $\mathbb{S}_5$. Here we deal with a quiver obtained from a Dynkin quiver Q' by adding two $\mathbb{A}_2$-arms in a vertex $x \in Q'_0$.



In our case $\mathbb{S}_5$, the subquiver Q' is of type $\mathbb{D}_4$. Again, we consider the maximal indecomposable representation N of Q' and note that in our case $\dim_k N_x = 2$. It follows that $\Delta(\mathcal{N})$ is of the following form

$$\begin{tikzcd}
 & {[S(y)]} \\
{[N]} \arrow[ur, bend left, no head] \arrow[ur, bend right, no head] \arrow[dr, bend left, no head] \arrow[dr, bend right, no head] & \\
 & {[S(z)]}
\end{tikzcd}$$

thus it contains a subquiver of type $\widetilde{\widetilde{\mathbb{A}}}_1$.

Appendix

4.7. Review of some known results from the theory of rings and modules.

We want to recall two basic results which concern modules of finite length over any ring R. The modules to be considered are R-modules.

Let M be a module. A *composition series* of M is a chain

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_t = M$$

of submodules (a "filtration" of M) such that all the factors M_i/M_{i-1} with $1 \leq i \leq t$ are simple. The number t is called the *length* of the composition series.

Jordan-Hölder Theorem. *Assume that two composition series of M are given:*

$$\begin{aligned} 0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_s = M, \\ 0 = M'_0 \subseteq M'_1 \subseteq M'_2 \subseteq \cdots \subseteq M'_t = M. \end{aligned}$$

Then $s = t$ and there is a permutation π of the set $\{1, 2, \ldots, s\}$ such that the modules M_i/M_{i-1} and $M'_{\pi(i)}/M'_{\pi(i)-1}$ are isomorphic, for $1 \leq i \leq s$.

In addition, for any filtration

$$0 = M''_0 \subseteq M''_1 \subseteq M''_2 \subseteq \cdots \subseteq M''_r = M.$$

with proper inclusions $M''_{i-1} \subset M''_i$ for all $1 \leq i \leq r$, we have $r \leq t$.

On the basis of this result, one introduces the following definitions: If M has a composition of length t, then one calls t the *length* of M and one calls the factors of a composition series the *composition factors* of M. If there is given a composition series

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_s = M$$

of M, and S a simple module, then one calls the number of factors M_i/M_{i-1} which are isomorphic to S the *Jordan-Hölder multiplicity of S in M.*

Theorem of Krull-Remak-Schmidt. *Let M be a module of finite length, and assume that there are given two direct decompositions:*

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_s, \qquad \text{and} \qquad M = M'_1 \oplus M'_2 \oplus \cdots \oplus M'_t,$$

such that all the modules M_i, M'_j with $1 \leq i \leq s, 1 \leq j \leq t$ are indecomposable. Then $s = t$ and there is a permutation π of the set $\{1, 2, \ldots, s\}$ such that the modules M_i and $M'_{\pi(i)}$ for $1 \leq i \leq s$ are isomorphic.

The proof is based on the following Lemma which is of independent interest:

Fitting Lemma. *Let M be an indecomposable module of finite length. Then the endomorphism ring of M is a local ring with nilpotent radical.*