Quasi-isomorphism versus chain homotopy equivalence

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Def: A <u>quasi-isomorphism</u> is a chain map $f: C \to D$ such that $H_n f: H_n C \to H_n D$ is an isomorphism for all $n \in \mathbb{Z}$.

Example: Let $f: C \to D$ be a chain homotopy equivalence, i.e. there is a chain map $g: D \to C$ such $f\circ g$ and $g \circ f$ are chain homotopic to the respective identity maps. Then

$$H_n f \circ H_n g = H_n(f\circ g) = H_n(Id_D) = Id_{H_n D}, \text{ and similarly, } H_n g \circ H_n f = Id_{H_n C}.$$

So f and g are quasi-isomorphism.

Example: Quasi-isomorphisms need not be chain homotopy equivalences:

$$\begin{tikzcd}
C = (\cdots & 0 \arrow[r] \arrow[d, "0"] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d, "0"] & \mathbb{Z} \arrow[r] \arrow[d, "proj"] & 0 \arrow[r] \arrow[d, "0"] & 0 \cdots) \\
D = (\cdots & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & 0 \cdots) \\
& 2 & 1 & 0 & -1 &
\end{tikzcd}$$

This is a quasi-isomorphism, but <u>not</u> a chain homotopy equivalence: The only chain map $g: D \to C$ is the trivial chain map!

$0 = H_n g$ is not an isomorphism, so f and g are not chain homotopy equivalences.

Thm: Let $f: C \to D$ be a quasi-isomorphism between complexes of free abelian group. Then f is a chain homotopy equivalence.

Proof. <u>Case 1</u>: Suppose the complex C is of the following very special form: $C_n = Z_n \oplus Z_{n-1}$ for some abelian groups Z_n, $n \in \mathbb{Z}$ and the differential $d_n: C_n = Z_n \oplus Z_{n-1} \longrightarrow Z_{n-1} \oplus Z_{n-2} = C_{n-1}$ is given by $d_n(x,y) = (y,0)$. Then the complex C is chain contractible, i.e. chain homotopy equivalent to the trivial complex.

Indeed, define $s_n: C_n = Z_n \oplus Z_{n-1} \longrightarrow Z_{n+1} \oplus Z_n = C_{n+1}$ by $s_n(x,y) = (0,x)$.

Then $d_{n+1} \circ s_n + s_{n-1} \circ d_n = Id_{C_n}$ for all $n \in \mathbb{Z}$, so $s = \{s_n\}_{n\in\mathbb{Z}}$ is a chain homotopy from the identity of C to the zero morphism.

<u>Case 2</u>: Let C be an acyclic complex of free abelian groups, i.e. $H_n C = 0$ for all $n \in \mathbb{Z}$. Then C is chain contractible.

Because C is acyclic, $Im(d_{n+1}: C_{n+1} \to C_n) = ker(d_n: C_n \to C_{n-1}) =: Z_n$.
Since C_{n-1} is free abelian, so is its subgroup Z_{n-1}. So the epimorphism $d_n: C_n \twoheadrightarrow Z_{n-1}$ admits an additive section $\sigma_n: Z_{n-1} \to C_n$, i.e. such that $d_n \circ \sigma_n = Id_{Z_{n-1}}$.

So σ_n splits the short exact sequence $0 \to Z_n \hookrightarrow C_n \underset{\sigma_n}{\overset{d_n}{\rightleftarrows}} Z_{n-1} \to 0$

So the map $Z_n \oplus Z_{n-1} \longrightarrow C_n$, $(x,y) \mapsto x + \sigma_n(y)$ is an isomorphism of abelian groups.

Moreover, the following squares commute:

$$\begin{tikzcd}
(x,y) \arrow[d, mapsto] & Z_n \oplus Z_{n-1} \arrow[r, "(x,y)\mapsto x+\sigma_n(y)", "\cong"'] \arrow[d] & C_n \arrow[d, "d_n"] \\
(y,0). & Z_{n-1} \oplus Z_{n-2} \arrow[r, "\cong", "(x,y)\mapsto x+\sigma_{n-1}(y)"'] & C_{n-1}
\end{tikzcd}$$

Indeed: $d_n(x + \sigma_n(y)) = d_n x + d_n(\sigma_n(y)) = y$

Upshot: C is isomorphic to a complex of the special type considered in Case 1. Hence C is chain contractible by case 1.

<u>Case 3</u>: Let $f: C \to D$ be a quasi-isomorphism between chain complexes of free abelian groups.

We define the <u>mapping cone</u> Cf, another chain complex, by $(Cf)_n = D_n \oplus C_{n-1}$ with differential

$$d_n: (Cf)_n = D_n \oplus C_{n-1} \longrightarrow D_{n-1} \oplus C_{n-2} = (Cf)_{n-1} \text{ defined as}$$

$$d_n(x,y) = (d_n x + (-1)^n \cdot f_{n-1}(y), d_{n-1} y)$$

This is indeed a chain complex: $d(d(x,y)) = d(dx + (-1)^n \cdot f_{n-1}(y), dy)$

$$= (\underbrace{ddx}_{=0} + \underbrace{(-1)^n d(f_{n-1}(y)) + (-1)^{n-1} \cdot f_{n-2}(dy)}_{\text{Cancel}}, \underbrace{ddy}_{=0}) = (0,0).$$

The mapping cone contains D as a subcomplex (dimension wise the first summand)
The projection to the second summands $p_n: (Cf)_n \longrightarrow C_{n-1}$ form a chain map from Cf to the <u>shift</u> $C[1]$ of C, i.e.

$(C[1]) = C$. $d^{C[1]} = d^C$

The projection to the second summands $p_n : (Cf)_n \longrightarrow C_{n-1}$ form a chain map from Cf to the shift $C[1]$ of C, i.e. $(C[1])_n = C_{n-1}$, $d_n^{C[1]} = d_{n-1}^{C}$.

So we get a short exact sequence of chain complexes

$$0 \longrightarrow D \xrightarrow{1^{st} \text{ summands}} Cf \xrightarrow{p} C[1] \longrightarrow 0$$

This short exact sequence gives rise to a long exact sequence of homology groups:

$$\cdots \longrightarrow H_{n+1}(C[1]) \xrightarrow{\partial} H_n D \longrightarrow H_n(Cf) \xrightarrow{H_n p} H_n(C[1]) \xrightarrow{\partial} \cdots$$

$$H_{n+1}(C[1]) = H_n C \xrightarrow{(-1)^{n+1} \cdot H_n f} H_n D, \qquad [x] \longmapsto [(-1)^{n+1} f_n(x)]$$

$$(Cf)_{n+1} \xrightarrow{p_{n+1}} (C[1])_{n+1} = C_n \ni x \quad \text{with } dx = 0$$

$$(Cf)_{n+1} = D_{n+1} \oplus C_n, \qquad (0,x) \longmapsto x$$

$$(0,x) \overset{d_{n+1}}{\longmapsto} ((-1)^{n+1} \cdot f_n(x), 0)$$

Because $H_n f$ is an isomorphism, $(-1)^{n+1} \cdot H_n f = \partial$ is an isomorphism, too. So by the long exact sequence, the groups $H_n(Cf)$ vanish, i.e. the mapping cone Cf is acyclic.
Since C and D consist of free abelian groups, Cf also consists of free abelian groups.
So Cf is chain contractible by case 2. We choose a contracting chain homotopy $s = \{s_n\}_{n \in \mathbb{Z}}$, $s_n : (Cf)_n \longrightarrow (Cf)_{n+1}$ such that $d_{n+1} \circ s_n + s_{n-1} \circ d_n = Id_{(Cf)_n}$.

To organize the following calculation, I'll use matrix notation for homomorphisms $\varphi : A \oplus B \longrightarrow A' \oplus B'$.
This way, composition becomes matrix multiplication. We write $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{21} \\ \varphi_{12} & \varphi_{22} \end{pmatrix}$ where $\varphi_{11} : A \longrightarrow A'$, $\varphi_{21} : B \longrightarrow A'$, $\varphi_{12} : A \longrightarrow B'$, $\varphi_{22} : B \longrightarrow B'$

$$\varphi(a,b) = (\varphi_{11}(a) + \varphi_{21}(b), \varphi_{12}(a) + \varphi_{22}(b))$$

In this notation, I'll expand the chain nullhomotopy $s_n : D_n \oplus C_{n-1} \longrightarrow D_{n+1} \oplus C_n$ as

$$s_n = \begin{pmatrix} \sigma & \alpha \\ \beta & \tau \end{pmatrix}$$

In this notation, $d_n : D_n \oplus C_{n-1} \longrightarrow D_{n-1} \oplus C_{n-2}$ is

$$d_n = \begin{pmatrix} d & (-1)^n \cdot f \\ 0 & d \end{pmatrix}$$

The relation $d_{n+1} \circ s_n + s_{n-1} \circ d_n = Id_{(Cf)_n}$ expands to:

$$\begin{pmatrix} d & (-1)^{n+1} f \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} \sigma & \alpha \\ \beta & \tau \end{pmatrix} + \begin{pmatrix} \sigma & \alpha \\ \beta & \tau \end{pmatrix} \cdot \begin{pmatrix} d & (-1)^n \cdot f \\ 0 & d \end{pmatrix} = \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix}$$

$$\| $$

$$\begin{pmatrix} d\sigma + (-1)^{n+1} f \beta & d\alpha + (-1)^{n+1} f \tau \\ d\beta & d\tau \end{pmatrix} + \begin{pmatrix} \sigma d & (-1)^n \sigma f + \alpha d \\ \beta d & (-1)^n \beta f + \tau d \end{pmatrix}$$

In the lower left corner, we get the identity $d\beta + \beta d = 0 : D_n \longrightarrow C_{n-1}$.
We define $g_n = (-1)^n \cdot \beta_n : D_n \longrightarrow C_n$; this is a chain map $g : D \longrightarrow C$.

In upper left corner, we get the relation $d\sigma + (-1)^{n+1} f \cdot \beta + \sigma d = Id \iff \sigma = \{\sigma_n\}$ is a chain homotopy between $f \circ g$ and Id_D.
In the lower right corner, we get:

$$d\tau + (-1)^n \beta \cdot f + \tau d = Id \iff \tau = \{\tau_n\} \text{ form a chain homotopy between } g \circ f \text{ and } Id_C. \qquad \square$$