Proof of the Eilenberg-Zilber theorem

This document might contain typos; if you happen to find any, please let me know so that I can correct them.

We write $C_*(X)$ for the integral chain complex of a simplicial set X. So for $n \geq 0$, the group $C_n(X) = \mathbb{Z}[X_n]$ is the free abelian group generated by the set of n-simplices of X, and the differentials are the alternating sums of the linearizations of the simplicial face maps. The complex $C_*(X)$ is trivial in negative dimensions.

Given a pair of simplicial sets X and Y, we can form the two chain complexes

$$C_*(X) \otimes C_*(Y) \qquad \text{and} \qquad C_*(X \times Y) \ .$$

The Eilenberg-Zilber theorem says that these two chain complexes are naturally chain homotopy equivalent, see Theorem 2 below.

For two sets S and T, the maps

$$\begin{aligned} \mathbb{Z}[S] \otimes \mathbb{Z}[T] &\longrightarrow \mathbb{Z}[S \times T] \ , \quad s \otimes t \longmapsto (s,t) \\ \mathbb{Z}[S \times T] &\longrightarrow \mathbb{Z}[S] \otimes \mathbb{Z}[T] \ , \quad (s,t) \longmapsto s \otimes t \end{aligned}$$

are mutually inverse isomorphisms of abelian groups. For $S = X_n$ and $T = Y_n$ and varying $n \geq 0$, these isomorphism identify $C_*(X \times Y)$ with the chain complex of the simplicial abelian group $\mathbb{Z}[X] \otimes \mathbb{Z}[Y]$, the dimensionwise tensor product of the linearizations of X and Y:

$$C_*(X \times Y) \ \cong \ C_*(\mathbb{Z}[X] \otimes \mathbb{Z}[Y])$$

The Eilenberg-Zilber map and the Alexander-Whitney map for the simplicial abelian groups $\mathbb{Z}[X]$ and $\mathbb{Z}[Y]$ thus specialize to chain maps

$$\mathrm{EZ} \ : \ C_*(X) \otimes C_*(Y) \ \longrightarrow \ C_*(X \times Y) \qquad \text{and} \qquad \mathrm{AW} \ : \ C_*(X \times Y) \ \longrightarrow \ C_*(X) \otimes C_*(Y) \ .$$

These chain maps are natural for morphisms of simplicial sets in X and in Y. We will prove that these two chain maps are mutually inverse chain homotopy equivalences by two applications of the theorem of acyclic models. The following proposition will provide the acyclicity hypotheses of this theorem. We write $\Delta^p = \Delta(-, [p])$ for the simplicial p-simplex.

Proposition 1. (i) *For all $p \geq 0$, the simplicial simplex Δ^p is contractible by a simplicial homotopy.*

(ii) *For all $p \geq 0$, the complex $C_*(\Delta^p)$ is chain homotopy equivalent to the complex consisting of a single copy of $\mathbb{Z}$ concentrated in dimension 0.*

(iii) *For all $p, q \geq 0$, the chain complex $C_*(\Delta^p) \otimes C_*(\Delta^q)$ is chain homotopy equivalent to the complex consisting of a single copy of $\mathbb{Z}$ concentrated in dimension 0. In particular, this complex is acyclic in positive dimensions.*

Proof. (i) We define a morphism of simplicial sets

$$H \ : \ \Delta^p \times \Delta^1 \ \longrightarrow \ \Delta^p$$

that contracts Δ^p to the 'last vertex', as follows. In dimension n, we define

$$H_n \ : \ \Delta([n],[p]) \times \Delta([n],[1]) \ \longrightarrow \ \Delta([n],[p])$$

by

$$H_n(\alpha, \beta)(i) \ = \ \begin{cases} \alpha(i) & \text{if } \beta(i) = 0, \text{ and} \\ p & \text{if } \beta(i) = 1. \end{cases}$$

We check that these maps indeed form a morphism of simplicial sets. To this end we let $\gamma : [m] \longrightarrow [n]$ be a morphism in Δ. Then

$$\begin{aligned} H_m(\gamma^*(\alpha,\beta))(j) &= H_m(\alpha\circ\gamma,\beta\circ\gamma)(j) \\ &= \begin{cases} \alpha(\gamma(j)) & \text{if } \beta(\gamma(j)) = 0, \text{ and} \\ p & \text{if } \beta(\gamma(j)) = 1. \end{cases} \\ &= H_n(\alpha,\beta)(\gamma(j)) \\ &= \gamma^*(H_n(\alpha,\beta))(j) \ . \end{aligned}$$

The morphism H is a simplicial homotopy from the identity of Δ^p to the morphism $c_* = c\circ - : \Delta^p \longrightarrow \Delta^p$, where $c : [p] \longrightarrow [p]$ is the constant morphism with $c(i) = p$ for all $0 \leq i \leq p$.

(ii) We showed in part (i) that the identity of Δ^p is simplicially homotopic to the morphism $c_* : \Delta^p \longrightarrow \Delta^p$, the 'constant morphism at the last vertex'. Equivalently, the unique morphism of simplicial sets

$$\pi \ : \ \Delta^p \ \longrightarrow \ \Delta^0$$

is simplicially homotopy inverse to the morphism

$$i \ : \ \Delta^0 \ \longrightarrow \ \Delta^p$$

induced by $[0] \longrightarrow [p]$ with image p. Passage to the chain complex of a simplicial set takes homotopic morphism to chain homotopic chain maps. So the two chain maps

$$C_*(\pi) \ : \ C_*(\Delta^p) \ \longrightarrow \ C_*(\Delta^0) \qquad \text{and} \qquad C_*(i) \ : \ C_*(\Delta^0) \ \longrightarrow \ C_*(\Delta^p)$$

are mutually inverse chain homotopy equivalences. Since the simplicial set Δ^0 is constant, its chain complex has the form

$$C_*(\Delta^0) \ \cong \ (\dots \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{Id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{Id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0) \ ,$$

and this complex is chain homotopy equivalent to the complex consisting of $\mathbb{Z}$ concentrated in dimension 0.

(iii) We write $\mathbb{Z}\{0\}$ for the chain complex consisting of the group $\mathbb{Z}$ in dimension 0, and such that $(\mathbb{Z}\{0\})_n = 0$ for all $n \neq 0$. Part (ii) provides a chain homotopy equivalence between $C_*(\Delta^p)$ and the complex $\mathbb{Z}\{0\}$. We tensor the data of the chain maps and chain homotopies with the complex $C_*(\Delta^q)$ to obtain chain homotopy equivalences between the complexes

$$C_*(\Delta^p)\otimes C_*(\Delta^q) \qquad \text{and} \qquad \mathbb{Z}\{0\}\otimes C_*(\Delta^q) \ .$$

The complex $\mathbb{Z}\{0\}\otimes C_*(\Delta^q)$ is isomorphic to $C_*(\Delta^q)$, and another application of part (ii) provides a chain homotopy equivalence between $C_*(\Delta^q)$ and $\mathbb{Z}\{0\}$. So altogether, we obtain a chain homotopy equivalence between the complex $C_*(\Delta^p)\otimes C_*(\Delta^q)$ and the complex $\mathbb{Z}\{0\}$. $\square$

There is only one $(0,0)$-shuffle, the identity of the empty set. So in chain complex dimension 0, the Eilenberg-Zilber map specializes to the isomorphism

$$\mathrm{EZ}_0 \ : \ (C_*(X)\otimes C_*(Y))_0 \ = \ \mathbb{Z}[X_0]\otimes\mathbb{Z}[Y_0] \ \longrightarrow \ \mathbb{Z}[X_0\times Y_0] \ = \ C_0(X\times Y)$$

sending $x\otimes y$ to (x,y). Similarly, in dimension $(0,0)$, we have $d_{\mathrm{front}} = d_{\mathrm{back}} = \mathrm{Id}_{[0]} : [0] \longrightarrow [0]$. So in chain complex dimension 0, the Alexander-Whitney map AW_0 specializes to the isomorphism inverse to EZ_0. We conclude that while the chain maps EZ and AW are complicated in general, they are the preferred mutually inverse isomorphisms in dimension 0.

Theorem 2. *The composite chain maps*

$$\mathrm{AW}\circ\mathrm{EZ} \ : \ C_*(X)\otimes C_*(Y) \ \longrightarrow \ C_*(X)\otimes C_*(Y) \quad \textit{and} \quad \mathrm{EZ}\circ\mathrm{AW} \ : \ C_*(X\times Y) \ \longrightarrow \ C_*(X\times Y)$$

are naturally chain homotopic to the respective identity maps.

Proof. We consider the natural chain map

$$\psi \;=\; \mathrm{AW} \circ \mathrm{EZ} - \mathrm{Id}_{C_*(X)\otimes C_*(Y)} \;:\; C_*(X) \otimes C_*(Y) \;\longrightarrow\; C_*(X) \otimes C_*(Y)\ .$$

We claim that the functor

$$C_*(-) \otimes C_*(-) \;:\; \mathrm{sset} \times \mathrm{sset} \;\longrightarrow\; \mathrm{Ch}_+$$

and the natural chain map ψ satisfy the hypotheses of the theorem of acyclic models.

The Eilenberg-Zilber and Alexander-Whitney maps are mutually inverse isomorphisms in chain dimension 0. So the transformation ψ_0 is trivial. For $n \geq 0$, the functor

$$(C_*(X) \otimes C_*(Y))_n \;=\; \bigoplus_{p=0}^{n} \mathbb{Z}[X_p] \otimes \mathbb{Z}[Y_{n-p}]$$

is the direct sum of $n+1$ functors, each of which is isomorphic to the functor

$$\mathrm{sset} \times \mathrm{sset} \;\longrightarrow\; \mathcal{A}b\ , \quad (X, Y) \;\longmapsto\; \mathbb{Z}[X_p \times Y_q]$$

for a pair of natural numbers (p, q). As we explained in the previous video, this functor is representable by the pair of simplicial sets (Δ^p, Δ^q). We showed in Proposition 1 that the homology groups of the complex $C_*(\Delta^p) \otimes C_*(\Delta^q)$ are trivial in positive dimensions. So the theorem of acyclic models applies, and it provides a natural chain nullhomotopy of the chain map ψ. This data is at the same time a a natural chain homotopy between the composite $\mathrm{AW} \circ \mathrm{EZ}$ and the identity chain map.

The argument for the other composite is very similar, but slightly different in the details. This time we consider the natural chain map

$$\varphi \;=\; \mathrm{EZ} \circ \mathrm{AW} - \mathrm{Id}_{C_*(X \times Y)} \;:\; C_*(X \times Y) \;\longrightarrow\; C_*(X \times Y)\ .$$

We claim that the functor

$$C_*(- \times -) \;:\; \mathrm{sset} \times \mathrm{sset} \;\longrightarrow\; \mathrm{Ch}_+$$

and the natural chain map φ also satisfy the hypotheses of the theorem of acyclic models. Since the Eilenberg-Zilber and Alexander-Whitney maps are mutually inverse isomorphisms in chain dimension 0, the transformation φ_0 is trivial. For $n \geq 0$, the functor

$$C_n(X \times Y) \;=\; \mathbb{Z}[X_n \times Y_n]$$

is representable by (Δ^n, Δ^n). We showed in Proposition 1 that the simplicial set Δ^n is simplicially homotopy equivalent to Δ^0; so $\Delta^n \times \Delta^n$ is simplicially homotopy equivalent to $\Delta^0 \times \Delta^0$, which in turn is isomorphic to Δ^0. Hence the complex $C_*(\Delta^n \times \Delta^n)$ is chain homotopy equivalent to the complex $C_*(\Delta^0)$, which is acyclic in positive dimensions. So the theorem of acyclic models applies, and it provides a natural chain nullhomotopy of the chain map φ. This data is at the same time a natural chain homotopy between the composite $\mathrm{EZ} \circ \mathrm{AW}$ and the identity chain map. $\square$