Topology II - Cohomology

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Reminder about homology:

$$\text{Top} \xrightarrow[\text{singular complex}]{\rho} \text{(simpicial sets)} \xrightarrow[\text{linearization}^{C(-,A)}]{} \text{(chain complex)} \xrightarrow[n\text{-th homology group}]{} \text{Ab}\,.$$

- For a space X, the singular complex $\rho(X)$ is the simplicial set with

$$\rho(X)_n = \text{map}^{\text{cpt}}(\nabla^n, X)$$

$$\nabla^n = \text{topological } n - \text{simplex} = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \;\;:\;\; x_n \geq 0, x_0 + ... + x_n = 1\}$$

- For a simplicial set Y and an abelian group, the linearization is the chain complex $C(Y; A)$ with

$$C_(Y; a) = A[Y_n] \quad A - \text{linarization of} Y_n \qquad (C_n(Y; A) = 0 \quad \text{for } n < 0)$$

- For a chain complex C and $n \in \mathbb{Z}$, the n-th homology group $H_n(C)$ is

$$\frac{\ker(d_n : C_n \to C_{n-1})}{\text{Im}(d_{n+1}) : C_{n+1} \to C_n}$$

0.1 Variation : Cohomology

Definition 0.1. A cochain coplex C consists of abelian groups C^n for $n \in \mathbb{Z}$ and homomorphisms $d^n : C^n \to C^{n+1}$ such that

$$d^{n+1} \circ d^n = 0 : C^n \to C^{n+2}.$$

A morphism $f : C \to D$ of cochain complexes (cochain map) consists of homomorphisms $f^n : C^n \to D^n$ such that $d_D^n \circ f^n = f^{n+1} \circ d_C^n$

TODOyellow!40

Diagram

The n-th cohomology group of a cochain complex C is

$$H^n C = \frac{\ker(d^n : C^n \to C^{n+1})}{\text{Im}(d^{n-1} : C^{n-1} \to C^n)}$$

The main tools and properties carry over from chain complexes to cochain complexs, with essentially the same proofs, such as:

- a morphism $f : C \to D$ of cochain complexes