Algebraic Geometry

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Lecture 3: Bezout's theorem and its applications

Bezout's theorem: motivation

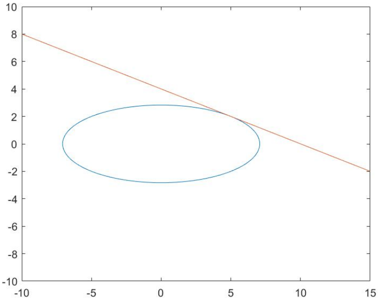
Lemma

Let L and L' be two lines in $\mathbb{P}^2_{\mathbb{C}}$ such that $L \neq L'$.
Then the intersection $L \cap L'$ consists of a single point.

Lemma

Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$. Let $\mathcal{C}$ be an irreducible conic in $\mathbb{P}^2_{\mathbb{C}}$. Then

- *the intersection $L \cap \mathcal{C}$ consists of one or two points,*
- $|L \cap \mathcal{C}| = 1 \iff$ *L is tangent to $\mathcal{C}$ at the point $L \cap \mathcal{C}$.*



Intersecting two conics

Let $\mathcal{C}$ and $\mathcal{C}'$ be two irreducible conics in $\mathbb{P}^2_{\mathbb{C}}$ such that $\mathcal{C} \neq \mathcal{C}'$.

Theorem

The intersection $\mathcal{C} \cap \mathcal{C}'$ consists of one, two, three or four points.

Proof.

We may assume that $\mathcal{C}$ is given by $xy = z^2$. Then $\mathcal{C}'$ is given by

$$\boxed{\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2 = 0}$$

for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ in $\mathbb{C}$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}) \neq (0, 0, 0, 0, 0, 0)$.

- Let L be the line $y = 0$. Then $L \cap \mathcal{C} \cap \mathcal{C}' \subset [1 : 0 : 0]$.
- One has $L \cap \mathcal{C} \cap \mathcal{C}' = [1 : 0 : 0] \iff \mathbf{a} = 0$.
- Let $U_y = \mathbb{P}^2_{\mathbb{C}} \setminus L$. Then $U_y \cap \mathcal{C} \cap \mathcal{C}'$ is given by

$$y - 1 = x - z^2 = \mathbf{a}z^4 + \mathbf{d}z^3 + (\mathbf{b} + \mathbf{f})z^2 + \mathbf{e}z + \mathbf{c} = 0.$$

If $\mathbf{a} = 0$, then $L \cap \mathcal{C} \cap \mathcal{C}' = [1 : 0 : 0]$ and $0 \leqslant |U_y \cap \mathcal{C} \cap \mathcal{C}'| \leqslant 3$.
If $\mathbf{a} \neq 0$, then $L \cap \mathcal{C} \cap \mathcal{C}' = \varnothing$ and $1 \leqslant |U_y \cap \mathcal{C} \cap \mathcal{C}'| \leqslant 4$. $\square$

Transversal intersection of two conics

Let $\mathcal{C}$ and $\mathcal{C}'$ be two irreducible conics in $\mathbb{P}^2_{\mathbb{C}}$.

Question

When the intersection $\mathcal{C} \cap \mathcal{C}'$ consists of 4 points?

Let P be a point in $\mathcal{C} \cap \mathcal{C}'$.

- $\exists$ unique line $L \subset \mathbb{P}^2_{\mathbb{C}}$ such that $P \in L$ and $|L \cap \mathcal{C}| = 1$.
- $\exists$ unique line $L' \subset \mathbb{P}^2_{\mathbb{C}}$ such that $P \in L'$ and $|L' \cap \mathcal{C}| = 1$.

The lines L and L' are tangent lines to $\mathcal{C}$ and $\mathcal{C}'$ at P, respectively.

Definition

We say that $\mathcal{C}$ intersects $\mathcal{C}'$ transversally at P if $L \neq L'$.

- The answer to the question above is given by

Theorem

The following two conditions are equivalent:

1. *the intersection $\mathcal{C} \cap \mathcal{C}'$ consists of 4 points,*
2. *$\mathcal{C}$ intersects $\mathcal{C}'$ transversally at every point of $\mathcal{C} \cap \mathcal{C}'$.*

The intersection of two conics: four points

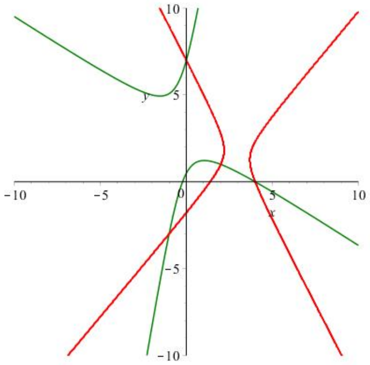
Let $\mathcal{C}$ be the irreducible conic

$$511x^2 + 709xy - 131y^2 - 1932xz + 981yz - 448z^2 = 0.$$

Let $\mathcal{C}$ be the irreducible conic

$$1217x^2 - 394xy - 541y^2 - 6555xz + 2823yz + 6748z^2 = 0.$$

Then $\mathcal{C} \cap \mathcal{C}$ consists of $[4 : 0 : 1]$, $[1 : 3 : -1]$, $[0 : 7 : 1]$, $[2 : 1 : 1]$.



The intersection of two conics: three points

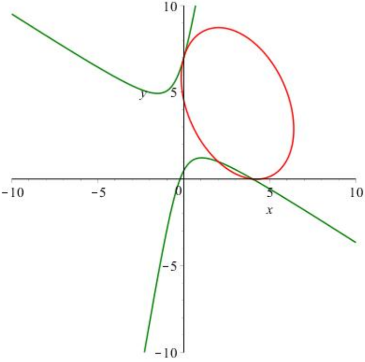
Let $\mathcal{C}$ be the irreducible conic

$$511x^2 + 709xy - 131y^2 - 1932xz + 981yz - 448z^2 = 0.$$

Let $\mathcal{C}$ be the irreducible conic

$$42049x^2 + 21271xy + 23536y^2 - 355005xz - 271500yz + 747236z^2 = 0.$$

Then $\mathcal{C} \cap \mathcal{C}$ consists of $[4:0:1]$, $2 \times [0:7:1]$, $[2:1:1]$.



The intersection of two conics: two points (2+2)

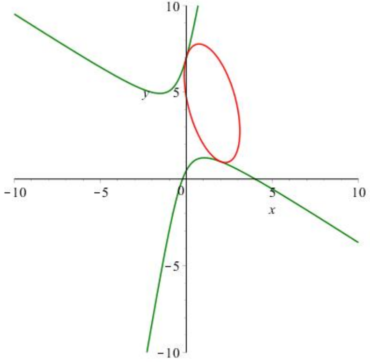
Let $\mathcal{C}$ be the irreducible conic $f(x, y, z) = 0$, where

$$f(x, y, z) = 511x^2 + 709xy - 131y^2 - 1932xz + 981yz - 448z^2.$$

Let $\mathcal{C}$ be the irreducible conic

$$(3031x-853y+5971z)(821x-3779y+2137z)-9700f(x, y, z) = 0.$$

Then $\mathcal{C} \cap \mathcal{C}$ consists of $2 \times [0 : 7 : 1]$ and $2 \times [2 : 1 : 1]$.



The intersection of two conics: two points (3+1)

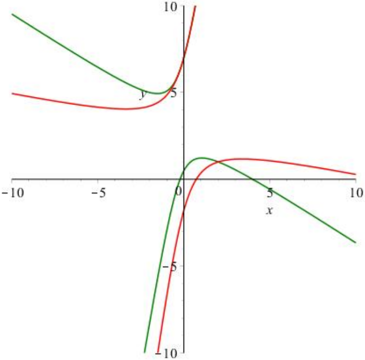
Let $\mathcal{C}$ be the irreducible conic $f(x,y,z)=0$, where

$$f(x,y,z) = 511x^2 + 709xy - 131y^2 - 1932xz + 981yz - 448z^2.$$

Let $\mathcal{C}$ be the irreducible conic

$$(3031x - 853y + 5971z)(6x + 2y - 14z) - 50f(x,y,z) = 0.$$

Then $\mathcal{C} \cap \mathcal{C}$ consists of $3 \times [0:7:1]$ and $[2:1:1]$.



The intersection of two conics: one points

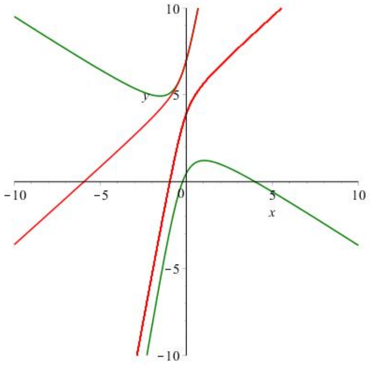
Let $\mathcal{C}$ be the irreducible conic $f(x, y, z) = 0$, where

$$f(x, y, z) = 511x^2 + 709xy - 131y^2 - 1932xz + 981yz - 448z^2.$$

Let $\mathcal{C}$ be the irreducible conic

$$(3031x - 853y + 5971z)^2 - 5000f(x, y, z) = 0.$$

Then $\mathcal{C} \cap \mathcal{C}$ consists of $4 \times [0 : 7 : 1]$.



Bezout's theorem: algebraic version

- Let $f(x, y, z)$ be a homogeneous polynomial of degree d.
- Let $g(x, y, z)$ be a homogeneous polynomial of degree $\widehat{d}$.

Consider the system of equations

$$\boxed{\begin{cases} f(x,y,z) = 0 \\ g(x,y,z) = 0 \end{cases}} \qquad (\bigstar)$$

Question

How many solutions in $\mathbb{P}^2_{\mathbb{C}}$ does $(\bigstar)$ has?

- Infinite if $f(x, y, z)$ and $g(x, y, z)$ have a common factor.

Theorem (Bezout)

Suppose that $f(x, y, z)$ and $g(x, y, z)$ have no common factors. Then the number of solutions to $(\bigstar)$ depends only on d and $\widehat{d}$.

- Here we should count solutions with multiplicities.

Intersection multiplicities

- Let $f(x,y,z)$ be a homogeneous polynomial of degree d.
- Let $g(x,y,z)$ be a homogeneous polynomial of degree $\widehat{d}$.

Suppose that $f(x,y,z)$ and $g(x,y,z)$ do not have common factors.

- Let C be the subset in $\mathbb{P}^2_{\mathbb{C}}$ that is given by $f(x,y,z)=0$.
- Let Z be the subset in $\mathbb{P}^2_{\mathbb{C}}$ that is given by $g(x,y,z)=0$.

For every $P \in C \cap Z$, define a positive integer $(f,g)_P$ as follows:

- Assume that $P \in U_z = \mathbb{C}^2$ with coordinates $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$.
- Let $\mathbf{R}$ be a subring in $\mathbb{C}(\overline{x},\overline{y})$ consisting of all fractions
$$\frac{a(\overline{x},\overline{y})}{b(\overline{x},\overline{y})}$$
with $a(\overline{x},\overline{y})$ and $b(\overline{x},\overline{y})$ in $\mathbb{C}[\overline{x},\overline{y}]$ such that $b(P) \neq 0$.
- Let $\mathbf{I}$ be the ideal in $\mathbf{R}$ generated by $f(\overline{x},\overline{y},1)$ and $g(\overline{x},\overline{y},1)$.
- Let $(f,g)_P = \dim_{\mathbb{C}}(\mathbf{R}/\mathbf{I}) \geqslant 1$.

Then Bezout's theorem says that
$$\sum_{P \in C \cap Z} \big(f,g\big)_P = d\widehat{d}.$$

Bezout's theorem: baby case

- Let $f(x, y, z)$ be a homogeneous polynomial of degree d.
- Let $g(x, y, z)$ be a homogeneous polynomial of degree 1.

Suppose that $g(x, y, z)$ does not divide $f(x, y, z)$.

- We may assume that $g(x, y, z) = z$.

We have to solve the system

$$\boxed{\begin{cases} z = 0, \\ f(x, y, z) = 0. \end{cases}}$$

Theorem (Fundamental Theorem of Algebra)

There are linear polynomials $h_1(x, y), \ldots, h_d(x, y)$ such that

$$f(x, y, 0) = \prod_{i=1}^{d} h_i(x, y).$$

- This gives d points in $\mathbb{P}^2_{\mathbb{C}}$ counted with multiplicities.

Basic properties of intersection multiplicities

- Let $f(x,y,z)$ be a homogeneous polynomial.
- Let $g(x,y,z)$ be a homogeneous polynomial.

Suppose that $f(x,y,z)$ and $g(x,y,z)$ do not have common factors. Fix $P \in \mathbb{P}^2_{\mathbb{C}}$ such that $f(P) = g(P) = 0$. Then

$$\boxed{\big(f,g\big)_P = \big(g,f\big)_P \geqslant 1.}$$

- Let $h(x,y,z)$ be a homogeneous polynomial.

Suppose that $f(x,y,z)$ and $h(x,y,z)$ do not have common factors.

- If $h(P) = 0$, then

$$\boxed{\big(f,gh\big)_P = \big(f,g\big)_P + \big(f,h\big)_P.}$$

- If $h(P) \neq 0$, then

$$\boxed{\big(f,gh\big)_P = \big(f,g\big)_P.}$$

Bezout's theorem: geometric version

- Let C be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by
$$f(x, y, z) = 0,$$
where f is a homogeneous irreducible polynomial of degree d.
- Let Z be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by
$$g(x, y, z) = 0,$$
where g is a homogeneous irreducible polynomial of degree $\widehat{d}$.

Theorem (Bezout)

Suppose that $f(x, y, z) \neq \lambda g(x, y, z)$ for any $\lambda \in \mathbb{C}^*$. Then

$$\boxed{1 \leqslant \big|C \cap Z\big| \leqslant \sum_{P \in C \cap Z} \big(C \cdot Z\big)_P = d\widehat{d}}$$

where $(C \cdot Z)_P = \big(f, g\big)_P$ is the intersection multiplicity.

Corollary

$C = Z \iff f(x, y, z) = \lambda g(x, y, z)$ *for some* $\lambda \in \mathbb{C}^*$.

Intersection of two cubics

Let $\mathcal{C}$ be the irreducible cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ given by

$$-5913252577x^3 + 30222000280x^2y - 21634931915xy^2 + \\ +5556266591y^3 - 73906985473x^2z + 102209537669xyz - 37300172365y^2z + \\ +1389517162xz^2 - 88423819400yz^2 + 204616284808z^3 = 0.$$

Let $\mathcal{C}$ be the irreducible cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ given by

$$-4844332x^3 - 8147864x^2y - 4067744xy^2 - \\ -1866029y^3 + 32668904x^2z - 28226008xyz + 41719157y^2z + \\ +252639484xz^2 + 126319742yz^2 - 960898976z^3 = 0$$

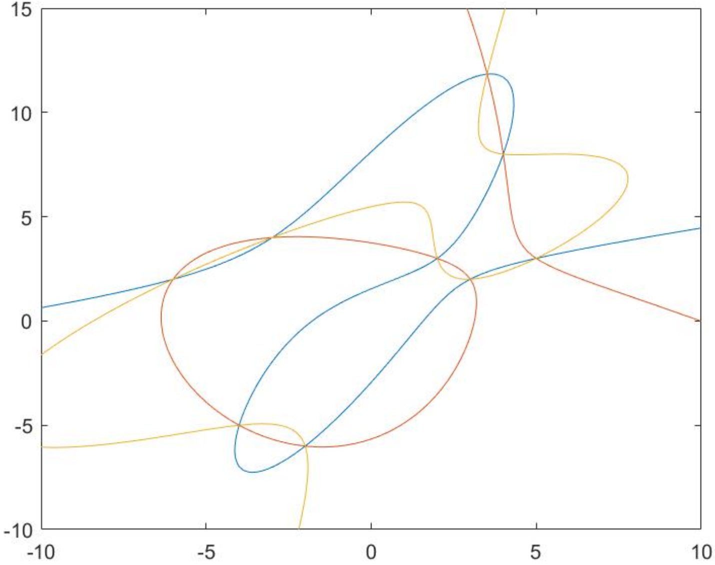
Then the intersection $\mathcal{C} \cap \mathcal{C}$ consists of the eight points

$$[2:3:1], [-3:4:1], [4:5:-1], [-6:2:1], [5:3:1], [3:2:1], [2:6:-11], [4:8:1]$$

and the ninth point

$$\left[1439767504290697562 : 4853460637572644276 : 409942054104759719\right].$$

Intersection of three cubics



How to find the intersection $\color{blue}{\mathcal{C}} \cap \color{red}{\mathcal{C}}$?

1. Let $f(x, y)$ be the polynomial

$$
\begin{aligned}
&-5913252577x^3 + 30222000280x^2y - 21634931915xy^2 + 5556266591y^3 - 73906985473x^2 + \\
&\quad + 102209537669xy - 37300172365y^2 + 1389517162x - 88423819400y + 204616284808.
\end{aligned}
$$

2. Let $g(x, y)$ be the polynomial

$$
\begin{aligned}
&-4844332x^3 - 8147864x^2y - 4067744xy^2 - 1866029y^3 + 32668904x^2 - \\
&\quad - 28226008xy + 41719157y^2 + 252639484x + 126319742y - 960898976.
\end{aligned}
$$

3. Consider $f(x, y)$ and $g(x, y)$ as polynomials in y with coefficients in $\mathbb{C}[x]$.
4. Their resultant $R(f, g, y)$ is the polynomial:

$$
\begin{aligned}
&319168411614335505141855887784472124841956719232716\!9x^9 - \\
&\quad - 8017907650232644802095920848553578107779291488585493x^8 - \\
&\quad - 19951895461883394788720945351923685301295332302821563\!3x^7 + \\
&\quad + 56880707484802669486621609640000274581156521359635915\!7x^6 + \\
&\quad + 388061426660860152303219450198457015206916475399893346\!4x^5 - \\
&\quad - 1170871430340388520426900204901359349819115417560887623\!2x^4 - \\
&\quad - 2793667817206367545025847395270310402043342406875801595\!2x^3 + \\
&\quad + 8667252653640632233373324200600241227745651744170592980\!8x^2 + \\
&\quad + 6160902638438975120413703773156220360186066368361917363\!2x - \\
&\quad - 1937017457229772774687302096721626128751162780061707993\!60.
\end{aligned}
$$

5. Its roots are 2, 3, 4, 5, -6, -4, -3, -2 and $\frac{1439767504290697562}{409942054104759719}$.

Resultant

One has $f(x, y) = a_3 y^3 + a_2 y^2 + a_1 y + a_0$, where

$$\begin{cases} a_3 = 5556266591, \\ a_2 = -21634931915x - 37300172365, \\ a_1 = 30222000280x^2 + 102209537669x - 88423819400, \\ a_0 = 5913252577x^3 - 73906985473x^2 + 1389517162x + 204616284808. \end{cases}$$

One has $g(x, y) = b_3 y^3 + b_2 y^2 + b_1 y + b_0$, where

$$\begin{cases} b_3 = -1866029, \\ b_2 = -4067744x + 41719157, \\ b_1 = -8147864x^2 - 28226008x + 126319742, \\ b_0 = -4844332x^3 + 32668904x^2 + 252639484x - 960898976. \end{cases}$$

The resultant of $f(x, y)$ and $g(x, y)$ (considered as polynomials in y) is

$$R(f, g, y) = \det \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 \end{pmatrix} = \det \begin{pmatrix} f(x, y) & a_1 & a_2 & a_3 & 0 & 0 \\ yf(x, y) & a_0 & a_1 & a_2 & a_3 & 0 \\ y^2 f(x, y) & 0 & a_0 & a_1 & a_2 & a_3 \\ g(x, y) & b_1 & b_2 & b_3 & 0 & 0 \\ yg(x, y) & b_0 & b_1 & b_2 & b_3 & 0 \\ y^2 g(x, y) & 0 & b_0 & b_1 & b_2 & b_3 \end{pmatrix}.$$

This shows that $R(f, g, y) = A(x, y)f(x, y) + B(x, y)g(x, y)$ for some polynomials $A(x, y)$ and $B(x, y)$.

Transversal intersections and intersection multiplicities

- Let C be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree d.
- Let Z be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree $\widehat{d}$.

Pick $P \in C \cap Z$.

Definition

We say that C intersects the curve Z transversally at P if

1. both curves C and Z are smooth at the point P,
2. and the tangent lines to C and Z are P are different.

Then $(C \cdot Z)_P = 1 \iff C$ intersects Z transversally at P.

Corollary

The following two conditions are equivalent:

1. $|C \cap Z| = d\widehat{d}$,
2. *C intersects Z transversally at every point of $C \cap Z$.*

Corollary

If $|C \cap Z| = d\widehat{d}$, then $\mathrm{Sing}(C) \cap Z = \varnothing = C \cap \mathrm{Sing}(Z)$.

Singular points and intersection multiplicities

- Let C be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree d.

Let $P = [0:0:1]$. Then C is given by the equation

$$z^d h_0(x,y) + z^{d-1}h_1(x,y) + z^{d-2}h_2(x,y) + \cdots + h_d(x,y) = 0,$$

where $h_i(x,y)$ is a homogenous polynomial of degree i. Let

$$\mathrm{mult}_P(C) = \min\Big\{ i \mid h_i(x,y) \text{ is not a zero polynomial} \Big\}$$

- $\mathrm{mult}_P(C) \geqslant 1 \iff P \in C$.
- $\mathrm{mult}_P(C) \geqslant 2 \iff P \in \mathrm{Sing}(C)$.

We say that C has multiplicity $\mathrm{mult}_P(C)$ at the point P.

- Let Z be another irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$.

Lemma

Suppose that $C \neq Z$ and $P \in C \cap Z$. Then

$$\big(C \cdot Z\big)_P \geqslant \mathrm{mult}_P\big(C\big)\mathrm{mult}_P\big(Z\big).$$

First application

Let $f(x, y, z)$ be a homogeneous polynomial of degree $d \geqslant 1$.

Lemma

Suppose that the system

$$\frac{\partial f(x, y, z)}{\partial x} = \frac{\partial f(x, y, z)}{\partial y} = \frac{\partial f(x, y, z)}{\partial z} = 0$$

has no solutions in $\mathbb{P}^2_{\mathbb{C}}$. Then $f(x, y, z)$ is irreducible.

Proof.

Suppose that $f(x, y, z)$ is not irreducible. Then

$$f\big(x, y, z\big) = g(x, y, z)h(x, y, z),$$

where g and h are homogeneous polynomials of positive degrees. There is $[a : b : c] \in \mathbb{P}^2_{\mathbb{C}}$ with $g(a, b, c) = h(a, b, c) = 0$. Then

$$\frac{\partial f(a, b, c)}{\partial x} = \frac{\partial g(a, b, c)}{\partial x} h(a, b, c) + g(a, b, c)\frac{\partial h(a, b, c)}{\partial x} = 0.$$

Second application

Let C be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree $d \geqslant 2$.

Theorem

Let P and Q be two different points in C. Then

$$\mathrm{mult}_P(C) + \mathrm{mult}_Q(C) \leqslant d.$$

Proof.

Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$ that passes through P and Q. Then

$$d = \sum_{O \in L \cap C} \big(L \cdot C\big)_O \geqslant \big(L \cdot C\big)_P + \big(L \cdot C\big)_Q \geqslant \mathrm{mult}_P(C) + \mathrm{mult}_Q(C).$$

□

Corollary

Let P be a point in C. Then $\mathrm{mult}_P(C) < d$.

Corollary

Suppose that $d = 3$. Then C has at most one singular point.

Third application

Let C be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree 4.

Lemma

The curve C has at most 3 singular points.

Proof.

Suppose that C has at least 4 singular points.
Denote four singular points of C as P_1, P_2, P_3, P_4.
Let Q be a point in C that is different from these 4 points.
There is a homogeneous polynomial $f(x, y, z)$ of degree 2 such that

$$f(P_1) = f(P_2) = f(P_3) = f(P_4) = f(Q) = 0.$$

Let Z the curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by $f(x, y, z) = 0$.
Since C is irreducible, we can apply Bezout's theorem to C and Z:

$$8 = \sum_{O \in C \cap Z} \left(C \cdot Z\right)_O \geqslant \sum_{i=1}^{4} \left(C \cdot Z\right)_{P_i} + \left(C \cdot Z\right)_Q \geqslant \sum_{i=1}^{4} \mathrm{mult}_{P_i}(C) + 1.$$

$\square$