The algebraic Künneth theorem

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E-Z-theorem $\Rightarrow$ $C_*(X;\mathbb{Z}) \otimes C_*(Y;\mathbb{Z}) \simeq C_*(X\times Y;\mathbb{Z})$

A, B spaces $\Rightarrow$ $S(A\times B) \xrightarrow{\cong} S(A)\times S(B)$ an isomorphism of simplicial sets

$\Rightarrow$

$$H_*(A\times B;\mathbb{Z}) = H_*(C_*(S(A\times B));\mathbb{Z})) \cong H_*\big(C_*(S(A);\mathbb{Z}) \otimes C_*(S(B);\mathbb{Z})\big)$$

$\updownarrow$???

$$H_*(A;\mathbb{Z}) \otimes H_*(B;\mathbb{Z})$$

Context for this video: R commutative ring. Let C and D be chain complexes of R-modules.

We define a new complex of R-modules $C \otimes_R D$ as follows:

$$(C\otimes_R D)_n = \bigoplus_{p+q=n} C_p \otimes_R D_q, \qquad d(x\otimes y) = (dx)\otimes y + (-1)^{|x|}\cdot x \otimes (dy)$$

Recall: For two sets S and T, the following are isomorphisms of R-modules:

$$R\otimes \mathbb{Z}[S] \cong R[S], \qquad r\otimes s \longmapsto r\cdot s$$

$$R[S]\otimes_R R[T] \xrightarrow{\cong} R[S\times T], \qquad (\Sigma\, r_s\cdot s)\otimes(\Sigma\, r_t \cdot t) \mapsto \Sigma\, (r_s\cdot r_t)\cdot(s,t)$$

The shuffle map for simplicial sets X and Y is a chain homotopy equivalence $\nabla: C_*(X;\mathbb{Z})\otimes C_*(Y;\mathbb{Z}) \longrightarrow C_*(X\times Y;\mathbb{Z})$

$R\otimes -$ yields a chain homotopy equivalence of complexes of R-modules

$$\nabla = R\otimes\nabla: \quad R\otimes C_*(X;\mathbb{Z})\otimes C_*(Y;\mathbb{Z}) \longrightarrow R\otimes C_*(X\times Y;\mathbb{Z})$$

$$\begin{array}{ccc} R\otimes C_*(X;\mathbb{Z})\otimes C_*(Y;\mathbb{Z}) & \longrightarrow & R\otimes C_*(X\times Y;\mathbb{Z}) \\ \| \wr & & \| \wr \\ (R\otimes C_*(X;\mathbb{Z}))\otimes_R (R\otimes C_*(Y;\mathbb{Z})) & & C_*(X\times Y;R) \\ \| \wr & \nearrow_{\nabla} & \\ C_*(X;R)\otimes_R C_*(Y;R) & & \end{array}$$

Aim: Understand $H_*(C\otimes_R D)$ in terms of $H_*(C)$ and $H_*(D)$

Example: One might hope that $H_n(C\otimes_R D)$ is isomorphic to $\bigoplus_{p+q=n} H_p(C)\otimes_R H_q(D)$; but this is not true in general:

$R=\mathbb{Z}$, $C = D = (\cdots 0 \longrightarrow \underset{1}{\mathbb{Z}} \xrightarrow{\cdot 2} \underset{0}{\mathbb{Z}} \longrightarrow 0)$ $\qquad H_0(C) = H_0(D) \cong \mathbb{Z}/2$, $H_n(C) = H_n(D) = 0$ for $n\neq 0$.

$$H_1(C\otimes D) = H_1\big(0 \longrightarrow \underset{2}{\mathbb{Z}} \xrightarrow{x\,\mapsto\,(2x,-2x)} \underset{1}{\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{(a,b)\,\mapsto\,2a+2b} \underset{0}{\mathbb{Z}} \longrightarrow 0\big) = \{(a,-a) : a\in\mathbb{Z}\} \big/ \{(2x,-2x) : x\in\mathbb{Z}\} \cong \mathbb{Z}/2.$$

Some homological algebra:

Def: Let R be a ring. A left R-module P is projective if for every surjective R-linear map $f: M\to N$ and every R-homomorphism $\alpha: P\longrightarrow N$, there is a R-homomorphism $\beta: P\longrightarrow M$ such that $f\circ\beta = \alpha$

$$\begin{tikzcd} & M \arrow[d, two heads, "f"] \\ P \arrow[ur, dashed, "\beta"] \arrow[r, "\alpha"'] & N \end{tikzcd}$$

Examples:
- Every free left R-module is projective. Let S be any set, and $\alpha: R[S]\longrightarrow N$ any homomorphism. Choose preimages $\beta(s)\in M$ of $\alpha(s)\in N$ for all $s\in S$, i.e. such that $f(\beta(s)) = \alpha(s)$. Then extend R-linearly to a homomorphism $\beta: R[S]\longrightarrow M$.
- If R is a field, every R-module is free, and hence every R-module is projective.
- Let P and Q be left R-modules such that $P\oplus Q$ is projective. Then P is projective:
  Let $\alpha: P\longrightarrow N$ be any R-linear map. Define $\tilde{\alpha}: P\oplus Q \longrightarrow N$ by $\tilde{\alpha}(p,q) = \alpha(p)$.
  Because $P\oplus Q$ is projective, there is a homomorphism $\tilde{\beta}: P\oplus Q \longrightarrow M$ such that $f\circ\tilde{\beta} = \tilde{\alpha}$.
  Define $\beta: P\longrightarrow M$ by $\beta(p) = \tilde{\beta}(p,0)$, then this a homomorphism such that $f\circ\beta = \alpha$.

  $\Rightarrow$ direct summands of projective modules are projective.
- Suppose that P is a projective left R-module. Let $R[P]$ be the free R-module on the underlying set of P.
  Let $f: R[P]\longrightarrow P$ be the unique homomorphism with $f(1\cdot p) = p$ for all $p\in P$.
  This a surjective R-linear map. Since P is projective, there is a homomorphism $\beta: P\longrightarrow R[P]$ such that $f\circ\beta = \mathrm{Id}_P$.

$$\begin{tikzcd} & R[P] \arrow[d, "f"] \\ P \arrow[ur, dashed, "\beta"] \arrow[r] & P \end{tikzcd}$$

... a surjective R-linear map ... Since P is projective, there is a homomorphism $\beta : P \to R[P]$ such that $f \circ \beta = Id_P$.

$$\begin{tikzcd} & R[P] \arrow[d, "f"] \\ P \arrow[ur, dashed, "\beta"] \arrow[r, "Id_P"'] & P \end{tikzcd}$$

So $R[P] \cong P \oplus \mathrm{Ker}(f)$, so P is a direct summand of the free R-module $R[P]$.

Summary: projective modules are precisely the direct summands of free module.

Example: Over $R = \mathbb{Z}/6$, the modules $\mathbb{Z}/2$ and $\mathbb{Z}/3$ are projective because $\mathbb{Z}/2 \oplus \mathbb{Z}/3 \cong \mathbb{Z}/6$.
But $\mathbb{Z}/2$ and $\mathbb{Z}/3$ are not free modules over $\mathbb{Z}/6$ because their cardinalities are not divisible by 6.

Remark: Over $R = \mathbb{Z}$ every projective module is free, but $\mathbb{Z}/2$ is not projective.

Prop: Let R be a commutative ring and let $0 \to I \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ be a short exact sequence of R-modules.
Then the sequence
$$P \otimes_R I \xrightarrow{P \otimes_R \alpha} P \otimes_R M \xrightarrow{P \otimes_R \beta} P \otimes_R N \to 0$$
is exact for every R-module P.

("tensor product is right exact"). If moreover P is projective, then $P \otimes_R \alpha$ is injective, and so
$$0 \to P \otimes_R I \xrightarrow{P \otimes_R \alpha} P \otimes_R M \xrightarrow{P \otimes_R \beta} P \otimes_R N \to 0$$
is exact.

("projectives modules are flat").

Proof: Because $(P \otimes_R \beta) \circ (P \otimes_R \alpha) = P \otimes_R (\beta\alpha) = 0$, we get an induced homomorphism
$$\gamma : \frac{P \otimes_R M}{\mathrm{Im}(P \otimes_R \alpha)} \longrightarrow P \otimes_R N, \quad \gamma(p \otimes m + \mathrm{Im}) = p \otimes \beta(m).$$
The first claim is equivalent to showing that γ is an isomorphism.
We will define an inverse homomorphism
$$\delta : P \otimes_R N \longrightarrow \frac{P \otimes_R M}{\mathrm{Im}}$$
Given $(p, n) \in P \times N$ we choose $\tilde{n} \in M$ with $\beta(\tilde{n}) = n$. Then
$$\delta(p \otimes n) = p \otimes \tilde{n} + \mathrm{Im}(P \otimes_R \alpha)$$
is independent of the choice of lift:

Let $\tilde{\tilde{n}}$ also satisfy $\beta(\tilde{\tilde{n}}) = n$. Then $\beta(\tilde{\tilde{n}} - \tilde{n}) = 0$, so $\tilde{\tilde{n}} - \tilde{n} = \alpha(i)$ for some $i \in I$.

Then $p \otimes \tilde{\tilde{n}} \equiv p \otimes \tilde{n}$ modulo $p \otimes \alpha(i) \in \mathrm{Im}(p \otimes_R \alpha)$, hence $p \otimes \tilde{\tilde{n}}$ and $p \otimes \tilde{n}$ are in the same residual class.

Claim: the assignment $(p, n) \longrightarrow p \otimes \tilde{n} + \mathrm{Im}$ is additive in p and n and sends (rp, n) and (p, rn) to the same image for all $r \in R$.

Suppose $\beta(\tilde{n}) = n$. Then $\beta(r \cdot \tilde{n}) = r \cdot \beta(\tilde{n}) = r \cdot n$, so $r \cdot \tilde{n}$ is a lift of $r \cdot n$. So
$$(rp, n) \longmapsto rp \otimes \tilde{n} + \mathrm{Im} = p \otimes r\tilde{n} + \mathrm{Im} \longleftarrow (p, rn)$$

Upshot: the assignment $(p, n) \longmapsto p \otimes \tilde{n} + \mathrm{Im}$ extends to a well-defined R-linear map δ on $P \otimes_R N$.

δ and γ are inverse to each other:
$$\gamma(\delta(p \otimes n)) = \gamma(p \otimes \tilde{n} + \mathrm{Im}) = p \otimes \beta(\tilde{n}) = p \otimes n$$
$$\delta(\gamma(p \otimes m + \mathrm{Im})) = \delta(p \otimes \beta(m)) = p \otimes m.$$
$p \in P,\ m \in M$

This ends the proof of right exactness.

Now suppose that the R-module P is projective. We need to show that then $P \otimes_R \alpha : P \otimes_R I \longrightarrow P \otimes_R M$ is injective.

Case 1: P is free, $P = R[S]$ for some set S. Then $P \otimes_R M = R[S] \otimes_R M \cong \bigoplus_{s \in S} M$
naturally for R-linear maps in M. $\quad \sum_{s \in S} s \otimes m_s \longleftarrow \{m_s\}_{s \in S}$

So we get a commutative square
$$\begin{tikzcd} P \otimes_R I \arrow[r, "P \otimes_R \alpha"] \arrow[d, phantom, "\cong" rotate=90] & P \otimes_R M \arrow[d, phantom, "\cong" rotate=90] \\ \bigoplus_{s \in S} P \arrow[r, "\bigoplus_{s \in S} \alpha"'] & \bigoplus_{s \in S} M \end{tikzcd}$$
$\Rightarrow P \otimes_R \alpha$ is injective.

injective because α is injective

Case 2: P any projective module. Then P is a direct summand of a free module F, i.e. there are homomorphisms
$P \xrightarrow{\lambda} F \xrightarrow{\mu} P$ such that $\mu \circ \lambda = \mathrm{Id}_P$.

We contemplate the commutative square

$$\begin{tikzcd} P\otimes_R I \arrow[r, "P\otimes_R \alpha"] \arrow[d, "\lambda\otimes_R I"'] & P\otimes_R M \arrow[d, "\lambda\otimes_R M"] \\ F\otimes_R I \arrow[r, "F\otimes_R \alpha"'] & F\otimes_R M \end{tikzcd}$$

injective because $\mu\otimes_R I$ is a retraction

injective by Case 1

$\Longrightarrow$ $P\otimes_R \alpha$ is injective. $\square$

Def: A ring R has global dimension ≤ 1 if every submodule of a projective module is projective.

Examples.
- Every field has global dimension ≤ 1.
- The ring $\mathbb{Z}$ of integers has global dimension ≤ 1.
- every principal ideal domain has global dimension ≤ 1 (commutative ring without zero divisors in which every ideal is generated by one element)

  Example:
  - $k[x]$ polynomial ring in one variable over a field k
  - $\mathbb{Z}[i]$ Gaussian integers
  - $\mathbb{Z}_p$ p-adic integers.

Def: Let R be a commutative ring of global dimension ≤ 1, M and N R-modules. We choose an R-linear surjection $p: F \to N$ from a free R-module F and set $K = \text{kernel}(p: F \to N)$. Then the Tor-group of M and N is

$$\mathrm{Tor}^R(M,N) = \ker(M\otimes_R \text{incl}: M\otimes_R K \longrightarrow M\otimes_R F)$$

Note: an exact sequence

$$0 \to \mathrm{Tor}^R(M,N) \to M\otimes_R K \xrightarrow{M\otimes_R \text{incl}} M\otimes_R F \xrightarrow{M\otimes_R p} M\otimes_R N \to 0$$

Construction: Let R be a commutative ring, C and D chain complexes of R-modules.
We define an R-linear map

$$\Phi: H_p(C)\otimes_R H_q(D) \longrightarrow H_{p+q}(C\otimes_R D)$$
$$[x]\otimes[y] \longmapsto [x\otimes y]$$

Well-definedness:
- if x and y are cycles, then $d(x\otimes y) = (\underbrace{dx}_{=0})\otimes y + (-1)^p \cdot x\otimes \underbrace{dy}_{=0} = 0$, so $x\otimes y$ is indeed a cycle.
- if x and y are cycles and $z \in C_{p+1}$, then
  $(x+dz)\otimes y = x\otimes y + (dz)\otimes y = x\otimes y \pm d(x\otimes z)$, so $[(x+dz)\otimes y] = [x\otimes y]$ in $H_{p+q}(C\otimes_R D)$.
- if x and y are cycles and $r\in R$, then $(rx)\otimes y = r\cdot(x\otimes y) = x\otimes ry$ $\Longrightarrow$ $[rx]\otimes[y]$ and $[x]\otimes[ry]$ have the same image, so Φ is well-defined on $H_p(C)\otimes_R H_q(D)$.

Theorem (Algebraic Künneth theorem) Let R be a commutative ring of global dimension ≤ 1, and let C and D be complexes of projective R-modules. Then the following map is split injective:

$$\sum_{p+q=n}\Phi : \bigoplus_{p+q=n} H_p(C)\otimes_R H_q(D) \longrightarrow H_n(C\otimes_R D)$$

Moreover, the cokernel of this homomorphism is naturally isomorphic to $\bigoplus_{p+q=n-1} \mathrm{Tor}^R(H_p(C), H_q(D))$.

Equivalently: there is a short exact sequence of R-modules

$$0 \to \bigoplus_{p+q=n} H_p(C)\otimes_R H_q(D) \xrightarrow{\Sigma\Phi} H_n(C\otimes_R D) \to \bigoplus_{p+q=n-1} \mathrm{Tor}^R(H_p(C), H_q(D)) \to 0$$

The short exact sequence is natural for R-linear chain maps in C and D, it splits, but there is no natural splitting in general.

Proof: We write $Z = \{Z_q\}_{q\in\mathbb{Z}}$ for the complex with trivial differential consisting of the cycle modules $Z_q = \ker(d: D_q \to D_{q-1})$ of D. We write $B = \{B_q, d=0\}$ for the complex of boundary modules $B_q = \text{image}(d: D_{q+1} \to D_q)$. We then have a short exact sequence of complexes of R-modules:

$$0 \to Z \xrightarrow{\text{incl}} D \xrightarrow{d} B[1] \to 0$$

$\curvearrowright$ here $d=0$

Since D is dimensionwise projective and $B_q \subseteq Z_q \subseteq D_q$ and R has global dimension ≤ 1, the R-modules B_q and Z_q are also projective. So the above short exact sequence splits R-linearly in every fixed chain complex dimension. We tensor this short exact sequence with C_p for some $p \in \mathbb{Z}$ and take the direct sum over all p to obtain a short exact sequence of chain complexes

$$0 \longrightarrow C \otimes_R Z \longrightarrow C \otimes_R D \longrightarrow C \otimes_R B[1] \longrightarrow 0$$

We obtain a long exact sequence of homology groups:

$$\cdots \longrightarrow H_n(C \otimes_R Z) \xrightarrow{H_n(C \otimes_R \mathrm{incl})} H_n(C \otimes_R D) \xrightarrow{H_n(C \otimes_R d)} H_n(C \otimes_R B[1]) \xrightarrow{\partial} H_{n-1}(C \otimes_R Z) \longrightarrow \cdots$$

Since Z has trivial differential and consists of projective R-modules, the following are isomorphisms:

$$H_n(C \otimes_R Z) = H_n\Big(\underbrace{\bigoplus_{q \in \mathbb{Z}} C[q] \otimes_R Z_q}_{\text{direct sum differential}}\Big) = \bigoplus_{q \in \mathbb{Z}} H_n(C[q] \otimes_R Z_q)$$

$$= \bigoplus_{q \in \mathbb{Z}} H_{n-q}(C \otimes_R Z_q)$$

$-\otimes_R Z_q$ is exact because Z_q is projective. $\rightsquigarrow$

$$\cong \bigoplus_{q \in \mathbb{Z}} H_{n-q}(C) \otimes_R Z_q$$

Similarly:

$$H_n(C \otimes_R B[1]) \cong \bigoplus_{q \in \mathbb{Z}} H_{n-q-1}(C) \otimes_R Z_q$$

So the long exact sequence becomes:

$$\cdots \longrightarrow \bigoplus_{p+q=n} H_p(C) \otimes_R Z_q \xrightarrow{\Phi} H_n(C \otimes_R D) \longrightarrow \bigoplus_{p+q=n-1} H_p(C) \otimes_R B_q \xrightarrow{\bigoplus_{p+q=n-1} H_p(C) \otimes_R \mathrm{incl}} \bigoplus_{p+q=n-1} H_p(C) \otimes_R Z_q \longrightarrow \cdots$$

Equivalently, we obtained short exact sequences of R-modules:

$$0 \longrightarrow \bigoplus_{p+q=n} \mathrm{Coker}\Big(H_p(C) \otimes_R B_q \xrightarrow{H_p(C) \otimes_R \mathrm{incl}} H_p(C) \otimes_R Z_q\Big) \longrightarrow H_n(C \otimes_R D) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Ker}\Big(H_p(C) \otimes_R B_q \xrightarrow{H_p(C) \otimes_R \mathrm{incl}} H_p(C) \otimes_R Z_q\Big) \longrightarrow 0$$

The short exact sequence

$$0 \longrightarrow B_q \xrightarrow{\mathrm{incl}} Z_q \xrightarrow{\mathrm{proj.}} H_q(D) \longrightarrow 0$$

is a projective resolution of $H_q(D)$ because Z_q is projective.

So we can use this resolution to calculate the Tor groups:

$$0 \longrightarrow \mathrm{Tor}^R(H_p(C), H_q(D)) \longrightarrow H_p(C) \otimes_R B_q \xrightarrow{H_p(C) \otimes_R \mathrm{incl}} H_p(C) \otimes_R Z_q \longrightarrow H_p(C) \otimes_R H_q(D) \longrightarrow 0$$

is exact.

So the previous short exact sequence is:

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C) \otimes_R H_q(D) \xrightarrow{\Phi} H_n(C \otimes_R D) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}^R(H_p(C), H_q(D)) \longrightarrow 0$$

<u>We still have to show that Φ admits an R-linear retraction.</u>

Since B_{q-1} is projective as an R-module, the short exact sequence splits: $0 \longrightarrow Z_q \xrightarrow{\mathrm{incl}} D_q \xrightarrow{d} B_{q-1} \longrightarrow 0$

So there is an R-linear map $r : D_q \longrightarrow Z_q$ such that $r(x) = x$ for all $x \in Z_q$.

The collection of R-linear maps

$$D_q \xrightarrow{r} Z_q \xrightarrow{\mathrm{proj.}} H_q(D)$$

form a chain map $r : D \longrightarrow H_*(D) = \{H_q(D), d = 0\}_{q \in \mathbb{Z}}$ that induces the identity on homology. Chain map property:

$$\begin{tikzcd}
D_{q+1} \arrow[d, "d"] \arrow[rrd, bend left, "0"] & & \\
B_q \arrow[rrd, "0"] & & \\
Z_q \arrow[u, phantom, "\cap"] \arrow[r, "= \mathrm{Id}"] & Z_q \arrow[r] & H_q(D) \\
D_q \arrow[u, phantom, "\cap"] \arrow[r, "r"] & Z_q \arrow[r, "\mathrm{proj.}"] & H_q(D).
\end{tikzcd}$$

Similarly, we obtain a chain map $s : C \longrightarrow H_*(C) = \{H_p(C), d = 0\}_{p \in \mathbb{Z}}$ that induces the identity on homology.

We define a chain map $g \otimes r : C \otimes_R D \longrightarrow H_*(C) \otimes_R H_*(D)$; on homology, this induces maps

$$H_n(C \otimes_R D) \longrightarrow H_n(\underbrace{H_*(C) \otimes_R H_*(D)}_{d=0\ !}) = \bigoplus_{p+q=n} H_p(C) \otimes_R H_q(C).$$

This R-linear map is a retraction to Φ: for cycles $x \in C_p$ and $y \in C_q$, we have

$$\Phi([x] \otimes [y]) = [x \otimes y] \xmapsto{H_n(g \otimes r)} [g(x)] \otimes [r(y)] = [x] \otimes [y].$$

$\square$

<u>Special Case</u> Let R be a field. Then every R-module is free and hence projective, and $\operatorname{Tor}^R(M,N) = 0$ for all R-modules M and N.

<u>Thm:</u> Let C and D be complexes of vector spaces over a field R. Then the map

$$\Phi : \bigoplus_{p+q=n} H_p(C) \otimes_R H_q(D) \longrightarrow H_n(C \otimes_R D), \quad [x] \otimes [y] \longmapsto [x \otimes y]$$

is an isomorphism.

<u>Special Case</u> ($R = \mathbb{Z}$): Let C and D be chain complexes of free abelian groups. Then there is naturally short exact sequence

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \longrightarrow H_n(C \otimes D) \longrightarrow \bigoplus_{p+q=n-1} \operatorname{Tor}(H_p(C), H_q(D)) \longrightarrow 0.$$

Moreover, this sequence splits.