The topological Künneth theorem in homology

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1 Construction (Exterior pairing)

Let R be a commutative ring and X and Y simplical sets. The homological exterior pairing

$$\begin{aligned}
\times : \quad & H_p(X;R) \otimes_R H_q(Y;R) = H_p(C_*(X;R)) \otimes_R H_q(C_*(Y;R)) \\
\xrightarrow[{[x]\otimes[y]\mapsto[x\otimes y]}]{\Phi} \quad & H_{p+q}(C_*(X;R) \otimes_R C_*(Y;R)) \\
\xrightarrow{H_{p+q}(\nabla)} \quad & H_{p+q}(C_*(X \times Y;R)) = H_{p+q}(X \times Y;R).
\end{aligned}$$

For two spaces A and B, the external homology pairing is the composite

$$\begin{aligned}
\times : \quad & H_p(A;R) \otimes_R H_q(B;R) = H_p(\rho(A);R) \otimes_R H_q(\rho(B);R) \\
\xrightarrow{\times} \quad & H_{p+q}(\rho(A) \times \rho(B);R) \\
\xrightarrow{H_{p+q}(c^{-1};R)} \quad & H_{p+q}(\rho(A \times B);R) = H_{p+q}(A \times B;R).
\end{aligned}$$

where $c = (\rho(p_1), \rho(p_2)) : \rho(A \times B) \xrightarrow[\cong]{\rho} (A) \times \rho(B)$ is the canonical isomorphism of simplical sets, where $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$ are the canonical projections.

Theorem 1.1. Let R be a field and X and Y be spaces or simpical spaces. Then the exterioir homology pairing provides a natural isomorphism of R-vector spaces

$$\bigoplus_{p+q=n} H_p(X;R) \otimes_R H_q(Y;R) \to H_n(X \times Y;R)$$

Proof. case of siplical sets: the map in question is the composite of two isomorphisms:

$$\bigoplus_{p+q=n} H_p(C_*(X;R)) \otimes_R H_q(C_*(Y;R)) \xrightarrow{\Phi} H_n(C_*(X;R) \otimes_R C_*(Y;R)) \xrightarrow{H_n(\nabla)} H_n(C_*(X \times Y;R)),$$

where the first map is an isomorphism by the algebraic Künneth theorem and the second is an isomorphism by the Elenberg-Zilber theorem. □

Simular arguments show:

Theorem 1.2. Let X and Y be spaces or siplical sets. THen the singular exterios homology pairing participate in a natural short exact sequence of abelian groups:

$$0 \to \bigoplus_{p+q=n} H_p(X;\mathbb{Z}) \otimes H_q(Y;\mathbb{Z}) \xrightarrow{\times} H_n(X \times Y;\mathbb{Z}) \to \bigoplus_{p+q=n} \mathrm{Tor}(H_p(X,\mathbb{Z}), H_q(Y,\mathbb{Z})) \to 0$$

The sequence splits (but the spliting cannot be chosen naturaly).