The Eilenberg-Zilber theorem

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<u>Overview</u>: Let A and B be simplicial abelian groups, and $C_*(A)$ the associated chain complex.

Then there are natural chain homotopy equivalences

$$C_*(A) \otimes C_*(B) \underset{\text{Alexander-Whitney map}}{\overset{\text{Eilenberg-Zilber map}}{\rightleftarrows}} C_*(A \otimes B)$$

tensor product of chain complexes — dimensionwise tensor product of simplicial abelian groups

<u>Applications</u>:
- Künneth theorem: relation between $H_*(X \times Y; \mathbb{Z})$, $H_*(X;\mathbb{Z})$, $H_*(Y;\mathbb{Z})$
- cellular interpretation of the cup product via "cellular approximation of the diagonal"

<u>Definition</u>: A <u>simplicial abelian group</u> is a functor $A : \Delta^{op} \longrightarrow Ab$ = (abelian groups, group homomorphisms)

$\Longleftrightarrow$ abelian groups A_n, $n \geq 0$, group homomorphisms $\alpha^* : A_n \longrightarrow A_m$ for all morphisms $\alpha : [m] \longrightarrow [n]$ in Δ, such that $(\alpha \circ \beta)^* = \beta^* \circ \alpha^*$ and $(Id_{[m]})^* = Id_{A_m}$.

$\Longleftrightarrow$ a simplicial set endowed with abelian group structures on A_n for all $n \geq 0$, such that all structure maps $\alpha^* : A_n \longrightarrow A_m$ are homomorphisms.

<u>Example</u>: Let X be a simplicial set and A an abelian group. Then the composite functor

$$\Delta^{op} \xrightarrow{X} (\text{sets}) \xrightarrow{A[-]} Ab$$

is a simplicial abelian group. We write $A[X]$ for this composite.

Explicitly: $(A[X])_n = A[X_n]$ and $\alpha^* : A[X_n] \longrightarrow A[X_m]$, $\alpha^* = A[\alpha^*]$

<u>Construction</u>: The chain complex $C_*(A)$ of a simplicial abelian group A is defined by

$$C_n(A) = \begin{cases} A_n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

with differential $d_n : A_n \longrightarrow A_{n-1}$ defined as $d_n = \sum_{i=0,\dots,n} (-1)^i d_i^*$

<u>Note</u>: The linearization functor

$$(\text{simplicial sets}) \xrightarrow{C_*(-;A)} (\text{chain complexes})$$

factors as the composite

$$\begin{tikzcd} (\text{simplicial sets}) \arrow[rr, "C_*(-;A)"] \arrow[dr, "A[-]"'] & & (\text{chain complexes}) \\ & (\text{simplicial abelian groups}) \arrow[ur, "C_*"'] & \end{tikzcd}$$

We recall the tensor product of two chain complexes C and D:

$$(C \otimes D)_n = \bigoplus_{\substack{p+q=n \\ p,q \in \mathbb{Z}}} C_p \otimes D_q \qquad n \in \mathbb{Z}$$

with differential the bilinear extension

$$d(x \otimes y) = (dx) \otimes y + (-1)^p \cdot x \otimes (dy) \quad \text{for } x \in C_p,\ y \in D_q.$$

<u>Construction</u>: Let A and B be two abelian groups. The <u>tensor product</u> $A \otimes B$ is the simplicial abelian group

$$\underbrace{\Delta^{op} \xrightarrow{diag} \Delta^{op} \times \Delta^{op} \xrightarrow{A \times B} Ab \times Ab \xrightarrow{\otimes} Ab}_{A \otimes B}$$

<u>Explicitly</u>: $(A \otimes B)_n = A_n \otimes B_n$ and $\alpha^*_{A \otimes B} = \alpha^*_A \otimes \alpha^*_B$.

<u>Construction of the comparison chain maps</u>: Let A, B be simplicial abelian groups.

The <u>Alexander-Whitney map</u>

$$AW : C_*(A \otimes B) \longrightarrow C_*(A) \otimes C_*(B)$$

is defined by

$$AW_n : A_n \otimes B_n \longrightarrow \bigoplus_{p+q=n} A_p \otimes B_q \quad \text{for all } n \geq 0$$

by

$$AW_n(a \otimes b) = \sum_{p+q=n} d_{front}^*(a) \otimes d_{back}^*(b)$$

in the (p,q)-summand: $d_{front} : [p] \longrightarrow [p+q] = [n]$

$p+q=n$ front back

in the (p,q)-summand: $d_{front}: [p] \longrightarrow [p+q] = [n]$

$d_{back}: [q] \longrightarrow [p+q]$

Check: for varying $n \geq 0$, the maps AW_n form a chain map. Proof: dualize the argument that proved the Leibniz formula for the cup product.

Def: A (p,q)-shuffle, for $p,q \geq 0$, is a permutation σ of the set $\{0,1,\dots,p+q-1\}$ such that the restriction of σ to $\{0,\dots,p-1\}$ is monotone and the restriction to the set $\{p,\dots,p+q-1\}$ is monotone.

Informally: (p,q)-shuffles leave the first p elements and the last q elements in order.

Examples:
- The only $(p,0)$-shuffle is the identity.
- The two permutations of the set $\{0,1\}$ are both $(1,1)$-shuffles
- The permutation of the set $\{0,1,2\}$ $\sigma: \left\{\begin{matrix} 0 \longmapsto 0 \\ 1 \longmapsto 2 \\ 2 \longmapsto 1 \end{matrix}\right\}$ is a $(2,1)$-shuffle, but not a $(1,2)$-shuffle.

Remarks: A (p,q)-shuffle $\sigma: \{0,\dots,p+q-1\} \longrightarrow \{0,\dots,p+q-1\}$ is uniquely determined by the set $\{\sigma(0),\dots,\sigma(p-1)\}$, and also by the set $\{\sigma(p),\dots,\sigma(p+q-1)\}$.

So (p,q)-shuffles $\cong$ p-element subsets of $\{0,\dots,p+q-1\}$ $\cong$ q-element subsets of $\{0,\dots,p+q-1\}$

So there are precisely $\binom{p+q}{p} = \binom{p+q}{q}$ (p,q)-shuffles.

Notation to be used in the definition of the shuffle map: let σ be a (p,q)-shuffle, we write

$$\mu_i = \sigma(i-1) \quad \text{for } 1 \leq i \leq p$$
$$\nu_i = \sigma(p+i-1) \quad \text{for } 1 \leq i \leq q$$

Then $0 \leq \mu_1 < \mu_2 < \dots < \mu_p \leq p+q-1$ and $0 \leq \nu_1 < \nu_2 < \dots < \nu_q \leq p+q-1$ by the shuffle condition.

Construction: Let A and B be simplicial abelian groups. Then the Eilenberg-Zilber map / shuffle map

$\nabla: C_*(A) \otimes C_*(B) \longrightarrow C_*(A \otimes B)$ has (p,q)-component

$$\nabla = \nabla_{p,q}: C_p(A) \otimes C_q(B) = A_p \otimes B_q \longrightarrow A_{p+q} \otimes B_{p+q} = C_{p+q}(A \otimes B)$$

defined as

$$a \nabla b = \sum_{\sigma: (p,q)\text{-shuffles}} \operatorname{sgn}(\sigma) \cdot \underbrace{(s_{\nu_1} \cdots s_{\nu_q})^*(a)}_{\in A_{p+q}} \otimes \underbrace{(s_{\mu_1} \cdots s_{\mu_p})^*(b)}_{\in B_{p+q}}$$

Here s_{ν_i} and s_{μ_j} are the simplicial degeneracy operators, so

$s_{\nu_1} \cdots s_{\nu_q}: [p+q] \longrightarrow [p]$

$s_{\mu_1} \cdots s_{\mu_p}: [p+q] \longrightarrow [q]$

Examples: - There is only one $(p,0)$-shuffle, the identity of $\{0,\dots,p-1\}$, and then $\mu_i = i-1$.

So for $a \in A_p$, $b \in B_0$, $a \nabla b = \nabla_{p,0}(a \otimes b) = a \otimes (s_0 \cdots s_{p-1})^*(b)$
$= a \otimes \bar{s}^*(b)$, where

$\bar{s}: [p] \longrightarrow [0]$ is the only map.

In particular, for $p=q=0$, $a \nabla b = a \otimes b$.

Example: $p=q=1$. There are two $(1,1)$-shuffles, the identity of $\{0,1\}$ and the map τ, $\tau(0)=1$, $\tau(1)=0$.

$\mathrm{Id} \rightsquigarrow \mu_1 = 0, \ \nu_1 = 1, \ \operatorname{sgn}(\mathrm{Id}) = +1$

$\tau \rightsquigarrow \mu_1 = 1, \ \nu_1 = 0, \ \operatorname{sgn}(\tau) = -1$

So for $a \in A_1$, $b \in B_1$, $a \nabla b = s_1^*(a) \otimes s_0^*(b) - s_0^*(a) \otimes s_1^*(b)$.

Claim: The shuffle products for varying (p,q) define a chain map, i.e.

$$d_{p+q}(\nabla_{p,q}(a \otimes b)) = \nabla_{p-1,q}(d_p(a) \otimes b) + (-1)^p \cdot \nabla_{p,q-1}(a \otimes d_q(b))$$

or shorter: $d(a \triangleright b) = (da) \triangleright b + (-1)^p \cdot a \triangleright (db)$.