HARMONIC MAPPINGS OF RIEMANNIAN MANIFOLDS.*

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Introduction. With any smooth mapping of one Riemannian manifold into another it is possible to associate a variety of invariantly defined functionals. Each such functional of course determines a class of extremal mappings, in the sense of the calculus of variations, and those extremals, in the very special cases thus far considered, play an important rôle in a number of familiar differential-geometric theories.

The present paper is devoted to a rather general study of a functional E of geometrical and physical interest, analogous to energy. Our central problem is that of deforming a given mapping into an extremal of E. Following an infinite-dimensional analogue of the Morse critical point theory, we construct gradient lines of E (in a suitable function space); and E is a decreasing function along those lines. With suitable metric and curvature assumptions on the target manifold (assumptions which cannot be entirely circumvented, in view of the examples of §§ 4E and 10D), we prove that the gradient lines do in fact lead to extremals (see Theorem 11A).

If $f\colon M \rightarrow M'$ is a smooth mapping of manifolds whose metrics are $g_{ij}dx^i dx^j$ resp. $g_{\alpha\beta}' dy^\alpha dy^\beta$, then the energy $E(f)$ is defined by the integral

$$E(f) = \tfrac{1}{2}\int_M g_{\alpha\beta}' \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j} g^{ij} * 1,$$

where the f^α are local coordinates of the point $f(x)$, $*1$ being the volume element of M (assumed compact). Thus $E(f)$ can be considered as a generalization of the classical integral of Dirichlet. The Euler-Lagrange equations for E are a system of non-linear partial differential equations of elliptic type:

$$\Delta f^\alpha + \Gamma_{\beta\gamma}'^\alpha \frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j} \quad g^{ij} = 0;$$

Δ is the Laplace-Beltrami operator on M and the $\Gamma_{\alpha\gamma}'^\alpha$ are the Christoffel symbols on M'. Although this system is suggestive of the simple equation $\Delta u + \phi(u) \cdot \mathrm{grad}^2 u = 0$ for one unknown, there is in general very little connection between the two because of the phenomenon of curvature.

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It has been necessary to go into the question of existence of solutions in rather great detail, owing to the want of general results for non-linear systems. Direct methods of the calculus of variations seem to lead to severe difficulties, and that is one reason why we have preferred to approach the problem through the gradient-line technique, which amounts to replacing the equations above by a system of parabolic equations whose relation to the elliptic system is analogous to that of Fourier's equation to Laplace's equation. This approach is of independent interest, in any case. Our methods are strongly potential-theoretic in nature. The local equations are first replaced by global equations of essentially the same form, embedding M' in a Euclidean space. A stability theorem is established showing that a solution of the resulting parabolic system does in fact produce a 1-parameter family of mappings of M into M'. Fundamental solutions of Laplace's equation and the heat equation on (compact) manifolds are used to establish *a priori* derivatives estimates and to construct solutions of the parabolic system, the latter being translated into a system of non-linear integro-differential equations of the Volterra type, following the method used by Milgram and Rosenbloom in a linear problem. Curvature enters in a manner not unlike that exploited by Bochner in [4].

Special cases of our extremal mappings go back to the very beginning of differential geometry. E. g. they include geodesics, harmonic functions, and minimal submanifolds. For minimal surfaces they were first studied locally by Bochner [2], in an explicitly Riemannian context. That work was carried to completion by Morrey [19]. In a report prepared by J. H. Sampson in February of 1954 at the Massachusetts Institute of Technology, the subject was taken up from a somewhat different point of view, and other geometrical applications were discussed. Since firm existence proofs were not then available, general publication of the results did not seem warranted. Shortly thereafter, J. Nash and, independently, F. B. Fuller [10] advanced the same definition as that on which this article is based, and Fuller described several examples. The problem has also been considered by E. Rauch.

The contents of the present paper are presented in the following order:

Added in proof: The theory of the energy functional (and its harmonic extremals) is the first-order case of a general theory of p-th order energy (and its polyharmonic extremals). See J. Eells, Jr. and J. H. Sampson, *L'énergie et les déformations en géometrie différentielle,* Colloque du CNRS (Proceding of a conference held in Grenoble in July, 1963) for a general formulation in fibre bundles.

Chapter I. The concepts of energy and tension.

1. The energy integral.

(A) Let M and M' denote complete Riemannian manifolds of dimensions n and m, respectively, and suppose further that M is closed (i. e., compact and without boundary) and oriented. In the interests of simplicity we assume that both manifolds and their Riemannian metrics are smooth (i. e., of class C^∞); however, it is not difficult to make minor modification to include differentiability class C^5. We will let $(x^1,\cdots,x^n)$ denote local coordinates on M in a neighborhood of a point P (said to be *centered at* P if $(0,\cdots,0)$ are its local coordinates), and $(y^1,\cdots,y^m)$ local coordinates on M'. Thus we can write the Reimannian metrics g and g' in these local coordinates as

$$ds^2 = g_{ij}dx^i dx^j, \qquad ds'^2 = g'_{\alpha\beta}dy^\alpha dy^\beta,$$

where we observe the summation convention; generally, when using local tensor calculus we follow the notations of Eisenhart [8]. We will denote covariant derivatives without the usual commas; e. g.,

$$f_i^\alpha = \partial f^\alpha/\partial x^i \qquad f_{ij}^\alpha = \partial^2 f^\alpha/\partial x^i \partial x^j - \Gamma_{ij}^k f_k^\alpha, \text{ etc.},$$

where Γ_{ij}^k denote the Christoffel symbols.

With any smooth map $f\colon M \to M'$ we assign a real number called its *energy,* as follows: For each point $P \in M$ we let $\langle\,,\rangle_P$ denote the inner product on the space of 2-covariant tensors of the tangent space $M(P)$ to M at P; thus if $(e^i)_{1\leq i\leq n}$ is a base for the cotangent space $M'(P)$ of $M(P)$ and $\alpha = \alpha_{ij}e^i \otimes e^j$ and α' are 2-covariant tensors of $M(P)$, then

$$\langle \alpha, \alpha' \rangle_P = \alpha_{ij} \alpha'_{pq} g^{ip} g^{jq},$$

where $g^{ik} g_{kj} = \delta_j{}^i$. Since the metric tensor g' of M' is 2-covariant, it induces through f a 2-covariant tensor field $f^* g'$ on M, whence we can define the function $P \to \langle g(P), (f^*g')(P) \rangle_P$ on M. We will call $e(f)P = \frac{1}{2} \langle g(P), (f^*g')(P) \rangle_P$ the *energy density of f at P*. Its dual differential n-form $e(f) * 1$ can then be integrated over M, and with an eye toward the physical concept of kinetic energy $(mv^2/2)$ we define *the energy of the map f* by

$$E(f) = \int_M e(f) * 1. \tag{1}$$

In a local coordinate representation we have

$$E(f) = \tfrac{1}{2} \int_M g^{ij} f_i{}^\alpha f_j{}^\beta g'_{\alpha\beta} [\det(g_{ij})]^{\frac{1}{2}} \, dx^1 \wedge \cdots \wedge dx^n,$$

where $f_i{}^\alpha = \partial f^\alpha / dx^i$. Observe that if the local coordinates centered at P and $f(P)$ are both chosen to be locally Euclidean at their centers, then

$$e(f)P = \tfrac{1}{2} \sum_{i,\alpha} [f_i{}^\alpha(P)]^2,$$

so that $e(f)$ is *non-negative, and $E(f)$ vanishes when and only when f is a constant map.*

(B) Although the present work is devoted primarily to the functional E and its extremals, there will be the indications (e.g. the example § 4E) that we will want ultimately to consider other types of energy of maps. We give here a general method of constructing these.

Starting with the manifold M and any smooth symmetric 2-covariant tensor field α on M, we fix a point $P \in M$ and form the proper values of α relative to the metric tensor g of M; i.e., the n real roots of the equation $\det(g_{ij}(P)\lambda - \alpha_{ij}(P)) = 0$. Apart from their order, these proper values are intrinsically associated with α and P; thus we are led to forming their symmetric functions.

Definition. Let σ be any symmetric real function of n variables. For any symmetric 2-covariant tensor field α on M we let $\sigma(\alpha) : M \to R$ be the function such that $\sigma(\alpha)P =$ the σ-function of the proper values of $\alpha(P)$ relative to $g(P)$. The *σ-integral* of α is the number

$$I_\sigma(\alpha) = \int_M \sigma(\alpha) * 1.$$

In particular, let σ_p denote the p-th elementary symmetric function.

Then setting $\mu_0 = 1$, $\mu_p = \sigma_p / \binom{n}{p}$ we have Newton's inequalities $\mu_{p-1}\mu_{p+1} \leqq (\mu_p)^2$, with equality if and only if all proper values λ_i are equal; furthermore, if all $\lambda_i \geqq 0$, then

(2) $$\mu_1 \geqq [\mu_2]^{1/2} \geqq [\mu_3]^{1/3} \geqq \cdots \geqq [\mu_n]^{1/n}, \text{ and}$$

$\mu_1/\mu_0 \geqq \mu_2/\mu_1 \geqq \cdots \geqq \mu_n/\mu_{n-1}$. These provide some sort of comparability of the various σ-integrals of α. We have

$$\sigma_1(\alpha)P = g^{ij}(P)\alpha_{ij}(P), \quad \sigma_n(\alpha)P = \det[\alpha_{ij}(P)]/\det[g_{ij}(P)],$$

and in general

$$\sigma_p(\alpha)P = (\det[g_{ij}])^{-1} \cdot \sum_{i_1 < \cdots < i_p} \begin{vmatrix} g_{11} \cdots g_{1n} \\ \alpha_{i_1 1} \cdots \alpha_{i_1 n} \\ \alpha_{i_p 1} \cdots \alpha_{i_p n} \\ g_{n1} \cdots g_{nn} \end{vmatrix}.$$

Remark. These integrals provide many variational problems of differential geometric interest. For example, if we take for α the Ricci tensor of M, then its proper values are the principal curvatures at a point, and $\sigma_1(\alpha)$ is the scalar curvature function. If we take for α a second fundamental form (for a given direction) of an immersion of M, then $\sigma_n(\alpha)$ is the Gauss-Kronecker curvature function (for that direction).

For the present purposes we take $\alpha = f^*g'$ for some map f. Then $(f^*g')_{ij} = f_i^\alpha f_j^\beta g'_{\alpha\beta}$, and $\sigma_1(f^*g') = \langle g, f^*g' \rangle$; we are led to the

Definition. For any map $f\colon M \to M'$ and any symmetric function σ of n variables, the *σ-energy of f* is the number

$$E_\sigma(f) = \int_M \sigma(f^*g') * 1.$$

We observe that *the energy of f is the $(\sigma_1/2)$-energy of f.*

Let $J_f(P)$ denote the Jacobian of f at P; i.e., the image under f of the unit n-vector of $M(P)$ defining the given orientation of M; thus

$$\sigma_n(f^*g')P = \langle J_f(P), J_f(P) \rangle / \det(g_{ij}(P)),$$

where the inner product is that of the space of n-vectors of $M'(f(P))$. The *volume of f* is the number

(3) $$V(f) = \int_M [\sigma_n(f^*g')]^{\frac{1}{2}} * 1.$$

Note that $V(f) = 0$ if $n > m$.

Remark. If we consider $g + \lambda f^* g' = ds_\lambda{}^2$ as a perturbation of the metric of M for $\lambda \geqq 0$, then the corresponding volume element is

$$*1_\lambda = [\det(g_{ij} + \lambda f_i{}^\alpha f_j{}^\beta g'_{\alpha\beta})]^{\frac{1}{2}} dx^1 \wedge \cdots \wedge dx^n.$$

A simple calculation shows that

$$E(f) = \frac{d}{d\lambda} V_\lambda(M)_{\lambda=0},$$

where $V_\lambda(M) = \int_M *1_\lambda$.

(C) We give here an interpretation of $E(f)$ as a measurement of dispersion; the calculations of this paragraph will not be used in any essential way later.

Let $r'(P', Q')$ denote the geodesic distance between the points $P', Q' \in M'$, and suppose $P, Q \in M$ are points such that $f(P) = P'$, $f(Q) = Q'$. For a fixed P we consider the function $Q \rightarrow r'^2(P', Q')$; it is elementary that, for Q sufficiently near P, that function is smooth.

Let $\Delta f = g^{ij} f_{ij}$ denote the Laplace-Beltrami operator on the function f. We calculate the Laplacean of r'^2 *qua* function of Q. In coordinates centered at P and P' we have

$$\Delta_Q r'^2(P', Q') = g^{ij} \frac{\partial^2 r'^2}{\partial x^i \partial x^j} - g^{ij} \Gamma_{ij}{}^k \frac{\partial^2 r'^2}{\partial x^k}.$$

A simple calculation gives

$$\Delta_Q r'^2 = g^{ij} \frac{\partial^2 r'^2}{\partial y^\alpha \partial y^\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} + \frac{\partial r'^2}{\partial y^\alpha} \Delta y^\alpha.$$

Now

$$\frac{\partial r'^2}{\partial y^\alpha} = O(r'), \qquad \frac{\partial^2 r'^2}{\partial y^\alpha \partial y^\beta} = 2g'_{\alpha\beta}(P') + O(r')$$

as $Q \rightarrow P$, whence

$$\Delta_Q r'^2(P', Q') = e(f)P + O(r').$$

For any smooth function u defined in a neighborhood of P let $\bar{u}_\epsilon(P)$ denote the average value $(\epsilon > 0)$

$$\bar{u}_\epsilon(P) = \int_{B_\epsilon} u*1/V(B_\epsilon),$$

where $B_\epsilon = \{Q \in M : f(P, Q) \leqq \epsilon\}$.

We require Maxwell's relation [16, p. 31. The formula is easily proved by expanding u in a Taylor's series at the center of a normal coordinate system.]:

$$\bar{u}_\epsilon(P) = u(P) + \frac{\epsilon^2}{2(n+2)} \Delta u(P) + O(\epsilon^3).$$

Applying this to $u(Q) = r'^2(P', Q')$ and noting that $u(P) = 0$ and

$$\Delta u(P) = \lim_{Q \to P} \Delta_Q r'^2(P', Q') = e(f)P,$$

we obtain

$$\int_{B\epsilon} r'^2(P', Q') * 1_Q / V(B_\epsilon) = \frac{2\epsilon^2}{n+2} e(f)P + O(\epsilon^3).$$

If $\phi_\epsilon(P, Q)$ is a smooth function which is zero for $r(P, Q) \geqq \epsilon$ and equals $1/V(B_\epsilon)$ for $r(P, Q) \leqq \epsilon$, then we obtain from our last equation

$$\int_M \int_M r'^2(P', Q') \phi_\epsilon(P, Q) * 1 = \frac{2\epsilon^2}{(n+2)} E(f) + O(\epsilon^3).$$

Naturally certain uniformity conditions must be fulfilled by the error terms for the validity of this formula, but since we make no essential use of the result, we shall not insist on that point. *The left member of the preceding equation represents the mean square infinitesimal dispersion of the image points on M produced by f, and that quantity is estimated by the energy of f.*

2. The tension field.

(A) In this section we examine the extremals of E, interpreted as the zeros of the Euler-Lagrange operator associated with E. For this purpose we let $\pi\colon \mathcal{T}(M') \to M'$ denote the tangent vector bundle. We let $\mathfrak{H}(M, M')$ denote the totality of smooth maps from M to M'; then for every $f \in \mathfrak{H}(M, M')$ the set of smooth maps $u\colon M \to \mathcal{T}(M')$ such that $\pi \circ u = f$ forms a vector space $\mathfrak{H}(f)$ with algebraic operations defined pointwise; such a u is called a *vector field along f*. We define an inner product in $\mathfrak{H}(f)$ by

$$\langle u, v \rangle_f = \int_M \langle u(P), v(P) \rangle_{f(P)} * 1.$$

For any vector field v along f the directional derivative of E in the direction v,

$$\nabla_v E(f) = \frac{d}{dt} [E(f_t)]_{t=0}, \text{ where } f_t(P) = \exp_{f(P)}(tv(P)),$$

is the endpoint of the geodesic segment in M' starting at $f(P)$ and determined in length and direction by the vector $tv(P) \in M'(f(P))$. We show in (B) below that *the Euler-Lagrange operator applied to a map f defines a vector field $\tau(f)$ along f which is the contravariant representative of the differential of E at f*; i.e.,

$$\nabla_v E(f) = -\langle \tau(f), v\rangle_f \text{ for all } v \in \mathfrak{H}(f).$$

Thus the maps f for which $\tau(f) = 0$ are (so to speak) the critical points of E.

(B) LEMMA. *Let $f_t\colon M \to M'$ be a smooth family of maps for t in some time interval $t_0 < t < t_1$. Then*

$$\frac{d}{dt}E(f_t) = -\int_M [\Delta f_t^\gamma + g^{ij}\Gamma'_{\alpha\beta}{}^\gamma f_{t,i}{}^\alpha f_{t,j}{}^\beta] g'_{\gamma\nu}\frac{\partial f_t{}^\nu}{\partial t} *1, \tag{4}$$

where Δ denotes the Laplace-Beltrami operator.

Proof. From the definition (1) we obtain for any $t \in (t_0, t_1)$

$$\frac{d}{dt}E(f_t) = \tfrac{1}{2}\int_M g^{ij}[2f_{t,i}{}^\alpha \frac{\partial^2 f_t{}^\beta}{\partial x^j \partial t} g'_{\alpha\beta} + f_{t,i}{}^\alpha f_{t,j}{}^\beta \frac{\partial g'_{\alpha\beta}}{\partial y^\nu}\frac{\partial f_t{}^\nu}{\partial t}]*1.$$

The quantities

$$\xi^j = g^{ij}\frac{\partial f_t{}^\alpha}{\partial x^i}\frac{\partial f_t{}^\beta}{\partial t} g'_{\alpha\beta}$$

are the components of a contravariant vector field ξ on M, whence by Green's divergence theorem

$$\int_M \xi^j{}_j *1 = 0.$$

I. e.,

$$\int_M g^{ij}[f_{t,ij}{}^\alpha \frac{\partial f_t{}^\beta}{\partial t} g'_{\alpha\beta} + f_{t,i}{}^\alpha \frac{\partial^2 f_t{}^\beta}{\partial x^j\partial t} g'_{\alpha\beta} + f_{t,i}{}^\alpha \frac{\partial f_t{}^\beta}{\partial t}\frac{\partial g'_{\alpha\beta}}{\partial y^\nu}\frac{\partial f_t{}^\nu}{\partial x^j}]*1 = 0.$$

A routine calculation now gives (4).

For any smooth map $f \in \mathfrak{H}(M, M')$ we set

$$\tau(f)^\gamma(P) = \Delta f^\gamma(P) + g^{ij}(P)\Gamma'_{\alpha\beta}{}^\gamma(f(P))f_i{}^\alpha(P)f_j{}^\beta(P). \tag{5}$$

It is clear that $\tau(f)^\gamma$ is unaffected by any transformation of the local coordinates on M near P, and for any such transformation of coordinates near $f(P)$ we see that $\tau(f)^\gamma(P)$ transforms as a contravariant vector in $M'(f(P))$; see § 3 for an invariant formulation. Thus with every $f \in \mathfrak{H}(M, M')$ we have a variation $\tau(f) \in \mathfrak{H}(f)$.

Definition. We call $\tau(f)$ *the tension field of the map f*, and we say that *f is a harmonic map if* $\tau(f) = 0$. Thus (4) becomes

$$\frac{d}{dt}E(f_t) = -\int_M \langle \tau(f), \frac{df_t}{dt}\rangle *1.$$

Suppose now that f is an extremal (with respect to small deformations) of E. If we apply (4) to suitably chosen local deformations (e.g., confined to small geodesic balls on M), we obtain from a standard argument that *the Euler-Lagrange equation for the energy functional E is $\tau(f)=0$. In local coordinates that equation is elliptic.*

Remark. It is a simple matter to modify these constructions to include the case that M has a boundary; we should require in Lemma 2B that f_t is a constant function of t on the boundary.

PROPOSITION. *Every map $f\colon M\to M'$ of class C^2 which satisfies $\tau(f)=0$ is smooth. If M and M' are both analytic Riemannian manifolds, then every such map is analytic.*

The proof of the first assertion is easily obtained by inductive application of [14, Theorem 3, p. 210]; analyticity follows directly from [22, p. 4].

(C) Our next step is to interpret the tension field $\tau(f)$ in terms of special coordinate systems on M and M'.

For any $P\in M$ we let $\exp_P\colon M(P)\to M$ be that map which assigns to each vector $u\in M(P)$ the end point of the geodesic segment emanating from P and determined in length and direction by u; it is elementary that $\exp_P$ is a smooth surjective map, carrying a neighborhood of $0\in M(P)$ diffeomorphically onto a neighborhood V of P in M. If we now choose an orthonormal frame in $M(P)$, then we can use the inverse of $\exp_P$ to refer each point $Q\in V$ to the components relative to that frame. These are called *normal coordinates in V centered at P*, and they form a coordinate system admissible for the differentiable structure of M.

In terms of P-centered normal coordinates we have

1) the metric tensor $g_{ij}(P)=\delta_{ij}$, since the frame is orthonormal;

2) the Christoffel symbols $\Gamma_{ij}^k(P)=0$ for all i, j, k; that is because the equations of the geodesics through P are linear, whence $\Gamma_{ij}^k(P)u^iu^j=0$ for all vectors $u\in M(P)$; see Eisenhart [8, p. 54]. In particular, $f_{ij}^\gamma(P)=\partial^2f^\gamma(P)/\partial x^i\partial x^j$, from which we obtain the

LEMMA. *For any $f\in\mathfrak{H}(M,M')$ and point $P\in M$ let us fix normal coordinate systems V and V' centered at P and $P'=f(P)$, respectively; in terms of these we have*

$$\tau(f)^\gamma(P)=\sum_{i=1}^n\frac{\partial^2f^\gamma(P)}{\partial(x^i)^2}. \qquad (1\leqq\gamma\leqq m)$$

Thus f is harmonic if and only if at each pair of points P, $f(P)$ there are such coordinates in which f satisfies the Euclidean-Laplace equation at P.

Example. Suppose M' is flat; i.e., its Riemannian curvature is zero at every point. It is well known that then M' admits a smooth coordinate covering such that in each coordinate chart we have $\Gamma'_{\alpha\beta}{}^{\gamma} \equiv 0$. Then $\tau(f)^{\gamma} = \Delta f^{\gamma}$, and in such local coordinates *the equation* $\tau(f) = 0$ *is linear.* In particular, if M' is the Euclidean space $\boldsymbol{R}^m$, then a map $f\colon M \to \boldsymbol{R}^m$ is harmonic if and only if it is constant, by the maximum principle.

Example. If M and M' are Lie groups with bi-invariant Riemann metrics, and if $f\colon M \to M'$ is a homomorphism, then f is harmonic. It suffices to verify this at the identity of M; taking canonical coordinates (which are normal; see Chevalley [5, p. 118]) at the neutral elements of M and M', we see that the representation of f is linear, whence $\tau(f) = 0$ by Lemma 2C.

Example. If M and M' are Kähler manifolds and $f\colon M \to M'$ is a holomorphic map, then f is harmonic relative to the associated real Riemann structures on M and M'. Namely, we take local holomorphic coordinates $z^j = x^j + \sqrt{-1}\, x^{h+j}$ and $w^{\alpha} = y^{\alpha} + \sqrt{-1}\, y^{k+\alpha}$ (where $\dim M = 2h$, $\dim M' = 2k$) centered at some points $P \in M$, $f(P) \in M'$. Because the w^{α} are holomorphic functions of the z^j and because M is Kähler, they satisfy Laplace's equation $\Delta w^{\alpha} = 0$. Now Δ is a real operator, which we have $\Delta y^{\alpha} = 0 = \Delta y^{k+\alpha}$ for $1 \leqq \alpha \leqq k$. Since M' is Kähler, the w^{α} can be chosen to be normal coordinates, whence all $\Gamma'_{\alpha\beta}{}^{\gamma} = 0$ at $f(P)$. Then $\tau(f)P = 0$.

(D) Suppose $f\colon M \to M'$ *is a Riemannian immersion.* Thus for each $P \in M$ the differential $f_*(P)$ of f at P maps $M(P)$ isometrically into $M'(f(P))$; i.e., $g = f^*g'$. If $\xi_{\sigma|}$ $(1 \leqq \sigma \leqq m - n)$ is an orthonormal frame in (some coordinate system) on M' orthogonal to M, then

$$b_{\sigma|ij} = (f_{ij}{}^{\gamma} + \Gamma'_{\alpha\beta}{}^{\gamma} f_i{}^{\alpha} f_j{}^{\beta})\, g'_{\gamma\nu}\xi_{\sigma}{}^{\nu}| = b_{\sigma|ji} \tag{6}$$

are the components of the second fundamental form $\beta(f)$ on M in M' relative to that frame (Eisenhart [8, § 50]). The vector field ξ along the map f defined by

$$\xi(P) = \sum_{i=1}^{m-n} g^{ij}(P)\, b_{\sigma|ij}(P)\, \xi_{\sigma|}(P) \tag{7}$$

for all $P \in M$ is independent of choice of frame. It is traditionally called *the mean normal curvature field of the immersion.* The following formula is a well known interpretation:

$$\xi(P) = -\sum_{\sigma=1}^{m-n} \operatorname{div}(\xi_{\sigma|}(P))\, \xi_{\sigma|}(P).$$

Note that the coefficient of each $\xi_{\sigma|}$ is the first elementary symmetric function of the proper values of $b_{\sigma|}$ relative to g; that indicates how the following results (and those of § 4C) will generalize when we replace energy by σ-energy.

A Riemannian immersion is said to be a *minimal immersion* if $\xi = 0$ on M. The minimal immersions are precisely those which are the extremals of the volume functional V (Eisenhart [8, p. 176]). Usually the notion of minimal immersion is taken in a somewhat broader context, as a smooth map which is an extremal of V. There are examples (even for $M' = \boldsymbol{R}^m$ and $n = 2$) of analytic extremals of V which are not strictly immersions, for their Jacobians vanish at isolated points.

From (7) we conclude that

$$\begin{aligned}\xi &= \sum_{\sigma=1}^{m-n} g^{ij}(f_{ij}{}^{\gamma} + \Gamma'_{\alpha\beta}{}^{\gamma} f_i{}^{\alpha} f_j{}^{\beta}) g'_{\gamma\nu} \xi_{\sigma|}{}^{\nu} \xi_{\sigma|} \\ &= \sum_{\sigma=1}^{m-n} \langle \tau(f), \xi_{\sigma|} \rangle \xi_{\sigma|}.\end{aligned}$$

But $(f_{ij}{}^{\gamma} + \Gamma'_{\alpha\beta}{}^{\gamma} f_i{}^{\alpha} f_j{}^{\beta}) g'_{\gamma\nu} f_k{}^{\nu} = 0$ $(1 \leqq k \leqq n)$ in general (Eisenhart [8, p. 160]); i. e., $\tau(f)(P)$ *is perpendicular to* $M(P)$.

PROPOSITION. *Let* $f: M \to M'$ *be a Riemannian immersion. Then* $e(f) = n/2$. *The tension field* $\tau(f)$ *coincides with the mean normal curvature field on* f. *In particular,* f *is harmonic if and only if it is minimal.*

Note that any isometry $f: M \to M$ is harmonic, and that any covering map is harmonic.

Remark. For any $u \in M'(f(P))$ perpendicular to $M(P)$ we have $\langle \tau(f), u \rangle = \text{trace}\,(\beta(f)u)$.

(E) For any map $f \in \mathfrak{H}(M, M')$ we define $F: M \to M \times M'$ by $F(P) = (P, f(P))$; then F is a smooth imbedding, but not isometric in the product metric on $M \times M'$. The following result is immediate:

PROPOSITION. *For any* $f \in \mathfrak{H}(M, M')$ *we have* $e(F) = n/2 + e(f)$. *Furthermore,* f *is harmonic if and only if* F *is harmonic.*

3. Invariant formulation.

(A) In this section we express our problem in terms of differential forms with values in a vector bundle. It turns out that in some sense the harmonic theory of such forms gives us an infinitesimal solution to our problem. We begin by summarizing the general theory; for details see Spencer [27] or Bochner [4].

1. Let $W \to M$ be a smooth vector bundle over M with fibre dimension m. If $\mathcal{J}^{[p]}(M) \to M$ denotes the bundle of p-covectors of M and if $W \otimes \mathcal{J}^{[p]}(M) \to M$ is the tensor product bundle, then the smooth sections of that latter bundle are called the *smooth p-forms on M with values in W*; their totality forms a vector space, which we denote by $A^p(M, W)$.

2. Suppose that W has a Riemannian structure; i. e., a given reduction of its structural group to the orthogonal group in dimension m. If $W \to W^*$. denotes the dual bundle of W, then we have a bundle isomorphism $\psi\colon W \to W^*$. Taken together with the canonical isomorphism $*\colon \mathcal{J}^{[p]}(M) \to \mathcal{J}^{[n-p]}(M)$ induced from the given Riemannian structure on M, we have the natural map $\psi \otimes *\colon A^p(M, W) \to A^{n-p}(M, W^*)$. In particular, the evaluation of W^* on W induces a bilinear pairing

$$\#\colon A^p(M, W) \times A^p(M, W) \to A^n(M),$$

the vector space of real valued n-forms on M. Thus $A^p(M, W)$ has the *inner product*

$$\langle \phi, \psi \rangle = \int_M \phi \# \psi. \tag{8}$$

3. Assume next that W has a connection which is compatible with its Riemannian structure; i. e., the covariant differential of the tensor field defining the orthogonal reduction of W is zero. We say that *W is a Riemannian-connected bundle*. Then we have a *linear map* $\tilde{d}\colon A^p(M, W) \to A^{p+1}(M, W)$, which can be described as follows: If $\mathcal{U}$ is a locally finite covering of M by coordinate systems, then for each $U \in \mathcal{U}$ the connection in W can be given by a certain $m \times m$ matrix $\theta^U = (\theta_\alpha{}^{U\beta})$ of 1-forms in U. Similarly any form $\phi \in A^p(M, W)$ defines an m-tuple $\phi^U = (\phi_\beta{}^U)$ of p-forms in U. If d denotes the exterior differential operator in U, then we define $\tilde{d}\phi$ by giving its representation in each U by the formula (In this section (3A) we violate our convention of lettering subscripts denote covariant differentiation. We adopt here a special symbol for that concept.)

$$\begin{aligned}(\tilde{d}\phi)^U &= d\phi^U + \theta^U \wedge \phi^U \\ &= (d\phi_1{}^U + \theta_1{}^{U\beta} \wedge \phi_\beta{}^U, \cdots, d\phi_m{}^U + \theta_m{}^{U\beta} \wedge \phi_\beta{}^U).\end{aligned}$$

4. We can now develop the theory of harmonic forms on M with values in a Riemannian-connected bundle. We have the adjoint map $\tilde{\delta}\colon A^p(M, W) \to A^{p-1}(M, W)$, characterized by the formula

$$\langle \tilde{d}\phi, \psi \rangle = \langle \phi, \tilde{\delta}\psi \rangle.$$

In particular, for any $\phi \in A^1(M, W)$ and any coordinate chart U we have 1-forms $\phi_\alpha{}^U$ $(1 \leqq \alpha \leqq m)$, which can be expressed

$$\phi_\alpha{}^U = \phi_{\alpha j}{}^U \, dx^j \text{ in the coordinates } (x^1, \cdots, x^n) \text{ in } U.$$

If we write similarly

$$\theta_\alpha{}^{U\beta} = T_{\alpha i}{}^\beta \, dx^i,$$

then

$$(9) \qquad \begin{aligned} (\tilde{d}\phi)_\alpha{}^U &= \Big(\frac{\partial \phi_{\alpha j}{}^U}{\partial x^i} + T_{\alpha i}{}^\beta \phi_{\beta j}{}^U\Big) dx^i \wedge dx^j, \\ (\tilde{\delta}\phi)_\alpha{}^U &= -g^{ij}(\nabla_i \phi_{\alpha j}{}^U + T_{\alpha i}{}^\beta \phi_{\beta j}{}^U), \end{aligned}$$

letting ∇_i denote covariant differentiation on M relative x^i. The *Dirichlet integral of* $\phi \in A^p(M, W)$ is $2\tilde{D}(\phi) = \langle \tilde{d}\phi, \tilde{d}\phi \rangle + \langle \tilde{\delta}\phi, \tilde{\delta}\phi \rangle$, and its associated *Laplace operator is* $\tilde{\Delta} = -(\tilde{d}\tilde{\delta} + \tilde{\delta}\tilde{d})$. *A form* $\phi \in A^p(M, W)$ *is harmonic if* $\tilde{\Delta}\phi = 0$; that condition is equivalent to the pair $\tilde{d}\phi = 0 = \tilde{\delta}\phi$.

It is known that a Green's form exists for the operator $\tilde{\Delta}$, from which it can be proved that there is a decomposition

$$(10) \qquad A^p(M, W) = H^p(M, W) \oplus [\tilde{d}A^{p-1}(M, W) + \tilde{\delta}A^{p+1}(M, W)],$$

where $H^p(M, W)$, the space of *harmonic p-forms with values in W,* is orthogonal to the other two summands.

5. The curvature of the connection can be expressed in U by an $m \times m$ matrix of 2-forms $\Theta^U = (\Theta_{\alpha\beta}{}^U)$ where $\Theta^U = d\theta^U + \theta^U \wedge \theta^U$. If ∇ denotes the covariant differential of M, then we define *the covariant differential* $\tilde{\nabla}$ on $\phi \in A^p(M, W)$ by

$$\tilde{\nabla}^U \phi^U = \nabla \phi^U + \theta^U \wedge \phi^U.$$

Letting $R_i{}^h$ denote $R_i{}^h I$, the diagonal $m \times m$ matrix each of whose diagonal terms is the Ricci tensor field $R_i{}^h$ (we follow the sign convensions in Eisenhart in defining $R_{ij} = R^h{}_{ijh}$), we have for any $\phi \in A^1(M, W)$ the following expression for the components of the Laplacean of ϕ in U:

$$(-\tilde{\Delta}\phi)_i{}^U = -(\tilde{\nabla}^U)^k (\tilde{\nabla}^U)_k \phi_i{}^U + {}^t[(\Theta^U)_i{}^h - R_i{}^h]\phi_h{}^U,$$

where t denotes the transposition of matrices.

If $\phi, \psi \in A^1(M, W)$, define the function $\phi \cdot \psi = *(\phi \# \psi)$ on M; then in U we have (usual Laplacean Δ of functions)

$$\begin{aligned} \Delta(\tfrac{1}{2}\phi \cdot \phi) = \phi^U \cdot (-\tilde{\Delta}\phi^U) &- (\tilde{\nabla}^U)_i (\phi^U)_k a^U (\tilde{\nabla}^U)^i (\phi^U)^k \\ &- {}^t[(\Theta^U)_i{}^h - R_i{}^h](\phi^U)_h a^U (\phi^U)^i, \end{aligned}$$

where a^U denotes the Riemannian structure of W in U. For each $\phi \in A^1(M, W)$ define the function

$$Q(\phi) = {}^t[(\Theta^U)_i{}^h - R_i{}^h](\phi^U)_h a^U (\phi^U)^i. \tag{11}$$

The matrix (of functions) of Q is

$$Q_{\alpha\beta}{}^{hi} = g^{hk} g^{ij} [(a_{\alpha\gamma}{}^U \Theta_{\gamma\beta}{}^U)_{kj} - a_{\alpha\beta}{}^U R_{kj}].$$

We consider this as an $nm \times nm$ matrix in the subscripts (αh), (βi); as such, it is symmetric: $Q_{\alpha\beta}{}^{hi} = Q_{\beta\alpha}{}^{ih}$.

The integral over M of $\Delta(\phi \cdot \phi/2)$ is always zero, by Green's theorem. Thus *if ϕ is harmonic we have*

$$\int_M Q(\phi) * 1 = - \int_M (\tilde{\nabla}^U)_i (\phi^U)_k a^U (\tilde{\nabla}^U)^i (\phi^U)^k * 1 \leqq 0. \tag{12}$$

(B) Given any $f \in \mathfrak{H}(M, M')$, let $f^{-1}\mathfrak{T}(M') \to M$ be the induced vector bundle; it is clearly Riemannian-connected. Let us interpret the preceding development for that bundle.

First of all, the elements of $A^0(M, f^{-1}\mathfrak{T}(M'))$ are canonically identified with the vector fields along f (i.e., with the elements of the space $\mathfrak{H}(f)$, in the notation of § 2A). Secondly, for any $P \in M$ the differential $f_*(P)$: $M(P) \to M'(f(P))$ is a linear map, to be considered as an element of $M'(f(P)) \otimes M^{[1]}(P)$; otherwise said, the assignment $P \to f_*(P)$ determines a specific 1-form $f_* \in A^1(M, f^{-1}\mathfrak{T}(M'))$. Thirdly, we have

$$E(f) = \tfrac{1}{2} \int_M f_* \# f_*.$$

LEMMA. *For any $f \in \mathfrak{H}(M, M')$ we have $\tilde{d} f_* = 0$; i.e., f_* is orthogonal to $\tilde{\delta} A^2(M, f^{-1}\mathfrak{T}(M'))$. Thus $\tilde{\Delta} f_* = - \tilde{d}\tilde{\delta} f_*$.*

Proof. Take a coordinate chart U on M, and write

$$\theta_\gamma{}^{U\beta} = \Gamma'_{\alpha\beta}{}^\gamma f_i{}^\alpha \, dx^i. \tag{13}$$

Then $(f_*{}^U)^\beta = f_j{}^\beta \, dx^j$, whence $(1 \leqq \alpha \leqq m)$

$$(\tilde{d} f_*{}^U)_\gamma = \frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} \, dx^i \wedge dx^j + \Gamma'_{\alpha\beta}{}^\gamma f_i{}^\alpha f_j{}^\beta \, dx^i \wedge dx^j,$$

which is zero, because both coefficients are symmetric in i, j.

Similarly, the variation $\tilde{\delta} f_* \in A^0(M, f^{-1}\mathfrak{T}(M'))$ has coordinate representation

$$(\tilde{\delta} f_*)_\gamma{}^U = - g^{ij} \{ \nabla_i \nabla_j f^\gamma + \Gamma'_{\alpha\beta}{}^\gamma f_i{}^\alpha f_j{}^\beta \} = - \tau(f)^\gamma.$$

PROPOSITION. *For any $f \in \mathcal{H}(M, M')$ its differential f_* is a closed 1-form. Its tension field $\tau(f) = -\tilde{\delta} f_*$, the divergence of its differential. The map f is harmonic if and only if its differential is a harmonic 1-form.*

Definition. For any map $f \in \mathcal{H}(M, M')$ its *fundamental form* $\beta(f)$ is the covariant differential $\tilde{\nabla} f_*$ of its differential. Thus $\beta(f)$ is the $f^{-1}\mathcal{T}(M')$-valued 2-covariant tensor field on M whose coordinate representation is

$$f_{;ij}{}^{\gamma} = f_{ij}{}^{\gamma} + \Gamma'_{\alpha\beta}{}^{\gamma} f_i{}^{\alpha} f_j{}^{\beta} = f_{;ji}{}^{\gamma}.$$

The tension field $\tau(f)$ is just the trace of $\beta(f)$; i.e., $\tau^{\gamma}(f) = g^{ij} f_{;ij}{}^{\gamma}$ $(1 \leqq \gamma \leqq m)$. It follows from § 2D that if f is a Riemannian immersion, then $\beta(f)$ is the second fundamental form of M in M'. Analogously, let us say that a map $f \in \mathcal{H}(M, M')$ *is totally geodesic* if $\beta(f) = 0$ on M; we will see as a consequence of Corollary 5A below that *totally geodesic maps map geodesics into geodesics.*

(C) Let us consider the function $Q(f_*)$; from the expression $(\Theta_{\alpha\beta}{}^{U})_i{}^{h} = g^{hk} R'_{\alpha\beta\gamma\delta} f_k{}^{\gamma} f_i{}^{\delta}$ we compute (11), taking into account the skew symmetry $\Theta_{\alpha\beta}{}^{U} = -\Theta_{\beta\alpha}{}^{U}$ to obtain the

LEMMA. *For any smooth map $f \colon M \to M'$ we have*

$$Q(f_*) = -R'_{\alpha\beta\gamma\delta} f_i{}^{\alpha} f_j{}^{\beta} f_k{}^{\gamma} f_l{}^{\delta} g^{ik} g^{jl} - R^{ij} f_i{}^{\alpha} f_j{}^{\beta} g'_{\alpha\beta}. \tag{14}$$

Its matrix (for arbitrary forms $\phi \in A^1(M, f^{-1}\mathcal{T}(M'))$) is

$$Q_{\alpha\beta}{}^{ij} = -R'_{\alpha\beta\gamma\delta} f_k{}^{\gamma} f_l{}^{\delta} g^{ik} g^{jl} - R^{ij} g'_{\alpha\beta}. \tag{15}$$

If f is harmonic, then

$$\Delta e(f) = |\beta(f)|^2 + Q(f_*). \tag{16}$$

We will refer to the matrix (15) as *the Ricci transformation on the tensor product bundle* $f^{-1}\mathcal{T}(M') \otimes \mathcal{T}^{[1]}(M)$. Observe that if f is a real-valued function on M, then that Ricci transformation is just that given by the Ricci tensor of M.

Remark. The above computations can of course be made without passing through the medium of vector-bundle-valued differential forms. One starts by applying the Ricci identities (Eisenhart [8, p. 30]) to the direct evaluation of $\Delta e(f)$, and then reads off the appropriate terms.

The next result follows the well known pattern of Bochner; in [2] Bochner has also applied the method in a special case for maps.

THEOREM. *If $f \colon M \to M'$ is a harmonic map, then*

$$\int_M Q(f_*)*1 \leqq 0,$$

and equality holds when and only when f is totally geodesic. Furthermore, if $Q(f_*) \geqq 0$ on M, then f is totally geodesic and has constant energy density $e(f)$.

Proof. This follows at once from Stokes' Theorem

$$\int_M \Delta e(f)*1 = 0,$$

applied to (16); for if $\Delta e(f) \geqq 0$, then $\Delta e(f) = 0$ everywhere, whence $e(f)$ is a constant function.

Following the conventions of Eisenhart [8], we say that the Ricci curvature of M is non-negative if at every point $P \in M$ the matrix $(-R_{ij}(P))$ is positive semi-definite.

COROLLARY. *Suppose that the Ricci curvature of M is non-negative and that the Riemannian curvature of M' is non-positive. Then a map $f\colon M \to M'$ is harmonic if and only if it is totally geodesic. Furthermore,*

1) *if there is at least one point of M at which its Ricci curvature is positive, then every harmonic map $f\colon M \to M'$ is constant;*

2) *if the Riemannian curvature of M' is everywhere negative, then every harmonic map $f\colon M \to M'$ is either constant or maps M onto a closed geodesic of M'.*

Proof. The theorem shows that

$$R'_{\alpha\beta\gamma\delta} f_i{}^\alpha f_j{}^\beta f_k{}^\gamma f_l{}^\delta g^{ik} g^{jl} + R^{ij} f_i{}^\alpha f_j{}^\beta g'_{\alpha\beta} \equiv 0.$$

If hypothesis 1) is satisfied at $P \in M$, then $f_*(P) = 0$, whence the constant $e(f) = 0$; i.e., f is a constant map. If hypothesis 2) is satisfied at $P' = f(P) \in M'$ and we take normal coordinates centered at P, then the $f_*(P)$-image of the tangent space $M(P)$ has dimension $\leqq 1$. If it has dimension 0 at any $f(P)$, then again $e(f) = 0$; otherwise, the image $f_*(P)M(P)$ has constant dimension 1. Because f is totally geodesic, the conclusion follows.

Example. If $f\colon M \to M'$ is a harmonic immersion, then $e(f) = n/2$, and $|\beta(f)|^2 + Q(f_*) = 0$. This relation also follows from Gauss's equations (Eisenhart [8, p. 162])

$$R_{ijkl} - R'_{\alpha\beta\gamma\delta} f_i{}^\alpha f_j{}^\beta f_k{}^\gamma f_l{}^\delta = \sum_{\sigma=1}^{m-n} (b_{\sigma|ik} b_{\sigma|jl} - b_{\sigma|il} b_{\sigma|jk})$$

by multiplying by $g^{ik}g^{jl}$ and summing. We obtain

$$R^{ij}g_{ij} + R'_{\alpha\beta\gamma\delta}f_i^{\alpha}f_j^{\beta}f_k^{\gamma}f_l^{\delta}g^{ik}g^{jl} = |\beta(f)|^2 - \sum_{\sigma=1}^{m-n}\mu_{\sigma}^2,$$

where $\mu_\sigma = g^{ij}b_{\sigma|ij}$ is the σ-th component of the mean normal curvature of the immersion (see (7)); each $\mu_\sigma = 0$ if f is harmonic. For instance, *if the Riemannian curvature of M' is non-positive and if the scalar curvature $R = R^{ij}g_{ij}$ of M is negative at some point, then there is no harmonic immersion of M in M'.*

4. Examples.

(A) *The case* $\dim M = 1$. Let us take for M the unit circle S^1, coordinatized by the central angle θ. For any $f \in (S^1, M')$ we have

$$e(f) = \tfrac{1}{2}\frac{df^\alpha}{dt}\frac{df^\beta}{dt}g'_{\alpha\beta}, \text{ whence } E(f) = \tfrac{1}{2}\int_M \left|\frac{df}{dt}\right|^2 *1.$$

The tension field is

$$\tau(f)^\gamma(t) = \frac{d^2f^\gamma}{dt^2} + \Gamma'_{\alpha\beta}{}^{\gamma}\frac{df^\alpha}{dt}\frac{df^\beta}{dt},$$

which (when the parameter of f is proportional to arc length) is often called *the curvature* (or *acceleration*) *of* f. We have $\langle\tau(f), \frac{df}{dt}\rangle = 0$, and $\tau(f)$ is proportional to the geodesic curvature vector field along f. Then f *is harmonic if and only if f defines a closed geodesic on M'.* It is well known that *if M' is compact, then in every homotopy class of maps $S^1 \to M'$ there is a harmonic map* (and furthermore, one which minimizes the *length* in that homotopy class.) On the other hand, without further restrictions on g' there are complete Riemannian manifolds M' and non-trivial homotopy classes of maps $S^1 \to M'$ having no harmonic representatives; see § 10 below.

(B) *The case* $\dim M = 2$. We established here certain relations showing the close connection of our problem with the Plateau problem, in its potential theoretic formulation (Morrey [19] and Bochner [2]); incidentally, we see that in our energy theory the cases $\dim M \leq 2$ are favored.

Recall that a map $h: M \to M'$ *is conformal* if there is a smooth function $\theta: M \to R$ such that

$$h^*g' = \exp(2\theta)g.$$

Thus the differential h_* preserves orthogonality and dilatates uniformly. Clearly such a map is a smooth immersion, and has energy density $e(h) = n\exp(2\theta)/2$.

PROPOSITION. *If* $\dim M = n = 2$ *and* $h\colon M \to M$ *is a conformal diffeomorphism, then for all* $f \in \mathfrak{H}(M, M')$ *we have* $E(f \circ h) = E(f)$. *Moreover,* h *is harmonic.*

Proof. First of all, $(f \circ h)_i{}^\alpha = f_p{}^\alpha h_i{}^p$, whence $2e(f \circ h) = g^{ij} h_i{}^p h_j{}^q f_p{}^\alpha f_q{}^\beta g'_{\alpha\beta}$. The conformaltity condition for h implies $g^{ij} h_i{}^p h_j{}^q = \exp(2\theta) g^{pq}$, and substituting gives

$$2e(f \circ h) = \exp(2\theta) g^{pq} f_p{}^\alpha f_q{}^\beta g'_{\alpha\beta} = 2\exp(2\theta) e(f).$$

Secondly, we have

$$\begin{aligned} h^*(*1) &= [\det(g_{pq} h_i{}^p h_j{}^q)/\det(g_{ij})]^{\frac{1}{2}} *1 \\ &= \exp(n\theta) *1, \end{aligned}$$

so that if $n = 2$ we have $h^*(e(f)*1) = e(f \circ h)*1$.

Finally, in suitable local coordinates on M we have

$$\Gamma_{ij}^k = \Gamma'{}_{ij}^k - (\delta_j^k \theta_i + \delta_i^k \theta_j - g^{kp} g_{ij} \theta_p),$$

and direct computation shows that

$$\tau(h)^i = (2 - n) g^{ij} \theta_j \quad . \qquad (1 \leqq i \leqq n).$$

PROPOSITION. *If* $\dim M = 2$, *then for any* $f \in \mathfrak{H}(M, M')$ *we have* $V(f) \leqq E(f)$. *Equality holds when and only when* f *is conformal.*

Proof. The first statement follows immediately from the inequalities (2). Suppose f is conformal; then for $n = 2$ we obtain

$$V(f) = \int_M \exp(2\theta) [g_{11} g_{22} - g_{12}{}^2]^{\frac{1}{2}} dx^1 dx^2 = \int_M e(f)*1 = E(f).$$

Conversely, if $V(f) = E(f)$ we conclude that at every point of M

$$2[\det(f^*g')_{ij}]^{\frac{1}{2}} = g^{ij} (f^*g')_{ij} [\det(g_{ij})]^{\frac{1}{2}}.$$

In isothermal coordinates on M we have

$$[(f^*g')_{11} - (f^*g')_{22}]^2 = -\{(f^*g')_{12}\}^2,$$

whence $(f^*g')_{11} - (f^*g')_{22} = 0 = (f^*g')_{12}$. Defining $\theta\colon M \to R$ by

$$\exp(2\theta) = [\det(f^*g')_{ij}/\det(g_{ij})]^{\frac{1}{2}},$$

we obtain $\exp(2\theta) g_{ij} = (f^*g')_{ij}$.

COROLLARY. *If a map* $f_0\colon M \to M'$ *minimizes* V *and is conformal, then* f_0 *minimizes* E.

For any $f \in \mathcal{H}(M, M')$ we have

$$E(f_0) = V(f_0) \leqq V(f) \leqq E(f).$$

Remark. If M is a Riemann surface, then its complex structure defines a conformal equivalence class of Riemann structures. The energy of any map $f\colon M \to M'$ therefore depends only on the complex structure of M.

(C) *Harmonic fibre maps.* For any map $f \in \mathcal{H}(M, M')$ and $P \in M$ we have the vector space $M_V(P) = \{u \in M(P) : f_*(P)u = 0\}$; the vectors in $M_V(P)$ are called *vertical.* Let $M_H(P)$ denote its orthogonal complement in $M(P)$. Suppose that for all $P \in M$ the differential $f_*(P)$ maps $M_H(P)$ isometrically onto $M'(f(P))$. Then f is a locally trivial surjective fibre map (see Hermann [13]); in particular, f determines an almost product structure on M (i. e., the structural group O_n of $\mathcal{J}(M)$ admits a reduction to the product group $O_m \times O_{n-m}$). We will call a map $f\colon M \to M'$ *a Riemannian fibration.*

Remark. There are smooth fibrations $f\colon M \to M'$ having no Lie structural group; e. g., there are non-trivial compact smooth fibrations over the 3-sphere S^3 (which cannot have a Lie structural group G, since $\pi_2(G) = 0$). It is a consequence of a theorem of Hermann [13] that the manifolds M, M' admit no Riemann structures compatible with f as above, for which the fibres are totally geodesic.

LEMMA. *If $f\colon M \to M'$ is a Riemannian fibration, then for any $P \in M$ there are coordinate charts U and U' centered at P and $f(P)$ respectively, in terms of which $f_i^\alpha(P) = \delta_i^\alpha$ $(1 \leqq i \leqq n, 1 \leqq \alpha \leqq m)$. Furthermore, the first m coordinates in U can be considered as normal coordinates in U', and the last $n - m$ coordinates are local coordinates for the fibres.*

Proof. Let $(e_i)_{1 \leqq i \leqq n}$ be an orthonormal base for $M(P)$ such that the first m vectors span $M_H(P)$, and the last $n - m$ vectors span $M_V(P)$. Then $f_*(P)e_i = e'_i$ $(1 \leqq i \leqq m)$ form an orthonormal base for $M'(f(P))$, and we can construct the associated normal coordinates in some neighborhood U'. According to Hermann [13] the unique horizontal lift to P of any geodesic of M' starting at $f(P)$ is a geodesic of M; one determined by e'_i lifts to one determined by e_i. We now use the local product structure to define a coordinate chart U in which the fibres have the desired property. Note that (unless the fibres are totally geodesic) we cannot generally require that the coordinates in U be normal.

PROPOSITION. *Let $f\colon M \to M'$ be a Riemannian fibration. Then* $e(f)$

$= n/2$. *If for any* $P \in M$ *we let* F_P *denote the fibre through* P *and* $i_P\colon F_P \to M$ *the inclusion map, then*

$$\tau(f)(Q) = -f_*(Q)\tau(i_P)(Q) \quad \textit{for all } Q \in F_P.$$

In particular, f *is harmonic if and only if all fibres are minimal submanifolds.*

Proof. In § 2*D* we have seen that the tension field of any Riemannian immersion is perpendicular to the submanifold. Thus for every $Q \in F_P$ we have $\tau(i_P)(Q) \in M_H(Q)$. If we use a split coordinate system as in the previous lemma, we see that

$$f_*\tau(i_P)(P)^\gamma = \sum_{k=1}^{m} \delta_k{}^\gamma \tau(i_P)^k(P).$$

The proposition follows by direct calculation.

Examples. All covering maps are harmonic; in particular, the identity map is harmonic. which amounts to saying that Cartan's *vitesse* is a harmonic 1-form. If $\mathcal{F}^r(M) \to M$ is the bundle of orthonormal r-frames of M, then it is known (Lichnerowicz [15]) that with its natural Riemannian structure on $\mathcal{F}^r(M)$ the fibres are minimal, whence that fibre map is harmonic. Vector bundle maps are harmonic. Every homogeneous Riemannian fibration is harmonic, for the fibres are always totally geodesic, and therefore minimal.

(D) *Maps into flat manifolds.* Let us take for M' the unit circle S^1; *we will construct harmonic representatives in every homotopy class of maps* $M \to S'$. First of all, it is well known that the set $[M, S^1]$ of these homotopy classes forms an abelian group canonically isomorphic to the first integral cohomology group $H^1(M)$. Secondly, every such cohomology class is canonically represented by a harmonic 1-form on M.

Suppose M is connected, and fix a point $P_0 \in M$. Given any such harmonic 1-form ω on M and any smooth path γ_P from P_0 to a point $P \in M$, we define the number

$$f(P) = \int_{\gamma_P} \omega.$$

A different choice $\tilde{\gamma}_P$ of γ_P may give a different number $\tilde{f}(P)$, but

$$\tilde{f}(P) - f(P) = \int_{\tilde{\gamma}_P - \gamma_P} \omega$$

is an integer since the periods of ω are integral; consequently, ω determines a well defined map $f_\omega\colon M \to S^1$ by letting $f_\omega(P)$ be the residue class modulo 1 of $f(P)$.

Now since f_ω is harmonic, every $P \in M$ has a neighborhood U in which $df = \omega$; thus $\Delta f = \delta df + d\delta f = \delta\omega = 0$ in U. It is easy to see that $\omega \to f_\omega$ establishes an isomorphism $[M, S^1] = H^1(M)$.

To define *harmonic representatives of the homotopy classes of maps* $M \to T^m$, *the flat m-torus,* we merely take m-fold products of harmonic maps $M \to S^1$, using § 5C below; the existence of these harmonic representatives was first proved by F. B. Fuller [10]. More generally, any compact flat manifold M' is covered by T^m, by a theorem of Bieberbach, and any homotopy class of maps $M \to M'$ which can be lifted to maps $M \to T^m$ has harmonic representatives obtained by composition with the projection $T^m \to M'$.

If M and M' are both flat, then the only harmonic maps $M \to M'$ are those which are locally linear, as can be seen from the maximum principle.

(E) *Maps of Euclidean spheres.* Let $S^n(r)$ denote the Euclidean n-sphere of radius r; write $S^n = S^n(1)$. Then the homotopy classes of maps of S^n into itself are classified by their degrees. We consider now the problem of constructing explicitly harmonic maps of a given degree k; it is a simple matter to modify the following remarks to include the case of maps $S^n(r) \to S^n(r')$.

If $(x^1, \cdots, x^{n+1})$ are Euclidean coordinates and $\eta\colon S^n(r) \to S^n(r')$ is a map given by $\eta(x^1, \cdots, x^{n+1}) = (x^1 r'/r, \cdots, x^{n+1} r'/r)$, then

$$e(\eta) = n/2\,(r'/r)^2.$$

We will henceforth refer points of S^n to coordinates (θ, ϕ), where θ denotes colatitude $(0 \leqq \theta \leqq \pi)$, and ϕ a point on the equator S^{n-1} of S^n. Furthermore, corresponding to the integer k let $\Phi\colon S^{n-1}(r) \to S^{n-1}(r')$ be the $(n-2)$-fold suspension of the map $S^1(r) \to S^1(r')$ defined by $\phi \to (\cos k\phi, \sin k\phi)$; since the degree is invariant under suspension, Φ has degree k. An elementary calculation gives

$$e(\Phi) = \frac{(k^2 + n - 2)}{2} (r'/r)^2. \tag{17}$$

We are interested in maps $f\colon S^n \to S^n$ of the form $(\theta, \phi) \to (\Theta, \Phi)$, where Θ is a function of θ alone, and Φ is defined for $r = r' = 1$. From (17) we see that on $S^{n-1}(\sin\theta)$ we have $2e(f) = (k^2 + n - 2)\sin^2\Theta/\sin^2\theta$, whence

$$\begin{aligned} E(f) &= \tfrac{1}{2}\int_0^\pi \int_{S^{n-1}(\sin\theta)} \left[\left(\frac{d\Theta}{d\theta}\right)^2 + (k^2 + n - 2)\frac{\sin^2\Theta}{\sin^2\theta}\right] ds^{n-1} d\theta. \\ &= \frac{V(S^{n-1})}{2}\int_0^\pi \sin^{n-1}\theta \left[\left(\frac{d\Theta}{d\theta}\right)^2 + (k^2 + n - 2)\frac{\sin^2\Theta}{\sin^2\theta}\right] d\theta. \end{aligned} \tag{18}$$

In case $n=1$ the existence and properties of harmonic maps of degree k is elementary.

We now consider the case $n=2$. We have $g_{11}=1$, $g_{12}=0=g_{21}$, $g_{22}=\sin^2\theta$. The tension field of any map f is given by its components

$$\tau(\Theta)=\Delta\Theta-\sin\Theta\cos\Theta\left[\frac{1}{\sin^2\theta}\left(\frac{\partial\Phi}{\partial\phi}\right)^2+\left(\frac{\partial\Phi}{\partial\theta}\right)^2\right],$$

$$\tau(\Phi)=\Delta\Phi+2\cot\Theta\left[\frac{1}{\sin^2\theta}\frac{\partial\Phi}{\partial\phi}\frac{\partial\Theta}{\partial\phi}+\frac{\partial\Phi}{\partial\theta}\frac{\partial\Theta}{\partial\theta}\right].$$

For the special maps under consideration we find that for a harmonic map f $\tau(\Phi)=\Delta\Phi=d^2\Phi/d\phi^2=0$, which means that the choice $\Phi=k\phi$ is compatible with our above selection. Substituting in the first equation we find

$$\tau(\Theta)=\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-k^2\frac{\sin\Theta\cos\Theta}{\sin^2\theta};$$

the only solution of $\tau(\Theta)=0$ regular at the poles is

$$\Theta=2\arctan\left[c(\tan\theta/2)^{\pm k}\right]\text{ with } c>0. \tag{19}$$

Then $d\Theta/d\theta=\pm k\sin\Theta/\sin\theta$, from which we can conclude that $E(f)=4\pi\,|\,k\,|$; note that it is independent of c. Finally, the integral formula for the degree of a map shows that degree $f=\pm k$. We observe that for $k=0$ and all $c>0$ the map is constant; for $k=\pm1$ and $c=1$ the map is the identity and the antipodal map, respectively.

Remark. Although the above construction does exhibit a harmonic map in every homotopy class, it does not begin to exhaust their topological interest. For instance, with the uniform topology on $\mathcal{H}(S^2,S^2)$, the component $\mathcal{H}_0(S^2,S^2)$ of those maps of degree 0 has infinite cyclic homology:

$$H_1(\mathcal{H}_0(S^2,S^2))=\pi_1(\mathcal{H}_0(S^2,S^2))=\pi_3(S^2)=Z,$$

generated by the Hopf map. A harmonic representative of that generator should have positive Morse index.

Consider now the case $n\geqq3$. We do not know how to construct harmonic maps of degree $\geqq2$. Incidentally, the suspensions of the above maps (and their "compressed" suspensions below) are generally not harmonic. We propose now to show that *the functional $E\colon\mathcal{H}(S^n,S^n)\to R$ does not have an absolute minimum on the component $\mathcal{H}_k(S^n,S^n)$ for $k\neq0$ and $n\geqq3$*; that this phenonmenon could be illustrated explicitly and simply by maps of spheres was suggested to us by C. B. Morrey.

Let $f_c(\theta,\phi)=(\Theta_c(\theta),\Phi)$ be the map of $S^n\to S^n$ defined using (19) with exponent $k \neq 0$. Thus for small $c > 0$ the map f_c compresses most of S^n into a small cell centered at the pole, and that compression takes place along longitudes.

The energy integral (18) now takes the form

$$E(f_c) = \frac{V(S^{n-1})(2k^2+n-2)}{2} \int_0^\pi \sin^{n-1}\theta \cdot \left(\frac{\sin \Theta_c(\theta)}{\sin\theta}\right)^2 d\theta.$$

For $n \geqq 3$ we have

$$E(f_c) \leqq \frac{V(S^{n-1})(2k^2+n-2)}{2} \int_0^\pi \sin^2 \Theta_c(\theta)\, d\theta.$$

The following lemma shows that $\lim_{c\to 0} E(f_c) = 0$; i.e., for $n \geqq 3$ there are maps in $\mathcal{H}_k(S^n, S^n)$ of arbitrarily small energy. But for $k \neq 0$ there is no map $f \in \mathcal{H}_k(S^n, S^n)$ with zero energy, for such an f is constant, and therefore has degree 0.

LEMMA. $\int_0^\pi \sin^2 \Theta_c(\theta)\, d\theta \to 0$ *as* $c \to 0$.

Proof. For any $\epsilon > 0$ let $\rho = \pi - \epsilon/2$. There is a number K such that $0 \leqq (\tan\theta/2) \leqq K$ for all $0 \leqq \theta \leqq \rho$. It follows that $\Theta_c(\theta) \leqq 2 \arctan(cK)$, whence there is a number $c_\epsilon > 0$ for which $0 \leqq \sin^2 \Theta_c(\theta) < \epsilon/2\rho$ if $0 < c \leqq c_\epsilon$. Thus

$$\int_0^\rho \sin^2 \Theta_c(\theta)\, d\theta + \int_\rho^\pi \sin^2 \Theta_c(\theta)\, d\theta \leqq \int_0^\rho \epsilon/2\rho\, d\theta + \int_\rho^\pi 1\, d\theta \leqq \epsilon.$$

5. The composition of maps.

(A) The following computation is elementary.

LEMMA. *If $f: M \to M'$ and $f': M' \to M''$ are any smooth maps, then their fundamental forms satisfy*

(20) $$(f' \circ f)_{;ij}{}^a = f_{;ij}{}^\gamma f'_\gamma{}^a + f'_{;\alpha\beta}{}^a f_i{}^\alpha f_j{}^\beta.$$

COROLLARY. *The composition of totally geodesic maps is totally geodesic. The inverse of a totally geodesic diffeomorphism is totally geodesic.*

COROLLARY.

(21) $$\tau(f' \circ f)^a = \tau(f)^\gamma f'_\gamma{}^a + g^{ij} f'_{;\alpha\beta}{}^a f_i{}^\alpha f_j{}^\beta.$$

If f is harmonic and f' totally geodesic, then $f' \circ f$ is harmonic.

In general, however, we do not expect the composition of harmonic maps to be harmonic, as the following example shows:

Example. Let T^2 be the flat 2-torus parametrized by the angles (θ, ϕ) with $0 \leqq \theta, \phi, < 2\pi$. Let $f'\colon T^2 \to S^3$ be defined by

$$f'(\theta, \phi) = (\cos\theta, \sin\theta, \cos\phi, \sin\phi)/\sqrt{2},$$

considered as a point in R^4. Then f' *defines a Riemannian imbedding of* T^2 *in* S^3, *which is a minimal but not a totally geodesic imbedding.* To see that f' is harmonic we show that $\tau(f')$ is perpendicular to $S^3(f'(P))$ in R^4, and then appeal to Proposition 5B below. Namely, because T^2 is flat we have

$$\tau(f')^a = \frac{\partial^2 f'^a}{\partial\theta^2} + \frac{\partial^2 f'^a}{\partial\phi^2} = -f'^a \qquad (1 \leqq a \leqq 4)$$

whence $\tau(f')(P)$ is directed along the radius of S^3 at $f(P)$. On the other hand, T^2 is not totally geodesic in S^3, for the map $f\colon S^1 \to T^2$ defined by $f(0) = (\theta, 0)$ is a geodesic of T^2; it does not lie in any 2-plane through 0 in R^4, and is therefore not a geodesic of S^3. In particular, $f\colon S^1 \to T^2$ *and* $f'\colon T^2 \to S^3$ *are both harmonic maps, and their composition* $f' \circ f$ *is not.*

(B) PROPOSITION. *If* $f'\colon M' \to M''$ *is a Riemannian immersion, then for any map* $f\colon M \to M'$ *we have* $E(f) = E(f' \circ f)$. *The tension field* $\tau(f)$ *is the projection on* M' *of the tension field* $\tau(f' \circ f)$.

Proof. The first statement follows from the equation

$$e(f' \circ f) = \tfrac{1}{2}\langle g, f^* \circ f'^* g''\rangle = e(f).$$

The second statement is a consequence of (21); for if f' is a Riemannian immersion, then the right member is the decomposition of $\tau(f' \circ f)$ into horizontal and vertical components because $(f'_{;\alpha\beta}{}^a)$ is the second fundamental form of f'.

COROLLARY. *A map* $f\colon M \to M'$ *is harmonic if and only if* $\tau(f' \circ f)$ *is perpendicular to* $M'(f(P))$ *for all* $P \in M$.

This generalizes the classical fact that a curve in M' is a geodesic if and only if its curvature vector in M'' is always perpendicular to M'.

(C) PROPOSITION. *Let* $f'\colon M' \to M''$ *be a Riemannian fibration with totally geodesic fibres. Then for any map* $f\colon M \to M'$ *we have*

$$\tau(f' \circ f) = f'_*(\tau(f)).$$

This is immediate, because we can in the present situation take split normal coordinates in Lemma 4C.

COROLLARY. *If* $M = M''$ *and* $f\colon M \to M'$ *is a section, then* $f' \circ f = 1$ *is harmonic, and therefore* $\tau(f)$ *is always vertical.*

Example. If we view a smooth r-form ω on M' as a section of the bundle $\mathcal{J}^{[r]}(M')$ of r-covectors of M', then the condition that ω be harmonic in the sense of de Rham-Hodge is generally different from the condition $\tau(\omega) = 0$, using (say) the Riemannian structure on $\mathcal{J}^{[r]}(M')$ of Sasaki [25]. However, these two concepts do coincide if M' is flat.

Example. A map $f\colon M \to M' \times M''$ into a Riemannian product has a canonical decomposition $f(P) = (f'(P), f''(P))$ for all $P \in M$. Then f is harmonic if and only if both components f', f'' are harmonic. For instance, seee Proposition 2E.

(D) Let us suppose that M' is a Riemannian submanifold of M'' and that the imbedding is proper; i. e., such that the inverse image of any compact subset of M'' is compact in M'. Since M' is complete, there is a positive smooth function $\rho\colon M' \to R$ such that for any $P' \in M'$ the set

$$\{P'' \in M''\colon r'(P', P'') \leqq \rho(P')\}$$

is geodesically convex in M''; if M' is compact, then of course we can suppose that ρ is a positive constant. For each $P' \in M'$ let $D_{P'}$ denote the closed ball of dimension $q - m$ ($q = \dim M''$) consisting of all geodesic segments of length $\leqq \rho(P')$ emanating from P' and perpendicular to $M'(P')$. The following result is well known and elementary.

LEMMA. *Let* $N = \cup \{D_{P'}\colon P' \in M'\}$; *then* N *is a neighborhood of* M' *in* M'', *and the obvious map* $\pi\colon N \to M'$ *defines a smooth fibre bundle over* M' *whose structure group is* O_{q-m} *and whose fibres are closed balls.*

Taking into account Proposition 5C we have the

PROPOSITION. *Let* $f'\colon M' \to M''$ *be a proper Riemannian imbedding and* $\pi\colon N \to M'$ *a normal tubular neighborhood. Then for any map* $f\colon M \to N$ *the composition* $\pi \circ f$ *is harmonic if and only if* $\tau(f)$ *is vertical.*

Chapter II. Deformations of Maps.

6. Deformations by the heat equation.

(A) This chapter is devoted to *the fundamental problem of deforming a given map into a harmonic map*; i. e., into a smooth map $f\colon M \to M'$ satisfying the nonlinear elliptic equation

$$\tau(f) = 0. \tag{1}$$

We begin by discussing general methods of attack.

The interpretation given in § 2A of the tension field $\tau(f)$ as the contravariant representative of the differential of E at f suggests that we try to invent gradient lines of E in a suitable function space of maps from M to M', and then to prove that these trajectories lead to critical points of E. We propose the following method for realizing such an attempt, which we now outline briefly. We do not pursue this method in the present paper, although the qualitative results are essentially those of the following sections.

Let $\mathfrak{H}^r(M,M')$ denote the function space of all maps from M to M' whose partial derivatives (relative to fixed coordinate coverings) of orders $\leqq r$ are square integrable. An inequality of Sobolev insures that if $2r > \dim M$ then the maps in $\mathfrak{H}^r(M,M')$ are continuous, and its topology is larger than the uniform topology. It can be shown that the space $\mathfrak{H}^r(M,M')$ admits an infinite dimensional Riemannian manifold structure modeled on a separable Hilbert space, and that $E\colon \mathfrak{H}^r(M,M') \to R$ is a differentiable function. If τ^r is its gradient field on $\mathfrak{H}^r(M,M')$; i.e., $\nabla_v E(f) = -\langle \tau^r(f), v\rangle$ for all vectors v in the tangent space at f, then the ordinary differential equation

$$\frac{df_t}{dt} = \tau^r(f_t)$$

has a local solution which is unique; furthermore, $E(f_t)$ is a decreasing function of t. Under suitable curvature restrictions the solutions are globally defined. If each trajectory f_t is relatively compact, then it has a limit point a harmonic map. Thus these trajectories define a canonical homotopy of the initial map onto a harmonic map; moreover, such trajectories enjoy the 1-parameter group property. We observe that the critical points of E are just the zeros of all τ^r (for any $r > \dim M/2$).

Now the function space $\mathfrak{H}^0(M,M')$ is *not* a manifold, although with every map f we have the Hilbert space $\mathfrak{H}^0(f)$ of vector fields along f defined in § 2A. In particular, $\tau^0(f) = \tau(f)$ is in $\mathfrak{H}^0(f)$. In analogy with the above outline we are led to consider the nonlinear parabolic equation

$$\frac{\partial f_t}{\partial t} = \tau(f_t) \qquad (t_0 < t < t_1). \tag{2}$$

The study of this equation is our primary object in the following sections. We will find that the properties of the trajectories of (2) include most of those mentioned as belonging to the trajectories of τ^r $(2r > \dim M)$. (There is one basic difference: The solutions of (2) are generally defined only for non-negative time, whereas the trajectories of τ^r are always defined for an open time interval around $t=0$.)

Remark. In the calculus of variations a standard method (Morrey [19]) of establishing the existence of a minimum of E for a given class of maps is to take the space $\mathcal{H}^1(M,M')$ and to introduce on it a weak topology relative to which 1) E is lower semi-continuous, and 2) there are sufficiently many compact sets. That approach works well for $\dim M=1$ or 2; however, the example given in § 4E shows that it will not work in general for $n \geqq 3$.

(B) PROPOSITION. *If $(t,P)\rightarrow f_t(P)$ is a map of $(t_0,t_1)\times M\rightarrow M'$ which is C^1 on the product manifold and C^2 on M for each t, and if that map satisfies (2), then it is C^∞.*

We will refer to such an f_t as a *solution* of (2).

This follows from Friedman [9, Th. 4 and 5] provided the second derivatives $f_{ij}{}^\alpha$ of the f^α are Hölder-continuous. But we can represent the local functions f^α by Green's formula, using the fundamental solution of the heat equation $\Delta u - \partial u/\partial t = 0$ as in §§ 9-10 below. The required Hölder-continuity is then established by standard techniques from the properties of the potentials involved (Pogorzelski [24], Dressel [7], Gevrey [11, No. 8]).

(C) Let $f_t\colon M\rightarrow M'$ satisfy (2); the subscript t refers to the deformation parameter (we will always indicate explicitly differentiation with respect to t). Then from (4) of § 2B we have

$$\frac{dE(f_t)}{dt} = -\int_M \langle \tau(f_t), \frac{\partial f_t}{\partial t}\rangle * 1 = -\int_M \left|\frac{\partial f_t}{\partial t}\right|^2 * 1.$$

If D/dt denotes covariant differentiation along paths in M', then for each $P\in M$ the curvature vector of the path $t\rightarrow f_t(P)$ is given by $D(\partial f_t/\partial t)/dt$.

PROPOSITION. *If $f_t\colon M\rightarrow M'$ satisfies (2), then the energy $E(f_t)$ is a strictly decreasing function; i.e., $dE(f_t)/dt<0$ except for those values of t for which $\tau(f_t)=0$. Furthermore, its second derivative expresses the average angle between the tension field and the curvature vectors of the deformation paths:*

$$\frac{d^2E(f_t)}{dt^2} = -2\int_M \langle \frac{D}{dt}\left(\frac{\partial f_t}{\partial t}\right), \tau(f_t)\rangle * 1.$$

LEMMA. *Let $f_t\colon M\rightarrow M'$ be an arbitrary deformation for $t\in(t_0,t_1)$. If we let*

$$\frac{D}{\partial x^i}\left(\frac{\partial f_t}{\partial t}\right)^\alpha = \frac{\partial^2 f_t{}^\alpha}{\partial x^i\partial t} + \Gamma'_{\mu\nu}{}^\alpha \frac{\partial f_t{}^\mu}{\partial x^i}\frac{\partial f_t{}^\nu}{\partial t},$$

then

$$\frac{d^2E(f_t)}{dt^2}=\int_M g^{ij}\langle \frac{D}{\partial x^j}\left(\frac{\partial f_t}{\partial t}\right), \frac{D}{\partial x^j}\left(\frac{\partial f_t}{\partial t}\right)\rangle *1$$

$$-\int_M g^{ij}R'_{\alpha\mu\beta\nu}f_{ti}{}^{\alpha}f_{tj}{}^{\beta}\frac{\partial f_t{}^{\mu}}{\partial t}\frac{\partial f_t{}^{\nu}}{\partial t}*1-\int_M\langle\frac{D}{\partial t}\left(\frac{\partial f_t}{\partial t}\right),\tau(f_t)\rangle *1.$$

This follows from a direct calculation of $\partial^2 e(f_t)/\partial t^2$ and an application of Green's divergence theorem.

Because the first integral of the right member is always non-negative, we have on appeal to the above proposition the

THEOREM. *If $f_t\colon M\to M'$ satisfies the heat equation (2), then*

$$\frac{1}{2}\frac{d^2E(f_t)}{dt^2}=\int_M g^{ij}\langle\frac{D}{\partial x^i}\left(\frac{\partial f_t}{\partial t}\right),\frac{D}{\partial x^j}\left(\frac{\partial f_t}{\partial t}\right)\rangle *1$$

(3)

$$-\int_M g^{ij}R'_{\alpha\mu\beta\nu}f_{ti}{}^{\alpha}f_{tj}{}^{\beta}\frac{\partial f_t{}^{\mu}}{\partial t}\frac{\partial f_t{}^{\nu}}{\partial t}*1.$$

In particular, if M' has non-positive sectional curvature, then $d^2E(f_t)/dt^2\geqq 0$. If t is a value for which equality holds, then $\tau(f_t)$ is a covariant constant; i. e.,

$$\frac{D}{\partial x^i}\left(\frac{\partial f_t(P)}{\partial t}\right)=0 \text{ for all } P\in M \text{ and } (1\leqq i\leqq n).$$

COROLLARY. *If M' has non-positive sectional curvature and if f_t satisfies (2) for all $t\geqq t_0$, then*

$$\frac{dE(f_t)}{dt}\to 0 \text{ as } t\to\infty.$$

(D) We have seen in § 5 that in supposing M' contained in a larger manifold M'' we do not alter the energy of a map $f\colon M\to M'$. That suggests that we still have control of the energy and the tension of deformations of f which take place in a normal tubular neighborhood N of M' in M''.

If $f'\colon M'\to M''$ denotes the imbedding, then the induced tangent vector bundle is $f'^{-1}\mathcal{T}(M'')=\mathcal{T}(M')\oplus\mathcal{N}(M'',M')$, where the second summand is the normal bundle of M' in M''. Then $\mathcal{N}\to M'$ is Riemannian-connected in the sense of § 3A. There is a canonical vector field $\rho\colon N\to\mathcal{N}$ covering the projection map $\pi\colon N\to M'$ defined by assigning to each $Q''\in N$ the unique vector $\rho(Q'')\in M''(\pi(Q''))$ such that $\exp_{\pi(Q'')}(\rho(Q''))=Q''$.

Suppose now that $f_t\colon M\to N$ is a smooth deformation $(t_0\leqq t<t_1)$. Then $\rho\circ f_t$ is a vector field on the map $\pi\circ f_t$, and using the harmonic integral

theory of § 3 applied to the vector bundle $\mathfrak{N} \to M'$, we can define its Laplacean $\tilde{\Delta}(\rho \circ f_t)$. Let

$$\tilde{L}(\rho \circ f_t) = \tilde{\Delta}(\rho \circ f_t) - \frac{\tilde{D}}{\partial t}(\rho \circ f_t),$$

where $\tilde{D}/\partial t$ is the covariant derivative in N along the path $\pi \circ f_t$.

We now establish the following stability property of deformations.

THEOREM. *Let N be a normal tubular neighborhood of M' in M'', and suppose that $f_t\colon M \to N$ is a smooth deformation $(t_0 \leqq t < t_1)$. If $\tilde{L}(\rho \circ f_t)$ is always horizontal in $\mathcal{J}(M'')$ and if $f_{t_0}\colon M \to M'$, then $f_t\colon M \to M'$ for all $t_0 \leqq t < t_1$.*

Proof. We apply Green's theorem to $u, v \in A^0(M, (\pi \circ f_t)^{-1}\mathfrak{N})$:

$$\int_M \langle u, \tilde{\Delta} v\rangle *1 = -\int_M \langle \tilde{d}u, \tilde{d}v\rangle *1.$$

The hypotheses imply that $\langle \rho \circ f_t, \tilde{L}(\rho \circ f_t)\rangle = 0$ for all t, so that

$$0 = \int_M \langle \rho \circ f_t, \tilde{\Delta}(\rho \circ f_t)\rangle *1 - \int_M \langle \rho \circ f_t, \frac{\tilde{D}}{\partial t}(\rho \circ f_t)\rangle *1.$$

Therefore

$$\tfrac{1}{2}\frac{d}{dt}\int_M \langle \rho \circ f_t, \rho \circ f_t\rangle *1 = \int_M \langle \rho \circ f_t, \tilde{\Delta}(\rho \circ f_t)\rangle *1$$

$$= -\int_M |\tilde{d}(\rho \circ f_t)|^2 *1 \leqq 0.$$

I. e., $\int_M |\rho \circ f_t|^2 *1$ is a non-negative, non-increasing function of t, and it is zero for $t = t_0$. We conclude that $\rho \circ f_t = 0$ for all $t_0 \leqq t < t_1$, which states precisely that every f_t maps M into M'.

7. Global equations.

(A) We now occupy ourselves with the problem of replacing equations (1) and (2), which in terms of local coordinates on M and M' are local systems of equations, by some much more tractable global systems.

Remark. Assume for the moment that we have an isometric imbedding $w\colon M' \to \boldsymbol{R}^q$ for some q, which we can always do by a theorem of Nash [21]. Then as in Proposition 5B we find that equation (2) is satisfied for a deformation $f_t\colon M \to M'$ when and only when the composition $W_t = w \circ f_t$ satisfies the condition that the vectors $LW_t = \Delta W_t - \dfrac{\partial W_t}{\partial t}$ are perpendicular to M'; see

Lemma 7B below. When expressed in terms of the coordinates of $\boldsymbol{R}^q$, that condition gives rise to a global parabolic system of equations of the type $(\tilde{2})$. On the other hand, the assumption of an isometric imbedding apparently affords no real simplification in our theory, and we will not make it. We proceed with an elementary imbedding convenient for our purposes.

Suppose that M' is smoothly and properly imbedded in some Euclidean space $\boldsymbol{R}^q$ by a map $w\colon M' \to \boldsymbol{R}^q$.

LEMMA. *Given such an imbedding of M' in $\boldsymbol{R}^q$, it is always possible to construct a smooth Riemannian metric $g'' = (g''_{ab})_{1 \leqq a, b \leqq q}$ on a tubular neighborhood N of M' so that N is Riemannian fibred.*

Proof. Let N be any tubular neighborhood of M' constructed using the Euclidean structure of $\boldsymbol{R}^q$; let $\pi\colon N \to M'$ be the projection map. It suffices to construct an appropriate smooth inner product in each space $\boldsymbol{R}^q(P')$ for all $P' \in M'$, for we can translate that tangent space to any point $Q' \in N$ along the straight line segment (necessarily contained in N) from $P' = \pi(Q')$ to Q'. We take g' in $M'(P')$ and the induced Euclidean metric in the (Euclidean) orthogonal complement of $M'(P')$ in $\boldsymbol{R}^q(P')$, and take their sum in $\boldsymbol{R}^q(P')$.

In terms of local coordinates $(y^1, \cdots, y^m)$ on M' that metric can be described as follows: Write $w(P') = (w^1(P'), \cdots, w^q(P'))$; then

$$\sum_{c=1}^{q} \frac{\partial w^c}{\partial y^\alpha} \frac{\partial w^c}{\partial y^\beta}$$

is the metric on M' induced from the Euclidean metric. Let $t_a{}^\alpha$ be the unique solution of

$$\sum_{c,\alpha} \frac{\partial w^c}{\partial y^\alpha} \frac{\partial w^c}{\partial y^\beta} t_a{}^\alpha = \frac{\partial w^a}{\partial y^\beta}.$$

We have the duality relation $\dfrac{\partial w^a}{\partial y^\alpha} t_a{}^\beta = \delta_\alpha{}^\beta$, and the metric tensor is

$$g''_{ab} = \delta_{ab} - \sum_{c=1}^{q} \frac{\partial w^c}{\partial y^\alpha} \frac{\partial w^c}{\partial y^\beta} t_a{}^\alpha t_b{}^\beta + g'_{\alpha\beta} t_a{}^\alpha t_b{}^\beta.$$

Then

$$g''_{ab} \frac{\partial w^a}{\partial y^\alpha} \frac{\partial w^b}{\partial y^\beta} = g'_{\alpha\beta},$$

so that g'' does induce g' on M'. For any vector $v \in \boldsymbol{R}^q(P')$ satisfying $\sum_{a=1}^{q} v^a \dfrac{\partial w^a}{\partial y^\alpha} = 0$ $(1 \leqq \alpha \leqq m)$ we have $\sum_{a=1}^{q} v^a t_a{}^\alpha = 0$ also, so that

$$g''_{ab} v^a \frac{\partial w^b}{\partial y^\alpha} = 0 \text{ and } g''_{ab} v^a v^b = \sum_a (v^a)^2.$$

Thus g'' has the desired properties.

(B) LEMMA. *Let $f_t\colon M\to M'$ be any smooth deformation, and let $W_t\colon M\to N$ be the composition $w\circ f_t = W_t$. Then $\tau(f_t) = \partial f_t/\partial t$ if and only if $\tau(W_t) - \partial W_t/\partial t$ is perpendicular to $M'(f_t(P))$ for all $P\in M$.*

Proof. The argument is essentially that of Proposition 5B; we choose local coordinates on M' and obtain

$$g'_{\gamma\lambda}(\tau(f_t)^\lambda - \frac{\partial f_t{}^\lambda}{\partial t}) = g''_{cd}\frac{\partial w^c}{\partial y^\gamma}\,\tau(W_t)^d - g'_{\gamma\lambda}\frac{\partial f_t{}^\lambda}{\partial t}$$

$$= g''_{cd}\frac{\partial w^c}{\partial y^\gamma}\,(\tau(W_t)^d - \frac{\partial W^d}{\partial t}).$$

In terms of the coordinates of R^q the differential of the projection map $\pi\colon N\to M'$ has components $\pi_a{}^c = \partial\pi^c/\partial w^a$. Its covariant differential in terms of the metric g'' is

$$\pi_{ab}{}^c = \frac{\partial^2\pi^c}{\partial w^a\partial w^b} - \Gamma''_{ab}{}^d\pi_d{}^c \qquad (1\leqq a,b,c\leqq q).$$

The map $\rho\colon N\to \boldsymbol{R}^q$ defined (as in § 6D) by $\rho(Q') = Q' - \pi(Q')$ assigns to each $Q'\in N$ a vector perpendicular to $M'(\pi(Q'))$, and $\rho_a{}^c + \pi_a{}^c = \delta_a{}^c$,

$$\rho_{ab}{}^c + \pi_{ab}{}^c + \Gamma''_{ab}{}^c = 0.$$

For its restriction to M' we obtain

$$-\rho_{ab}{}^c w^a w^b = \frac{\partial^2\pi^c}{\partial w^a\partial w^b}\; w_\alpha{}^a w_\beta{}^b = \rho_b{}^c\,\frac{\partial^2\pi^b}{\partial y^\alpha\partial y^\beta}\;,$$

where $w_\alpha{}^a = \partial w^a/\partial y^\alpha$. For each α, β the right member defines a vector perpendicular to M'; because the $w_\alpha{}^a$ span the tangent space to M' in which they lie, we obtain the

LEMMA. *For any vectors $u, v\in M'(Q')$ the vector whose components are $\rho_{ab}{}^c u^a v^b$ is perpendicular to $M'(Q')$.*

LEMMA. *For any map $W_t\colon M\to N$ with image in M' let ξ be the vector with components*

$$\xi^c = LW_t{}^c - \pi_{ab}{}^c W_{ti}{}^a W_{tj}{}^b g^{ij},$$

where $L = \Delta - \partial/\partial t$ is the heat operator on M. Then ξ is tangent to M'.

Proof. As in Proposition 5B we have

$$L(\rho(W_t))^d = \rho_c{}^d L(W_t)^c + \frac{\partial^2\rho^d}{\partial w^a\partial w^b}\,W_{ti}{}^a W_{tj}{}^b g^{ij}$$

(5)

$$= \rho_c{}^d L(W_t)^c - \frac{\partial^2 \pi^d}{\partial w^a \partial w^b} W_{ti}{}^a W_{tj}{}^b g^{ij}$$

But $\rho(W_t) \equiv 0$, whence the left member vanishes. Since the second term in the right member is a normal vector, we have

$$\rho_c{}^d L(W_t)^c = \rho_e{}^d \frac{\partial^2 \pi^e}{\partial w^a \partial w^b} W_{ti}{}^a W_{tj}{}^b g^{ij}.$$

It follows that

$$\rho_c{}^d \xi^c = (\rho_e{}^d \frac{\partial^2 \pi^e}{\partial w^a \partial w^b} - \rho_e{}^d \pi_{ab}{}^e) W_{ti}{}^a W_{tj}{}^b g^{ij} = 0$$

for $1 \leqq d \leqq q$. Thus ξ has normal component 0.

PROPOSITION. *A map $f\colon M \to M'$ satisfies* (1) *if and only if the composition $W = w \circ f$ satisfies*

$(\tilde{1})$ $$\Delta W^c = \pi_{ab}{}^c W_i{}^a W_j{}^b g^{ij}$$

in terms of local coordinates on M.

A deformation $f_t\colon M \to M'$ $(t_0 < t < t_1)$ satisfies (2) *if and only if $W_t = w \circ f_t$ satisfies*

$(\tilde{2})$ $$L(W_t)^c = \pi_{ab}{}^c W_{ti}{}^a W_{tj}{}^b g^{ij} \qquad (t_0 < t < t_1).$$

Proof. It suffices to establish the equivalence of (2) and $(\tilde{2})$. For that we take any deformation f_t and compute

$$\begin{aligned}(\tau(W_t)^c - \frac{\partial W_t{}^c}{\partial t}) g''_{cd} w_\gamma{}^d &= (L(W_t)^c + \Gamma''_{ab}{}^c W_{ti}{}^a W_{tj}{}^b g^{ij}) g''_{cd} w_\gamma{}^d \\ &= (L(W_t)^c - \pi_{ab}{}^c W_{ti}{}^a W_{tj}{}^a g^{ij}) g''_{cd} w_\gamma{}^d,\end{aligned}$$

by the second lemma. If W_t satisfies $(\tilde{2})$, then $\tau(W_t)(P) - \partial W_t(P)/\partial t$ is perpendicular to $M'(f_t(P))$, whence by the first lemma f_t satisfies (2). Conversely, if f_t satisfies (2) and we define ξ as in the third lemma, then our equation shows that $\xi(P)$ is perpendicular to $M'(f_t(P))$, whence it must be 0.

(C) The following result is an application of Theorem 6D. In order to have a proof avoiding the use of vector-bundle-valued harmonic forms, we can start with (5) and substitute $(\tilde{2})$ to obtain

$$L(\rho(W_t))^c = -\pi_d{}^c (\frac{\partial^2 \pi^d}{\partial w^a \partial w^b} + \rho_e{}^d \Gamma''_{ab}{}^e) W_{ti}{}^a W_{tj}{}^b g^{ij},$$

using the projection relation $\rho_d{}^c \pi_e{}^d = \pi_d{}^c \rho_e{}^d$. It follows that $\rho^c \cdot L(\rho(W_t))^c = 0$, so that

$$\sum_c \rho^c \cdot \Delta(\rho(W_t))^c = \tfrac{1}{2}\frac{\partial}{\partial t}\sum_c (\rho(W_t)^c)^2.$$

Applying Green's identity (as in Theorem 6D) we find that

$$\tfrac{1}{2}\frac{d}{dt}\int_M \sum_c (\rho(W_t)^c)^2 *1 = -\int_M (\operatorname{grad}(W_t)^2 *1,$$

and the conclusion follows:

THEOREM. *Let $W_t\colon M \to N$ be a smooth deformation satisfying $(\tilde{2})$ for $t_0 \leqq t < t_1$. If W_{t_0} maps M into M', then so does every $W_t (t_0 \leqq t < t_1)$.*

8. Derivative bounds for the elliptic case.

(A) Under suitable curvature and metric restrictions on M' we shall establish derivative estimates important for the solution of (1) and (2), or equivalently, of $(\tilde{1})$ and $(\tilde{2})$. Our starting point is the following result, essentially established in § 3C.

LEMMA. *Any solution f_t of (2) has energy density $e(f_t)$ satisfying*

$$\Delta e(f_t) - \frac{\partial e(f_t)}{\partial t} = |\beta(f_t)|^2 - R'_{\alpha\beta\gamma\delta} f_{ti}{}^{\alpha} f_{tj}{}^{\beta} f_{tk}{}^{\gamma} f_{tl}{}^{\delta} g^{ik} g^{jl} - g'_{\alpha\beta} f_{ti}{}^{\alpha} f_{tj}{}^{\beta} R^{ij}, \tag{6}$$

where $\beta(f_t)$ is the fundamental form of f_t and where $R'_{\alpha\beta\gamma\delta}$ and R^{ij} are the components of the Riemannian curvature tensor on M' and of the contravariant Ricci tensor on M, respectively.

(B) For the elliptic systems (1), $(\tilde{1})$ we invoke Green's formula for the operator Δ on M. Let $r(P,Q)$ denote the geodesic distance between points P, Q of M. Since M is compact (without boundary), there is a constant $a > 0$ such that $r^2(P,Q)$ is of class C^∞ for $r^2(P,Q) < 3a$. Let $\phi(\lambda)$ be a monotone C^∞ function in $0 \leqq \lambda < \infty$ with $\phi(\lambda) = \lambda$ in $0 \leqq \lambda \leqq a$ and $\phi(\lambda) =$ constant for $\lambda > 2a$. Set $\rho^2(P,Q) = \phi(r^2(P,Q))$; this is positive for $P \neq Q$ and is of class C^∞. The function

$$F(P,Q) = \kappa \cdot \rho(P,Q)^{-n+2} \qquad \left(= \frac{-1}{2\pi} \log \rho(P,Q) \text{ for } n = 2\right)$$

is a parametrix for the operator Δ, where $1/\kappa$ is $(n-2)$ times the surface of the unit $(n-1)$-sphere. Green's formula is

$$u(P) = \int_M [u(Q) \cdot \Delta_Q F(P,Q) - F(P,Q) \cdot \Delta u(Q)] *1_Q, \tag{7}$$

this holding for any function $u(P)$ of class C^2 on M (see Giraud [12], Bidal-de Rham [1]).

Consider now a solution of (1), and suppose that the Riemannian curvature of M' is non-positive. From (6), in which it is merely necessary to suppress the term $\partial e(f_t)/\partial t$, there follows at once

$$e(f) \geqq -\mathrm{Re}(f),$$

where R is a constant (independent of the solution in question). Since $F(P,Q) \geqq 0$ (a and ϕ being suitably chosen for $n=2$), (7) applied to $e(f)$ yields

$$e(f)(P) \leqq \int_M [\Delta_Q F(P,Q) + R \cdot F(P,Q)] e(f)(Q) {*1_Q}.$$

By using the osculating Euclidean metric at a point of M one can show that $|\Delta_Q F(P,Q)| \leqq \text{const.} \times \rho(P,Q)^{-n+2}$. Hence for some constant A,

$$e(f)(P) \leqq A \int_M F(P,Q) e(f)(Q) {*1_Q}.$$

Iterating this $k-1$ times we obtain

$$e(f)(P) \leqq A^k \int_M F_k(P,Q) e(f)(Q) {*1_Q}.$$

where the F_k are defined inductively by $F_1 = F$ and

$$F_k(P,Q) = \int_M F_{k-1}(P,Z) \cdot F(Z,Q) {*1_Z} \qquad (k>1).$$

If $k > n/2$, then F_k is bounded (see Giraud [12]), and we have the following

THEOREM. *If M' has non-positive Riemannian curvature, then there is a constant C such that $e(f) < C \cdot E(f)$ for any harmonic mapping $f\colon M \to M'$.*

(C) Green's function $G(P,Q)$ for Δ can be written in the form $G(P,Q) = F(P,Q) + F'(P,Q)$, where F' is of class C^∞ for $P \neq Q$ and has a singularity of order lower than that of F for $P = Q$ (Giraud [12], Bidal-de Rham [1]). $G(P,Q)$ is symmetric and of class C^∞ for $P \neq Q$ and satisfies $\Delta_P G(P,Q) = \Delta_Q G(P,Q) = V^{-1}$, where V is the volume of M. Green's formula (7), with G in place of F, is

$$(8) \qquad u(P) = V^{-1} \int_M u(Q) {*1} - \int_M G(P,Q) \cdot \Delta u(Q) {*1_Q}.$$

Now let $\bar{U}$ be a compact coordinate neighborhood on M, U its interior, and let $P \in U$ have coordinates $(x^1, \cdots, x^n)$. Write

$$G_i(P,Q) = \partial G(P,Q)/\partial x^i; \quad G_{i,j}(P,Q) = \partial^2 G(P,Q)/\partial x^i \partial x^j$$

Using normal coordinates on M one can show that there is a constant C such that

$$\begin{aligned} &| G_i(P,Q) | \leqq C \cdot r(P,Q)^{-n+1} \\ (9) \qquad &| G_{i,j}(P,Q) | \leqq C \cdot r(P,Q)^{-n} \\ &| G_i(P,Z) - G_i(Q,Z) | \leqq C \cdot r(P,Q)^{\alpha} [r(P,Z)^{-n+1-\alpha} + r(Q,Z)^{-n+1-\alpha}] \end{aligned}$$

for $P, Q \in U$, α being an arbitrary but fixed number with $0 < \alpha < 1$.

If u is a solution of $\Delta u = f$ on M for some function f, then from (8) we have (for $P \in U$)

$$u_i(P) = -\int_M G_i(P,Z) \cdot f(Z) * 1_Z.$$

Hence

$$| u_i(P) - u_i(Q) | \leqq \int_M | G_i(P,Z) - G_i(Q,Z) | \cdot | f(Z) | * 1_Z$$

for $P, Q \in U$. Using the last inequality of (9) we obtain

$$(10) \qquad | u_i(P) - u_i(Q) | \cdot r(P,Q)^{-\alpha} \leqq C' \sup_M | f |,$$

where

$$C' = C \cdot \sup \int_M [r(P,Z)^{-n+1-\alpha} + r(Q,Z)^{-n+1-\alpha}] * 1_Z.$$

Suppose now that f is Hölder-continuous with exponent α and Hölder modulus $M_\alpha(f)$. I. e., $M_\alpha(f) = \sup_M | f(P) - f(Q) | \cdot r(P,Q)^{-\alpha}$. It is a classical result of potential theory that the function

$$\int_M G_i(P,Z) \cdot [f(P_0) - f(Z)] * 1_Z \qquad (P, P_0 \in U)$$

has a derivative with respect to x^j at the point P_0, given by

$$\int_M G_{i,j}(P_0, Z) \cdot [f(P_0) - f(Z)] * 1_Z,$$

and by (9), this integral is majorized by

$$C \cdot M_\alpha(f) \cdot \int_M r(P_0, Z)^{-n+\alpha} * 1_Z.$$

On the other hand, the function $\phi(P) = \int_M G(P,Z) * 1_Z$ is a constant, since $\Delta\phi = 0$, and so $\int_M G_i(P,Z) \cdot f(P_0) * 1_Z = 0$. We conclude that

(11) $$|\partial^2 u/\partial x^i \partial x^j| \leqq C' \cdot M_\alpha(f) \text{ in } U.$$

(This is an interior estimate of the type given by Hopf and Schauder (Hopf [14], Schauder [26], Miranda [18]), but differs from them in that the magnitude of u is not involved.)

(D) We apply these results to a harmonic mapping $f\colon M \to M'$. Denote the right member of $(\tilde{1})$ by F^c:

$$F^c = \pi_{ab}{}^c W_i{}^a W_j{}^b g^{ij}.$$

At this point *we impose some boundedness conditions on the embedding of M' in $\boldsymbol{R}^q$, conditions which are automatically fulfilled if M' is compact. Namely we assume that*

(12) $$\begin{gathered} |\pi_{ab}{}^c| \leqq C_0, \ |\partial \pi_{ab}{}^c/\partial w^d| \leqq C_0 \quad \text{on } M', \\ A_1 ds_0{}^2 \leqq ds'^2 \leqq A_2 ds_0{}^2, \end{gathered}$$

where C_0, A_1, A_2 denote positive constants and where $ds_0{}^2$ denotes the line element induced on M' by the usual metric in $\boldsymbol{R}^q$.

Again let U be the interior of a compact coordinate neighborhood on M, and let P, Q be points of U. From (12) and an elementary calculation involving the Schwarz inequality for quadratic forms and the equality $e(f)P = g''_{ab} W_i{}^a W_j{}^b g^{ij}$,

(13) $$\begin{aligned} &|F^c(P) - F^c(Q)| \cdot r(P,Q)^{-\alpha} \\ &\qquad \leqq B \cdot [\bar{e} + \bar{e}^{\frac{1}{2}} \sup_{a,i} | W_i{}^a(P) - W_i{}^a(Q)| \cdot r(P,Q)^{-\alpha}], \end{aligned}$$

where B is a constant and $\bar{e} = \bar{e}(f) = \sup\{e(f)(P) : P \in M\}$. From (1) and (10),

$$|W_i{}^a(P) - W_i{}^a(Q)| \cdot r(P,Q)^{-\alpha} \leqq C' \sup_M |F^a|,$$

and plainly $|F^a| \leqq \text{const.} \times \bar{e}$, in virtue of (12). From the compactness of M there follows the estimate

(14) $$M_\alpha(F^c) \leqq B'(\bar{e} + \bar{e}^{\frac{3}{2}})$$

for the Hölder-modulus of F^c, B' denoting a constant. Referring to (11), we have the

THEOREM. *Suppose that M' satisfies the embedding conditions* (12). *Let $U(x^1, \cdots, x^n)$ be the interior of a compact coordinate neighborhood on M. Then there is a constant C such that*

$$|\partial^2 W^a/\partial x^i \partial x^j| \leqq C(\bar{e}(f) + \bar{e}(f)^{\frac{3}{2}}) \text{ in } U, \ (1 \leqq a \leqq q),$$

for any harmonic mapping $f: M \to M'$, *where* $\bar{e}(f) = \sup\{e(f)(P) : P \in M\}$.

Remark. We point out that second derivative estimates can be obtained from linear theory in another way. Namely, if we write our equation $(\tilde{1})$ in the form

$$\Delta W^c + A_b{}^{cj} W_j{}^b = 0, \text{ where } A_b{}^{cj} = -\pi_{ab}{}^c W_i{}^a g^{ij},$$

then we have a linear system with bounded coefficients (in compact coordinate neighborhoods), by Theorem 8B. A Hölder-modulus for the $W_i{}^c$ will then give us a Hölder-modulus for the coefficients of the linear system, and we can apply Theorem 1 of Douglis and Nirenberg [6] to deduce second derivative estimates and analogous estimates on all higher derivatives as well. The second derivative estimates we obtain here are somewhat sharper, in certain respects; i.e., they do not involve *a priori* estimates on the magnitudes of the solutions W^c.

9. Bounds for the parabolic case.

(A) We now embark upon some analogous computations for the operator $L = \Delta - \partial/\partial t$. The function

$$K(P, Q, t) = (2\sqrt{\pi})^{-n} t^{-n/2} \exp(-\rho^2(P, Q)/4t) \tag{15}$$

is a parametrix for the operator L (ρ^2 as in the preceding paragraph). Put

$$N_1(P, Q, t) = L_P K(P, Q, t) = (\Delta_P - \partial/\partial t) K(P, Q, t)$$

and

$$N_k(P, Q, t) = \int_0^t d\tau \int_M N_{k-1}(P, Z, t-\tau) \cdot N(Z, Q, \tau) * 1_Z \qquad (k > 1).$$

It is well known that there exists a fundamental solution H for the heat operator L on any compact Riemannian manifold M, which can be expressed in the form

$$H(P, Q, t) = K(P, Q, t) + \int_0^t d\tau \int_M K(P, Z, t-\tau) \cdot N(Z, Q, \tau) * 1_Z, \tag{16}$$

where

$$N(P, Q, t) = \sum_{k=1}^{\infty} N_k(P, Q, t).$$

(See Milgram-Rosenbloom [17], Pogorzelski [24]). The function $H(P, Q, t)$ is symmetric in P, Q and is positive. It is of class C^∞ except for $P = Q$, $t = 0$;

and it satisfies $L_P H(P,Q,t) = L_Q H(P,Q,t) = 0$. Its spectral decomposition is

$$H(P,Q,t) = V^{-1} + \sum_{\nu=1}^{\infty} \exp(-\lambda_\nu t)\phi_\nu(P)\phi_\nu(Q),$$

where the λ_ν are the non-zero eigenvalues of Δ, the $\phi_\nu(P)$ being the corresponding orthonormal eigenfunctions.

Green's formula (analogous to 8) is

$$\begin{aligned}(17)\qquad u(P,\mathrm{t}) = &-\int_{t_0}^{t} d\tau \int_M H(P,Q,t-\tau)\cdot Lu(Q,\tau)*1_Q \\ &+\int_M H(P,Q,t-t_0)\cdot u(Q,t_0)*1_Q \quad (t_0 < t < t_1),\end{aligned}$$

this holding for any function $u(P,t)$ on M which is of class C^2 in P and C^1 in t for $t_0 \leqq t < t_1$.

(B) Suppose now that we have a solution $f_t\colon M \to M'$ of (2) defined in $0 < t < t_1$, and let M' have non-positive Riemannian curvature. According to Lemma 8A we have then $Le(f_t) \geqq -Re(f_t)$, R being a positive constant (independent of the solution in question). Since $H > 0$, there follows from (17)

$$(18)\qquad e(f_t)(P) \leqq R\int_{t_0}^{t} d\tau \int_M H(P,Q,t-\tau)\cdot e(f_\tau)(Q)*1_Q + e_0(f_t)(P)$$

where

$$(19)\qquad e_0(t_t)(P) = \int_M H(P,Q,t-t_0)\, e(f_{t_0})(Q)*1_Q$$

and $0 < t_0 < t < t_1$. Iterating (18) $k-1$ times we obtain

$$\begin{aligned}(20)\qquad e(f_t)(P) \leqq\, &R^k \int_{t_0}^{t} d\tau \int_M H_k(P,Q,t-\tau)\, e(f_\tau)(Q)*1_Q \\ &+ e_0(f_t)(P) + \sum_{\nu=1}^{k-1} R^\nu \int_{t_0}^{t} d\tau \int_M H_\nu(P,Q,t-\tau)\, e_0(f_\tau)(Q)*1_Q,\end{aligned}$$

where the H_k are defined by $H_1 = H$ and

$$H_k(P,Q,t) = \int_0^t d\tau \int_M H_{k-1}(P,Z,t-\tau) H(Z,Q,\tau)*1_Z \qquad (k > 1).$$

From the integral representation it can be shown (see Pogorzelski [24]) that

$$H(P,Q,t) \leqq \text{const.} \times t^{-\alpha} r(P,Q)^{-n+2\alpha} \qquad (0 \leqq t \leqq 1).$$

where α is an arbitrary but fixed positive number less than 1. Therefore H_k is bounded for $k > n/2$ $(0 \leqq t \leqq 1, k > 1)$.

Consider now our solution f_t for $t > 1$. Putting $t-1$ for t_0 in (20) we have

$$e(f_t)(P) \leqq \text{const.} \int_{t-1}^{t} d\tau \int_M e(f_\tau)(Q)*1 \tag{21}$$
$$+ \sup_M e_0(f_\tau)(P) \cdot [1 + \sum_{\nu=1}^{k-1} R^\nu \int_0^1 d\tau \int_M H_\nu(P,Q,\tau)*1_Q].$$

For the case at hand, e_0 is given from (19) by

$$e_0(f_t)(P) = \int_M H(P,Q,1) \cdot e(f_{t-1})(Q)*1_Q.$$

Since $H(P,Q,1)$ is bounded, we have $e_0(f_t)(P) \leqq \text{const.} \int_M e(f_{t-1})(Q)*1_Q$. Recalling that $\int_M e(f_t)(Q)$ is a decreasing function, we obtain finally from (21)

$$e(f_t)(P) \leqq \text{const.} \int_M e(f_{t-1})(Q)*1 \qquad (t > 1).$$

Any smaller value can be put in for $t-1$ on the right, for example zero if $e(f_t)(P)$ is continuous at $t=0$.

Making that assumption, we now obtain an estimate for the range $0 \leqq t \leqq 1$. In (20) we now put $t_0 = 0$, getting

$$e(f_t)(P) \leqq \text{const.} [\int_0^t d\tau \int_M e(f_\tau)(Q)*1 + \bar{e}_0(f_t)], \tag{22}$$

where now we have

$$e_0(f_t)(P) = \int_M H(P,Q,t)\, e(f_0)(Q)*1_Q,$$

and where $\bar{e}_0(f_t) = \sup\{e_0(f_t)(P) : P \in M\}$; we define $\bar{e}(f_t)$ similarly. But this function is precisely the solution of $Le_0 = 0$ that reduces at $t=0$ to $e(f_0)(Q)$. From general principles it follows that $e_0(f_t)(P) \leqq \bar{e}(f_0)$ for $t \geqq 0$, and so (22) gives at once

$$e(f_t)(P) \leqq \text{const.}\, \bar{e}(f_0) \qquad (0 \leqq t \leqq 1).$$

We have then the

THEOREM. *Let $f_t : M \to M'$ be a family of mappings for $0 \leqq t < t_1$ satisfying (2) for $0 < t < t_1$ and such that the energy density $e(f_t)(P)$ is* *continuous at* $t=0$. *Suppose that* M' *has non-positive Riemannian curvature. Then*

$$e(f_t)(P) \leqq C \cdot \int_M e(f_0)(Q) * 1 \qquad \textit{for } 1 \leqq t < t_1$$

and

$$e(f_t)(P) \leqq C \cdot \sup\{e(f_0)(Q) : Q \in M\} \qquad \textit{for } 0 \leqq t \leqq 1,$$

C *denoting a constant which does not depend on the particular solution* f_t *of* (5).

Remark. Under certain circumstances much sharper estimates can be obtained. With the hypotheses of the preceding theorem, assume further that the Ricci tensor of M is positive definite at every point. From (6) it is clear that $\partial e(f_t)/\partial t \leqq -Ae(f_t)$ at any maximum point of $e(f_t)$ on M, A denoting a positive constant. It follows easily that $e(f_t)(P) \leqq \text{const.}\, e^{-At}$.

(C) Now let $(x^1, \cdot\cdot\cdot, x^n)$ be the coordinates of P in the interior U of a compact coordinate neighborhood $\bar{U}$ on M. And suppose that our solution of (2) and the first-order space derivatives are continuous at $t=0$. Then from $(\tilde{2})$, (17) we have

$$\begin{aligned} W_t^c(P) &= W^c(P,t) \\ &= -\int_0^t d\tau \int_M H(P,Q,t-\tau) \cdot F^c(Q,\tau) * 1_Q + W_0^c(P,t) \qquad (23) \\ &= V^c(P,t) + W_0^c(P,t), \end{aligned}$$

where

$$W_0^c(P,t) = \int_M H(P,Q,t) W^c(Q,0) * 1_Q,$$

the $F^c(P,t)$ being the functions on the right of $(\tilde{2})$.

The first integral $V^c(P,t)$ has Hölder continuous first-order space derivatives (Pogorzelski [23], Theorem 5):

$$| V_i^c(P,t) - V_i^c(P',t') | \leqq \text{const.} \sup | F^c | \cdot [r(P,P')^\alpha + | t-t' |^{\alpha/2}],$$

α being an arbitrary positive number less than 1, the points P and P' both in U. If we continue with the assumption (12), we shall have $| F^c | \leqq \text{const.} \times \bar{e}$. The integral $W_0^c(P,t)$ can be differentiated under the integral sign (for $t>0$) and the derivatives tend exponentially to zero, as is quickly seen from the spectral formula for H. Hence, if the hypotheses of Theorem 9B hold, it follows that the functions $W^c(P,t)$ have first-order space derivatives which are Hölder-continuous in P, t, uniformly so for $t \geqq \epsilon > 0$:

$$| W_i^c(P,t) - W_i^c(P',t') | \leqq \text{const.} [\bar{e}(f_0) + \sup_M | W^c(Q,0) |] \cdot [r(P,P')^\alpha + | t - t' |^{\alpha/2}].$$

(The constant depends upon ϵ because of the behavior of $W_0^c(P,t)$ for small t.)

Referring to (13) we see that

$$(24) \qquad | F^c(P,t) - F^c(Q,t) | \cdot r(P,Q)^{-\alpha} \leqq \text{const.}\, \bar{e}(f_0) [1 + \bar{e}(f_0)^{\frac{1}{2}} + \sup_M | W^c(Q,0) |]$$

for $t \geqq \epsilon > 0$.

Now from (17) it is clear that $\int_0^t d\tau \int_M H(P,Q,t-\tau)\psi(\tau) * 1_Q$ is a function of t alone for arbitrary ψ. Hence for the second derivatives $V_{i,j}^c(P,t) = \partial^2 V^c(P,t)/\partial x^i \partial x^j$ we can write

$$V_{i,j}(P,t) = \frac{\partial^2}{\partial x^i \partial x^j} \int_0^t d\tau \int_M H(P,Q,t-\tau) [F^c(P_0,\tau) - F^c(Q,\tau)] * 1_Q.$$

From (Pogorzelski [24], Th. 3) this is

$$V_{i,j}^c(P,t) = \int_0^t d\tau \int_M H_{i,j}(P,Q,t-\tau) [F^c(P_0,\tau) - F^c(Q,\tau)] * 1_Q$$
$$= \int_{t-\epsilon}^t + \int_0^{t-\epsilon} = I_1 + I_2,$$

where we assume $t \geqq 2\epsilon > 0$. The integral I_1 is improper but uniformly convergent. Now $| H_{i,j}(P,Q,t) | < \text{const.} \times t^{-\beta} r(P,Q)^{-n-2+2\beta}$ (arbitrary β, $0 < \beta < 1$). Using (24) and putting $P = P_0$ in I_1 (we assume $(P, P_0 \in U)$, we obtain an absolutely convergent integral if α, β are chosen properly, and there results

$$| I_1 | < \text{const.}\, \bar{e}(f_0) [1 + \sup_M | W^c(Q,0) | + \bar{e}(f_0)^{\frac{1}{2}}] \text{ if } t \geqq 2\epsilon,$$

since then τ in I_1 will be $\geqq \epsilon$. For I_2 we have $| H_{i,j}(P,Q,t) | < \text{const.}\, e^{-\gamma t}$ for some positive γ and for $t \geqq \epsilon > 0$. Hence

$$| I_2 | < \text{const.} \sup | F^c | \cdot \int_0^{t-\epsilon} e^{-\gamma(t-\tau)}\, d\tau$$
$$< \text{const.} \sup | F^c | < \text{const.}\, \bar{e}(f_0),$$

using Theorem 9B.

THEOREM. *Suppose that M' satisfies the imbedding conditions* (12). *Let f_t satisfy the conditions of Theorem* 9B, *and let* $(x^1, \cdots, x^n)$ *be the coordinates of a point P in the interior of a compact coordinate neighborhood $\bar{U}$ on M. Given $\epsilon > 0$, there is a constant C, independent of the solution f_t of* (2), *such that*

$$\left| \frac{\partial^2 W^c(P,t)}{\partial x^i \partial x^j} \right| < C \cdot \bar{e}(f_0)[1 + \bar{e}(f_0)^{\frac{1}{2}} + \sup\{|W^c(Q,0)| : Q \in M\}$$

for $t \geqq \epsilon$, where $\bar{e}(f_0) = \sup\{e(f_0)(P) : P \in M\}$.

Remark. Since $\int_M H(P,Q,t) * 1_Q$ is a constant, the functions $W^c(Q,0)$ appearing in the foregoing estimates can be altered by arbitrary additive constants without affecting the validity of the estimates. For example, one could replace $W^c(Q,0)$ by the function minus its average value, say $\hat{W}^c$, with the result that the term $\sup |W^c(Q,0)|$ in Theorem 9C would be replaced by $\sup |W^c(Q,0) - \hat{W}^c|$.

10. Successive approximations.

(A) Let $W(P,t)$ and $W'(P,t)$ be two solutions of $(\tilde{2})$ in $0 \leqq t < t_1$, both continuous along with their first order space derivatives at $t = 0$; and suppose that $W(P,0) = W'(P,0) \in M'$ for all $P \in M$. From (17) with $t_0 = 0$,

$$W^c(P,t) - W'^c(P,t)$$
$$= -\int_0^t d\tau \int_M H(P,Q,t-\tau)[F^c(Q,\tau) - F'^c(Q,\tau)] * 1_Q,$$

where F^c, F'^c are the respective right members of $(\tilde{2})$. Set

$$X(t) = \sup_{M,c} |W^c(P,t) - W'^c(P,t)| + [\sup_{M,c} (W_i^c - W_i'^c)(W_j^c - W_j'^c) g^{ij}]^{\frac{1}{2}}.$$

From the constitution of F^c and F'^c it is easily verified (by an argument similar to that for (13)), account taken of (12), that

$$|F^c - F'^c| < \text{const.}\, X(t) \cdot U(t),$$

where

$$U(t) = \bar{e}(f_t) + \bar{e}(f'_t) + \bar{e}(f_t)^{\frac{1}{2}} + \bar{e}(f'_t)^{\frac{1}{2}}.$$

For $0 \leqq t \leqq 1$ we can write $H(P,Q,t) < \text{const.}\, t^{-\alpha} r(P,Q)^{-n+2\alpha}$ and

$$|\partial H(P,Q,t)/\partial x^i| < \text{const.}\, t^{-\alpha} r(P,Q)^{-n-1+2\alpha} \qquad (0 < \alpha < 1),$$

where in the latter the constant depends upon the particular choice of local coordinates x^i, of course. Now let A denote an upper bound for the quantity $U(t)$ in some fixed time interval $0 \leqq t \leqq t_2$. From the integral expression above there follows easily, for $\frac{1}{2} < \alpha < 1$,

$$X(t) < \text{const.} \times A \int_0^t (t-\tau)^{-\alpha} X(\tau)\, d\tau,$$

and we conclude that $X(t)$ vanishes for small t. Hence the

THEOREM. *Let f_t and f'_t be two solutions of (2), both continuous along with their first-order space derivatives at $t=0$. If $f_0 = f'_0$, then the two solutions coincide for all (relevant) $t > 0$.*

COROLLARY. *Any solution of (2) enjoys the semi-group property along the trajectory of each point $P \in M$. That is, if we write $f_t(P) = T_t(f\ )$, then*

$$T_{t+\tau}(f_0) = T_t(f_\tau) = T_t(T_\tau(f_0)).$$

(B) For the solution of (2) we now turn to the system of non-linear integro-differential equations (23) associated with $(\tilde{2})$.

Let $f\colon M \to M'$ be a mapping of class C^1, given by global mapping functions $W = (W^1, \cdots, W^q)$. For $\nu \geqq 0$ define $W^\nu = (W^{\nu,1}, \cdots, W^{\nu,q})$ by

$$W^{0,c}(P,t) = \int_M H(P,Q,t)\, W^c(Q) * 1_Q \tag{25}$$

and

$$W^{\nu,c}(P,t) = -\int_0^t d\tau \int_M H_i(P,Q,t-\tau) \cdot F^{\nu-1,c}(Q,\tau) * 1_Q + W_i^{0,c}(P,t), \tag{26}$$

where

$$F^{\nu,c}(P,t) = \pi_{ab}{}^c(W^\nu) \cdot W_i^{\nu,a} W_j^{\nu,b} g^{ij},$$

the functions $\pi_{ab}{}^c$ as in $(\tilde{2})$. Set

$$y_\nu = \sup_M \Big[\sum_{c=1}^q W_i^{\nu,c} W_j^{\nu,c} g^{ij} \Big]^{\frac{1}{2}}.$$

From (26)

$$W_i^{\nu,c}(P,t) = -\int_0^t d\tau \int_M H(P,Q,t-\tau) \cdot F^{\nu-1,c}(Q,\tau) * 1_Q + W^{0,c}(P,t), \tag{27}$$

where the subscript i denotes differentiation with respect to a system of local coordinates x^i at P. We recall the estimate

$$|H_i(P,Q,t)| < At^{-\alpha}r(P,Q)^{-n-1+2\alpha} \qquad (0 \leqq t \leqq 1; \tfrac{1}{2} < \alpha < 1),$$

A denoting a constant which depends in general on the local coordinate system. Now let B denote an upper bound for the quantities $|\pi_{ab}{}^c|$ in some compact neighborhood U' of the image $f(M)$ in the tubular neighborhood $N \subset \boldsymbol{R}^q$. If $W^{\nu-1} \in U'$ for $0 \leqq t \leqq \epsilon$, then from (27) we conclude that

$$y_\nu < BC \int_0^t (t-\tau)^{-\alpha} y^2_{\nu-1}(\tau)\, d\tau + y_0(t) \qquad (0 \leqq t \leqq \epsilon),$$

where C is a constant which does not depend upon the given mapping f. Put

$$\bar{y}_\nu = \sup y_\nu, \qquad 0 \leqq t \leqq \epsilon.$$

Then

$$\bar{y}_\nu < K\epsilon^{1-\alpha}\bar{y}_{\nu-1}{}^2 + \bar{y}_0, \qquad K = BC/(1-\alpha).$$

If $K\epsilon^{1-\alpha}\bar{y}_{\nu-1} \leqq \frac{1}{2}$ and $K\epsilon^{1-\alpha}\bar{y}_0 \leqq \frac{1}{4}$, then

$$K\epsilon^{1-\alpha}\bar{y}_\nu \leqq (K\epsilon^{1-\alpha}\bar{y}_{\nu-1})^2 + K\epsilon^{1-\alpha}\bar{y}_0 \leqq \tfrac{1}{2}.$$

From a transparent induction it follows that, for sufficiently small positive ϵ, we shall have

$$W^\nu(P,t) \in U' \text{ and } \bar{y}_\nu \leqq \tfrac{1}{2}K\epsilon^{1-\alpha} \text{ for } 0 \leqq t \leqq \epsilon,\ \nu = 0, 1, 2, \cdots.$$

In particular, the defining equation (26) makes sense for all ν, provided $0 \leqq t \leqq \epsilon$.

Now put

$$X_\nu(t) = \sup_{M,c} |W^{\nu,c}(P,t) - W^{\nu-1,c}(P,t)| + [\sup_{M,c}(W_i^{\nu,c} - W_i^{\nu-1,c})(W_j^{\nu,c} - W_j^{\nu-1,c})g^{ij}]^{\frac{1}{2}}.$$

From the definition above,

$$F^{\nu,c} - F^{\nu-1,c} = \pi_{ab}{}^c(W^{\nu-1})(W_i^{\nu,a}W_j^{\nu,b} - W_i^{\nu-1,a}W_j^{\nu-1,b})g^{ij} + [\pi_{ab}{}^c(W^\nu) - \pi_{ab}{}^c(W^{\nu-1})]W_i^{\nu,a}W_j^{\nu,b}g^{ij}.$$

We can suppose that the constant B occurring above is also an upper bound for the quantities $|\partial\pi_{ab}{}^c/\partial w^d|$ in the neighborhood U'. The preceding formula then gives us

$$|F^{\nu,c}-F^{\nu-1,c}|<C'BX_\nu(\bar{y}_\nu+\bar{y}_{\nu-1}+\bar{y}_\nu{}^2),$$

C' a constant independent of the given mapping f. Hence,

$$|F^{\nu,c}-F^{\nu-1,c}|<C'BX_\nu[1/K\epsilon^{1-\alpha}+1/4K^2\epsilon^{2-2\alpha}]=C''X_\nu(t).$$

From (26), (27) and the estimates cited for $H(P,Q,t)$ and $H_i(P,Q,t)$ we obtain

$$X_{\nu+1}(t)<DC''\int_0^t X_\nu(\tau)\cdot(t-\tau)^{-\alpha}\,d\tau \qquad (0\leqq t\leqq\epsilon),$$

where D is a new constant. Thus if

$$\bar{X}_\nu(t)=\sup X_\nu(\tau) \qquad (0\leqq\tau\leqq t),$$

then

$$\bar{X}_\nu(t)<\left(\frac{DC''}{1-\alpha}t^{1-\alpha}\right)^\nu\bar{X}_0(t) \qquad (\nu=1,2,\cdots);$$

and so the series

$$\sum_0^\infty \bar{X}_\nu(t)$$

converges for all sufficiently small t. This shows that our successive approximations W^ν and their first-order space derivatives converge uniformly on M (for small t). Hence, the $F^{\nu,c}$ also converge. Set $W^c=\lim W^{\nu,c}$ and $F^c=\lim F^{\nu,c}$ $(\nu\to\infty)$. Thus $W^c(P,t)$ has continuous first-order space derivatives W_i^c (for sufficiently small t, of course), and $W_i^{\nu,c}\to W_i^c$ $(\nu\to\infty)$, so that

$$F^c(P,t)=\pi_{ab}{}^c(W)\,W_i^aW_j^bg^{ij}.$$

From (26) there follows at once

$$(28) \qquad W^c(P,t)=-\int_0^t d\tau\int_M H(P,Q,t-\tau)\cdot F^c(Q,\tau)*1_Q+W^{0,c}(P,t).$$

We conclude that the functions W^c have Hölder-continuous first order space derivatives—uniformly Hölder-continuous for any small closed t-interval not containing zero. Therefore the functions $F^c(P,t)$ are Hölder-continuous with respect to the space variable, and so the functions W^c of (28) satisfy equation $(\tilde{2})$ for all positive t in the interval in which the successive approximations converge (see Pogorzelski [24]). The W^c are moreover visibly continuous with their first order space derivatives at $t=0$, and $W^c(P,0)=W^c(P)$. From Theorem 7C we recall that the point $W(P,t)=(W^1,\cdots,W^q)$ must lie on the manifold M' for all t in the interval of convergence.

THEOREM. *Let M'' be a compact subset of M'. Then, for any continuously differentiable mapping $f\colon M\to M'$ such that $f(M)$ lies in M'', there is a positive constant t_1 depending only on M'' and the energy density $e(f)$ such that* (2) *has a solution f_t for $0\leqq t\leqq t_1$ which is continuous at $t=0$, along with its first order space derivatives, and which coincides with f at $t=0$.*

It only remains for us to inquire into the length of the t-interval for which the successive approximations converge. First of all, the upper bound B for $|\pi_{ab}{}^c|$ and $|\partial\pi_{ab}{}^c/\partial w^d|$ figuring in the foregoing proof can be taken once for all to be valid in some neighborhood U'' of M''. Therefore the constant K can be fixed, and α as well, of course. The ϵ must satisfy $K\epsilon^{1-\alpha}\bar{y}_0\leqq\frac{1}{4}$, and to see what this entails we must look briefly into $\bar{y}_0$.

Consider then a solution u of the heat equation on M, as a map $u\colon M\to\boldsymbol{R}$. We suppose that u and its first order space derivatives are continuous at $t=0$. The argument of Theorem 9B holds in this situation. If we put $p=g^{ij}u_iu_j$, then $p(P,t)\leqq C\sup\{p(Q,0):Q\in M\}$, where C is a constant depending only on M. Applying this to the functions $W^{0,c}$, we conclude that there is a constant C_1 such that $\bar{y}_0<C_1\cdot\bar{e}(f)$. Thus ϵ depends only on M'' and the magnitude of $e(f)$, and the same is true of the quantity C'' involved in the estimates of the X_ν. The assertion of the theorem then follows at once from those estimates.

In § 2B we described the harmonic character of C^2 maps in terms of their tension fields. For C^1 maps we have the

COROLLARY. *Let $f\colon M\to M'$ be a continuously differentiable mapping for which the energy is a minimum with respect to small variations. Then f is harmonic.*

For let f_t be the corresponding solution of (2) guaranteed by the preceding theorem. The energy $E(f_t)$ is continuous at $t=0$ and by assumption it must be non-decreasing for small t. But $E(f_t)$ is always non-increasing, so that $dW/dt=0$ for small t. Thus $(\tilde{2})$ reduces to $(\tilde{1})$.

(C) THEOREM. *Suppose that M' has non-positive Riemannian curvature, and that it satisfies the embedding restrictions* (12). *Then for any continuously differentiable mapping $f\colon M\to M'$ there is a unique solution f_t*

of (2), *defined for all* $t \geqq 0$, *which is continuous along with its first order space derivatives at* $t = 0$ *and which coincides with* f *at* $t = 0$.

Such a solution exists for small t, by Theorem 10B, and it is unique, by Theorem 10A. Let t_1 be the largest number such that a solution of the required sort exists for $0 \leqq t < t_1$, and supposes that t_1 is finite. From Theorem 9B it is clear that the right members of (28) cannot become unbounded for $0 \leqq t < t_1$ and consequently the images $f_t(M)$ $(0 \leqq t < t_1)$ all lie in a compact subset of M'; we recall that M' is always assumed to be complete. On the other hand, Theorem 9B shows that the energy density $e(f_t)$ remains bounded, and therefore by Theorems 10A and 10B there is a fixed positive number ϵ_1 such that any f_t can be continued as a solution of (2) into the interval $(t, t+\epsilon_1)$. This contradicts the definition of t_1.

(D) If M' is not compact, then solutions of (2) may very well become unbounded as $t \to \infty$, as in the

Example. Let M' be the manifold obtained by revolving the graph of a positive strictly decreasing smooth function $v = v(u)$ around the u-axis; let ϕ denote the revolution angle. For a map $f\colon S^1 \to M'$ of a circle S^1 parametrized by the central angle θ our heat equation is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \theta^2} + \frac{v'v''}{1+(v')^2}\left(\frac{\partial u}{\partial \theta}\right)^2 - \frac{vv'}{1+(v')^2}\left(\frac{\partial \phi}{\partial \theta}\right)^2$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial \theta^2} + 2\,\frac{v'}{v}\,\frac{\partial u}{\partial \theta}\,\frac{\partial \phi}{\partial \theta}.$$

If f satisfies initial conditions $\partial u/\partial\theta = 0$, $\phi = \theta$ when $t = 0$, then so does the solution f_t for any subsequent time. If we take $v(u) = 1 + e^{-u}$, then $R'_{1212} = -(e^u+1)/(e^{2u}+1) < 0$, and the heat equation reduces to

$$\frac{\partial u}{\partial t} = \frac{e^u+1}{e^{2u}+1}.$$

Thus $e^u + u - 2\log(e^u+1) = t + \text{const.}$; in particular, $u \to \infty$ as $t \to \infty$.

We note in passing that there are no non-trivial closed geodesics on M', so that there are no harmonic representatives in any non-trivial homotopy classes of maps $S^1 \to M'$.

The following result shows that solutions must remain bounded if M' satisfies certain conditions at infinity.

THEOREM. *Let M' be as in the preceding theorem and suppose further that* $|w|\cdot\pi_{ab}{}^c(w)\to 0$ *uniformly as* $|w|\to\infty$, *where* $|w|=\sup_c|w^c|$. *Then every solution of* $(\tilde{2})$ *is bounded.*

Set

$$\bar{W}(t)=\sup_{M,c}|W^c(P,t)|,\ \underline{W}(t)=\inf_{M,c}|W^c(P,t)|,\ U(t)=\int_M\sum_c(W^c)^2*1.$$

In virtue of Theorem 9B, the difference $\bar{W}(t)-\underline{W}(t)$ is bounded. Hence if our solution is unbounded as $t\to\infty$, then that is true of all three of the quantities above. Supposing that to be the case, let us denote by λ_k the set of all t for which $U(t)>k$. The λ_k are then all non-empty and each λ_k must contain at least one $t=t_k$ at which $dU/dt\geqq 0$. Now from $(\tilde{2})$ we have

$$\sum_c W^c\Delta W^c-\sum_c W^c\frac{dW^c}{dt}=\sum W^c\pi_{ab}{}^cW_i{}^aW_j{}^bg^{ij}.$$

Hence by Green's Theorem

$$\tfrac{1}{2}dU/dt=-\int_M(\operatorname{grad}W)^2*1-\sum_c\int_M W^c\pi_{ab}{}^cW_i{}^aW_j{}^bg^{ij}*1.$$

For large values of t_k we have a plain contradiction, since the right-hand side must be negative.

11. Harmonic mappings.

(A) We can now apply some of the results established above to prove the existence of harmonic mappings, even though we do not know whether the solutions of the parabolic system (2) converge in general as $t\to\infty$.

THEOREM. *Let M' have non-positive Riemannian curvature and let* $f_t\colon M\to M'$ *be a bounded solution of* (2), $0<t<\infty$. *Then there is a sequence* $t_1,t_2,t_3,\cdots$ *of t-values such that the mappings* f_{t_k} *converge uniformly, along with their first order space derivatives, to a harmonic mapping f.*

From Theorems 9B and 9C it is clear that the mappings f_t and their first order space derivatives form equicontinuous families. Hence there exists a sequence $t_1,t_2,\cdots$ such that the mappings $f_k=f_{t_k}$ converge uniformly, with their first order space derivatives, to a continuously differentiable mapping f. From $(\tilde{2})$ and (8) we can represent the f_k by the formula

$$W^c(P,t_k)=V^{-1}\int_M W^c(Q,t_k)*1-\int_M G(P,Q)\left[F^c(Q,t_k)+\frac{dW^c}{dt}(Q,t_k)\right]*1_Q,$$

where as usual F^c stands for the right member of $(\tilde{2})$. Now fix c and temporarily put

$$u_k(P) = \int_M G(P,Q) \frac{dW^c}{dt} (Q, t_k)*1_Q.$$

The $dW^c(P,t_k)/dt$ are bounded as $k \to \infty$, by Theorems 9B and 9C, and so the u_k and their first derivatives are bounded. Hence the u_k form an equicontinuous family, and we can suppose that the sequence $t_1, t_2, \cdots$ is chosen so that the u_k converge uniformly, say to u. Now let G_ν denote the ν-th iterate of the Green's function G. We have

$$\begin{aligned}\int_M G_\nu(P,Q)u(Q)*1 &= \lim_k \int_M G_\nu(P,Q)u_k(Q)*1 \\ &= \lim_k \int_M G_{\nu+1}(P,Q) \frac{dW^c(Q,t_k)}{dt} *1.\end{aligned}$$

If $\nu + 1 > n/2$, then $G_{\nu+1}$ is bounded. But the $dW^c(P,t_k)/dt$ converge in the mean to zero as $k \to \infty$, by Corollary 6C. Thus

$$\int_M G_\nu(P,Q)u(Q)*1 = 0,$$

and so $u = 0$ because of the positive-definite character of G. Therefore, passing to the limit in the equation above, we get for the limit mapping f the formula

$$W^c(P) = V^{-1} \int_M W^c(Q*1 - \int_M G(P,Q) \cdot F^c(Q)*1_Q,$$

where

$$F^c(Q) = \lim_k F^c(Q,t_k) = \pi_{ab}{}^c(W) W_i{}^a W_j{}^b g^{ij}.$$

From this it follows (as in § 10B) that $W^c(P)$ has Hölder-continuous first derivatives, and therefore f^c is Hölder-continuous. Consequently the $W^c(P)$ satisfy $(\tilde{1})$.

COROLLARY. *Let M' have non-positive Riemannian curvature and let $f\colon M \to M'$ be a continuously differentiable mapping. Let f_t be the solution of* (2) *which reduces to f at $t = 0$. If f_t is bounded as $t \to \infty$, then f is homotopic to a harmonic mapping f' for which $E(f') \leqq E(f)$. In particular, if M' is compact or satisfies the conditions of Theorem* 10D, *then every continuous mapping $M \to M'$ is homotopic to a harmonic mapping.*

COROLLARY. *If M' is compact and has non-positive Riemannian curvature, then every homotopy class of mappings $M \to M'$ contains a harmonic mapping whose energy is an absolute minimum.*

For in any homotopy class we can choose a minimizing sequence of *harmonic* mappings f_1, f_2, etc., by the preceding corollary. From Theorems 8B and 8D it follows that we can select a subsequence (same notation) which converges uniformly along with first derivatives to a continuously differentiable mapping f. Then $E(f) = \lim E(f_k)$, and f is harmonic by Corollary 10B.

(B) THEOREM. *Let M have non-negative Ricci curvature and let M' have non-positive Riemannian curvature. Suppose that M' is compact or that it satisfies the conditions of Theorem* 10D. *Then any continuous map $f\colon M \to M'$ is homotopic to a totally geodesic map. Furthermore,*

1) *if there is at least one point of M at which its Ricci curvature is positive, then every continuous map from M to M' is null-homotopic;*

2) *if the Riemannian curvature of M' is everywhere negative, then every continuous map from M to M' is either null-homotopic or is homotopic to a map of M onto a closed geodesic of M'.*

This is a combination of Theorem 3C and Corollary 11A.

COROLLARY. *Let M be a compact smooth manifold admitting a Riemann structure g' with non-positive Riemannian curvature. Then M does not admit any Riemannian structure g with non-negative Ricci curvature unless that curvature vanishes everywhere.*

Proof. It suffices (by passing to the two leaved orientable cover if necessary) to consider the case that M is orientable. If there were such a metric g, then the identity map $(M, g) \to (M, g')$ would be homotopic to a totally geodesic map. That map has degree one, and therefore M cannot have any point of positive Ricci curvatuıe relative to g.

Remark. A special case of Theorem 11B (Part 1) can be obtained without the use of harmonic theory. Namely, assume

1) that M has positive Ricci curvature everywhere; then by a theorem of S. Myers [20], the fundamental group $\pi_1(M, m)$ is finite;

2) that M' is any complete Riemannian manifold with non-positive Riemannian curvature. Then the homotopy groups $\pi_i(M', m') = 0$ for $i \neq 1$, and $\pi_1(M', m')$ has no elements of finite order.

It is well known that the homotopy classes of maps of any arcwise connected space M into M' are in natural 1-1 correspondence with the conjugacy classes of homomorphisms $\pi_1(M, m) \to \pi_1(M', m')$. But clearly in the situation at hand every such homomorphism is trivial, whence every continuous map $M \to M'$ is null homotopic.

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