Cohomology

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Reminder about homology: $\mathrm{Top} \xrightarrow[\text{singular complex}]{\mathcal{S}} (\text{simplicial sets}) \xrightarrow[\text{linearization}]{C(-;A)} (\text{chain complexes}) \xrightarrow[\substack{n\text{-th homology}\\ \text{groups}}]{H_n} (\text{abelian groups})$

- For a space X, the singular complex $\mathcal{S}(X)$ is the simplicial set with $\mathcal{S}(X)_n = \mathrm{map}^{cts}(\nabla^n, X)$

$$\nabla^n = \text{topological } n\text{-simplex} = \{ (x_0,\dots,x_n) \in \mathbb{R}^{n+1} : x_i \geq 0,\ x_0 + \dots + x_n = 1 \}$$

- For a simplicial set Y and an abelian group, the linearization is the chain complex $C(Y;A)$ with

$$C_n(Y;A) = A[Y_n] \quad A\text{-linearization of } Y_n \qquad (C_n(Y;A) = 0 \text{ for } n < 0)$$

$$d_n : \sum_{i=0,\dots,n} (-1)^i \cdot d_i^* : A[Y_n] \longrightarrow A[Y_{n-1}]$$

- For a chain complex C and $n \in \mathbb{Z}$, the n-th homology group $H_n(C)$ is $\dfrac{\mathrm{Ker}(d_n : C_n \to C_{n-1})}{\mathrm{Im}(d_{n+1} : C_{n+1} \to C_n)}$

Variation: Cohomology

Def: A cochain complex C consists of abelian groups C^n for $n \in \mathbb{Z}$ and homomorphisms $d^n : C^n \longrightarrow C^{n+1}$ such that $d^{n+1} \circ d^n = 0 : C^n \longrightarrow C^{n+2}$. A morphism $f : C \to D$ of cochain complexes (cochain map) consists of homomorphisms $f^n : C^n \longrightarrow D^n$ such that $d^n_D \circ f^n = f^{n+1} \circ d^n_C$

$$\begin{tikzcd} C^n \arrow[r, "f^n"] \arrow[d, "d^n"'] & D^n \arrow[d] \\ C^{n+1} \arrow[r, "f^{n+1}"'] & D^{n+1} \end{tikzcd}$$

The n-th cohomology group of a cochain complex C is

$$H^n C = \frac{\mathrm{Ker}(d^n : C^n \longrightarrow C^{n+1})}{\mathrm{Im}(d^{n-1} : C^{n-1} \longrightarrow C^n)}$$

A cochain homotopy between two morphisms $f, g : C \to D$ of cochain complexes consists of homomorphisms $s^n : C^n \longrightarrow D^{n-1}$ for all $n \in \mathbb{Z}$ such that

$$d^{n-1} \circ s^n + s^{n+1} \circ d^n = f^n - g^n \quad \text{for all } n \in \mathbb{Z}.$$

The main tools and properties carry over from chain complexes to cochain complexes, with essentially the same proofs, such as:

- a morphism $f : C \to D$ of cochain complexes induces a homomorphism $H^n f : H^n C \to H^n D$ of cohomology groups by $(H^n f)[x] = [f^n(x)]$ $\quad x \in \mathrm{Ker}(d^n : C^n \to C^{n+1})$
- cochain homotopic morphisms $f, g : C \to D$ between cochain complexes induce the same map in cohomology, i.e. $H^n f = H^n g$.
- every short exact sequence of cochain complexes $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ gives rise to a long exact sequence of cohomology groups

$$\dots \longrightarrow H^n A \xrightarrow{H^n f} H^n B \xrightarrow{H^n g} H^n C \xrightarrow{\partial} H^{n+1}(A) \longrightarrow \dots$$

where the connecting homomorphism ∂ is defined as follows:
given $x \in C^n$ with $d^n(x) = 0$, choose $\tilde{x} \in B^n$ such that $g^n(\tilde{x}) = x$,
then $g^{n+1}(d^n_B(\tilde{x})) = d^n_C(g^n(\tilde{x})) = d^n_C(x) = 0$, so there is a unique $y \in A^{n+1}$ such that $f^{n+1}(y) = d^{n+1}_B(\tilde{x})$. Set

$$\partial[x] = [y] \in H^{n+1}(A).$$

An isomorphism of categories:

We define a functor $D : \begin{pmatrix}\text{chain complexes,} \\ \text{chain maps}\end{pmatrix} \xrightarrow{\cong} \begin{pmatrix}\text{cochain complexes} \\ \text{and cochain maps}\end{pmatrix}$

On objects: $(DC)^n = C_{-n}, \quad d^n_{DC} = d^C_{-n} : \underset{\substack{\parallel \\ (DC)^n}}{C_{-n}} \longrightarrow C_{-n-1} = (DC)^{n+1}$

On morphisms: let $f : C \to C'$ be a chain map. Then

$Df : DC \longrightarrow DC'$ is the cochain map with $(Df)^n = f_{-n} : \underset{\substack{\parallel \\ (DC)^n}}{C_{-n}} \longrightarrow C'_{-n} = (DC')^n$.

The process is reversible on the nose (and not just up to natural isomorphism), so we have defined an isomorphism of categories.

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Moreover: $H^n(DC) = H_{-n}(C)$

$f, g: C \to C'$ are chain homotopic $\iff$ $Df, Dg : DC \to DC'$ are cochain homotopic

Construction: Let C be a chain complex and A an abelian group. We define a cochain complex $\mathrm{Hom}(C,A)$ by

$$\mathrm{Hom}(C,A)^n = \mathrm{Hom}(C_n, A) \quad \text{with differential}$$

$$\begin{array}{ccc} d^n: \mathrm{Hom}(C,A)^n & \longrightarrow & \mathrm{Hom}(C,A)^{n+1} \\ \shortparallel & & \shortparallel \\ \mathrm{Hom}(d_{n+1},A): \mathrm{Hom}(C_n,A) & \longrightarrow & \mathrm{Hom}(C_{n+1},A) \end{array}$$

Then: $d^{n+1} \circ d^n = \mathrm{Hom}(d_{n+2},A) \circ \mathrm{Hom}(d_{n+1},A)$
$= \mathrm{Hom}(d_{n+1} \circ d_{n+2}, A) = \mathrm{Hom}(0,A) = 0.$

This construction becomes a contravariant functor from chain complexes to cochain complexes

$$\mathrm{Hom}(-,A) : (\text{chain complexes})^{op} \longrightarrow (\text{cochain complexes})$$

On chain morphisms $f: C \to C'$, $\mathrm{Hom}(f,A): \mathrm{Hom}(C',A) \longrightarrow \mathrm{Hom}(C,A)$ is given by

$$\mathrm{Hom}(f,A)^n = \mathrm{Hom}(f_n, A) : \mathrm{Hom}(C'_n, A) \longrightarrow \mathrm{Hom}(C_n, A)$$

Lemma: Let $f, g: C \to C'$ be chain homotopic chain maps. Then the cochain maps $\mathrm{Hom}(f,A), \mathrm{Hom}(g,A): \mathrm{Hom}(C',A) \to \mathrm{Hom}(C,A)$ are cochain homotopic.

Proof: Suppose that $s = \{ s_n : C_n \to C'_{n+1} \}_{n \in \mathbb{Z}}$ is chain homotopy, then

$$\{ \underset{\mathrm{Hom}(C',A)^{n+1} \to \mathrm{Hom}(C,A)^n}{\mathrm{Hom}(s_n, A) : \mathrm{Hom}(C'_{n+1}, A) \longrightarrow \mathrm{Hom}(C_n, A)} \}_{n \in \mathbb{Z}}$$

is a cochain homotopy between $\mathrm{Hom}(f,A)$ and $\mathrm{Hom}(g,A)$. $\square$

The singular cohomology of spaces and simplicial sets

Def: Let Y be a simplicial set and A an abelian group. The cohomology of Y with coefficients in A is

$$H^n(Y;A) = H^n(\mathrm{Hom}(C(Y;\mathbb{Z});A))$$

If $Y' \subseteq Y$ is a simplicial subset, the relative cohomology of the pair (Y, Y') is

$$H^n(Y, Y'; A) = H^n\left(\mathrm{Hom}\left(\frac{C(Y;\mathbb{Z})}{C(Y';\mathbb{Z})}, A\right)\right).$$

If X is a space, the cohomology with coefficients in A is

$$H^n(X;A) = H^n(\mathcal{S}(X), A) = H^n(\mathrm{Hom}(C(\mathcal{S}(X), \mathbb{Z}), A)).$$

If X' is a subspace of X, the relative cohomology $H^n(X, X'; A)$ is the relative cohomology of the pair $(\mathcal{S}(X), \mathcal{S}(X'))$.

The definition can be made more concrete / remember [illegible], as follows:

Construction: Let Y be a simplicial set, A an abelian group. We define a cochain complex $C^*(Y;A)$ by

$$C^n(Y;A) = \mathrm{map}(Y_n, A) = \text{abelian group of maps } f: Y_n \to A \text{ under pointwise addition.}$$

The differential is defined by:

$$d^n(f)(y) = \sum_{i=0,\dots,n+1} (-1)^i \cdot f(d_i^*(y)) \qquad \text{for } y \in Y_{n+1}$$

Omitted: verification that

$$d^{n+1}(d^n(f)) = 0.$$

If Y' is a simplicial subset of Y, we define $C^*(Y, Y'; A)$ as follows:

$$C^n(Y, Y'; A) = \{ f: Y_n \to A \; : \; f(Y'_n) = 0 \}$$

Omitted: this defines a sub-cochain complex of $C^*(Y;A)$.

Lemma: Let be a simplicial set and A an abelian group. Then there is an isomorphism of cochain complexes

$$\mathrm{Hom}(C(Y;\mathbb{Z}), A) \cong C^*(Y;A)$$, hence an induced isomorphism of cohomology groups

$H^n(Y;A) \cong H^n(C^*(Y;A))$. These isomorphisms are natural for morphisms of simplicial sets in . For a simplicial subset Y' of Y there is an isomorphism of cochain complexes

$$\mathrm{Hom}\left(\frac{C(Y;\mathbb{Z})}{C(Y';\mathbb{Z})}, A\right) \cong C^*(Y,Y';A),$$ natural for morphisms of pairs of simplicial sets.

<u>Proof</u>: In the absolute case: define a specific isomorphism $\varphi : \mathrm{Hom}(C(Y;\mathbb{Z});A) \longrightarrow C^*(Y;A)$ in dimension $n \geq 0$

$$\varphi^n : \mathrm{Hom}(C_n(Y;\mathbb{Z}),A) = \mathrm{Hom}(\mathbb{Z}[Y_n],A) \xrightarrow{\cong} \mathrm{map}(Y_n,A)$$
$$f \longmapsto f|_{Y_n}$$

evaluation at the generators of the free abelian group $\mathbb{Z}[Y_n]$, is an isomorphism of groups.
Omitted: the φ^n's form a morphism of cochain complexes. So $\varphi = \{\varphi^n\}$ is an isomorphism of cochain complexes. □

<u>Schematically</u>:

$$\begin{tikzcd}
\mathrm{Top} \arrow[r, "S"] \arrow[rrrr, bend left, "H_n(-;A)"] \arrow[drrrr, bend right, "H^n(-;A)"'] & (\text{simplicial sets}) \arrow[r, "C(-{,}\mathbb{Z})"] & (\text{chain complexes}) \arrow[r, "A\otimes -"] \arrow[dr, "\mathrm{Hom}(-{,}A)"'] & (\text{chain complexes}) \arrow[r, "H_n"] & \mathrm{Ab} \\
 & & & (\text{cochain complexes})^{op} \arrow[r, "H^n"] & (\mathrm{Ab})^{op}
\end{tikzcd}$$

the key properties of singular homology all have analogs for cohomology:

<u>Homotopy invariance</u>: Let $f,g : X \to Y$ be homotopic continuous maps. Then for all $n \geq 0$ and all abelian groups A,
$$H^n(f;A) = H^n(g;A) : H^n(Y;A) \longrightarrow H^n(X;A).$$

<u>Proof</u>: Since f and g are homotopic, $C(f,\mathbb{Z}), C(g,\mathbb{Z}) : C(X;\mathbb{Z}) \longrightarrow C(Y;\mathbb{Z})$ are chain homotopic.
By an earlier lemma, $\mathrm{Hom}(C(f,\mathbb{Z});A)$ and $\mathrm{Hom}(C(g,\mathbb{Z});A)$ are cochain homotopic, so they induce the same map on cohomology groups. □

<u>Long exact sequence</u>: Let Y' be a simplicial subset of a simplicial set Y. Then we have a short exact sequence of cochain complexes

$$0 \to C^*(Y,Y';A) \to C^*(Y,A) \xrightarrow{\text{restriction}} C^*(Y',A) \to 0$$

$\Rightarrow$ long exact sequence of cohomology groups

$$\cdots \longrightarrow H^n(Y,Y';A) \longrightarrow H^n(Y;A) \longrightarrow H^n(Y',A) \xrightarrow{\partial} H^{n+1}(Y,Y';A) \to \cdots$$

For a subspace X' of a space X, we can apply this to the pair $(S(X), S(X'))$ to get a long exact sequence of singular cohomology groups.

<u>Excision</u>: Let (X,Y,U) be an excisive triple of spaces, i.e. $U \subseteq Y \subseteq X$ and $\bar{U} \subseteq \mathring{Y}$.
In the proof of excision for homology we showed that the inclusions induce a quasi-isomorphism of chain complexes

$$i : \frac{C(S(X\setminus U);\mathbb{Z})}{C(S(Y\setminus U);\mathbb{Z})} \longrightarrow \frac{C(S(X),\mathbb{Z})}{C(S(Y),\mathbb{Z})},$$ i.e. it induces an isomorphism of all homology groups.

<u>Prop</u>: Let $f : C \to D$ be a quasi isomorphism of chain complexes of free abelian groups. Then f is a chain homotopy equivalence.
<u>Proof</u>: Deferred to a separate video.

Since i is a chain homotopy equivalence, $$\mathrm{Hom}(i,A) : \mathrm{Hom}\left(\frac{C(S(X),\mathbb{Z})}{C(S(Y),\mathbb{Z})}, A\right) \longrightarrow \mathrm{Hom}\left(\frac{C(S(X\setminus U),\mathbb{Z})}{C(S(Y\setminus U),\mathbb{Z})}, A\right)$$ is a cochain homotopy equivalence.

So $\mathrm{Hom}(i,A)$ induces isomorphisms of cohomology groups

$$H^n(X,Y;A) = H^n\left(\mathrm{Hom}\left(\frac{C(S(X),\mathbb{Z})}{C(S(Y),\mathbb{Z})}, A\right)\right) \xrightarrow{\cong} H^n\left(\mathrm{Hom}\left(\frac{C(S(X\setminus U),\mathbb{Z})}{C(S(Y\setminus U),\mathbb{Z})}, A\right)\right)$$
$$\cong \qquad\qquad H^n(X\setminus U, Y\setminus U;A)$$

$$\cong \quad H^n(X\setminus U, Y\setminus U; A)$$

inclusion*

Given the same fundamental formal properties, the basic calculations for singular homology can be repeated in much the same way for singular cohomology: for $n \geq 1$

$$H^m(S^n; A) \cong \begin{cases} A & \text{for } m = 0, n \\ 0 & \text{otherwise} \end{cases}$$

$$H^m(D^n, S^{n-1}; A) \cong \begin{cases} A & \text{for } m = n \\ 0 & \text{otherwise.} \end{cases}$$

Reminder: Let X be an absolute CW-complex with skeleta $\{X_n\}_{n\geq 0}$. The cellular chain complex is given by

$$C_n^{cell}(X, \mathbb{Z}) = H_n(X_n, X_{n-1}; \mathbb{Z}) \quad \text{, with cellular differential defined as the composite}$$

$$C_n^{cell}(X, \mathbb{Z}) = H_n(X_n, X_{n-1}; \mathbb{Z}) \xrightarrow{\partial} H_{n-1}(X_{n-1}; \mathbb{Z}) \longrightarrow H_{n-1}(X_{n-1}, X_{n-2}; \mathbb{Z}) = C^{cell}_{n-1}(X, \mathbb{Z})$$

We define the <u>cellular cochain complex</u> of X with coefficients in A as

$$C^*_{cell}(X; A) = \mathrm{Hom}(C_*^{cell}(X, \mathbb{Z}), A)$$

<u>Thm</u>: There is an isomorphism $H^n(C^*_{cell}(X, A)) \cong H^n(X; A)$ that is moreover natural for cellular maps in X.

<u>Proof</u>: Copy for the proof for homology. □

<u>Example</u>: Let X be a CW-complex with no cells in any odd dimension.
Then $C^{cell}_*(X, \mathbb{Z})$ has trivial differentials. So $\mathrm{Hom}(C_*^{cell}(X; \mathbb{Z}), A)$ has trivial differentials and so

$$H^n(X; A) \cong H^n(C^*_{cell}(X, A)) = \underbrace{\mathrm{Hom}(C_n^{cell}(X, \mathbb{Z}), A)}_{= C^*_{cell}(X, A)}$$

$$\cong \mathrm{Hom}(\mathbb{Z}[J_n], A) \cong \mathrm{map}(\underbrace{J_n}_{\text{set of } n\text{-cells}}, A).$$

<u>Example</u> $\mathbb{C}P^\infty$ has a CW-structure with exactly 1 cell in every even dimension and no cells in odd dimensions.

$$H^n(\mathbb{C}P^\infty, A) \cong \begin{cases} A & \text{for } n \geq 0 \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$