The cohomology ring of $\mathbb{C}P^2$

In this section I explain a way to use the cohomological Künneth theorem, plus a degree calculation, to determine the cohomology ring of $\mathbb{C}P^2$. The complex projective space $\mathbb{C}P^2$ has a CW-structure with exactly one cell in dimensions 0, 2 and 4, and with no cells in any other dimension. So the cellular cochain complex has trivial differential, the group $H^n(\mathbb{C}P^2;\mathbb{Z})$ is free of rank 1 for $n = 0, 2, 4$, and trivial in all other dimensions. This does not leave much room for the multiplicative structure: we let

$$x \in H^2(\mathbb{C}P^2;\mathbb{Z})$$

be one of the two generators. The only open questions is how divisible the cup square $x^2 \in H^4(\mathbb{C}P^2;\mathbb{Z})$ is. We will now show that in fact, x^2 is not divisible at all, i.e., x^2 is a generator of $H^4(\mathbb{C}P^2;\mathbb{Z})$.

Construction 1. We use homogeneous coordinate notation to describe elements of $\mathbb{C}P^n$, i.e., for $x_0, \dots, x_n \in \mathbb{C}$, not all equal to 0, we write $[x_0 : \dots : x_n] \in \mathbb{C}P^n$ for the complex line spanned by the vector $(x_0, \dots, x_n)$. In this notation, $[x_0 : \dots : x_n] = [\lambda x_0 : \dots : \lambda x_n]$ for all non-zero complex scalars λ. We define a continuous map

$$\mu \ : \ \mathbb{C}P^1 \times \mathbb{C}P^1 \ \longrightarrow \ \mathbb{C}P^2$$

by

$$\mu([v : w], [x : y]) \ = \ [vx : vy + wx : wy] \ .$$

We take the point $e = [1 : 0]$ as a basepoint of $\mathbb{C}P^1$. Then

$$\mu(e, [x : y]) \ = \ \mu([x : y], e) \ = \ [x : y : 0] \ .$$

So the maps

$$\mu(e, -) \ , \mu(-, e) \ : \ \mathbb{C}P^1 \ \longrightarrow \ \mathbb{C}P^2$$

are the standard embeddings that we often use to consider $\mathbb{C}P^1$ as a subspace of $\mathbb{C}P^2$

The next proposition essentially says that the map $\mu : \mathbb{C}P^1 \times \mathbb{C}P^1 \longrightarrow \mathbb{C}P^2$ has degree 2. Strictly speaking, we have so far only defined the degree for continuous selfmaps of spheres. But as we discuss in more detail later, one can more generally define the degree for a continuous map between oriented, compact, connected manifolds of the same dimension. This makes sense for $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2$, which are both 4-dimensional, orientable, compact connected manifolds.

Proposition 2. *The map*

$$\mu^* \ : \ H^4(\mathbb{C}P^2;\mathbb{Z}) \ \longrightarrow \ H^4(\mathbb{C}P^1 \times \mathbb{C}P^1;\mathbb{Z})$$

is injective and its image has index 2.

Proof. For the course of the proof we simplify the notation by omitted the coefficient ring $\mathbb{Z}$ from the notation for cohomology groups.

The continuous map

$$\pi \ : \ \mathbb{C}^2 \ \longrightarrow \ \mathbb{C}P^2 \ , \quad (a, b) \ \longmapsto \ [a^2 - b : 2a : 1]$$

parameterizes the open 4-cell of the standard CW-structure, i.e., it is an open embedding onto the complement of $\mathbb{C}P^1$ inside $\mathbb{C}P^2$. It will become clear below why we use this particular parameterization. So this map induces an isomorphism of relative cohomology groups

$$H^4(\mathbb{C}P^2 \setminus \mathbb{C}P^1, \mathbb{C}P^2 \setminus (\mathbb{C}P^1 \cup [0 : 0 : 1])) \ \xrightarrow{\cong} \ H^4(\mathbb{C}^2, \mathbb{C}^2 \setminus (0, 0)) \ .$$

Excision provides an isomorphism

$$H^4(\mathbb{C}P^2, \mathbb{C}P^2 \setminus [0 : 0 : 1]) \ \xrightarrow{\cong} \ H^4(\mathbb{C}P^2 \setminus \mathbb{C}P^1, \mathbb{C}P^2 \setminus (\mathbb{C}P^1 \cup [0 : 0 : 1])) \ .$$

Since $\mathbb{C}P^2 \setminus [0 : 0 : 1]$ is homotopy equivalent to $\mathbb{C}P^1$, its cohomology groups in dimension 3 and 4 are trivial, and so the map

$$H^4(\mathbb{C}P^2, \mathbb{C}P^2 \setminus [0 : 0 : 1]) \ \longrightarrow \ H^4(\mathbb{C}P^2)$$

from relative to absolute cohomology is an isomorphism in dimension 4. When combined, we arrive at an isomorphism

$$H^4(\mathbb{C}P^2) \;\cong\; H^4(\mathbb{C}^2, \mathbb{C}^2 \setminus (0,0)) \ .$$

Similarly, the continuous map

$$\pi' \;:\; \mathbb{C}^2 \;\longrightarrow\; \mathbb{C}P^1 \times \mathbb{C}P^1 \ , \quad (a,b) \;\longmapsto\; ([a+b:1],[a-b:1])$$

parameterizes the open 4-cell of the product CW-structure, i.e., it is an open embedding onto the complement of $\mathbb{C}P^1 \vee \mathbb{C}P^1$ inside $\mathbb{C}P^1 \times \mathbb{C}P^1$. As similar reasoning as for $\mathbb{C}P^2$ above yields an composite isomorphism

$$H^4(\mathbb{C}P^1 \times \mathbb{C}P^1) \;\cong\; H^4(\mathbb{C}^2, \mathbb{C}^2 \setminus (0,0)) \ .$$

We define

$$\nu \;:\; \mathbb{C}^2 \;\longrightarrow\; \mathbb{C}^2 \quad \text{by} \quad \nu(a,b) \;=\; (a,b^2) \ .$$

The following square commutes by direct inspection – this is why the two parameterizations were chosen in this particular form:

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[r, "\nu"] \arrow[d, "\pi'"'] & \mathbb{C}^2 \arrow[d, "\pi"] \\
\mathbb{C}P^1 \times \mathbb{C}P^1 \arrow[r, "\mu"'] & \mathbb{C}P^2
\end{tikzcd}$$

Hence the following square of cohomology groups commutes as well:

$$\begin{tikzcd}
H^4(\mathbb{C}P^2) \arrow[r, "\mu^*"] \arrow[d, "\cong"'] & H^4(\mathbb{C}P^1 \times \mathbb{C}P^1) \arrow[d, "\cong"] \\
H^4(\mathbb{C}^2, \mathbb{C}^2 \setminus (0,0)) \arrow[r, "\nu^*"'] & H^4(\mathbb{C}^2, \mathbb{C}^2 \setminus (0,0))
\end{tikzcd}$$

This reduces the claim to showing that the lower horizontal map ν^* is multiplication by 2.

Since the map ν is the product of the identity on the first factor of $\mathbb{C}^2$ and the squaring map in the second factor, the following square commutes:

$$\begin{tikzcd}
H^2(\mathbb{C}, \mathbb{C} \setminus 0) \times H^2(\mathbb{C}, \mathbb{C} \setminus 0) \arrow[r, "\mathrm{Id} \times (b \mapsto b^2)^*"] \arrow[d, "\times"', "\cong"] & H^2(\mathbb{C}, \mathbb{C} \setminus 0) \times H^2(\mathbb{C}, \mathbb{C} \setminus 0) \arrow[d, "\cong"', "\times"] \\
H^4(\mathbb{C}^2, \mathbb{C}^2 \setminus (0,0)) \arrow[r, "\nu^*"'] & H^4(\mathbb{C}^2, \mathbb{C}^2 \setminus (0,0))
\end{tikzcd}$$

The vertical maps are the relative exterior cup product pairings, which are isomorphisms in this particular situation. For every $m \in \mathbb{Z} \setminus 0$, the map selfmap of $\mathbb{C}$ sending b to b^m induces multiplication by m on $H^2(\mathbb{C}; \mathbb{C} \setminus 0)$, so this concludes the proof. □

Theorem 3. *Let $x \in H^2(\mathbb{C}P^2; \mathbb{Z})$ be an additive generator. Then the class x^2 is a generator of the group $H^4(\mathbb{C}P^2; \mathbb{Z})$. So the integral cohomology ring of $\mathbb{C}P^2$ is a truncated polynomial algebra:*

$$H^*(\mathbb{C}P^2; \mathbb{Z}) \;=\; \mathbb{Z}[x]/(x^3) \ .$$

Proof. For the course of the proof we simplify the notation by omitted the coefficient ring $\mathbb{Z}$ from the notation for cohomology groups. We write $i : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^2$ for the standard embedding, i.e., $i[u:v] = [u:v:0]$. Then the class $i^*(x) \in H^2(\mathbb{C}P^1)$ is a generator, and the integral cohomology of $\mathbb{C}P^1$ is free of rank 2 with basis 1 and $i^*(x)$. Since the integral cohomology of $\mathbb{C}P^1$ is finitely generated and free in every dimension, the cohomological Künneth theorem shows that the exterior product map is an isomorphism of graded rings

$$\times \ : \ H^*(\mathbb{C}P^1) \otimes H^*(\mathbb{C}P^1) \ \longrightarrow \ H^*(\mathbb{C}P^1 \times \mathbb{C}P^1) \ .$$

We define

$$\begin{aligned} a \ &= \ p_1^*(i^*(x)) \ \in \ H^2(\mathbb{C}P^1 \times \mathbb{C}P^1) \\ b \ &= \ p_2^*(i^*(x)) \ \in \ H^2(\mathbb{C}P^1 \times \mathbb{C}P^1) \ , \end{aligned}$$

so that

$$H^*(\mathbb{C}P^1 \times \mathbb{C}P^1) \ = \ \mathbb{Z}\{1, a, b, ab\} \ ,$$

is a free abelian group of rank 4. The multiplicative structure is determined by the relations $a^2 = b^2 = 0$ and $ba = ab$. Since $\mathbb{C}P^1$ is homeomorphic to S^2, this is essentially a special case of the discussion from the previous video.

We claim that the relation

$$\mu^*(x) \ = \ a + b \tag{4}$$

holds in $H^2(\mathbb{C}P^1 \times \mathbb{C}P^1)$.

We recall that $e = [1 : 0]$ is the basepoint of $\mathbb{C}P^1$, and that the two maps $\mu(e, -)$ and $\mu(-, e)$ both coincide with the standard embedding $i : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^2$. So we deduce the relation

$$\begin{aligned} (e, -)^*(a + b) \ &= \ (e, -)^*(p_1^*(i^*(x))) + (e, -)^*(p_2^*(i^*(x))) \\ &= \ (i \circ p_1 \circ (e, -))^*(x) + (i \circ p_2 \circ (e, -))^*(x) \\ &= \ i^*(x) \ = \ \mu(e, -)^*(x) \ = \ (e, -)^*(\mu^*(x)) \end{aligned}$$

in the group $H^2(\mathbb{C}P^2)$. The third equation uses that $p_2 \circ (e, -)$ is the identity of $\mathbb{C}P^1$; and $p_1 \circ (e, -)$ is a constant map, and hence induces the trivial map in H^2. Similarly, $(-, e)^*(a + b) = (-, e)^*(\mu^*(x))$.

The product $\mathbb{C}P^1 \times \mathbb{C}P^1$ can be obtained from the wedge $\mathbb{C}P^1 \vee \mathbb{C}P^1$ by attaching a 4-cell. So the inclusion $\mathbb{C}P^1 \vee \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1$ induces an isomorphism in second cohomology. Together with additivity of cohomology on wedges this shows that the map

$$((e, -)^*, (-, e)^*) \ : \ H^2(\mathbb{C}P^1 \times \mathbb{C}P^1) \ \longrightarrow \ H^2(\mathbb{C}P^1) \times H^2(\mathbb{C}P^1)$$

is an isomorphism. We have just shown that the relation (4) holds after applying this isomorphism. So we have proved the relation (4).

Proposition 2 shows that the map $\mu^* : H^4(\mathbb{C}P^2) \longrightarrow H^4(\mathbb{C}P^1 \times \mathbb{C}P^1)$ i injective and its image has index 2. Since the target is generated by the class ab, exactly one of the two additive generators y of $H^4(\mathbb{C}P^2)$ satisfies $\mu^*(y) = 2ab$. Then $x^2 = ny$ for a unique integer $n \in \mathbb{Z}$. We square the relation (4) in the cohomology ring of $\mathbb{C}P^1 \times \mathbb{C}P^1$ and obtain

$$2n \cdot ab \ = \ n \cdot \mu^*(y) \ = \ \mu^*(ny) \ = \ \mu^*(x^2) \ = \ (\mu^*(x))^2 \ = \ (a + b)^2 \ = \ 2ab \ .$$

In the last step we have exploited that a and b commute and that $a^2 = b^2 = 0$. Since ab is a generator of the torsion free abelian group $H^4(\mathbb{C}P^1 \times \mathbb{C}P^1)$, this relation forces $n = 1$. This concludes the proof that $x^2 = y$ is an additive generator of $H^4(\mathbb{C}P^2)$. □

We will show later that this calculation is just the beginning of a whole pattern: more generally the integral cohomology ring of $\mathbb{C}P^m$ polynomial algebra truncated at height $m + 1$, i.e.,

$$H^*(\mathbb{C}P^m; \mathbb{Z}) \ = \ \mathbb{Z}[x]/(x^{m+1}) \ ;$$

and $H^*(\mathbb{C}P^\infty; \mathbb{Z}) = \mathbb{Z}[x]$ is a polynomial algebra, where always x is an additive generator of the cohomology in dimension 2.