The fundamental class

Sunday, 13 June 2021 11:48

Homology of some connected compact manifolds: $\quad H_n(S^n;\mathbb{Z}) \cong H_{2n}(\mathbb{CP}^n;\mathbb{Z}) = H_{4n}(\mathbb{HP}^n;\mathbb{Z}) \cong \mathbb{Z}$

$$H_n(\mathbb{RP}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even}, \ n>0. \end{cases}$$

We will show today that for a connected, compact, non-empty n-manifold M, $\quad H_n(M;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } M \text{ is orientable} \\ 0 & \text{if } M \text{ is not orientable.} \end{cases}$

<u>Thm</u>: Let M be an oriented n-manifold and let K be a compact subset of M. Then there is a unique class $\mu_K \in H_n(M|K) = H_n(M, M\setminus K;\mathbb{Z})$ such that

$$r^K_x(\mu_K) = \mu_x = \text{given local orientation in } H_n(M|x) \text{ for all } x\in K.$$

<u>Important special case</u>: If M is itself compact, we can take $K=M$; there is thus a unique class $\mu_M = [M] \in H_n(M;\mathbb{Z})$, <u>the fundamental class</u> such that $r^M_x[M] = \mu_x$ in $H_n(M|x)$ for all $x\in M$.

We will show later that if M is moreover connected and non-empty then $H_n(M,\mathbb{Z}) \cong \mathbb{Z}$ with generator $[M]$.

<u>Remark</u>: Let M be oriented, compact, connected. Let $\bar{M}$ be M with the opposite orientation. Then

$$[\bar{M}] = -[M] \quad \text{because both have the same image in } H_n(M, M\setminus\{x\};\mathbb{Z}) \text{ for all } x\in M.$$

<u>Remark</u>: Compactness is necessary: let M be an n-manifold and $u \in H_n(M;A)$ (A: some abelian group) such that $r^M_x(u) \neq 0$ in $H_n(M|x;A)$ for all $x\in M$. Then M is compact.

Indeed, write $u=[x]$ for some n-cycle $x\in C_n(M;A)$. Write $x = \sum_{\text{finite}} a_i \cdot (f_i : D^n \to M)$, and set $L = \operatorname{supp}(x) = \bigcup f_i(D^n)$, some compact subset of M.

If M is not compact, there is a point $x\in M\setminus L$. So $x\in C_n(M\setminus\{x\};A)$ is a cycle in $M\setminus\{x\}$. So $u=[x]$ has zero image in $H_n(M, M\setminus\{x\};A) = H_n\left(C_*(M;A)/C_*(M\setminus\{x\};A)\right)$.

This contradicts the assumption on u.

<u>Proof</u> of the theorem (following the proof of Theorem A.8 in Appendix A of Milnor-Stasheff's "Characteristic Classes"):

Uniqueness of μ_K follows from the detection property of classes in $H_n(M|K)$ from the previous video.

<u>Step 1</u>: Suppose K is contained in an open subset U of M with $U \cong \mathbb{R}^n$. So K is contained in a local ball B (in the sense defined when defining orientations). Let $x,y\in K$ be any two points. We get a commutative diagram:

$$\begin{tikzcd}
 & H_n(M|B) \arrow[d, "r^B_K"] \arrow[ddl, bend right, "r^B_x"', "\cong"] \arrow[ddr, bend left, "r^B_y", "\cong"'] & \\
 & H_n(M|K) \arrow[dl, "r^K_x"'] \arrow[dr, "r^K_y"] & \\
\mu_x \in H_n(M|x) & & H_n(M|y) \ni \mu_y
\end{tikzcd}$$

The continuity in the choice of local orientations means that μ_x and μ_y have the same preimage μ_B in $H_n(M|B)$.

Then $\mu_K = r^B_K(\mu_B)$ has the desired property.

<u>Step 2</u>: Suppose that $K = K_1 \cup K_2$ for K_1, K_2 compact subsets that satisfy the conclusion.

Let $\mu_{K_1} \in H_n(M|K_1)$ and $\mu_{K_2} \in H_n(M|K_2)$ be the two classes. Then

$$r^{K_1}_{K_1\cap K_2}(\mu_{K_1}),\ \mu^{K_2}_{K_1\cap K_2}(\mu_{K_2}) \in H_n(M|K_1\cap K_2) \text{ are two classes with the same}$$

restriction to $H_n(M|x)$ for all $x\in K_1\cap K_2$. So these two classes agree in $H_n(M|K_1\cap K_2)$ by the uniqueness property of the previous video. The Mayer-Vietoris sequence from the previous video contains the following part:

$$0 \longrightarrow H_n(M|K) \xrightarrow{(r^K_{K_1},\, r^K_{K_2})} H_n(M|K_1) \oplus H_n(M|K_2) \longrightarrow H_n(M|K_1\cap K_2)$$

$$(u,v) \longmapsto r^{K_1}_{K_1\cap K_2}(u) - r^{K_2}_{K_1\cap K_2}(v)$$

$$\mu_K \longmapsto (\mu_{K_1}, \mu_{K_2}) \longmapsto 0$$

Exactness provides a class $\mu_K \in H_n(M|K)$ with $r^K_{K_1}(\mu_K) = \mu_{K_1}$, $r^K_{K_2}(\mu_K) = \mu_{K_2}$.

Since every $x\in K$ is contained in K_1 or in K_2, $\quad r^K_x(\mu_K) = \begin{cases} r^{K_1}_x(r^K_{K_1}(\mu_K)) & x\in K_1 \\ r^{K_2}_x(r^K_{K_2}(\mu_K)) & x\in K_2 \end{cases}$

$= \mu_x$ for all $x\in K$.

<u>Step 3</u>: M and K are general. As in Step 6 of the proof in the previous video, we write $K = K_1 \cup \dots \cup K_m$ where each K_i is compact and contained in an open set homeomorphic to $\mathbb{R}^n$. Then μ_{K_i} exists for all $i=1,\dots,m$ by Step 1. Induction on m and Step 2 provide the class $\mu_K \in H_n(M|K)$. $\square$

Step 3: K in general. As in step 3 of the proof in the previous video, we write $K = K_1 \cup \dots \cup K_m$ where each K_i is compact and contained in an open set homeomorphic to $\mathbb{R}^n$. Then μ_{K_i} exists for all $i = 1, \dots, m$ by Step 1. Induction on m and Step 2 provide the class $\mu_K \in H_n(M|K)$. $\square$

Corollary: Let M be an orientable, compact, connected n-manifold. Then for all $x \in M$, the map
$$r^M_x : H_n(M;\mathbb{Z}) \longrightarrow H_n(M, M \setminus \{x\}; \mathbb{Z}) \quad \text{is an isomorphism.}$$
So if M is non-empty, $H_n(M;\mathbb{Z})$ is free of rank 1, and the fundamental class of the two orientations are the two generators.

Proof: Let $\alpha \in H_n(M)$ be any class. Then the set $\{x \in M : r^M_x(\alpha) = 0 \text{ in } H_n(M|x)\}$ is open and closed.
Indeed let $x \in M$ be such that $r^M_x(\alpha) = 0$ (or $r^M_x(\alpha) \neq 0$). Let B be a local ball in M containing x.
Then we consider the commutative diagram

$$\begin{tikzcd}
 & H_n(M) \ni \alpha \arrow[dl, bend right, "r^M_x"'] \arrow[d, "r^M_B"] \arrow[dr, bend left, "r^M_y"] & \\
H_n(M|x) & H_n(M|B) \arrow[l, "r^B_x", "\cong"'] \arrow[r, "r^B_y"', "\cong"] & H_n(M|y)
\end{tikzcd}$$

Let $y \in B$ be another point. So if $r^M_x(\alpha) = 0$ (or $r^M_x(\alpha) \neq 0$) then also $r^M_y(\alpha) = 0$ (or $r^M_y(\alpha) \neq 0$) for all $y \in B$.

So the sets $\{x \in M : r^M_x(\alpha) = 0\}$ and $\{x \in M : r^M_x(\alpha) \neq 0\}$ are both open, and hence also closed.

We show next that for all $x \in M$, $r^M_x : H_n(M) \longrightarrow H_n(M|x)$ is injective, using connectedness of M.

Indeed suppose that $\alpha \in \ker(r^M_x)$. Then the set $\{y \in M : r^M_y(\alpha) = 0\}$ is open, closed and non-empty.
Since M is connected, this set is all of M. So $\alpha = 0$ by the detection property.

The map $r^M_x : H_n(M) \longrightarrow H_n(M|x)$ is also surjective: choose an orientation, then the fundamental class $[M] \in H_n(M)$ restricts to a generator of $H_n(M|x)$. $\square$

Corollary: Let M be a compact connected n-manifold that is not orientable. Then $H_n(M;\mathbb{Z}) = 0$.

Proof: As in the previous proof we show that:

- for all $\alpha \in H_n(M;\mathbb{Z})$, the set $\{x \in M : r^M_x(\alpha) = 0 \text{ in } H_n(M|x;\mathbb{Z})\}$ is open and closed.
- the map $r^M_x : H_n(M;\mathbb{Z}) \longrightarrow H_n(M|x;\mathbb{Z}) \cong \mathbb{Z}$ is injective for all $x \in M$.

So the group $H_n(M;\mathbb{Z})$ is torsion free.

Let $p : \hat{M} \to M$ be the orientation covering. Then $\hat{M}$ is orientable (comes with a tautological orientation).
Since M is not orientable, $\hat{M}$ is connected. So $H_n(\hat{M};\mathbb{Z}) \cong \mathbb{Z}$, generated by $[\hat{M}]$.
Let $\tau : \hat{M} \to \hat{M}$ be the non-identity deck transformation. Then τ is orientation reversing, and hence
$$\tau_*[\hat{M}] = -[\hat{M}].$$
Then $p_*[\hat{M}] \underset{p\tau = p}{=} p_*(\tau_*[\hat{M}]) = -p_*[\hat{M}]$ in $H_n(M;\mathbb{Z})$.

So $2 \cdot p_*[\hat{M}] = 0$, hence $p_*[\hat{M}] = 0$ (because $H_n(M;\mathbb{Z})$ is torsion free).

So $p_* = 0 : H_n(\hat{M};\mathbb{Z}) \longrightarrow H_n(M;\mathbb{Z})$ is the zero homomorphism.

The composite
$$\underbrace{H_n(M;\mathbb{Z}) \xrightarrow{\text{transfer}} H_n(\hat{M};\mathbb{Z}) \xrightarrow{p_* = 0} H_n(M;\mathbb{Z})}_{\times 2}$$

So $\times 2$ is zero on the group $H_n(M;\mathbb{Z})$; since $H_n(X;\mathbb{Z})$ is also torsion free, we conclude that $H_n(M;\mathbb{Z}) = 0$. $\square$