Algebraic Geometry

Vanya Cheltsov

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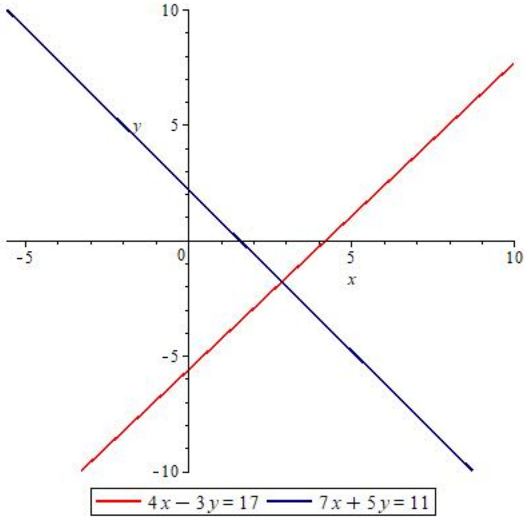
Lecture 1: complex projective plane

What is a line?



Complex plane

Definition

A line in $\mathbb{C}^2$ is a subset that is given by

$$\boxed{\mathbf{a}x + \mathbf{b}y + \mathbf{c} = 0}$$

for some complex numbers $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ such that $(\mathbf{a}, \mathbf{b}) \neq (0, 0)$.

- Here x and y are coordinates on $\mathbb{C}^2$.

Lemma

There is a unique line in $\mathbb{C}^2$ passing through two distinct points.

Proof.

Let $(\mathbf{x}_1, \mathbf{y}_1)$ and $(\mathbf{x}_2, \mathbf{y}_2)$ be two distinct points. Then

$$\Big(\mathbf{y}_2 - \mathbf{y}_1\Big)\Big(x - \mathbf{x}_1\Big) = \Big(\mathbf{x}_2 - \mathbf{x}_1\Big)\Big(y - \mathbf{y}_1\Big)$$

defines the line that contains $(\mathbf{x}_1, \mathbf{y}_1)$ and $(\mathbf{x}_2, \mathbf{y}_2)$. $\square$

Intersection of two lines

- Let L_1 be a line in $\mathbb{C}^2$ that is given by
$$\mathbf{a}_1 x + \mathbf{b}_1 y = \mathbf{c}_1,$$
where $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{c}_1$ are complex numbers and $(\mathbf{a}_1, \mathbf{b}_1) \neq (0, 0)$.
- Let L_2 be a line in $\mathbb{C}^2$ that is given by
$$\mathbf{a}_2 x + \mathbf{b}_2 y = \mathbf{c}_2$$
where $\mathbf{a}_2$, $\mathbf{b}_2$, $\mathbf{c}_2$ are complex numbers and $(\mathbf{a}_2, \mathbf{b}_2) \neq (0, 0)$.

Lemma

Suppose that $L_1 \neq L_2$. Then $L_1 \cap L_2$ consists of at most one point.

Proof.

If $\mathbf{a}_1\mathbf{b}_2 - \mathbf{a}_2\mathbf{b}_1 \neq 0$, then $L_1 \cap L_2$ consists of the point
$$\boxed{\left(\frac{\mathbf{b}_2\mathbf{c}_1 - \mathbf{b}_1\mathbf{c}_2}{\mathbf{a}_1\mathbf{b}_2 - \mathbf{a}_2\mathbf{b}_1}, \frac{\mathbf{a}_1\mathbf{c}_2 - \mathbf{a}_2\mathbf{c}_1}{\mathbf{a}_1\mathbf{b}_2 - \mathbf{a}_2\mathbf{b}_1}\right).}$$

If $\mathbf{a}_1\mathbf{b}_2 - \mathbf{a}_2\mathbf{b}_1 = 0$, then $L_1 \cap L_2 = \varnothing$. $\square$

Conics

Definition

A conic in $\mathbb{C}^2$ is a subset that is given by

$$\boxed{\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}x + \mathbf{e}y + \mathbf{f} = 0,}$$

where **a**, **b**, **c**, **d**, **e**, **f** are complex numbers and $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \neq (0, 0, 0)$.

The conic is said to be *irreducible* if the polynomial

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}x + \mathbf{e}y + \mathbf{f}$$

is *irreducible*. Otherwise the conic is called *reducible*.

- If $\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}x + \mathbf{e}y + \mathbf{f}$ is *reducible*, then

  $$\boxed{\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}x + \mathbf{e}y + \mathbf{f} = (\alpha x + \beta y + \gamma)(\alpha' x + \beta' y + \gamma')}$$

  for some complex numbers α, β, γ, α', β', γ'.
- In this case the conic is a union of two lines.

Matrix form

Let C be a conic in $\mathbb{C}^2$ that is given by

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}x + \mathbf{e}y + \mathbf{f} = 0,$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ are complex numbers and $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \neq (0, 0, 0)$.

- We can rewrite the equation of the conic C as

$$\boxed{\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a} & \frac{\mathbf{b}}{2} & \frac{\mathbf{d}}{2} \\ \frac{\mathbf{b}}{2} & \mathbf{c} & \frac{\mathbf{e}}{2} \\ \frac{\mathbf{d}}{2} & \frac{\mathbf{e}}{2} & \mathbf{f} \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0.}$$

- Denote this 3×3 matrix by M.

Lemma

The conic C is irreducible if and only if $\det(M) \neq 0$.

Proof.

We will prove this on Thursday. $\square$

Intersection of a line and a conic

Let L be a line in $\mathbb{C}^2$. Let C be an *irreducible* conic in $\mathbb{C}^2$.

Lemma

The intersection $L \cap C$ consists of at most 2 points.

Proof.

The line L is given by

$$\alpha x + \beta y + \gamma = 0$$

for some complex numbers α, β, γ such that $(\alpha, \beta) \neq (0, 0)$.
The conic C is given by a polynomial equation

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}x + \mathbf{e}y + \mathbf{f} = 0,$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ are complex numbers and $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \neq (0, 0, 0)$.
Then the intersection $L \cap C$ is given by

$$\begin{cases} \alpha x + \beta y + \gamma = 0, \\ \mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}x + \mathbf{e}y + \mathbf{f} = 0. \end{cases}$$

Five points determine a conic

Let P_1, P_2, P_3, P_4, P_5 be distinct points in $\mathbb{C}^2$.

- Suppose that no 4 points among them are collinear.

Theorem

There is a unique conic in $\mathbb{C}^2$ that contains P_1, P_2, P_3, P_4, P_5.

Proof.

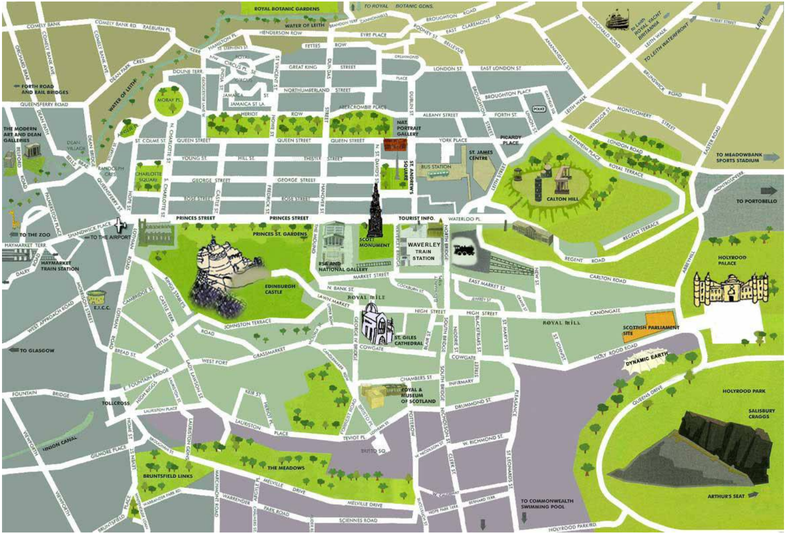
Let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, $P_3 = (x_3, y_3)$, $P_4 = (x_4, y_4)$, $P_5 = (x_5, y_5)$.
Find complex numbers **a**, **b**, **c**, **d**, **e**, **f** such that

$$\begin{cases} \mathbf{a}x_1^2 + \mathbf{b}x_1y_1 + \mathbf{c}y_1^2 + \mathbf{d}x_1 + \mathbf{e}y_1 + \mathbf{f} = 0, \\ \mathbf{a}x_2^2 + \mathbf{b}x_2y_2 + \mathbf{c}y_2^2 + \mathbf{d}x_2 + \mathbf{e}y_2 + \mathbf{f} = 0, \\ \mathbf{a}x_3^2 + \mathbf{b}x_3y_3 + \mathbf{c}y_3^2 + \mathbf{d}x_3 + \mathbf{e}y_3 + \mathbf{f} = 0, \\ \mathbf{a}x_4^2 + \mathbf{b}x_4y_4 + \mathbf{c}y_4^2 + \mathbf{d}x_4 + \mathbf{e}y_4 + \mathbf{f} = 0, \\ \mathbf{a}x_5^2 + \mathbf{b}x_5y_5 + \mathbf{c}y_5^2 + \mathbf{d}x_5 + \mathbf{e}y_5 + \mathbf{f} = 0. \end{cases}$$

Then the conic containing P_1, P_2, P_3, P_4, P_5 is given by

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}x + \mathbf{e}y + \mathbf{f} = 0.$$

What is a plane?



What is a plane?

What is a plane?

What is a plane?

What is a plane?

Complex projective line

- Let $\sim$ be a relation on $\mathbb{C}^2 \setminus (0,0)$ such that

$$\boxed{\big(x,y\big) \sim \big(x',y'\big) \iff \exists\, \lambda \in \mathbb{C}^\star \;\big|\; \big(x,y\big) = \big(\lambda x', \lambda y'\big)}$$

  for any (x,y) and (x',y') in $\mathbb{C}^2 \setminus (0,0)$.
- Then $\sim$ is an equivalence relation.

Definition

The complex projective line $\mathbb{P}^1_{\mathbb{C}}$ is $(\mathbb{C}^2 \setminus (0,0))/\sim$.

- We refer to the elements of the set $\mathbb{P}^1_{\mathbb{C}}$ as points.
- We denote by $[x:y]$ the equivalence of $(x,y) \neq (0,0)$.

We consider elements of $\mathbb{P}^1_{\mathbb{C}}$ as 2-tuples $[x:y]$ such that

$$\boxed{\big[x:y\big] = \big[x':y'\big] \iff \exists\, \lambda \in \mathbb{C}^\star \;\big|\; \big(x,y\big) = \big(\lambda x', \lambda y'\big)}$$

excluding the 2-tuple $[0:0]$ (bad point)!

A point at infinity

Put $P = [1:0] \in \mathbb{P}^1_{\mathbb{C}}$ and $U = \mathbb{P}^1_{\mathbb{C}} \setminus P$. Then

$$[x:y] = \begin{cases} \left[1 : \dfrac{y}{x}\right] & \text{if } x \neq 0 \\ \left[\dfrac{x}{y} : 1\right] & \text{if } y \neq 0 \end{cases}$$

for every point $[x:y] \in \mathbb{P}^1_{\mathbb{C}}$.

Corollary

The map $U \to \mathbb{C}$ given by

$$[x:y] \mapsto \frac{x}{y}$$

is a bijection.

Thus, we can identify $U = \mathbb{C}$ with coordinate $\overline{x} = \frac{x}{y}$.

- We can refer to P as a point at infinity.

Complex projective plane (formal definition)

- Let $\sim$ be a relation on $\mathbb{C}^3 \setminus (0,0,0)$ such that

$$\boxed{(x,y,z) \sim (x',y',z') \iff \exists\, \lambda \in \mathbb{C}^\star \mid (x,y,z) = (\lambda x', \lambda y', \lambda z')}$$

  for any (x,y,z) and (x',y',z') in $\mathbb{C}^3 \setminus (0,0,0)$.
- Then $\sim$ is an equivalence relation.

Definition
The projective plane $\mathbb{P}^2_{\mathbb{C}}$ is $(\mathbb{C}^3 \setminus (0,0,0))/\sim$.

- We refer to the elements of the set $\mathbb{P}^2_{\mathbb{C}}$ as points.
- We denote by $[x : y : z]$ the equivalence class of (x,y,z).

We consider points in $\mathbb{P}^2_{\mathbb{C}}$ as 3-tuples $[x : y : z]$ such that

$$\boxed{[x : y : z] = [x' : y' : z'] \iff \exists\, \lambda \in \mathbb{C}^\star \mid (x,y,z) = (\lambda x', \lambda y', \lambda z'),}$$

excluding the 3-tuple $[0 : 0 : 0]$ (bad point)!

Complex projective plane (informal definition)

- Let (x, y, z) be a point in $\mathbb{C}^3$ such that $(x, y, z) \neq (0, 0, 0)$.
- Let $[x : y : z]$ be the subset in $\mathbb{C}^3$ such that

$$\big(a, b, c\big) \in \big[x : y : z\big] \iff \begin{cases} a = \lambda x \\ b = \lambda y \\ c = \lambda z \end{cases}$$

  for some non-zero complex number λ.

Definition

The projective plane $\mathbb{P}^2_{\mathbb{C}}$ is the set of all possible $[x : y : z]$.

- We refer to the elements of $\mathbb{P}^2_{\mathbb{C}}$ as points.
- We have $[1 : 2 : 3] = [7 : 14 : 21] = [2 - i : 4 - 4i : 3 - 3i]$.
- We have $[1 : 2 : 3] \neq [3 : 2 : 1]$ and $[0 : 0 : 1] \neq [0 : 1 : 0]$.
- Remember, there is no such point as $[0 : 0 : 0]$.

How to live in projective plane?

Let U_z be the subset in $\mathbb{P}^2_{\mathbb{C}}$ consisting of points $[x : y : z]$ with $z \neq 0$.

Lemma

The map $U_z \to \mathbb{C}^2$ given by

$$\left[x : y : z\right] = \left[\frac{x}{z} : \frac{y}{z} : 1\right] \mapsto \left(\frac{x}{z}, \frac{y}{z}\right)$$

is a bijection (one-to-one and onto).

- Thus, we can identify $U_z = \mathbb{C}^2$.
- Put $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$.
- Then we can consider $\overline{x}$ and $\overline{y}$ as coordinates on $U_z = \mathbb{C}^2$.

Question

What is $\mathbb{P}^2_{\mathbb{C}} \setminus U_z$?

- The subset in $\mathbb{P}^2_{\mathbb{C}}$ consisting of points $[x : y : 0]$.
- We can identify $\mathbb{C}^2 \setminus U_z$ and $\mathbb{P}^1_{\mathbb{C}}$.
- This is a line at infinity.

A line at infinity

What is a line?

Definition

A line in $\mathbb{P}^2_{\mathbb{C}}$ is the subset given by

$$Ax + By + Cz = 0$$

for some (fixed) point $[A : B : C] \in \mathbb{P}^2_{\mathbb{C}}$.

Example

Let $P = [5 : 0 : -2]$. Let $Q = [1 : -1 : 1]$. Then the line

$$2x - 3y + 5z = 0$$

contains P and Q. It is the only line in $\mathbb{P}^2_{\mathbb{C}}$ that contains P and Q.

Example

Let L be the line in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$x + 2y + 3z = 0.$$

Let L' be the line given by $x - y = 0$. Then $L \cap L' = [1 : 1 : -1]$.

Lines and points in projective plane

- Let P and Q be two points in $\mathbb{P}^2_{\mathbb{C}}$ such that $P \neq Q$.

Theorem

There is a unique line in $\mathbb{P}^2_{\mathbb{C}}$ that contains P and Q.

Proof.

Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$ that is given by $Ax + By + Cz = 0$.
If $P = [a : b : c] \in L$ and $Q = [a' : b' : c'] \in L$, then

$$\begin{cases} Aa + Bb + Cc = 0, \\ Aa' + Bb' + Cc' = 0. \end{cases}$$

The rank–nullity theorem implies that L exists and is unique. $\square$

- Let L and L' be two lines in $\mathbb{P}^2_{\mathbb{C}}$ such that $L \neq L'$.

Theorem

The intersection $L \cap L'$ consists of one point in $\mathbb{P}^2_{\mathbb{C}}$.

How to find a line passing through two points?

Let us find the line in $\mathbb{P}^2_{\mathbb{C}}$ that contains $[11 : -7 : 1]$ and $[2 : 5 : 1]$. We have to solve the system of linear equations

$$\begin{cases} 11A - 7B + C = 0, \\ 2A + 5B + C = 0. \end{cases}$$

The solutions of this system form a one-dimensional vector space. One solution is $(A, B, C) = (4, 3, -23)$. Thus, the required line is given by $4x + 3y - 23z = 0$. Here is the Maple's code we used:

```
solution:=solve([11*A-7*B+C=0,2*A+5*B+C=0,C=23],{A,B,C}):
L:=eval([A,B,C],solution):
x*L[1]+y*L[2]+z*L[3];
```

We can find the same line using explicit determinant equation

$$\det \begin{pmatrix} 11 & -7 & 1 \\ 2 & 5 & 1 \\ x & y & z \end{pmatrix} = 0.$$

Where parallel lines meet?

Parallel lines

Let U_z be the complement in $\mathbb{P}^2_{\mathbb{C}}$ to the line $z = 0$. Identify

$$\boxed{U_z = \mathbb{C}^2}$$

with coordinates $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$.

- Let $\overline{L}$ be the line in $U_z = \mathbb{C}^2$ given by $2\overline{x} - 3\overline{y} + 5 = 0$.
- Let $\overline{L}'$ be the line in $U_z = \mathbb{C}^2$ given by $2\overline{x} - 3\overline{y} + 7 = 0$.
- Then the intersection $\overline{L} \cap \overline{L}'$ is empty.

Question

Where do $\overline{L}$ and $\overline{L}'$ meet?

- Let L be the line in $\mathbb{P}^2_{\mathbb{C}}$ given by $2x - 3y + 5z = 0$.
- Let L' be the line in $\mathbb{P}^2_{\mathbb{C}}$ given by $2x - 3y + 7z = 0$.
- Then the lines L and L' meet at $[3 : 2 : 0]$.

Conics

Definition

A conic in $\mathbb{P}^2_{\mathbb{C}}$ is a subset that is given by

$$\boxed{\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2 = 0}$$

for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ in $\mathbb{C}$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}) \neq (0, 0, 0, 0, 0, 0)$.

The conic is said to be *irreducible* if the polynomial

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2$$

is *irreducible*. Otherwise the conic is called *reducible*.

If $\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2$ is *reducible*, then

$$\boxed{\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2 = (\alpha x + \beta y + \gamma z)(\alpha' x + \beta' y + \gamma' z)}$$

for some complex numbers α, β, γ, α', β', γ'.

In this case the conic is a union of two lines.

Matrix form

Let $\mathcal{C}$ be a conic in $\mathbb{P}^2_{\mathbb{C}}$. Then $\mathcal{C}$ that is given by

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2$$

for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ in $\mathbb{C}$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}) \neq (0,0,0,0,0,0)$.

- Rewrite the equation of the conic $\mathcal{C}$ in the matrix form:

$$\boxed{\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} \mathbf{a} & \frac{\mathbf{b}}{2} & \frac{\mathbf{d}}{2} \\ \frac{\mathbf{b}}{2} & \mathbf{c} & \frac{\mathbf{e}}{2} \\ \frac{\mathbf{d}}{2} & \frac{\mathbf{e}}{2} & \mathbf{f} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.}$$

- Denote this 3×3 matrix by M.

Lemma

The conic $\mathcal{C}$ is irreducible if and only if $\det(M) \neq 0$.

Example

The conic $xy - z^2 = 0$ is *irreducible*.

Intersection of a line and a conic

Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$. Let C be an *irreducible* conic in $\mathbb{P}^2_{\mathbb{C}}$.

Lemma

The intersection $L \cap C$ consists of 2 points (counted with multiplicities).

Proof.

The line L is given by

$$\alpha x + \beta y + \gamma z = 0$$

for complex numbers α, β, γ such that $(\alpha, \beta, \gamma) \neq (0, 0, 0)$.
The conic C is given by a polynomial equation

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2 = 0$$

for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ in $\mathbb{C}$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}) \neq (0, 0, 0, 0, 0, 0)$.
Then the intersection $L \cap C$ is given by

$$\begin{cases} \alpha x + \beta y + \gamma z = 0, \\ \mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2 = 0. \end{cases}$$

How to find an intersection of a line and a conic?

Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$ given by $2x+7y-5z=0$.
Let $\mathcal{C}$ be a conic in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$2x^2-3xy+7y^2-5xz+11yz-8z^2=0.$$

The intersection $L_z\cap L\cap\mathcal{C}$ is empty, since the system

$$\begin{cases} 2x^2-3xy+7y^2-5xz+11yz-8z^2=0,\\ 2x+7y-5z=0,\\ z=0, \end{cases}$$

does not have solutions in $\mathbb{P}^2_{\mathbb{C}}$.
Hence, to find $L\cap\mathcal{C}$, we have to solve the following system:

$$\begin{cases} 2x^2-3xy+7y^2-5xz+11yz-8z^2=0,\\ 2x+7y-5z=0,\\ z=1. \end{cases}$$

Solving this system, we see that $L\cap\mathcal{C}$ consists of two points

$$\Big[161\pm 7\sqrt{385}:14\mp 2\sqrt{385}:84\Big].$$

Five points determine a conic

Let P_1, P_2, P_3, P_4, P_5 be distinct points in $\mathbb{P}^2_{\mathbb{C}}$.

- Suppose that no 4 points among them are collinear.

Theorem

There is a unique conic in $\mathbb{P}^2_{\mathbb{C}}$ that contains P_1, P_2, P_3, P_4, P_5.

Proof.

Let $[x_1 : y_1 : z_1]$, $[x_2 : y_2 : z_2]$, $[x_3 : y_3 : z_3]$, $[x_4 : y_4 : z_4]$, $[x_5 : y_5 : z_6]$ be our points. Find complex numbers **a**, **b**, **c**, **d**, **e**, **f** such that

$$\begin{cases} \mathbf{a}x_1^2 + \mathbf{b}x_1y_1 + \mathbf{c}y_1^2 + \mathbf{d}x_1z_1 + \mathbf{e}y_1z_1 + \mathbf{f}z_1^2 = 0, \\ \mathbf{a}x_2^2 + \mathbf{b}x_2y_2 + \mathbf{c}y_2^2 + \mathbf{d}x_2z_1 + \mathbf{e}y_2z_1 + \mathbf{f}z_1^2 = 0, \\ \mathbf{a}x_3^2 + \mathbf{b}x_3y_3 + \mathbf{c}y_3^2 + \mathbf{d}x_3z_1 + \mathbf{e}y_3z_1 + \mathbf{f}z_1^2 = 0, \\ \mathbf{a}x_4^2 + \mathbf{b}x_4y_4 + \mathbf{c}y_4^2 + \mathbf{d}x_4z_1 + \mathbf{e}y_4z_1 + \mathbf{f}z_1^2 = 0, \\ \mathbf{a}x_5^2 + \mathbf{b}x_5y_5 + \mathbf{c}y_5^2 + \mathbf{d}x_5z_1 + \mathbf{e}y_5z_1 + \mathbf{f}z_1^2 = 0. \end{cases}$$

Then the conic containing P_1, P_2, P_3, P_4, P_5 is given by

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2 = 0.$$

$\square$

How to find a conic passing through five points?

The conic in $\mathbb{P}^2_{\mathbb{C}}$ containing

$$[3:4:1], [-3:4:1], [-4:-5:1], [-6:2:1], [5:3:1].$$

is given by the following equation:

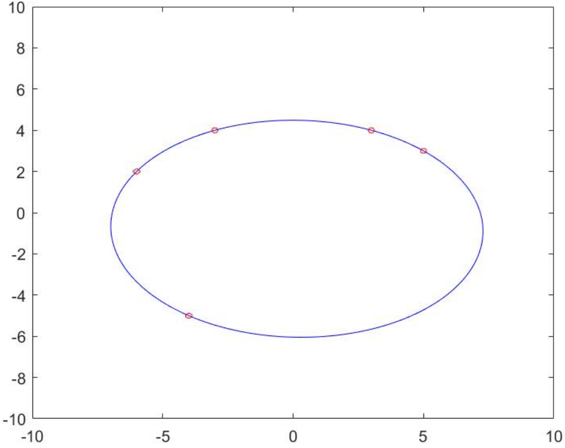
$$-\frac{711}{35389}x^2 - \frac{xy}{823} - \frac{2609}{70778}y^2 + \frac{4}{823}xz - \frac{4059}{70778}yz + z^2 = 0.$$

This can be checked by running the following Maple's script:

```
f:=A1*x^2+A2*x*y+A3*y^2+A4*x*z+A5*y*z+A6*z^2:
P1:=[3,4,1]: P2:=[-3,4,1]: P3:=[-4,-5,1]: P4:=[-6,2,1]: P5:=[5,3,1]:
L1:=subs([x=P1[1],y=P1[2],z=P1[3]],f):
L2:=subs([x=P2[1],y=P2[2],z=P2[3]],f):
L3:=subs([x=P3[1],y=P3[2],z=P3[3]],f):
L4:=subs([x=P4[1],y=P4[2],z=P4[3]],f):
L5:=subs([x=P5[1],y=P5[2],z=P5[3]],f):
solution:=solve([L1=0,L2=0,L3=0,L4=0,L5=0,A6=1],{A1,A2,A3,A4,A5,A6}):
C1:=eval([A1,A2,A3,A4,A5,A6], solution):
f1:=subs([A1=C1[1],A2=C1[2],A3=C1[3],A4=C1[4],A5=C1[5],A6=C1[6]],f);
```

Conic passing through five points

We can plot the real part of the above conic in the chart $z \neq 0$.



The dots are the points $(3, 4)$, $(-3, 4)$, $(-4, -5)$, $(-6, 2)$, $(5, 3)$.