Algebraic Geometry

Vanya Cheltsov

17th January 2019

Lecture 2: projective curves and projective transformations

Complex projective plane

$\mathbb{P}^2_{\mathbb{C}}$ consists of 3-tuples $[x : y : z]$ with $(x, y, z) \neq (0, 0, 0)$ such that

★ $[x : y : z] = [\lambda x : \lambda y : \lambda z]$ for every non-zero $\lambda \in \mathbb{C}$.

Let U_x be the complement in $\mathbb{P}^2_{\mathbb{C}}$ to the line $x = 0$.

- Then $U_x = \mathbb{C}^2$ with coordinates $\widetilde{y} = \frac{y}{x}$ and $\widetilde{z} = \frac{z}{x}$.

Let U_y be the complement in $\mathbb{P}^2_{\mathbb{C}}$ to the line $y = 0$.

- Then $U_y = \mathbb{C}^2$ with coordinates $\widehat{x} = \frac{x}{y}$ and $\widehat{z} = \frac{z}{y}$.

Let U_z be the complement in $\mathbb{P}^2_{\mathbb{C}}$ to the line $z = 0$.

- Then $U_z = \mathbb{C}^2$ with coordinates $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$.

Then $\mathbb{P}^2_{\mathbb{C}}$ is $U_x = \mathbb{C}^2$, $U_y = \mathbb{C}^2$, $U_z = \mathbb{C}^2$ patched together by

$$\boxed{\widetilde{y} = \frac{1}{\widehat{x}} = \frac{\overline{y}}{\overline{x}}, \widetilde{z} = \frac{\widehat{z}}{\widehat{x}} = \frac{1}{\overline{x}}}$$

$$\boxed{\widehat{x} = \frac{\overline{x}}{\overline{y}} = \frac{1}{\widetilde{y}}, \widehat{z} = \frac{1}{\overline{y}} = \frac{\widetilde{z}}{\widetilde{y}}}$$

$$\boxed{\overline{x} = \frac{1}{\widetilde{z}} = \frac{\widehat{z}}{\widehat{x}}, \overline{y} = \frac{\widetilde{y}}{\widetilde{z}} = \frac{1}{\widehat{z}}}$$

Lines and conics

Definition

A line in $\mathbb{P}^2_{\mathbb{C}}$ is the subset given by

$$\boxed{\mathbf{a}x + \mathbf{b}y + \mathbf{c}z = 0}$$

for complex numbers $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \neq (0, 0, 0)$.

Definition

A conic in $\mathbb{P}^2_{\mathbb{C}}$ is a subset that is given by

$$\boxed{\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2 = 0}$$

for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ in $\mathbb{C}$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}) \neq (0, 0, 0, 0, 0, 0)$.

- The conic is said to be irreducible if the polynomial

  $$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2$$

  is irreducible.
- Otherwise the conic is said to be reducible.

Complex irreducible plane curves

Definition

An irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree $d \geqslant 1$ is a subset given by

$$f(x, y, z) = 0$$

for an irreducible homogeneous polynomial $f(x, y, z)$ of degree d.

Let us give few examples. The equation

$$2x^2 - y^2 + 2z^2 = 0$$

defines an irreducible conic in $\mathbb{P}^2_{\mathbb{C}}$. The equation

$$zy^2 - x(x - z)(x + z) = 0$$

defines an irreducible cubic curve in $\mathbb{P}^2_{\mathbb{C}}$. The equation

$$\Big(2x^2 - y^2 + 2z^2\Big)\Big(zy^2 - x(x - z)(x + z)\Big) = 0$$

defines the union of the two curves above.

Projective transformations

Let $\mathbf{M}$ be a complex 3×3 matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Let $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ be the map given by

$$\left[x : y : z\right] \mapsto \left[a_{11}x+a_{12}y+a_{13}z : a_{21}x+a_{22}y+a_{23}z : a_{31}x+a_{32}y+a_{33}z\right].$$

Recall that there is no such point in $\mathbb{P}^2_{\mathbb{C}}$ as $[0 : 0 : 0]$.

Question

When ϕ is well-defined?

The map ϕ is well-defined $\iff \det(\mathbf{M}) \neq 0$.

Definition

If $\det(\mathbf{M}) \neq 0$, we say that ϕ a projective transformation.

Projective linear group

Projective transformations of $\mathbb{P}^2_{\mathbb{C}}$ form a group.

- Let $\mathbf{M}$ be a matrix in $\mathrm{GL}_3(\mathbb{C})$.
- Denote by $\phi_{\mathbf{M}}$ the corresponding projective transformation.

Question

When $\phi_{\mathbf{M}}$ is an identity map?

The map $\phi_{\mathbf{M}}$ is an identity map $\iff$ $\mathbf{M}$ is scalar.
Recall that $\mathbf{M}$ is said to be scalar if

$$M = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

for some complex number λ.

Corollary

Let $\mathbf{G}$ be a subgroup in $\mathrm{GL}_3(\mathbb{C})$ consisting of scalar matrices. The group of projective transformations of $\mathbb{P}^2_{\mathbb{C}}$ is isomorphic to

$$\mathrm{PGL}_3(\mathbb{C}) = \mathrm{GL}_3(\mathbb{C})/\mathbf{G}.$$

Rational maps

Let $\mathbf{M}$ be a complex matrix with non-zero determinant

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Let ϕ be the projective transformation:

$$[x : y : z] \mapsto [a_{11}x + a_{12}y + a_{13}z : a_{21}x + a_{22}y + a_{23}z : a_{31}x + a_{32}y + a_{33}z].$$

Then ϕ maps the line $a_{31}x + a_{32}y + a_{33}z = 0$ to the line $z = 0$.

- Let U_z be the subset in $\mathbb{P}^2_{\mathbb{C}}$ given by $z \neq 0$.
- Identify $U_z = \mathbb{C}^2$ with coordinates $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$.

Then ϕ induces the rational map $U_z \dashrightarrow U_z$ given by

$$(\overline{x}, \overline{y}) \mapsto \left(\frac{a_{11}\overline{x} + a_{12}\overline{y} + a_{13}}{a_{31}\overline{x} + a_{32}\overline{y} + a_{33}}, \frac{a_{21}\overline{x} + a_{22}\overline{y} + a_{23}}{a_{31}\overline{x} + a_{32}\overline{y} + a_{33}} \right).$$

When is this rational map $U_z \dashrightarrow U_z$ well-defined?

Four points in the plane

Let P_1, P_2, P_3, P_4 be four points in $\mathbb{P}^2_{\mathbb{C}}$ such that

- no three points among them are collinear.

Then there is a projective transformation $\mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that

$$P_1 \mapsto [1:0:0], P_2 \mapsto [0:1:0], P_3 \mapsto [0:0:1], P_4 \mapsto [1:1:1].$$

Let $P_1 = [a_{11} : a_{12} : a_{13}]$, $P_2 = [a_{21} : a_{22} : a_{23}]$, $P_3 = [a_{31} : a_{32} : a_{33}]$.
Let ϕ be the projective transformation

$$\big[x : y : z\big] \mapsto \big[a_{11}x + a_{21}y + a_{31}z : a_{12}x + a_{22}y + a_{32}z : a_{13}x + a_{23}y + a_{33}x\big].$$

Then $\phi([1:0:0]) = P_1$, $\phi([0:1:0]) = P_2$, $\phi([0:0:1]) = P_3$.
Let ψ be the inverse of the map ϕ. Write $\psi(P_4) = [\alpha : \beta : \gamma]$.
Let τ be the projective transformation

$$[x : y : z] \mapsto \left[\frac{x}{\alpha} : \frac{y}{\beta} : \frac{z}{\gamma}\right] = \big[\beta\gamma x : \alpha\gamma y : \alpha\beta z\big].$$

Then $\tau \circ \psi$ is the required projective transformation.

Conics and their tangent lines

Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$, and let $\mathcal{C}$ be an irreducible conic in $\mathbb{P}^2_{\mathbb{C}}$.

Question

When $|L \cap \mathcal{C}| = 1$?

We may assume that $[0:0:1] \in L \cap \mathcal{C}$. Then $\mathcal{C}$ is given by

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz = 0$$

for some $[\mathbf{a}:\mathbf{b}:\mathbf{c}:\mathbf{d}:\mathbf{e}] \in \mathbb{P}^4_{\mathbb{C}}$.
We may assume that L is given by $x = 0$. Then

$$L \cap \mathcal{C} = [0:0:1] \cup [0:\mathbf{e}:-\mathbf{c}].$$

Thus, we have $|L \cap \mathcal{C}| = 1 \iff \mathrm{e} = 0$.

- Let U_z be the complement in $\mathbb{P}^2_{\mathbb{C}}$ to the line $z = 0$.
- Identify U_z and $\mathbb{C}^2$ with coordinates $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$.

Then $U_z \cap \mathcal{C}$ is given by $\mathbf{a}\overline{x}^2 + \mathbf{b}\overline{xy} + \mathbf{c}\overline{y}^2 + \mathbf{d}\overline{x} + \mathbf{e}\overline{y} = 0$.

- $\mathbf{d}\overline{x} + \mathbf{e}\overline{y} = 0$ is the tangent line to $U_z \cap \mathcal{C}$ at $(0,0)$.
- $\mathbf{d}x + \mathbf{e}y = 0$ is the tangent line to $\mathcal{C}$ at $[0:0:1]$.

Then $|L \cap \mathcal{C}| = 1 \iff L$ is tangent to $\mathcal{C}$ at the point $L \cap \mathcal{C}$.

Smooth complex plane curves

Let C be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree d given by

$$f(x, y, z) = 0,$$

where $f(x, y, z)$ is a homogeneous polynomial of degree d.

Definition

A point $[a : b : c] \in \mathbb{P}^2_{\mathbb{C}}$ is a singular point of the curve C if

$$\boxed{\frac{\partial f(a,b,c)}{\partial x} = \frac{\partial f(a,b,c)}{\partial y} = \frac{\partial f(a,b,c)}{\partial z} = 0.}$$

- Denote by $\mathrm{Sing}(C)$ the set of singular points of the curve C.
- Non-singular points of the curve C are called smooth.
- The curve C is said to be smooth if $\mathrm{Sing}(C) = \varnothing$

Example

1. If $f = zx^{d-1} - y^d$ and $d \geqslant 3$, then $\mathrm{Sing}(C) = [0 : 0 : 1]$.
2. If $f = x^d + y^d + z^d$, then $\mathrm{Sing}(C) = \varnothing$.

Tangent lines

- Let C be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree d given by
$$f(x, y, z) = 0,$$
where $f(x, y, z)$ is a homogeneous polynomial of degree d.
- Let $P = [\alpha : \beta : \gamma]$ be a smooth point in C. Then the line
$$\boxed{\frac{\partial f(\alpha, \beta, \gamma)}{\partial x} x + \frac{\partial f(\alpha, \beta, \gamma)}{\partial y} y + \frac{\partial f(\alpha, \beta, \gamma)}{\partial z} z = 0}$$
is the tangent line to the curve C at the point P.

Remark

We may assume that $P = [0 : 0 : 1]$. Then

$$\boxed{f(x, y, z) = z^{d-1} h_1(x, y) + z^{d-2} h_2(x, y) + \cdots + z h_{d-1}(x, y) + h_d(x, y) = 0,}$$

where $h_i(x, y)$ is a homogenous polynomial of degree i.

Then $\boxed{h_1(x, y) = 0}$ is the tangent line to C at the point P.

Conics and projective transformation

Let $\mathcal{C}$ be a conic in $\mathbb{P}^2_{\mathbb{C}}$. Then $\mathcal{C}$ is given by

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2 = 0$$

for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ in $\mathbb{C}$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}) \neq (0, 0, 0, 0, 0, 0)$.

Theorem

There is a projective transformations ϕ such that $\phi(\mathcal{C})$ is given by

1. *either $xy = z^2$ (an irreducible smooth conic),*
2. *or $xy = 0$ (a union of two lines in $\mathbb{P}^2_{\mathbb{C}}$),*
3. *or $x^2 = 0$ (a line in $\mathbb{P}^2_{\mathbb{C}}$ taken with multiplicity 2).*

Example

Let $\mathcal{C}$ be a conic in $\mathbb{P}^2_{\mathbb{C}}$ given by $(x - 3y + z)(x + 7y - 5z) = 0$.
Let $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ be a projective transformations given by

$$[x : y : z] \mapsto \big[x - 3y + z : x + 7y - 5z : z\big].$$

Then $\phi(\mathcal{C})$ is a conic in $\mathbb{P}^2_{\mathbb{C}}$ that is given by $xy = 0$.

Irreducible conics

Let $\mathcal{C}$ be a conic in $\mathbb{P}^2_{\mathbb{C}}$. Then $\mathcal{C}$ that is given by

$$\boxed{\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} \mathbf{a} & \frac{\mathbf{b}}{2} & \frac{\mathbf{d}}{2} \\ \frac{\mathbf{b}}{2} & \mathbf{c} & \frac{\mathbf{e}}{2} \\ \frac{\mathbf{d}}{2} & \frac{\mathbf{e}}{2} & \mathbf{f} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.}$$

for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ in $\mathbb{C}$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}) \neq (0,0,0,0,0,0)$.

- Denote this 3×3 matrix by $\mathcal{M}$.

Lemma

The conic $\mathcal{C}$ is irreducible if and only if $\det(\mathcal{M}) \neq 0$.

Proof.

Let $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ be a projective transformation given by matrix $\mathbf{M}$. Let $\mathbf{N} = \mathbf{M}^{-1}$. Then the conic $\phi(\mathcal{C})$ is given by

$$\boxed{\begin{pmatrix} x & y & z \end{pmatrix} \mathbf{N}^T \begin{pmatrix} \mathbf{a} & \frac{\mathbf{b}}{2} & \frac{\mathbf{d}}{2} \\ \frac{\mathbf{b}}{2} & \mathbf{c} & \frac{\mathbf{e}}{2} \\ \frac{\mathbf{d}}{2} & \frac{\mathbf{e}}{2} & \mathbf{f} \end{pmatrix} \mathbf{N} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.}$$

Classification of irreducible conics

Let $\mathcal{C}$ be an irreducible conic in $\mathbb{P}^2_{\mathbb{C}}$ given by

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2 = 0$$

for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ in $\mathbb{C}$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}) \neq (0, 0, 0, 0, 0, 0)$.

1. Pick a point in $\mathcal{C}$ and map it to $[0 : 0 : 1]$. This kills $\mathbf{f}$.
2. Map the tangent line $\mathbf{d}x + \mathbf{e}y = 0$ to $x = 0$. This kills $\mathbf{e}$.
3. Map the line $z = 0$ to the line

   $$z + \alpha y + \beta z$$

   for appropriate α and β to kill $\mathbf{a}$ and $\mathbf{b}$.
4. Scale x, y, and z appropriately to get $\mathbf{b} = 1$ and $\mathbf{c} = -1$.

This gives a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that

$$\boxed{xz = y^2}$$

defines the curve $\phi(\mathcal{C})$.

The conic $x^2 + y^2 - 2xy + xz - 3yz + 2z^2 = 0$

Let $\mathcal{C}$ be the conic in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$\boxed{x^2 + y^2 - 2xy + xz - 3yz + 2z^2 = 0.}$$

1. Note that $[0:1:1] \in \mathcal{C}$. Let $\mathbf{y} = y - z$. Then $\mathcal{C}$ is given by
$$x^2 + \mathbf{y}^2 - \mathbf{y}z - 2x\mathbf{y} - xz = 0.$$
2. To map the tangent line $x + \mathbf{y} = 0$ to the line $x = 0$, let
$$\mathbf{x} = x + \mathbf{y}.$$
Then $\mathcal{C}$ is given by $\mathbf{x}^2 + 4\mathbf{y}^2 - 4\mathbf{x}\mathbf{y} - \mathbf{x}z = 0$.
3. Let $\mathbf{z} = z + \mathbf{x} - 4\mathbf{y}$. Then $\mathcal{C}$ is given by $4\mathbf{y}^2 - \mathbf{x}\mathbf{z} = 0$.

Since $\mathbf{x} = x + y - z$, $\mathbf{y} = y - z$, and $\mathbf{z} = x - 3y + 4z$, the matrix

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & -3 & 4 \end{pmatrix}$$

gives a projective transformation that maps $\mathcal{C}$ to $xz = 4y^2$.

Intersecting two conics

Let $\mathcal{C}$ and $\mathcal{C}'$ be two irreducible conics in $\mathbb{P}^2_{\mathbb{C}}$ such that $\mathcal{C} \neq \mathcal{C}'$.

Theorem

One has $1 \leqslant |\mathcal{C} \cap \mathcal{C}'| \leqslant 4$.

Proof.

We may assume that $\mathcal{C}$ is given by $xy = z^2$. Then $\mathcal{C}'$ is given by

$$\boxed{\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2 = 0}$$

for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ in $\mathbb{C}$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}) \neq (0, 0, 0, 0, 0, 0)$.

- Let L be the line $y = 0$. Then $L \cap \mathcal{C} \cap \mathcal{C}' \subset [1 : 0 : 0]$.
- One has $L \cap \mathcal{C} \cap \mathcal{C}' = [1 : 0 : 0] \iff \mathbf{a} = 0$.
- Let $U_y = \mathbb{P}^2_{\mathbb{C}} \setminus L$. Then $U_y \cap \mathcal{C} \cap \mathcal{C}'$ is given by

$$y - 1 = x - z^2 = \mathbf{a}z^4 + \mathbf{d}z^3 + (\mathbf{b} + \mathbf{f})z^2 + \mathbf{e}z + \mathbf{c} = 0.$$

If $\mathbf{a} = 0$, then $L \cap \mathcal{C} \cap \mathcal{C}' = [1 : 0 : 0]$ and $0 \leqslant |U_y \cap \mathcal{C} \cap \mathcal{C}'| \leqslant 3$.
If $\mathbf{a} \neq 0$, then $L \cap \mathcal{C} \cap \mathcal{C}' = \varnothing$ and $1 \leqslant |U_y \cap \mathcal{C} \cap \mathcal{C}'| \leqslant 4$. □

Transversal intersection of two conics

Let $\mathcal{C}$ and $\mathcal{C}'$ be two irreducible conics in $\mathbb{P}^2_{\mathbb{C}}$.

Question

When the intersection $\mathcal{C} \cap \mathcal{C}'$ consists of 4 points?

Let P be a point in $\mathcal{C} \cap \mathcal{C}'$.

- $\exists$ unique line $L \subset \mathbb{P}^2_{\mathbb{C}}$ such that $P \in L$ and $|L \cap \mathcal{C}| = 1$.
- $\exists$ unique line $L' \subset \mathbb{P}^2_{\mathbb{C}}$ such that $P \in L'$ and $|L' \cap \mathcal{C}| = 1$.

The lines L and L' are tangent lines to $\mathcal{C}$ and $\mathcal{C}'$ at P, respectively.

Definition

We say that $\mathcal{C}$ intersects $\mathcal{C}'$ transversally at P if $L \neq L'$.

- The answer to the question above is given by

Theorem

The following two conditions are equivalent:

1. *the intersection $\mathcal{C} \cap \mathcal{C}'$ consists of 4 points,*
2. *$\mathcal{C}$ intersects $\mathcal{C}'$ transversally at every point of $\mathcal{C} \cap \mathcal{C}'$.*