The topological Künneth theorem in cohomology

Now we can proceed to prove the topological Künneth theorem for cohomology. We let R be a commutative ring, and we let X and Y be two simplicial sets, or two spaces. The *external cup product* is defined as

$$\times \ : \ H^p(X;R) \times H^q(Y;R) \ \longrightarrow \ H^{p+q}(X \times Y;R) \ , \quad x \times y \ = \ p_1^*(x) \cup p_2^*(y) \ ,$$

where $p_1 : X \times Y \longrightarrow X$ and $p_2 : X \times Y \longrightarrow Y$ are the projections to the factors. The exterior product map is additive in x and y, and it satisfies

$$(r \cdot x) \times y \ = \ r \cdot (x \times y) \ = \ x \times (r \cdot y)$$

for all $r \in R$. So the exterior product map extends to an R-linear map defined on the tensor product $H^p(X;R) \otimes_R H^q(Y;R)$. We will abuse notation and use the same symbol '$\times$' for this extension.

In the proof of the cohomological Künneth theorem we need a different perspective on the exterior cup product, as explained in the next proposition. In the statement and proof of the following proposition, we abbreviate the integral chain complex $C_*(X;\mathbb{Z})$ of a simplicial set X to $C_*(X)$, i.e., we drop integral coefficients from the notation; moreover,

$$AW \ : \ C_*(X \times Y) \ \longrightarrow \ C_*(X) \otimes C_*(Y)$$

denotes the Alexander-Whitney map.

Proposition 1. *Let X and Y be two simplicial sets, and let R be a commutative ring. Then the composite*

$$\begin{aligned} H^p(X;R) \otimes_R H^q(Y;R) \ &\xrightarrow{[f]\otimes[g] \mapsto [f \otimes g]} \ H^{p+q}(\mathrm{Hom}(C_*(X),R) \otimes_R \mathrm{Hom}(C_*(Y),R)) \\ &\xrightarrow{H^{p+q}(\bullet)} \ H^{p+q}(\mathrm{Hom}(C_*(X) \otimes C_*(Y),R)) \\ &\xrightarrow{H^{p+q}(\mathrm{Hom}(AW,R))} \ H^{p+q}(X \times Y;R) \ . \end{aligned}$$

agrees with the exterior cup product.

Proof. The proof is a variation of an earlier argument in the alternative proof of the commutativity of the cup product. We let $f : X_p \longrightarrow R$ and $g : Y_q \longrightarrow R$ be cocycles in the cochain complexes $C^*(X;R)$ and $C^*(Y;R)$, respectively. Then

$$\begin{aligned} (p_1^*(f) \cup p_2^*(g))(x,y) \ &= \ p_1^*(f)(d^*_{\text{front}}(x,y)) \cdot p_2^*(g)(d^*_{\text{back}}(x,y)) \\ &= \ f(d^*_{\text{front}}(p_1(x,y))) \cdot g(d^*_{\text{back}}(p_2(x,y))) \\ &= \ f(d^*_{\text{front}}(x)) \cdot g(d^*_{\text{back}}(y)) \\ &= \ (f \bullet g) \left(\sum_{j=0}^{p+q} d^*_{\text{front}}(x) \otimes d^*_{\text{back}}(y) \right) \\ &= \ ((f \bullet g) \circ \mathrm{AW}_{p+q})(x,y) \end{aligned}$$

for all $(x,y) \in X_{p+q} \times Y_{p+q} = (X \times Y)_{p+q}$. Hence

$$\begin{aligned} H^{p+q}(\mathrm{Hom}(\mathrm{AW},R) \circ \bullet)([f \otimes g]) \ &= \ [(f \bullet g) \circ \mathrm{AW}_{p+q}] \\ &= \ [p_1^*(f) \cup p_2^*(g))] \ = \ p_1^*[f] \cup p_2^*[g] \ = \ [f] \times [g] \ . \end{aligned}$$

$\square$

Theorem 2. *Let R be a commutative ring of global dimension at most 1. Let X and Y be spaces or simplicial sets such that the abelian group $H_n(X;\mathbb{Z})$ is finitely generated for every $n \geq 0$. Then the total exterior product map*

$$\bigoplus_{p+q=n} H^p(X;R) \otimes_R H^q(Y;R) \ \longrightarrow \ H^n(X \times Y;R)$$

is injective, and its cokernel is naturally R-linearly isomorphic to the R-module

$$\bigoplus_{p+q=n+1} \mathrm{Tor}^R(H^p(X;R), H^q(Y;R)) \ .$$

Proof. We give the argument in the case where X and Y are simplicial sets. The version for spaces follows by applying it to the singular complexes, and exploiting that the singular complex functor takes products of spaces to products of simplicial sets.

Again we abbreviate $C_*(X;\mathbb{Z})$ to $C_*(X)$, i.e., we drop integral coefficients from the notation. The Eilenberg-Zilber theorem shows that the Alexander-Whitney map

$$\mathrm{AW} \ : \ C_*(X \times Y) \ \longrightarrow \ C_*(X) \otimes C_*(Y)$$

is a chain homotopy equivalence. Applying the functor $\mathrm{Hom}(-, R)$ yields an R-linear cochain homotopy equivalence

$$\mathrm{Hom}(\mathrm{AW}, R) \ : \ \mathrm{Hom}(C_*(X) \otimes C_*(Y), R) \ \longrightarrow \ \mathrm{Hom}(C_*(X \times Y), R) \ ,$$

and this induces an isomorphism of cohomology groups. Proposition 1 shows that the composite

$$\bigoplus_{p+q=n} H^p(X;R) \otimes_R H^q(Y;R) \ \xrightarrow{[f]\otimes[g]\mapsto[f\bullet g]} \ H^n(\mathrm{Hom}(C_*(X) \otimes C_*(Y), R))$$
$$\xrightarrow[\cong]{H^n(\mathrm{Hom}(AW,R))} \ H^n(X \times Y; R)$$

agrees with the total exterior cup product. The homology groups $H_n(C_*(X)) = H_n(X;\mathbb{Z})$ are finitely generated by assumption, so the algebraic cohomological Künneth theorem from the previous video shows that the first map is a monomorphism and has the desired cokernel. This proves the claim. $\square$

Remark 3 (CW-complexes of finite type)**.** A sufficient finiteness conditions that is particularly relevant in applications of the cohomological Künneth theorem is when the space X admits a CW-structure of finite type, i.e., a CW-structure with finitely many cells in every fixed dimension. Here X may be infinite dimensional, so that the total number need not be finite.

If X admits a CW-structure of finite type, then the group $H^n(X;\mathbb{Z})$ can be calculated from the cellular cochain complex, which is finitely generated in every dimension. So the group $H^n(X;\mathbb{Z})$ is also finitely generated for all $n \geq 0$.

We make two particularly important special cases of the cohomological Künneth theorem explicit. First we recall that the external cup product becomes a homomorphism of graded rings

$$\times \ : \ H^*(X;R) \otimes_R H^*(X;R) \ \longrightarrow \ H^*(X \times Y;R) \ .$$

Here the source is the tensor product of graded R-algebras, whose component of degree n is

$$(H^*(X;R) \otimes_R H^*(Y;R))^n \ = \ \bigoplus_{p+q=n} H^p(X;R) \otimes_R H^q(Y;R) \ ;$$

the multiplication in this graded ring is given on homogeneous elements by the rule

$$(x_1 \otimes y_1) \cdot (x_2 \otimes y_2) \ = \ (-1)^{q_1 \cdot p_2} (x_1 \cup x_2) \otimes (y_1 \cup y_2) \qquad \text{in} \qquad H^{p_1+p_2+q_1+q_2}(X \times Y; R)$$

where $x_i \in H^{p_i}(X;R)$ and $y_i \in H^{q_i}(Y;R)$. The proof that the exterior cup product is multiplicative for this product was the content of Exercise 1 on exercise sheet 4.

Over a field every module (i.e., vector space) is free, hence projective, and so over fields all Tor groups vanish. Hence the Künneth theorem specializes to:

Corollary 4. *Let R be a field. Let X and Y be spaces or simplicial sets such that the abelian group $H_n(X;\mathbb{Z})$ is finitely generated for every $n \geq 0$. Then the total exterior product map is an isomorphism of graded R-algebras*

$$H^*(X;R) \otimes_R H^*(Y;R) \ \longrightarrow \ H^*(X \times Y;R) \ .$$

For a finitely generated abelian group A, the vanishing of $\mathrm{Tor}(A, B)$ for all abelian groups B already implies that A is free. So the relevant specialization of the Künneth theorem for the ring $R = \mathbb{Z}$ looks as follows:

Corollary 5. *Let X and Y be spaces or simplicial sets such that the abelian group $H_n(X;\mathbb{Z})$ is finitely generated and free for every $n \geq 0$. Then the total exterior product map is an isomorphism of graded rings*

$$H^*(X;\mathbb{Z}) \otimes H^*(Y;\mathbb{Z}) \longrightarrow H^*(X \times Y;\mathbb{Z}) .$$

Proof. Since the group $H_{p-1}(X;\mathbb{Z})$ is free, the Ext term in the universal coefficient theorem for the group $H^p(X;\mathbb{Z})$ vanishes. So by the universal coefficient theorem, the group $H^p(X;\mathbb{Z})$ is isomorphic to $\mathrm{Hom}(H_p(X;\mathbb{Z}),\mathbb{Z})$. Since $H_p(X;\mathbb{Z})$ is finitely generated and free, the group $\mathrm{Hom}(H_p(X;\mathbb{Z}),\mathbb{Z})$ is finitely generated and free, too. Hence the group $H^p(X;\mathbb{Z})$ is free, and so the Tor terms in the cohomological Künneth theorem (Theorem 2) vanish. □

Remark 6. As before we let X and Y be two simplicial sets or two spaces. I want to mention – without proof – a variation of the Künneth theorems for homology and cohomology in which R can be any commutative ring. The price to pay for the extra generality is the stronger assumption that all homology groups $H_n(X;R)$ are *flat* as R-modules, i.e., the functor $H_n(X;R) \otimes_R -$ on the category of R-modules is exact. A sufficient condition for flatness is that $H_n(X;R)$ is free or projective as an R-module. If the R-module $H_n(X;R)$ is flat for all $n \geq 0$, then the exterior homology pairing

$$\bigoplus_{p+q=n} H_p(X;R) \otimes_R H_q(Y;R) \longrightarrow H_n(X \times Y;R)$$

is an isomorphism. Under the assumption that the homology R-modules are free (as opposed to just flat), this statement can be found as Theorem 9.8.2 in tom Dieck's 'Algebraic Topology'.

For the analogous cohomological statement we again need additional hypothesis: if the R-module $H^n(X;R)$ is finitely generated and projective for all $n \geq 0$, then the external cup product pairing

$$\bigoplus_{p+q=n} H^p(X;R) \otimes_R H^q(X;R) \longrightarrow H^n(X \times Y;R)$$

is an isomorphism. A proof of this statement under the additional assumption that the space X admits a CW-structure and the cohomology modules are free (as opposed to projective) can be found as Theorem 3.16 in Hatcher's 'Algebraic Topology'.

If R happens to have global dimension at most 1, then the flatness or projectivity hypotheses imply the vanishing of all Tor terms in the Künneth theorems; so for commutative rings of global dimension at most 1, the results are special cases of the versions of the Künneth theorem that we have proved.

Now we use the cohomological Künneth theorem to determine a few more cohomology rings.

Example 7. For every $k \geq 1$, the integral homology and cohomology groups $H_n(S^k;\mathbb{Z})$ and $H^n(S^k;\mathbb{Z})$ are finitely generated and free. So the Künneth theorem in the form of Corollary 5 applies to $X = S^k$. Hence for every space Y, the total exterior cup product map is an isomorphism of graded rings

$$H^*(S^k;\mathbb{Z}) \otimes H^*(Y;\mathbb{Z}) \longrightarrow H^*(S^k \times Y;\mathbb{Z}) .$$

If we let $Y = S^l$ be another sphere, we obtain an isomorphism of graded rings

$$\times \ : \ H^*(S^k;\mathbb{Z}) \otimes H^*(S^l;\mathbb{Z}) \longrightarrow H^*(S^k \times S^l;\mathbb{Z}) .$$

We let $e_k \in H^k(S^k;\mathbb{Z})$ be one of the two generators. Then

$$H^*(S^k;\mathbb{Z}) \ = \ \mathbb{Z}\{1, e_k\} \ = \ \mathbb{Z}[e_k]/(e_k^2)$$

is additively freely generated by the e_k and the multiplicative unit 1, and the multiplicative structure is that of an exterior algebra, i.e., $e_k^2 = 0$. We define

$$\begin{aligned} a &= p_1^*(e_k) \in H^k(S^k \times S^l;\mathbb{Z}) \\ b &= p_2^*(e_l) \in H^l(S^k \times S^l;\mathbb{Z}) \ . \end{aligned}$$

The above isomorphism then shows that

$$H^*(S^k \times S^l;\mathbb{Z}) = \mathbb{Z}\{1, a, b, a \cdot b\} \ ,$$

is a free abelian group of rank 4, with basis the classes 1, a, b and ab. The multiplicative structure is determined by the relations $a^2 = 0$, $b^2 = 0$ and

$$ba = (-1)^{kl} \cdot ab \ .$$

So when k or l is even, then $H^*(S^k \times S^l;\mathbb{Z})$ is a truncated polynomial algebra

$$\mathbb{Z}[a, b]/(a^2, b^2) \ ;$$

and when k and l are both odd, then $ab = -ba$ and $H^*(S^k \times S^l;\mathbb{Z})$ is an exterior algebra on two generators a and b.

Example 8. By applying the previous examples iteratively we deduce that the iterated exterior cup product map

$$\underbrace{H^*(S^1;\mathbb{Z}) \otimes \ldots \otimes H^*(S^1;\mathbb{Z})}_{m} \longrightarrow H^*((S^1)^m;\mathbb{Z})$$

is an isomorphism of graded rings, where $(S^1)^m = S^1 \times \cdots \times S^1$ is the m-dimensional torus. For $1 \le i \le m$ we write

$$a_i = p_i^*(e_1) \in H^1((S^1)^m;\mathbb{Z}) \ .$$

We then conclude that $H^*((S^1)^m;\mathbb{Z})$ is an exterior algebra on the classes $a_1, \ldots, a_m$; in other words, this cohomology ring is additively free of rank 2^m on the classes

$$a_{n_1} \cdot \ldots \cdot a_{n_k} \in H^k((S^1)^m;\mathbb{Z})$$

for all $0 \le k \le m$ and all tuples $1 \le n_1 < \cdots < n_k \le m$. The multiplicative structure is determined by the relations $a_i^2 = 0$ and $a_i a_j = -a_j a_i$ for $i \ne j$.