Topology II - Cohomology

Tor Gjone October 4, 2021

Notation:

- Coefficients for homology will always be $\mathbb{Z}$, omitted from the notation.
- If $X$is a space and $Y \subset X$, we write $H_n(X|Y) = H_n(X, X \setminus Y; \mathbb{Z})$ "local homology at Y".
  Note: if $Y \subset U \subset X$ and U is a neighbourhood of Y, then excision provides an isomorphism

$$H_n(U|Y) = H_n(U, U \setminus Y) \xrightarrow{\cong} H_n(X, X \setminus Y) = H_n(X|Y).$$

For n-manifold M and $x \in M$,

$$H_n(M|x) := H_n(M, M \setminus \{x\}; \mathbb{Z}) \cong \mathbb{Z}.$$

A <u>local orientation</u> of M at x is a generator of $H_n(M|x)$. There are exactly two local orientations.

Heuristically: an orientation of $M$is a "continues choice" of local orientations.

<u>Construction: (The orientation covering)</u>

Let M be an n-manifold. Set

$$\widetilde{M} = \{(x, \mu) \ : \ x \in M, \mu \in H_n(M|x) \text{ a local orientation}\}.$$

The map $p : \widetilde{M} \to M$, defined by $p(x, \mu) = x$ is surjective and every point has exactly two pre-images.

We endow $\widetilde{M}$ with a topology that makes p into a two fold covering. A subset B of M is a local ball if B is open and there is a homeomorphism $\phi : \mathbb{R}^n \to U$, for some open neighbourhood U of B, such that $\phi(\mathring{D}^n) = B$, where $\mathring{D}^n = \{x \in \mathbb{R}^n \ : \ |x| < 1\}$.

Note: the inclusion $M \setminus B \to M \setminus \{x\}$ is a homotopy equivalence for all $x \in B$. So it induces an isomorphism

$$r_x^B : H_n(M|B) \xrightarrow{\cong} H_n(M|x) \cong \mathbb{Z}.$$

We let $\mu \in H_n(M|B)$ be a generator and define

$$U(B, \mu) = \{(x, r_x^B(\mu)) \ : \ x \in B\} \subseteq \widetilde{M}.$$

Theorem 0.1. Let M be an n-manifold.

1. As (B, μ) varies over all local balls B and all generators of $H_n(M|B)$, the sets $U(B, \mu)$ form the basis of a topology on $\widetilde{M}$.
2. For this topology, the map $p : \widetilde{M} \to M$, defined by $p(x, \nu) = x$, is a twofold covering map, the <u>orientation covering</u> M.
3. $\widetilde{M}$ is an n-manifold.

Proof. 1. We show that $U(B, \mu) \cap U(B', \mu')$ is a union of basis sets.

Let $(x, \nu) \in U(B, \mu) \cap U(B', \mu')$. Then $x \in B \cap B'$ and $\nu = r_x^B(\mu) = r_x^{B'}(\mu')$. We choose another local ball B'' with $x \in B'' \subseteq B \cap B'$.

We obtain a diagram of local homology groups

$$\begin{tikzcd}
H_n(M|B) \arrow[dr] \arrow[drr, bend left] \arrow[drrr, bend left] & & & \\
& H_n(M|B \cap B') \arrow[r] & H_n(M|B'') \arrow[r] & H_n(M|x) \\
H_n(M|B') \arrow[ur] \arrow[urr, bend right] \arrow[urrr, bend right] & & &
\end{tikzcd}$$

Where all the maps are isomorphisms. Since μ and μ' restrict to the same generator of $H_n(M|x)$, they also restrict to the same generator $\mu'' = r^B_{B''}(\mu) = r^{B'}_{B''}(\mu')$ of $H_n(M|B'')$.

Hence:

$$(x, \nu) \in U(B'', \mu'') \subseteq U(B, \mu) \cap U(B', \mu').$$

So the sets $U(B, \mu)$ form a basis for a topology on $\widetilde{M}$.

2. Because M is an n-manifold, the local balls form a basis of the topology of M. Moreover $p^{-1}(B) = U(B, \mu) \dot{\cup} U(B, -\mu)$ where $\pm\mu$ are the two generators of $H_n(M|B)$.

   So p is continues. Morover, the restriction $p|_{U(B,\mu)} : U(B, \mu) \to B$ is a bijective continues map. The map is also open (and hence a homeomorphism) because a basis of the subspace topology of $U(B, \mu)$ is given by the sets $U(B'', \mu'')$ for local balls $B'' \subseteq B$ and $\mu'' = r^B_{B''}(\mu)$. Because $p(U(B'', \mu'')) = B''$is open in M, the restriction of p to $U(B, \mu)$ is an open map.

   So $p^{-1}(B) \cong B \coprod B$ is homeomorphic in a way that matches $p$with the local map $B \coprod B \to B$. So p is a twofold covering map.

3. By design, every point $(x, \nu) \in \widetilde{M}$ has an open neighbourhood $U(B, \mu)$, which is homeomorphic to $B \cong \mathring{D}^n \cong \mathbb{R}^n$. So $\widetilde{M}$ is locally euclidean of dimension n. Since M is Hausdorff and $p : \widetilde{M} \to M$ a covering, $\widetilde{M}$ is Hausdorff.

□

Definition 0.2. An <u>orientation</u> of an n-manifold M is a continues section $s : M \to \widetilde{M}$ of the covering $p : \widetilde{M} \to M$. The manifold M is <u>orientable</u> is there exists an orientation of M.

Remark 0.3. Because manifolds are locally euclidean, their path components are open. So manifolds are the topological disjoint union of their path components. For many purposes one can restrict to connected manifolds by considering each path component separately.

Corollary 0.4. A connected orientable manifold has exactly 2 orientations. An orientable manifold with n components has 2^n orientations.

Proof. If M is orientable, then $\widetilde{M} \cong M \coprod M$, taking $p : \widetilde{M} \to M$ to the be fold map. So there are exactly two continues sections if M is connected. In general, you can independently choose an orientation of each path component. □

Note 0.5. If M is connected, $p : \widetilde{M} \to M$ is a product cover $\iff$ there is a continues section of p, $\iff$ M is orientable.

Corollary 0.6. Let M be a connected n-manifold such that for some (hence ant) $x \in M$, the group $\pi_1(M, x)$ does <u>not</u> have a subgroup of index 2. Then $M$is orientable. In particular, all simply connected manifolds are orientable.

Proof. Let $\tilde{x} \in \widetilde{M}$ be any point over x. If M is not orientable, then $p : \widetilde{M} \to M$ is not a product cover, and $\widetilde{M}$ would be connected. So $p_* : \pi_1(\widetilde{M}, \tilde{x}) \to \pi_1(M, x)$ is an injective group homomorphism whose image has index 2 in $\pi_1(M, x)$. This contradicts the hypothesis, so M is orientable. □

Example 0.7. S^n is simply connected for $n \geq 2$, and hence orientable. For all $n \geq 1$, $\mathbb{CP}^n$ and $\mathbb{HP}^n$ are simply connected, and hence orientable.

Example 0.8. Let M be an n-manifold that also admits the structure of a topological group, i.e. a gorup structure such that the mulriplication and iverse maps are continues in the given topology. Then M is orientable. Examples: S^1, $\varnothing(n)$, $\mathrm{U}(n)$, $\mathrm{Sp}(n)$ and $\mathrm{SU}(n)$.

Proof. Let $m : M \times M \to M$ be the group structure and let $e \in M$ be the natural element. We choose a local orientation $\mu_0 \in H_n(M|e)$. For any $x \in M$, the map $m(x, -) : M \to M$ is a homeomorphism, with inverse $m(x^{-1}, -)$, that sends e to m. So it induces an isomorphism of local homology groups

$$m(x, -)_* : H_(M|e) \xrightarrow{\cong} H_n(M|x); \quad \mu_0 \mapsto \mu_x.$$

We define $s : M \to \widetilde{M}$ by $s(x) = \mu_x = m(x, -)_*(\mu_0)$, this is continues and hence an orientation of M. $\square$

Proposition 0.9. Let M be an n-manifold

(i) The manifold $\widetilde{M}$ is orientable and the map $\tau : \widetilde{M} \to \widetilde{M}$, $\tau(x, \nu) = (x, -\nu)$ reverses the local orientation of $\widetilde{M}$.

(ii) Suppose that $q : N \to M$ is a 2-fold covering and N an orientable manifold. Moreover, suppose that the non-identity dichtransformation $\tau : N \to N$ reverses the local orientations. Then $q : N \to M$ is isomorphic, as a covering, to $p : \widetilde{M} \to M$.

Proof. (i) Let $\tilde{x} = (x, \mu) \in \widetilde{M}$. Since $p : \widetilde{M} \to M$ is a local homeomorphism, it induces an isomorphism

$$p_0 : H_n(\widetilde{M}, \tilde{x}) \xrightarrow{\cong} H_n(M|x) = H_n(M|p(\tilde{x})).$$

We let $p_*^{-1}(\mu)$ be the "tautological" local orientation of $\widetilde{M}$ at $\tilde{x}$. This defines a continues (!) map

$$\widetilde{M} \to \widetilde{\widetilde{M}}; \quad \tilde{x} = (x, \mu) \mapsto (\tilde{x}, p_*^{-1}(\mu)),$$

hence an orientation of $\widetilde{M}$.

$$\begin{aligned}
\tau_* : H_n(\widetilde{M}|\tilde{x}) &\to H_n(\widetilde{M}|\tau(\tilde{x})) \\
(\tilde{x}, p_*^{-1}) &\mapsto (\tau(\tilde{x}), \tau_*(p_*^{-1}(\mu))) \\
&= ((x, -\mu), p_*^{-1}(\mu)) \\
&= ((x, -\mu), -p_*^{-1}(-\mu)) \\
&\neq ((x, -\mu), p_*(-\mu))
\end{aligned}$$

So τ reverts the local orientation of $\widetilde{M}$.

(ii) Let $q : N \to M$ be any 2-fold covering such that N is orientable and $\tau : N \to N$ is orientation reverting. Let $\{\mu_y\}_y$ be an oreintation of N. We define $f : N \to \widetilde{M}$ by $f(y) = (q(y, q_*(\mu_y)) \in \widetilde{M})$, where $q_* : H_n(N, y) \xrightarrow{\cong} H_n(M, q(y))$. The continuity of the local orientations $\{\mu_y\}$ ipmiles the continuity of $f(!)$. Because $\tau : N \to N$ reverses the orientation, f is compatible with the non-trivial dich transformations:

$$\begin{aligned}
f(\tau y) &= (q(y), q_*(\mu_{\tau y})) \\
&= (q(y), q_*(-\tau_*|\mu_y)), \quad \tau : N \to N \text{ reverts the orientation} \\
&= (q(y), -q_*(\mu_y)), \qquad q\tau = q \\
&= \tau(q(y), q_*(\mu_y)) \\
&= \tau(f(y))
\end{aligned}$$

So $f$is a continues bijection between two fold coverings over the same base M, so f is also open, and hence an isomorphism of coverings.

$\square$

Example 0.10. The antipital map $A : S^n \to S^n$, defined by $A(x) = -x$, has degree $(-1)^{n+1}$. LEt $e \in H_n(S^n; \mathbb{Z})$ be any generator and orient S^n by the image of e in $H_n(S^n|x)$ for all $x \in S^n$. So $A$is orientation reversing if and only if n is even. The projection $q : S^n \to \mathbb{RP}^n$, $q(x) = \mathbb{R} \cdot x$, is a 2-fold covering with orientable total space.

If n is even, the non-identity dechtransformation A reverses the orientation. So for n even, $q : S^n \to \mathbb{RP}^n$ is isomorphic to the orientatin covering of $\mathbb{RP}^n$. So for all n even, $\mathbb{RP}^n$ is <u>not</u> orientable.

Example 0.11. For n odd, $\mathbb{RP}^n$ is orientable. We construct an orientation of $\mathbb{RP}^n$ by choosing a generator $e \in H_n(S^n;\mathbb{Z})$ and define the local orientation at $\mathbb{R} \cdot x \in \mathbb{RP}^n$ as the image of e under the isomorphism.

$$H_n(S^n;\mathbb{Z}) \xrightarrow{\cong} H_n(S^n|x) \xrightarrow[q]{\cong} H_n(\mathbb{RP}^n|\mathbb{R} \cdot x); e \mapsto \mu_{\mathbb{R}\cdot x}$$

Because the antipital map has degree $+$, this yields the same orientation for x and $-x$. In short all the local orientations arise from one class in $H_n(\mathbb{RP}^n;\mathbb{Z})$, so they vary continuesly in $\mathbb{R}x \in \mathbb{RP}^n$.