Covariant functoriality of compactly supported cohomology

As mentioned before, the proof of Poincaré duality that we will give uses a bootstrap argument: the given compact manifold is covered by finitely many euclidean open subsets, and then 'local' version of Poincaré duality are suitably patched together. For this argument we need to compare the compactly supported cohomology of an open subset with that of the ambient manifold. The comparison map is a special case of the covariant functoriality of compactly supported cohomology for open embeddings. In this section we introduce and study this covariant functoriality systematically.

We start with a useful way to construct homomorphisms out of compactly supported cohomology groups. In slightly fancy categorical language, the statement is that the group $H^n_{\text{comp}}(X;A)$ is the filtered colimit, over the poset of compact subsets K of X, of the relative cohomology groups $H^n(X, X\setminus K;A)$. Since I will not assume that all participants of this class are familiar with this categorical language, I will now formulate a down-to-earth version of this statement. If you happen to know what colimits in categories are, you will recognize Proposition 1 as the universal property that characterizes this particular colimit.

If K is a compact subset of a topological space X, then every singular cochain that is supported on K is in particular compactly supported. In other words, the relative cochain complex $C^*(\mathcal{S}(X), \mathcal{S}(X\setminus K);A)$ is a subcomplex of $C^*_{\text{comp}}(X;A)$. The inclusion gives a homomorphism of cohomology groups

$$\lambda_K \ : \ H^n(X, X\setminus K;A) \ \longrightarrow \ H^n_{\text{comp}}(X;A) \ .$$

If K is contained in another compact subset L of X, then

$$C^*(\mathcal{S}(X), \mathcal{S}(X\setminus K);A) \ \subseteq \ C^*(\mathcal{S}(X), \mathcal{S}(X\setminus L);A) \ \subseteq \ C^*_{\text{comp}}(X;A) \ ,$$

and so the following triangle commutes:

$$\begin{tikzcd} H^n(X, X\setminus K;A) \arrow[r, "j^L_K"] \arrow[dr, "\lambda_K"'] & H^n(X, X\setminus L;A) \arrow[d, "\lambda_L"] \\ & H^n_{\text{comp}}(X;A) \end{tikzcd}$$

The homomorphism j^L_K is induced by the inclusion of $X\setminus L$ into $X\setminus K$.

Proposition 1. *Let X be a Hausdorff space and let A and B be abelian groups. For every compact subset K of X, let $\alpha_K : H^n(X, X\setminus K;A) \longrightarrow B$ be a homomorphism, and suppose that triangle*

$$\begin{tikzcd} H^n(X, X\setminus K;A) \arrow[r, "j^L_K"] \arrow[dr, "\alpha_K"'] & H^n(X, X\setminus L;A) \arrow[d, "\alpha_L"] \\ & B \end{tikzcd}$$

commutes for all nested pairs of compact subsets $K \subset L$ of X. Then there is a unique homomorphism $\alpha : H^n_{\text{comp}}(X;A) \longrightarrow B$ such that $\alpha \circ \lambda_K = \alpha_K$ for all compact subsets K of X.

Proof. As usual we drop the coefficient group A from the notation to simplify the exposition. We start with the uniqueness property, so we let $\alpha : H^n_{\text{comp}}(X) \longrightarrow B$ be a homomorphism that satisfies $\alpha \circ \lambda_K = \alpha_K$ for all compact subsets K of X. We let $f \in C^n_{\text{comp}}(X)$ be any compactly supported n-cocycle of X. If K is a compact subset of X on which f is supported, then f also a cocycle of the complex $C^*(\mathcal{S}(X), \mathcal{S}(X\setminus K))$, and hence

$$\alpha[f] \ = \ \alpha(\lambda_K[f]) \ = \ \alpha_K[f] \ .$$

So α is uniquely determined by the homomorphisms α_K.

Conversely, this argument also shows how to construct α from the homomorphisms α_K. Given any n-cocycle $f \in C^n_{\mathrm{comp}}(X)$ supported on some compact subset K of X, we *define*

$$\alpha[f] \;=\; \alpha_K[f] \;\in B\ .$$

The main point is the verification that this does not depend on the supporting set K and on the representative for the class in $H^n_{\mathrm{comp}}(X)$. So we let K' be another compact subset on which f is supported. Then $L = K \cup K'$ is yet another compact subset on which f is supported, and so

$$\alpha_K[f] \;=\; \alpha_L(j^L_K[f]) \;=\; \alpha_L(j^L_{K'}[f]) \;=\; \alpha_{K'}[f]\ .$$

The second equation holds because both $j^L_K[f]$ and $j^L_{K'}[f]$ are represented by f, cosidered as a cocycle of the complex $C^*(\mathcal{S}(X), \mathcal{S}(X \setminus L))$. So $\alpha[f]$ does not depend on the supporting subset.

Now we let $g \in C^{n-1}(X)$ be any compactly supported cochain, and we let K' be a compact subset on which g is supported. Then f and g are both supported on the compact set $L = K \cup K'$, and hence $f + dg$ is supported on L, too. Hence $[f] = [f + dg]$ in $H^n(X, X \setminus L)$, and so

$$\alpha[f] \;=\; \alpha_L[f] \;=\; \alpha_L[f + dg] \;=\; \alpha[f + dg]\ .$$

This concludes the verification that the map $\alpha : H^n_{\mathrm{comp}}(X) \longrightarrow B$ is well-defined. The relations $\alpha \circ \lambda_K = \alpha_K$ were hard wired into the definition of α.

The final thing to check is that the map α is a group homomorphism. So we let $f, f' \in C^n_{\mathrm{comp}}(X)$ be two compactly supported cocycles. By taking the union of the supports, if necessary, we can assume that f and f' are supported on the same compact subset L of X. Because α_L and λ_L are group homomorphisms, we conclude that

$$\begin{aligned} \alpha[f + f'] \;&=\; \alpha_L(\lambda_L[f + f']) \;=\; \alpha_L(\lambda_L[f] + \lambda_L[f']) \\ &=\; \alpha_L(\lambda_L[f]) + \alpha_L(\lambda_L[f']) \;=\; \alpha[f] + \alpha[f']\ . \end{aligned}$$

□

We will now use Proposition 1 to define the covariant functoriality of compactly supported cohomology for open embeddings. Later we will use Proposition 1 again to define the duality homomorphism for non-compact manifolds.

Construction 2. We let Y be an open subspace of a Hausdorff topological space X, and we let A be an abelian group. We let K be any compact subset of Y. Because Y is open in X, and K is closed in X, the triple (X, Y, K) is excisive. We can thus define a homomorphism

$$\alpha_K \;:\; H^n(Y, Y \setminus K; A) \;\longrightarrow\; H^n_{\mathrm{comp}}(X; A)$$

as the composite

$$H^n(Y, Y \setminus K; A) \;\underset{\cong}{\xleftarrow{\text{excision}}}\; H^n(X, X \setminus K; A) \;\xrightarrow{\lambda_K}\; H^n_{\mathrm{comp}}(X; A)$$

of the inverse of the restriction isomorphism and the 'canonical' maps λ_K. We claim that these homomorphims satisfy the compatibilities necessary to apply Proposition 1. Indeed, if K is contained in a larger compact subset L of Y, then the left square in the following diagram commutes by functoriality of singular cohomology:

$$\begin{tikzcd}
H^n(Y,Y\setminus K;A) \arrow[d, "j_K^L"'] \arrow[rrr, bend left, "\alpha_K"] & H^n(X,X\setminus K;A) \arrow[l, "\text{excision}", "\cong"'] \arrow[d, "j_K^L"'] \arrow[dr, "\lambda_K"] & \\
H^n(Y,Y\setminus L;A) \arrow[rr, bend right, "\alpha_L"'] & H^n(X,X\setminus L;A) \arrow[l, "\text{excision}", "\cong"'] \arrow[r, "\lambda_L"'] & H^n_{\mathrm{comp}}(X;A)
\end{tikzcd}$$

Hence the entire diagram commutes, so that $\alpha_L \circ j_K^L = \alpha_K$. We can now appeal to Proposition 1, which provides us with a unique homomorphism

$$\iota_Y^X \ : \ H^n_{\mathrm{comp}}(Y;A) \ \longrightarrow \ H^n_{\mathrm{comp}}(X;A)$$

such that for all compact subsets K of Y, the composite

$$H^n(X,X\setminus K;A) \xrightarrow[\cong]{\text{excision}} H^n(Y,Y\setminus K;A) \xrightarrow{\lambda_K} H^n_{\mathrm{comp}}(Y;A) \xrightarrow{\iota_Y^X} H^n_{\mathrm{comp}}(X;A)$$

coincides with the homomorphism $\lambda_K : H^n(X,X\setminus K;A) \longrightarrow H^n_{\mathrm{comp}}(X;A)$.

Remark 3. Depending on your mathematical taste, you might or might not like the abstract procedure by which we constructed the homomorphisms ι_Y^X. However, no matter how the definition is set up, it will one way or the other use that for K compact in Y, the restriction homomorphism

$$H^n(X,X\setminus K;A) \ \longrightarrow \ H^n(Y,Y\setminus K;A)$$

is an isomorphism by excision. So at some point, a relative cohomology class must be extended from $(Y,Y\setminus K)$ to $(X,X\setminus K)$ for some compact subset K of Y. Excision guarantees that this can be done uniquely in cohomology, but it does not provide an explicit way or a formula for how to do this. In summary: as far as I can see, the covariant functoriality of compactly supported cohomology cannot be made completely explicit at the cochain level.

We still need to show that the maps ι_Y^X are indeed covariantly functorial. Our approach via universal properties makes this almost tautological.

Proposition 4. *Let X be a Hausdorff topological space and A an abelian group.*

(i) *The homomorphism $\iota_X^X : H^n_{\mathrm{comp}}(X;A) \longrightarrow H^n_{\mathrm{comp}}(X;A)$ is the identity.*
(ii) *If $Z\subset Y$ are two nested open subsets of X, then*

$$\iota_Y^X \circ \iota_Z^Y \ = \ \iota_Z^X \ : \ H^n_{\mathrm{comp}}(Z;A) \ \longrightarrow \ H^n_{\mathrm{comp}}(X;A) \ .$$

Proof. As usual we drop the coefficient group A from the notation to simplify the exposition.

(i) For every compact subset K of X, the composite

$$H^n(X,X\setminus K) \ \xrightarrow{\lambda_K} \ H^n_{\mathrm{comp}}(X) \ \xrightarrow{\iota_X^X} \ H^n_{\mathrm{comp}}(X)$$

coincides with the homomorphism $\lambda_K : H^n(X,X\setminus K) \longrightarrow H^n_{\mathrm{comp}}(X)$, by the very construction of ι_X^X. The identity of $H^n_{\mathrm{comp}}(A)$ also satisfies $\mathrm{Id}\circ\lambda_K = \lambda_K$ for all compact K. So the uniqueness clause in Proposition 1 guarantees that $\iota_X^X = \mathrm{Id}$.

(ii) For every compact subset K of Z, the two inner rectangles of the diagram

$$\begin{tikzcd}
H^n(X, X \setminus K) \arrow[r, "\text{excision}"'] \arrow[d, "\lambda_K"'] \arrow[rr, bend left, "\text{excision}"] & H^n(Y, Y \setminus K) \arrow[r, "\text{excision}"'] \arrow[d, "\lambda_K"] & H^n(Z, Z \setminus K) \arrow[d, "\lambda_K"] \\
H^n_{\mathrm{comp}}(X) & H^n_{\mathrm{comp}}(Y) \arrow[l, "\iota^X_Y"] & H^n_{\mathrm{comp}}(Z) \arrow[l, "\iota^Y_Z"]
\end{tikzcd}$$

commute by the defining properties of ι^X_Y and ι^Y_Z. So the composite $\iota^X_Y \circ \iota^Y_Z$ enjoys the property that characterizes the homomorphism ι^X_Z. The uniqueness clause in Proposition 1 thus guarantees that $\iota^X_Y \circ \iota^Y_Z = \iota^X_Z$. $\square$