WORKSHOP SHEET 2

ALGEBRAIC GEOMETRY [MATH11120]

Try to do some exercises before tutorial.

Each exercise has ($\star$)-part. They are for self-study only.

Exercise 1. Let C be the curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$6x^3 - 7x^2y - 7x^2z + 3xy^2 + 4xyz + 2xz^2 - y^3 - y^2z = 0.$$

Let O be the point $[0 : 1 : -1]$. Then C contains O, and the curve C is smooth at O. Moreover, the point O is an inflection point of the curve C.

(a) Find a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi(C)$ is given by

$$zy^2 = x(x - z)(x - \lambda z)$$

for some $\lambda \in \mathbb{C}$ such that $\lambda \neq 0$ and $\lambda \neq 1$. Deduce that the curve C is smooth.

(b) Find a projective transformation $\psi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\psi(C)$ is given by

$$zy^2 = x^3 + axz^2 + bz^3$$

for some a and $b \in \mathbb{C}$. The number $1728\frac{4a^3}{4a^3+27b^2}$ is called the **j**-invariant of the curve C. Check that it is equal to $256\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$.

($\star$) Equip C with the structure of an abelian group such that O is the identity element. Find all points of order 2 on the curve C.

Exercise 2. Let C be the curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$x^3 + y^3 + z^3 + 4xyz = 0.$$

Let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Denote by Σ the subset in $\mathbb{P}^2_{\mathbb{C}}$ consisting of the following 9 points:

$$[1 : -1 : 0], [1 : -\omega : 0], [1 : -\omega^2 : 0],$$
$$[1 : 0 : -1], [1 : 0 : -\omega], [1 : 0 : -\omega^2],$$
$$[0 : 1 : -1], [0 : 1 : -\omega], [0 : 1 : -\omega^2].$$

It follows from Exercise 3 in the second worksheet that the curve C is smooth and $\Sigma \subset C$. Let $O = [0 : 1 : -1]$, $P = [-1 : 2 : 1]$ and $Q = [-1 : 0 : 1]$. Then C contains O, P and Q. Equip C with the structure of an abelian group such that O is the identity element.

(a) Show that the finite set Σ consists of all inflection points of the smooth cubic curve C. Prove that the finite set $\Sigma \setminus O$ consists of all points of the group C that have order 3. Deduce that no line in $\mathbb{P}^2_{\mathbb{C}}$ contains exactly two points of the set Σ.

(b) Compute $P + Q$, $2P$, $3P$, $-P$ and $2Q$.

($\star$) Show that P has infinite order.

Workshop 2 will be on Friday 8th February 2019 at 12:10pm at room 4325C at JCMB.

Exercise 3. Let C be the cubic curve in $\mathbb{P}^2$ that is given by

$$zy^2 - x^3 - xz^2 - z^3 = 0.$$

Let $P_1 = [0 : 1 : 1]$, $P_2 = [0 : 1 : 0]$, $P_3 = [2 : 9 : 8]$, $P_4 = [72 : 611 : 1]$ and

$$P_5 = \big[-10332 : 40879 : 46656\big].$$

Then C is smooth, and the curve C contains the points P_1, P_2, P_3, P_4, P_5.

(a) Equip C with the structure of an abelian group such that P_1 is the identity element. Compute $P_4 + P_5$, $2P_4$, $-P_4$, $P_3 + P_4$, $P_3 + P_5$, $2P_3$, $2P_5$, $-P_3$, $-P_5$.

(b) Equip C with the structure of an abelian group such that P_2 is the identity element. Compute $P_4 + P_5$, $2P_4$, $-P_4$, $P_3 + P_4$, $P_3 + P_5$, $2P_3$, $2P_5$, $-P_3$, $-P_5$.

($\star$) In the assumptions of (b), prove that P_3 has an infinite order.

Exercise 4. Let C be the curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$zy^2 = x^3 - 3267xz^2 + 45630z^3.$$

Let $O = [0 : 1 : 0]$, $P = [-21 : 324 : 1]$, $Q = [15 : 0 : 1]$. Then C contains O, P and Q. Equip C with the structure of an abelian group such that O is the identity element.

(a) Compute $Q + P$, $2P$ and $-P$. Show that $2Q = O$.

(b) Compute $-2P$, $4P$, $8P$ and $16P$. Show that P has an infinite order.

($\star$) Prove that O and Q are the only points of C defined over $\mathbb{Q}$ that have finite order.

Exercise 5. Let C be the curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by $f(x, y, z) = 0$, where

$$f(x, y, z) = zy^2 + xy^2 - y^3 - x^3 - x^2z.$$

Lt P be the point $[0 : 0 : 1]$. Then $P \in C$ and C is singular at the point P.

(a) Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$\alpha x + \beta y = 0$$

for $[\alpha : \beta] \in \mathbb{P}^1_{\mathbb{C}}$. Check that $P \in L$ and find $L \cap C$. Show that $f(x, y, z)$ is irreducible.

(b) Find coprime homogeneous polynomials $u(s, t)$, $v(s, t)$, $w(s, t)$ of degree 3 such that

$$f\Big(u(\alpha, \beta), v(\alpha, \beta), w(\alpha, \beta)\Big) = 0$$

for every $[\alpha : \beta] \in \mathbb{P}^1_{\mathbb{C}}$. Decide whether the map $\mathbb{P}^1_{\mathbb{C}} \to C$ given by

$$[\alpha : \beta] \mapsto \big[u(\alpha, \beta) : v(\alpha, \beta) : w(\alpha, \beta)\big]$$

is bijective or not.

($\star$) Put $g(x, y, z) = x^2y^2 - 2x^2z^2 + y^2z^2$. Show that the polynomial $g(x, y, z)$ is irreducible. Let Z be the curve in $\mathbb{P}^2_{\mathbb{C}}$ given by $g(x, y, z) = 0$. Show that Z is singular at the points

$$[0 : 0 : 1], [0 : 1 : 0], [1 : 0 : 0],$$

and these points are all singular points of the curve Z. Check that Z contains $[1 : 1 : 1]$. Find coprime homogeneous polynomials $u(s, t)$, $v(s, t)$, $w(s, t)$ of degree 4 such that

$$g\Big(u(\alpha, \beta), v(\alpha, \beta), w(\alpha, \beta)\Big) = 0$$

for every $[\alpha : \beta] \in \mathbb{P}^1_{\mathbb{C}}$.