Topology II - The mod-2 fundamental class

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Theorem 0.1. Let M be a compact connected orientable n-manifold. Then for all $x \in M$ and all coefficient groups A, the restriction map $r_x^M : H_n(M; A) \to H_n(M|x; A) \cong A$ is an isomorphism.

Proof. □

Note 0.2. The proof showed that the map $H_n(M;\mathbb{Z}) \otimes A \to H_n(M;A)$ from the universal coefficient theorem is an isomorphism. Hence its cokernel $\mathrm{Tor}(H_{n-1}(H;\mathbb{Z}), A)$ is zero for all abelian groups A. In particular, for all $k \geq 1$,

$$k\text{-torsion in } H_{n-1}(M;\mathbb{Z}) \cong \mathrm{Tor}(H_{n-1}(M;\mathbb{Z}), \mathbb{Z}/k) = 0.$$

So the group $H_{n-1}(M;\mathbb{Z})$ is torsion free for every compact, connected orientable n-manifold M.

Next: "in mod-2 homology there are no orientability issues because $\mathbb{F}_2$ has only one unit"

Theorem 0.3. Let M be an n-manifold and K a compact subset of M. Then there is a unique class $\nu_K \in H_n(M|K;\mathbb{F}_2)$ such that for all $x \in K$, the class $r_x^K(\mu_K)$ is non-zero, and hence a generator of $H_n(M|x;\mathbb{F}_2) \cong \mathbb{F}_2$.

Corollary 0.4. Let M be a compact connected n-manifold. Then for very $x \in M$ the map

$$r_x^M : H_n(M;\mathbb{F}_2) \to H_n(M|x;\mathbb{F}_2) \cong \mathbb{F}_2$$

is an isomorphism. If $M \neq \emptyset$, then in particular $H_n(M;\mathbb{F}_2) \cong \mathbb{F}_2$.

Proof. Exactly as in the previous video for $\mathbb{Z}$-coefficients and orientable M. □

Remark 0.5. Let M be a connected, compact, non-orientable manifold. Then $H_n(M;\mathbb{Z}) = 0$ by the previous video. The universal coefficient theorem provides and isomorphism

$$\mathbb{F}_2 \cong H_n(M;\mathbb{F}_2) \xrightarrow{\cong} \mathrm{Tor}(H_{n-1}(M;\mathbb{Z}), \mathbb{F}_2) = 2\text{-torsion in } H_{n-1}(M;\mathbb{Z}).$$