Topology

Lecture Notes From Lectures of Dr. Schwede, University of Bonn

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Contents

Part I
Topology 1

1 Lecture 1: Cell Attachments

We can think of CW-complexes as “nice” space for the purposes of homotopy theory. They are built by attaching cells ($D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$).

1.1 Cells

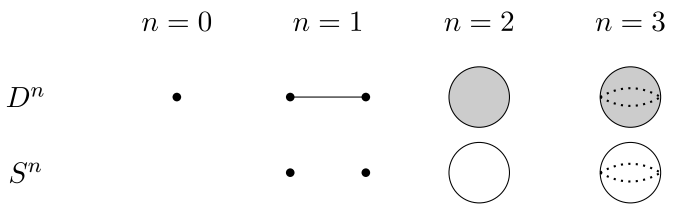
The set

$$D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$$

is called an n-cell, n-ball. We call the $(n-1)$-sphere

$$S^{n-1} = \partial D^n = \{x \in \mathbb{R}^n : |x| = 1\}$$

the boundary of the n-cell. i.e.



1.2 Construction of CW-complexes

For $n \geq 0$ let $f : S^{n-1} \to X$ be a continuous map, the attaching map. We form the quotient space

$$X \cup_{f,\partial D^n} D^n = X \cup_f D^n = X \cup_{\partial D^n} D^n = X \sqcup D^n / \sim$$

where the equivalence relation $\sim$ is generated by $x \sim f(x)$ for all $x \in S^{n-1} = \partial D^n$.

Examples:

- $X \cup_f D^0 = X \sqcup \{x\}$
- $\{*\} \cup_{S^{n-1}} D^n \cong S^n$maybe add a picture here as well
- $S^{n-1} \cup_f D^n \cong D^n$ via the inclusion of S^{n-1} into D^n
- $S^{n-1} \cup_f D^n \cong S^{n-1} \vee S^n$ the wedge product/ one point union i.e. $S^{n-1} \sqcup S^n / x \sim x'$ for f some constant map $f : S^{n-1} \to S^{n-1}$.

1.3 Simultaneous Attatching of Several Cells

We want to avoid that cells of the same dimension are attached to the interior. We let J be indexing set for the n-cells. Note that J can be empty. Give J the discrete topology and give $J \times D^n$ the product topology. This can also be described by

$$J \times D^n = \bigsqcup_{j \in J} \{j\} \times D^n$$

as topological spaces. Where this has the disjoint union topology. With the universal property

$$\{J \times D^n \overset{\text{cont.}}{\to} X\} \overset{\cong}{\to} J \text{ indexed families of cont. maps } D^n \to X$$
$$f : J \times D^n \to X \mapsto \{f_j : D^n \to X; x \to (j,x)\}$$

Now let $f : J \times \partial D^n \to X$ be a continuous map (the attaching map). Now we can attach n-cells indexed by J to X along f as follows

$$X \cup_{f, J\times\partial D^n} J \times D^n = X \sqcup J \times D^n/(x \sim f(x))$$

We will write $p : X \sqcup J \times D^n \to X \cup_f J \times D^n$ to be the quotient projection. The attaching has the following properties:

- Universal property: Given continuous maps $g : X \to Y$ and $f_j : D^n \to Y$ for every $j \in J$ such that $g(f_j(x)) = \psi_j(x)$ for all $j \in J, x \in \partial D^n$, then there is a unique continuous map $\psi : X \cup_f J \times D^n \to Y$ such that $\psi \circ p = g + \sqcup_j f_j : X \sqcup J \times D^n \to Y$.
- A subset N of $X \cup_f J \times D^n$ is open (resp. closed) if and only if $p^{-1}(N)$ is open (closed) in $X \sqcup J \times D^n$.
- Suppose $M \subset X \sqcup J \times D^n$ is saturated (i.e. $p^{-1}(p(M)) = M$). If M is saturated and open (saturated and closed), then $p(M)$ is open (closed) in $X \cup_f J \times D^n$.

Proposition 1.1. *1. The composite $X \hookrightarrow X \sqcup J \times D^n \overset{p}{\to} X \cup_z J \times D^n$ is injective and a homeomorphism onto its image which is closed in $X \cup_f J \times D^n$.*

2. The composite $J \times \mathring{D}^n \hookrightarrow X \sqcup J \times D^n \overset{p}{\to} X \cup_f J \times D^n$ is injective, and a homeomorphism onto its image $p(J \times \mathring{D}^n)$, which is an open subset.

3. As a set, $X \cup_f J \times D^n$ is the disjoint union of $p(X)$ and $p(J \times \mathring{D}^n)$.

Remark. Warning: $X \cup_f J \times D^n$ is not the topological disjoint union of these two subspaces.

Proof. long and boring and technical proof ew □

2 Lecture 2: Compactness of Cell Attachments

Fix a space X, for $n \geq 0$, $f : J \times \partial D^n \to X$ continuous.

Definition 2.1. A topological space X is compact if it is Hausdorff and quasi-compact.

Theorem 2.1. *1. If X is Hausdorff, then so is $X \cup_f J \times D^n$.*

2. If X is compacr and J is finite, then $X \cup_{J\times\partial D^n} J \times D^n$ is compact.

3. Let K be a quasi-compact subspace of $X \cup_{J\times\partial D^n} J \times D^n$. Then $K \cap (\{j\} \times \mathring{D}^n) = \emptyset$ for almost all $j \in J$. In particular, if J is infinite then $X \cup_{J\times\partial D^n} J \times D^n$ is not compact.

Lemma 2.2. *There is a open neighbourhood V of X in $X \cup_{J\times\partial D^n} J \times D^n$ and a retraction $r : V \to X$ i.e. continuous map with $r(x) = x$ for $x \in X$.*

Proof. (Of Lemma) do this later □

Proof. (Of Theorem) do this later □

3 Lecture 3: Hawaiian Earrings

hmm this is a picturey example and i dont feel like doing that sorry

4 Lecture 4: CW-Complexes

Definition 4.1. A relative CW-complex is a space X equippped with a nested sequence of closed subspaces

$$A = X_{-1} \subset X_0 \subset X_1 \subset ... \subset X_n \subset X$$

such that

1. For every $n \geq 0$ the space X_n can be obtained by attaching n-cells to X_{n-1}
2. $X = \cup_{n \leq 0} X_n$, and X has the weak topology for this filtration.

Remark. In more detail:

- The following data must exist
  - An indexing set J
  - A continuous map $f : J \times \partial D^n \to X_{n-1}$
  - A homeomorphism $X_{n-1} \cup_f J \times D^n \xrightarrow{\cong} X_n$ that is the inclusion on X_{n-1}
- X has the weak topology i.e. a subset O of X is open $\iff$ $O \cap X_n$ is open in X_n for $n \geq 0$ (equivalently a subset A of X is closed $\iff$ $A \cap X_n$ is closed in X_n for all $n \geq 0$).

Remark. Terminology:

- We might say (X, A) is a relative CW-complex
- For $A = \emptyset$ we speak of absolute CW-complexes
- The subspace X_n is the n-skeleton of the CW-structure
- A CW-complex is finite dimensional is there is an $m \geq 0$ such that $X_m = X$
- A CW-complex is finite if the total number of cells is finite.
- Once chosen a homeomorphism $X_{n-1} \cup_{J \times \partial D^n} J \times D^n \cong X_n$, we call the composite

  $$D^n \to X_{n-1} \cup_{J \times \partial D^n} J \times D^n \to X_n$$

  the characteristic map for the j-th n-cell. The restriction to the boundary of the disk is the attaching map for the j-th cell.

Remark. The space $X_n \setminus X_{n-1}$ is homemorphic to $(X_{n-1} \cup_{J \times \partial D^n} J \times D^n) \setminus X_{n-1} \cong J \times \mathring{D}^n$ so the indexing set J bijects with the set of path components of $X_n \setminus X_{n-1}$. For every path component of $X_n \setminus X_{n-1}$ there exists a homeomorphism $f : \mathring{D}^n \to X_n \setminus S_{n-1}$ that extends continuously to a map $\overline{f} : D^n \to X_n$.

Example:

- Let $z \in S^n$ be any point. Then there is an absolute CW-structure on $X = S^n$ with skeleta:

  $$\begin{aligned} X_{-1} &= \emptyset \\ X_0 &= \{z\} = X_1 = X_2 = ... = X_{n-1} \\ X_n &= X_{n+1} = ... = S^n \end{aligned}$$

  one 0-cell, one n-cell. Think of an n-disk glued around the boundary to the point.

- $X = S^n$ for $n \geq 2$, another CW-structure is:

$$\begin{aligned} X_{-1} &= \emptyset \\ X_0 = X_1 = ... = X_{n-2} &= \{(1, 0, ..., 0)\} \\ X_{n-1} &= \text{equator} = \{(x, 0) : x \in S^{n-1}\} \\ X_n = X_{n+1} = ... &= S^n \end{aligned}$$

so one 0-cell, one $(n-1)$-cell, two n-cells. Think of the above construction to give the $(n-1)$-sphere as the boundary of the n-sphere and then 2 n-disks glued around the equator to give S^n.

- $X = S^1$, choose points $x_1, ..., x_m \in S^1$. Let

$$X_{-1} = \emptyset, X_0 = \{x_1, ..., x_m\}, X_1 = X_2 = ... = S^1$$

with m 0-cells and m 1 cells.

5 Lecture 5: Elementary Properties of CW-Complexes

Theorem 5.1. *Let (X, A) be a relative CW-complex*

1. *If A is Hausdorff, then so is X.*
2. *If A is compact and (X, A) is finite, then X is compact.*

Proof. 1. Because $X_{-1} = A$ is Hausdorff and X_n is obtained from X_{n-1} by attaching cells, X_n is Hausdorff by induction.
Claim: Let O_n and P_n be disjoint open subsets of X_n. Then there are disjoint open subsets such that $O_{n+1} \cap X_n = O_n$ and $P_{n+1} \cap X_n = P_n$.
Proof of claim: There is an open neighbourhood V of X_n in X_{n+1} and a continuous retraction $r : V \to X_n$. We set $O_{n+1} = r^{-1}(O_n)$, $P_{n+1} = r^{-1}(P_n)$.
Back to proof that X is Hausdorff: Let $x, y \in X$ be distinct points. Suppose that $x, y \in X_n$ then since X_n is Hausdorff we can find O_n, P_n open and disjoint in X_n with $x \in O_n$ and $y \in P_n$. Inductively we find that O_{n+k}, P_{n+k} open and disjoint in X_{n+k}, such that $O_{n+k+1} \cap X_{n+k} = O_{n+k}$ and the same for P_{n+k+1}. Then $O = \cup O_{n+k}$, $P = \cup P_{n+k}$ are disjoint open subsets of X and $x \in O, y \in P$.

2. Induction on n shows that X_n is compact for $n \geq 0$. Since $X = X_n$ for sufficiently large n, X is compact.

$\square$

Remark. From now on we will assume that A is Hausdorff.

Theorem 5.2. *Let (X, A) be a relative CW-complex*

1. *The closure of every open n-cell (a path component of $Xn \setminus X_{n-1}$) is compact.*
2. *Let U be a subset of X with $A \subset U$. Suppose that the intersection of U with the closure of every open cell is closed in X. Then U is closed in X.*

Proof. 1. By design the open cell has a characteristic map i.e. a continuous map $f : D^n \to X_n \subset X$ such that $f(mathringD^n)$ is the closure of the open cell. Since D^n is compact, so $f(D^n)$ is quasi compact. Since X is Hausdorff, $f(D^n)$ is closed. So $f(D^n)(\subset \overline{f(\mathring{D})}$ is quasi compact, hence compact.

2. It suffices to show that $U \cap X_n$ is closed in X_n. We ca argue by induction on n.
For $n = -1$: $U \cap X_{-1} = U \cap A$ is closed A.
For $n \geq 0$: We choose a homeomorphism $X_n \cong X_{n-1} \cup_{J \times \partial D^n} J \times D^n$. Let $p : X_{n-1} \sqcup J \times D^n \to X_{n-1} \cup_{J \times \partial D^n} J \times D^n \cong X_n$ be the quotient map. Then $p^{-1}(U \cap X_n) = U \cap X_{n-1} \sqcup \bigsqcup_j p^{-1}(U \cap$ closure of jth$n$-cell). Which is closed by hypothesis hence $U \cap X_n$ is closed.

□

6 Lecture 6: CW-Subcomplex

Proposition 6.1. *Let A be a Hausdorff space and let $X = A \cup_f J \times D^n$ be obtained from A by attaching n-cells. Let $Y \subset X$ be a subspace. Suppose that*

- *$Y \cap A$ is closed in A*
- *Y can be obtained from $A \cap Y$ by attaching n-cells*
- *$Y \cap (J \times \mathring{D}^n)$ is a union of path components of $J \times \mathring{D}^n$*

then Y is closed in X.

Proof. do this proof later □

Theorem 6.2. *Let $(X < AA)$ be a relative CW-complex and Y a closed subset of X with $A \subset Y$. Suppose that all $n \geq 0$, $Y \cap (X_n \setminus X_{n-1})$ is a union of path components of $X_n \setminus X_{n-1}$. Then (Y, A) is a relative CW-complex with respect to the filtration of subspaces $Y_n = Xn \cap Y$.*

Proof. finish proof later □

Definition 6.1. A CW-subcomplex of a relative CW-complex (X, A) is a closed subspace Y of X such that $A \subset Y$ and $Y \cap (X_n \setminus X_{n-1})$ is a union of path components of $X_n \setminus X_{n-1}$.

Theorem 6.3. *Let (X, A) be a relative CW-complex.*

1. *The closure of every cell is contained in a finite subcomplex of (X, A).*
2. *Every compact subset of X is contained in a finite subcomplex of (X, A).*

Proof. 1. The closure of any n-cell is in the image of D^n and the charateristic map hence quasi compact , hence compact, so 1 is actually just a special case of 2.

2. We let K be a compact subset of X.
Claim: There is an $n \geq 1$ such that $K \subset X_n$.
Proof of Claim: We will argue by contradiction. Suppose this were not true. There there are points $x_1, x_2, ... \in K$ such that $x_i \in X_{n_i} \setminus X_{n_i - 1}$ for $n_1 < n_2 < ....$ Set $D = \{x_1, x_2, ...\}$ an infinite subset of K.
Subclaim: Every subset S of D is closed in X.
Proof of Subclaim: $S \cap$ (closure of ak-cell) $\subset D_n \cap X_k$ is finite, hence closed in X, hence in the closure of the cell. So S is a closed subset of X. So D is closed, hence compact. D has the discrete topology and D is infinite.
Now we argue by induction on the number n such that $K \leq X_n$. For $n = -1$ if $K \subset X_{-1} = A$, we are done because (A, A) is a finite subcomplex of (X, A). For $n \geq 0$ we write $X_n = X_{n-1} \cup_{J \times \partial D^n} J \times D^n$. We showed last time that K only meets finitely many open n-cells. Set $I = \{j \in J : K \cap (j \times \mathring{D}^n) \neq \emptyset\}$, a finite subset of J. St $L = K \cup \bigcup_{j \in I}$(closure of jth$n$-cell), this is again compact. Since X_{n-1} is closed in X_n, $L \cap X_{n-1}$ is compact. By induction there is a finite subcomplex Y of $X_{n-1}, A)$ such that $L \cap X_{n-1} \subset Y$. The desired subcomplex. The desired subcomplex containing K is then $Y \cup_{I \times \partial D^n} I \times D^n$.

□

7 Lecture 7: Cellular Approximation Theorem I

Definition 7.1. Let (X, A) and (Y, B) be CW-complexes. A continuous map $f : X \to Y$ is cellular if $f(X_n) \subset f(Y_n)$ for all $n \leq -1$.

Theorem 7.1. *(Cellular Approximation Theorem) Let (X, A) and (Y, B) be CW-complexes. Let $f : X \to Y$ be map such that $f(A) \subset f(B)$. Then f is homotopic rel A to a cellular map.*

Remark. Recall: A homotopy $H : X \times [0, 1] \to Y$ is a homotopy relative A if $H(a, 0) = H(a, t)$ for all $a \in A, t \in [0, 1]$.

Example: Consider the "minimal" CW-structure on S^n with $A = X_{-1} = X_0 = ... = X_{m-1} = \{z\}, X_m = X_{m+1} = ... = XS^n$. Give S^m the same CW-structure for m. Let $m < n$. Then $f : (S^m, z) \to (S^n, z)$ is cellular iff f is the constant map with value z.

Corollary 7.2. *Every continuous map $f : S^m \to S^n$ with $m \leq n$ is homotopic rel any given point to a constant map.*

Remark. A consequence of this corollary is that for any $m < n$ we have that $\pi_m(S^n, \{z\}) = 0$. (We will see this later when we talk about homotopy groups.)

We will not prove the whole cellular approximation theorem in one lecture we will just start it here. We can prove this following special case of the cellular approximation theorem.

Theorem 7.3. *Suppose that $Y = B \cup_{\partial D^n} D^n$. Then for all $m < n$, any map of pairs $f : (D^m, \partial D^m) \to (Y, B)$ is homotopic rel ∂D^m to a map with image in B.*

Proof. We will argue by induction on n. For $n = 1$, $m = 0$, $\partial D^m = \partial D^0 = \emptyset$ (as $D^0 = *$). We use that D^m is path connected so every point of $Y = B \cup_{\partial D^n} D^n$ can be connected by a path to a point in B.
Now suppose hat $n \geq 2$, and the theorem is known for smaller values of n.
Fact 1:
For all $p < n - 1$, every continuous map $h : S^p \to S^{n-1}$ is homotopic to a constant map.
Proof of Fact 1: By the inductive hypothesis, the composite $D^p \to D^p/\partial D^p \cong S^P \to S^{n-1}$ is homotopic rel ∂D^p ti a constant map. Let $H : D^p \times [0, 1] \to S^{n-1}$ be such a homotopy . This descends to a map which is constant.

$$\begin{array}{ccc} D^p \times [0,1] & \xrightarrow{H} & S^{n-1} \\ \downarrow & & \uparrow \\ D^p/\partial D^p \times [0,1] & \xrightarrow{\cong} & S^p \times [0,1] \end{array}$$

Fact 2:
For $p < n-1$, every continuous map $h = (h_1, h_2) : S^p \to S^{n-1} \times (a, b)$ is homotopic to a constant map.
Proof of Fact 2: We use homotopy $H_1 : S^p \times [0, 1] \to S^{n-1}$ from h_1 to a continuous (which exists by fact 1). And a linear homotopy $H_2 : S^p \times \to (a, b)$ to a constant map. Together these yield the desired homotopy $H = (H_1, H_2) : S^p \times [0, 1] \to S^{n-1} \times (a, b)$.
Fact 3:
For $q < n$, every continuous map $h : \partial D^q \to S^{n-1} \times (a, b)$ admits a continuous extension to D^q.
Proof of Fact 3: The map $\partial D^q \to D^q; (x, t)) \mapsto x \cdot t$ is a quotient projection. Let H be a homotopy from a constant map to h (which exits by fact 2). Then $\underline{H}$ extends h. $\square$

$$\begin{tikzcd} \partial D^q \times [0,1] \arrow[r, "H"] \arrow[d] & S^m \times (a,b) \\ D^q \arrow[ur, dotted, "\overline{H}"] & \end{tikzcd}$$

8 Lecture 8: Cellular Approximation Theorem II

In this lecture we will continue to prove the cellular approximation theorem.

Theorem 8.1. *Let $m \leq n$. Let $f : D^m \to Y = B \cup_{\partial D^n} D^n$ be a continuous map such that $f(\partial D^m) \subset B$. Then f is homotopic relative ∂D^m to a map with image in B.*

Theorem 8.2. *Let (Y, B) be a relative CW-complex and $f : D^m \to Y$ a continuous map with $f(\partial D^m) \subset B$. Then f is homotopic relative ∂D^m to a map with image in Y_m.*

Proof. come back here later □

Theorem 8.3. *Let X be obtained from a subspace A by attaching m-cells. Let (Y, B) be a relative CW-complex. Let $f : X \to Y$ be a continuous map such that $f(A) \subset B$. Then f is homotopic relative A, to a map with image in Y_m.*

Proof. nah □

9 Lecture 9: Products with Locally Compact Spaces

Definition 9.1. A space X is locally compact if every neighbourhood of every point contains a compact neighbourhood.

Lemma 9.1. *Let X be a space such that every point has compact neighbourhood. Then X is locally compact. In paticular, compact spaces are locally compact.*

Example: $\mathbb{R}^n$ is locally compact but not compact for $n \geq 1$.

Proof. Let U be a neighbourhood of $x \in X$. There is an open neighbourhood by $U' \subset U$. Let K be a compact neighbourhood of x in X. Then $\{x\}$ and $K \setminus U'$ are disjoint closed subsets of the compact subspace K. Since compact spaces are normal i.e.there are open subsets W_1, W_2 of K with $x \in W_1$, $K \setminus U' \subset W_2$ with $W_1 \cap W_2 = \emptyset$. Then $K \setminus W_2$ is closed in K , and compact. Since W_1 is a neighbourhood of x in K, and K is a neighbourhood of x in X, W_1 is also a neighbouhood of x in X. Hence $x \in W_1 \subset K \setminus W_2 \subset U' \subset U \subset X$. □

Lemma 9.2. *Let X and Y be spaces and K a compact subset of Y. Let $x \in X$ and W an open subset of $X \times Y$ such that $\{x\} \times K \subset W$. Then there is an open subset V of X with $x \in V$ and $V \times K \subset W$.*

Proof. nope □

Theorem 9.3. *Let $f : X \to Y$ be a quotient map, i.e. a surjective map such that $O \subset Y$ is open if and only if $f^{-1}(O) \subset X$ is open. Then for every locally compact space Z, the map $f \times z : X \times Z \to Y \times Z$ is also a quotient map.*

Proof. nope □

Corollary 9.4. *Let $X = A \cup_{J \times \partial D^n} J \times D^n$ be obtained be attaching n-cells to A. Then for every locally compact space Z,*

$$(A \times Z) \sqcup (J \times D^n \times Z \to (A \cup_{J \times \partial D^n} J \times D^n) \times Z = X \times Z$$

is a quotient map.

Proof. The map factors as the composite

$$(A \times Z) \sqcup (J \times D^n \times Z \xrightarrow{\cong} (A \sqcup J \times D^n) \times Z \to (A \cup_{J \times \partial D^n} J \times D^n) \times Z$$

where this second part is a quotient map. □

Corollary 9.5. *Let (X, A) be a relative CW-complex and Z a locally compact space. For every subset O of $X \times Z$ the following are equivalent*

1. *The set O is open in $X \times Z$.*

2. *For every $n \leq -1$, $O \cap (X_n \cap Z)$ is open in $X \times Z$.*

3. *For every finite subcomplex (Y, A) of (X, A), the set $) \cap (Y \times Z)$ is open in $Y \times Z$.*

Proof. nope □

Corollary 9.6. *Let (X, A) be a relative CW-complex and Z a locally compact space. Let $f : X \times Z \to Y$ be a map to another space Y. Then the following are equivalent*

1. *f is continuous.*

2. *For every $n \geq -1$, the map $f|_{X_n \times Z} : X_n \times Z \to Y$ is continuous.*

Proof. nope □

10 Lecture 10: Homotopy Extension Property

Definition 10.1. Let X be a space and A a subspace of X. Then (X, A) has the <u>homotopy extension property</u> (HEP) if the following holds: for every continuous map $f : X \to Y$ to some other space Y and every homotopy $H : A \times [0, 1] \to Y$ starting with $f|_A : A \to Y$ there exists a homotopy $\overline{H} : X \times [0, 1] \to Y$ starting with f and extending to H.

$$\begin{tikzcd}
A \arrow[r, "id_A \times \{0\}"] \arrow[d, hook, "i"] & A \times I \arrow[d, "i \times id_I"] \arrow[ddr, bend left, "H"] & \\
X \arrow[r, "id_X \times \{0\}"] \arrow[drr, bend right, "f"'] & X \times I \arrow[dr, dotted, "\exists \overline{H}"] & \\
& & Y
\end{tikzcd}$$

Lemma 10.1. *A pair (X, A) has the homotopy extension property i and only if for every continuous map $g : X \cup_A A \times [0, 1] \to Y$ there i a continuous map $\overline{H} : X \times [0, 1] \to Y$ extending g.*

Here $X \cup_A A \times [0, 1] = (X \sqcup A \times [0, 1])/(a \sim (0, 0)$ for $a \in A)$ with the quotient space topology.

Proof. no thank you □

Lemma 10.2. *Let A be a <u>closed</u> subspace of X. Then the canonical map $X \cup_A A \times [0, 1] \to X \cup \{0\} \cup A \times [0, 1]$ is a homomorphism.*

Proof. similarly no thank you □

Corollary 10.3. *Let A be a closed subspace of X. Then (X, A) has the HEP if and only if every continuous map defined on $X \times 0 \cup A \times [0, 1]$ admits a continuous extension to $X \times [0, 1]$.*

Lemma 10.4. *Let B be a subspace of a sace Z. Then the following are equivalent:*

1. *Every continuous map $B \to Y$ admits a continuous extension to Z.*
2. *B is a retract of Z i.e. there is a continuous map $r : Z \to B$ such that $r|_B = id_B$.*

Proof. $\underline{(1 \implies 2)}$
2 is the special case of 1 for $Y = B$ and id_B.
$\underline{(2 \implies 1)}$
We define the desired extension as the following composite $Z \xrightarrow{r} B \to Y$. □

Corollary 10.5. *Let A be a closed subspace of X. Then (X, A) has the homotopy extension property if and only id $X \times 0 \cup A \times [0, 1]$ is a retract of $X \times [0, 1]$.*

Proposition 10.6. *For every $m \leq 0$, the pair $(D^m, \partial D^m)$ has the HEP.*

Proof. We check the retract criterion. For $x, t) \in D^m \times [0, 1]$ the line through (x, t) and $(0, 2)$ meets the subspace $D^m \times 0 \cup \partial D^m \times [0, 1]$ is exactly one point. This point varies continuously with (x, t) and this defines a retraction. □

Proposition 10.7. *Let X be a space obtained from a subspace A by attaching m-cells. Then (X, A) has the HEP.*

Proof. later □

Theorem 10.8. *Every relative CW-complex has the HEP.*

Proof. later □

Example:
A non-example. Let $X = [-1, 0] \cup \{\frac{1}{n} : n \in \mathbb{N}_{\geq 1}\}$ and $A = [-1, 0]$.
Claim: (X, A) does not have the HEP.
The homotopy

$$H : A \times 0, 1] \to A$$

defined by $H(a, t) = (1-t) \cdot a - 1$ defined $[-1, 0]$ linearly onto -1. Suppose there was a homotopy

$$\overline{H} : X \times [0, 1] \to X$$

that starts with id_X and extends H. Then $\overline{H}$ would have been constant and each isolated point $\frac{1}{n}$ for $n \leq 1$. So $\overline{H}$ must also be constant on 0. But H is not constant on 0, so such $\overline{H}$ does not exist.

11 Lecture 11: Cellular Approximation Theorem III

12 Lecture 12: Products of CW-Complexes

Cells multiply:
There is a homeomorphism $D^n \times D^m \cong D^{m+n}$, taking $(\partial D^m) \times D^n \cup D^m \cup (\partial D^n)$ to ∂D^{m+1}.
This suggests that if X, Y are CW-compleces, then $X \times Y$ should be a CW-complex with skeleta

$$(X \times Y)_k = \bigcup_{m+n=k} X_m \times Y_n.$$

Proposition 12.1. *Let X be a Hausdorff space, J_k sets for $k \leq 0$ and $q : \bigsqcup_{k \geq 0} J_k \times D^n \to X$ a continuous map. Suppose that:*

1. *For every $n \geq 0$, the restriction of q to $J_n \times \mathring{D}^n$ is injective, and the underlying set of X is the disjoint union of the sets $q(J_n \times \mathring{D}^n)$ for $n \geq 0$.*

2. *For all $k \geq 0$ and all $j \in J_k$, the set $q(j \times \partial D^k)$ is contained in a finite union of sets of the form $q(i \times D^l)$ for certain $l \leq k$ and $i \in J_l$.*

3. *A subset A of X is closed in X if and only if $A \cap q(j \times D^n)$ is closed in $q(j \times D^k)$ for all $k \geq 0$, $j \in J_k$.*

Then X is a CW-comples with respect to the skeleta $X_n = q(\bigsqcup_{k=1,\dots,n} J_k \times D^n)$.

Remark. Notation: We let $e_j^k = q(j \times \mathring{D}^k)$ for $k \geq 0$, $j \in J_k$, the jth open k-cell.

Proof. come back later □

Theorem 12.2. *Let X and Y be a CW-complex one of which is locally compact. Then the above defines a CW-structure on $X \times Y$, the product CW-structure. The k-cell in the product CW-structure are indexed by points of cells in X and Y whose dimensions add up to k.*

Proof. long, finish later □

Remark. A CW-complex Y is locally compact for example if:

- Y is finite
- Y is locally finite i.e. every point has a finite subcomplex as a neighbourhood

13 Lecture 13: Whiteheads Theorem

Theorem 13.1. *Let $f : X \to Y$ be a continuous map between path connected CW-complexes. Suppose that the map $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, f(x))$ is a group isomorphism for all $n \geq 1$ and some $x \in X$. Then f is a homotopy equivalence.*

Definition 13.1. Let Y be a space and B a subspace of Y. Then (Y, B) is <u>n-connected</u> for $n \geq 0$ if the following holds: for every $0 \leq m \leq n$ and every continuous map $g : D^m \to Y$ with $g(\partial D^m) \subset B$ there is a homotopy relative ∂D^m, from g to a map with image in B. The inclusion $B \hookrightarrow Y$ is a <u>weak homotopy equivalence</u> if (Y, B) is n-connected for all $n \geq 0$.

Examples:

- n-connected $\implies$ $(n-1)$-connected
- (Y, B) is 0-connected if and only if every point in Y is connected by a path in Y to a point in B. (equivalently if the map $\pi_0(B) \to \pi_0(Y)$ is surjective.
- (Y, B) is 1-connected if and only if
  - the map $\pi_0(B) \to \pi_0(Y)$ is bijective
  - the map $\pi_1(B, b) \to \pi_1(Y, b)$ is surjective for all $b \in B$.
- Suppose that B is deformation retract of Y, i.e. there is a homotopy $H : Y \times [0,1] \to Y$ relative B, from the identity og Y to a map with image in B. Then $B \hookrightarrow Y$ is a weak homotopy equivalence. Indeed, given $g : D^m \to Y$ continuous with $g(\partial D^m) \hookrightarrow B$. Then the composite
$$D^m \times \xrightarrow{g \times \mathrm{id}} Y \times [0,1] \xrightarrow{H} Y$$
is a relative homotopy to a map with image in B.

Theorem 13.2. *Let (X, A) be a relative CW-complex and (Y, B) an n-connected space pair for $n \leq 0$. Let $f : X \to Y$ be a continuous map with $f(A) \subset B$. Then there is a homotopy, relative A, from f to a map $g : X \to Y$ with $g(X_n) \subset B$.*

Proof. no thank you □

14 Lecture 14: Homotopy groups

Homotopy groups generalise the idea of the fundamental group. We write π_n for the n-th homotopy groups where π_1 is the fundamental group.

Remark. Notation: The n-cube is the set

$$I^n = [0,1]^n = \{(x_1, ..., x_n) \in R^n : 0 \leq x_i \leq 1\}$$

with boundary

$$\partial I^n = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_i \in \{0,1\} \text{ for some } 1 \leq i \leq n\}$$

we will often identify I^{n-1} with a particular face of I^n via

$$(x_1, ..., x_{n-1}) \mapsto (x_1, ..., x_{n-1}, 0)$$

and let $J^{n-1} =$ union of all faces of I^n except $I^{n-1} = \overline{(\partial I^n \setminus I^{n-1})}$ Then $I^{n-1} \cup J^{n-1} = \partial I^n$ and $I^{n-1} \cap J^{n-1} = \partial I^{n-1}$.

Definition 14.1. Let X be a space, A as a subspace of X and $x \in A$. For $n \geq 1$ we define

$$\pi_n(X, A, x)$$

as the set of homotopy classes of maps of space triples

$$(I^n, \partial I^n, J^{n-1}) \to (X, A, \{x\}).$$

In more detail: elements are represented by continuous maps $f : I^n \to X$ with $f(\partial I^n) \subset A$ and $f(J^{n-1}) = \{x\}$. Two such maps f and f' represent the same class in $\pi_n(X, A, x)$ is there is a homotopy

$$H : I^n \times [0,1] \to X$$

with $H(-,0) = f, H(-,1) = f'$ and $H(\partial I^n \times [0,1]) \subset A$ and $H(J^{n-1} \times [0,1]) = \{x\}$. We call this is <u>$n$-th relative homotopy group</u> of (X, A, x).

Remark.
- Special case: For $A = \{x\}$, we abbreviate notation to $\pi_n(X, x) = \pi_n(X, \{x\}, x)$, the absolute n-th homotopy group of X based at x.
- For $n =$,

$$\begin{aligned} \pi_1(X, x) &= \{\text{homotopy classes of pair maps } ([0,1], \{0,1\}) \to (X, \{x\})\} \\ &= \{\text{homotopy classes of loops based at } x\} \\ &= \text{fundamental group.} \end{aligned}$$

14.1 Construction of Homotopy groups

How does the group structure arise from the n-th homotopy group?
Let $n \geq 2$. Let $f, g : I^n \to X$ be continuous maps that send ∂I^n to A and J^{n-1} to $\{x\}$. We define $(f+g) : I^n \to X$ by the following

$$(f+g)(x_1, ..., x_n) = \begin{cases} f(2x_1, x_2, ..., x_n) & 0 \leq x_1 \leq \frac{1}{2} \\ g(2x_1 - 1, x_2, ..., x_n) & \frac{1}{2} \leq x_1 \leq 1 \end{cases}$$

this is well defined at $x_1 = \frac{1}{2}$ as here we take $f(1, x_2, ..., x_n) = g(0, x_2, ..., x_n) = x$ as we have the point is in J^{n-1} and so both f and g send J^{n-1} to the basepoint. So we can also observe that this is continuous as we define by finitely many case distinctions of continuous things on closed subspaces that overlap.
We can stack triple homotopies in the in the same way using the first coordinate so this descends to a well defined binary operation

$$+ : \pi_n(X, A, x) \times \pi_n(X, A, x) \to \pi_n(X, A, x); [f] + [g] \mapsto [f+g]$$

Remark. We can see that for $n = 1$, $A = \{x\}$ the same procedure in $\pi_n(X, x)$ which is exactly the concatenation of loops that gives the group structure on the fundamental group.

Theorem 14.1. *Let X be a space, A a subspace and $x \in A$. For every $n \geq 2$ the operation $+$ makes the set $\pi_n(X, A, x)$ into a group. Moreover, for $n \geq 3$, (and $n = 2$ with $A = \{x\}$), the group structure is abelian.*

Proof. For the group structure we can argue the same as for the fundamental group $\pi_1(X, x)$, using the first coordinate for the homotopies and treating the remaining coordinates as dummies. The neutral element is given by the class of the constant map $I^n \to X$ to the basepoint, x. Then inverse of $[f]$ is represented by the map $\overline{f}(x_1, ..., x_n) = f(1 - x_1, ..., x_n)$.
For the proof of commutativity: For the rest of the proof we suppose that $n \geq 3$ (or $n = 1$ and $A = \{x\}$) We can then also use he second coordinate to define another binary operation

$$(f \diamond g)(x_1, x_2, ..., x_n) = \begin{cases} f(x_1, 2x_{2},_{3}, ..., x_n) & 0 \leq x_2 \leq \frac{1}{2} \\ g(x_1, 2x_2 - 1,_{3}, ..., x_n) & \frac{1}{2} \leq x_2 \leq 1 \end{cases}$$

This also descends to a well defined binary operation

$$\diamond : \pi_n(X, A, x)^2 \to \pi_n(X, A, x).$$

The operations $+$ and $\diamond$ satisfy the interchange law i.e. $(x + y) \diamond (z + w) = (x \diamond z) + (y \diamond w)$ for all $x, y, z, w \in \pi_n(X, A, x)$ i.e. we stack x and y, z and w horizontally and then stack those vertically but we can change the order and still have the same thing i.e. Moreover,

z	w
x	y

both $+$ and $\diamond$ share the same neutral element $0 =$ class of the constant map Then $y \diamond z = (0 + y) \diamond (z + 0) = (0 \diamond z) + (y \diamond 0) = z + y$. But we can see these operations are actually the same by $y \diamond z = (y + 0) \diamond (0 + z) = (y \diamond 0) + (0 \diamond z) = y + z$. Hence $z + y = y + z$. ☐

15 Lecture 15: Homotopy Groups Via Spheres

The relative n homotopy group is

$$\pi_n(X,A,x) = \{(I^n, \delta I^n, J^{n-1}) \overset{\text{cont.}}{\to} (X,A,x)\}/\text{homotopy}$$

And the absolute homotopy group is

$$\pi_n(X,x) = \pi_n(X,\{x\},x) = \{(I^n,\delta I^n) \to (X,\{x\})\}/\text{homotopy}$$

we choose a quotient map $\psi : I^n \to D^n$ such that $\psi(\partial I^n) = \partial D^n$ and $\psi(J^{n-1}) = z = (1,0,...,0)$, and such that the induced map

$$I^n/J^{n-1} \overset{\cong}{\to} D^n$$

is a homeomorphism. Then thus restricts to a homeomorphism

$$\partial I^n/J^{n-1} \overset{\cong}{\to} \partial D^n$$

Then precomposition with ψ is a bijection

$$\left\{(D^n, \partial D^n, z) \overset{\text{cont.}}{\to} (X,A,x)\right\} \overset{\cong}{\to} \left\{(I^n, \delta I^n) \overset{\text{cont.}}{\to} (X,\{x\})\right\}$$
$$f \mapsto f \circ \psi$$

and so we have

$$\left\{(D^n, \partial D^n, z) \overset{\text{cont.}}{\to} (X,A,x)\right\}/\text{homotopy} \overset{\cong}{\to} \pi_n(X,A,x)$$
$$f \mapsto f \circ \psi$$

Further simplification in the absolute case we chose a quotient map $\phi : I^n \to S^n = \partial D^{n+1}$ such that $\phi(\partial I^n) = z = (1,0,...,0)$ and such that the induced map

$$I^n/\partial I^n \to S^n$$

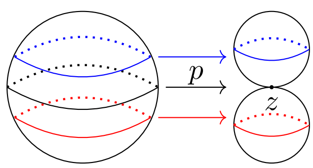
is a homeomorphism. Precomposition with ϕ induces a bijection

$$\left\{(S^n, z) \overset{\text{cont.}}{\to} (X,x)\right\} \overset{\cong}{\to} \pi_n(X,x)$$

For the group structure on the left hand side we choose a pinch map which is a continuous map $p : S^n \to S^n \vee_z S^n$ that collapses the equator to the based point and induces a homeomorphism where

$$\text{upper hemisphere/equator} \overset{\cong}{\to} \text{1-st copy of } S^n$$
$$\text{lower hemisphere/equator} \overset{\cong}{\to} \text{2-nd copy of } S^n$$

as in the picture. which we can use to define a binary operator on $\pi'_n(X,x)$ by



$$[f] + [g] = [S^n \overset{p}{\to} S^n \vee S^n \overset{f \vee g}{\to} X].$$

We show associativity by considering where these maps are not the same but they are homotopic rel z.

$$\begin{tikzcd}
S^n \arrow[r, "p"] \arrow[d, "p"] & S^n \vee S^n \arrow[d, "\mathrm{id}\vee p"] \\
S^n \vee S^n \arrow[r, "p\vee \mathrm{id}"] & S^n \vee S^n \vee S^n
\end{tikzcd}$$

16 Lecture 16: Functorality and the Role of the Basepoint

16.1 Functoriality of Homotopy Groups

Remark. Recall:

- For a continuous map $f : X \to Y$ induces a group homomorphism $f_* : \pi_1(X, x) \to \pi_1(Y, f(x)$.
- For $w : [0,1] \to X$ a path, we have an induced isomorphism $w_* : \imath_1(X, w(0) \to \pi_1(X, w(1))$.

The aim of this video is to generalise these facts for higher homotopy groups.
Let $f : X \to Y$ be continuous with $f(A) \subset B$ and $f(x) = y \in B$. Then we have the map

$$f_* : \pi_n(X, A, x) \to \pi_n(Y, B, y)$$

defined by $f_X[g : I^n \to X] = [f \circ g : I^n \to Y]$. It still remains to show that this is well defined i.e. $[f \circ g] \in \pi_n(Y, B, y)$ and also that taking a different representative of $[g]$ yields the same class as $f \circ g$.

Lemma 16.1. *(Functoriality of homotopy groups) Let $f : (X, A, x) \to (Y, B, y)$ be a continuous map of triples. Then*

$$f_* : \pi_n(X, A, x) \to \pi_n(Y, B, y)$$

is a group homomorphism for $n \geq 2$.
If $\overline{f} : (Y, B, y) \to (Z, C, z)$ is another continuous map of triples, then $(\overline{f} \circ f)_* = \overline{f}_* \circ f_*$. Moreover, $(id_X)_* = id_{\pi_n(X,A,x)}$.

Proof. All properties already hold on the level of representatives i.e. $g_1, g_2 : (I^n, \partial I^n, J^{n-1}) \to (X, A, x)$ then

$$f \circ (g_1 + g_2) = (f \circ g_1) + (f \circ g_2)$$

gives the group homomorphism property for representatives of homotopy class.

$$\overline{f} \circ (f \circ g) = (\overline{f} \circ f) \circ g_1$$

so we have that $\overline{f}_* \circ f = (\overline{f} \circ f)$. □

16.2 The Role of the Basepoint

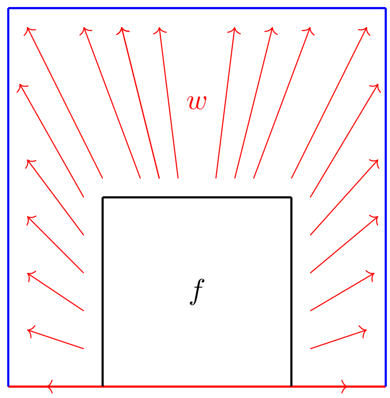
The relative case: Given (X, A, X). Let $w : [0,1] \to A$ be a path, et $x_0 = w(0)$ and $x_1 = w(1)$. We define a map

$$w_* : \pi_n(X, A, x_0) \to \pi_n(X, A, X_1)$$

by sending the class $[f : I^n \to X]$ to the class of the following map $w \times f$ as in the following picture Note: If f is homotopic, as maps of triples, to f' then $w \times f'$ is homotopic to $w \times f'$ as triples. Hence this construction is well defined on homotopy classes.

Proposition 16.2. *Let (X, A) be a space pair and $w : [0,1] \to A$ a path, set $x_0 = w(0), x_1 = w(1)$, for $n \leq 2$ we have*

1. *The map $w_* : \pi_n(X, A, x_0) \to \pi_n(X, A, x_1)$ is a group homomorphism*



2. *Let w' be homotopic rel end points to w. Then $w'_* = w_*$*

3. *Let $v : [0,1] \to A$ be a path with $v(0) = x_1 = w(1)$. Then*

$$v * (w * [f]) = (v * w) * [f] \in \pi_n(X, A, x_2)$$

   where $x_2 = v(1)$. (Where $*$ is concatenation of paths).

4. *Let $c : [0,1] \to A$ be the constant path at x_0. Then $c_* = id_{\pi_n(X,A,x_0)}$*

5. *The homomorphism w_* is an isomorphism of groups.*

Proof. by pictures, won't include here □

In the absolute case we have $w : [0,1] \to A$ induce a map $w_* : \pi_n(A, x_0) \to \pi_n(A, x_1)$ where $x_i = w(i)$. By sending f to the class represented by $w \times f$ as above in the relative case.

Proposition 16.3. *In this situation*

$$w_* : \pi_n(A, x_0) \to \pi_n(A, x_1)$$

is a group isomorphism that only depends on the homology class relative he end points of w. If $v : [0,1] \to A$ is another path with $v(0) = w(1)$, then $v_* \circ w_* = (v * w)_*$. Moreover $c_* = id$.

Remark. A space X is simply connected if it is path connected and for some base point $x_0 \in X$, the group $\pi_1(X, x_0)$ is trivial. Then any two points x_0, x_1 can be connected by a path $w : [0,1] \to X$, and w is unique up to homotopy relative end points. So the isomorphism w_* is cannonical i.e. independent of any choice.

17 Lecture 17: The Long Exact Sequence of Homotopy Groups

Definition 17.1. A sequence of groups and homomorphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if $\text{im}(f) = \ker(g)$. A sequence of maps of based sets

$$(A, a) \xrightarrow{f} (B, b) \xrightarrow{g} (C, c)$$

exact if $\text{im}(f) = \{b \in B : g(b) = c\}$.

Example:

- A sequence of based maps of based sets

$$A \xrightarrow{f} B \xrightarrow{g} \{*\}$$

  is exact iff f is surjective.

- A sequence of group homomorphisms

$$0 \xrightarrow{f} B \xrightarrow{g} C$$

  is exact iff g is injective.

17.1 Construction and proof of the Long Exact Sequence of Homotopy Groups

Let X be a space, A a subspacee of X and $x \in A$. We will construct a long exact sequence of homotopy groups

$$... \to \pi_n(A,x) \xrightarrow{i_*} \pi_n(X,x) \xrightarrow{j_*} \pi_n(X,A,x) \xrightarrow{\partial} \pi_{n-1}(A,x) \xrightarrow{i_*} \pi_{n-1}(X,x) \to ...$$

$$... \xrightarrow{i_*} \pi_1(X,x) \xrightarrow{j_*} \pi_1(X,A,x) \xrightarrow{\partial} (A,x) \xrightarrow{i_*} \pi_0(X,x)$$

Remark. At the end we do not have well defined groups. $\pi_1(X,A,x)$ is a based set and does not have natural groups although we define it in the same way. Similarly the sets of path components $\pi_0(X,x), \pi_0(A,x)$ are now based sets with the base points the oath component containing x.

Here:

- $i_* : \pi_n(A,x) \to \pi_n(X,x)$ is the homomorphism induced by the inclusion $i : X \hookrightarrow X$
- $j_* : \pi_n(X,x) \overset{\cong}{\to} \pi_n(X,\{x\},x) \to \pi_n(X,A,x)$ is induced by inclusion $j : (X,\{x\}) \hookrightarrow (X,A)$
- $\partial : \pi_n(X,A,x) \to \pi_{n-1}(A,x)$ is the map $\partial[f : (I^n, \partial I^n, J^{n-1}) \to (X,A,x)] = [f|_{I^{n-1}} : (I^{n-1}, \partial I^{n-1}) \to (A,x)]$

Theorem 17.1. *For every space pair (X,A) and $x \in A$ the sequence of homotopy groups is long exact i.e. every pair of consecutive maps is exact.*

Proposition 17.2. *(Compression criterion). Let $f : (I^n, \partial I^n, J^{n-1}) \to (X,A,x)$ be a continuous map of triples. Then the following are equivalent: (for $n \geq 1$)*

1. *f represents a trivial element in $\pi_n(X,A,x)$*
2. *f is homotopic relative ∂I^n, to a map with image A.*

Proof. (Of Proposition) ($1 \implies 2$):
If f represents the trivial elemet, there is a homotopy $H : I^n \times [0,1] \to X$ with

- $H(-,0) = f$, $H(-,1)$ the constant map
- $H(\partial I^n \times [0,1]) \subset A$
- $H(J^{n-1} \times [0,1]) = \{x\}$

We reparametrize the homotopy H. picture here of how we reparametrized We use a continuous map

$$Q : I^n \times [0,1] \to I^n \times [0,1]$$

finish proof later if time ∎

Proof. (Of Theorem) <u>For exactness at $\pi_n(A,x) \xrightarrow{i_*} \pi_n(X,x) \xrightarrow{j_*} \pi_n(X,A,x)$:</u>
$\mathrm{im}(i_*) \subset \ker(j_*) \iff j_*(i_*[f : (I^n, \partial I^n) \to (A,x)]) = *$ this comes from $2 \implies 1$ in the compression criterion.
$\mathrm{im}(i_*) \supset \ker(j_*)$
Suppose $f : (I^n, \partial I^n) \to (X,x)$ is such that $j_*[f] = *$. Then by $1 \implies 2$ of the compression criterion we have that f is homotopic rel ∂I^n to a map f' with image in A. But $[f] \in \pi_n(A,x)$ and $i_*[f] = [f]$.
<u>For exactness at $\pi_n(X,x) \xrightarrow{j_*} \pi_n(X,A,x) \xrightarrow{\partial} \pi_{n-1}(A,x)$:</u>
$\mathrm{im}(j_*) \subset \ker(\partial)$:
this is true $\iff \partial(j_*[f : (I^n, \partial I^n) \to (X,x)]) = *$ where $\partial(j_*[f : (I^n, \partial I^n) \to (X,x)]) = [f|_{I^{n-1}} : (I^{n-1}, \partial I^{n-1}) \to (A,x)] = *$.
$\mathrm{im}(j_*) \supset \ker(\partial)$:
Let $f : (I^n, \partial I^n, J^{n-1}) \to (X,A,x)$ represent an element in the kernl of ∂. There is thus a homotopy $H : I^n \times [0,1] \to A$ from $f|_{I^{n-1}}$, relative ∂D^{n-1} to the constant map at x. We extend H to a homotopy

$$H' : (\partial I^n) \times [0,1] = (I^{n-1} \times [0,1]) \cup (J^{n-1} \times [0,1]) \xrightarrow{H \cup c_x} A$$

The pair $(I^n, \partial I^n)$ has the homotopy extension property so we can extend H' to a homotopy

$$K : I^n \times [0,1] \to X$$

starting with f. Then K is a homotopy of triples $(I^n, \partial I^n, J^{n-1}) \to (X,A,x)$ from f to $f' = K(-,1)$. Since $f;(\partial I^n) = \{x\}$, f' represents a class in $\pi_n(X,x)$ and $j_*[f'] = [f]$.
<u>Exactness at $\pi_n(X,A,x) \xrightarrow{\partial} \pi_{n-1}(A,x) \xrightarrow{i_*} \pi_{n-1}(X,x)$:</u>
$\mathrm{im}(\partial) \subset \ker(i_*)$:
This is true iff $i_*(\partial[f : (I^n, \partial I^n, J^{n-1}) \to (X,A,x)]) = *$ where $i_*(\partial[f : (I^n, \partial I^n, J^{n-1}) \to (X,A,x)]) = [f|_{I^{n-1}} : (I^{n-1}, \partial I^{n-1}) \to (A,x)]$ we interpret f as a homotopy $f : I^{n-1} \times [0,1] = I^n \to X$ from $f|_{I^{n-1}}$ to the constant map rel ∂I^{n-1}. Hence we have the above.
$\mathrm{im}(\partial) \supset \ker(i_*)$:
Let $g : (I^{n-1}, \partial I^{n-1}) \to (A,x)$ represent an element in the kernal of i_*. Let $H : I^{n-1} \times [0,1] \to X$ be a homotopy from g relative ∂I^{n-1} to the constant map at x. We view H as a map $I^n \to X$, as such, it sends ∂I^n to A and sends J^{n-1} to x. So H represents a class in $\pi_N(X,A,x)$ with $\partial(H) = [H|_{I^{n-1}}] = [g]$. □

17.2 Application of Long Exactness of Homotopy groups

Corollary 17.3. *Let X be a path connected space, A a subspace of X, $x \in A$. Then the following are equivalent:*

1. $\pi_n(X,A,x) = *$ *for all* $n \geq 1$
2. *A is path connected and the map i_* :*
   $pi_n(A,x) \to \pi_n(X,x)$ is a group isomorphism

Proof. ($1 \implies 2$)
The exact sequence of based sets

$$\{*\}\pi_x(X,A,x) \xrightarrow{\partial} \pi_0(A,x) \xrightarrow{i_*} \pi_0(X,x) = \{*\}$$

hence $\pi_0(A,x) = \ker(i_*) = \mathrm{im}(\partial) = \{*\}$ i.e. A is path connected. For $n \leq 1$ we have exact sequences

$$\{*\} = \pi_{n+1}(X,A,x) \xrightarrow{\partial} \pi_n(A,x) \xrightarrow{i_*} \pi_n(X,x) \xrightarrow{j_*} \pi(X,A,x) = \{*\}$$

where $\pi_{n+1}(X,A,x) = \{*\}$ so i_* is injective and $\pi_n(X,A,x) = \{*\}$ so i_* is injective.
$(2 \implies 1)$
We have exact sequences

$$\pi_n(A,x) \xrightarrow{i_*} \pi_n(X,x) \xrightarrow{j_*} \pi_n(X,A,x) \xrightarrow{\partial} \pi_{n-1}(A,x) \xrightarrow{i_*} \pi_{n-1}(X,x)$$

where both i_* are isomorphisms. So $\mathrm{im}(\partial) = ker(i_*) = \{*\}$ as i_* is injective. Hence ∂ is the constant map to the basepoint. Hence $\ker(\partial) = \pi_n(X,A,x)$. Also $\ker(j_*) = \mathrm{im}(i_*) = \pi_n(X,x)$ as $\imath_*$ is surjective. Hence j_* is also the constant map at the basepoint. So $\pi_n(X,A,x) = \ker(\partial) = \mathrm{im}(j_*) = \{*\}$. $\square$

Exercise:
Let X be a path connected space, $A \subset X$ and $x \in A$, $m \geq 1$. Then the following are equivalent

1. $\pi_n(X,A,x) = \{*\}$ for all $1 \leq n \leq m$
2. Th pair (X,A) is m-connected.

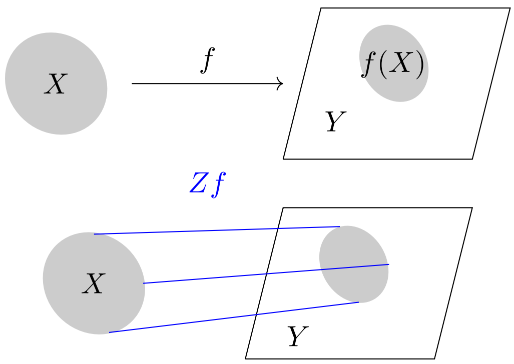
18 Lecture 18: Mapping Cylinders

18.1 Construction of the Mapping Cylinder

Let $f : X \to Y$ be a continuous map. Then the mapping cylinder of f is the space

$$Zf = X \times [0,1] \cup_f Y$$

i.e the quotient space of $X \times [0,1] \sqcup Y$ by the equivalence relation generated by $(x,1) \sim f(x)$ for all $x \in X$. There are continuous maps



$$\begin{aligned}
i : &X \to X \times [0,1] \cup_f Y \\
&x \to (x,0) \\
p : &Zf = X \times [0,1] \cup Y \to Y \\
&(x,t) \to f(x) \\
&y \to y \\
j : &Y \to X \times [0,1] \cup_f Y \\
&y \to y
\end{aligned}$$

where i, j is a closed embedding (i.e. injective, closed). Note that $p \circ i = f$.

Proposition 18.1. *The map $p : Zf \to Y$ is a homotopy equivalence.*

Proof. We show that $j : Y \to Zf$ embeds Y as a deformation retract of Zf. Note that $p \circ j = \mathrm{id}_Y$. A homotopy $Zf \times [0,1] \to Zf$ is defined by $(x,t,s) \mapsto (x, (1-s)t + s)$ for $x \in X, t, s \in [0,1]$ and $(y,s) \mapsto y$ for $y \in Y$. This homotopy starts with $j \circ p$ and ends with the identity of Zf. $\square$

18.2 Gluing CW-complexes

Let X and Y be CW-complexes and A a subcomplex of X. Let $f : A \to Y$ be a cellular continuous map. Set $Z = X \cup_{A,f} Y = X \sqcup Y/(a \sim f(a), \text{ for } a \in A)$

Proposition 18.2. *The subspaces $Z_n = X_n \cup_{A,f} Y_n$ of Z define a CW structure on Z.*

Proof. First we wish to show Z_n is obtained from Z_{n-1} by attatching n-cells

$$Z_{n-1} = X_{n-1} \cup_{A_{n-1},f} Y_{n-1} \subset X_{n-1} \cup_{A_{n-1},f} Y_n \subset X_n \cup_{A_n,f} Y_n = Z_n$$

we show both inclusions are attatchments of n-cells and all cells in the second inclusion are not attached to interiors of first batch of n-cells.
For the first inclusion: We choose a presentation $Y_n \cong Y_{n-1} \cup_{\partial D^n} J \times D^n$, and use it to define a homeomorphism

$$X_{n-1} \cup_{A_{n-1},f} Y_n \cong X_{n-1} \cup_{A_{n-1},f} (Y_{n-1} \cup_{\partial D^n} J \times D^n) \cong (X_{n-1}] \cup_{A_{n-1},f} Y_{n-1}) \cup_{\partial D^n} J \times D^n = Z_{n-1} \cup_{\partial D^n} J \times D^n.$$

For the second inclusion: We choose a presentation $X_n \cong X_{n-1} \cup_{\partial D^n} I \times D^n$, let $I' \subset I$ be the subset that indexed the n-cells of A. Then we get a chain of homeomorphisms

$$\begin{aligned} Z_n &= X_n \cup_{A_n,f} Y_n \\ &\cong (X_{n-1} \cup_{\partial D^n} I \times D^n) \cup_{A_n,f} Y_n \\ &\cong ((X_{n-1} \cup_{\partial D^n} I' \times D^n) \cup_{I \setminus I', \partial D^n} (I \setminus I') \times D^n) \cup_{A_n,f} Y_n \\ &\cong (X_{n-1} \cup_{A_{n-1},f} Y_n) \cup_{I \setminus I', \partial D^n} (I \setminus I') \times D^n \end{aligned}$$

Note: the n-cells of Z are the n-cells of Y disjoint union with the n-cells of X that are not in A.
Now we wish to show Z has the weak topology with respect to Z_n.
We can do this by considering the below commutative diagra of quotient maps □

$$\begin{tikzcd} (\sqcup_n X_n) \sqcup (\sqcup_n Y_n) = \sqcup_n (X_n \sqcup Y_n) \arrow[r] \arrow[d] & X \sqcup Y \arrow[d] \\ \sqcup_n (X_n \cup_{A_n,f} Y_n) = \sqcup_n Z_n \arrow[r] & X \cup_{A,f} Y = Z \end{tikzcd}$$

18.3 CW-Structure on the Mapping Cylinder

Let $f : X \to Y$ be a continuous cellular map between CW-complexes. We give $[0,1]$ the minimal CW-structure with 0-skeleton= $\{0,1\}$ and 1-skeleton=$[0,1]$. We give $X \times [0,1]$ the product CW-structure then the

$$n-\text{cells of } X \times [0,1] = (n-\text{cells of } X) \times 0 \sqcup ((n-1)-\text{cells of } X) \times [0,1] \sqcup (n-\text{cells of } X) \times 1$$

so $X \times \{1\}$ is a CW-structure is the product CW-structure, and $f : X \times \{1\} \to Y; (x,1) \to f(x)$ is cellular, so we obtained the glued CW-structure $X \times [0,1] \cup_{X \times 1, f} Y = Zf$. Note:

$$n-\text{cells of } Zf = (n-\text{cells of } X) \times 0 \sqcup ((n-1)-\text{cells of } X) \times (0,1) \sqcup (n-\text{cells of } Y)$$

19 Lecture 19: Proof of Whiteheads Theorem

Theorem 19.1. *(Whiteheads theorem) Let $f : X \to Y$ be a continuous map between path connected CW-complexes such that $f_* : \pi_n(X,x) \to \pi_n(Y, f(x))$ is a group isomorphism for some $x \in X$ and all $n \geq 1$. Then f is a homotopy equivalence.*

$$\begin{tikzcd} \pi_n(X, x_0) \arrow[r, "f_*"] \arrow[d, "w_*"] & \pi_n(Y, f(x_0)) \arrow[d, "(f\circ w)_*"] \\ \pi_n(X, x_1) \arrow[r, "f_*"] & \pi_n(Y, f(x_1)) \end{tikzcd}$$

Remark. Since X is path connected, any two points can be joined by a path $w : [0,1] \to X$, $x_0 = w(0)$, $x_1 = w(1)$. Then the following diagram of group homomorphisms commutes by inspection: hence in whiteheads theorem we can replace for some x with for all x.

Proposition 19.2. *Let Y be a space and B a subspace of Y, and $m \geq 1$. Then the following are equivalent:*

1. *Every continuous map $f : D^m \to Y$ with $f(\partial D^m) \subset B$ is homotopic, relatuve ∂D^m to a map with image in B.*
2. *For all $y \in B$, $\pi_m(Y, B, y) = \{*\}$.*

Proof. long boi no thank □

Corollary 19.3. *Let Y be a path connected space and B a subspace of Y. Then the following are equivalent.*

1. *B is path connected and $i_* : \pi_n(B, y) \to \pi_n(Y, y)$ is an isomorphism for all $n \geq 1$ and some $y \in B$.*
2. *$\pi_n(Y, B, y) = \{*\}$ for all $y \in Y$, $n \geq 1$*
3. *The iclustion $i : B \hookrightarrow Y$ us a weak homotopy equivalence i.e. for all $m \geq 0$ every $f : D^m \to Y$ with $f(\partial D^m) \subset B$ is homotopic relative ∂D^n to a map with image in B.*

Proposition 19.4. *Let $H : X \times [0,1] \to Y$ be a homotopy from $f = H(-,0)$ to $g = H(-,1)$. For $x \in X$ with $H_x : [0,1] \to Y$ for the path $H(x,-)$ from $f(x)$ to $g(x)$. Then the diagram of group homomorphisms commutes*

$$\begin{tikzcd} \pi_n(X, x) \arrow[r, "f_*"] \arrow[dr, "g_*"] & \pi_n(Y, f(x)) \arrow[d, "(H_x)_*"] \\ & \pi_n(Y, g(x)) \end{tikzcd}$$

Proof. booo hiss □

Proof. (Of Whiteheads theorem) long □

20 Lecture 20: The Road to Homology

Vaguely homology is a functor, $H_n(-, A)$, from the category of topological spaces (**Top**) to the category of abelian groups (**Ab**). In contrast to homotopy groups, homology groups are tricky

$$\begin{tikzcd} \mathbf{Top} \arrow[r, "\text{sing.}", "\text{complex}"'] \arrow[rrr, bend left, "H_n(-,A)"] & \mathbf{sSets} \arrow[r, "\text{linearisation}"] & \mathbf{Ch} \arrow[r, "\text{homology}", "\text{groups}"'] & \mathbf{Ab} \end{tikzcd}$$

to define but fairly straightforward to calculate. We will split the steps of defining homology up as in the diagram above. We will show that homology is a functor. Every step in the above construction is a functor. This means that for a continuous map $f : X \to Y$, this induces a group homomorphism $f_* : H_n(X, A) \to H_n(Y, A)$ which is compatible with composition and identities.
Some key properties of homology:

- Homotopy invariance: Let $f, g : X \to Y$ be homotopic continuous maps, then $f_* = g_*$
- Long exact sequence: Let B be a subspace. There are relative homology groups $H_n(X, B, A)$ and a long exact sequence

$$... \to H_n(B, A) \xrightarrow{i_*} H)n(X, A) \xrightarrow{p_*} H_n(X, B, A) \xrightarrow{\partial} H_{n-1}(B, A) \to ....$$

- Excision: Let $B \subset X$ be a subspace, let $U \subset B$ be such that $\overline{U} \subset \mathring{B}$. Then the inclusion $(X \setminus U, B \setminus U) \hookrightarrow (X, B)$ induces an isomorphism $H_n(X \setminus U, X \setminus B, A) \cong H_n(X, B, A)$.

It will also be useful to note that CW-structures give effective ways to calculate groups via "cellular homology".

21 Lecture 21: Categories and Functors

Definition 21.1. A category $\mathscr{C}$ consists of a class of objects $\mathrm{Ob}(\mathscr{C})$ and a class of morphisms $\mathrm{Mor}(\mathscr{C})$ and maps

$$\begin{aligned} s, t : \mathrm{Mor}(\mathscr{C}) &\to \mathrm{Ob}(\mathscr{C}) && \text{“source” and “target”} \\ 1 : \mathrm{Ob}(\mathscr{C}) &\to \mathrm{Mor}(\mathscr{C}) && \text{“identity”} \\ \circ : \{(f, g) \in \mathrm{Mor}(\mathscr{C})^2 : s(f) = t(g)\} &\to \mathrm{Mor}(\mathscr{C}) && \text{“composition”} \end{aligned}$$

which satisfy the following:

- For all objects X, Y og $\mathscr{C}$, the maps from X to Y, $\mathscr{C}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y) = \{f \in \mathrm{Mor}(\mathscr{C}) : s(f) = X, t(f) = Y\}$ forms a set and we note $f : X \to Y$ as a morphism f, with source X and target Y.
- For all objects X, the morphism 1_X satisfies $s(1_X) = t(1_X) = X$, and 1_X is a two sided identity for composition i.e. for all $f : X \to Y$ and $g : W \to X$, $f \circ 1_X = f$ and $1_X \circ g = g$.
- For all $f : X \to Y, g : W \to X$, $s(f \circ g) = s(g)$, $t(f \circ g) = t(f)$.
- Composition is associative i.e. for all $f : X \to Y, g : W \to X, h : V \to W$,

$$(f \circ g) \circ h = f \circ (g \circ h) \in \mathscr{C}(V, Y) = \mathrm{Hom}_{\mathscr{C}}(V, Y).$$

Examples:

- The category of groups with objects groups and morphisms group homomorphisms
- The category, **Top**, of topological spaces with objects topological spaces and morphisms continuous maps.

Definition 21.2. A morphism $f : X \to Y$ in a category $\mathscr{C}$ is an isomorphism if there is a morphism $g : Y \to X$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$. Here g is called the inverse of f.

Remark. • If an inverse exists it is unique.

- In the category of topological spaces isomorphisms are the homeomorphisms.

Example:
The homotopy category of spaces, Ho(**Top**) has all spaces as objects, morphisms are given by

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{Top})}(X,Y) = \text{ homotopy classes of continuous maps } f: X \to Y.$$

i.e. $1_x = [\mathrm{id}_X]$, $[f] \circ [g] = [f \circ g]$.
Let $f : X \to Y$ be a continuous map. Then the foloowing are equivalent.

1. $[f]$ is an isomorphism in Ho(**Top**).
2. f is a homotopy equivalent.

Example:
A category $\mathscr{C}$ is given by $\mathrm{Ob}(\mathscr{C}) = \{*\}$, $\mathscr{C}(*,*) = \{1_*, t\}$ with $t \circ t = 1_*$.

Definition 21.3. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A (covariant) functor $F : \mathscr{C} \to \mathscr{D}$ consists of a map $F : \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{D})$ and $F : \mathrm{Mor}(\mathscr{C}) \to \mathrm{Mor}(\mathscr{D})$ such that for any $f : X \to Y$, $g : W \to X$ morphism in $\mathscr{C}$

$$\begin{aligned} s(Ff) &= F(s(f)) \\ t(Ff) &= F(t(f)) \\ F(1_X) &= 1_{FX} \\ F(f \circ g) &= Ff \circ Fg \end{aligned}$$

i.e. for $f : X \to Y$ in $\mathscr{C}$ we get a morphism $Ff : FX \to FY$ in $\mathscr{D}$.

Now what do we mean when we talk about functoriality?
Example:

$$\pi_0 : \mathbf{Top} \to \mathbf{Sets}; X \mapsto \pi_0(X)$$

is a functor with $f : X \to Y$ a morphism of topological spaces sent to $f_* = \pi_0(f) : \pi_0(X) \to \pi_*(Y)$ with $\pi_(f)[x] = [f(x)]$.
Example:

$$\pi_1 : \mathbf{Top}_x \to \mathbf{Groups}; (X,x) \to \pi_1(X,x)$$

(where $\mathbf{Top}_x$ is the category of based topological spaces and based maps) is a functor where a based map $f : X \to Y$ is taken to $f_* : \pi_1(f) : \pi_1(X,x) \to \pi_1(Y,y)$ for $f(x) = y$.
Example:
For $n \geq 2$

$$\pi_n : \mathbf{Top} \to \mathbf{Ab}.$$

in the way that you think.

22 Lecture 22: Simplicial Sets

Definition 22.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A contravariant functor $F : \mathscr{C} \to \mathscr{D}$ consists of maps $F : \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{D})$ and $F : \mathrm{Mor}(\mathscr{C}) \to \mathrm{Mor}(\mathscr{D})$ such that

$$\begin{aligned} s(Ff) &= F(t(f)) \\ t(Ff) &= F(s(f)) \\ F(1_X) &= 1_{FX} \\ F(f \circ g) &= F(g) \circ F(f) \end{aligned}$$

i.e. for a morphism $f : X \to Y$ in $\mathscr{C}$ we have a morphism $Ff : FY \to FX$ in $\mathscr{D}$.

Remark. Let $\mathscr{C}$ be a category. The opposite category, $\mathscr{C}^{\mathrm{op}}$ has $\mathrm{Ob}(\mathscr{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathscr{C})$ and $\mathrm{Mor}(\mathscr{C}^{\mathrm{op}}) = \mathrm{Mor}(\mathscr{C})$ with

$$\begin{aligned} s^{\mathrm{op}}(f) &= t(f), \\ t^{\mathrm{op}}(f) &= s(f), \\ 1_X^{\mathrm{op}} &= 1_X, \\ g \circ_{\mathrm{op}} f &= f \circ g. \end{aligned}$$

$\mathscr{C}^{\mathrm{op}}(X,Y) = \mathscr{C}(Y,X)$. I.e. a contravariant functor $F : \mathscr{C} \to \mathscr{D}$ is a covariant functor $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$.

Definition 22.2. The category Δ has objects the set $[n] = \{0, 1, ..., n\}$, for all $n \geq 0$. And morphisms $f : [m] \to [n]$ weekly monotone maps i.e. $f(i) \leq f(i+1)$ for all $0 \leq i \leq m$.

Definition 22.3. A simplicial set is a contravariant functor X from Δ to **Sets**. Equivalently, a covariant functor $\Delta^{\mathrm{op}} \to \mathbf{Sets}$.

Remark. (Notation) Let $X : \Delta^{\mathrm{op}} \to \mathbf{Sets}$ be a simplicial set. We write $X_n = X([n])$ and call this the "set of n-simplicies" of X. For a morphism $\alpha : [m] \to [n]$ in Δ, we write $\alpha^* = X(\alpha) : X_n \to X_m$.

Example:
For $k \geq 0$, the simplicial k-simplex is the simplicial set Δ^k defined by

$$(\Delta^k)_m = \Delta^k([m]) = \Delta([m],[k]) = \text{ the set of weakly monotone maps } [m] \to [k].$$

For $\alpha : [m] \to [n]$ in $\mathrm{Mor}(\Delta)$, $\Delta^k(\alpha) = \alpha^* : \Delta([n],[k]) \to \Delta([m],[k])$ is defined by $\alpha^*(f) = f \circ \alpha$.
Example: (Represented functors, contravariant version)
Let $\mathscr{C}$ be a category and Z an object of $\mathscr{C}$. The functor represented by Z

$$\mathscr{C}(-,Z) : \mathscr{C}^{\mathrm{op}} \to \mathbf{Sets}$$

is defined by for $X \in \mathrm{Ob}(\mathscr{C})$

$$\mathscr{C}(-,Z)(X) = \mathscr{C}(X,Z) = \text{ set of } \mathscr{C}\text{-morphisms from X to Z.}$$

and for $f : X \to Y$ in $\mathrm{Mor}(\mathscr{C})$

$$\mathscr{C}(-,Z)(f) : \mathscr{C}(Y,Z) \to \mathscr{C}(X,Z)$$

is defined by $\mathscr{C}(-,Z)(f)(g) = g \circ f$.
Example:(The singular complex of a space)
We define a functor $\nabla : \Delta \to \mathbf{Top}$ as follows:

$$\nabla([m]) = \nabla^m = \text{topological } m\text{−simplex} = \{(x_0, x_1, ..., x_m) \in \mathbb{R}^{m+1} : \forall x_i \leq 0, x_0 + x_1 + ... + x_m = 1\}$$

In general, ∇^m is the convex hull of the $m+1$ vectors in the standard basis of $\mathbb{R}^{m+1}$.
Let $\alpha : [m] \to [n]$ be a morphism in Δ. Then $\nabla(\alpha) = \alpha_* : \nabla^m \to \nabla^n$ is defined by $\alpha_*(x_1, ..., x_m) = (y_0, ..., y_n)$ where $y_i = \sum_{\alpha(j)=i} x_j$, and α_* is affine linear.
Note: $\alpha_*(e_i) = e_{\alpha(i)}$, where $e_i = (0, ..., 0, 1, 0, ..., 0)$ with the 1 in the $(i+1)$th slot.

Remark. Functors can be composed. Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{E}$ be functors. Then composite functor $G \circ F : \mathscr{C} \to \mathscr{E}$ is given an objects and morphism by composing F and G.

Definition 22.4. The singular complex $S(X)$ of a space X is the simplicial set given by the contravariant functor More explicitly: $S(X)_n = S(X)([m]) = \mathbf{Top}(\triangledown^m, X) =$ set of continuous maps $f : \triangledown^m \to X$. For a morphism $\alpha : [m] \to [n]$ in Δ the map $\alpha^* = S(X)(\alpha) : \mathbf{Top}(\triangledown^n, X) \to \mathbf{Top}(\triangledown^m, X)$ is given by $\alpha^*(g : \triangledown^n \to X) = g \circ \alpha_* : \triangledown^m \to X$.

$$
\begin{tikzcd}
\Delta \arrow[r, "\nabla"] \arrow[rr, bend right, "S(X)"'] & \mathbf{Top} \arrow[r, "\mathbf{Top}(-,X)"] & \mathbf{Sets}
\end{tikzcd}
$$

23 Lecture 23: Face and Degeneracy Maps

In this lecture we will show that a simplicial set $X : \Delta^{\text{op}} \to \mathbf{Sets}$ is determined by

- The sets $X_n = X([n])$ for all $n \geq 0$,
- certain face maps $d_i^* : X_n \to X_{n-1}$,
- certain degeneracy maps $s_i^* : X_{n-1} \to X_n$.

Reminder: Δ = Category with objects $[n] = \{0, ..., n\}$, $n \geq 0$ and morphisms all weakly monotone maps.

Definition 23.1. For $0 \leq i \leq n$, the morphism $d_i : [n-1] \to [n]$ is defined by

$$d_i(j) = \begin{cases} j & \text{for } 0 \leq j \leq i-1 \\ j+1 & \text{for } i \leq j \leq n-1 \end{cases}$$

Remark.
- d_i is injective
- the number i is not in the image of d_i
- these two properties uniquely determine d_i.

Definition 23.2. For $0 \leq i \leq n-1$, the morphism $s_i : [n] \to [n-1]$ is defined by

$$s_i(j) = \begin{cases} j & \text{for } 0 \leq j \leq i \\ j+1 & \text{for } i+1 \leq j \leq n \end{cases}$$

Remark.
- s_i is surjective
- the number $i \in [n-1]$ has two preimages
- these to properties uniquely determine s_i.

Remark. We introduced a functor $\nabla : \Delta \to \mathbf{Top}$ with $\nabla^n = \nabla([n])$= top n-simplex= $\{(x_0, ..., x_n) \in \mathbb{R}^{n+1} : x_i \geq 0, \sum x_i = 1\}$. Then

$$(d_i)_*(x_0, ..., x_{n-1}) = (x_0, ..., x_{i-1}, 0, x_{i+1}, ..., x_n), (d_i)_* : \nabla^{n-1} \to \nabla^n$$

$\implies$ $(d_i)_*$ identifies ∇^{n-1} with the face of ∇^n that is opposite e_i.

$$\delta(\nabla^n) = \bigcup_{i=0,...,n} \text{im}((d_i)_*)$$

Also $(s_i)_* : \nabla^n \to \nabla^{n-1}$ is given by $(s_i)_*(x_0, ..., x_n) = (x_0, ..., +x_{i+1}, ..., x_n)$.

Still need to finish this lecture

24 Lecture 24: Reconstructing simplicial sets from faces and degeneracies

Proofs omitted from this lecture, fill in later

Remark. Recall: $d_i[n-1] \to [n]$ is the unique morphism in Δ that is injective and satisfies $i = \mathrm{im}(d_i)$ and $s_j : [n] \to [n-1]$ is the unique morphism in Δ that is surjective and such that $s_j^{-1}(j) = \{j, j+1\}$. For a simplicial set $X : \Delta^{op} \to \mathbf{Sets}$,

$$d_i^* = X(d_i) : X_n \to X_{n-1}$$

is the i-th face map $0 \leq i \leq n$ and

$$s_j^* = X(s_j) : X_{n-1} \to X_n$$

is the j-th degeneracy map for $0 \leq j \leq n$. Then

$$d_j \circ d_i = d_i \circ d_{j-1} \text{ for } i < j$$

$$s_j \circ d_i = \begin{cases} d_i \circ s_{j-1} & \text{for } i < j \\ \mathrm{id} & \text{for } i = j, j+1 \\ d_{i-1} \circ s_j & \text{for } i > j+1 \end{cases}$$

$$s_j \circ s_i = s_i \circ s_{j+1} \text{ for } i \leq j$$

in the category of Δ which implies

$$d_i^* \circ d_j^* = d_{j-1}^* \circ d_i^* \text{ for } i < j$$

$$d_i^* \circ s_j^* = \begin{cases} s_{j-1}^* \circ d_i^* & \text{for } i < j \\ \mathrm{id} & \text{for } i = j, j+1 \\ s_j^* \circ d_{i-1}^* & \text{for } i > j+1 \end{cases}$$

$$s_i^* \circ s_j^* = s_{j+1}^* \circ s_i^* \text{ for } i \leq j$$

between the face and degeneracy map of X. These are fundamental relations.

Proposition 24.1. *Every morphism $\alpha : [m] \to [n]$ in Δ can be uniquely factored as a composite $\alpha = \delta \circ \sigma$ for an injective morphism δ and a surjective morphism σ in Δ.*

Proposition 24.2. *Every injective morphism $\delta : [m] \to [n]$ in Δ can uniquely be factored as $\delta = d_{i_1} \circ ... \circ \delta_{i_p}$ with $p = n - m$, and $n \geq i_1 > i_2 > ... > i_p \geq 0$.*

Proposition 24.3. *Every surjective morphism sigma $: [m] \to [n]$ is Δ can uniquely be factored as $\sigma = s_{i_1} \circ ... \circ s_{i_q}$, with $q = m - n$ and $j_1 < j_2 < ... < j_q$.*

Corollary 24.4. *Every morphism $\alpha : [m] \to [n]$ in Δ in Δ can uniquely be factored as*

$$\alpha = d_{i_1} \circ ... \circ d_{i_p} \circ s_{j_1} \circ ... \circ s_{j_q}$$

subject to the conditions $i_1 > ... > i_p$ and $j_1 < ... < j_q$. This is called the canonical factorisation. Moreover: $p = n + 1 - \#im(\alpha)$, $q = m + 1 - \#im(\alpha)$.

$$\{i_p, i_{p-1}, ..., i_1\} = [n] \setminus im(\alpha)$$

$$\{j_1, ..., j_q\} = \{l \in \{0, ..., m-1\}, \alpha(l) = \alpha(l+1)\}$$

Theorem 24.5. *Suppose we are given*

- *sets X_n for $n \geq 0$,*
- $d_i^* : X_n \to X_{n-1}$, $0 \leq i \leq n$
- $s_j^* : X_{n-1} \to X_n$, $0 \leq j \leq n-1$

such that the fundamental relations hold. Then there is a unique simplicial set $X : \Delta^{op} \to \mathbf{Sets}$ such that $X_n = X([n])$, $d_i^* = X(d_i)$ and $s_j^* = X(s_j)$.

25 Lecture 25: Linearisation of Simplicial Sets

25.1 Construction: Linearisation of Simplicial Sets

$$C(-, A) : \mathbf{sSet} \to \mathbf{ch}$$

With A an abelian group called the coefficient group. Note: I'm using the notation **sSet** to denote the category of simplicial sets and **ch** to denote the category of chain complexes.

Definition 25.1. (The construction of linearisation of sets) Let A be an abelian group and X a set. The $\underline{A\text{-linearisation of } X}$ is the abelian group

$$A[X] = \{f : X \to A : f(x) = 0 \text{ for almost all } x \in X\}.$$

Where we 'almost all" means all except finitely many, if X is finite then all maps $f : X \to A$ are in $A[X]$. We can also give $A[X]$ the structure of an abelian group with addition defined by $(f + g)(x) = f(x) + g(x)$.

Remark. Notation: We may write elemets of $A[X]$ as finite sums of points in X with coefficients in A.

$$f \in A[X] \iff \underbrace{\sum_{x \in X} f(x) \cdot x}_{\text{notation}}$$

where we ca assume this sum is finite on the right as f is 0 for all but finitely many x_i. We may also have an expression like this

$$a_1 \cdot x_1 + ... + a_n \cdot x_n \text{ for } a_i \in A, x_i \in X \text{ and } x_i \neq x_j$$

then this corresponds to the function

$$y \mapsto \begin{cases} a_i & \text{if } y = x_i \\ 0 & \text{if } y \notin \{_1, .., x_n\} \end{cases}$$

25.2 Extending Linearisation to a Functor

$$A[-] : \mathbf{Sets} \to \mathbf{ab}$$

i.e. for a map $\phi : X \to Y$ we define $\phi_* = A[\phi] : A[X] \to A[Y]$ by $\phi_*(f)(y) = \sum_{x \in X, \phi(x) = y} f(x)$ for $y \in Y$.
We still need to check that

- ϕ_* is really a group homomorphism
- for $\psi : Y \to Z$, we have $(\psi \circ \phi)_* = \psi_* \circ \phi_* : A[X] \to A[Z]$ and $(\mathrm{id}_X)_* = \mathrm{id}_A[X]$.

Remark. Suppose that A is a ring. Then $A[X]$ becomes a left A-module by pointwise scalar multiplication.

$$(a \cdot f)(x) = a \cdot f(x) \text{ for } a \in A, f \in A[X], a \in X.$$

In this situation,

$$\phi_* : A[X] \to A[Y]$$

is a homomorphism of A-modules. So A linearisation lifts to a functor

$$A[-] : \mathbf{Sets} \to A - \mathbf{Mod}$$

Given $x \in X$, let $\underline{x} \in A[X]$ be the indicator functor i.e.

$$\underline{x}(y) = \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases}.$$

Then

$$f = \sum_{x \in X} f(x) \cdot \underline{x} \in A[X]$$

so the elements of $\underline{x}$ for $x \in X$ form a basis of $A[X]$ as a left A-module.

Definition 25.2. A chain complex C consists of abelian groups C_n for $n \in \mathbb{Z}$ and group homomorphisms $d_n : C_n \to C_{n-1}$ such that $d_{n-1} \circ d_n = 0 : C_n \to C_{n-2}$ (for $n \in \mathbb{Z}$) i.e.

$$\begin{tikzcd} \dots \arrow[r, "d_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[rr, bend right, "0"'] & C_{n-1} \arrow[r, "d_{n-1}"] & C_{n-2} \arrow[r, "d_{n-2}"] & \dots \end{tikzcd}$$

Remark. Terminology:

- d_n is called the n-th differential boundary map
- often, the dimension n is supressed from the notation, writing only d. Then the relation becomes $d^2 = 0$.

Remark. We can construct the linearisation of a simplicial set using chain complexes. Let A be an abelian group and X a simplicial set. Then the chain complex $C(X, A)$ is defined by $C_n(X, A) = A[X_n]$, the differential $d_n : A[X_n] \to A[X_{n-1}]$ is defined by

$$d_n = \sum_{i=0,\dots,n} (-1)^i \cdot d_i^*.$$

Where $d_i^* : X_n \to X_{n-1}$ is the i-th face map, and also its A-linearisation.
(We are actually using an abuse of notation as d_i^* is a map of sets and in the sum we should really write $A[d_i^*]$ or $(d_i^*)_*$ the group homomorphism induced by this map.)

Lemma 25.1. $d_{n-1} \circ d_n = 0$ *i.e. the above construction is a chain complex.*

Remark. Let Δ_{inj} denote the non-full subcategory of Δ with all objects but only the injective monotone maps as morphism. A semi simplicial set is a functor $\Delta_{\text{inj}}^{\text{op}} \to \mathbf{Sets}$. The linearisations makes sense with the same construction for semisimplicial sets.

Geometric motivation for the differential: Let Y be a space and $So(Y)$ its singular complex. Then let $(f : \nabla^n \to Y) \in S(Y)_n$. Then $f \in \mathbb{Z}[S(Y)_n] = C_n(S(Y), \mathbb{Z})$ and the boundary $d_n(f) = \sum_{i=0,\dots,n} (-1)^i \cdot d_i^*(f) = \sum_{0 \leq i \leq n} (-1)^* \cdot (f \circ (d_i)_*)$.

26 Lecture 26: Singular Homology

For topological spaces we can construct homology from topological spaces to simplicial sets to chain complexes to abelian groups (the n-th homology group).

$$\mathbf{Top} \xrightarrow{S(-)} \mathbf{sSets} \xrightarrow{C(-,A)} \mathbf{Ch} \xrightarrow{H_n} \mathbf{ab}$$

Construction: Let $C = (C_n, d_n : C_n \to C_{n-1})_{n \in Z}$ be a chain complex. Then $d_n \circ d_{n+1} = 0$ equivalently $\text{im}(d_{n+1}) \subset \ker(d_m)$, both are subgroups of C_n.

Definition 26.1. The n-th homology group of a chain complex C is the abelian group

$$H_n(C) = \frac{\ker(d_n : C_n \to C_{n-1})}{\operatorname{im}(d_{n+1} : C_{n+1} \to C_n)}$$

(Where this is the algebraic quotient of groups)

Definition 26.2. The nth homology group of a space Y with coefficients in an abelian subgroup A is

$$H_n(Y, A) = H_n(C(S(Y), A))$$

for $n \in \mathbb{Z}$ where this is 0 for $n < 0$.

Example: (Homology of discrete spaces)
A simplicial set X is constant if for every $n \geq 0$ the map $q_n^* : X_0 \to X_n$ is bijective, where $q_n : [n] \to [0]$ is the unique morphism in Δ.

Lemma 26.1. *Let X be a constant simplicial set. Then:*

1. *For every morphism $\alpha : [m] \to [n]$ in Δ, the map $\alpha^* : X_n \to X_m$ is bijective.*
2. *For all morphisms $\alpha, \beta : [m] \to [n]$ in Δ, $\alpha^* = \beta^*$.*

proof ommitted

Proposition 26.2. *For every discrete space Y, $H_0(Y, A)$ is isomorphic to $A[Y]$, and $H_n(Y, A)$ is trivial for $n \neq 0$.*

Example: (Homology in dimension 0).
First for $A = \mathbb{Z}$. We let Y be any space. We define two homomorphisms

$$H_0(Y, \mathbb{Z}) \underset{\psi}{\overset{\phi}{\leftrightarrows}} \mathbb{Z}[\pi_0(Y)] = \text{ free abelian group generated by } \pi_0(Y)$$

construction of ψ and ϕ ommitted

Proposition 26.3. *For every space Y, the map $H_0(Y, A) \overset{\cong}{\to} A[\pi_0(Y)]; [\sum a_i \cdot y_i] \mapsto \sum a_i[y_i]$ is an isomorphism of groups.*

27 Lecture 27: Homology as a Functor

Recall the construction of homology from previous lecture This lecture we will consider (f :

$$\mathbf{Top} \xrightarrow{S(-)} \mathbf{sSets} \xrightarrow{C(-,A)} \mathbf{Ch} \xrightarrow{H_n} \mathbf{ab}$$

$X \to Y$) a continuous map then

$$H_n(f, A) : H_n(X, A \to H_n(Y, A)$$

a homomorphism of homology groups.

$$
\begin{array}{ccc}
F(X) & \xrightarrow{\zeta_X} & G(X) \\
\downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\
F(Y) & \xrightarrow{\zeta_Y} & G(Y)
\end{array}
$$

27.1 Natural Transformations

Simplicial sets are functors $X : \Delta^{\mathrm{op}} \to \mathbf{Sets}$; in general, functors have a notion of morphisms called natural transformations.

Definition 27.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F, G : \mathscr{C} \to \mathscr{D}$ functors. A natural transformations $\zeta : F \to G$ consists of a $\mathscr{D}$-morphism

$$\zeta_X : F(X) \to G(X)$$

for all objects X in $\mathscr{C}$, such that for all $\mathscr{C}$-morphisms $f : X \to Y$ the following commutes in $\mathscr{D}$:

Example: The dual vector space construction is a functor $()^* : \mathbf{Vect}_k^{\mathrm{op}} \to \mathbf{Vect}_k, V \mapsto V^* = \mathrm{Hom}_k(V, k)$. The double dual is the composite

$$()^* \circ ()^* = ()^{**} : \mathbf{Vect}_k \to \mathbf{Vect}_k, V \mapsto V^{**}$$

The evaluation map

$$\zeta_V : V \to V^{**}$$

such that $\zeta_V(v)(\psi : V \to k) = \psi(v)$ is a linear map, and as V varies these maps form a natural transformation $\zeta : \mathrm{id} \to ()^{**}$. So for every k-linear map $\phi : V \to W$, the following commutes

$$
\begin{array}{ccc}
V & \xrightarrow{\zeta_V} & V^{**} \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi^{**}} \\
W & \xrightarrow{\zeta_W} & W^{**}
\end{array}
$$

Example: Let X be a space. Write $\delta(X) = X \times X$, $\delta(f : X \to Y) = f \times f : X \times X \to Y \times Y$, this defines a functor

$$\delta : \mathbf{Top} \to \mathbf{Top}.$$

The maps $\zeta_X : X \to X \times X, \zeta_X(x) = (x, x)$ form a natural transformation $\mathrm{id} \to \delta$.

Remark. Construction: Let $\mathscr{C}$ and $\mathscr{D}$ be categories. The category of functors $\mathbf{Fun}(\mathscr{C}, \mathscr{D})$ has all fuctors $F : \mathscr{C} \to \mathscr{D}$ as objects and all natural transformations as morphisms. The composition of a natural transformation $\zeta : F \to G$ and $\nu : G \to H$ is defined objectwise i.e.

$$(\nu \circ \zeta)_X = \nu_X \circ \zeta_X : F(X) \to H(X)$$

Moreover $\mathrm{id}_F = \{\mathrm{id}_F(X)\}$.

27.2 The Category of Simplicial Sets

Definition 27.2. Let $X, Y : \Delta^{\mathrm{op}} \to \mathbf{Sets}$ be simplicial sets. A morphism $f : X \to Y$ is a natural transformation of functors. The category of simplicial sets is the functor category $\mathbf{Fun}(\Delta^{\mathrm{op}}, \mathbf{Sets})$.

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \alpha^*} \\ X_m & \xrightarrow{f_m} & Y_m \end{array}$$

Remark.

- **More explicitly:** A morphism $f : X \to Y$ of simplicial sets consists of maps $f_n : X_n \to Y_n$ for all $n \geq 0$ such that the morphisms $\alpha : [m] \to [n]$ in Δ, the following square commutes:
- For $f = \{f_n : X_n \to Y_n\}$ to be a morphism of simplicial sets it suffices that the following diagram commutes for all $0 \leq i \leq n$:

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} \\ {\scriptstyle s_i^*}\uparrow & & {\scriptstyle s_i^*}\uparrow \\ X_n & \xrightarrow{f_n} & Y_n \\ \downarrow{\scriptstyle d_i^*} & & \downarrow{\scriptstyle d_i^*} \\ X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} \end{array}$$

because every morphism can be factored as we have seen previously.

27.3 Functoriality of a Singular Complex

Let $f : X \to Y$ be a continuous map between topological spaces. We define a morphism of simplicial sets $f_* = S(f) : S(X) \to S(Y)$ by $S(f)_n : X_n \to Y_n$ such that

$$S(f)_n(\psi : \nabla^n \to X) = f \circ \psi : \nabla^n \to Y$$

These maps form a morphism of simplicial sets: let $\alpha : [m] \to [n]$ be a morphism in Δ

$$(\alpha^* \circ S(f)_n)(\psi : \nabla^n \to X) = \alpha^*(S(f)_n(\psi)) = \alpha^*(f \circ \psi) = (f \circ \psi) \circ \alpha_* = f \circ (\psi \circ \alpha_*) = (S(f)_n \circ \alpha_*)(\psi)$$

for all ϕ in $S(X)_n$, hence we can conclude that $\alpha^* \circ S(f)_n = S(f)_m \alpha^*$.

Proposition 27.1. *The assignments $X \mapsto S(X)$ and $f \mapsto S(f)$ defines a functor $S :$ **Top** $\to$ **sSet**.*

27.4 The Category of Chain Complexes

Let $C = \{C_n, d_n\}_{n \in \mathbb{Z}}$ and $D = \{C_n, d_n\}_{n \in \mathbb{Z}}$ be chain complexes. A chain map $f : C \to D$ consist of group homomorphisms $f_n : C_n \to D_n$ such that for every $n \in \mathbb{Z}$ the relations $d_n^D \circ f_n = f_{n-1} \circ d_n^C$ holds. I.e. the following commutes

$$\begin{array}{ccc} C_n & \xrightarrow{f_n} & D_n \\ \downarrow{\scriptstyle d_n^C} & & \downarrow{\scriptstyle d_n^D} \\ C_{n-1} & \xrightarrow{f_{n-1}} & D_{n-1} \end{array}$$

Composition of chain maps is dimension wise: $f : C \to D, g : D \to E$ compose by

$$(g \circ f)_n = g_n \circ f_n : C_n \to E_n$$

with $(\mathrm{id}_C)_n = \mathrm{id}_{C_n}$. The chain maps are the morphisms in the category of chain complexes.

27.5 Linearisation as a Functor

Let $f : X \to Y$ be a morphism of simplicial sets. We define a chain map

$$f_* = C(f, A) : C(X, A) \to C(Y, A).$$

We set

$$C_n(f, A) = \begin{cases} A[f_n] : A[X_n] \to A[Y_n] & n \geq 0 \\ 0 & n < 0 \end{cases}$$

This is indeed a chain map for $n \geq 1$:

$$\begin{aligned} d_n \circ C_n(f, A) &= (\sum_{i=0,\ldots,n} (-1)^i \cdot A[d_i^*]) \circ A[f_n] \\ &= \sum_{i=0,..,n} (-1)^i A[d_i^*] \circ A[f_n] \\ &= \sum_{i=0,\ldots,n} (-1)^i \cdot A[d_i^* \circ f_n] && \text{by functoriality of } A[-] \\ &= \sum_{i=0,\ldots,n} (-1)^i \cdot A[f_{n-1} \circ d_i^*] && \text{as } f \text{ is a morphism in } \mathbf{sSets} \\ &= \sum_{i=0,\ldots,n} (-a)^i \cdot A[f_{n-1}] \circ A[d_i^*] && \text{by functiality of } A[-] \\ &= A[f_{n-1}] \circ (\sum_{i=1,\ldots,n} (-1)^i \cdot A[d_i^*]) \\ &= C_{n-1}(f, A) \circ d_n \end{aligned}$$

Proposition 27.2. *The assignments $X \mapsto C(X, A)$ and $f \mapsto C(f, A)$ defines a functor*

$$C(-, A) : \mathbf{sSets} \to \mathbf{Ch}.$$

Proof. Remains to check

$$C(g, A) \circ C(f, A) = C(g \circ f, A) : C(X, A) \to C(Z, A)$$

for all composable morsphisms of simplicial sets and $C(\mathrm{id}_X, A) = \mathrm{id}_{C(X,A)}$. □

27.6 Homology as a Functor

$$H_n : \mathbf{Ch} \to \mathbf{Ab}$$

for $n \in \mathbb{Z}$.
Let $f : C \to D$ be a chain map.
Note: for $x \in C_n$ with $d_n^C(x) = 0$ we have $d_n^D(f_n(x)) = f_n - 1(d_n^C(x)) = f_{n-1}(0) = 0$ i.e. $f_n(\ker(d_n^C)) \subset \ker(d_n^D)$. Also $f_n(\mathrm{im}(d_{n+1}^C)) = \mathrm{im}(f_n \circ d_{n+1}^C) = \mathrm{im}(d_{n+1}^D \circ f_{n+1}) \subset \mathrm{im}(d_{n+1}^D)$ and so f_n induces a group homomorphism on the factor groups

$$f_* = H_n(f) : H_n(C) = \frac{\ker(d_n^C : C_n \to C_{n-1})}{\mathrm{im}(d_{n+1}^C : C_{n+1} \to C_n)} \to H_n(D) = \frac{\ker(d_n^D : D_n \to D_{n-1})}{\mathrm{im}(d_{n+1}^D : D_{n+1} \to D_n)}$$
$$[x] = x + \mathrm{im}(d_{n+1}^C) \mapsto [f_n(x)] = f_n(x) + \mathrm{im}(d_{n+1}^D)$$

Proposition 27.3. *The assignments $C \to H_n(C)$, $f \mapsto H_n(f)$ define a functor*

$$H_n : \mathbf{Ch} \to \mathbf{Ab}$$

Construction: The functor

$$H_n(-,A) : \mathbf{Top} \to \mathbf{Ab}$$

is defined as the composite

$$H_n(-,A) = H_n \circ C(-,A) \circ S.$$

Corollary 27.4. *For every homeomorphism $f : X \to Y$ the map $H_n(f,A) : H(X,A) \to H_n(Y,A)$ is an isomorphism of abelian groups.*

Proof. Functors take isomorphisms to isomorpshims and we have homeomorphism of topological spaces as isomrphism in the category **Top** and group isomorphisms as isomorphisms in the category of **Ab**. □

28 Lecture 28: Homotopy for Simplicial Sets

Let $f : X \to Y$ be a continuous map of topological spaces. Then $H_n(f,A) : H_n(X,A) \to H_n(Y,A)$ is a group homomorphism.

Theorem 28.1. *Let $f, g : X \to Y$ be homotopic continuous maps. Then $H_n(f,A) = H_n(g,A)$.*

$$\begin{matrix}\text{homotopic} & & \text{homotopic morphisms} & & \text{chain homotopic} & & \\ \text{continuous maps} & \xrightarrow{S} & \text{of simplicial set} & \xrightarrow{C(-,A)} & \text{chain maps} & \xrightarrow{H_n} & \text{same maps}\end{matrix}$$

Definition 28.1. Let X and Y be simplicial sets. The <u>product</u> $X \times Y$ is the simplicial set defined by $X \times Y)_n = X_n \times Y_n$, $\alpha^*_{X\times Y} = \alpha^*_X \times \alpha^*_Y$.
Equivalently: $X \times Y$ is the following composite functor

$$\begin{aligned} \Delta^{\mathrm{op}} \overset{(X,Y)}{\longrightarrow} \mathbf{Sets} \times \mathbf{Sets} &\overset{\times}{\longrightarrow} \mathbf{Sets} \\ [n] \mapsto (X_n, Y_n) \\ \alpha \mapsto (\alpha^*_X, \alpha^*_Y) \end{aligned}$$

this product comes with the two morphisms

$$p_X : X \times Y \to X, p_Y : X \times Y \to Y$$

which are the dimensionwise projection to the two factors.

Proposition 28.2. *(Universal property of product) Let A, X and Y simplicial sets. For every pair of morphism $f : A \to X$, $g : A \to Y$ there is a unique norphism $(f,g) : A \to X \times Y$ such that $p_X \circ (f,g) = f$ and $p_Y \circ (f,g) = g$.*

Remark. Recall: The separated simplicial sets $\Delta^n = \Delta(-,[n])$ with

$$(\Delta^n)_k = \Delta)[k],[n])$$

and

$$\alpha^* = \Delta(\alpha,[n]).$$

Later we will see "geometric realisation"

$$|-| : \mathbf{sSets} \to \mathbf{Top}$$

where $|\Delta^n| \cong \nabla^n$ and $\nabla^1 \cong [0,1]$, so Δ^1 is the simplicial analog of the interval $[0,1]$.
Hence we define two morphisms of a simplicial set X $i_0, i_1 : X \to X \times \Delta^1$ by

$$\begin{aligned} (i_0)_n, (i_1)_n : &X_n \to X_n \times \Delta([n],[1]) \\ (i_0)_n(x) &= (x, \text{constant map } [n] \to [1] \text{ with value } 0) \\ (i_1)_n(x) &= (x, \text{constant map } [n] \to [1] \text{ with value } 1) \end{aligned}$$

we can check these are indeed morphisms of simplicial sets i.e. $i_1, i_1 : \mathrm{id} \to (-) \times \Delta^1$ are natural transformations between endo functors on the category of simplicial sets.

Definition 28.2. Let $f, g : X \to Y$ be morphisms of simplicial sets. A homotopy from f to g is a morphism of simplicial sets

$$H : X \times \Delta^1 \to Y$$

such that $H \circ i_0 = f$ and $H \circ i_1 = g$.

Remark. WARNING: Homotopy for simplicial sets is not an equivalence relation unlike what we are ued to for topological homotopy. It is in general neither symmetric nor transitive.

Proposition 28.3. *Let $f : X \to Y$ be a morphism of simplicial sets. Then the following are equivalent*

1. *f is an isomorphism.*
2. *For all $n \geq 0$, $f_n : X_n \to Y_n$ is bijective.*

Proof. **proof omitted but come back to later** □

Remark. Using a variant of this proof we can show a more general statement for functors.

Proposition 28.4. *Let $\mathscr{C}, \mathscr{D}$ be categories, $F, G : \mathscr{C} \to \mathscr{D}$ be functors and $\zeta : F \to G$ a natural transformation. Then the following are equivalent:*

1. *ζ is an isomorphism in the category $\mathbf{Fun}(\mathscr{C}, \mathscr{D})$.*
2. *For all $\mathscr{C}$-objects X, the morphism $\zeta_X : F(X) \to G(X)$ is an isomorphism in $\mathscr{D}$.*

Proposition 28.5. *(Singular complex preserves products) Let X and Y be topological spaces, and let $p_x : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the projections. Then the morphism*

$$(S(p_X), S(p_Y)) : S(X \times Y) \to S(X) \times S(Y)$$

is an isomorphism.

Proof. **proof omitted cant be bothered, looks short come back later** □

Theorem 28.6. *Let $f, g : X \to Y$ be homotopic continuous maps. Then the morphisms if simplicial sets $S(f), S(g) : S(X) \to S(Y)$ are homotopic.*

Proof. Because f is homotopic to g there is a continuous map

$$H : X \times \nabla^1 \to Y$$

such that the **finih proof later** □

29 Lecture 29: Homotopy Invariance of Singular Homology

29.1 Defining Chain Homotopy

Definition 29.1. Let $C = \{C_n, d_n : C_n \to C_{n-1}\}_{n \in \mathbb{Z}}$ and $D = \{D_n, d_n : D_n \to D_{n-1}\}_{n \in \mathbb{Z}}$ be chain complexes and let $f, g : C \to D$ be chain maps. A chain homotopy consists of group homomorphisms $\sigma = \{\sigma_n : \mathbb{C}_n \to D_{n+1}\}_{n \in Z}$ such that

$$d^D_{n+1} \circ \sigma_n + \sigma_{n-1} \circ d^C_n = g_n - f_n.$$

(Notation: $d\sigma + \sigma d = g - f$)

$$\begin{tikzcd}
C_{n+1} \arrow[r, "f_{n+1},g_{n+1}"] \arrow[d, "d^C_{n+1}"] & D_{n+1} \arrow[d, "d^D_{n+1}"] \\
C_n \arrow[r, "f_n,g_n"] \arrow[ur, "\sigma_n"] \arrow[d, "d^C_n"] & D_n \arrow[d, "d^D_n"] \\
C_{n-1} \arrow[r, "f_{n-1},g_{n-1}"] \arrow[ur, "\sigma_{n-1}"] & D_{n-1}
\end{tikzcd}$$

Lemma 29.1. *Let $f, g : C \to D$ be chain homotopic maps. Then*

$$H_n(f) = H_n(g) : H_n(C) \to H_n(D)$$

for all $n \in \mathbb{Z}$.

Proof. We let $x \in C_n$ be an n-cycle, i.e. $d_n(x) = 0$. Then

$$\begin{aligned} H_n(g)[x] = [g_n(x)] &= [f_n(x) + d_{n+1}(\phi_n(x)) + \phi_{n-1}(d_n(x))] \\ &= [f_n(x) + d_{n+1}(\sigma_n(x))] && \text{by } d_{n+1}(\sigma_n(x)) \in \mathrm{im}(d_{n+1}) \\ &= [f_n(x)] = H_n(f)[x] \end{aligned}$$

□

Proposition 29.2. *Let $f, g : X \to Y$ be homotopic morphisms of simplicial sets. Then the chain maps $C(f, A), C(g, A) : C(X, A) \to C(Y, A)$ are chain homotopic.*

Proof. **proof omitted he said it was long maybe we ccome back** □

29.2 Homotopy Invariance

Theorem 29.3. *Let $f, g : X \to Y$ be homotopic continuous maps of spaces. Then*

$$H_n(f, A) = H_n(g, A) : H_n(X, A) \to H_n(Y, A).$$

Corollary 29.4. *Let $f : X \to Y$ be a homotopy equivalence of spaces. Then $H_n(f, A) : H_n(X, A) \to H_n(Y, A)$ is an is an isomorphism.*

Proof. Let $g : Y \to X$ be a homotopy inverse of spaces i.e. such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. Then

$$H_n(g, A) \circ H_n(f, A) = H_n(g \circ f, A) = H_n(\mathrm{id}_X, A) = \mathrm{id}_{H_n(X,A)}$$

similarly

$$H_n(f, A) \circ H_n(g, X) = \mathrm{id}_{H_n(Y,A)}.$$

So $H_n(g, A)$ is inverse to $H_n(f, A)$. □

Example:
Let X be a contractible topological space. Then X is homotopy equivalent to $*$, so

$$H_n(X, A) \cong H_n(*, A) = \begin{cases} A & n = 0 \\ 0 & n > 0 \end{cases}.$$

29.3 Motivation for the Formula of the Chain Homotopy

- By naturality of the chain homotopy, and because simplicial sets are "built from Δ^ns" it suffices to understand the prosm operator/ chain homotopy for $X = \Delta^n$ for $n \geq 0$.

he said this wasnt necessary for understanding of lectures so again i may come back to this

30 Lecture 30: The Long Exact Sequence of Homology Groups

Definition 30.1. Let $C = \{C_n, d_n\}_{n\in\mathbb{Z}}$ be a chain complex. A subcomplex C' of C consists of subgroups C'_n of C_n for all $n \in \mathbb{Z}$ such that $d_n(C'_n) \subset C'_{n-1}$. I.e.

$$\begin{tikzcd} C'_n \arrow[r, hook] \arrow[d, "d'_n"] & C_n \arrow[d, "d_n"] \\ C'_{n-1} \arrow[r, hook] & C_{n-1} \end{tikzcd}$$

Remark.
- The groups $\{C'_n\}$ form a chain complex with respect to $d'_n = d_n|_{C'_n} : C'_n \to C'_{n-1}$.
- The inclusions $C'_n \hookrightarrow C_n$ for $n \in \mathbb{Z}$ forma a chain map $\iota : C' \hookrightarrow C$.
- **Warning**: The induced homomorphism of homology groups $\iota_* = H_n(i) : H_n(C') \to H_n(C)$ need not be injective!

Definition 30.2. Let C' be a subcomplex of a chain complex C. The quotient complex C/C' is defined by $(C/C')_n = C_n/C'_n$ with the induced differential

$$\overline{d_n} : C_n/C'_n \to C_{n-1}/C'_{n-1}$$

given by $\overline{d_n}(x + C'_n) = d_n(x) + C'_{n-1}$.

We can check this is well defined and satisfies $\overline{d_{n-1}} \circ \overline{d_n} = 0$, and the projections $C - n \to C_n/C'_n$ for $n \in \mathbb{Z}$ forms a chain map $p : C \to C/C'$.

30.1 Homology of a Quotient Complex

Remark. The homology $H_n(C/C')$ involves 2 kinds of residues:

- $(C/C')_n$ id modulo C'_n. Here we use notation $x + C'_n$.
- Homology is taken modulo the boundary map i.e. $\overline{d_n} : C_{n+1}/C'_{n+1} \to C_n/C'_n$. Here we will use the notation [..].
- Combine this notation to get $[x + C_n]$.
- The notation $[x+C_n]$ makes sense whenever $x+C_n$ is an n-cycle in C/C' i.e. $d_n(x)+C'_{n-1} = 0$ in $C_{n-1}/C'_{n=1}$ i/e/ $d_n(x) \in C'_{n-1}$.

Definition 30.3. Let X be a space and Y a subsapce of X. Then $S(Y)$ is a simplicial subset of $S(X)$. Hence $C_n(X, A) = A[S(X)_n]$ contains $C_n(Y, A) = A[S(Y)_n]$ as a subgroup, invariant under the differential. In other words $C(Y, A)$ is a subcomplex of $C(X, A)$. So we can form the quotient complex

$$C(S(X), A)/C(S(Y), A)$$

The relative homology groups are defined as

$$H_n(X, Y, A) = H_n(\frac{C(S(X), A)}{C(S(Y), A)}).$$

Remark. **Warning:** $H_n(X, Y, A)$ is typically different from $H_n(X/Y, A)$ and $\frac{H_n(X,A)}{\mathrm{im} H_n(Y,A)}$.

30.2 Construction of the Long Exact Sequence

Let C' be a subcomplex of a chain complex C. We define a connecting homomorphism

$$\delta : H_n(C/C') \to H_{n-1}(C')$$

by $\delta[x + C'_n] = [d_n(x)]$. Here $x \in C_n$ is such that $x + C'_n$ is an n-cycle in C/C'. This does not necesarily mean that $d_n(x) = 0$ in C'_{n-1}. This only means that $d_n(x)$ is in C'_{n-1}. Also $d'_{n-1}(d_n(x)) = 0$, so $d_n(x)$ is really an n-cycle in C. Hence $[d_n(x)] \in H_{n-1}(C')$.

Lemma 30.1. *The connecting homomorphism is well defined and a group homomorphism.*

Proof. **Well defined:** We let $x, y \in C_n$ be such that $x + C'_n \in C_n/C'_n$ are n-cycles in C/C'. And moreover that $[x + C'_n] = [y + C_n;]$ in $H_n(C/C')$. This means that there is $z \in C_{n+1}$ such that

$$y + C'_n = (x + C'_n) + (d_{n+1}(z) + C'_n) \in C_n/C'_n$$

or equivalently

$$x + d_{n+1}(z) - y \in C'_n.$$

Hence $d_n(x) - d_n(y) = d_n(x + d_{n+1}(z) - y) \in C'_n$ and so $\delta[x + C'_n] = [d_n(x)] = [d_n(y)] = \delta[y + C'_n]$ in $H_{n-1}(C')$. □

Theorem 30.2. *Let C' be a subcomplex of a chain complex C. Then the following sequence of abelian groups is exact:*

$$... \to H_n(C') \xrightarrow{\iota_*} H_n(C) \xrightarrow{p_*} H_n(C/C') \xrightarrow{\delta} H_{n-1}(C') \to ...$$

Proof. **Exactness at** $H_n(C') \xrightarrow{\iota_*} H_n(C) \xrightarrow{p_*} H_n(C/C')$:
$\mathrm{im}(\iota_*) \subset \ker(p_*)$:
Let $y \in C'_n$ be an n-cycle of C' i.e. $d'_n(y) = 0$. Then $p_*(\iota_*[y]) = \underbrace{[y + C'_n]}_{=0 \in C_n/C'_n} = 0 \in H_n(C/C')$.
$\ker(p_*) \subset \mathrm{im}(\iota_*)$:
Let $x \in C'_n$ be an n-cycle in C i.e. $d_n(x) = 0$ and such that $0 = p_*[x] = [x + C'_n]$. Hence $x + C'_n \in \mathrm{im}(\overline{d_{n+1}})$, i.e. there is a $z \in C_{n+1}$ such that $x + C'_n = d_{n+1}(z) + C'_n$, i.e. $y = x - d_{n+1}(z) \in C'_n$. Moreover $d_n(y) = d_n(x - d_{n+1}(z)) = d_n(x) - d(d_{n+1}(z)) = 0$. So y is an n-cycle of the subcomplex C', and so $[y] \in H_n(C')$. Finally $\iota_*[y] = [y] = [x] \in H_n(C)$. We have shown that $[x] \in \mathrm{im}(\iota_*)$.

Exactness at $H_n(C) \xrightarrow{p_*} H_n(C/C') \xrightarrow{\delta} H_{n-1}(C')$
$\mathrm{im}(p_*) \subset \ker(\delta)$:
Let $x \in C_n$ be an n-cycle of C. Then $d_n(x) = 0$ and so $\delta(p_*[x]) = \delta[x + C'_n] = [d_n(x)] = 0$.
$\ker(\delta) \subset \mathrm{im}(p_*)$:
Let $x \in C_n$ be such that $x + C'_n$ is an n-cycle in C/C' i.e. $d_n \in C'_{n-1}$, and such that $\delta[x + C'_n] = [d_n(x)] = 0 \in H_{n-1}(C')$. Then $d_n(x) = d_n(z)$ for some $z \in C'_n$. Hence $d_n(x - z) = 0$, so $x - z \in C_n$ is an n-cycle of C, and moreover $p_*[x - z] = [(x - z) + C'_n] = [x + C'_n]$. So $[x + C'_n] \in \mathrm{im}(p_*)$.

Exactness at $H_n(C, C') \xrightarrow{\delta} H_{n-1}(C') \xrightarrow{\iota_*} H_{n-1}(C)$
$\mathrm{im}(\delta) \subset \ker(\iota_*)$:
Let $x \in C_n$ be such that $d_n(x) \in C'_n$. Then $\iota_*(\delta[x + C'_n]) = \iota[d_n(x)] = \underbrace{[d_n(x)]}_{\in \mathrm{im}(d_n)} = 0 \in H_n(C)$.
$\ker(\iota_*) \subset \mathrm{im}(\delta)$:
We let $y \in C'_{n-1}$ be an $(n-1)$-cycle of C' such that $0 = \iota_*[y] = [y] \in H_{n-1}(C)$ so there is an $x \in C_n$ such that $y = d_n(x)$. So $\delta[x + C'_n] = [d_n(x)] = [y] \in H_{n-1}(C')$ i,e, $[y] \in \mathrm{im}(\delta)$.

□

30.3 Application to Relative Homology Groups

Let X be a space and Y a subspace of X. We apply the theorem to $C = C(S(X), A)$ and the subcomplex $C' = C(S(Y), A)$. The result is a long exact sequence:

$$... \to H_n(C(S(Y), A)) \to H_n(C(S(X), A) \to H_n(\frac{C(S(X), A)}{C(S(Y), A)}) \to H_{n-1}(C(S(Y), A)) \to ...$$

31 Lecture 31: Barycentric Subdivision

31.1 Motivation for barycentric subdivision

Barycentric subdivision is a key geometric ingredient for excision: we use barycentric subdivision to replace singular simplicies by smaller ones in the same homology class. The barycentre of is the centre of mass of a geometric object: i.e. for a simplex ∇^n the barycentre is $\beta_n = \frac{1}{n+1}(1, ..., 1)$.

31.2 Linear extension to

i give up with this lecture

32 Lecture 32: Excision

32.1 Construction

Let X be a space and $O = \{O_i\}$ a set of subsets of X. A singular n-simplex $\psi : \nabla^n \to X$ is O-small if $\psi(\nabla^n) \subset O_i$ for some $i \in I$.The O-small simplicies form a simplicial subset $S_O(X)$ of the singular complex.

Definition 32.1. A set $O = \{O_i\}_{i \in I}$ of subsets of a space X is an admissible cover if $X = \bigcup_{i \in I} \mathring{O}_i$.

Theorem 32.1. *Let $O = \{O_i\}_{i \in I}$ be an admissible cover of a spcace X. Then the inclusion $S_O(X) \to S(X)$ induces isomorphisms*

$$H_n(C(S_O(X), A)) \xrightarrow{\cong} H_n(C(S(X), A)) = H_n(X, A)$$

for all $n \geq 0$, all abelian groups A.

Proof. Recall homological barycentric subdivision: there is a chain map $U : C(X) = C(S(X), A) \to C(X)$ and a chain homotopy D from $\mathrm{id}_{C(X)}$ to U with the following properties:

1. For every $\psi : \nabla^n \to X$, $U_n(\phi)$ is a linear combination of singular n-simplicies $\phi_i : \nabla^n \to X$ with $\mathrm{im}(\phi_i) \subset \mathrm{im}(\psi)$.
2. For every $\psi : \nabla^n \to X$, $D_n(\psi)$ is a linear combination of singular $(n+1)$-simplicies $k_j : \nabla^{n+1} \to X$ with $\mathrm{im}(k_j) \subset \mathrm{im}(\psi)$.
3. $U_n^k(\mathrm{id}_{\nabla^n})$ is an alternating sum of affine linear n-simplicies in

need to finish this proof after watching 31 properly ☐

Lemma 32.2. *Let C' be a subcomplex of a chain complex C such that $H_n(C/C') = 0$ for all $n \in \mathbb{Z}$. Then the inclusion $C' \hookrightarrow C$ induces isomorphrsims $H_n(C') \xrightarrow{\cong} H_n(C)$ for all $n \in \mathbb{Z}$.*

Proof. We consider the long exact sequence

$$\dots \to \underbrace{H_{n+1}(C/C')}_{=0} \xrightarrow{\delta} H_n(C') \xrightarrow{\iota_*} H_n(C) \xrightarrow{p_*} \underbrace{H_n(C/C')}_{=0} \xrightarrow{\delta} \dots$$

So the ι_* is both injective and surjective. □

Relative homology is a functor of pairs of spaxes. Let (X, Y) and (X', Y') be pairs of spaces. Let $f : X \to X'$ be a continuous map with $f(Y) \subset Y'$; we call such an f a map of space pairs. Then the morphism of simplicial sets $S(f) : S(X) \to S(X')$ sets $S(Y)$ into $S(Y')$. So the chain map $C(S(f), A) : C(S(X), A) \to C(S(X'), A)$ sends the subcomplex $C(S(Y), A)$ into $C(S(Y'), A)$. So it descends to a chain map between the quotient complexes

$$\overline{f} : C(S(X), A)/C(S(Y), A) \to C(S(X'), A)/C(S(Y'), A)$$

So $\overline{f}$ induces a homomorphism of homology groups

$$H_n(\overline{f}) : H_n(X, Y, A) \to H_n(X', Y', A).$$

So altogether this makes relative homology into a functor

$$H_n(-, -, A) : \mathbf{Top}^{\text{pairs}} \to \mathbf{Ab}.$$

Theorem 32.3. *(Excision) Let (X, Y, U) be an excisive triple of spaces, i.e. $\overline{U} \subset \mathring{Y}$. Then for every $n \geq 0$ and all abelian groups A, the inclusion $X \setminus U \to X$ induces an isomorphism*

$$H_n(X \setminus U, Y \setminus U, A) \to H_n(X, Y, A).$$

Proof. We drop the coefficient group A from the notation for simplicity. Because $\underline{U} \subset \mathring{Y}$ means that $O = \{X \setminus U, Y\}$ is n admissible cover of X. By the theorem of small simplicies, the inclusion $S_O(X) \to S(X)$ induces isomorphisms of homology groups. Because $S(Y)$ is a simplicial subset of $S_O(X)$, the complex $C(S(Y))$ is a subcomplex of $C(S_O(X))$. We get a commutative diagram of abelian groups with exact rows.

$$\begin{tikzcd}
\dots \arrow[r] & H_n(C(S(Y))) \arrow[r, "i_*"] \arrow[d, "\mathrm{id}"] & H_n(C(S_O(X))) \arrow[r, "p_*"] \arrow[d] & H_n\left(\frac{C(S_O(X))}{C(S(Y))}\right) \arrow[r, "\delta"] \arrow[d] & H_{n-1}(C(S(Y))) \arrow[r] \arrow[d, "\mathrm{id}"] & \dots \\
\dots \arrow[r] & {}_n(C(S(Y))) \arrow[r, "i_*"] & H_n(C(S(X))) \arrow[r, "p_*"] & H_n\left(\frac{C(S(X))}{C(S(Y))}\right) \arrow[r, "\delta"] & H_{n-1}(C(S(Y))) \arrow[r] & \dots
\end{tikzcd}$$

So by the 5 lemma we have that $H_n\left(\frac{C(S_O(X))}{C(S(Y))}\right) \to H_n\left(\frac{C(S(X))}{C(S(Y))}\right)$ is an isomorphism. (Where $H_n\left(\frac{C(S(X))}{C(S(Y))}\right) = H(X, Y, A)$). Now we can observe that $S_n(X \setminus U) \cup S_n(Y) = S_{O,n}(X)$ and $S_n(X \setminus U) \cap S_n(Y) = S_n((X \setminus U) \cap Y) = S_n(Y \setminus U)$ so the chain map of quotient complexes is an isomorphism i.e.

$$\frac{C(S(X \setminus U))}{C(S(Y \setminus U))} \xrightarrow{\cong} \frac{C(S_O(X))}{C(S(Y))}.$$

So it induces an isomorphism of homology groups

$$H_n(X \setminus U, Y \setminus U, A) = H_n\left(\frac{C(S(X \setminus U))}{C(S(Y \setminus U))}\right) \xrightarrow{\cong} H_n\left(\frac{C(S_O(X))}{C(S(Y))}\right) \xrightarrow{\cong} H_n\left(\frac{C(S(X))}{C(S(Y))}\right) = H_n(X, Y, A).$$

□

33 Lecture 33: Relative Homology Vs. Quotient Homology

The aim for this section is to find sufficient conditions on a space pair (X, Y) such that the quotient map $p : X \to X/Y$ induces isomorphisms

$$p_* : H_n(X, Y, A) \to H_n(X/Y, Y/Y, A).$$

Proposition 33.1. *Let Y be a closed subspace of a space X such that (X, Y) has the homotopy extension property. Let $f : Y \to A$ be any continuous map. The composite*

$$A \to A \sqcup X \to A \cup_{Y,F} X$$

is a closed embedding and the pair $(A \cup_f X, A)$ also has the homotopy extension property.

Proof. The proof that $A \to A \cup_f X$ is a closed embeffind is the same as in the special case of a cell attachment $(X, Y) = (D^m, \partial D^n)$. For the HEP we consider a continuous map $\psi : A \cup_f X \to T$ and a homotopy $H : A \times [0, 1] \to T$ such that $H(-, 0)$ is the restriction of ψ to A. We consider the commutative diagram of spaces where g is the composite map from X to T

$$\begin{tikzcd}
V \arrow[r, hook] \arrow[d, "f"] & X \arrow[d] \arrow[ddr, bend left, "g"] & \\
A \arrow[r, hook] \arrow[drr, bend right, "\psi|_A"'] & A \cup_f X \arrow[dr, "\psi"] & \\
 & & T
\end{tikzcd}$$

Then the continuous map $g : X \to T$ and the homotopy $K = H \circ (f \times \mathrm{id}_{[0,1]}) : Y \times [0, 1] \to T$ define a homotopy extension problem for the pair (X, Y). So there is a homotopy

$$\overline{K} : X \times [0, 1] \to T$$

that extends K and that starts with g.
Hence we obtain a continuous map

$$A \times [0, 1] \cup_{Y \times [0,1], f \times \mathrm{id}} X \times [0, 1] \overset{H \cup \overline{K}}{\longrightarrow} T$$

is a homeomorphism. Composing the inverse homeomorphism with $H \cup \overline{K}$ yields a homotopy $(A \cup_f X) \times [0, 1] \to T$ that satisfies the original HEP i.e. it extends H and begins with ψ.

□

Theorem 33.2. *Let Y be a closed subspace of a space X such that (X, Y) has the HEP.*

1. *The pair $\left(\frac{X \times \{0\} \cup Y \times [0,1]}{Y \times \{1\}}, \frac{Y \times [0,1]}{Y \times \{1\}}\right)$ has the HEP. (Where $\frac{X \times \{0\} \cup Y \times [0,1]}{Y \times \{1\}}$ is the mapping cone of the inclusion $Y \hookrightarrow X$).*
2. *The continuous map $\pi_1 : \frac{X \times \{0\} \cup Y \times [0,1]}{Y \times \{1\}} \to X/Y; [x, t] \mapsto [x]$ is a homotopy equivalence.*
3. *For all $n \geq 0$ and all abelian groups A, the projection $p : X \to X/Y$ induces an isomorphism*
$$p_* : H_n(X, Y, A) \to H_n(X/Y, Y/Y, A).$$

Proof. 1. We form the space $\frac{Y\times[0,1]}{Y\times\{1\}}\cup_{Y,f} X$, using the continuous map $f: Y \to \frac{Y\times[0,1]}{Y\times\{1\}}$ defined as the composite $Y \overset{(-,0)}{\to} Y\times[0,1] \overset{\text{quotient}}{\longrightarrow} \frac{Y\times[0,1]}{Y\times\{1\}}$. By the previous proposition the pair $\left(\frac{X\times\{0\}\cup Y\times[0,1]}{Y\times\{1\}}, \frac{Y\times[0,1]}{Y\times\{1\}}\right)$ has the HEP. Because Y is closed in X, the cannonical map $Yt\times[0,1]cupX \to X\cup\{0\}\cup Y\times[0,1]$ is a homeomorphism so collapsing $Y\times\{1\}$ on both sides yields another homeomorphism

$$\frac{Y\times[0,1]\cup_f X}{Y\times\{1\}} \overset{\cong}{\to} \frac{X\times\{0\}\cup Y\times[0,1]}{Y\times\{1\}}$$

So also the pair in the statement (1) has a homotopy extension property.

2. Because the pair $\left(\frac{X\times\{0\}\cup Y\times[0,1]}{Y\times\{1\}}, \frac{Y\times[0,1]}{Y\times\{1\}}\right)$ has the HEP and $\frac{Y\times[0,1]}{Y\times\{1\}}$ us contractible to the point $\frac{Y\times\{1\}}{Y\times\{1\}}$. Using exercise 9 on sheet 3 we have that the quotient map

$$\frac{X\times\{0\}\cup Y\times[0,1]}{Y\times\{1\}} \to \frac{X\times\{0\}\cup Y\times[0,1]}{Y\times\{1\}}\Big/\frac{Y\times[0,1]}{Y\times\{1\}}$$

Up to homeomorphism the target can be rewritten as follows:

$$\frac{X\times\{0\}\cup Y\times[0,1]}{Y\times\{1\}}\Big/\frac{Y\times[0,1]}{Y\times\{1\}} \cong \frac{X\times\{0\}\cup Y\times[0,1]}{Y\times[0,1]} \overset{\cong}{\to} X/Y$$

Where the isomorphism $X/Y \overset{\cong}{\to} \frac{X\times\{0\}\cup Y\times[0,1]}{Y\times[0,1]}; [x]\mapsto[x,0]$.

3. Do this proof later

$\square$

34 Lecture 34: Homology of Spheres- Calculations

The sphere $S^0=\{\pm 1\}$ is discrete so

$$H_n(S^0,A)\cong\begin{cases} A\oplus A & n=0\\ 0 & n>0\end{cases}$$

Theorem 34.1. *Let $m>0$. Then*

$$H_n(S^m,A)=\begin{cases} A & n=0,m\\ 0 & otherwise\end{cases}$$

Theorem 34.2. *Let $m>0$. Then*

$$H_n(D^m,S^{m-1},A)=\begin{cases} A & n=m\\ 0 & otherwise\end{cases}$$

Proof. We will prove both of the above by an induction.
Start with the relative case (Theorem 34.2) for the case where $m=1$. We know that D^1 is contractible. Hence $H_n(D^1,A)=\begin{cases} A & n=0\\ 0 & n>0\end{cases}$. Now consider the long exact sequence

$$\ldots\to \underbrace{H_n(D^1,A)}_{=0\text{ for }n\geq 1}\to H_n(D^1,S^0,A)\overset{\delta}{\to}\underbrace{H_{n-1}(S^0,A)}_{=0\text{ for }n\geq 1}\to\underbrace{H_{n-1}(D^1,A)}_{=0\text{ for }n\geq 1}\ldots$$

So we can see that $H_n(D^1, S^0, A) = 0$ for $n \geq 1$. And now we can consider the part around $n = 0, 1$

$$... \to \underbrace{H_1(D^1, A)}_{=0} \to H_1(D^1, S^0, A) \xrightarrow{\delta} \underbrace{H_0(S^0, A)}_{\cong A[S^0] \cong A \oplus A} \to \underbrace{H_0(D^1, A)}_{A[\pi_0(D^1)] \cong A} \to H_0(D^1, S^0, A) \to 0$$

Where we know that $A[S^0] = A \oplus A$ by S^0 discrete and made of two points and $A[\pi_0(D^1)] = A$ by D^1 path connected. The natural isomorphism of the 0 homology groups gives us the natural group homomorhism $A \oplus A \to A; (a, b) \mapsto a + b$. So we have that $H_1(D^1, S^0, A) = \ker(A \oplus A \to A) \cong A$. And also that $H_0(D^1, S^0, A) = \operatorname{coker}(A \oplus A \to A) = A/A = 0$. Hence we have the relative homology case for $n = 1$.

The relative (Theorem 34.2) case for the m-th homology group implies the absolute (Theorem 34.1) case for the m-th homology group. Because we choose a homeomorphism $\psi : D^m/S^{n-1} \cong S^m$. Write $z = \psi(S^{m-1}/S^{m-1}) \in S^m$. We can see (D^m, S^{m-1}) has the homotopy extension property, so the projection $p : D^m \to D^m/S^{m-1}$ induces isomorphisms

$$H_n(D^m, S^{m-1}, A) \xrightarrow{p_*} H_n(D^m/S^{m-1}, S^{m-1}/S^{m-1}, A) \underset{\psi_*}{\overset{\cong}{\to}} H_n(S^m, \{e\}, A).$$

here he goes through a direct argument of this for spheres

We can also show that the absolute (Theorem 34.1) case for m implies the relative case for $m+1$. We can study the long exact sequence of the pair (D^{m+1}, S^m), since D^{m+1} is contractible we know that $H_n(D^{m+1}, A) = 0$ for $n > 0$. So for all $n > 0$ the connecting homomorphism $\delta : H_{n+1}(D^{m+1}, S^m, A) \overset{\cong}{\to} H_n(S^m, A)$ is an isomorphism and an exact sequence

$$0 \to H_1(D^{m+1}, S^m, A) \xrightarrow{\delta} \underbrace{H_0(S^m, A)}_{\cong A} \xrightarrow{i_*} \underbrace{H_0(D^{m+1}, A)}_{\cong} \to H_0(D^{m+1}, S^m, A) \to 0$$

ans do $H_1(D^{m+1}, S^m, A) \cong H_0(D^{m+1}, S^m, A) = 0$. Hence $H_{n+1}(D^{m+1}, S^m, A) = \begin{cases} A & n = m \\ A & n \neq m \end{cases}$.

□

35 Lecture 35: Homology of Spheres- Application

Some cool geometric facts we will hang out with in this lecture:

- S^{m-1} is not retract of D^m.
- Brower fixed point theorem
- Topological invariance of dimension

Theorem 35.1. *Let $m \geq 1$. There is no continuous map $r : D^m \to S^{m-1}$ that is the identity on S^{m-1}.*

Proof. Write $i : S^{m-1} \to D^m$ for the inclusion. Suppose such a continuous retraction $r : D^m \to S^{m-1}$ exists. then $r \circ i = \mathrm{id}_{S^{m-1}}$. Let us apply $H_{m-1}(-, \mathbb{Z})$.

$$r_* \circ i_* = (r \circ i)_* = \mathrm{id}_{H_{m-1}(S^{m-1}, \mathbb{Z})}$$

which implies that $\mathrm{id}_{H_{m-1}(S^{m-1}, Z)} = 0$ which contradicts that $H_{m-1}(S^{m-1}, \mathbb{Z}) \cong \mathbb{Z} \neq 0$ for $m > 1$. □

$$\begin{tikzcd}
H_{m-1}(S^{m-1},\mathbb{Z}) \arrow[r, "i_*"] \arrow[dr, "\mathrm{id}"'] & H_{m-1}(D^m,\mathbb{Z}) = 0 \arrow[d, "r_*"] \\
& H_{m-1}(S^{m-1},\mathbb{Z})
\end{tikzcd}$$

Theorem 35.2. *(Brower Fixed point theorem) Let $m \geq 1$. Then ever continuous self map of D^m has a fixed point.*

Proof. Suppose for contradiction there is a continuous map $g : D^m \to D^m$ without a fixed point. We define a continuous map $r : D^m \to S^{m-1}$ by $r(x) =$ the point on S^{m-1} on the ray from $g(x)$ to x. put this picture in here pls its useful Then $r(x) = x$ for $x \in S^{m-1}$. Such a continuous map r does not exist by the previous theorem, hence g does not exist. $\square$

Let $U \subset \mathbb{R}^n$ be an open subset and $x \in U$. We calculate the "local homology of U in x" as follows.
Let $\epsilon > 0$ be such that $D(x,\epsilon) \subset U$. Then

$$H_n(U, U\backslash\{x\},\mathbb{Z}) \overset{\cong}{\to} H_n(D(x,\epsilon, D(x,\epsilon\backslash\{x\},\mathbb{Z}) \cong H_n(D^m, D^m\backslash\{0\},\mathbb{Z}) \cong H_n(D^m, S^{m-1},\mathbb{Z}) = \begin{cases} \mathbb{Z} & n = m \\ 0 & n \neq m \end{cases}$$

Theorem 35.3. *(Topological invariance of dimension) Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ open. If U and V are homeomorphic then $m = n$.*

Proof. Let $\psi : U \to V$ be a homeomorphism, choose any $x \in U$, set $y = \psi(x) \in V$. Then

$$\mathbb{Z} \cong H_m(U, U \setminus \{x\},\mathbb{Z}) \underset{\psi_*}{\overset{\cong}{\to}} H_m(V, V \setminus \{y\}, Z)$$

and so $n = m$. $\square$

36 Lecture 36: Degree of Sphere maps (Part 1)

36.1 Definition of Degree

Remark. (Recall): Let A be a free abelian group of rank 1 i.e. $A \cong \mathbb{Z}$. Then every endomorphism $f : A \to A$ is of the form $f(x) = d \cdot x$ for a unique integer $d \in \mathbb{Z}$.

Definition 36.1. Let $m \geq 1$. Then the degree of a continuous map $f : S^m \to S^m$ is the unique integer $\deg(f) \in \mathbb{Z}$ such that $f_*(x) = \deg(f) \cdot x$ for all $x \in H_m(S^m,\mathbb{Z})$.

36.2 Properties of Degree

Proposition 36.1. *Let $m \geq 1$ and let $f, g : S^m \to S^m$ be continuous maps.*

1. *If f is homotopic to g then* $\deg(f) = \deg(g)$*,*
2. $\deg(id_{S^n}) = 1$*,*
3. *If f is not surjective then* $\deg(f) = 0$*,*
4. $\deg(f \cdot g) = \deg(f) \cdot \deg(g)$*,*
5. *If f is a homotopy equivalence then* $\deg(f) \in \{\pm 1\}$*.*

Proof. 1. If $f \simeq g$ then $f_* = g_* : H_m(S^m,\mathbb{Z}) \to H_m(S^m,\mathbb{Z})$ so $\deg(f) = \deg(g)$.

2. $\mathrm{id}_*(d) = \mathrm{id}(x) = x = 1 \cdot x$ for all $x \in H_m(S^m, \mathbb{Z})$, so $\deg(\mathrm{id}) = 1$.
3. Let $z \in S^m$ be any point that is not in the image of f. Then f factors as the composite

$$S^m \xrightarrow{f} S^m \setminus \{x\} \hookrightarrow S^m$$

Where $S^m \setminus \{x\}$ is contractible so $f_*(x)$ factors as the composite

$$H_m(S^m, \mathbb{Z}) \to \underbrace{H_m(S^m \setminus \{x\}, \mathbb{Z})}_{\cong 0} \to H_m(S^n, \mathbb{Z})$$

so $f_*(x) = 0$ for all $x \in H_n(S^m, \mathbb{Z})$ hence $\deg(f) = 0$.
4. For all $x \in H_n(S^m, \mathbb{Z})$ we have that $(f \circ g)_*(x) = f_* g_*(x) = f_*(\deg(g) \cdot x) = \deg(f) \cdot \deg(g) \cdot x$ Hence $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.
5. Let $g : S^m \to S^m$ be a homotopy inverse to f, i.e. $g \circ f = \mathrm{id}$. Hence $\deg(g) \cdot \deg(f) = \deg(g \circ f) = \deg(\mathrm{id}) = 1$ hence $\deg(f), \deg(g)$ must be units in $\mathbb{Z}$ for which we have only ± 1.

$\square$

36.3 Additivity of the Degree

We chose a basepoint $z \in S^m$. A pinch map is a continuous map $p : S^m \vee S^m \to S^m$ (where the wedge $S^m \vee S^m$ is the disjoint union with the two basepoints identified) whose composite with the two projections $q^1, q^2 : S^m \vee S^m \to S^m$ is based homotopic to the identity. Then given two continuous maps $f, g : S^m \to S^m$, we define $f + g : S^m \to S^m$ by $(f \vee g) \circ p$ where $f \vee p : S^m \vee S^m \to S^m; \begin{cases} f & \text{in the first copy of } S^m \\ g \text{in the second copy of } S^m \end{cases}$.

Proposition 36.2. *Let $m \geq 1$, $f, g : S^m \to S^m$ based continuous maps. Then* $\deg(f + g) = \deg(f) + \deg(g)$.

Proof. (Note: During this prove we drop the coefficients for homology for ease but assume were in $\mathbb{Z}$.)
Claim 1: We write $i^1, i^2 : S^m \to S^m \vee S^m$ for the two wedge summand inclusions. Then

$$i^1_* + i^2_* : H_m(S^m) \oplus H_m(S^m) \overset{\cong}{\to} H_m(S^m \vee S^m); (x, y) \to i^1_*(x) + i^2_*(y)$$

is an isomorphism.
Proof of Claim 1: We factor the homomorphisms i^1_*, i^2_* as the following composite:

$$H_m(S^m) \oplus H_m(S^m) \underset{i^1_*, i^2_*}{\overset{\cong}{\to}} H_m(S^m \sqcup S^m) \underset{\pi_*}{\overset{\cong}{\to}} H_m((S^m \sqcup S^m)/B) \cong H_m(S^m \vee S^m)$$

Where B is the union of the two basepoints of $S^m \sqcup S^m$. Note that i^1_*, i^2_* are isomorphisms by exercise 26 and π_* is also an isomorphism by the long exact sequence of the pair $(S^m \sqcup S^m, B)$ and $H_n(B) = 0$ for all $n > 1$

$$\begin{tikzcd} H_m(S^m \sqcup S^m) \arrow[r, "\cong"] \arrow[d, "\cong"] & H_m(S^m \sqcup S^m, B) \arrow[d, "\cong \text{ by HEP}"] \\ H_n((S^m \sqcup S^m)/B) \arrow[r, "\cong"] & H_m((S^m \sqcup S^m)/B, B/B) \end{tikzcd}$$

Claim 2: The continuous maps $i^1, i^2, p : Sm \to S^m \vee S^m$ satisfy:

$$p_* = i^1_* + i^2_* : H_m(S^m) \to H_m(S^m \vee S^m)$$

Proof of Claim 2:
By claim 1, for $x \in H_m(S^m)$ there are $y, y' \in H_m(S^m)$ such that $p_*(x) = i^1_*(y) \oplus i^2_*(y')$.
Recall: $q^1, q^2 : S^m \vee S^m \to S^m$ are the projections. Then

$$x = \underbrace{(q^1 \circ p)_*}_{=\mathrm{id}}(x) = q^1_*(p_*(x) = q^1_*(i^1_*(y) + i^2_*(y')) = \underbrace{(q^1 \circ \imath^1)_*}_{=\mathrm{id}}(y) + \underbrace{(q^1 \circ \imath^2)_*}_{=*}(y') = y$$

and similarly with q^2 we get $x = y'$.
To finish the proof: for all $x \in H_m(S^m)$ we have that

$$\begin{aligned}(f+g)_*(x) &= ((f \vee g) \circ p)_*(x) \\ &= (f \vee g)_*(p_*(x)) \\ &= (f \vee g_*(i^1_*(x) + i^2_*(x)) \\ &= (f \vee g) \circ i^1)_*(x) + ((f \vee g) \circ i^2)_*(x) \\ &= f_*(x) = g_*(x) \\ &= \deg(f) \cdot x + \deg(g) \cdot x \\ &= (\deg(f) + \deg(g)) \cdot x\end{aligned}$$

Hence the degree of $f + g$ is the sum of the degrees. □

37 Lecture 37: Degree of Sphere maps (Part 2)

From the previous lecture we know that the degree of a continuous sphere map $f : S^m \to S^m$ is the unique integer such that $f_* = \deg(f) \cdot - : H_m(S^m, \mathbb{Z}) \to H_m(S^m, \mathbb{Z})$

Theorem 37.1. *Let $r : S^m \to S^m$ be defined by $r(x_1, ..., x_{m+1}) = (-x_1, x_2, ..., x_{m+1})$. Then the* $\deg(r) = -1$.

Corollary 37.2. *For every $d \in \mathbb{Z}$, there is a continuous self map of S^n of degree d.*

Proof. (Of Corollary)
For $d \geq 1$: Consider

$$\deg(\underbrace{(((\mathrm{id} \vee \mathrm{id}) \vee \mathrm{id}) \vee ...) \vee \mathrm{id}}_{d}) = \underbrace{\deg(\mathrm{id}) + ... \deg(\mathrm{id})}_{d} = d$$

For $d \leq -1$: Consider

$$\deg(r \circ \underbrace{(((\mathrm{id} \vee \mathrm{id}) \vee \mathrm{id}) \vee ...) \vee \mathrm{id}}_{d}) = \deg(r) \cdot \underbrace{(\deg(\mathrm{id}) + ... \deg(\mathrm{id}))}_{d} = -d$$

Then in the case where $d = 1$ we can consider any surjective map. □

Proof. (Of Theorem)
By induction on m, (we will also proving a relative version at the same time): Write $r : D^n \to D^n$ for the map $r(x_1, ..., x_m) = (-x_1, ..., x_m)$. hen $r_* = -\mathrm{id} : H_m(D^n, S^{m-1}, \mathbb{Z}) \to H_m(D^m, S^{m-1}, \mathbb{Z})$. finish proof later □

Corollary 37.3. *For a matrix $A \in \mathfrak{o}(m+1)$ (the group of orthogonal matrices),* $\deg(A|^m_S) = \det(A)$.

Proof. The general orthogonal group only has two path components and they are decided by determinant. Every orthogonal matrix has determinant ± 1.

If $\det(A) = 1$ then A lies in the same path component of $\mathfrak{o}(m+1)$ as the matrix $\begin{pmatrix} 1 & ... & 0 \\ \vdots & \ddots & \vdots \\ 0 & ... & 1 \end{pmatrix}$ and so $A|_{S^m} \simeq \mathrm{id}_{S^m}$ and so $\deg(A|_{S^m}) = \deg(\mathrm{id}) = 1$.

If $\det(A) = -1$ then A is in the same path component of $\mathfrak{o}(m+1)$ as the matrix $\begin{pmatrix} -1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & ... & 0 & 1 \end{pmatrix}$

So $A|_{S^m} \simeq r$ hence $\deg(A|_{S^m}) = \deg(r) = -1$. □

Examples:

- Let $R \in \mathfrak{o}(m+1)$ be the reflection in any hyperplane, then $\deg(R|_{S^m}) = -1$.
- Let $A : S^m \to S^m$ be the antipodal map, i.e.c$A(x) = -x$. Then A is the restriction to S^m of the linear map $M = \begin{pmatrix} -1 & ... & 0 \\ \vdots & \ddots & \vdots \\ 0 & ... & -1 \end{pmatrix}$. So $\deg(A) = \det(M) = (-1)^{m+1}$.

Proposition 37.4. *Let $f : S^m \to S^m$ be a continuous map without a fixed point. Then f is homotopic to the antipodal map and hence* $\deg(f) = (-1)^{m+1}$.

Proof. Becuase $f(x) \neq x$ for all $x \in S^m$, so the line from $f(x)$ to $-x$ does not pass through the origin of $\mathbb{R}^{m+1}$. So

$$\begin{aligned} H :& S^m \times [0,1] \to \mathbb{R}^{m+1} \setminus \{0\} \\ & (x,t) \mapsto (1-t)f(x) - tx \end{aligned}$$

is a homotopy from f to the antipodal map. Composing with normalisation $\mathbb{R}^{m+1} \setminus \{0\} \to S^m; y \mapsto \frac{y}{|y|}$ yields a homotopy in S^m from f to A. Hence $\deg(f) = \deg(A) = (-1)^{m+1}$. □

Definition 37.1. A continuous vector field on S^m is a continuous map $v : S^m \to \mathbb{R}^{m+1}$ such that $\langle v(x), x \rangle = 0$, i.e. $v(x)$ is orthogonal to x for all $a \in S^m$.

Theorem 37.5. *The sphere S^m admits a nowhere vanishing continuous vector field if and only if m is odd.*

Proof. Suppose m is odd. Let $m = 2k-1$. Then

$$v(x_1, ..., x_{2k}) = (x_2, -x_1, x_4, -x_3, ..., x_{2k}, -x_{2k-1})$$

which is continuous and nowhere 0.
Suppose conversely that $v : S^n \to \mathbb{R}^{m+1}$ is a nowhere vanishing continuous vector field then $w : S^m \to S^m$ defined by $w(x) = \frac{v(x)}{|v(x)|}$ is another one. Hence $H : S^m[0,1] \to S^m$, $H(x,t) = \cos(t)x + \sin(t)w(x)$ is a homotopy from id_{S^m} to the antipodal map A. Hence $(-1)^{m+1} = \deg(A) = \deg(\mathrm{id}) = 1$, so this exists only when m is odd. □

A specific example of the above theorem is the hairy ball theorem.

38 Lecture 38: Long Exact Sequence of a Triple

Proposition 38.1. *Let (X, Y, Z) be a nested triple of spaces. Define a homomorphism δ : $H_n(X, Y, A) \to H_{n-1}(Y, Z, A)$ as the composite*

$$H_n(X, Y, A) \xrightarrow{\partial} H_{n-1}(Y, A) \xrightarrow{p_*} H_{n-1}(Y, Z, A).$$

Then the following long sequence of abelian groups is exact

$$\ldots \to H_n(Y, Z, A) \to H_n(X, Z, A) \to H_n(X, Y, A) \xrightarrow{\delta} H_{n-1}(Y, Z, A) \to \ldots$$

Remark. Note that if $Z = \emptyset$ then $H_n(Y, Z, A) = H_n(Y, \emptyset, A = H_n(Y, A)$ and $H_n(X, \emptyset, A) = H_n(X, A)$ and this sequence specialises to the long exact sequence of the pair (X, Y).

Proof. The space triple (X, Y, Z) yields a nested sequence of subcomplexes (again for ease we do not write A but assume coefficients in A for duration of this proof).

$$C(S(Z)) \hookrightarrow C(S(Y)) \hookrightarrow C(S(X)).$$

We can divide out $C(S(Z))$ to give a subcomplex $\frac{C(S(Y))}{C(S(Z))}$ of $\frac{C(S(X))}{C(S(Z))}$ which yields a long exact sequence of homology groups

$$\ldots \to \underbrace{H_n\left(\frac{C(S(Y))}{C(S(Z))}\right)}_{=H_n(Y,Z)} \xrightarrow{i_*} \underbrace{H_n\left(\frac{C(S(X))}{C(S(Z))}\right)}_{=H_n(X,Z)} \xrightarrow{p_*} \underbrace{H_n\left(\frac{C(S(X))}{C(S(Z))}\Big/\frac{C(S(Y))}{C(S(Z))}\right)}_{\cong H_n(C(S(X))/C(S(Y)))=H_n(X,Y)} \xrightarrow{\partial} \underbrace{H_{n-1}\left(\frac{C(S(Y))}{C(S(Z))}\right)}_{=H_n(Y,Z)} \to \ldots$$

which we can see gives us the required long exact sequence. ☐

39 Lecture 39: Cellular Homology- Abstract Theory

39.1 Overview

Let X be a CW-complex and A an abelian group of coefficients. Then we define a chain complex $C^{\text{cell}}(X, A)$ and an isomorphism $H_n(C^{\text{cell}}(X, A) \cong H_n(X, A)$. The complex $C^{\text{cell}}(X, A)$ is functorial for <u>cellular</u> maps in X, and the isomorphism is natural for cellular maps. In the next lecture we will see a concrete interpretation of this i.e. $C^{\text{cell}}(X, A)$ is isomorphic to the A-linearisation of the set of n-cells of X. In terms of this bases, the differential $d_n^{\text{cell}} : C_n^{\text{cell}}(X, A) \to C_{n-1}^{\text{cell}}(X, A)$ is given by the degree of the attaching maps of n-cells o the $(n-1)$-cells.

39.2 Construction of Cellular Homology

Let X be a CW-complex with skeleta $\{X_n\}_{n \geq 0}$, let A be an abelian group. The <u>Cellular chain complex</u> $C^{\text{cell}}(X, A)$ has terms

$$C_n^{\text{cell}}(X, A) = H_n(X_n, X_{n-1}, A).$$

The differential $d_n^{\text{cell}} : C_n^{\text{cell}}(X, A) \to C_{n-1}^{\text{cell}}(X, A)$ is the connecting homomorphism in the long exact sequence of the triple (X_n, X_{n-1}, X_{n-2}) i.e. d_n^{cell} is the composite:

$$C_n^{\text{cell}}(X, A) = H_n(X_n, X_{n-1}, A) \xrightarrow{\partial} H_{n-1}(X_{n-1}, A) \xrightarrow{p_*} H_{n-1}(X_{n-1}, X_{n-2}, A) = C_{n-1}^{\text{cell}}(X, A)$$

This is indeed a chain complex: the composite $d_{n-1}^{\text{cell}} \circ d_n^{\text{cell}}$ is the composite of four homomorphisms, the middle two of which are

$$H_{n-1}(X_{n-1}, A) \xrightarrow{p_*} H_{n-1}(X_{n-1}, X_{n-2}, A) \xrightarrow{\partial} H_{n-1}(X_{n-1}, A).$$

Our next aim is to find an isomorphism between $H_n(C^{\text{cell}}(X, A)) \cong H_n(X, A)$. This composite occurs as two adjacent maps in the long exact sequence of the pair (X_{n-1}, X_{n-2}) so it is 0 and hence $d_{n-1}^{\text{cell}} \circ d_n^{\text{cell}} = 0$.

Proposition 39.1. *Let X be a CW-complex with $\{X_n\}_{n\geq 0}$, $0 \leq m \leq n$ and let A be any abelian group, Then $H_k(X_n, X_m, A) = 0$ for all $k > n, k \leq m$.*

Proof. finish proof later □

Proposition 39.2. *Let X be a CW-complex. Then for all $n \geq 0$ the inclusion induces isomorphisms*

$$H_n(X_{n+1}, A) \overset{\cong}{\to} H_n(X_{n+1}, A) \overset{\cong}{\to} ... \overset{\cong}{\to} H_n(X_{n+q}, A) \overset{\cong}{\to} ...$$

and

$$H_n(X_{n+1}, A) \overset{\cong}{\to} H_n(X, A).$$

Proof. finish proof later □

Proposition 39.3. *Let X be a CW-complex and $n \geq 0$. Then the map $p_* : H_n(X_{n+1}, A) \to H_n(X_{n+1}, X_{n-2}, A)$ is an isomorphism.*

Corollary 39.4.

$$H_n(X, A) \cong H_n(X_{n+1}, A) \cong H_n(X_{n+1}, X_{n-2}, A)$$

Proof. (Of Proposition) A part of the long exact sequence of the pair (X_{n+1}, X_{n-2}) looks as follows

$$0 = H_n(X_{n-2}) \overset{i_*}{\to} H_n(X_{n+1}) \overset{p_*}{\to} H_n(X_{n+1}, X_{n-1}) \overset{\partial}{\to} H_{n-1}(X_{n-2}) = 0.$$

Hence we have that p_* is an isomorphism. □

Construction of an isomorphism:

$$H_n(C^{\text{cell}}(X, A)) \cong H_n(X_{n+1}, X_{n-2}, A)$$

We consider the long exact sequences of the triples (X_n, X_{n-1}, X_{n-2}) (horiontal), (X_{n+1}, X_n, X_{n-2}) (left vertical), (X_{n+1}, X_n, X_{n-1}) (right vertical)

$$\begin{tikzcd}
 & H_{n+1}(X_{n+1}, X_n) \arrow[r, leftrightarrow, "\mathrm{id}", "="'] \arrow[d, "\partial"] & H_{n+1}(X_{n+1}, X_n) \arrow[d, "\partial"] & \\
0 = H_n(X_{n-1}, X_{n-2}) \arrow[r] & H_n(X_n, X_{n-2}) \arrow[r] \arrow[d] & H_n(X_n, X_{n-1}) \arrow[r, "\partial"] \arrow[d] & H_{n-1}(X_{n-1}, X_{n-2}) \\
 & H_n(X_{n+1}, X_{n-2}) \arrow[r] \arrow[d] & H_n(X_{n+1}, X_{n-1}) \arrow[d] & \\
 & 0 = H_n(X_{n+1}, X_n) \arrow[r, leftrightarrow, "\mathrm{id}", "="'] & H_n(X_{n+1}, X_n) &
\end{tikzcd}$$

The diagram can thus be simplified to So

$$H_n(C^{\text{cell}}(X)) = \frac{\ker(d_n^{\text{cell}})}{\text{im}(d_{n+1}^{\text{cell}})} \cong H_n(X_{n+1}, X_{n-2}) \cong H_n(X)$$

(as in the corollary).

$$\begin{tikzcd}
 & C_{n+1}^{\text{cell}}(X) \arrow[r, leftrightarrow, "\text{id}", "="'] \arrow[d, "\partial"] & C_{n+1}^{\text{cell}}(X) \arrow[d, "d_n^{\text{cell}}"] & \\
0 \arrow[r] & H_n(X_n, X_{n-2}) \arrow[r] \arrow[d] & C_n^{\text{cell}}(X) \arrow[r, "d_n^{\text{cell}}"] \arrow[d] & C_{n-1}^{\text{cell}}(X) \\
 & H_n(X_{n+1}, X_{n-2}) \arrow[r] \arrow[d] & H_n(X_{n+1}, X_{n-1}) \arrow[d] & \\
 & 0 & 0 &
\end{tikzcd}$$

39.3 Functorality and Naturality

Let $f : X \to Y$ be continuous cellular map between CW-complex. Then f restricts to a map of pairs

$$f : (X_n, X_{n-1}) \to (Y_n, Y_{n-1})$$

and it induces a homomorphism

$$d_n^{\text{cell}}(f) = f_* : \underbrace{H_n(X_n, X_{n-1}, A)}_{=C_n^{\text{cell}}(X,A)} \to \underbrace{H_n(Y_n, Y_{n-1}, A)}_{=C_n^{\text{cell}}(X,A)}$$

The following diagram commutes: The homomorphism $d_n^{\text{cell}}(f)$ for varying n form a chain map

$$\begin{tikzcd}
H_n(X_n, X_{n-1}, A) \arrow[r, "\partial"] \arrow[d, "f_*"] \arrow[rr, bend left, "d_n^{\text{cell}}"] & H_{n-1}(X_{n-1}, A) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_{n-1}(X_{n-1}, X_{n-2}, A) \arrow[d, "f_*"] \\
H_n(Y_n, Y_{n-1}, A) \arrow[r, "\partial"] \arrow[rr, bend right, "d_n^{\text{cell}}"'] & H_{n-1}(Y_{n-1}, A) \arrow[r, "p_*"] & H_{n-1}(Y_{n-1}, Y_{n-2}, A)
\end{tikzcd}$$

$d_n^{\text{cell}}(f) : C^{\text{cell}}(X < A) \to C^{\text{cell}}(Y, A)$.
It remains to check

- $C^{\text{cell}}(-, A)$ defined a functor from the category of CW-complexes and cellular maps to the category of chain complexes with chain maps.
- the isomorphism $H_n(C^{\text{cell}}(X, A)) \cong H_n(X, A)$ are natural for cellular maps of CW-complexes.

40 Lecture 40: Cellular Homology- Concrete Description

The cellular chain complex of a CW- complex is given by $C_n^{\text{cell}}(X, A) = H_n(X_n, X_{n-1}, A)$ and with differential maps $d_n^{\text{cell}} : C_n^{\text{cell}}(X, A) \to C_{n-1}^{\text{cell}}(X, A)$ the connecting homomorphisms of the triple (X_n, X_{n-1}, X_{n-2}). We established an isomorphism

$$H_n(C^{\text{cell}}(X, A)) \cong H_n(X, A)$$

which is natural for cellular maps.
Now we want to describe $C^{\text{cell}}(X, A)$ in terms of the cells of the CW structure and the degrees of certain attaching maps.
We write $J_n = \pi_0(X_n \setminus X_{n-1})$ for the set of open n-cells, so an n-cell is a path component of $X_n \setminus X_{n-1}$. Each element of J_n is homeomorphic to S^n. We choose characteristic maps $\chi_j : D^n \to X_n$ for all $n \geq 0$, all $j \in J_n$. In particular χ_j is a map of pairs $(D^n, S^{n-1}) \to (X_n, X_{n-1})$ so it induces a homomorphism $(\chi_j)_* : H_n(D^n, S^{n-1}, A) \to H_n(X_n, X_{n-1}, A)(= C_n^{\text{cell}}(X, A))$.

Theorem 40.1. *With the notation as above, the map*

$$\begin{aligned} \sum (\chi_j)_* : \bigoplus_{j \in J_n} H_n(D^n, S^{n-1}, A) &\to H_n(X_n, X_{n-1}, A) \\ (x_j)_{j \in J_n} &\mapsto \sum_{j \in J} (\chi_j)_*(x_j) \end{aligned}$$

is an isomorphism. So in particular, $C_n^{cell}(X, A)$ is isomorphic to $A[J_n]$.

Proof. We consider the commutative diagram where this bottom left isomorphism is from ex-

$$\begin{tikzcd}
 & \bigoplus_{j \in J_n} H_n(D^n, S^{n-1}) \arrow[r, "\sum(\chi_j)_*"] \arrow[d, "\cong \text{ by htp invar.}"] & H_n(X_n, X_{n-1}) \arrow[d, "\cong \text{ by htp invar.}"] \\
 & \bigoplus_{j \in J} H_n(D^n, D^n \setminus \{0\}) \arrow[r, "\sum(\chi_j)_*"] & H_n(X_n, X_n \setminus \cup_j \{\chi_j(0)\}) \\
H_n(\mathring{D}^n, \mathring{D}^n \setminus \{0\}) \arrow[r, "\cong"] & H_n(\sqcup_j \mathring{D}^n, \sqcup(\mathring{D}^n \setminus \{0\})) \arrow[r, "(\sqcup \chi_j)_*"] \arrow[u, "\cong \text{ by excision}"'] & H_n(X_n \setminus X_{n-1}, X_n \setminus (X_{n-1} \cup \chi_j(0)) \arrow[u, "\cong \text{ by excision}"']
\end{tikzcd}$$

ercise 26, sheet 7. Since all other maps are isomorphisms, so is the one in the statement. $\square$

Definition 40.1. A CW-complex X is sparse if for every $n \geq 0$, $X_{n-1} = X_n$ or $X_n = X_{n+1}$. (Informally: A CW-complex is sparse iff there is at least one n-cell for some n then there is no $n + 1$-cell.)

Examples:

- The minimal CW-structure on S^n (i.e. one 0-cell and one n-cell) for $n \geq 2$.

- A CW- structure with only even dimensional cells.

Proposition 40.2. *Let X be a sparse CW-complex. Then*

$$H_n(X, A) = A[J_n]$$

for all $n \geq 0$.

Proof. By sparseness, for every $n \geq 0$, $C_n^{\text{cell}}(X, A) = H_n(X_n, X_{n_1}, A) = 0$ or $C_{n+1}^{\text{cell}}(X, A) = H_{n+1}(X_{n+1}, X_n, A) = 0$ so for every d_{n+1}^{cell} either the source or the target is the zero group. So $d_n^{\text{cell}} = 0$ for all n. So

$$H_n(X, A) \cong H_n(C^{\text{cell}}(X, A)) = C_n^{\text{cell}}(X, A) \cong A[J_n].$$

$\square$

Example: The complex projective space $\mathbb{CP}^m$ has a $2m$-dimensional CW-structure with skeleta

$$(\mathbb{CP}^m)_n = \begin{cases} \mathbb{CP}^k & n = 2k, 2k+1 \\ \mathbb{CP}^m & n \geq 2m \end{cases}$$

This is a sparse structure with one cell in every even dimension. Hence

$$H_n(\mathbb{CP}^m, A) \cong \begin{cases} A & n \text{ even}, n \leq 2m \\ 0 & \text{otherwise} \end{cases}$$

and similarly

$$H_n(\mathbb{CP}^\infty, A) \cong \begin{cases} A & n \text{ even} \\ 0 & \text{otherwise} \end{cases}.$$

Now we interpret d_n^{cell}, for the rest of this lecture we will restrict to the case where $A = \mathbb{Z}$ (without a huge loss of generality as $C^{\text{cell}}(X, A) \cong C^{\text{cell}}(X, \mathbb{Z}) \otimes A$ as we will see later).

40.1 Inductive Definition of Specific Generators

Consider

$$\begin{aligned} &\tau_n \in H_n(D^n, S^{n-1}, \mathbb{Z}) && n \geq 1 \\ &\sigma_n \in H_n(S^n, \mathbb{Z}) \end{aligned}$$

We will start by defining τ_1

$$\tau_1 = [\nabla^1 \overset{\cong}{\to} D^1; (x, 1-x) \mapsto 2x - 1]$$

Now we choose a homeomorphism $\psi_n : D^n/S^{n-1} \overset{\cong}{\to} S^n$, and we define σ_n on the image of τ_n under the isomorphism

$$\underset{\tau_n \in}{H_n(D^n, S^{n-1}, \mathbb{Z})} \underset{p_*}{\overset{\cong}{\to}} H_n(D^n/S^{n-1}, *, \mathbb{Z}) \underset{(\psi_n)_*}{\overset{\cong}{\to}} \underset{\sigma_n \in}{H_n(S^n, \mathbb{Z})}$$

Then we define τ_{n+1} as the preimage of σ_n under the isomorphism

$$\partial : H_{n+1}(D^{n+1}, S^n, \mathbb{Z}) \overset{\cong}{\to} H_n(S^n, \mathbb{Z}); \tau_{n+1} \mapsto \sigma_n$$

Again, let $\chi_j : (D^n, S^{n-1}) \to (X_n, X_{n-1})$ be the characteristic map of the jth n-cell. Define $e_j = (\chi_j)_*(\tau_n) \in H_n(X_n, X_{n-1}, \mathbb{Z}) = C_n^{\text{cell}}(X, \mathbb{Z})$. For $A = \mathbb{Z}$, the earlier theorem says:

Theorem 40.3. *The group $H_n(X_n, X_{n-1}, \mathbb{Z}) = C_n^{cell}(X, \mathbb{Z})$ is the free abelian group with basis given by the classes e_j for $j \in J_n$.*

Now we express d_n^{cell} in terms of these bases by an integer matric whose coefficients are certain degrees. We let $k \in J_{n+1}$ and $j \in J_n$. We define a continuous map $\alpha(k, j) : S^n \to S^n$ by

$$S^n \overset{(\chi_k)|_{S^n}}{\to} X_n \overset{q_j}{\to} D^n/S^{n-1} \underset{\psi_n}{\overset{\cong}{\to}} S^n$$

where q_j is the "collapsing away from j-th cell" i.e.

$$q_j = \begin{cases} \chi_j^{-1}(x) & x \in j \\ * & x \notin j \end{cases}$$

Theorem 40.4. *In terms of the bases of $C^{cell}(X, \mathbb{Z})$ specified above, the differential d_{n+1}^{cell} : $C_{n+1}^{cell}(X, Z) \to C_n^{cell}(X, \mathbb{Z})$ is determined by*

$$d_n^{cell}(e_k) = \sum_{j \in J_n} \deg(\alpha(k, j)) \cdot e_j$$

Proof. long proof, too hard not sure how necessary it is □

41 Lecture 41: Homology of the Torus

Remark. Recall:

- $\tau_n \in H_n(D^n, S^{n-1}, \mathbb{Z})$, $n \geq 1$ specific generator
- $J_n = (X_n \setminus X_{n-1})$ the set of n-cells of a CW-complex X, $n \geq 0$
- for all $n \geq 0$, $j \in J_n$, choose characteristic maps $\chi_j : (D^n, S^{n-1}) \to (X_n, X_{n-1})$ for the jth n-cell
- $e_j = (\chi_j)_*(\tau_n) \in H_n(X_n, X_{n-1}, \mathbb{Z}) = C_n^{\text{cell}}(X, \mathbb{Z})$
- For $n \geq 0$, $\{e_j\}_{j \in J_n}$ is a basis of the free abelian group $C_n^{\text{cell}}(X, Z)$
- For $n \geq 1$, $d_n^{\text{cell}} : C_{n+1}^{\text{cell}}(X, \mathbb{Z}) \to C_n^{\text{cell}}(X, \mathbb{Z})$ is given in these bases by

  $$d_{n+1}^{\text{cell}}(e_k) = \sum_{j \in J_n} \deg(\alpha(k, j)) \cdot e_j$$

  where $k \in J_{n+1}$ and $\alpha(k, j) : S^n \to S^n$ is the attaching map of the k-th, $(n+1)$-cell to te jth n-cell.

Proposition 41.1. *Let X be a CW-complex. Then*

$$d_1^{cell} : C_1^{cell}(X, \mathbb{Z}) \to C_0^{cell}(X, \mathbb{Z})$$

is given by $d_1^{cell}(e_k) = [\chi_k(-1)] - [\chi_k(1)]$ for $k \in J_1$. Where $\chi_k : D^1 \to X_1; \{\pm 1\} \mapsto X_0 = J_0$.

Proof. The class τ_1
$inH_1(D^1, S^0, \mathbb{Z})$ is represented by the relative 1-cycle

$$f : \nabla^1 \xrightarrow{\cong} D^1; (x, 1 \cdot x) \mapsto 2x - 1$$

and $\tau_1 = [1 \cdot f + C_1(S^0, Z)]$. Hence $e_k = (\chi_k)_*(\tau_1)$ is represented by $\chi_k \circ f : \nabla^1 \to X_1$.

$$\begin{aligned}
d_n^{\text{cell}} &= \partial[\chi_k \circ f : D^1 \to X_1] \\
&= [d_1^{\text{sing}}(\chi_k \circ f)] = [d_0^*(\chi_k \circ f) - d_1^*(\chi_k \circ f)] \\
&= [\chi_k \circ f \circ (d_0)_* - \chi_k \circ f \circ (d_1)_*] \\
&= [\chi_k(f(0,1))] - [\chi_k(f(1,0))] \\
&= [\chi_k(-1)] - [\chi_k(1)]
\end{aligned}$$

□

41.1 Homology of the Torus $S^1 \times S^1$

We endow S^1 with the CW-structure

$$S^1 = \{x \in \mathbb{C} : |z| = 1\}$$

with $(S^1)_0 = \{1\}, (S^1)_n = S^1$ for $n \geq 1$. We endow $S^1 \times S^1$ with the product CW-structure

$$\begin{aligned}
&(S^1 \times S^1)_0 = \{(1, 1)\} \\
&(S^1 \times S^1)_1 = (S^1 \times \{1\}) \cup (\{1\} \times S^1) \\
&(S^1 \times S^1)_n = S^1 \times S^1 && \text{for } n \geq 1
\end{aligned}$$

so there is one 0-cell (1×1), two 1-cells $(E \times 1, 1 \times E)$, one 2-cell $(E \times E)$. The two 1-cells have the same start and end point so

$$d_1^{\text{cell}}(E \times 1) = d_1^{\text{cell}}(1 \times E) = 0$$

The attaching map for the one 2-cell is a map

$$S^1 \to (S^1 \times 1) \cup (1 \times S^1)$$

so $C^{\text{cell}}(S^1 \times S^1, \mathbb{Z})$ has trivial differential, and so

$$H_n(S^1 \times S^1, \mathbb{Z}) \cong \mathbb{Z}[J_n] \cong \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z}^2 & n = 1 \\ 0 & n \geq 3 \end{cases}$$

42 Lecture 42: Homology of Real Projective Spaces

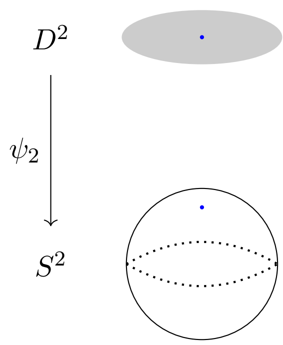
In this lecture we will do an auxiliary calculation of homology of spheres via a CW-structure with 2-cells in each dimension. We endow S^m with a CW-structure chose n-skeleton for $0 \leq n \leq m$ in S^n, interpreted on the subspace of S^m defined by $\{(x_1, ..., x_{n+1}) : \sum_{i=1}^{n+1} x_i^2 = 1\}$ i.e. the subspace of those tuples with $x_{n+2} = ... = x_{m+1} = 0$.
This CW-structure has exactly 2 n-cells, namely

- $\{(x_1, ..., x_{n+1}) \in S^n : x_{n+1} > 0\}$ “northern hemisphere”
- $\{(x_1, ..., x_{n+1}) \in S^n : x_{n+1} < 0\}$ “southern hemisphere”

We will use a specifies homomorphism $\psi_n : D^n/S^{n-1} \overset{\cong}{\to} S^n$, namely

$$\psi_n(x) = \Bigg(\underbrace{2x \cdot \sqrt{\frac{1}{|x|} - 1}}_{\in \mathbb{R}^n}, \underbrace{1 - 2|x|}_{\in [-1,1] \subset \mathbb{R}}\Bigg)$$

Note that $\psi_n(S^{n-1}) = \{(0, -1)\}$ e.g. And we specify characteristic maps for the cells of the



above CW-structure. For $0 \leq n \leq m$

$$\begin{aligned} \chi_+^n : & D^n \to S^n \\ & x \mapsto \left(x \cdot \sqrt{\frac{2}{|x|} - 1}, 1 - |x|\right) \\ \chi_-^n = & A \circ \chi_+^n \end{aligned}$$

for $A : S^n \to S^n$ the antipodal map (i.e. $A(y) = -y$). χ_+^n is a homeomorphism onto the closed nothern hemisphere and similarly for χ_-^n. We set

$$e_+^n = (\chi_+^n)_*(\tau_n)$$
$$e_-^n = (\chi_-^n)_*(\tau_n)$$

which form a basis of $H_n(S^n, S^{n-1}, \mathbb{Z}) = C_n^{\text{cell}}(S^m, \mathbb{Z})$

Proposition 42.1. *In the cellular chain complex of S^m, the following relation holds:*

$$d_{n+1}^{cell}(e_+^{n+1}) = e_+^n + (-1)^{n+1} \cdot e_-^n$$

Proof. no thank $\square$

Homology of $\mathbb{R}P^m$:
We equip $\mathbb{R}P^m$ with the CW-structure chose n-skeleton for $0 \le n \le m$ is $\mathbb{R}P^n$, considered as the subspace of those $[x_1 : x_2 : ... : x_{m+1}] \in \mathbb{R}P^m$ such that $x_{n+2} = ... = x_{m+1} = 0$. This CW-structure has exactly one cell in dimensions $0, 1, ...m$. A characteristic map for the n-cell is the composite

$$\begin{aligned} \chi_n : D^n \overset{\chi_+^n}{\to} S^n &\overset{p}{\to} \mathbb{R}P^n \\ x &\mapsto \mathbb{R} \cdot x \\ (x_1, ..., x_{m+1}) &\mapsto [x_1 : ... : x_{m+1}] \end{aligned}$$

so

$$e^n = \chi_*^n(\tau_n) = p_*(e_+^n) \in H_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}, \mathbb{Z})$$

is a generator.

$$\begin{aligned} d_{n+1}^{\text{cell}}(e^{n+1}) &= d_{n+1}^{\text{cell}}(p_*(e_+^{n+1})) \\ &= p_*(d_{n+1}^{\text{cell}}(e_+^{n+1})) \\ &= p_*(e_+^n + (-1)^{n+1} \cdot e_-^n) \\ &= p_*(e_+^n) = (-1)^{n+1} \cdot p_*(e_-^n) \\ &= e^n + (-1)^{n+1} \cdot e^n \\ &= (1 + (-1)^{n+1}) \cdot e^n \end{aligned}$$

We have that $C^{\text{cell}}(\mathbb{R}P^m, \mathbb{Z})$ is

$$... \underset{m+1}{0} \to \underset{m}{\mathbb{Z}} \overset{1+(-1)^m}{\to} \underset{m-1}{\mathbb{Z}} \overset{1+(-1)^{m-1}}{\to} ... \overset{0}{\to} \underset{3}{\mathbb{Z}} \overset{2}{\to} \underset{2}{\mathbb{Z}} \overset{0}{\to} \underset{1}{\mathbb{Z}}$$

and so

$$H_n(\mathbb{R}P^m, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2\mathbb{Z} & n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

For a general coefficient group we have that $C^{\text{cell}}(\mathbb{R}P^m, A)$ is

$$...0 \to \underset{m}{A} \overset{1+(-1)^m}{\to} \underset{m-1}{A} \overset{1+(-1)^{m-1}}{\to} ... \overset{0}{\to} \underset{2}{A} \overset{2}{\to} \underset{1}{A} \overset{0}{\to} \underset{0}{A} \to 0$$

and so

$$H_n(\mathbb{R}P^m, A) \cong \begin{cases} A & m = 0 \\ A/2A & n \text{ odd}, 1 \le n \le m \\ \{a \in A : 2a = 0\} & n \text{ even}, 2 \le n \le m \\ 0 & \text{otherwise} \end{cases}$$

43 Lecture 43: Euler Characteristic

43.1 Homological Formula for Euler Characteristic

Theorem 43.1. *The surface of every convex polyhedron in $\mathbb{R}^3$ satisfies*

$$\#verticies - \#edges + \#faces = 2.$$

This formula also holds for all CW-structures on S^2 i.e.

$$\#0 - \text{cells} - \#1 - \text{cells} + \#2 - \text{cells} = 2.$$

Definition 43.1. The <u>Euler characteristic</u> of a finite CW-complex is

$$\chi(X) = \sum_{n\geq 0}(-1)^n \cdot \#n - \text{cells}.$$

The Euler characteristic is independent of CW-structure.

Remark. Let R ve a ring.

- For every set X, the R-linearisation $R[X]$ becomes a left R-module via $r \cdot (\sum s_i \cdot x_i) = \sum (r \cdot s_i) \cdot x_i$ for $r, s_i \in R, x_i \in X$
- fore every map $f : X \to Y$, the map $R[f] : R[X] \to R[Y]$ is a homomorphism of R-modules.
- for every simplicial set X the differential $d_n : C_n(X,R) = R[X_n] \to X_{n-1}(X,R) = R[X_{n-1}]$ is $d_n = \sum_{r=0,\dots,n}(-1)^i R[d_i^*]$ is an alternating sum og R-module homomorphisms hence is itself an R-module homomorphism so its kernel and image are R-submodules.
- So $H_n(X,R) = \ker(d_n)/\mathrm{im}(d_{n+1})$ inherits an R-module structure by scalar multiplication by $r \cdot [X] = [r \cdot x]$

In particular, when R is a field and $X = S(Y)$ for some space Y, then the groups $H_n(Y,R)$ becomes R-vectors space.

Theorem 43.2. *Let X be a finite CW-complex and k a field. Then:*

- $H_n(X,k) = 0$ *for* $n \leq \dim(X)$
- $H_n(X,k)$ *is finite dimensional over* k *for all* $n \geq 0$.

Moreover

$$\chi(X) = \sum_{n\geq 0}(-1)^n \cdot \dim_k(H_n(X,k))$$

In particular, the Euler characeristic of X is independent of the CW-structure and only depends on the underlying space.

Remark. Warning: For a fixed n, $\dim_k(H_n(X,k))$ is typically different from the number of n-cells of X, and it depends on the field. Only the alternating sum of the dimension is invariant and equals $\chi(X)$.

Example:
$\mathbb{R}P^n$ has a CW structure with one cell in dimensions $0, 1$ and 2. So $\chi(\mathbb{R}P^2) = 1$. By the last video

$$\begin{aligned}
&H_0(\mathbb{R}P^2, k) \cong k \\
&H_1(\mathbb{R}P^2, k) \cong k/2k \\
&H_2(\mathbb{R}P^2, k) \cong = \{x \in k : 2x = 0\} \\
&H_n(\mathbb{R}P^2, k) \cong 0 && \text{for } n \geq 3
\end{aligned}$$

For k a field of characteristic $\neq 2$ we have that $H_1(\mathbb{R}P^2, k) = H_2(\mathbb{R}P^2, k) = 0$, for $\mathrm{char}(k) = 2$ we have that $H_1(\mathbb{R}P^2, k) = H_2(\mathbb{R}P^2, k) = k$. Either way we conclude $\chi(\mathbb{R}P^2) = 1$ by the theorem.

Proposition 43.3. *Let (X, Y) be a space pair such that X is obtained from Y by attaching an m-cell. Then*

$$H_n(X, Y, A) \cong \begin{cases} A & n = m \\ 0 & n \neq m \end{cases}.$$

Proof. (Of Proposition) Let $x \in X \setminus Y$ be any point. Then $X \setminus \{x\}$ deformation retracts onto Y. So we get isomorphisms $H_n(X, Y, A) \xrightarrow{\cong} H_n(X, X \setminus \{x\}, A)$ by homotopy invariance and $H_n(\mathring{D}^m, \mathring{D}^m \setminus \{x\}, A) \cong H_n(X \setminus Y, X \setminus (\{x\} \cup Y), A) \xrightarrow{\cong} H_n(X, X \setminus \{x\}, A)$ by excision. And hence

$$H_n(X, Y, A) = \begin{cases} A & n = m \\ 0 & n \neq \end{cases}$$

□

Proof. (Of Theorem) come back later □

43.2 Simple Formulas for $\chi(X)$

Let X and Y be CW-complexes. Then $X \sqcup Y$ becomes a CW-complex with skeleta

$$X \sqcup Y)_n = X_n \sqcup Y_n$$

and n-cells of $(X \sqcup Y) = \pi_0((X \sqcup Y)_n \setminus (X \sqcup Y)_{n-1}) = \pi_0((X_n \setminus X_{n-1}) \sqcup (Y_n \setminus Y_{n-1}))$. And

$$\#n - \text{cells of } (X \sqcup Y) = \#n - \text{cells of } (X) + \#n - \text{cells of } (Y)$$

so

$$\chi(X \sqcup Y) = \chi(X) + \chi(Y)$$

If X and Y are finite CW-complexes, then so $X \times Y$ with n-cells of $(X \sqcup Y) = \chi(X \times Y)$.

Proposition 43.4. *Let $p : E \to X$ be a covering space:*

1. *If X has a CW-structure with skeleta $\{X_n\}_{n \geq 0}$, then the spaces $E_n = p^{-1}(X_n)$ define a CW-structure on E.*

2. *If X has a finite CW-structure, then the CW-structure on E is also finite.*

3. *If p is a k-skeleta covering for some $k \geq 0$ and X is finite CW-complex, then*

$$\chi(E) = k \cdot \chi(X).$$

Proof. proooooooof nooooo □

44 Lecture 44: Algebraic Theory of The Universal Coefficient theorem

Informally: $H_n(X, \mathbb{Z})$ is the universal homology and $H_n(X, A)$ can be algebraically determined from $H_n(X, \mathbb{Z}), H_{n-1}(X, \mathbb{Z})$ and A.
Let C be a chain complex and A an abelian group. We define a chain complex $A \otimes C$ by $(A \otimes C)_n = A \otimes C_n$, with $d_n^{A \otimes C} = A \otimes d_n^C$. We define a homomorphism

$$\Phi : A \otimes H_n C \to H_n(A \otimes C)$$

by $\Phi(a \otimes [x]) = [a \otimes x]$.

Lemma 44.1. *The map $\Phi : A \otimes H_nC \to H_n(A \otimes C)$ is well defined homorphism and natural in A and C.*

Proof. Suppose that $x \in C_n$ is a cycle and $a \in A$. Then $d_n^{A\otimes C}(a\otimes x) = a\otimes d_n x = a\otimes 0 = 0$. Let $x' \in C_n$ be another cycle in the same homology class as x. So $x' = x + dy$ for some $y \in C_{n+1}$. Then

$$d_{n+1}^{A\otimes C}(a \otimes y) = a \otimes d_{n+1}y = a \otimes (x' - x) = a \otimes x' - a \otimes x.$$

So $[a \otimes x'] = [a \otimes x]$.
For natruality:
Let $f : A \to B$ be a group homomorphism. Then the following square commutes

$$\begin{array}{ccc} A \otimes H_nC & \xrightarrow{\Phi} & H_n(A \otimes C) \\ \big\downarrow{\scriptstyle f\otimes H_nC} & & \big\downarrow{\scriptstyle H_n(f\otimes C)} \\ B \otimes H_nC & \xrightarrow{\Phi} & H_n(B \otimes C) \end{array}$$

Let $\phi : C \to C'$ be a chain map. Then $A\otimes\phi : A\otimes C \to A\otimes C'$ is a chain map and the following square commutes ☐

$$\begin{array}{ccc} A \otimes H_nC & \xrightarrow{\Phi} & H_n(A \otimes C) \\ \big\downarrow{\scriptstyle A\otimes H_n\phi} & & \big\downarrow{\scriptstyle H_n(A\otimes\phi)} \\ A \otimes H_nC' & \xrightarrow{\Phi} & H_n(A \otimes C') \end{array}$$

44.1 Reminder on Tor Groups

Facts from algebra: every subgroup of a free abelian group is free.
Let A be a abelian group. We choose a free abelian group F and an epimorphism $p : F \to A$. Then $R = \ker(p)$ is also a free abelian group and we have a short exact sequence

$$0 \to R \xrightarrow{i} F \xrightarrow{p} A \to 0$$

which we call the "free resolution of A".

Definition 44.1. Let B be another abelian group. Then the Tor group of A and B is

$$\mathrm{Tor}(A, B) = \ker(i \otimes B : R \otimes B \to F \otimes B).$$

Remark. A non-trivial fact from homological algebra:

- $\mathrm{Tor}(A, B)$ is independent of the choice of F and p up to canonical isomorphism .
- $\mathrm{TTor}(A, B)$ can also be calculated by interchanging the roles of A and B.

Examples:

1. Suppose that A is a free abelian group. Then we can choose $F = A$ and $p = \mathrm{id}$. So then $R = 0$ and $R \otimes B = 0$ and $\mathrm{Tor}(A, B) = 0$.
2. The projection $p : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ has kernel $m\mathbb{Z}$, so

   $\mathrm{Tor}(\mathbb{Z}/m\mathbb{Z}, B) = \ker(i \otimes B : m\mathbb{Z} \otimes B \to \mathbb{Z} \otimes B) \cong \ker(m : B \to B) = \{x \in B : m \cdot x = 0\}$
3. If A is a finitely generated abelian group, then A is isomorphic to a direct sum of finitely many copies of $\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$ for varying m. Since $\mathrm{Tor}(A\oplus A', B) \cong \mathrm{Tor}(A, B)\oplus\mathrm{Tor}(A', B)$

Remark. 1-3 determine $\mathrm{Tor}(A, B)$ for all finitely generated A.

44.2 Algebraic Universal Coefficient Theorem

Theorem 44.2. *(Universal Coefficient Theorem) Let C be a chain complex such that C_n is a free abelian group for all $nin\mathbb{Z}$. Let A be an abelian group. Then there is a short exact sequence*

$$0 \to A \otimes H_n C \xrightarrow{\Phi} H_n(A \otimes C) \to Tor(A, H_{n-1}(C)) \to 0$$

Moreover the sequence is natural in A and C and the sequence splits: there is an additive retraction to Φ.

Remark. Warning: The splitting $r : H_n(A \otimes C) \to A \otimes H_n(C)$ with $r \otimes \Phi = \text{id}$ means that the splitting cannot be chosen naturally

Proof. We define two sub-chain complexes Z and $B[1]$ of C as follows.

$$Z_n = \ker(d_n : C_n \to C_{n-1})$$

and

$$B[1]_n = B_{n-1} = \text{im}(d_n(C_n \to C_{n-1})$$

both Z and B are subcomplexes with trivial differential. We now obtain an exact sequence of abelian groups

$$0 \to Z_n \overset{\iota}{\hookrightarrow} C_n \xrightarrow{d_n} B_{n-1} = B[1]_n \to 0$$

for $n \in \mathbb{Z}$ varying , these define a short exact sequence of chain complexes

$$0 \to Z \hookrightarrow C \xrightarrow{d} B[1] \to 0$$

since C_n is free for every $n \in \mathbb{Z}$, so are its subgroups Z_n and B_n. Si in every fixed dimension, the short exact sequence splits. So $A \otimes -$ yields another short exact sequence

$$0 \to A \otimes Z_n \to A \otimes C_n \to A \otimes B_{n-1} \to 0.$$

For varying $n \in Z$ so we have another short exact sequence of chain complexes.

$$0 \to A \otimes Z \to A \otimes C \to A \otimes B[1] \to 0.$$

By some exercise we have a long exact sequence of homology groups: So we obtain a short exact

$$\ldots \longrightarrow H_{n+1}(A \otimes B[1]) \xrightarrow{\partial} H_n(A \otimes Z) \longrightarrow H_n(A \otimes C) \longrightarrow H_n(A \otimes B[1]) \longrightarrow \ldots$$

sequence finish proof ☐

45 Lecture 45: Consequences of The Universal Coefficient theorem

Theorem 45.1. *(Universal coefficient theorem) Let X be a space and A an abelian group. Then there is a short exact sequence*

$$0 \to A \otimes H_n(A, \mathbb{Z}) \xrightarrow{\Phi} H_n(X, A) \to Tor(A, H_{n-1}(X, Z)) \to 0$$

Moreover:

- *the sequence is natural for group homomorphisms in A and for continuous maps in X.*

- *the sequence splits; in particular te group $H_n(X,A)$ is isomorphic to $(A\otimes H_n(X,Z))\oplus Tor(A,H_{n-1}(X,\mathbb{Z}))$.*

Proof. For every set S, the map $A\otimes\mathbb{Z}[S]\to A[S]; a\otimes\sum b_is_i\mapsto\sum(a\cdot b_i)s_i$ is an isomorphism of groups. For every simplicial set Y, the isomorphisms $A\otimes\mathbb{Z}[Y_n]\to A[Y_n]$ are composite with the differentials and hence constitute an isomorphism of chain complexes

$$A\otimes C(Y,\mathbb{Z})\xrightarrow{\cong}C(Y,A).$$

this applies in particular to
finish proof ☐

Example: We showed that

$$H_n(\mathbb{R}P^\infty,\mathbb{Z})=\begin{cases}\mathbb{Z} & n=0\\ \mathbb{Z}/2\mathbb{Z} & n\text{ odd}\\ 0 & \text{otherwise}\end{cases}$$

We can now calculate this for the general coefficient using the universal coefficient theorem.

$$\begin{aligned}H_n(\mathbb{R}P^\infty,A)&=(A\otimes H_n(\mathbb{R}P^\infty,\mathbb{Z}))\otimes\mathrm{Tor}(A,H_{n-1}(\mathbb{R}P^\infty,\mathbb{Z}))\\&=\begin{cases}A & n=0\\ A\otimes\mathbb{Z}/2\mathbb{Z}=A/2A & n\text{ odd}\\ \mathrm{Tor}(A,\mathbb{Z}/2\mathbb{Z})\cong\{a\in A:2a=0\} & \text{otherwise}\end{cases}\end{aligned}$$

Example: $n\geq1$ since $H_{n-1}(S^n,\mathbb{Z})$ is either $\mathbb{Z}$ or 0, it is free and so the homomorphism $\Phi:A\otimes H_n(S^n,\mathbb{Z})\to H_n(S^n,A)$ is an isomorphism because Φ is natural in X, for every continuous $f:S^n\to S^n$, the following square commutes So we conclude that $H_n(f,A)$ is

$$\begin{tikzcd}A\otimes H_n(S^n,\mathbb{Z}) \arrow[r, "\Phi"] \arrow[d, "A\otimes H_n(f,\mathbb{Z})"] & H_n(S^n,A) \arrow[d, "H_n(f,A)"]\\ A\otimes H_n(S^n,\mathbb{Z}) \arrow[r, "\Phi"] & H_n(S^n,A)\end{tikzcd}$$

multiplication by the degree of f.

Definition 45.1. An abelian group A is torsion free if the following holds. For every $n\geq1$ and $a\in A$ such that $n\cdot a=0$, we have $a=0$.

Example: Every free abelian group is torsion free.

Proposition 45.2. *Let A and B be abelian groups one of which is torsio free. Then $Tor(A,B)=0$.*

Proof. finish proof ☐

Since $\mathbb{Q}$ is torsion free, $\mathrm{Tor}(\mathbb{Q},B)=0$ for all abelian groups B. By thr universal coefficient theorem, for every space X the map

$$\Phi:\mathbb{Q}\otimes H_n(X,\mathbb{Z})\to H_n(X,\otimes\mathbb{Q})$$

is an isomorphism. Every finitely generated abelian group A is isomorphic $\mathbb{Z}^r\oplus T$ for some $r\leq0$ and a finite torsion group T. The number r is the rank of A. Since

$$\mathbb{Q}\otimes A\cong\mathbb{Q}\otimes(\mathbb{Z}^r\oplus T)\cong(\mathbb{Q}\otimes\mathbb{Z}^r)\oplus(\mathbb{Q}\otimes T)\cong\mathbb{Q}^r$$

so $\mathrm{rank}(A)=\dim_{\mathbb{Q}}(\mathbb{Q}\otimes A)$.

Corollary 45.3. *Let X be a finite CW-complex. Then $H_n(X, \mathbb{Z}) = 0$ for $n > \dim(X)$, and $H_n(X, \mathbb{Z})$ is finitely generated for all $n \geq 0$. Moreover,*

$$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot rank(H_n(X < \mathbb{Z}))$$

Proof. do it later □

46 Things to Take away from Worksheets

46.1 Worksheet 1

From **Question 2** we have that $\mathbb{R}P^{n+1} \cong \mathbb{R}P^n \cup_{\partial D^{n+1}} \mathbb{R}P^n$ and hence

Theorem 46.1. *The real projective space admits an absolute CW structure*

$$\emptyset = \mathbb{R}P^{-1} \cup \mathbb{R}P^0 \subset \mathbb{R}P^1 \subset ... \subset \mathbb{R}P^m.$$

And similarly from **Question 3** we have that we can obtain $\mathbb{C}P^{m+1}$ from $\mathbb{C}P^m$ by attaching a $2(m+1)$ cell and so

Theorem 46.2. *$X = \mathbb{C}P^m$ is a CW structure with $X_0 = X_1 = *$, $X_2 = X_3 = \mathbb{C}P^1$, $X_4, X_5 = \mathbb{C}P^2$ etc.*

46.2 Worksheet 2

46.3 Worksheet 3

46.4 Worksheet 4

Theorem 46.3. *(Lifting theorem for covering spaces) Let $p : (E, e) \to (X, x_0)$ be a based covering map and $f : (Y, y_0) \to (X, x_0)$ a based map with Y connected and locally path connected then there is a unique lift $\tilde{f} : (Y, y_0) \to (E, e_0)$, such that $p \circ \tilde{f} = f$ if and only if $f_*(\pi_1(Y, y_0)) \subset p(\pi_1(E, e))$*

Part II

Topology 2