Functions of Several Complex Variables

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Disclaimer: I hereby confirm that much of this document is not my own words, but the words of the professor for the course, Dr. Ingo Lieb, one of the authors in the bibliography, or somewhere random where I saw an insight. What is and is not my own words will not be mentioned, because this text is only meant to be for my own learning and possibly that of friends.

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CONTENTS

INTRODUCTION

The main focus of this course will be holomorphic functions and quotients of holomorphic functions, i.e. meromorphic functions, with several complex variables. Their study is justified not only by the deep and fascinating theory surrounding them, but also because they appear naturally in many areas of math and science. In these lecture notes we will only see the very beginnings of a rich theory; namely, first we will see how results from the single variable theory extend to the multiple variable setting, then we will study analytic functions in the context of algebraic geometry, and we will finally close with an investigation of the link between partial differential equations and functions of several complex variables. For the time being, we quickly observe the aforementioned relevance to a number of subjects.

In algebraic geometry we consider solutions to polynomial equations, i.e. zero sets, vanishing ideals, and the underlying geometry. For example,

$$\{(z_1, z_2, z_3) : z_1^2 + z_2^3 + z_3^5 = 0\} \subset \mathbb{C}^3$$

is a 2D hypersurface in $\mathbb{C}^3$. It is an example of an affine algebraic variety (it is obviously the zero set of a polynomial although irreducibility is harder to show). This concept of course also extends to zero sets for systems of polynomials. There is a rich theory using just this basic concept, but as mathematicians, we generalize. A variety X has a generating set $(f_1, \ldots, f_k) = \mathfrak{a} \subset \mathbb{C}[z_1, \ldots, z_n]$, which then allows us to define the coordinate ring, $\mathbb{C}[z_1, \ldots, z_n]/\mathfrak{a}$, which can often be studied instead of the original variety, X. Replacing $\mathbb{C}$ by another algebraically closed field (or even a general one) and the coordinate ring by a general ring leads to the idea of a scheme. In particular, in complex analysis, we wish to replace polynomials by holomorphic functions. In particular, we say X is an analytic set in $U \subset \mathbb{C}^n$ if the set $\{z : f_1(z) = \cdots = f_k(z) = 0\}$ for $(f_j)_{j=1}^k$ holomorphic on U. Consult [**grauert2012coherent**] or chapter 1 for details.

Holomorphic functions of several complex variables also show up naturally in differential equations. Suppose we wish to study a differential equation in just one variable, e.g.

$$\begin{cases} w' = f(z, w) \\ w = \varphi(z). \end{cases}$$

Then we will likely want at least a condition along the lines of $f(z, w)$ being holomorphic, hence we have a function with two complex variables. If we consider some initial value problem by adding the condition $\varphi(z_0) = w_0$, then we further have $w = \varphi(z; z_0, w_0)$, so we may even achieve three complex variables here. More concretely, one of the most important families of differential equations is the so-called hypergeometric differential equations:

$$z(z-1)w'' + (c - (a+b+1)z)w' - abw = 0.$$

The solution, $w = F(z; a, b, c)$ is the hypergeometric series, and is a function of four complex variables.

The relation to PDE is even more apparent if one has studied the single variable theory for complex variables. Namely, as was in the single variable case, we have the so-called Cauchy-Riemann equations relating holomorphic functions to harmonic functions. Explicitly, a function $f(z_1, \ldots, z_n)$ is holomorphic if and only if

$$\frac{\partial f}{\partial \bar{z}_\nu} = 0$$

for $\nu = 1, \ldots, n$. Note that this is an overdetermined elliptic system (if it were not then holomorphicity would be a rather loose and useless concept). In any case, this displays one way in which the study of holomorphic functions can give information about PDE. See [**range1998holomorphic**] or chapter 2 for details.

A problem that should look familiar is the problem of establishing the following antiderivative:

$$\int \frac{dx}{\sqrt{y}},$$

where y is a quadratic polynomial in x. We of course know that these are linked to the trigonometric functions, or, more specifically, the inverses of the trigonometric functions. This begs the question of what happens when y is a cubic polynomial? Or even a polynomial of higher degree? This question goes all the way back to Euler and Fagnano before the year 1800 and it was further studied by such famous mathematicians as Abel, Jacobi, Gauss, and Legendre. The clever idea is to look not at the actual antiderivatives, but their inverses (as the inverses are much clearer to us, at least in the case where y is quadratic). In the cubic and quartic case, the integrals are called elliptic integrals and their inverse functions are called elliptic functions—one may show that they are meromorphic and doubly periodic. Furthermore, by identifying sides of the parallelograms created by double periodicity, we may think of elliptic functions as functions on a torus, which is a compact Riemann surface.

Abelian integrals are then just the integrals given by having y be of degree greater than 4. For example, when they are of degree 5 or 6, after inversion we run into a problem; we can show that they will have to have 4 independent periods, which obviously cannot happen in $\mathbb{C}$. This is the famous Jacobi inversion problem, which was solved by Riemann. As one may guess from the results with cubic and quartic polynomials, meromorphic functions on the 2-torus (i.e. 4 real dimensions) give information on these integrals. Consult [**forster2013riemannsche**] for more details and the general statements of these problems.

Finally, as an honorable mention, functions of several complex variables are pervasive in the sciences, especially, as one may guess, in physics. One concept for the interested reader to investigate along these lines is that of the Feynman integral in quantum electrodynamics. We will not make any further mention of such applications in this course.

CHAPTER
ZERO

HOLOMORPHIC FUNCTIONS

In this chapter we will establish first concepts in the study of functions of several complex variables, i.e. we will decide on the degrees of freedom that such functions can have that we would like to understand. These will include concepts of the unit ball, of differentiability and holomorphicity, and the theory that these concepts entail. Special attention will be given to those theorems which may be extended to the several variable setting from the single variable context.

0.0 Preliminary concepts

0.0.1 Topology and norms

We will be working in the ambient domain $\mathbb{C}^n = \{z = (z_1, \ldots, z_n) : z_\nu \in \mathbb{C}\}$, which is an n-dimensional complex vector space (i.e. a $2n$-dimensional real vector space). We can further write each $z_\nu = x_\nu + iy_\nu$ to think of our complex numbers in terms of real numbers. In this case we will write

$$z = (z_1, \ldots, z_n) = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n).$$

Of course, we could choose some other orientation, but this will be the convenient one. In terms of topology, we will take the usual one (from $\mathbb{R}^{2n}$), so we have the normal concepts of convergence, continuity, compactness, etc.

It is also important to think about the size of numbers in our complex space. All norms on a finite dimensional vector space are, as we know, equivalent, but the simplifications in computation and the perspective that a different norm may lend will be of use. For this reason, we will think of the size of $z \in \mathbb{C}^n$ in the following two ways:

$$\|z\|_\infty = \max_{\nu=1,\ldots,n} |z_\nu| \quad \text{and} \quad \|z\|_2^2 = \sum_{\nu=1}^{n} |z_\nu|^2.$$

The most apparent change that this yields in our perspective is in how we think of the unit ball, which one may recall is of fundamental importance in the single variable theory. Using the maximum norm above, we shall write

$$\mathbb{D}^n = \{z : \|z\|_\infty < 1\} = \{|z_\nu| < 1 \text{ for all } \nu\} = \mathbb{D}^1 \times \cdots \times \mathbb{D}^1.$$

This notion of a unit ball will be called an n-dimensional unit polydisc (we may omit the n and just write $\mathbb{D}$ if the dimension of the ambient domain is clear from context). On the other hand, we will use the Euclidean norm to define what we will call the unit ball of dimension $2n$:

$$\mathbb{B}^n = \{\|z\|_2 = 1\}.$$

Turning back to the polydiscs, we shall write, more generally, for $a \in \mathbb{C}^n$ and $r = (r_1, \ldots, r_n)$ with $r_\nu > 0$

$$D_r(a) = \{z : |z_\nu - a_\nu| < r_\nu\},$$

which is a neighborhood base of a. Most of the time, we will just fix $r = (\varepsilon, \ldots, \varepsilon)$ and write $D_\varepsilon(a) = \{z : |z_\nu - a_\nu| < \varepsilon\}$. If the components are not clear, then they may safely be assumed to all be the same.

Moving back to topology, we have the topological notion of the boundary for $M \subset \mathbb{C}^n$, i.e. $\partial M = \overline{M} \setminus \mathring{M}$. For example, the boundary of the polydisc is

$$\begin{aligned}\partial\mathbb{D} &= \{z : \|z\|_\infty = 1\} \\ &= \{(z_1, \ldots, z_n) : |z_\nu| \leqslant 1 \text{ for all } \nu \text{ and there exists } \nu \text{ with } |z_\nu| = 1\},\end{aligned}$$

which is not a smooth boundary. Some authors prefer the notation bM for the boundary due to the possibility of confusion with derivatives, but this author finds this notation ugly enough that the confusion is worth it.

In an attempt to consider some notion of a boundary, but maintaining smoothness, we define the distinguished boundary of a polydisc as

$$\partial_0 \mathbb{D} = \{z : |z_\nu| = 1 \text{ for all } \nu\} = S^1 \times \cdots \times S^1 = \mathbb{T}^n \subset \partial\mathbb{D},$$

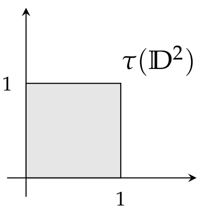
where $\mathbb{T}^n$ is the n-torus. We are in a simpler situation with the ball:

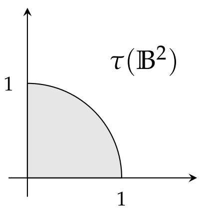
$$\partial\mathbb{B} = \{\|z\|_2 = 1\} = S^{2n-1},$$

which is smooth.

0.0.2 Visualization

We define the map $\tau : \mathbb{C}^n \to \mathbb{R}^n_+ = \{r = (r_1, \ldots, r_n) \in \mathbb{R}^n : r_j \geqslant 0 \text{ for all } j\}$ taking $z \mapsto (|z_1|, \ldots, |z_n|)$. It will, as one may wonder, be important that we include $\mathbb{R}^{n-1}$ in this so-called absolute space. It is simple to prove the continuity of τ as well as to show that it is an open map. We merely define this map here for the purposes of visualization. As a remark, it is important in the theory, e.g. in defining Reinhardt domains. To give some intuition, we illustrate where it sends some familiar objects, like $\mathbb{D}^2$ and $\mathbb{B}^2$.



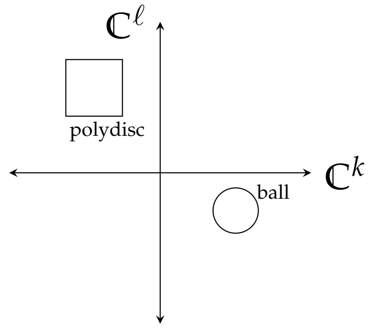


In general, the fiber of $r \in \mathbb{R}^n_+$ is

$$\tau^{-1}(r) = \{(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}) : 0 \leqslant \theta_j \leqslant 2\pi\},$$

which is an n-dimensional real torus.

One last remark on the ambient space and its topology should be given on visualization. Since $\mathbb{C}^2$ already has 4 real dimensions, this poses a problem for any kind of mental image. For that reason, we sometimes, in $\mathbb{C}^{k+\ell} = \mathbb{C}^k \times \mathbb{C}^\ell$, purely for intuition, draw polydiscs as squares and balls as circles in a plane where one axis is $\mathbb{C}^k$ and the other is $\mathbb{C}^\ell$. See below.



0.0.3 Complex differentiability

Now if we consider a map $F : M \subset \mathbb{C}^n \to \mathbb{C}^m$ with $F = (f_1, \ldots, f_m)$ where $f_\nu : M \to \mathbb{C}$, then as mentioned before, the normal notions of continuity, etc. apply. By further writing $f_\nu = g_\nu + ih_\nu$ where $g_\nu, h_\nu : M \to \mathbb{R}$ we can say that F is (real) differentiable if all of the f_ν are, which we define to be exactly when each of g_ν, h_ν are.

Now we wish to replicate a theorem from basic analysis to describe differentiability. Recall that if $f : [a, b] \to \mathbb{R}$ and $x_0 \in [a, b]$ then the following are equivalent:

(i) f is differentiable at x_0 and

(ii) there exists $\alpha \in \mathbb{R}$ and an error function $\Delta : [a, b] \to \mathbb{R}$ such that Δ is uniquely determined at x_0 by f and continuous at x_0 satisfying

$$f(x) - f(x_0) = \Delta(x)(x - x_0).$$

This will of course be more complicated in our context, but we see below that the spirit is maintained. In particular, suppose $U \subset \mathbb{C}^n$ is open and $f : U \to \mathbb{C}$. Take $z^0 \in U$. Then we say that f is differentiable at z^0 if there exist $2n$ functions, $\Delta_1, \ldots, \Delta_n, E_1, \ldots, E_n$ all continuous at z^0 such that

$$f(z) - f(z^0) = \sum_{\nu=1}^{n} \Delta_\nu(z)(z_\nu - z_\nu^0) + \sum_{\nu=1}^{n} E_\nu(z)(\overline{z}_\nu - \overline{z}_\nu^0).$$

Note that, as in the more simple case, the functions Δ_ν, E_ν are uniquely determined by f only at z^0. The above can easily be transfered to a somewhat more familiar definition of differentiability using difference quotients, but we prefer this one since it skips any messy division. The simple proof is left to the reader.

Other simple observations to make (whose proof we will not give) are that

$$\Delta_\nu(z^0) = \frac{\partial f}{\partial z_\nu}(z^0) \quad \text{and} \quad E_\nu(z^0) = \frac{\partial f}{\partial \bar{z}_\nu}(z^0).$$

We of course define the Wirtinger derivatives above as in the single variable theory:

$$\frac{\partial f}{\partial z_\nu} = \frac{1}{2}\left(\frac{\partial f}{\partial x_\nu} - i\frac{\partial f}{\partial y_\nu}\right) \quad \text{and} \quad \frac{\partial f}{\partial \overline{z_\nu}} = \frac{1}{2}\left(\frac{\partial f}{\partial x_\nu} + i\frac{\partial f}{\partial y_\nu}\right).$$

The above derivatives should be interpreted in complex components, i.e. $f = g + ih$ and

$$\frac{\partial f}{\partial x_\nu} = \frac{\partial g}{\partial x_\nu} + i\frac{\partial h}{\partial x_\nu},$$

etc.

With these definitions in hand, we make a few notes that are useful for computations. We have

$$\frac{\partial z_\nu}{\partial z_\mu} = \delta_{\nu\mu}$$

and the three other obvious similar identities with the conjugate derivatives. We also can easily show that the Wirtinger derivatives commute and that

$$\frac{\partial f}{\partial z_\nu} = \overline{\frac{\partial \overline{f}}{\partial \bar{z}_\nu}}.$$

The chain rule is as one may expect. Finally, we will write $C^k(U)$ to denote the functions on an open set $U \subset \mathbb{C}^n$ which are k-times real differentiable. See
[**kaup2011holomorphic**] for more details on some of these basic considerations.

0.1 Holomorphic functions

0.1.1 Definitions and basic theorems

Definition 0.1.1. *Let $U \subset \mathbb{C}^n$ be open and $f : U \to \mathbb{C}$. We say that f is complex differentiable at z^0 if there are n functions $\Delta_1, \ldots, \Delta_n$, continuous at z^0, such that*

$$f(z) - f(z^0) = \sum_{\nu=1}^{n} \Delta_\nu(z)(z_\nu - z_\nu^0),$$

where Δ_ν is uniquely determined by f at z^0. Further, we say f is holomorphic in U if f is complex differentiable on all of U.

The salient point to note is that holomorphicity requires an open set, whereas differentiability can occur at merely a point. Remark also, that the striking difference above to real differentiability is that we disallow, in a way, f to have a conjugate part, which one may recall is what we do in the single variable theory as well. It is, thus, plain to see that complex differentiable implies real differentiable. The converse is obviously not true; one need only look at the conjugate of a complex differentiable function to see that it is still real differentiable but not complex differentiable (e.g. z_ν is complex differentiable, but $\overline{z_\nu}$ is only real differentiable).

We may define differentiability in a more familiar way as well. Suppose $U \subset \mathbb{C}^n$ is open, $f : U \to \mathbb{C}^m$, and $a \in U$. We use $\| \cdot \|$ to mean any norm on $\mathbb{C}^n$. In any case, f is said to be complex differentiable at a if for every $\varepsilon > 0$, there exists $\delta = \delta_{\varepsilon,a} > 0$ and a $\mathbb{C}$-linear mapping $Df(a) : \mathbb{C}^n \to \mathbb{C}^m$ such that for all $z \in U$ with $\|z - a\| < \delta$, the inequality

$$\|f(z) - f(a) - Df(a)(z - a)\| \leqslant \varepsilon \|z - a\|$$

holds. If $Df(a)$ exists, it is called the complex derivative of f at a. The equivalence of this to our other definition is easy and left to the reader.

Definition 0.1.2. *For $U \subset \mathbb{C}^n$ open and $F : U \to \mathbb{C}^n$, we say F is holomorphic if all components are holomorphic.*

Briefly take note that all of the normal rules of calculus will follow in much the same way as they do in the real variable theory (e.g. product rule, sums of differentiable functions are differentiable, etc.).

Now that we have defined the types of functions that we are interested in, we proceed to that part of the theory which may be extended from the single variable case. Because holomorphicity requires an open set, if we say f is holomorphic on $E \subset \mathbb{C}^n$, we mean it is holomorphic in some open neighborhood (if E is already open, then we just take E itself).

Theorem 0.1.3. *We may define holomorphicity differently: f is holomorphic if and only if it is real differentiable and satisfies the Cauchy-Riemann equations in each variable, i.e.*

$$\frac{\partial f}{\partial \bar{z}_\nu} = 0$$

for $\nu = 1, \ldots, n$. Furthermore, if f is holomorphic in $(z_1, \ldots, z_n)$ then it is also holomorphic in each variable individually (and each subset of variables).

Remark 0.1.1. We can, somewhat surprisingly, reverse the last statement. In particular, Osgood's lemma says that a continuous function of several complex variables, which is holomorphic individually in each variable, is in fact holomorphic. Further, Hartogs' theorem on separate holomorphicity says that a function of several complex variables, which is holomorphic in each variable separately, is continuous. Combining these gives the result. The former is not too hard to show, but the latter is a rather deep result. See **[hormander1973introduction]**.

Unfortunately, Hartogs' result, while deep, has no interesting applications whatsoever in complex analysis; it is just a precious pearl for its own sake.

0.1.2 Results from single variable theory

Moving on, the first theorem that one may wish to have is Cauchy's integral formula. At least in the case of the polydisc, we may simply extend the result from the single variable theory. Suppose that f is holomorphic in an open neighborhood of $\overline{D}_r(a)$. Since holomorphicity implies that a function is holomorphic on each variable, we have, for $z \in D_r(a)$

$$\begin{aligned}
f(z_1, \dots, z_n) &= \frac{1}{2\pi i} \int_{|\zeta_1 - a_1| = r_1} \frac{f(\zeta_1, z_2, \dots, z_n)}{\zeta_1 - z_1} \, d\zeta_1 \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{|\zeta_1 - a_1| = r_1} \int_{|\zeta_2 - a_2| = r_2} \frac{f(\zeta_1, \zeta_2, z_3, \dots, z_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \, d\zeta_2 d\zeta_1 \\
&= \left(\frac{1}{2\pi i}\right)^n \int_{\partial_0 D_r(a)} \frac{f(\zeta)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \, d\zeta.
\end{aligned}$$

We thus, for notational convenience, define the Cauchy kernel,

$$C(\zeta, z) = \left(\frac{1}{2\pi i}\right)^n \frac{d\zeta}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}$$

for $\zeta_\nu \neq z_\nu$. We have, hence, proven the following theorem.

Theorem 0.1.4. *Suppose f is holomorphic on an open neighborhood of $\overline{D}_r(a)$. Then*

$$f(z) = \int_{\partial_0 D_r(a)} f(\zeta) C(\zeta, z) \, d\zeta$$

for $z \in D_r(a)$. The order of integration does not matter. The theorem, in fact, holds if f is holomorphic on $D_r(a)$ and continuous up to the boundary. Conversely, if $D_r(a)$ is a polydisc and h is a continuous function on $\partial_0 D_r(a)$ with

$$f(z) = \int_{\partial_0 D_r(a)} h(\zeta) C(\zeta, z) \, d\zeta$$

for $z \in D_r(a)$, then f is holomorphic in $D_r(a)$.

The only step in the proof for the converse is in justifying differentiation under the integral sign. Using this, we can prove some Cauchy estimates as in the real case. See
[range1998holomorphic].

Remark 0.1.2. We can, as in the single variable case, differentiate Cauchy's integral formula. Namely, if f is holomorphic in an open neighborhood of $\overline{\mathbb{D}}$, then for $z \in \mathbb{D}$ and the multiindex, $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}_0^n$,

$$\partial^\nu f(z) = \frac{\nu!}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}} \frac{f(\zeta)}{(\zeta - z)^{\nu+1}} \, d\zeta.$$

A number of other results follow easily from the single variable theory. The following lemma is one of the essential observations in proving these extensions.

Lemma 0.1.5. *Let $U \subset \mathbb{C}^n$ be open, $a \in U$, f holomorphic in U, and $b \in \mathbb{C}^n$. Define*

$$V_{a,b;U} = \{t \in \mathbb{C} : a + tb \in U\}.$$

Then V is open in $\mathbb{C}$, contains 0, and $g_{a,b} : V \to \mathbb{C}$ defined by $t \mapsto f(a + tb)$ is holomorphic.

Proof. That $0 \in V$ is immediate since $a \in U$. Suppose $b \neq 0$ (otherwise $V = \mathbb{C}$, so it is open). Take $t_0 \in V$ and call $z_0 = a + t_0 b \in U$. We know U is open, so there exists an ε-ball, $\mathbb{B}_\varepsilon(z_0) \subset U$. Putting $z_t = a + tb$, we note $|z_0 - z_t| = \|b\| |t_0 - t| < \varepsilon$ for all t with $|t_0 - t| < \varepsilon / \|b\|$. In other words,

$$B_{\varepsilon/\|b\|}(t_0) \subset V,$$

i.e. V is open. Holomorphicity is obvious, since g is merely a composition of a linear mapping with a holomorphic function. That is, use the chain rule. □

Theorem 0.1.6. *The following results hold. Suppose $G \subset \mathbb{C}^n$ is a region (nonempty, open, connected).*

(i) (Identity theorem) If f is holomorphic in G and identically zero on a nonempty, open subset of G, then f is the zero function on G.

(ii) Holomorphicity implies infinite differentiability and all derivatives are holomorphic.

(iii) (Weierstrass) If $(f_j)_{j \in \mathbb{N}}$ are holomorphic and $f_j \to f$ uniformly on compacta, then f is holomorphic and all derivatives tend to the derivatives of f uniformly on compacta.

(iv) If f is holomorphic on an open set U, then f can be decomposed locally into a power series about each $a \in U$.

(v) (Liouville) If f is holomorphic in $\mathbb{C}^n$ and bounded then f is constant.

(vi) (Open mapping theorem) Let $G \subset \mathbb{C}^n$ be a region and $U \subset G$ be open. If $f : G \to \mathbb{C}$, holomorphic on G, is nonconstant, then $f(U)$ is open, i.e. every holomorphic function is an open mapping. In particular, $f(G)$ is a region in $\mathbb{C}$.

(vii) (Maximum principle) If f is holomorphic in G, and there exists $a \in G$ such that $|f(a)|$ is a local maximum for $|f|$, then f is constant.

Proof. We will not prove all of these statements. For proofs of (i), (vi), and (vii) see
[scheidemann2005introduction] (they also make for good exercises). For (ii), (iii), and (iv), the proofs are essentially the same as in the case $n = 1$, so they are omitted.

To prove Liouville's theorem (and demonstrate a technique in a simple context), we first take $a, b \in \mathbb{C}^n$. The function $g_{a,b-a}$ from lemma 0.1.5 is holomorphic on $\mathbb{C}$ and satisfies

$$g_{a,b-a}(0) = f(a), \quad g_{a,b-a}(1) = f(b).$$

Further, $g_{a,b-a}(\mathbb{C}) \subset f(\mathbb{C}^n)$. Thus, since f is bounded, so is $g_{a,b-a}$, which means we can apply the one variable version of Liouville's theorem to $g_{a,b-a}$ to say it is constant. But in this case,

$$f(a) = g_{a,b-a}(0) = g_{a,b-a}(1) = f(b).$$

Since $a, b \in \mathbb{C}^n$ were arbitrary, we are done. □

Remark 0.1.3. The open mapping theorem does not extend to $f : \mathbb{C}^n \to \mathbb{C}^m$ holomorphic. For example,

$$f : \mathbb{C}^2 \to \mathbb{C}^2, \quad (z, w) \mapsto (z, zw)$$

is holomorphic and not open.

Remark 0.1.4. Equivalent to uniform convergence on compacta is that for all $a \in U$ there exists an open neighborhood V of a so that we attain uniform convergence on V (using notation from the theorem statement).

Remark 0.1.5. We can easily show that the maximum modulus principle holds for bounded regions in the obvious way. That is, if $G \subset \mathbb{C}^n$ is a bounded region and $f : \overline{G} \to \mathbb{C}$ is holomorphic on G and continuous up to the boundary, then $|f|$ attains a maximum on the boundary ∂G. The proof is just that $\overline{G}$ is compact, so $|f|$ attains a maximum and if that maximum is attained in the interior then f is constant on G, but by continuity then on $\overline{G}$.

Recall that in the single variable setting, the identity theorem states that we only need to know that the zero set has a cluster point; this is no longer sufficient here. For example, the holomorphic function $f : \mathbb{C}^2 \to \mathbb{C}$ mapping $(z, w) \mapsto zw$ is not identically zero, yet its zero set clearly has cluster points.

0.1.3 The space $\mathcal{O}(G)$ and Hartogs' Kugelsatz

Moving on from these basic results, suppose G is a region. Then we define $\mathcal{O}(G)$ to be all holomorphic functions on G. One may show without much difficulty that this is a $\mathbb{C}$-algebra. We would like to show that $\mathcal{O}(G)$ is a Fréchet algebra, i.e. an associative algebra over $\mathbb{C}$, which is also a Fréchet space where the multiplication operation is required to be jointly continuous. We review the concept of a Fréchet space a bit before proceeding.

Definition 0.1.7. *A topological vector space, X, is said to be a Fréchet space if it satisfies the following three conditions*

(i) X is Hausdorff,

(ii) the topology of X can be induced by a countable family of seminorms, $(\|\cdot\|_k)_{k\in\mathbb{N}}$, i.e. $U \subset X$ is open if for every $u \in U$ there exists $K \in \mathbb{N}$ and $r > 0$ so that $\{v \in X : \|v - u\|_k < r \text{ for all } k \geqslant K\} \subset U$, and

(iii) X is complete with respect to the family of seminorms.

Alternatively, a Fréchet space is a vector space that is a complete metrizable topological space so that the vector space operations are continuous.

Of course, a sequence converges in X above if and only if it converges with respect to each seminorm. A useful fact is that a family of seminorms, $\mathscr{P}$, generates a Hausdorff topological space if and only if

$$\bigcap_{\|\cdot\|\in\mathscr{P}} \{x \in X : \|x\| = 0\} = \{0\}.$$

See **[conway2019course]** for more details.

Note that if we take $K \subset G$ to be compact, then

$$|f|_K = \max_{z\in K} \|f(z)\|_\infty$$

defines a seminorm on $\mathcal{O}(G)$. This transforms $\mathcal{O}(G)$ into a topological $\mathbb{C}$-algebra. In fact, we can define a basis. A neighborhood of zero in $\mathcal{O}(G)$ is

$$V(K, \varepsilon) = \{f \in \mathcal{O}(G) : |f|_K < \varepsilon\}$$

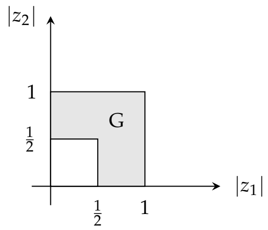
with K compact. Letting K compact and $\varepsilon > 0$ vary we have that $\{V(K, \varepsilon)\}_{K,\varepsilon}$ is a basis for $\mathcal{O}(G)$.

Theorem 0.1.8. *$\mathcal{O}(G)$ is a Fréchet algebra where convergence is uniform convergence on compacta.*

Proof. The set of seminorms is of countable type since the topology on G has a countable basis. Thus, there is an exhaustion, $K_1 \subset K_2 \subset \cdots$, of compact subsets of G. In particular, every compact subset of G is then contained in some K_i (and convergence with respect to one of these seminorms takes place if and only if we have uniform convergence on the compact set given in the seminorm). Convergence of Cauchy sequences then follows since a sequence of holomorphic functions converging uniformly on compacta gives a holomorphic function. Details are omitted. □

Note also that $\mathcal{O}(G)$ is complete (exercise) and an integral domain (by the identity theorem). In the latter, it is of course essential that G is connected. Of course, $\mathcal{O}(G)$ is also metrizable (since all Fréchet spaces are metrizable).

With this definition we make one new observation that does not hold in the case of one variable to close the section. Take $\mathbb{D} \subset \mathbb{C}^2$ and let $G = \mathbb{D} \setminus \overline{D}_{1/2}(0)$, pictured symbolically below.



Warning: Note that the diagram above can look a bit deceiving, since it seems that neither variable can get close to zero, but in fact they both can; it is just that $\max(|z_1|, |z_2|)$ needs to stay larger than $1/2$. This is often the case that diagrams drawn for several complex variables are misleading, so one should be careful.

In any case, for $f \in \mathcal{O}(G)$, we can write

$$f(z_1, z_2) = \frac{1}{2\pi i} \int_{|\zeta_2|=1-\varepsilon} \frac{f(z_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 =: J(z_1, z_2).$$

Note that, while f is only defined on G, J is defined for all $|z_1| < 1$, $|z_2| < 1 - \varepsilon$ and is holomorphic. In any case, we may therefore extend f to all of $\mathbb{D}$. We have the following deep generalization.

Theorem 0.1.9 (Hartogs' Kugelsatz). *Let $U \subset \mathbb{C}^n$ be open, $n > 1$, and $K \subset U$ compact such that $U \setminus K$ is connected. Then $\mathcal{O}(U \setminus K) = \mathcal{O}(U)$. More precisely, the restriction map $\rho : \mathcal{O}(U) \to \mathcal{O}(U \setminus K)$ is an isomorphism of $\mathbb{C}$-algebras.*

Morally, among other things, this theorem is saying that holomorphic functions (for $n > 1$) have no isolated singularities (or zeros by looking at reciprocals). Perhaps more suggestively put, the zero set of a holomorphic function is never compact (unless it is empty). For this, we need that if $G \subset \mathbb{C}^n$ is a region and A is an analytic subset of G with codimension at least 1, then $G \setminus A$ is connected. Then combined with the fact that the zero set of a holomorphic function of more than one variable has codimension 1, we get the result. We will return to Hartogs' Kugelsatz and see a proof in chapter 2. For now, check
[**scheidemann2005introduction**] for proofs of some consequences and more exposition on this powerful theorem.

Remark 0.1.6. The statement of the consequence does not hold for real analytic functions. For example, $f : \mathbb{C}^n \to \mathbb{C}$ mapping $z \mapsto \|z\|_2^2 - 1$ has zero set exactly equal to the compact unit sphere in $\mathbb{C}^n$.

0.2 Holomorphic maps

0.2.1 Jacobians and biholomorphicity

Definition 0.2.1. *Take $U \subset \mathbb{C}^n$ open and $F : U \to \mathbb{C}^m$. Then F is called a holomorphic map if for a holomorphic function f on a subset of $F(U)$, we always have $f \circ F$ holomorphic on U. Reformulated, $F = (f_1, \ldots, f_m)$ where f_ν is holomorphic for each ν.*

Indeed, if $m = 1$, then a holomorphic map is just a holomorphic function.

Definition 0.2.2. *We say that $F : U \to V$ where $U, V \in \mathbb{C}^n$ is biholomorphic if F^{-1} exists and is holomorphic.*

Consider some $F : U \to V$ with $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$, which is real differentiable. We write $F = (f_1, \ldots, f_m)$ and further $f_\mu = g_\mu + ih_\mu$. Then we have several Jacobians to look at. Namely,

$$J_F^{\mathbb{R}} = \begin{pmatrix} \dfrac{\partial g_\mu}{\partial x_\nu} & \dfrac{\partial h_\mu}{\partial x_\nu} \\ \\ \dfrac{\partial g_\mu}{\partial y_\nu} & \dfrac{\partial h_\mu}{\partial y_\nu} \end{pmatrix} \in \mathbb{R}^{2m \times 2n}, \quad J_F^{\mathbb{C}} = \begin{pmatrix} \dfrac{\partial f_\mu}{\partial z_\nu} & \dfrac{\partial \overline{f_\mu}}{\partial z_\nu} \\ \\ \dfrac{\partial f_\mu}{\partial \overline{z_\nu}} & \dfrac{\partial \overline{f_\mu}}{\partial \overline{z_\nu}} \end{pmatrix} \in \mathbb{C}^{2m \times 2n},$$
$$J_F^h = \left(\frac{\partial f_\mu}{\partial z_\nu} \right) \in \mathbb{C}^{m \times n}.$$

The final one we only use for F holomorphic (hence the h). A few observations are in order. Take $m = n$. Then $\mathrm{rk}(J_F^{\mathbb{R}}) = \mathrm{rk}(J_F^{\mathbb{C}})$ and $\det(J_F^{\mathbb{R}}) = \det(J_F^{\mathbb{C}})$ (these are easy computations, but good exercises). Further, if F is holomorphic, then

$$J_F^{\mathbb{C}} = \begin{pmatrix} J_F^h & 0 \\ 0 & \overline{J_F^h} \end{pmatrix},$$

which is a simple consequence of the Cauchy-Riemann equations. Hence,

$$\det(J_F^{\mathbb{C}}) = \det(J_F^h)\det(\overline{J_F^h}) = \det(J_F^h)\overline{\det(J_F^h)} = |\det(J_F^h)|^2 \geqslant 0.$$

Thus, biholomorphic maps respect orientation. Additionally, we note that $J_F^{\mathbb{R}}$ and $J_F^{\mathbb{C}}$ essentially carry the same information, as one may assume. We also have the following preliminary proposition, which we will improve later.

Proposition 0.2.3. *Suppose $F : U \to V$ is holomorphic ($U, V \subset \mathbb{C}^n$ open), bijective, and J_F^h is everywhere regular (nonzero determinant). Then F is biholomorphic.*

Proof. From real analysis results, we get that F^{-1} is differentiable. Now,

$$J_F^{\mathbb{C}} = \begin{pmatrix} J_F^h & 0 \\ 0 & \overline{J_F^h} \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & \overline{M} \end{pmatrix}.$$

Write

$$J_{F^{-1}}^{\mathbb{C}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then we note

$$J_F^{\mathbb{C}} J_{F^{-1}}^{\mathbb{C}} = \begin{pmatrix} M & 0 \\ 0 & \overline{M} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix}.$$

Simple computations show that $B = C = 0$. Since C encodes the Cauchy-Riemann equations for F^{-1}, we are done. □

Now we recall the implicit function theorem and the submanifold theorem in our context.

Theorem 0.2.4 (Implicit function theorem). *Take $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ both open and containing zero. Let $F : U \times V \to \mathbb{C}^m$ be holomorphic (we will say z is the U variable and w the V variable). Assume $F(0,0) = 0$ and $J_{F,w}^h(0,0)$ is regular ($J_{F,w}^h$ means only w derivatives). Then there exists $U' \subset U$ containing 0 and $V' \subset V$ containing 0 with a map $\varphi : U' \to V'$ holomorphic so that $F(z,w) = 0$ for $z, w \in U' \times V'$ if and only if $w = \varphi(z)$, i.e. $F(z, \varphi(z)) = 0$.*

In other words, we are implicitly taking w as a function of z. This does, of course, differ from the form many people would have in mind by the word holomorphic, but the proof works precisely the same.

Theorem 0.2.5 (Submanifold theorem). *Take $A \subset \mathbb{C}^n$ (not necessarily open) and let $0 \leqslant k \leqslant n$. Then the following are equivalent.*

(i) (Parameter) There exists U depending on $a \in A$, $V \subset \mathbb{C}^k$, and $\varphi : V \to \mathbb{C}^n$ with $\varphi(0) = a$ and $\varphi : V \to U \cap A$ is a homeomorphism with $\mathrm{rk}(J^h_\varphi(0)) = k$. In essence, we have a local holomorphic parameterization of A.

(ii) (Zero set) There exists U depending on $a \in A$, $F : U \to \mathbb{C}^{n-k}$ with $\mathrm{rk}(J^h_F(a)) = n - k$ such that $A \cap U = \{z \in U : F(z) = 0\}$. In essence, A is the zero set of some holomorphic map F.

(iii) There exists U depending on $a \in A$, $W \subset \mathbb{C}^n$ a neighborhood of the origin, and $\varphi : U \to W$ biholomorphic such that $\varphi(a) = 0$ and $\varphi(A \cap U) = \{w_{k+1} = w_{k+2} = \cdots = w_n = 0\}$. In essence, A is linear and k-dimensional in W (and $a \in A$ gets moved to zero).

See [**forster1976lokale**] for a proof.

Definition 0.2.6. *We say that A is a local analytic submanifold of dimension k at $a \in A$ if one of (i), (ii), (iii) holds above. We say A is a k-dimensional analytic submanifold of U if it is a local analytic submanifold everywhere and closed in U.*

Theorem 0.2.7. *Suppose $F : U \to V$ is holomorphic and bijective ($U, V \subset \mathbb{C}^n$ open). Then F is biholomorphic and J^h_F is regular everywhere.*

Historical quip 0.2.1. The theorem above is sometimes called Osgood's theorem, because it was due to the American mathematician William Osgood. Mathematical folklore says that he used to publish papers and books about several complex variables in German because it was too complicated of a subject to discuss in any other language. The proof we will see is, however, quite clean (especially in comparison to the original given by Osgood).

In order to prove theorem 0.2.7 we will actually prove the following.

Theorem 0.2.8. *Take $U \subset \mathbb{C}^n$ open and suppose $F : U \to \mathbb{C}^n$ is a holomorphic map that is injective. Then J_F^h is everywhere regular.*

This theorem combined with proposition 0.2.3 gives the desired result. We will, somewhat surprisingly, apply induction to this problem. First, we need two technical lemmas.

Lemma 0.2.9. *Suppose $G \subset \mathbb{C}^n$ is a region, $f \in \mathcal{O}(G)$ and $V(f) = \{f = 0\} =: M \neq \emptyset$. Furthermore, take f not identically zero. Then there exists $a \in M$ such that M is a locally analytic submanifold of dimension $n-1$ close to a.*

Proof. If there is a point $a \in M$ and $j \in \{1, \dots, n\}$ with

$$\frac{\partial f}{\partial z_j}(a) \neq 0,$$

then the submanifold theorem, 0.2.5, applies and we are done, so it only remains to apply this to the general case. We consider

$$\Lambda = \{\lambda \in \mathbb{N} : D^\alpha f(z) = 0 \text{ for all } z \in M \text{ and } |\alpha| \leqslant \lambda\}$$

—the higher order derivatives of f. Since f is not identically zero, the identity theorem implies that Λ is finite. Hence, we may choose β, a multiindex, to be maximal in Λ, i.e. $|\beta| = \max \Lambda$, such that $d(D^\beta f)(a) \neq 0$ (this is the total differential, i.e. the set of all derivatives, so the statement is that one of the complex derivatives is nonzero) for some $a \in M$ and $V(f) \subset V(D^\beta f)$.

By the submanifold theorem, every sufficiently small neighborhood U of a, the set $V(D^\beta f, U)$ is an $(n-1)$-dimensional complex submanifold of U. We complete the proof by showing that U may be chosen so that $V(f, U) = V(D^\beta f, U)$. We can use the submanifold theorem to find a change of variables such that $a = 0$ and

$$V(D^\beta f, W) = \{(w', w_n) \in W : w_n = 0\}$$

for some neighborhood, W, of 0. Choose $\delta_n > 0$ sufficiently small so that $f(0', w_n)$ has a zero of some positive order k at $w_n = 0$ and no other zero on $\Delta = \{|w_n| \leqslant \delta_n\}$ (possible by the identity theorem). By continuity of f and Rouché's theorem, there exists $\delta' > 0$ such that the number of zeros of $f(w', \cdot)$ in Δ is constant for $w' \in \mathbb{D}_{\delta'}(0')$, i.e. equals $k > 0$. We clearly may assume that $U = \mathbb{D}_{(\delta', \delta_n)}(0) \subset W$. Thus, for each $w' \in \mathbb{D}_{\delta'}(0')$, there is at least one $w_n \in \Delta$ with $(w', w_n) \in V(f, U)$. Moreover, since $V(f, G) \subset V(D^\beta f, G)$ and $V(D^\beta f, W) = \{(w', w_n) \in W : w_n = 0\}$, if $(w', w_n) \in V(f, U)$, then $w_n = 0$. Hence, we have shown that

$$V(f, U) = \{w \in U : w_n = 0\} = V(D^\beta f, U),$$

which completes the proof. □

Proof of theorem 0.2.8. The proof will, as mentioned, be an induction on the number of variables, n. The classical case $n = 1$ we just assume (see **[ahlfors1953complex]**, theorem 4.11). We must first prove the following technical lemma.

Lemma 0.2.10. *Under the assumptions of theorem 0.2.8, if $J_F^h(a) \neq 0$ at any point $a \in G$, then* $\det(J_F^h(a)) \neq 0$.

Proof. After permuting variables, we may assume $F = (f_1, \dots, f_n)$ and

$$\frac{\partial f_n}{\partial z_n}(a) \neq 0.$$

If $w(z) = (z_1, \dots, z_{n-1}, f_n(z))$, then

$$\det\left(\frac{\partial w_k}{\partial z_j}(a)\right) \neq 0,$$

so that $w = (w_1, \dots, w_n)$ defines holomorphic coordinates in a neighborhood of a. In these coordinates, $\widetilde{F} = F \circ w^{-1}$ is given by

$$\widetilde{F}(w) = (g_1(w), \dots, g_{n-1}(w), w_n)$$

with $g_1, \dots, g_{n-1}$ holomorphic at $b = w(a)$. We write $w = (w', w_n)$ and define $G(w') = (g_1(w', b_n), \dots, g_{n-1}(w', b_n))$. Then G is an injective, holomorphic map in $(n-1)$ variables in a neighborhood of $b' = (b_1, \dots, b_{n-1})$ so that, by inductive assumption, $\det(J_{G'}^h(b')) \neq 0$. But this and the definition of $\widetilde{F}$ imply that $\det(J_{\widetilde{F}}^h(b)) \neq 0$, and hence $\det(J_F^h(a)) \neq 0$ as well. □

Returning to the proof of the theorem, notice that $h = \det(J_F^h) \in \mathcal{O}(G)$. Suppose $V(h) \neq \emptyset$. It then follows from lemma 0.2.9 that $V(h)$ contains a complex submanifold $M \neq \emptyset$ of dimension $n-1 > 0$. By lemma 0.2.10, this means $J_F^h(z) = 0$ for all $z \in V(h)$ and hence J_F^h is identically zero on M. But this implies that F is locally constant on M (just express F in terms of local parameterizations of M) and since $\dim M > 0$, F could not be injective. Thus, $V(h)$ must be empty. □

Remark 0.2.1. It is crucial that $m = n$ for theorem 0.2.8. For example, $f(z) = (z^2, z^3)$ from $\mathbb{C}$ to $\mathbb{C}^2$ is injective, but J_f^h is singular at 0.

0.2.2 A remark on the Riemann mapping theorem

Before finishing the section, we make one last remark on a natural question. Namely, are the two unit balls that we have defined, $\mathbb{D}$ and $\mathbb{B}$, "essentially identical?" In more precise terms, are they biholomorphically equivalent? One may suspect from the single variable theory that they are, but in fact they are not.

Theorem 0.2.11. *Suppose $n \neq 1$. Then $\mathbb{D}$ and $\mathbb{B}$ are not biholomorphically equivalent.*

The case $n = 2$ is originally due to Poincaré, who computed the automorphism groups for each. Recall that we can express the automorphisms of $\mathbb{D}^1$ in terms of Blaschke factors:

$$\mathrm{Aut}(\mathbb{D}^1) = \left\{ w = e^{i\theta} \frac{z-a}{1-\bar{a}z} : \theta \in \mathbb{R}, a \in \mathbb{D} \right\},$$

which is of (real) dimension 3. Poincaré's original proof was based on a computation and comparison of the groups of holomorphic automorphisms of $\mathbb{B}$ and $\mathbb{D}$ which fix the origin. In the former case he found the dimension to be 8 and the latter 6. See [**poincare1907fonctions**] for the original proof or, e.g., [**range1998holomorphic**] for a different approach. Thus, the Riemann mapping theorem does not extend to the case $n > 1$ even in the simplest case.

CHAPTER
ONE

COMPLEX SPACES AND ANALYTIC SHEAVES

One may wonder why in complex analysis we care about algebra; the idea that the two are intimately linked goes back at least to Weierstraß when he said “Je mehr ich über die Principien der Funktionentheorie nachdenke—und ich thue dies unablässig—um so fester wird meine Überzeugung, dass diese auf dem Fundamente algebraischer Wahrheiten aufgebaut werden muss.” In this chapter, we investigate this deep connection. The main reference will be **[grauert2012coherent]**.

1.0 Power series

1.0.1 Definitions and basic results

We consider the formal power series

$$f = \sum_{\nu \in \mathbb{N}_0^n} a_\nu z^\nu = \sum_{j=0}^{\infty} f_j$$

for $a_\nu \in \mathbb{C}$ and where f_j is a homogeneous polynomial of degree j. We can endow this with a ring structure by defining componentwise addition and the Cauchy product as multiplication. In particular, if g

is another formal power series

$$g = \sum_{\nu \in \mathbb{N}_0^n} b_\nu z^\nu = \sum_{j=0}^{\infty} g_j,$$

then

$$f + g = \sum_{\nu \in \mathbb{N}_0^n} (a_\nu + b_\nu) z^\nu, \quad fg = \sum_{\ell=0}^{\infty} \sum_{j+h=\ell} f_j g_h.$$

This will be denoted $\mathbb{C}[[z]]$ and is, in fact, a $\mathbb{C}$-algebra.

Definition 1.0.1. *If f is a formal power series as above, then* $\operatorname{ord}(f) = \min\{j : f_j \neq 0\}$ *is the order of f. We say* $\operatorname{ord}(0) = +\infty$.

Lemma 1.0.2. *We have the following simple relations on the order.*

(i) $\infty \geqslant \operatorname{ord}(f) \geqslant 0$ *with equality to infinity if and only if* $f = 0$.

(ii) $\operatorname{ord}(f+g) \geqslant \min(\operatorname{ord}(f), \operatorname{ord}(g))$.

(iii) $\operatorname{ord}(fg) = \operatorname{ord}(f) + \operatorname{ord}(g)$.

(iv) $\operatorname{ord}(f) = 0$ *if and only if f is a unit in $\mathbb{C}[[z]]$ (i.e. f has a nonzero constant term).*

We leave the (simple) proof to the reader. One should note while doing the proof, though, that there is no special property of $\mathbb{C}$ over any other field that helps prove the above lemma, i.e. $\mathbb{C}$ can be replaced by any field and the lemma holds. Another observation to make is that, because of (iii) combined with (i), the ring of formal power series, $\mathbb{C}[[z]]$, does not have any zero divisors. We now define convergence in this context.

Definition 1.0.3. *Suppose $c_\nu \in \mathbb{C}$. Then we say*

$$\sum_{\nu \in \mathbb{N}_0^n} c_\nu = c$$

if for all $\varepsilon > 0$, there exists a finite set $J_0 \subset \mathbb{N}_0^n$ such that for all finite $J \subset \mathbb{N}_0^n$ with $J_0 \subset J$ we have

$$\left| c - \sum_{\nu \in J} c_\nu \right| < \varepsilon.$$

Proposition 1.0.4. *If $\sum c_\nu$ as above converges then the convergence is absolute and unconditional (i.e. we can permute the sum and get the same result). In particular,*

$$\sum_{\nu \in \mathbb{N}_0^n} c_\nu = \sum_{\nu_1=0}^{\infty} \cdots \sum_{\nu_n=0}^{\infty} c_{\nu_1 \cdots \nu_n},$$

i.e. we can take each sum individually.

Of course, the most important example of a convergent sum is the geometric series. If $q \in \mathbb{D}$, then, due to the above proposition,

$$\sum_{\nu \in \mathbb{N}_0^n} q^\nu = \prod_{j=1}^{n} \frac{1}{1-q_j}.$$

We can also replace the constants c_ν with functions.

Definition 1.0.5. *Suppose $M \subset \mathbb{C}^n$ and $c_\nu : M \to \mathbb{C}$ for each $\nu \in \mathbb{N}_0^n$. Then*

$$\sum_{\nu \in \mathbb{N}_0^n} c_\nu = c$$

uniformly on M if for all $\varepsilon > 0$, there exists a finite set $J_0 \subset \mathbb{N}_0^n$ such that for all $J \supset J_0$ finite and all $x \in M$, one has

$$\left| \sum_{\nu \in J} c_\nu(x) - c(x) \right| < \varepsilon.$$

Similarly we can define local uniform convergence (and, of course, pointwise convergence is as in convergence for constant functions).

We have essentially the exact same theorem as for the single variable case regarding convergence of these power series.

Proposition 1.0.6. *Suppose*

$$f(z) = \sum_{\nu \in \mathbb{N}_0^n} a_\nu z^\nu$$

and there exists $z^0 \in \mathbb{C}^n$ with $z_j^0 \neq 0$ for all j such that $|a_\nu (z^0)^\nu| \leqslant S < \infty$. Then the series converges uniformly on compacta, $K \subset D_{\|z^0\|_\infty}(0)$. We also get pointwise convergence on the full $D_{\|z^0\|_\infty}(0)$.

Proof. As in the single variable case, we compare with the geometric series. □

Take f to be a formal power series converging uniformly in a neighborhood, $U = U(0)$, of 0. Since the partial sums converge uniformly, f is holomorphic in U (the partial sums are holomorphic since they are polynomials). Looking at the theorem above, one may suspect that, just as in the single variable case, power series converge in polydiscs, but the domain of definition for a power series is, in fact, *not* in general a polydisc. For example, consider

$$\sum_{\nu=0}^{\infty} z_2 z_1^{\nu}.$$

Then convergence occurs, for instance, in $\{|z_1| < 1, z_2 \in \mathbb{C}\}$ and $\{z_2 = 0, z_1 \in \mathbb{C}\}$. The domains of convergence are actually log-convex, complete, Reinhardt domains, but we will not concern ourselves with this in the course. See
[fritzsche1974einfuhrung].

1.0.2 The structure sheaf for $\mathbb{C}^n$

Definition 1.0.7. *We denote by $\mathbb{C}\{z\} \subset \mathbb{C}[[z]]$ the $\mathbb{C}$-algebra of convergent power series (around the origin—one could write $\mathbb{C}\{z - z_0\}$ for the convergent power series around z_0, but they are algebraically the same).*

Proposition 1.0.8. *The units in $\mathbb{C}\{z\}$ are those $f \in \mathbb{C}\{z\}$ with* $\operatorname{ord}(f) = 0$*.*

Proof. Note $\operatorname{ord}(f) = 0$ means $f(0) \neq 0$, which, in turn, implies $1/f$ is holomorphic in a neighborhood of zero. In other words, it can be written as a convergent power series around 0; we will call it g. Then $fg = 1$ and we are done. □

Remark 1.0.1. Both $\mathbb{C}\{z\}$ and $\mathbb{C}[[z]]$ are local rings where the unique maximal ideal is

$$\mathfrak{m} = \{f : \operatorname{ord}(f) > 0\}.$$

Furthermore, we could replace $\mathbb{C}$ by another field and this would work out the same way.

Definition 1.0.9. *Suppose f, g are holomorphic on a neighborhood of z_0. We write $f \sim_{z_0} g$ whenever there exists $V = V(z_0)$ with $f \equiv g$ on V. The equivalence classes, denoted by f_{z_0}, are called the germ of a holomorphic function. Denote by $\mathcal{O}_{z_0}$ the set of germs, which is a $\mathbb{C}$-algebra.*

Note that $\mathcal{O}_{z_0} \cong \mathbb{C}\{z - z_0\} \cong \mathbb{C}\{z\} =: H_n$. We will write

$$\mathcal{O} = \coprod_{z \in \mathbb{C}^n} \mathcal{O}_z$$

—the set of all germs at all points in $\mathbb{C}^n$. We also define $\pi : \mathcal{O} \to \mathbb{C}^n$ to be the natural projection. With this, we turn to defining a topology on $\mathcal{O}$ such that π is continuous. It is known as the étale space, although we will not use this terminology.

Definition 1.0.10. *We call $\mathcal{O}$ the structure sheaf of $\mathbb{C}^n$ and $\mathcal{O}_z$ its stalks. Write $\mathcal{O}_U$ to be the structure sheaf of $U \subset \mathbb{C}^n$ open (i.e. the part of $\mathcal{O}$ lying over U).*

Definition 1.0.11. *Let $U \subset \mathbb{C}^n$ be open. A continuous section in $\mathcal{O}$ above U is $\sigma : U \to \mathcal{O}$ such that*

(i) $\sigma(z) \in \mathcal{O}_z$ and

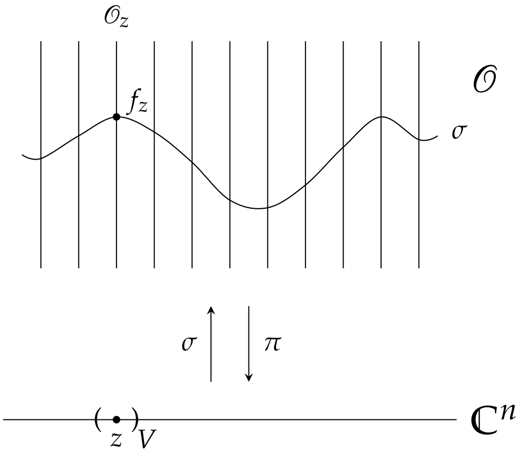
(ii) there exists a holomorphic function, f, on U such that $\sigma(z) = f_z$.

Written with a diagram, a section is such that

$$\begin{tikzcd} & \mathcal{O} \arrow[d, "\pi"] \\ U \arrow[ur, dashed, "\sigma"] \arrow[r, hook] & \mathbb{C}^n \end{tikzcd}$$

commutes. We define the strong topology on $\mathcal{O}$ by letting $\sigma(U)$ be a base (with σ any continuous section and $U \subset \mathbb{C}^n$ any open set).

Before proceeding, we give some intuition (that will also apply to other sheaves once we define them in section 1.3). Throughout this intuitive explanation, σ will represent a section (but not a specific one). Sheaves can be visualized in the following way. We have some underlying topological space, X (in our case $X = \mathbb{C}^n$), and above each point, e.g. $z \in \mathbb{C}^n$, we have $\sigma(z)$ on the stalk $\mathcal{O}_z$ (this is what (i) is saying). What (ii) gives us is some "continuity" of σ. That is, each z is associated to a germ, f_z, which is a class of functions equal on some open set V around z. That is, without the second condition, σ could be wildly changing from point to point (intuitively, in a "discontinuous" way). Indeed, as one may wonder from this, (ii) does imply (i), but not the converse. For a visual of the above description, see below.



The more subtle point from all of this is that (ii) guarantees that the map taking $\mathbb{C}^n$ to $\mathbb{C}$ by mapping z to its germ, f_z, and then evaluating a representative, f, at z, is a holomorphic function. In other words, we have a correspondence between sections of $\mathcal{O}$ on U and holomorphic functions on U. A concrete realization that comes from this is that analytically continuing a function can in a way be seen as just scooting along the curve given by σ. The above image also gives some clarity as to the naming of sheaves and stalks (each line is a stalk—think, of hay—and all together the stalks give a sheaf of hay).

In any case, we move to some consequences of this definition. Most obviously, we see that, tautologically, the continuous sections are continuous; in fact they are also open. Furthermore, as we wished, the projection, π, is continuous and even a local homeomorphism (since for $f \in \mathcal{O}_{z_0}$ and $U = U(z_0)$ a neighborhood of z_0 where f converges, the set $\{f_z : z \in U\}$ is a neighborhood of f_{z_0} and π is an injection between it and U).

Additionally, $\mathcal{O}_z = \pi^{-1}(z)$, i.e. discrete subsets of $\mathcal{O}$, are closed. Finally, algebraic operations are continuous, i.e. if we take the fiber product, $\mathcal{O} \times_\pi \mathcal{O}$ and map to $\mathcal{O}$ through multiplication or addition, then the map is continuous (note $\mathcal{O} \times_\pi \mathcal{O}$ has the induced product topology). Later, using this as the prototype for a sheaf, we will require some of these properties in the definition of a more general sheaf.

Now that we have the definitions out of the way, we will work with them a bit by proving the following simple (and nice) proposition.

Proposition 1.0.12. *The topology of $\mathcal{O}$ is Hausdorff.*

Proof. This is a consequence of the identity theorem. Indeed, since $\mathbb{C}^n$ is Hausdorff, germs in different stalks can always be separated by disjoint neighborhoods, so we can restrict our attention to points (i.e. germs) in the same stalk, say $\mathcal{O}_{z_0}$. If we take $f_{z_0} \neq g_{z_0}$ to be two germs above z_0 with s, t two sections representing the germs, say on U_1, U_2 respectively.

Say that $U \subset U_1 \cap U_2$ is an open, connected neighborhood of z_0. We continue to write s, t, but mean the sections each restricted to U. Then two open disjoint neighborhoods of f_{z_0} and g_{z_0}, respectively, are

$$\{s_z : z \in U\}, \quad \{t_z : z \in U\}.$$

Indeed, if there were a germ, say above z', in their intersection, then there would be a neighborhood $W \subset U$ of z' such that s, t further restricted to W are equal, and thus, by the identity theorem, would be equal on U. In particular, we get that $f_{z_0} = g_{z_0}$, which is a contradiction. $\square$

Warning: There is some possibility for confusion in notation. If f is holomorphic on U, then $\sigma_f(z) = f_z \in \mathcal{O}_z$ and $f(z) \in \mathbb{C}$. We often identify f and σ_f (if we know one we know the other), but one should be careful to distinguish $f(z)$, the complex number, and f_z, the germ. Even worse, we have two interpretations of $\mathcal{O}(U)$; first, as the set of holomorphic functions on U, and second, as the set of continuous sections $\sigma : U \to \mathcal{O}$. These, as we will see, are intimately related and the notation being the same is somewhat justified, however one should be careful with them.

To close this section, we briefly remark that one could replace holomorphic functions above by continuously differentiable functions (by C^0, C^k, or C^∞). Then we get sheaves of germs of k-differentiable, continuous functions. Explicitly, if X is a topological space, $x \in X$, and $f, g : U(x) \to \mathbb{C}$, then $f \sim_x g$ whenever there exists $V = V(x) \subset U(x)$ such that $f = g$ on $V(x)$. Equivalence classes are still called germs.

Note that, since we do not have the identity theorem for any of these spaces, we cannot prove that the topology is Hausdorff, and it is, in fact, not. For example, for germs of continuous functions on $\mathbb{R}$, the germs of $f(x) = x$ and $g(x) = |x|$ at 0 have no disjoint neighborhoods.

1.1 The Weierstrass theorem

1.1.1 The Vorbereitungssatz and Formel

We look at $H_n = \mathbb{C}\{z\}$. Before proceeding, we will need the following way of writing its elements. If $f \in H_n$, then there exist $f_\nu \in H_{n-1}$ such that

$$f = \sum_{\nu=0}^{\infty} f_\nu z_n^\nu.$$

We will not prove it, but it should be, at least, fairly intuitively obvious. We use a prime symbol, e.g. z', to denote a variable with its last component truncated (so if $z \in \mathbb{C}^n$ is $(z_1, \ldots, z_n)$ then $z' \in \mathbb{C}^{n-1}$ is just $(z_1, \ldots, z_{n-1})$).

Definition 1.1.1. *Say that f is z_n-regular of order $k \geqslant 0$ if, using the expression for f above, $f_\nu(0') = 0$ for $\nu = 0, \ldots, k-1$ and $f_k(0') \neq 0$. In other words, f_k is the first unit. We will sometimes instead use the words k-regular or k-general.*

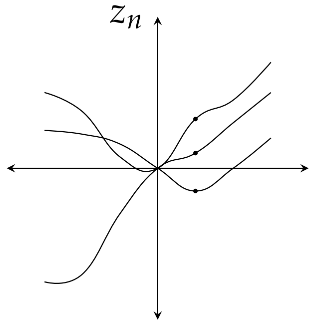
Note that $f(z_1, z_2) = z_1 z_2$ is neither z_1- nor z_2-regular for any $k \geqslant 0$.

Theorem 1.1.2 (Weierstraßsche Vorbereitungssatz). *Let $f \in H_n$ be k-regular. Then there are $e, \omega \in H_n$ with* *(i)* $f = e\omega$ *and*

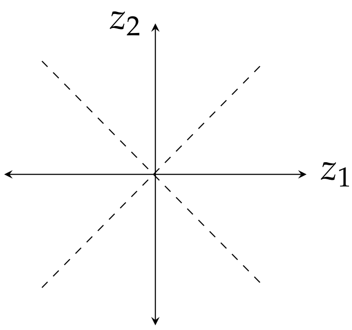
(ii) e *is a unit,* $\omega(z', z_n) = z_n^k + a_{k-1}(z')z_n^{k-1} + \cdots + a_0(z')$, *i.e.* $\omega \in H_{n-1}[z_n]$ *and is monic.*

Both e and ω *are unique and, moreover,* $a_\nu(0') = 0$ *for all* $\nu = 0, \ldots, k-1$. *We call* ω *the Weierstrass polynomial (i.e. a monic polynomial in* $H_{n-1}[z_n]$ *with the coefficients vanishing at the origin*[1]*).*

Before proceeding to the proof, we give some intuition. Since $f = e\omega$ and e is a unit, we can say that $V(f) = V(\omega) =: M$ around 0. Say, for example, that $\omega = z_n^3 + a_2(z')z_n^2 + a_1(z')z_n + a_0(z')$. Thus, for each z', we have a cubic polynomial of z_n with three zeros, pictured below (i.e. the zeros are graphed and pictured below, not the cubic polynomials). Three zeros for a fixed z' are labeled with dots.



Thus, one may think of M as a 3-fold branched covering of $\mathbb{C}^{n-1}$. Generally now, the theorem says that if f is z_n-regular of order k, then it gives rise to a k-fold branch covering of $\mathbb{C}^{n-1}$ through its zeros for each fixed $z' \in \mathbb{C}^{n-1}$. Looking at our counterexample, $f(z_1, z_2) = z_1 z_2$, we only get the axes.



[1]In the case $k = 0$, we have $\omega = 1$ (since then the Weierstrass polynomial should be $\omega(z) = z_n^0 = 1$).

The only way to get a 2-fold branched covering would be to change variables to rotate everything (which is what the dashed lines represent—a rotation by $\pi/4$).

Proof of the Weierstraßsche Vorbereitungssatz. The case $k = 0$ is trivial, so take $k \geqslant 1$. Suppose that

$$f(z) = \sum_{\nu=0}^{k-1} c_\nu z_n^\nu + z_n^k + \sum_{\nu=k+1}^{\infty} c_\nu z_n^\nu, \tag{1.1}$$

which is possible after multiplication by a unit on H_{n-1} since f is z_n regular of order k. Now we factor out the z_n^k:

$$f(z) = z_n^k \Bigg(1 + \underbrace{\sum_{\nu=0}^{k-1} c_\nu z_n^{\nu-k}}_{=:A} + \underbrace{\sum_{\nu=k+1}^{\infty} c_\nu z_n^{\nu-k}}_{=:B} \Bigg).$$

Further, we will name $W := A + B$. Thus, in total, we have written $f(z) = z_n^k(1 + W)$. Now, say that f converges in the polydisc of radius R. Choose $\rho > 0$ so small that for $|z_n| < \rho$, we have the bound $|B| < 1/2$ (which is possible by convergence of the power series). Now, A is not bounded at zero since we factored out z_n^k, but it is bounded in $\rho/2 < |z_n| < \rho$. We may choose R' so small so that $|A| < 1/2$ on $\{z \in \mathbb{C}^n : \|z'\|_\infty \leqslant R' \text{ and } \rho/2 < |z_n| < \rho\}$. In total, on this set, $|W|$ is strictly bounded by 1.

The Laurent series (see appendix B for the relevant theorem) for $\log(1 + W)$ viewed as a function just of z_n exists and is holomorphic on the annulus $\rho/2 < |z_n| < \rho$ (note that $|W| < 1$ places us comfortably in the radius of convergence for the expansion below). We may write

$$\log(1 + W) = W - \frac{W^2}{2} + \frac{W^3}{3} - \cdots = C + D,$$

where C is the Taylor part (containing nonnegative powers of z_n) and D is the principal part (containing the negative powers of z_n). Then, we in fact have

$$f(z) = z_n^k(1 + W) = z_n^k \exp(\log(1 + W)) = z_n^k \exp(C + D).$$

Note C is holomorphic, so composing it with the exponential yields a holomorphic function, in fact, it yields a unit. We are left with

$$f(z)e^{-C} = z_n^k e^D,$$

where

$$\begin{aligned} z_n^k e^D &= z_n^k \left(1 + D + \frac{D^2}{2} + \cdots\right) \\ &= z_n^k + a_{k-1}(z')z_n^{k-1} + \cdots + a_0(z') + (\text{negative powers of } z_n). \end{aligned}$$

However, $f(z)e^{-C}$ contains only nonnegative powers of z_n and the Laurent series expansion is unique on annuli like $\rho/2 < |z_n| < \rho$, so the right hand side, $z_n^k e^D$ must also contain only nonnegative powers, i.e. no negative powers, i.e. that in the parentheses in the last line above is zero. Of course, this leaves us with

$$f(z)e(z)^{-1} = \omega(z),$$

where $e(z)^{-1} = e^{-C}$ and $\omega(z) = z_n^k + a_{k-1}(z')z_n^{k-1} + \cdots + a_0(z')$, as we wished.

By comparing coefficients, we can see that $a_\nu(0') = 0$ for each $\nu = 1, \ldots, k-1$. Because f is z_n-regular of order k, we know that the coefficients of f are not units before z_n^k in the expansion of f. In other words, they have constant term equal to zero. Thus, still writing f as in (1.1),

$$\begin{aligned} f(0', z_n) &= z_n^k + c_{k+1}z_n^{k+1} + \cdots = e(0', z_n)\omega(0', z_n) \\ &= (1 + \cdots)(z_n^k + a_{k-1}(0')z_n^{k-1} + \cdots + a_0(0')). \end{aligned}$$

Hence, $\omega(0', z_n) = z_n^k$; otherwise, we would, e.g., get terms like z_n^{k-1} on the right hand side after multiplying it out. Seeing as there are no terms below the order z_n^k on the left hand side, this cannot happen.

It remains to show uniqueness, which follows quite easily. In a neighborhood of 0, $V(f) = V(\omega)$. A monic polynomial, such as $\omega(z)$ (in the variable z_n) is completely determined by its zero set. Thus, $\omega(z)$ is unique and further, $e(z)$ must also be unique. $\square$

The proof above is due to Stickelberger [**stickelberger1887ueber**] published in the year 1887. Before him, Weierstrass [**weierstrass1895mathematische**] gave a proof first printed in 1886, although he claimed to have been displaying the theorem in his lectures since 1860. In 1927 Wirtinger [**wirtinger1927weierstrass**] gave a significant simplification of Weierstrass' proof without realizing that Stickelberger had given essentially the same proof already in 1887. This proof was also discovered by Hartogs [**hartogs1909elementare**] in 1909. Furthermore, in 1929, Späth [**spath1929weierstrass**] gave a new proof of the theorem only using local theory of power series. He also used this to show the so-called Weierstraßsche Formel or Weierstraßsche Divisionssatz (see below), which turns out to be equivalent to the Vorbereitungssatz.

One can also prove the Vorbereitungssatz by comparing coefficients (although this method is very tedious), which was done first by Späth in the above mentioned paper. Or so he thought. It was, in fact, Cartan [**cartan1966theoreme**] in 1966 (on Weierstrass' 150th birthday) that noticed that Brill [**brill1910weierstrassschen**] had already given the same proof in 1910 and further, in 1905, Lasker [**lasker1905theorie**] (the chess champion) had already given the sketch of this proof without providing details. Brill [**brill1891ueber**] had, in fact, already done the proof for $n = 1$ in the year 1891. Neither Brill nor Lasker cited each other, so one assumes these discoveries were independent of each other. This is all to say that the theorem is, as one hopefully now expects, fundamental. So fundamental, that it has been discovered, forgotten, and discovered again many times (not all rediscoveries of the theorem are mentioned above).

The advantage of the proof given by Späth is that, since it only uses local properties of power series, it can be extended to the case where the power series takes coefficients from a completely valued field (defined in the next section) of characteristic zero. This is, of course, not possible with the proof we gave, since it relies on the uniqueness of Laurent series expansions, which cannot be proven through purely local investigations of power series. See **[siegel1979gesammelte]** for the particulars along with some of the other proofs mentioned above. For a proof that it works for all completely valued fields, see **[grauert2012coherent]**.

Before proceeding to the Weierstraßsche Formel, we state a lemma that is interesting in its own right.

Lemma 1.1.3. *Let $\omega \in H_{n-1}[z_n]$ be a Weierstrass polynomial and let $g \in H_n$ be such that*

$$p = g\omega \in H_{n-1}[z_n].$$

Then $g \in H_{n-1}[z_n]$.

Proof idea. First reduce to the case where the degree of p (as a polynomial in z_n) is smaller than that of ω (use polynomial long division). In this case, the lemma is proved by showing $g = 0$. This follows by considering the zeros of ω and the implications they have on the zeros of p. $\square$

Remark 1.1.1. As examples like

$$(1 - z_n)(1 + z_n + \cdots) = 1$$

demonstrate, the assumption that ω is a Weierstrass polynomial is essential.

Theorem 1.1.4 (Weierstraßsche Formel). *Let $g \in H_n$ be z_n-regular of order k. Let $f \in H_n$ as well. Then there are unique $q \in H_n$ and $r \in H_{n-1}[z_n]$ with $\deg r < k$ such that $f = gq + r$.*

Proof. From the Vorbereitungssatz, theorem 1.1.2, we can write $g = e\omega$, where ω is a monic polynomial of degree k in z_n. Without loss of generality, we just take $g = \omega$. We look at the Laurent expansion of f/ω. Namely,

$$\frac{f}{\omega} = q + H,$$

where q is the Taylor part and H the principal part. Multiplying by ω, we see $f = q\omega + H\omega$ and thus (using the usual trick)

$$f - q\omega = H\omega =: r.$$

The left hand side is a power series, so the right hand side can contain no negative terms. Furthermore, since the highest possible power in H is z_n^{-1}, the highest possible degree for r is $k-1$; in particular, $r \in H_{n-1}[z_n]$ has $\deg r < k$. Behold,

$$f = qg + r,$$

where q and r have the desired properties.

Uniqueness remains. Set $f = 0$ and find q, r with the assumed properties so that $0 = qg + r$. Then, by the lemma, $q \in H_{n-1}[z_n]$ (we have used the Weierstrass Vorbereitungssatz on g to show this). Then comparing degrees, we see that $q = r = 0$. Of course, now if there were two representations for any other f, then the uniqueness of the representation for zero would be contradicted after subtraction. $\square$

Remark 1.1.2. We have shown that the Weierstraßsche Vorbereitungssatz implies the Weierstraßsche Formel; however, the implication can also be reversed. The sketch is the following. Take f to be z_n-regular of order k and use the formula to find q and r with $z_n^k = qf + r$. Then, define $\omega = z_n^k - r$ and use that f is z_n-regular to show that q is a unit. Then $f = q^{-1}\omega$ and we have the Vorbereitungssatz back.

Remark 1.1.3. Two observations are in order for the case where f and g are polynomials in z_n. They are exercises for the reader.

(i) In the Weierstrass formula, if f and g are both polynomials in z_n, i.e. $f, g \in H_{n-1}[z_n]$, of degree m and k, respectively, then q is a polynomial of degree $m - k$.

(ii) In the Weierstrass Vorbereitungssatz, if f is z_n-regular of order k and is a polynomial of order m in z_n, then $f = e\omega$ where e is a polynomial of degree $m - k$.

Remark 1.1.4. Thanks to the Weierstrass theorems, proofs in analytic sheaf theory (and other areas of several complex variables) often proceed by induction on the number of complex variables. It is for this reason that the sheaf theoretic approach to function theory is sometimes considered pedestrian and called the "one variable at a time" approach.

1.1.2 Extending the Weierstraß theorems with shears

We return to our favorite counterexample. Namely, $f(z_1, z_2) = z_1 z_2$, which is neither z_1- nor z_2-regular. Thus, we may not apply the theorems 1.1.2 and 1.1.4. We remedy this by introducing new coordinates (as we did in the intuition following the Vorbereitungssatz):

$$\begin{cases} z_1 = w_1 \\ z_2 = w_1 + w_2. \end{cases}$$

Thus, $z_1 z_2 = w_1(w_1 + w_2) = w_1^2 + w_1 w_2$, which is w_1-regular of order 2. Let us generalize this.

If $f = f_1 f_2 \cdots f_r$, then f is z_n-regular if and only if f_s is z_n-regular for each $1 \leqslant s \leqslant r$. Suppose $c \in \mathbb{C}^{n-1}$ and define $\sigma_c : H_n \to H_n$ by

$$\begin{cases} z_\nu \mapsto z_\nu + c_\nu z_n & \text{for } \nu = 1, \dots, n-1 \\ z_n \mapsto z_n. \end{cases}$$

If $f \in H_n$ we, somewhat confusingly, write $\sigma_c(f) = f \circ \sigma_c$; this, i.e. σ_c, is called a shear.

The shears form a group with composition that we will denote by Σ. Each element of Σ is completely determined by the fixed $c \in \mathbb{C}^{n-1}$, so $\Sigma \cong (\mathbb{C}^{n-1}, +)$. Indeed,

$$\sigma(f+g) = \sigma f + \sigma g \quad \text{and} \quad \sigma(fg) = \sigma f \sigma g.$$

Let $f \in H_n$ and suppose $\operatorname{ord}(f) = k$. Then

$$f = \sum_{j=k}^{\infty} f_j,$$

where f_k is not identically zero. If $\sigma \in \Sigma$, then

$$\sigma f = \sum_{j=k}^{\infty} \sigma f_j,$$

with (written explicitly)

$$f_k = \sum_{\nu_1+\cdots+\nu_n=k} a_{\nu_1\cdots\nu_n} z_1^{\nu_1}\cdots z_n^{\nu_n}.$$

Thus,

$$\sigma f_k = \sum_{\nu_1+\cdots+\nu_n=k} a_{\nu_1\cdots\nu_n}(z_1+c_1z_n)^{\nu_1}\cdots(z_{n-1}+c_{n-1}z_n)^{\nu_{n-1}}z_n^{\nu_n}.$$

Hence, $\sigma f_k(0', z_n) = f_k(c,1)z_n^k$. If $f_k(c_1,\ldots,c_{n-1},1) \neq 0$, then σf is z_n-regular of order k, so the question is, can we find a c so that this is true? In fact, this will hold for "most" $c \in \mathbb{C}^{n-1}$.

Theorem 1.1.5. *Let $f_1,\ldots,f_r$ be given and not identically zero. Then there exists a shear transformation σ such that all of the f_ν are z_n-regular of some order.*

Thus, the theorem above lifts the limitation that we had before from the Weierstrass Vorbereitungssatz and formula at the cost of a linear change of variables.

1.2 Algebraic properties of H_n

We begin with some recollections and remarks. For more review of concepts from algebra, see appendix A. Two important examples of local rings are $H_n = \mathbb{C}\{z\}$ and $\mathbb{C}[[z]]$. The maximal ideal in both cases will be $\mathfrak{m} = \{f : \operatorname{ord}(f) > 0\}$. We also have that $\mathscr{C}_0^\infty$, the ring of germs of C^∞ functions at 0, is local. The maximal ideal in this case is the set of $f \in \mathscr{C}_0^\infty$ that vanish at 0. Another desirable property is being noetherian. For example, consider $\mathbb{C}\{z\}$ and

$$\mathfrak{a} = \bigcap_{\ell \geqslant 1} \mathfrak{m}^\ell.$$

Then $\mathfrak{a} \subset (0) + \mathfrak{m}^k(1)$, which implies by the Krull lemma A.1.6 (if H_n is noetherian), that $\mathfrak{a} \subset (0)$, i.e. $\mathfrak{a} = (0)$. Since this is, indeed the case (the set of functions above is the set of convergent power series with null derivative at zero for every order derivative, which in the complex case gives only the zero function), we are encouraged that H_n may be noetherian—we will show later that this is the case.

Remark 1.2.1. We can show using this that $\mathscr{C}_0^\infty$ is definitely not noetherian. Study again

$$\bigcap_{\ell \geqslant 1} \mathfrak{m}^\ell = \{f : \partial^k f = 0 \text{ for all } k\}.$$

This is now not the zero set as functions like $f(x) = e^{-1/x^2}$ demonstrate. Thus, $\mathscr{C}_0^\infty$ is not noetherian, otherwise the Krull lemma would give a false result.

To move further, we must give an equivalent characterization of what it means for a power series to converge. Suppose $f \in \mathbb{C}[[z]]$ is given by

$$f(z) = \sum_{\nu \in \mathbb{N}_0^n} a_\nu z^\nu.$$

Then f converges exactly when there exists an $r \in \mathbb{R}_+^n$ with

$$\sum_{\nu \in \mathbb{N}_0^n} |a_\nu| r^{|\nu|} < \infty. \tag{1.2}$$

The utility of this equivalent definition is that it is more readily generalizable to arbitrary valued fields. Recall the following.

Definition 1.2.1 (Valuation). *Let K be a field. A map $v : K \to [0, \infty)$ is called a valuation if the following conditions are fulfilled: for $a, b \in K$*

(i) $v(a + b) \leqslant v(a) + v(b)$ and

(ii) $v(ab) = v(a)v(b)$.

If a field has a valuation, we call it a valued field. A completely valued field is a field such that Cauchy sequences converge with respect to the valuation on the field.

Remark 1.2.2. There is another definition of a valuation that one may see in books. If we replace the triangle inequality above by an ultrametric inequality (i.e. $v(a + b) \leqslant \max(v(a), v(b))$), then we can use exponentials and logarithms to map between the two definitions. The triangle inequality is, of course, weaker than the ultrametric inequality, so we have slightly weakened the other definition of a valuation here.

With this definition in hand, say K is a field with $|\cdot| : K \to [0, \infty)$ a valuation. Then we will say that $f \in K[[z]]$ converges if (1.2) happens. With this, we may define $K\{z\}$ as one may expect. A trivial example of a valuation is

$$|a| := \begin{cases} 1 & a \neq 0 \\ 0 & a = 0. \end{cases}$$

One may verify that, in this case, $K[[z]] = K\{z\}$.

As we noted before, the Weierstrass theorems hold for completely valued fields. One may, as in the case of $K = \mathbb{C}$ use shears to extend the Weierstrass theorems even to all of $K\{z\}$. Remark that in the case of finite fields, one may need to use nonlinear shears (i.e. $z_\nu \mapsto z_\nu + z_n^{c_\nu}$ for suitable c_ν).

Moving on, we take K to be a field and $(A, \mathfrak{m})$ a local K-algebra paired with its maximal ideal. We have the ring morphisms

$$K \xhookrightarrow{\iota} A \xrightarrow{\pi} A/\mathfrak{m},$$

where ι is the inclusion $a \mapsto a \cdot 1$ and π the projection $b \mapsto b + \mathfrak{m}$. Then ideally $\sigma = \pi \circ \iota$ would be an isomorphism; at the very least, homomorphisms between fields are always injective, so what we want (and cannot always have) is surjectivity. In our case, $H_n = \mathbb{C}\{z\}$ is a $\mathbb{C}$-algebra and

$$\mathbb{C} \xhookrightarrow{\iota} H_n \xrightarrow{\pi} H_n/\mathfrak{m};$$

here, π takes $f \mapsto f(0) + \mathfrak{m}$. Notice that $H_n/\mathfrak{m} \cong \mathbb{C}$ because the maximal ideal in H_n is (as in any local ring) $H_n \setminus H_n^\times$, which is those elements which have zero constant term. Thus quotienting by $\mathfrak{m}$ just leaves $\mathbb{C}$ and we see that σ is just the identity map from $\mathbb{C}$ to itself.

In the case where σ is surjective we can also consider the following:

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[rr, bend right, "\rho"'] & A/\mathfrak{m} \arrow[r, "\sigma^{-1}"] & K. \end{tikzcd}$$

For our current case, $A = H_n$, we just have $\rho = \sigma^{-1} \circ \pi = \pi$, so, in fact, ρ takes in f and gives $f(0)$. We now lift ρ to $H_n[t]$ in the natural way,

$$A[t] \xrightarrow{\rho[t]} K[t].$$

In our case, this translates to taking $\omega = \omega(z;t) \in H_n[t]$ to $\rho[t](\omega) = \omega(0;t) \in \mathbb{C}[t]$. For brevity, we will just write ρ and not $\rho[t]$. It is with this that we give the following important definition.

Definition 1.2.2 (Hensel ring). *Let R be a local ring with maximal ideal $\mathfrak{m}$. We say R is henselian if the following holds. If $\omega(t) \in R[t]$ is a monic polynomial, then any factorization of its image in $(R/\mathfrak{m})[t]$ into a product of coprime monic polynomials can be lifted to a factorization in $R[t]$.*

Remark 1.2.3. We can write this differently (and here explicitly) when σ is surjective. In this case $A/\mathfrak{m} \cong K$. We say that a local K-algebra, A, is henselian (also a Hensel algebra) if whenever $\omega(t) \in A[t]$ is a monic polynomial with $\Omega(t) = \rho(\omega) \in K[t]$ and $\Omega(t) = \Omega_1(t)\Omega_2(t)$ is a decomposition into monic coprime polynomials, then we have $\omega_1(t), \omega_2(t) \in A[t]$ with

$$\omega(t) = \omega_1(t)\omega_2(t)$$

and

$$\rho(\omega_1) = \Omega_1, \quad \rho(\omega_2) = \Omega_2.$$

Remark 1.2.4. Henselian rings behave quite well. In particular, homomorphic images of Hensel rings are again Hensel. There are many other similar examples (e.g. integral extensions of henselian rings are henselian). Furthermore, valuation rings are always henselian (in fact, valuation theory is where the term henselian originates).

We ask now, more broadly, what nice algebraic properties H_n has. In the case $n = 0$, we have $H_0 = \mathbb{C}$, which obviously has many of the properties that we would want—noetherian, UFD, etc. When $n = 1$, we can easily prove that H_1 is a PID and all of its ideals are of the form (z^j). Which of these will extend to a general n?

Theorem 1.2.3. *H_n is noetherian.*

Proof. The cases $n = 0$ and $n = 1$ are well-known (and easy); we proceed by induction. Let $\mathfrak{a} \subset H_n$ be an ideal with $\mathfrak{a} \neq (0)$ and choose some $f \in \mathfrak{a} \setminus \{0\}$. There is an automorphism taking

$$f \mapsto \widetilde{f}, \quad \mathfrak{a} \mapsto \widetilde{\mathfrak{a}},$$

such that $\widetilde{f}$ is z_n-regular of order k by theorem 1.1.5. Thus, without loss of generality, we assume f is z_n-regular of order k. We also define $A = H_n/(f)$ and the natural projection $\pi : H_n \to H_n/(f)$.

By the Weierstraßsche Formel, for $h \in H_n$, there exists $q \in H_n$ and $r \in H_{n-1}[z_n]$ with $\deg_{z_n} r < k$ such that $h = qf + r$. We will write

$$r = a_0(z') + a_1(z')z_n + \cdots + a_{k-1}(z')z_n^{k-1},$$

where $a_j \in H_{n-1}$ for each j. We have the following:

$$H_n \xrightarrow{\pi} H_n/(f) = A \xrightarrow{\cong} H_{n-1}^k,$$

where "$\cong$" above represents an isomorphism of H_{n-1}-modules, where the associated isomorphism is $h \mapsto (a_0, \ldots, a_{k-1})$. Thus, since by the induction hypothesis, H_{n-1} is noetherian, so is H_{n-1}^k (easy exercise). In total, A is also noetherian.

Let $\pi(\mathfrak{a}) = \overline{\mathfrak{a}} \subset A$. Then $\overline{\mathfrak{a}}$ is finitely generated, so there are $f_1, \ldots, f_m \in \mathfrak{a}$ such that $(\overline{f}_1, \ldots, \overline{f}_m) = \overline{\mathfrak{a}}$. Hence,

$$\mathfrak{a} = (f, f_1, \ldots, f_m)$$

is finitely generated; in other words, H_n is noetherian. □

Theorem 1.2.4. *H_n is a UFD.*

Proof. We again use induction and also again, the cases $n = 0$ and $n = 1$ are obvious (and well-known). First, observe that every $f \in H_n$ is a product of irreducibles by definition of irreducible essentially: if f is not irreducible and $\operatorname{ord}(f) = k$, then $f = f_1 f_2$ with $\operatorname{ord}(f_1), \operatorname{ord}(f_2) < k$. If f_1, f_2 are irreducible, we are done; otherwise, we continue splitting each until they are written as a product of irreducibles. This process will, indeed, terminate since the order is reduced each time (so at worst it terminates once the order reaches zero).

So now assume that $f \in H_n$ is irreducible and $f | gh$ for some $g, h \in H_n$. We may, by theorem 1.1.5, assume f, g, h are all z_n-regular of some order. By the Weierstraßsche Vorbereitungssatz, we write

$$f = e_0\omega_0, \quad g = e_1\omega_1, \quad h = e_2\omega_2,$$

where $\omega_j \in H_{n-1}[z_n]$ are Weierstraß polynomials for $j = 0, 1, 2$ and $e_j \in H_n$ are units for each j. Now that we have set up this scenario, since $f | gh$, we observe $\omega_0 | \omega_1 \omega_2$ in H_n. By lemma 1.1.3, we deduce that $\omega_0 | \omega_1 \omega_2$ in $H_{n-1}[z_n]$. By the induction hypothesis, H_{n-1} is a UFD and further by Gauss' lemma A.2.4 $H_{n-1}[z_n]$ is also a UFD. We thus deduce that $\omega_0 | \omega_1$ or $\omega_0 | \omega_2$ in $H_{n-1}[z_n]$. Thus, either $f | g$ or $f | h$ in H_n, so every irreducible is prime. Hence, the claim. □

Lemma 1.2.5 (Hensel's lemma). *Let $\omega \in H_n[t]$ be a monic polynomial with $\Omega(t) = \omega(0; t) \in \mathbb{C}[t]$ having $\deg \Omega = s$. We write*

$$\Omega(t) = \prod_{\lambda=1}^{\ell} (t - c_\lambda)^{s_\lambda},$$

where $c_\lambda \neq c_\mu$ if $\lambda \neq \mu$ and $\sum s_\lambda = s$. Then there are uniquely determined, mutually prime, and monic $\omega_\lambda(z; t) \in H_n[t]$ for $\lambda = 1, \ldots, \ell$ such that $\deg \omega_\lambda = s_\lambda$, $\omega_\lambda(0; t) = (t - c_\lambda)^{s_\lambda}$, and

$$\prod_{\lambda=1}^{\ell} \omega_\lambda(z; t) = \omega(z; t).$$

Proof. This is once again an induction, this time on ℓ. The base case, $\ell = 1$, is obvious. We do two cases: $c_1 = 0$ and $c_1 \neq 0$.

First, suppose $c_1 = 0$. Then

$$\omega(0; t) = \Omega(t) = t^{s_1} \prod_{\lambda=2}^{\ell} (t - c_\lambda)^{s_\lambda}.$$

Thus, $\omega(z; t) \in H_{n+1}$ is t-regular of order s_1, so the Weierstraßsche Vorbereitungssatz gives that

$$\omega(z; t) = \omega_1(z; t) e(z; t),$$

where $\omega_1 \in H_n[t]$ is a Weierstrass polynomial of degree s_1 and $e \in H_{n+1}$ is a unit. By lemma 1.1.3 we observe $e(z; t) \in H_n[t]$ and is of degree $s - s_1$ by remark 1.1.3. Obviously e is monic (since both ω, ω_1 are). In total, we can easily conclude that

$$\omega_1(0; t) = t^{s_1}, \quad e(0; t) = \prod_{\lambda=2}^{\ell} (t - c_\lambda)^{s_\lambda}.$$

Hence, applying the induction hypothesis, we have $e = \omega_2 \cdots \omega_\ell$ where $\omega_\lambda \in H_n[t]$ is monic of degree s_λ and $\omega_\lambda(0;t) = (t - c_\lambda)^{s_\lambda}$. This finishes the existence proof in the case $c_1 = 0$.

For the case $c_1 \neq 0$, we just apply these considerations to $\omega'(z;t) := \omega(z; t + c_1)$ so that the first case gives the result for ω' and then we extend it to ω. Details are omitted. We also omit the proof of mutual primality since we will not need this assertion.

For uniqueness, if $\omega = \omega_1 \cdots \omega_\ell$ is any decomposition of ω with the stated properties, then, supposing $c_1 = 0$, ω_1 is a Weierstrass polynomial in $H_n[t]$ since $\omega_1(0,t) = t^{s_1}$ and $e = \omega_2 \cdots \omega_\ell$ is a unit in H_{n+1} since $e(0,0) \neq 0$. The Vorbereitungssatz then applied to $\omega = \omega_1 e$ as a t-regular element of H_{n+1} of order s_1 yields the uniqueness of ω_1. An analogous argument gives the uniqueness for the other factors and shifting by c_1 gives the general result. $\square$

Directly from the lemma above, we get the following corollary.

Corollary 1.2.6. *H_n is henselian.*

We can extend this to a broader family of $\mathbb{C}$-algebras.

Definition 1.2.7. *A $\mathbb{C}$-algebra of the form $H_n/\mathfrak{a}$ is called an analytic algebra.*

In general, one may check that analytic algebras are local noetherian henselian $\mathbb{C}$-algebras. However, they are not necessarily integral domains (since $\mathfrak{a}$ does not need to be a prime ideal) and they are never factorial.

Remark 1.2.5. Each stalk of the structure sheaf $\mathscr{O}_{\mathbb{C}^n}$ is isomorphic to an analytic algebra. In fact, each stalk of a general complex space, see section 1.5, is an analytic algebra. See the end of section 1.5 for more exposition on this.

1.3 Sheaves

We will now define sheaves more generally. Intuitively speaking, sheaves are used to systematically track data attached to the open sets of a topological space and defined locally with regard to them. One could, for example, consider the ring of all continuous, smooth, or holomorphic functions defined on a certain open set. Indeed, sheaf theory was made as a general tool for handling questions which involve local solutions and global patching.

Some recommended literature for further reading is as follows: [**cartan2011faisceaux**, **grauert2012coherent**, **hirzebruch1966topological**, **serre1955faisceaux**].

1.3.1 An explicit construction

Definition 1.3.1 (Sheaf of sets). *A sheaf of sets on a topological space X is a pair $(\mathscr{A}, \pi)$ where $\mathscr{A}$ is a topological space and $\pi : \mathscr{A} \to X$ is a surjective local homeomorphism. The fibers $\pi^{-1}(x) =: \mathscr{A}_x$ are called the stalks. We could write (the other way around)*

$$\mathscr{A} = \coprod_{x \in X} \mathscr{A}_x,$$

where π takes some $\mathscr{A}_x$ to $x \in X$.

Definition 1.3.2 (Section). *Let $M \subset X$. A section in $\mathscr{A}$ over M is a continuous map $s : M \to \mathscr{A}$ with $s(x) \in \mathscr{A}_x$. In other words, $\pi \circ s = \mathrm{id}_M$. If U is open, then we will write $\Gamma(U, \mathscr{A}) = \mathscr{A}(U)$ to be the set of sections over U.*

Remark 1.3.1. For the sheaf, $\mathscr{A}$, restricted to U, we write $\mathscr{A}_U$. Later, we will identify this with $\mathscr{A}(U)$ (which we will also call the canonical[2] presheaf), so the notations will often be interchanged.

Before proceeding with further definitions, we make some observations. Any of these would be good easy exercises for grappling with the definitions (most are due to π being a local homeomorphism).

(i) Sections are open maps.

(ii) The sets $s(U)$ with U open in X and $s \in \mathscr{A}(U)$ form a base for the topology of $\mathscr{A}$.

(iii) The stalks, $\mathscr{A}_x$, are discrete spaces (i.e. they carry the discrete topology).

[2]Dr. Lieb does not define the word "canonical." He leaves that to the pope.

(iv) It need not be the case that $\mathscr{A}$ is Hausdorff even if X is.

(v) Suppose that $s, t \in \Gamma(U, \mathscr{A})$ and $s(x_0) = t(x_0)$. Then there is a neighborhood $V = V(x_0)$ such that $s \equiv t$ on V. This in a sense gives compatibility of sections.

(vi) We can also think of $\mathscr{A}_x$ as the set of germs of sections in x. Sometimes along these lines we will write $s(x) = s_x$.

Definition 1.3.3 (Subsheaf). *We say that $\mathscr{B} \subset \mathscr{A}$ is a subsheaf, $(\mathscr{B}, \pi_{\mathscr{B}}) \subset (\mathscr{A}, \pi_{\mathscr{A}})$, if*

(i) $\mathscr{B}$ is a sheaf,

(ii) $\mathscr{B} \subset \mathscr{A}$ is open, and

(iii) $\pi_{\mathscr{B}} = \pi_{\mathscr{A}}|_{\mathscr{B}}$.

In particular, we will have $\mathscr{B}_x \subset \mathscr{A}_x$ and $\mathscr{B}(U) \subset \mathscr{A}(U)$ for U open.

Take care to note that a subsheaf carries the same underlying topological space as the sheaf in which it is contained. Thus, for example, if $M \subset X$ is arbitrary (need not be open), then

$$\pi^{-1}(M) =: (\mathscr{A}_M, \pi|_{\pi^{-1}(M)})$$

is again a sheaf—the restriction of $\mathscr{A}$ to M—but it is not a subsheaf, because now the underlying space is M, not X.

Definition 1.3.4. *We call $\alpha : \mathscr{A} \to \mathscr{B}$ a sheaf morphism if it is continuous, $\alpha_x := \alpha|_{\mathscr{A}_x} : \mathscr{A}_x \to \mathscr{B}_x$, and the following diagram commutes:*

$$\begin{tikzcd}
\mathscr{A} \arrow[rr, "\alpha"] \arrow[dr, "\pi_{\mathscr{A}}"'] & & \mathscr{B} \arrow[dl, "\pi_{\mathscr{B}}"] \\
& X &
\end{tikzcd}$$

We call α an isomorphism if it is a homeomorphism.

The obvious consequences of this is that $\alpha(\mathscr{A})$ is a subsheaf of $\mathscr{B}$. That $\alpha(\mathscr{A})$ is open follows since α must be open, due to $\pi_{\mathscr{B}} \circ \alpha = \pi_{\mathscr{A}}$ (and both projections are local homeomorphisms). An easy example of a sheaf morphism is the identity, id : $\mathscr{A} \to \mathscr{A}$ (this makes the sheaves over X into a category). A potentially more useful example arises as follows. Suppose $\alpha : \mathscr{A} \to \mathscr{B}$ is a sheaf morphism and so is $\beta : \mathscr{B} \to \mathscr{C}$. Then in

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\alpha"] \arrow[rr, bend right, "\beta \circ \alpha"'] & \mathscr{B} \arrow[r, "\beta"] & \mathscr{C}, \end{tikzcd}$$

the map $\beta \circ \alpha : \mathscr{A} \to \mathscr{C}$ is also a sheaf morphism, i.e. sheaf morphisms are stabile under composition, as one may desire. We now turn to considering sheaves with some algebraic structure.

Definition 1.3.5 (Sheaf of abelian groups). *We say that $\mathscr{A}$ is a sheaf of abelian groups over a topological space X if*

(i) $\mathscr{A}$ is a sheaf,

(ii) each stalk $\mathscr{A}_x$ carries the structure of an abelian group,

(iii) for each $U \subset X$ open and $s, t \in \mathscr{A}(U)$ implies $(s-t)(x) = s(x) - t(x)$ is a section, and

(iv) the map $0 : X \to \mathscr{A}$ (the zero section) taking $x \mapsto 0_x \in \mathscr{A}_x$ is a continuous global section.

Of course, all sets can be given some kind of group structure, which may make one question the usefulness of (ii) above. Indeed, (ii) is only in place to allow for the important data, i.e. (iii) and (iv), since a group action on the stalk gives rise to a group action on arbitrary set-theoretic sections (we defined the group operation on the stalks, which gives one automatically for the sections). That this group structure given by (ii) is useful (that it varies continuously from stalk to stalk) is given by (iii); explicitly, the subtraction operation maps pairs of continuous sections to a continuous section. Item (iv) is not strictly necessary as it is implied by (iii), but it is important, since it guarantees that the additive identity element varies continuously from stalk to stalk.

We could make this intuition even more explicit by replacing (ii) and (iii) above with the following condition.

(ii,iii)′ We have a subtraction operation, $-$, which is continuous on the fiber product

$$\mathscr{A} \times_\pi \mathscr{A} = \{(a,b) \in \mathscr{A} \times \mathscr{A} : \pi(a) = \pi(b)\},$$

i.e. the map $\mathscr{A} \times_\pi \mathscr{A} \xrightarrow{-} \mathscr{A}$ taking $(a,b) \mapsto a - b$ is continuous.

Another comment is that for $M \subset X$ arbitrary, $\Gamma(M, \mathscr{A})$ is an abelian group (M need not be open). Now, all of the definitions from before extend as one may expect.

Definition 1.3.6. *We call $\alpha : \mathscr{A} \to \mathscr{B}$ a sheaf morphism of abelian groups if*

(i) it is a sheaf morphism and

(ii) $\alpha_x : \mathscr{A}_x \to \mathscr{B}_x$ is a morphism of abelian groups.

We can write

$$\alpha = \{\alpha_x : \mathscr{A}_x \to \mathscr{B}_x \textit{ morphisms}\}.$$

From here we can also define isomorphisms, composition of morphisms, and subsheaves of sheaves of abelian groups. Sometimes we will also write $\alpha_U : \mathscr{A}(U) \to \mathscr{B}(U)$. In particular, if $U \subset X$ is open and $s \in \mathscr{A}(U)$, then $\alpha_U \circ s \in \mathscr{B}(U)$. We can also define

$$\ker \alpha = \{\ker \alpha_x : x \in X\},$$

where $\ker \alpha_x = \{a \in \mathscr{A}_x : \alpha_x(a) = 0\}$. This way, one may check that $\ker \alpha$ is a subsheaf of $\mathscr{A}$. Also, $\operatorname{im} \alpha \subset \mathscr{B}$ is a subsheaf of $\mathscr{B}$.

In the following, we will always be considering commutative rings with unity (without mention).

Definition 1.3.7 (Sheaf of rings). *A sheaf of (commutative) rings $(\mathscr{R}, \pi)$ fulfills the following properties:*

(i) $\mathscr{R}$ is a sheaf of abelian groups,

(ii) $\mathscr{R}_x$ has a ring structure,

(iii) if $s, t \in \mathscr{R}(U)$, then $(s \cdot t)(x) = s(x)t(x)$ is a section, and

(iv) the map taking X to $\mathscr{R}$ given by $x \mapsto 1_x$ (the unity section) is a global continuous section.

Once again, the third condition is to guarantee that the ring structure on each stalk agrees with that on others. This could be reformulated as with the sheaf of abelian groups by requiring continuity of the map from $\mathscr{A} \times_\pi \mathscr{A} \to \mathscr{A}$ taking $(a, b) \mapsto a \cdot b$, which provides each stalk $\mathscr{A}_x$ with the structure of a commutative ring. In addition, with this definition, one may show that $\Gamma(M, \mathscr{R})$ is a ring for each $M \subset X$ arbitrary (not necessarily open). These are the definitions we will need, so with that, we turn to some remarks and then some relevant examples.

Remark 1.3.2. The case $1_x = 0_x$ above is *not* excluded, but this will not be the case for $\mathscr{A}_x \neq 0_x$ (exercise).

Remark 1.3.3. Returning briefly to the idea of analytic continuation, if s, t are sections in any sheaf over an open set U, then $\{x \in U : s_x = t_x\}$ is always open in U. If the topology on the sheaf is Hausdorff, then the set is also closed in U. Thus, if two sections of a Hausdorff sheaf have the same germ at one point, then these sections coincide in the connected component of that point. This is a kind of principle of analytic continuation for sheaves. That is, it is a kind of identity theorem for sections on Hausdorff sheaves.

Historical quip 1.3.1. It is at this point, after all of these definitions that require so much intuitive explanation, that one wonders what is wrong with mathematicians. Goethe apparently once said that mathematicians are a type of Frenchman: when told something they immediately translate it to their own language at which point it means something completely different.

1.3.2 Examples

First, we already defined the structure sheaf $\mathscr{O}$ in section 1.0. The stalks are $\mathscr{O}_{z_0}$ for each $z_0 \in \mathbb{C}^n$ where $\mathscr{O}_{z_0} = \mathbb{C}\{z - z_0\}$ and

$$\mathscr{O} = \coprod_{z \in \mathbb{C}^n} \mathscr{O}_z.$$

Now, let $U \subset \mathbb{C}^n$ be open and let f be holomorphic on U. Then we define $V(U, f) := \{f_x : x \in U\} \subset \mathcal{O}$. Then $V(U, f)$ is a basis for $\mathcal{O}$. This structure sheaf of $\mathbb{C}^n$, namely $(\mathcal{O}, \pi)$, is a sheaf of rings.

Furthermore, if U is open, then there is a ring isomorphism between the ring of holomorphic functions on U and the ring of sections over U in $\mathcal{O}$. Both of these rings we denote by $\mathcal{O}(U)$. If $f \in \mathcal{O}(U)$, then "the value of f at $x \in U$" can unambiguously be regarded as $f(x) \in \mathbb{C}$ or $f_x \in \mathcal{O}_x$.

Recall also that the structure sheaf is Hausdorff (because of the identity theorem), which distinguishes it from many other sheaves. In particular, none of the following sheaves are Hausdorff if $X = \mathbb{C}^n$ below:

(i) the germs of continuous functions from a topological space X to $\mathbb{C}$, denoted $\mathscr{C}^0_X$,

(ii) the germs of k-times continuously differentiable functions, denoted $\mathscr{C}^k_X$, and

(iii) the germs of infinitely differentiable functions, denoted $\mathscr{C}^\infty_X$.

We may restrict any of these to an open set, e.g. $\mathscr{C}^0_X(U)$ is the sheaf of germs of continuous functions on $U \subset X$.

A very simple example of a sheaf is as follows. Take X to be a topological space. Then each of $X \times \mathbb{C}$, $X \times \mathbb{R}$, and $X \times \mathbb{Z}$ with $\mathbb{C}, \mathbb{R}, \mathbb{Z}$ carrying the discrete topology, give a sheaf. The sections on these sheaves are the locally constant functions. Another simple example is if X is a singleton. Then a sheaf of rings over X is just a ring.

Another example is the meromorphic functions. Recalling that $\mathcal{O}_x$ is an integral domain for each x, we may consider its field of fractions, $\mathscr{M}_x := \operatorname{Quot}(\mathcal{O}_x)$, and define

$$\mathscr{M} = \coprod_{x \in \mathbb{C}^n} \mathscr{M}_x.$$

Observe that $\mathcal{O} \subset \mathscr{M}$ is a subsheaf.

If we consider $U \subset \mathbb{C}^n$ open and $f = g/h$ with g, h holomorphic, h not identically zero (i.e. it is not the zero germ), then the sets

$$V(U, f) = \left\{ f_x = \frac{g_x}{h_x} : x \in U \right\}$$

form a base of the topology of $\mathscr{M}$, which is a sheaf of fields. We also denote by $\mathscr{O}_x^\times$ the multiplicative group of units of $\mathscr{O}_x$. Then $\mathscr{O}^\times$ is the sheaf of multiplicative groups. We may define $\mathscr{M}^\times$ correspondingly.

Remark 1.3.4. If $h \in \mathscr{M}$ is a quotient f/g of holomorphic functions, $f, g \in \mathscr{O}$, then clearly $V(h) \subset V(f)$ and the poles of h are a subset of the zeros of g. In general, h cannot be expressed, even locally, as such a quotient where the zeros and poles exactly line up with those zeros of f and g. The trouble is that the zero set and pole set of h, when in the several variable setting, may have points in common (e.g. for $z/w \in \mathscr{M}_{\mathbb{C}^2}$). These points are a major problem that we will avoid in these notes. They are called points of indeterminacy.

1.3.3 Construction of sheaves via presheaves

We now define sheaves through presheaves. Some would say this is the only correct way to define sheaves. In practice, this definition, as we will see, is very slick, since it will allow us to define each type of sheaf at once. This elegance, however, comes at the price of usability, i.e. actually doing anything with the definition can at times be a challenge.

Let X be a topological space; as mentioned in appendix A (where one can review some category theory), the set of open sets in X form a category, $\mathscr{T}$, whose objects are the open sets and the morphisms are inclusions. Viewed differently, one may think of the morphisms as inclusion maps, i.e.

$$\mathrm{Hom}(U, V) = \begin{cases} \emptyset, & \text{if } U \not\subset V \\ \iota : U \hookrightarrow V, & \text{if } U \subset V. \end{cases}$$

Definition 1.3.8. *A presheaf $\overline{\mathscr{A}}$ is a contravariant functor from $\mathscr{T}$ into some category, e.g. sets, abelian groups, rings, etc.*

To translate this into some more explicit terms, if $\overline{\mathscr{A}}$ is a presheaf of abelian groups, then first, $\overline{\mathscr{A}}$ associates each open set $U \subset X$ to an abelian group $\overline{\mathscr{A}}(U)$. Secondly, it associates to an open set inclusion, $\iota : U \to V$, a group homomorphism, $\rho_U^V : \overline{\mathscr{A}}(V) \to \overline{\mathscr{A}}(U)$ (since the presheaf is a contravariant functor, the domain and codomain are reversed), such that

(i) $\rho^U_U = \mathrm{id}_{\overline{\mathscr{A}}(U)}$ and

(ii) given open set inclusions $U \subset V \subset W$, we have that

$$\begin{tikzcd} \overline{\mathscr{A}}(W) \arrow[r, "\rho^W_V"] \arrow[rr, bend right, "\rho^W_U"'] & \overline{\mathscr{A}}(V) \arrow[r, "\rho^V_U"] & \overline{\mathscr{A}}(U) \end{tikzcd}$$

commutes. Of course, commutes here means $\rho^W_U = \rho^V_U \circ \rho^W_V$.

By convention, the morphism ρ^V_U is called the restriction homomorphism of V to U (this name will make sense when we explicitly look again at the structure sheaf). At least intuitively, we observe that this is consistent with the image of a sheaf that we drew following the definition of the structure sheaf of $\mathbb{C}^n$.

Definition 1.3.9. *A homomorphism of presheaves $\overline{\mathscr{A}}, \overline{\mathscr{B}}$ is a natural transformation $F : \overline{\mathscr{A}} \to \overline{\mathscr{B}}$.*

Once again, we translate this into more concrete terms. Consider presheaves of abelian groups. For each open $U \subset X$, there is a group homomorphism

$$F_U : \overline{\mathscr{A}}(U) \to \overline{\mathscr{B}}(U)$$

called the component of F at U such that for all open set inclusions $U \subset V$, the diagram

$$\begin{tikzcd} \overline{\mathscr{A}}(U) \arrow[r, "F_U"] & \overline{\mathscr{B}}(U) \\ \overline{\mathscr{A}}(V) \arrow[u, "\rho^V_U"] \arrow[r, "F_V"'] & \overline{\mathscr{B}}(V) \arrow[u, "\rho^V_U"'] \end{tikzcd}$$

commutes; the ρ^V_U on the left corresponds to the presheaf $\overline{\mathscr{A}}$ and that on the right to $\overline{\mathscr{B}}$ (i.e. they are not the same map).

For example, consider some sheaf $\mathscr{A}$ and set $\overline{\mathscr{A}}(U) = \Gamma(U, \mathscr{A})$. For a section, s on U, we have the restriction $\rho^U_V s = s|_V$. Then $\overline{\mathscr{A}}$ is a presheaf; in fact, we will call this presheaf a canonical presheaf of $\mathscr{A}$

(we can do this more generally, see **[grauert2012coherent]**). Considering, in particular, the structure sheaf $\mathscr{O}$ for $\mathbb{C}^n$, we have

$$\overline{\mathscr{O}}(U) = \Gamma(U, \mathscr{O}) \cong \mathscr{O}(U).$$

We thus obtain the presheaf of holomorphic functions

$$\mathscr{O}(U) = \{\text{holomorphic functions on } U\}, \quad \rho_V^U = \text{restriction of } U \text{ to } V.$$

Since $\overline{\mathscr{O}}$ and $\mathscr{O}$ are isomorphic as presheaves, we will use the same notation for the structure sheaf and the canonical presheaf. Indeed, for reasons discussed momentarily, we will usually identify a sheaf and its associated canonical presheaf.

We have seen the easy direction (as indicated by the law of good names for definitions): obtaining a presheaf from a sheaf is simple (just consider the presheaf of sections). We now construct a sheaf from a presheaf.

Let $\overline{\mathscr{A}}$ be a presheaf (of sets, abelian groups, rings, another mathematical object of preference) over a topological space X. Fix $x \in X$ and define

$$\widetilde{\mathscr{A}_x} = \coprod_{\substack{U \subset X \text{ open} \\ x \in U}} \overline{\mathscr{A}}(U). \tag{1.3}$$

We introduce the following equivalence relation on $\widetilde{\mathscr{A}_x}$. Given $s \in \overline{\mathscr{A}}(U)$ and $t \in \overline{\mathscr{A}}(V)$, we say that s and t are x-equivalent (or equivalent at x) and write $s \sim_x t$ if there is an open neighborhood $W \subset U \cap V$ of x such that

$$\rho_W^U(s) = \rho_W^V(t).$$

One may easily check that this defines an equivalence relation. Now we define the stalks $\mathscr{A}_x$ to be

$$\mathscr{A}_x = \widetilde{\mathscr{A}_x} / \sim_x .$$

One should notice the similarity between this definition and that of a germ (which we used to construct the structure sheaf in the first place). Intuitively, what we have done here is carelessly taken all of the information we have around x from the presheaf in (1.3) and then thrown out overlap with the equivalence relation.

We cannot, however, define a sheaf purely from its stalks, because we need a topology. To that end, for each open set $U \subset X$ with $x \in U$, we have a canonical map

$$\overline{\mathscr{A}}(U) \xrightarrow{\rho_x^U} \mathscr{A}_x$$

given by the composite function

$$\overline{\mathscr{A}}(U) \hookrightarrow \widetilde{\mathscr{A}_x} \longrightarrow \mathscr{A}_x,$$

where the latter map is the natural projection. Now for the topology, consider $U \subset X$ open and $s \in \overline{\mathscr{A}}(U)$. Define

$$V(U,s) = \{\rho_x^U s : x \in U\};$$

these form a basis for our topology. Thus, to the presheaf $\overline{\mathscr{A}}$, we have associated a sheaf, $\mathscr{A}$.

Moreover, there is a canonical morphism between presheaves and the canonical presheaf (i.e. a presheaf gives rise to a sheaf, which gives rise to a canonical sheaf, and there is a canonical morphism between these presheaves). To be more explicit, if $\mathscr{A}$ is a sheaf constructed from a presheaf $\overline{\mathscr{A}}$, then for each element $s \in \overline{\mathscr{A}}(U)$, we have the section,

$$s_U : U \to \mathscr{A}, \quad x \mapsto s_x := \rho_x^U s$$

of $\mathscr{A}$ over U. In this way, we get a map $F_U : \overline{\mathscr{A}}(U) \to \mathscr{A}(U)$. It is easily verified that the family (F_U) is a presheaf map from $\overline{\mathscr{A}}$ into the canonical presheaf of $\mathscr{A}$.

In general, F_U is neither injective nor surjective, but in the case when all the maps F_U are bijective, we call $\overline{\mathscr{A}}$ a canonical presheaf. One may show that a morphism of sheaves, F, is an isomorphism if and only if every F_U is bijective. As mentioned before, we will just call the presheaf $\overline{\mathscr{A}}$ a sheaf if the canonical morphism taking $\overline{\mathscr{A}}$ to the canonical presheaf is an isomorphism, which is the case when the presheaf is a canonical presheaf.

To sum up, we have the following for $\overline{\mathscr{A}}$, a presheaf, and $\mathscr{A}$, its associated sheaf.

(i) There exists a canonical homomorphism from $\overline{\mathscr{A}}$ into the canonical presheaf of $\mathscr{A}$.

(ii) If $\overline{\mathscr{A}}$ is the canonical presheaf of $\mathscr{A}$, then there is a canonical isomorphism between $\mathscr{A}$ and the sheaf constructed by $\overline{\mathscr{A}}$.

(iii) A presheaf is just called a sheaf if it is isomorphic to the canonical presheaf of its associated sheaf.

(iv) A homomorphism of presheaves, $F : \overline{\mathscr{A}} \to \overline{\mathscr{B}}$ gives rise to a homomorphism, $\mathscr{A} \to \mathscr{B}$, between their associated sheaves.

Remark 1.3.5. In the language of category theory we can shorten this construction a bit. Let $\overline{\mathscr{A}}$ be a presheaf on X. For each point $x \in X$ the subsystem $(\overline{\mathscr{A}}(U), \rho^U_V, x \in U)$ is directed with respect to inclusion. Thus, the direct limit

$$\mathscr{A}_x := \varinjlim_{x \in U} \overline{\mathscr{A}}(U)$$

and the maps $\rho^U_x : \overline{\mathscr{A}}(U) \to \mathscr{A}_x$ are well-defined.

1.4 Coherent sheaves

As we mentioned before, sheaf theory is a tool for handling questions with local solutions and global patching. Now, here, we implement the idea of coherence, which makes it possible to pass from point-properties to local properties. For the purposes of demonstration (so we will not give the proof), a typical example is the following.

Let $\mathscr{A}' \xrightarrow{\varphi} \mathscr{A} \xrightarrow{\psi} \mathscr{A}''$ *be a sequence of coherent sheaves on a topological space X. If, for a certain point* $x \in X$*, the sequence* $\mathscr{A}'_x \xrightarrow{\varphi_x} \mathscr{A}_x \xrightarrow{\psi_x} \mathscr{A}''_x$ *is exact, then the same holds for all points sufficiently near to x.*

In a vague sense, coherence is a local principle of analytic continuation. In any case, we will shortly see that it is a difficulty of the theory of coherent sheaves that there are no nontrivial examples to give immediately. Proving that a sheaf is coherent is among the most difficult of problems facing complex analysts. As a reward, though, the fruit of the labor we will go through to prove coherence is some of the most delicious.

1.4.1 Definitions

Let $\mathscr{R}$ be a sheaf of rings. We can look at modules over $\mathscr{R}$. As usual, the underlying topological space will be called X.

Definition 1.4.1. *An $\mathscr{R}$-module sheaf (or sheaf of $\mathscr{R}$-modules) is a sheaf, $\mathscr{A}$, of abelian groups such that each stalk, $\mathscr{A}_x$, is an $\mathscr{R}_x$-module and so that for $U \subset X$ open, given $f \in \mathscr{R}(U)$ and $a \in \mathscr{A}(U)$, the function $fa : U \to \mathscr{A}$ defined as $fa(x) = f(x)a(x)$ is a continuous section.*

Equivalently, the set $\mathscr{A}(U)$ is an $\mathscr{R}(U)$-module and for each open set inclusion $V \subset U$, we have

$$\rho_V^U(fa) = (\rho_V^U f)(\rho_V^U a).$$

In other words, there is a sheaf morphism

$$\mathscr{R} \times \mathscr{A} \longrightarrow \mathscr{A}$$

such that at each open set $U \subset X$, the component

$$\mathscr{R}(U) \times \mathscr{A}(U) \longrightarrow \mathscr{A}(U)$$

is an $\mathscr{R}(U)$-module structure on $\mathscr{A}(U)$. It is left to the reader to define homomorphisms of $\mathscr{R}$-modules (it is a good exercise in understanding what they are).

Definition 1.4.2. *Let $\mathscr{A}, \mathscr{B}$ be $\mathscr{R}$-module sheaves. We define $\mathscr{A} \oplus \mathscr{B}$ as the $\mathscr{R}$-module sheaf whose sections at $U \subset X$ are*

$$(\mathscr{A} \oplus \mathscr{B})(U) = \mathscr{A}(U) \oplus \mathscr{B}(U).$$

This implies $(\mathscr{A} \oplus \mathscr{B})_x = \mathscr{A}_x \oplus \mathscr{B}_x$.

A simple example is

$$\underbrace{\mathscr{R} \oplus \mathscr{R} \oplus \cdots \oplus \mathscr{R}}_{k \text{ times}} =: \mathscr{R}^{\oplus k} =: \mathscr{R}^k.$$

Definition 1.4.3. *Let $\alpha : \mathscr{A} \to \mathscr{B}$ be a homomorphism of $\mathscr{R}$-modules. Then* $\ker \alpha$ *is the sheaf of $\mathscr{R}$-modules defined as*

$$(\ker \alpha)(U) = \{a \in \mathscr{A}(U) : \alpha_U(a) = 0\}.$$

Of course, $\ker \alpha \subset \mathscr{A}$ is a subsheaf (it is not obvious that the presheaf $\ker \alpha$ is a sheaf; this is left as an exercise). This implies that

$$(\ker \alpha)_x = \ker(\alpha_x : \mathscr{A}_x \to \mathscr{B}_x).$$

Definition 1.4.4. *The image is defined as*

$$(\operatorname{im} \alpha)_x = \operatorname{im}(\alpha_x : \mathscr{A}_x \to \mathscr{B}_x).$$

One may check that this defines a subsheaf $\operatorname{im} \alpha \subset \mathscr{B}$. We could define the image by the presheaf $(\operatorname{im} \alpha)(U) = \operatorname{im} \alpha_U$, but this presheaf will not turn out to be a sheaf in general.

Definition 1.4.5. *We will say that a morphism of sheaves, $\alpha : \mathscr{A} \to \mathscr{B}$ is injective if* $\ker \alpha = 0$ *and surjective if* $\operatorname{im} \alpha = \mathscr{B}$.

Remark 1.4.1. Note that one may prove that a morphism of sheaves, $\alpha : \mathscr{A} \to \mathscr{B}$, is injective if and only if the corresponding morphisms, $\alpha_x : \mathscr{A}_x \to \mathscr{B}_x$, are all injective. The analogous statement for surjectiveness does not hold, however.

We now consider $\mathscr{R}$-modules, $\mathscr{A} \subset \mathscr{B}$.

Definition 1.4.6. *The sheaf associated to the presheaf*

$$\overline{\mathscr{B}/\mathscr{A}}(U) = \mathscr{B}(U)/\mathscr{A}(U)$$

is called the quotient sheaf, denoted by $\mathscr{B}/\mathscr{A}$.

Warning: One may prove that $(\mathscr{B}/\mathscr{A})_x = \mathscr{B}_x/\mathscr{A}_x$, but be warned that the presheaf $\overline{\mathscr{B}/\mathscr{A}}$ is not a sheaf, so in general

$$(\mathscr{B}/\mathscr{A})(U) \neq \mathscr{B}(U)/\mathscr{A}(U),$$

although we will always have $(\mathscr{B}/\mathscr{A})(U) \supset \mathscr{B}(U)/\mathscr{A}(U)$.

We can also define exact sequences with these $\mathscr{R}$-modules.

Definition 1.4.7. *A sequence*

$$\mathscr{A} \xrightarrow{\alpha} \mathscr{B} \xrightarrow{\beta} \mathscr{C}$$

of sheaves of $\mathscr{R}$-modules is called exact at $\mathscr{B}$ if $\operatorname{im}\alpha = \ker\beta$. Analogously we can define this for longer sequences. A sequence is exact if it is exact at each $\mathscr{R}$-module other than the first and last in the sequence (since exactness is not defined there).

Lemma 1.4.8. *We have the following conditions for exactness (which should be familiar).*

(i) $0 \longrightarrow \mathscr{A} \xrightarrow{\alpha} \mathscr{B}$ *is exact if and only if α is injective.*

(ii) $\mathscr{A} \xrightarrow{\alpha} \mathscr{B} \longrightarrow 0$ *is exact if and only if α is surjective.*

(iii) $0 \longrightarrow \mathscr{A} \xrightarrow{\alpha} \mathscr{B} \xrightarrow{\beta} \mathscr{C} \longrightarrow 0$ *is exact if and only if α is injective and β is surjective. In this case*

$$\mathscr{C} \cong \mathscr{B}/\operatorname{im}\alpha = \mathscr{B}/\ker\beta.$$

Of course, here 0 is the zero sheaf (or the sheaf of 0-modules).

This lemma would make for a good exercise to play with the definitions if one is not comfortable with them.

Finally, before defining coherence, we must define finiteness. Suppose we have a ringed space $(X, \mathscr{R})$, i.e. a sheaf of rings, $\mathscr{R}$, over a topological space, X. Further, let $\mathscr{A}$ be an $\mathscr{R}$-module and $s_1, \ldots, s_k \in \mathscr{A}_U$ be finitely many sections. Then we define the $\mathscr{R}_U$-homomorphism

$$\sigma : \mathscr{R}_U^k \to \mathscr{A}_U, \quad (r_{1,x}, \ldots, r_{k,x}) \mapsto \sum_{\kappa=1}^{k} r_{\kappa,x} s_{\kappa,x}, \quad x \in U.$$

We say that $\mathscr{A}_U$ is generated by the sections $s_1, \ldots, s_k$ if σ is surjective. Expressed differently, each element of the stalk $\mathscr{A}_x$ for $x \in U$ is a linear combination of the germs $s_{1,x}, \ldots, s_{k,x}$ with coefficients in $\mathscr{R}_x$.

Definition 1.4.9. *An $\mathscr{R}$-module $\mathscr{A}$ is said to be finite (or finitely generated) at $x \in X$ if there is an open neighborhood U of x such that $\mathscr{A}_U$ is generated by finitely many sections in $\mathscr{A}$ over U. We say that $\mathscr{A}$ is finite on X if it is finite at each point $x \in X$.*

Indeed, if $\mathscr{A}$ is finite on X, all stalks $\mathscr{A}_x$ are finite (i.e. finitely generated) $\mathscr{R}_x$-modules, however being finite means much more for $\mathscr{A}$, as we will see (at least initially, take care to notice that finiteness is a local property, not just a point property). Some examples are in order.

(i) All sheaves $\mathscr{R}^k$ for $1 \leqslant k < \infty$ are finite on X. The sections $e_i = (0, \ldots, 1, \ldots, 0)$ for $1 \leqslant i \leqslant k$ generate $\mathscr{R}^k$ over X.

(ii) The sheaf $\mathscr{M}_G$ of germs of meromorphic functions on a region G (recall, region means G is open and connected) in $\mathbb{C}^n$, $n \geqslant 1$, is nowhere finite (since denominators can have arbitrarily large orders and a meromorphic function cannot have an arbitrarily large order in the denominator, see appendix B).

(iii) Subsheaves of finite sheaves are not necessarily finite. Take a region $G \subset \mathbb{C}^n$ and $U \subsetneq G$ open. Consider the $\mathscr{O}_G$-submodule $\mathscr{I}$ of $\mathscr{O}_G$ given by $\mathscr{I}_z := \mathscr{O}_z$ for $z \in U$ and $\mathscr{I}_z := 0_z$ for $z \in G \setminus U$. At all points of $\partial U \cap G$, this sheaf is not of finite type (exercise). Finiteness is, however, preserved by restriction since it is a local property anyway.

Take $U \subset X$ open and $V \subset U$ also open. Now suppose that $\sigma : \mathscr{R}_U^k \to \mathscr{A}_U$ is an $\mathscr{R}_U$-homomorphism determined by the sections $s_1, \ldots, s_k \in \mathscr{A}(U)$.

Definition 1.4.10. *The sheaf of relations of $s_1, \ldots, s_k$ is defined by the presheaf*

$$\mathscr{R}(s_1, \ldots, s_k)(V) := \left\{ (r_1, \ldots, r_k) \in \mathscr{R}^k(V) : \sum_{\kappa=1}^{k} r_\kappa s_\kappa = 0 \right\}.$$

We could write

$$\mathscr{R}(s_1, \ldots, s_k) = \ker \sigma = \bigcup_{x \in U} \left\{ (r_{1,x}, \ldots, r_{k,x}) \in \mathscr{R}_x^k : \sum_{\kappa=1}^{k} r_{\kappa,x} s_{\kappa,x} = 0 \right\}.$$

One can also prove that

$$\mathscr{R}(s_1, \ldots, s_k)_x = \left\{ (r_1, \ldots, r_k) \in \mathscr{R}_x^k : \sum_{\kappa=1}^{k} r_\kappa s_{\kappa,x} \right\}.$$

Obviously, $\mathscr{R}(s_1, \ldots, s_k)$ is an $\mathscr{R}_U$-submodule of $\mathscr{R}_U^k$. One intuitive perspective is that the sheaf of relations is in some way giving information about how linearly dependent sections are. If they are "too dependent," i.e. the relation sheaf is not finite, then problems arise, i.e. we will not get coherence, see below. Another perspective can be found in remark 1.4.4 below.

Definition 1.4.11 (Coherent sheaf). *An $\mathscr{R}$-module, $\mathscr{A}$, is called coherent if it is finite and all relation sheaves are finite. The sheaf of rings, $\mathscr{R}$, is called coherent if it is a coherent $\mathscr{R}$-module.*

These (above) are both local conditions, i.e. $\mathscr{A}$ is coherent if it is coherent at every point, i.e. if every $x \in X$ has an open neighborhood U such that $\mathscr{A}_U$ is coherent.

It is not very clear from the definition that coherent sheaves exist, so we give two trivial examples. These will be the only two we can give for some time.

(i) The zero sheaf, $\mathscr{A} = 0$, is coherent.

(ii) If $X = \{p\}$ and $\mathscr{R} = \mathbb{C}$, then the $\mathscr{R}$-modules are $\mathbb{C}$-vector spaces. Finite in this case corresponds to finite dimensional, which then also is the same as coherence.

One should not be discouraged by the lack of examples. Our aim for the coming sections will be to develop the calculus for coherent sheaves independent of the existence issue. This will be followed by the example of a coherent sheaf that we want: $\mathscr{O}$.

1.4.2 Properties of finiteness and coherence

We will mostly just give an exposé of some properties; we will avoid proofs that are overly annoying and not enlightening. See **[grauert2012coherent]** for the proofs.

Lemma 1.4.12. *We have that*

(i) if $\mathscr{A} \longrightarrow \mathscr{B} \longrightarrow 0$ *is exact, then* $\mathscr{A}$ *finite implies* $\mathscr{B}$ *is finite and*

(ii) $\mathscr{A}, \mathscr{B}$ *being finite implies that* $\mathscr{A} \oplus \mathscr{B}$ *is finite.*

Proof. Part (i) is trivial. For (ii), it is good practice to come up with an analogous version of (i) for a short exact sequence; then use this to prove (ii). □

Remark 1.4.2. For a converse to (i) we would need some control of the kernel of the map from $\mathscr{A}$ to $\mathscr{B}$. In particular, $\ker(\mathscr{A} \to \mathscr{B})$ being finite would work.

Proposition 1.4.13. *Suppose* $\mathscr{A}$ *is finite at* $x \in X$*,* $U \subset X$ *is open with* $x \in U$*, and* $s_1, \ldots, s_k \in \mathscr{A}(U)$ *are chosen so that they generate* $\mathscr{A}_x$ *over* $\mathscr{R}_x$ *(i.e. the germs* $s_{\kappa,x}$ *are generators for* $\mathscr{A}_x$ *over* $\mathscr{R}_x$*). Then there exists an open neighborhood* $V \subset U$ *of* x*, such that* $s_{\kappa,y}$ *generates generates* $\mathscr{A}_y$ *over* $\mathscr{R}_y$ *for all* $y \in V$*.*

Proof. Since $\mathscr{A}$ is finite at x, there exists an open neighborhood W of x and sections $t_1, \ldots, t_\ell \in \mathscr{A}(W)$ generating $\mathscr{A}_W$. Since $s_{1,x}, \ldots, s_{k,x}$ generate $\mathscr{A}_x$, we can find sections r_{ij} in a neighborhood of x such that

$$t_{j,x} = \sum_{i=1}^{k} r_{ij,x} s_{i,x}, \quad 1 \leqslant j \leqslant \ell.$$

Therefore, by definition of a germ, we may choose $V \subset W$ in such a way that

$$t_j|_V = \sum_{i=1}^{k} r_{ij} s_i, \quad 1 \leqslant j \leqslant \ell.$$

Since $t_1|_V, \ldots, t_\ell|_V$ generate $\mathscr{A}_V$ these equations imply that $s_1, \ldots, s_k$ also generate $\mathscr{A}_V$. $\square$

Note: The above proof would not work if we had infinitely many t_j. This is because we had to pick $V \subset W$ for *each* t_j. That is, we pick $V_1 \subset W$ for t_1 then $V_2 \subset W$ for t_2 and so on. Then $V_1 \cap \cdots \cap V_\ell$ is open and contains x. If there were infinitely many, we would have potentially shrunk to just the point x. This will be a recurring theme and is essentially the reason that finiteness allows us to take point properties to local properties.

Definition 1.4.14. *The support of a sheaf is*

$$\operatorname{supp} \mathscr{A} = \{x \in X : \mathscr{A}_x \neq 0_x\}.$$

Corollary 1.4.15. *If $\mathscr{A}$ is finite, then* $\operatorname{supp} \mathscr{A}$ *is closed.*

Proof. If $x \notin \operatorname{supp} \mathscr{A}$ then $\mathscr{A}_x = 0_x$ and 0 generates $\mathscr{A}$ at x, so also in $U = U(x)$ by proposition 1.4.13. $\square$

Lemma 1.4.16. *Suppose $\mathscr{A}$ is coherent and $\mathscr{B} \subset \mathscr{A}$ is a subsheaf. Then $\mathscr{B}$ is coherent if and only if it is finite.*

Theorem 1.4.17 (Serre). *Suppose we are given a short exact sequence*

$$0 \longrightarrow \mathscr{A} \xrightarrow{\alpha} \mathscr{B} \xrightarrow{\beta} \mathscr{C} \longrightarrow 0$$

of $\mathscr{R}$-modules. If two of the sheaves are coherent, so is the third. More precisely, we have the following.

(i) If $\mathscr{A}, \mathscr{B}$ are coherent with $\mathscr{A} \xrightarrow{\alpha} \mathscr{B} \xrightarrow{\beta} \mathscr{C} \longrightarrow 0$ exact, then $\mathscr{C}$ is coherent.

(ii) *If $\mathscr{B}, \mathscr{C}$ are coherent with* $0 \longrightarrow \mathscr{A} \xrightarrow{\alpha} \mathscr{B} \xrightarrow{\beta} \mathscr{C}$ *exact, then $\mathscr{A}$ is coherent.*

(iii) *If $\mathscr{A}, \mathscr{C}$ are coherent with* $0 \longrightarrow \mathscr{A} \xrightarrow{\alpha} \mathscr{B} \xrightarrow{\beta} \mathscr{C} \longrightarrow 0$ *exact, then $\mathscr{B}$ is coherent.*

Proof. This is a matter of studying the definitions. See **[serre1955faisceaux]**. □

Apparently one should not bother to read proofs of results like the one above from any source other than Serre, because most proofs follow his anyway and lack the didactic flare and skill that Serre showed in his proofs.

Remark 1.4.3. We could have defined the notion of coherence in linear algebra with modules over rings and the same theorem above would hold (with virtually the same proof). It is a good intuition to keep in mind that many of these concepts are almost the same as the corresponding ones from linear algebra.

Also just in general this is one main purpose of exact sequences: if we can put an object in an exact sequence with objects we know better, then commonly we can show some nice properties of the object in question.

We turn to some consequences of the above theorem.

Proposition 1.4.18. *The Whitney sum of finitely many coherent sheaves is again coherent.*

Proof. If $\mathscr{A} = \mathscr{A}' \oplus \mathscr{A}''$ with $\mathscr{A}', \mathscr{A}''$ coherent, then one need only consider the canonically induced exact sequence

$$0 \longrightarrow \mathscr{A}' \longrightarrow \mathscr{A} \longrightarrow \mathscr{A}'' \longrightarrow 0$$

and apply theorem 1.4.17. Induction gives the result from here. □

Proposition 1.4.19. *Suppose $\mathscr{A}, \mathscr{B}$ are coherent and $\alpha : \mathscr{A} \to \mathscr{B}$ is a sheaf homomorphism. Then* $\ker \alpha, \operatorname{im} \alpha$, *and* $\operatorname{coker} \alpha$ *are also coherent.*

Proof. We simply study the following two exact sequences

$$0 \longrightarrow \ker \alpha \hookrightarrow \mathscr{A} \longrightarrow \operatorname{im} \alpha \longrightarrow 0$$

$$0 \longrightarrow \operatorname{im} \alpha \hookrightarrow \mathscr{B} \longrightarrow \operatorname{coker} \alpha \longrightarrow 0.$$

Recall the cokernel is the quotient $\operatorname{coker} \alpha = \mathscr{B} / \operatorname{im} \alpha$. Now, $\mathscr{A}$ is finite, so $\operatorname{im} \alpha$ is finite too. But then a finite subsheaf of a coherent sheaf, like $\mathscr{B}$, is coherent. Thus $\operatorname{im} \alpha$ is coherent. Then using theorem 1.4.17, the kernel, $\ker \alpha$, must also be coherent. Likewise, $\operatorname{coker} \alpha$ must also be. $\square$

One may show just from the definition of coherence that if $\mathscr{A}$ is a coherent $\mathscr{R}$-module, then for every $x \in X$, there exists a neighborhood U of x and an exact $\mathscr{R}_U$-sequence

$$\mathscr{R}_U^p \longrightarrow \mathscr{R}_U^q \longrightarrow \mathscr{A}_U \longrightarrow 0$$

for some $1 \leqslant p, q < \infty$. With theorem 1.4.17 one may also reverse this implication. That is, if for every point $x \in X$ there exists $U = U(x)$ and positive integers p, q with an exact sequence as above, then $\mathscr{A}$ is coherent (this is because the sheaves $\mathscr{R}^p$ and $\mathscr{R}^q$ are coherent). We will call this sequence a local presentation of $\mathscr{A}$.

Remark 1.4.4. Another intuition for the relation sheaves now is that we only want to talk about finitely presented sheaf modules, that is, quotients of $\mathscr{R}^\ell$ by finite type ideals (just as we do for linear algebra modules).

Proposition 1.4.20. *Let $\mathscr{A}, \mathscr{B}$ be coherent and $x \in X$. If $\varphi : \mathscr{A}_x \to \mathscr{B}_x$ is a homomorphism, then there is a neighborhood $U = U(x)$ and an extension $\Phi : \mathscr{A}_U \to \mathscr{B}_U$ with $\Phi_x = \varphi$.*

Proof. Choose $U \ni x$ and sections $a_1, \dots, a_k$ that generate $\mathscr{A}$ over U. Call $\beta_\kappa = \varphi(a_\kappa(x)) \in \mathscr{B}_x$ and take $b_\kappa \in \mathscr{B}(U)$ with $b_\kappa(x) = \beta_\kappa$. Define Φ in the natural way through $\Phi(a_\kappa) = b_\kappa$ for each $\kappa = 1, \dots k$.

We need to check that Φ is well-defined. Since $\mathscr{R}(a_1, \dots, a_k)$ is finite, we can choose a finite system of generators over U, say $f^\lambda = (f_1^\lambda, \dots, f_k^\lambda)$ for $\lambda = 1, \dots, \ell$ with

$$\sum_{\kappa=1}^{k} f_\kappa^\lambda a_\kappa = 0, \quad \lambda = 1, \dots, \ell.$$

In particular, the above equation holds at x, so

$$\sum_{\kappa=1}^{k} f_\kappa^\lambda(x)\beta_\kappa = 0,$$

which, by the definition of a germ, gives

$$\sum_{\kappa=1}^{k} f_\kappa^\lambda b_\kappa = 0$$

on $V \subset U$. Thus, if $\sum g_\kappa a_\kappa = 0$, then

$$\Phi\left(\sum_{\kappa=1}^{k} g_\kappa a_\kappa\right) = \sum_{\kappa=1}^{k} g_\kappa b_\kappa = 0.$$

This completes the proof. □

Take a second to look back at the above proof and assure yourself of why it was important that the relation sheaf was finite.

Especially this result, but also the others, should continue to illuminate why we claimed earlier that coherence allows us to move from point-properties to local properties. We finish this section with some collected facts about sheaves, coherence, finiteness, etc. For more details, see
[grauert2012coherent, serre1955faisceaux].

We take $\mathscr{A}, \mathscr{B}$ to be coherent sheaves. Then

$$U \mapsto H(U) = \mathrm{Hom}_{\mathscr{R}|_U}(\mathscr{A}|_U, \mathscr{B}|_U)$$

is a presheaf, which also turns out to be a sheaf. We denote the associated sheaf by $\mathscr{H}om_{\mathscr{R}}(\mathscr{A}, \mathscr{B})$. It is the sheaf of germs of homomorphisms from $\mathscr{A}$ to $\mathscr{B}$. There is a canonical morphism

$$\rho_x : \mathscr{H}om_{\mathscr{R}}(\mathscr{A}, \mathscr{B})_x \to \mathrm{Hom}_{\mathscr{R}_x}(\mathscr{A}_x, \mathscr{B}_x).$$

It need not be an isomorphism. We do, however, have the following.

Proposition 1.4.21. *If $\mathscr{R}, \mathscr{A}, \mathscr{B}$ are coherent, then $\mathscr{H}om_{\mathscr{R}}(\mathscr{A}, \mathscr{B})$ is coherent. In this case ρ_x is an isomorphism.*

To prove this, use the canonically induced Hom-sequence. Of course, the above implies (for coherent sheaves $\mathscr{R}, \mathscr{A}, \mathscr{B}$)

$$\mathscr{H}om_{\mathscr{R}}(\mathscr{A}, \mathscr{B})(U) = \operatorname{Hom}_{\mathscr{R}|_U}(\mathscr{A}|_U, \mathscr{B}|_U),$$
$$\mathscr{H}om_{\mathscr{R}}(\mathscr{A}, \mathscr{B})_x = \operatorname{Hom}_{\mathscr{R}_x}(\mathscr{A}_x, \mathscr{B}_x).$$

Definition 1.4.22. *If $\mathscr{A}$ is an $\mathscr{R}$-module, then we define*

$$(\operatorname{Ann}\mathscr{A})_x = \operatorname{Ann}\mathscr{A}_x = \{r_x \in \mathscr{R}_x : r_x \mathscr{A}_x = 0_x\}$$

and call the associated sheaf, with the above as stalks, the annihilator of $\mathscr{A}$.

In the case where $\mathscr{A}$ is finite, the annihilator of $\mathscr{A}$ turns out to be an ideal sheaf, which we define presently.

Definition 1.4.23. *An $\mathscr{R}$-submodule of $\mathscr{R}$ itself is called a sheaf of ideals or, for short, an ideal (sheaf). An $\mathscr{R}$-submodule of an $\mathscr{R}$-module $\mathscr{A}$ is a subsheaf $\mathscr{B} \subset \mathscr{A}$ that is itself an $\mathscr{R}$-module when we restrict the action of $\mathscr{R}$ from $\mathscr{A}$ to $\mathscr{B}$.*

To show that the annihilator is an ideal one must only show that $\operatorname{Ann}\mathscr{A} \subset \mathscr{R}$ is open in $\mathscr{R}$; this is the case if $\mathscr{A}$ is of finite type. Details are omitted.

Another way of thinking about the annihilator is as follows. Consider the $\mathscr{R}_x$-module morphism

$$\iota_x : \mathscr{R}_x \to \operatorname{Hom}_{\mathscr{R}_x}(\mathscr{A}_x, \mathscr{A}_x), \quad r_x \mapsto (s_x \mapsto r_x s_x).$$

Then $\ker \iota_x = \operatorname{Ann}\mathscr{A}_x$. One then sees the following from this observation (which, in principle, needs a proof).

Proposition 1.4.24. *If $\mathscr{R}, \mathscr{A}$ are coherent, then* $\operatorname{Ann}\mathscr{A}$ *is coherent.*

Similar to the annihilator, one may consider the following.

Definition 1.4.25. *Suppose $\mathscr{A} \subset \mathscr{B}$ are $\mathscr{R}$-modules. Then $(\mathscr{A}_x : \mathscr{B}_x) := \{r_x \in \mathscr{R}_x : r_x \mathscr{B}_x \subset \mathscr{A}_x\}$. Define the sheaf*

$$(\mathscr{A} : \mathscr{B}) = \coprod_{x \in X} (\mathscr{A}_x : \mathscr{B}_x).$$

Once again, if $\mathscr{A}, \mathscr{B}$ are finite then $(\mathscr{A} : \mathscr{B})$ is an ideal sheaf.

Proposition 1.4.26. *If $\mathscr{R}, \mathscr{A}, \mathscr{B}$ are coherent, then so is $(\mathscr{A} : \mathscr{B})$.*

Proof. Observe that

$$(\mathscr{A} : \mathscr{B}) = \mathrm{Ann}(\mathscr{B}/\mathscr{A})$$

and apply the previous results. $\square$

One last concept we will need before moving on is that of the zero set of an ideal sheaf, $\mathscr{I}$. This is a general device for producing new $\mathbb{C}$-ringed spaces from a given one. We will write

$$V(\mathscr{I}) = \{x \in X : \mathscr{I}_x \neq \mathscr{R}_x\}.$$

In other words, $V(\mathscr{I})$ consists of those $x \in X$ such that $\mathscr{I}_x$ does not contain the multiplicative identity, 1_x. One other view (which sheds some light on the name) is

$$V(\mathscr{I}) = \mathrm{supp}(\mathscr{R}/\mathscr{I}),$$

which immediately implies that $V(\mathscr{I})$ is closed, since its complement is $\{x \in X : 1_x \in \mathscr{I}_x\}$, which is evidently open in X.

1.5 Complex Spaces

From the beginning it was clear that the notion of a complex manifold was not general enough. For example in $\mathbb{C}^3$ zero sets of quadratics like $w^2 - z_1 z_2$ are not complex manifolds. In order to include such algebraic varieties one needs a category of "local models" larger than that of the open sets.

There is a long history of trying to solve this problem; after many ideas from great mathematicians such as H. Behnke, K. Stein, and H. Cartan, it was finally Serre in 1955 that allowed all analytic sets (i.e. zero sets of holomorphic functions) as local models. Today however, local models are all analytic sets, A, together with structure sheaves, $\mathscr{O}_A$. Intuitively, one should think of complex spaces as complex manifolds that are allowed to have singularities.

Take $U \subset \mathbb{C}^n$ and $\mathscr{O}_U$, the structure sheaf on U. If $\mathscr{I} \subset \mathscr{O}_U$ is an ideal sheaf, then recall that $V(\mathscr{I})$ is closed in U.

Definition 1.5.1. *An analytic set $A \subset U$ is a set such that for all $x \in U$ there exists a neighborhood $V = V(x)$ and a finite ideal sheaf, $\mathscr{I}$, on $V(x)$ such that $A \cap V(x) = V(\mathscr{I})$. Analytic sets are closed sets that are locally the zero sets of finitely generated (i.e. coherent) ideals.*

Remark 1.5.1. The zero set of sections $s_1, \ldots, s_k \in \mathscr{O}(U)$ is defined as the zero set of the ideal $\mathscr{O}s_1 + \cdots + \mathscr{O}s_k$ generated by $s_1, \ldots, s_k$. We thus have

$$V(s_1, \ldots, s_k) = \{x \in U : s_{1x}, \ldots, s_{kx} \in \mathfrak{m}(\mathscr{O}_x)\} = V(s_1) \cap \cdots \cap V(s_k).$$

Since, as discussed before, $s_x \in \mathfrak{m}(\mathscr{O}_x)$ if and only if $s(x) = 0$, we see that calling this the zero set makes sense. Reframing this, if $f_1, \ldots, f_k$ are generators on $V(x)$ for $\mathscr{I}$, then $x' \in A$ if and only if $f_\kappa(x') = 0$ for all $x' \in V$.

We have two extreme cases:

(i) $\mathscr{I} = (0)$, in which case $V(\mathscr{I}) = U$, and

(ii) $\mathscr{I} = \mathscr{O}_U$, which corresponds to $V(\mathscr{I}) = \emptyset$.

In all other cases, A is closed, nowhere dense, has Lebesgue measure zero, and is nowhere separating (i.e. if W is open and connected, then $W \setminus A$ is connected). For intuition it can help to know that analytic sets have complex codimension 1 (for example, in $\mathbb{C}$, analytic sets are points).

Definition 1.5.2. *To each set $M \subset \mathbb{C}^n$ (or a general topological space) we can attach a family of ideals*

$$\mathscr{I}_M(U) := \{f \in \mathscr{O}(U) : V(f) \supset M \cap U\}.$$

This is the analytic presheaf of ideals, in fact it is a canonical presheaf, i.e. already a sheaf. We will call this analytic ideal sheaf $\mathscr{I}_M$.

This definition will be particularly important if the underlying set is analytic. Some quick remarks on this definition are below.

(i) If A is analytic, then $V(\mathscr{I}_A) = A$.

(ii) If $\mathscr{I}$ is defined on W and $V(\mathscr{I}) = W \cap A$, then $\mathscr{I} \subset \mathscr{I}_A|_W$, so $\mathscr{I}_A$ is the maximal ideal sheaf of A.

(iii) It will turn out that $\mathcal{I}_A$ is finite, although we will not be able to prove this for some time.

Definition 1.5.3. *Let $A \subset U$ be an analytic set and $\mathcal{I}$ a finite ideal sheaf such that $A = V(\mathcal{I})$. A function $f : A \to \mathbb{C}$ is said to be holomorphic if for all $x \in A$, there exists $V = V(x)$ and a holomorphic function $\widetilde{f}$ on U such that*

$$\widetilde{f}|_{A\cap V} = f|_{A\cap V}.$$

In other words, we define $\mathcal{O}_{A,a} = (\mathcal{O}_{U,a}/\mathcal{I}_a)|_A$ for each $a \in A$.

Remark 1.5.2. In the definition above, we have actually discretely given two definitions of $\mathcal{O}_A$; it is an exercise to show they are isomorphic. The fact that the lift is well-defined is reflected in the quotient by $\mathcal{I}$.

Remark 1.5.3. Of course, just given an analytic set A there are many $\mathcal{O}_A$ that could be constructed from it by choosing different ideals $\mathcal{I}$ with $A = V(\mathcal{I})$. Thus, if at any time we do not mention the ideal, then it may be assumed that there is one fixed which gives the $\mathcal{O}_A$ we are considering. In fact, in most contexts it makes most sense to just give A and $\mathcal{O}_A$ (and this fixes an ideal).

Note that $\mathcal{O}_A$ is a sheaf of rings (in fact it is a sheaf of $\mathbb{C}$-algebras).

Definition 1.5.4. *A ringed space is a pair $(X, \mathcal{R})$ where X is a topological space and $\mathcal{R}$ is a sheaf of rings over X. A ringed space is called reduced if $\mathcal{R}$ is a subsheaf of $\mathcal{C}^0_X$, the sheaf of complex-valued continuous functions on X. A ringed space, $(X, \mathcal{R})$, is called a $\mathbb{C}$-ringed space if $\mathcal{R}$ is a sheaf of local $\mathbb{C}$-algebras.*

All previous examples are reduced ringed spaces. We now introduce a complex model space. If $G \subset \mathbb{C}^n$ is a region and $\mathcal{I}$ is an ideal sheaf in $\mathcal{O}_G$, which is of finite type on G, say with generators $f_1, \ldots, f_k$ near z, then the quotient sheaf $\mathcal{O}_G/\mathcal{I}$ is a sheaf of rings on G. Write $A = \operatorname{supp}(\mathcal{O}_G/\mathcal{I})$, i.e. the set where $(\mathcal{O}_G/\mathcal{I})_z \neq 0$. Clearly in a neighborhood U of z we have $A \cap U = V(f_1, \ldots, f_k)$, so locally, A is the zero set of finitely many holomorphic functions (i.e. it is an analytic set). As above, we write $\mathcal{O}_A = (\mathcal{O}_G/\mathcal{I})|_A$.

Definition 1.5.5. *The ringed space $(A, \mathcal{O}_A)$ is called a complex model space (in G).*

As we alluded to at the beginning of this section, locally, complex spaces should look like complex model spaces. Before we can even think of saying this, though, we need the notion of an isomorphism.

A familiar situation is if $\varphi : X \to Y$ is a continuous map between topological spaces, every continuous function $f : V \to \mathbb{C}$ on an open set $V \subset Y$ can be "lifted" by φ to the function $f \circ \varphi$, which is continuous in the open set $\varphi^{-1}(V)$ of X. Thus, for $V \subset Y$, we have induced $\mathbb{C}$-algebra homomorphisms

$$\widetilde{f}_V : C^0(V) \to C^0(f^{-1}(V)),$$

which clearly commute with restrictions. Similarly, we come up with the following definition.

Definition 1.5.6. *Suppose $(X, \mathscr{R}_X), (Y, \mathscr{R}_Y)$ are reduced ring spaces. A map $(\varphi, \widetilde{\varphi}) : (X, \mathscr{R}_X) \to (Y, \mathscr{R}_Y)$ is a morphism of reduced ringed spaces if $\varphi : X \to Y$ is continuous and for each $V \subset Y$ open and $f \in \mathscr{R}_Y(V)$, we have $f \circ \varphi \in \mathscr{R}_X(\varphi^{-1}(V))$. That is, we have a morphism of sheaves over Y*

$$\widetilde{\varphi} : \mathscr{R}_Y \to \varphi_* \mathscr{R}_X,$$

where $\varphi_* \mathscr{R}_X(V) = \mathscr{R}_X(\varphi^{-1}(V))$ is the pushforward (see section 1.6)

We sometimes omit the $\widetilde{\varphi}$ for brevity. It should be clear that the composition of morphisms of reduced ringed spaces is again a morphism of reduced ringed spaces and so are the identities. Thus, we have a category and, therefore, we have a notion of isomorphism of reduced ringed spaces.

Definition 1.5.7. *Let $(X, \mathscr{O}_X)$ be a $\mathbb{C}$-ringed space. We call $(X, \mathscr{O}_X)$ a complex space if every point of X has an open neighborhood U such that the open $\mathbb{C}$-ringed subspace $(U, \mathscr{O}_U)$ of $(X, \mathscr{O}_X)$ is isomorphic to a complex model space.*

Remark 1.5.4. We will always assume that the topology of a complex space is Hausdorff without further mention. This avoids some nasty examples like the complex line with double origin.

Remark 1.5.5. In particular, $(X, \mathscr{O}_X)$ is a complex manifold of dimension n if it is a ringed space locally isomorphic to $(U, \mathscr{O}_U)$ where $U \subset \mathbb{C}^n$ is open. That is, the model spaces come from just open sets.

Definition 1.5.8. *A sheaf, $\mathscr{A}$, is called analytic if it is an $\mathcal{O}_X$-module for some complex space $(X, \mathcal{O}_X)$.*

The obvious examples of complex spaces are complex model spaces and complex manifolds.

Definition 1.5.9. *Let $(X, \mathcal{O}_X)$ be a complex space. Then $x \in X$ is said to be regular if there exists $U = U(x)$ such that $(U, \mathcal{O}_U)$ is a complex manifold. We call $x \in X$ singular if it is not regular.*

Remark that the set of regular points is an open set in X. We present two examples to demonstrate what these singular and regular points are.

(i) In $\mathbb{C}^2$ we have that $V(zw)$ gives the coordinate axes and there is one singular point (the origin).

(ii) In $\mathbb{C}^2$ the set $V(z^2 - w^3) = A$ is called the Neil parabola, pictured below, which is homeomorphic to $\mathbb{C}^1$. The origin is a singular point. draw

Remark 1.5.6. Let $U \subset \mathbb{C}^n$ and $f_1, \dots, f_{n-k}$ be holomorphic functions on U. If we write $A = V(f_1, \dots, f_{n-k})$ and $\operatorname{rk} J^h_{(f_1, \dots, f_{n-k})}(x) = n - k$ for each $x \in A$, then $(A, \mathcal{O}_A)$ is a complex manifold and is called an analytic submanifold of U. A potentially more traditional definition of regular is that a point $x \in A$ (here A is an analytic set) is regular if there is an open neighborhood U of x with $A \cap U$ being an analytic submanifold of U. Recall definition 0.2.6.

Definition 1.5.10. *A complex space $(X, \mathcal{O}_X)$ is called reduced at x if the stalk $\mathcal{O}_x$ is a reduced ring, i.e. does not contain nonzero nilpotent elements. Likewise a space is called irreducible at x if the stalk $\mathcal{O}_x$ is an integral domain, otherwise it is called reducible.*

It need not be the case that $(A, \mathcal{O}_A)$ be reduced for analytic sets, A. For example, the so-called n-fold point is a complex model space defined in $\mathbb{C}$, with complex coordinate z, by the monomial z^n for $n \geqslant 1$. Here $V(z^n)$ is just the origin, call it p, and $\mathcal{O}_{\{p\}} = (\mathcal{O}_{\mathbb{C}}/z^n\mathcal{O}_{\mathbb{C}})|_p$ is a local $\mathbb{C}$-algebra with n generators $1, \varepsilon, \dots, \varepsilon^{n-1}$ and $\varepsilon^n = 0$. In the case $n > 1$ there are nonzero nilpotent germs on the n-fold point, so this is not reduced. If $n = 1$, then we have $(p, \mathcal{O}_{\{p\}}) = (p, \mathbb{C})$.

As we have discussed before, sections can be identified with complex valued functions. If $\mathscr{A}$ is an arbitrary sheaf of local $\mathbb{C}$-algebras on X we can attach to any $s \in \mathscr{A}(U)$, $U \subset X$, a $\mathbb{C}$-valued function taking U to $\mathbb{C}$. For every point $y \in U$, the germ $s_y \in \mathscr{A}_y = \mathbb{C} \oplus \mathfrak{m}(\mathscr{A}_y)$ can be uniquely written in the form

$$s_y = c_y + t_y$$

with $c_y \in \mathbb{C}$ and $t_y \in \mathfrak{m}(\mathscr{A}_y)$. We call c_y the complex value of s at y, i.e. we have induced a function from s, call it f. The map $s \mapsto f$ is a $\mathbb{C}$-algebra homomorphism of $\mathscr{A}(U)$ into the $\mathbb{C}$-valued functions on U. For complex spaces one can prove the induced function is continuous, i.e. we have a sheaf map $\mathscr{O}_X \to \mathscr{C}^0_X$. Beware that this map is not, in general injective; one can see this, for example, through the nilpotent elements.

For a germ, s_x, that is nilpotent, the function induced by s will actually be null in a neighborhood of x; with this, we see that for non-reduced complex spaces, sections are more than just functions; nevertheless, it is customary to call them holomorphic functions. Furthermore, morphisms of complex spaces are called holomorphic maps and isomorphisms are called biholomorphic maps.

It is at this point that we will need some basics in algebraic geometry. For a review, see appendix A.

1.6 Direct and inverse images

Here we record some results and definitions about the direct image functor and the inverse image functor, i.e. the pushforward and pullback functors. This will mostly be without proof; for more details one should check
[grauert2012coherent] or the Stacks Project.

1.6.1 The pushforward sheaf

We let $\varphi : X \to Y$ be a continuous map of topological spaces and $\mathscr{A}$ a sheaf of rings over X. Recall that we are always restricting ourselves to Hausdorff topological spaces.

Definition 1.6.1. *Let $V \subset Y$ be open. Then the pushforward along φ*

$$V \mapsto (\varphi_* \mathscr{A})(V) = \mathscr{A}(\varphi^{-1}(V))$$

is a presheaf, which is also a sheaf. It is called the direct image sheaf of $\mathscr{A}$ with respect to φ and φ_* is called the direct image functor.

With this definition, we want to define a structure on $\varphi_* \mathscr{A}$ induced by φ. If $\varphi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of complex spaces and $\mathscr{A}$ is an analytic sheaf over X (i.e. $\mathscr{A}$ is an $\mathscr{O}_X$-module), then we define the structure as follows. Take $V \subset Y$ open and $s \in \varphi_* \mathscr{A}(V) = \mathscr{A}(\varphi^{-1}(V))$. If $f \in \mathscr{O}_Y(V)$ then we say $f \circ \varphi$ is holomorphic in $\mathscr{O}_X(\varphi^{-1}(V))$ and define

$$fs := (f \circ \varphi)s \in \mathscr{A}(\varphi^{-1}(V)) = \varphi_* \mathscr{A}(V).$$

With this definition we see the following.

Lemma 1.6.2. *With the definition above, $\varphi_* \mathscr{A}$ is an analytic sheaf on Y.*

Furthermore, we observe that φ_* is a covariant additive functor. Here, additive means that we can pass φ_* through finite direct sums, i.e. if $|J| < \infty$ is an index set and

$$\mathscr{A} = \bigoplus_{j \in J} \mathscr{A}_j,$$

then

$$\varphi_* \mathscr{A} = \bigoplus_{j \in J} \varphi_* \mathscr{A}_j.$$

Equivalently, φ_* is additive on morphisms, i.e.

$$\varphi_*(\alpha + \beta) = \varphi_* \alpha + \varphi_* \beta.$$

The proof we omit; is is not immediate, but not very difficult.

Lemma 1.6.3. *If $(\varphi, \widetilde{\varphi}) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is biholomorphic (i.e. an isomorphism of reduced ringed spaces), then an analytic sheaf $\mathscr{A}$ over X is coherent if and only if $\varphi_* \mathscr{A}$ is coherent.*

Definition 1.6.4. *We recall the following important concepts.*

(i) We say φ is proper if for each $K \subset Y$ compact, $\varphi^{-1}(K)$ is compact.

(ii) We say φ is discrete if the fiber $\varphi^{-1}(y)$ is a discrete topological space for all $y \in Y$.

(iii) We call φ finite if it is compact and discrete.

Remark 1.6.1. The definition of finite above is not exactly standard and in fact differs from the one found in
[grauert2012coherent], which defines finite maps to be closed and quasifinite (i.e. finite on the fibers). One can prove without too much trouble that if $\varphi : X \to Y$ is a continuous map of topological spaces, where X, Y are Hausdorff and Y is locally compact, then finite in the sense above is equivalent to finite in the sense given in
[grauert2012coherent]. This is an easy exercise in topology.

Due to the remark above, we will be assuming without further mention from now on that arbitrary topological spaces X, Y are not only Hausdorff, but also that Y is locally compact, i.e. the codomain of a continuous function will be assumed to be locally compact.

In any case, the most salient part of either definition of finiteness that we will be using repeatedly is quasifiniteness, i.e. finiteness on the fibers, which is always obviously implied by the definition of finite given above. The converse is not, in general, true.

Lemma 1.6.5. *Let $\varphi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a finite holomorphic map of complex spaces. Let $\mathscr{A}$ be an analytic sheaf on X and let $y \in Y$. Label $\varphi^{-1}(y) = \{x_1, \ldots, x_\ell\}$. Then*

$$(\varphi_* \mathscr{A})_y \xrightarrow{\cong} \bigoplus_{\lambda=1}^{\ell} \mathscr{A}_{x_\lambda}$$

as $\mathscr{O}_{Y,y}$-modules.

Proof. Suppose V is an open neighborhood of y. From finiteness we may write

$$\varphi^{-1}(V) = U_1 \cup \cdots \cup U_\ell,$$

where U_λ is an open neighborhood of x_λ and they are pairwise disjoint. Now, if $\sigma \in (\varphi_*\mathscr{A})_y$, then we can find $s \in (\varphi_*\mathscr{A})(V)$ with $s(y) = \sigma$. But we can write

$$(\varphi_*\mathscr{A})(V) = \mathscr{A}(\varphi^{-1}(V)) = \bigoplus_{\lambda=1}^{\ell} \mathscr{A}(U_\lambda).$$

Thus, we define the map $\sigma \mapsto (s(x_\lambda))_{\lambda=1}^{\ell}$ and check that it is an isomorphism (exercise). □

Lemma 1.6.6. *Suppose we are given a homomorphism $\alpha : \mathscr{A} \to \mathscr{B}$ of analytic sheaves over a complex space $(X, \mathscr{O}_X)$ and $\varphi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a finite morphism of complex spaces. Pick $y \in Y$. The diagram*

$$\begin{tikzcd}
(\varphi_*\mathscr{A})_y \arrow[r, "(\varphi_*\alpha)_y"] \arrow[d, "\cong"'] & (\varphi_*\mathscr{B})_y \arrow[d, "\cong"] \\
\bigoplus \mathscr{A}_{x_\lambda} \arrow[r, "\bigoplus \alpha_{x_\lambda}"'] & \bigoplus \mathscr{B}_{x_\lambda}
\end{tikzcd}$$

commutes. In other words, the isomorphism from the last lemma is natural.

Lemma 1.6.7. *If $\varphi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is finite, then the direct image functor $\varphi_* : \mathrm{Mod}(\mathscr{O}_X) \to \mathrm{Mod}(\mathscr{O}_Y)$ is exact. (Here, $\mathrm{Mod}(\mathscr{O}_X)$ is the category of $\mathscr{O}_X$-modules, i.e. analytic sheaves)*

Proof. Let

$$\mathscr{A} \xrightarrow{\alpha} \mathscr{B} \xrightarrow{\beta} \mathscr{C}$$

in $\mathrm{Mod}(\mathscr{O}_X)$ be exact at $\mathscr{B}$. Then we have

$$\begin{tikzcd}
(\varphi_*\mathscr{A})_y \arrow[r, "(\varphi_*\alpha)_y"] \arrow[d, "\cong"] & (\varphi_*\mathscr{B})_y \arrow[r, "(\varphi_*\beta)_y"] \arrow[d, "\cong"] & (\varphi_*\mathscr{C})_y \arrow[d, "\cong"] \\
\bigoplus_\lambda \mathscr{A}_{x_\lambda} \arrow[r, "\bigoplus_\lambda \alpha_{x_\lambda}"'] & \bigoplus_\lambda \mathscr{B}_{x_\lambda} \arrow[r, "\bigoplus_\lambda \beta_{x_\lambda}"'] & \bigoplus_\lambda \mathscr{C}_{x_\lambda}.
\end{tikzcd}$$

Since the bottom row is componentwise exact, it is exact, and thus, so is the upper row. □

1.6.2 The pullback sheaf

Suppose X, Y are topological spaces and $\varphi : X \to Y$ is continuous. If $\mathscr{A}$ is a sheaf on Y, then we define the pullback sheaf of $\mathscr{A}$ along φ as

$$\varphi^*\mathscr{A}(U) = \{s : U \to \mathscr{A} \text{ continuous with } s(x) \in \mathscr{A}_{\varphi(x)}\}$$

for $U \subset X$ open. In other words, $\varphi^*\mathscr{A}(U)$ is the set of liftings

$$\begin{tikzcd} & \mathscr{A} \arrow[d] \\ U \arrow[ur, "s"] \arrow[r, "\varphi|_U"'] & Y. \end{tikzcd}$$

Definition 1.6.8. *The inverse image of $\mathscr{A}$ is $\varphi^*\mathscr{A}$. It is a sheaf on X and $(\varphi^*\mathscr{A})_x = \mathscr{A}_{\varphi(x)}$ (this needs a proof).*

For example, let $\varphi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a holomorphic map of reduced complex spaces. Then $(\varphi^*\mathscr{O}_Y)_x = \mathscr{O}_{Y,\varphi(x)}$. In this case $\varphi^*\mathscr{O}_Y \subset \mathscr{O}_X$; in particular, there is an injective map of sheaves $\varphi^*\mathscr{O}_Y \to \mathscr{O}_X$. We recall the definition of a morphism of ringed spaces, but this time flipping the domain and codomain of $\widetilde{\varphi}$.

Definition 1.6.9. *Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be ringed spaces. A morphism $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of ringed spaces is a pair of maps $(\varphi, \widetilde{\varphi})$ where $\varphi : X \to Y$ is a continuous map of topological spaces and $\widetilde{\varphi} : \varphi^*\mathscr{O}_Y \to \mathscr{O}_X$, a sheaf homomorphism.*

Remark 1.6.2. The identity $(\mathrm{id}, \mathrm{id})$ taking $(X, \mathscr{O}_X)$ to itself is a morphism (where $\mathrm{id}^* \mathscr{O}_X \cong \mathscr{O}_X$). Also, since composition is definable, ringed spaces form a category. It follows that a morphism $(\varphi, \widetilde{\varphi}) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of ringed spaces is an isomorphism if and only if φ is a homeomorphism and $\widetilde{\varphi}$ is an isomorphism.

1.7 The Weierstrass isomorphism

We continue our investigation now by presenting a more general version of the Weierstrass preparation (Vorbereitung) theorem. All local function theory sprouts from this preparation theorem, since it expresses the fundamental fact that the zero set of a holomorphic function displays, at least locally in the right coordinates, an "algebraic" and hence "finite" character.

That is, the preparation theorem reduces the study of zero sets of holomorphic function in n variables to the study of zero sets of Weierstrass polynomials. Naturally, the advantage of this is that the projection onto $\mathbb{C}^{n-1}$ of zero sets of such polynomials are finite maps. We have already seen a bit in the last section that finite maps and their induced functors are quite nice. In fact, this, along with the theory we have built for coherent sheaves, will allow us to give a relatively simple proof of Oka's coherence theorem later. In particular, we will show the existence of a sheaf isomorphism, which will give us the coherence lemma we need to prove the coherence of the structure sheaf for $\mathbb{C}^n$.

In this section $U \subset \mathbb{C}^n$ will be a connected, open set without further reference. We let $\omega = \omega(z;w) = w^k + a_{k-1}(z)w^{k-1} + \cdots + a_0(z)$ be in $\mathcal{O}^n(U)[w]$ (where $\mathcal{O}^n$ is the structure sheaf for $\mathbb{C}^n$). Note that

$$\mathcal{O}^n(U)[w] \subset \mathcal{O}^{n+1}|_{U\times\mathbb{C}}.$$

We ultimately want to think about $A = V(\omega) \subset U \times \mathbb{C}$, so we define the projection $p : U \times \mathbb{C} \to U$ taking $(z,w) \mapsto z$ and write $\pi = p|_A$. Then π is surjective and finite. Furthermore, we observe that

$$1 \leqslant \#\pi^{-1}(y) \leqslant k,$$

so we may write $\pi^{-1}(z_0) = \{\zeta_1, \ldots, \zeta_\ell\}$ for some $z_0 \in U$ fixed with $1 \leqslant \ell \leqslant k$. Of course, these will be of the form

$$\zeta_\lambda = (z_0, w_\lambda)$$

due to the definition of π as a projection. We therefore observe that

$$\omega_{\zeta_\lambda} \in \mathcal{O}^n_{z_0}[w - w_\lambda] \subset \mathcal{O}^{n+1}_{\zeta_\lambda}.$$

With this setup in mind, we can state the theorem.

Theorem 1.7.1 (General Weierstrass formula). *Pick $f_\lambda \in \mathcal{O}^{n+1}_{\zeta_\lambda}$ for each $\lambda = 1, \ldots, \ell$. Then there are uniquely determined $q_\lambda \in \mathcal{O}^{n+1}_{\zeta_\lambda}$ and a polynomial $r \in \mathcal{O}^n_{z_0}[w]$ of degree less than k (in w) such that*

$$f_\lambda = q_\lambda \omega_{\zeta_\lambda} + r_{\zeta_\lambda}.$$

Of course, in the case $\ell = 1$ we just get the normal Weierstrass formula.

Proof. By the Hensel property, there exist monic polynomials $\omega_1, \ldots, \omega_\ell \in \mathcal{O}^n_{z_0}[w]$ such that

$$\omega_{\zeta_\lambda} = \omega_{1,\zeta_\lambda} \cdots \omega_{\ell,\zeta_\lambda} \quad \text{with} \quad \omega_\lambda(z_0, w) = (w - w_\lambda)^{k_\lambda}$$

and $\sum_{\lambda=1}^{\ell} k_\lambda = k$. In particular, the germ $\omega_{\lambda,\zeta_\lambda} \in \mathcal{O}^{n+1}_{\zeta_\lambda}$ is $(w - w_\lambda)$-regular of order k_λ for each $\lambda = 1, \ldots, \ell$. We define the polynomials

$$e_\kappa = \prod_{\lambda \neq \kappa} \omega_\lambda \in \mathcal{O}^n_{z_0}[w]$$

for $\kappa = 1, \ldots, \ell$ and observe that e_{κ,ζ_κ} is a unit in $\mathcal{O}^{n+1}_{\zeta_\kappa}$ since

$$e_\kappa(\zeta_\kappa) = \prod_{\lambda \neq \kappa} (w_\kappa - w_\lambda)^{k_\lambda} \neq 0.$$

We have now set ourselves up perfectly to prove existence. From the usual Weierstrass formula, every germ $f_\lambda e^{-1}_{\lambda,\zeta_\lambda} \in \mathcal{O}^{n+1}_{\zeta_\lambda}$ has a decomposition

$$f_\lambda e^{-1}_{\lambda,\zeta_\lambda} = \omega_{\lambda,\zeta_\lambda} q'_\lambda + r_{\lambda,\zeta_\lambda},$$

where $q'_\lambda \in \mathcal{O}^{n+1}_{\zeta_\lambda}$ and $r_\lambda \in \mathcal{O}_{z_0}[w - w_\lambda]$ with $\deg r_\lambda < k_\lambda$. Now setting

$$e_{\iota\kappa} = \prod_{\lambda \neq \iota,\kappa} \omega_\lambda \in \mathcal{O}_{z_0}[w]$$

for $\iota \neq \kappa$, we define

$$q_\lambda = q'_\lambda - \sum_{\kappa \neq \lambda} r_{\kappa,\zeta_\kappa} e_{\kappa\lambda,\zeta_\lambda}$$

for $1 \leqslant \lambda \leqslant \ell$. We also define

$$r = r_1 e_1 + \ldots + r_\ell e_\ell \in \mathcal{O}_{z_0}[w],$$

so, in particular, $\deg r < k$. It then follows easily that

$$f_\lambda = \omega_{\zeta_\lambda} q_\lambda + r_{\zeta_\lambda}$$

for $1 \leqslant \lambda \leqslant \ell$ since

$$\omega_{\lambda,\zeta_\lambda} e_{\lambda,\zeta_\lambda} = \omega_{\zeta_\lambda} \quad \text{and} \quad e_{\kappa\lambda,\zeta_\lambda} \omega_{\lambda,\zeta_\lambda} = e_{\kappa,\zeta_\lambda}.$$

This finishes the proof of existence.

For uniqueness, it is enough to show $r = 0$ if

$$0 = \omega_{\zeta_\lambda} q_\lambda + r_{\zeta_\lambda}$$

with q_λ, r as in the theorem statement. Assume $r \neq 0$ and put $p_\lambda = r \cdot (\omega_1 \omega_2 \cdots \omega_\lambda)^{-1} \neq 0$. Of course, $r = p_1 \omega_1$ and for $\lambda = 2, \ldots, \ell$ we have $p_{\lambda-1} = p_\lambda \omega_\lambda$. These along with lemma 1.1.3 give successively that $p_1, \ldots, p_\ell$ are polynomials in w. Since $r = p_\ell \omega$, we have a contradiction to $\deg r < \deg \omega$. Thus, $r = 0$ and the proof is complete. □

We recall some concepts for ease of reading and to set notation before proceeding. Keeping the same setting as that described before the general Weierstrass formula, we will consider the ideal sheaf $\mathscr{I}$ generated by ω, i.e. $\mathscr{I} = \omega \mathscr{O}^{n+1}$, i.e. a sheaf coming from $A = V(\omega)$, which will be a finite ideal sheaf (it is generated by just one function). As usual, we write $\mathscr{O}_A = \mathscr{O}^{n+1}/\mathscr{I}|_A$. and remark that

$$\operatorname{supp}(\mathscr{O}^{n+1}/\mathscr{I}) = V(\mathscr{I}) = V(\omega) = A.$$

We will be interested in the pushforward $\pi_* \mathscr{O}_A$, which is an $\mathscr{O}^n$-module on our open set $U \subset \mathbb{C}^n$. Namely, we recall that if $V \subset U$ is open and we have $f \in \mathscr{O}^n(V)$ and $s \in (\pi_* \mathscr{O}_A)(V) = \mathscr{O}_A(\pi^{-1}(V))$, we define $fs \in (\pi_* \mathscr{O}_A)(V)$ in the usual way by

$$fs = (f \circ \pi \bmod \mathscr{I}) s = (f \circ \pi \bmod \omega) s \in \mathscr{O}_A(\pi^{-1}(V)) = (\pi_* \mathscr{O}_A)(V).$$

In this way $\pi_* \mathscr{O}_A$ is an $\mathscr{O}^n$-module on U.

Finally, as before we fix $z \in U$ and write $\pi^{-1}(z) = \{\zeta_1, \ldots, \zeta_\ell\}$ for some $0 \leqslant \ell \leqslant k$, then $(\pi_* \mathscr{O}_A)_z$ is an $\mathscr{O}^n_z$-module. Further, we know from lemma 1.6.5 that

$$(\pi_* \mathscr{O}_A)_z \cong \bigoplus_{\lambda=1}^{\ell} \mathscr{O}_{A,\zeta_\lambda}$$

as $\mathscr{O}^n_z$-modules. With all of this in mind, we have the following important theorem.

Theorem 1.7.2. *We have $\pi_* \mathcal{O}_A \cong \mathcal{O}^{n,\oplus k}$. In particular, the homomorphism constructed below will turn out to be an isomorphism and we call it the Weierstrass isomorphism.*

Proof. As above we take $V \subset U$ to be open. Let $(r_0, \dots, r_{k-1}) \in \mathcal{O}^{n,\oplus k}(V)$ and define

$$r(z; w) = r_0(z) + r_1(z)w + \cdots + r_{k-1}(z)w^{k-1}.$$

Taking now

$$\bar{r} = r \circ \pi \bmod \mathscr{I}(V) \in \mathcal{O}_A(\pi^{-1}(V))$$

defines a homomorphism from $\mathcal{O}^{n,\oplus k}(V)$ to $(\pi_* \mathcal{O}_A)(V)$.

These components assemble into a morphism

$$\alpha : \mathcal{O}^{n,\oplus k} \to \pi_* \mathcal{O}_A,$$

which we now show to be an isomorphism. As we recalled above,

$$(\pi_* \mathcal{O}_A)_z \cong \bigoplus_{\lambda=1}^{\ell} \mathcal{O}_{A,\zeta_\lambda}$$

where $\pi^{-1}(z) = \{\zeta_1, \dots, \zeta_\ell\}$. Given $f_\lambda \in \mathcal{O}^{n+1}_{\zeta_\lambda}$ for $\lambda = 1, \dots, \ell$, there is exactly one set of data (q_λ, r) such that $f_\lambda = q_\lambda \omega_{\zeta_\lambda} + r_{\zeta_\lambda}$ for each λ (recall we constructed this homomorphism by taking a function modulo ω), where r is a polynomial in w of degree less than k. In the previous sentence, of course "is" proves the map is surjective and "exactly one" proves it is injective, i.e. α is an isomorphism at the stalk at z, and so, α is an isomorphism of sheaves. We omit the details. □

1.8 Coherence of the structure sheaf

The fundamental problem of coherence was first posed in 1944 by Cartan while studying ideals of holomorphic functions and subsequently it was proven in 1948 by the Japanese mathematician K. Oka, see [**oka1950fonctions**]. However, Oka did not have the language of sheaves at his disposal, so he put the problem of finding locally finitely many generators for relation sheaves in the form "find a finite pseudobasis." Two years later, in 1950, Cartan simplified Oka's proof and introduced the term "coherent sheaf."

To be specific, the goal of this section is the following remarkable theorem.

Theorem 1.8.1 (Oka's coherence theorem). *The sheaf $\mathcal{O}_{\mathbb{C}^n}$ is coherent.*

1.8.1 Consequences of Oka's theorem

Before giving the proof the coherence theorem, we discuss some of the many corollaries spawning from it.

Corollary 1.8.2. *If $(X, \mathcal{O}_X)$ is a complex space, then $\mathcal{O}_X$ is coherent.*

The above corollary is due to the fact that coherence is a local property. In particular, note that if $\mathscr{A}$ is a sheaf over X such that each $x \in X$ has a neighborhood U with $\mathscr{A}|_U$ coherent, then $\mathscr{A}$ is coherent. This is exactly the situation for a complex space (write it out as an exercise).

Corollary 1.8.3. *An ideal sheaf $\mathscr{I} \subset \mathcal{O}_X$ where $(X, \mathcal{O}_X)$ is a complex space, is coherent if and only if it is finite.*

Corollary 1.8.4. *Locally free sheaves, $\mathscr{S}$ over $\mathcal{O}_X$ where $(X, \mathcal{O}_X)$ is a complex space, are coherent. (Recall that locally free means that for each point $x \in X$ there is an open neighborhood $U \subset X$ of x such that $\mathscr{S}_U \cong \mathcal{O}_X^{\oplus k}|_U$.)*

Corollary 1.8.5. *If $\mathscr{A}$ is a coherent analytic sheaf on $U \subset \mathbb{C}^n$, then $\operatorname{supp} \mathscr{A}$ is an analytic set. Namely,*

$$\operatorname{supp} \mathscr{A} = V(\operatorname{Ann} \mathscr{A}),$$

where $\operatorname{Ann} \mathscr{A}$ is a coherent ideal sheaf.

In particular, if $f : X \to Y$ is a holomorphic map of complex manifolds, and $\mathscr{S}$ is a coherent sheaf over $(X, \mathcal{O}_X)$ and we suppose that $f_*\mathscr{S}$ is coherent, then $\operatorname{supp}(f_*\mathscr{S})$ is an analytic set in Y. Further, if $f_*\mathcal{O}_X$ is coherent, then $f(X)$ is an analytic set in Y since

$$\operatorname{supp}(f_*\mathcal{O}_X) = \{y \in Y : y = f(x) \text{ for some } x \in X\} = f(X).$$

It will turn out that $f_*\mathcal{O}_X$ is coherent if f is a finite map.

1.8.2 Proving Oka's theorem

Now we begin the proof of the theorem with a technical lemma known as Oka's lemma.

Definition 1.8.6. *We will say that an $\mathscr{R}$-module $\mathscr{A}$ over X is coherent at $x \in X$ if there exists $U = U(x) \subset X$ such that $\mathscr{A}|_U$ is coherent.*

Lemma 1.8.7 (Oka's lemma). *Suppose X is a topological space and $\mathcal{O}$ is a sheaf of rings with the following properties:*

(i) the stalks, $\mathcal{O}_x$, are integral domains and

(ii) the sheaf $\mathcal{O}$ is Hausdorff.

Then $\mathcal{O}$ is coherent if and only if the Oka criterion holds. That is, for any open $U \subset X$ and any section $s \in \mathcal{O}(U)$ the sheaf of rings $\mathcal{O}_U / s\mathcal{O}_U$ is coherent at every point $x \in U$ where $s_x \neq 0$.

Proof. The forward implication is easy, since the sheaf $\mathcal{O}/s\mathcal{O}$ is the cokernel of the morphism of coherent sheaves $\alpha : \mathcal{O} \to \mathcal{O}$ with image being the sheaf of ideals generated by the section s. That is, use proposition 1.4.19.

The other direction is where the challenge lies. Suppose $U \subset X$ is open and $s_1, \ldots, s_k \in \mathcal{O}(U)$. We will show that if

$$\varphi : \mathcal{O}^k \to \mathcal{O}, \quad (f_1, \ldots, f_k) \mapsto \sum_{\kappa=1}^{k} f_\kappa s_\kappa,$$

then $\ker \varphi$ is finite (i.e. $\mathcal{O}$ is finite relation type, which is all we need to show, since $\mathcal{O}$ is obviously of finite type over itself).

Suppose $s_{\kappa,x} = 0_x$ for each $\kappa = 1, \ldots, k$, i.e. there is an open neighborhood for each κ, say $V_\kappa = V_\kappa(x)$ such that $s_\kappa = 0$ on V_κ. We call $V = V_1 \cap \cdots \cap V_k$, which is open and nonempty. On V,

$$\ker \varphi|_V = \mathcal{O}_V^k,$$

which is of finite type at x. Thus, we may assume that $s_{1,x} \neq 0$. Furthermore, since $\mathcal{O}$ is Hausdorff, we may as well restrict to a small enough neighborhood, still call it V, such that $s_{1,x} \neq 0$ on V. Define $\overline{\mathcal{O}} = \mathcal{O}/s_1\mathcal{O}$; then we have the following commutative diagram of sheaves and sheaf maps:

$$\begin{tikzcd} \mathscr{O}^k \arrow[r, "\varphi"] \arrow[d, "\pi^k"'] & \mathscr{O} \arrow[d, "\pi"] \\ \overline{\mathscr{O}}^k \arrow[r, "\overline{\varphi}"'] & \overline{\mathscr{O}}. \end{tikzcd}$$

We are restricting to V above, but we have not notated it to avoid messy notation. Just keep in mind throughout the proof that we are restricting to smaller neighborhoods potentially, but only ever doing so finitely many times. Coherence being a local property, this is all irrelevant, so we omit the notation.

Before moving to the main step of the proof, we note that $\ker(\pi \circ \varphi) = \ker(\overline{\varphi} \circ \pi^k)$ is finite. Indeed, first note $\ker \overline{\varphi}$ is finite by the Oka criterion. Since π^k is surjective, there is a neighborhood, $V = V(x)$, and a finitely generated $\mathscr{O}_V$-submodule $\mathscr{S}$ of $\mathscr{O}_V^k$ such that

$$\pi^k(\mathscr{S}) = \ker \overline{\varphi}|_V.$$

Thus, again omitting restriction to V,

$$\ker(\pi \circ \varphi) = \ker(\overline{\varphi} \circ \pi^k) = (\pi^k)^{-1}(\ker \overline{\varphi}) = \mathscr{S} + \ker \pi^k = \mathscr{S} + s_1\mathscr{O}^k.$$

Hence, $\ker \pi \circ \varphi$ is of finite type at x by lemma 1.4.12.

We now exploit this discovery by finding a surjective map $\chi : \ker(\pi \circ \varphi) \to \ker(\varphi)$, thus, $\ker \varphi$, being the surjective image of a finite sheaf, is finite, and we will be done. Recall that, since $\mathscr{O}$ is Hausdorff, we can restrict to a neighborhood V of x with $s_{1,y} \neq 0$ for $y \in V$. Fix $y \in V$ and pick $a \in (\ker \pi \circ \varphi)_y$, so $\varphi(a) \in \ker \pi = s_1\mathscr{O}$. There exists $r \in \mathscr{O}_y$ such that $\varphi(a) = rs_1$ and it is uniquely determined since $\mathscr{O}_y$ is an integral domain and $s_1 \neq 0$ at $y \in V$. Thus, we have a well-defined map

$$\psi : \ker(\pi \circ \varphi) \to \mathscr{O}, \quad \psi(a) = r.$$

Denote by $j : \mathscr{O} \to \mathscr{O}^k$ the map $r \mapsto (r, 0, \ldots, 0)$ and define

$$\chi = \mathrm{id} - j \circ \psi.$$

We check that if $a \in \ker(\pi \circ \varphi)$, then

$$\varphi \circ \chi(a) = \varphi(a) - \varphi(j \circ \psi(a)) = \varphi(a) - \varphi(r, 0, \ldots, 0) = \varphi(a) - rs_1 = 0,$$

so $\chi(a) \in \ker\varphi$, i.e. χ does indeed map from $\ker(\pi \circ \varphi)$ to $\ker\varphi$. Moreover, if $a \in \ker\varphi \subset \ker(\pi \circ \varphi)$, then

$$\chi(a) = a - (0, 0, \ldots, 0) = a.$$

Hence, $\chi : \ker \pi \circ \varphi \to \ker\varphi$ satisfies $\chi|_{\ker\varphi} = \mathrm{id}$, so χ is surjective. This completes the proof. $\square$

We can finally give the proof of the celebrated Oka's theorem.

Proof of Oka's coherence theorem. Since $\mathcal{O} = \mathcal{O}_{\mathbb{C}^n}$ is Hausdorff and all of the stalks are integral domains, we need only verify the Oka criterion. We proceed by induction on n. The case $n = 0$ is clear (as it is one of the only two examples of coherent sheaves we were able to give from the beginning). So, we take $n > 0$ and assume that the statement holds for n.

Let $W \subset \mathbb{C}^{n+1}$ be open and $f \in \mathcal{O}^{n+1}(W)$ with $f \not\equiv 0$. Choose $x \in W$ such that $f_x \neq 0_x$. Since we can localize, we can assume that

$$W = U \times V \subset \mathbb{C}^n \times \mathbb{C},$$

where $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}$ are open; we will write z for the U variables and w for the V variables. Without loss of generality, we will say $x = 0 \in U \times V$. By a linear change of coordinates, we may assume that f is w-regular of order k, so by the Weierstrass preparation theorem, $f = e\omega$ where e is a unit and

$$\omega(z, w) = w^k + a_{k-1}(z)w^{k-1} + \cdots + a_0(z)$$

is a Weierstrass polynomial. Of course, we may further simplify by just taking $f = \omega$, since multiplication by units does not change the situation.

Now we define $A = V(\omega)$ and, as usual, write $\mathcal{O}_A = \mathcal{O}^{n+1}/\omega\mathcal{O}^{n+1}|_A$. We also take $\pi : A \to U$ to be the natural projection, which, again, as usual, is finite. Thus, what we have done so far is reduce the Oka criterion to showing that $\mathcal{O}_A$ is coherent. With induction in hand, our goal is to show that $\mathcal{O}_A$ is coherent given $\mathcal{O}^n$ is coherent, which we do, as in the lemma, by showing that if $\varphi : \mathcal{O}_A^m \to \mathcal{O}_A$ is any $\mathcal{O}_A$-homomorphism, then $\ker\varphi$ is finite. This will complete the proof.

We examine the direct image functor. We have by lemma 1.6.6 that

$$\begin{array}{ccc}
(\pi_*\mathcal{O}_A)^m & \xrightarrow{\pi_*\varphi} & \pi_*\mathcal{O}_A \\
\cong\downarrow & & \downarrow\cong \\
\mathcal{O}_A^{n,\oplus km} & \xrightarrow[\widetilde{\varphi}]{} & \mathcal{O}_A^{n,\oplus k}.
\end{array}$$

By the induction hypothesis and the forward direction of Oka's lemma, $\ker\widetilde{\varphi}$ is finite, so $\ker(\pi_*\varphi)$ is finite as well.

Let $s_1,\ldots,s_r$ be sections generating $\ker\pi_*\varphi$ on U, so

$$s_\rho \in (\pi_*\mathcal{O}_A)^m(U) = \mathcal{O}_A^m(\pi^{-1}(U)), \quad \rho = 1,\ldots,r.$$

For any $z\in U$, we may write $\pi^{-1}(z) = \{\zeta_1,\ldots,\zeta_\ell\}\subset A$ and the diagram

$$\begin{array}{ccc}
(\pi_*\mathcal{O}_A)_z^m & \xrightarrow{\cong} & \oplus_{\lambda=1}^{\ell}\mathcal{O}_{A,\zeta_\lambda}^m \\
{\scriptstyle (\pi_*\varphi)_z}\downarrow & & \downarrow{\scriptstyle \oplus_\lambda\varphi_{\zeta_\lambda}} \\
(\pi_*\mathcal{O}_A)_z & \xrightarrow[\cong]{} & \oplus_{\lambda=1}^{\ell}\mathcal{O}_{A,\zeta_\lambda}
\end{array}$$

commutes. This means that s_{ρ,ζ_λ} generates the kernel of $\varphi_{\zeta_\lambda}:\mathcal{O}^n_{A,\zeta_\lambda}\to\mathcal{O}_{A,\zeta_\lambda}$, so $\ker\varphi$ is finite. which is what we wanted. □

Remark 1.8.1. There is an analogue of Oka's theorem in algebraic geometry, which can be found at the beginning of Serre's famous paper, **[serre1955faisceaux]**.

1.8.3 Remarks on sheaf theory

Before proceeding to a different topic, we make some general remarks on sheaf theory that may be useful or enlightening. go through these with someone who knows what they're talking about... I may move some of them around in these notes, since they are very out of place here

(i) Let $\varphi: R\to S$ be a homomorphism of rings. This induces a functor from R-modules to S-modules called the "restriction of scalars functor." The name comes from the case when φ is injective, although φ may not be, in general, injective.

(ii) Let M be an R-module. That is, M is an abelian group along with a ring action $R \times M \to M$. Equivalently, a homomorphism of rings $R \to \mathrm{End}(M)$ (where $\mathrm{End}(M)$ is the noncommutative ring of abelian group homomorphisms on M). The annihilator of M is an R-module, by definition, and is the kernel of the ring homomorphism $R \to \mathrm{End}(M)$. Thus, if $I \subset R$ is an ideal such that $I \subset \mathrm{Ann}_R M$, then M gets a natural R/I-module structure

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \mathrm{End}(M) \\ R/I. \arrow[ur, dashed] & \end{tikzcd}$$

(iii) Suppose $U \subset \mathbb{C}^n$ and $\omega \in \mathcal{O}(U)$. Define $A = V(\omega)$. If $\mathcal{S}$ is an $\mathcal{O}_A$-module, then the trivial extension to $\mathcal{O}$ is

$$\mathcal{S}' = \begin{cases} 0 & x \notin A \\ \mathcal{S}_x & x \in A. \end{cases}$$

The topology on $\mathcal{S}'$ is defined as "a local section $\sigma : \widetilde{V} \to \mathcal{S}'$ is continuous if and only if $\sigma|_A$ is continuous," where $\widetilde{V} \subset U$ is open in U. Then $\mathcal{S}'$ is an $\mathcal{O}$-module and if $f \in \mathcal{O}_x$, $\sigma \in \mathcal{S}'_x$ then $f\sigma = (f \bmod \omega)\sigma$.

(iv) Further, with the trivial extension, $\mathcal{S}$ is a finite $\mathcal{O}_A$-module if and only if $\mathcal{S}'$ is a finite $\mathcal{O}$-module. Indeed, if V is open in A, let $s_1, \ldots, s_m \in \mathcal{S}(V)$ be generators for $\mathcal{S}$ over $\mathcal{O}_A$. Let $\widetilde{V}$ be open in U with $\widetilde{V} \cap A = V$. Let s'_μ be the trivial extensions to $\widetilde{V}$. Claim: the s'_μ generate $\mathcal{S}'$ over $\widetilde{V}$. To see this, for $x \in A$ and $\sigma \in \mathcal{S}_x$ we have

$$\sigma = \sum_{\mu=1}^{m} f_\mu s_{\mu x}$$

for certain $f_\mu \in \mathcal{O}_{A,x}$. Choose $\widetilde{f_\mu} \in \mathcal{O}_x$, representatives of f_μ and

$$\sum_{\mu=1}^{m} \widetilde{f_\mu} s_{\mu x} = \sum_{\mu=1}^{m} f_\mu s_{\mu x} = \sigma.$$

(v) The same result as in the previous point holds for coherence.

1.9 Finite holomorphic maps

The goal of this section is to systematically exploit the techniques that we have developed with the Weierstrass theorems. We will be discussing the relationship of coherence and finite holomorphic maps, which we define presently.

1.9.1 The finite mapping theorem and related results

Definition 1.9.1. *A holomorphic map of complex spaces is called finite if the underlying map on topological spaces is finite (recall definition 1.6.4).*

The big result of this section is the following. Proving this will be our goal.

Theorem 1.9.2 (Finite mapping theorem). *Let $\mathscr{A}$ be a coherent analytic sheaf on a complex space $(X, \mathscr{O}_X)$, i.e. $\mathscr{A}$ is an $\mathscr{O}_X$-module that is coherent. Let $(\varphi, \widetilde{\varphi}) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a finite holomorphic map of complex spaces. Then $\varphi_*\mathscr{A}$ is a coherent $\mathscr{O}_Y$-module.*

We will prove this at the end of the section. First, as a consequence, we see that since $\varphi_*\mathscr{A}$ is, in particular, finite, we deduce that $\operatorname{supp} \varphi_*\mathscr{A}$ is closed in Y. Moreover, $\operatorname{supp} \varphi_*\mathscr{A}$ is what is known as a closed complex subspace defined by $\operatorname{Ann} \varphi_*\mathscr{A}$. In general, clearly if $(X, \mathscr{R})$ is a $\mathbb{C}$-ringed space with coherent structure sheaf, $\mathscr{R}$, we have for all $\mathscr{R}$-coherent sheaves $\mathscr{A}$, the equation

$$\operatorname{supp} \mathscr{A} = V(\operatorname{Ann} \mathscr{A}),$$

i.e. the support of any coherent sheaf is the zero set of a coherent ideal.

Definition 1.9.3. *Let $(Y, \mathscr{O}_Y)$ be a complex space. Say $A = V(\mathscr{I})$, where $\mathscr{I} \subset \mathscr{O}_Y$ is a finite ideal in $\mathscr{O}_Y$ and $\mathscr{O}_A = (\mathscr{O}_Y/\mathscr{I})|_A$. Then $(A, \mathscr{O}_A)$ is called a closed complex subspace of $(Y, \mathscr{O}_Y)$. That is, closed complex subspaces are analytic sets.*

Remark 1.9.1. The definition above comes from the fact that the injection morphism $(\iota, \widetilde{\iota}) : (A, \mathscr{O}_A) \to (Y, \mathscr{O}_Y)$ turns out to be a holomorphic map.

We also have a vast generalization of the finite mapping theorem, which we also do not have the time to prove. See [**grauert2012coherent**] for the proof (fair warning, it will take a lot of work and lead up to get to this proof).

Theorem 1.9.4 (Grauert's direct image theorem). *Let $(\varphi, \widetilde{\varphi}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a proper holomorphic map between complex spaces. Then, for each coherent analytic sheaf $\mathcal{S}$ on X, the sheaf $\varphi_* \mathcal{S}$ is coherent on Y.*

From this, if $\varphi : X \to Y$ is a proper holomorphic map between arbitrary complex spaces, then the sheaf $\varphi_* \mathcal{O}_X$ is a coherent analytic sheaf on Y. Since

$$\varphi(X) = \operatorname{supp} \varphi_* \mathcal{O}_X = V(\operatorname{Ann} \varphi_* \mathcal{O}_X),$$

we immediately get another big important theorem. Namely, the following.

Theorem 1.9.5 (Remmert's proper mapping theorem). *Let $(\varphi, \widetilde{\varphi}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a proper holomorphic map of complex spaces. Then the image set $\varphi(X) \subset Y$ is a closed complex subspace of $(Y, \mathcal{O}_Y)$.*

Remark 1.9.2. The theorem actually says that $\varphi(X)$ is an analytic set, but, in view of work we will do in section 1.12, closed complex subspaces and analytic sets are the same concept in complex spaces.

Legend has it that Remmert told Dr. Lieb that he came up with his proper mapping theorem while thinking about the problem below. His original proof, obtained in 1954 for the proper mapping theorem relied on an extension theorem for analytic sets. Indeed, the problem below is an application of Remmert's proper mapping theorem.

To Riemann circa 1860, Riemann surfaces were just what we would call 1 dimensional complex manifolds. Although, complex manifolds were not formally defined until 1939 by Teichmüller, although the definition began by saying that the definition was already well-known and accepted. In any case, the compact Riemann surfaces can be described through their genus, g, pictured below. draw later :)

When $g = 0$, we take X_0 as the Riemann sphere and call $K(X_0)$ the field of meromorphic functions, in this case that would be $\mathbb{C}(z)$, which is an algebraic function field of algebraic dimension 1 (here by algebraic dimension, we mean transcendence degree), i.e. a finite algebraic extension of a rational function field.

If $g = 1$ then we get that $K(X_1)$ is the elliptic functions (mentioned in the introduction to these notes). Generally, with X_g, the compact Riemann surface of genus g, there is an associated torus $T_g = \mathbb{C}^g/\Gamma$, where Γ is a $2g$-rank lattice. With this we get an algebraic function field of algebraic dimension g. It was known at the time that there were examples of tori "without" meromorphic functions, i.e. its meromorphic functions must be constant.

The question is, if X is an n dimensional compact complex manifold, is $K(X)$ an algebraic function field? That is, is it a finite field extension of $\mathbb{C}(z_1, \ldots, z_n)$. In 1939, Thimm answered in the affirmative given a certain additional assumption in his thesis **[thimm1939uber]**. It was done in general in 1953.

Historical quip 1.9.1. Siegel and Serre gave a proof as well with certain additional assumptions; according to Dr. Lieb, Serre once expressed to him that he was unhappy with being underappreciated given this result.

We, for the sake of interest and applying some of what we have learned, partially give the proof that Remmert came up with for the above problem using the proper mapping theorem. We will freely use facts not discussed in the course; the proof below is just for demonstration purposes.

Remmert's proof. We say that meromorphic functions $f_1, \ldots, f_\ell$ on X are analytically dependent if there is a holomorphic function $h \not\equiv 0$ on $\mathbb{C}^\ell$ such that $h(f_1, \ldots, f_\ell) \equiv 0$. Equivalentely, the matrix

$$\left(\frac{\partial f_\lambda}{\partial z_\nu}\right)_{\lambda,\nu}$$

in local coordinates is of rank less than ℓ. Let X be an n-dimensional compact complex manifold. If $f_1, \ldots, f_{n+1}$ are meromorphic functions, then they must be analytically dependent.

If one is a bit careless, a meromorphic function, f, can be thought of as $f : X \to \widehat{\mathbb{C}} = \mathbb{P}^1$. This is sloppy since meromorphic functions may have points of indeterminacy; the difficulty was circumvented by Remmert. From here, we define $\varphi : X \to (\mathbb{P}^1)^{n+1}$ by $\varphi(x) =$

$(f_1(x), \ldots, f_{n+1}(x))$ and note that it is not surjective since $f_1, \ldots, f_{n+1}$ are analytically dependent why? and why do we need that?. Of course φ is proper since X is compact, so we can apply Remmert's proper mapping theorem to conclude that $\varphi(X)$ is a closed complex subspace of $(\mathbb{P}^1)^{n+1}$.

Now, a theorem of Chow, see **[remmert1957holomorphe]**, says that closed complex subspaces of $(\mathbb{P}^1)^n$ are algebraic, i.e. there exists a polynomial $p \not\equiv 0$ such that $p(f_1, \ldots, f_n) \equiv 0$. Thus, $K(X)$ is an algebraic function field of transcendence degree less than or equal to n. □

1.9.2 Proof of the finite mapping theorem

We begin by proving a projection lemma. Note that if X, Y are arbitrary complex spaces and $f : X \to Y$ is a finite holomorphic map, then each point $x \in X$ is an isolated point of the fiber $f^{-1}(f(x))$. The purpose of the projection lemma below is that it allows us to prove a converse to this, which will be useful in constructing finite holomorphic maps locally. not sure I understand this

Lemma 1.9.6 (Projection lemma). *Suppose $\mathscr{A}$ is a coherent analytic sheaf on $U = U(0)$. Write $U \subset \mathbb{C}^m \times \mathbb{C}^n$; for $\mathbb{C}^m$ we use w and for $\mathbb{C}^n$ we use z. Suppose that $\operatorname{supp} \mathscr{A} \cap (\mathbb{C}^m \times \{0\}^n)$ contains the origin as an isolated point. Then there exist neighborhoods $W \subset \mathbb{C}^m$ and $Z \subset \mathbb{C}^n$ of the origin such that if $p : \mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}^n$ is the natural projection then*

(i) $\pi = p|_{\operatorname{supp} \mathscr{A}}$ is finite and

(ii) $p_*(\mathscr{A}|_{W \times Z})$ is coherent over Z.

Sloppy proof. As the name suggests, this proof will be a bit sloppy; the reader may consult **[grauert2012coherent]** for a fully correct proof. It is a good exercise to pick out the points in the following proof where we are being sloppy. this whole proof needs to be made better... I basically just copied it and don't fully understand it yet

Now, since $\mathscr{A}$ is a coherent analytic sheaf, $\operatorname{supp} \mathscr{A} = V(\operatorname{Ann} \mathscr{A})$. Further, the condition $\operatorname{supp} \mathscr{A} \cap \mathbb{C}^m \times \{0\}^n)$ contains $(0,0)$ as an isolated point means that there is an $f \in \operatorname{Ann} \mathscr{A}_0$ that depends on w that is not identically zero. By a linear change of coordinates we may assume that f is w_1-regular of order k so that, by the Weierstrass preparation theorem, $f = e\omega$, where $e \in \mathscr{O}_0$ is a unit and ω is a Weierstrass polynomial. In particular,

$$\omega = w_1^k + a_{k-1}(w', z)w_1^{k-1} + \cdots + a_0(w', z),$$

where $w' = (w_2, \ldots, w_m)$. Of course, $\omega_{(0,0)} \in \operatorname{Ann} \mathscr{A}_0$. We will now write

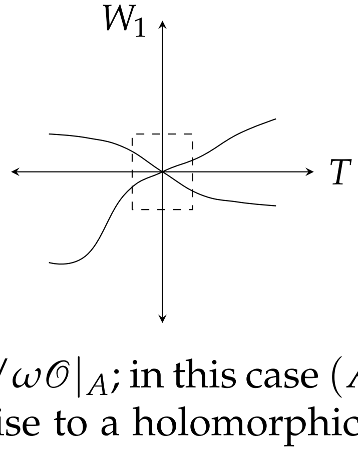
$$\mathbb{C}^m \times \mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{m-1} \times \mathbb{C}^n.$$

Choose a neighborhood W_1 of $0 \in \mathbb{C}$ and another neighborhood $T \subset \mathbb{C}^{m-1} \times \mathbb{C}^n$ of 0 such that everything converges and set

$$A = V(\omega) \subset W_1 \times T;$$

note that $\operatorname{supp} \mathscr{A} \subset A$.

We now use that fact that ω has a k-fold zero at the origin to choose T sufficiently small such that if $q : W_1 \times T \to T$ is the natural projection, then $\rho = q|_A$ is finite. We want ρ to be a k-fold branch covering; such a T can be shown to exist with a continuity argument. In essence, we want the polydisc below (the dashed line box) to be such that the curves come out of the left and right sides (but not the top or bottom).



Now, we write $\mathscr{O}_A = \mathscr{O}/\omega\mathscr{O}|_A$; in this case $(A, \mathscr{O}_A)$ is a complex space and $\rho : A \to T$ gives rise to a holomorphic map $(\rho, \widetilde{\rho}) : (A, \mathscr{O}_A) \to (T, \mathscr{O}_T)$ where the map $\widetilde{\rho}$ is essentially a quotient map modulo ω.

Now, we have that $q_*\mathscr{A} = \rho_*\mathscr{A}$. Further, since $\mathscr{A}$ is coherent in $W_1 \times T$, we also know $\mathscr{A}_A = \mathscr{A}|_A$ is a coherent $\mathscr{O}_A$-module and therefore has a local presentation:

$$\mathcal{O}_A^r \longrightarrow \mathcal{O}_A^s \longrightarrow \mathscr{A}_A \longrightarrow 0.$$

Since ρ_* is exact and additive,

$$(\rho_*\mathcal{O}_A)^r \longrightarrow (\rho_*\mathcal{O}_A)^s \longrightarrow \rho_*\mathscr{A}_A \longrightarrow 0$$

is exact. Thus, $\rho_*\mathscr{A}_A$ is the cokernel of the morphism of coherent sheaves $(\rho_*\mathcal{O}_A)^r \to (\rho_*\mathcal{O}_A)^s$ and is thus coherent. To achieve the full statement of the theorem we repeat the above argument with one less varaible. □

With this lemma, we can now prove the finite mapping theorem.

Proof of theorem 1.9.2. Maybe type to proposition above this in Coherent Analytic Sheaves with a proof sketch then type this one as Lieb did it. □

1.10 Rückert's Nullstellensatz

At the end of section 1.5, we discussed the sheaf map $\mathcal{O}_X \to \mathscr{C}_X^0$ given by taking a section and defining its complex value at a point based on the germ at that point. We will see in this section that the kernel of this map is the nilradical of $\mathcal{O}_X$. Thus, reduced complex spaces, $\mathcal{O}_X$, can always be considered as a subsheaf of $\mathscr{C}_X^0$ and consequently sections may be identified with their functions.

1.10.1 Recollections from algebraic geometry

Recall the following from classical algebraic geometry. Suppose K is an algebraically closed field and $R = K[x_1, \ldots, x_n]$. For an ideal $\mathfrak{a} \subset R$, one defines

$$V(\mathfrak{a}) = \{x \in K^n : f(x) = 0 \text{ for all } f \in \mathfrak{a}\}.$$

For a set $M \subset K^n$, one defines

$$I(M) = \{f \in R : f|_M \equiv 0\}.$$

Theorem 1.10.1 (Nullstellensatz). *For an ideal $\mathfrak{a} \subset K[x_1, \ldots, x_n]$, one has*

$$I(V(\mathfrak{a})) = \operatorname{rad}(\mathfrak{a}),$$

where $\operatorname{rad}(\mathfrak{a})$ *is the usual radical ideal of the ideal* $\mathfrak{a}$.

This incredibly important theorem was first proved by Hilbert in 1893 following his seminal paper in 1890 where he proved his, possibly even more famous, basis theorem. The obvious application of this is that if $\mathfrak{m} \subset R$ is a maximal ideal, then it is prime and thus radical, so

$$I(V(\mathfrak{m})) = \mathfrak{m}.$$

Hence, there is a point $a \in K^n$ such that $f(a) = 0$ for all $f \in \mathfrak{m}$. Actually, $\mathfrak{m}$ turns out to be $\mathfrak{m} = (a_1 - x_1, \ldots, a_n - x_n)$. The assignments $V(\cdot)$ and $I(\cdot)$ give a one-to-one correspondence between maximal ideals of R and points of K^n.

In modern algebraic geometry, we associate to any ring R, a geometric object X, such that R is the ring of functions on X. This space is $X = \operatorname{Spec} R = \{\mathfrak{p} \subset R : \mathfrak{p} \text{ is a prime ideal}\}$. The monumental work to build the theory of modern algebraic geometry was done by A. Grothendieck, who had a very interesting life; see W. Scharlau **[scharlau2008alexander]** for a biography.

1.10.2 The Nullstellensatz for sheaves

It is now our goal to formulate and prove this fundamental theorem from algebraic geometry in the language of coherent sheaves.

Theorem 1.10.2 (Rückert Nullstellensatz). *Let $(X, \mathcal{O}_X)$ be a complex space and consider a coherent analytic sheaf on X, say $\mathcal{S}$, and a coherent ideal sheaf, $\mathcal{I} \subset \mathcal{O}_X$. Suppose*

$$\operatorname{supp} \mathcal{S} \subset V(\mathcal{I}).$$

Then for each $x \in X$, there is an integer $s \in \mathbb{N}$ and a neighborhood $U = U(x)$ such that on U,

$$\mathcal{I}^s \subset \operatorname{Ann} \mathcal{S}.$$

Before proving this, we examine a corollary which more clearly displays the relation to our above discussion of classical algebraic geometry.

Definition 1.10.3. *Recall that, for any ideal, $\mathfrak{a}$, in a commutative ring, R, we write*

$$\operatorname{rad}\mathfrak{a} = \{r \in R : r^m \in \mathfrak{a} \text{ for some } m \in \mathbb{N}\}$$

for the radical of $\mathfrak{a}$ in R. If $(X, \mathscr{A})$ is a ringed space, then for every ideal sheaf $\mathscr{I}$ in $\mathscr{A}$, we write

$$\operatorname{rad}\mathscr{I} = \coprod_{x \in X} \operatorname{rad}\mathscr{I}_x$$

for the radical of $\mathscr{I}$ in $\mathscr{A}$.

If $f \in \mathscr{A}(X)$ is a section and $f_p \in \operatorname{rad}\mathscr{I}_p$, then clearly $f_x \in \operatorname{rad}\mathscr{I}_x$ for x sufficiently close to p. Thus, the radical ideal sheaf of any ideal sheaf is itself an ideal sheaf.

Corollary 1.10.4. *Let $(X, \mathscr{O}_X)$ be a complex space and $\mathscr{I} \subset \mathscr{O}_X$ a coherent sheaf of ideals with zero set $A = V(\mathscr{I})$. Recall the definition of $\mathscr{I}_A$ (definition 1.5.2). Then*

$$\mathscr{I}_A = \operatorname{rad}\mathscr{I}.$$

Proof. type □

With that, we first prove a lemma, which can be seen as a preliminary version of the Rückert Nullstellensatz. We will write $(z, w) = (z_1, \ldots, z_{n-1}, w) \in \mathbb{C}^n$.

Lemma 1.10.5. *Say $\mathscr{S}$ is a coherent analytic sheaf in $U = U(0) \subset \mathbb{C}^n$ with*

$$\operatorname{supp}\mathscr{S} \subset \{(z, w) \in U : w = 0\}.$$

Then there exists a positive integer, $s \in \mathbb{N}$, such that $w^s \mathscr{S}_0 = 0$.

Proof. type □

Proof of theorem 1.10.2. type □

1.11 Germs and ideals

This section will hopefully be a bit of a respite for those lost in the swamp of sheaf theory, since this theory is fairly elementary and essentially just a complex analytic interpretation of well-known results from basic algebra and algebraic geometry. We begin by quickly developing the theory from algebra.

1.11.1 The Lasker-Noether decomposition

For this subsection, R will be a commutative ring with unity without mention of these properties.

Definition 1.11.1. *We recall some basic definitions.*

(i) A prime ideal of R is an ideal $\mathfrak{p} \subsetneq R$ such that for all $f, g \in R$ with $fg \in \mathfrak{p}$, we must have $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.

(ii) A primary ideal of R is an ideal $I \subset R$ such that for all $f, g \in R$ with $fg \in I$ and $g \notin I$, there is $\ell \geqslant 1$ such that $f^\ell \in I$. In particular, all prime ideals are primary, but not the converse.

(iii) Given an ideal $I \subset R$, the radical of I, denoted by $\operatorname{rad}(I)$ or sometimes $\sqrt{I}$ is

$$\operatorname{rad}(I) = \{f \in R : \text{there exists } \ell \geqslant 1 \text{ with } f^\ell \in I\}.$$

One can easily check that $\operatorname{rad}(I)$ is an ideal and that $I \subset \operatorname{rad}(I)$. An ideal $I \subset R$ is said to be radical if $I = \operatorname{rad}(I)$. One may also check that the radical ideal of I is the smallest radical ideal that contains I.

(iv) The ideal quotient of two ideals $\mathfrak{a}, \mathfrak{b} \subset R$ is

$$(\mathfrak{a} : \mathfrak{b}) = \{f \in R : f\mathfrak{b} \subset \mathfrak{a}\}.$$

For example, in $\mathbb{Z}$, $((6) : (2)) = (3)$.

(v) Let M be an R-module. The annihilator of M is

$$\operatorname{Ann} M = \{f \in R : fM = \{0\}\}.$$

(vi) *An ideal $\mathfrak{a}$ is said to be irreducible if whenever $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$, we must have that either $\mathfrak{b}$ or $\mathfrak{c}$ is exactly $\mathfrak{a}$. That is, $\mathfrak{a}$ cannot be written as the intersection of two strictly larger ideals.*

It is well known that we may write

$$\operatorname{rad} \mathfrak{a} = \bigcap_{\mathfrak{p} \supset \mathfrak{a} \text{ prime}} \mathfrak{p}$$

and in particular, the nilradical, $\mathfrak{N} = \operatorname{rad}(0)$, is

$$\mathfrak{N} = \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p}.$$

Lemma 1.11.2. *Let $\mathfrak{a}$ be an irreducible ideal in a noetherian ring, R. Then $\mathfrak{a}$ is primary.*

Proof. Take $a, b \in R$ such that $ab \in \mathfrak{a}$ but $b \notin \mathfrak{a}$. For each $n \in \mathbb{N}$, consider the ideal

$$\mathfrak{a}_n := (\mathfrak{a} : (a^n)) = \{r \in R : ra^n \in \mathfrak{a}\}.$$

Observe that $(\mathfrak{a}_n)_{n \in \mathbb{N}}$ is an ascending chain of ideals, which implies that it becomes stationary, i.e. there exists an $n \in \mathbb{N}$ such that $\mathfrak{a}_m = \mathfrak{a}_n$ for every $m \geqslant n$. Now consider the ideals $I = (a^n) + \mathfrak{a}$ and $J = (b) + \mathfrak{a}$. Clearly

$$\mathfrak{a} \subset I \cap J.$$

We now argue for the reverse inclusion. Suppose $y \in I \cap J$ and $y = ra^n + c$ for some $r \in R$ and $c \in \mathfrak{a}$. Since $aJ \subset \mathfrak{a}$, it follows that $ay \in \mathfrak{a}$. Hence, $ra^{n+1} = ay - ac \in \mathfrak{a}$. This then implies that $r \in \mathfrak{a}_{n+1} = \mathfrak{a}_n$, so $y = ra^n + c \in \mathfrak{a}$. We have therefore shown that $\mathfrak{a} = I \cap J$.

Since $\mathfrak{a}$ is irreducible, we must have that $\mathfrak{a} = I$ or $\mathfrak{a} = J$. The fact that $b \notin \mathfrak{a}$ ensures that $\mathfrak{a} \neq J$, so $\mathfrak{a} = I$, which implies that $a^n \in \mathfrak{a}$. Hence, $\mathfrak{a}$ is primary, as desired. □

Lemma 1.11.3 (Primary decomposition). *If R is noetherian, then each proper ideal is a non-redundant, i.e. each ideal is necessary, finite intersection of primary ideals.*

Proof. By virtue of the previous lemma, it suffices to argue that every proper ideal of R is the finite intersection of irreducible ideals. Suppose, for the sake of contradiction, that this is not the case and denote by S the set of all ideals in R that cannot be written as a finite intersection of irreducible ideals. Since R is noetherian and S is nonempty, it must contain a maximal element, J. Since $J \in S$, it is not irreducible and so there exist ideals I_1, I_2 both properly containing J such that $J = I_1 \cap I_2$. The maximality of J now implies that I_1 and I_2 can be written as finite intersections of irreducible ideals in R. However, this then immediately implies that J also can be written in this way, which is a contradiction to J being in S.

Furthermore, obviously any primary decomposition can be turned into a non-redundant primary decomposition by dropping unnecessary primary ideals from the intersection and successively replacing all primary ideals with the same radical by their intersection. □

Of course, the natural question to ask any time we get some kind of existence is whether or not we also have uniqueness of any kind. Before answering this, note that the radical of any primary ideal is (obviously) prime. If $\mathfrak{q}$ is primary and $\operatorname{rad}(\mathfrak{q}) = \mathfrak{p}$, which is prime, then we will say $\mathfrak{q}$ is $\mathfrak{p}$-primary.

Lemma 1.11.4. *Suppose* $\operatorname{rad}(\mathfrak{q}) = \mathfrak{p}$. *Then we have the following.*

(i) If $f \in \mathfrak{q}$ *then* $(\mathfrak{q} : f) = R$.

(ii) If $f \notin \mathfrak{q}$ *then* $(\mathfrak{q} : f)$ *is* $\mathfrak{p}$*-primary. If* $f \in \mathfrak{p}$*, then* $(\mathfrak{q} : f) = \mathfrak{q}$.

Proof. Point (i) is trivial. type (ii) □

Lemma 1.11.5. *Suppose* R *is noetherian with* $\mathfrak{a} \subset R$ *an ideal. Furthermore, say that*

$$\mathfrak{a} = \bigcap_{i=1,\ldots,\ell} \mathfrak{q}_i,$$

where the $\mathfrak{q}_i$ *(which are primary) are chosen so that each is necessary for equality above, i.e. the decomposition is non-redundant. Define* $\mathfrak{p}_i = \operatorname{rad}(\mathfrak{q}_i)$. *Then the* $\mathfrak{p}_i$ *are the ideals among* $\operatorname{rad}(\mathfrak{a} : f)$ *for* $f \notin \mathfrak{a}$. *In particular,* ℓ *and the* $\mathfrak{p}_i$ *are determined by* $\mathfrak{a}$.

Proof. type □

Lemma 1.11.6. *If $\mathfrak{a} \subset R$ is an ideal,*

$$\mathfrak{a} = \bigcap_{i=1,\dots,\ell} \mathfrak{q}_i$$

has no redundancy (as in the lemma above), and again $\mathfrak{p}_i = \operatorname{rad}(\mathfrak{q}_i)$ (recall we are taking $\mathfrak{q}_i$ to be primary), then all of the minimal primes containing $\mathfrak{a}$ occur among the $\mathfrak{p}_i$ (recall a prime is minimal over an ideal if there is no smaller prime still containing that ideal).

Proof. type □

With all of these lemmas in hand, we have made it to the goal of this subsection.

Theorem 1.11.7 (Lasker-Noether decomposition). *Say $\mathfrak{a} \subset R$ is an ideal and R is noetherian. As usual take*

$$\mathfrak{a} = \bigcap_{i=1,\dots,\ell} \mathfrak{q}_i$$

to be non-redundant and say $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. The $\mathfrak{p}_i$ are uniquely determined by $\mathfrak{a}$ (i.e. they do not depend on the primary decomposition), so also ℓ is determined. Further, if $\mathfrak{p} \supset \mathfrak{a}$ is a minimal prime over $\mathfrak{a}$, then $\mathfrak{p}$ occurs in the $\mathfrak{p}_i$. If $\mathfrak{a}$ is actually radical, then

$$\mathfrak{a} = \bigcap_{\mathfrak{p} \supset \mathfrak{a} \ \textit{minimal}} \mathfrak{p}.$$

We call the $\mathfrak{p}_i$ the prime components of $\mathfrak{a}$. Those that are minimal are called the minimal associated primes of $\mathfrak{a}$ and the others in the prime components are said to be embedded.

A historical note is that Lasker was the chess champion for 27 years, the longest reign as of the writing of these notes. He proved this theorem for his Ph.D. but only for polynomial rings. The general version is due to the great Emmy Noether.

1.11.2 Analytic set germs

In this subsection we familiarize ourselves with a concept we will need while proving Cartan's coherence theorem in the next section. We conclude by using the Lasker-Noether decomposition to decompose the set germs that we define below. The main reference will be **[forster1976lokale]**.

Intuitively speaking, we sometimes want to treat elements of $\mathcal{O}_{z_0}$ as honest functions and in this case, we may want to think about their zero sets. This, however, would depend on the open neighborhood we chose around z_0. To remedy this lack of well-definedness, we consider the zero sets locally as well. That is, we will define the germ of a set and then be able to rigorously talk about the zero set of a germ in $\mathcal{O}_{z_0}$.

Definition 1.11.8. *Say X is a topological space and $a \in X$. Suppose $M, N \subset X$. Then we say $M \sim_a N$ when there exists $U = U(a)$ such that $U \cap M = U \cap N$ and we write $M_a = N_a$ for the associated equivalence class and call these set germs.*

By passing to representatives, we can define the notions of inclusion, intersection, and union, i.e. the symbols $M_a \subset N_a$, $M_a \cap N_a$, and $M_a \cup N_a$ all make sense with the appropriate interpretation. We make this explicit.

(i) We say $M_a \subset N_a$ if there exist representatives $M \in M_a$ and $N \in N_a$ such that $M \subset N$ in the traditional sense.

(ii) The only sensible way to define intersection and union is that taking the germ commutes with these. That is, for any representatives $M \in M_a$ and $N \in N_a$, we have

$$M_a \cap N_a = (M \cap N)_a, \quad M_a \cup N_a = (M \cup N)_a.$$

Clearly this does not depend on the choice of representative.

At this point, to proceed, we take the topological space in question to be $\mathbb{C}^n$.

Definition 1.11.9. *The set germ of the zero set of functions in $\mathcal{O}_a$ will be called an analytic germ. That is, if $(f_1, \ldots, f_k) = \mathfrak{a} \subset \mathcal{O}_a$ is an ideal, we can locally define its zero set, $V(\mathfrak{a})_a = X_a$.*

Explicitly, the $f_j \in \mathfrak{a}$ are each defined in an open neighborhood, U_j of a, so if we consider $U = U_1 \cap \cdots \cap U_k$, then we can write $V(f_1, \ldots, f_k)$ to mean the zero set of representatives of $f_1, \ldots, f_k$ all restricted to U. Then this gives a well-defined object when we write $V(\mathfrak{a})_a$, since any other set we could have chosen for the above work will clearly be in the same equivalence class.

Definition 1.11.10. *Let $X_a \subset \mathbb{C}^n$ be a germ at $a \in \mathbb{C}^n$. Then the vanishing ideal, $I(X_a)$, is the set of all elements $f \in \mathcal{O}_a$ with $X_a \subset V(f)_a$. This defines an ideal in $\mathcal{O}_a$.*

With these basic definitions out of the way, we give some examples.

(i) Around any point, we have the full germ, i.e. the germ corresponding to the whole space. In this equivalence class we will just have the open neighborhoods of that point. Likewise there is also an empty germ.

(ii) In $\mathbb{C}^3$, say X_1 is the z_1-axis, X_2 is the z_2z_3-plane, and $X = (X_1 \cup X_2)_0$. Note that $(X_1)_0$ is (by the above item) the open neighborhoods of the origin when restricting to the z_1-axis (open in the relative topology). Likewise for $(X_2)_0$. Thus, since

$$X = (X_1 \cup X_2)_0 = (X_1)_0 \cup (X_2)_0,$$

X will consist of unions of open neighborhoods of the origin on the z_1-axis and those on the z_2z_3-plane (both open in the relative topology). Stated differently, we could think of this as a cartesian product.

At this point (until otherwise mentioned at the end of this section), we will cease to write the subscript to denote a set germ; all set germs will be assumed to be around the same point.

Proposition 1.11.11. *We have many identities familiar from algebraic geometry. Namely, for X, Y, analytic germs at $a \in \mathbb{C}^n$, and any ideals $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_a$*

(i) $V(I(X)) = X$,

(ii) $\mathrm{rad}(I(X)) = I(X)$,

(iii) $V(\mathfrak{a}) = V(\mathrm{rad}(\mathfrak{a}))$,

(iv) the Nullstellensatz, $I(V(\mathfrak{a})) = \mathrm{rad}(\mathfrak{a})$,

(v) $\mathfrak{a} \subset \mathfrak{b}$ *implies* $V(\mathfrak{a}) \supset V(\mathfrak{b})$,

(vi) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$,

(vii) $V(\mathfrak{a} + \mathfrak{b}) = V(\mathfrak{a}) \cap V(\mathfrak{b})$,

(viii) $X \subset Y$ *implies* $I(X) \supset I(Y)$,

(ix) $I(X \cap Y) \supset I(X) + I(Y)$, *and*

(x) $I(X \cup Y) = I(X) \cap I(Y)$.

Proof. All but the Nullstellensatz are easy exercises (which may be helpful in understanding the definitions). □

Proposition 1.11.12. *A descending chain of analytic germs (all at a) becomes stationary.*

Proof. Since $\mathcal{O}_a$ is noetherian, the corresponding ascending sequence of ideals becomes stationary. □

Definition 1.11.13. *We say that an analytic germ, X (at $a \in \mathbb{C}^n$), is irreducible if $X = Y \cup Z$ (both Y and Z are analytic germs) implies that either Y or Z is exactly X.*

Due to the following proposition, we use prime and irreducible interchangeably in this context. It also is convenient for thinking about the Lasker-Noether decomposition to use these words interchangeably.

Proposition 1.11.14. *An analytic germ, X, is irreducible if and only if $I(X) = \mathfrak{p}$ is prime. For any prime $\mathfrak{p}$, $V(\mathfrak{p})$ is irreducible.*

We omit the proof. As an example, consider $\mathbb{C}^3$ and the function $f(x, y, z) = x^2 - zy^2$. Then the germ f_0 is irreducible in $\mathcal{O}_0$, but for every $a = (0, 0, a_3)$ with $a_3 \neq 0$, f_a is reducible in $\mathcal{O}_a$. Suppose first that

f_0 is reducible. Then there exist non-units $\varphi_0, \psi_0 \in \mathcal{O}_0$ with $f_0 = \varphi\psi$. We will choose representatives and write

$$\varphi = \sum_{\nu=1}^{\infty} p_\nu, \quad \psi = \sum_{\nu=1}^{\infty} q_\nu,$$

where p_ν and q_ν are homogeneous polynomials in (x, y, z) of degree ν. Then we have

$$x^2 - zy^2 = \sum_{\mu=2}^{\infty} \sum_{\lambda=1}^{\mu-1} p_\lambda q_{\mu-\lambda}.$$

Then we can find, by comparing coefficients that $x|zy^2$, which is a contradiction. For $a = (0, 0, a_3)$, we choose a branch of the square root through a_3 and find that

$$f_a = ((x - y\sqrt{z})(x + y\sqrt{z}))_a.$$

Thus, in particular, if $X = V(f)$, then $I(X_0)$ is prime, but $I(X_a)$ is not.

In the following theorem we will use the term non-redundant in the analogous sense to how we were using it in the previous subsection. That is, the union

$$X = \bigcup_{j=1,\ldots,\ell} X_j$$

is non-redundant if there is no i such that

$$X_i \subset \bigcup_{j \neq i} X_j,$$

where $X_1, \ldots, X_\ell$ are germs. With this we can state and prove the theorem that we have been working towards.

Theorem 1.11.15. *If X is an analytic germ, then there are uniquely determined prime germs X_j such that*

$$X = \bigcup_{j=1,\ldots,\ell} X_j$$

is non-redundant.

Proof. Pass to the ideal $\mathfrak{a} = I(X)$ and use the Lasker-Noether decomposition to write

$$\mathfrak{a} = \bigcap_{j=1,\ldots,\ell} \mathfrak{q}_j.$$

Then we go back:

$$X = V(\mathfrak{a}) = \bigcup_{j=1,\ldots,\ell} V(\mathfrak{q}_j)$$

and define $X_j = V(\mathfrak{q}_j)$ to finish. $\square$

We now begin writing the subscript again to avoid confusion, since we will be talking about sets and set germs together.

Definition 1.11.16. *Let X_a be an analytic germ at $a \in \mathbb{C}^n$. Then write $\mathcal{O}_{X_a} = \mathcal{O}_a / I(X_a)$ and call it the analytic algebra associated to X_a.*

One may wonder what the relation of $\mathcal{O}_{X_a}$ is to $\mathcal{O}_{X,a}$ (the stalk at a of the sheaf $\mathcal{O}_X$). To answer this, we just look at the definitions. Note that, since X is an analytic set, we have the corresponding radical ideal $\mathcal{I}_X$ with $X = V(\mathcal{I}_X)$. Then $I(X_a) = \mathcal{I}_{X,a}$ (they are isomorphic as rings), so we actually have $\mathcal{O}_{X_a} = \mathcal{O}_{X,a}$ if $\mathcal{O}_X$ is the sheaf defined through $\mathcal{I}_X$.

Note that $\mathcal{O}_{X_a}$ above is a reduced ring (i.e. no nilpotent elements) since $I(X_a)$ is a radical ideal. This corresponds to $I(X_a)$ being isomorphic to the radical ideal $\mathcal{I}_{X,a}$ and $\mathcal{O}_{X_a} = \mathcal{O}_{X,a}$. Furthermore, as we know from basic commutative algebra, X_a is a prime germ if and only if $\mathcal{O}_{X_a}$ is an integral domain.

We make some general comments and reflections on analytic sets. If X is an analytic set, $a \in X$, then we say a is regular, or X_a is regular (this will be equivalent to definition 1.5.9), if $\mathcal{O}_{X_a}$ is isomorphic to some power series ring $\mathbb{C}\{z_1, \ldots, z_\ell\}$. This is the case whenever there is a neighborhood of a in X which is biholomorphically equivalent to an open set in $\mathbb{C}^\ell$ (this is just a different view of the normal definition of a point being regular). Of course, we say a point is irregular if it is not regular. For example, points where X is reducible are irregular. That is, X is an analytic set, so $(X, \mathcal{O}_X)$ is a complex space. Recall a complex space is called irreducible at a point a if the stalk

$\mathcal{O}_{X,a}$ is an integral domain, otherwise X is reducible at a. Note that $\mathbb{C}\{z_1, \ldots, z_\ell\}$ is an integral domain, so points where X is reducible are irregular.

The definition of an analytic set being irreducible is as one may expect. Namely, if X is an analytic set and $X = Y \cup Z$ for analytic sets Y, Z, then it is called irreducible if either $X = Y$ or $X = Z$ for all such choices of Y, Z. One view of this in the current context, which is well outside the scope of these notes, is that an analytic set, X, is irreducible if and only if the set of regular points, X_{reg}, is connected.

1.11.3 Dimension theory for set germs

In this short section we survey some of the ways one can define the dimension of a set germ.

Definition 1.11.17 (Remmert-Stein). *Let X_0 be an analytic set germ at $0 \in \mathbb{C}^n$. The codimension of X is the maximal dimension of a plane $E \subset \mathbb{C}^n$ through 0 for which $X_0 \cap E_0 = \{0\}$. The Remmert-Stein dimension of X_0 is $n - codim(X_0)$.*

This was the original definition, but we present two potentially more useful equivalent definitions—one purely geometric and one purely algebraic.

Proposition 1.11.18. *The Krull dimension of an analytic set germ at a is the Krull dimension of $\mathcal{O}_{X_a}$. This agrees with the Remmert-Stein definition. Furthermore, the germ of a locally analytic, k-dimensional submanifold X of $\mathbb{C}^n$ with respect to $a \in X$ has Krull dimension k.*

Thus, we leave no ambiguity when we write $\dim(X_0)$ for the dimension.

Theorem 1.11.19. *If X_0 is an analytic set germ at $0 \in \mathbb{C}^n$ and*

$$X_0 = X_0^1 \cup \cdots \cup X_0^\ell$$

is its non-redundant decomposition into irreducible components, then we have

$$\dim X_0 = \max_{1 \leqslant j \leqslant \ell} \dim X_0^j.$$

Definition 1.11.20. *If X is an analytic set (not a germ), then we can define its dimension at $a \in X$ through the set germ* $\dim_a X = \dim(X_a)$.

A trivial example where we can see the utility of this geometric definition is if X_0 is the germ at $0 \in \mathbb{C}^3$ of the set defined by the union of the z_3-axis and the $z_1 z_2$-plane. Then $\dim(X_0) = 2$.

Remark 1.11.1. Interestingly, the function $\dim_a(X)$ for an analytic germ X on an open set, $U \subset \mathbb{C}^n$ is upper semicontinuous as a function of a. That is, for each $a \in X$ there is a neighborhood $V \subset U$ of a with

$$\dim_x X \leqslant \dim_a X$$

for all $x \in V \cap X$. If the germ is irreducible at a, then the dimension is locally constant (i.e. we get the above statement with an equal sign).

1.12 Coherence of the ideal sheaf

The goal of this final section will be to prove Cartan's coherence theorem, which says that the ideal sheaf of an analytic set is coherent. This is one of the four fundamental coherence theorems in the area of several complex variables. We will not get to the other two in these notes, but we will mention them in section 1.13. Again, the main reference for this section will be
[**forster1976lokale**]. The following theorem is our goal.

Theorem 1.12.1 (Cartan's coherence theorem). *If X is an analytic set, then the ideal sheaf $\mathscr{I}_X$ is finite and thus coherent.*

It will take us some time to get to the proof. Say $U \subset \mathbb{C}^n$ is open and $X \subset U$ is an analytic set. Recall that we may define the presheaf

$$\mathscr{I}_X(V) = \{f \in \mathscr{O}(V) : f|_{X \cap V} = 0\},$$

which turns out to be a sheaf. This is the ideal sheaf of X. Recall that, since X is analytic, there exists a finite ideal sheaf $\mathscr{J}$ with $X = V(\mathscr{J})$. By corollary 1.10.4, we see that $\mathscr{I}_X = \operatorname{rad}(\mathscr{J})$, which implies $\mathscr{I}_X$ is a radical ideal, i.e. $\mathscr{I}_X = \operatorname{rad}(\mathscr{I}_X)$.

1.12.1 Branched coverings

Suppose $f \in H_n$ is z_n-regular of order k and recall from the Weierstrass preparation theorem that then $f = e\omega$ where e is a unit and ω is a Weierstrass polynomial. It is the case that e is not zero near the origin, so

$$V(f)_0 = V(\omega)_0.$$

In this short section we want to make the ideas about branched coverings from section 1.1 more precise. We will first need some language to talk about these branched coverings. Consider a continuous map $\pi : X \to Y$ between locally compact Hausdorff topological spaces.

(i) We will call π a covering map if it is open and discrete.

(ii) We will say π is unramified/unbranched if it is a local homeomorphism.

(iii) We say that π is an unlimited, unramified covering map if the following holds. For each $y \in Y$ there is an open neighborhood $V \subset Y$ and pairwise disjoint open subsets $U_i \subset X$, $i \in I$, with $\pi|_{U_i} \to V$ a homeomorphism and

$$\pi^{-1}(V) = \bigcup_{i \in I} U_i.$$

The U_i are called sheets.

The set I above has a fixed cardinality, however in more complicated scenarios, e.g. the one we are about to discuss, the map π may be branched (i.e. not unbranched), which means at least intuitively that the U_i touch. The drawing below after we describe our setting should clarify this.

We will be interested in Weierstrass polynomials of degree k. In particular, we could decompose a Weierstrass polynomial into prime factors. Since any power of a prime factor vanishes at the same points as the prime factor, we will not be losing much to just assume that the prime factors are all simple (i.e. all have multiplicity one). This will only be of technical importance (it will mean that the discriminant, which is a holomorphic function in $n-1$ variables, will not vanish identically).

Let us state the relevant theorem and then we will see a diagram, which should explain why it is true.

Theorem 1.12.2. *Let $\omega \in \mathcal{O}(U)[z_n]$ be a Weierstrass polynomial of degree k where U is an open, connected neighborhood of zero in $\mathbb{C}^{n-1}$. Assume, in addition, that the prime factors of ω are all simple. Call $V(\omega) = X$ and let the discriminant of ω be $\Delta_\omega \in \mathcal{O}(U)$. We define the natural projection $\pi : X \to U$ taking $(z', z_n) \mapsto z'$. Now define*

$$S = V(\Delta_\omega), \quad U' = U \setminus S, \quad X' = X \setminus \pi^{-1}(S).$$

Then we have the following three results.

(i) The restriction $\pi|_{X'} : X' \to U'$ is an unbranched, unlimited covering of order k, i.e. the fibers have cardinality exactly k.

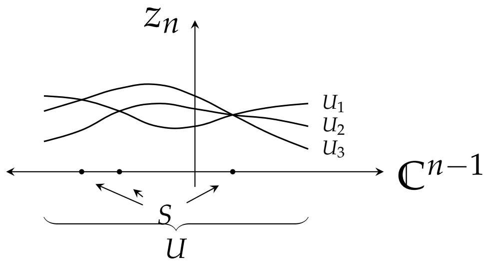
(ii) The set X' is locally the graph of a holomorphic function

$$z_n = \varphi(z_1, \ldots, z_{n-1});$$

in particular, it is an $(n-1)$-dimensional local analytic submanifold of $\mathbb{C}^n$ (recall definition 0.2.6).

(iii) The set X' is connected if and only if ω is irreducible in $\mathcal{O}(U)[z_n]$.

We do not give the proof, but it should be intuitively obvious from the picture below that the theorem holds. Below we consider a Weierstrass polynomial of degree 3.



The points in S are called branch points; as mentioned before, they are where the U_i touch. Of course, $X = U_1 \cup U_2 \cup U_3$ using the notation from above. We now take a look at a more general version of the above theorem: a covering presentation for prime germs.

Theorem 1.12.3 (Canonical covering). *Suppose X_0 is an irreducible, k-dimensional analytic set germ at $0 \in \mathbb{C}^n$. Then there is a coordinate system $z_1, \ldots, z_n$ that allows for the following description. There is a neighborhood basis of $0 \in \mathbb{C}^n$ consisting of open sets of the special form*

$$U = U_0 \times U' \times U''$$

(above, U_0 is not a set germ, the only set germ in this theorem is X_0) with $U_0 \subset \mathbb{C}^k$ a polydisc, $U' \subset \mathbb{C}$ a disk, and $U'' \subset \mathbb{C}^{n-k-1}$ a polydisc. There is an irreducible Weierstrass polynomial $F \in \mathcal{O}(U_0)[z_{k+1}]$ and polynomials $G_{k+2}, \ldots, G_n \in \mathcal{O}(U_0)[z_{k+1}]$ whose degrees are smaller than that of F and X_0 has a representative $X \subset U$ with the following properties.

(i) Define the prime $\mathfrak{p} = I(X_0)$; then the germs $(z_j \Delta_F - G_j)_0, F_0 \in \mathfrak{p}$ where $\Delta_F \in \mathcal{O}(U_0)$ is the discriminant of F.

(ii) Define the projection, $p : \mathbb{C}^n \to \mathbb{C}^k$, taking

$$(z_1, \ldots, z_k, z_{k+1}, \ldots, z_n) \mapsto (z_1, \ldots, z_k).$$

Then $p|_X : X \to U_0$ is finite and surjective.

(iii) Define the projection $p_1 : U \to U_0 \times U'$ taking

$$(z_1, \ldots, z_{k+1}, z_{k+2}, \ldots, z_n) \mapsto (z_1, \ldots, z_{k+1}).$$

Under this projection, X is mapped surjectively to the analytic set

$$X' := \{(z_1, \ldots, z_{k+1}) \in U_0 \times U' : F(z_1, \ldots, z_{k+1}) = 0\}.$$

For every $a = (a_1, \ldots, a_k) \in U_0$, all zeros of the polynomial $F(a; z_{k+1})$ are in U'.

Now define

$$S_0 := \{(z_1, \ldots, z_k) \in U_0 : \Delta_F(z_1, \ldots, z_k) = 0\},$$
$$S' := S_0 \times U', \quad S := S' \times U''.$$

Then we also have the following

(iii) Define the projection $p_2 : U_0 \times U' \to U_0$ taking $(z_1, \ldots, z_{k+1}) \mapsto (z_1, \ldots, z_k)$. The restriction $p_2|_{X' \setminus S'} : X' \setminus S' \to U_0 \setminus S_0$ is an unramified, unlimited, connected covering map whose order is the same as the degree of F.

(iv) *The restriction* $p_1|_{X\setminus S} : X \setminus S \to X' \setminus S'$ *is a biholomorphism and it has the following inverse:*

$$\begin{aligned}&(z_1, \ldots, z_{k+1})\\ &\qquad\mapsto \left(z_1, \ldots, z_{k+1}, \frac{G_{k+2}(z_1, \ldots, z_{k+1})}{\Delta_F(z_1, \ldots, z_k)}, \ldots, \frac{G_n(z_1, \ldots, z_{k+1})}{\Delta_F(z_1, \ldots, z_k)}\right).\end{aligned}$$

In particular, $X \setminus S$ *is a connected, k-dimensional, local analytic submanifold of* $\mathbb{C}^n$*.*

(v) *We have* $X = \overline{X \setminus S}$ *(the closure is taken in* U*).*

After a choice of coordinates as above and the construction of F *and* G_j*, there exists arbitrarily small polydiscs around 0 with the theorem above holding on them.*

As was with the last theorem, this one has a nice visual, see below. draw

Remark 1.12.1. If $k = 0$ in the above theorem, then $\mathfrak{p} = \mathfrak{m} \subset H_n$ and $X_0 = V(\mathfrak{m})_0 = \{0\}$. If $k = n$ then $\mathfrak{p} = (0)$ and $X_0 = V((0)) = \mathbb{C}^n_0$. Finally, if $k = n - 1$, then theorem 1.12.3 boils down to just theorem 1.12.2, so it is, indeed, a generalization.

1.12.2 Gap sheaves

Gap sheaves were introduced by Thimm in 1962 **[thimm1962luckengarben]** while investigating different holomorphic structures set on the same analytic set. It was only later that it was discovered that these would give a fairly easy proof of Cartan's coherence theorem.

Definition 1.12.4. *Suppose* $A \subset B$ *is closed in* B*, which is an open subset of* $\mathbb{C}^n$*. Further, say* $\mathcal{S}$ *is a submodule of* $\mathcal{O}_B^m$*. Then for* $U \subset B$ *open, we define the presheaf*

$$\mathcal{S}[A](U) := \{f \in \mathcal{O}_B^m : f|_{U\setminus A} \in \mathcal{S}(U \setminus A)\},$$

which turns out to be a sheaf. This is an $\mathcal{O}_B^m$*-module called the gap sheaf of* $\mathcal{S}$ *with gap* A*. Intuitively, we are adding in those elements that are "almost in* $\mathcal{S}$*," i.e. they would be in* $\mathcal{S}$ *if it were not for* A*.*

As is plain to see $\mathcal{S} \subset \mathcal{S}[A] \subset \mathcal{O}_B^m$. If $A = \emptyset$, we have no elements to add, so $\mathcal{S} = \mathcal{S}[A]$. We are interested in analytic sets and, in particular, zero sets of ideal sheaves; thus, it is natural to ask how the zero sets of gap sheaves relate to their corresponding sheaf.

Proposition 1.12.5. *Let $B \subset \mathbb{C}^n$ be open and consider $\mathcal{I} \subset \mathcal{O}_B$, a coherent ideal sheaf over B. Suppose $A \subset B$ is closed. Then*

$$V(\mathcal{I}[A]) = \overline{V(\mathcal{I}) \setminus A}.$$

The closure is with respect to the relative topology on B.

Proof. We will write $X = V(\mathcal{I})$ and $Y = V(\mathcal{I}[A])$. Since

$$\mathcal{I}|_{B \setminus A} = \mathcal{I}[A]|_{B \setminus A},$$

we see that $X \setminus A = Y \setminus A$. Since Y is closed, we see that $X \setminus A \subset Y$ implies $\overline{X \setminus A} \subset Y$. For the other inclusion, suppose $z \in B \setminus \overline{X \setminus A}$ is given. There exists an open neighborhood V of z in $B \setminus \overline{X \setminus A}$. Thus, $V \cap (X \setminus A) = \emptyset$, so $\mathcal{I}[A]|_V = \mathcal{O}_B^m|_V$ and thus $z \notin V(\mathcal{I}[A]) = Y$. □

We now have one of the fundamental observations that will help us prove Cartan's coherence theorem.

Theorem 1.12.6. *Suppose $\mathcal{I}, \mathcal{J} \subset \mathcal{O}_B$ are finite ideal sheaves over $B \subset \mathbb{C}^n$ and define $A = V(\mathcal{J})$, an analytic subset of B. Then the gap sheaf, $\mathcal{I}[A]$ is also finite.*

Proof sketch. For a given $z_0 \in B$, there exists a neighborhood of z_0, which we may assume is just B, so that

$$A = \{z \in B : g_1(z) = \cdots = g_k(z) = 0\}$$

for finitely many $g_1, \ldots, g_k \in \mathcal{O}(B)$. Since $\mathcal{O}_{B,z_0}$ is noetherian, the increasing sequence of ideals,

$$(\mathcal{I}_{z_0} : (g_1, \ldots, g_k)_{z_0}^r), \quad r \in \mathbb{N},$$

must become stationary, so there exists $r \in \mathbb{N}$ so that

$$(\mathcal{I}_{z_0} : (g_1, \ldots, g_k)_{z_0}^r) = (\mathcal{I}_{z_0} : (g_1, \ldots, g_k)_{z_0}^{r+1}).$$

We can use this along with standard techniques with finiteness to extend it to an open neighborhood $V \subset B$ of z_0. We may then show that

$$\mathscr{I}[A] = \mathscr{I} : (g_1, \ldots, g_k)^r$$

over V, which can then finally be used to show that $\mathscr{I}[A]$ is finite. $\square$

1.12.3 Proof of Cartan's coherence theorem

We begin with a simple case in the form of a lemma.

Lemma 1.12.7. *Let $U \subset \mathbb{C}^n$ be open. Suppose $X = V(f_1, \ldots, f_k)$ for some $f_i \in \mathscr{O}(U)$ and*

$$\operatorname{rk}\left(J_f^h(x)\right) = k$$

for every

$$x \in X := \{x \in U : f_1(x) = \cdots = f_k(x) = 0\}.$$

Then $f_1, \ldots, f_k$ generate $\mathscr{I}_X$ over U.

Proof. type; this is on page 128 of Forster and is very quick $\square$

Finally, with this in hand, we can prove Cartan's coherence theorem.

Proof of theorem 1.12.1. type; this is on page 129 of Forster but is quite long (and it involves TeXing a copmlicated matrix with tikz) $\square$

1.12.4 Consequences of Cartan's coherence theorem

Finally, we explore some consequences of Cartan's coherence theorem related to the study of singular and regular points.

Lemma 1.12.8. *Let X be an analytic set in the open set $U \subset \mathbb{C}^n$. Take $x \in X$ and finitely many $f_1, \ldots, f_\ell$ that generate $\mathscr{I}_{X,x}$ in a neighborhood of x. If X has constant dimension, k, then*

$$X_{reg} = \left\{x \in X : \operatorname{rk}\left(J_f^h(x)\right) = n - k\right\}.$$

Proof. type; this is Hilfssatz 1 on page 131 of Forster and is quite long □

If X is an analytic subset of an open set $U \subset \mathbb{C}^n$, then we will say that $S \subset X$ is an analytic subset of X if it is analytic in U. From the lemma above, we see that X_{sing}, the singular set of X, is an analytic subset of X if X is purely k-dimensional. In fact, we have more.

Theorem 1.12.9. *If X is an analytic subset of an open set $U \subset \mathbb{C}^n$, then X_{sing} is a nowhere dense[3], analytic subset of X and it holds*

$$X = \overline{X \setminus X_{sing}}.$$

Proof. type; we only proved the part that says X_{sing} is an analytic subset... this is Satz 3 on page 133 of Forster, it is quick □

1.13 Summary and outlook

type

[3] Recall, M is nowhere dense in N if $\mathring{\overline{M}} = \emptyset$ where the closure is taken in N.

CHAPTER

TWO

THE CAUCHY-RIEMANN EQUATIONS AND CONVEXITY

In this chapter, we display the beginnings of a deep and interesting area of complex analysis that reaches far beyond the purview that one may expect from the Cauchy-Riemann equations. As we will discover, studying these equations will lead us to a quick proof of Hartogs' Kugelsatz and through more questions around extending holomorphic functions. Indeed, solving the Cauchy-Riemann equations will be a deciding ingredient in our continued attempts to understand holomorphic functions. The main references for this chapter will be
[**hormander1973introduction**, **range1998holomorphic**][1].

2.0 Preliminaries

2.0.1 Basic differential geometry

We take $U \subset \mathbb{C}^n$ and E_0 to be the ring of smooth functions on U. Recall that a 1-form looks like

$$a = \sum_{j=1}^{n} a_j\, dz_j + \sum_{j=1}^{n} b_j\, d\overline{z}_j$$

[1] According to Dr. Lieb, Hörmander was the only mathematician to never make a mistake, so it is likely a good idea to read his work.

for $a_j, b_j \in E_0$. If $u \in E_0$ then we have

$$du = \sum_{j=1}^{n} \frac{\partial u}{\partial z_j}\, dz_j + \sum_{j=1}^{n} \frac{\partial u}{\partial \bar{z}_j}\, d\bar{z}_j =: \partial u + \bar{\partial} u.$$

As usual, we will be using $d : E_0 \to E_1$ as above to denote the exterior derivative. It will sometimes be helpful to further decompose E_1 as

$$E_1 = E_{10} \oplus E_{01},$$

where E_{10} and E_{01} are the E_0-submodules generated by (dz_j) and $(d\bar{z}_j)$, respectively.

We now recall one of the main objects of interest in this chapter: the Cauchy-Riemann equations. They say, very simply and elegantly, that a function $u \in E_0$ is holomorphic if and only if $\bar{\partial} u = 0$. This poses two interesting questions. First, if

$$f = \sum_{j=1}^{n} a_j\, dz_j + \sum_{j=1}^{n} b_j\, d\bar{z}_j$$

is a 1-form, then is there a $u \in E_0$ such that $du = f$? And is there a u such that

$$\bar{\partial} u = \sum_{j=1}^{n} b_j\, d\bar{z}_j?$$

Of course, as should be expected, the answer is, in general, no. But the more interesting question is, when is the answer yes?

Before moving on to some results and progress towards answering this question, we need to expand a bit on the definitions we have above. Suppose

$$\begin{aligned} f &= \sum_{\substack{1 \leqslant i_1 \leqslant \cdots \leqslant i_p \leqslant n \\ 1 \leqslant j_1 \leqslant \cdots \leqslant j_q \leqslant n}} f_{i_1,\dots,i_p,j_1,\dots,j_q}\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \\ &= \sum_{|I|=p,|J|=q} f_{IJ}\, dz_I \wedge d\bar{z}_J, \end{aligned}$$

where we have drastically simplified above using multiindices. We call f above a (p, q) form and E_{pq} is the E_0-module of (p, q) forms. Note that

$$\mathrm{rk}(E_{pq}) = \binom{n}{p}\binom{n}{q}.$$

We can write

$$E_k = \bigoplus_{p+q=k} E_{pq}.$$

Then

$$\mathrm{rk}(E_k) = \binom{2n}{k}.$$

The exterior derivative $d : E_k \to E_{k+1}$ works as it usually does. So does the conjugate differential $\overline{\partial} : E_{pq} \to E_{p,q+1}$. As is standard knowledge from differential geometry, $d \circ d = 0$, which implies that $\overline{\partial} \circ \overline{\partial} = 0$ (and also that $\partial \circ \partial = 0$). We also have

$$\partial\overline{\partial} + \overline{\partial}\partial = 0.$$

These are all either the well-known Poincaré lemma or trivial corollaries of it.

2.0.2 Poincaré and Dolbeault cohomology

Before giving a few more definitions, we turn back to the question of interest. Given $f \in E_{p,q+1}$ with $\overline{\partial} f = 0$, we wish to solve $\overline{\partial} u = f$ for $u \in E_{pq}$. A particular case is $p = q = 0$, where $f \in E_{01}$ with $\overline{\partial} f = 0$, and we wish to find a smooth function $u \in E_0$ such that $\overline{\partial} u = f$.

Remark 2.0.1. One should quickly (i.e. in one sentence) convince oneself that we require $\overline{\partial} f = 0$ out of necessity. That is, we know there are no solutions, u, otherwise.

We are motivated by a result from real analysis. Take $U \subset \mathbb{R}^n$ (no special properties for the time being); we denote by $E_0(U)$ the ring of $\mathbb{F}$-valued (here $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$) smooth functions on U and by $E_k(U)$

the smooth k-forms on U over $\mathbb{F}$. We can, similar to above, pose the problem of finding $u \in E_k(U)$ satisfying $du = f$ for some $f \in E_{k+1}$ with $df = 0$. If $U \subset \mathbb{R}^n$ is a convex open set, then this is actually solvable. This will lead us to defining a notion of "convexity" for our situation that makes the problem solvable.

We now return to some more useful definitions and to U being an open subset of $\mathbb{C}^n$. We will say that

(i) f is d-closed if $df = 0$ and

(ii) f is d-exact if $f = du$ for some u.

Of course, since $d \circ d = 0$, exactness implies closedness. The converse is not true in general. In fact, we now define cohomology, which in essence is defining "the failure of the converse."

To be precise, we will write

$$Z_k(U) = \text{closed } k\text{-forms}, \quad B_k(U) = \text{exact } k\text{-forms}$$

and define the k-th Poincaré cohomology of U:

$$H_P^k(U) = \frac{Z_k(U)}{B_k(U)}.$$

Remark 2.0.2. Returning to the real analysis case for a brief remark: from before, if U is convex and open, then $H^k(U, \mathbb{F}) = 0$ (the corresponding cohomology, called singular cohomology) for $k \geqslant 1$ since the converse never fails. Moreover $H_P^k(U) \cong H^k(U, \mathbb{F})$ for all $U \subset \mathbb{C}^n$ open.

In a similar fashion, we define Dolbeault cohomology. We write

$$Z_{pq}(U) = \{f \in E_{pq}(U) : \overline{\partial} f = 0\} =: \overline{\partial}\text{-closed}, \quad B_{pq}(U) = \{f = \overline{\partial} u : u \in E_{p,q-1}\}$$

Again and for the same reason, exact implies closed but not the converse. We thus define the Dolbeault cohomology

$$H^{pq}(U) = \frac{Z_{pq}(U)}{B_{pq}(U)}.$$

To expand on our questions with this new language, we wonder for which $U \subset \mathbb{C}^n$ it holds that $H^{pq}(U)$ is zero (our previous question) or even just finite dimensional. Furthermore, as before, is there some kind of "convexity condition?"

Moving on, recall that we may change coordinates with the following results. Suppose $V \subset \mathbb{C}^m$, $U \subset \mathbb{C}^n$, and $F : V \to U$ a holomorphic map (we assume this throughout the subsection). If

$$f = \sum_{\substack{1 \leqslant i_1 \leqslant \cdots \leqslant i_p \leqslant n \\ 1 \leqslant j_1 \leqslant \cdots \leqslant j_q \leqslant n}} f_{i_1,\ldots,i_p,j_1,\ldots,j_q}\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$$

then composition is defined as

$$f \circ F = \sum_{\substack{1 \leqslant i_1 \leqslant \cdots \leqslant i_p \leqslant n \\ 1 \leqslant j_1 \leqslant \cdots \leqslant j_q \leqslant n}} (f_{i_1,\ldots,i_p,j_1,\ldots,j_q} \circ F)\, d(z_{i_1} \circ F) \wedge \cdots \wedge d(z_{i_p} \circ F) \wedge d(\bar{z}_{j_1} \circ F) \wedge \cdots$$

and $f \circ F \in E_{pq}(V)$. Furthermore, we have the following lemma.

Lemma 2.0.1. *The exterior derivative applied to composition gives*

$$d(f \circ F) = df \circ F$$

Similarly, $\partial(f \circ F) = \partial f \circ F$ *and* $\bar{\partial}(f \circ F) = \bar{\partial} f \circ F$.

2.0.3 Čech cohomology

It will be another goal of ours in this chapter to generalize the famous Mittag-Leffler theorem from single variable complex analysis. For the statement and some exposition, see appendix B. This also serves as some motivation for defining Čech cohomology.

We consider a distribution of principal parts on $U \subset \mathbb{C}^n$. That is, say $\mathfrak{U} = \{U_i : i \in I\}$ is an open covering of U and f_i are meromorphic on U_i such that on $U_{ij} = U_i \cap U_j$,

$$f_i - f_j = f_{ji}$$

is holomorphic.

The goal is to find a meromorphic function f on U such that $f - f_i$ is holomorphic on U_i for each i. If such an f exists, we say that the distribution of principal parts is solvable. The aforementioned Mittag-Leffler theorem is the case $n = 1$ and $U = \mathbb{C}$.

Theorem 2.0.2. *If $H^{01}(U) = 0$, then the distribution of principal parts is solvable.*

Proof. We assume that $\mathfrak{U}$ is locally finite and $U_i \Subset U$ (i.e. $\overline{U_i} \subset U$), which we may do since $\mathbb{C}^n$ is a paracompact Hausdorff space. Take a partition of unity corresponding to $\mathfrak{U}$, call it $\{\varphi_i : i \in I\}$, where $\varphi_i \in C^\infty(U)$, $\operatorname{supp} \varphi_i \subset U_i$, $0 \leqslant \varphi_i \leqslant 1$, and $\sum \varphi_i = 1$. Consider

$$g_i = \sum_{j \in I} f_{ji} \varphi_j,$$

which is smooth on U_i. Furthermore,

$$\begin{aligned} g_i - g_k &= \sum_{j \in I} f_{ji} \varphi_j - \sum_{j \in I} f_{jk} \varphi_j = \sum_{j \in I} \varphi_j (f_i - f_j - f_k + f_j) \\ &= (f_i - f_k) \sum_{j \in I} \varphi_j = f_i - f_k. \end{aligned}$$

Thus, $\overline{\partial} g_i - \overline{\partial} g_k = \overline{\partial}(g_i - g_k) = 0$ on U_{ik}. Defining $\gamma = \overline{\partial} g_i$ on U_i (which is well-defined by the above work) we see that $\gamma \in E_{01}(U)$ and $\overline{\partial}\gamma = 0$. Since $H^{01}(U) = 0$, there exists $u \in C^\infty(U)$ with $\gamma = \overline{\partial} u$.

All that remains to show is that

$$f = f_i - g_i + u, \quad \text{on } U_i$$

solves. Just note that $f - f_i = -g_i + u$ is holomorphic on U_i since $\overline{\partial}(-g_i + u) = -\gamma + \overline{\partial} u = 0$, so we are done. $\square$

Theorem 2.0.3. *Suppose $U \subset \mathbb{C}^n$ is such that $H^{01}(U) = 0$. Then, if (U_i) is a covering of U and $b_{ij} \in \mathcal{O}(U_{ij})$ is such that*

(i) $b_{ii} = 0$,

(ii) $b_{ij} = -b_{ji}$, *and*

(iii) $b_{ij} + b_{jk} + b_{ki} = 0$ *wherever this sum is defined,*

then there are functions $b_i \in \mathcal{O}(U_i)$ such that $b_i - b_j = b_{ji}$.

Definition 2.0.4. *The conditions (i), (ii), (iii) above are called the cocycle conditions.*

One could state this previous theorem in fancy language if familiar with sheaf cohomology:

$$H^{01}_{\text{Dol}}(U) \cong H^1(U, \mathcal{O}).$$

To do this, we will need some definitions.

Let X be an arbitrary topological space and $\mathcal{S}$ an abelian sheaf on X. Suppose $\mathfrak{U} = \{U_i : i \in I\}$ is an open covering of X. Suppose that for each i we are given a section $a_i \in \mathcal{S}(U_i)$.

Definition 2.0.5. *We call $a = \{a_i : i \in I\}$ a 0-cochain. The set $C^0(\mathfrak{U}, \mathcal{S})$ of 0-cochains of $\mathcal{S}$ over the open cover $\mathfrak{U}$ is an abelian group with index-wise addition.*

Now suppose that $U_{ij} = U_i \cap U_j$ and $b_{ij} \in \mathcal{S}(U_{ij})$.

Definition 2.0.6. *We say that $b = \{b_{ij} : (i, j) \in I \times I\}$ is a 1-cochain if*

(i) $b_{ii} = 0$ *and*

(ii) $b_{ij} = -b_{ji}$.

Now we have a group of 1-cochains, call it $C^1(\mathfrak{U}, \mathcal{S})$.

If $a \in C^0(\mathfrak{U}, \mathcal{S})$, we define $\delta a \in C^1(\mathfrak{U}, \mathcal{S})$ as

$$(\delta a)_{ij} = (a_j - a_i)|_{U_{ij}}.$$

Then $\delta : C^0(\mathfrak{U}, \mathcal{S}) \to C^1(\mathfrak{U}, \mathcal{S})$ is additive. We call δa the coboundary of a.

Definition 2.0.7. *(i) A 1-cochain of the form δa for some 0-cochain, a, is called a 1-coboundary. Denote by $B^1(\mathfrak{U}, \mathcal{S})$ the set of 1-coboundaries. This set is a subgroup of $C^1(\mathfrak{U}, \mathcal{S})$.*

(ii) A 1-cochain $b \in C^1(\mathfrak{U}, \mathcal{S})$ is called a 1-cocycle if

$$b_{ij} + b_{jk} + b_{ki} = 0$$

on $U_{ijk} = U_i \cap U_j \cap U_k$. Denote $Z^1(\mathfrak{U}, \mathcal{S})$ to the set of 1-cocycles. This set is also a subgroup of $C^1(\mathfrak{U}, \mathcal{S})$.

As an easy exercise, one should convince oneself that every 1-coboundary is a 1-cocycle, i.e.

$$B^1(\mathfrak{U}, \mathscr{S}) \subset Z^1(\mathfrak{U}, \mathscr{S}).$$

With this in mind, we define

$$H^1(\mathfrak{U}, \mathscr{S}) = \frac{Z^1(\mathfrak{U}, \mathscr{S})}{B^1(\mathfrak{U}, \mathscr{S})},$$

the first cohomology group with respect to $\mathfrak{U}$ in $\mathscr{S}$.

Remark 2.0.3. Returning to the intuition that cohomology describes the "failure of closedness to imply exactness" we can also think of it as "obstructions" to solving $\delta a = b$ for b a coboundary. Or in our previous case as obstructions to solving $\bar{\partial} u = f$ given $\bar{\partial} f = 0$. For more on this one should look into obstruction theory.

With this out of the way, we reformulate theorem 2.0.3. Consider a region $G \subset \mathbb{C}^n$ (i.e. open and connected) with $H^{01}(G) = 0$ such that it is possible to find meromorphic functions on G with prescribed principal parts. As a side note, if $n = 1$, we actually have that $H^{01}(G) = 0$ for all regions $G \subset \mathbb{C}$.

Theorem 2.0.8. *Suppose $G \subset \mathbb{C}^n$ is a region such that $H^{01}_{Dol}(G) = 0$. Then for each open covering $\mathfrak{U}$ of G,*

$$H^1(\mathfrak{U}, \mathscr{O}) = 0$$

It is a good exercise in the definitions to untie the above restatement and explain why it is in fact a restatement. We have the following (completely trivial) corollary.

Corollary 2.0.9. *If $G \subset \mathbb{C}^n$ is a region with $H^{01}_{Dol}(G) = 0$ and $\mathscr{K}$ is an analytic sheaf isomorphic to $\mathscr{O}$, then $H^1(\mathfrak{U}, \mathscr{K}) = 0$.*

As an application of this, we consider a region $G \subset \mathbb{C}^2$ containing the origin such that $H^{01}_{\mathrm{Dol}}(G) = 0$. Let $\mathscr{I}$ be the ideal sheaf associated with the origin, i.e. $\mathscr{I} = (z_1, z_2)\mathscr{O}$. Then

$$\mathscr{I}(G) = \{f \in \mathscr{O}(G) : f(0) = 0\}.$$

We claim that $\mathscr{I}(G)$ is generated by z_1 and z_2. In other words, if $f \in \mathscr{O}(G)$ is such that $f(0) = 0$, then $f = z_1 f_1 + z_2 f_2$ for some $f_1, f_2 \in \mathscr{O}(G)$.

Proof of claim. Consider the following exact sequence: is this even exact?

$$0 \longrightarrow \mathscr{K} \hookrightarrow \mathscr{O} \oplus \mathscr{O} \xrightarrow{\varphi} \mathscr{I} \longrightarrow 0,$$

where φ maps $(f_1, f_2) \mapsto z_1 f_1 + z_2 f_2$ and $\mathscr{K} = \ker \varphi$. We show that $\mathscr{K} \cong \mathscr{O}$, since, in this case, $H^1(\mathfrak{U}, \mathscr{K}) = 0$, which we will use to show the claim.

Consider the sheaf morphism $\mathscr{O} \to \mathscr{K}$ taking $h \in \mathscr{O}_x$ to $(hz_2, -hz_1) \in \mathscr{K}_x$. Since this is clearly injective, we just need to show it is surjective. Suppose $(f_1, f_2) \in \mathscr{K}(U(x))$, so on $U(x)$,

$$f_1 z_1 + f_2 z_2 = 0.$$

Observe that $\frac{f_1}{z_2} z_1 = -f_2$ is holomorphic. Using this, we note that if $z_0 \neq 0$, then f_1/z_2 is holomorphic. Further if $z_1 \neq 0$, then f_1/z_2 is holomorphic since is a quotient of holomorphic functions:

$$\frac{f_1}{z_2} = \left(\frac{f_1}{z_2} z_1\right) \frac{1}{z_1}.$$

The only point that remains is the origin, but recall that holomorphic functions for $n > 1$ cannot have isolated singularities by Hartogs' extension principle. Thus, $f_1/z_2 =: h$ is holomorphic. Then

$$(f_1, f_2) = (hz_2, -hz_1),$$

so our map is also surjective.

As mentioned, this combined with the previous corollary gives $H^1(\mathfrak{U}, \mathscr{K}) = 0$, which we will presently use to find functions $a, b \in \mathscr{O}(G)$ for a given $f \in \mathscr{I}$ such that $f = az_1 + bz_2$. We consider an open covering $\mathfrak{U} = \{U_i : i \in I\}$ of G. this proof makes no sense at this point □

What happens if $n > 2$? Then one may think that $\mathscr{I}(G)$ would be generated by $z_1, \ldots, z_n$. This is however not, in general, the case, although it will be if $H^{0q}_{\mathrm{Dol}}(G) = 0$ for each $q \geqslant 1$.

Before moving on, just to illustrate how complicated this study can be, recall that we mentioned if $G \subset \mathbb{C}$ is a region, then $H^{01}(G) = 0$ (one uses Runge approximation to show this). But we could take a very simple region in $\mathbb{C}^2$ like $G = \mathbb{C}^2 \setminus \{0\}$, and we no longer have $H^{01}(G) \neq 0$.

2.1 Local solutions

We wish to generalize the following well-known result from single variable theory.

Theorem 2.1.1. *Let $G \Subset \mathbb{C}$ be a region (recall, $U \Subset V$ means U is compactly contained in V, so $\overline{U} \subset V$ is compact). Suppose also that the boundary ∂G is "reasonable" (to be made precise). Let $f \in C^1(\overline{G})$ (continuously real differentiable). Then for $z \in G$,*

$$f(z) = \frac{1}{2\pi i} \int_{\partial G} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_G \frac{\partial f / \partial \overline{\zeta}(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta}.$$

Our first concern when making such a claim is whether or not the integrals even make sense. Here it is easy to see. Since f is continuously differentiable up to the boundary and G is compact, we can simply take the L^∞ norm of f from the first integral and that of $\partial f / \partial \overline{\zeta}$ from the latter. Then it reduces to the fact that in $\mathbb{R}^N$,

$$\int_{B_a(0)} \frac{dx}{|x|^k} < \infty \quad \text{for } k < N.$$

For $k \geqslant N$ the integral is infinite, of course. Thus far, we do not need the boundary to be anything special other than measurable, which it is as the boundary of an open set.

What we do need a nice boundary for is the application of Stokes' theorem, which, as one should recall, says

$$\int_{\partial K} \omega = \int_K d\omega$$

for a differential form, ω and some region $K \subset \mathbb{R}^N$. Specifically, ω is an $(N-1)$-form defined on $\overline{K}$, which is continuously differentiable.

This is the only result where we need a "reasonable" boundary, so for example, a piecewise smooth boundary would work. We will not concern ourselves with the details of choosing the most general boundary that will work, so we just will continue to say reasonable to mean that Stokes' theorem will work in our setting.

We need to define the Bochner-Martinelli kernel, $B_n(\zeta, z)$ for our next theorem. First, say

$$\beta(\zeta, z) = \sum_{j=1}^{n} (\overline{\zeta}_j - \overline{z}_j)\, d\zeta_j.$$

Then

$$\overline{\partial}_\zeta \beta(\zeta, z) = \sum_{j=1}^{n} d\overline{\zeta}_j \wedge d\zeta_j,$$

which is a volume form in $\mathbb{C}^n$. Then

$$B_n(\zeta, z) = \left(\frac{1}{2\pi i}\right)^n \frac{1}{\|\zeta - z\|_2^{2n}} \beta \wedge (\overline{\partial}_\zeta \beta(\zeta, z))^{n-1}.$$

In the case $n = 1$, we get

$$\frac{1}{2\pi i} \frac{1}{|\zeta - z|^2} (\overline{\zeta} - \overline{z})\, d\zeta = \frac{1}{2\pi i} \frac{d\zeta}{\zeta - z},$$

which gives the previous theorem when plugged into the theorem below.

Theorem 2.1.2 (Bochner-Martinelli formula). *Suppose $G \Subset \mathbb{C}^n$ is a region, ∂G is reasonable, $f \in C^1(\overline{G})$, and $z \in G$. Then*

$$f(z) = \int_{\partial G} f(\zeta) B(\zeta, z) - \int_G \overline{\partial} f(\zeta) \wedge B(\zeta, z).$$

Proof. As before, we can easily see the integrals exist. We examine the latter integral. Consider a fixed $z \in G$ and an $\varepsilon > 0$ small enough so that $\mathbb{B}_\varepsilon(z) \Subset G$. We will call $G_\varepsilon = G \setminus \overline{\mathbb{B}}_\varepsilon(z)$. Using Stokes' theorem, we see

$$\begin{aligned} \int_{G_\varepsilon} \overline{\partial} f(\zeta) \wedge B(\zeta, z) &= \int_{G_\varepsilon} d(f(\zeta) B(\zeta, z)) \\ &= \int_{\partial G} f(z) B(\zeta, z) - \int_{S_\varepsilon} f(z) B(\zeta, z), \end{aligned}$$

where S_ε is the boundary of $\mathbb{B}_\varepsilon(z)$. It remains to show that as $\varepsilon \to 0$, the latter integral tends to $f(z)$ and the left hand side to an integral over G. We show the convergence to $f(z)$; the other limit is an easy exercise.

Note that

$$\lim_{\varepsilon\to 0}\int_{S_\varepsilon} f(\zeta)B(\zeta,z) = \lim_{\varepsilon\to 0}\left(\int_{S_\varepsilon}(f(\zeta)-f(z))B(\zeta,z) + \int_{S_\varepsilon} f(z)B(\zeta,z)\right).$$

The former integral clearly goes to zero. Briefly, $f(\zeta)-f(z)$ will act like ε, $B(\zeta,z)$ like $1/\varepsilon^{2n-1}$, and $\operatorname{vol}(S_\varepsilon)$ like ε^{2n-1}, so altogether they act like ε. This is not hard to make precise. For the latter, we may factor out $f(z)$ and transform the integral to be around $z=0$. So, using Stokes' theorem again,

$$\begin{aligned}
\int_{S_\varepsilon(0)} B(\zeta,0) &= \frac{1}{\varepsilon^{2n}}\int_{S_\varepsilon(0)}\left(\frac{1}{2\pi i}\right)^n \beta(\zeta,0)\wedge(\bar\partial_\zeta\beta)^{n-1}\\
&= \frac{1}{\varepsilon^{2n}}\left(\frac{1}{2\pi i}\right)^n\int_{\mathbb{B}_\varepsilon(0)}\bar\partial\beta(\zeta,0)^n\\
&= \frac{1}{\varepsilon^{2n}}\left(\frac{1}{2\pi i}\right)^n\int_{\mathbb{B}_\varepsilon(0)} n!\,d\bar\zeta_1\wedge d\zeta_1\wedge\cdots\wedge d\bar\zeta_n\wedge d\zeta_n\\
&= \frac{1}{\varepsilon^{2n}}\left(\frac{1}{2\pi i}\right)^n n!(2i)^n\int_{\mathbb{B}_\varepsilon(0)} d\mathrm{Vol}(\zeta)\\
&= \frac{1}{\varepsilon^{2n}}\left(\frac{1}{2\pi i}\right)^n n!(2i)^n\frac{\pi^n}{n!}\varepsilon^{2n}\\
&= 1,
\end{aligned}$$

where we have used that

$$dz\wedge d\bar z = (dx+idy)\wedge(dx-idy) = 2idx\wedge dy.$$

This completes the proof. □

With this we finally make meaningful progress on the main question of this chapter that we have mentioned a number of times.

Theorem 2.1.3. *Let $f \in C^1_{01}(\mathbb{C}^n)$ be a $(0,1)$-form, continuously differentiable, with $\bar\partial f = 0$, and having compact support. Then there is a $u \in C^1(\mathbb{C}^n)$ with $\bar\partial u = f$. If $n > 1$, then u can be chosen with compact support.*

Proof. Consider the following idea. If there were such a u with compact support, then the Bochner-Martinelli formula would give

$$u(z) = \int_{\partial B} u(\zeta)B(\zeta, z) - \int_B f(\zeta)B(\zeta, z),$$

where B is some ball. If we pick the ball with sufficiently large radius, then $\operatorname{supp} u$ would be contained inside of it and we would have

$$u(z) = -\int_B f(\zeta)B(\zeta, z).$$

With this intuition in mind, we define the function

$$u(z) = -\int_{\mathbb{C}^n} f(\zeta)B(\zeta, z)$$

and will show that $\overline{\partial} u(z) = f(z)$. First note that f is a $(0, 1)$ form, so we can write it as

$$f(\zeta) = \sum_{j=1}^{n} f_j(\zeta)\, d\overline{\zeta}_j.$$

With this, we can define

$$u_j(z) = \int_{\mathbb{C}^n} f_j(\zeta)\, d\overline{\zeta}_j \wedge B(\zeta, z),$$

so that

$$u(z) = \sum_{j=1}^{n} u_j(z).$$

We will write C for a constant that we do not care to write in the following computations. We show that u_j is differentiable. Expanding u_j, we get

$$\begin{aligned} u_j(z) &= C\int_{\mathbb{C}^n} f_j(\zeta)(\overline{\zeta}_j - \overline{z}_j)\frac{1}{\|\zeta - z\|_2^{2n}}\, dV(\zeta) \\ &= C\int_{\mathbb{C}^n} \frac{f_j(z+w)\overline{w}_j}{\|w\|_2^{2n}}\, dV(w), \end{aligned}$$

where the second equality follows from the transformation $w = \zeta - z$. One may show (easy exercise) using results from Lebesgue integration theory that if L_z is a linear first order differential operator with constant coefficients, i.e.

$$L_z = \sum_{j=1}^{n} \lambda_j \frac{\partial}{\partial z_j} + \mu_j \frac{\partial}{\partial \bar{z}_j},$$

then we can pass it through the integral sign above. That is,

$$L_z u_j(z) = C \int_{\mathbb{C}^n} \frac{L_z f_j(z+w)\overline{w}_j}{\|w\|_2^{2n}}\, dV(w).$$

From here it is a simple matter to verify that $u \in C^1(\mathbb{C}^n)$.

Next we show that

$$\frac{\partial u}{\partial \bar{z}_j} = f_j.$$

Consider $j = 1$. We have

$$\frac{\partial u}{\partial \bar{z}_1} = -\int_{\mathbb{C}^n} \frac{\partial f}{\partial \bar{\zeta}_1} \wedge B(\zeta, z) = -\int_{\mathbb{C}^n} \sum_{j=1}^{n} \frac{\partial f_j}{\partial \bar{\zeta}_1}\, d\zeta_j \wedge B(\zeta, z).$$

Since $\bar{\partial} f = 0$, we see that

$$\frac{\partial f_j}{\partial \bar{\zeta}_1} = \frac{\partial f_1}{\partial \bar{\zeta}_j},$$

which turns the latter integral above into

$$\frac{\partial u}{\partial \bar{z}_1} = -\int_{\mathbb{C}^n} \bar{\partial} f_1 \wedge B(\zeta, z) = f_1(z),$$

since f has compact support (recall the work we did in the idea at the beginning of the proof). We may repeat this for each j, so in the end we achieve $\bar{\partial} u = f$.

Now, on the unbounded component of $(\operatorname{supp} f)^c$, which contains complex lines when $n > 1$, we have $\bar{\partial} u = f = 0$, which means that

u is holomorphic there as a $C^1(\mathbb{C}^n)$ function satisfying the Cauchy-Riemann functions[2]. Thus, by Liouville's theorem, we see that u is identically zero on the unbounded component of $(\operatorname{supp} f)^c$. That is, since f had compact support, u must have compact support. $\square$

The question one is likely left with is, what happens in the case $n = 1$. If u did have compact support, then for $R > 0$ big enough,

$$\begin{aligned} 0 &= \int_{|z|=R} u(z)\,dz = \int_{|z|\leqslant R} d(u(z)\,dz) \\ &= \int_{|z|\leqslant R} \frac{\partial u}{\partial \bar{z}}\,d\bar{z} \wedge dz = -\int_{\mathbb{C}} f(z)\,dz \wedge d\bar{z}. \end{aligned}$$

This is not the case for f in general, so the theorem fails.

We can once again clean up the statement of the theorem and state it very succinctly with some fancy language. We introduce the compact support Dolbeault cohomology:

$$H_c^{pq}(G) = \frac{Z_c^{pq}(G)}{B_c^{pq}(G)};$$

as one may guess, the c refers to the compact support of the elements in $Z_c^{pq}(G)$ and $B_c^{pq}(G)$. With this, theorem 2.1.3 becomes $H_c^{01}(\mathbb{C}^n) = 0$ if $n > 1$.

Remark 2.1.1. One also has $H_c^{pq}(\mathbb{C}^n) = 0$ for $1 \leqslant q \leqslant n-1$ and $H_c^{pn}(\mathbb{C}^n) \neq 0$.

As an application, we can finally make good on the promise from chapter 0 to prove Hartogs' Kugelsatz, which we state again here for posterity.

Theorem (Hartogs' Kugelsatz). *Suppose $n > 1$, $U \subset \mathbb{C}^n$ is open, and $K \subset U$ is compact such that $U \setminus K$ is connected. Then the map*

$$\mathcal{O}(U) \to \mathcal{O}(U \setminus K)$$

given by restriction is surjective.

[2]This naturally works on all of $(\operatorname{supp} f)^c$, but we only care about the unbounded component

Proof. Suppose χ is a compactly supported smooth bump function on U which is identically 1 on a neighborhood of K and between 0 and 1 otherwise. If f is holomorphic on $U \setminus K$, we see that $\widetilde{f} = (1-\chi)f \in C^\infty(U)$. Write $F = \overline{\partial}\widetilde{f}$ and notice that $\operatorname{supp} F \subset \operatorname{supp}\chi$, which is compact, so we may solve $\overline{\partial}u = F$ such that u has compact support.

Note that $u \equiv 0$ on the unbounded component of $(\operatorname{supp} F)^c$. Now define $\widehat{f} = \widetilde{f} - u$, which satisfies $\overline{\partial}\widehat{f} = 0$, so $\widehat{f} \in \mathcal{O}(U)$. Now, since the boundary of the unbounded component of $(\operatorname{supp} F)^c$ is contained in $U \setminus K$, there are open sets in $U \setminus K$ where $u \equiv 0$. Thus, $\widehat{f} = f$ in a nonempty open set. By the connectedness assumption, this extends to $U \setminus K$. □

Moving back to theorem 2.1.3, we can actually find an integral representation of the solution u when $n = 1$. One may naively expect that this is nice, because it is nice to explicitly write down a solution, and this is true, but further, oftentimes in analysis it is useful to have such an integral representation, because integrals behave better than derivatives (especially with estimates) and are in many cases easier to work with.

Theorem 2.1.4. *Suppose $G \Subset \mathbb{C}$ is a region and $f \in C^1(G)$. As usual, ∂G is taken to be reasonable. Then the function*

$$u(z) = \frac{1}{2\pi i}\int_G \frac{f(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta}$$

is C^1 and satisfies $\frac{\partial u}{\partial \overline{z}} = f(z)$ for all $z \in G$.

Proof. We single out an arbitrary $z_0 \in G$ and consider an $\varepsilon > 0$ small enough so that the open ball around it $D_{2\varepsilon}(z_0)$ is compactly contained in G. Now we define a standard bump function satisfying

$$\chi = \begin{cases} 1 & z \in D_\varepsilon(z_0) \\ 0 & z \notin D_{2\varepsilon}(z_0); \end{cases}$$

on the annulus that we have neglected we only require that $0 \leqslant \chi \leqslant 1$. As usual we are also requiring $\chi \in C^\infty(\mathbb{C})$. Now, we rewrite u from the theorem:

$$u(z) = \frac{1}{2\pi i}\int_G \frac{(\chi f)(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} + \frac{1}{2\pi i}\int_G \frac{((1-\chi)f)(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$
$$=: u_1(z) + u_2(z).$$

Note that

$$2\pi i u_1(z) = \int_{\mathbb{C}} \frac{(\chi f)(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta},$$

so we can repeat work we have already done in the proof of the Bochner-Martinelli formula to show that $\partial u_1/\partial \bar{z} = f$ finish this when you are not lazy, really should be easy □

Moreover, if f is assumed to be more regular, then u inherits that extra regularity (i.e. if f is C^k, so is u). If f depends differentiably on parameters, the u depends in the same way on these parameters. That is, if $f(z; t_1, \ldots, t_n)$ for $t_j \in \mathbb{R}$ then

$$u(z; t_1, \ldots, t_n) = \frac{1}{2\pi i}\int_G \frac{f(\zeta; t_1, \ldots, t_n)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}.$$

Differentiation under the integral sign with respect to any of the t_j is fully justified. For more on this, see
[lieb2012cauchy].

Proposition 2.1.5. *Let $q \geqslant 1$ and $D \subset \mathbb{C}^n$ be a polydisc. Suppose $D' \Subset D$ is another polydisc. Then for each $f \in C^\infty_{0q}(D)$ with $\bar{\partial} f = 0$, there is a $u \in C^\infty_{0,q-1}(D')$ with $\bar{\partial} u = f$.*

Proof. type □

Remark 2.1.2. If X is a complex manifold, $U \subset X$ open, and f is a $(0, q)$-form on U with $\bar{\partial} f = 0$, then for each $x \in U$, there exists $u \in C^\infty_{0,q-1}(V(x))$ with $\bar{\partial} u = f$.

Theorem 2.1.6. *Let D be a polydisc in $\mathbb{C}^n$. Suppose $f \in C^\infty_{0q}(D)$ with $\bar{\partial} f = 0$. Then there is a $u \in C^\infty_{0,q-1}(D)$ with $\bar{\partial} u = f$.*

Proof. type □

Some remarks are in order.

(i) It is somehow typical that the proof for $q = 1$ is harder and must be done separately. Above we needed an approximation theorem—Taylor's theorem being the most basic example—, which is also typical.

(ii) $H^{0q}(D) = 0$ for $q \geqslant 1$.

(iii) $H^{pq}(D) = 0$ for $q \geqslant 1$ as well. A (p,q)-form is the same as an $\binom{n}{p}$-tuple of $(0,q)$-forms and this equivalence preserves exactness and closedness. This only works for $\mathbb{C}^n$ and not for general complex manifolds due to the ability to choose a global coordinate system.

(iv) For the case $n = 1$, it is true that $H^{01}(G) = 0$ for an arbitrary region $G \subset \mathbb{C}$. The proof is based on Runge approximation, which is a stronger (and deeper) approximation theorem than Taylor's.

(v) We can take D to have infinite radius in the theorem above, i.e. $D = \mathbb{C}^n$.

2.2 Holomorphic convexity

The phenomenon of simultaneous extension of all holomorphic functions from one domain to a strictly larger one raises the question of characterizing those domains for which this phenomenon does not occur. Since Hartogs' pioneering work in 1906 this has been the source of a great deal of developments in the theory of several complex variables. This will lead into the next section, where we introduce the "curious restriction" referred to by Oka, which is today known as pseudoconvexity.

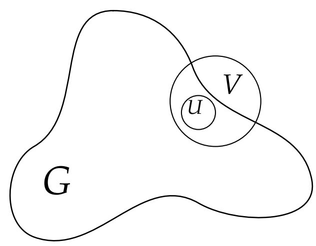
We make some initial comments on extending functions to start. If $G \subset \mathbb{C}$ is a region then there exists $f \in \mathcal{O}(G)$ not extendable beyond any point in ∂G. For example, consider $G = \mathbb{D} = \{|z| < 1\}$ and

$$f(z) = \sum_{j=0}^{\infty} z^{j!}.$$

One may show that this cannot be extended up to or past the boundary at any point.

For $n \geqslant 1$ and $G \subset \mathbb{C}^n$ a convex region, we have that for all $a \in \partial G$ there exists a function $f \in \mathcal{O}(G)$ that does not extend beyond a. To show this, one picks a hyperplane at a that intersects G only at a (which is possible since G is convex). The hyperplane is such that G is contained completely in one of the half-spaces defined by the hyperplane. Then we draw a line $\ell : \mathbb{C}^n \to \mathbb{C}$ such that $\ell(a) = 0$ and define $f(z) = 1/\ell(z)$.

Definition 2.2.1. *If $G \subset \mathbb{C}^n$ is a region, then an extension pair of G is a pair (U, V) of open sets in $\mathbb{C}^n$ with $\varnothing \neq U \subset V \cap G$ and $V \not\subset G$, see below.*



Definition 2.2.2. *Suppose $f \in \mathcal{O}(G)$ (as usual $G \subset \mathbb{C}^n$ is a region). A holomorphic extension of f is a triple $(U, V, \widetilde{f})$ where*

(i) (U, V) is an extension pair of G,

(ii) $\widetilde{f} \in \mathcal{O}(V)$, and

(iii) $\widetilde{f}|_U = f|_U$.

Definition 2.2.3. *(i) G is called a domain of holomorphy if it is a region $G \subset \mathbb{C}^n$ such that for all extension pairs (U, V) of G, there is $f \in \mathcal{O}(G)$ without holomorphic extensions $(U, V, \widetilde{f})$.*

(ii) G is called a domain of existence if there is a holomorphic function $f \in \mathcal{O}(G)$ not extendable for any extension pair.

Of course, (ii) implies (i) above. Somewhat surprisingly, the converse is also true, which is the content of a soon-to-come theorem. The obvious examples are that $\mathbb{C}^n$ is a domain of existence and convex domains are domains of holomorphy (explained at the beginning of the section).

Definition 2.2.4. *Suppose $G \subset \mathbb{C}^n$ is a region and $K \subset G$ is compact. Then we have the following definitions.*

(i) The set

$$\begin{aligned}\widehat{K} &= \{z \in G : \|f(z)\|_\infty \leqslant |f|_K \\ &= \max_{z' \in K} \|f(z')\|_\infty \textit{ for all } f \in \mathcal{O}(G) \\ &= \bigcap_{f \in \mathcal{O}(G)} \|f\|_\infty^{-1}([0, |f|_K]).\end{aligned}$$

is called the holomorphic convex hull of K.

(ii) We say K is holomorphically convex if $K = \widehat{K}$.

(iii) We say G is holomorphically convex if $\widehat{K}$ is compact for all K compact in G.

Some remarks are in order.

(i) Obviously $K \subset \widehat{K}$.

(ii) $\widehat{K}$ is closed in G.

(iii) $\widehat{\widehat{K}} = \widehat{K}$.

(iv) If we choose $f(z) = \exp(z \cdot \zeta)$ we can show that $\widehat{K}$ is contained in the convex hull of K, which is bounded. Thus, $\widehat{K}$ is also bounded. Carefully note that this does not imply in general that $\widehat{K}$ is compact in G, and in fact, it is not. In fact, we see from this that $\widehat{K}$ is compact if and only if it is a positive distance from the boundary of G (defined below).

(v) This notion of convexity is modelled on Euclidean convexity. If we substitute in the definition of $\widehat{K}$ the set $\mathcal{O}(G)$ by the algebra of linear functions on G, then the definition is equivalent to Euclidean convexity. More generally, one can define abstract notions of convexity by substituting $\mathcal{O}(G)$ by any algebra of continuous functions.

Theorem 2.2.5. *Suppose $G \subset \mathbb{C}^n$ is a region. Then the following are equivalent.*

(i) G is a domain of existence.

(ii) G is a domain of holomorphy.

(iii) G is holomorphically convex.

We hold off on the proof for now; we first need some definitions and another theorem.

Definition 2.2.6. *Let $G \subset \mathbb{C}^n$ be a region and $z \in G$. The function*

$$\delta(z) = \delta_G(z) = \sup\{r \in \mathbb{R}_+ : D_r(z) \subset G\}$$

is called the boundary distance.

Note that $\delta(z) = +\infty$ only when $G = \mathbb{C}^n$. Furthermore, δ_G is continuous; this is a general fact about metric spaces. In particular, one can show that if (X, d) is a metric space and $A \subset X$ is bounded, then $\delta : X \to \mathbb{R}_+$ given by $x \mapsto d(x, A) = \sup_{a\in A} d(x, a)$ is continuous.

If $K \subset G$ is compact, then $\delta|_K$ has a positive minimum, i.e.

$$0 < r < \delta(K) := \min_{z\in K} \delta(z)$$

for some $r > 0$. In this case, we will write

$$K_r = \bigcup_{z\in K} D_r(z) \Subset G.$$

Theorem 2.2.7 (Lemma of Cartan/Thullen). *As usual, $G \subset \mathbb{C}^n$ is a region. Let $K \subset G$ be compact and $z_1 \in \widehat{K}$. Let $u \in \mathcal{O}(G)$ and write $P_u(z)$ for the power series of u about z_1. Then $P_u(z)$ converges at least in a polydisc, $D_{\delta(K)}(z_1)$. Further, it converges uniformly on compacta of $D_{\delta(K)}(z_1)$.*

Proof. For $G = \mathbb{C}^n$ the result is immediate, so suppose otherwise. Then let $r < \delta(K) < \infty$ and pick $z_0 \in K$. We write

$$P_u(z) = \sum_{\nu\in\mathbb{N}_0^n} a_\nu (z - z_0)^\nu, \quad a_\nu = \frac{\partial^\nu u(z_0)}{\nu!}$$

and we may write

$$\partial^\nu u(z_0) = \frac{\nu!}{(2\pi i)^n} \int_{|\zeta_1 - w_1|<r} \cdots \int_{|\zeta_n - w_n|<r} \frac{u(\zeta)}{(\zeta - w)^{\nu+1}}\, d\zeta.$$

We thus have the Cauchy estimates, with the compact set $S = \overline{K}_r$,

$$|\partial^\nu u(z_0)| \leqslant \frac{\nu!}{(2\pi)^n}(2\pi r)^n \frac{1}{r^{|\nu|+n}}|u|_S = \frac{\nu!}{r^{|\nu|}}|u|_S.$$

These will hold for $z \in D_r(z_0)$. Hence,

$$|a_\nu| \leqslant \frac{|u|_{K_r}}{r^{|\nu|}}$$

and

$$|a_\nu||z - z_0|^\nu \leqslant |u|_{K_r} < \infty,$$

since K_r is bounded. The terms of the power series are uniformly bounded, so by proposition 1.0.6, we get convergence of $P_u(z)$ in $D_r(z_0)$ for each $r < \delta(K)$. this proof needs some explanation about $z_1 \in \widehat{K}$... should be a good exercise in understanding the definition of the holomorphic hull to figure it out. □

With this, we can finally return and prove theorem 2.2.5.

Proof of theorem 2.2.5. As we already mentioned, (i) clearly implies (ii). Furthermore, we omit the proof of (iii) implies (i) as we will not be needing it, see
[hormander1973introduction] for a proof. That only leaves (ii) implies (iii).

Check Hormander... proof makes no sense. □

Remark 2.2.1. If G is a domain of holomorphy and $K \subset G$ is compact with $\delta(\widehat{K}) = \delta(K)$, then $\widehat{K}$ is compact, since, in that case $\delta(\widehat{K}) > 0$.

2.3 Pseudoconvexity

Pseudoconvexity is one of the most central concepts in several complex variables, since it relates to the very core of holomorphic functions. It is intimitely intertwined with power series, the identity theorem, and analytic continuation. It has its roots in Hartogs' amazing discovery in 1906 that we have discussed.

2.3.1 Subharmonic and plurisubharmonic functions

We will need some results about subharmonic functions to continue. Recall first that a function $h : \mathbb{C} \to \mathbb{R}$ is called harmonic if $-\Delta h = 0$. We have the following two results from the single variable theory.

(i) If $D \subset \mathbb{C}$ is a disk and $\Gamma = \partial D$, then, given a function h, continuous on Γ, there exists a unique $\widetilde{h}$ continuous on $\overline{D}$ that is harmonic in D and satisfies $\widetilde{h} = h$ on Γ.

(ii) **Fejér's theorem.** Let $\Gamma \subset \mathbb{C}$ be a circle. The functions $\Re(P(z))$, where P is a complex polynomial, are uniformly dense in $C^0(\Gamma)$.

Using (i) as motivation, one may be inclined to define a weakened version of harmonicity. We do this now.

Definition 2.3.1. *Let $U \subset \mathbb{C}$ be open and $\varphi : U \to \mathbb{R} \cup \{-\infty\}$ be a function. We say that φ is subharmonic if the following hold.*

(i) The function φ is upper semicontinuous.

(ii) For each $z_0 \in U$, there is a disc $D_{r_0}(z_0) \subset U$ with the property that whenever h is a continuous function on $\overline{D_r(z_0)}$ with $r \leqslant r_0$, and harmonic in $D_r(z_0)$ with

$$\varphi|_{\partial D_r(z_0)} \leqslant h|_{\partial D_r(z_0)}$$

then $\varphi \leqslant h$ on all of D_r.

Remark 2.3.1. One could equivalently say

(ii′) For each disk, $D_r(z_0) \Subset G$, and each continuous, real-valued function h defined on $\overline{D_r(z_0)}$ which is harmonic in $D_r(z_0)$, if $\varphi \leqslant h$ on $\partial D_r(z_0)$, then $\varphi \leqslant h$ on $\overline{D_r(z_0)}$.

Remark 2.3.2. A further definition in the case that $u \in C^2(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open is that u is subharmonic if $-\Delta u \leqslant 0$ in Ω. These definitions do turn out to be equivalent, although we will not go into the details.

competing notation and stuff above... fix

One may easily verify that in one real variable, harmonic functions are the linear functions and subharmonic functions are the convex functions. Also, another easy exercise is that when $n = 2$ (as in 2 real dimensions), we can write

$$\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}}.$$

We record some useful facts about subharmonic functions. The proofs can be found in
[**hormander1973introduction**] or really any book on elliptic differential equations.

Theorem 2.3.2. *(i) For all $z_0 \in \mathbb{C}$, there exists $r_0 > 0$ such that for all $0 < r < r_0$, the mean value inequality holds*

$$\varphi(z_0) \leqslant \frac{1}{2\pi}\int_0^{2\pi} \varphi(z_0 + re^{i\theta})\, d\theta.$$

Here $\varphi : \mathbb{C} \to \mathbb{R}$ is continuous.

(ii) The mean value inequality holds for all r_0 such that $D_{r_0}(z_0) \Subset U$ and this actually characterizes subharmonic functions.

(iii) Subharmonicity is a local property.

(iv) We have a local maximum principle coming from the mean value property. That is, at any point, the function is no greater than the average of the values in a circle around it (this is the intuition, at least).

(v) If φ is subharmonic and not $-\infty$, then it is locally integrable.

(vi) If φ, ψ are both subharmonic, then $\varphi + \psi$ and $c\varphi$ for $c \geqslant 0$ are subharmonic. That is, subharmonic functions form a convex cone.

Remark 2.3.3. All of these can be extended to higher dimensions. See any PDE text for details.

As one can see, subharmonic functions are rather nice. In addition to these theorems above, we also have that subharmonicity is preserved by some limits.

Theorem 2.3.3. *Suppose φ_j is a sequence of subharmonic functions such that $\varphi_j \to \varphi$ locally uniformly. Then φ is subharmonic. If instead, we just have that $\varphi_j \to \varphi$ pointwise, then φ will be subharmonic given the additional assumption that the sequence is (pointwise) monotone decreasing.*

One should keep in mind the relation between holomorphic functions and harmonic functions. In terms of subharmonic functions we also have that if f is harmonic then, e.g. $\log |f|$ (defining it to be $-\infty$ at the zeros) and $|f|$ are both subharmonic. In fact, if f has no zeros in U, then $\log |f|$ is just harmonic in U.

The so-called Levi conditions, which we will discuss soon involves the complex Hessian (or Levi form). To prepare for this, we will single out those functions that turn out to have positive semidefinite complex Hessians. It may not be obvious for a while why this is the case.

Definition 2.3.4. *Suppose $G \subset \mathbb{C}^n$ is open and $\varphi : G \to \mathbb{R} \cup \{-\infty\}$. We say φ is plurisubharmonic if it satisfies*

(i) φ is upper semicontinuous and

(ii) for each complex line L in $\mathbb{C}^n$, $\varphi|_{L\cap G}$ is subharmonic.

Explicitly, the second condition says that if $L = \{a + tb : t \in \mathbb{C}\}$ for some $a, b \in \mathbb{C}^n$ and $b \neq 0$, then $\varphi(a + tb)$ is subharmonic in t wherever it is defined.

Remark 2.3.4. Of course, if $n = 1$, then plurisubharmonic is the same as subharmonic, hence the prefix.

Remark 2.3.5. Plurisubharmonicity is a local property. That is, a function is plurisubharmonic globally if and only if it is plurisubharmonic in a neighborhood of each point (exercise).

We record some facts about plurisubharmonic functions; we omit the proofs, they are all either trivial themselves or trivial consequences of facts we already listed about subharmonic functions.

Theorem 2.3.5. *Denote by $\mathscr{P}(G)$ the set of plurisubharmonic functions on G.*

(i) Constant functions, $c \in \mathbb{R} \cup \{-\infty\}$ are plurisubharmonic.

(ii) *If* $\varphi, \psi \in \mathscr{P}(G)$, *then* $\max(\varphi, \psi) \in \mathscr{P}(G)$.

(iii) *If* $c \geqslant 0$ *and* $\varphi \in \mathscr{P}(G)$, *then* $c\varphi \in \mathscr{P}(G)$.

(iv) *If* $(\varphi_j)_{j=1}^{\infty} \subset \mathscr{P}(G)$ *and* $\varphi_j \to \varphi$ *locally uniformly, then* $\varphi \in \mathscr{P}(G)$.

(v) *If* $(\varphi_j)_{j=1}^{\infty} \subset \mathscr{P}(G)$ *and* $\varphi_j \to \varphi$ *is a monotone decreasing sequence, then* φ *is plurisubharmonic.*

(vi) *If* $\varphi, \psi \in \mathscr{P}(G)$ *then so is* $\varphi + \psi$ *(note that* $\mathscr{P}(G)$ *is not a vector space, but a cone).*

(vii) *A local maximum principle holds for plurisubharmonic functions.*

We prove one fact about plurisubharmonic functions in order to see how they work a bit.

Proposition 2.3.6. *Let* G *be a region. If* $\varphi : G \to [-\infty, \infty)$ *is plurisubharmonic and not identically* $-\infty$, *then* φ *is locally integrable.*

Proof. We prove this for $n = 1$ (i.e. where φ is just subharmonic) and then Fubini's theorem gives the result for general n.

Pick $z_0 \in G$ with $\varphi(z_0) > -\infty$ and $r > 0$ so that $D = D_r(z_0) \Subset G$. Let h be a continuous function on $\overline{D}$ with $\varphi \leqslant h$. Then using the mean value property,

$$\begin{aligned}\int_D h(z)\, d\lambda(z) = \int_0^r \int_0^{2\pi} \rho h(z_0 + \rho e^{i\theta})\, d\theta d\rho &\geqslant \int_0^r \int_0^{2\pi} \rho\varphi(z_0 + \rho e^{i\theta})\, d\theta d\rho \\ &\geqslant \int_0^r 2\pi\rho\varphi(z_0)\, d\rho = \pi r^2 \varphi(z_0).\end{aligned}$$

Thus, φ is integrable over $\overline{D}$. This is because of the following fact: if u is upper semicontinuous on G and $K \Subset G$ then u is Lebesgue integrable on K if and only if

$$\int_K u(x)\, d\lambda(x) > -\infty.$$

This then extends easily to the same fact, but where we test that for all h continuous on K with $h \geqslant \varphi$, we have

$$\int_K h\, d\lambda \geqslant C$$

for some constant C not depending on h. The proofs of these facts are exercises.

With this information in hand, we set

$$M = \{z \in G : \varphi \text{ is integrable in a neighborhood of } z\},$$

which is open. But $G \setminus M \subset \{\varphi = -\infty\}$, which is also open (because of what we just showed). Thus, $M = G$ since G is connected and $M \neq \emptyset$. This completes the proof. $\square$

Remark 2.3.6. An example is, as before, if f is analytic then $\log|f|$ is plurisubharmonic (again defining $\log(0) = -\infty$).

2.3.2 The Levi form

We fix some notation to simplify our lives a bit. Say $U \subset \mathbb{C}^n$ is open and φ is twice continuously (real) differentiable on U. We will write

$$\varphi_j := \partial_{z_j}\varphi, \quad \varphi_{\bar{k}} := \partial_{\bar{z}_k}\varphi.$$

Definition 2.3.7. *The Levi matrix at z of φ is $(\varphi_{\nu\bar{\mu}}(z))$ and the function*

$$\sum_{\nu,\mu=1}^{n} \varphi_{\nu\bar{\mu}}(z) t_\nu \bar{t}_\mu = L_\varphi(z;t), \quad t \in \mathbb{C}^n$$

is called the Levi form.

Note that L is hermitian, i.e. self-adjoint, i.e. it is equal to its own conjugate transpose. Under coordinate changes, the Levi form acts in the expected way. That is, if $F : U \to V$ is biholomorphic and $\varphi : V \to \mathbb{R}$, then

$$L_{\varphi\circ F}(w) = (J_F^h)' L_\varphi(F(w)) \overline{J_F^h}$$

More abstractly, $\mathbb{C}^n$ is the tangent space to $\mathbb{C}^n$ at z. Defining $F : \mathbb{C}^n \to \mathbb{C}^n$ and

$$t = J_F^h s =: F_* s,$$

we see that $L_{\varphi\circ F}(w; s) = L_\varphi(F(w); F_* s)$.

In particular, the rank and index of the Levi matrix is unvariant under biholomorphic transformations. Also, the numbers of positive, negative, and null eigenvalues are invariant under biholomorphic transformations (recall that hermitian matrices have real eigenvalues).

With this, we make good on the promise that the plurisubharmonic functions are those with positive semidefinite Levi forms.

Theorem 2.3.8. *If φ is twice continuously (real) differentiable and plurisubharmonic, then $L_\varphi \geqslant 0$ (this is notation for saying L_φ is positive semidefinite).*

Proof. In the case $n = 1$ we already know that if φ is subharmonic and C^2 then $\Delta\varphi \geqslant 0$. For general n, we fix $z_0 \in G$ and $w \in \mathbb{C}^n \setminus \{0\}$. Then

$$\Delta_t \varphi(z_0 + tw) \geqslant 0,$$

so by a remark we made about the Laplacian when $n = 2$, we see

$$\frac{1}{4}\Delta_t \varphi(z_0 + tw) = \frac{\partial^2}{\partial t \partial \bar{t}} \varphi(z_0 + tw) = \sum_{\nu,\mu=1}^{n} \varphi_{\nu\overline{\mu}}(z_0 + tw) w_\nu \overline{w}_\mu.$$

At $t = 0$, this is exactly the inequality we want. □

This naturally gives rise to the idea of strict plurisubharmonicity.

Definition 2.3.9. *We say φ is strictly plurisubharmonic if $L_\varphi > 0$.*

For example, $\varphi(z) = \|z\|_2^2$ is strictly plurisubharmonic. Further, if f is holomorphic, then $|f|^2$ is plurisubharmonic.

Remark 2.3.7. One may define pluriharmonic functions by saying they have zero Levi form. In this case one may show that a function is pluriharmonic if and only if it is locally the real or imaginary part of a holomorphic function.

2.3.3 Pseudoconvex domains

Finally, with the background of the previous two subsections, we may define the idea of pseudoconvexity.

Definition 2.3.10. *We say G is (Hartogs) pseudoconvex if G has a smooth strictly plurisubharmonic exhaustion function, $\varphi : G \to \mathbb{R}$.*

For a review, a function $\varphi : G \to \mathbb{R}$ is called an exhaustion function if for all $c \in \mathbb{R}$, the set

$$K_c = \{z \in G : \varphi(z) < c\}$$

is compactly contained in G.

For example, $G = \mathbb{C}^n$ is trivially pseudoconvex since it has a smooth strictly plurisubharmonic exhaustion function, namely $\varphi(z) = \|z\|_2^2$. Furthermore, any Euclidean convex domain is pseudoconvex.

Remark 2.3.8. One can (and often does in the literature) define pseudoconvexity through a continuous plurisubharmonic exhaustion function. It turns out, however, that on an open pseudoconvex domain (with this definition using just continuity and regular plurisubharmonicity), there is always a smooth strictly plurisubharmonic exhaustion function on this domain, so the definitions are equivalent (and this one is nicer, since the functions are nicer to deal with).

There is another common characterization of pseudoconvexity.

Theorem 2.3.11. *Let $G \subset \mathbb{C}^n$ be an open set with C^2 boundary. Suppose ρ is C^2 in a neighborhood of $\overline{G}$ satisfying $G = \{z : \rho(z) < 0\}$ and $\nabla\rho \neq 0$ on ∂G. Then G is pseudoconvex if and only if the following two conditions hold*

(i) $\displaystyle\sum_{\nu,\mu=1}^{n} \frac{\partial^2 \rho}{\partial z_\nu \partial \overline{z}_\mu} w_\nu \overline{w}_\mu \geqslant 0$ *when* $z \in \partial G$ *and*

(ii) $\displaystyle\sum_{\nu=1}^{n} \frac{\partial \rho}{\partial z_\nu} w_\nu = 0.$

Remark 2.3.9. This condition above does not come from nowhere; there is a characterization of regular convexity that is very similar. That is, if $G \subset \mathbb{R}^n$ is a region and $f : G \to \mathbb{R}$ is C^2 then f is convex if and only if for all $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n$ and $x \in G$ we have

$$\sum_{\nu,\mu=1}^{n} \frac{\partial^2 f}{\partial x_\nu \partial x_\mu}(x) \delta_\nu \delta_\mu \geqslant 0.$$

This then implies that a region $G \subset \mathbb{R}^n$ is convex if and only if there is a convex exhaustion function for G.

The condition above is called the Levi condition and one may define Levi pseudoconvexity through it. Of course, Hartogs pseudoconvexity can be defined on more general types of domains, but on any domain where they are both defined, it is a fact that they are equivalent (by the above theorem), so it never hurts to just say pseudoconvex. One advantage to this definition is it allows us to define strict pseudoconvexity: that condition (i) has a strict inequality so long as $w \neq 0$. This characterization also more clearly shows that pseudoconvexity is a local property and really just a property of the boundary. See
[**hormander1973introduction**, **lebl2019tasty**] for more exposition and proofs.

A natural question to ask is how pseudoconvexity relates to Euclidean convexity. The explicit formulations are clearly complex analogues of corresponding characterizations of convexity. In particular, convexity implies pseudoconvexity. Furthermore, it is elementary, but nontrivial, to show that a domain, G, is strictly pseudoconvex near a point $p \in \partial G$ if and only if it is strictly Euclidean convex with respect to suitable local holomorphic coordinates centered at p. That is, strict pseudoconvexity is locally the biholomorphically invariant version of strict convexity. This neat characterization breaks down already in the case of general pseudoconvex domains.

In any case, we have now defined two new notions of convexity (holomorphic convexity and pseudoconvexity), so the natural question to ask is how they are related. We will see below that it is fairly easy to show that holomorphic convexity implies pseudoconvexity; the other implication also turns out to be true, but is a significantly harder problem to solve—more on this later.

Theorem 2.3.12. *If G is holomorphically convex, then it is pseudoconvex.*

Proof. type □

In the case $n = 1$, every open subset of $\mathbb{C}$ is a domain of holomorphy and thus also pseudoconvex, which is why this concept does not come up in the single variable theory.

Some remarks are now in order in terms of intuitions and historical development. The question of whether or not one can reverse the above theorem was a long lasting problem in the area of several complex variables. It is known as the Levi problem, posed in 1911. It was Blumenthal in 1912,
[**blumenthal1912bemerkungen**], who first gave an answer, which was negative. In fact he proved that $\mathbb{C}^2 \setminus H$, where

$$H = \{(z_1, z_2) : x_1 = 0, x_2 \geqslant 0\}$$

with $z_j = x_j + iy_j$, is not a domain of holomorphy and is pseudoconvex. And this stood as problem solved for 15 years before Behnke in 1927,
[**behnke1978semesterberichte**], found a mistake in Blumenthal's proof and concluded that the Levi problem perhaps has a positive answer. He did give some conditions under which the Levi problem is true, but did not solve it in general.

In 1934, Behnke and Thullen published their influential "Ergebnisbericht,"
[**behnke1934theorie**], which collected all of the important unsolved problems in complex analysis; among these was the Levi problem.

In 1929 a young Japanese mathematician, Kiyoshi Oka came to Paris on scholarship to study the newest developments in analysis. After returning to Japan, he studied the Ergebnisbericht and seemingly decided to resolve all of these problems put forward by Thullen and Behnke. Of course, it commonly happens that a young scientist makes such an adventurous resolution, but it seldomly actually happens. Oka, however, did exactly that. In a sequence of works that stretched about two decades, he solved every single problem, including the Levi problem. In the course of this, he also opened the road to many other interesting questions (and answered many of those).

With a positive answer given by Oka, see
[**hormander1973introduction**] for a proof, we see that pseudoconvexity is that local analytic/geometric property of the boundary of a domain in $\mathbb{C}^n$ which characterizes domains of holomorphy. It is not clear at all that this global property should allow such a purely local characterization, which is part of the reason these results are so influential. Indeed, one does not need to know anything about the

holomorphic functions on a domain to check if it is pseudoconvex. To be explicit, there are lots of plurisubharmonic functions and they are easy to construct. We can even construct them locally and patch them together by taking maxima; contrastively, there are few holomorphic functions and we certainly cannot construct them locally and expect them to fit together at all.

Historical quip 2.3.1. It was said of Oka that students rather liked him, because he was a very mild examiner.

We unfortunately do not have time to see the proof of the Levi problem, but we briefly describe how it is done. First, we prove one final interesting theorem.

Theorem 2.3.13. *Suppose $G \subset \mathbb{C}^n$ is a region such that $H^{0q}(G) = 0$ for each $q \geqslant 1$. Then G is holomorphically convex.*

Proof. type □

Thus, if one can solve the Cauchy-Riemann equations for $q \geqslant 1$ on pseudoconvex domains, then they will automatically be holomorphically convex. This is what L. Hörmander does in [**hormander1973introduction**]. Briefly, one solves these in the sense of distributions and then studies the regularity of these solutions through mostly well-known methods in regularity theory to see that they can be taken to be smooth. Oka, on the other hand, only worked with the first cohomology, $q = 1$, when working with the Levi problem, which, as one may imagine, drastically complicated the situation.

2.4 Summary and outlook

type; this was not a part of the class, I just think it would be nice to have

Recently, people have been studying the Cauchy-Riemann equations on complex spaces, i.e. spaces with singularities allowed by I think by defining them off of the singularities and then meaningfully extending them

APPENDIX

A

ALGEBRA REVIEW

We review some basic algebra. Let R be a commutative ring with unity and let M, N be R-modules. See **[atiyah2018introduction, dummit2004abstract]** for proofs of many of these results and some more review. The exposition below is largely unorganized and is just meant to fill in any ideas one may not recall from basic commutative algebra.

A.1 Properties of rings and modules

Definition A.1.1. *We say that M is noetherian if one of the following equivalent conditions hold.*

(i) Each submodule $N \subset M$ is finitely generated.

(ii) Each ascending sequence of submodules

$$N_1 \subset N_2 \subset N_3 \subset \cdots \subset M$$

becomes stationary.

(iii) Each non-empty set of submodules has maximal elements.

Note that R is noetherian if and only if R is a noetherian R-module. Some examples include R being a field or principal ideal domain (PID).

Theorem A.1.2 (Hilbert). *If R is noetherian, then $R[x]$ is noetherian.*

Of course, by induction, this means that if R is noetherian, then $R[x_1, \ldots, x_n]$ is noetherian.

Proposition A.1.3. *Take*

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

to be a short exact sequence. Then if two of the modules, M, N, P above are noetherian, so is the third.

Proposition A.1.4. *Let R be noetherian. Then M is a noetherian R-module if and only if M is finite (i.e. finitely generated).*

Definition A.1.5. *We say R is a local ring if R has exactly one maximal ideal.*

Lemma A.1.6 (Krull lemma). *Let $(R, \mathfrak{m})$ be a local noetherian ring and its maximal ideal. Let M be a noetherian R-module and $M_1, M_2 \subset M$ submodules. If, for each k, we have $M_1 \subset M_2 + \mathfrak{m}^k M$, then $M_1 \subset M_2$.*

A.2 Divisibility

We recall some basic concepts of divisibility. The setting for this discussion is an integral domain, R.

Definition A.2.1 (Irreducibility and primes). *We say that $x \in R \setminus \{0\}$ is irreducible if $x \notin R^\times$ and $x = yz$ implies that y or z is a unit. Furthermore, we say that x is prime if it is not a unit and $p|ab$ implies $p|a$ or $p|b$.*

Of course, as one may recall from commutative algebra, primality implies divisibility (but the reverse implication is false). Using the idea of a prime we recall some properties of unique factorization domains (UFD).

Definition A.2.2. *R is called a UFD or factorial if every non-zero element is a product of primes, unique up to units.*

Proposition A.2.3. *R is factorial if and only if*

(i) every element is a product of irreducible elements and

(ii) every irreducible element is prime.

For example, principal ideal domains are factorial (like $\mathbb{Z}$ and $K[X]$). Euclidean domains are also examples of UFD.

Lemma A.2.4 (Gauss). *If R is a UFD, then $R[X]$ is also a UFD and the primes in $R[X]$ consist of primes in R and those primitive polynomials that are irreducible over the field of fractions,* $\mathrm{Quot}(R)$*.*

Primitive above, of course, means that the greatest common denominators of the coefficients is unity.

A.3 Category theory

We review some of the basics of category theory. Only definitions relevant to the course are given below, so there is obviously a lot missing.

Definition A.3.1 (Category). *A category $\mathscr{C}$ is a class of objects $\mathrm{Ob}(\mathscr{C})$ and sets of morphisms (arrows) $\mathrm{Ar}(\mathscr{C})$ between the objects with the following properties:*

(i) for every pair (A, B) of objects, there is a set $\mathrm{Hom}(A, B)$ of morphisms (possibly empty) and

(ii) for every triple (A, B, C) of objects, there is a map

$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C), \quad (f, g) \mapsto gf$$

such that the following hold

- *for $A \neq C$ or $B \neq D$, $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(C, D)$ are disjoint,*
- *$h(gf) = (hg)f$ for $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, and $h \in \mathrm{Hom}(C, D)$, and*
- *for all objects A there is an identity morphism $1_A \in \mathrm{Hom}(A, A)$ with $f1_A = f$ for all $f \in \mathrm{Hom}(A, B)$ and $1_A g = g$ for all $g \in \mathrm{Hom}(B, A)$.*

A morphism $f \in \mathrm{Hom}(A, B)$ is an isomorphism if there exists $g \in \mathrm{Hom}(B, A)$ with $fg = 1_B$ and $gf = 1_A$.

Some examples are in order. Take care to note that, despite many of the examples making it look this way, morphisms do not need to be some kind of "mapping."

(i) Sets with maps.

(ii) Groups with group homomorphisms.

(iii) Topological spaces with continuous functions.

(iv) The open sets of a topological space with inclusions.

(v) Open subsets of $\mathbb{C}^n$ and holomorphic functions.

(vi) The empty category with no objects and no morphisms[1].

Definition A.3.2 (Covariant Functor). *Let $\mathscr{C}, \mathscr{D}$ be categories. A covariant functor $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ assigns to every $A \in \mathrm{Ob}(\mathscr{C})$ an $\mathscr{F}(A) \in \mathrm{Ob}(\mathscr{D})$ and to every $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ an $\mathscr{F}(f) \in \mathrm{Hom}_{\mathscr{D}}(\mathscr{F}(A), \mathscr{F}(B))$ such that $\mathscr{F}(1_A) = 1_{\mathscr{F}(A)}$ for all $A \in \mathrm{Ob}(\mathscr{C})$ and furthermore $\mathscr{F}(gf) = \mathscr{F}(g)\mathscr{F}(f)$ for all morphisms f, g with compatible domains of definition.*

Definition A.3.3 (Contravariant Functor). *A contravariant functor $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ assigns to every $A \in \mathrm{Ob}(\mathscr{C})$ an $\mathscr{F}(A) \in \mathrm{Ob}(\mathscr{D})$ and to every $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ an $\mathscr{F}(f) \in \mathrm{Hom}_{\mathscr{D}}(\mathscr{F}(B), \mathscr{F}(A))$ so that $\mathscr{F}(1_A) = 1_{\mathscr{F}(A)}$ for all $A \in \mathrm{Ob}(\mathscr{C})$ and $\mathscr{F}(gf) = \mathscr{F}(f)\mathscr{F}(g)$ for all morphisms f, g with compatible domains of definition. In essence, a contravariant functor turns the arrows around.*

Again, we give some examples. The main relevant example, however, is a presheaf as defined in section 1.3.

(i) The identity functor.

(ii) The forgetful functor, for example mapping the category of groups to the category of sets.

(iii) The covariant functor $\mathrm{Hom}(A, \cdot)$ which gives every morphism, f, the mapping $\varphi \mapsto f \circ \varphi$.

(iv) The contravariant functor $\mathrm{Hom}(\cdot, A)$.

[1] for true category theory connoisseurs

Definition A.3.4 (Natural transformation). *Let $\mathscr{C}, \mathscr{D}$ be categories and $\mathscr{F}, \mathscr{G} : \mathscr{C} \to \mathscr{D}$ be (covariant) functors. A natural transformation from $\mathscr{F}$ to $\mathscr{G}$ is a map η that assigns to every $A \in \mathrm{Ob}(\mathscr{C})$ a morphism $\eta_A \in \mathrm{Hom}_{\mathscr{D}}(\mathscr{F}A, \mathscr{G}A)$ such that for all $A, B \in \mathrm{Ob}(\mathscr{C})$ and $f \in \mathrm{Hom}(A, B)$, the diagram*

$$\begin{array}{ccc} \mathscr{F}A & \xrightarrow{\eta_A} & \mathscr{G}A \\ {\scriptstyle \mathscr{F}f}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{G}f} \\ \mathscr{F}B & \xrightarrow[\eta_B]{} & \mathscr{G}B \end{array}$$

commutes

A simple example is between the categories of commutative rings and groups. For $n \in \mathbb{N}$, look at the functor $\mathscr{F}_n : R \to \mathrm{Gl}_n(R)$, where for $f : R \to S$, the map $\mathscr{F}(f) : \mathrm{Gl}_n(R) \to \mathrm{Gl}_n(S)$ is defined pointwise. Then the determinant is a natural transformation from $\mathscr{F}_n$ to $\mathscr{F}_1$.

Definition A.3.5 (Equivalence of categories). *Two categories $\mathscr{C}, \mathscr{D}$ are called equivalent if there are functors $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ and $\mathscr{G} : \mathscr{D} \to \mathscr{C}$ and natural transformations η from $\mathscr{F}\mathscr{G}$ to $\mathscr{I}_{\mathscr{D}}$ (identity functor) and ρ from $\mathscr{G}\mathscr{F}$ to $\mathscr{I}_{\mathscr{C}}$, so that every $\eta_D \in \mathrm{Hom}_{\mathscr{D}}(\mathscr{F}\mathscr{G}D, D), \rho_C \in \mathrm{Hom}_{\mathscr{C}}(\mathscr{G}\mathscr{F}C, C)$ is an isomorphism.*

A.4 Algebraic geometry

Before reviewing this, one should be familiar with sheaves, which are discussed in chapter 1.

For the following discussion, R will be a commutative ring with unity. We introduce the set called the spectrum of R, denoted $\mathrm{Spec}\, R$. It is simply the set of prime ideals of R. We can endow this set with the Zariski topology; we write this out explicitly. For $\mathfrak{a} \subset R$ an ideal, we define the zero set

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}\, R : \mathfrak{p} \supset \mathfrak{a}\}.$$

Then the sets $V(\mathfrak{a})$ satisfy the axioms of closed sets in a topology (this is not too hard to check), so $\mathrm{Spec}\, R$ becomes a topological space. If

$f \in R$ and $\mathfrak{p} \in X = \operatorname{Spec} R$, then we define the sheaf of rings

$$f(\mathfrak{p}) = f \bmod \mathfrak{p} \in R/\mathfrak{p} \subset \operatorname{Quot}(R/\mathfrak{p}).$$

For each $f \in R$, we define

$$D_f = \{\mathfrak{p} \in X : f \notin \mathfrak{p}\} = X \setminus V(\langle f \rangle),$$

which is open in X by definition of the Zariski topology. The D_f sets constitute a basis of X.

We can also define the following presheaf on X:

$$\mathcal{O}(D_f) = R_f = \left\{ \frac{a}{f^k} : a \in R, k \geqslant 0 \right\} \subset \operatorname{Quot}(R),$$

i.e. the localization of R at f (we are assuming R is an integral domain, so there are no nilpotent elements). The $\subset$ above just means as a ring.

If $D_f \subset D_g$ then there is an induced ring morphism $R_g \to R_f$. Then $\mathcal{O}$ is a presheaf of rings defined on a basis of X. This can (not trivial) be extended to a sheaf over X in such a way that

(i) $\mathcal{O}_\mathfrak{p} = R_\mathfrak{p}$ (here $\mathcal{O}_\mathfrak{p}$ is the stalk at $\mathfrak{p} \in X$ and $R_\mathfrak{p}$ is the localization of R at $\mathfrak{p}$),

(ii) $\mathcal{O}(X) = R$.

In this case, $(X, \mathcal{O})$ is a (non-reduced) ringed space.

Some simple examples include the following.

(i) The case $R = \mathbb{Z}$. Then $\operatorname{Spec} \mathbb{Z} = \{(0), (2), (3), (5), \ldots\}$ and

$$\operatorname{Quot}(\mathfrak{p}) = \begin{cases} \mathbb{F}_\mathfrak{p} & \mathfrak{p} = (p),\ p \text{ prime} \\ \mathbb{Q} & \mathfrak{p} = (0). \end{cases}$$

(ii) If K is an algebraically closed field and $R = K[x_1, \ldots, x_n]$, then Hilbert's Nullstellensatz tells us that

$$\operatorname{Spec}_{\max}(K[x_1, \ldots, x_n]) \cong K^n$$

with maximal ideals $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ for points $a = (a_1, \ldots, a_n) \in K^n$. (Here $\operatorname{Spec}_{\max}$ indicates that we are just taking maximal ideals.) So if $f \in R$ then $f(\mathfrak{m}) = f(a)$.

APPENDIX

B

ANALYSIS AND GEOMETRY REVIEW

We review some basic analysis results. This will be largely unorganized—essentially a list of relevant theorems written here instead of in some textbook for convenience. See **[ahlfors1953complex, conway1973functions, rudin1974real]** for more.

B.1 Big results from single variable complex analysis

Theorem B.1.1 (Laurent series development). *Let f be analytic in the annulus $r < |z-a| < R$. Then*

$$f(z) = \sum_{\nu=-\infty}^{\infty} a_\nu (z-a)^\nu,$$

where the convergence is absolute and uniform on compacta. Also, the coefficients, a_ν, are given by the formula

$$a_\nu = \frac{1}{2\pi i} \int_{|z-a|=r_0} \frac{f(z)}{(z-a)^{\nu+1}}\, dz,$$

where $r < r_0 < R$. Moreover, this series is unique.

Remark that the Laurent series development can be extended to the several complex variables situation, although we will not that in these notes; see **[range1998holomorphic]**. Also, recall that we can classify the singularities of a function based on its Laurent series.

Proposition B.1.2. *let $z = a$ be an isolated singularity of f and let*

$$f(z) = \sum_{\nu=-\infty}^{\infty} a_n (z-a)^{\nu}$$

be its Laurent series Expansion around a. Then

(i) $z = a$ is a removable singularity if and only if $a_\nu = 0$ for $\nu \leqslant -1$,

(ii) $z = a$ is a pole of order μ if and only if $a_{-\mu} \neq 0$ and $a_\nu = 0$ for $\nu \leqslant -(\mu+1)$, and

(iii) $z = a$ is an essential singularity if and only if $a_\nu \neq 0$ for infinitely many negative integers, ν.

Now we turn to two very important (and rather deep) theorems from the single variable theory that are of interest in chapter 2. In the theorem below, we write S^2 to mean the Riemann sphere. For proofs of the following two theorems, see [**rudin1974real**].

Theorem B.1.3 (Runge's approximation theorem). *Let Ω be an open set in $\mathbb{C}$ and A a set which has one point in each connected component of $S^2 \setminus \Omega$. Assume $f \in \mathcal{O}(\Omega)$. Then there is a sequence (R_n) of rational functions with poles only in A such that $R_n \to f$ uniformly on compacta of Ω.*

In the special case where $S^2 \setminus \Omega$ is connected, we may take $A = \{\infty\}$ and thus obtain a polynomial approximation.

Remark B.1.1. It may even be the case that $S^2 \setminus \Omega$ has uncountably many components; for example, we may have $S^2 \setminus \Omega = \{\infty\} \cup C$, where C is a Cantor set. It is with this (and the many amazing applications of this theorem), that we see how powerful it is.

Remark B.1.2. There is a great need in complex analysis (and analysis in general) for approximation theorems. Runge's theorem is just the beginning. For example, Mergelyan's theorem is a generalization of Runge's and Arakelyan's theorem is a generalization of Mergelyan's. The list goes on.

With this result, one can prove that meromorphic functions can be constructed with arbitrarily preassigned poles. Precisely, we have the following theorem.

Theorem B.1.4 (Mittag-Leffler theorem). *Suppose $\Omega \subset \mathbb{C}$ is open and $A \subset \Omega$. Suppose further that A has no cluster points in Ω. To each $\alpha \in A$ we associate a positive integer $m(\alpha)$ and a rational function,*

$$P_\alpha(z) = \sum_{j=1}^{m(\alpha)} \frac{c_{j,\alpha}}{(z-\alpha)^j}.$$

Then there exists a meromorphic function f in Ω whose principal part at each $\alpha \in A$ is P_α and which has no other poles in Ω.

This is, as one may note, a sister theorem to the classical Weierstrass factorization theorem.

B.2 Differential geometry

review of wedge product, maybe Poincaré lemma or at least a bit about the differential, etc.