Tensor product versus Hom

The overall strategy for the cohomological Künneth theorem is the same as for the homological version:

- for a pair of simplicial sets X and Y, the Eilenberg-Zilber theorem provides a chain homotopy equivalence between $C_*(X;\mathbb{Z}) \otimes C_*(Y;\mathbb{Z})$ and $C_*(X \times Y;\mathbb{Z})$
- applying the functor $\mathrm{Hom}(-, R)$ for a commutative ring R provides a cochain homotopy equivalence between $\mathrm{Hom}(C_*(X;\mathbb{Z}) \otimes C_*(Y;\mathbb{Z}), R)$ and $\mathrm{Hom}(C_*(X \times Y;\mathbb{Z}), R) = C^*(X \times Y; R)$
- in favorable cases, one can relate the cochain complex $\mathrm{Hom}(C_*(X;\mathbb{Z}) \otimes C_*(Y;\mathbb{Z}), R)$ to the cochain complex $\mathrm{Hom}(C_*(X;\mathbb{Z}), R) \otimes_R \mathrm{Hom}(C_*(Y;\mathbb{Z}), R) = C^*(X; R) \otimes_R C^*(Y; R)$
- the algebraic Künneth theorem expresses the cohomology of $C^*(X; R) \otimes_R C^*(Y; R)$ in terms of $H^*(C^*(X; R)) = H^*(X; R)$ and $H^*(C^*(Y; R)) = H^*(Y; R)$

The extra complication in the cohomological argument, and the subtle point in the above strategy, is to understand the relationship between the cochain complexes

$$\mathrm{Hom}(C_*(X;\mathbb{Z}) \otimes C_*(Y;\mathbb{Z}), R) \qquad \text{and} \qquad \mathrm{Hom}(C_*(X;\mathbb{Z}), R) \otimes_R \mathrm{Hom}(C_*(Y;\mathbb{Z}), R) \ .$$

These complexes are *not* in general quasi-isomorphic; but they are cochain homotopy equivalence if we add suitable finiteness hypotheses on the homology of one of the two simplicial sets, as we shall now discuss.

To clarify what is really going on, we will replace $C_*(X;\mathbb{Z})$ and $C_*(Y;\mathbb{Z})$ by arbitrary complexes C and D of abelian groups, we define a natural chain map

$$\bullet \ : \ \mathrm{Hom}(C, R) \otimes_R \mathrm{Hom}(D, R) \ \longrightarrow \ \mathrm{Hom}(C \otimes D, R) \ ,$$

and we derive sufficient conditions for this map to be a chain homotopy equivalence.

Construction 1. We let A and B be abelian groups, and we let R be a commutative ring. The set $\mathrm{Hom}(A, R)$ of additive maps $A \longrightarrow R$ becomes an R-module under pointwise addition and scalar multiplication, i.e., by

$$(f + g)(a) \ = \ f(a) + g(a) \qquad \text{and} \qquad (r \cdot f)(a) \ = \ r \cdot f(a)$$

for $f, g, \in \mathrm{Hom}(A, R)$, $a \in A$ and $r \in R$. We define a map

$$\bullet \ : \ \mathrm{Hom}(A, R) \times \mathrm{Hom}(B, R) \ \longrightarrow \ \mathrm{Hom}(A \otimes B, R) \quad \text{by} \quad (f \bullet g)(a \otimes b) \ = \ f(a) \cdot g(b) \ .$$

This assignment is additive in f and g, and it satisfies

$$(r \cdot f) \bullet g \ = \ r \cdot (f \bullet g) \ = \ f \bullet (r \cdot g)$$

for all $r \in R$. So the map extends to a homomorphism of R-modules

$$\bullet \ : \ \mathrm{Hom}(A, R) \otimes_R \mathrm{Hom}(B, R) \ \longrightarrow \ \mathrm{Hom}(A \otimes B, R) \tag{2}$$

sending an elementary tensor $f \otimes g$ to $f \bullet g$.

Proposition 3. *Let A and B be abelian groups and R a commutative ring. If A is finitely generated and free, then the map*

$$\bullet \ : \ \mathrm{Hom}(A, R) \otimes_R \mathrm{Hom}(B, R) \ \longrightarrow \ \mathrm{Hom}(A \otimes B, R)$$

is an isomorphism of R-modules.

Proof. We let $\mathcal{A}$ denote the class of abelian groups A such that the map $\bullet$ is an isomorphism for all abelian groups B. In a first step we show that $\mathbb{Z} \in \mathcal{A}$. To this end we consider the diagram of R-modules

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{Z}, R) \otimes_R \mathrm{Hom}(B, R) \arrow[r, "\bullet"] \arrow[d, "\mathrm{ev}_1 \otimes_R \mathrm{Id}"'] & \mathrm{Hom}(\mathbb{Z} \otimes B, R) \arrow[d, "\mathrm{Hom}(\kappa, R)"] \\
R \otimes_R \mathrm{Hom}(B, R) \arrow[r, "r \otimes g \mapsto r \cdot g"'] & \mathrm{Hom}(B, R)
\end{tikzcd}$$

Here

$$\mathrm{ev}_1 \ : \ \mathrm{Hom}(\mathbb{Z}, R) \ \longrightarrow \ R \ , \quad \mathrm{ev}_1(f) \ = \ f(1)$$

is evaluation at the generator 1 of $\mathbb{Z}$. This map is an isomorphism of R-modules because $\mathbb{Z}$ is freely generated by 1. Also

$$\kappa \ : \ B \ \longrightarrow \ \mathbb{Z} \otimes B \ , \quad \kappa(b) \ = \ 1 \otimes b$$

is an isomorphism of abelian groups. The relation

$$((f \bullet g) \circ \kappa)(b) \ = \ (f \bullet g)(1 \otimes b) \ = \ f(1) \cdot g(b) \ = \ \mathrm{ev}_1(f) \otimes g(b)$$

shows that the square commutes. Since both vertical and the lower horizontal homomorphisms are isomorphisms, the upper horizontal morphisms is an isomorphism, too. This concludes the proof that the abelian group $\mathbb{Z}$ is contained in the class $\mathcal{A}$.

Now we show that the class $\mathcal{A}$ is closed under direct sum of abelian groups. So we suppose that A and A' belong to the class, and we let B be any abelian group. The functors

$$\mathrm{Hom}(-, R) \ , \quad - \otimes_R \mathrm{Hom}(B, R) \quad \text{and} \quad - \otimes B$$

all preserves direct sums. So the canonical horizontal maps in the commutative square

$$\begin{tikzcd}
\mathrm{Hom}(A \oplus A', R) \otimes_R \mathrm{Hom}(B, R) \arrow[r] \arrow[d, "\bullet"'] & (\mathrm{Hom}(A, R) \otimes_R \mathrm{Hom}(B, R)) \oplus (\mathrm{Hom}(A', R) \otimes_R \mathrm{Hom}(B, R)) \arrow[d, "\bullet \oplus \bullet"] \\
\mathrm{Hom}((A \oplus A') \otimes B, R) \arrow[r] & \mathrm{Hom}(A \otimes B, R) \oplus \mathrm{Hom}(A' \otimes B, R)
\end{tikzcd}$$

are isomorphisms. The right vertical map is an isomorphism because A and A' belong to the class $\mathcal{A}$. So the left vertical map is an isomorphism, and so $A \oplus A'$ belongs to the class $\mathcal{A}$.

Since the class $\mathcal{A}$ contains $\mathbb{Z}$ and is closed under direct sums, it contains the group $\mathbb{Z}^m$ for every $m \geq 0$. Since $\mathcal{A}$ is clearly closed under isomorphisms of abelian groups, it contains all finitely generated free abelian groups. □

Now we give two examples to illustrate how the conclusion of Proposition 3 can fail if we drop the finite generation or the freeness property.

Example 4. To illustrate that the finiteness hypothesis in Proposition 3 is necessary we take $R = \mathbb{F}_2$ and $A = B = \mathbb{Z}[\mathbb{N}]$, the free abelian group with basis given by the set of natural numbers. Evaluating at the basis identifies $\mathrm{Hom}(\mathbb{Z}[\mathbb{N}], \mathbb{F}_2)$ with the set $\mathrm{map}(\mathbb{N}, \mathbb{F}_2)$ of functions from $\mathbb{N}$ to $\mathbb{F}_2$, with pointwise $\mathbb{F}_2$-vector space structure. Similarly, $\mathrm{Hom}(\mathbb{Z}[\mathbb{N}] \otimes \mathbb{Z}[\mathbb{N}], \mathbb{F}_2)$ is isomorphic to $\mathrm{map}(\mathbb{N} \times \mathbb{N}, \mathbb{F}_2)$, by evaluating at elementary tensor of basis elements. Under these identifications, the map

$$\bullet \ : \ \mathrm{Hom}(\mathbb{Z}[\mathbb{N}], \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathrm{Hom}(\mathbb{Z}[\mathbb{N}], \mathbb{F}_2) \ \longrightarrow \ \mathrm{Hom}(\mathbb{Z}[\mathbb{N}] \otimes \mathbb{Z}[\mathbb{N}], \mathbb{F}_2)$$

becomes the map

$$\bullet \ : \ \mathrm{map}(\mathbb{N}, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathrm{map}(\mathbb{N}, \mathbb{F}_2) \ \longrightarrow \ \mathrm{map}(\mathbb{N} \times \mathbb{N}, \mathbb{F}_2) \ , \quad (f \bullet g)(m, n) \ = \ f(m) \cdot g(n) \ .$$

While this map is injective, it is *not* surjective. Indeed, the characteristic function of the diagonal $\chi : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{F}_2$ defined by

$$\chi(m, n) \ = \ \begin{cases} 1 & \text{if } m = n, \text{ and} \\ 0 & \text{if } m \neq n, \end{cases}$$

is not in the image.

Example 5. The next example illustrates how the conclusion of Proposition 3 can fail for finitely generated abelian groups that are not free. Here we take $A = B = \mathbb{Z}/2$, the cyclic group of order 2, and we take $R = \mathbb{Z}/4$, the ring of integers modulo 4. There is exact one non-trivial homomorphism $i : \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4$, the one defined by

$$i(m + 2\mathbb{Z}) \ = \ 2m + 4\mathbb{Z} \ .$$

So the abelian groups $\mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}/4)$, $\mathbb{Z}/2 \otimes \mathbb{Z}/2$, $\mathrm{Hom}(\mathbb{Z}/2 \otimes \mathbb{Z}/2, \mathbb{Z}/4)$ and $\mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}/4) \otimes_{\mathbb{Z}/4} \mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}/4)$ are all cyclic groups of order 2. In particular, in this example the source and target of the map (2) are isomorphism. However, the map (2) is *not* an isomorphism. Indeed, the group $\mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}/4) \otimes_{\mathbb{Z}/4} \mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}/4)$ is generated by the non-zero element $i \otimes i$. But

$$(i \bullet i)((m + 2\mathbb{Z}) \otimes (n + 2\mathbb{Z})) \ = \ i(m + 2\mathbb{Z}) \cdot i(n + 2\mathbb{Z}) \ = \ (2m + 4\mathbb{Z}) \cdot (2n + 4\mathbb{Z}) \ = \ 4mn + 4\mathbb{Z} \ = \ 0 \ .$$

Hence $i \bullet i = 0$ in the group $\mathrm{Hom}(\mathbb{Z}/2 \otimes \mathbb{Z}/2, \mathbb{Z}/4)$.

Construction 6. We let C and D be chain complexes of abelian groups, and we let R be a commutative ring. As before we write $\mathrm{Hom}(C, R)$ for the cochain complex of R-modules with

$$\mathrm{Hom}(C, R)^n \ = \ \mathrm{Hom}(C_n, R) \qquad \text{and} \qquad d^n \ = \ \mathrm{Hom}(d_{n+1}, R) \ .$$

The tensor product of *co*chain complexes of R-modules is defined in much the same way a for chain complexes, just that differential now increase the dimension.

We define a natural cochain map

$$\bullet \ : \ \mathrm{Hom}(C, R) \otimes_R \mathrm{Hom}(D, R) \ \longrightarrow \ \mathrm{Hom}(C \otimes D, R) \tag{7}$$

by summing over the homomorphisms (2). More precisely, in cochain dimension n, we define $\bullet$ as the map

$$\bigoplus_{p+q=n} \mathrm{Hom}(C_p, R) \otimes_R \mathrm{Hom}(D_q, R) \ \longrightarrow \ \mathrm{Hom}\left(\bigoplus_{p+q=n} C_p \otimes D_q, R \right)$$

by sending $f \otimes g$ for $f \in \mathrm{Hom}(C_k, R)$ and $g \in \mathrm{Hom}(D_{n-k}, R)$ to the composite

$$\bigoplus_{p+q=n} C_p \otimes D_q \ \xrightarrow{\text{project}} \ C_k \otimes D_{n-k} \ \xrightarrow{f \bullet g} \ R \ .$$

We omit the verification that this map is a cochain map.

Proposition 8. *Let C and D be chain complexes of abelian groups such that $C_n = 0$ and $D_n = 0$ for all $n < 0$. Suppose moreover that C_n is finitely generated and free for every $n \geq 0$. Then the cochain map* (7)

$$\bullet \ : \ \mathrm{Hom}(C, R) \otimes_R \mathrm{Hom}(D, R) \ \longrightarrow \ \mathrm{Hom}(C \otimes D, R)$$

is an isomorphism of cochain complexes of R-modules.

Proof. We unravel the definitions and apply Proposition 3. In more detail: because the complexes C and D vanish in negative dimensions, source and target vanish in cochain dimension n for $n < 0$. For $n \geq 0$, the group

$$\mathrm{Hom}^n(C \otimes D, R) \ = \ \mathrm{Hom}(\bigoplus_{p=0}^{n} C_p \otimes D_{n-p}, R)$$

is isomorphic to

$$\bigoplus_{p=0}^{n} \mathrm{Hom}(C_p \otimes D_{n-p}, R)$$

because the direct sum is over finitely many summands. Under this identification, the degree n component of the cochain map $\bullet$ becomes the map

$$\bigoplus_{p=0}^{n} \mathrm{Hom}(C_p, R) \otimes_R \mathrm{Hom}(D_{n-p}, R) \;\longrightarrow\; \bigoplus_{p=0}^{n} \mathrm{Hom}(C_p \otimes D_{n-p}, R)$$

that is the direct sum, over $0 \leq q \leq n$, of instances of the map $\bullet$ map for the abelian groups C_p and D_{n-p}. Since C_p is finitely generated and free, each of these maps is an isomorphism by Proposition 3. So the cochain map $\bullet$ is an isomorphism. □

Since we are ultimately interested in the cohomology groups of topological spaces, we want to eventually apply our discussion to the chain complex $C_*(\mathcal{S}(X);\mathbb{Z})$, the integral chain complex of the singular complex of a space X. While the n-th chain group $C_n(\mathcal{S}(X);\mathbb{Z}) = \mathbb{Z}[S(X)_n]$ of this complex is free by design, it is almost never finitely generated (only when X has finitely many points). So it might seem that we have a problem. The next proposition provides a remedy: it allows us pass from complexes that are dimensionwise finitely generated to the much more general class of complexes whose homology is finitely generated.

Proposition 9. *Let C be a chain complex of free abelian groups such that $C_n = 0$ for $n < 0$. Suppose moreover that that group $H_n(C)$ is finitely generated for all $n \geq 0$. Then there is a subcomplex B of C such that*

- *the group B_n is finitely generated and free for every $n \geq 0$, and*
- *the inclusion $B \longrightarrow C$ is a chain homotopy equivalence.*

Proof. We choose finitely generated subgroups B_n of C_n by induction on $n \geq 0$, such that

- the differential $d : C_n \longrightarrow C_{n-1}$ takes B_n to B_{n-1}, and
- the inclusion of the subcomplex

$$\ldots 0 \longrightarrow B_n \xrightarrow{d} B_{n-1} \xrightarrow{d} \ldots \xrightarrow{d} B_0 \longrightarrow 0 \longrightarrow \ldots$$

into C induces an isomorphism on H_k for all $0 \leq k \leq n-1$, and an epimorphism on H_n.

To start the induction we choose finitely many elements $x_1, \ldots, x_m$ of C_0 whose homology classes generate the group $H_0(C) = \mathrm{coker}(d : C_1 \longrightarrow C_0)$. This is possible because the homology group $H_0(C)$ is finitely generated.

Now we suppose that $B_0, \ldots, B_{n-1}$ have already been chosen subject to the conditions above. We choose finitely many elements $x_1, \ldots, x_m$ of C_n that are cycles and whose homology classes generate the group $H_n(C)$ This is possible because the homology group $H_n(C)$ is finitely generated. We set

$$Z \;=\; \ker(d : B_{n-1} \longrightarrow B_{n-2}) \cap \mathrm{image}(d : C_n \longrightarrow C_{n-1}) \;.$$

Since B_{n-1} is finitely generated, so is its subgroup Z. We choose finitely elements $z_1, \ldots, z_k$ that generate Z, and then we choose preimage $y_1, \ldots, y_k$ in C_n such that $z_i = d(y_i)$ for all $1 \leq i \leq k$. We let B_n be the subgroup of C_n generated by the elements $x_1, \ldots, x_m$ and the elements $y_1, \ldots y_k$. Since the elements x_i are cycles and the elements y_i are taken by the differential into B_{n-1}, this group satisfies $d(B_n) \subset B_{n-1}$.

We write $B_{<n}$ for the subcomplex of C spanned by $B_0, \ldots, B_{n-1}$. Similarly, we write $B_{\leq n}$ for the subcomplex of C spanned by $B_0, \ldots, B_{n-1}, B_n$. The inclusions

$$B_{<n} \;\longrightarrow\; B_{\leq n} \;\longrightarrow\; C$$

are chain maps. By the inductive hypothesis, the inclusion $B_{<n} \longrightarrow C$ induces isomorphisms on H_k for $0 \leq k \leq n-2$ and a surjection on H_{n-1}. Since $B_{<n}$ and $B_{\leq n}$ agree up to dimension $n-1$, they have the same homology up to dimension $n-2$. So the inclusion $B_{\leq n} \longrightarrow C$ also induces isomorphisms on H_k for $0 \leq k \leq n-2$.

Since $H_{n-1}(B_{<n}) \longrightarrow H_{n-1}(C)$ is surjective, the map $H_{n-1}(B_{\leq n}) \longrightarrow H_{n-1}(C)$ is also surjective. Now we let $x \in B_{n-1}$ by a cycle whose homology class lies in the kernel of $H_{n-1}(B_{\leq n}) \longrightarrow H_{n-1}(C)$. Then x

belongs to the group Z defined in the inductive construction of B_n. Hence x is a linear combination of the classes $z_1, \ldots, z_k$, and hence a boundary of a linear combination of the classes $y_1, \ldots, y_k$. So x is in the image of $d : B_n \longrightarrow B_{n-1}$, and so the homology class $[x] \in H_{n-1}(B_{\leq n})$ is trivial. This shows that the map $H_{n-1}(B_{\leq n}) \longrightarrow H_{n-1}(C)$ is also injective, and hence bijective.

Finally, since B_n contains the cycles $x_1, \ldots, x_m$ whose classes generate $H_n(C)$, the map $H_n(B_{\leq n}) \longrightarrow H_n(C)$ is surjective. This completes the induction construction of the finitely generated groups B_n, and thus the construction of the subcomplex B of C.

By construction, the inclusion $B \longrightarrow C$ induces isomorphisms of all homology groups, i.e., it is a quasi-isomorphism. Since the groups C_n are free, their subgroups B_n are also free. So B_n is a complex of free abelian groups. I showed in an earlier video that every quasi-isomorphism between chain complexes of free abelian groups is a chain homotopy equivalence. This applied to the inclusion $B \longrightarrow C$, so we have proved the proposition. □

Corollary 10. *Let R be a commutative ring or global dimension at most 1. Let C and D be chain complexes of abelian groups such that $C_n = 0$ and $D_n = 0$ for all $n < 0$. Suppose moreover that for every $n \geq 0$, the group C_n is free and the group $H_n(C)$ is finitely generated. Then for every $n \geq 0$, the map*

$$\bigoplus_{p+q=n} H^p(\mathrm{Hom}(C,R)) \otimes_R H^q(\mathrm{Hom}(D,R)) \ \longrightarrow\ H^{p+q}(\mathrm{Hom}(C \otimes D, R))$$

sending $[f] \otimes [g]$ to $[f \bullet g]$ for cocycles $f : C_p \longrightarrow R$ and $g : D_q \longrightarrow R$ is injective, and its cokernel is naturally R-linearly isomorphic to the R-module

$$\bigoplus_{p+q=n+1} \mathrm{Tor}^R(H^p(\mathrm{Hom}(C,R)), H^q(\mathrm{Hom}(D,R))) \ .$$

Proof. In a first step we consider the special case where for all $n \geq 0$, the group C_n is not just free, but also finitely generated. Then the cochain map (7)

$$\bullet \ :\ \mathrm{Hom}(C,R) \otimes_R \mathrm{Hom}(D,R) \ \longrightarrow\ \mathrm{Hom}(C \otimes D, R)$$

is an isomorphism, by Proposition 8. So we may equivalently prove the statement for the map

$$\bigoplus_{p+q=n} H^p(\mathrm{Hom}(C,R)) \otimes_R H^q(\mathrm{Hom}(D,R)) \ \longrightarrow\ H^{p+q}(\mathrm{Hom}(C,R) \otimes_R \mathrm{Hom}(D,R))$$

sending $[f] \otimes [g]$ to $[f \otimes g]$. We observe that the cochain complex $\mathrm{Hom}(C,R)$ consists of free R-modules. Indeed, since C_n is isomorphic to $\mathbb{Z}^k$ for some $k \geq 0$, the R-module $\mathrm{Hom}(C_n, R)$ is isomorphic to

$$\mathrm{Hom}(\mathbb{Z}^k, R) \ \cong\ \mathrm{Hom}(\mathbb{Z}, R)^k \ \cong\ R^k \ .$$

So if R has global dimension at most 1, we can apply the algebraic Künneth theorem to the cochain complexes $\mathrm{Hom}(C,R)$ and $\mathrm{Hom}(D,R)$. To be completely honest, there are two small caveats, namely:

- We must apply the algebraic Künneth theorem to the 'dual' chain complexes obtained by inverting the gradings, i.e., the chain complex that has the group $\mathrm{Hom}(C_n, R)$ in degree $-n$, and similarly for $\mathrm{Hom}(D,R)$. This 'inversion' of the degrees has the effect that the Tor terms in the cohomological version are summed up over tuples with $p + q = n + 1$ (as opposed to $p + q = n - 1$).
- We must use a slight generalization of the algebraic Künneth theorem that we proved earlier. Indeed, I had assumed earlier that both chain complexes of R-modules are dimensionwise projective. However, the part of the proof of the algebraic Künneth theorem that establishes the natural short exact sequence and identifies the cokernel with the sum of Tor terms only needs one of the two complexes to be dimensionwise projective. (The proof of the splitting of the short exact sequence used that both complexes consist of projective modules, so we cannot conclude a splitting in our situation).

Now we treat the general case. We apply Proposition 9 to obtain a subcomplex B of C that is dimensionwise finitely generated free, and such that the inclusion $i : B \longrightarrow C$ is a chain homotopy equivalence. The functor $-\otimes D$ preserves chain homotopies and chain homotopy equivalences; and the functor $\mathrm{Hom}(-,R)$ takes chain homotopies and chain homotopy equivalences to cochain homotopies and cochain homotopy equivalences, respectively. So the two cochain maps

$$\begin{array}{rcl} \mathrm{Hom}(i,R) & : & \mathrm{Hom}(C,R) \longrightarrow \mathrm{Hom}(B,R) \qquad \text{and} \\ \mathrm{Hom}(i\otimes D,R) & : & \mathrm{Hom}(C\otimes D,R) \longrightarrow \mathrm{Hom}(B\otimes D,R) \end{array}$$

are R-linear cochain homotopy equivalences. In the commutative square

$$\begin{tikzcd} \bigoplus_{p+q} H^p(\mathrm{Hom}(C,R)) \otimes_R H^q(\mathrm{Hom}(D,R)) \arrow[r, "{[f]\otimes[g]\mapsto[f\bullet g]}"] \arrow[d, "\bigoplus H^p(\mathrm{Hom}(i,R))\otimes_R H^q(\mathrm{Hom}(D,R))"', "\cong"] & H^n(\mathrm{Hom}(C\otimes D,R)) \arrow[d, "H^n(\mathrm{Hom}(i\otimes D,R))", "\cong"'] \\ \bigoplus_{p+q=n} H^p(\mathrm{Hom}(B,R)) \otimes_R H^q(\mathrm{Hom}(D,R)) \arrow[r, "{[f]\otimes[g]\mapsto[f\bullet g]}"'] & H^n(\mathrm{Hom}(B\otimes D,R)) \end{tikzcd}$$

the vertical maps are thus isomorphism. The lower horizontal morphism is injective by the special case treated above; so the upper horizontal morphism is also injective. Also by the special case, the cokernel of the lower horizontal morphism is isomorphic to the direct sum, over $p+q=n+1$, of the R-modules

$$\mathrm{Tor}^R(H^p(\mathrm{Hom}(B,R)), H^q(\mathrm{Hom}(D,R)))\ .$$

Since $H^p(\mathrm{Hom}(i,R)) : H^p(\mathrm{Hom}(C,R)) \longrightarrow H^p(\mathrm{Hom}(B,R))$ is an isomorphism, and since the cokernels of the two horizontal maps are isomorphic, this proves the claim about the cokernel. $\square$