The duality map

Now we have all ingredients at hand to define the *duality map*

$$D_M \ : \ H^i_{\mathrm{comp}}(M;\mathbb{Z}) \ \longrightarrow \ H_{n-i}(M;\mathbb{Z})$$

for an oriented n-manifold M. If M happens to be compact, then compactly supported cohomology specializes to singular cohomology, i.e., $H^i_{\mathrm{comp}}(M;\mathbb{Z}) = H^i(M;\mathbb{Z})$. In this case, the duality map becomes cap product with the fundamental class:

$$D_M \ = \ [M] \cap - \ : \ H^i(M;\mathbb{Z}) \ \longrightarrow \ H_{n-i}(M;\mathbb{Z}) \ .$$

Construction 1. We let (M,μ) be an oriented n-manifold, i.e., $\mu = \{\mu_x\}_{x\in M}$ is a continuous choice of local orientations $\mu_x \in H_n(M|x;\mathbb{Z})$ for all points in M. In an earlier lecture we had constructed the orientation classes

$$\mu_K \ \in \ H_n(M, M\setminus K;\mathbb{Z}) \ = \ H_n(M|K;\mathbb{Z})$$

for compact subsets K of M. The class μ_K is uniquely characterized by the property that

$$r^K_x(\mu_K) \ = \ \mu_x \ \in \ H_n(M|x;\mathbb{Z})$$

for all points $x \in K$. If M itself is compact, then $[M] = \mu_M$ is the *fundamental class* in $H_n(M;\mathbb{Z})$.

Cap product with the class μ_K is a homomorphism

$$\mu_K \cap - \ : \ H^i(M, M\setminus K;\mathbb{Z}) \ \longrightarrow \ H_{n-i}(M;\mathbb{Z}) \ .$$

We claim that these homomorphisms satisfy the compatibilities necessary to assemble into a homomorphism from $H^i_{\mathrm{comp}}(M;\mathbb{Z})$. Indeed, if K is contained in a larger compact subset L of M, then the orientation class of L restricts to the orientation class of K, i.e.,

$$i_*(\mu_L) \ = \ \mu_K \ ,$$

where $i : X\setminus L \longrightarrow X\setminus K$ denotes the inclusion. Naturality of the cap product for the map of space pairs $(\mathrm{Id}, i) : (M, M\setminus L) \longrightarrow (M, M\setminus K)$ thus specializes to the relation

$$\mu_K \cap \alpha \ = \ i_*(\mu_L)\cap\alpha \ = \ \mu_L \cap i^*(\alpha)$$

for all classes $\alpha \in H^i(M, M\setminus K;\mathbb{Z})$. This precisely means that the following diagram commutes:

$$\begin{tikzcd}
H^i(M, M\setminus K;\mathbb{Z}) \arrow[d, "i^*=j^L_K"'] \arrow[drr, bend left, "\mu_K\cap-"] & & \\
H^i(M, M\setminus L;\mathbb{Z}) \arrow[rr, "\mu_L\cap-"'] & & H_{n-i}(M;\mathbb{Z})
\end{tikzcd}$$

The universal property of the group $H^i_{\mathrm{comp}}(M;\mathbb{Z})$ as the colimit of the groups $H^i(M, M\setminus K;\mathbb{Z})$ thus provides a unique homomorphism

$$D_M \ : \ H^i_{\mathrm{comp}}(M;\mathbb{Z}) \ \longrightarrow \ H_{n-i}(M;\mathbb{Z})$$

characterized by the property that for all compact subset K of M, the composite

$$H^i(M, M\setminus K;\mathbb{Z}) \ \xrightarrow{\lambda_K} \ H^i_{\mathrm{comp}}(M;\mathbb{Z}) \ \xrightarrow{D_M} \ H_{n-i}(M;\mathbb{Z})$$

is cap product with the orientation class μ_K. If M itself happens to be compact, then $\lambda_M : H^i(M;\mathbb{Z}) \longrightarrow H^i_{\mathrm{comp}}(M;\mathbb{Z})$ is the identity, and hence in this case $D_M(\alpha) = [M]\cap\alpha$.

The duality map of M depends on the chosen orientation μ of M, but we are not recording this dependence in the notation. The justification for this carelessness is that the dependence is relatively minor. For example, if $\bar{\mu} = \{-\mu_x\}_{x\in M}$ is the *opposite* orientation to μ, then $\bar{\mu}_K = -\mu_K$ for all compact K in M. Hence the duality map associated to μ is the *negative* of the duality map associated to the opposite orientation $\bar{\mu}$.

If M is not connected, there are more choices of orientations – we can reverse the orientation in only some of the path components of M. Source and target of D_M decompose as a direct sum indexed by the path components of M, and the duality map gets negated on those summands on which the orientation is reversed. In any case: if the duality map is an isomorphism for some orientation of M, then it is an isomorphism for all orientations of M.

The next proposition verifies a special case of Poincaré duality; it is in fact the base case for the upcoming boostrap argument towards the general Poincaré duality.

Proposition 2. *For any orientation of $M = \mathbb{R}^n$ and all $i \geq 0$, the duality map*

$$D_{\mathbb{R}^n} \ : \ H^i_{\mathrm{comp}}(\mathbb{R}^n;\mathbb{Z}) \ \longrightarrow \ H_{n-i}(\mathbb{R}^n;\mathbb{Z})$$

is an isomorphism.

Proof. We showed earlier that the group $H^i_{\mathrm{comp}}(\mathbb{R}^n;\mathbb{Z})$ is trivial for $i \neq n$, and free of rank 1 for $i = n$. Because $\mathbb{R}^n$ is contractible, the group $H_{n-i}(\mathbb{R}^n;\mathbb{Z})$ is also trivial for $i \neq n$, and free of rank 1 for $i = n$. Thus the duality map is automatically an isomorphism whenever $i \neq n$.

The real issue is thus to show that $D_{\mathbb{R}^n}$ is an isomorphism for $i = n$. We let

$$\Phi \ : \ H^n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\};\mathbb{Z}) \ \longrightarrow \ \mathrm{Hom}(H_n(\mathbb{R}^n|0;\mathbb{Z}),\mathbb{Z})$$

denote the evaluation homomorphism from the universal coefficient theorem; the map simply evaluates a representing cocycle on a representing cycle. The evaluation map Φ is surjective by the universal coefficient theorem. We can thus choose a cohomology class $\alpha \in H^n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\};\mathbb{Z})$ so that

$$\Psi(\alpha)(\mu_0) \ = \ 1 \ ,$$

where $\mu_0 \in H_n(\mathbb{R}^n|0;\mathbb{Z})$ is the given local orientation at the origin.

Since $\mathbb{R}^n$ is path connected, every point $x \in \mathbb{R}^n$ represents the same class $e = [x]$ in $H_0(\mathbb{R}^n;\mathbb{Z})$, and this class generates the 0-th homology group. We let $f \in C^n(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n\setminus\{0\});\mathbb{Z})$ be a relative singular n-cocycle that represents the cohomology class α. We let

$$\sum a_i\cdot\psi_i \ \in C_n(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n\setminus\{0\});\mathbb{Z})$$

represent the local orientation μ_0, where $\psi_i : \nabla^n \longrightarrow \mathbb{R}^n$ are singular simplices. Then

$$\begin{aligned}\mu_0\cap\alpha \ &= \ [\sum a_i\cdot\psi_i]\cap[f] \ = \ \sum a_i\cdot[\psi_i\cap f] \\ &= \ \sum a_i\cdot f(\psi_i)\cdot[\psi_i(0,\dots,0,1)] \ = \ \Big(\sum a_i\cdot f(\psi_i)\Big)\cdot e \ = \ \Psi(\alpha)(\mu_0)\cdot e\end{aligned}$$

in the group $H_0(\mathbb{R}^n;\mathbb{Z})$. Because $\Psi(\alpha)(\mu_0) = 1$, we conclude that the class $\mu_0\cap\alpha$ generates $H_0(\mathbb{R}^n;\mathbb{Z})$.

To conclude, we contemplate the commutative triangle:

$$\begin{tikzcd} H^n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\};\mathbb{Z}) \arrow[r, "\lambda_{\{0\}}"] \arrow[dr, bend right, "\mu_0\cap-"'] & H^n_{\mathrm{comp}}(\mathbb{R}^n;\mathbb{Z}) \arrow[d, "D_{\mathbb{R}^n}"] \\ & H_0(\mathbb{R}^n;\mathbb{Z}) \end{tikzcd}$$

The composite cap product map is surjective because $\mu_0\cap\alpha$ generates $H_0(\mathbb{R}^n;\mathbb{Z})$. So the duality morphism $D_{\mathbb{R}^n}$ is surjective, too. Since the source and target of $D_{\mathbb{R}^n}$ are two free abelian groups of rank 1, the map $D_{\mathbb{R}^n}$ must be an isomorphism. $\square$