The mod-2 fundamental class

Monday, 14 June 2021 10:59

Thm: Let M be a compact connected orientable n-manifold.
Then for all $x \in M$ and all coefficient groups A, the restriction map $r_x^M : H_n(M;A) \longrightarrow H_n(M|x;A) \cong A$ is an isomorphism.

Proof: As in the previous video, one shows:
- for all $\alpha \in H_n(M;A)$, the set $\{x \in M : r_x^M(\alpha) = 0\}$ is open and closed in M.
- for all $x \in M$ the restriction map $r_x^M : H_n(M;A) \longrightarrow H_n(M|x;A)$ is injective.

The following square commutes:

$$\begin{tikzcd}
H_n(M;\mathbb{Z}) \otimes A \arrow[r, tail] \arrow[d, "r_x^M \otimes A", "\cong"'] & H_n(M;A) \arrow[d, "r_x^M"] \\
H_n(M|x;\mathbb{Z}) \otimes A \arrow[r, tail, "\cong"'] & H_n(M|x;A)
\end{tikzcd}$$

injective by the UCT

injective!

isomorphism by previous video

isomorphism by UCT

$\Rightarrow$ all four maps are isomorphisms. □

Note: The proof showed that the map $H_n(M;\mathbb{Z}) \otimes A \longrightarrow H_n(M;A)$ from the universal coefficient theorem is an isomorphism.
Hence its cokernel $\mathrm{Tor}(H_{n-1}(M;\mathbb{Z}), A)$ is zero for all abelian groups A. In particular, for all $k \geq 1$

k-torsion in $H_{n-1}(M;\mathbb{Z}) \cong \mathrm{Tor}(H_{n-1}(M;\mathbb{Z}), \mathbb{Z}/k) = 0$. So the group $H_{n-1}(M;\mathbb{Z})$ is torsion free for every compact, connected orientable n-manifold M.

Next: "in mod-2 homology there are no orientability issues because $\mathbb{F}_2$ has only one unit"

Thm: Let M be an n-manifold and K a compact subset of M. Then there is a unique class $\nu_K \in H_n(M|K;\mathbb{F}_2)$ such that for all $x \in K$, the class $r_x^K(\nu_K)$ is non-zero, and hence a generator of $H_n(M|x;\mathbb{F}_2) \cong \mathbb{F}_2$.

Proof: analogous to the proof in the previous video for $\mathbb{Z}$-coefficients and oriented manifolds.
The only difference is in Step 1: K is contained in a local ball B;

Last video:

$$\begin{tikzcd}
& H_n(M|B;\mathbb{Z}) \arrow[dl, "r_x^B"', "\cong"] \arrow[d, "r_K^B"] \arrow[dr, "r_y^B", "\cong"'] & \\
\mu_x \in H_n(M|x;\mathbb{Z}) & H_n(M|K;\mathbb{Z}) \arrow[l, "r_x^K"] \arrow[r] & H_n(M|y;\mathbb{Z}) \ni \mu_y
\end{tikzcd}$$

Now:

$$\begin{tikzcd}
H_n(M|B;\mathbb{F}_2) \arrow[d] \arrow[dd, bend right, "\cong"'] \\
H_n(M|K;\mathbb{F}_2) \arrow[d] \\
H_n(M|x;\mathbb{F}_2)
\end{tikzcd}$$

contains a unique non-zero element

contains a unique non-zero element.

The rest of the argument is the same.

Alternative reformulation of what is going on with mod 2 coefficients:
if we were to introduce the "mod 2 homology covering" $p_{\mathbb{F}_2} : \tilde{M}_{\mathbb{F}_2} \longrightarrow M$ this would be a surjective 1-sheeted covering, hence a homeomorphism. □

If M is compact, then we can take $K = M$. Then the $\nu_M \in H_n(M;\mathbb{F}_2)$ is the mod-2 fundamental class.

If M is compact and orientable, then ν_M is the image of the fundamental class of any orientation under map of coefficient reductions

$$\begin{array}{ccc}
H_n(M;\mathbb{Z}) & \longrightarrow & H_n(M;\mathbb{F}_2) \\
\cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\
[M] & \longmapsto & \nu_M
\end{array}$$

induced by the ring homomorphism $\mathbb{Z} \longrightarrow \mathbb{F}_2$

Proof: The mod-2 reduction of $[M]$ has the properties that characterize the mod-2 fundamental class:

$$\begin{tikzcd}
{[M]} \in H_n(M;\mathbb{Z}) \arrow[r] \arrow[d, "r_x^M"] & H_n(M;\mathbb{F}_2) \arrow[d, "r_x^M"] & {[M] \bmod 2} \\
\mu_x \in H_n(M|x;\mathbb{Z}) \arrow[r] & H_n(M|x;\mathbb{F}_2) & \text{non-zero class} \\
\mathbb{Z} \arrow[u, phantom, "\cong"] \arrow[r, two heads] & \mathbb{F}_2 \arrow[u, phantom, "\cong"] &
\end{tikzcd}$$

Cor: Let M be a compact connected n-manifold. Then for any $x \in M$ the map

$$r_x^M : H_n(M;\mathbb{F}_2) \longrightarrow H_n(M|x;\mathbb{F}_2) \cong \mathbb{F}_2$$ is an isomorphism.

If $M \neq \emptyset$, then in particular $H_n(M;\mathbb{F}_2) \cong \mathbb{F}_2$.

Proof: exactly as in the previous video for $\mathbb{Z}$-coefficients and oriented M.

Remark: M connected compact non-orientable manifold. Then $H_n(M;\mathbb{Z}) = 0$ by the previous video.
The universal coefficient theorem provides an isomorphism
$M \neq \emptyset$.

$$\mathbb{F}_2 \cong H_n(M;\mathbb{F}_2) \xrightarrow{\cong} \operatorname{Tor}(H_{n-1}(M;\mathbb{Z});\mathbb{F}_2) = 2\text{-torsion in } H_{n-1}(M;\mathbb{Z}) .$$