The coboundary formula for the $\cup_1$-product

Let X be a simplicial set and R a commutative ring.
The $\cup_1$-product in the cochain complex $C^*(X;R)$ is

$$\cup_1 \ : C^n(X;R) \times C^m(X;R) \longrightarrow C^{n+m-1}(X;R)$$

$$(f \cup_1 g)(x) = \sum_{i=0}^{n-1} (-1)^{(n-i)(m+1)} f((d_i^{\text{out}})^* x) \cdot g((d_i^{\text{inn}})^* x)$$

where

$$d_i^{\text{out}} : [n] \longrightarrow [n+m-1] \quad \text{and} \quad d_i^{\text{inn}} : [m] \longrightarrow [n+m-1]$$

are the injective monotone maps with images

$$\begin{aligned} \text{image}\left(d_i^{\text{out}}\right) &= \{0,\ldots,i\} \cup \{i+m,\ldots,n+m-1\} \\ \text{image}\left(d_i^{\text{inn}}\right) &= \{i,\ldots,i+m\} \end{aligned}$$

To simplify notation, we will leave out the argument $x \in X_{n+m}$ in the formulas that follow.

Proposition (Coboundary formula)

$$\begin{aligned} d(f \cup_1 g) \;&=\; (df) \cup_1 g \;+\; (-1)^n f \cup_1 (dg) \\ &\quad -\; (-1)^{n+m} f \cup g \;-\; (-1)^{(n+1)(m+1)} (g \cup f)\,. \end{aligned}$$

Proof. We expand

$$\begin{aligned} &(df) \cup_1 g \;-\; (-1)^{n+m} \cdot f \cup g \;-\; (-1)^{(n+1)(m+1)} \cdot g \cup f \\ &\quad = \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{(n+1-i)(m+1)+j}\, f(d_j^*(d_i^{\mathrm{out}})^*) \cdot g((d_i^{\mathrm{inn}})^*) \\ &\qquad -\; (-1)^{n+m} \qquad\quad \cdot f(d_{\mathrm{front}}^*) \cdot g(d_{\mathrm{back}}^*) \\ &\qquad -\; (-1)^{(n+1)(m+1)} \cdot f(d_{\mathrm{back}}^*) \cdot g(d_{\mathrm{front}}^*) \end{aligned}$$

Two terms cancel because

$$d_0^{\mathrm{out}} d_0 = d^{\mathrm{back}}, \quad d_0^{\mathrm{inn}} = d^{\mathrm{front}}, \quad d_n^{\mathrm{out}} d_{n+1} = d^{\mathrm{front}}, \quad d_n^{\mathrm{inn}} = d^{\mathrm{back}}$$

$$(\dots) \;=\; \sum_{\mathcal{I}} (-1)^{(n+1-i)(m+1)+j}\, f(d_j^*(d_i^{\mathrm{out}})^*) \cdot g((d_i^{\mathrm{inn}})^*)$$

with

$$\mathcal{I} \;=\; \{(i,j) \;:\; 0 \le i \le n,\; 0 \le j \le n+1\} \setminus \{(0,0),(n,n+1)\}$$

$$
\begin{aligned}
(df) \cup_1 g \;-\; (-1)^{n+m} \cdot f \cup g \;-\; (-1)^{(n+1)(m+1)} \cdot g \cup f \\
= \sum_{\mathcal{I}} (-1)^{(n+1-i)(m+1)+j} f(d_j^*(d_i^{\text{out}})^*) \cdot g((d_i^{\text{inn}})^*)
\end{aligned}
$$

Relations in the category Δ:

$$
\begin{array}{llll}
d_i^{\text{out}} d_j = d_j d_{i-1}^{\text{out}} & \text{and} & d_i^{\text{inn}} = d_j d_{i-1}^{\text{inn}} & \text{for } 0 \leq j < i \leq n \\
d_i^{\text{out}} d_j = d_{i-1}^{\text{out}} & \text{and} & d_i^{\text{inn}} = d_{i-1}^{\text{inn}} d_0 & \text{for } 0 < i = j \\
d_i^{\text{out}} d_j = d_i^{\text{out}} & \text{and} & d_i^{\text{inn}} = d_i^{\text{inn}} d_{m+1} & \text{for } i+1 = j \leq n \\
d_i^{\text{out}} d_j = d_{j+m-1} d_i^{\text{out}} & \text{and} & d_i^{\text{inn}} = d_{j+m-1} d_i^{\text{inn}} & \text{for } i+1 < j \leq n+1
\end{array}
$$

We split the sum over $\mathcal{I}$ according to this four cases:

$$
\begin{aligned}
&(S1) \quad \sum\nolimits_{0 \leq j < i \leq n} (-1)^{(n+1-i)(m+1)+j} f((d_{i-1}^{\text{out}})^* d_j^*) \cdot g((d_{i-1}^{\text{inn}})^* d_j^*) \\
&(S2) + \sum\nolimits_{i=1}^{n} (-1)^{(n+1-i)(m+1)+i} f((d_{i-1}^{\text{out}})^*) \cdot g(d_0^*(d_{i-1}^{\text{inn}})^*) \\
&(S3) + \sum\nolimits_{i=0}^{n-1} (-1)^{(n+1-i)(m+1)+i+1} f((d_i^{\text{out}})^*) \cdot g(d_{m+1}^*(d_i^{\text{inn}})^*) \\
&(S4) + \sum_{i+2 \leq j \leq n+1} (-1)^{(n+1-i)(m+1)+j} f((d_i^{\text{out}})^* d_{j+m-1}^*) \cdot g((d_i^{\text{inn}})^* d_{j+m-1}^*)
\end{aligned}
$$

$$(df)\cup_1 g \;-\; (-1)^{n+m}\cdot f\cup g \;-\; (-1)^{(n+1)(m+1)}\cdot g\cup f$$
$$= (S1) \;+\; (S2) \;+\; (S3) \;+\; (S4)$$

Variable substitution in (S1): $i \rightsquigarrow i-1$
Variable substitution in (S4): $j \rightsquigarrow j+m-1$

$$(S1)+(S4) \;=$$
$$\sum\nolimits_{0\le j\le i\le n-1} (-1)^{(n-i)(m+1)+j}\, f((d_i^{\mathrm{out}})^* d_j^*)\cdot g((d_i^{\mathrm{inn}})^* d_j^*)$$
$$+ \sum_{i+m+1\le j\le n+m} (-1)^{(n-i)(m+1)+j}\, f((d_i^{\mathrm{out}})^* d_j^*)\cdot g((d_i^{\mathrm{inn}})^* d_j^*)$$

This is a sub-sum of the expanded term

$$d(f\cup_1 g) \;=\; \sum_{i=0}^{n-1}\sum_{j=0}^{n+m} (-1)^{(n-i)(m+1)+j}\, f((d_i^{\mathrm{out}})^* d_j^*)\cdot g((d_i^{\mathrm{inn}})^* d_j^*)$$

$$= \;(S1)+(S4)+\sum_{i=0}^{n-1}\sum_{j=i+1}^{i+m} (-1)^{(n-i)(m+1)+j}\, f((d_i^{\mathrm{out}})^* d_j^*)\cdot g((d_i^{\mathrm{inn}})^* d_j^*)$$

$$
\begin{aligned}
(df) \cup_1 g \;-\; (-1)^{n+m} \cdot f \cup g \;-\; (-1)^{(n+1)(m+1)} \cdot g \cup f
&= (S1) \;+\; (S2) \;+\; (S3) \;+\; (S4) \\
&= (S2) \;+\; (S3) + d(f \cup_1 g) \\
&\quad - \sum_{i=0}^{n-1} \sum_{j=i+1}^{i+m} (-1)^{(n-i)(m+1)+j} f((d_i^{\mathrm{out}})^* d_j^*) \cdot g((d_i^{\mathrm{inn}})^* d_j^*) \\
{}_{[j \rightsquigarrow j+i]} &= (S2) \;+\; (S3) + d(f \cup_1 g) \\
&\quad - \sum_{i=0}^{n-1} \sum_{j=1}^{m} (-1)^{(n-i)(m+1)+j+i} f((d_i^{\mathrm{out}})^* d_{j+i}^*) \cdot g((d_i^{\mathrm{inn}})^* d_{j+i}^*)
\end{aligned}
$$

using $d_{j+i} d_i^{\mathrm{out}} = d_i^{\mathrm{out}}$ and $d_{j+i} d_i^{\mathrm{inn}} = d_i^{\mathrm{inn}} d_j$ for $1 \leq j \leq m$:

$$
\begin{aligned}
(\dots) &= (S2) \;+\; (S3) + d(f \cup_1 g) \\
&\quad - \sum_{i=0}^{n-1} \sum_{j=1}^{m} (-1)^{(n-i)(m+1)+j+i} f((d_i^{\mathrm{out}})^*) \cdot g(d_j^* (d_i^{\mathrm{inn}})^*)
\end{aligned}
$$

Equivalently:

$$d(f \cup_1 g) - (df) \cup_1 g \; + \; (-1)^{n+m} \cdot f \cup g \; + \; (-1)^{(n+1)(m+1)} \cdot g \cup f$$

$$= \sum_{i=0}^{n-1} \sum_{j=1}^{m} (-1)^{(n-i)(m+1)+j+i} f((d_i^{\text{out}})^*) \cdot g(d_j^*(d_i^{\text{inn}})^*)$$

$$- (S2) - (S3)$$

Variable substitution in (S2): $i \rightsquigarrow i+1$

$$(\ldots) = \sum_{i=0}^{n-1} \Bigg[\sum_{j=1}^{m} (-1)^{(n-i)(m+1)+j+i} f((d_i^{\text{out}})^*) \cdot g(d_j^*(d_i^{\text{inn}})^*)$$

$$+ (-1)^{(n-i)(m+1)+i} f((d_i^{\text{out}})^*) \cdot g(d_0^*(d_i^{\text{inn}})^*)$$

$$+ (-1)^{(n+1-i)(m+1)+i} f((d_i^{\text{out}})^*) \cdot g(d_{m+1}^*(d_i^{\text{inn}})^*) \Bigg]$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{m+1} (-1)^{(n-i)(m+1)+j+i} f((d_i^{\text{out}})^*) \cdot g(d_j^*(d_i^{\text{inn}})^*)$$

$$= (-1)^n \cdot f \cup_1 (dg) \qquad \square$$