The cap product

The Poincaré duality isomorphism for a compact oriented n-manifold M is given by a specific pairing with the fundamental class $[M]$ in $H_n(M;\mathbb{Z})$, namely the *cap product*

$$[M]\cap - \ : \ H^i(M;\mathbb{Z}) \ \longrightarrow \ H_{n-i}(M;\mathbb{Z}) \ .$$

Similarly, the mod-2 Poincaré duality isomorphism for a compact n-manifold M (not necessarily orientable) is given by

$$\nu_M\cap - \ : \ H^i(M;\mathbb{F}_2) \ \longrightarrow \ H_{n-i}(M;\mathbb{F}_2) \ ,$$

the cap product with the mod-2 fundamental class in $H_n(M;\mathbb{F}_2)$. The purpose of this section is to introduce the cap product and prove some of its basic properties.

Construction 1 (Cap product)**.** Let X be a simplicial set, Y a simplicial subset of X, and R a commutative ring. The *cap product* is an R-bilinear pairing

$$\cap \ : \ H_n(X,Y;R)\times H^i(X,Y;R) \ \longrightarrow \ H_{n-i}(X;R)$$

that is defined for all $0\leq i\leq n$. An important special case is of course when X is the singular complex of a topological space, and Y is the singular complex of some subspace. One can think of the cap product as 'partial evaluation of a singular cochain on a singular chain'.

The cap product arises from the map

$$\cap \ : \ X_n\times C^i(X,Y;R) \ \longrightarrow \ C_{n-i}(X;R) \ , \quad x\cap f \ = \ f(x[0,i])\cdot x[i,n] \ ,$$

where

$$x[0,i] \ = \ d^*_{\mathrm{front}}(x) \qquad \text{and} \qquad x[i,n] \ = \ d^*_{\mathrm{back}}(x)$$

are the front and back faces of x as in the definition of the cup product, i.e., $d_{\mathrm{front}} : [i]\longrightarrow [n]$ and $d_{\mathrm{back}} : [n-i]\longrightarrow [n]$ are the morphisms in the category Δ defined by

$$d_{\mathrm{front}}(j) \ = \ j \qquad \text{and} \qquad d_{\mathrm{back}}(j) \ = \ i+j \ .$$

We extend this map R-linearly in the first variable to an R-bilinear map

$$\cap \ : \ R[X_n]\times C^i(X,Y;R) \ \longrightarrow \ C_{n-i}(X;R) \ , \quad (\sum r_j x_j)\cap f \ = \ \sum r_j f(x[0,i])\cdot x[i,n] \ .$$

Because the cochain f vanishes on all simplices in Y_i, this last pairing vanishes on $R[Y_n]\times C^i(X,Y;R)$, so it descends to a well-defined and R-bilinear map

$$\cap \ : \ C_n(X,Y;R)\times C^i(X,Y;R) \ \longrightarrow \ C_{n-i}(X;R) \ ,$$

where

$$C_n(X,Y;R) \ = \ C_n(X;R)/C_n(Y;R) \ = \ R[X_n]/R[Y_n]$$

is the R-module of relative n-chains.

Remark 2 (Beware the conventions)**.** There are two essentially different conventions for defining the cap product, depending on whether the cochains are evaluated on the front or on the back face of the singular simplices. We follow the convention that is used in Hatcher's 'Algebraic topology'. Our convention is *different* from the conventions used in tom Dieck's 'Algebraic topology' and in Appendix A of Milnor-Stasheff's 'Characteristic classes'.

Example 3. For $i=n$ the cap product is induced by the map

$$\cap \ : \ X_n\times C^n(X,Y;R) \ \longrightarrow \ C_0(X;R) \ , \quad x\cap f \ = \ f(x)\cdot x[n,n] \ ,$$

where $x[n,n]=d_0^*(\dots(d_0^*(x)))\in X_0$ is the 'last vertex' of the n-simplex x. For $i=0$ the cap product is induced by the map

$$\cap \ : \ X_n\times C^0(X,Y;R) \ \longrightarrow \ C_n(X;R) \ , \quad x\cap f \ = \ f(x[0,0])\cdot x \ ,$$

where $x[0,0] = d_1^*(\dots(d_n^*(x))) \in X_0$ is the 'first vertex' of x.

Proposition 4. *Let Y be a simplicial subset of a simplicial set X, and let R be a commutative ring.*

(i) *The cap product satisfies the boundary formula*
$$d(x \cap f) \;=\; (-1)^i \cdot ((dx) \cap f - x \cap (df))$$
for all $x \in C_n(X,Y;R)$ and $f \in C^i(X,Y;R)$.

(ii) *The cap product descends to a well-defined R-bilinear map*
$$\cap \;:\; H_n(X,Y;R) \times H^i(X,Y;R) \;\longrightarrow\; H_{n-i}(X;R)\ , \quad [x] \cap [f] \;=\; [x \cap f]\ .$$

(iii) *Suppose that $Y = \emptyset$. The relation*
$$(\xi \cap \alpha) \cap \beta \;=\; \xi \cap (\alpha \cup \beta)$$
holds for all $\xi \in H_n(X;R)$ and all homogeneous cohomology classes $\alpha, \beta \in H^*(X;R)$ with $|\alpha|+|\beta| \leq n$.

(iv) *For $Y = \emptyset$, the relation $\xi \cap 1 = \xi$ holds for all $\xi \in H_n(X;R)$.*

(v) *Let $\psi : X \longrightarrow X'$ be a morphism of simplicial sets such that $\psi(Y) \subset Y'$. Then for all classes $\xi \in H_n(X,Y;R)$ and $\alpha \in H^i(X',Y';R)$, the relation*
$$\psi_*(\xi) \cap \alpha \;=\; \psi_*(\xi \cap \psi^*(\alpha))$$
holds in the group $H_{n-i}(X',Y';R)$.

Proof. (i) For all $x \in X_n$ and $0 \leq j \leq n$, we have
$$(d_j^*(x))[0,i] = \begin{cases} d_j^*(x[0,i+1]) & \text{if } j \leq i, \\ x[0,i] & \text{if } j > i; \end{cases}$$
and
$$(d_j^*(x))[i,n-1] \;=\; \begin{cases} x[i+1,n] & \text{if } j \leq i, \\ d_{j-i}^*(x[i,n]) & \text{if } j > i. \end{cases}$$
Hence
$$\begin{aligned}
(dx) \cap f \;&=\; \sum_{j=0}^{n} (-1)^j \cdot d_j^*(x) \cap f \\
&=\; \sum_{j=0}^{n} (-1)^j \cdot f((d_j^*(x))[0,i]) \cdot (d_j^*(x))[i,n-1] \\
&=\; \sum_{j=0}^{i+1} (-1)^j \cdot f(d_j^*(x[0,i+1])) \cdot x[i+1,n] \quad + \sum_{j=i}^{n} (-1)^j \cdot f(x[0,i]) \cdot d_{j-i}^*(x[i,n]) \\
&=\; (df)(x[0,i+1])) \cdot x[i+1,n] \quad + \quad (-1)^i \cdot f(x[0,i]) \cdot \sum_{j=0}^{n-i} (-1)^j \cdot d_j^*(x[i,n]) \\
&=\; x \cap (df) \quad + \quad (-1)^i \cdot d(x \cap f)
\end{aligned}$$

In the third equation, a cancellation is happening: the term with $j = i+1$ in the first sum cancels the term with $j = i$ in the second sum.

(ii) If $x \in C_n(X,Y;R)$ is an n-cycle and $f \in C^i(X,Y;R)$ is an i-cocycle, then
$$d(x \cap f) \;=\; (-1)^i \cdot ((dx) \cap f - x \cap (df)) \;=\; 0\ ,$$

i.e., $x \cap f$ is a cycle. For every $(n+1)$-chain y we have

$$(x+dy) \cap f \;=\; x \cap f \;\pm\; d(y \cap f)\ ,$$

so the homology class of $x \cap f$ only depends on the homology class of x. Similarly, for every $(i-1)$-cochain g we have

$$x \cap (f+dg) \;=\; x \cap f \;\pm\; d(x \cap g)\ ,$$

so the homology class of $x \cap f$ only depends on the cohomology class of f.

(iii) We let $x \in C_n(X;R)$ represent the homology class ξ; and we let $a : X_i \longrightarrow R$ and $b : X_k \longrightarrow R$ represent the cohomology classes α and β, respectively. With this notation

$$\begin{aligned}
(x \cap a) \cap b =\ & (a(x[0,i]) \cdot x[i,n]) \cap b \\
=\ & a(x[0,i]) \cdot (x[i,n] \cap b) \\
=\ & a(x[0,i]) \cdot b\,((x[i,n])[0,k]) \cdot (x[i,n])[k,n-i] \\
=\ & a(x[0,i]) \cdot b(x[i,i+k]) \cdot x[i+k,n] \\
=\ & a((x[0,i+k])[0,i]) \cdot b((x[0,i+k])[i,i+k]) \cdot x[i+k,n] \\
=\ & (a \cup b)(x[0,i+k]) \cdot x[i+k,n] \\
=\ & x \cap (a \cup b)\ \ .
\end{aligned}$$

Passing to homology classes yields the desired relation.

(iv) The cohomology class $1 \in H^0(X;R)$ is represented by the constant function $\underline{1} : X_0 \longrightarrow R$ with value $1 \in R$. So if $x \in C_n(X,Y;R)$ represents the homology class ξ, then

$$1 \cap \xi \;=\; [\underline{1} \cap x] \;=\; [\underline{1}(x[n,n]) \cdot x] \;=\; [x] \;=\; \xi\ . \qquad \square$$

(v) We let $x \in X_n$ be an n-simplex of X, and we let $f \in C^i(X',Y';R)$ represent the cohomology class α. Then

$$\begin{aligned}
\psi_*(x) \cap f \;&=\; \psi_n(x) \cap f \\
&=\; f((\psi_n(x))[0,i]) \cdot (\psi_n(x))[i,n] \\
&=\; f(\psi_i(x[0,i])) \cdot \psi_{n-i}(x[i,n]) \\
&=\; \psi_*\left(f(\psi_i(x[0,i])) \cdot x[i,n]\right) \\
&=\; \psi_*(x \cap (f \circ \psi_i)) \\
&=\; \psi_*(x \cap \psi^*(f))\ .
\end{aligned}$$

By R-linearity, the same formula holds when $x \in C_n(X,Y;R)$ is an R-linear combination of n-simplices, i.e., a relative n-chain with coefficients in R. Passing to homology classes yields the desired relation.