A Mayer-Vietoris sequence for compactly supported cohomology

In this section we let X be a locally Hausdorff space, and we let U and V be two open subsets of X such that $X = U \cup V$. The aim is to construct a long exact sequence of Mayer-Vietoris type that relates the compactly supported cohomology groups of X, U, V and $U \cap V$. Manifolds are locally compact, so this discussion in particular applies to them.

We let A be an abelian group that we will use as coefficient group throughout this section. To simplify the notation we will drop A completely from the notation, even in the statements. So all singular and compactly supported cohomology groups are to be understood with coefficients in A.

We begin with a somewhat technical lemma.

Lemma 1. *Let X be a locally compact Hausdorff space, and let U and V be open subsets of X such that $X = U \cup V$. Then every compact subset of X is of the form $K \cup L$ for some compact subset K of U and some compact subset L of V.*

Proof. We let C be any compact subset of X. Because $X = U \cup V$, every point $x \in C$ is contained in U or in V. If $x \in U$, then U is in particular a neighborhood of x; because X is locally compact Hausdorff, there is a compact neighborhood N_x of x contained in U. Similarly, if $x \in V$, then there is a compact neighborhood N_x of x contained in V. Since C is compact, finitely many of the compact neighborhoods N_x suffice to cover C, say $C \subset N_{x_1} \cup \dots N_{x_m}$, and each N_{x_i} is contained in U or in V. We let $\bar{K}$ and $\bar{L}$ be the union of all those N_{x_i} that are contained in U or in V, respectively. Then $K = C \cap \bar{K}$ and $L = K \cap \bar{L}$ are compact $K \subset U$, $L \subset V$, and $C = K \cup L$. $\square$

Construction 2 (Connecting homomorphism)**.** We let X be a locally compact Hausdorff space, and we let U and V be open subsets of X such that $X = U \cup V$. We construct a homomorphism

$$\partial \ : \ H^i_{\mathrm{comp}}(X) \ \longrightarrow \ H^{i+1}_{\mathrm{comp}}(U \cap V) \tag{3}$$

that will serve as the connecting homomorphism in the upcoming Mayer-Vietoris sequence. We define compatible homomorphisms

$$\partial_C \ : \ H^i(X, X \setminus C) \ \longrightarrow \ H^{i+1}_{\mathrm{comp}}(U \cap V)$$

for all compact subsets C of X, and then appeal to the colimit universal property of compactly supported cohomology. Lemma 1 lets us choose a compact subset K of U and a compact subset L of V such that $C = K \cup L$. Then we define ∂_C as the composite

$$\begin{aligned} H^i(X, X \setminus C) \ = H^i(X, X \setminus (K \cup L)) \ &\xrightarrow{\ \partial\ } \ H^{i+1}(X, X \setminus (K \cap L)) \\ \xrightarrow[\cong]{\text{excision}} H^{i+1}(U \cap V, (U \cap V) \setminus (K \cap L)) \ &\xrightarrow{\lambda_{K \cap L}} \ H^{i+1}_{\mathrm{comp}}(U \cap V) \ . \end{aligned}$$

Here the second map is the connecting homomorphism of the Mayer-Vietoris sequence in relative singular cohomology for the open covering

$$X \setminus (K \cap L) \ = \ (X \setminus K) \cup (X \setminus L)$$

for which the intersection satisfies

$$(X \setminus K) \cap (X \setminus L) \ = \ X \setminus (K \cup L) \ .$$

We claim that

(a) the definition of ∂_C does not depend on the choices of K and L, and

(b) if $\bar{C}$ is another compact subset of X with $C \subset \bar{C}$, then the following diagram commutes:

$$\begin{tikzcd}
H^i(X, X \setminus C) \arrow[d, "j_C^{\bar{C}}"'] \arrow[dr, bend left, "\partial_C"] & \\
H^i(X, X \setminus \bar{C}) \arrow[r, "\partial_{\bar{C}}"'] & H^{i+1}_{\mathrm{comp}}(U \cap V)
\end{tikzcd}$$

To prove both claims we consider nested compact subsets $K \subset \bar{K}$ of U and nested compact subset $L \subset \bar{L}$ of V. The connecting homomorphism in the Mayer-Vietoris sequence for relative singular cohomology is natural with respect to refinements of coverings. So we obtain a commutative diagram:

$$\begin{tikzcd}
H^i(X, X \setminus (K \cup L)) \arrow[rr, "j_{K \cup L}^{\bar{K} \cup \bar{L}}"] \arrow[d, "\partial"] & & H^i(X, X \setminus (\bar{K} \cup \bar{L})) \arrow[d, "\partial"'] \\
H^{i+1}(X, X \setminus (K \cap L)) \arrow[rr, "j_{K \cap L}^{\bar{K} \cap \bar{L}}"'] \arrow[d, "\text{excision}", "\cong"'] & & H^{i+1}(X, X \setminus (\bar{K} \cap \bar{L})) \arrow[d, "\text{excision}", "\cong"'] \\
H^{i+1}(U \cap V, (U \cap V) \setminus (K \cap L)) \arrow[rr, "j_{K \cap L}^{\bar{K} \cap \bar{L}}"'] \arrow[dr, bend right, "\lambda_{K \cap L}"'] & & H^{i+1}(U \cap V, (U \cap V) \setminus (\bar{K} \cap \bar{L})) \arrow[dl, bend left, "\lambda_{\bar{K} \cap \bar{L}}"] \\
& H^{i+1}_{\mathrm{comp}}(U \cap V) &
\end{tikzcd}$$

Claim (a): We suppose that $C = K \cup L = K' \cup L'$ for compact subsets K, K' of U and compact subset L, L' of V. Then we obtain yet another admissible covering of C by compact subsets from U and from V as $C = \bar{K} \cup \bar{L}$ by setting $\bar{K} = K \cup K'$ and $\bar{L} = L \cup L'$. So by comparing each of the coverings to this common enlarging, the above commutative diagram shows the two ways to cover C lead to the same homomorphism ∂C.

Claim (b): Given nested compact subsets $C \subset \bar{C}$ of X, we write $\bar{C} = \bar{K} \cup \bar{L}$ for compact subsets $\bar{K}$ of U and $\bar{L}$ of V. We set $K = C \cap \bar{K}$ and $L = C \cap \bar{L}$; then $C = K \cup L$, and the above commutative diagram precisely shows that $\partial_{\bar{C}} \circ j_C^{\bar{C}} = \partial_C$.

Given claims (a) and (b), the universal property of the group $H^i_{\mathrm{comp}}(X)$ as the colimit of the groups $H^i(X, X \setminus C)$ thus provides a unique homomorphism (3) characterized by the property that for all compact subset C of X, the composite

$$H^i(X, X \setminus C) \xrightarrow{\lambda_C} H^i_{\mathrm{comp}}(X) \xrightarrow{\partial} H^{i+1}(U \cap V)$$

is the homomorphism ∂_C.

Proposition 4. *Let U and V be open subsets of a locally compact Hausdorff space X such that $X = U \cup V$. Then the following long sequence of compactly supported cohomology groups is exact:*

$$\ldots \longrightarrow H^i_{\mathrm{comp}}(U \cap V) \xrightarrow{(\iota^U_{U \cap V}, \iota^V_{U \cap V})} H^i_{\mathrm{comp}}(U) \oplus H^i_{\mathrm{comp}}(V) \xrightarrow{\begin{pmatrix} \iota^X_U \\ -\iota^X_V \end{pmatrix}} H^i_{\mathrm{comp}}(M) \xrightarrow{\partial} H^{i+1}_{\mathrm{comp}}(U \cap V) \longrightarrow \ldots$$

Proof. In this proof I will freely use the concept of filtered colimits of abelian groups, and the fact that such filtered colimits are exact. If you have not seen these concepts before, you can find the definition on page 241 in Section 3.3 of Hatcher's *Algebraic Topology* (but beware that Hatcher uses the terminology 'directed sets'). The exactness property is Hatcher's Exercise 17 at the end of Section 3.3. If you don't want to do the exercise yourself, you can find a proof in the *Stacks Project* at the tag `https://stacks.math.columbia.edu/tag/07N7`.

We consider the set of pairs (K, L) such that K is a compact subset of U, and L is a compact subset V. This is a filtered partially ordered set ('directed set') under inclusion, i.e., $(K, L) \leq (K', L')$ if and only if $K \subset K'$ and $L \subset L'$. For every pair (K, L) in $\mathcal{P}$, the set $X \setminus (K \cap L)$ is the union of its open subsets $X \setminus K$ and $X \setminus L$. So there is a long exact Mayer-Vietoris sequence in relative singular cohomology of the form

$$\ldots \longrightarrow H^i(X, X \setminus (K \cap L)) \xrightarrow{(j^K_{K\cap L}, j^L_{K\cap L})} H^i(X, X \setminus K) \oplus H^i(X, X \setminus L) \xrightarrow{\begin{pmatrix} j^{K\cup L}_K \\ -\iota^{K\cup L}_L \end{pmatrix}} H^i(X, X \setminus (K \cup L)) \xrightarrow{\partial} \ldots$$

The connecting homomorphism is the same as in the definition of ∂_C above. For $(K, L) \leq (K', L')$ in $\mathcal{P}$, the homomorphisms induced by the various inclusions (i.e., maps of the kind $j^{K'}_K$ etc.) form a commutative diagram from of the Mayer-Vietoris sequence for (K, L) to the one for (K', L'). Since colimits of abelian groups over filtered posets are exact, we obtain a long exact sequence in the colimit over $\mathcal{P}$. It remains to identify the terms and morphisms in the colimit exact sequence with the desired Mayer-Vietoris sequence for compactly supported cohomology.

• The colimit of the groups $H^i(X, X \setminus (K \cap L))$: Excision provides a system of restriction isomorphisms

$$H^i(X, X \setminus (K \cap L)) \xrightarrow{\cong} H^i(U \cap V, (U \cap V) \setminus (K \cap L))$$

that are compatible as (K, L) vary in the poset $\mathcal{P}$. In the colimit, this provides an isomorphism

$$\mathrm{colim}_{(K,L)\in\mathcal{P}} H^i(X, X \setminus (K \cap L)) \xrightarrow{\cong} \mathrm{colim}_{(K,L)\in\mathcal{P}} H^i(U \cap V, (U \cap V) \setminus (K \cap L)) \ .$$

As (K, L) run through $\mathcal{P}$, intersection $K \cap L$ exhausts all compact subsets of $U \cap V$; so the universal colimit property of compactly supported cohomology identifies the right hand side with $H^i_{\mathrm{comp}}(U \cap V)$:

$$\mathrm{colim}_{(K,L)\in\mathcal{P}} H^i(U \cap V, (U \cap V) \setminus (K \cap L)) \cong \mathrm{colim}_{C\subset U\cap V \text{ compact}} H^i(U \cap V, (U \cap V) \setminus C) \cong H^i_{\mathrm{comp}}(U \cap V) \ .$$

• The colimit of the groups $H^i(X, X \setminus K)$: Excision provides a system of restriction isomorphisms

$$H^i(X, X \setminus K) \xrightarrow{\cong} H^i(U, U \setminus K)$$

that are compatible as (K, L) vary in the poset $\mathcal{P}$. Since the right hand side does not depend on L, in the colimit we obtain an isomorphism

$$\begin{aligned} \mathrm{colim}_{(K,L)\in\mathcal{P}} H^i(X, X \setminus K) &\xrightarrow{\cong} \mathrm{colim}_{(K,L)\in\mathcal{P}} H^i(U, U \setminus K) \\ &\cong \mathrm{colim}_{K\subset U \text{compact}} H^i(U, U \setminus K) \cong H^i_{\mathrm{comp}}(U) \ . \end{aligned}$$

Reversing the roles of U and V, and the roles of K and L provides an analogous isomorphism

$$\mathrm{colim}_{(K,L)\in\mathcal{P}} H^i(X, X \setminus L) \cong H^i_{\mathrm{comp}}(V) \ .$$

• The colimit of the groups $H^i(X, X \setminus (K \cup L))$: By Lemma 1, the unions $K \cup L$ exhaust all compact subsets of X as (K, L) varies in the poset $\mathcal{P}$. So the universal colimit property of compactly supported cohomology identifies the colimit with the compactly supported cohomology of X:

$$\mathrm{colim}_{(K,L)\in\mathcal{P}} H^i(X, X \setminus (K \cup L)) \cong \mathrm{colim}_{C\subset X \text{ compact}} H^i(X, X \setminus C) \cong H^i_{\mathrm{comp}}(X) \ .$$

This concludes the construction of the Mayer-Vietoris sequence. To be completely honest, we also have to check that the colimits of the morphisms in the individual Mayer-Vietoris sequences become the morphisms in the statement of the proposition. Since this is a routine, albeit somewhat tedious, verification from the definitions, we omit it. □