Algebraic Topology

based on notes by Andrew Ranicki

2018–19

Contents

Introduction

The aim of this course is to develop the basic notions of algebraic topology, such as homotopy and the fundamental group. We shall relate these notions to other areas of mathematics — geometry, analysis and algebra, and see applications.

DRPS entry

This course will introduce students to essential notions in algebraic topology, such as compact surfaces, homotopies, fundamental groups and covering spaces.

Intended learning outcomes:

1. Construct homotopies and prove homotopy equivalence for simple examples.

2. Calculate fundamental groups of simple topological spaces, using generators and relations or covering spaces as necessary.

3. Calculate simple homotopy invariants, such as degrees and winding numbers.
4. State and prove standard results about homotopy, and decide whether a simple unseen statement about them is true, providing a proof or counterexample as appropriate.
5. Provide an elementary example illustrating specified behaviour in relation to a given combination of basic definitions and key theorems across the course.

Intuitively, two spaces are homotopy equivalent if they can be made homeomorphic by shrinking and expanding. For example, a solid ball is homotopy equivalent to a point; a Möbius strip is homotopy equivalent to a standard strip, since both are homotopy equivalent to a circle.

Recommended books

There are many good books on topology in the Library covering the material in this course. Amongst them are:

J.R. Munkres, Topology, a first course (Prentice-Hall) 1975, esp. Ch. 8, which deals with the fundamental group. [In the second edition, 2000, this is Ch. 9.]

C. Kosniowski, A First Course in Algebraic Topology, (CUP) 1980, esp. Chs. 12–17.

K. Jänich, Topology (Springer UTM) 1984, esp., Chs. 8 and 10. (Very good pictures but few exercises).

Beware that other books titled "Algebraic topology" will often focus on more advanced material than we have time to cover. For instance, we don't cover anything beyond chapter 1 of Hatcher's Algebraic Topology (and not all of that), which is an affordable text recommended for anyone wishing to read further material.

There are also many relevant online resources, such as Harpreet Bedi's excellent youtube lectures on Elementary Homotopy, which cover much of the course.

Lecture Notes

Lecture notes are posted on the course website, and lectures will be given on the assumption that you have tried to look at them in advance. It helps if you can come armed with questions on the notes. I might make some changes to the notes as the course progresses, but if so, this should be at least a week before the affected lecture. Also feel free to email me any questions you have on the notes (but bear in mind I might not have seen your email in time for the lecture).

0.1 Glossary

We will sometimes use different notation or terminology for concepts encountered in General Topology. They will be listed here as they come to light.

1. "map" = "continuous function"
2. $I = [0, 1]$, the closed unit interval with subspace topology in $\mathbb{R}$
3. $D^n = \{x \in \mathbb{R}^n \mid 0 \leq \|x\| \leqslant 1\}$, the closed unit n-ball.

Lecture 1

1 Recap on path-connectedness

Definition 1.1. The *path relation* on X is

$$x_0 \sim x_1 \text{ if there exists a path } \alpha : I \to X \text{ from } \alpha(0) = x_0 \in X \text{ to } \alpha(1) = x_1 \in X.$$

The path relation is an equivalence relation. To see this, we proceed as follows:

Definition 1.2. The **constant path** at $x \in X$ is the path

$$\alpha_x \ : \ I \to X \ ; \ t \mapsto x$$

from $\alpha_x(0) = x \in X$ to $\alpha_x(1) = x \in X$.

Definition 1.3. The **reverse** of a path $\alpha : I \to X$ is the path

$$-\alpha \ : \ I \to X \ ; \ t \mapsto \alpha(1-t)$$

retracing α backwards, with

$$-\alpha(0) = \alpha(1) \xrightarrow{\ -\alpha\ } -\alpha(1) = \alpha(0)$$

Definition 1.4. The **concatenation** of paths $\alpha : I \to X$, $\beta : I \to X$ with

$$\alpha(1) \ = \ \beta(0) \in X$$

is the path

$$\alpha \bullet \beta \ : \ I \to X \ ; \ t \mapsto \begin{cases} \alpha(2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ \beta(2t-1) & \text{if } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

which starts at $\alpha(0)$, follows along α at twice the speed in the first half, switching at $\alpha(1) = \beta(0)$ (at half-time) to follow β at twice the speed in the second half.

$$\alpha \bullet \beta(0) = \alpha(0) \xrightarrow{\ \alpha\ } \alpha(1) = \beta(0) \xrightarrow{\ \beta\ } \beta(1) = \alpha \bullet \beta(1)$$

Proposition 1.5. *The path relation defined on a space X by $x_0 \sim x_1$ if there exists a path $\alpha : I \to X$ from $\alpha(0) = x_0$ to $\alpha(1) = x_1$ is an equivalence relation.*

Proof. The path relation is reflexive by 1.2, symmetric by 1.3 and transitive by 1.4. $\square$

Definition 1.6. Let X be a topological space.
(i) The **path components** of X are the equivalence classes of the path equivalence relation $\sim$, i.e. the subspaces

$$\begin{aligned}[x] &= \{y \in X \mid y \sim x\} \\ &= \{y \in X \mid \text{there exists a path } \alpha : I \to X \text{ from } \alpha(0) = x \text{ to } \alpha(1) = y\}\end{aligned}$$

(ii) The **set of path components** (which may be infinite) is denoted by

$$X/\!\sim \;=\; \pi_0(X)\ .$$

The function

$$X \to \pi_0(X)\ ;\ x \mapsto [x] = \{\text{equivalence class of } x\}$$

is surjective.

Proposition 1.7. *The cardinality of set of path components $\pi_0(X)$ of a topological space X is a topological invariant, i.e. if X and Y are homeomorphic, there is a bijection between $\pi_0(X)$ and $\pi_0(Y)$.*

Remark 1.8. However, a bijection between $\pi_0(X)$ and $\pi_0(Y)$ does *not* necessarily imply that X and Y are homeomorphic.

Proof. This is "obvious" as ultimately the whole structure of the set of path components of X depends only on the open set structure of X. (Or, if X and Y are homeomorphic, so are their respective quotient spaces $\pi_0(X)$ and $\pi_0(Y)$.) But to prepare for the sorts of arguments which will be regularly used in this course, we also give a "functorial" proof.

A continuous map $f : X \to Y$ induces a function

$$f_* \;:\; \pi_0(X) \to \pi_0(Y)\ ;\ [x] \mapsto [f(x)]\ .$$

The composition of maps $f : X \to Y$, $g : Y \to Z$ is a map $g \circ f : X \to Z$ which induces the composition

$$(g \circ f)_* \;=\; g_* \circ f_* \;:\; \pi_0(X) \to \pi_0(Z)\ .$$

The identity map $1 : X \to X$ induces the identity function

$$1_* \;=\; 1 \;:\; \pi_0(X) \to \pi_0(X)\ .$$

A homeomorphism $f : X \to Y$ induces a bijection $f_* : \pi_0(X) \to \pi_0(Y)$ with inverse

$$(f_*)^{-1} \;=\; (f^{-1})_* \;:\; \pi_0(Y) \to \pi_0(X)\ .$$

$\square$

2 Homotopy theory

Definition 2.1. A **homotopy** between maps $f, g : X \to Y$ is a map $h : X \times I \to Y$ such that

$$h(x, 0) = f(x) \ , \ h(x, 1) = g(x) \in Y \quad (x \in X) \ ,$$

The maps f, g are **homotopic**, and the homotopy is denoted by

$$h \ : \ f \simeq g \ : \ X \to Y \ .$$

Example 2.2. If $X = \{x\}$ is a space with one element x, a map $f : X \to Y$ is the same as an element $f(x) \in Y$. A homotopy $h : f \simeq g : X \to Y$ is the same as a path $h : I \to Y$ with initial point $h(0) = f(x) \in Y$ and terminal point $h(1) = g(x) \in Y$. A homotopy $h : f \simeq f : X \to Y$ is the same as a closed path $h : I \to Y$.

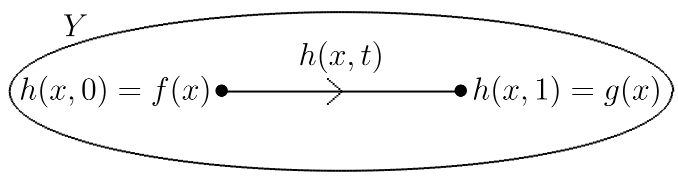
A homotopy $h : f \simeq g : X \to Y$ deforms the map f continuously to g in the space Y^X of maps from X to Y, with the 'compact-open' topology on Y^X (defined in the non-examinable appendix to this lecture). For each $t \in I$ there is defined a map

$$h_t \ : \ X \to Y \ ; \ x \mapsto h_t(x) \ = \ h(x, t)$$

with $h_0 = f$, $h_1 = g$. Equivalently, there is a continuous choice of path for each $x \in X$

$$h(x, -) \ : \ I \to Y \ ; \ t \mapsto h(x, -)(t) \ = \ h(x, t)$$

which starts at $h(x, 0) = f(x)$ and ends at $h(x, 1) = g(x)$.



Proposition 2.3. *For fixed X, Y the notion of homotopy is an equivalence relation on the set of maps $f : X \to Y$.*

Proof. (i) For every map $f : X \to Y$ define the **constant homotopy** $h : f \simeq f : X \to Y$ by

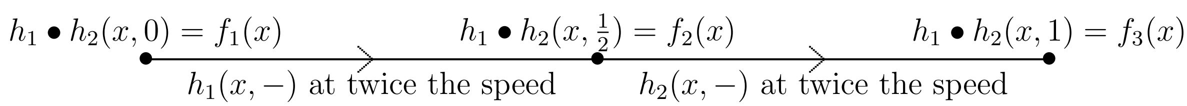
$$h \ : \ X \times I \to Y \ ; \ (x, t) \mapsto f(x)$$

(ii) Given a homotopy $h : f \simeq g : X \to Y$ define the **reverse homotopy** $-h : g \simeq f : X \to Y$ by

$$-h \ : \ X \times I \to Y \ ; \ (x, t) \mapsto h(x, 1 - t) \ .$$

(iii) Given homotopies $h_1 : f_1 \simeq f_2 : X \to Y$ and $h_2 : f_2 \simeq f_3 : X \to Y$ define the **concatenation homotopy** $h_1 \bullet h_2 : f_1 \simeq f_3 : X \to Y$ by

$$h_1 \bullet h_2 \ : \ X \times I \to Y \ ; \ (x, t) \mapsto \begin{cases} h_1(x, 2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ h_2(x, 2t - 1) & \text{if } \frac{1}{2} \leqslant t \leqslant 1 \ . \end{cases}$$



$\square$

2.1 Nonexaminable appendix on function spaces

Definition 2.4. (i) For any spaces X, Y let Y^X be the set of maps $f : X \to Y$.

(ii) The **compact-open** topology on Y^X has basis the subspaces

$$\mathcal{B}(K, V) \;=\; \{f \,|\, f(K) \subseteq V\} \subseteq Y^X$$

defined for compact $K \subseteq X$ and open $V \subseteq Y$.

Remark 2.5. (pp. 286-289 of Munkres "Topology").

(i) The compact-open topology has the key property that the function defined for any spaces X, Y, Z by

$$Y^{X\times Z} \to (Y^X)^Z \;;\; (h : X \times Z \to Y) \mapsto (H : Z \to Y^X; z \mapsto (x \mapsto h(x, z)))$$

is continuous. If X is compact Hausdorff this function is a homeomorphism. The special case $Z = I$ identifies homotopies $h : f \simeq g : X \to Y$ with paths $H : I \to Y^X$ from $H(0) = f$ to $H(1) = g$.

(ii) If X is compact and (Y, d) is a metric space the compact-open topology on Y^X is the topology determined by the metric

$$d(f, g) \;=\; \sup\{d(f(x), g(x)) \,|\, x \in X\} \quad (f, g \in Y^X)\;.$$

Lecture 2

Constructing homotopies

The construction of homotopies depends on being able to construct paths, and this is particularly easy in convex subspaces of $\mathbb{R}^n$, by the construction of straight line paths. (Recall that a subspace $Y \subseteq \mathbb{R}^n$ is convex if the line segment in $\mathbb{R}^n$ joining any two points in Y is wholly in Y).

Proposition 2.6. *Any two maps $f, g : X \to Y$ to a convex subspace $Y \subseteq \mathbb{R}^n$ are homotopic.*

Proof. The map

$$h \ : \ X \times I \to Y \ ; \ (x,t) \mapsto (1-t)f(x) + tg(x)$$

defines a homotopy $h : f \simeq g : X \to Y$. □

N.B. $\mathbb{R}^n$ is a convex subspace of $\mathbb{R}^n$.

Example 2.7. (i) Any two paths $f, g : I \to \mathbb{R}^n$ are homotopic, with

$$h \ : \ I \times I \to \mathbb{R}^n \ ; \ (s,t) \mapsto (1-t)f(s) + tg(s)$$

a homotopy $h : f \simeq g$.

(ii) Given a path $f : I \to \mathbb{R}^n$ define the straight line path

$$g \ : \ I \to \mathbb{R}^n \ ; \ s \mapsto (1-s)f(0) + sf(1)$$

with $g(0) = f(0)$ and $g(1) = f(1)$. In this case the homotopy $h : f \simeq g$ of (i)

$$h \ : \ I \times I \to \mathbb{R}^n \ ; \ (s,t) \mapsto (1-t)f(s) + t((1-s)f(0) + sf(1))$$

is such that the paths

$$h_t \ : \ I \to \mathbb{R}^n \ ; \ s \mapsto h(s,t) \ \ (0 \leqslant t \leqslant 1)$$

'slide' continuously from the 'curved' path $h_0 = f : I \to \mathbb{R}^n$ to the 'straight' path $h_1 = g : I \to \mathbb{R}^n$, keeping the endpoints fixed:

$$h_t(0) \ = \ f(0) \ = \ g(0) \ , \ h_t(1) \ = \ f(1) \ = \ g(1) \in \mathbb{R}^n \ \ (0 \leqslant t \leqslant 1) \ .$$

In general, geometry is used to construct homotopies, and algebra is used to show that homotopies with certain properties cannot exist.

Definition 2.8. Two spaces X, Y are **homotopy equivalent** if there exist maps $f : X \to Y$, $g : Y \to X$ and homotopies

$$h \ : \ gf \simeq 1_X \ : \ X \to X \ , \ k \ : \ fg \simeq 1_Y \ : \ Y \to Y \ .$$

A map $f : X \to Y$ is a **homotopy equivalence** if there exist such g, h, k. The maps f, g are **inverse homotopy equivalences**.

The relation defined on the set of topological spaces by

$$X \simeq Y \text{ if } X \text{ is homotopy equivalent to } Y$$

is an equivalence relation.

Definition 2.9. A space X is **contractible** if it is homotopy equivalent to the space $\{0\}$ with one point.

A contractible space X has a point $x_0 \in X$ and maps

$$f : X \to \{0\} ; x \mapsto 0 , g : \{0\} \to X ; 0 \mapsto x_0$$

as well as homotopies

$$h : gf \simeq 1_X : X \to X , k : fg \simeq 1_{\{0\}} : \{0\} \to \{0\} .$$

The essential feature is the continuous choice of paths

$$h(x,-) : I \to X ; t \mapsto h(x,t) \ (x \in X)$$

from $h(x,0) = gf(x) = x_0$ to $h(x,1) = x$.

Proposition 2.10. *The unit line $I = [0,1]$ is contractible.*

Proof. The maps

$$f : I \to \{0\} ; t \mapsto 0 , g : \{0\} \to I ; 0 \mapsto 0$$

are related by the homotopies $h : gf \simeq 1_I$, $k : fg \simeq 1_{\{0\}}$ defined by

$$\begin{aligned} h &: I \times I \to I ; (s,t) \mapsto st , \\ k &: \{0\} \times I \to \{0\} ; (0,t) \to 0 . \end{aligned}$$

□

Remark 2.11. In Proposition 2.6 it was proved that for a convex subspace $Y \subseteq \mathbb{R}^n$ any two maps $f, g : X \to Y$ are homotopic. In fact, for any contractible space Y any two maps $f, g : X \to Y$ are homotopic.

A non-compact space can be homotopy equivalent to a compact space:

Example 2.12. The non-compact space $D^n \backslash \{0\} = \{x \in \mathbb{R}^n \,|\, 0 < \|x\| \leqslant 1\}$ is homotopy equivalent to the compact space $S^{n-1} = \{x \in \mathbb{R}^n \,|\, \|x\| = 1\}$, since the maps

$$\begin{aligned} f &: D^n \backslash \{0\} \to S^{n-1} ; x \mapsto x/\|x\| , \\ g &: S^{n-1} \to D^n \backslash \{0\} ; x \mapsto x \end{aligned}$$

are such that $fg = 1 : S^{n-1} \to S^{n-1}$, and

$$h : D^n \backslash \{0\} \times I \to D^n \backslash \{0\} ; (x,t) \mapsto tx + (1-t)x/\|x\|$$

defines a homotopy $h : gf \simeq 1 : D^n \backslash \{0\} \times I \to D^n \backslash \{0\}$.

Proposition 2.13. *Homeomorphic spaces are homotopy equivalent.*

Proof. If $f : X \to Y$ is a homeomorphism then the actual inverse $g = f^{-1} : Y \to X$ is also a homotopy inverse, with the maps

$$h \ : \ X \times I \to X \ ; \ (x, t) \mapsto x \ ,$$
$$k \ : \ Y \times I \to Y \ ; \ (y, t) \mapsto y$$

defining the constant homotopies $h : gf \simeq 1_X : X \to X$, $k : fg \simeq 1_Y : Y \to Y$. □

Homotopy equivalent spaces have the same homotopy classes of maps. More precisely, if $F : X' \to X$ and $G : Y \to Y'$ are both homotopy equivalences then the function

$$\{\text{homotopy classes of maps } f : X \to Y\} \to \{\text{homotopy classes of maps } f' : X' \to Y'\} \ ;$$
$$(f : X \to Y) \mapsto (GfF : X' \to Y')$$

is a bijection. Homotopy theory regards homotopy equivalent spaces as being isomorphic.

Proposition 2.14. *A homotopy equivalence $f : X \to Y$ induces a bijection of the path-component sets*

$$f_* \ : \ \pi_0(X) \to \pi_0(Y) \ ; \ [x] \mapsto [f(x)] \ .$$

Proof. Let $g : Y \to X$ be a homotopy inverse for f, so that there exist homotopies

$$h \ : \ gf \simeq 1_X \ : \ X \to X \ , \ k \ : \ fg \simeq 1_Y \ : \ Y \to Y \ .$$

For any $x \in X$ there exists a path in X from $gf(x)$ to x

$$h(x, -) \ : \ I \to X \ ; \ t \mapsto h(x, t) \ ,$$

so that $[x] = [gf(x)] \in \pi_0(X)$. Similarly, for any $y \in Y$ there exists a path in Y from $fg(y)$ to y

$$k(y, -) \ : \ I \to Y \ ; \ t \mapsto k(y, t) \ ,$$

so that $[y] = [fg(y)] \in \pi_0(Y)$. It follows that the function

$$g_* \ : \ \pi_0(Y) \to \pi_0(X) \ ; \ [y] \mapsto [g(y)]$$

is an inverse of f_*, so that f_* is a bijection. □

Example 2.15. It is immediate from 2.14 that if X, Y are spaces such that X is path-connected and Y is not path-connected then X is not homotopy equivalent to Y. Simplest example: $X = \{0\}$ and $Y = \{0, 1\}$ with the discrete topology (= every subset is open). Indeed, two finite spaces X, Y with the discrete topology are homotopy equivalent if and only if they have the same number of elements.

Lecture 3

Definition 2.16. (i) A **retraction** of a topological space X onto a subspace $Y \subseteq X$ is a map $r : X \to Y$ such that

$$r(y) \ = \ y \in Y \text{ for all } y \in Y \ .$$

The subspace Y is a **retract** of X.

(ii) A **deformation retraction** of a topological space X onto a subspace $Y \subseteq X$ is a map $h : X \times I \to X$ such that

$$\begin{aligned} & h(x,0) \in Y \ , \ h(x,1) \ = \ x \in X \ , \ \text{for all } x \in X \ , \\ & h(y,0) \ = \ y \in Y \subseteq X \text{ for all } y \in Y \ . \end{aligned}$$

The subspace Y is a **deformation retract** of X.

Note that for a deformation retraction h of X onto Y the maps

$$i \ = \ \text{inclusion} \ : \ Y \to X \ , \ j \ : \ X \to Y \ ; \ x \mapsto h(x,0)$$

are such that

$$ji \ = \ 1_Y \ : \ Y \to Y \ ,$$

so that j is a retraction of X onto Y. Moreover, i, j are inverse homotopy equivalences, with

$$c \ : \ Y \times I \to Y \ ; \ (y,t) \mapsto y$$

and h defining homotopies

$$\begin{aligned} c \ : \ ji \ \simeq \ 1_Y \ : \ Y \to Y \ , \\ h \ : \ ij \ \simeq \ 1_X \ : \ X \to X \ . \end{aligned}$$

Example 2.17. $\{0\}$ is a deformation retract of $[0,1]$, with deformation retraction

$$h \ : \ I \times I \to I \ ; \ (x,t) \mapsto tx \ .$$

While every deformation retract is a retract, not every retract is a deformation retract:

Example 2.18. Let X be any non-empty space which is not contractible, for example $\{1,2\}$ with the discrete topology (every subset is open). For any $x_0 \in X$ the subspace $Y = \{1\}$ is a retract which is not a deformation retract.

Example 2.19. If X is a non-empty space with the trivial topology $\{\emptyset, X\}$ then for any $x_0 \in X$ the one-point subspace $\{x_0\} \subseteq X$ is a deformation retract, with deformation retraction

$$h \ : \ X \times I \to X \quad ; \quad (x,t) \mapsto \begin{cases} x_0 & \text{if } t = 0 \\ x & \text{if } 0 < t \leqslant 1 \ . \end{cases}$$

(Recall that any function $Y \to X$ to a space with the trivial topology is continuous). In particular, X is contractible.

Definition 2.20. A subspace $X \subseteq \mathbb{R}^n$ is **star-shaped** at $x \in X$ if for each $y \in X$ the straight line segment

$$[x, y] \ = \ \{(1-t)x + ty \,|\, 0 \leqslant t \leqslant 1\} \subset \mathbb{R}^n$$

is wholly contained in X, $[x, y] \subseteq X$.

Remark 2.21. (i) A subspace $X \subseteq \mathbb{R}^n$ is convex if it is star-shaped at every $x \in X$.
(ii) Construct an example of a star-shaped subspace of $\mathbb{R}^n$ which is not convex!

Proposition 2.22. *If $X \subseteq \mathbb{R}^n$ is star-shaped at $x_0 \in X$ then $\{x_0\} \subseteq X$ is a deformation retract. In particular, X is contractible.*

Proof. The map

$$h \ : \ X \times I \to X \ ; \ (x, t) \mapsto x_0 + t(x - x_0)$$

is a deformation retraction. □

Example 2.23. (i) A non-empty path-connected subspace $X \subseteq \mathbb{R}$ is either $\mathbb{R}$ itself or $\{x\}$ $(x \in \mathbb{R})$ or one of the intervals $[a, \infty)$, (a, ∞), $(-\infty, a]$, (∞, a), $[a, b]$, $[a, b)$, $(a, b]$, (a, b) $(a < b \in \mathbb{R})$. In each case X is convex, and hence contractible.
(ii) For any $n \geqslant 1$ the subspaces $\mathbb{R}^n, D^n \subseteq \mathbb{R}^n$ are convex, and hence contractible.

Definition 2.24. The **cone** on a non-empty space X is the quotient space obtained from $X \times I$ by collapsing $X \times \{0\}$ to a point

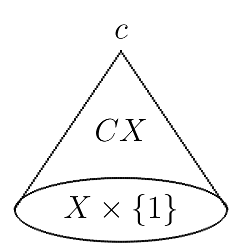
$$CX \ = \ (X \times I)/(X \times \{0\}) \ .$$

Explicitly, $CX = (X \times I)/\sim$ where $(x, 0) \sim (y, 0)$ for all $x, y \in X$.

The point $c = [X \times \{0\}] \in CX$ is the **cone point**, and the subspace

$$X = X \times \{1\} \subset CX$$

is the **cone base**.



For any point $[x, s] \in CX$ in a cone there is an obvious path

$$\alpha \ : \ I \to CX \ ; \ t \mapsto [x, st]$$

along the ray from the cone point $\alpha(0) = c$ to $\alpha(1) = [x, s]$. Here, obvious means that the path varies continuously with $[x, s]$.

This is a very special property for CX to have — by contrast, try to find an obvious choice of path $\alpha : I \to S^1$ from $x_0 = (1,0) \in S^1$ to any $x \in S^1$. If $x \neq -x_0$, you can just choose the shorter of the two arcs from x_0 to x. But for $x = -x_0$ the two arcs are of equal length. It is a non-trivial theorem that there is no obvious (= continuous) choice of path from x_0 to x for all $x \in S^1$. [In general, if (X,d) is a metric space such that between any two points $x, y \in X$ there is a unique path of shortest length, then X is contractible.]

Proposition 2.25. *(i) For any space X the cone point $\{c\} \subset CX$ is a deformation retract of the cone CX, so that CX is contractible.*

(ii) A map $f : X \to Y$ is homotopic to a constant map $g : X \to Y; x \mapsto y_0$ (for some $y_0 \in Y$) if and only if there exists a map $F : CX \to Y$ such that

$$F[x,1] \;=\; f(x) \;,\; F[x,0] \;=\; y_0 \in Y \quad (x \in X) \;.$$

Proof. (i) The paths along rays from the cone point c define a deformation retraction of CX onto $\{c\} \subset CX$

$$h \;:\; CX \times I \to CX \;;\; ([x,s],t) \mapsto [x,st] \;.$$

so that the maps

$$f \;:\; CX \to \{0\} \;;\; [x,s] \mapsto 0 \;,$$
$$g \;:\; \{0\} \to CX \;;\; 0 \mapsto [x,0]$$

are inverse homotopy equivalences.

(ii) Given a homotopy $h : f \simeq g : X \to Y$ define a map

$$F \;:\; CX \;=\; (X \times I)/(X \times \{0\}) \to Y \;;\; [x,t] \mapsto h(x,1-t)$$

such that

$$F[x,1] \;=\; h(x,0) \;=\; f(x) \;,\; F[x,0] \;=\; h(x,1) \;=\; g(x) \;=\; y_0 \in Y \quad (x \in X) \;.$$

Conversely, given a map $F : CX \to Y$ such that $F[x,1] = f(x)$ let $y_0 = F[x,0] \in Y$, and define a homotopy $h : f \simeq g : X \to Y$ by

$$h \;:\; X \times I \to Y \;;\; (x,t) \mapsto F[x,1-t] \;.$$

□

Example 2.26. For any $n \geqslant 0$ there is a homeomorphism

$$CS^n \to D^{n+1} \;;\; [x,t] \mapsto tx$$

sending the cone point c to $0 \in D^{n+1}$ and the cone base $S^n = S^n \times \{1\}$ to $S^n \subset D^{n+1}$ by the identity. The inverse homeomorphism is given by

$$y \mapsto \begin{cases} [y/\|y\|, \|y\|] & y \neq 0 \\ c & y = 0 \end{cases} .$$

By 2.25 (ii) a map $f : S^n \to Y$ is homotopic to a constant map if and only if there exists a map $F : D^{n+1} \to Y$ such that

$$F(x) \;=\; f(x) \in Y \quad (x \in S^n) \;.$$

Removing the cone point $c = [X \times \{0\}] \in CX$ from a cone results in a space $CX \backslash \{c\}$ for which there is a homeomorphism

$$X \times (0,1] \to CX \backslash \{c\} \ ; \ (x,t) \mapsto [x,t] \ .$$

In particular, $X = X \times \{1\} \subset CX \backslash \{c\}$ is a deformation retract.

Proposition 2.27. *Extend an equivalence relation $\sim$ on a space X to an equivalence relation $\sim$ on the cone CX by the identity, i.e.*

$$(x,s) \sim (y,t) \text{ if } (x,s) = (y,t) \text{ or } s = t = 0 \text{ or } (s = t = 1 \text{ and } x \sim y) \ .$$

Then $X/{\sim} = (X \times \{1\})/{\sim}$ is a deformation retract of $Y = (CX \backslash \{c\})/{\sim}$, and Y is homotopy equivalent to the quotient space $X/{\sim}$.

Proof. The maps

$$\begin{aligned} i \ &: \ X/{\sim} \to Y \ ; \ [x] \mapsto [x,1] \ , \\ j \ &: \ Y \to X/{\sim} \ ; \ [x,t] \mapsto [x] \end{aligned}$$

are such that $ji = 1 : X/{\sim} \to X/{\sim}$ and $ij([x,t]) = [x,1]$. The map

$$h \ : \ Y \times I \to Y \ ; \ ([x,t],s) \mapsto [x, 1 - s + st]$$

is a deformation retraction, defining a homotopy $h : ij \simeq 1 : Y \to Y$ which is constant on $X/{\sim} \subset Y$ (where $t = 1$). $\square$

Lecture 4

3 Some quotient spaces of I^2 and ∂I^2

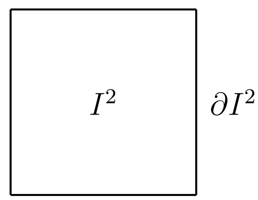
We shall now deal with the scissors and paper constructions of Workshop 3 last semester in a more abstract way.

The unit square

$$I^2 \; = \; I \times I \; = \; \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leqslant x \leqslant 1 \text{ and } 0 \leqslant y \leqslant 1\}$$

has boundary

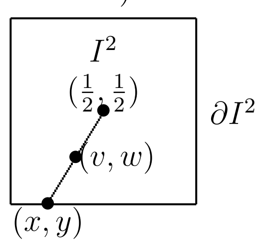
$$\partial I^2 \; = \; \{(x,y) \in \mathbb{R}^2 \,|\, \text{either } x=0 \text{ or } x=1 \text{ or } y=0 \text{ or } y=1\} \; .$$



In this chapter we shall consider the cone of $X = \partial I^2$, noting that the map

$$\beta : \; CX \to I^2 \; ; \; [(x,y),t] \mapsto (v,w) = (1-t)(\tfrac{1}{2},\tfrac{1}{2}) + t(x,y)$$

is a homeomorphism sending the cone point $c \in CX$ to the midpoint $\beta(c) = (\frac{1}{2},\frac{1}{2}) \in I^2$ (similar to the homeomorphism $CS^1 \cong D^2$ of 2.26, which sends the cone point to $(0,0) \in D^2$, but for I^2 the inverse is fiddly to write down).



The straight line 'ray' from the cone point c to a point $[(x,y),1] \in X \times \{1\}$ is sent by β to the line segment in I^2 from $\beta(c) = (\frac{1}{2},\frac{1}{2})$ to $(x,y) \in X$. Make sure you understand this homeomorphism, before proceeding further!

By Proposition 2.27, for any equivalence relation $\sim$ on ∂I^2 extended by the identity to an equivalence relation $\sim$ on I^2 the space obtained from $I^2/{\sim}$ by puncturing (i.e. removing) $(\frac{1}{2},\frac{1}{2})$

$$Y \; = \; (I^2 \setminus \{(\tfrac{1}{2},\tfrac{1}{2})\})/{\sim}$$

contains the quotient space $\partial I^2/{\sim}$ as a deformation retract.

Specifically, we have a deformation retraction

$$h \ : \ Y \times I \to Y \ ; \ ([v,w],s) \mapsto \beta[(x,y),1-s+st]$$

with $[(x,y),t] = \beta^{-1}([v,w]) \in C\partial I^2$. The maps

$$\begin{aligned} f \ &: \ Y \to \partial I^2/{\sim} \ ; \ [v,w] \mapsto [x,y] \ , \\ g \ &: \ \partial I^2/{\sim} \to Y \ ; \ [x,y] \mapsto [x,y] \end{aligned}$$

are inverse homotopy equivalences, such that $h : gf \simeq 1$, $fg = 1$. For any $(v,w) \neq (\frac{1}{2},\frac{1}{2}) \in I^2$ we have $gf[v,w] = [x,y]$ with

$$I \to Y \ ; \ s \mapsto h([v,w],s) \ = \ \beta[(x,y),1-s+st]$$

a path along the straight line segment from $h([v,w],0) = [x,y]$ to $h([v,w],1) = [v,w]$.

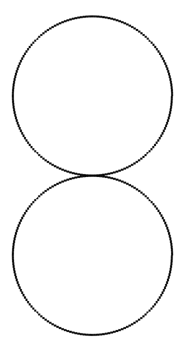
3.1 The figure eight

Terminology: the *one-point union* $X \vee Y$ of two spaces X, Y with base points $x_0 \in X$, $y_0 \in Y$ is the quotient space $(X \sqcup Y)/{\sim}$ of the disjoint union $X \sqcup Y$, with $x_0 \sim y_0$. Equivalently

$$X \vee Y \ = \ (X \times \{y_0\}) \cup (\{x_0\} \times Y) \subseteq X \times Y \ .$$

The figure eight is the one-point union of two circles

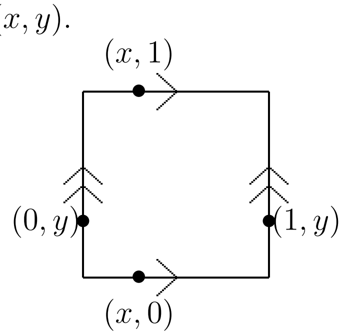
$$8 \ = \ S^1 \vee S^1$$



Define an equivalence relation $\sim$ on ∂I^2 by

$$(x,0) \sim (x,1) \ , \ (0,y) \sim (1,y) \text{ for all } x,y \in I$$

as well as the standard $(x,y) \sim (x,y)$.



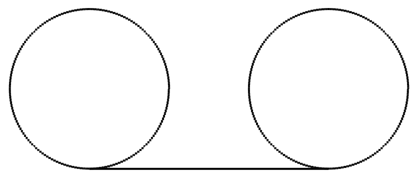
The quotient space is the figure 8, in the precise sense that the function

$$\partial I^2/{\sim} \to 8 \ ; \ \begin{cases} [x,0]=[x,1] \mapsto e^{2\pi i x} \text{ in first } S^1 \\ [0,y]=[1,y] \mapsto e^{2\pi i y} \text{ in second } S^1 \end{cases}$$

is a homeomorphism.

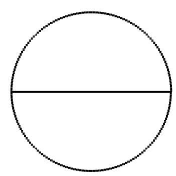
Later on, we shall use without proof the following facts:

(i) The space Υ obtained by joining two circles by a line



is homotopy equivalent to 8 – the projection $\Upsilon \to 8$ collapsing the diameter to a point is a homotopy equivalence.

(ii) The space Θ defined by a circle with a diameter

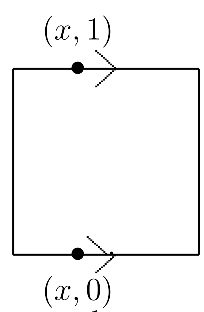


is homotopy equivalent to 8 – the projection $\Theta \to 8$ collapsing the diameter to a point is a homotopy equivalence.

3.2 The cylinder

The cylinder is the quotient space

$$I \times S^1 \ = \ I^2/{\sim} \ \ , \ (x,0) \sim (x,1)$$



The cylinder $I \times S^1$ contains the circle S^1 as a deformation retract, with the inclusion

$$S^1 \to I \times S^1 \ ; \ z \mapsto (0,z)$$

a homotopy equivalence. (Exercise: prove this!)

The quotient space $\partial I^2/{\sim}$ is homeomorphic to Υ (prove this!). The punctured cylinder $(I \times S^1) \setminus \{(\frac{1}{2},\frac{1}{2})\}$ contains Υ as a deformation retract, with the inclusion

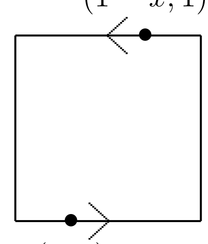
$$\Upsilon \;=\; \partial I^2/\!\sim\, \to (I \times S^1) \setminus \{(\tfrac{1}{2}, \tfrac{1}{2})\}$$

a homotopy equivalence. The punctured cylinder is thus homotopy equivalent to the figure 8.

3.3 The Möbius band

The Möbius band is the quotient space

$$M \;=\; I^2/\!\sim \;\;,\;\; (x,0) \sim (1-x,1)$$



The Möbius band M contains the circle $S^1 = I/(0 \sim 1)$ as a deformation retract, with the inclusion as the midpoints of the lines joining $[0,y]$ to $[1,y]$

$$S^1 \to M \;;\; [y] \mapsto [\tfrac{1}{2}, y]$$

a homotopy equivalence. (Exercise: prove this!). Note that this is *not* the boundary

$$\partial M \;=\; (\{0,1\} \times I)/\{(0,0) \sim (1,1),\; (1,0) \sim (0,1)\} \subset M$$

(draw a picture). The boundary is homeomorphic to S^1 (prove this!), but the inclusion $\partial M = S^1 \to M$ is not a homotopy equivalence, as will be proved later on in the course using the fundamental group.

The quotient space $\partial I^2/\!\sim$ is homeomorphic to Υ (prove this!). The punctured Möbius band contains Θ as a deformation retract, with the inclusion

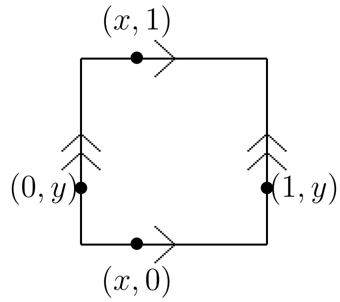
$$\Theta \;=\; \partial I^2/\!\sim\, \to M \setminus \{(\tfrac{1}{2}, \tfrac{1}{2})\}$$

a homotopy equivalence. The punctured Möbius band is thus homotopy equivalent to the figure 8.

3.4 The torus

The torus is the quotient space

$$S^1 \times S^1 \;=\; I^2/\!\sim \;\;,\;\; (x,0) \sim (x,1) \;,\; (0,y) \sim (1,y)$$



The quotient space $\partial I^2/\!\sim$ is homeomorphic to the figure 8 (prove this!). The punctured torus thus contains 8 as a deformation retract, with the inclusion

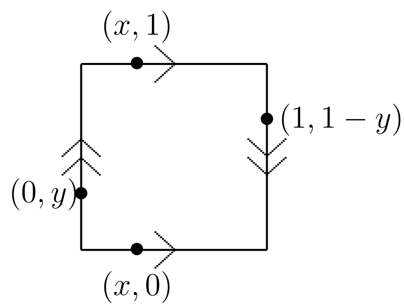
$$\partial I^2/\!\sim\, = \ 8 \to (S^1 \times S^1) \setminus \{(\tfrac{1}{2}, \tfrac{1}{2})\}$$

a homotopy equivalence.

3.5 The Klein bottle

The Klein bottle is the quotient space

$$K \ = \ I^2/\!\sim \ , \ (x,0) \sim (x,1) \ , \ (0,y) \sim (1,1-y)$$



The quotient space $\partial I^2/\!\sim$ is homeomorphic to the figure 8 (prove this!). The punctured Klein bottle thus contains 8 as a deformation retract, with the inclusion

$$\partial I^2/\!\sim\, = \ 8 \to K \setminus \{(\tfrac{1}{2}, \tfrac{1}{2})\}$$

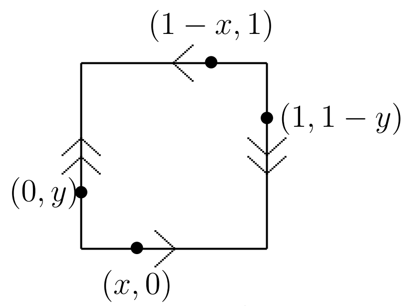
a homotopy equivalence.

Lecture 5

3.6 The projective plane

The projective plane is the quotient space

$$\mathbb{RP}^2 \;=\; I^2/\!\sim \;\;,\;\; (x,0) \sim (1-x,1) \;,\; (0,y) \sim (1,1-y)$$



There are four equivalent (i.e. homeomorphic) ways of defining the projective plane $\mathbb{RP}^2$, as a quotient space of I^2, S^2, D^2, $\mathbb{R}^3$:

(i) $\mathbb{RP}^2 = I^2/\!\sim$, as above.

(ii) $\mathbb{RP}^2 = S^2/\!\sim$ with $(x,y,z) \sim (-x,-y,-z)$.

(iii) $\mathbb{RP}^2 = D^2/\!\sim$ with $(x,y) \sim (-x,-y)$ for $(x,y) \in S^1$.

(iv) $\mathbb{RP}^2 = (\mathbb{R}^3\backslash\{(0,0,0)\})/\!\sim$ with $(x,y,z) \sim (\lambda x, \lambda y, \lambda z)$ for $\lambda \neq 0 \in \mathbb{R}$.

The quotient space $\partial I^2/\!\sim$ is homeomorphic to the circle S^1, and is a deformation retract of $\mathbb{RP}^2 \setminus \{(\frac{1}{2},\frac{1}{2})\}$ (prove this!). The punctured projective plane is thus homotopy equivalent to a circle:

$$\mathbb{RP}^2 \setminus \{(\tfrac{1}{2},\tfrac{1}{2})\} \;\simeq\; \partial I^2/\!\sim \;=\; S^1 \;.$$

Remark 3.1. The circle, the figure eight, the torus and the Klein bottle are not homotopy equivalent to each other: the proofs require the 'fundamental group' which will be developed later in the course. Homotopy equivalent spaces have isomorphic fundamental groups, and these spaces have non-isomorphic fundamental groups.

4 Cutting and pasting paths

The fundamental group $\pi_1(X,x_0)$ will be defined in section 5 for any space X and base point $x_0 \in X$ to be the set of 'rel $\{0,1\}$ homotopy classes' of paths $\alpha : [0,1] \to X$ such that

$$\alpha(0) \;=\; \alpha(1) \;=\; x_0 \in X \;,$$

with appropriate group law and inversion. For this purpose it is necessary to be able to paste together paths by concatenation. For nontrivial applications it is also necessary to improve paths, by cutting them into conveniently smaller pieces and if possible straightening each piece!

What does 'rel $\{0,1\}$' mean? Keeping the endpoints $\alpha(0), \alpha(1) \in X$ of a path $\alpha : I \to X$ fixed during the homotopy.

Definition 4.1. If $f, g : A \to X$ are maps and $B \subseteq A$ is a subspace such that

$$f(b) \;=\; g(b) \in X \quad (b \in B)$$

then a **homotopy rel** B (or **relative** to B) is a homotopy $h : f \simeq g : A \to X$ such that

$$h(b,t) \;=\; f(b) \;=\; g(b) \in X \quad (b \in B, t \in I) \;.$$

We shall be particularly concerned with the special case

$$(A, B) \;=\; (I, \{0,1\})$$

Example 4.2. A homotopy rel $\{0,1\}$ of two paths $\alpha_0, \alpha_1 : I \to X$ with the same endpoints

$$\alpha_0(0) \;=\; \alpha_1(0) \;,\; \alpha_0(1) \;=\; \alpha_1(1) \in X$$

is a collection of paths $h_t : I \to X$ $(0 \leqslant t \leqslant 1)$ with the same endpoints

$$h_t(0) \;=\; \alpha_0(0) \;=\; \alpha_1(0) \;,\quad h_t(1) \;=\; \alpha_0(1) \;=\; \alpha_1(1)$$

such that $h_0 = \alpha_0$, $h_1 = \alpha_1$ and such that the function

$$h \;:\; I \times I \to X \;;\; (s,t) \mapsto h_t(s)$$

is continuous.

It will turn out that any two paths $\alpha_0, \alpha_1 : I \to X$ with the same endpoints are homotopic rel $\{0,1\}$ for $X = \mathbb{R}^n$ or S^n for $n \geqslant 2$, but not necessarily so for $X = S^1$: while this may be intuitively obvious, it is quite hard to prove.

As was already seen in General Topology it is convenient to be able to glue together paths, using the following construction:

Definition 4.3. The **concatenation** at $\lambda \in (0,1)$ of paths $\alpha, \beta : I \to X$ such that $\alpha(1) = \beta(0) \in X$ is the path

$$\alpha \bullet_\lambda \beta \;:\; I \to X \;;\; s \mapsto \begin{cases} \alpha(\dfrac{s}{\lambda}) & \text{if } 0 \leqslant s \leqslant \lambda \\ \beta(\dfrac{s-\lambda}{1-\lambda}) & \text{if } \lambda \leqslant s \leqslant 1 \;. \end{cases}$$

with

$$\alpha \bullet_\lambda \beta(0) \;=\; \alpha(0) \;,\; \alpha \bullet_\lambda \beta(\lambda) \;=\; \alpha(1) \;=\; \beta(0) \;,\; \alpha \bullet_\lambda \beta(1) \;=\; \beta(1) \;.$$

Note that the image of a concatenation is the union of the images

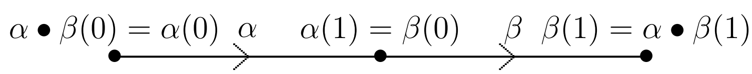
$$(\alpha \bullet_\lambda \beta)(I) \;=\; \alpha(I) \cup \beta(I) \subseteq X \;.$$

Example 4.4. (i) The concatenation of two paths $\alpha : I \to X$, $\beta : I \to X$ with $\alpha(1) = \beta(0) \in X$ defined in Definition 1.4 is the concatenation of 4.3 with $\lambda = \frac{1}{2}$

$$\alpha \bullet \beta \;=\; \alpha \bullet_{\frac{1}{2}} \beta \;:\; I \to X \;;\; t \mapsto \begin{cases} \alpha(2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ \beta(2t-1) & \text{if } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

which starts at $\alpha(0)$, follows along α at twice the speed in the first half, switching at $\alpha(1) = \beta(0)$ (at half-time) to follow β at twice the speed in the second half.

(ii) The concatenation of paths as in (i) was used in Proposition 2.3 to prove that homotopy is a transitive relation.



□

Proposition 4.5. *The rel* $\{0,1\}$ *homotopy class of* $\alpha \bullet_\lambda \beta : I \to X$ *is independent of* λ.

Proof. For any $\lambda, \mu \in (0,1)$ the function

$$\nu \;:\; I \to (0,1) \;;\; t \mapsto \nu(t) = (1-t)\lambda + t\mu$$

is such that

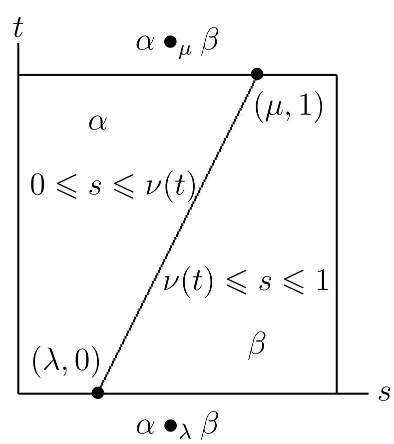
$$h \;:\; I \times I \to X \;;\; (s,t) \mapsto (\alpha \bullet_{\nu(t)} \beta)(s)$$

defines a rel $\{0,1\}$ homotopy

$$h \;:\; \alpha \bullet_\lambda \beta \;\simeq\; \alpha \bullet_\mu \beta \;:\; I \to X \;,$$

with

$$h(s,0) \;=\; (\alpha \bullet_\lambda \beta)(s) \;,\; h(s,1) \;=\; (\alpha \bullet_\mu \beta)(s) \;.$$



This is a picture of the unit square I^2 in the (s,t)-plane subdivided according to how h is defined: the line joining $(\lambda,0)$ to $(\mu,1)$ is $s=\nu(t)$ and

$$h(s,t) \;=\; (\alpha \bullet_{\nu(t)} \beta)(s) \;=\; \begin{cases} \alpha(\dfrac{s}{\nu(t)}) & \text{if } 0 \leqslant s \leqslant \nu(t) \\ \beta(\dfrac{s-\nu(t)}{1-\nu(t)}) & \text{if } \nu(t) \leqslant s \leqslant 1\ . \end{cases}$$

(The picture is perhaps more memorable than the formula.) □

More generally:

Definition 4.6. Suppose given N paths $\alpha_1,\alpha_2,\dots,\alpha_N : I \to X$ which adjoin each other, so that

$$\alpha_1(1)=\alpha_2(0)\ ,\ \alpha_2(1)=\alpha_3(0)\ ,\ \dots\ ,\ \alpha_{N-1}(1)=\alpha_N(0)\in X\ ,$$

and denote this collection of paths by α. Suppose given also a partition λ of I as a union of intervals

$$I \;=\; [\lambda_0,\lambda_1]\cup[\lambda_1,\lambda_2]\cup\dots\cup[\lambda_{N-1},\lambda_N]$$

with

$$\lambda_0=0<\lambda_1<\lambda_2<\dots<\lambda_{N-1}<\lambda_N=1\ .$$

The **N-tuple concatenation** path is

$$\alpha_\lambda \;=\; \alpha_1 \bullet_{\lambda_1} \alpha_2 \bullet \dots \bullet_{\lambda_{N-1}} \alpha_N \;:\; I \to X\ ;\ s \mapsto \alpha_i(\frac{s-\lambda_{i-1}}{\lambda_i-\lambda_{i-1}}) \text{ if } s\in[\lambda_{i-1},\lambda_i]$$

from $\alpha_\lambda(0)=\alpha_1(0)$ to $\alpha_\lambda(1)=\alpha_N(1)\in X$.

Proposition 4.7. *The rel $\{0,1\}$ homotopy class of an N-tuple concatenation $\alpha_\lambda : I \to X$ is independent of λ.*

Proof. For any partitions λ,μ of I the map

$$h \;:\; I\times I \to X\ ;\ (s,t)\mapsto(\alpha_{(1-t)\lambda+t\mu})(s)$$

defines a rel $\{0,1\}$ homotopy

$$h \;:\; \alpha_\lambda \;\simeq\; \alpha_\mu \;:\; I \to X\ .$$

□

Example 4.8. (i) Proposition 4.5 is the special case of Proposition 4.7 with $N=2$.

(ii) Consider the special case $N=3$, writing $\alpha_1,\alpha_2,\alpha_3$ as α,β,γ. Let then $\alpha,\beta,\gamma : I\to X$ be three adjoining paths, such that

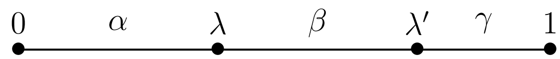
$$\alpha(1) \;=\; \beta(0)\ ,\ \beta(1) \;=\; \gamma(0)\in X\ .$$

The concatenation of α, β, γ for $0 < \lambda < \lambda' < 1$ is the path

$$\delta \;=\; \alpha \bullet_\lambda \beta \bullet_{\lambda'} \gamma \;:\; I \to X \;;\; s \mapsto \begin{cases} \alpha(s/\lambda) & \text{if } 0 \leqslant s \leqslant \lambda \\ \beta((s-\lambda)/(\lambda'-\lambda)) & \text{if } \lambda \leqslant s \leqslant \lambda' \\ \gamma((s-\lambda')/(1-\lambda')) & \text{if } \lambda' \leqslant s \leqslant 1 \;. \end{cases}$$

such that

$$\delta(0) \;=\; \alpha(0) \;,\; \delta(1) \;=\; \gamma(1) \in X \;.$$



The path starts by going along α at $1/\lambda$ the speed on $[0, \lambda]$, followed by going along β at $1/(\lambda' - \lambda)$ the speed on $[\lambda, \lambda']$, and finishes by going along γ at $1/(1-\lambda')$ the speed on $[\lambda', 1]$. The homotopy class rel $\{0, 1\}$ of $\delta : I \to X$ is independent of the choice of λ, λ'.

Lecture 6

5 The fundamental group $\pi_1(X)$

The fundamental group is the fundamental algebraic object associated to a topological space! It was first defined by Poincaré around 1900, and is the key to using algebra to prove theorems in topology. Why should one want to do this? Roughly speaking, geometric constructions tell us what can be done in topology, while algebraic obstructions tell us what cannot be done. Homeomorphic spaces have isomorphic groups. Given two spaces X, Y it may be possible to construct a homeomorphism $X \to Y$ by geometry, or it may be possible to prove algebraically that their fundamental groups are not isomorphic, so that X, Y are not homeomorphic.

The fundamental group $\pi_1(X, x_0)$ of a space X at a base point $x_0 \in X$ is a geometrically defined group of the 'based homotopy classes' of maps $\omega : S^1 \to X$ such that $\omega(1,0) = x_0$. The isomorphism class of $\pi_1(X, x_0)$ depends only on the homotopy type of the path-component of the base point $x_0 \in X$. Homotopy equivalent path-connected spaces have isomorphic fundamental groups. The fundamental group is used to distinguish non-homeomorphic spaces: path-connected spaces with non-isomorphic fundamental groups are not homotopy equivalent, and (a fortiori) not homeomorphic.

Definition 5.1. (i) A **based space** (X, x_0) is a space with a base point $x_0 \in X$.

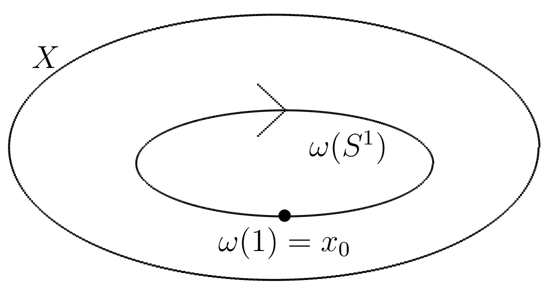
(ii) A **based map** $f : (X, x_0) \to (Y, y_0)$ is a map $f : X \to Y$ such that $f(x_0) = y_0 \in Y$.

(iii) A **based homotopy** $h : f \simeq g : (X, x_0) \to (Y, y_0)$ is a homotopy $h : f \simeq g : X \to Y$ such that $h(x_0, t) = y_0 \in Y$ $(t \in I)$.

Proposition 5.2. *For any based spaces (X, x_0), (Y, y_0) based homotopy is an equivalence relation on the set of based maps $f : (X, x_0) \to (Y, y_0)$.*

Proof. Exactly as for unbased homotopy. □

Definition 5.3. A **based loop** is a based map $\omega : (S^1, 1) \to (X, x_0)$ where $1 = (1, 0) \in S^1$.



In view of the homeomorphism

$$I/\{0 \sim 1\} \to S^1 \ ; \ [t] \mapsto (\cos 2\pi t, \sin 2\pi t)$$

a loop $\omega : S^1 \to X$ at $\omega(1) = x_0 \in X$ is essentially the same as a closed path $\alpha : I \to X$ such that

$$\alpha(0) = \alpha(1) = x_0 \in X ,$$

with α and ω related by

$$\alpha(t) = \omega(\cos 2\pi t, \sin 2\pi t) \in X \quad (t \in I)$$

The closed path α is the composite

$$\alpha : I \xrightarrow{\text{projection}} S^1 \xrightarrow{\omega} X .$$

The rel $\{0,1\}$ homotopy classes of closed paths $\alpha : I \to X$ such that $\alpha(0) = \alpha(1) = x_0 \in X$ are in one-one correspondence with the rel $\{1\}$ homotopy classes of loops $\omega : S^1 \to X$ with $\omega(1) = x_0 \in X$.

Homotopy theory uses the topological properties of closed paths $I \to X$ and loops $S^1 \to X$ and the algebraic properties of groups to decide whether topological spaces are homotopy equivalent. Since I is contractible (2.10) two paths $\alpha, \beta : I \to X$ are homotopic if and only if $\alpha(I), \beta(I) \subseteq X$ are in the same path component of X. In order to use the homotopy classes of paths $I \to X$ to detect more than just the path components of X, it is necessary to keep the endpoints fixed!

Definition 5.4. The **fundamental group** $\pi_1(X, x_0)$ at a base point $x_0 \in X$ is the set of rel $\{0,1\}$ homotopy classes $[\alpha]$ of closed paths $\alpha : I \to X$ such that

$$\alpha(0) = \alpha(1) = x_0 \in X$$

with multiplication by the concatenation of paths (4.3)

$$\pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0) \ ; \ ([\alpha], [\beta]) \mapsto [\alpha][\beta] = [\alpha \bullet \beta] ,$$

and inverses by path reversal (1.3)

$$\pi_1(X, x_0) \to \pi_1(X, x_0) \ ; \ [\alpha] \mapsto [\alpha]^{-1} = [-\alpha]$$

and neutral element $[e_{x_0}] \in \pi_1(X, x_0)$ the class of the constant path

$$e_{x_0} : I \to X \ ; \ t \mapsto x_0 .$$

It is of course also possible to regard $\pi_1(X, x_0)$ as the set of rel $\{1\}$ homotopy classes $[\omega]$ of loops $\omega : S^1 \to X$ such that $\omega(1) = x_0 \in X$. The path formulation is more convenient for algebra, while the loops are more geometric.

Theorem 5.5. *The fundamental group $\pi_1(X, x_0)$ is a group.*

Proof. I. $[e_{x_0}]$ is a unit: $[\alpha][e_{x_0}] = [\alpha] \in \pi_1(X, x_0)$.

Define a rel $\{0,1\}$ homotopy

$$h \;:\; \alpha \bullet e_{x_0} \simeq \alpha \;:\; I \to X$$

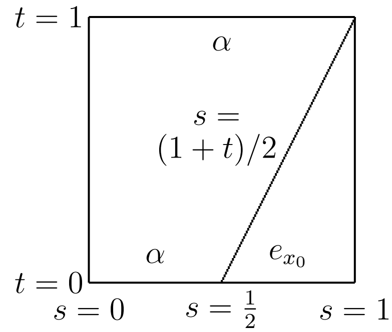
by

$$h \;:\; I \times I \to X \;;\; (s,t) \mapsto \begin{cases} \alpha(2s/(1+t)) & \text{if } s \leqslant (1+t)/2 \\ x_0 & \text{if } s \geqslant (1+t)/2 \;. \end{cases}$$

To make sense of this formula draw the unit square in the (s,t)-plane and join the point $(\frac{1}{2}, 0)$ to the point $(1,1)$ by the line $s = (1+t)/2$. Think what happens at each time $t \in I$: the path

$$h_t \;=\; \alpha \bullet_{(1+t)/2} e_{x_0} \;:\; I \to X \;;\; s \mapsto h_t(s) = h(s,t)$$

starts by going along α at $2/(1+t)$ the speed on $[0,(1+t)/2]$, and then stays put at x_0 on $[(1+t)/2, 1]$. The homotopy h starts at $h_0 = \alpha \bullet e_{x_0}$ and ends at $h_1 = \alpha$.



(Work out the corresponding formula for $[e_{x_0}][\alpha] = [\alpha] \in \pi_1(X, x_0)$.)

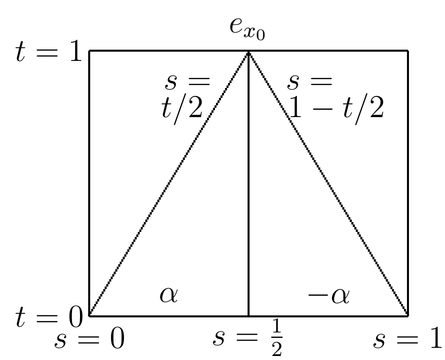
II. Inverses: $[\alpha][-\alpha] = [e_{x_0}] \in \pi_1(X, x_0)$.

Define a rel $\{0,1\}$ homotopy

$$h \;:\; \alpha \bullet -\alpha \simeq e_{x_0} \;:\; I \to X$$

by

$$h \;:\; I \times I \to X \;;\; (s,t) \mapsto \begin{cases} x_0 & \text{if } 0 \leqslant s \leqslant t/2 \\ \alpha(2s - t) & \text{if } t/2 \leqslant s \leqslant \frac{1}{2} \\ \alpha(2 - 2s - t) & \text{if } \frac{1}{2} \leqslant s \leqslant 1 - t/2 \\ x_0 & \text{if } 1 - t/2 \leqslant s \leqslant 1 \;. \end{cases}$$



Again, think what happens at each time $t \in I$: the path

$$h_t \ : \ I \to X \ ; \ s \to h_t(s) = h(s,t)$$

is constant on $[0, t/2]$, goes along the restriction $\alpha| : [0, 1-t] \to X$ (**i.e. using only a part of** α) at twice the speed on $[t/2, \frac{1}{2}]$, then along the restriction $-\alpha| : [t, 1] \to X$ at twice the speed on $[\frac{1}{2}, 1-t/2]$, and stays constant on $[1-t/2, 1]$. Note that $\alpha(1-t) = -\alpha(t)$ is essential for continuity. The homotopy h starts at $h_0 = \alpha \bullet -\alpha$ and ends at $h_1 = e_{x_0}$. (Work out the corresponding formula for $[-\alpha][\alpha] = [e_{x_0}]$.)

III. Associativity of multiplication: $([\alpha][\beta])[\gamma] = [\alpha]([\beta][\gamma]) \in \pi_1(X, x_0)$.

Let $\alpha, \beta, \gamma : I \to X$ be paths which send each endpoint to $x_0 \in X$. For $0 < \lambda < \mu < 1$ let

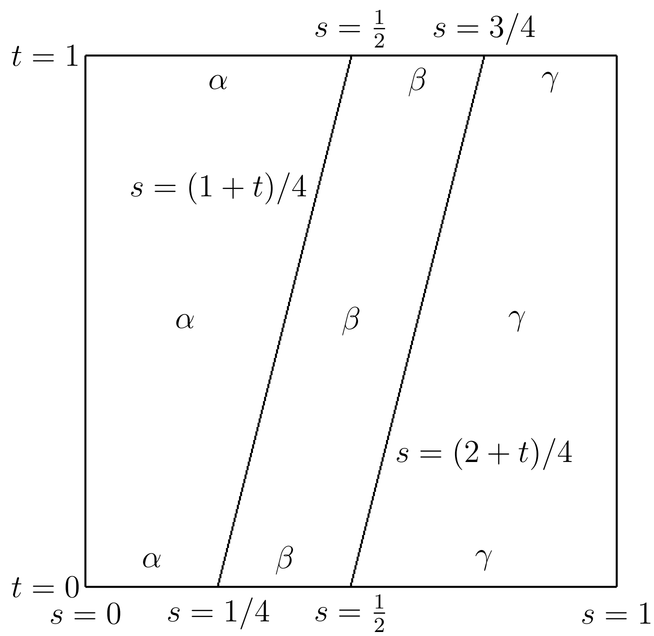
$$\delta(\lambda, \mu) \ = \ \alpha \bullet_\lambda \beta \bullet_\mu \gamma \ : \ I \to X$$

be the triple concatenation of 4.6. From the definitions

$$([\alpha][\beta])[\gamma] \ = \ \delta(1/4, \tfrac{1}{2}) \ : \ I \to X \ ; \ s \mapsto \begin{cases} \alpha(4s) & \text{if } 0 \leqslant s \leqslant 1/4 \\ \beta(4s-1) & \text{if } 1/4 \leqslant s \leqslant \frac{1}{2} \\ \gamma(2s-1) & \text{if } \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

and

$$[\alpha]([\beta][\gamma]) \ = \ \delta(\tfrac{1}{2}, 3/4) \ : \ I \to X \ ; \ s \mapsto \begin{cases} \alpha(2s) & \text{if } 0 \leqslant s \leqslant \frac{1}{2} \\ \beta(4s-2) & \text{if } \frac{1}{2} \leqslant s \leqslant 3/4 \\ \gamma(4s-3) & \text{if } 3/4 \leqslant s \leqslant 1 \ . \end{cases}$$



Finally, construct a homotopy rel $\{0, 1\}$

$$h \ : \ ([\alpha][\beta])[\gamma] \simeq [\alpha]([\beta][\gamma]) \ : \ I \to X$$

by

$$\begin{aligned} h_t &= \delta((1-t)/4 + t/2, (1-t)/2 + t(3/4)) \\ &= \delta((1+t)/4, (2+t)/4) \ : \ I \to X \end{aligned}$$

with $h_0 = \delta(1/4, \frac{1}{2})$, $h_1 = \delta(\frac{1}{2}, 3/4)$. $\square$

Remark 5.6. The following table gives the fundamental groups of some path-connected spaces (which will be calculated in subsequent chapters):

X	$\pi_1(X)$
$\mathbb{R}^n$	$\{1\}$
S^1	$\mathbb{Z}$
S^n $(n \geqslant 2)$	$\{1\}$
$\mathbb{RP}^n$ $(n \geqslant 2)$	$\mathbb{Z}_2$
$S^1 \times S^1$	$\mathbb{Z} \oplus \mathbb{Z}$
8	$\mathbb{Z} * \mathbb{Z}$

(i) $\pi_1(\mathbb{R}^n) = \{1\}$ by the convexity of $\mathbb{R}^n$.

(ii) The fundamental group $\pi_1(S^1) = \mathbb{Z}$ is generated by the identity loop $1 : S^1 \to S^1$. The homotopy class of a loop $f : S^1 \to S^1$ is the degree of f, the number $\deg(f) \in \mathbb{Z}$ introduced in Workshop 2.

(iii) The proof of $\pi_1(S^n) = \{1\}$ for $n \geqslant 2$ is easy once it is known that every loop $\omega : S^1 \to S^n$ is homotopic to a non-surjective one, i.e. that a space-filling closed path in S^n is homotopic to one which misses at least one point of S^n.

(iv) The n-dimensional projective space $\mathbb{RP}^n$ is the quotient of S^n by the equivalence relation

$$x \sim y \text{ if either } x = y \text{ or } x = -y \ ;$$

for $n \geqslant 2$ $\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$ is the cyclic group of order 2, generated by the square root loop

$$\sigma \ : \ S^1 \to \mathbb{RP}^n \ ; \ (\cos 2\pi t, \sin 2\pi t) \mapsto [\cos \pi t, \sin \pi t, 0, \ldots, 0] \ .$$

(v) The fundamental group of the torus $S^1 \times S^1$ is the free abelian group on two generators $\pi_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$. The generators $(1,0), (0,1)$ are the based homotopy classes of the meridian and longitude based loops:

$$\begin{aligned} (1,0) \ &: \ S^1 \to S^1 \times S^1 \ ; \ x \mapsto (x, 1) \ , \\ (0,1) \ &: \ S^1 \to S^1 \times S^1 \ ; \ y \mapsto (1, y) \ . \end{aligned}$$

(vi) The figure eight space 8 is the one-point union of two copies of S^1. The fundamental group $\pi_1(8) = \mathbb{Z} * \mathbb{Z}$ is the nonabelian free group on two generators. Elements of this group are all words on the alphabet $a, b, a^{-1} b^{-1}$, but cancelling expressions like aa^{-1}. In this group, note that $ab \neq ba$, and $aba^{-1}b^{-1} \neq 1$. The generators of $pi_1(8)$ are the based homotopy classes of the two based loops defined by the images of the two circles.

Lecture 7

The fundamental group $\pi_1(X, x_0)$ of a space X at a base point $x_0 \in X$ is defined geometrically, in terms of paths $\alpha : I \to X$ such that $\alpha(0) = \alpha(1) = x_0$, or equivalently in terms of loops $\omega : S^1 \to X$ such that $\omega(1) = x_0 \in X$. A calculation of $\pi_1(X, x_0)$ is an algebraic description. In general, it is quite difficult to compute $\pi_1(X, x_0)$, unless there is a geometric reason for it to be the trivial group $\{1\}$.

Definition 5.7. A based space (X, x_0) is **simply-connected** if X is path-connected and $\pi_1(X, x_0) = \{1\}$.

Example 5.8. The space $X = \{x_0\}$ with a single point $x_0 \in X$ is simply-connected.

In Theorem 5.15 below it will be proved that if X, Y are homotopy equivalent path-connected spaces then the fundamental groups $\pi_1(X, x_0)$, $\pi_1(Y, y_0)$ are isomorphic, for any $x_0 \in X$, $y_0 \in Y$. In particular, the fundamental group $\pi_1(X, x_0)$ of a contractible space X is trivial, so that X is simply-connected.

In dealing with the fundamental group $\pi_1(X, x_0)$ of a path-connected space X it is usual to just write $\pi_1(X)$, since the the isomorphism class of $\pi_1(X, x_0)$ is independent of the base point $x_0 \in X$. (This is the special case of Theorem 5.15 below with $X = Y$).

A space determines a group. (In fact, every group arises as the fundamental group of a space). A map of spaces determines a group homomorphism.

Proposition 5.9. (i) *A map $f : X \to Y$ induces a group homomorphism*

$$f_* \ : \ \pi_1(X, x_0) \to \pi_1(Y, f(x_0)) \ ; \ [\alpha] \mapsto [f\alpha]$$

for any base point $x_0 \in X$.

(ii) *The identity map $1 : X \to X$ induces the identity homomorphism*

$$1_* \ = \ 1 \ : \ \pi_1(X, x_0) \to \pi_1(X, x_0) \ .$$

(iii) *The composite $gf : X \to Z$ of maps $f : X \to Y$, $g : Y \to Z$ induces the composite group homomorphism*

$$(gf)_* \ = \ g_* f_* \ : \ \pi_1(X, x_0) \to \pi_1(Y, f(x_0)) \to \pi_1(Z, gf(x_0)) \ .$$

Proof. Easy consequences of the definitions! □

Definition 5.10. Let X be a space with a base point $x_0 \in X$. A map $f : X \to Y$ is a **homotopy equivalence rel** $\{x_0\}$ if there exists a map $g : Y \to X$ such that $g(f(x_0)) = x_0$, a homotopy rel $\{x_0\}$ denoted $h : gf \simeq 1_X : X \to X$ (with $h(x_0, t) = x_0$ for $t \in I$) and a homotopy rel $\{f(x_0)\}$ denoted $K : fg \simeq 1_Y : Y \to Y$ (with $K(f(x_0), t) = f(x_0)$ for $t \in I$).

Proposition 5.11. (i) *If $f_1, f_2 : X \to Y$ are maps which are related by a rel $\{x_0\}$ homotopy $h : f_1 \simeq f_2 : X \to Y$ then*

$$(f_1)_* \ = \ (f_2)_* \ : \ \pi_1(X, x_0) \to \pi_1(Y, f_1(x_0)) \ .$$

(ii) *If* $f : X \to Y$ *is a homotopy equivalence rel* $\{x_0\}$ *then* f_* *is an isomorphism, with inverse*

$$(f_*)^{-1} \;=\; g_* \;:\; \pi_1(Y, f(x_0)) \to \pi_1(X, x_0)\ ,$$

for g *as in Definition 5.10.*

Proof. Easy consequences of the definitions! □

In fact, the isomorphism class of the fundamental group $\pi_1(X, x_0)$ is independent of the choice of the base point $x_0 \in X$ within its path component. (Recall that the **path component** of $x_0 \in X$ is the set of all $x \in X$ such that there exists a path $\gamma : I \to X$ from $\gamma(0) = x_0$ to $\gamma(1) = x$.) Also, if $f : X \to Y$ is any homotopy equivalence (not necessarily rel $\{x_0\}$) then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.

Proposition 5.12. *(i) A path* $\gamma : I \to X$ *determines an isomorphism of groups*

$$\gamma_\# \;:\; \pi_1(X, \gamma(0)) \to \pi_1(X, \gamma(1))\ ;\ [\alpha] \mapsto [-\gamma \bullet \alpha \bullet \gamma]$$

with $-\gamma : I \to X; t \mapsto \gamma(1-t)$ *the reverse path from* $-\gamma(0) = \gamma(1)$ *to* $-\gamma(1) = \gamma(0)$*. The reverse path* $-\gamma$ *determines the inverse isomorphism*

$$(-\gamma)_\# \;=\; (\gamma_\#)^{-1} \;:\; \pi_1(X, \gamma(1)) \to \pi_1(X, \gamma(0))\ .$$

(ii) If $\gamma = e_{x_0}$ *is the constant path at* $x_0 \in X$ *(with* $\gamma(t) = x_0$*) then* $[\gamma] = [e_{x_0}] \in \pi_1(X, x_0)$ *is the unit element and*

$$\gamma_\# \;=\; 1 \;:\; \pi_1(X, x_0) \to \pi_1(X, x_0)\ ;\ [\alpha] \mapsto [\alpha]$$

is the identity automorphism.

(iii) The isomorphism $\gamma_\#$ *depends only on the rel* $\{0,1\}$ *homotopy class of* γ*.*

(iv) If $\gamma = \gamma_1 \bullet \gamma_2 : I \to X$ *is the concatenation of paths* $\gamma_1, \gamma_2 : I \to X$ *with* $\gamma_1(1) = \gamma_2(0) \in X$ *then*

$$\gamma_\# \;=\; (\gamma_2)_\#(\gamma_1)_\# \;:\; \pi_1(X, \gamma_1(0)) \to \pi_1(X, \gamma_2(1))\ .$$

Proof. (i) Define a homotopy rel $\{0,1\}$

$$h \;:\; \gamma \bullet -\gamma \;\simeq\; e_{\gamma(0)} \;:\; I \to X$$

(with e_{x_0} the constant path at x_0) by

$$h(s,t) \;=\; \begin{cases} \gamma(0) & \text{if } 0 \leqslant s \leqslant t/2 \\ \gamma(2s-t) & \text{if } t/2 \leqslant s \leqslant \frac{1}{2} \\ \gamma(2-2s-t) & \text{if } \frac{1}{2} \leqslant s \leqslant 1-t/2 \\ \gamma(0) & \text{if } 1-t/2 \leqslant s \leqslant 1\ . \end{cases}$$

It follows that if $\alpha, \beta : I \to X$ are closed paths at $\gamma(0)$ (i.e. such that $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = \gamma(0) \in X$) then

$$\begin{aligned}\gamma_\#([\alpha][\beta]) &= \gamma_\#([\alpha \bullet \beta]) \\ &= [-\gamma \bullet (\alpha \bullet \beta) \bullet \gamma] \\ &= [(-\gamma \bullet \alpha \bullet \gamma) \bullet (-\gamma \bullet \beta \bullet \gamma)] \\ &= \gamma_\#([\alpha])\gamma_\#([\beta]) \in \pi_1(X, \gamma(1))\end{aligned}$$

so that $\gamma_\#$ preserves multiplications. Also

$$\gamma_\#[e_{\gamma(0)}] \ = \ [-\gamma \bullet e_{\gamma(0)} \bullet \gamma] \ = \ [e_{\gamma(1)}] \in \pi_1(X, \gamma(1)) \ ,$$

so $\gamma_\#$ preserves the units. Therefore $\gamma_\#$ is a group homomorphism.

It follows from the existence of rel $\{0, 1\}$ homotopies $\gamma \bullet -\gamma \simeq e_{\gamma(0)}$ and $-\gamma \bullet \gamma \simeq e_{\gamma(1)}$ that $\gamma_\#$, $-\gamma_\#$ are inverse isomorphisms.

(ii) If γ is constant there exists a rel $\{0, 1\}$ homotopy $-\gamma \bullet \alpha \bullet \gamma \simeq \alpha$.

(iii) A rel $\{0, 1\}$ homotopy $\delta : \gamma \simeq \gamma' : I \to X$ determines a rel $\{0, 1\}$ homotopy

$$-\delta \bullet 1 \bullet \delta \ : \ -\gamma \bullet \alpha \bullet \gamma \ \simeq \ \gamma' \bullet \alpha \bullet \gamma' \ : \ I \to X \ .$$

(iv) There exists a rel $\{0, 1\}$ homotopy $-\gamma \simeq -\gamma_2 \bullet -\gamma_1$ and hence a rel $\{0, 1\}$ homotopy

$$-\gamma \bullet \alpha \bullet \gamma \ \simeq -\gamma_2 \bullet (-\gamma_1 \bullet \alpha \bullet \gamma_1) \bullet \gamma_2 \ : \ I \to X \ .$$

□

Example 5.13. A closed path $\gamma : I \to X$ with $\gamma(0) = \gamma(1) = x_0 \in X$ determines the conjugation automorphism

$$\gamma_\# \ : \ \pi_1(X, x_0) \to \pi_1(X, x_0) \ ; \ [\alpha] \mapsto [-\gamma \bullet \alpha \bullet \gamma] = [\gamma]^{-1}[\alpha][\gamma]$$

(which is the identity if $\pi_1(X, x_0)$ is abelian).

Lecture 8

Proposition 5.14. *Given maps $F, G : X \to Y$, a homotopy $H : F \simeq G : X \to Y$ and a base point $x_0 \in X$ define a path*

$$\gamma \ : \ I \to Y \ ; \ t \mapsto H(x_0, t)$$

from $\gamma(0) = F(x_0)$ to $\gamma(1) = G(x_0)$, so that there is defined an isomorphism

$$\gamma_\# \ : \ \pi_1(Y, F(x_0)) \to \pi_1(Y, G(x_0)) \ .$$

The induced homomorphisms of fundamental groups

$$F_* \ : \ \pi_1(X, x_0) \to \pi_1(Y, F(x_0)) \ , \ G_* \ : \ \pi_1(X, x_0) \to \pi_1(Y, G(x_0))$$

are such that

$$G_* \ = \ \gamma_\# F_* \ : \ \pi_1(X, x_0) \to \pi_1(Y, G(x_0)) \ .$$

Proof. For $t \in I$ define the path $\gamma_t : I \to Y; s \to H(x_0, st)$ from $\gamma_t(0) = F(x_0)$ to $\gamma_t(1) = H(x_0, t) = \gamma(t)$. For any closed path $\alpha : I \to X$ with $\alpha(0) = \alpha(1) = x_0 \in X$ the triple concatenations

$$\gamma_t \bullet_{1/3} (H_t \alpha) \bullet_{2/3} (-\gamma_t) \ : \ I \to Y \quad (t \in I)$$

define a rel $\{0, 1\}$ homotopy $(e_{Fx_0})_\# F_*(\alpha) \simeq (-\gamma)_\# G_*(\alpha)$. By the inverse and identity relations, we deduce that $G_* \simeq \gamma_\# F_*$ □

Theorem 5.15. *If $f : X \to Y$ is a homotopy equivalence then*

$$f_* \ : \ \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

is an isomorphism of groups for any base point $x_0 \in X$.

Proof. Let $g : Y \to X$ be a homotopy inverse of f, so that there exist homotopies

$$h \ : \ 1_X \ \simeq \ gf \ \ : \ X \to X \ \ , \ \ h' \ : \ 1_Y \ \simeq \ fg \ : \ Y \to Y \ .$$

Consider the group homomorphisms

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, gf(x_0)) \xrightarrow{f'_*} \pi_1(Y, fgf(x_0))$$

where f' is also induced by f. Define the path $\gamma : I \to X; t \to h(x_0, t)$ from $\gamma(0) = x_0$ to $\gamma(1) = gf(x_0)$. By Proposition 5.14 applied to the homotopy $H = h : F = 1_X \simeq G = gf : X \to X$

$$G_* \ = \ g_* f_* \ = \ \gamma_\# \ : \ \pi_1(X, x_0) \to \pi_1(X, gf(x_0))$$

is an isomorphism, so that $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is one-one. Similarly, it follows from $h' : fg \simeq 1_Y$ that $f'_* g_*$ is an isomorphism, so that g_* is also one-one. It follows from $g_* f_*$ being onto that g_* is also onto, and hence an isomorphism. Finally, note that the composite

$$e \ = \ (g_* f_*)^{-1} g_* \ : \ \pi_1(Y, f(x_0)) \to \pi_1(X, gf(x_0)) \to \pi_1(X, x_0)$$

is an isomorphism such that $ef_* = 1 : \pi_1(X, x_0) \to \pi_1(X, x_0)$, so that f_* is an isomorphism with inverse $(f_*)^{-1} = e$. □

Example 5.16. A contractible space X is simply-connected; $\pi_1(X, x_0) = \{1\}$ for any base point $x_0 \in X$.

Example 5.17. (i) If $Y \subseteq X$ is a deformation retract (Definition 2.16), i.e. if the inclusion $i : Y \to X$ is a homotopy equivalence, then $i_* : \pi_1(Y, y_0) \to \pi_1(X, iy_0)$ is an isomorphism for any $y_0 \in Y$.

(ii) The maps

$$\begin{aligned} i &= \text{inclusion} : S^{n-1} \to \mathbb{R}^n \backslash \{0\} , \\ j &: \mathbb{R}^n \backslash \{0\} \to S^{n-1} ; \ x \mapsto \frac{x}{\|x\|} \end{aligned}$$

are inverse homotopy equivalences, with $ji = 1 : S^{n-1} \to S^{n-1}$ and

$$h : \mathbb{R}^n \backslash \{0\} \times I \to \mathbb{R}^n \backslash \{0\} ; \ (x, t) \mapsto (1-t)\tfrac{x}{\|x\|} + tx$$

defining a homotopy $h : ij \simeq 1 : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}$. Thus i induces an isomorphism

$$i_* : \pi_1(S^{n-1}) \cong \pi_1(\mathbb{R}^n \backslash \{0\}) .$$

Remark 5.18 (Non-examinable). For any space X the cone CX is contractible (and in particular path-connected) so $\pi_1(CX) = \{1\}$. For any equivalence relation $\sim$ on X let $p : X \to X/\sim$ be the projection, let $\sim$ be the extended equivalence relation on CX as in Proposition 2.27, with $(x, 1) \sim (x', 1)$ if $x \sim x'$. The quotient space $Y = CX/\sim$ is path-connected, but not in general simply-connected.

The van Kampen theorem (which we will see briefly and non-examinably at the end of the course) shows that for path-connected X, the fundamental group of Y is the quotient group

$$\pi_1(Y) = \pi_1(X/\sim)/N$$

with N the normal subgroup generated by $\mathrm{im}(p_* : \pi_1(X) \to \pi_1(X/\sim))$.

It should be clear that a loop $\omega : S^1 \to X \times \{\frac{1}{2}\} \subset Y$ is homotopic to the constant loop at the cone point of Y. This expression for $\pi_1(Y)$ can be used to compute the fundamental groups of all the quotient spaces $I^2/\sim$ in Chapter 8, with $X = \partial I^2 \cong S^1$, $CX \cong I^2 \cong D^2$ (and indeed for all cell complexes). We shall use a different method in the course. The first step for both methods is to compute $\pi_1(S^1) = \mathbb{Z}$.

5.1 Nonexaminable appendix on the higher homotopy groups

Remark 5.19. Let X be a space with a base point $x_0 \in X$.

(i) Let $\Omega_{x_0}(X) \subset X^{S^1}$ be the subspace consisting of the maps $\omega : S^1 \to X$ such that $\omega(1) = x_0 \in X$ with $1 = (1, 0) \in S^1$, called the **based loops** at x_0. The path-component set of $\Omega_{x_0}(X)$ is in bijective correspondence with the fundamental group $\pi_1(X, x_0)$ of homotopy classes of based loops

$$\pi_0(\Omega_{x_0}(X)) = \pi_1(X, x_0)$$

(about which more later) since a path $I \to \Omega_{x_0}(X)$ is the same as a rel 1 homotopy of based loops in X.

(ii) More generally, for any $n \geqslant 1$ let $\Omega_{x_0}(S^n, X) \subset X^{S^n}$ be the subspace of the maps $\omega : S^n \to X$ such that $\omega(1) = x_0 \in X$ with $1 = (1, 0, \ldots, 0) \in S^n$. The set

$$\pi_n(X, x_0) \ = \ \pi_0(\Omega_{x_0}(S^n, X))$$

can be identified with the set of homotopy classes of such maps ω keeping the image of $1 \in S^n$ fixed at $x_0 \in X$. The set $\pi_n(X, x_0)$ can be given a group structure, which is abelian for $n \geqslant 2$; a map of based spaces $f : (X, x_0) \to (Y, y_0)$ induces group homomorphisms $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ $(n \geqslant 2)$ which are isomorphisms if $f : X \to Y$ is a homotopy equivalence, just like for $n = 1$.

(iii) The standard proof of Brouwer's theorem on the topological invariance of dimension (that $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$) uses the one-point compactification $(\mathbb{R}^m)^\infty = S^m$. It is immediate from the computation

$$\pi_n(S^m, 1) \ = \ \begin{cases} \mathbb{Z} & \text{if } n = m \\ 0 & \text{if } n < m \end{cases}$$

that S^m is homeomorphic to S^n if and only if $m = n$. (The computation of $\pi_n(S^m, 1)$ for $n \leqslant m = 1$ will be worked out in the course, using the $\mathbb{Z}$-valued degree of maps $f : S^1 \to S^1$ for $n = m = 1$. The general case $n < m$, $m \geqslant 2$ is not much more difficult, using the $\mathbb{Z}$-valued degree of maps $f : S^n \to S^n$ for $n = m$). If $n < m$ and S^n were homeomorphic to S^m then the infinite cyclic group $\pi_n(S^n, 1) = \mathbb{Z}$ would be isomorphic to the zero group $\pi_n(S^m, 1) = 0$ – a contradiction!

Lecture 9

6 Compact surfaces and their classification

We will now look at more examples of topological spaces with interesting fundamental groups, namely compact surfaces. Although we are not in a position to compute the fundamental groups yet, by taking them on trust we'll see that they can be used to distinguish between different surfaces.

Definition 6.1. (i) A Hausdorff topological space M is called an *n-dimensional (topological) manifold* if it admits a countable cover by open subsets $U \subseteq M$ such that each U is homeomorphic to the Euclidean n-space $\mathbb{R}^n$.

(ii) A 2-dimensional manifold is called a *surface*.

It is true, but not obvious, that if $m \neq n$ then an m-dimensional manifold cannot be homeomorphic to an n-dimensional manifold. This follows from the invariance of dimension: the Euclidean spaces $\mathbb{R}^m, \mathbb{R}^n$ are homeomorphic if and only if $m = n$.

Example 6.2. (i) $\mathbb{R}^n$ is an n-dimensional manifold: take $U = \mathbb{R}^n$!

(ii) S^n is an n-dimensional manifold.

(iii) The torus T^2 and the Klein bottle K are surfaces.

Definition 6.3.

1. A surface M is **orientable** if there is no subspace $N \subset M$ which is homeomorphic to a Möbius band.
2. A surface M is **nonorientable** if there is a subspace $N \subset M$ which is homeomorphic to a Möbius band.

Example 6.4.

1. The plane $\mathbb{R}^2$, the sphere S^2 and the torus T^2 are orientable. (This is not obvious).
2. The Klein bottle K and the projective plane $\mathbb{RP}^2$ are not orientable. (Although not exactly obvious, it is much easier to verify that there is a subspace which is homeomorphic to a Möbius band than that there is no such subspace).

Remark 6.5. A **surface with boundary** $(M, \partial M)$ is a space M together with a subspace ∂M (the **boundary**) such that $M \backslash \partial M$ is a surface (the **interior**), and every $x \in \partial M$ has an open neighbourhood $U \subset M$ such that U is homeomorphic to the upper half-plane $\mathbb{R}^2_+ = \{(y, z) \in \mathbb{R}^2 \,|\, z \geq 0\}$, with $U \cap \partial M$ homeomorphic to the real line.

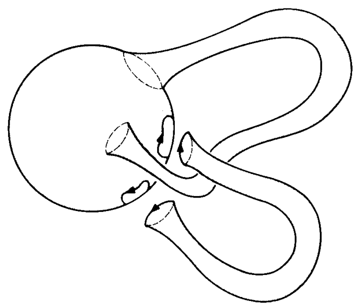
For example, (D^2, S^1) is a surface with boundary. Also, The Möbius band M defines a surface with boundary (M, S^1).

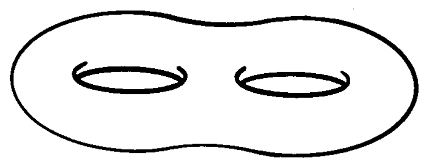
Definition 6.6. Let $g \geq 0$ be an integer. The **sphere with** g **handles** $H(g)$ (or the **g-holed torus**) is the orientable surface obtained from the 2-sphere S^2 by punching out $2g$ disjoint copies of D^2 and joining up the $2g$ holes by g **handles**, each a copy of the cylinder $S^1 \times [0, 1]$.

Example 6.7. 1. $H(0) = S^2$.

2. $H(1) = T^2 = S^1 \times S^1$, the 2-torus.

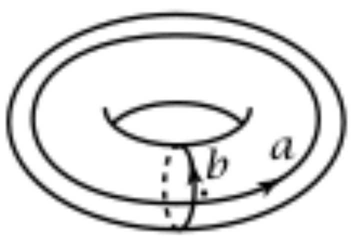
3. Here are two pictures of $H(2)$ (taken from the books of Armstrong and Stillwell):

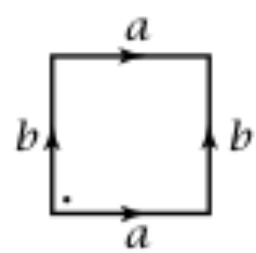


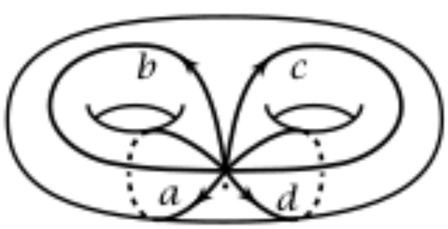


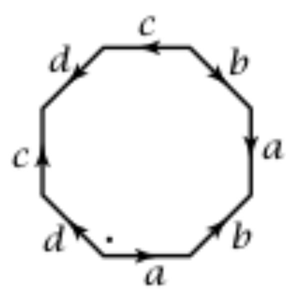
4. Here are the orientable surfaces $H(g)$ of genus $g = 0, 1, 2, 3, \ldots$:

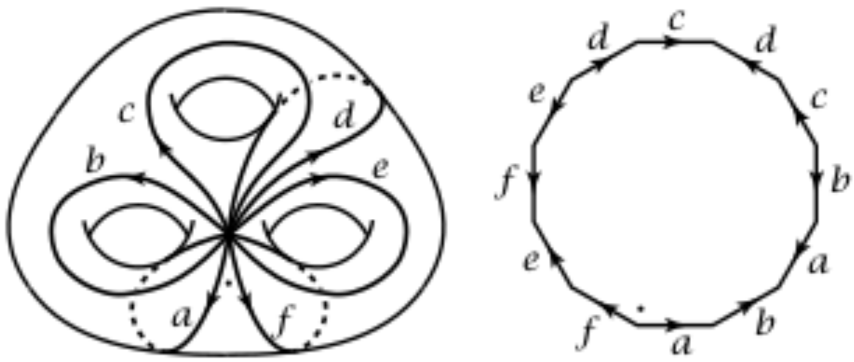
Poincaré's fundamental polygons give us a way to study surfaces, by describing them as quotient spaces of polygons with respect to an equivalence relation on the boundary. Here are the diagrams for $H(1), H(2), H(3)$, taken from Hatcher's book:



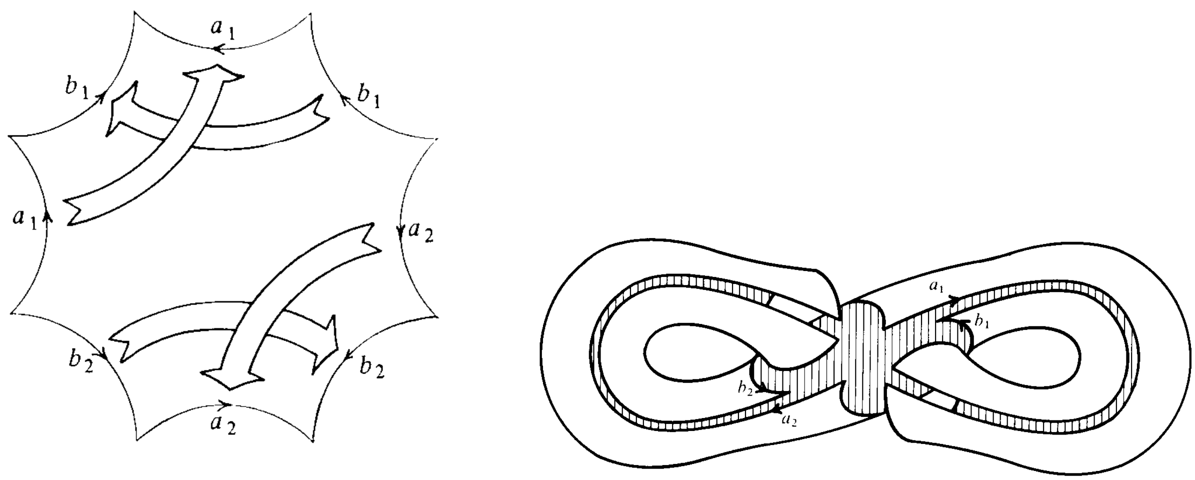








Here are pictures, taken from Stillwell's book, showing how the correspondence works for $H(2)$:



There is an algebraic convention for summarising the equivalence relations. Starting from the dot, the expressions for $H(1)$ and $H(2)$ as $aba^{-1}b^{-1}$ and $aba^{-1}b^{-1}cdc^{-1}d^{-1}$.

In this vein, each space $H(g)$ can be obtained as a quotient space of a $(4g)$-gon, where the equivalence relation on the boundary corresponds to the expression

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}.$$

These descriptions of the equivalence relations for the fundamental polygons are very closely related to the fundamental groups of these surfaces!

If we puncture $H(g)$, can you see what the result is homotopy equivalent to?

Definition 6.8. Let $g \geq 1$ be an integer. The **sphere with** g **cross-caps** $M(g)$ is the nonorientable surface obtained from the 2-sphere S^2 by punching out g disjoint copies of D^2 and replacing each of the g holes by glueing in g copies of the Möbius band.

There also exist fundamental polygons for these non-orientable surfaces. The space $M(g)$ can be obtained as a quotient space of a $(2g)$-gon, with the equivalence relation on the boundary corresponding to the expression

$$a_1 a_1 a_2 a_2 \cdots a_g.$$

If we puncture $M(g)$, can you see what the result is homotopy equivalent to?

Theorem 6.9 (Classification theorem for compact surfaces (non-examinable))**.**

1. *Every connected compact orientable surface M is homeomorphic to $H(g)$ for a unique $g \geq 0$.*

2. *Every connected compact nonorientable surface M is homeomorphic to $M(g)$ for a unique $g \geq 1$.*

3. *Connected compact surfaces M, M' are homeomorphic if and only if there exists a group isomorphism $\pi_1(M) \cong \pi_1(M')$.*

Proof. See Armstrong (pp. 16–18, Chapter 7). □

Example 6.10.

1. A connected compact surface M is homeomorphic to S^2 if and only if $\pi_1(M) = \{1\}$.

2. A connected compact surface M is homeomorphic to $\mathbb{RP}^2$ if and only if $\pi_1(M) = \mathbb{Z}_2$.

Every connected compact surface M is homeomorphic to either $H(g)$ for some $g \geq 0$ (if M is orientable) or to $M(g)$ for some $g \geq 1$ (if M is nonorientable).

Definition 6.11. The number g is the **genus** of M.

Roughly speaking, g is the number of 'holes' in M.
How does the fundamental group $\pi_1(M)$ determine the genus?

Definition 6.12.

1. The **commutator** of any two elements $a, b \in G$ in a group G is

$$[a, b] \ = \ aba^{-1}b^{-1} \in G \ .$$

2. The **abelianisation** of a group G is the abelian quotient group

$$G^{ab} \ = \ G/[G, G] \ ,$$

with $[G, G] \triangleleft G$ the normal subgroup generated by the commutators $[a, b]$ $(a, b \in G)$.

The appendix of Armstrong (pp. 241–243) is a brief account of the presentation of groups in terms of generators and relations. For a more detailed account see pp.40–51 of Stillwell's **Classical topology and combinatorial group theory** (51.57 Sti).

Proposition 6.13 (Non-examinable)**.** *We have:*

1. *The abelianisation of* $\pi_1(H(g))$ *is the free abelian group of rank* $2g$

$$\pi_1(H(g))^{ab} \; = \; \mathbb{Z}^{2g} \; .$$

2. *The abelianisation of* $\pi_1(M(g))$ *is the quotient of the free abelian group* $\mathbb{Z}^g$ *by the cyclic subgroup generated by* $(2, 2, \ldots, 2)$

$$\pi_1(M(g))^{ab} \; = \; \mathbb{Z}^g/\{(2, 2, \ldots, 2)\} \; \cong \; \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2 \; .$$

Proof. See Armstrong p.168 and Stillwell, pp.82–84. □

Note that Proposition 6.13 only describes the abelianisations of the fundamental groups. We will not describe the fundamental groups themselves until the end of the course.

Lecture 10

7 The fundamental group of S^n is $\pi_1(S^n) = \{1\}$ for $n \geqslant 2$

For any $n \geqslant 1$, a path $\alpha : I \to S^n$ can be space-filling, i.e. onto, with $\alpha(I) = S^n$ (Peano, 1890). We will show that every path $\alpha : I \to S^n$ is homotopic rel $\{0,1\}$ to the concatenation of paths which are not onto. For $n \geqslant 2$ it follows that α itself is homotopic rel $\{0,1\}$ to a path which is not onto, which suffices to prove that $\pi_1(S^n) = \{1\}$ for $n \geqslant 2$. This is definitely not true for $X = S^1$, but at least every path $\alpha : I \to S^1$ is homotopic rel $\{0,1\}$ to the concatenation of paths which are not onto, allowing the computation $\pi_1(S^1) = \mathbb{Z}$ (in the next section).

Lemma 7.1. *The spaces*

$$\begin{aligned} U_+ &= S^n - \{(0,\dots,0,1)\} = \{(x_1,x_2,\dots,x_{n+1}) \in S^n \,|\, x_{n+1} \neq 1\}, \\ U_- &= S^n - \{(0,\dots,0,-1)\} = \{(x_1,x_2,\dots,x_{n+1}) \in S^n \,|\, x_{n+1} \neq -1\} . \end{aligned}$$

are both homeomorphic to $\mathbb{R}^n$.

Proof (non-examinable). Explicit homeomorphisms are given by the stereographic projection maps

$$\begin{aligned} \phi_+ &: U_+ \to \mathbb{R}^n \; ; \; x = (x_1,x_2,\dots,x_{n+1}) \mapsto \phi_+(x) = \frac{1}{1-x_{n+1}}(x_1,x_2,\dots,x_n) \; , \\ \phi_- &: U_- \to \mathbb{R}^n \; ; \; x = (x_1,x_2,\dots,x_{n+1}) \mapsto \phi_-(x) = \frac{1}{1+x_{n+1}}(x_1,x_2,\dots,x_n) \; . \end{aligned}$$

(The formula for $\phi_+(x)$ is obtained by vector analysis, from the requirement that the vectors $(0,\dots,0,1)$, x, $(\phi_+(x),0) \in \mathbb{R}^{n+1}$ be collinear:

$$(\phi_+(x),0) = sx + (1-s)(0,\dots,0,1) \text{ with } s = \frac{2}{1-x_{n+1}} \; .$$

Similarly for $\phi_-(x)$.) Their inverses are

$$\begin{aligned} (\phi_+)^{-1}(a_1,\dots,a_n) &= (2a_1,2a_2,\dots,2a_n,\textstyle\sum a_i^2 - 1)/(1+\sum_i a_i^2) \\ (\phi_-)^{-1}(a_1,\dots,a_n) &= (2a_1,2a_2,\dots,2a_n,1-\textstyle\sum a_i^2)/(1+\sum_i a_i^2). \end{aligned}$$

□

Proposition 7.2. *(i) Every map $\alpha : [s_0,s_1] \to \mathbb{R}^n$ is homotopic rel $\{s_0,s_1\}$ to a linear map*

$$\beta : [s_0,s_1] \to \mathbb{R}^n \; ; \; s \mapsto \frac{(s_1-s)\alpha(s_0) + (s-s_0)\alpha(s_1)}{s_1-s_0} \; .$$

with image the straight line segment from $\alpha(s_0)$ to $\alpha(s_1) \in \mathbb{R}^n$

$$\beta([s_0,s_1]) = [\alpha(s_0),\alpha(s_1)] \subset \mathbb{R}^n \; .$$

(ii) For $n \geqslant 2$ every path $\alpha : I \to S^n$ is homotopic rel $\{0,1\}$ to a path $\beta : I \to S^n$ which is not onto.

Proof (details non-examinable). (i) Straightforward: define a rel $\{0,1\}$ homotopy

$$\gamma \ : \ \alpha \ \simeq \ \beta \ : \ I \to \mathbb{R}^n$$

by a straight line homotopy

$$\gamma \ : \ I \times I \to \mathbb{R}^n \ ; \ (s,t) \mapsto (1-t)\alpha(s) + t\beta(s) \ .$$

(ii) Let $\mathcal{U} = \{U_+, U_-\}$ be the open cover of S^n

$$S^n \ = \ U_+ \cup U_-$$

defined by the complements of two antipodal points.

The unit line $I = [0,1]$ is a compact metric space and the open subsets $\{\alpha^{-1}(U_+), \alpha^{-1}(U_-)\}$ define an open cover $\mathcal{F}$ of I. For any integer $N > 0$ define a sequence $t_0, \dots, t_N$ by $t_i = \frac{i}{N}$ so the intervals $[t_i, t_{i+1}]$ have diameter $1/N$ and subdivide I.

By the Lebesgue Covering Lemma (GT Lemma B7.9) there is an $N > 0$ so large that each $[t_i, t_{i+1}]$ is contained in some $U \in \mathcal{F}$, and so $\alpha[t_i, t_{i+1}]$ is contained in either U_+ or U_- (possibly both).

If $\alpha[t_i, t_{i+1}] \subset U_+$, then as in (i), the composite

$$\phi_+ \circ \alpha|_{[t_i, t_{i+1}]} \ : \ [t_i, t_{i+1}] \to \mathbb{R}^n$$

with the stereographic projection map is homotopic rel $\{t_i, t_{i+1}\}$ to a linear map.

Applying ϕ_+^{-1}, this gives us a homotopy γ_i rel $\{t_i, t_{i+1}\}$ between $\alpha|_{[t_i, t_{i+1}]}$ and a map $\beta_i : [t_i, t_{i+1}] \to U_+$ for which $\phi_+ \circ \beta_i$ is linear.

On the other hand, if $\alpha[t_i, t_{i+1}] \not\subset U_+$, then $\alpha[t_i, t_{i+1}] \subset U_-$, and there is a homotopy γ_i rel $\{t_i, t_{i+1}\}$ between $\alpha|_{[t_i, t_{i+1}]}$ and a path β_i which becomes a straight line segment under the stereographic projection ϕ_-.

Concatenating these gives a 'piecewise-linear' path

$$\beta \ : \ I \to S^n \ ; \ s \mapsto \beta_i(s) \text{ if } t_i \leqslant s \leqslant t_{i+1}$$

and a 'piecewise-linear' homotopy rel $\{0,1\}$

$$\gamma : \ I \times I \to S^n \ ; \ (s,t) \mapsto \gamma_i(s,t) \ \ (\text{if } t_i \leqslant t \leqslant t_{i+1}) \ .$$

between α and β.

It remains to show that β is not onto. Consider the union of the inverse stereographic projections

$$f \ = \ (\phi_+)^{-1} \cup (\phi_-)^{-1} \ : \ \mathbb{R}^n \cup \mathbb{R}^n \to S^n \ .$$

For $n \geqslant 2$, S^n is not the image $f(C_1 \cup C_2)$ with $C_1, C_2 \subset \mathbb{R}^n$ subsets defined by the union of a finite collection of line segments. (One way to see this is that $C_1 \cup \phi_+(\phi_-)^{-1}(C_2 \setminus \{0\})$ is a finite union of lines and circles.) $\square$

Note that the proof of Proposition 7.2 (ii) is quite false for $n = 1$: S^1 is a union of two doubly-overlapping arcs, so the map

$$f \;=\; (\phi_+)^{-1} \cup (\phi_-)^{-1} \;:\; \mathbb{R} \cup \mathbb{R} \to S^1$$

can send a union of two line segments (one in each $\mathbb{R}$) onto S^1.

Theorem 7.3. *$\pi_1(S^n) = \{1\}$ for $n \geqslant 2$.*

Proof. Since S^n is path-connected there is no need to specify a base point, although it is conventional to select $1 = (1, 0, \ldots, 0) \in S^n$ as the base point. Represent an arbitrary element $[\alpha] \in \pi_1(S^n, 1)$ by a path $\alpha : I \to S^n$ such that

$$\alpha(0) \;=\; \alpha(1) \;=\; 1 \in S^n \;.$$

By Proposition 7.2(ii), α is homotopic rel $\{0, 1\}$ to a path $\beta : I \to S^n$ which is not onto. Choose $x \in S^n \backslash \beta(I)$, noting that $x \neq 1 \in S^n$ and $\beta(I) \subseteq S^n \backslash \{x\}$. The inclusion $i : S^n \backslash \{x\} \to S^n$ induces a homomorphism of groups

$$i_* \;:\; \pi_1(S^n \backslash \{x\}, 1) \to \pi_1(S^n, 1)$$

such that

$$i_*[\beta] \;=\; [i\beta] \;=\; [\alpha] \in \pi_1(S^n, 1) \;.$$

But $S^n \backslash \{x\}$ is homeomorphic to $\mathbb{R}^n$, so it is contractible (= homotopy equivalent to a point) and

$$\pi_1(S^n \backslash \{x\}, 1) \;=\; \{1\} \;.$$

Thus $[\beta] = 1 \in \pi_1(S^n \backslash \{x\}, 1)$ and

$$[\alpha] \;=\; i_*(1) \;=\; 1 \in \pi_1(S^n, 1) \;,$$

so that $\pi_1(S^n, 1) = \{1\}$. □

Corollary 7.4. *For $n \geqslant 3$*

$$\pi_1(\mathbb{R}^n \backslash \{0\}) \;=\; \{1\} \;.$$

Proof. By Example 5.17(ii) $\pi_1(\mathbb{R}^n \backslash \{0\}) \cong \pi_1(S^{n-1})$, for any $n \geqslant 1$. □

Lecture 11

8 The fundamental group of the circle is $\pi_1(S^1) = \mathbb{Z}$

We shall prove that $\pi_1(S^1) = \mathbb{Z}$ by defining and studying the homotopy theoretic properties of the degree of a loop $\omega : S^1 \to S^1$, which is an integer degree$(\omega) \in \mathbb{Z}$ counting the number of anticlockwise turns of ω.

Regard the circle as the space of complex numbers of unit length

$$S^1 \;=\; \{z \in \mathbb{C} \,|\, |z| = 1\} \;.$$

Proposition 8.1. *For each $N \in \mathbb{Z}$ let*

$$\omega_N \;:\; S^1 \to S^1 \;;\; z \mapsto z^N$$

be the map winding the circle around itself N times in the anticlockwise direction, with $\omega_N(1) = 1$. The function

$$\mathbb{Z} \to \pi_1(S^1) \;;\; N \mapsto \omega_N$$

is a homomorphism of groups.

Proof. It suffices to show that for any $N\, in \mathbb{Z}$ we have $\omega_N = (\omega_1)^N$, since that automatically implies $\omega_{M+N} = (\omega_1)^M \cdot (\omega_1)^N$.

The composite of the projection

$$p \;:\; I \to S^1 \;;\; t \mapsto e^{2\pi i t} \;=\; \cos(2\pi t) + i\sin(2\pi t)$$

and ω_N is the closed path

$$\alpha_N \;=\; \omega_N \circ p \;:\; I \to S^1 \;;\; t \mapsto e^{2\pi i N t} \;=\; \cos(2\pi N t) + i\sin(2\pi N t)$$

with $\alpha_N(0) = \alpha_N(1) = 1$.

For $N \geqslant 1$,

$$\alpha_N \;=\; \alpha_1 \bullet_{1/N} \alpha_1 \bullet_{2/N} \alpha_1 \cdots \bullet_{(N-1)/N} \alpha_1 \;:\; I \to S^1$$

in the terminology defined above, corresponding to the partition of I into N (equal) subintervals

$$I \;=\; [0, 1/N] \cup [1/N, 2/N] \cup \cdots \cup [(N-1)/N, 1] \;.$$

By Proposition 4.7, the rel $\{0, 1\}$ homotopy class of a concatenation is independent of the choice of partition of $I = [0, 1]$, so

$$\omega_N \;=\; [\alpha_N] = [\alpha_1]^N = (\omega_1)^N \in \pi_1(S^1) \;.$$

To generalise this to negative integers, we note that α_{-N} is the reverse of α_N

$$\alpha_{-N} = \overline{\alpha}_N : I \to S^1 \; ; \; t \mapsto e^{-2\pi i N t} = e^{2\pi i N(1-t)},$$

so

$$\omega_{-N} = (\omega_N)^{-1} = ((\omega_1)^N)^{-1} = (\omega_1)^{-N}.$$

□

We shall prove that $\mathbb{Z} \to \pi_1(S^1); N \mapsto \omega_N$ is an isomorphism of groups, using the topology of 'angle maps' to define the inverse isomorphism

$$\text{degree} : \pi_1(S^1) \to \mathbb{Z} \; ; \; \omega \mapsto \text{degree}(\omega)$$

which counts the number of times a loop $\omega : S^1 \to S^1$ winds around S^1 in the anticlockwise sense. For differentiable ω the degree can be computed quite easily by counting (with signs) the number of points in $\omega^{-1}(z)$ for any $z \in S^1$ with $\omega^{-1}(z)$ finite (see Remark 8.9 below).

In general ω is not differentiable, although it is homotopic rel $\{1\}$ to a differentiable loop, so we take a different approach. Our definition of degree will make use of the projection

$$p : \mathbb{R} \to S^1 \; ; \; x \mapsto e^{2\pi i x}$$

and its key properties:

(i) For every $x \in \mathbb{R}$ the open subset $U = (x, x+1) \subset \mathbb{R}$ is such that p restricts to a homeomorphism

$$p| : U \to p(U) \; ; \; u \mapsto p(u)$$

with $p(U) = S^1 \backslash \{e^{2\pi i x}\}$.

(ii) p is onto, with the inverse image of $y = p(x) \in S^1$ given by

$$p^{-1}(y) = \{x + n \,|\, n \in \mathbb{Z}\} \subset \mathbb{R} \, .$$

Any real numbers $x_0, x_1 \in \mathbb{R}$ with $p(x_0) = p(x_1) \in S^1$ differ by an integer $x_0 - x_1 \in \mathbb{Z} \subset \mathbb{R}$.

We shall show that for every path $\alpha : I \to S^1$ there exists a 'lift' path $\theta : I \to \mathbb{R}$ such that $\alpha = p\theta : I \to S^1$. If α is closed

$$p\theta(0) = \alpha(0) = \alpha(1) = p\theta(1) \in S^1 \; ,$$

allowing the degree of α to be defined by

$$\text{degree}(\alpha) = \theta(1) - \theta(0) \in \mathbb{Z} \; .$$

Definition 8.2. An **angle map** or **lift** of a map $\alpha : [t_0, t_1] \to S^1$ is a map $\theta : [t_0, t_1] \to \mathbb{R}$ such that

$$\alpha(t) \;=\; e^{2\pi i\theta(t)} \in S^1 \quad (t \in [t_0, t_1]) \;,$$

with $\theta(t_0) = \theta_0 \in \mathbb{R}$ the **initial angle** and $\theta(t_1) = \theta_1 \in \mathbb{R}$ the **terminal angle**.

It is an essential ingredient of the definition that the angle map $\theta : [t_0, t_1] \to \mathbb{R}$ be continuous, i.e. that θ be a map. The angle map fits into a commutative diagram

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[t_0, t_1]} \arrow[ur, "\theta"] \arrow[r, "\alpha"'] & S^1 \end{tikzcd}$$

Proposition 8.3. *Given a map $\alpha : [t_0, t_1] \to S^1$ and $\theta_0 \in \mathbb{R}$ such that*

$$\alpha(t_0) \;=\; e^{2\pi i\theta_0} \in S^1$$

there is a unique angle map $\theta : [t_0, t_1] \to \mathbb{R}$ for α with initial angle

$$\theta(t_0) \;=\; \theta_0 \in \mathbb{R} \;.$$

Proof (details non-examinable). Consider first the special case when α is not onto. Choose an $x \in \mathbb{R}$ such that

$$e^{2\pi ix} \in S^1\backslash\alpha([t_0, t_1]) \;,\; \theta_0 \in (x, x+1) \;,$$

and note that the restriction of p defines a homeomorphism

$$q \;=\; p| \;:\; (x, x+1) \to S^1\backslash\{e^{2\pi ix}\} \;;\; y \mapsto e^{2\pi iy}$$

such that

$$q(\theta_0) \;=\; e^{2\pi i\theta_0} \;=\; \alpha(t_0) \in S^1 \;.$$

The composite

$$\theta \;:\; [t_0, t_1] \xrightarrow{\alpha} S^1\backslash\{e^{2\pi ix}\} \xrightarrow{q^{-1}} (x, x+1) \xrightarrow{incl.} \mathbb{R}$$

is an angle map θ with $\theta(t_0) = \theta_0 \in \mathbb{R}$.

If $\theta' : [t_0, t_1] \to \mathbb{R}$ is another angle map such that $\theta'(t_0) = \theta_0 \in \mathbb{R}$ then $\theta'([t_0, t_1]) \subset (x, x+1)$ and it is immediate from the commutative triangle

$$\begin{tikzcd} & (x, x+1) \arrow[d, "q"] \\ {[t_0, t_1]} \arrow[ur, "\theta'"] \arrow[r, "\alpha"'] & S^1\backslash\{e^{2\pi ix}\} \end{tikzcd}$$

that

$$\theta' \;=\; q^{-1} \circ \alpha \;=\; \theta \;:\; [t_0, t_1] \to \mathbb{R} \;.$$

Next, consider the general case of a map $\alpha : [t_0, t_1] \to S^1$ which may not be onto, with $t_0 < t_1$. As in the proof of Proposition 7.2 (ii) define an open cover $\mathcal{U} = \{U_+, U_-\}$ of S^1 by

$$U_+ \;=\; S^1\backslash\{1\}\ ,\ U_- \;=\; S^1\backslash\{-1\}\ ,$$

so that $\alpha^{-1}\mathcal{U} = \{\alpha^{-1}(U_+), \alpha^{-1}(U_-)\}$ is an open cover of the compact metric space $[t_0, t_1]$.

By the Lebesgue Covering Lemma (GT Lemma B7.9) there exists an integer $N \geqslant 1$ so large that for

$$s_j := t_0 + \tfrac{j(t_1 - t_0)}{N},$$

the N subintervals $[s_j, s_{j+1}] \subset [t_0, t_1]$ have diameter $(t_1 - t_0)/N$ so small that

$$[s_j, s_{j+1}] \subset \alpha^{-1}(U_+) \text{ or } \alpha^{-1}(U_-) \ \ (0 \leqslant j \leqslant N-1)\ .$$

None of the restrictions $\alpha|_{[s_j, s_{j+1}]}$ is onto. Apply the special case to obtain an angle map $\theta|_{[s_0, s_1]} : [s_0, s_1] \to \mathbb{R}$ for $\alpha|_{[s_0, s_1]}$ with initial angle θ_0. Next, apply the special case to obtain an angle map $\theta|_{[s_1, s_2]} : [s_1, s_2] \to \mathbb{R}$ for $\alpha|_{[s_1, s_2]}$ with initial angle $\theta|_{[s_0, s_1]}(s_1) \in \mathbb{R}$ the previous terminal angle. And so on, until each $\alpha|_{[s_j, s_{j+1}]}$ has an angle map $\theta|_{[s_j, s_{j+1}]} \to \mathbb{R}$ with initial angle $\theta|_{[s_j, s_{j+1}]}(s_j) = \theta|_{[s_{j-1}, s_j]}(s_j) \in \mathbb{R}$ the previous terminal angle.

Piecing together the angle maps $\theta|_{[s_j, s_{j+1}]}$ for $\alpha|_{[s_j, s_{j+1}]}$, we obtain an angle map

$$\theta \ :\ [t_0, t_1] \to \mathbb{R}\ ;\ t \mapsto \theta|_{[s_j, s_{j+1}]}(t) \text{ if } s_j \leqslant t \leqslant s_{j+1}$$

for α with initial angle θ_0. $\square$

Lecture 12

Definition 8.4. (i) The **degree** of a closed path $\alpha : I \to S^1$ is the difference between the terminal and initial angles in any angle map $\theta : I = [0,1] \to \mathbb{R}$ for α

$$\text{degree}(\alpha) \;=\; \theta(1) - \theta(0) \in \mathbb{Z}\ ,$$

using $\alpha(0) = \alpha(1) \in S^1$ to ensure that this is an integer.

(ii) The **degree** of a loop $\omega : S^1 \to S^1$ is

$$\text{degree}(\omega) \;=\; \text{degree}(\alpha) \in \mathbb{Z}$$

with α the closed path

$$\alpha \;:\; I \to S^1 \;;\; t \mapsto \omega(e^{2\pi i t})\ .$$

Example 8.5. The closed path defined for $N \in \mathbb{Z}$ by

$$\alpha_N \;:\; I \to S^1 \;;\; t \mapsto e^{2\pi i N t}$$

has angle map

$$\theta_N \;:\; I \to \mathbb{R} \;;\; t \mapsto Nt$$

so that the degree is

$$\text{degree}(\alpha_N) \;=\; \theta_N(1) - \theta_N(0) \;=\; N \in \mathbb{Z} \subset \mathbb{R}\ .$$

The loop

$$\omega_N \;:\; S^1 \to S^1 \;;\; z = e^{2\pi i t} \mapsto z^N = e^{2\pi i N t}$$

thus has

$$\text{degree}(\omega_N) \;=\; N \in \mathbb{Z}\ .$$

Example 8.6. The antipodal map is a loop

$$\omega \;:\; S^1 \to S^1 \;;\; z \mapsto -z\ .$$

The corresponding closed path

$$\alpha \;:\; I \to S^1 \;;\; t \mapsto (\cos(2\pi(t+\tfrac{1}{2})), \sin(2\pi(t+\tfrac{1}{2})))$$

has angle map

$$\theta \;:\; I \to \mathbb{R} \;;\; t \mapsto t + \tfrac{1}{2}$$

so

$$\text{degree}(\omega) \;=\; \text{degree}(\alpha) \;=\; \theta(1) - \theta(0) \;=\; 1 \in \mathbb{Z}\ .$$

We shall also need angle maps for homotopies:

Proposition 8.7. *(i) Let $\alpha, \beta : [t_0, t_1] \to S^1$ be paths related by a homotopy $\gamma : \alpha \simeq \beta : [t_0, t_1] \to S^1$. Given an initial angle θ_0 for α there exists a unique homotopy $\psi : \theta \simeq \phi : [t_0, t_1] \to \mathbb{R}$ such that*

$$\gamma(s,t) \;=\; e^{2\pi i\psi(s,t)} \in S^1 \;,\; \psi(t_0,0) \;=\; \theta_0 \in \mathbb{R} \; (s \in [t_0,t_1],\, t \in I) \;,$$

with

$$\theta \;:\; [t_0,t_1] \to \mathbb{R} \;;\; s \mapsto \psi(s,0) \;,\; \phi \;:\; [t_0,t_1] \to \mathbb{R} \;;\; s \mapsto \psi(s,1)$$

angle maps for α, β

$$\alpha(s) \;=\; e^{2\pi i\theta(s)} \;,\; \beta(s) \;=\; e^{2\pi i\phi(s)} \quad (s \in [t_0,t_1]) \;.$$

(ii) If

$$\alpha(t_0) \;=\; \beta(t_0) \;,\; \alpha(t_1) \;=\; \beta(t_1) \in S^1$$

and $\gamma : \alpha \simeq \beta$ is a homotopy rel $\{t_0, t_1\}$ then ψ is a homotopy rel $\{t_0, t_1\}$.

Proof (details non-examinable). (i) As for angle maps in Proposition 8.3. First consider the special case when γ is not onto: choose $x \in \mathbb{R}$ such that

$$e^{2\pi ix} \in S^1 \backslash \gamma([t_0,t_1] \times I) \;,\; \theta_0 \in (x, x+1) \;,$$

and define $\psi = q^{-1} \circ \gamma$ to fit into a commutative triangle

$$\begin{tikzcd} & (x,x+1) \arrow[d, "q"] \\ {[t_0,t_1] \times I} \arrow[ur, "\psi"] \arrow[r, "\gamma"] & S^1\backslash\{e^{2\pi ix}\} \end{tikzcd}$$

with $\psi(t_0, 0) = \theta_0 \in \mathbb{R}$.

Next, consider the general case of a homotopy $\gamma : [t_0, t_1] \times I \to S^1$ which may not be onto, with $t_0 < t_1$. Define an open cover $\mathcal{U} = \{U_+, U_-\}$ of S^1 by

$$U_+ \;=\; S^1\backslash\{1\} \;,\; U_- \;=\; S^1\backslash\{-1\} \;,$$

so that $\gamma^{-1}\mathcal{U} = \{\gamma^{-1}(U_+), \gamma^{-1}(U_-)\}$ is an open cover of the compact metric space $[t_0, t_1] \times I$.

By GT Lemma B7.9 there exists a Lebesgue number $N \geqslant 1$ so large that the N^2 subrectangles

$$[s_j, s_{j+1}] \times [\tfrac{k}{N}, \tfrac{k+1}{N}] \subset [t_0, t_1] \times I,$$

for $s_j := t_0 + \dfrac{j(t_1 - t_0)}{N}$ have diameter $\sqrt{1 + (t_1 - t_0)^2}/N$ so small that

$$[s_j, s_{j+1}] \times [\tfrac{k}{N}, \tfrac{k+1}{N}] \subset \alpha^{-1}(U_+) \text{ or } \alpha^{-1}(U_-) \quad (0 \leqslant j, k \leqslant N-1) \;.$$

None of the restrictions

$$\gamma|_{[s_j,s_{j+1}]\times[\frac{k}{N},\frac{k+1}{N}]} : [s_j,s_{j+1}]\times[\tfrac{k}{N},\tfrac{k+1}{N}] \to S^1 \ \ (0 \leqslant j,k \leqslant N-1)$$

is onto, so we may apply the special case N^2 times and piece them together to obtain ψ.

(ii) A special case of (i), noting that the functions

$$I \to \mathbb{Z} \ ; \ t \mapsto \psi(t_i,t) - \psi(t_i,0) \ \ (i = 0,1)$$

are continuous, taking the value 0 at $t = 0$, so that

$$\psi(t_i,t) \ = \ \psi(t_i,0) \in \mathbb{R} \ (i = 0,1, \ t \in [0,1])$$

and ψ is a homotopy rel $\{t_0,t_1\}$. □

Theorem 8.8. *The fundamental group of the circle S^1 is the infinite cyclic group, with inverse group homomorphisms*

$$\begin{aligned} &\mathbb{Z} \to \pi_1(S^1) \ ; \ N \mapsto \omega_N \ , \\ &\text{degree} \ : \ \pi_1(S^1) \to \mathbb{Z} \ ; \ \omega \mapsto \text{degree}(\omega) \ . \end{aligned}$$

Proof. In the first instance, let us check that $\text{degree}(\omega) \in \mathbb{Z}$ depends only on the rel 1 homotopy class of a loop $\omega : S^1 \to S^1$ based at $1 \in S^1$, i.e. that $\pi_1(S^1) \to \mathbb{Z}$ is well-defined.

Suppose that $\alpha, \beta : I \to S^1$ are closed paths such that

$$\alpha(0) \ = \ \beta(0) \ = \ \alpha(1) \ = \ \beta(1) \ = \ 1 \in S^1$$

and

$$[\alpha] \ = \ [\beta] \ = \ [\omega] \in \pi_1(S^1) \ .$$

There exists a rel $\{0,1\}$ homotopy $\gamma : I \times I \to S^1$ with the closed paths

$$\gamma_t \ : \ I \to S^1 \ ; \ s \mapsto \gamma(s,t) \ \ (s \in I)$$

such that

$$\gamma_t(0) \ = \ \gamma_t(1) \ = \ 1 \ , \ \gamma_0 \ = \ \alpha \ , \ \gamma_1 \ = \ \beta \ .$$

By Proposition 8.7 there exists an angle map $\psi : I \times I \to \mathbb{R}$ for γ such that

$$\gamma_t(s) \ = \ \gamma(s,t) \ = \ e^{2\pi i\psi(s,t)} \in S^1 \ \ (s,t \in I)$$

with $\psi(0,0) = 0$. Thus for each $t \in I$ there is defined an angle map

$$\psi_t \ : \ I \to \mathbb{R} \ ; \ s \mapsto \psi(s,t)$$

for γ_t, and

$$\text{degree}(\gamma_t) \ = \ \psi_t(1) - \psi_t(0) \in \mathbb{Z} \ .$$

(In fact, $\psi_t(0) = 0$). The degree function

$$I \to \mathbb{Z} \; ; \; t \mapsto \text{degree}(\gamma_t)$$

is continuous, I is connected, and $\mathbb{Z}$ is disconnected. The degree function must therefore be constant, and

$$\text{degree}(\alpha) \;=\; \text{degree}(\gamma_0) \;=\; \text{degree}(\gamma_1) \;=\; \text{degree}(\beta) \in \mathbb{Z} \; .$$

Next, we verify that degree : $\pi_1(S^1) \to \mathbb{Z}$ is a group homomorphism: if $\alpha, \beta : I \to S^1$ are closed paths with angle maps $\theta, \phi : I \to \mathbb{R}$ such that $\theta(0) = \phi(0) = 0$ then

$$\text{degree}(\alpha) \;=\; \theta(1) \; , \; \text{degree}(\beta) \;=\; \phi(1) \in \mathbb{Z} \; .$$

The concatenation $\alpha \bullet \beta : I \to S^1$ has angle map

$$\lambda \; : \; I \to \mathbb{R} \; ; \; t \mapsto \begin{cases} \theta(2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ \theta(1) + \phi(2t-1) & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

The degree of $\alpha \bullet \beta$ is

$$\begin{aligned} \text{degree}(\alpha \bullet \beta) &= \lambda(1) \\ &= \theta(1) + \phi(1) \\ &= \text{degree}(\alpha) + \text{degree}(\beta) \in \mathbb{Z} \; , \end{aligned}$$

so degree is a group homomorphism. For each $n \in \mathbb{Z}$ the standard path α_n defined above has degree$(\alpha_n) = n$, so degree is surjective.

It remains to prove that degree is injective. Let then $\alpha : I \to S^1$ be a closed path with $\alpha(0) = \alpha(1) = 1$, such that degree$(\alpha) = 0 \in \mathbb{Z}$. There exists an angle map $\theta : I \to \mathbb{R}$ for α with $\theta(0) = \theta(1) = 0 \in \mathbb{Z}$, and the map

$$H \; : \; I \times I \to S^1 \; ; \; (s,t) \mapsto e^{2\pi i \theta(s) t}$$

defines a rel $\{0,1\}$ homotopy $H : \alpha_0 \simeq \alpha : I \to S^1$, with

$$\begin{aligned} &H(s,0) \;=\; e^{2\pi i 0} \;=\; 1 \;=\; \alpha_0(s) \; , \; H(s,1) \;=\; e^{2\pi i \theta(s)} \;=\; \alpha(s) \; , \\ &H(0,t) \;=\; e^{2\pi i \theta(0) t} \;=\; 1 \; , \; H(1,t) \;=\; e^{2\pi i \theta(1) t} \;=\; 1 \; . \end{aligned}$$

So $[\alpha] = [\alpha_0] = [e_1] = 1 \in \pi_1(S^1)$ is the identity element, and degree is injective. $\square$

Remark 8.9 (Non-examinable). It is possible to compute degree$(\alpha) \in \mathbb{Z}$ for a differentiable closed path

$$\alpha \; : \; I \to S^1 \; ; \; t \mapsto (\alpha_1(t), \alpha_2(t))$$

as follows. The tangent vector at $t \in I$

$$\alpha'(t) \;=\; (\alpha_1'(t), \alpha_2'(t)) \in \mathbb{R}^2$$

is orthogonal to $\alpha(t)$, as is clear geometrically, or by differentiating

$$\alpha_1(t)^2 + \alpha_2(t)^2 \;=\; 1$$

to obtain

$$2(\alpha_1(t)\alpha_1'(t) + \alpha_2(t)\alpha_2'(t)) \;=\; 2\,\alpha(t)\cdot\alpha'(t) \;=\; 0 \in \mathbb{R}\ .$$

Define a sign

$$\epsilon(t) \;=\; \begin{cases} +1 \\ -1 \\ 0 \end{cases} \text{if} \begin{cases} \alpha_1(t)\alpha_2'(t) > \alpha_2(t)\alpha_1'(t) \\ \alpha_1(t)\alpha_2'(t) < \alpha_2(t)\alpha_1'(t) \\ \alpha_1(t)\alpha_2'(t) = \alpha_2(t)\alpha_1'(t) \end{cases}$$

according to whether $\alpha'(t)$ is pointing anticlockwise/clockwise, or is 0. Then for any $z \in S^1$ such that $\alpha^{-1}(z) \subset I$ is finite

$$\text{degree}(\alpha) \;=\; \sum_{t \in \alpha^{-1}(z)} \epsilon(t) \in \mathbb{Z}\ .$$

Lecture 13

Example 8.10. Consider the Möbius band

$$M \ = \ I \times I/\{(0,t) \sim (1,1-t)\} \ .$$

(i) The inclusion of the middle circle

$$f_1 \ : \ S^1 \to M \ ; \ e^{2\pi i s} \mapsto [s, \tfrac{1}{2}]$$

is a homotopy equivalence, inducing an isomorphism of groups $(f_1)_* : \pi_1(S^1) \to \pi_1(M)$ with inverse $(f_1)_*^{-1} = g_*$ induced by the homotopy inverse

$$g \ : \ M \to S^1 \ ; \ [s,t] \mapsto e^{2\pi i s} \ .$$

(ii) The inclusion of the boundary circle

$$f_2 \ : \ S^1 \to M \ ; \ e^{2\pi i s} \mapsto \begin{cases} (2s, 0) & \text{if } 0 \leqslant s \leqslant \frac{1}{2} \\ (2s-1, 1) & \text{if } \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

is not a homotopy equivalence: the composite

$$gf_2 \ : \ S^1 \to S^1 \ ; \ z = e^{2\pi i s} \mapsto z^2 = e^{4\pi i s}$$

has degree 2, so that the induced homomorphism of groups

$$(f_1)_*^{-1}(f_2)_* \ = \ (gf_2)_* \ = \ 2 \ : \ \pi_1(S^1) = \mathbb{Z} \to \pi_1(S^1) = \mathbb{Z}$$

is not an isomorphism.

Remark 8.11. (i) The definition of degree$(\omega) \in \mathbb{Z}$ works for any loop $\omega : S^1 \to S^1$ (or closed path $\alpha : I \to S^1$), without reference to a base point in S^1. The proof of Theorem 8.8 is easily extended to also prove that the degree depends only on the homotopy class of ω, and that two loops $\omega, \omega' : S^1 \to S^1$ are homotopic if and only if

$$\text{degree}(\omega) \ = \ \text{degree}(\omega') \in \mathbb{Z} \ .$$

(ii) In particular, if a loop $\omega : S^1 \to S^1$ extends to a map $\delta\omega : D^2 \to S^1$ then degree$(\omega) = 0 \in \mathbb{Z}$.

Example 8.12. For any map $f : \mathbb{C} \to \mathbb{C}\backslash\{0\}$ the loop

$$\omega \ : \ S^1 \to S^1 \ ; \ z \mapsto f(z)/\|f(z)\|$$

extends to the map

$$\delta\omega \ : \ D^2 \to S^1 \ ; \ z \mapsto f(z)/\|f(z)\| \ ,$$

identifying $D^2 = \{z \in \mathbb{C} \,|\, \|z\| \leqslant 1\}$. It follows from Remark 8.11 (ii) that degree$(\omega) = 0 \in \mathbb{Z}$.

Definition 8.13. The **winding number around** 0, denoted $W(\alpha) \in \mathbb{Z}$, of a closed path

$$\alpha \ : \ I \to \mathbb{C}\backslash\{0\} \ \ (\alpha(0) = \alpha(1))$$

is defined to be the degree of the loop

$$\omega \ : \ S^1 \to S^1 \ ; \ e^{2\pi i t} \mapsto \tfrac{\alpha(t)}{|\alpha(t)|} \ ,$$

that is

$$W(\alpha) \ = \ \text{degree}(\omega : S^1 \to S^1) \ = \ \theta(1) - \theta(0) \in \mathbb{Z}$$

for any angle map $\theta : I \to \mathbb{R}$ such that

$$\alpha(t) \ = \ |\alpha(t)| e^{2\pi i \theta(t)} \in \mathbb{C}\backslash\{0\} \ \ (t \in I) \ .$$

Remark 8.14 (Non-examinable). (i) For analytic α the winding number can be computed as the complex contour integral in Cauchy's theorem

$$W(\alpha) \ = \ \tfrac{1}{2\pi i} \oint_\omega \tfrac{dz}{z} \ .$$

It is possible to prove directly that the winding number function

$$\pi_1(\mathbb{C}\backslash\{0\}, 1) \to \mathbb{Z} \ ; \ [\alpha] \mapsto W(\alpha)$$

is an isomorphism.

However, a better way would be to use the calculation $\pi_1(S^1) = \mathbb{Z}$ of Theorem 8.8 and the isomorphism $\pi_1(S^1) \cong \pi_1(\mathbb{R}^2\backslash\{0\})$ given by Example 5.17(ii). The inclusion $S^1 \to \mathbb{R}^2\backslash\{0\} = \mathbb{C}\backslash\{0\}$ is a homotopy equivalence with homotopy inverse

$$j \ : \ \mathbb{C}\backslash\{0\} \to S^1 \ ; \ z \mapsto \tfrac{z}{|z|}$$

inducing an isomorphism

$$j_* \ : \ \pi_1(\mathbb{C}\backslash\{0\}) \to \pi_1(S^1) = \mathbb{Z} \ ; \ \alpha \mapsto W(\alpha) \ .$$

(ii) The winding number $W(\alpha) \in \mathbb{Z}$ around 0 of a differentiable closed path $\alpha : I \to \mathbb{C}\backslash\{0\}$ can be calculated by looking at each crossing point of the curve and a radius drawn from $0 \in \mathbb{C}$ to which the curve is not tangent, assigning $+1$ (resp. -1) where the curve is turning anticlockwise (resp. clockwise) and adding up all these $+1$'s and -1's, in the manner of Remark 8.9.

Theorem 8.15. *(The Fundamental Theorem of Algebra) For $n \geqslant 1$, a degree n polynomial with complex coefficients*

$$p(z) \ = \ \sum_{k=0}^{n} a_k z^k \ \ (a_k \in \mathbb{C}, \ a_n \neq 0)$$

has at least one root, an element $z \in \mathbb{C}$ such that $p(z) = 0 \in \mathbb{C}$.

Proof (details non-examinable). Replacing $p(z)$ by $p(z)/a_n$ we can take $a_n = 1$. Assume that $p(z) \neq 0$ for all $z \in \mathbb{C}$. The loop

$$p \ : \ S^1 \to \mathbb{C}\backslash\{0\} \ ; \ z \mapsto p(z)$$

extends to a map

$$\delta p \ : \ D^2 = \{z \in \mathbb{C} \,|\, |z| \leqslant 1\} \to \mathbb{C}\backslash\{0\} \ ; \ z \mapsto p(z)$$

so that $W(p) = 0$ (by Example 8.12). We shall obtain a contradiction to the assumption that $p(z) \neq 0$ for all $z \in \mathbb{C}$ by proving that $W(p) = n \neq 0$. For any $r > 0$ define

$$q_r \ : \ \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\} \ ; \ z \mapsto rz \ .$$

The following analysis ensures that for sufficiently large $r > 0$ the map

$$h \ : \ S^1 \times I \to \mathbb{C} \ ; \ (z,t) \mapsto (1-t)p(rz) + t(rz)^n$$

never takes the value 0, in which case it defines a homotopy

$$h_1 \ : \ pq_r \simeq (q_r)^n \ : \ S^1 \to \mathbb{C}\backslash\{0\}$$

with

$$(q_r)^n \ : \ S^1 \to \mathbb{C}\backslash\{0\} \ ; \ z \mapsto (rz)^n \ .$$

The maps defined for any $r > 0$ by

$$\begin{aligned} h_2 \ &: \ S^1 \times I \to \mathbb{C}\backslash\{0\} \ ; \ (z,t) \mapsto p((1-t+rt)z) \ , \\ h_3 \ &: \ S^1 \times I \to \mathbb{C}\backslash\{0\} \ ; \ (z,t) \mapsto ((1-t+rt)z)^n \end{aligned}$$

are homotopies

$$h_2 \ : \ p \ \simeq \ pq_r \ , \ h_3 \ : \ \omega_n \ \simeq \ (q_r)^n : \ S^1 \to \mathbb{C}\backslash\{0\} \ ,$$

so that there is a concatenation homotopy

$$h_2 \bullet h_1 \bullet -h_3 \ : \ p \ \simeq \ \omega_n \ : \ S^1 \to \mathbb{C}\backslash\{0\}$$

and

$$W(p) \ = \ W(\omega_n) \ = \ n \in \mathbb{Z} \ .$$

Now, for $r > 0$ sufficiently large and any $z \in S^1$

$$|(rz)^n - p(rz)| \ = \ |\sum_{k=0}^{n-1} a_k(rz)^k| \leqslant \sum_{k=0}^{n-1} |a_k| r^k < r^n \ = \ |(rz)^n|$$

and by the triangle inequality

$$\begin{aligned} 0 &< |(rz)^n| - (1-t)|(rz)^n - p(rz)| \\ &\leqslant |(1-t)(p(rz) - (rz)^n) + (rz)^n| \ = \ |(1-t)p(rz) + t(rz)^n| \ . \end{aligned}$$

$\square$

Remark 8.16 (Non-examinable). The image of an injective map $i : S^1 \to \mathbb{R}^2$ is a closed curve

$$K = i(S^1) \subset \mathbb{R}^2 .$$

The **Jordan curve theorem** (Google it!) proves that the complement $\mathbb{R}^2\backslash K$ has two path-components, a one inside K with compact closure, and one outside K with non-compact closure.

The question of whether $x \in \mathbb{R}^2\backslash K$ is inside K or outside K is decided by the degree of the loop

$$\omega_x : S^1 \to S^1 ; \ y \mapsto \frac{i(y) - x}{\|i(y) - x\|}$$

which sends $y \in S^1$ to the unit vector $\omega_x(y) \in S^1 \subset \mathbb{R}^2$ parallel to the directed straight line segment joining x to $i(y) \in K$.

$$[x, i(y)] = \{(1-t)x + ti(y) \,|\, t \in I\} \subset \mathbb{R}^2 .$$

Specifically,

$$|\mathrm{degree}(\omega_x)| = \begin{cases} 1 & \text{if } x \text{ is inside } K \\ 0 & \text{if } x \text{ is outside } K . \end{cases}$$

For any $z \in S^1$ define the ray from x in the direction of z

$$[x, z(= \{x + tz \,|\, t \geqslant 0\} \subset \mathbb{R}^2 ,$$

so that

$$(\omega_x)^{-1}(z) = \{y \in S^1 \,|\, i(y) \in [x, z(\}$$

is the set of points of K which can be "seen" from $x \in \mathbb{R}^2\backslash K$ by looking in the direction z. If i is differentiable and $[x, z($ is not tangent to K, or if K is a polygon and $[x, z($ does not contain any edges of K, then the intersection $[x, z(\cap K$ is finite, and

$$|\mathrm{degree}(\omega_x)| = \begin{cases} 1 & \text{if } [x, z(\cap K \text{ has an odd number of points} \\ 0 & \text{if } [x, z(\cap K \text{ has an even number of points} . \end{cases}$$

In particular, if there exists $z \in S^1$ such that $[x, z(\cap K$ is empty (i.e. you cannot see any point of K in the direction z from x) then x is outside K.

Lecture 14

9 The n-dimensional projective space $\mathbb{RP}^n$

We shall now define an interesting family of n-dimensional manifolds.

Definition 9.1. The *n-dimensional real projective space* is the quotient space of the n-sphere

$$\mathbb{RP}^n \;=\; S^n/{\sim}$$

with $\sim$ the antipodal equivalence relation

$$v \sim w \text{ if either } v \;=\; w \text{ or } v \;=\; -w \in S^n \;.$$

Let $p : S^n \to \mathbb{RP}^n; v \mapsto [v]$ be the natural projection. For every $x \in \mathbb{RP}^n$ the inverse image is the equivalence class

$$p^{-1}(x) \;=\; \{\widetilde{x}, -\widetilde{x}\} \subset S^n$$

consisting of an unordered pair of antipodal points in S^n. By the definition of a quotient space a subset $U \subseteq \mathbb{RP}^n$ is open if and only if $p^{-1}(U) \subseteq S^n$ is open.

A point $x \in \mathbb{RP}^n$ can be viewed in several different ways:

(i) as a subset $\{\widetilde{x}, -\widetilde{x}\} \subset S^n$, with $x = p(\widetilde{x}) = p(-\widetilde{x}) \in \mathbb{RP}^n$,

(ii) as a line $L \subset \mathbb{R}^{n+1}$ through the origin $\mathbf{0}$, with $x = p(\dfrac{v}{\|v\|}) \in \mathbb{RP}^n$ for any $v \neq \mathbf{0} \in L$,

(iii) as an equivalence class of non-zero vectors $v \neq \mathbf{0} \in \mathbb{R}^{n+1}$ with

$$v \sim v' \text{ if } v' = \lambda v \text{ for some } \lambda \neq 0 \in \mathbb{R} \;,$$

and $x = p(\dfrac{v}{\|v\|}) = p(\dfrac{-v}{\|v\|}) \in \mathbb{RP}^n$ for any v in the class.

Remark 9.2. The 1-dimensional real projective space $\mathbb{RP}^1$ is homeomorphic to S^1 via the 'square root' homeomorphism

$$q \;:\; S^1 \to \mathbb{RP}^1 \;;\; z = e^{i\theta} \to \sqrt{z} = \{e^{i\theta/2}, -e^{i\theta/2}\}$$

such that $p = qr : S^1 \to \mathbb{RP}^1$ with $r : S^1 \to S^1; z = e^{i\theta} \mapsto z^2 = e^{2i\theta}$.

The 2-dimensional real projective space $\mathbb{RP}^2$ is also known as the *projective plane*.

Exercise 9.3. (i) Prove that $\mathbb{RP}^2$ is homeomorphic to the quotient space $I^2/{\sim}$ with

$$(x, 0) \sim (1 - x, 1) \;\;,\;\; (0, y) \sim (1, 1 - y) \;\; (0 \leqslant x, y \leqslant 1) \;.$$

(ii) Prove that $\mathbb{RP}^2$ is homeomorphic to the quotient space of the Möbius band obtained by identifying all the points of the boundary circle to a single point.

Exercise 9.4. Prove that $\mathbb{RP}^n$ is the space obtained from $X_1 = \mathbb{R}^n$ by attaching an $(n-1)$-sphere $X_2 = S^{n-1}$ 'at infinity', with antipodal points on the $(n-1)$-sphere identified.)

The n-dimensional projective space $\mathbb{RP}^n$ is a quotient space of a compact path-connected space S^n, so it is also compact and is path-connected We can thus write $\pi_1(\mathbb{RP}^n)$ without specifying a base point.

We shall prove in Theorem 9.12 below that $\pi_1(\mathbb{RP}^n)$ is isomorphic to the infinite cyclic group $\mathbb{Z}$ for $n = 1$, and to the cyclic group $\mathbb{Z}_2$ of order 2 for $n \geqslant 2$.

Definition 9.5. The **homogeneous coordinates** of a point in $\mathbb{RP}^n$ is the equivalence class $[x_0, x_1, \ldots, x_n] \in \mathbb{RP}^n$ of non-zero vectors $(x_0, x_1, \ldots, x_n) \neq (0, 0, \ldots, 0) \in \mathbb{R}^{n+1}$, with

$$\begin{aligned}&(x_0, x_1, \ldots, x_n) \sim (y_0, y_1, \ldots, y_n)\\ &\text{if } (y_0, y_1, \ldots, y_n) = \lambda(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \text{ for some } \lambda \neq 0 \in \mathbb{R}\end{aligned}$$

The projection

$$p \ : \ S^n \to \mathbb{RP}^n \ ; \ x \ = \ (x_0, x_1, \ldots, x_n) \mapsto [x] \ = \ [x_0, x_1, \ldots, x_n]$$

has the key property that for any $x, y \in S^n$ with $p(x) = p(y) \in \mathbb{RP}^n$ either $x = y$ or $x = -y$.

Every point on the unit circle

$$z \ = \ \cos(2\pi t) + i\sin(2\pi t) \in S^1$$

(regarded as a complex number) has two square roots

$$\sqrt{z} \ = \ \pm\left(\cos(\pi t) + i\sin(\pi t)\right) \in S^1 \ .$$

Definition 9.6. The **square root loop** in $\mathbb{RP}^n$ is

$$\sigma \ : \ S^1 \to \mathbb{RP}^n \ ; \ z = \cos(2\pi t) + i\sin(2\pi t) \mapsto \sqrt{z} = [\cos(\pi t), \sin(\pi t), 0, \ldots, 0] \ .$$

It is convenient to write the cyclic group of 2 elements as

$$\mathbb{Z}_2 \ = \ \{-1, +1\} \ ,$$

with the group operation given by multiplication.

Proposition 9.7. *(i) For $n = 1$ the square root loop*

$$\sigma \ : \ S^1 \to \mathbb{RP}^1 \ ; \ z = e^{2\pi i t} = cos(2\pi t) + isin(2\pi t) \mapsto \sqrt{z} = [cos\,\pi t, sin\,\pi t]$$

is a homeomorphism, so

$$\pi_1(\mathbb{RP}^1) \ = \ \pi_1(S^1) \ = \ \mathbb{Z} \ .$$

(ii) The square of the square root $\sigma \in \pi_1(\mathbb{RP}^n)$ is

$$[\sigma]^2 \ = \ [qp] \in \pi_1(\mathbb{RP}^n)$$

with $p : S^1 \to \mathbb{RP}^1$ *the projection and*

$$q \ : \ \mathbb{RP}^1 \to \mathbb{RP}^n \ ; \ [x_1, x_2] \mapsto [x_1, x_2, 0, \dots, 0]$$

the inclusion.

(iii) For $n \geqslant 2$

$$[\sigma]^2 \ = \ 1 \in \pi_1(\mathbb{RP}^n)$$

and there is defined a group homomorphism

$$[\sigma] \ : \ \mathbb{Z}_2 \to \pi_1(\mathbb{RP}^n) \ ; \ -1 \mapsto [\sigma] \ .$$

Proof. (i) By construction.

(ii) The paths

$$\begin{aligned} \widetilde{\sigma}_1 \ &: \ I \to S^n \ ; \ t \mapsto (\cos(\pi t), \sin(\pi t), 0, \dots, 0) \ , \\ \widetilde{\sigma}_2 \ &: \ I \to S^n \ ; \ t \mapsto (\cos(\pi(t+1)), \sin(\pi(t+1)), 0, \dots, 0) \end{aligned}$$

are such that

$$\begin{aligned} &\widetilde{\sigma}_1(0) \ = \ \widetilde{\sigma}_2(1) \ = \ (1, 0, \dots, 0) \ , \ \widetilde{\sigma}_1(1) \ = \ \widetilde{\sigma}_2(0) \ = \ (-1, 0, \dots, 0) \in S^n \ , \\ &\widetilde{\sigma}_1 \bullet \widetilde{\sigma}_2 \ : \ I \to S^n \ ; \ t \mapsto [\cos(2\pi t), \sin(2\pi t), 0, \dots, 0] \ , \\ &p\widetilde{\sigma}_1(t) \ = \ p\widetilde{\sigma}_2(t) \ = \ \sigma(e^{2\pi i t}) \in \mathbb{RP}^n \ . \end{aligned}$$

The concatenation of the square root loop σ with itself is thus the composite of the projection $p : S^1 \to \mathbb{RP}^1$ and the inclusion $\mathbb{RP}^1 \subset \mathbb{RP}^n$

$$\sigma \bullet \sigma \ = \ qp \ : \ S^1 \to \mathbb{RP}^n \ ; \ (x_1, x_2) \mapsto [x_1, x_2, 0, \dots, 0] \ .$$

(iii) As $n \geqslant 2$ it is possible to extend $\sigma \bullet \sigma$ to the map

$$D^2 \to \mathbb{RP}^n \ ; \ (x_1, x_2) \mapsto [x_1, x_2, \sqrt{1 - (x_1)^2 - (x_2)^2}, 0, \dots, 0]$$

and $\pi_1(D^2) = \{1\}$, so

$$[\sigma]^2 \in \mathrm{im}(\pi_1(D^2) \to \pi_1(\mathbb{RP}^n)) \ = \ \{1\} \ .$$

$\square$

We shall use topology to define a group homomorphism

$$\mathrm{sign} \ : \ \pi_1(\mathbb{RP}^n) \to \mathbb{Z}_2$$

which will be shown in Theorem 9.12 below to be an isomorphism for $n \geqslant 2$. While $\mathbb{RP}^1$ is homeomorphic to S^1, $\mathbb{RP}^n$ is not homeomorphic to S^n for $n \geqslant 2$, since $\pi_1(S^n) = \{1\}$ (Theorem 7.3) and $\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$.

Lecture 15

Definition 9.8. A path-connected subset $U \subset \mathbb{RP}^n$ is **small** if $p^{-1}(U) \subset S^n$ has 2 path-components U_+, U_- and the restrictions

$$p_+ \;=\; p| \;:\; U_+ \to U \;,\; p_- \;=\; p| \;:\; U_- \to U$$

are homeomorphisms.

Example 9.9. For any $x \in S^n$ and $\epsilon < 1$ the open subset

$$U \;=\; p(B_\epsilon(x)) \subset \mathbb{RP}^n$$

(homeomorphic to $\mathbb{R}^n$) is small, with

$$B_\epsilon(x) \;=\; \{y \in S^n \,|\, \|x-y\| < \epsilon\} \;,\; p^{-1}(U) \;=\; B_\epsilon(x) \cup -B_\epsilon(x) \subset S^n \;.$$

If $y \in B_\epsilon(x) \cap -B_\epsilon(x)$ then $y, -y \in B_\epsilon(x)$ and

$$\|y-(-y)\| \;=\; 2\|y\| \;=\; 2 \leqslant \|x-y\| + \|x+y\| < 2\epsilon \;,$$

a contradiction.

Definition 9.10. A **lift** of a map $\alpha : [t_0, t_1] \to \mathbb{RP}^n$ is a map $\theta : [t_0, t_1] \to S^n$ such that

$$\alpha(t) \;=\; p(\theta(t)) \in \mathbb{RP}^n \quad (t \in [t_0, t_1]) \;.$$

$$\begin{tikzcd} & S^n \arrow[d, "p"] \\ {[t_0, t_1]} \arrow[ur, "\theta"] \arrow[r, "\alpha"'] & \mathbb{RP}^n \end{tikzcd}$$

If α is closed

$$p\theta(0) \;=\; \alpha(0) \;=\; \alpha(1) \;=\; p\theta(1) \in S^n \;,$$

allowing the **sign** of α to be defined by

$$\mathrm{sign}(\alpha) \;=\; \begin{cases} +1 & \text{if } \theta(0) = \theta(1) \\ -1 & \text{if } \theta(0) = -\theta(1) \;. \end{cases}$$

Proposition 9.11. *Given a map $\alpha : [t_0, t_1] \to \mathbb{RP}^n$ and $\theta_0 \in S^n$ such that*

$$\alpha(t_0) \;=\; p(\theta_0) \in \mathbb{RP}^n$$

there is a unique lift $\theta : [t_0, t_1] \to S^n$ for α with initial point

$$\theta(t_0) \;=\; \theta_0 \in S^n \;.$$

Proof. The method of proof is essentially the same as for Proposition 8.3, which proved the existence of an angle map for each path in S^1. Consider first the special case when $\alpha([t_0,t_1]) \subset U$ for a small open subset $U \subset \mathbb{RP}^n$, so that

$$\theta_0 \in p^{-1}(U) \; = \; U_+ \cup U_- \; .$$

Let $\epsilon \in \{+,-\}$ be the unique sign such that $\theta_0 \in U_\epsilon$. The composite

$$\theta \; = \; (p_\epsilon)^{-1} \circ \alpha \; : \; [t_0,t_1] \to U \to (p_\epsilon)^{-1}(U_\epsilon) \subset S^n$$

is the unique lift of α with $\theta(t_0) = \theta_0$.

Next, consider the general case of a map $\alpha : [t_0,t_1] \to \mathbb{RP}^n$ such that $\alpha([t_0,t_1])$ may not be contained in a single small open subset $U \subset \mathbb{RP}^n$. Define an open cover $\mathcal{U} = \{U\}$ of $\mathbb{RP}^n$ consisting of all the small open subsets $U \subset \mathbb{RP}^n$. By the Lebesgue Covering Lemma (GT Lemma B7.9) there exists an integer $N \geqslant 1$ so large that the N subintervals in the partition

$$[t_0,t_1] \; = \; \bigcup_{j=0}^{N-1} [s_j, s_{j+1}]$$

with

$$s_0 = t_0 < s_1 = t_0 + (t_1 - t_0)/N < \cdots < s_j = t_0 + j(t_1 - t_0)/N < \cdots < s_N = t_1$$

have diameter $(t_1 - t_0)/N$ so small that

$$\alpha([s_j, s_{j+1}]) \subset U_j \subset \mathbb{RP}^n \;\; (0 \leqslant j \leqslant N-1)$$

for small open $U_j \subset \mathbb{RP}^n$. Let

$$\alpha(j) \; = \; \alpha| \; : \; [s_j, s_{j+1}] \to \mathbb{RP}^n \; .$$

Apply the special case to construct the unique lift $\theta(0) : [s_0,s_1] \to S^n$ for $\alpha(0)$ with $\theta(0)(s_0) = \theta_0$. Next, apply the special case to construct the unique lift $\theta(1) : [s_1,s_2] \to S^n$ for $\alpha(1)$ with $\theta(1)(s_1) = \theta(0)(s_1) \in \mathbb{RP}^n$. And so on $\dots$, until each $\alpha(j)$ has a lift $\theta(j) : [s_j, s_{j+1}] \to S^n$. Piecing together the lifts $\theta(0), \theta(1), \dots, \theta(N-1)$ for $\alpha(0), \alpha(1), \dots, \alpha(N-1)$ there is obtained the unique lift

$$\theta \; = \; \theta(0) \cup \cdots \cup \theta(N-1) \; : \; [t_0,t_1] \to S^n \; ; \; t \to \theta(j)(t) \text{ if } s_j \leqslant t \leqslant s_{j+1}$$

for $\alpha = \alpha(0) \cup \cdots \cup \alpha(N-1)$ with $\alpha(t_0) = \theta_0$. □

For a closed path $\alpha : I \to \mathbb{RP}^n$ and for a choice of $\theta_0 \in p^{-1}(\alpha(0)) \subset S^n$ the endpoints $\theta(0) = \theta_0$, $\theta(1) \in S^n$ of the lift $\theta : I \to S^n$ are such that

$$p(\theta(0)) \; = \; \alpha(0) \; = \; \alpha(1) \; = \; p(\theta(1)) \in \mathbb{RP}^n \; .$$

There are two mutually exclusive possibilities:

$$\text{either } \theta(0) = \theta(1) \in S^n \text{ or } \theta(0) = -\theta(1) \in S^n$$

or equivalently

$$\text{the lift } \theta \text{ is/is not a closed path in } S^n.$$

The possibilities are independent of the choice of θ_0.

Theorem 9.12. *(i) The function*

$$sign \ : \ \pi_1(\mathbb{RP}^n) \to \mathbb{Z}_2 \ = \ \{+1,-1\} \ ; \ [\alpha] \mapsto \begin{cases} +1 & \text{if } \theta(0) = \theta(1) \\ -1 & \text{if } \theta(0) = -\theta(1) \end{cases}$$

is a surjective group homomorphism which takes the value -1 on the square root loop $\sigma \in \pi_1(\mathbb{RP}^n)$.

(ii) For $n = 1$

$$sign \ : \ \pi_1(\mathbb{RP}^1) = \mathbb{Z} \to \mathbb{Z}_2 \ ; \ N \mapsto \begin{cases} +1 & \textit{if } N \textit{ is even} \\ -1 & \textit{if } N \textit{ is odd} \end{cases}$$

is the surjection of the infinite cyclic group onto the cyclic group of order 2.

(iii) For $n \geqslant 2$ $sign : \pi_1(\mathbb{RP}^n) \to \mathbb{Z}_2$ is an isomorphism of groups inverse to $\sigma : \mathbb{Z}_2 \to \pi_1(\mathbb{RP}^n)$.

Proof. The method of proof is essentially the same as for Theorem 8.8.

(i) Adapting Proposition 8.7, we can use the Lebesgue Covering Lemma to lift homotopies, and thus show that $\mathrm{sign}(\omega) \in \mathbb{Z}$ depends only on the rel 1 homotopy class of a loop $\omega : S^1 \to \mathbb{RP}^n$.

The function $\mathrm{sign} : \pi_1(\mathbb{RP}^n) \to \mathbb{Z}_2$ is a group homomorphism, since a concatenation of closed paths lifts to the concatenation of paths. The square root loop $\sigma : S^1 \to \mathbb{RP}^n$ lifts to the path

$$\theta \ : \ I \to S^n \ ; \ t \mapsto (\cos(\pi t), \sin(\pi t), 0, \dots, 0)$$

with

$$\theta(0) \ = \ (1, 0, \dots, 0) \neq \theta(1) \ = \ (-1, 0, \dots, 0) \in S^n$$

so that $\mathrm{sign}(\sigma) = -1 \in \mathbb{Z}_2$.

(ii) There is only one surjective group homomorphism $\mathbb{Z} \to \mathbb{Z}_2$.

(iii) The sign function is injective for $n \geqslant 2$ since for any closed path $\alpha : I \to \mathbb{RP}^n$ at $\alpha(0) = \alpha(1)$ with a lift $\theta : I \to S^n$ to a closed path at $\theta(0) = \theta(1)$

$$[\alpha] \ = \ p_*[\theta] \ = \ 1 \in \pi_1(\mathbb{RP}^n)$$

on account of $[\theta] = 1 \in \pi_1(S^n) = \{1\}$ (Theorem 7.3). $\square$

Example 9.13. The projection $p : S^1 \to \mathbb{RP}^1$ corresponds to the closed path

$$\alpha \ : \ I \to \mathbb{RP}^1 \ ; \ t \mapsto p(e^{2\pi i t}) = [\cos(2\pi t), \sin(2\pi t)] \ .$$

The lift of α

$$\theta \ : \ I \to S^1 \ ; \ t \mapsto e^{2\pi i t}$$

has $\theta(0) = \theta(1) = (1, 0)$, so

$$\mathrm{sign}(p) \ = \ +1 \in \mathbb{Z}_2 \ .$$

This is in agreement with Theorem 9.12 (ii), since composition of p with the inverse of the square root homeomorphism $\sigma : S^1 \to \mathbb{RP}^1$ is the squaring function

$$\sigma^{-1}p \ : \ S^1 \to S^1 \ ; \ z \mapsto z^2 \ ,$$

so that

$$\mathrm{degree}(p) \ = \ \mathrm{degree}(\sigma^{-1}p) \ = \ 2 \in \pi_1(\mathbb{RP}^1) \ = \ \mathbb{Z} \ .$$

Example 9.14. For any $a, b \in S^2$ with $a \cdot b = 0 \in \mathbb{R}$ the path

$$\theta \ : \ I \to S^2 \ ; \ t \mapsto b\cos\pi t + (a \times b)\sin\pi t$$

traces out half the great circle $a^\perp \subset S^2$ from $\theta(0) = b$ to $\theta(1) = -b$. Then $\alpha = p\theta : I \to \mathbb{RP}^2$ is a closed path tracing out the projective line

$$L(a) \ = \ \{p(c) \,|\, c \in S^2, \ a \cdot c = 0 \in \mathbb{R}\} \subset \mathbb{RP}^2$$

with $[\alpha] = -1 \in \pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$. Any path $\widetilde{\beta} : I \to S^2$ with

$$\widetilde{\beta}(0) \ = \ b \ , \ \widetilde{\beta}(1) \ = \ -b \in S^2$$

is homotopic rel $\{0, 1\}$ to θ, for any $a \in b^\perp$. Thus $\beta = p\widetilde{\beta} : I \to \mathbb{RP}^2$ is a closed path with $\beta(0) = \beta(1) = p(b) \in \mathbb{RP}^2$ such that β is homotopic rel $\{0, 1\}$ to α and

$$[\beta] \ = \ [\alpha] \ = \ -1 \in \pi_1(\mathbb{RP}^2, b) \ = \ \mathbb{Z}_2 \ .$$

Remark 9.15. The sign of a loop $\omega : S^1 \to \mathbb{RP}^n$ is $+1$ or -1, depending on whether the space

$$S(\omega) \ = \ \{(x, y) \in S^1 \times S^n \,|\, \omega(x) = p(y) \in \mathbb{RP}^n\}$$

is homeomorphic to $S^1 \sqcup S^1$ or S^1.

Lecture 16

10 Fixed points and non-retraction

Definition 10.1. A **fixed point** of a map $f : X \to X$ is point $x \in X$ such that $f(x) = x$.

Topology is very good at predicting which maps $f : X \to X$ have fixed points, although not very good at actually finding them! The fixed point theorems of topology have applications not only in mathematics (e.g. differential equations, dynamical systems, analysis, ...) but also in economics and game theory. An equilibrium of some system is a fixed point of some map f; in order to actually find such fixed points it is necessary to employ non-topological methods such as analysis or linear programming, which are adapted to f.

Example 10.2. (i) Every $x \in X$ is a fixed point of the identity map

$$f = 1 : X \to X \; ; \; x \mapsto x \; .$$

(ii) The rotation of $\mathbb{R}^2$ through an angle θ with $0 < \theta < 2\pi$

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$$

has only $(0, 0)$ as fixed point.

(iii) The fixed points of the reflection of $\mathbb{R}^2$ through the line $y = x\tan\phi$

$$S_\phi = \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$$

are precisely the points on the line.

(iv) A translation of $\mathbb{R}^n$ by $a \neq 0 \in \mathbb{R}^n$

$$T : \mathbb{R}^n \to \mathbb{R}^n \; ; \; x \mapsto x + a$$

does not have fixed points.

(v) The antipodal map

$$T : S^n \to S^n \; ; \; x \mapsto -x$$

does not have fixed points.

For certain spaces X every map $f : X \to X$ can be shown to have a fixed point. The **Brouwer fixed point theorem** (dating from 1910) is that every map $f : D^n \to D^n$ has a fixed point, for every $n \geqslant 1$. This is a consequence of the **non-retraction theorem**: there is no map $g : D^n \to S^{n-1}$ which is the identity on $S^{n-1} \subset D^n$. We shall only consider the cases $n = 0, 1$, although the method extends to all $n \geqslant 2$.

Definition 10.3. Let $i : A \to X$ be the inclusion of a subspace $A \subset X$. The subspace A is a **retract** of X if there exists a map $j : X \to A$ such that $j \circ i = \text{identity} : A \to A$. The map j is a **retraction** of X onto A.

Example 10.4. (i) For every $a \in X$ the subspace $A = \{a\} \subset X$ is a retract.

(ii) Let $A = \{a, b\} \subseteq X$, with $a \neq b$, and X Hausdorff. There exist disjoint open subsets $U, V \subset X$ such that $a \in U$, $b \in V$ so that A has the discrete topology, with both $\{a\} = A \cap U$ and $\{b\} = A \cap V$ open in A. The subspace A is a retract of X if and only if a, b are in different components of X.

(iii) For any $n \geqslant 1$ $S^{n-1} \subset D^n \backslash \{0\}$ is a retract, with

$$j \ : \ D^n \backslash \{0\} \to S^{n-1} \ ; \ x \mapsto \frac{x}{\|x\|}$$

a retraction of $D^n \backslash \{0\}$ onto S^{n-1}.

Proposition 10.5. *Let X be a path-connected space, and let $i : A \to X$ be the inclusion of a path-connected subspace A.*

(i) *If A is a retract of X, $i : A \to X$ is the inclusion and $j : X \to A$ is a retraction the induced group homomorphisms*

$$i_* \ : \ \pi_1(A) \to \pi_1(X) \ , \ j_* \ : \ \pi_1(X) \to \pi_1(A)$$

are such that

$$j_* \circ i_* \ = \ \text{identity} \ : \ \pi_1(A) \xrightarrow{i_*} \pi_1(X) \xrightarrow{j_*} \pi_1(A) \ .$$

Thus $i_* : \pi_1(A) \to \pi_1(X)$ is injective, and $j_* : \pi_1(X) \to \pi_1(A)$ is surjective.

(ii) *If $\pi_1(A) \neq \{1\}$ and $\pi_1(X) = \{1\}$ then A cannot be a retract of X.*

Proof. (i) Immediate from

$$(j \circ i)_* \ = \ j_* \circ i_* \ , \ \text{identity}_* \ = \ \text{identity}$$

and the fact that homotopic maps induce the same homomorphisms of the fundamental groups.

(ii) Immediate from (i). □

Remark 10.6. If X is a connected space and $A \subset X$ is a retract then A must be connected: Proposition 10.5 is the π_1-analogue.

Proposition 10.7. *If there exists a map $f : D^n \to D^n$ without fixed points then the inclusion $i : S^{n-1} \to D^n$ admits a retraction.*

Proof. Define a retraction of i

$$j \ : \ D^n \to S^{n-1} \ ; \ x \mapsto j(x)$$

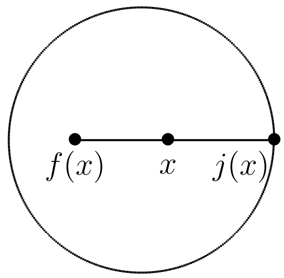
by sending $x \in D^n$ to the unique point

$$j(x) \ = \ (1-t)f(x) + tx \in S^{n-1}$$

with $t \geqslant 1$. Explicitly,

$$t = 1 + \frac{x\cdot(f(x)-x)+\sqrt{(x\cdot(f(x)-x))^2-4\|f(x)-x\|^2(\|x\|^2-1)}}{2\|f(x)-x\|^2}$$

If $x \in D^n \backslash S^{n-1}$ then $f(x), x, j(x) \in D^n$ are distinct collinear points. If $x \in S^{n-1}$ then $j(x) = x$.



□

Theorem 10.8. *(Non-retraction for $n = 1, 2$)*
(i) $S^0 \subset D^1$ is not a retract.
(ii) $S^1 \subset D^2$ is not a retract.

Proof. (i) Suppose $j : D^1 \to S^0$ is a retraction of the inclusion $i : S^0 \to D^1$, so that

$$ji \ = \ 1 \ : \ S^0 \to S^0 \ .$$

Now j is onto, D^1 is connected, S^0 is disconnected : a contradiction!

(ii) Apply Proposition 10.5: suppose $j : D^2 \to S^1$ is a retraction of the inclusion $i : S^1 \to D^2$, so that

$$ji \ = \ 1 \ : \ S^1 \to S^1 \ .$$

induces

$$(ji)_* \ = \ j_* i_* \ = \ 1 \ = \ 0 \ : \ \pi_1(S^1) \ = \ \mathbb{Z} \to \pi_1(D^2) \ = \ \{0\} \to \pi_1(S^1) \ = \ \mathbb{Z}$$

- a contradiction! □

Corollary 10.9. *(Brouwer fixed point theorem for $n = 1, 2$)*
Every map $f : D^n \to D^n$ has a fixed point.

Proof. Immediate from Proposition 10.7 and Theorem 10.8. □

Remark 10.10 (Non-examinable). (i) The non-retraction Theorem 10.8 for $n = 1$ is the intermediate value theorem in disguise.

(ii) The non-retraction Theorem 10.8 generalizes to all $n \geqslant 3$: $S^{n-1} \subset D^n$ is not a retract. The standard proof uses the computations

$$\pi_{n-1}(S^{n-1}) \ = \ \mathbb{Z} \ , \ \pi_{n-1}(D^n) \ = \ \{0\}$$

of the higher homotopy groups (cf. Remark 5.19). The proof of the Brouwer fixed point theorem for $n \geqslant 3$ follows as in the cases $n = 1, 2$.

(iii) The proof of the Brouwer fixed point theorem is non-constructive, i.e. does not provide an algorithm for actually finding a fixed point. This is very ironic, since Brouwer subsequently developed 'intuitionistic logic' in which only constructive proofs are allowed!

What about the fixed points of maps $f : S^n \to S^n$?

Proposition 10.11. *A map $f : S^n \to S^n$ without fixed points is homotopic to the antipodal map $a : S^n \to S^n; x \mapsto -x$.*

Proof. The map

$$h \ : \ S^n \times I \to S^n \ ; \ (x,t) \mapsto \tfrac{(1-t)f(x)-tx}{\|(1-t)f(x)-tx\|}$$

defines a homotopy $f \simeq a$. (If $(1-t)f(x) - tx = 0 \in \mathbb{R}^{n+1}$ the unit vectors $f(x) \neq x \in S^n$ are parallel, so that $f(x) = -x$ and $0 = (1-t)f(x) - tx = f(x)$, a contradiction). $\square$

Example 10.12. The rotation of S^1 through an angle $\theta \in [0, 2\pi)$

$$R_\theta \ : \ S^1 \to S^1 \ ; \ z \mapsto ze^{i\theta}$$

has fixed points if $\theta = 0$ (since $R_0 = 1$) and no fixed points if $\theta > 0$, since $e^{i\theta} \neq 1$. The map

$$S^1 \times I \to S^1 \ ; \ (z,t) \mapsto ze^{t\theta i}$$

defines a homotopy $R_0 = 1 \simeq R_\theta$, so

$$\mathrm{degree}(R_\theta) \ = \ \mathrm{degree}(1) \ = \ 1 \in \mathbb{Z} \ .$$

Lecture 17

Proposition 10.13. *(i) A map $f : S^1 \to S^1$ without fixed points has degree 1.*
(ii) A map $f : S^1 \to S^1$ with degree $\neq 1$ has a fixed point.

Proof. (i) By Proposition 10.11 f is homotopic to the antipodal map

$$a \;=\; R_\pi \;:\; S^1 \to S^1 \;;\; z \mapsto -z \;.$$

By Example 10.12

$$\text{degree}(f) \;=\; \text{degree}(a) \;=\; \text{degree}(1) \;=\; 1 \in \mathbb{Z} \;.$$

(ii) This is just the contrapositive of (i). □

Remark 10.14 (Non-examinable). (i) It can be shown that the antipodal map $a : S^n \to S^n$ is homotopic to the identity if and only if n is odd. (This may be deduced for example from the higher homotopy group computation

$$a_* \;=\; (-1)^{n+1} \;:\; \pi_n(S^n) \;=\; \mathbb{Z} \to \pi_n(S^n) \;=\; \mathbb{Z}$$

also due to Brouwer).

(ii) A **vector field** on S^n is a map $v : S^n \to \mathbb{R}^{n+1}$ such that

$$x \cdot v(x) = 0 \in \mathbb{R} \text{ for all } x \in S^n \;,$$

so that $v(x)$ is tangent to $S^n \subset \mathbb{R}^{n+1}$ at $x \in S^n$.

If $v(x) \neq 0$ for all $x \in S^n$, the maps

$$\begin{aligned} f &\;:\; S^n \to S^n \;;\; x \mapsto \frac{v(x)}{\|v(x)\|} \;, \\ h &\;:\; S^n \times I \to S^n \;;\; (x,t) \mapsto \frac{(1-t)v(x) + tx}{\|(1-t)v(x) + tx\|} \end{aligned}$$

are such that $h : f \simeq 1 : S^n \to S^n$, so that f is homotopic to the identity.

It follows from Proposition 10.11 and (i) that for even $n = 2m$ f must have a fixed point $x = f(x) \in S^n$. But for such a fixed point $v(x) \neq 0 \in \mathbb{R}^{n+1}$ would be both parallel and orthogonal to x. So there is no such v. This is the **Hairy Ball Theorem** (first proved by Poincaré for $m = 1$, but not called that by him): every vector field v on S^{2m} has at least one $x \in S^{2m}$ with $v(x) = 0 \in \mathbb{R}^{2m+1}$.

On the other hand, for odd $n = 2m+1$ there exist non-zero vector fields on S^{2m+1}, for example

$$\begin{aligned} & v \;:\; S^{2m+1} \to \mathbb{R}^{2m+2} \backslash \{0\} \;; \\ & (x_1, x_2, \ldots, x_{2m+2}) \mapsto (-x_{m+1}, -x_{m+2}, \ldots, -x_{2m+2}, x_1, x_2, \ldots, x_m) \;. \end{aligned}$$

11 Covering spaces

It is practically impossible to actually compute the fundamental group $\pi_1(X, x_0)$ directly from the definition! However, in many cases it is possible to compute it geometrically, as the 'group of covering translations' of a 'covering space' of X. For any topological space $\widetilde{X}$ the set $\text{Homeo}(\widetilde{X})$ of all homeomorphisms $h : \widetilde{X} \to \widetilde{X}$ is a group, with the composition of homeomorphisms as the group law.

In favourable circumstances it is possible to compute $\pi_1(X, x_0)$ for a path-connected space X using the **universal covering projection** $p : \widetilde{X} \to X$ with $\widetilde{X}$ simply-connected (= path connected and $\pi_1(\widetilde{X}) = 0$), such that each inverse image $p^{-1}(x) \subset \widetilde{X}$ $(x \in X)$ is a copy of $\pi_1(X, x_0)$ as a discrete subspace.

The fundamental group $\pi_1(X, x_0)$ is isomorphic to the subgroup $\text{Homeo}_p(\widetilde{X})$ of $\text{Homeo}(\widetilde{X})$ consisting of the homeomorphisms $h : \widetilde{X} \to \widetilde{X}$ such that $ph = p : \widetilde{X} \to X$, called the **covering translations**. This correspondence (which will take us a few lectures) generalises the degree homomorphism on $\pi_1(S^1)$: we lift loops $\alpha : S^1 \to X$ to paths $\tilde{\alpha} : I \to \tilde{X}$, and find that the homotopy class $[\alpha]$ is determined by the difference between $\tilde{\alpha}(0)$ and $\tilde{\alpha}(1)$.

Recall that a topological space F is *discrete* if it has the topology in which every subset is open.

Definition 11.1. A *covering space* of a space X with *fibre* the discrete space F is a space $\widetilde{X}$ with a *covering projection* map $p : \widetilde{X} \to X$ such that for each $x \in X$ there exists an open subset $U \subseteq X$ with $x \in U$, and with a homeomorphism

$$\phi \ : \ F \times U \to p^{-1}(U)$$

such that

$$p\phi(a, u) \ = \ u \in U \subseteq X \quad (a \in F, u \in U) \ .$$

In particular, for each $x \in X$, $p^{-1}(x)$ is homeomorphic to F.

A covering projection $p : \widetilde{X} \to X$ is a 'local homeomorphism': for each $\widetilde{x} \in \widetilde{X}$ there exists an open subset $U \subseteq \widetilde{X}$ such that $\widetilde{x} \in U$ and

$$U \to p(U) \ ; \ u \mapsto p(u)$$

is a homeomorphism, with $p(U) \subseteq X$ an open subset.

Definition 11.2. Given a covering projection $p : \widetilde{X} \to X$ let $\text{Homeo}_p(\widetilde{X})$ be the subgroup of $\text{Homeo}(\widetilde{X})$ consisting of the homeomorphisms $h : \widetilde{X} \to \widetilde{X}$ such that

$$ph \ = \ p \ : \ \widetilde{X} \to X \ ,$$

i.e. such that the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "h"] \arrow[dr, "p"'] & & \widetilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

is commutative. Such h are called *covering translations*.

Example 11.3. A map $p : \widetilde{X} \to X$ is a homeomorphism if and only if it is a covering projection with fibre $F = \{1\}$.

Definition 11.4. A covering projection $p : \widetilde{X} \to X$ with fibre F is *trivial* if there exists a homeomorphism

$$\phi \ : \ F \times X \to \widetilde{X}$$

such that

$$p\phi(a,x) \ = \ x \in X \quad (a \in F, \ x \in X) \ .$$

A particular choice of ϕ is a *trivialisation* of p.

Example 11.5. For any space X and discrete space F the covering projection

$$p \ : \ \widetilde{X} \ = \ F \times X \to X \ ; \ (a,x) \mapsto x$$

is trivial, with the identity trivialization $\phi = 1 : F \times X \to \widetilde{X}$.

Example 11.6. Let $p : \widetilde{X} \to X$ be a covering projection, with fibre F.

(i) For any subset $U \subseteq X$ the restriction $p| : p^{-1}(U) \to U$ is a covering projection, with the same fibre F. By Definition 11.1, X has an open cover $\{U \subset X\}$ such that the restrictions $p| : p^{-1}(U) \to U$ are trivial. But p itself need not be trivial – see Example 11.7 below.

(ii) If p is trivial and $\phi_1, \phi_2 : F \times X \to \widetilde{X}$ are two trivialisations there is a unique function

$$\sigma \ : \ X \to \mathrm{Homeo}(F)$$

such that

$$\phi_2(a,x) \ = \ \phi_1(\sigma(x)(a),x) \in \widetilde{X} \quad (a \in F, x \in X) \ .$$

Both F and $\mathrm{Homeo}(F)$ are given the discrete topology, and the homeomorphisms (= bijections in this case) $\sigma(x) : F \to F$ vary continuously with $x \in X$, so the function σ is constant on each path-component of X.

(iii) If p is trivial and ϕ is a trivialization, a covering translation $h : \widetilde{X} \to \widetilde{X}$ is necessarily of the form

$$h \ : \ \widetilde{X} \to \widetilde{X} \ ; \ \phi(a,x) \mapsto \phi(\sigma(x)(a),x)$$

with the bijections $\sigma(x) : F \to F$ varying continuously with $x \in X$ as in (ii). For path-connected X the $\sigma(x)$'s are all the same, and the group of covering translations is

$$\mathrm{Homeo}_p(\widetilde{X}) \ = \ \mathrm{Homeo}(F)$$

the group of all bijections $F \to F$.

Example 11.7. (i) The map

$$p \ : \ \mathbb{R} \to S^1 \ ; \ x \mapsto e^{2\pi i x}$$

is a covering projection with fibre $\mathbb{Z}$. Note that p is not trivial, since $\mathbb{R}$ is not homeomorphic to $\mathbb{Z} \times S^1$. The group of covering translations is the infinite cyclic group $\mathbb{Z}$ (see Example 11.9 below for details).

(ii) For any $n \neq 0 \in \mathbb{Z}$ the map

$$p_n \ : \ S^1 \to S^1 \ ; \ z \mapsto z^n$$

is a covering projection.

If $|n| = 1$ then $p_n : S^1 \to S^1$ is a homeomorphism (= the identity map for $n = 1$, the reflection in the real axis for $n = -1$), so it has fibre $F = \{1\}$ and group of covering translations $\mathrm{Homeo}_{p_n}(S^1) = \{1\}$.

If $|n| \geqslant 2$ then $p_n : S^1 \to S^1$ has fibre $F = \{1, 2, \ldots, |n|\}$. The covering translation defined by anticlockwise rotation through the angle $2\pi/n$ (= the antipodal map for $|n| = 2$)

$$h_n \ : \ S^1 \to S^1 \ ; \ z \mapsto ze^{2\pi i/n}$$

generates the group of covering translations

$$\mathrm{Homeo}_{p_n}(S^1) \ = \ \{(h_n)^j \,|\, 0 \leqslant j \leqslant |n| - 1\} \ = \ \mathbb{Z}_{|n|} \subset \mathrm{Homeo}(S^1) \ ,$$

a cyclic group of order $|n|$.

Lecture 18

Here is a very general construction of covering projections:

Theorem 11.8. *Given a space $\widetilde{X}$ and a subgroup $G \subseteq \mathrm{Homeo}(\widetilde{X})$ define an equivalence relation $\sim$ on $\widetilde{X}$ by*

$$\widetilde{x}_1 \sim \widetilde{x}_2 \text{ if there exists } g \in G \text{ such that } \widetilde{x}_2 = g(\widetilde{x}_1)$$

and write

$$X \;=\; \widetilde{X}/\!\sim \;=\; \widetilde{X}/G\ ,$$

with quotient map $p\ :\ \tilde{X} \to X$.

Suppose that for each $\widetilde{x} \in \widetilde{X}$ there exists an open neighbourhood U of $\widetilde{x}$ such that

$$g(U) \cap U = \emptyset \text{ for } g \in G \setminus \{1\}\ .$$

Then $p : \widetilde{X} \to X$ is a covering projection with fibre G. Furthermore, if $\widetilde{X}$ is path-connected then so is X, and the group of covering translations of p is

$$\mathrm{Homeo}_p(\widetilde{X}) \;=\; G \subset \mathrm{Homeo}(\widetilde{X})\ .$$

Proof. If we have $g_1u_1 = g_2u_2 \in g_1(U) \cap g_2(U)$, then $u_2 = gu_1 \in g(U) \cap U$ for $g = (g_2)^{-1}g_1$, so $g = 1 \in G$ and $g_1 = g_2$. Thus the subsets $g(U) \subset p^{-1}p(U)$ are disjoint and hence clopen, so

$$p^{-1}p(U) \;=\; \{gu \,|\, g \in G, u \in U\} \;=\; \bigsqcup_{g\in G} g(U) \subseteq \widetilde{X}$$

is a disjoint union and the surjective map

$$G \times U \to p^{-1}p(U)\ ;\ (g,u) \mapsto gu$$

is a homeomorphism.

By definition of the quotient topology, p is an open map, and the surjective open map $p|_U\ :\ U \to p(U)$ is injective by the argument above, hence a homeomorphism, so we may combine these maps to give the required homeomorphism

$$\phi\ :\ G \times p(U) \to p^{-1}p(U)\ ;\ (g, p(u)) \mapsto gu.$$

If $h \in \mathrm{Homeo}_p(\widetilde{X})$, then for any $\widetilde{x} \in \widetilde{X}$ there is a unique $g_{\widetilde{x}} \in G$ such that

$$h(\widetilde{x}) \;=\; g_{\widetilde{x}}(\widetilde{x}) \in \widetilde{X}\ .$$

For the open subset $W = h^{-1}(g_{\widetilde{x}}U) \cap U$, we have $h(W) \cap g(W) = \emptyset$ if $g \neq g_{\widetilde{x}}$, so we must also have $g_y = g_{\widetilde{x}}$ for all $y \in W$. Thus the function

$$\widetilde{X} \to G\ ;\ \widetilde{x} \to g_{\widetilde{x}}$$

is continuous. If $\widetilde{X}$ is path-connected, the function is constant (since G is discrete), so $h = g_{\widetilde{x}}$ and thus

$$\mathrm{Homeo}_p(\widetilde{X}) \;=\; G \; .$$

□

Example 11.9. For each $n \in \mathbb{Z}$ the translation of $\mathbb{R}$ by n units to the right defines a homeomorphism

$$h_n \;:\; \mathbb{R} \to \mathbb{R} \; ; \; x \mapsto x + n$$

with $h_n h_m = h_{m+n}$. The infinite cyclic subgroup

$$G \;=\; \{h_n \,|\, n \in \mathbb{Z}\} \subset \mathrm{Homeo}(\mathbb{R})$$

satisfies the hypothesis of Theorem 11.8, so that

$$p \;:\; \mathbb{R} \to \mathbb{R}/G \;=\; \mathbb{R}/\mathbb{Z} \;=\; S^1 \; ; \; x \mapsto e^{2\pi i x}$$

is a covering projection with fibre $G = \mathbb{Z}$ and group of covering translations

$$\mathrm{Homeo}_p(\mathbb{R}) \;=\; G \;=\; \mathbb{Z} \subset \mathrm{Homeo}(\mathbb{R}) \; .$$

Example 11.10. Let $p : \widetilde{X} \to X$ be a covering projection. A *lift* of a map $f : Y \to X$ is a map $\widetilde{f} : Y \to \widetilde{X}$ such that

$$p(\widetilde{f}(y)) \;=\; f(y) \in X \quad (y \in Y)$$

so that there is defined a commutative diagram

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Example 11.11. For the trivial covering projection $p : \widetilde{X} = F \times X \to X$ of Example 11.5 define a lift of any map $f : Y \to X$ by choosing a point $a \in F$ and setting

$$\widetilde{f}_a \;:\; Y \to \widetilde{X} \;=\; F \times X \; ; \; y \mapsto (a, f(y)) \; .$$

If Y is path-connected every lift of f is of this type, and the function $a \to \widetilde{f}_a$ defines a one-one correspondence between the points $a \in F$ and the lifts $\widetilde{f}$ of f.

Theorem 11.12. (Path lifting property) *Let $p : \widetilde{X} \to X$ be a covering projection with fibre F. Let $x_0 \in X$, $\widetilde{x}_0 \in \widetilde{X}$ be such that $p(\widetilde{x}_0) = x_0 \in X$.*

(i) *Every path $\alpha : I \to X$ with $\alpha(0) = x_0 \in X$ has a unique lift to a path $\widetilde{\alpha} : I \to \widetilde{X}$ such that $\widetilde{\alpha}(0) = \widetilde{x}_0 \in \widetilde{X}$.*

(ii) *Let $\alpha, \beta : I \to X$ be paths with*

$$\alpha(0) \ = \ \beta(0) \ = \ x_0 \in X \ , \ \alpha(1) \ = \ \beta(1) \in X \ .$$

and let $\widetilde{\alpha}, \widetilde{\beta} : I \to \widetilde{X}$ be the lifts with $\widetilde{\alpha}(0) = \widetilde{\beta}(0) = \widetilde{x}_0 \in \widetilde{X}$ given by (i). *Every rel $\{0,1\}$ homotopy*

$$h \ : \ \alpha \ \simeq \ \beta \ : \ I \to X$$

has a unique lift to a rel $\{0,1\}$ homotopy

$$\widetilde{h} \ : \ \widetilde{\alpha} \ \simeq \ \widetilde{\beta} \ : \ I \to \widetilde{X}$$

and in particular

$$\widetilde{\alpha}(1) \ = \ \widetilde{h}(1,t) \ = \ \widetilde{\beta}(1) \in \widetilde{X} \quad (t \in I) \ .$$

Proof. (Nonexaminable) (i) For each $s \in I$ there exists an open subset $U_s \subseteq X$ with $\alpha(s) \in U_s$, and with a homeomorphism $\phi_s : F \times U_s \to p^{-1}(U_s)$ such that

$$p\phi_s(a,u) \ = \ u \in U_s \subseteq X \quad (a \in F, u \in U_s) \ .$$

By the Lebesgue Covering Lemma (GT Lemma B7.9) there exists an integer $N \geqslant 1$ and $s_1, s_2, \dots, s_N \in I$ such that

$$[\tfrac{k}{N}, \tfrac{k+1}{N}] \subset \alpha^{-1}(U_{s_k}) \ (0 \leqslant k < N) \ .$$

If the lift $\widetilde{\alpha}$ has already been defined on $[0, \frac{k}{N}]$ and

$$(\phi_{s_k})^{-1}\widetilde{\alpha}(\tfrac{k}{N}) \ = \ (f_k, \alpha(\tfrac{k}{N})) \in F \times U_{s_k}$$

there is a unique extension to a lift on $[0, \frac{k+1}{N}]$ by

$$\widetilde{\alpha} \ : \ [\tfrac{k}{N}, \tfrac{k+1}{N}] \to \widetilde{X} \ ; \ s \to \phi_{s_k}(f_k, \alpha(s)) \ .$$

with

$$(\phi_{s_{k+1}})^{-1}\widetilde{\alpha}(\tfrac{k+1}{N}) \ = \ (f_{k+1}, \alpha(\tfrac{k+1}{N})) \in F \times U_{s_{k+1}} \ .$$

(ii) For each $t \in I$ there is defined a path

$$h_t \ : \ I \to X \ ; \ s \to h(s,t)$$

with

$$\begin{aligned} h_0(s) \ &= \ \alpha(s) \ , \ h_1(s) \ = \ \beta(s) \ , \\ h_t(0) \ &= \ h(s,0) \ = \ x_0 \ , \ h_t(1) \ = \ \alpha(1) \ = \ \beta(1) \in X \ . \end{aligned}$$

Let $\widetilde{h}_t : I \to \widetilde{X}$ be the unique lift of h_t with $\widetilde{h}_t(0) = \widetilde{x}_0 \in \widetilde{X}$ given by (i), and set

$$\widetilde{h} \ : \ I \times I \to \widetilde{X} \ ; \ (s,t) \mapsto \widetilde{h}_t(s) \ .$$

$\square$

Definition 11.13. Given a covering projection $p : \widetilde{X} \to X$ and a path $\alpha : I \to X$ define the *fibre transport* bijection

$$\alpha_{\#} \ : \ p^{-1}(\alpha(0)) \to p^{-1}(\alpha(1)) \ ; \ \widetilde{x} \mapsto \widetilde{\alpha}_{\widetilde{x}}(1)$$

where $\widetilde{\alpha}_{\widetilde{x}} : I \to \widetilde{X}$ is the unique lift of α given by Theorem 11.12 (i) with

$$\widetilde{\alpha}_{\widetilde{x}}(0) \ = \ \widetilde{x} \in \widetilde{X} \ .$$

Remark 11.14. In a covering $p : \widetilde{X} \to X$, the inverse image $p^{-1}(x) \subseteq \widetilde{X}$ of any $x \in X$ is homeomorphic to F, but not in a particular way: a choice of open subset $U \subseteq X$ and a homeomorphism $\phi : F \times U \to p^{-1}(u)$ with $x \in U$ and $p\phi(a,u) = p(u)$ determine a homeomorphism

$$F \to p^{-1}(x) \ ; \ (a,u) \mapsto \phi(a,u)$$

but different choices of U, ϕ will give a different homeomorphism $F \cong p^{-1}(x)$.

If $x, y \in X$ are joined by a particular path $\alpha : I \to X$ with $\alpha(0) = x$, $\alpha(1) = y \in X$ the fibre transport is a particular bijection $\alpha_{\#} : p^{-1}(x) \to p^{-1}(y)$. By Theorem 11.12 (ii) $\alpha_{\#}$ depends only on the rel $\{0,1\}$ homotopy class of α. If $\alpha : I \to X$ is a closed path with $\alpha(0) = \alpha(1) = x \in X$ then

$$\alpha_{\#} \ : \ p^{-1}(x) \to p^{-1}(x)$$

is a self-homeomorphism (= permutation) of $p^{-1}(x)$.

Lecture 19

Recall that a space X is *simply-connected* if it is path-connected and

$$\pi_1(X) \;=\; \{1\}\ .$$

Proposition 11.15. *Every covering projection $p : \widetilde{X} \to X$ of a simply-connected space X is trivial.*

Proof. (Nonexaminable) Let F be the fibre. Choose a base point $x_0 \in X$, and an open neighbourhood $U_0 \subseteq X$ of x_0 with a trivialisation

$$\phi_0\ :\ F \times U_0 \to p^{-1}(U_0)$$

of $p| : p^{-1}(U_0) \to U_0$, i.e. a homeomorphism such that

$$p\phi_0(a,u) \;=\; u \in X \quad (a \in F, u \in U_0)\ .$$

In particular, there is defined a bijection

$$F \to p^{-1}(x_0)\ ;\ a \mapsto \phi_0(a,x_0)\ .$$

For each $x \in X$ choose a path $\alpha_x : I \to X$ from $\alpha_x(0) = x_0$ to $\alpha_x(1) = x$, and use fibre transport to define a homeomorphism

$$\phi\ :\ F \times X \to \widetilde{X}\ ;\ (a,x) \mapsto (\alpha_x)_\#(\phi_0(a,x_0))\ .$$

The condition $\pi_1(X) = \{1\}$ is needed to prove that ϕ is independent of the choices of paths α_x. $\square$

Example 11.16. (i) By Proposition 11.15 every covering $p : \widetilde{I} \to I$ is trivial, with a homeomorphism $\phi : F \times I \to \widetilde{I}$ such that $p\phi(a,x) = x$.

(ii) For any discrete space F and bijection $\sigma : F \to F$ define a covering $p : \widetilde{S}^1 \to S^1$ with fibre F by

$$p\ :\ \widetilde{S}^1 \;=\; F \times I/\{(x,0) \sim (\sigma(x),1)\} \to S^1 = I/\{0 \sim 1\}\ ;\ [x,t] \mapsto [t]\ .$$

In fact, every covering $p : \widetilde{S}^1 \to S^1$ arises in this way: define the closed path

$$\alpha\ :\ I \to S^1\ ;\ t \mapsto e^{2\pi i t}$$

with $\alpha(0) = \alpha(1) = 1 \in S^1$, and note that the fibre transport is a bijection

$$\alpha_\#\ :\ F \;=\; p^{-1}(1) \to F \;=\; p^{-1}(1)$$

such that

$$p\ :\ \widetilde{S}^1 \;=\; F \times I/\{(x,0) \sim (\alpha_\#(x),1)\} \to S^1 = I/\{0 \sim 1\}\ ;\ [x,t] \mapsto [t]\ .$$

(iii) For the covering projection $p : \mathbb{R} \to S^1; x \mapsto e^{2\pi i x}$ of Example 11.7 (i) the fibre transport bijection in (ii) is just the shift map

$$\sigma \ : \ F \ = \ \mathbb{Z} \to F \ = \ \mathbb{Z} \ ; \ x \mapsto x+1 \ ,$$

with a homeomorphism

$$\mathbb{Z} \times I/\{(x,0) \simeq (x+1,1)\} \to \mathbb{R} \ ; \ [y,t] \mapsto y-t \ .$$

(iv) For the covering projection $p_n : S^1 \to S^1; z \mapsto z^n$ of Example 11.7 (ii) with $n \geqslant 1$ the fibre transport bijection is the cyclic permutation

$$\alpha_{\#} \ = \ (1\ 2\ \dots\ n) \ : \ F = \{1,2,\dots,n\} \to F = \{1,2,\dots,n\} \ ; \ k \mapsto k+1\,(\mathrm{mod}\,n) \ .$$

(v) A permutation $\sigma \in \Sigma_n$ determines a covering map

$$p \ : \ \widetilde{S}^1 \ = \ \{1,2,\dots,n\} \times I/\{(x,0) \sim (\sigma(x),1)\} \to S^1 = I/\{0 \sim 1\} \ ; \ (x,t) \mapsto t$$

with fibre $F = \{1,2,\dots,n\}$ and fibre transport bijection $\alpha_{\#} = \sigma$. Exercise: prove that $\widetilde{S}^1$ has one path-component for each cycle $(i_1\ i_2\ \dots\ i_m)$ in the cycle decomposition of σ, and that $\mathrm{Homeo}_p(\widetilde{S}^1)$ is isomorphic to the subgroup of Σ_n consisting of the permutations τ such that $\tau\sigma = \sigma\tau$, called the **centralizer** of σ.

Definition 11.17. A covering projection $p : \widetilde{X} \to X$ of a path-connected space X is *universal* if $\widetilde{X}$ is simply-connected.

Example 11.18. (i) The covering projection $p : \mathbb{R} \to S^1$ is universal.

(ii) The covering projection $p : S^n \to \mathbb{RP}^n$ is universal for $n \geqslant 2$.

Recall that a space X is *locally path connected* if for each $x \in X$ and for each open subset $U \subseteq X$ with $x \in U$ there is a path-connected open subset $V \subseteq U$ with $x \in V$. (Main example: open subsets of $\mathbb{R}^n$).

Theorem 11.19. *Let X be a path-connected locally path-connected space with a universal covering projection $p : \widetilde{X} \to X$. Let $x_0 \in X$, $\widetilde{x}_0 \in \widetilde{X}$ be base points such that $p(\widetilde{x}_0) = x_0$.*
(i) *The function*

$$\pi_1(X,x_0) \to p^{-1}(x_0) \ ; \ \alpha \to \alpha_{\#}(\widetilde{x}_0)$$

is a bijection.
(ii) *For each $y \in p^{-1}(x_0)$ there is a unique covering translation $h_y \in \mathrm{Homeo}_p(\widetilde{X})$ such that*

$$h_y(\widetilde{x}_0) \ = \ y \in \widetilde{X} \ .$$

The function

$$p^{-1}(x_0) \to \mathrm{Homeo}_p(\widetilde{X}) \ ; \ y \to h_y$$

is a bijection, with inverse $h \to h(\widetilde{x}_0)$. The composite bijection

$$\pi_1(X,x_0) \to p^{-1}(x_0) \to \mathrm{Homeo}_p(\widetilde{X})$$

is an isomorphism of groups.

Proof. (*Nonexaminable*) (i) The function $\pi_1(X, x_0) \to p^{-1}(x_0)$ is well-defined and injective, by the construction of the fibre transport $\alpha_\#$ (11.13) and its rel $\{0, 1\}$ homotopy invariance (11.14). Since $\widetilde{X}$ is simply-connected, for every $\widetilde{x}_1 \in p^{-1}(x_0)$ there is a unique rel $\{0, 1\}$ homotopy class of paths $\widetilde{\alpha} : I \to \widetilde{X}$ with

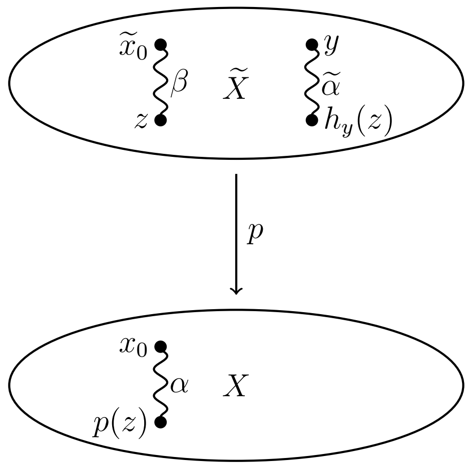
$$\widetilde{\alpha}(0) \;=\; \widetilde{x}_0 \;,\; \widetilde{\alpha}(1) \;=\; \widetilde{x}_1 \in \widetilde{X}$$

so that

$$\widetilde{x}_1 \;=\; (p\widetilde{\alpha})_\#(\widetilde{x}_0) \in p^{-1}(x_0)$$

is the image of $p\widetilde{\alpha} \in \pi_1(X, x_0)$, and $\pi_1(X, x_0) \to p^{-1}(x_0)$ is surjective.

(ii) By the path-connectedness of $\widetilde{X}$ for each $z \in \widetilde{X}$ there exists a path $\beta : I \to \widetilde{X}$ from $\beta(0) = \widetilde{x}_0$ to $\beta(1) = z$.



There is a unique lift of $\alpha = p\beta : I \to X$ to a path $\widetilde{\alpha} : I \to \widetilde{X}$ such that $\widetilde{\alpha}(0) = y$; define

$$h_y(z) \;=\; \alpha_\#(y) \;=\; \widetilde{\alpha}(1) \in \widetilde{X}$$

with

$$ph_y(z) \;=\; p\widetilde{\alpha}(1) \;=\; \alpha(1) \;=\; p\beta(1) \;=\; p(z) \in X \;.$$

The construction of $h_y(z)$ is independent of the choice of β, by the simple-connectedness of $\widetilde{X}$. The function

$$h_y \;:\; \widetilde{X} \to \widetilde{X} \;;\; z \to h_y(z)$$

is continuous because X (and hence $\widetilde{X}$) is locally path-connected. For every $t \in I$ the restriction $\widetilde{\alpha}| : [0, t] \to \widetilde{X}$ is the unique lift of the restriction $\alpha| : [0, t] \to X$ such that $\widetilde{\alpha}(0) = y$, so that

$$\widetilde{\alpha}(t) \;=\; h_y(\beta(t)) \in \widetilde{X}$$

and in particular

$$y \;=\; \widetilde{\alpha}(0) \;=\; h_y(\beta(0)) \;=\; h_y(\widetilde{x}_0) \in \widetilde{X} \;.$$

□

Example 11.20. It is possible to apply Theorem 11.19 to the covering projection $p : \mathbb{R} \to S^1$ of Example 11.9, since $\mathbb{R}$ is simply-connected. It is precisely the path-lifting Theorem 11.12 (i) which is used to prove that for every closed path $\alpha : I \to S^1$ there exists a lift $\widetilde{\alpha} : I \to \mathbb{R}$, allowing the degree to be defined by

$$\mathrm{degree}(\alpha) \ = \ \alpha_{\#}(0) \ = \ \widetilde{\alpha}(1) - \widetilde{\alpha}(0) \in \mathbb{Z} \ ,$$

with Theorem 11.12 (ii) showing that the degree depends only on the rel $\{0,1\}$ homotopy class of α. The bijection of Theorem 11.19 (i) is given by the degree map

$$\pi_1(S^1,1) \to p^{-1}(1) \ = \ \mathbb{Z} \ ; \ \alpha \to \mathrm{degree}(\alpha) \ = \ \widetilde{\alpha}(1) \ ,$$

since $\widetilde{\alpha}(0) = 0$. The bijection of Theorem 11.19 (ii) is given by

$$p^{-1}(1) \ = \ \mathbb{Z} \to \mathrm{Homeo}_p(\mathbb{R}) \ ; \ n \to h_n$$

with $h_n : \mathbb{R} \to \mathbb{R}; x \to x + n$ the unique covering translation such that $h_n(0) = n$. The isomorphism of Theorem 11.19 (iii) is given by

$$\pi_1(S^1,1) \to \mathrm{Homeo}_p(\mathbb{R}) \ ; \ \alpha \to h_n$$

with $\widetilde{\alpha}(1) = n$. In particular, we have that $\pi_1(S^1) = \mathbb{Z}$ is the infinite cyclic group generated by $1 : S^1 \to S^1$.

Lecture 20

Example 11.21. Define homeomorphisms $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$A(x,y) \;=\; (x+1,y)\;,\;\; B(x,y) \;=\; (x,y+1)$$

such that

$$AB \;=\; BA \;:\; \mathbb{R}^2 \to \mathbb{R}^2\;;\; (x,y) \mapsto (x+1,y+1)$$

and

$$A^rB^s(x,y) \;=\; (x+r,y+s) \text{ for any } r,s \in \mathbb{Z}\;.$$

Let $G \subset \text{Homeo}(\mathbb{R}^2)$ be the subgroup generated by A, B, that is

$$G \;=\; \{A^rB^s \,|\, r,s \in \mathbb{Z}\} \subset \text{Homeo}(\mathbb{R}^2)$$

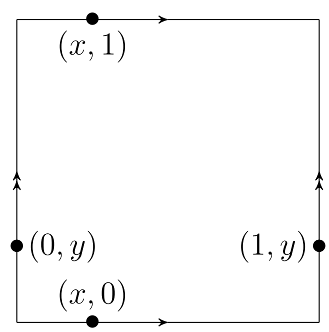
with multiplication by

$$(A^rB^s)(A^tB^u) \;=\; A^{r+t}B^{s+u} \in G\;.$$

The quotient space $X = \widetilde{X}/G$ is obtained from the unit square $I^2 = I \times I$ by identifying

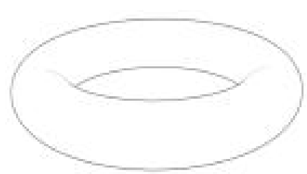
$$(0,y) \sim A(0,y) = (1,y) \text{ and } (x,0) \sim B(x,0) = (x,1)\;.$$

The quotient space



is a torus

$$X \;=\; \widetilde{X}/G \;=\; I^2/\!\sim \;=\; T \;=\; S^1 \times S^1\;.$$



Theorems 11.8, 11.19 apply to the universal covering projection

$$p \;:\; \widetilde{X} \;=\; \mathbb{R}^2 \to X \;=\; \widetilde{X}/G \;=\; T\;.$$

The fundamental group of the torus is thus the free abelian group on 2 generators

$$\pi_1(T) \;=\; \text{Homeo}_p(\widetilde{X}) \;=\; G \;=\; \mathbb{Z} \oplus \mathbb{Z}\;.$$

Example 11.22. Define homeomorphisms $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$A(x,y) \;=\; (x+1,y)\;,\; B(x,y) \;=\; (-x+1,y+1)$$

such that

$$AB \;=\; BA^{-1} \;:\; \mathbb{R}^2 \to \mathbb{R}^2\;;\; (x,y) \mapsto (-x+2,y+1)$$

and

$$A^rB^s(x,y) \;=\; ((-1)^s x + r + \frac{1+(-1)^{s+1}}{2}, y+s) \text{ for any } r,s \in \mathbb{Z}\;.$$

Let $G \subset \mathrm{Homeo}(\mathbb{R}^2)$ be the (nonabelian) subgroup generated by A, B, that is

$$G \;=\; \{A^rB^s \,|\, r,s \in \mathbb{Z}\} \subset \mathrm{Homeo}(\mathbb{R}^2)$$

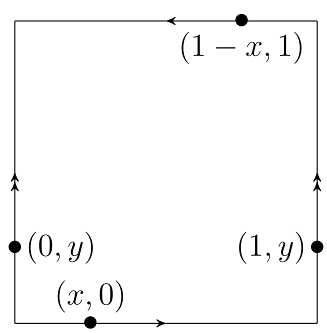
with multiplication by

$$(A^rB^s)(A^tB^u) \;=\; A^{r+t}B^{s(-1)^t+u} \in G\;.$$

The quotient space $X = \widetilde{X}/G$ is obtained from the unit square I^2 by identifying

$$(0,y) \sim A(0,y) = (1,y) \text{ and } (x,0) \sim B(x,0) = (1-x,1)\;.$$

The quotient space



is a Klein bottle

$$X \;=\; \widetilde{X}/G \;=\; I^2/{\sim} \;=\; K\;.$$

Theorems 11.8, 11.19 apply to the universal covering projection

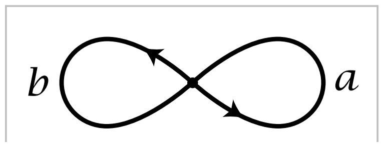
$$p \;:\; \widetilde{X} \;=\; \mathbb{R}^2 \to X \;=\; \widetilde{X}/G \;=\; K\ .$$

The fundamental group of the Klein bottle is thus

$$\pi_1(K) \;=\; \mathrm{Homeo}_p(\widetilde{X}) \;=\; G\ .$$

Example 11.23. The figure 8 space

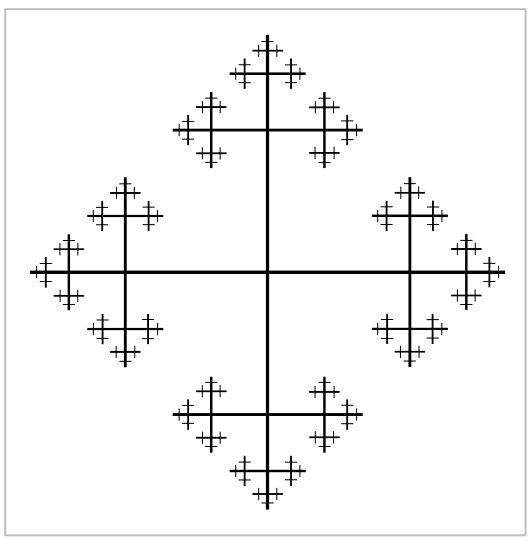
$$X \;=\; S^1 \times \{1\} \cup \{1\} \times S^1 \subset S^1 \times S^1$$



has universal cover $\widetilde{X}$ an infinite tree with one vertex for each word in two letters a, b

$$w \;=\; a^{m_1}b^{n_1}a^{m_2}b^{n_2}\dots a^{m_k}b^{n_k} \quad (m_j, n_j \in \mathbb{Z}, k \geqslant 0)$$

and one edge joining w, w' if and only if $w^{-1}w' \in \{a, a^{-1}, b, b^{-1}\}$.



The pictures are taken from pages 57,59 of Hatcher's *Algebraic Topology*, which can be downloaded from
http://www.math.cornell.edu/ hatcher/AT/ATpage.html

The projection $p : \widetilde{X} \to X$ sends each vertex to $1 \in S^1$, and maps each edge onto S^1 (one-one on the interior). For each word w let $h_w : \widetilde{X} \to \widetilde{X}$ be the unique covering translation with $h_w(w') = ww'$, so that $h_{w'w} = h_{w'}h_w$. The fundamental group of X is the free nonabelian group on two generators

$$\pi_1(X) \;=\; \mathrm{Homeo}_p(\widetilde{X}) \;=\; \{h_w \,|\, w\} \;=\; \mathbb{Z} * \mathbb{Z} \subset \mathrm{Homeo}(\widetilde{X})\ .$$

Remark 11.24. If $p : \widetilde{X} \to X$ is a universal covering projection satisfying the hypothesis of Theorem 11.19, then for any subgroup

$$G \subseteq \pi_1(X) \;=\; \mathrm{Homeo}_p(\widetilde{X})$$

there is defined a universal covering projection

$$q \;:\; \widetilde{Y} \;=\; \widetilde{X} \to Y \;=\; \widetilde{X}/G$$

satisfying the hypothesis of Theorem 11.19, with

$$\pi_1(Y) \;=\; \mathrm{Homeo}_q(\widetilde{Y}) \;=\; G \; .$$

If $G \subseteq \pi_1(X)$ is a normal subgroup the projection $r : Y \to X$ is a covering projection with

$$\mathrm{Homeo}_r(Y) \;=\; \pi_1(X)/G \; .$$

Remark 11.25. Theorem 11.19 gives a geometric method for computing the fundamental group of a path-connected space X which admits a universal covering $p : \widetilde{X} \to X$, namely

$$\pi_1(X, x_0) \;=\; \mathrm{Homeo}_p(\widetilde{X}) \;=\; p^{-1}(x_0) \; .$$

For any path-connected space X and $x_0 \in X$, let $\widetilde{X}$ be the topological space of equivalence class of paths $\alpha : I \to X$ such that $\alpha(0) = x_0$, with $\alpha \sim \alpha'$ if there exists a rel $\{0,1\}$ homotopy $\beta : \alpha \simeq \alpha' : I \to X$, and

$$p \;:\; \widetilde{X} \to X \;;\; \alpha \mapsto \alpha(1) \; .$$

It is a theorem that p is the universal covering projection of X with fibre $F = p^{-1}(x_0) = \pi_1(X, x_0)$ if X is *semi-locally simply-connected*, meaning that for every $x \in X$ there exists an open subset $U \subseteq X$ with $x \in U$ such that the inclusion $i : U \to X$ induces the trivial homomorphism $i_* = 1 : \pi_1(U, x) \to \pi_1(X, x)$ (in which case $p^{-1}(U)$ is homeomorphic to $U \times \pi_1(X, x)$).

In general, this is too synthetic a construction of the universal cover to be of use in the computation of $\pi_1(X)$. In practice, a geometrically interesting space X has a geometrically interesting universal cover $\widetilde{X}$, and this can be used to compute $\pi_1(X)$.

Lecture 21

12 Fundamental groups of surfaces and van Kampen's theorem — non-examinable

This section is a brief outline of the application of the fundamental group to the classification of compact surfaces (see Definition 6.1).

Remark 12.1. A **double cover** of a space X is a covering projection $p : Y \to X$ such that the fibre $F = \{-1, +1\}$ is the discrete space with two points. A nonorientable surface M has an **orientable double cover** $p : \widetilde{M} \to M$, with $\widetilde{M}$ an orientable surface. For example, $\widetilde{K} = T^2$, $\widetilde{\mathbb{RP}}^2 = S^2$.

Example 12.2.

1. $M(1) = \mathbb{RP}^2$, the projective plane, with orientable double cover $\widetilde{M}(1) = H(0) = S^2$.
2. $M(2) = K$, the Klein bottle, with orientable double cover $\widetilde{M}(2) = H(1) = T^2$.

The classification theorem 6.9 states that the following conditions on connected compact surfaces M, M' are equivalent:

1. M,M' are homeomorphic,
2. M and M' are both orientable or both nonorientable, and have the same genus,
3. the fundamental groups $\pi_1(M), \pi_1(M')$ are isomorphic.

Proposition 12.3. *If we write FS for the free abelian group generated by a set S, and $\langle x\rangle$ for the normal subgroup generated by x, then we have:*

1. *For the orientable surface $H(g)$ with $g \geq 1$, there exists a covering projection $p : \mathbb{R}^2 \to H(g)$ with*

$$\pi_1(H(g)) \;=\; \mathrm{Homeo}_p(\mathbb{R}^2) \;=\; F\{a_1, b_1, \ldots, a_g, b_g\}/\langle [a_1, b_1][a_2, b_2] \ldots [a_g, b_g]\rangle \;.$$

2. *For the nonorientable surface $M(g)$ with $g \geq 2$, there exists a covering projection $q : \mathbb{R}^2 \to M(g)$ with*

$$\pi_1(M(g)) \;=\; \mathrm{Homeo}_q(\mathbb{R}^2) \;=\; F\{c_1, c_2, \ldots, c_g\}/\langle (c_1)^2(c_2)^2 \ldots (c_g)^2\rangle \;.$$

3. *The orientable double cover (4.3) of $M(g)$ is*

$$\widetilde{M}(g) \;=\; H(g-1) \;.$$

Proof. See Armstrong p.168 and Stillwell, pp.82–84. $\square$

Surfaces have interesting geometric properties:

Example 12.4. 1. The expression of the torus as an quotient space

$$H(1) \;=\; T^2 \;=\; \mathbb{R}^2/\{(x,y)\sim(x+1,y)\sim(x,y+1)\}$$

makes apparent the covering projection $p:\mathbb{R}^2\to H(1)=T$. The homeomorphisms

$$\begin{aligned} a_1 &: \mathbb{R}^2 \to \mathbb{R}^2\ ;\ (x,y) \to (x+1,y)\ ,\\ b_1 &: \mathbb{R}^2 \to \mathbb{R}^2\ ;\ (x,y) \to (x,y+1) \end{aligned}$$

generate $\mathrm{Homeo}_p(\mathbb{R}^2)$, with the relation $a_1b_1=b_1a_1$ (or equivalently $[a_1,b_1]=1$).

2. The expression of the Klein bottle as an quotient space

$$M(2) \;=\; K \;=\; \mathbb{R}/\{(x,y)\sim(x+1,y)\sim(1-x,y+\tfrac{1}{2})\}$$

makes apparent the covering projection $q:\mathbb{R}^2\to M(2)=K$. The homeomorphisms

$$\begin{aligned} c_1 &: \mathbb{R}^2 \to \mathbb{R}^2\ ;\ (x,y) \to (1-x,y-\tfrac{1}{2})\ ,\\ c_2 &: \mathbb{R}^2 \to \mathbb{R}^2\ ;\ (x,y) \to (2-x,y+\tfrac{1}{2}) \end{aligned}$$

generate $\mathrm{Homeo}_q(\mathbb{R}^2)$, with the relation $(c_1)^2(c_2)^2=1$.

3. Note that q factors as $q:\mathbb{R}^2\xrightarrow{p}T^2\xrightarrow{r}K$ with $r:T^2\to K$ a covering projection with fibre $\mathbb{Z}_2$, and

$$a_1 \;=\; c_2c_1\ \ ,\ \ b_1 \;=\; (c_1)^{-2}\ :\ \mathbb{R}^2 \to \mathbb{R}^2\ ,$$

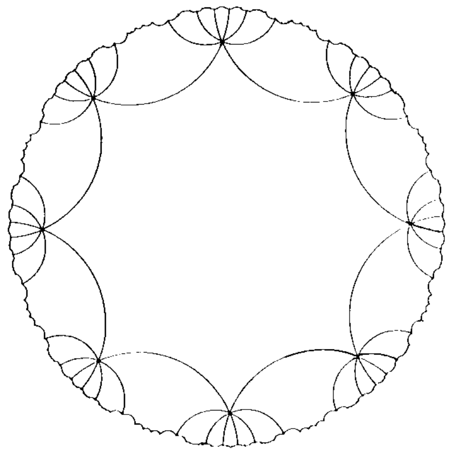
with $\pi_1(T^2)=\mathrm{Homeo}_p(\mathbb{R}^2)\triangleleft\pi_1(K)=\mathrm{Homeo}_q(\mathbb{R}^2)$ a normal subgroup of index 2. Thus $T^2=\widetilde{K}$ is the orientable double cover (4.3) of K.

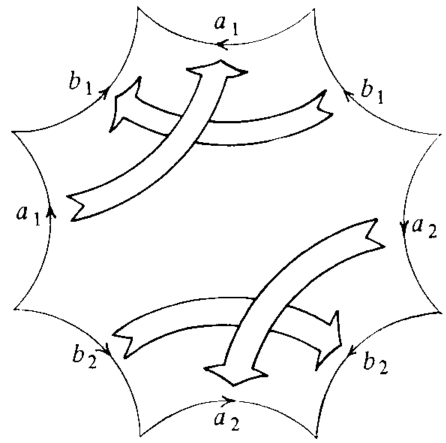
Remark 12.5. See Stillwell's **Geometry of surfaces** (51.51 Sti) for a beautiful account of the connections between surfaces and hyperbolic geometry: the groups of homeomorphisms $\pi_1(H(g))=\mathrm{Homeo}_p(\mathbb{R}^2)$, $\pi_1(M(g))=\mathrm{Homeo}_q(\mathbb{R}^2)$ in 12.3 for $g\geq 2$ are groups of isometries of the hyperbolic plane $\mathbb{H}^2$ (= the topological space $\mathbb{R}^2$ with a hyperbolic geometry). It is possible to realize each connected compact surface M of genus $g\geq 2$ as an quotient space $M=\mathbb{H}^2/\pi_1(M)$ with $\pi_1(M)$ acting on $\mathbb{H}^2$ by symmetries of a tessellation of $\mathbb{H}^2$. (The projection $\mathbb{H}^2\to M$ is a covering as in Definition 11.1).

Example 12.6. The Poincaré D^2-model for the hyperbolic plane $\mathbb{H}^2$ is the open disc $(D^2)^\circ$, in which the 'lines' are the circular arcs orthogonal to the boundary circle (at infinity) S^1. The orientable surface $H(2)$ of genus 2 is the 'hyperbolic surface'

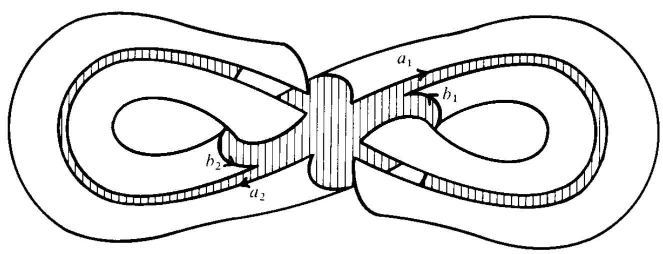
$$H(2) \;=\; \mathbb{H}^2/\pi_1(H(2))$$

with $\pi_1(H(2))=\{a_1,b_1,a_2,b_2\,|\,[a_1,b_1][a_2,b_2]\}$ acting on $\mathbb{H}^2$ by the symmetries of a tessellation of $\mathbb{H}^2$ by regular (hyperbolic) octagons, as in the first picture:





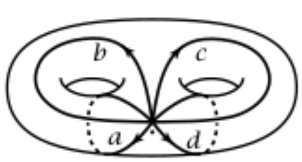
to a neighbour.
Here is a picture of $H(2)$ cut open along the loops a_1, b_1, a_2, b_2

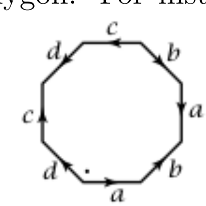


The pictures are from Stillwell's books.

12.1 Fundamental groups via van Kampen's theorem

If we look at the fundamental polygon associated to a surface, then each edge gives a closed loop on the surface, and hence an element of the fundamental group. The path which traces the whole boundary of the polygon is homotopic to a constant, by shrinking it to the centre of the polygon. For instance, for the orientable surface $H(2)$ of genus 2,





we have $[a], [b], [c], [d] \in \pi_1 H(2)$, with the class of the boundary path $[aba^{-1}b^{-1}cdc^{-1}d^{-1}] = 1$. We therefore have a group homomorphism

$$F\{a, b, c, d\}/\langle aba^{-1}b^{-1}cdc^{-1}d^{-1}\rangle \to \pi_1 H(2),$$

and this is precisely how the group isomorphisms of Proposition 12.3 arise. The following theorem establishes isomorphisms of this kind:

Definition 12.7. Given groups $G_i = FS_i/\langle T_i \rangle$ for $i = 1, 2, 3$, and group homomorphisms $\theta : G_2 \to G_1$ and $\phi : G_2 \to G_3$, the *pushout* $G_1 *_{G_2} G_3$ is given by

$$F(S_1 \cup S_3)/\langle T_1 \cup T_3 \cup \{\phi(s)\theta(s)^{-1} : s \in S_2\}\rangle.$$

This is a very ad hoc definition, which we have not even shown is well-defined (i.e. independent of the choices of generators and relations). The pushout is characterised by the following universal property, which ensures it is well-defined up to group isomorphism:

Proposition 12.8. *Given groups G_1, G_2, G_3 as above and a group H, the set of group homomorphisms $\alpha : G_1 *_{G_2} G_3 \to H$ is isomorphic to the set of pairs*

$$(\alpha_1 : G_1 \to H, \alpha_3 : G_3 \to H)$$

or group homomorphisms satisfying $\alpha_1 \circ \theta = \alpha_3 \circ \phi : G_2 \to H$.

For the natural group homomorphisms $\gamma_1 : G_1 \to G_1 *_{G_2} G_3$ and $\gamma_1 : G_3 \to G_1 *_{G_2} G_3$, the homomorphisms α_1, α_3 are given by $\alpha_i = \alpha \circ \gamma_i$.

Theorem 12.9 (van Kampen (very non-examinable))**.** *Let X be a topological space which is the union of two open and path connected subspaces U, W. Suppose $U \cap W$ is path connected and nonempty, and let x_0 be a point in $U \cap W$ that will be used as the base of all fundamental groups. Then*

$$\pi_1 X \cong (\pi_1 U) *_{\pi_1(U \cap W)} (\pi_1 W).$$

In particular, if W is contractible, then

$$\pi_1 X \cong (\pi_1 U)/\langle j_* \pi_1(U \cap W)\rangle,$$

for $j : U \cap W \to U$.

Proof. See Theorem 1.20 in Hatcher's book. □

Now given a compact surface $X = P/\sim$ for a fundamental $2n$-gon P, we let W be the interior of P, and $U = (P \setminus \{0\})/\sim$. Thus W is contractible, U is homotopy equivalent to n circles joined together at a single point, and $U \cap W$ is a punctured open polygon, so homotopy equivalent to a circle. Repeatedly applying van Kampen's theorem, we see that $\pi_1 U$ is the free group on n generators, and we see that j_* sends the generator of $\pi_1(U \cap W)$ to the expression corresponding to the equivalence relation on the boundary.

For instance the relation $aba^{-1}b^{-1}$ for $H(1)$ in Proposition 12.3 exactly corresponds to our standard representation for the torus, while the description of $M(2)$ relates to our usual representation of the Klein bottle by setting $a = c_1, b = c_1c_2$, so $(c_1)^2(c_2)^2 = aba^{-1}b$,

Lecture 22

13 A lightning introduction to homology (non-examinable)

Although we saw in §5.1 that higher homotopy groups $\pi_n X$ are quite easy to define, they can be very difficult to compute. A major open problem is to complete the calculation of the higher homotopy groups of n-spheres.

There is another series of groups, the homology groups $H_n(X)$ $(n \geq 0)$, which are quite easy to compute, but tricky to define — see for instance Ch. 2 of Hatcher's book. We actually saw the first homology group $H_1(X)$ in disguise in Proposition 6.13, since it is the abelianisation of $\pi_1 X$. The 0th homology group $H_0 X$ is even easier to describe: it is the free abelian group generated by the set $\pi_0 X$.

Instead of giving a definition of homology groups, I will now explain how they are computed. We start with four basic properties:

1. The homology groups of the empty space $\emptyset$ are given by $H_i(\emptyset) = 0$ for all $i \geq 0$.
2. The homology groups of a point $*$ are given by $H_0(*) = \mathbb{Z}$ and $H_i(*) = 0$ for all $i > 0$.
3. A map $f : X \to Y$ induces homomorphisms $f_* : H_n X \to H_n Y$.
4. If f is a homotopy equivalence, then f_* is an isomorphism.

On their own, these properties only suffice to describe homology of contractible and empty spaces, but there is an analogue of homology of van Kampen's theorem, called the Mayer–Vietoris theorem. We begin with a definition.

Definition 13.1. A sequence $A \to B \to C \to D \to \dots$ of group homomorphisms is called *exact* if the image of each map is the kernel of the next.

Theorem 13.2 (Mayer-Vietoris)**.** *Let X be a topological space and A, B be two subspaces whose interiors cover X (the interiors need not be disjoint). Denote the inclusion maps by $I : A \to X$ and $j : B \to X$. Then there is a long exact sequence*

$$\begin{aligned} \dots \to H_n(A \cap B) &\xrightarrow{(j_*, i_*)} H_n A \oplus H_n B \xrightarrow{i_* - j_*} H_n X \to H_{n-1}(A \cap B) \xrightarrow{(j_*, i_*)} \dots \\ \dots &\xrightarrow{i_* - j_*} H_1 X \to H_0(A \cap B) \xrightarrow{(j_*, i_*)} H_0 A \oplus H_0 B \xrightarrow{i_* - j_*} H_0 X \to 0. \end{aligned}$$

Example 13.3. The simplest application of the theorem comes when the subspaces A, B are disjoint, so $X = A \sqcup B$. In that case, the long exact sequence breaks down into exact sequences

$$0 \to H_n A \oplus H_n B \xrightarrow{i_* - j_*} H_n X \to 0,$$

giving an isomorphism $H_n(A \sqcup B) \cong H_n A \oplus H_n B$.

Remark 13.4. The Mayer–Vietoris theorem does not necessarily determine H_*X from H_*A and H_*B. This is because there can exist short exact sequences with the same ends but a different term in the middle, such as:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\;2\;} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$
$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Exercise 13.5.

1. By expressing S^n as a union of contractible open subspaces, prove inductively that for $n \geq 1$,
$$H_i(S^n) \cong \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{else.} \end{cases}$$

2. Now try some other spaces:

   For instance, the figure of 8 has homology groups $\mathbb{Z}, \mathbb{Z}^2, 0, 0, \ldots$, and in general $\underbrace{S^1 \vee \ldots \vee S^1}_{n}$ has homology groups $\mathbb{Z}, \mathbb{Z}^n, 0, 0, \ldots$.

   The orientable surfaces $H(g)$ have homology groups $\mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z}, 0, 0, \ldots$, while the non-orientable surfaces $M(g)$ have homology groups $\mathbb{Z}, \mathbb{Z}^g \oplus \mathbb{Z}/2\mathbb{Z}, 0, 0, \ldots$.

Exercise 13.6. Can you see how to use homology to show that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ for any $m \neq n$.

Exercise 13.7. Can you see how to use homology to prove the higher-dimensional **non-retraction theorem**: that for $n \geq 1$ there is no map $g : D^n \to S^{n-1}$ which is the identity on $S^{n-1} \subset D^n$ (cf. Theorem 10.8, which used the fundamental group). A corollary is the Brouwer fixed point theorem, that every map $f : D^n \to D^n$ has a fixed point.