On the Geometry of Singular Cubic Surfaces

Emily Luk, Ilyas Nicholson

Year 4 Project
School of Mathematics
University of Edinburgh
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Abstract

It is well established that isolated singularities can be categorised into *ADE* types. In a 1979 paper, *On the Classification of Cubic Surfaces*, J.W. Bruce and C.T.C. Wall give a table of results outlining the number of lines contained in cubic surfaces which were smooth away from some isolated singularities. This report proves these results of Bruce and Wall for each *ADE* cubic surface and demonstrates explicitly how these cubic surfaces can be categorised into *ADE* types from the Dynkin diagram patterns exhibited by their resolution.

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Chapter 1

Introduction

1.1 Motivation

> Wake an algebraic geometer in the dead of night, whispering: "27". Chances are, he will respond: "lines on a cubic surface."
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>
> *Ron Donagi*
> *Roy Smith*

There is a very famous result in algebraic geometry that says that any smooth cubic surface contains exactly 27 lines. Anyone who has ever studied algebraic geometry can recite this result like a 13 year old reciting Pythagoras' theorem. But what happens when we remove the condition of smoothness? Do all non-smooth cubic surfaces act similarly to one another? Is there a magic number that unifies non-smooth surfaces in the same way that 27 seems to do for all smooth surfaces?

In a 1979 paper, *On the Classification of Cubic Surfaces* [1], J.W. Bruce and C.T.C. Wall explored this question and compiled a table demonstrating the different cases of cubic surfaces which were smooth away from some isolated points and how many lines there were in each case.

1.2 Outline of Report

The idea of this report is to explicitly prove the number of lines in each case of an *ADE* singular cubic surface as given by Bruce and Wall. Along the way we will give some examples of how we can categorise singularities into their *ADE* types by resolving them using blow up and give an example of how we can find the 27 lines in one example of a smooth surface.
The broad structure of this report is as follows:

- **Basic Algebraic Geometry:**
  Chapter 2 gives the basic algebraic geometry which is necessary to under-

stand this report including defining projective space and subsets of that and stating Bezout's Theorem and other related results;

- **Singular Cubic Surfaces:**
  Chapter 3 gives more of the specific algebraic geometry which will be relevant to this report including defining singular cubic surfaces, presenting the theory of blow up and how exactly Dynkin diagrams relate to singularities;

- **Resolving and categorising singularities:**
  Chapter 4 gives examples of how to resolve and categorise ADE singularities using blow up;

- **27 lines:**
  Chapter 5 gives an example of a smooth surface in the Fermat cubic and describes the 27 lines contained in it;

- **Reconstructing Bruce and Wall:**
  Chapters 6 and 7 outline the proofs of the result for each case of Bruce and Wall's table.

Chapter 2

Basic Algebraic Geometry

In this chapter we will outline the background knowledge of algebraic geometry necessary to follow the rest of the report. This will include defining necessary objects such as projective space and lines, conics and curves within projective spaces. We will also highlight theorems and results used later in the report for proofs. Much of this section will follow Miles Reid's textbook *Undergraduate Algebraic Geometry* [13] and Ivan Cheltsov's lecture notes for the course titled *Algebraic Geometry* [2].

2.1 Projective Space

In this report we will be talking specifically about surfaces in complex projective 3-space, $\mathbb{P}^3$. In this section we will define the relevant projective space; the complex projective space $\mathbb{P}^n_{\mathbb{C}}$.

Remark. Although there exists projective space over any field this report will only work with projective space over $\mathbb{C}$. We will often drop the subscript $\mathbb{C}$ for ease and use $\mathbb{P}^n$ to mean complex projective n-space.

Definition 2.1.1. We define $\mathbb{C}^3$ as being the set of all triplets (x, y, z) where $x, y, z \in \mathbb{C}$.

Clearly here we have that two triplets $(x, y, z), (x', y', z') \in \mathbb{C}^3$ are only considered to be equivalent if $x = x', y = y'$ and $z = z'$.

There is a natural relation, $\sim$, on $\mathbb{C}^3 \setminus (0, 0, 0)$ defined by

$$(x, y, z) \sim (x', y', z') \iff (x, y, z) = \lambda(x', y', z')$$

for some non-zero $\lambda \in \mathbb{C}$. We can easily spot that $\sim$ is an equivalence relation so we can naturally define the equivalence classes of $\sim$. For every point $(x, y, z) \in \mathbb{C}^3 \setminus (0, 0, 0)$, the equivalence class of (x, y, z) is denoted by $[x : y : z]$. This is the set consisting of all the points $\lambda(x, y, z)$, where $\lambda \in \mathbb{C}^3 \setminus (0, 0, 0)$.

Definition 2.1.2. The **complex projective plane** is defined as the set consisting of all the equivalence classes $[x : y : z]$, where $(x, y, z) \in \mathbb{C}^3 \setminus (0, 0, 0)$. Formally, this is denoted by $\mathbb{P}^2_{\mathbb{C}}$, the projective plane associated with $\mathbb{C}^3$.

We can notice here that we can informally think of $\mathbb{P}^2$ as 3 copies of $\mathbb{C}^2$ glued together. For example consider the sets U_x, U_y and U_z defined as the subsets of $\mathbb{P}^2$ where $x \neq 0, y \neq 0$ and $z \neq 0$ respectively. Clearly the union of these sets cover $\mathbb{P}^2$.

We can see that each of these is a copy of $\mathbb{C}^2$ embedded in $\mathbb{P}^2$. Consider the chart U_z which is defined by $z \neq 0$ then this is all the points

$$\left[\frac{x}{z} : \frac{y}{z} : 1\right]$$

for any $x, y \in \mathbb{C}$. So clearly $U_z \cong \mathbb{C}^2$. It is possible to extend this definition to n dimensions. We will now consider the relation $\sim$ on $\mathbb{C}^n$.

Similar to our definition of $\mathbb{C}^3$ above we can define $\mathbb{C}^n$ as the set of all the n- tuples $(x_1, ..., x_n)$ such that $x_i \in \mathbb{C}$ for all $1 \leq i \leq n$. Then we can define the relation $\sim$ as follows

$$(x_1, ..., x_n) \sim (x'_1, ..., x'_n) \iff (x_1, ..., x_n) = \lambda(x'_1, ..., x'_n)$$

for some non-zero $\lambda \in \mathbb{C}$. This is also an equivalence relation. The equivalence classes of this relation are also denote by $[x_1 : ... : x_n]$. So it is a straightforward extension of the case concerning $\mathbb{C}^3$ by which we reach the definition of $\mathbb{P}^n$.

Definition 2.1.3. We can define **complex projective n-space** as the set consiting of all the equivalence classes $[x_1 : ... : x_{n+1}]$, where $(x_1, ..., x_{n+1}) \in \mathbb{C}^{n+1} \setminus 0$. This is the projective space denoted $\mathbb{P}^n_{\mathbb{C}}$ associated with $\mathbb{C}^{n+1}$.

Remark. Similarly to $\mathbb{P}^2$, we can view any $\mathbb{P}^n$ as $n+1$ copies of $\mathbb{C}^n$ glued together.

2.2 Subsets of Projective space

Now that we have defined what we mean by projective space, it is necessary to define objects within this. Objects in projective space such as lines, planes, curves and surfaces are defined by multivariate homogeneous polynomials.

2.2.1 Hyperplanes

A hyperplane in $\mathbb{P}^n$ is the space defined by a degree 1 polynomial.

Definition 2.2.1. A **hyperplane** in $\mathbb{P}^n$ is the subset of $\mathbb{P}^n$ defined by the equation

$$a_1x_1 + ...a_{n+1}x_{n+1} = 0$$

for $a_1, ..., a_n \in \mathbb{C}$ such that at least one a_i is non-zero.

A hyperplane in $\mathbb{P}^n$ is a space with dimension $n - 1$. So we can think of a hyperplane in $\mathbb{P}^n$ as being the space $\mathbb{P}^{n-1}$ embedded in $\mathbb{P}^n$. In fact, a hyperplane in $\mathbb{P}^2$ is just a line as we will see in the next section and a hyperplane in $\mathbb{P}^3$ is a plane.

2.2.2 Lines

For this report we are considering lines contained inside of surfaces in $\mathbb{P}^3$. Therefore it will be important to define lines in $\mathbb{P}^2$ and $\mathbb{P}^3$.

Definition 2.2.2. A **line in** $\mathbb{P}^2$ is some subset of $\mathbb{P}^2$ defined by the equation

$$ax + by + cz = 0$$

for $a, b, c \in \mathbb{C}$ such that $(a, b, c) \neq (0, 0, 0)$.

We can define a line in $\mathbb{P}^3$ nicely in the sense that the definition follows what our intuition suggests from our knowledge of how planes intersect in real space.

Definition 2.2.3. Consider distinct planes Π and Π' in $\mathbb{P}^3$. The subset defined by $\Pi \cap \Pi'$ is a **line in** $\mathbb{P}^3$.

We can also define a line by 2 distinct points in $\mathbb{P}^3$.

Proposition 2.2.4. *For any 2 distinct points $P, Q \in \mathbb{P}^3$ the equation*

$$\lambda P + \mu Q,$$

*where λ and μ are variables in $\mathbb{C}$, also defines a unique **line** in $\mathbb{P}^3$.*

2.2.3 Planar Curves and Conics

The space defined by one homogeneous multivariate polynomial in $\mathbb{P}^2$ is called a **curve**. Similarly the space defined by one polynomial in $\mathbb{P}^3$ is called a **surface**. A surface intersected with a plane in $\mathbb{P}^3$ will be the space defined by one polynomial on a copy of $\mathbb{P}^2$ so we have a planar curve on a plane in $\mathbb{P}^3$.

Definition 2.2.5. Let F be the curve defined by the polynomial $f = 0$ and G be the curve defined by the polynomial $g = 0$. We say that F and G do not share a common component if and only if f and g are coprime.

An important example of a planar curve is a conic curve of degree 2.

Definition 2.2.6. A **conic in** $\mathbb{P}^2$ is a subset of $\mathbb{P}^2$ given by the equation

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$

for $a, b, c, d, e, f \in \mathbb{C}$ such that $(a, b, c, d, e, f, g) \neq 0$.

We can represent the conic in the above definition by a matrix which we will call M.

$$M = \begin{pmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{pmatrix}.$$

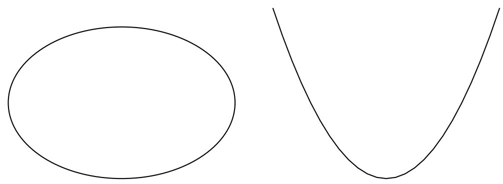
The equation $\begin{pmatrix} x & y & z \end{pmatrix} M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ is the same as the equation in the definition.

Definition 2.2.7. We say that a curve defined by $f(x, y, z) = 0$ in $\mathbb{P}^2$ is **irreducible** if the polynomial $f(x, y, z)$ is an irreducible polynomial.

Lemma 2.2.8. *A conic is irreducible if and only if the determinant of its corresponding matrix M is non-zero.*

Remark. A reducible conic is sometimes called a degenerate conic.

Non-degenerate conics are nice and familiar as they are analogous to subsets of $\mathbb{R}^2$ that we are used to working with.



Ellipse Parabola

Figure 2.1: Real analogues of Irreducible (non-degenerate) Conics.

2.2.4 Tangents

Given a point on a space we can construct a tangent space to that point. It is important to think about tangents when we talk about how objects intersect, as will be demonstrated later in the report.

Definition 2.2.9. Consider the space X in $\mathbb{P}^n$ defined by

$$f(x_1, ..., x_{n+1}) = 0.$$

The **tangent space** at a point $[a_1 : ... : a_{n+1}] \in X$ is the hyperplane

$$\frac{\partial f}{\partial x_1}(a, ..., a_{n+1})x_1 + ... + \frac{\partial f}{\partial x_{n+1}}(a, ..., a_{n+1})x_{n+1} = 0.$$

2.3 Bézout's Theorem

Now that we have defined objects such as conics, lines and curves we can ask the question: how do these objects intersect in projective space?

Theorem 2.3.1 (Bézout's Theorem)**.** *Let F and G be projective plane curves of degree m and n respectively. Assume F and G have no component in common. Then*

$$\sum_p (F \cdot G)_p = mn.$$

Remark. Here we are using the notiation $(F \cdot G)_p$ to denote the multiplicity of the intersection of F and G at the point p and sum over all points p in the intersection of F and G. A precise definition of intersection multiplicity will not be necessary for this report however the interested reader should consult [9, Chapter 7].

Bezout's Theorem is a result that can be found in most texts on algebraic geometry as it is the most important result for telling us about how objects live in the projective plane. Due to this, statements of the theorem vary slightly. The one used above is taken almost directly from [8, Section 5.3]. This text also contains an easy to follow proof of this statement.

Definition 2.3.2. For 2 curves C and Z we say that C intersects Z **transversally** at $P \in C \cap Z$ if the following two conditions are satisfied:

1. both curves C and Z are smooth at the point P;

2. the lines in $\mathbb{P}^2$ that are tangent to C and Z at the point P are different.

We will define exactly what it means to be 'smooth' in Section 3.1

Remark. A point where curves intersect transversally has intersection multiplicity 1.

2.4 Corollaries of Bézout's Theorem

There are some obvious consequences of Bézout's Theorem which will be important to us which we will highlight in this section. However, many of these results can actually be proved only using the Fundamental Theorem of Algebra.

Lemma 2.4.1. *For L and L' which are distinct lines in $\mathbb{P}^2_{\mathbb{C}}$, the intersection $L \cap L'$ consists of one point.*

Lemma 2.4.2. *For a line L and a plane Π in $\mathbb{P}^3_{\mathbb{C}}$, the intersection $L \cap \Pi$ contains exactly one point or $L \subset \Pi$.*

As we can see, a line in the projective complex plane is clearly analogous to a line in complex space. Similarly, a non-degenerate conic in projective complex space is analogous to a conic section in the complex plane (e.g. parabola, circle, ellipse). With this information we can imagine a picture like the figure below.

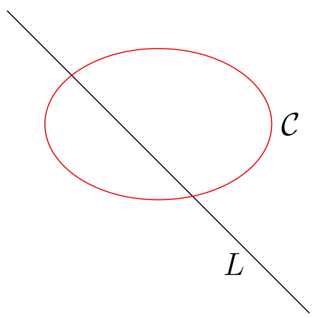


Figure 2.2: An intuitive picture of a line and a conic in projective space.

It feels intuitive that a conic would intersect a line at 1 or 2 points.

Lemma 2.4.3. *For any line L and irreducible conic $\mathcal{C}$ in $\mathbb{P}^2$, $\mathcal{C} \cap L$ consists of two points counted with multiplicities.*

Lemma 2.4.4. *For an irreducible conic $\mathcal{C}_2$ and a degree 3 curve $\mathcal{C}_3$, if $\mathcal{C}_2$ and $\mathcal{C}_3$ intersect transversally at every point in $\mathcal{C}_2 \cap \mathcal{C}_3$, then they intersect at exactly 6 points.*

Chapter 3

Singular Cubic Surfaces

3.1 Defining Cubic Surfaces and Singularities

Definition 3.1.1. A **cubic surface in** $\mathbb{P}^3$ is a subset of $\mathbb{P}^3$ defined by

$$f(x, y, z, t) = 0$$

where f is a homogeneous, degree 3 polynomial in t, x, y, z.

Definition 3.1.2. A cubic surface $S \subset \mathbb{P}^n$ is said to be **singular** if there exists a point $P = [x_1 : y_1 : z_1 : t_1] \in S$ such that

$$\begin{cases} f_x(x_1, y_1, z_1, t_1) = 0, \\ f_y(x_1, y_1, z_1, t_1) = 0, \\ f_z(x_1, y_1, z_1, t_1) = 0, \\ f_t(x_1, y_1, z_1, t_1) = 0, \end{cases}$$

and such a point P is called a **singularity** of S. In other words the singular points of a surface are those points at which all the partial derivatives simultaneously vanish.

Definition 3.1.3. We say that a surface $S \subset \mathbb{P}^3$ is **smooth** if it has no singular points.

Now we have the notion of a smooth surface we can state an important theorem that motivates this report.

Theorem 3.1.4. *A smooth cubic surface contains exactly 27 lines.*

A proof of this theorem is tangential to the scope of this report however we will give an example of a smooth surface and find all 27 lines later. An accessible proof of this fact is given in chapter 7 of Miles Reid's *Undergraduate Algebraic Geometry* [13].

We can define singularities more generally for any subsets of $\mathbb{P}^n$ given in chapter 2. Any subset defined by $f = 0$ for some polynomial f is singular where all its partial derivatives vanish. If no such point exists then it is smooth.

Most of the surfaces we will consider in this report will only contain 'isolated' singularities. Formally we can define what it means to be isolated as follows.

Definition 3.1.5. A singularity P of some cubic surface $S \in \mathbb{P}^3$ is considered **isolated** if there is some Zariski open subset U of $\mathbb{P}^3$ such that $P \subset U$ and there are no other singularities of S contained in U.

A precise understanding of the Zariski topology will not be important for the arguments within this project. For a rigorous explanation of the Zariski topology see chapter 6 of Fulton's *Algebraic curves, An Introduction to Algebraic Geometry* [8].

Remark. The important thing to take away from this definition is that intuitively we can see that in the case where we have only finitely many singularities we cannot have singularities infinitesimally close together. Therefore some such open set exists. So these are clearly isolated. In the case where we have a line of singularities, they are clearly not isolated.

3.2 How Can We Resolve Singularities?

3.2.1 Blow Up

In order to resolve our singularities we will use a technique called blow up.
Blow up is a useful tool in algebraic geometry. It is possible to blow up any algebraic variety but we will specifically be working over complex spaces. The following definition of blow up is tailored from [11, p29] and [15, p13] specifically for $\mathbb{C}^n$.

Let us start with the example of $\mathbb{C}^2$. Consider $\mathbb{C}^2_{x,y} \times \mathbb{P}^1_{[a:b]}$. Note that the projective space $\mathbb{P}^1_{[a:b]}$ can be covered by 2 charts given by $a \neq 0$ and by $b \neq 0$. Let us look at the chart U_b, where $b \neq 0$. Under this consider

$$\begin{aligned}
[x : y] &= [a : b], \\
\Rightarrow \left[\frac{x}{y} : 1\right] &= \left[\frac{a}{b} : 1\right], \\
\Rightarrow \frac{x}{y} &= \frac{a}{b}.
\end{aligned}$$

Rearranging the above equation results in the following

$$xb = ya.$$

Consider the set $X \subseteq \mathbb{C}^2_{x,y} \times \mathbb{P}^1_{[a:b]}$ defined by the equation $xb = ya$. Then we have a natural map π such that

$$\pi : X \to \mathbb{C}^2_{x,y},$$
$$((x,y),[a:b]) \mapsto (x,y).$$

The set X is the blow up of $\mathbb{C}^2$ at the point O.

We can define this in general for $\mathbb{C}^n$.

Definition 3.2.1. We can define the **blow up of** $\mathbb{C}^n$ **at the point** $O = (0,...,0)$ to be the subset $X = \{x_i y_j = x_j y_i : i,j = 1,...n\} \subset \mathbb{C}^n \times \mathbb{P}^{n-1}$ where x_i are the coordinates of $\mathbb{C}^n$ and y_i are the coordinates of $\mathbb{P}^n$.

We now have the natural map π

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "\pi"'] & \mathbb{C}^n \times \mathbb{P}^{n-1} \arrow[d, "pr_1"] \\ & \mathbb{C}^n \end{tikzcd}$$

which is a restriction of the projection map. We can use π to define the blow up of subspaces of $\mathbb{C}^n$.

Definition 3.2.2. If Y is a closed subspace of $\mathbb{C}^n$ passing through O, we define the **blow up of** Y at the point O to be $\tilde{Y} = (\pi^{-1}(Y \setminus \{O\}))$. To blow up at any other point P of $\mathbb{C}^n$, make a linear change of coordinates sending P to O.

A full understanding of the topology will again not be necessary and we can take for granted that all our surfaces are closed and so we can use π^{-1} to find the blow up of our surfaces as in the definition above.

We can see that this map π is an isomorphism away from $O \in \mathbb{C}^n$.

Definition 3.2.3. The **exceptional divisor** of the blow up, E, is defined as

$$E = \pi^{-1}(O).$$

We will later use the blow up of $\mathbb{C}^3$ to resolve singularities on a cubic surface S. In this case our exceptional divisor will be a surface, E and the intersection $S \cap E$ will give **exceptional curves**.

3.2.2 How we use Blow Up to Resolve Singularities

Continuing with our example of $\mathbb{C}^2$ in section 3.2.1 we can construct an intuitive picture of how the technique is helpful when resolving singularities, as can be seen in Figure (3.1) below.

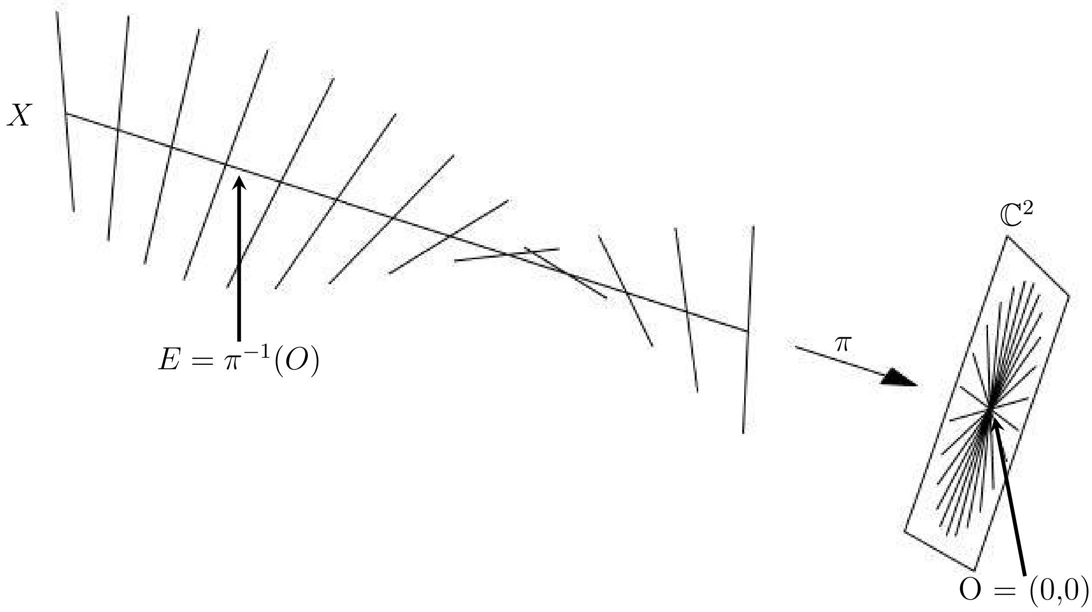


Figure 3.1: blow up of $\mathbb{C}^2$ [7].

On the right hand side of the figure we consider a plane curve C which lies in $\mathbb{C}^2$ and has an isolated singularity at O. Then the blow up at the origin is $X \subset \mathbb{C}^2_{x,y} \times \mathbb{P}^1_{[a,b]}$, where X satisfies $xb = ya$. We know that we can cover $\mathbb{P}^1_{[a,b]}$ with 2 charts given by $a \neq 0$ and $b \neq 0$. Let us arbitrarily choose to work in the chart given by $a \neq 0$, allowing us to set $a = 1$. Let us call the resulting transformation π^*. Then the full transform of C is denoted as $\pi^*(C)$. We can then observe that $\pi^*(C) = E \cup \tilde{C}$, where E is the exceptional curve and $\tilde{C}$ is the proper transform of C.

From the figure we can then observe that blow up essentially separates the parts of the plane curve which passed through the singularity, resulting in a smooth curve and thus resolving the singularity. The effects of blowing-up $\mathbb{C}^3$ are more difficult to picture, but it can be considered to be similar to what happens in the $\mathbb{C}^2$ case above.

3.3 What are the Types of Singularities Present in Cubics?

In October 1934 Dr. Patrick Du Val published 3 papers entitled *On isolated singularities of surfaces which do not affect the conditions of adjunction* [4], [5],[6] in which he discussed these isolated singularities on a complex plane to which he gives his name. These 'Du Val' singularities can be categorised into ADE-types.

By representing each exceptional curve as a vertex and connecting all the intersecting vertex by an edge we can produce a graph. These graphs are Dynkin diagrams [12].

The corresponding Dynkin diagram for each of the ADE type singularities can be seen below.

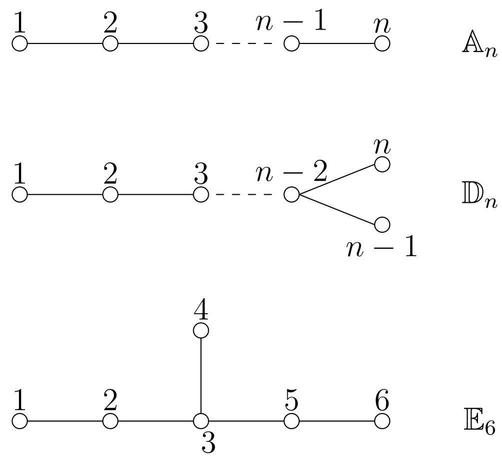


Figure 3.2: $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_n$ Dynkin diagrams

Cubic surfaces can contain singularities of $\mathbb{A}_n$ with $n \leq 5$, $\mathbb{D}_4$, $\mathbb{D}_5$ and $\mathbb{E}_6$ types.

In Chapter 4 we will show how we can resolve a singularity on a cubic surface and determine the type of singularity by the process of resolving it via blow up. Later, in chapter 6, we will demonstrate one of the differences in properties of singular surfaces with different types of singularity.

Remark. We will use the notation $\mathbb{A}_1 2\mathbb{A}_2$ for example, to mean a surface containing exactly one $\mathbb{A}_1$ singularity and two $\mathbb{A}_2$ singularities.

Chapter 4

Resolving And Categorising Cubic Singularities

Now that we have defined what the general technique of blow up is, we wish to apply it specifically to surfaces in $\mathbb{C}^3$. Much of the theory we present in this section has been adapted from [7], [11] and [13]. After this theoretical set-up we will then demonstrate explicitly how to resolve singularities with some examples.

4.1 Blowing Up $\mathbb{C}^3$

Let us work in the space $\mathbb{C}^3_{x,y,z} \times \mathbb{P}^2_{[a:b:c]}$ The space $\mathbb{P}^2_{[a:b:c]}$ can be covered 3 charts, i.e. the cases defined by $a \neq 0, b \neq 0$ and $c \neq 0$ which we will denote by U_a, U_b and U_c respectively. Working in the chart U_c we have $c \neq 0$. Then consider

$$\begin{aligned} [x : y : z] &= [a : b : c], \\ \Rightarrow \left[\frac{x}{z} : \frac{y}{z} : 1\right] &= \left[\frac{a}{c} : \frac{b}{c} : 1\right], \\ \Rightarrow \frac{x}{z} = \frac{a}{c} &\text{ and } \frac{y}{z} = \frac{b}{c}. \end{aligned}$$

By rearranging and substitution, we can derive the system of equations

$$xb = ya, \tag{4.1}$$
$$yc = bz, \tag{4.2}$$
$$xc = za, \tag{4.3}$$

or equivalently

$$\operatorname{rank} \begin{pmatrix} x & y & z \\ a & b & c \end{pmatrix} \leq 1.$$

Now, let us consider a set $X \subseteq \mathbb{C}^3_{x,y,z} \times \mathbb{P}^2_{[a:b:c]}$ such that everything in X satisfies the above system of equations. Then we have a natural map π, where π is a

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, "\pi"'] & \mathbb{C}^3_{x,y,z} \times \mathbb{P}^2_{[a:b:c]} \arrow[d, "pr_1"] \\
& \mathbb{C}^3_{x,y,z}
\end{tikzcd}$$

restriction of our projection map to X and is defined by

$$\pi : X \to \mathbb{C}^3_{x,y,z},$$
$$((x, y, z), [a : b : c]) \mapsto (x, y, z).$$

This set X is considered the blow up of $\mathbb{C}^3$ at O. The method of blow up enables us to consider how a singular surface S acts locally about the point $O = (0, 0, 0)$.

The space $\mathbb{C}^3 \times \mathbb{P}^2$ is difficult to work in, so in order to simplify it we use our 3 charts which we defined earlier. To gain some understanding of how these charts simplifies things, let us first look at the chart of $\mathbb{P}^2$, U_c. So we are working in $\mathbb{C}^3 \times U_c$. Let us denote $X \cap (\mathbb{C}^3 \times U_c)$ as X_c.

We begin by noticing that $U_c = \mathbb{C}^2$ which is easily seen if we define a new set of coordinates $(\overline{x}, \overline{y})$ as

$$\overline{x} = \frac{a}{c}, \qquad \overline{y} = \frac{b}{c}$$

where $c \neq 0$. Our space is now reduced such that $X_c \subset \mathbb{C}^3_{x,y,z} \times \mathbb{C}^2_{\overline{x},\overline{y}}$. Applying this change of coordinates to the system defined by equations (4.1), (4.2) and (4.3) we obtain a new system given by

$$x\overline{y} = y\overline{x}, \qquad y = \overline{y}z, \qquad x = \overline{x}z,$$

Observe that it is possible to reduce this system of equations significantly. Ultimately we have that $X_c \subset \mathbb{C}^5_{x,y,z,\overline{x},\overline{y}}$. The system of equations becomes

$$y = \overline{y}z, \tag{4.4}$$
$$x = \overline{x}z. \tag{4.5}$$

For consistency we can set $z = \overline{z}$ and our system of equations becomes $x = \overline{xz}, y = \overline{yz}$ and $z = \overline{z}$.

We can substitute this new system of coordinates into the original equation of S to obtain to blow up of S at $O = (0, 0, 0)$.

4.2 Resolving $\mathbb{A}_1$

Sakamaki gives an example of a cubic surface and claims it is an $\mathbb{A}_1$ type singularity, the equation of which can be seen in Appendix B.1. In this section we will prove explicitly that this is the case, demonstrating how to resolve singularities.

Using Sakamaki's notation, let us define

$$
\begin{aligned}
f_2 &= xz - y^2, \\
f_3 &= (x - ay)(-x + (b+1)y - bz)(y - cz),
\end{aligned}
$$

where a, b, c are distinct elements in $\mathbb{C} \setminus \{0, 1\}$. Choose $a = -1, b = 2$ and $c = 3$, then let the surface $S \subset \mathbb{P}^3$ be described by $f(x, y, z, t) = tf_2 - f_3 = 0$, i.e.

$$
f = txz - ty^2 + x^2y - 3x^2z - 2xy^2 + 8xyz - 6xz^2 - 3y^3 + 11y^2z - 6yz^2.
$$

Taking partial derivatives we see

$$
\begin{aligned}
f_x &= tz + 2xy - 6xz - 2y^2 + 8yz - 6z^2, \\
f_y &= -2ty + x^2 - 4xy + 8xz - 9y^2 + 22yz - 6z^2, \\
f_z &= tx - 3x^2 + 8xy - 12xz + 11y^2 - 12yz, \\
f_t &= xz - y^2.
\end{aligned}
$$

This is complicated to solve by hand, so we employed the use of Maple to find the solution to $f_x = f_y = f_z = f_t = 0$. The code for this can be found in Appendix C.1. Then we can observe that $P = [0 : 0 : 0 : 1]$, a point in $\mathbb{P}^3$, is the only singular point for S.

Let us then work in the chart defined by $t \neq 0$. Informally, this is equivalent to us setting $t = 1$. Then we may rewrite our equation describing $S \subset \mathbb{C}^3$ as $f = xz - y^2 + x^2y - 3x^2z - 2xy^2 + 8xyz - 6xz^2 - 3y^3 + 11y^2z - 6yz^2 = 0$, where the singular point is given by $(0, 0, 0)$. To resolve this singularity we can blow up this surface. Working in the chart given by $z \neq 0$ for the blow up , let us use the coordinate system given by given by

$$
\begin{aligned}
x &= \overline{xz}, \\
y &= \overline{yz}, \\
z &= \overline{z}.
\end{aligned}
$$

Then the full transform $\pi^*(S)$ can then be described by

$$
\begin{aligned}
&\overline{xz}^2 - \overline{y}^2\overline{z}^2 + \overline{x}^2\overline{yz}^3 - 3\overline{x}^2\overline{z}^3 - 2\overline{xy}^2\overline{z}^3 + 8\overline{xyz}^3 - 6\overline{xz}^3 - 3\overline{y}^3\overline{z}^3 + 11\overline{y}^2\overline{z}^3 - 6\overline{yz}^3 = 0, \\
&\Rightarrow \overline{z}^2(\overline{x} - \overline{y}^2 + \overline{x}^2\overline{yz} - 3\overline{x}^2\overline{z} - 2\overline{xy}^2\overline{z} + 8\overline{xyz} - 6\overline{xz} - 3\overline{y}^3\overline{z} + 11\overline{y}^2\overline{z} - 6\overline{yz}) = 0.
\end{aligned}
\tag{4.6}
$$

Let us define $\tilde{S} : \overline{x} - \overline{y}^2 + \overline{x}^2\overline{yz} - 3\overline{x}^2\overline{z} - 2\overline{xy}^2\overline{z} + 8\overline{xyz} - 6\overline{xz} - 3\overline{y}^3\overline{z} + 11\overline{y}^2\overline{z} - 6\overline{yz} = 0$, i.e. the proper transform of S. To see what our exceptional curves are we consider the intersection $\{\overline{z} = 0\} \cap \tilde{S}$. This yields the single exceptional curve, F, described by

$$F = \begin{cases} \overline{z} = 0, \\ \overline{x} - \overline{y}^2 = 0. \end{cases}$$

Using Maple again as in Appendix C.1 we can also check that $\tilde{S}$ has no singular points, and therefore this is the minimal amount of blow ups required to resolve this singularity. Hence we can determine that the corresponding Dynkin diagram looks like the figure shown below.

$$\overset{F}{\circ}$$

From Figure (3.3) we observe that this corresponds to an $\mathbb{A}_1$ singularity, implying that the point P on our original surface S is an $\mathbb{A}_1$ singularity.

4.3 Resolving $\mathbb{A}_2$

In this section we further consolidate our knowledge of blow ups by considering another relatively simple example of a surface which has an $\mathbb{A}_2$ singularity.

Consider the surface $S \subset \mathbb{C}^3$ described by $f(x, y, z) = x^2 + y^2 + z^3 = 0$. Taking partial derivatives we see

$$f_x = 2x, \qquad f_y = 2y, \qquad f_z = 3z^2, \qquad f = 0.$$

By observation, the only singular point for this surface is $P = (0, 0, 0)$. We can resolve this using blow up.

Working in the chart defined by $z \neq 0$ on the blow up, let us use the change of coordinates outlined previously:

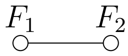
$$\begin{aligned} x &= \overline{xz}, \\ y &= \overline{yz}, \\ z &= \overline{z}. \end{aligned}$$

In this new coordinate system, after division by $\overline{z}^2$ we see that our proper transform of S is given by $\tilde{S} : \overline{x}^2 + \overline{y}^2 + \overline{z} = 0$. In this case our exceptional curves, F_1 and F_2, can also be obtained by the intersection $\{\overline{z} = 0\} \cap \tilde{S}$, yielding the following two sets of equations

$$F_1 = \begin{cases} \overline{z} = 0, \\ \overline{x} + i\overline{y} = 0, \end{cases} \qquad \text{and} \qquad F_2 = \begin{cases} \overline{z} = 0, \\ \overline{x} - i\overline{y} = 0. \end{cases}$$

We can observe that F_1 and F_2 intersect at the point (0,0,0).

We can clearly see that the partial derivative of $\tilde{S}$ with respect to $\overline{z}$ is equal to 1. Since this always equals 1 and can never equal 0, it follows that $\tilde{S}$ is smooth. Therefore this is the minimal amount of blow ups necessary to resolve the singularity. Combining this with the knowledge that we have exactly 2 exceptional surfaces which intersect each other, the corresponding Dynkin diagram will look like the figure below.



From Figure (3.3) we observe that this corresponds to an $\mathbb{A}_2$ singularity, showing that the point P on our original cubic surface S is in fact an $\mathbb{A}_2$ singularity.

4.4 Resolving $\mathbb{A}_5$

For a more complex example which requires multiple blow ups before resolution is achieved, with only one singularity in each iteration, we look at the equation Sakamaki outlines for a surface with an $\mathbb{A}_5$ singularity. The equation defining this surface was adapted from Appendix B.1.

Using Sakamaki's notation, let us define

$$\begin{aligned} f_2 &= xy, \\ f_3 &= x^3 + y^3 - yz^2. \end{aligned}$$

Let our surface $S \subset \mathbb{P}^3$ be described by $f(x, y, z, t) = tf_2 - f_3 = 0$, i.e.

$$f = xyt - x^3 - y^3 + yz^2 = 0.$$

Then taking partial derivatives of this, we obtain

$$f_x = y - 3x^2, \qquad f_y = x - 3y^2 + z^2, \qquad f_z = 2yz, \qquad f_t = xy.$$

We want to solve $f_x = f_y = f_z = f_t = 0$. We observe that the only singular point of our surface is $P = [0 : 0 : 0 : 1]$. As before let us work in the chart defined by $t \neq 0$. Hence let us rewrite our equation as $f = xy - x^3 - y^3 + yz^2 = 0 \subset \mathbb{C}^3$, where the singular point is given by $(0, 0, 0)$.

To resolve this singularity we blow up. Working in the chart given by $z \neq 0$ on the blow up, let us use the coordinate system given by

$$\begin{aligned} x &= \overline{xz}, \\ y &= \overline{yz}, \\ z &= \overline{z}. \end{aligned}$$

Substitution into our original equation for S results in the full transform of S, denoted $\pi^*(S)$, which is given by

$$\begin{aligned} \overline{xyz}^2 - \overline{x}^3\overline{z}^3 - \overline{y}^3\overline{z}^3 + \overline{yz}^3, \\ \Rightarrow \overline{z}^2(\overline{xy} - \overline{x}^3\overline{z} - \overline{y}^3\overline{z} + \overline{yz}) = 0. \end{aligned} \tag{4.7}$$

Let us define the proper transform of S by $\tilde{S}$ which is given by $\tilde{f} = \overline{xy} - \overline{x}^3\overline{z} - \overline{y}^3\overline{z} + \overline{yz} = 0$. Then by observing the intersection $\{\overline{z} = 0\} \cap \tilde{S}$ we can see that there are two exceptional curves given by

$$F_1 = \begin{cases} \overline{z} = 0, \\ \overline{x} = 0, \end{cases} \qquad F_2 = \begin{cases} \overline{z} = 0, \\ \overline{y} = 0. \end{cases}$$

We can see that these two intersect, i.e. $F_1 \cap F_2 = (0,0,0)$. Taking partial derivatives of $\tilde{f}$, we get

$$\begin{aligned} \tilde{f}_x &= \overline{y} - 3\overline{x}^2\overline{z}, \\ \tilde{f}_y &= \overline{x} - 3\overline{y}^2\overline{z} + \overline{z}, \\ \tilde{f}_z &= \overline{y} - \overline{x}^3 - \overline{y}^3. \end{aligned}$$

We may observe that in this case there is a singular point at $(0,0,0)$, hence we need to blow up again to resolve the surface $\tilde{S}$ further. Let us work once again in the chart defined by $\overline{z} \neq 0$ on the blow up, $\tilde{S}$, this time using the coordinate system defined by

$$\begin{aligned} \overline{x} &= ac, \\ \overline{y} &= bc, \\ \overline{z} &= c. \end{aligned}$$

Then the full transform $\pi^*(\tilde{S})$ can be described by

$$\begin{aligned} -a^3c^4 - b^3c^4 + abc^2 + bc^2 = 0, \\ \Longrightarrow c^2\left(-a^3c^2 - b^3c^2 + ab + b\right) = 0. \end{aligned}$$

Defining the proper transform of $\tilde{S}$ as $\tilde{S}_1 : -a^3c^2 - b^3c^2 + ab + b = 0$, we can observe that there are also two exceptional curves for this equation given by

$$F_3 = \begin{cases} c = 0, \\ b = 0. \end{cases} \qquad F_4 = \begin{cases} c = 0, \\ a + 1 = 0. \end{cases}$$

We can note that F_1 can also be described by the two equations $\overline{z} = 0$ and $\overline{x} + \overline{z} = 0$. Then in the new coordinate system we also have that the proper transforms of F_1 and F_2 respectively are

$$\tilde{F}_1 : a + 1 = 0, \qquad \text{and} \qquad \tilde{F}_2 : b = 0.$$

We can make the observation that $F_1 \subset \tilde{F}_1$ and $F_2 \subset \tilde{F}_2$. Then we have that

$F_3 \cap F_4 = (-1,0,0)$. We can also observe that $\tilde{F_2} \cap F_3$ and $\tilde{F_1} \cap F_4$. Taking partial derivatives a third time, we can observe that $\tilde{S_1}$ is still singular and requires a further blow up. However, is should be noted that this singular point is at $(-1,0,0)$ and not the origin. We have only defined the blow up of a point at (0,0,0), so let us consider another change of coordinates;

$$\tilde{a} = a + 1, \qquad \tilde{b} = b, \qquad \tilde{c} = c.$$

Using this change of coordinates we have that

$$\tilde{S_1} : -\tilde{a}^3\tilde{c}^2 - \tilde{b}^3\tilde{c}^2 + 3\,\tilde{a}^2\tilde{c}^2 - 3\,\tilde{a}\tilde{c}^2 + \tilde{a}\tilde{b} + \tilde{c}^2.$$

Then we have that in this coordinate system, the singular point is at $(0,0,0)$ as required. Continuing with the resolution of the singularity, let us work in the chart given by $\tilde{a} \neq 0$ on the blow up $\tilde{S_1}$. This time let us use the coordinate system defined by

$$\tilde{a} = \overline{a}, \qquad \tilde{b} = \overline{a}\overline{b}, \qquad \tilde{c} = \overline{a}\overline{c}.$$

Then the full transform of $\tilde{S_1}$, denoted $\pi^*(\tilde{S_1})$, can be described by

$$\begin{aligned} -\overline{a}^5\overline{c}^2 - \overline{a}^5\overline{b}^3\overline{c}^2 + 3\overline{a}^4\overline{c}^2 - 3\overline{a}^3\overline{c}^2 + \overline{a}^2\overline{b} + \overline{a}^2\overline{c}^2 &= 0, \\ \overline{a}^2\left(-\overline{a}^3\overline{c}^2 - \overline{a}^3\overline{b}^3\overline{c}^2 + 3\overline{a}^2\overline{c}^2 - 3\overline{a}\overline{c}^2 + \overline{b} + \overline{c}^2\right) &= 0. \end{aligned}$$

Consequently the proper transform of $\tilde{S_1}$ can be defined as

$$\tilde{S_2} : -\overline{a}^3\overline{c}^2 - \overline{a}^3\overline{b}^3\overline{c}^2 + 3\overline{a}^2\overline{c}^2 - 3\overline{a}\overline{c}^2 + \overline{b} + \overline{c}^2 = 0.$$

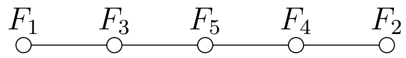
Then the corresponding exceptional curve can be described by

$$F_5 := \begin{cases} \overline{a} = 0, \\ \overline{b} + \overline{c}^2 = 0. \end{cases}$$

In this coordinate system we can also see that the proper transforms of F_3 and F_4 respectively can be described by

$$\tilde{F_3} := \begin{cases} \overline{c} = 0, \\ \overline{b} = 0, \end{cases} \qquad \text{and} \qquad \tilde{F_4} := \overline{c} = 0.$$

Note that $F_3 \subset \tilde{F_3}$ and $F_4 \subset \tilde{F_4}$. Then we can observe that $\tilde{F_4} \cap F_5$ and $\tilde{F_3} \cap F_5$. We can make the further observation that $\tilde{S_2}$ has no singular points, i.e. $\tilde{S_2}$ is smooth. Therefore this is the minimal amount of blow ups necessary to resolve the singularity. Based off of the intersection of exceptional surfaces that we have noted throughout the process, we can draw the corresponding Dynkin diagram as shown below.



From Figure (3.3) we can see that this corresponds to an $\mathbb{A}_5$ singularity, showing that the point P on our original surface S is an $\mathbb{A}_5$ singularity.

4.5 Resolving $\mathbb{D}_4$

In this section we consider an example of a surface which has a $\mathbb{D}_4$ singularity, the equation of which is adapted Appendix B.1.

Using Sakamaki's notation, let us define

$$\begin{aligned} f_2 &= x^2, \\ f_3 &= y^3 + z^3. \end{aligned}$$

Let our cubic surface $S \subset \mathbb{P}^3$ be described by $f(x, y, z, t) = tf_2 - f_3 = 0$, i.e. $f = x^2t - y^3 - x^3 = 0$. Taking partial derivatives of f we then obtain the following set of equations;

$$f_x = 2xt, \qquad f_x = -3y^2, \qquad f_x = -3z^2, \qquad f_x = x^2.$$

Setting the partial derivatives equal to zero, we can use Maple to observe that there is only 1 singular point which lies in S; $P = [0 : 0 : 0 : 1]$. The code for this lies in Appendix C.3. Let us work in the chart defined by $t \neq 0$. To resolve the singularity P in S we blow up. Let us consider the chart given by $z \neq 0$ on the blow up. Then we can define a new coordinate system given by the set of equations

$$\begin{aligned} x &= \overline{x}\overline{z}, \\ y &= \overline{y}\overline{z}, \\ z &= \overline{z}. \end{aligned}$$

Substituting these new coordinates into S in combination with $t = 1$ we then obtain the full transform $\pi^*(S)$ given by

$$\begin{aligned} -\overline{y}^3\overline{z}^3 + \overline{z}^2\overline{x}^2 - \overline{z}^3 &= 0, \\ \implies \overline{z}^2\left(-\overline{y}^3\overline{z} + \overline{x}^2 - \overline{z}\right) &= 0. \end{aligned}$$

Let us define the proper transform of S by $\tilde{S} : -\overline{y}^3\overline{z} + \overline{x}^2 - \overline{z} = 0$. The only exceptional curve, F, is obtained by the intersection $\{\overline{z} = 0\} \cap \tilde{S}$. Thus we have that

$$F = \begin{cases} \overline{z} = 0, \\ \overline{x} = 0. \end{cases}$$

Taking partial derivatives of the defining polynomial of $\tilde{S}$ we can further observe that $\tilde{S}$ is non-singular, so our resolution is incomplete. In fact, we can observe that $\tilde{S}$ has a total of three singular points, P_1, P_2 and P_3, which need to be resolved. To solve this we take a blow up of each of the three points.

First let us consider the singular point $P_1 = (0, -1, 0)$. To transform this point to $(0, 0, 0)$ let us use a change of coordinates defined by

$$a_1 = \overline{x}, \qquad b_1 = \overline{y} + 1, \qquad c_1 = \overline{z}.$$

Under this transformation $\tilde{S}$ may then be described by

$$-{b_1}^3 c_1 + 3\,{b_1}^2 c_1 + {a_1}^2 - 3\,b_1 c_1 = 0.$$

From here we then work in the chart given by $c_1 \neq 0$ on the blow up $\tilde{S}$, allowing us to use the coordinate system defined by

$$a_1 = \overline{a_1 c_1}, \qquad b_1 = \overline{b_1 c_1}, \qquad c_1 = \overline{c_1}.$$

Substituting these new coordinates into our equation for $\tilde{S}$, we have that $\pi^*(\tilde{S})$ can be described by

$$\begin{aligned} &-\overline{b_1}^3\overline{c_1}^4 + 3\,\overline{b_1}^2\overline{c_1}^3 + \overline{a_1}^2\overline{c_1}^2 - 3\,\overline{b_1}\overline{c_1}^2 = 0, \\ \implies &\overline{c_1}^2\left(-\overline{b_1}^3\overline{c_1}^2 + 3\,\overline{b_1}^2\,\overline{c_1} + \overline{a_1}^2 - 3\,\overline{b_1}\right) = 0. \end{aligned}$$

Let us define the proper transform of $\tilde{S}$ by $\tilde{S}_1 : -\overline{b_1}^3\overline{c_1}^2 + 3\,\overline{b_1}^2\,\overline{c_1} + \overline{a_1}^2 - 3\,\overline{b_1} = 0$. We may then observe that any exceptional curves can be obtained by looking at the intersection $\{\overline{c_1} = 0\} \cap \tilde{S}_1$. We therefore only have one exceptional surface, G_1. This exceptional surface and the proper transform of F, denoted F_1, in these new coordinates can be described by

$$G_1 = \begin{cases} \overline{c_1} = 0, \\ \overline{a_1}^2 - 3\overline{b_1} = 0, \end{cases} \qquad F_1 = \left\{ \overline{a_1} = 0. \right.$$

Note that $F \subset F_1$. We can observe that $F_1 \cap G_1$, and we can check partial derivatives to confirm that $\tilde{S}_1$ is smooth. This means that this singularity has been resolved but there are still two more.

Now let's consider the singular point $P_2 = (0, \frac{1}{2} - \frac{i\sqrt{3}}{2}, 0) = (0, \alpha, 0)$. Similarly to before, let's use a change of coordinates defined by

$$a_2 = \overline{x}, \qquad b_2 = \overline{y} - \alpha, \qquad c_3 = \overline{z}.$$

Then we have that $\tilde{S}$ may be described by the equation

$$-\alpha^3 c_2 - 3\alpha^2 b_2 c_2 - 3\alpha {b_2}^2 c_2 - {b_2}^3 c_2 + {a_2}^2 - c_2 = 0.$$

Working in the chart given by $c_2 \neq 0$ on the blow up $\tilde{S}$ allows us to use the coordinate system defined by

$$a_2 = \overline{a_2 c_2}, \qquad b_2 = \overline{b_2}\overline{c_2}, \qquad c_2 = \overline{c_2}.$$

Substituting these new coordinates into our equation for $\tilde{S}$ and noting that $\alpha^3 = -1$, we obtain the full transform $\pi^*(\tilde{S})$ which can be described by

$$\begin{aligned} -3\alpha^2\overline{b_2}\overline{c_2}^2 - 3\alpha\overline{b_2}^2\overline{c_2}^3 - \overline{b_2}^3\overline{c_2}^4 + \overline{a_2}^2\overline{c_2^2} &= 0, \\ \implies \overline{c_2}^2(-3\alpha^2\overline{b_2} - 3\alpha\overline{b_2}^2\overline{c_2} - \overline{b_2}^3\overline{c_2}^2 + \overline{a_2}^2) &= 0. \end{aligned}$$

Then we have that the proper transform of $\tilde{S}$ in these coordinates is given by $\tilde{S_2} : -3\alpha^2\overline{b_2} - 3\alpha\overline{b_2}^2\overline{c_2} - \overline{b_2}^3\overline{c_2}^2 + \overline{a_2}^2 = 0$. Looking at the intersection $\{\overline{c_2} = 0\} \cap \tilde{S_2}$ we can see that there is only one exceptional surface, G_2. We note that G_2 and the proper transform of F, denoted F_2, in this coordinate system can be described by

$$G_2 = \begin{cases} \overline{c_2} = 0, \\ -3\alpha^2\overline{b_2} + \overline{a_2}^2 = 0, \end{cases} \qquad F_2 = \begin{cases} \overline{a_2} = 0. \end{cases}$$

Note that $F \subset F_2$. Clearly the intersection of these consists of only one point, i.e. $G_2 \cap F_2$, and it can be shown that $\tilde{S_2}$ is smooth, so the resolution of the second singularity of S is complete.

The third and final singularity of S is $P_3 = (0, \frac{1}{2} + \frac{i\sqrt{3}}{2}, 0) = (0, \beta, 0)$. The resolution of this point is very similar to that of P_2, so the finer details are left out. Let's define a change of coordinates

$$a_3 = \overline{x}, \qquad b_3 = \overline{y} - \beta, \qquad c_3 = \overline{z}.$$

We can substitute this into $\tilde{S}$ and then work in the chart given by $c_3 \neq 0$ on the blow up $\tilde{S}$, giving a new coordinate system defined by

$$a_3 = \overline{a_3 c_3}, \qquad b_3 = \overline{b_3}\overline{c_3}, \qquad c_3 = \overline{c_3}.$$

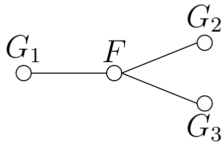
Substituting again into $\tilde{S}$ and noting that $\beta^3 = 1$ yields the full transform $\pi^*(\tilde{S})$ given by

$$\begin{aligned} -3\beta^2\overline{b_3}\overline{c_3}^2 - 3\beta\overline{b_3}^2\overline{c_3}^3 - \overline{b_3}^3\overline{c_3}^4 + \overline{a_3}^2\overline{c_3^2} &= 0, \\ \implies \overline{c_3}^2(-3\beta^2\overline{b_3} - 3\beta\overline{b_3}^2\overline{c_3} - \overline{b_3}^3\overline{c_3}^2 + \overline{a_3}^2) &= 0. \end{aligned}$$

Defining the surface $\tilde{S_3} : -3\beta^2\overline{b_3} - 3\beta\overline{b_3}^2\overline{c_3} - \overline{b_3}^3\overline{c_3}^2 + \overline{a_3}^2 = 0$ to be the proper transform of $\tilde{S}$ and looking at the intersection $\{\overline{c_3} = 0\} \cap \tilde{S_3}$, we observe only one exceptional surface. This is denoted by G_3. Then in this coordinate system we have that G_3 and the proper transform of F, denoted by F_3, can be described by

$$G_3 = \begin{cases} \overline{c_3} = 0, \\ -3\beta^2\overline{b_3} + \overline{a_3}^2 = 0, \end{cases} \qquad\qquad F_3 = \left\{ \overline{a_3} = 0. \right.$$

Note that $F \subset F_3$. We may observe again that the intersection $F_3 \cap G_3$ consists of one point, $(0,0,0)$, and that $\tilde{S}_3$ is smooth. Thus this is the minimal amount of blow ups necessary to resolve our surface. From the intersections of the exceptional surfaces which we have noted with each blow up iteration, we can construct the corresponding Dynkin diagram as shown below.



We can then make the final observation from Figure (3.3) that this diagram corresponds to a $\mathbb{D}_4$ singularity, implying that the point P on our surface S is a $\mathbb{D}_4$ singularity.

Chapter 5

27 Lines

One of the best known results in algebraic geometry is that every smooth cubic surface contains exactly 27 lines. In this section we will outline an example of a smooth cubic surface and demonstrate that there are indeed 27 lines present, the proof of which is inspired by [10].

5.1 The Fermat Cubic

The Fermat Cubic [10] is the cubic surface in $\mathbb{P}^3$ defined by

$$f = t^3 + x^3 + y^3 + z^3 = 0.$$

Any singular points on the surface would have to be a solution to the following

$$\begin{cases} f_t = 3t^2 = 0, \\ f_x = 3x^2 = 0, \\ f_y = 3y^2 = 0, \\ f_z = 3z^2 = 0. \end{cases}$$

Clearly there is no point in $\mathbb{P}^3$ which satisfies this system so there are no singular points on this surface. Hence we determine that the Fermat cubic is smooth.

From Theorem 3.1.4, it follows that the Fermat cubic contains exactly 27 lines. We can find these 27 lines explicitly as follows.

Let $a, b \in \mathbb{C}$ such that $a^3 = -1$ and $b^3 = -1$. Consider the Fermat cubic in $\mathbb{P}^3_{[x:y:z:t]}$. Any line in $\mathbb{P}^3$ can be described by a system of two linear equations. Consider the three lines $L_1^{(a,b)}, L_2^{(a,b)}$ and $L_3^{(a,b)}$ in $\mathbb{P}^3_{[x:y::t]}$ described by

$$L_1^{(a,b)} = \begin{cases} t = ax, \\ y = bz, \end{cases} \qquad L_2^{(a,b)} = \begin{cases} t = ay, \\ x = bz, \end{cases} \qquad L_3^{(a,b)} = \begin{cases} t = az, \\ x = by. \end{cases}$$

To check if each line lies within the Fermat cubic, we can substitute the system of equations of each line into f and observe the resulting equation. For L_1 we obtain

$$f = (ax)^3 + x^3 + (bz)^3 + z^3 = 0,$$

since $a^3 = b^3 = -1$. A similar result follows for $L_2^{(a,b)}$ and $L_3^{(a,b)}$. Different choices of a and b give distinct lines $L_i^{(a,b)}$. Since there are $3 \times 3 = 9$ possible pairs of (a, b), we get $9 \times 3 = 27$ different lines. Then we have that these 27 lines all lie in the Fermat cubic. We claim that these 27 lines up to a permutation of coordinates are the only lines that lie in the Fermat cubic. To prove this, let us assume that there exists a line, L, in the Fermat cubic such that L is not one of the permutations of $L_1^{(a,b)}, L_2^{(a,b)}$ or $L_3^{(a,b)}$. Since any line in $\mathbb{P}^3$ can be described by a system of 2 linear equations, then this new line can be described by

$$L = \begin{cases} x = c_1 z + c_2 t, \\ y = d_1 z + d_2 t, \end{cases}$$

for some $c_1, c_2, d_1, d_2 \in \mathbb{C}$, up to a permutation of coordinates. For L to lie in the Fermat cubic we must have

$$(c_1 z + c_2 t)^3 + (d_1 z + d_2 t)^3 + z^3 + t^3 = 0,$$

which we can expand to obtain

$$d_1^3 z^3 + 3d_1^2 d_2 t z^2 + 3d_1 d_2^2 t^2 z + d_2^3 t^3 + c_1^3 z^3 + 3c_1^2 c_2 t z^2 + 3c_1 c_2^2 t^2 z + c_2^3 t^3 + z^3 + t^3 = 0.$$

We may then compare coefficients to obtain

$$d_1^3 + c_1^3 = -1, \tag{5.1}$$
$$d_2^3 + c_2^3 = -1, \tag{5.2}$$
$$d_1^2 d_2 = -c_1^2 c_2, \tag{5.3}$$
$$d_2^2 d_1 = -c_2^2 c_1. \tag{5.4}$$

Using these 4 equations, let us further assume that c_1, c_2, d_1 and d_2 are all non-zero. Then by taking the square of equation (5.3) and dividing by equation (5.4), we obtain $d_1^3 = -c_1^3$. However, if we substitute this into equation (5.1) we get $0 = -1$ which is a contradiction. Therefore one of the constants must be zero.

Without loss of generality, let us then assume $c_1 = 0$. Substitution yields $d_1^3 = -1$, $d_2 = 0$, and $c_2^3 = -1$. Using these values, our line may then be described by

$$L = \begin{cases} x = c_2 t, \\ y = d_1 z, \end{cases}$$

where $d_1^3 = c_2^3 = -1$. A permutation of coordinates implies that L is one of the permutations of $L_1^{(a,b)}$, which is a contradiction. Similar results follow when a different permutation of coordinates is used to describe L. Therefore the 27 lines described previously are the only lines contained within in the Fermat cubic.

Chapter 6

How Many Lines are Contained in Singular Cubic Surfaces?

We know that there are 27 lines contained in any smooth cubic surface, but how many lines are there in singular cubic surfaces? The answer to this depends only on the types of singularity within that surface. This is presented in the table in Bruce and Wall's *On the Classification of Cubic Surfaces* [1]. In this chapter we will explicitly prove the results summarised in [1].

We will start with the easiest case to understand; the case with 1 $\mathbb{E}_6$ singularity. We will prove that this surface only contains one line, making it the easiest proof to follow, so this case is outlined first.

6.1 $\mathbb{E}_6$

According to Bruce and Wall's table in Appendix A.1 the equation of a normal cubic surface with 1 $\mathbb{E}_6$ singularity has 0 parameters; that is, it is unique up to projective transformation. Thus we can directly determine the number of lines that fall in this unique surface. The table in Appendix B.1 gives the equation to be

$$tx^2 - xz^2 - y^3 = 0.$$

We can immediately see that the line

$$L_1 = \begin{cases} x = 0 \\ y = 0 \end{cases} \tag{6.1}$$

lies inside of this surface.

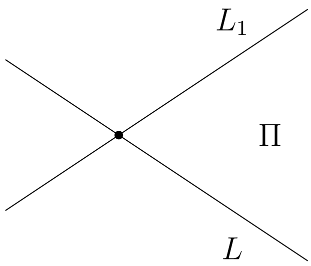
<u>Claim:</u> The line in $\mathbb{P}^3$ defined in equation (6.1) is the only line contained in the surface defined by $tx^2 - xz^2 - y^3 = 0$.
Proof. Let us call our surface S. Let us assume for contradiction that there is some line $L \in S$ such that $L \neq L_1$.

Let us look at the plane defined by $x = 0$. We have

$$S \cap \{x = 0\} = \begin{cases} -y^3 = 0 \\ x = 0 \end{cases}$$
$$= 3L_1$$

Where the notiation $3L_1$ means that we have the line L_1 counted 3 times with multiplicity.
We know that for any line and any plane, their intersection is either 1 point or the line is contained in the plane by Lemma 2.4.2. Looking at $L \cap \{x = 0\}$ we can see that as $L \subset S$ this is the same as $L \cap S \cap \{x = 0\} = L \cap L_1$. So we have that either $L \subset L_1 \implies L = L_1$ which contradicts our assumption that $L \neq L_1$ or $L \cap L_1$ intersect at 1 point.

From this we know that there is some unique plane, Π, such that $L \subset \Pi$ and $L_1 \subset \Pi$. As $L_1 \subset \Pi$ we know that Π is either the plane defined by $y = 0$ or the plane defined by $y = \lambda x$ for some non zero $\lambda \in \mathbb{C}$. We have that $L, L_1 \subset S$ and $L, L_1 \subset \Pi$ so $L, L_1 \subset \Pi \cap S$.
In the case where Π is defined by $y = \lambda x$ we have that



$$\Pi \cap S = \begin{cases} y = \lambda x \\ x(tx - z^2 - \lambda^3 x^2) = 0 \end{cases}$$
$$= L_1 \cup \{tx - z^2 - \lambda^3 x^2 = 0, y = \lambda x\}$$

So we have that this is the space defined by the line L_1 and the conic $tx - z^2 - \lambda^3 x^2 = 0$ in the plane $y = \lambda x$. This conic is represented by the matrix

$$\begin{pmatrix} \lambda^3 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

This matrix has determinant $\frac{1}{2}$ for all values of λ so this conic is irreducible and hence by Lemma 2.4.3 we know that there are no lines contained inside this.
Similarly for the case where Π is defined by $y = 0$ we have that

$$\Pi \cap S = \begin{cases} y = 0 \\ x(tx - z^2) = 0 \end{cases} \qquad = L_1 \cup \{tx - z^2, y = 0\}$$

so this intersection is defined by the line L_1 and the conic $(tx - z^2) = 0$ on the plane $y = 0$. As this conic is irreducible again by the same logic as above we know that this does not contain any new lines. Hence L cannot exist. $\square$

Theorem 6.1.1. *A cubic surface which has one $\mathbb{E}_6$ singularity and is smooth elsewhere contains exactly one line.*

Proof. From the proof of our above claim we know that $tz^2 - xz^2 - y^3 = 0$ contains exactly 1 line. As this is the unique cubic surface (up to projective transformation) with exactly 1 $\mathbb{E}_6$ singularity we know that any cubic surface with exactly 1 $\mathbb{E}_6$ singularity contains exactly 1 line. $\square$

We can see that Theorem 6.1.1 is consistent with the conclusions of Bruce and Wall [1].

6.2 $\mathbb{A}_1$

A surface with one $\mathbb{A}_1$ singularity has 3 parameters according to [1]. Up to projective transformation,the equation of any such surface is given by

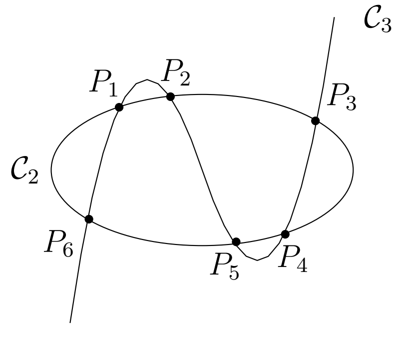
$$t(xz - y^2) - (x - ay)(-x + (b + 1)y - bz)(y - cz) = 0$$

according to Appendix B.1 where a, b, c are all distinct elements of $\mathbb{C} \setminus \{0, 1\}$. For ease, let us call this surface S. It is possible to find all the lines in this explicitly and prove we have found the complete set as we will do for other cases in this chapter. However, as there are lots of parameters, we will outline a slightly different proof of the number of lines in this surface.

Theorem 6.2.1. *A cubic surface which has exactly one $\mathbb{A}_1$ singularity and is smooth elsewhere contains exactly 21 lines.*

Proof. We can see that the singularity in S is at $O = [0 : 0 : 0 : 1]$.
We know S is of the form $tf_2(x, y, z) + f_3(x, y, z) = 0$ where f_2 is a homogeneous polynomial of degree 2 and f_3 is a homogeneous polynomial of degree 3.

Let us consider the conic $f_2(x, y, z) = 0$, which denote by $\mathcal{C}_2$, and the cubic curve $f_3(x, y, z) = 0$ which denote by $\mathcal{C}_3$ on the plane $t = 0$. Where both f_3 and f_2 are 0 we can see that these points are contained in S. As f_2 and f_3 are co-prime they should intersect 6 times with multiplicity. If all the intersections are transversal, then by the Corollary 2.4.4 of Bezout's Theorem we will have 6 distinct points. Let us assume for contradiction that there is a point $Q \in \mathcal{C}_2 \cap \mathcal{C}_3$

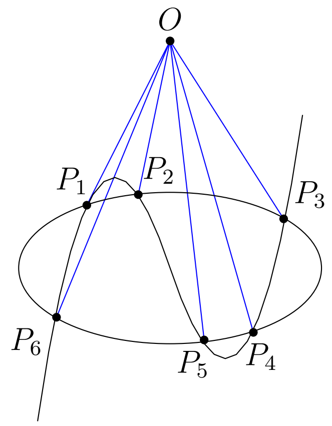


on the curve at which $\mathcal{C}_2$ and $\mathcal{C}_3$ do not intersect transversally. By the classification of conics [2, p60] there is a projective transformation ϕ that takes $\mathcal{C}_2$ to any other smooth conic and Q to any other point on $\phi(\mathcal{C}_2)$. Thus we can assume $Q = [1 : 0 : 0]$ on $xz - y^2 = 0$. Then we must have no x^3 term in f_3 as this would not satisfy $f_3(Q) = 0$ so we can write $f_3 = x^2h_1(y, z) + xh_2(y, z) + h_3(y, z)$.

If $h_1(y,z)=0$ at Q then this cubic is singular at Q. If $\mathcal{C}_3$ is smooth at Q then $h_1(x,z)=0$ is the equation of the tangent. We know the tangent to $\mathcal{C}_2$ at Q is $z=0$ from Definition 2.2.9. So this intersction is not transversal if and only if we have that $h_1(y,z)=\lambda z$ for some $\lambda \in \mathbb{C}$. However the surface defined by

$$t(xz-y^2)+\lambda x^2 z + x h_2(y,z)+h_3(y,z)=0$$

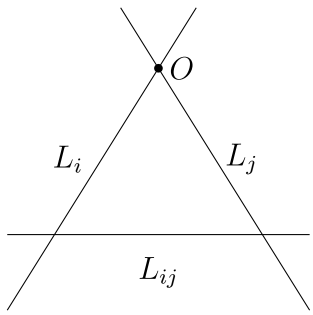
contains 2 singularities at $[0:0:0:1]$ and $[1:0:0:-\lambda]$ which contradicts that our surface only has 1 singularity. Therefore $\mathcal{C}_2$ and $\mathcal{C}_3$ only intersect transversally.



So we know there are 6 points $P_1,...,P_6 \in \mathcal{C}_2 \cap \mathcal{C}_3$. All 6 points in this intersection are contained in S as they clearly satisfy $tf_2+f_3=0$ and have $t=0$. Each of these defines a line $\mu P_i + \nu O$ which is contained in S as $P_i \in S, O \in S$ and $O=[0:0:0:1]$ and $P=[a:b:c:0]$ for some $(a,b,c) \in \mathbb{C}$. Let us denote each line $\mu P_i+\nu O$ by L_i. We know that

$$\{f_3=0\}\cap S = \{f_2=0\}\cap S=\{f_3=0\}\cap\{f_2=0\}=L_1\cup ... \cup L_6.$$

No three of these lines are contained in the same plane because the points $P_1,...P_6$ lie on an irreducible conic so no 3 of them lie in the same line. Each L_i, L_j define a unique plane Π_{ij}.
The intersection $S\cap \Pi_{ij}$ contains both L_i and L_j. As S is a cubic surface and our plane is a linear equation we have that $S\cap\Pi_{ij}=L_i+L_j+L_{ij}$ where L_{ij} is some line in Π.



We will show that this line L_{ij} is distinct from both L_i and L_j. Again by the classification of conics there is some projective transformation such that the 2 points P_i and P_j are sent to $[1:0:0]$ and $[0:0:1]$ on $\mathcal{C}_2$ so our lines L_i and L_j are defined

$$L_i : y = z = 0 \qquad\qquad L_j : x = y = 0$$

and the plane Π_{ij} is defined by $y = 0$. Then there is no x^3 or z^3 term in our equation for S so it becomes $t(xz - y^2) + y^2h_1(x,z) + yh_2(x,z) + xz(ax + bz) = 0$ for some $a, b \in \mathbb{C}$. Hence

$$\begin{aligned} S \cap \Pi_{ij} &= \begin{cases} y = 0 \\ t(xz - y^2) + y^2h_1(x,z) + yh_2(x,z) + xz(ax + bz) = 0 \end{cases} \\ &= \begin{cases} y = 0 \\ xz(t + ax + bz) \end{cases} \\ &= L_i + L_j + L_{ij} \end{aligned}$$

where L_{ij} is the line defined by $y = t + ax + bz = 0$ which is clearly not the same as L_i or L_j. So we have a line, L_{ij} for each combination of i and j which is a total of $\binom{6}{2} = 15$. With our original 6 L_is that is 21 lines.

So far we have shown there are at least 21 lines in S. Now we need to show that this is all of the lines contained in S.
We know $L_1, ..., L_6$ are all the lines that intersect O. We can show that we have found all the lines that intersect one of these initial six lines. Say there is some line $L \subset S$ such that $L \cap L_i \neq \emptyset$. Then L is either the line L_i itself or intersects L_i at 1 point. In the latter case we have a plane Π that contains L and L_i. The intersection of Π with S will be a cubic that factors to give L and L_i so it must have some third linear term, a line which we will call L'. As S is singular at O the cubic $\Pi \cap S$ is also singular at O and so at least two of our lines from L_i, L and L' must intersect at O. This means that one of our lines L' or L is L_j and so the other must be the line L_{ij}. Hence we have found all the lines in S that intersects any L_i.
Any line intersects the space $f_2 = 0$ at least at 1 point. So any line in S must intersect $S \cap \{f_2 = 0\} = L_1 \cup ... \cup L_6$. Thus any line in S must intersect at least one of our L_i. From the above we know that we have already found any such line. Hence there are exactly 21 lines in S. □

6.3 $\mathbb{A}_2$

A surface with an $\mathbb{A}_2$ singularity has 2 parameters according to Appendix A.1. In Appendix B.1 the equation of any such surface is given by

$$txy - z(x + y + z)(dx + ey - dez) = 0$$

where $d, e \in \mathbb{C} \setminus \{0, -1\}$. Let us call this surface S.

Theorem 6.3.1. *A cubic surface which has exactly one $\mathbb{A}_2$ singularity and is* *smooth elsewhere contains exactly 15 lines.*

Proof. Here we will use a similar style of proof to that for an $\mathbb{A}_1$ surface.
We can see that the singularity in S is at $O = [0:0:0:1]$. S is of the form $tf_2(x,y,z) - f_3(x,y,z) = 0$ where f_2 is a degree 2 polynomial and f_3 is a homogeneous polynomial of degree 3.

Let us call $f_2 = 0$ the conic $\mathcal{C}_2$ and $f_3 = 0$ the cubic curve $\mathcal{C}_3$ on $t = 0$.
In this case $f_2 = xy = 0$ is a reducible conic so we have 2 lines. By the Fundamental Theorem of Algebra we know that $\mathcal{C}_3$ intersects each of these lines 3 times with multiplicity. We can see that the point where x and y are equal to 0 does not satisfy $f_3 = 0$ so $\mathcal{C}_3$ does not intersect the lines $y = 0$ and $x = 0$ at the point where these 2 lines intersects. So there are a total of 6 points of intersection of $\mathcal{C}_2$ and $\mathcal{C}_3$ counted with multiplicity.

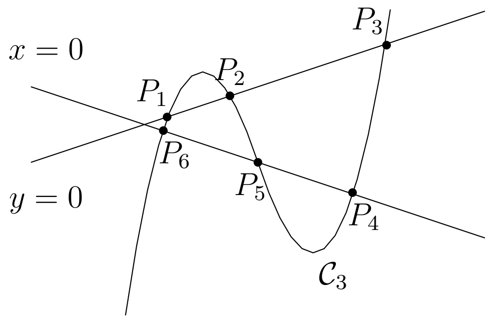


Figure 6.1: $\mathcal{C}_2$ intersecting $\mathcal{C}_3$

We can show that there are exactly 6 different intersection points as shown in Figure 6.1. Clearly there are no intersections with multiplicity higher than 1 provided there are no points such that $x = 0$ or $y = 0$ is a tangent to $\mathcal{C}_3$.
If there is a point such that $x = 0$ is a tangent to $\mathcal{C}_3$ at that point, it must be on the line $x = 0$ and satisfy $\frac{\partial f_3}{\partial y} = 0, \frac{\partial f_3}{\partial} z = 0$. So it must satistfy

$$\begin{cases} x = 0 \\ \frac{\partial f_3}{\partial y} = (d+e)xz + 2eyz + (e - de)z^2 = 0 \\ \frac{\partial f_3}{\partial z} = dx^2 + (d+e)xy + ey^2 + 2(d-de)xz + 2(e-de)yz - 3dez^2 = 0 \end{cases}$$

and attempting to solve this we see that there are no solutions in $\mathbb{P}^2$. Hence $\mathcal{C}_3$ is nowhere tangent to $x = 0$. We can similarly see that any point where $\mathcal{C}_3$ is tangent to $y = 0$ must satisfy the system

$$\begin{cases} y = 0 \\ \frac{\partial f_3}{\partial x} = 2dxz + (d+e)yz + (d - de)z^2 = 0 \\ \frac{\partial f_3}{\partial z} = dx^2 + (d+e)xy + ey^2 + 2(d-de)xz + 2(e-de)yz - 3dez^2 = 0 \end{cases}$$

which also has no solutions in $\mathbb{P}^2$ so $\mathcal{C}_3$ is nowhere tangent to $y = 0$. Hence there are 6 distinct points (let us call them P_i for i=1,...,6) in $\mathcal{C}_2 \cap \mathcal{C}_3$. Any of these points are clearly contained in the surface S for any value of t. Hence any line,

L_i, defined by $\mu P_i + \nu O$ is contained in S.

Similar to the proof that there are 21 lines in an $\mathbb{A}_1$ surface we can see that for any 2 lines L_i and L_j they define a plane Π_{ij} which when intersected with S should give 3 lines counted with multiplicities. We can use the same proof as we did there to show that none of these planes intersect S at lines with a multiplicity bigger than 1.

However in the case where you have L_1, L_2 we know that the 3rd line in that plane will be L_3 as L_1, L_2, L_3 are all contained in the plane $x = 0$ as P_1, P_2, P_3 (labeled as in Figure 6.1) were all on the line $x = t = 0$ when we found these points in the first instance. Similarly we have that L_4, L_5, L_6 all lie in the same plane. So the only pairs of lines L_i, L_j that will yield a third line will be ones such that P_i, P_j did not both lie on the same line of $x = t = 0$ and $y = t = 0$. Hence there are 9 possible pairs. So in total we have $9 + 6 = 15$ lines.
We know we have indeed counted all of the possible lines because any line $L \subset S$ must intersect the conic $\mathcal{C}_2$ at some point. However, $S \cap \mathcal{C}_2 = L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5 \cup L_6$ so L intersects at least one L_i, but from the above we know that we have found all the lines that intersect L_i for every i. $\square$

6.4 $2\mathbb{A}_1$

According to Appendix B.1, the equation of any $2\mathbb{A}_1$ surface up to projective transformation is

$$t(xz - y^2) - (x - 2y + z)(x - ay)(y - bz) = 0$$

for distinct $a, b \in \mathbb{C} \setminus \{0, 1\}$. Let us call this surface S. We can see the lines

$$\begin{aligned}
L_1 &= \begin{cases} x = 0 \\ y = 0 \end{cases} & L_2 &= \begin{cases} y = 0 \\ z = 0 \end{cases} \\
L_3 &= \begin{cases} y = 0 \\ t + bx + bz = 0 \end{cases} & L_4 &= \begin{cases} t = 0 \\ x - 2y + z = 0 \end{cases} \\
L_5 &= \begin{cases} t = 0 \\ x - ay = 0 \end{cases} & L_6 &= \begin{cases} t = 0 \\ y - bz = 0 \end{cases} \\
L_7 &= \begin{cases} x = y \\ y = z \end{cases} & L_8 &= \begin{cases} x = y \\ t + (a-1)y + (a-1)bz = 0 \end{cases} \\
L_9 &= \begin{cases} x = ay \\ y = az \end{cases} & L_{10} &= \begin{cases} x = by \\ y = bz \end{cases} \\
L_{11} &= \begin{cases} x = by \\ t + (b-a)(b-2)y + (b-a)z = 0 \end{cases} & L_{12} &= \begin{cases} y = z \\ t + (b-1)x - a(b-1)y = 0 \end{cases} \\
L_{13} &= \begin{cases} y = az \\ t - (a-b)x + (a-b)(2a-1)z = 0 \end{cases} & L_{14} &= \begin{cases} t + bx + bz = 2y \\ x + abz = (a+b)y \end{cases} \\
L_{15} &= \begin{cases} t + bx + bz = (b - a + ab + 1)y \\ -x + (a+1)y - az = 0 \end{cases} & L_{16} &= \begin{cases} t + bx + bz = (b - a + ab + 1)y \\ x - (b+1)y + bz = 0 \end{cases}
\end{aligned} \tag{6.2}$$

are contained in S.

Theorem 6.4.1. *A cubic surface which contains two $\mathbb{A}_1$ singularities and is smooth elsewhere contains exactly 16 lines.*

Proof. In order to prove that any $2\mathbb{A}_1$ surface contains 16 lines we only need to show that S contains exactly 16 lines. We have listed 16 unique lines in equation 6.2 so it remains to show that these are all the lines contained in S.

Let L be a line contained in S. Consider the plane $y = 0$. The line L is either contained in $y = 0$ or intersects it at 1 point.

$$\begin{aligned} S \cap \{y = 0\} &= \begin{cases} y = 0 \\ t(xz - y^2) - (x - 2y + z)(x - ay)(y - bz) = 0 \end{cases} \\ &= \begin{cases} y = 0 \\ xz(t + bx + bz) = 0 \end{cases} \\ &= L_1 \cup L_2 \cup L_3 \end{aligned}$$

so L is either one of the lines L_1, L_2, L_3 or intersects at least 1 of these lines.

Let us consider the case where L intersects L_1. Then there is a plane Π such that $L, L_1 \subset \Pi$. Π is either $y = 0$ or $x = \lambda y$ for some $\lambda \in \mathbb{C}$. We have already seen the case where Π is defined by $y = 0$ so consider the case where Π is $x = \lambda y$ such that

$$\begin{aligned} S \cap \Pi &= \begin{cases} x = \lambda y \\ y(\lambda tz - ty - (\lambda - a)((\lambda - 2)y + z)(y - bz)) = 0 \end{cases} \\ &= L_1 \cup \{x = \lambda y, \lambda tz - ty - (\lambda - a)((\lambda - 2)y + z)(y - bz) = 0\} \end{aligned}$$

where $\lambda tz - ty - (\lambda - a)((\lambda - 2)y + z)(y - bz) = 0$ is a conic on $x = \lambda y$ which is irreducible for $\lambda \neq 1, a, b$. When $\lambda = 1$ the plane Π is defined by $x = y$ so

$$\begin{aligned} S \cap \Pi &= \begin{cases} x = y \\ y(z - y)(t + (a - 1)y + (a - 1)bz) = 0 \end{cases} \\ &= L_1 \cup L_7 \cup L_8 \end{aligned}$$

so L is L_1, L_7 or L_8.
In the case that $\lambda = a$ the plane Π is defined by $x = ay$. So

$$\begin{aligned} S \cap \Pi &= \begin{cases} x = ay \\ ty(az - y) = 0 \end{cases} \\ &= L_1 \cup L_5 \cup L_9 \end{aligned}$$

so L is L_1, L_5 or L_9 in the case that $\lambda = b$ we have Π is defined by $x = by$

$$\begin{aligned} S \cap \Pi &= \begin{cases} x = by \\ y(bz - y)(t + (b - a)(b - 2)y + (b - a)z) = 0 \end{cases} \\ &= L_1 \cup L_{10} \cup L_{11} \end{aligned}$$

so L is L_1, L_{10} or L_{11}.

So L intersects L_2 or L_3. Let us consider the case where L intersects L_2. So there exists a plane Π such that $L, L_2 \subset \Pi$. Then Π is $y = 0$ or $z = \lambda y$. Suppose

Π is defined by $z = \lambda y$ then

$$S \cap \Pi = \begin{cases} z = \lambda y \\ y(\lambda t x - t y + (\lambda b - 1)(x^2 + (\lambda - a - 2)xy - a(\lambda - 2)y^2)) = 0 \end{cases}$$

which is L_2 and the conic $(\lambda b - 1)(x^2 + (\lambda - a - 2)xy - a(\lambda - 2)y^2) = 0$ on $z = \lambda y$ which is irreducible for $\lambda \neq 1, \frac{1}{a}, \frac{1}{b}$. When $\lambda = 1$ we have that Π is defined by $y = z$ then

$$\begin{aligned} S \cap \Pi &= \begin{cases} z = y \\ y(x - y)(t + (b - 1)x - a(b - 1)y) = 0 \end{cases} \\ &= L_2 \cup L_7 \cup L_{12} \end{aligned}$$

so L is L_2, L_7 or L_{12}. In the case where $\lambda = \frac{1}{a}$ the plane Π is defined by $az = y$ so

$$\begin{aligned} S \cap \Pi &= \begin{cases} az = y \\ z(x - a^2 z)(t - (a - b)x + (a - b)(2a - 1)z) = 0 \end{cases} \\ &= L_2 \cup L_9 \cup L_{13} \end{aligned}$$

so L is L_2, L_9 or L_{13}.
In the case where $\lambda = \frac{1}{b}$ we have that Π is defined by $y = bz$ so

$$\begin{aligned} S \cap \Pi &= \begin{cases} y = zb \\ tz(x - b^2 z) = 0 \end{cases} \\ &= L_2 \cup L_6 \cup L_{10}. \end{aligned}$$

So L is L_2, L_6 or L_{10}.
So L intersects L_3. So there is some plane Π such that $L, L_3 \subset S$. Π is either $y = 0$ or $t + bx + bz = \lambda y$ for some $\lambda \in \mathbb{C}$. Let us consider the latter case. Then

$$S \cap \Pi = \begin{cases} t + bx + bz = \lambda y \\ y(-(\lambda + 2a)y^2 + (a + b + 2)xy + (b + 2ab + a)yz - x^2 - abz^2 + \\ (\lambda - ab - 2b - 1)xz) = 0 \end{cases}$$

which is union of the line L_3 with the conic $-(\lambda + 2a)y^2 + (a + b + 2)xy + (b + 2ab + a)yz - x^2 - abz^2 + (\lambda - ab - 2b - 1)xz = 0$ on $t + bx + bz = \lambda y$ which is irreducible for $\lambda \neq 2b, b - a + ab + 1$.
When $\lambda = 2b$ we have that Π is defined by $t + bx + bz = 2by$ or equivalently $t = -b(x - 2y + z)$ so

$$\begin{aligned} S \cap \Pi &= \begin{cases} t = -b(x - 2y + z) \\ y(x - 2y + z)(x + abz - (a + b)y) = 0 \end{cases} \\ &= L_3 \cup L_4 \cup L_{14}. \end{aligned}$$

In the case where $\lambda = b - a + ab + 1$ we have that Π is defined by $t + bx + bz = (b - a + ab + 1)y$ so

$$\begin{aligned} S \cap \Pi &= \begin{cases} t + bx + bz = (b - a + ab + 1)y \\ y(-x + (a+1)y - az)(x - (b+1)y + bz) = 0 \end{cases} \\ &= L_3 \cup L_{15} \cup L_{16} \end{aligned}$$

so these are all the lines in L.

□

6.5 $\mathbb{A}_1\mathbb{A}_2$

According to Appendix B.1, the equation of any $\mathbb{A}_1\mathbb{A}_2$ surface up to projective transformation is

$$t(xz - y^2) + (x - y)(y - z)(x - (a+1)y + az) = 0$$

for some $a \in \mathbb{C} \setminus \{0, 1\}$. Let us call this surfaces S.
We can see that the lines

$$\begin{aligned} L_1 &= \begin{cases} t = 0 \\ x = y \end{cases} & L_2 &= \begin{cases} t = 0 \\ y = z \end{cases} \\ L_3 &= \begin{cases} t = 0 \\ x - (a+1)y + az = 0 \end{cases} & L_4 &= \begin{cases} x = 0 \\ y = 0 \end{cases} \\ L_5 &= \begin{cases} y = 0 \\ z = 0 \end{cases} & L_6 &= \begin{cases} x = y \\ y = z \end{cases} \\ L_7 &= \begin{cases} y = 0 \\ t - x - az = 0 \end{cases} & L_8 &= \begin{cases} y = az \\ t + (a-1)x - a(a-1)z = 0 \end{cases} \\ L_9 &= \begin{cases} y = az \\ x - a^2 z = 0 \end{cases} & L_{10} &= \begin{cases} t = x + az - (a+1)y \\ x + z - 2y = 0 \end{cases} \\ L_{11} &= \begin{cases} x = ay \\ t + (a-1)y + (a-1)z = 0 \end{cases} \end{aligned} \tag{6.3}$$

are contained in S.

Theorem 6.5.1. *A surface with one $\mathbb{A}_1$ singularity, one $\mathbb{A}_2$ singularity and which is smooth elsewhere contains exactly 11 lines.*

Proof. To prove that any $\mathbb{A}_1\mathbb{A}_2$ surface contains 16 lines we only need to show that S contains exactly 16 lines. We have listed 16 unique lines in equation 6.3 so it remains to show that these are all the lines contained in S.

Let L be a line in S. Consider the plane $y = 0$. L either intersects $y = 0$

or is contained in it.

$$
\begin{aligned}
S \cap \{y=0\} &= \begin{cases} y=0 \\ t(xz-y^2)+(x-y)(y-z)(x-(a+1)y+az)=0 \end{cases} \\
&= \begin{cases} xz(t-x-az)=0 \end{cases} \\
&= L_4 \cup L_5 \cup L_7
\end{aligned}
$$

so L is either L_4, L_5, L_6 or intersects at least one of them. Let us assume that L intersects L_4, then there is some plane Π such that $L, L_4 \subset \Pi$ then Π is either $y=0$ or $x=\lambda y$ for $\lambda \in \mathbb{C}$. We know that in the case that Π is defined by $y=0$ we have that $S \cap \Pi = L_4 \cup L_5 \cup L_7$ so there is no such line L in this case. So Π must be defined by $x=\lambda y$ and

$$
\begin{aligned}
S \cap \Pi &= \begin{cases} x=\lambda y \\ t(xz-y^2)+(x-y)(y-z)(x-(a+1)y+az)=0 \end{cases} \\
&= \begin{cases} x=\lambda y \\ y(\lambda tz-ty+(\lambda-1)(y-z)((\lambda-a-1)y+az))=0 \end{cases} \\
&= L_4 \cup \{x=\lambda y, \lambda tz-ty+(\lambda-1)(y-z)((\lambda-a-1)y+az)=0\}
\end{aligned}
$$

where $\lambda tz-ty+(\lambda-1)(y-z)((\lambda-a-1)y+az=0$ is a conic on $x=\lambda y$ which is irreducible for $\lambda \neq 1, a$. When $\lambda=1$ we have

$$
\begin{aligned}
S \cap \Pi &= \begin{cases} x=y \\ ty(z-y)=0 \end{cases} \\
&= L_1 \cup L_4 \cup L_6
\end{aligned}
$$

so L is either L_1, L_4 or L_6 in this case.
In the case where $\lambda=a$ we have

$$
\begin{aligned}
S \cap \Pi &= \begin{cases} x=ay \\ y(az-y)(t+(a-1)y+(a-1)z)=0 \end{cases} \\
&= L_4 \cup L_9 \cup L_{11}
\end{aligned}
$$

so L is either L_4, L_9 or L_{11} in this case.
We are left with the cases where L intersects L_5 or L_7.
Let us assume that L intersects L_5. Then there is a plane Π such that $L, L_5 \subset \Pi$.

This Π must be $z = \lambda y$. So

$$\begin{aligned}
S \cap \Pi &= \begin{cases} z = \lambda y \\ t(xz - y^2) + (x-y)(y-z)(x-(a+1)y+az) = 0 \end{cases} \\
&= \begin{cases} z = \lambda y \\ y(\lambda tx - ty + (1-\lambda)(x-y)(x+(a\lambda - a - 1)y)) = 0 \end{cases} \\
&= L_5 \cup \{z = \lambda y, \lambda tx - ty + (1-\lambda)(x-y)(x+(a\lambda - a - 1)y) = 0\}
\end{aligned}$$

where $\lambda tx - ty + (1-\lambda)(x-y)(x+(a\lambda-a-1)y) = 0$ is an irreducible conic on $z = \lambda y$ for $\lambda \neq 1, \frac{1}{a}$. When $\lambda = 1$ we have

$$S \cap \Pi = \begin{cases} z = y \\ ty(x-y) = 0 \end{cases} \qquad = L_2 \cup L_5 \cup L_6$$

So L is either L_2, L_5 or L_6 in this case. When $\lambda = \frac{1}{a}$ we have that

$$\begin{aligned}
S \cap \Pi &= \begin{cases} az = y \\ t(xz - a^2 z) + (a-1)z(x-az)(x-a^2 z) = 0 \end{cases} \\
&= \begin{cases} az = y \\ z(t-(a-1)x - a(a-1)z)(x-a^2 z) = 0 \end{cases} \\
&= L_5 \cup L_8 \cup L_9
\end{aligned}$$

so L is either L_5, L_8 or L_9 in this case.
So we have left the case where L intersects L_7. In this case there exists a plane Π such that $L, L_7 \subset \Pi$. Π must be $t - x - az = \lambda y$ for some $\lambda \in \mathbb{C}$. So

$$\begin{aligned}
S \cap \Pi &= \begin{cases} t - x - az = \lambda y \\ t(xz - y^2) + (x-y)(y-z)(x-(a+1)y+az) = 0 \end{cases} \\
&= \begin{cases} t - x - az = \lambda y \\ y((a-\lambda+1)y^2 - (a+3)xy - (3a+1)yz + x^2 + az^2 + (2a+\lambda+2=0)xz) = 0 \end{cases}
\end{aligned}$$

which is the line L_7 and the conic $(a-\lambda+1)y^2 - (a+3)xy - (3a+1)yz + x^2 + az^2 + (2a+\lambda+2=0)xz = 0$ on $t - x - az = \lambda y$ which is irreducible for all $\lambda \neq -a-1$. When $\lambda = -a-1$ the plane Π is defined by $t = x + az - (a+1)y$

in which case

$$S \cap \Pi = \begin{cases} t = x + az - (a+1)y \\ t(xz - y^2) + (x-y)(y-z)(x - (a+1)y + az) = 0 \end{cases}$$
$$= \begin{cases} t = x + az - (a+1)y \\ y(x + az - (a+1)y)(x + z - 2y) = 0 \end{cases}$$
$$= L_3 \cup L_7 \cup L_{10}$$

so L is L_3, L_7 or L_{10}. Hence the only L are given in equation 6.3 so there are exactly 11 lines in S. □

6.6 $3\mathbb{A}_1$

According to Appendix B.1, equation of a $3\mathbb{A}_2$ surface is

$$t(xz - y^2) - xz(x - (a+1)y + az) = 0$$

for $a \in \mathbb{C} \setminus \{0, 1\}$. Let us call this surface S.
We can see that the following lines are contained in S.

$$\begin{aligned} L_1 &= \begin{cases} t = 0 \\ x = 0 \end{cases} & L_2 &= \begin{cases} t = 0 \\ z = 0 \end{cases} \\ L_3 &= \begin{cases} x = 0 \\ y = 0 \end{cases} & L_4 &= \begin{cases} y = 0 \\ z = 0 \end{cases} \\ L_5 &= \begin{cases} t = 0 \\ x - (a+1)y + az = 0 \end{cases} & L_6 &= \begin{cases} x = y \\ y = z \end{cases} \\ L_7 &= \begin{cases} y = 0 \\ t - x - az = 0 \end{cases} & L_8 &= \begin{cases} x = y \\ t = az \end{cases} \\ L_9 &= \begin{cases} x = ay \\ y = az \end{cases} & L_{10} &= \begin{cases} x = ay \\ t = az \end{cases} \\ L_{11} &= \begin{cases} t = x \\ y = az \end{cases} & L_{12} &= \begin{cases} t = x \\ y = z \end{cases} \end{aligned} \tag{6.4}$$

Theorem 6.6.1. *A cubic surface with three $\mathbb{A}_1$ singularities and is smooth elsewhere contains exactly 12 lines.*

Proof. To prove that any $3\mathbb{A}_1$ surface contains 12 lines we only need to show that S contains exactly 12 lines. We have listed 12 unique lines in equation 6.4 so it remains to show that these are all the lines contained in S.

Let L be a line in S. Consider the plane $x = 0$.

$$\begin{aligned} S \cap \{x = 0\} &= \begin{cases} x = 0 \\ t(xz - y^2) - xz(x - (a+1)y + az) = 0 \end{cases} \\ &= \begin{cases} x = 0 \\ -ty^2 = 0 \end{cases} \\ &= L_1 \cup 2L_3 \end{aligned}$$

so L is either L_1, L_3 or intersects at least one of L_1 or L_3.
Let us start with the case where L intersects L_1, then there is some plane Π such that $L, L_1 \subset \Pi$. This Π contains L_1 so is either $x = 0$ or $t = \lambda t$ for some $\lambda \in \mathbb{C}$. When Π is defined by $x = 0$ we know what happens so consider $\Pi : t = \lambda x$. We have that

$$\begin{aligned} S \cap \Pi &= \begin{cases} t = \lambda x \\ t(xz - y^2) - xz(x - (a+1)y + az) = 0 \end{cases} \\ &= \begin{cases} t = \lambda x \\ \lambda x(xz - y^2) - xz(x - (a+1)y + az) = 0 \end{cases} \\ &= \begin{cases} t = \lambda x \\ x(\lambda xy - \lambda y^2 - xz + (a+1)yz - az^2) = 0 \end{cases} \\ &= L_1 \cup \{t = \lambda x, \lambda xy - \lambda y^2 - xz + (a+1)yz - az^2 = 0\} \end{aligned}$$

where $\lambda xy - \lambda y^2 - xz + (a+1)yz - az^2 = 0$ is a conic on $t = \lambda x$ which is irreducible for all $\lambda \neq 0, 1$. When $\lambda = 0$ the plane Π is defined by $t = 0$ so

$$\begin{aligned} S \cap \Pi &= \begin{cases} t = 0 \\ t(xz - y^2) - xz(x - (a+1)y + az) = 0 \end{cases} \\ &= \begin{cases} t = 0 \\ xz(x - (a+1)y + az) = 0 \end{cases} \\ &= L_1 \cup L_2 \cup L_5 \end{aligned}$$

so L is one of L_1, L_2, L_5. In the case where $\lambda = 1$, Π is defined by $t = x$. Then

$$\begin{aligned} S \cap \Pi &= \begin{cases} t = x \\ t(xz - y^2) - xz(x - (a+1)y + az) = 0 \end{cases} \\ &= \begin{cases} t = x \\ x(-y + az)(y + z) = 0 \end{cases} \\ &= L_1 \cup L_{11} \cup L_{12} \end{aligned}$$

so L is one of L_1, L_{11}, L_{12} in this case.

We are left with the case where L intersects L_3. Hence there is some plane Π such that $L, L_3 \subset \Pi$. Then Π is defined by either $x = 0$ or $y = \lambda x$ for some $\lambda \in \mathbb{C}$. Consider $\Pi : y = \lambda x$. Then

$$\begin{aligned} S \cap \Pi &= \begin{cases} y = \lambda x \\ t(xz - y^2) - xz(x - (a+1)y + az) = 0 \end{cases} \\ &= \begin{cases} y = \lambda x \\ x(tz - \lambda^2 tx + (\lambda(a+1) - 1)xz - az^2) = 0 \end{cases} \\ &= L_3 \cup \{tz - \lambda^2 tx + (\lambda(a+1) - 1)xz - az^2 = 0\} \end{aligned}$$

where $tz - \lambda^2 tx + (\lambda(a+1) - 1)xz - az^2 = 0$ is a conic on $y = \lambda x$ which is irreducible for $\lambda = 0, 1, \frac{1}{a}$. When $\lambda = 0$, the plane Π is defined by $y = 0$ and

$$S \cap \Pi = \begin{cases} y = 0 \\ xz(t - x - az) = 0 \end{cases}$$
$$= L_3 \cup L_4 \cup L_7$$

so L is one of L_3, L_4, L_7 in this case. When $\lambda = 1$ the plane Π is defined by $x = y$ and

$$\begin{aligned} S \cap \Pi &= \begin{cases} x = y \\ x(-t + az)(x - z) = 0 \end{cases} \\ &= L_3 \cup L_6 \cup L_8 \end{aligned}$$

so L is one of L_3, L_6, L_8 in this case. When $\lambda = \frac{1}{a}$ the plane Π is defined by $ay = x$ so

$$\begin{aligned} S \cap \Pi &= \begin{cases} x = ay \\ y(y - az)(-a + az) = 0 \end{cases} \\ &= L_3 \cup L_9 \cup L_{10} \end{aligned}$$

so L is one of L_3, L_9, L_{10} in this case. Hence equation 6.4 gives a complete list of lines in S. So there are exactly 12 lines in any $3\mathbb{A}_1$ surface. $\square$

6.7 $\mathbb{A}_1\mathbb{A}_3$

The equation of any $\mathbb{A}_1\mathbb{A}_3$ surface up to projective transformation is

$$t(xz - y^2) + (x - y)(y - z)(x - 2y + z) = 0$$

according to Appendix B.1. Let's call this surface S.
We can see that the lines

$$
\begin{aligned}
L_1 &= \begin{cases} t = 0 \\ x - y = 0 \end{cases} & L_2 &= \begin{cases} t = 0 \\ y - z = 0 \end{cases} \\
L_3 &= \begin{cases} t = 0 \\ x - 2y + z = 0 \end{cases} & L_4 &= \begin{cases} x - z = 0 \\ y - z = 0 \end{cases} \\
L_5 &= \begin{cases} x = 0 \\ y = 0 \end{cases} & L_6 &= \begin{cases} y = 0 \\ z = 0 \end{cases} \\
L_7 &= \begin{cases} y = 0 \\ t - x - z = 0 \end{cases}
\end{aligned}
\tag{6.5}
$$

are contained in S.

Theorem 6.7.1. *A cubic surface with one $\mathbb{A}_1$ singularity and one $\mathbb{A}_3$ singularity which is smooth elsewhere contains exactly 7 lines.*

Proof. To prove that any $\mathbb{A}_1\mathbb{A}_3$ surface contains 7 lines we only need to show that S contains exactly 7 lines. We have listed 7 unique lines in equation 6.5 so it remains to show that these are all the lines contained in S.

Let L be some line in S. Consider the plane $y = 0$. Then

$$
\begin{aligned}
S \cup \{y = 0\} &= \begin{cases} y = 0 \\ t(xz - y^2) + (x - y)(y - z)(x - 2y + z) = 0 \end{cases} \\
&= \begin{cases} y = 0 \\ txz - xz(x + z) = 0 \end{cases} \\
&= L_5 \cup L_6 \cup L_7
\end{aligned}
$$

so we have that L can be one of L_5, L_6, L_7 or intersects at least one of L_5, L_6 or L_7 at 1 point.

Consider the case where L intersects L_5. There is a unique plane Π such that $L, L_5 \subset \Pi$. This Π will be either $y = 0$ or $x = \lambda y$ for some $\lambda \in \mathbb{C}$. In the case that Π is $y = 0$ we have that $S \cap \Pi = L_5 \cup L_6 \cup L_7$ as we have already seen.

In the case where Π is defined by $x = \lambda y$ we have

$$\begin{aligned} S \cap \Pi &= \begin{cases} x = \lambda y \\ t(xz - y^2) + (x-y)(y-z)(x-2y+z) = 0 \end{cases} \\ &= \begin{cases} x = \lambda y \\ y(\lambda tz - ty + (\lambda-1)(y-z)((\lambda-2)y+z)) = 0 \end{cases} \\ &= L_5 \cup \{x = \lambda y, \lambda tz - ty + (\lambda-1)(y-z)((\lambda-2)y+z) = 0\} \end{aligned}$$

where $\lambda tz - ty + (\lambda-1)(y-z)((\lambda-2)y+z = 0$ is a conic on $x = \lambda y$ which is irreducible for $\lambda \neq 1$. When $\lambda = 1$ we have that

$$S \cup \Pi = \begin{cases} x = \lambda y \\ ty(z-y) = 0 \end{cases} \qquad = L_1 \cup L_4 \cup L_5$$

so L is L_1, L_4 or L_5 in this situation.

In the case where L intersects L_6 there is a plane Π such that $L, L_6 \subset \Pi$. Since Π contains L_6 we know that Π is either $y = 0$ or $z = \lambda y$ for $\lambda \in \mathbb{C}$. We know what happens when $y = 0$ so let us consider the case where Π is defined by $z = \lambda y$. In this case we have that

$$\begin{aligned} S \cap \Pi &= \begin{cases} z = \lambda y \\ t(xz - y^2) + (x-y)(y-z)(x-2y+z) = 0 \end{cases} \\ &= \begin{cases} z = \lambda y \\ y(\lambda tx - ty - (\lambda-1)(x-y)(x+(\lambda-2)y)) = 0 \end{cases} \\ &= L_6 \cup \{z = \lambda y, \lambda tx - ty - (\lambda-1)(x-y)(x+(\lambda-2)y) = 0\} \end{aligned}$$

where $\lambda tx - ty - (\lambda-1)(x-y)(x+(\lambda-2)y = 0$ is a conic on $z = \lambda y$ which is irreducible for $\lambda \neq 1$. When $\lambda = 1$ we have that

$$S \cap \Pi = \begin{cases} z = y \\ ty(x-y) = 0 \end{cases} \qquad = L_2 \cup L_4 \cup L_6$$

so L is one of L_2, L_4, L_6 in this case.

In the case where L intersects L_7 there exists a plane Π such that $L, L_7 \in \Pi$. So Π is either $y = 0$ or $t - x - z = \lambda y$. We know what happens when Π is defined by $y = 0$. In the case where Π is defined by $t - x - z = \lambda y$ we have

$$\begin{aligned}
S \cap \Pi &= \begin{cases} t - x - z = \lambda y \\ t(xz - y^2) + (x - y)(y - z)(x - 2y + z) = 0 \end{cases} \\
&= \begin{cases} t - x - z = \lambda y \\ (x - \lambda y + z)(xz - y^2) + (x - y)(y - z)(x - 2y + z) = 0 \end{cases} \\
&= \begin{cases} t - x - z = \lambda y \\ y((\lambda + 4)xz - (\lambda - 2)y^2 - 4xy - 4yz + x^2 + z^2) = 0 \end{cases} \\
&= L_7 \cup \{t - x - z = \lambda y, (\lambda + 4)xz - (\lambda - 2)y^2 - 4xy - 4yz + x^2 + z^2 = 0\}
\end{aligned}$$

where $(\lambda + 4)xz - (\lambda - 2)y^2 - 4xy - 4yz + x^2 + z^2 = 0$ is a conic on the plane $t - x - z = \lambda y$. The matrix associated with this conic is

$$\begin{pmatrix} 1 & -2 & \frac{\lambda+4}{2} \\ -2 & -(\lambda - 2) & -2 \\ \frac{\lambda+4}{2} & -2 & 1 \end{pmatrix}$$

the determinant of which is $(\lambda+2)^3$ so this conic is irreducible for $\lambda \neq -2$. When $\lambda = -2$ we have

$$S \cup \Pi = \begin{cases} t - x - z = -2y \\ y(x - 2y + z)^2 = 0 \end{cases} \qquad = 2L_3 \cup L_7$$

so L is L_3 or L_7 in this case.
Hence our L_i are the only lines contained in S.

$\square$

6.8 $2\mathbb{A}_1\mathbb{A}_2$

According to Appendix B.1 the equation of any $2\mathbb{A}_1\mathbb{A}_2$ surface up to projective transformation is

$$t(xz - y^2) - y^2(x - y) = 0.$$

Let's call this surface S.
We can see that the lines

$$L_1 = \begin{cases} t = 0 \\ y = 0 \end{cases} \qquad L_2 = \begin{cases} t = 0 \\ x - y = 0 \end{cases}$$
$$L_3 = \begin{cases} x - y = 0 \\ y - z = 0 \end{cases} \qquad L_4 = \begin{cases} x = 0 \\ t - y = 0 \end{cases} \tag{6.6}$$
$$L_5 = \begin{cases} x = 0 \\ y = 0 \end{cases} \qquad L_6 = \begin{cases} y = 0 \\ z = 0 \end{cases}$$
$$L_7 = \begin{cases} z = 0 \\ t + x - y = 0 \end{cases} \qquad L_8 = \begin{cases} t - y = 0 \\ t - z = 0 \end{cases}$$

are contained in S.

Theorem 6.8.1. *A cubic surface with two $\mathbb{A}_1$ singularities and one $\mathbb{A}_2$ singularity which is smooth everywhere else contains exactly 8 lines.*

Proof. To prove that any $2\mathbb{A}_1\mathbb{A}_2$ surface contains 8 lines we only need to show that S contains exactly 8 lines. We have listed 8 unique lines in equation 6.6 so it remains to show that these are all the lines contained in S.

Let L be a line in S. Consider the plane $t = 0$. The line L intersects $t = 0$ at one point or is contained in $t = 0$.

$$\begin{aligned} S \cap \{t = 0\} &= \begin{cases} t = 0 \\ t(xz - y^2) - y^2(x - y) = 0 \end{cases} \\ &= \begin{cases} t = 0 \\ y^2(x - y) = 0 \end{cases} \\ &= 2L_1 \cup L_2 \end{aligned}$$

so L is either L_1 or L_2, or L intersects either L_1 or L_2.

In the case where L intersects L_1 we have that they define a unique plane Π such that $L, L_1 \subset \Pi$. Given that L_1 is contained in Π we know that Π is either $t = 0$ or $y = \lambda t$.
If Π is defined by $t = 0$ we know that $S \cap \Pi = 2L_1 \cup L_2$ so again L is either L_1 or L_2.
When Π is defined by $y = \lambda t$ we have

$$\begin{aligned} S \cap \Pi &= \begin{cases} y = \lambda t \\ t(xz - \lambda^2 tx + (\lambda^3 - \lambda^2)t^2) = 0 \end{cases} \\ &= L_1 \cup \{y = \lambda t, xz - \lambda^2 tx + (\lambda^3 - \lambda^2)t^2 = 0\} \end{aligned}$$

where $xz - \lambda^2 tx + (\lambda^3 - \lambda^2)t^2 = 0$ is a conic on $y = \lambda t$. This is an irreducible

conic for $\lambda \neq 0, 1$.
When $\lambda = 0$ we have that

$$S \cap \Pi = \begin{cases} y = 0 \\ txz = 0 \end{cases}$$
$$= L_1 \cup L_5 \cup L_6$$

so L is one of L_1, L_5 or L_6 in this case.
If $\lambda = 1$ we have

$$S \cap \Pi = \begin{cases} y = t \\ tx(z-t) = 0 \end{cases}$$
$$= L_1 \cup L_2 \cup L_8$$

so L is L_1, L_2 or L_8 in this case.

Now consider the case where L intersects L_2. Then there is some unique plane Π which contains both L and L_2. Π is either of the form $x - y = 0$ or $t = \lambda(x - y)$. In the case that Π is defined by $x - y = 0$ we have

$$\begin{aligned} S \cap \Pi &= \begin{cases} x = y \\ t(xz - y^2) - y^2(x - y) = 0 \end{cases} \\ &= \begin{cases} x = y \\ ty(z - y) = 0 \end{cases} \\ &= L_2 \cup L_4 \cup L_5 \end{aligned}$$

so L is L_2, L_4 or L_5 in this case.
Where Π is defined by $t = \lambda(x - y)$ we have

$$\begin{aligned} S \cap \Pi &= \begin{cases} t = \lambda(x - y) \\ t(xz - y^2) - y^2(x - y) = 0 \end{cases} \\ &= \begin{cases} t = \lambda(x - y) \\ (x - y)(\lambda xz - (\lambda + 1)y^2) = 0 \end{cases} \\ &= L_2 \cup \{t = \lambda(x - y), \lambda xz - (\lambda + 1)y^2 = 0\} \end{aligned}$$

where $\lambda xz - (\lambda + 1)y^2 = 0$ is a conic on $t = \lambda(x - y)$ which is irreducible for $\lambda \neq -1, 0$.
When $\lambda = 0$ we have that Π is defined by $t = 0$ which we have already seen.
When $\lambda = -1$

$$S \cup \Pi = \begin{cases} t + x - y = 0 \\ -(x - y)xz = 0 \end{cases} \quad = L_2 \cup L_4 \cup L_7$$

so L is L_2, L_4 or L_7 in this case. So we have found all the lines in S. $\square$

6.9 $4\mathbb{A}_1$

Appendix B.1 gives the equation of any $4\mathbb{A}_1$ up to projective transformation as

$$t(xz - y^2) - (x - y)(y - z)y = 0.$$

Let's call this surface S.

We can see that the following lines are contained in S.

$$\begin{aligned}
L_1 &= \begin{cases} t = 0 \\ y = 0 \end{cases} & L_2 &= \begin{cases} t = 0 \\ x - y = 0 \end{cases} \\
L_3 &= \begin{cases} t = 0 \\ y - z = 0 \end{cases} & L_4 &= \begin{cases} x = 0 \\ y = 0 \end{cases} \\
L_5 &= \begin{cases} y = 0 \\ z = 0 \end{cases} & L_6 &= \begin{cases} x = 0 \\ -t + y - z = 0 \end{cases} \\
L_7 &= \begin{cases} x - y = 0 \\ t = 0 \end{cases} & L_8 &= \begin{cases} x - y = 0 \\ y - z = 0 \end{cases} \\
L_9 &= \begin{cases} z = 0 \\ t - x + y = 0 \end{cases}
\end{aligned} \tag{6.7}$$

Theorem 6.9.1. *A cubic surface with four $\mathbb{A}_1$ singularities which is smooth everywhere else contains exactly 9 lines.*

Proof. To prove that any $4\mathbb{A}_1$ surface contains 9 lines we only need to show that S contains exactly 9 lines. We have listed 9 unique lines in equation 6.7 so it remains to show that these are all the lines contained in S.
Let L be a line in S. Consider the plane defined by $y = 0$ then

$$\begin{aligned}
S \cap \{y = 0\} &= \begin{cases} y = 0 \\ txz = 0 \end{cases} \\
&= L_1 \cup L_4 \cup L_5
\end{aligned}$$

so, either L is one of L_1, L_4 or L_5, or L intersects at least one of L_1, L_4 or L_5.
In the case where L intersects L_1 there exists a plane Π such that $L, L_1 \subset \Pi$. Since Π contains L_1 it must be defined by $y = 0$ or $t = \lambda y$ for some $\lambda \in \mathbb{C}$. In the case where Π is defined by $y = 0$ we know from above that $S \cap \Pi = L_1 \cup L_4 \cup L_5$ so again L can be one of L_1, L_4 or L_5.

In the case that Π is defined by $t = \lambda y$ we have

$$\begin{aligned} S \cap \Pi &= \begin{cases} t = \lambda y \\ t(xz - y^2) - (x-y)(y-z)y = 0 \end{cases} \\ &= \begin{cases} t = \lambda y \\ y(\lambda xz - \lambda y^2 - (x-y)(y-z)) = 0 \end{cases} \\ &= L_1 \cup \{t = \lambda y, \lambda xz - \lambda y^2 - (x-y)(y-z) = 0\} \end{aligned}$$

where $\lambda xz - \lambda y^2 - (x-y)(y-z) = 0$ is a conic on $t = \lambda y$. This conic is irreducible for $\lambda \neq 0$. When $\lambda = 0$ we have that

$$\begin{aligned} S \cap \Pi &= \begin{cases} t = 0 \\ y(x-y)(y-z) = 0 \end{cases} \\ &= L_1 \cup L_2 \cup L_3 \end{aligned}$$

so L can be one of L_1, L_2 or L_3.

In the case where L intersects L_4 we have that there is some unique plane Π such that $L, L_4 \subset \Pi$. As $L_4 \subset \Pi$ it must be defined by $y = 0$ or $x = \lambda y$. Again we know about the case where $y = 0$. In the case where Π is defined by $x = \lambda y$ we have

$$\begin{aligned} S \cap \Pi &= \begin{cases} x = \lambda y \\ t(xz - y^2) - (x-y)(y-z)y = 0 \end{cases} \\ &= \begin{cases} x = \lambda y \\ y(\lambda tz - ty - (\lambda - 1)y(y-z)) = 0 \end{cases} \\ &= L_4 \cup \{x = \lambda y, \lambda tz - ty - (\lambda - 1)y(y-z) = 0\} \end{aligned}$$

where $\lambda tz - ty - (\lambda - 1)y(y-z) = 0$ is a conic on $x = \lambda y$ which is irreducible for $\lambda \neq 0, 1$. For $\lambda = 0$ we have

$$\begin{aligned} S \cap \Pi &= \begin{cases} x = 0 \\ y^2(-t + (y-z)) = 0 \end{cases} \\ &= 2L_4 \cup L_6 \end{aligned}$$

and for $\lambda = 1$ we have

$$\begin{aligned} S \cap \Pi &= \begin{cases} x = y \\ ty(z-y) = 0 \end{cases} \\ &= L_4 \cup L_7 \cup L_8 \end{aligned}$$

so L could be one of L_4, L_6, L_7 or L_8.

Where L_5 intersects L we have that there is some unique plane Π such that

$L, L_5 \subset \Pi$. As $L_5 \subset \Pi$ it must be defined by $y = 0$ or $z = \lambda y$. Again we have already seen what happens for the case where Π is defined by $y = 0$. In the case when Π is defined by $z = \lambda y$ we have

$$\begin{aligned} S \cap \Pi &= \begin{cases} z = \lambda y \\ t(xz - y^2) - (x - y)(y - z)y = 0 \end{cases} \\ &= \begin{cases} z = \lambda y \\ y(\lambda tx - ty + (\lambda - 1)y(x - y)) = 0 \end{cases} \\ &= L_5 \cup \{z = \lambda y, \lambda tx - ty + (\lambda - 1)y(x - y) = 0\} \end{aligned}$$

where $\lambda tx - ty + (\lambda - 1)y(x - y) = 0$ is a conic on $z = \lambda y$. This cubic is irreducible for $\lambda \neq 0, 1$. Where $\lambda = 0$ we have

$$\begin{aligned} S \cap \Pi &= \begin{cases} z = 0 \\ y^2(t - (x - y)) = 0 \end{cases} \\ &= 2L_5 \cup L_9 \end{aligned}$$

and for $\lambda = 1$ we have

$$\begin{aligned} S \cap \Pi &= \begin{cases} y = z \\ ty(x - y) = 0 \end{cases} \\ &= L_3 \cup L_5 \cup L_8. \end{aligned}$$

So L can be one of L_3, L_5, L_8 or L_9. Hence we have found all the lines in S. $\square$

6.10 $\mathbb{A}_1\mathbb{A}_4$

From the table in Appendix B.1 we can see that any $\mathbb{A}_1\mathbb{A}_4$ surface is defined, up to projective transformation, by the equation

$$t(xz - y^2) - x^2y = 0.$$

Let us call this surface S.

By observation we can see that the following four lines are contained in this surface.

$$L_1 = \begin{cases} x = 0 \\ t = 0 \end{cases} \qquad L_2 = \begin{cases} y = 0 \\ t = 0 \end{cases} \tag{6.8}$$

$$L_3 = \begin{cases} z = 0 \\ y = 0 \end{cases} \qquad L_4 = \begin{cases} x = 0 \\ y = 0 \end{cases}$$

Theorem 6.10.1. *A cubic surface which has one $\mathbb{A}_1$ singularity, one $\mathbb{A}_4$ singularity and is smooth elsewhere contains exactly 4 lines.*

Proof. To prove that any $\mathbb{A}_1\mathbb{A}_4$ surface contains 4 lines we only need to show that S contains exactly 4 lines. We have listed 4 unique lines in equation 6.8 so it remains to show that these are all the lines contained in S.
Yet again let L be some line in S. Consider the plane $t = 0$. We have

$$\begin{aligned} S \cap \{t = 0\} &= \begin{cases} t = 0 \\ t(xz - y^2) - x^2 y = 0 \end{cases} \\ &= \begin{cases} t = 0 \\ x^2 y = 0 \end{cases} \\ &= 2L_1 \cup L_2 \end{aligned}$$

so either, L is one of L_1 or L_2, or L intersects at least one of L_1 or L_2.

Let us start by assuming that L intersects L_1. Then L and L_1 uniquely define some plane Π. As Π contains L_1 we know it is of the form $t = 0$ or $x = \lambda t$ for some $\lambda \in \mathbb{C}$.
We have seen the case where Π is defined by $t = 0$ so let us consider case where Π is defined by $x = \lambda t$ we have that

$$\begin{aligned} S \cap \Pi &= \begin{cases} x = \lambda t \\ t(\lambda t z - y^2 - \lambda^2 t y) = 0 \end{cases} \\ &= L_1 \cup \{x = \lambda t, \lambda t z - y^2 - \lambda^2 t y = 0\} \end{aligned}$$

where $\lambda t z - y^2 - \lambda^2 t y = 0$ is a conic on $x = \lambda t$. This conic is irreducible (and so contains no lines) where $\lambda \neq 0$. When $\lambda = 0$ we have that

$$\begin{aligned} S \cap \Pi &= \begin{cases} x = 0 \\ t y^2 \end{cases} \\ &= L_1 \cup L_4 \end{aligned}$$

so L can be L_1 or L_4.

Now let us assume that L intersects L_2. Then L and L_2 uniquely define some plane Π. As Π contains L_2 it must be defined by either $t = 0$ or $y = \lambda t$ for some $\lambda \in \mathbb{C}$.
We have seen the case where $\Pi : t = 0$ so consider the case where Π is defined by $y = \lambda t$. Then

$$\begin{aligned} S \cap \Pi &= \begin{cases} y = \lambda t \\ t(xz - \lambda^2 t^2 - \lambda x^2) \end{cases} \\ &= L_2 \cup \{y = \lambda t, xz - \lambda^2 t^2 - \lambda x^2 = 0\} \end{aligned}$$

where $xz - \lambda^2 t^2 - \lambda x^2 = 0$ is an irreducible conic for $\lambda \neq 0$. When $\lambda = 0$ we have that Π is defined by $y = 0$ and so

$$S \cap \Pi = \begin{cases} y = 0 \\ txz = 0 \end{cases}$$
$$= L_2 \cup L_3 \cup L_4$$

So L can be L_2, L_3 or L_4. Hence we have all the lines in S. $\square$

6.11 $2\mathbb{A}_1\mathbb{A}_3$

Referring again to the table in Appendix B.1 we can see that any $2\mathbb{A}_1\mathbb{A}_3$ surface up to projective transformation is defined by the equation

$$t(xz - y^2) - xy^2 = 0.$$

By observation we can see that the following 5 lines are contained in this surface.

$$\begin{aligned} L_1 &= \begin{cases} x = 0 \\ t = 0 \end{cases} & L_2 &= \begin{cases} y = 0 \\ t = 0 \end{cases} \\ L_3 &= \begin{cases} z = 0 \\ y = 0 \end{cases} & L_4 &= \begin{cases} x = 0 \\ y = 0 \end{cases} \\ L_5 &= \begin{cases} z = 0 \\ t = x \end{cases} \end{aligned} \tag{6.9}$$

Theorem 6.11.1. *A cubic surface which has two $\mathbb{A}_1$ singularities and one $\mathbb{A}_3$ singularity and is smooth elsewhere contains exactly 5 lines.*

Proof. To prove that any $2\mathbb{A}_1\mathbb{A}_3$ surface contains 5 lines we only need to show that S contains exactly 5 lines. We have listed 5 unique lines in equation 6.9 so it remains to show that these are all the lines contained in S. Let L be some line in S. Consider the plane $t = 0$. The intersection of this plane with S is the space defined by $xy^2 = 0$ and $t = 0$ which consists of the lines L_1 and L_2. Then L is not contained in this plane and so must intersect this plane at one point. So either L is L_1 or L_2 or intersects at least one of L_1 or L_2.

Let us start by assuming that L intersects L_1. Then L and L_1 uniquely define some plane Π. As Π contains L_1 we know it is of the form $x = 0$ or $t = \lambda x$ for some $\lambda \in \mathbb{C}$.
In the case where Π is defined by $x = 0$ we have

$$S \cap \Pi = \begin{cases} x = 0 \\ ty^2 = 0 \end{cases}$$

which is the space defined by the lines L_1 and L_4 so L can be L_1 or L_4.

In the case where Π is defined by $t = \lambda x$ we have that

$$S \cap \Pi = \begin{cases} t = \lambda x \\ x(\lambda xz - (\lambda + 1)y^2) = 0 \end{cases}$$

which is the space defined by the line L_1 and the conic $\lambda xz - (\lambda - 1)y^2 = 0$ on $t = \lambda x$. This conic is irreducible apart from where $\lambda = -1$ and $\lambda = 0$. When $\lambda = 1$ we have

$$\begin{aligned} S \cap \Pi &= \begin{cases} t = -x \\ x^2 z = 0 \end{cases} \\ &= 2L_1 \cup L_5 \end{aligned}$$

so L can be L_1 or L_5. When $\lambda = 0$ we have

$$\begin{aligned} S \cap \Pi &= \begin{cases} t = 0 \\ xy^2 = 0 \end{cases} \\ &= L_1 \cup 2L_2 \end{aligned}$$

so L could be L_1 or L_2.

Now let us assume that L intersects L_2. Then L and L_2 uniquely define some plane Π. As Π contains L_2 it must be defined by either $y = 0$ or $t = \lambda y$ for some $\lambda \in \mathbb{C}$.
Take first the case where Π is defined by $y = 0$ we have

$$S \cap \Pi = \begin{cases} y = 0 \\ txz = 0 \end{cases}$$

which is clearly the space comprised of the three lines L_2, L_3 and L_4.
In the second case where Π is defined by $t = \lambda y$ we have

$$S \cap \Pi = \begin{cases} t = y\lambda \\ y(\lambda xz - \lambda y^2 - xy) = 0 \end{cases}$$

which is the space comprised of the line L_2 and the conic $\lambda xz - \lambda y^2 - xy = 0$ on the plane $t = \lambda y$. We can show this conic is irreducible away from $\lambda = 0$. When $\lambda = 0$ we have that Π is defined by $t = 0$ which is a case we have already seen. So we have that L is one of L_1, L_2, L_3, L_4 or L_5. So there are only 5 lines in a $2\mathbb{A}_1\mathbb{A}_3$ surface. $\square$

6.12 $\mathbb{A}_1 2\mathbb{A}_2$

According to Appendix B.1 the equation of an $\mathbb{A}_1 2\mathbb{A}_2$ surface up to projective transformation is

$$t(xz - y^2) - y^3.$$

Let us call this surface S.
The lines

$$L_1 = \begin{cases} t = 0 \\ y = 0 \end{cases} \qquad L_2 = \begin{cases} y = 0 \\ x = 0 \end{cases}$$
$$L_3 = \begin{cases} y = 0 \\ z = 0 \end{cases} \qquad L_4 = \begin{cases} x = 0 \\ t + y = 0 \end{cases} \tag{6.10}$$
$$L_5 = \begin{cases} z = 0 \\ t + y = 0 \end{cases}$$

are contained in S.

Theorem 6.12.1. *A cubic surface which has one $\mathbb{A}_1$ singularity, two $\mathbb{A}_2$ singularities and is smooth elsewhere contains exactly 5 lines.*

Proof. In order to prove that any $\mathbb{A}_1 2\mathbb{A}_2$ surface contains 5 lines we only need to show that S contains exactly 5 lines. We have listed 5 unique lines in equation 6.10 so it remains to show that these are all the lines contained in S. Let L be a line in S. Now consider the plane defined by $t = 0$. Then

$$\begin{aligned} S \cap \{t = 0\} &= \begin{cases} t = 0 \\ t(xz - y^2) - y^3 = 0 \end{cases} \\ &= \begin{cases} t = 0 \\ y^3 = 0 \end{cases} \\ &= 3L_1 \end{aligned}$$

so L must intersect this plane at 1 point or be contained in this plane. So from the above we know that either L is L_1 or L intersects L_1. In the case where L and L_1 intersect, they uniquely define a plane Π such that $L, L_1 \subset \Pi$. Π contains L so must be either $t = 0$ or $y = \lambda t$ for some $\lambda \in \mathbb{C}$.

In the first case where $\Pi : t = 0$ we have already seen that $S \cap \Pi = 3L_1$ from the above so again L can be L_1.

In the second case where $\Pi : y = \lambda t$ we have that

$$\begin{aligned} S \cap \Pi &= \begin{cases} y = \lambda t \\ t(xz - y^2) - y^3 = 0 \end{cases} \\ &= \begin{cases} y = \lambda t \\ t((xz - \lambda^2 t^2) - \lambda^3 t^2) = 0 \end{cases} \\ &= L_1 \cup \{y = \lambda t, xz - (\lambda^2 + \lambda^3)t^2 = 0\} \end{aligned}$$

where $xz-(\lambda^2+\lambda^3)t^2 = 0$ is a conic on the plane $y = \lambda t$. This conic is irreducible and hence contains no lines for $\lambda \neq 0, -1$.
In the case where $\lambda = 0$ we have that

$$S \cap \Pi = L_1 \cup L_2 \cup L_3$$

and in the case whee $\lambda = -1$ we have that

$$S \cap \Pi = L_1 \cup L_4 \cup L_5$$

so L is either L_1, L_2, L_3, L_4 or L_5. Hence there are only 5 lines in S. □

6.13 $\mathbb{A}_1\mathbb{A}_5$

Appendix B.1 gives the equation of an $\mathbb{A}_1\mathbb{A}_5$ surface up to projective transformation as

$$t(xz - y^2) - x^3 = 0.$$

Let us call this S.
By observation we can see that the lines

$$L_1 = \begin{cases} t = 0 \\ x = 0 \end{cases} \qquad L_2 = \begin{cases} x = 0 \\ y = 0 \end{cases} \tag{6.11}$$

are contained in S.

Theorem 6.13.1. *A cubic surface which has one $\mathbb{A}_1$ singularity, one $\mathbb{A}_5$ singularity and is smooth elsewhere contains exactly 2 lines.*

Proof. To prove that any $\mathbb{A}_1\mathbb{A}_5$ surface contains 2 lines we only need to show that S contains exactly 2 lines. We have listed 2 distinct lines in equation 6.11 so it remains to show that these are the only lines contained in S. Let L be some line in S. Consider the plane defined by $t = 0$.

$$\begin{aligned} S \cap \{t = 0\} &= \begin{cases} t = 0 \\ t(xz - y^2) - x^3 = 0 \end{cases} \\ &= \begin{cases} t = 0 \\ x^3 = 0 \end{cases} \\ &= 3L_1 \end{aligned}$$

so L must intersect this plane at 1 point or be contained in this plane. So from the above we know that either L is L_1 or L intersects L_1 at one point. So L and L_1 uniquely define some plane Π such that $L, L_1 \subset S$. This plane contains L_1 so must be either $t = 0$ or $x = \lambda t$. We have aready seen what happens when Π is defined by $t = 0$.

In the case where Π is defined by $x = \lambda t$ we have

$$\begin{aligned} S \cap \Pi &= \begin{cases} x = \lambda t \\ t(xz - y^2) - x^3 = 0 \end{cases} \\ &= \begin{cases} x = \lambda t \\ t(\lambda tz - y^2 - \lambda^3 t^2) = 0 \end{cases} \\ &= L_1 \cup \{x = \lambda t, \lambda tz - y^2 - \lambda^3 t^2 = 0\} \end{aligned}$$

where $\lambda tz - y^2 - \lambda^3 t^2 = 0$ defines a conic on $x = \lambda t$. This conic is irreducible for $\lambda \neq 0$. Where $\lambda = 0$ we can see that Π becomes $x = 0$. Then

$$\begin{aligned} S \cap \Pi &= \begin{cases} x = 0 \\ ty^2 \end{cases} \\ &= L_1 \cup 2L_2 \end{aligned}$$

so L is either L_1 or L_2. So we have found all the lines. $\square$

6.14 $2\mathbb{A}_2$

According to Appendix B.1 the equation of a $2\mathbb{A}_2$ surface up to projective transformation is

$$txy + z(y + z)(y - dz) = 0.$$

where $d \in \mathbb{C} \setminus \{0,1\}$. Let us call this surface S.
We can see that the lines

$$\begin{aligned} L_1 &= \begin{cases} t = 0 \\ z = 0 \end{cases} & L_2 &= \begin{cases} x = 0 \\ z = 0 \end{cases} \\ L_3 &= \begin{cases} y = 0 \\ z = 0 \end{cases} & L_4 &= \begin{cases} t = 0 \\ y + z = 0 \end{cases} \\ L_5 &= \begin{cases} t = 0 \\ y - dz = 0 \end{cases} & L_6 &= \begin{cases} x = 0 \\ y + z = 0 \end{cases} \\ L_7 &= \begin{cases} x = 0 \\ y - dz = 0 \end{cases} \end{aligned} \tag{6.12}$$

are contained in S.

Theorem 6.14.1. *A cubic surface which has two $\mathbb{A}_2$ singularities and is smooth elsewhere contains exactly 7 lines.*

Proof. To prove that any $2\mathbb{A}_2$ surface contains 7 lines we only need to show that S contains exactly 7 lines. We have listed 7 distinct lines in equation 6.11 so it remains to show that these are the only lines contained in S. Let L be some line in S. Consider the plane $y = 0$. Then L is either contained in $y = 0$ or it intersects it at 1 point.

$$\begin{aligned} S \cap \{y = 0\} &= \begin{cases} y = 0 \\ txy + z(y+z)(y - dz) = 0 \end{cases} \\ &= \begin{cases} y = 0 \\ -dz^3 = 0 \end{cases} \\ &= 3L_3 \end{aligned}$$

so either L is L_3 or L intersects L_3. In the latter case there is some plane Π such that $L, L_3 \subset \Pi$. As $L_3 \subset \Pi$ we know that Π is either $y = 0$ or $z = \lambda y$. In the case that Π is defined by $y = 0$ we know that $S \cap \Pi = 3L_3$ so again L can be L_3. In the case that Π is defined by $z = \lambda y$ we have

$$\begin{aligned} S \cap \Pi &= \begin{cases} z = \lambda y \\ txy + z(y+z)(y - dz) = 0 \end{cases} \\ &= \begin{cases} z = \lambda y \\ y(tx - \lambda(\lambda+1)(1 - d\lambda)y^2) \end{cases} \\ &= L_3 \cup \{z = \lambda y, tx - \lambda(\lambda+1)(1-d\lambda)y^2 = 0\} \end{aligned}$$

where $tx - \lambda(\lambda+1)(1-d\lambda)y^2 = 0$ is a conic on $z = \lambda y$ which is irreducible for $\lambda \neq -1, 0, \frac{1}{d}$.

In the case that $\lambda = -1$ we have that

$$S \cap \Pi = \begin{cases} y + z = 0 \\ txy = 0 \end{cases} = L_3 \cup L_4 \cup L_6.$$

In the case that $\lambda = 0$ we have that Π is defined by $z = 0$ so we have that

$$S \cap \Pi = \begin{cases} z = 0 \\ txy = 0 \end{cases} = L_1 \cup L_2 \cup L_3.$$

In the case that $\lambda = \frac{1}{d}$ we have that Π is defined by $y - dz = 0$ so

$$S \cap \Pi = \begin{cases} y - dz = 0 \\ dtxz = 0 \end{cases} = L_3 \cup L_5 \cup L_7.$$

Hence L can be only $L_1, L_2, L_3, L_4, L_5, L_6$ or L_7. So there are only 7 lines in S. $\square$

6.15 $3\mathbb{A}_2$

Any $3\mathbb{A}_2$ surface is defined up to transformation by the equation

$$txy - z^3 = 0$$

according to Appendix B.1.
We can see immediately 3 lines contained in this as listed below.

$$L_1 = \begin{cases} t = 0 \\ z = 0 \end{cases} \qquad L_2 = \begin{cases} x = 0 \\ z = 0 \end{cases} \tag{6.13}$$

$$L_3 = \begin{cases} y = 0 \\ z = 0 \end{cases}$$

Theorem 6.15.1. *A cubic surface which has three $\mathbb{A}_2$ singularities and is smooth elsewhere contains exactly 3 lines.*

Proof. To prove that any $3\mathbb{A}_2$ surface contains 3 lines we only need to show that S contains exactly 3 lines. We have listed 3 distinct lines in equation 6.11 so it remains to show that these are the only lines contained in S. Let L be some line in S. Consider the plane defined by $t = 0$.

$$S \cap \{t = 0\} = \begin{cases} t = 0 \\ txy - z^3 \end{cases}$$
$$= 3L_1.$$

so either L is L_1 or L intersects L_1 at exactly 1 point.
When L intersects L_1 there is a unique plane Π such that $L, L_1 \subset \Pi$. As Π

contains L_1 it is defined by $t = 0$ or $z = \lambda t$ for $\lambda \in \mathbb{C}$. Then L and L_1 are contained in the intersection $S \cap \Pi$.
We have already seen the case where Π is $t = 0$, so let us assume that Π is defined by $z = \lambda t$. Then

$$\begin{aligned} S \cap \Pi &= \begin{cases} z = \lambda t \\ t(xy - \lambda^3 t^2) = 0 \end{cases} \\ &= L_1 \cup \{z = \lambda t, xy - \lambda^3 t^2 = 0\} \end{aligned}$$

where $xy - \lambda^3 t^2$ is a conic on $z = \lambda t$ which is irreducible for $\lambda \neq 0$. When $\lambda = 0$ we have that Π is defined by $z = 0$ so

$$\begin{aligned} S \cap \Pi &= \begin{cases} z = 0 \\ txy = 0 \end{cases} \\ &= L_1 \cup L_2 \cup L_3 \end{aligned}$$

so L can only be L_1, L_2 or L_3. Therefore there are only 3 lines contained in S. $\square$

6.16 $\mathbb{A}_3$

According to Appendix B.1 the equation of a cubic surface with one $\mathbb{A}_3$ singularity up to projective transformation is

$$txy - z(x + y + z)(x - uy) = 0$$

where $u \in \mathbb{C} \setminus \{0\}$. Let us call this surface S. We can see that the lines

$$\begin{aligned} L_1 &= \begin{cases} t = 0 \\ z = 0 \end{cases} & L_2 &= \begin{cases} t = 0 \\ x + y + z = 0 \end{cases} \\ L_3 &= \begin{cases} t = 0 \\ x - uy = 0 \end{cases} & L_4 &= \begin{cases} x = 0 \\ z = 0 \end{cases} \\ L_5 &= \begin{cases} x = 0 \\ y + z = 0 \end{cases} & L_6 &= \begin{cases} x = 0 \\ y = 0 \end{cases} \\ L_7 &= \begin{cases} y = 0 \\ z = 0 \end{cases} & L_8 &= \begin{cases} y = 0 \\ x + z = 0 \end{cases} \\ L_9 &= \begin{cases} x + z = 0 \\ t + x - uy = 0 \end{cases} & L_{10} &= \begin{cases} y + z = 0 \\ t + x + uz = 0 \end{cases} \end{aligned} \tag{6.14}$$

are contained in S.

Theorem 6.16.1. *A cubic surface which has an $\mathbb{A}_3$ singularity and is smooth elsewhere contains exactly 10 lines.*

Proof. To prove that any $\mathbb{A}_3$ surface contains 10 lines we only need to show that

S contains exactly 10 lines. We have listed 10 distinct lines in equation 6.14 so it remains to show that these are the only lines contained in S. Let L be some line in S. Consider the plane $z = 0$. Then L either intersects $z = 0$ once or is contained in $z = 0$.

$$\begin{aligned} S \cap \{z = 0\} &= \begin{cases} z = 0 \\ txy - z(x + y + z)(x - uy) = 0 \end{cases} \\ &= \begin{cases} z = 0 \\ txy = 0 \end{cases} \\ &= L_1 \cup L_4 \cup L_7 \end{aligned}$$

so either L is L_1, L_4 or L_7 or intersects at least one of L_1, L_4 or L_7.

Let us first assume that L intersects L_1 then there is some plane Π such that $L, L_1 \subset \Pi$. This Π is either $z = 0$ or $t = \lambda z$ for some $\lambda \in \mathbb{C}$. We know that in the case that Π is defined by $z = 0$ we have $S \cap \Pi = L_1 \cup L_4 \cup L_7$. So when Π is defined by $t = \lambda z$ we have

$$\begin{aligned} S \cup \Pi &= \begin{cases} t = \lambda z \\ txy - z(x + y + z)(x - uy) = 0 \end{cases} \\ &= \begin{cases} t = \lambda z \\ z(\lambda xy - (x + y + z)(x - uy)) = 0 \end{cases} \\ &= L_1 \cup \{t = \lambda z, \lambda xy - (x + y + z)(x - uy) = 0\} \end{aligned}$$

where $xy - (x + y + z)(x - uy) = -x^2 - uy^2 + (\lambda + u - 1)xy - xz + uyz = 0$ is a conic on $t = \lambda z$ which is irreducible for $\lambda \neq 0$. When $\lambda = 0$ we have

$$\begin{aligned} S \cap \Pi &= \begin{cases} t = 0 \\ z(x + y + z)(x - uy) = 0 \end{cases} \\ &= L_1 \cup L_2 \cup L_3 \end{aligned}$$

so L can be L_1, L_2 or L_3.

In the case where L intersects L_4 we have that there is some plane Π such that $L, L_4 \subset \Pi$. So Π is defined by $z = 0$ or $x = \lambda z$ for some $\lambda \in \mathbb{C}$. When Π is defined by $x = \lambda x$ we have

$$\begin{aligned} S \cap \Pi &= \begin{cases} x = \lambda z \\ z(\lambda ty - ((\lambda + 1)z + y)(\lambda z - uy)) = 0 \end{cases} \\ &= L_4 \cup \{x = \ \lambda z, \lambda ty - ((\lambda + 1)z + y)(\lambda - uy) = 0\} \end{aligned}$$

where $\lambda ty - ((\lambda+1)z+y)(\lambda-uy) = 0$ is a conic which is irreducible for $\lambda \neq 0, -1$.

When $\lambda = 0$ we have that

$$S \cap \Pi = \begin{cases} x = 0 \\ uyz(y+z) = 0 \end{cases}$$
$$= L_4 \cup L_5 \cup L_6$$

and when $\lambda = -1$ we have

$$S \cap \Pi = \begin{cases} x + z = 0 \\ yz(t + x - uy) = 0 \end{cases}$$
$$= L_4 \cup L_6 \cup L_9$$

so L is L_4, L_5, L_6 or L_9.

In the case that L intersects L_7 we have that there is a plane Π such that $L, L_7 \subset \Pi$. This plane is of the form $y = \lambda z$ so

$$S \cap \Pi = \begin{cases} y = \lambda z \\ z(\lambda tx - (x + (\lambda+1)z)(x - u\lambda z)) = 0 \end{cases}$$
$$= L_7 \cup \{y = \lambda z, \lambda tx - (x + (\lambda+1)z)(x - u\lambda z) = 0\}$$

where $\lambda tx - (x + (\lambda+1)z)(x - u\lambda y) = 0$ is a conic on $y = \lambda t$ which is irreducible for $\lambda \neq 0, -1$. When $\lambda = 0$ we have that

$$S \cap \Pi = \begin{cases} y = 0 \\ xz(x+z) = 0 \end{cases}$$
$$= L_6 \cup L_7 \cup L_8$$

and when $\lambda = -1$ we have that

$$S \cap \Pi = \begin{cases} y + z = 0 \\ xz(t + x + uz) \end{cases}$$
$$= L_5 \cup L_7 \cup L_{10}$$

so L can be L_6, L_7, L_8 or L_{10}.
Hence we have found all the lines in S. So there are only 10 lines in any $\mathbb{A}_3$ surface. $\square$

6.17 $\mathbb{A}_4$

Any cubic surface with an $\mathbb{A}_4$ singularity up to projective transformation is given by

$$txy - (x^2z + y^3 - yz^2) = 0$$

according to Appendix B.1. Let's call this surface S.

By inspection we can see that the 6 lines

$$
\begin{aligned}
L_1 &= \begin{cases} x = 0 \\ y = 0 \end{cases} & L_2 &= \begin{cases} z = 0 \\ y = 0 \end{cases} \\
L_3 &= \begin{cases} y - z = 0 \\ x = 0 \end{cases} & L_4 &= \begin{cases} y + z = 0 \\ x = 0 \end{cases} \\
L_5 &= \begin{cases} t + x = 0 \\ y + z = 0 \end{cases} & L_6 &= \begin{cases} t - x = 0 \\ y - z = 0 \end{cases}
\end{aligned} \tag{6.15}
$$

are contained in the surface defined above.

Theorem 6.17.1. *A cubic surface which has one $\mathbb{A}_4$ singularity and is smooth elsewhere contains exactly 6 lines.*

Proof. To prove that any $\mathbb{A}_4$ surface contains 6 lines we only need to show that S contains exactly 6 lines. We have listed 6 distinct lines in equation 6.15 so it remains to show that these are the only lines contained in S. Let L be some line in S. Either $L \subset \{y = 0\}$ or L intersects this plane at exactly 1 point.

$$
\begin{aligned}
S \cap \{y = 0\} &= \begin{cases} y = 0 \\ x^2 z = 0 \end{cases} \\
&= 2L_1 \cup L_2.
\end{aligned}
$$

So either $L = L_1$,$L = L_2$ or L intersects one of L_1, L_2.

Let us consider the case where L intersects L_1 then there is a unique plane Π which contains both of these lines. Π must be defined by either $y = 0$ or $x = \lambda y$ for some $\lambda \in \mathbb{C}$. We have already seen the case where Π is defined by $y = 0$.
In the case where Π is defined by $x = \lambda y$ we have

$$
\begin{aligned}
S \cap \Pi &= \begin{cases} x = \lambda y \\ y(\lambda t y - \lambda^2 y z - y^2 + z^2) = 0 \end{cases} \\
&= L_1 \cup \{\lambda t y - \lambda^2 y z - y^2 + z^2 = 0\}
\end{aligned}
$$

where $\lambda t y - \lambda^2 y z - y^2 + z^2 = 0$ is a conic on $x = \lambda y$ which is irreducible for all $\lambda \neq 0$. In the case where $\lambda = 0$ we have

$$
\begin{aligned}
S \cap \Pi &= \begin{cases} x = 0 \\ y(z - y)(z + y) \end{cases} \\
&= L_1 \cup L_3 \cup L_4
\end{aligned}
$$

o L can be one of L_1, L_3 or L_4.

Now consider the case where L intersects L_2. Then there is a unique plane Π that contains both L and L_2. This plane must be either $y = 0$ or $z = \lambda y$. Again we know the case where Π is defined by $y = 0$ so let us consider the case where Π is defined by $z = \lambda y$. In this case

$$\begin{aligned} S \cap \Pi &= \begin{cases} z = \lambda y \\ y(tx - \lambda x^2 + (\lambda^2 - 1)y^2) = 0 \end{cases} \\ &= L_2 \cup \{tx - \lambda x^2 + (\lambda^2 - 1)y^2 = 0\} \end{aligned}$$

where $tx - \lambda x^2 + (\lambda^2 - 1)y^2 = 0$ is a conic on $z = \lambda y$ which is irreducible for $\lambda \neq -1, 1$. In the case where $\lambda = -1$ we have

$$\begin{aligned} S \cap \Pi &= \begin{cases} z = -y \\ xy(t + x) = 0 \end{cases} \\ &= L_1 \cup L_2 \cup L_5. \end{aligned}$$

In the case where $\lambda = 1$ we have

$$\begin{aligned} S \cap \Pi &= \begin{cases} z = y \\ xy(t - x) \end{cases} \\ &= L_1 \cup L_2 \cup L_6. \end{aligned}$$

So L could be L_1, L_2, L_5 or L_6. Hence we have found all the lines contained in S. □

6.18 $\mathbb{A}_5$

Any cubic surface with one A_5 singularity is defined in Appendix B.1 by the equation

$$txy - x^3 - y^3 + yz^2 = 0.$$

Let us call this surface S.
By observation we can see the lines

$$L_1 = \begin{cases} x = 0 \\ y = 0 \end{cases} \qquad L_2 = \begin{cases} x = 0 \\ y - z = 0 \end{cases} \tag{6.16}$$

$$L_3 = \begin{cases} x = 0 \\ y + z = 0 \end{cases}$$

are contained in S.

Theorem 6.18.1. *A cubic surface which has one $\mathbb{A}_5$ singularity and is smooth elsewhere contains exactly 3 lines.*

Proof. To prove this it is sufficient to prove that S contains exactly 3 lines. In order to do this we need to prove that the lines given by equation 6.16 are the only lines in S. Let L be a line in S. Consider the plane $y = 0$.

$$
\begin{aligned}
S \cap \{y = 0\} &= \begin{cases} y = 0 \\ txy - x^3 - y^3 + yz^2 = 0 \end{cases} \\
&= \begin{cases} y = 0 \\ -x^3 = 0 \end{cases} \\
&= 3L_1.
\end{aligned}
$$

So either $L = L_1$ or L intersects L_1. In the case where L intersects L_1 we know that L and L_1 uniquely define some plane Π. So Π is either defined by $y = 0$ or $x = \lambda y$ for some $\lambda \in \mathbb{C}$.
In the case where Π is defined by $y = 0$ we know from above that the only line in $S \cap \Pi$ is L_1.
In the case where Π is defined by $x = \lambda y$ we have that

$$
\begin{aligned}
S \cap \Pi &= \begin{cases} x = \lambda y \\ y(\lambda t y - (\lambda^3 + 1)y^2 + z^2) = 0 \end{cases} \\
&= L_1 \cup \{x = \lambda y, \lambda^3 + 1)y^2 + z^2 = 0\}
\end{aligned}
$$

where $(\lambda^3 + 1)y^2 + z^2 = 0$ is a conic on $x = \lambda y$. This conic is irreducible for all $\lambda \neq 0$. In the case where $\lambda = 0$ we have that

$$
\begin{aligned}
S \cap \Pi &= \begin{cases} x = 0 \\ y(y - z)(y + z) = 0 \end{cases} \\
&= L_1 \cup L_2 \cup L_3.
\end{aligned}
$$

So L can only be one of L_1, L_2 or L_3. Hence there are only 3 lines in S. $\square$

6.19 $\mathbb{D}_4$

Any surface with a $\mathbb{D}_4$ surface is given by either $tx^2 - y^3 - z^3 = 0$ or $tx^2 - y^3 - z^3 - xyz = 0$ up to projective transformation. Sakimaki calls this first surface $\mathbb{D}_4(1)$ and the second $\mathbb{D}_4(2)$ in Appendix B.1.

Let's consider $\mathbb{D}_4(1)$. Let S be the surface defined by $tx^2 - y^3 - z^3 = 0$. Looking at the equation

$$tx^2 - y^3 - z^3 = 0$$

we can see that we can split the $y^3 + z^3$ part into $(y + z)(y^2 - yz + z^2)$ and with a little more work this becomes $y^3 + z^3 = (y + z)(y - \frac{1+\sqrt{3}i}{2}z)(y - \frac{1-\sqrt{3}i}{2}z)$. So S becomes

$$tx^2 - (y + z)(y - \frac{1 + \sqrt{3}i}{2}z)(y - \frac{1 - \sqrt{3}i}{2}z) = 0.$$

So S contains the lines below.

$$
\begin{aligned}
L_1 &= \begin{cases} t = 0 \\ y + z = 0 \end{cases} & L_2 &= \begin{cases} x = 0 \\ y + z = 0 \end{cases} \\
L_3 &= \begin{cases} t = 0 \\ y - \frac{1+\sqrt{3}i}{2} z = 0 \end{cases} & L_4 &= \begin{cases} x = 0 \\ y - \frac{1+\sqrt{3}i}{2} z = 0 \end{cases} \\
L_5 &= \begin{cases} t = 0 \\ y - \frac{1-\sqrt{3}i}{2} z = 0 \end{cases} & L_6 &= \begin{cases} x = 0 \\ y - \frac{1-\sqrt{3}i}{2} z = 0 \end{cases}
\end{aligned} \tag{6.17}
$$

Theorem 6.19.1. *A cubic surface with a $\mathbb{D}_4$ singularity which is smooth everywhere else has exactly 6 lines.*

Proof. To prove that $\mathbb{D}_4$ surface contains exactly 6 lines we can prove that the cases $\mathbb{D}_4(1)$ and $\mathbb{D}_4(2)$ contain exactly 6 lines.

To show that the $\mathbb{D}_4(1)$ surface only contains 6 lines we need to show that the lines given by equation 6.17 are the only 6 lines in S. Let L be a line in S. Consider the plane $y + z = 0$.

$$
\begin{aligned}
S \cap \{y + z = 0\} &= \begin{cases} y + z = 0 \\ tx^2 - y^3 - z^3 = 0 \end{cases} \\
&= \begin{cases} y + z = 0 \\ tx^2 = 0 \end{cases} \\
&= L_1 \cup 2L_2
\end{aligned}
$$

so either $L = L_1$, $L = L_2$ or L intersects at least one of L_1 or L_2.

Let us first assume that L intersects L_1, then there is some plane Π such that $L, L_1 \subset \Pi$. Then Π is either $y+z = 0$ or $t = \lambda(y+z)$. For Π defined by $y+z = 0$ we have from the above that $S \cap \Pi = L_1 \cup 2L_2$. Then consider the case where Π is defined by $t = \lambda(y + z)$ for some $\lambda \in \mathbb{C}$ in which case

$$
\begin{aligned}
S \cap \Pi &= \begin{cases} t = \lambda(y + z) \\ tx^2 - y^3 - z^3 = 0 \end{cases} \\
&= \begin{cases} t = \lambda(y + z) \\ (y + z)(\lambda x^2 - y^2 + yz - z^2) = 0 \end{cases} \\
&= L_1 \cup \{t = \lambda(y + z), \lambda x^2 - y^2 + yz - z^2 = 0\}
\end{aligned}
$$

where $\lambda x^2 - y^2 + yz - z^2 = 0$ is a conic on $t = \lambda(y + z)$ which is irreducible for $\lambda \neq 0$. In the case where $\lambda = 0$ we have that Π is defined by $t = 0$ in which case

$$
S \cap \Pi = \begin{cases} t = 0 \\ tx^2 - y^3 - z^3 = 0 \end{cases} = \begin{cases} t = 0 \\ -y^3 - z^3 = 0 \end{cases} = L_1 \cup L_3 \cup L_5
$$

so L can be one of L_1, L_3 or L_5.

Now consider the case where L intersects L_2. In this case there exists a plane Π such that $L, L_2 \subset \Pi$. So Π is either $y+z=0$ or $x=\lambda(y+z)$ for some $\lambda \in \mathbb{C}$. In the latter case

$$\begin{aligned} S \cap \Pi &= \begin{cases} x = \lambda(y+z) \\ tx^2 - y^3 - z^3 = 0 \end{cases} \\ &= \begin{cases} x = \lambda(y+z) \\ (y+z)(\lambda^2 ty + \lambda^2 tz) - y^2 + yz - z^2) = 0 \end{cases} \\ &= L_2 \cup \{x = \lambda(y+z), \lambda^2 ty + \lambda^2 tz) - y^2 + yz - z^2 = 0\} \end{aligned}$$

where $\lambda^2 ty + \lambda^2 tz) - y^2 + yz - z^2 = 0$ is a conic on $x = \lambda(y+z)$ which is irreducible for $\lambda \neq 0$. When $\lambda = 0$ we have that Π is defined by $x = 0$ and hence

$$\begin{aligned} S \cap \Pi &= \begin{cases} x = 0 \\ -y^3 - z^3 \end{cases} \\ &= L_2 \cup L_4 \cup L_6 \end{aligned}$$

so L can be one of L_2, L_4 or L_6. We have exhausted all possible cases so this is all the lines in S.

We can use a very simialry proof to show the same for $\mathbb{D}_4(2)$. $\square$

6.20 $\mathbb{D}_5$

According to Appendix B.1, a cubic surface with one $\mathbb{D}_5$ singularity is defined up to projective transformation by the equation

$$tx^2 - xz^2 - y^2 z = 0.$$

Let us call this surface S.
By observation, S contains the lines

$$L_1 = \begin{cases} x = 0 \\ y = 0 \end{cases} \qquad L_2 = \begin{cases} x = 0 \\ z = 0 \end{cases} \qquad L_3 = \begin{cases} t = 0 \\ z = 0 \end{cases} \tag{6.18}$$

Theorem 6.20.1. *A cubic surface which has one $\mathbb{D}_5$ singularity and is smooth elsewhere contains exactly 3 lines.*

Proof. To prove this we must show that S contains exactly 3 lines. To do this we need to show that all the lines in S are given by equation 6.18. Let L be any line in S. Consider the plane $x = 0$. We have

$$S \cap \{y = 0\} = \begin{cases} x = 0 \\ y^2 z = 0 \end{cases}$$
$$= 2L_1 \cup L_2.$$

So either $L = L_1$ or $L = L_2$, or L intersects at least one of L_1 or L_2.

Consider the case where L intersects L_1. Then there is some unique plane Π such that $L, L_1 \subset \Pi$. Π must defined by either $x = 0$ or $y = \lambda x$.
In the case where Π is defined by $x = 0$ we know by the above that $S \cap \Pi = L_1 \cap L_2$.
In the case where Π is defined by $y = \lambda x$ we have that

$$S \cap \Pi = \begin{cases} y = \lambda x \\ x(tx - z^2 - \lambda^2 xz) = 0 \end{cases}$$
$$= L_1 \cup \{y = \lambda x, tx - z^2 - \lambda^2 xz = 0\}$$

where $tx - z^2 - \lambda^2 xz = 0$ is a conic on $y = \lambda x$. This conic is irreducible for all λ.

Now consider the case where L intersects L_2. Then there is some unique plane Π such that $L, L_2 \subset \Pi$. Π must defined by either $x = 0$ or $z = \lambda x$. We look at the case where Π is defined by $z = \lambda x$. In this case

$$S \cap \Pi = \begin{cases} z = \lambda x \\ x(tx - \lambda^2 x^2 - \lambda y) = 0 \end{cases} \qquad = L_2 \cup \{z = \lambda x, tx - \lambda^2 x^2 - \lambda y = 0\}$$

where $tx - \lambda^2 x^2 - \lambda y = 0$ is a conic on $z = \lambda x$. This conic is irreducible for all $\lambda \neq 0$. When $\lambda = 0$ we have

$$S \cap \Pi = \begin{cases} z = 0 \\ tx^2 = 0 \end{cases} \qquad = 2L_2 \cup L_3$$

so L can be one of L_2 or L_3. Hence L is one of L_1, L_2 or L_3 and so 6.18 gives a complete list of lines in S.

$\square$

Chapter 7

Special Cases of Singular Cubic Surfaces

There are some types of singularities that cannot be categorised by the *ADE* classification outlined in chapter 3. There are two such cases, both of which will be outlined in this chapter. The first is named $\tilde{\mathbb{E}}_6$ by Bruce and Wall [1]. This case is essentially the cone in $\mathbb{P}^3$. The second case is where we have non isolated singularities e.g. a surface which is singular along a line. The table in Appendix B.1, from which the rest of the sections of Chapter 6 and 7 take their equations, is taken from Sakamaki's paper [14]. This paper only deals with cubic surfaces with isolated singularities so for this section we have constructed our own example which contains a line of singularities.

7.1 $\tilde{\mathbb{E}}_6$

Consider the surface defined in $\mathbb{P}^3$ by the equation

$$f = y^2z - x(x-z)(x-az) = 0$$

for some $a \in \mathbb{C} \setminus \{0, 1\}$.
Taking partial derivatives we can find the singularities;

$$\begin{cases} \frac{\partial f}{\partial t} = 0, \\ \frac{\partial f}{\partial x} = -3x^2 + 2(a+1)xz - az^2 = 0, \\ \frac{\partial f}{\partial y} = 2yz = 0, \\ \frac{\partial f}{\partial z} = y^2 + (a+1)x^2 - 2axz = 0. \end{cases}$$

The only solution to this system is where $x = y = z = 0$ and the system does not depend on t. Hence the only singularity of this surface is at $O = [0 : 0 : 0 : 1]$. We can see that this surface has infinitely many lines contained within it.

This is a cubic surface in $\mathbb{P}^3$ in 3 variables, that is, it does not depend on t. So consider the chart $t \neq 0$. Here we have a copy of $\mathbb{C}^3$. The cubic curve in $\mathbb{C}^3$ sketched out by $y^2z - x(x-z)(x-az) = 0$ (which we can call $\mathcal{C}_3$) gives a cone

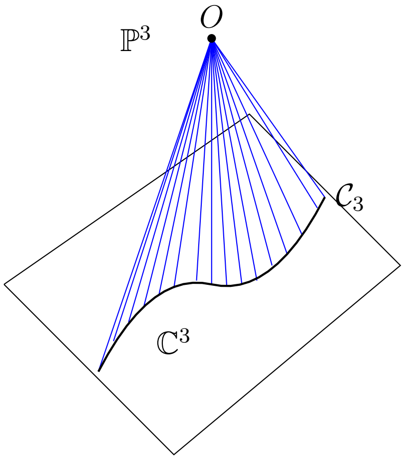


Figure 7.1: Our intuition of the cone in $\mathbb{P}^3$.

over $\mathbb{P}^3$. We can imagine this as in figure 7.1.

Explicitly the lines contained in S are

$$\mu P + \nu O$$

where $P = [a : b : c : 0]$ for $(a, b, c) \in \mathcal{C}_3$. We can see that any such line is contained in S as if $(a, b, c) \in \mathcal{C}_2 \cap \mathcal{C}_3$ then the point $[a : b : c : t] \in S$ satisfies $f_2 = 0$ and $f_3 = 0$ for any value of t. We can see that, as there are infinitely many points on $\mathcal{C}_3$, there are infinitely many such lines.

Theorem 7.1.1. *A cubic surface with an $\tilde{\mathbb{E}}_6$ singularity contains infinitely many lines.*

7.2 Infinitely Many Singularities

Consider the surface S in $\mathbb{P}^3$ defined by the equation

$$f = tx^2 + y^2 z = 0.$$

Taking partial derivatives we can find singularities;

$$\begin{cases} \frac{\partial f}{\partial t} = x^2 = 0, \\ \frac{\partial f}{\partial x} = 2tx = 0, \\ \frac{\partial f}{\partial y} = 2yz = 0, \\ \frac{\partial f}{\partial z} = y^2 = 0. \end{cases}$$

This system is satisfied for all $x = y = 0$ so this surface is singular along the line below.

$$L_1 = \begin{cases} x = 0 \\ y = 0 \end{cases}$$

This is an example of a surface with non-isolated singularities. Other such surfaces exist.

The following are all the lines contained in S

$$L_1 = \begin{cases} x = 0 \\ y = 0 \end{cases} \qquad L_2 = \begin{cases} z = 0 \\ t = 0 \end{cases} \qquad L_3 = \begin{cases} ax + by = 0 \\ a^2 z + b^2 t = 0 \end{cases} \tag{7.1}$$

for any $a, b \in \mathbb{C}$. So clearly there are infinitely many lines contained in S. In fact for any surface with a line of singularities there are infinitely many lines contained in it.

In fact we can see that these are all the lines in S. The proof of this follows a very similar structure to many of those given in Chapter 6.
Claim: All the lines in S are given by the equation 7.1.

Proof. We can check that the lines defined in equation 7.1 are in fact contained in S. It is clear that L_1 and L_2 are subsets of S but let us consider L_3. When $a = 0$ we have that L_3 is the line $y = t = 0$ which is clearly contained in S. When $a \neq 0$ we can see that L_3 is as follows.

$$L_3 = \begin{cases} x = -\frac{b}{a} y \\ z = -(\frac{b}{a})^2 t \end{cases} .$$

We can see that

$$f = tx^2 + y^2 z = t\Big(- \frac{b}{a} y\Big)^2 + y^2 \Big(- \Big(\frac{b}{a}\Big)^2 \Big) t = 0$$

so this line is contained in S for any $a, b \in \mathbb{C}$. Now it remains to show that these are all the lines in S.

Let L be any line in S. Consider the plane $x = 0$. Then

$$S \cap \{x = 0\} = \begin{cases} x = 0 \\ y^2 z = 0 \end{cases}$$

so L can be the line $x = y = 0$ (which is L_1) or L can be $x = z = 0$ (which is L_3 for $b = 0$), or L intersects the line $x = y = 0$ or $x = z = 0$.
Let us consider the case such that L intersects L_1. Then there is some plane Π such that L and L_1 are contained in Π. As L_1 is contained in Π, it is either defined by $x = 0$ or $y = \lambda x$ for some $\lambda \in \mathbb{C}$. We already have seen the case where Π is defined by $x = 0$ so consider Π defined by $y = \lambda x$

$$S \cap \Pi = \begin{cases} y = \lambda x \\ x^2(t + \lambda^2 z) = 0 \end{cases}$$

which is the union of the lines L_1 and L_3 for $b \neq 0$.
Let us consider the case where L intersects the line $x = z = 0$. Again there is a plane Π that contains both L and $x = z = 0$. Here Π is defined by either $x = 0$ or $z = \lambda x$. Consider the case where Π is $z = \lambda x$. Then we have

$$S \cap \Pi = \begin{cases} z = \lambda x \\ x(tx + \lambda y^2) \end{cases} = L_1 \cup \{z = \lambda x, tx + \lambda y^2\}$$

where $tx + \lambda y^2 = 0$ is a conic on $z = \lambda x$ which is only irreducible for $\lambda = 0$. When $\lambda = 0$

$$S \cap \Pi = \begin{cases} z = 0 \\ tx^2 = 0 \end{cases}$$

which is the union of the lines L_2 and L_3 for $b = 0$. Hence these are all the lines in S. $\square$

Chapter 8

Conclusion

In this report the theory of blow up is presented in a more approachable manner and the technique was employed to successfully demonstrate how to explicitly resolve singularities. This highlighted the connection between these resolution patterns and Dynkin diagrams which we have then used to categorise singularities, following the work of Du Val [4].

We have proved how many lines are contained in any cubic surface with only isolated singularities. Bruce and Wall's table summarises these results succinctly and can be reconstructed by compiling the theorems at the end of each section of chapter 6 and 7.

We have also given one example of the case where there are non-isolated singularities within a cubic surface, however our report is heavily focused on isolated singularities. We would then suggest that a natural pathway for future research could be the geometry of cubic surfaces with non-isolated singularities and how to find lines within these.

Appendix A

Bruce-Wall Table

Type	Nonsing.	A_1	$2A_1$	A_2	$3A_1$	A_1A_2	A_3	$4A_1$	A_22A_1	A_3A_1
Codi- mension	0	1	2	2	3	3	3	4	4	4
Class	12	10	8	9	6	7	8	4	5	6
No. of lines	27	21	16	15	12	11	10	9	8	7

$2A_2$	A_4	D_4	A_32A_1	$2A_2A_1$	A_4A_1	A_5	D_5	$3A_2$	A_5A_1	E_6
4	4	4	5	5	5	5	5	6	6	6
6	7	6	4	4	5	6	5	3	4	4
7	6	6	5	5	4	3	3	3	2	1

Figure A.1: Table from Bruce-Wall paper summarising the number of lines contained within each type of singularity [1].

Appendix B

Sakamaki Table

Singularities	$f_2(x_0, x_1, x_2)$	$f_3(x_0, x_1, x_2)$
A_1	$x_0x_2 - x_1^2$	$(x_0 - ax_1)(-x_0 + (b+1)x_1 - bx_2)(x_1 - cx_2)$
$2A_1$	$x_0x_2 - x_1^2$	$(x_0 - 2x_1 + x_2)(x_0 - ax_1)(x_1 - bx_2)$
A_1A_2	$x_0x_2 - x_1^2$	$(x_0 - x_1)(-x_1 + x_2)(x_0 - (a+1)x_1 + ax_2)$
$3A_1$	$x_0x_2 - x_1^2$	$x_0x_2(x_0 - (a+1)x_1 + ax_2)$
A_1A_3	$x_0x_2 - x_1^2$	$(x_0 - x_1)(-x_1 + x_2)(x_0 - 2x_1 + x_2)$
$2A_1A_2$	$x_0x_2 - x_1^2$	$x_1^2(x_0 - x_1)$
$4A_1$	$x_0x_2 - x_1^2$	$(x_0 - x_1)(x_1 - x_2)x_1$
A_1A_4	$x_0x_2 - x_1^2$	$x_0^2x_1$
$2A_1A_3$	$x_0x_2 - x_1^2$	$x_0x_1^2$
A_12A_2	$x_0x_2 - x_1^2$	x_1^3
A_1A_5	$x_0x_2 - x_1^2$	x_0^3
A_2	x_0x_1	$x_2(x_0 + x_1 + x_2)(dx_0 + ex_1 - dex_2)$
$2A_2$	x_0x_1	$x_2(x_1 + x_2)(-x_1 + dx_2)$
$3A_2$	x_0x_1	x_2^3
A_3	x_0x_1	$x_2(x_0 + x_1 + x_2)(x_0 - ux_1)$
A_4	x_0x_1	$x_0^2x_2 + x_1^3 - x_1x_2^2$
A_5	x_0x_1	$x_0^3 + x_1^3 - x_1x_2^2$
$D_4(1)$	x_0^2	$x_1^3 + x_2^3$
$D_4(2)$	x_0^2	$x_1^3 + x_2^3 + x_0x_1x_2$
D_5	x_0^2	$x_0x_2^2 + x_1^2x_2$
E_6	x_0^2	$x_0x_2^2 + x_1^3$
$\tilde{E}_6$	0	$x_1^2x_2 - x_0(x_0 - x_2)(x_0 - ax_2)$

Figure B.1: Table from Sakamaki paper showing how to construct the normal forms of each singular cubic surface [14].

Figure (B.1) was used to construct singular cubic surfaces with specified singularities. In particular, each surface is constructed such that they have one singularity at $[0 : 0 : 0 : 1] \in \mathbb{P}^3$. Using the appropriate value of f_2 and f_3, each cubic surface, S, can be constructed via $S := x_3f_2(x_0, x_1, x_2) - f_3(x_0, x_1, x_2)$. It should also be noted that in figure (B.1), $a, b, c \in \mathbb{C} \setminus \{0, 1\}$ and are distinct from each other. Also we have that $d, e \in \mathbb{C} \setminus \{0, -1\}$ and $u \in \mathbb{C} \setminus \{0\}$.

Appendix C

Maple Code

C.1 $\mathbb{A}_1$ Code

```
> resolve1 := proc(f2,f3)
> local  partials1, stuff1, Ans1, f:
> f := expand(t*f_2 - f_3);
> partials1 := [diff(f,x), diff(f,y), diff(f,z), diff(f,t)];
> stuff1 := [op(partials1), f];
> Ans1 := solve(stuff1, allsolutions = true);
> return([partials1, Ans1, f]);
> end proc:

> New := proc(S)
> local S_New, fact:
> S_New := subs([x=x*z, y=y*z, z=z, t=1], S);
> fact := factor(S_New);
> return([S_New, fact]);
> end proc:

> f_2 := x*z - y^2;
> f_3 := expand((x + y)*(-x + 3*y - 2*z)*(y - 3*z));
> SingCheck1 := resolve1(f2,f3):
> partials := SingCheck1[1];
> singular_points := SingCheck1[2];
> original_eqn := SingCheck1[3];
> #Has a singular point at [0:0:0:1], so need to resolve.
> Blowup_1 := New(factor(original_eqn));
> new_eqn := Blowup_1[1];
> factorised_new_eqn := Blowup_1[2];

> resolve2 := proc(S)
> local g, partials1, stuff1, Ans1:
> g := S/z^2;
> partials1 := [diff(g,x), diff(g,y), diff(g,z)];
> stuff1 := [op(partials1), g];
> Ans1 := solve(stuff1, allsolutions = true);
> return([partials1, Ans1]);
> end proc:
```

```
> SingCheck2 := resolve2(factorised_new_eqn);
> #No singular point, so smooth.
```

C.2 $\mathbb{A}_5$ Code

```
> resolve1 := proc(f2,f3)
> local  partials1, stuff1, Ans1:
> global f:
> f := expand(t*f_2 - f_3);
> partials1 := [diff(f,x), diff(f,y), diff(f,z), diff(f,t)];
> stuff1 := [op(partials1), f];
> Ans1 := solve(stuff1, allsolutions = true);
> return([partials1, Ans1,f]);
> end proc:

> Newx := proc(S)
> global S_New, fact:
> S_New := subs([x=x, y=y*x, z=z*x, t=1], S);
> fact := factor(S_New);
> return([S_New, fact]);
> end proc:

> Newz := proc(S)
> global S_New, fact:
> S_New := subs([x=x*z, y=y*z, z=z, t=1], S);
> fact := factor(S_New);
> return([S_New, fact]);
> end proc:

> f_2 := x*y;
> f_3 := x^3 + y^3 - y*z^2;
> SingCheck1 := resolve1(f2,f3):
> partials := SingCheck1[1];
> singular_points := SingCheck1[2];
> original_eqn := SingCheck1[3];
> #Has a singular point at [0:0:0:1], so need to resolve.
> Blowup_1 := Newz(original_eqn):
> new_eqn := Blowup_1[1];
> factorised_new_eqn := Blowup_1[2];

> resolvez := proc(S)
> local partials1, stuff1, Ans1, g:
> g := S/z^2;
> partials1 := [diff(g,x), diff(g,y), diff(g,z)];
> stuff1 := [op(partials1), g];
> Ans1 := solve(stuff1, allsolutions = true);
> return([g, partials1, Ans1]);
> end proc:
```

```
> resolvex := proc(S)
> local partials1, stuff1, Ans1, g:
> g := S/x^2;
> partials1 := [diff(g,x), diff(g,y), diff(g,z)];
> stuff1 := [op(partials1), g];
> Ans1 := solve(stuff1, allsolutions = true);
> return([g, partials1, Ans1]);
> end proc:

> SingCheck2 := resolvez(Blowup_1[2]);
> #Has a singular point at (0,0,0), so need to resolve again.

> Blowup_2:= Newz(SingCheck2[1]);
> SingCheck3:= resolvez(Blowup_2[2]);
> #Has a singular point at (-1,0,0).  Change coords then resolve
again.

> Coords1 := expand(subs([x=X-1, y=Y, z=Z], SingCheck3[1]));
> Coords2 := expand(subs([X=x,Y=y, Z=z], Coords1));
> Blowup_3 := Newx(Coords2);
> SingCheck4 := resolvex(Blowup_3[2]);
> #No singular point, so smooth.
```

C.3 $\mathbb{D}_4$ Code

```
> resolve1 := proc(f2,f3)
> local  partials1, stuff1, Ans1, f:
> f := expand(t*f_2 - f_3);
> partials1 := [diff(f,x), diff(f,y), diff(f,z), diff(f,t)];
> stuff1 := [op(partials1), f];
> Ans1 := solve(stuff1, allsolutions = true);
> return([partials1, Ans1, f]);
> end proc:

> Newz := proc(S)
> local S_New, fact:
> S_New := subs([x=x*z, y=y*z, z=z, t=1], S);
> fact := factor(S_New);
> return([S_New, fact]);
> end proc:

> f_2 := x^2;
> f_3 := y^3 + z^3;
> SingCheck1 := resolve1(f2,f3):
> partials := SingCheck1[1];
> singular_points := SingCheck1[2];
> original_eqn := SingCheck1[3];
> #Has a singular point at [0:0:0:1], so need to resolve.
> Blowup_1 := Newz(original_eqn):
> new_eqn := Blowup_1[1];
> factorised_new_eqn := Blowup_1[2];
```

```
> resolvez := proc(S)
> local partials1, stuff1, Ans1, g:
> g := S/z^2;
> partials1 := [diff(g,x), diff(g,y), diff(g,z)];
> stuff1 := [op(partials1), g];
> Ans1 := solve(stuff1, allsolutions = true);
> return([g, partials1, Ans1]);
> end proc:

> SingCheck2 := resolvez(Blowup_1[2]);
> #Has 3 singular ponits, need to resolve 3 more times then at minimum.
> # Consider (0,-1,0) first.  Need to change coordinates to get
to (0,0,0).
> map(allvalues,SingCheck2[4]);

> #Singularity at (0,-1,0).
> Coords1 := expand(subs([x=X, y=Y-1, z=Z], SingCheck2[1]));
> Coords2 := factor(expand(subs([X=x,Y=y, Z=z], Coords1)));
> Blowup_2 := Newz(Coords2);
> SingCheck3 := resolvez(Blowup_2[2]);
> #No singular points, so smooth.

> #Singularity at (0, alpha, 0).
> Coords3 := expand(subs([x=X, y=Y + 1/2 - (1/2)*(I)*(sqrt(3)),
z=Z], SingCheck2[1]));
> Coords4 := factor(expand(subs([X=x,Y=y, Z=z], Coords3)));
> Blowup_3 := Newz(Coords4);
> SingCheck4 := resolvez(Blowup_3[2]);
> #No singular points, so smooth.

> #Singularity at (0, beta, 0).
> Coords5 := expand(subs([x=X, y=Y + 1/2 + (1/2)*(I)*(sqrt(3)),
z=Z], SingCheck2[1]));
> Coords6 := factor(expand(subs([X=x,Y=y, Z=z], Coords5)));
> Blowup_4 := Newz(Coords6);
> SingCheck5 := resolvez(Blowup_4[2]);
> #No singular points, so smooth.
```

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