Graßmannian manifolds

We consider natural numbers $0 \leq k \leq n$. The *Grassmannian* of k-planes in $\mathbb{R}^n$ is

$$Gr(k,n) \;=\; \{L \subset \mathbb{R}^n \colon L \text{ is a vector subspace of dimension } k\} \;.$$

Another common notation for $Gr(k,n)$ is $Gr_k(\mathbb{R}^n)$. As before we let $V_{k,n}$ be the Stiefel manifold of k-frames in $\mathbb{R}^n$, with the subspace topology of $(\mathbb{R}^n)^k$. Every k-plane in $\mathbb{R}^n$ has an orthonormal basis, so the map

$$V_{k,n} \;\longrightarrow\; Gr(k,n) \;, \quad (v_1,\dots,v_k) \;\longmapsto\; \mathrm{span}(v_1,\dots,v_k)$$

that sends a frame to its $\mathbb{R}$-linear space is surjective. We endow the Grassmannian $G(k,n)$ with the quotient topology of $V_{k,n}$ through this surjective map.

Example 1. Since $\mathbb{R}^n$ has only one 0-dimensional vector subspace and only one n–dimensional vector subspace, the Grassmannian $G(0,n)$ and $G(n,n)$ consists of a single point each. The Grassmannian $G(1,n)$ is the projective space $\mathbb{R}P^{n-1}$.

Theorem 2. *For every $0 \leq k \leq n$, the Grassmannian $G(k,n)$ is a compact manifold of dimension $k\cdot(n-k)$.*

Proof. To show that $G(k,n)$ is a Hausdorff space, we exhibit an injective continuous map to another Hausdorff space. Given $L \in G(k,n)$, we let $p_L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ denote the orthogonal projection onto the subspace L. Since L can be recovered from p_L as the image, the map

$$p \;:\; Gr(k,n) \;\longrightarrow\; \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n,\mathbb{R}^n) \;, \quad L \;\longmapsto\; p_L$$

is injective. The image of this map can be characterized as the set of those linear endomorphisms $q : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ that are self-adjoint (i.e., $q^* = q$), idempotent (i.e., $q \circ q = q$) and whose trace is k.

If $(v_1,\dots,v_k)$ is any orthonormal basis of the k-plane L, then the orthogonal projection is given by the formula

$$p_L(x) \;=\; \langle v_1, x\rangle \cdot v_1 \;+\; \dots \;+\; \langle v_k, x\rangle \cdot v_k \;.$$

So the composite

$$V_{k,n} \xrightarrow{\text{span}} Gr(k,n) \xrightarrow{p} \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n,\mathbb{R}^n) \;, \quad (v_1,\dots,v_k) \;\longmapsto\; \sum_{i=1}^{k} \langle v_k, -\rangle \cdot v_k$$

is continuous, hence so is the map p. Since p is continuous and injective, and since $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n,\mathbb{R}^n)$ is a Hausdorff space, the Grassmannian $Gr(k,n)$ is a Hausdorff space, too. Since the Stiefel manifold $V_{k,n}$ is compact, its quotient space $Gr(k,n)$ is quasi-compact; so altogether, we have shown that the Grassmannian is a compact space.

To show that $Gr(k,n)$ is a manifold we first parameterize a neighborhood of the particular k-plane $\mathbb{R}^k \oplus 0^{n-k}$ by the linear space $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^k,\mathbb{R}^{n-k})$. This will also show that $Gr(k,n)$ has dimension $k\cdot(n-k)$, the dimension of the vector space $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^k,\mathbb{R}^{n-k})$. We define

$$U \;=\; \{L \in Gr(k,n) \colon L \cap (0^k \times \mathbb{R}^{n-k}) = \{0\}\} \;.$$

This set contains $\mathbb{R}^k \oplus 0^{n-k}$, and we will now show that it is open inside the Grassmannian.

We write

$$q \;:\; \mathbb{R}^n \;\longrightarrow\; \mathbb{R}^k \;, \quad q(x_1,\dots,x_n) \;=\; (x_1,\dots,x_k)$$

for the orthogonal projection onto the first k coordinates. The condition $L \cap (0^k \times \mathbb{R}^{n-k}) = \{0\}$ is equivalent to demanding that the restricted projection $q|_L : L \longrightarrow \mathbb{R}^k$ has a trivial kernel; since L and $\mathbb{R}^k$ have the same dimension, this is equivalent to requiring $q|_L$ to be an isomorphism. If $(v_1,\dots,v_k) \in V_{k,n}$ is any k-frame in $\mathbb{R}^n$ that spans L, this in turn is equivalent to the requirement that the linear map

$$\mathbb{R}^k \;\longrightarrow \mathbb{R}^k \;, \quad (x_1,\dots,x_k) \;\longmapsto\; q(x_1 v_1 + \dots + x_k v_k)$$

is an isomorphism. This last map is described by the $(k \times k)$-matrix with columns $q(v_1), \dots, q(v_k)$; so we conclude that a k-plane L belongs to U if and only if for some (and hence any) k-frame $(v_1, \dots, v_k)$ that spans L the $(k \times k)$-matrix $(q(v_1), \dots, q(v_k))$ is invertible. In other words:

$$\text{span}^{-1}(U) \;=\; \{(v_1, \dots, v_k) \in V_{k,n} \colon (q(v_1), \dots, q(v_k)) \in GL_k(\mathbb{R})\} \; . \tag{3}$$

Since the assignment

$$V_{k,n} \;\longrightarrow\; M(k \times k; \mathbb{R}) \; , \quad (v_1, \dots, v_k) \longmapsto (q(v_1), \dots, q(v_k))$$

to the space of $k \times k$ square matrices is continuous, and since $GL_k(\mathbb{R})$ is open inside $M(k \times k, \mathbb{R})$, this shows that the set $\text{span}^{-1}(U)$ is open in $V_{k,n}$. So the set U is open in $Gr(k, n)$.

Now we exhibit a homeomorphism from the linear space $\text{Hom}_R(\mathbb{R}^k, \mathbb{R}^{n-k})$ to the open set U. To this end we 'split' $\mathbb{R}^n$ as $\mathbb{R}^k \oplus \mathbb{R}^{n-k}$ and use the map

$$\begin{aligned} \Gamma \;:\; \text{Hom}_{\mathbb{R}}(\mathbb{R}^k, \mathbb{R}^{n-k}) \;&\longrightarrow\; Gr(k, n) \\ \Gamma(f) \;=\; \text{graph of } f \;&=\; \{(x, f(x)) \;:\; x \in \mathbb{R}^k\} \end{aligned}$$

that sends a linear map $f : \mathbb{R}^k \longrightarrow \mathbb{R}^{n-k}$ to its graph. This map factors as the composite

$$\text{Hom}_{\mathbb{R}}(\mathbb{R}^k, \mathbb{R}^{n-k}) \;\xrightarrow{\text{GS}}\; V_{k,n} \;\xrightarrow{\text{span}}\; G(k, n) \; ,$$

where the first map sends f to the Gram-Schmidt orthonormalization of the linearly independent k-tuple $(e_1, f(e_1)), \dots, (e_k, f(e_k))$; since the Gram-Schmidt process is continuous, so is the graph map. The image of Γ is clearly contained in the open set U. Also, for every $x \in \mathbb{R}^k$ we have

$$\Gamma(f) \cap (\{x\} \times \mathbb{R}^{n-k}) \;=\; \{(x, f(x))\} \; ,$$

so the linear map f can be recovered from its graph, and hence the map Γ is injective.

To show that Γ is a homeomorphism onto U, we recall from (3) that a k-frame $(v_1, \dots, v_k)$ spans a k-plane in U if and only if the square matrix $(q(v_1), \dots, q(v_k))$ is invertible. We write

$$\bar{q} \;:\; \mathbb{R}^n \;\longrightarrow\; \mathbb{R}^{n-k} \; , \quad q(x_1, \dots, x_n) \;=\; (x_{k+1}, \dots, x_n)$$

for the orthogonal projection onto the last $n-k$ coordinates. Because matrix multiplication and the inversion map of the general linear group $GL_k(\mathbb{R})$ are continuous, the map

$$\Psi \;:\; \text{span}^{-1}(U) \;\longrightarrow\; \text{Hom}_{\mathbb{R}}(\mathbb{R}^k, \mathbb{R}^{n-k}) \; , \quad \Psi(v_1, \dots, v_k) = (\bar{q}(v_1), \dots, \bar{q}(v_k)) \cdot (q(v_1), \dots, q(v_k))^{-1}$$

is continuous. We will show that Ψ descends to a continuous inverse to the graph map Γ. To this end we consider any vector $x \in \mathbb{R}^k$ and we set $y = (q(v_1), \dots, q(v_k))^{-1}(x) \in \mathbb{R}^k$. Then

$$\begin{aligned} (x, \Psi(v_1, \dots, v_k)(x)) \;&=\; (((q(v_1), \dots, q(v_k))(y), (\bar{q}(v_1), \dots, \bar{q}(v_k))(y)) \\ &=\; (q(y_1 v_1 + \cdots + y_k v_k), \bar{q}(y_1 v_1 + \cdots + y_k v_k)) \\ &=\; y_1 v_1 + \cdots + y_k v_k \; . \end{aligned}$$

So the graph of $\Psi(v_1, \dots, v_n)$ is given by

$$\Gamma(\Psi(v_1, \dots, v_k)) \;=\; \text{span}(v_1, \dots, v_k) \; .$$

Since Γ is injective, this shows in particular that the linear map $\Psi(v_1, \dots, v_k)$ only depends on the span of $(v_1, \dots, v_k)$. The restriction of the quotient map $\text{span} : V_{k,n} \longrightarrow Gr(k, n)$ is another quotient map

$$\text{span} \;:\; \text{span}^{-1}(U) \;\longrightarrow\; U \; .$$

So the map Ψ descends to a well-defined and continuous map

$$\bar{\Psi} \;:\; U \;\longrightarrow\; \text{Hom}_{\mathbb{R}}(\mathbb{R}^k, \mathbb{R}^{n-k}) \; , \quad \bar{\Psi}(\text{span}(v_1, \dots, v_k)) \;=\; (\bar{q}(v_1), \dots, \bar{q}(v_k)) \cdot (q(v_1), \dots, q(v_k))^{-1}$$

that is right inverse to the graph map Γ. Since the graph map is injective, the map $\bar{\Psi}$ is also left inverse to the graph map. Hence we have shown that the graph map Γ and the map $\bar{\Psi}$ are mutually inverse homeomorphisms.

Finally, we exploit the homogeneity of $Gr(k,n)$ to exhibit a euclidean neighborhood of a general k-plane L. We choose an orthonormal basis $(v_1,\dots,v_k)$ of L and extend it to an orthonormal basis $(v_1,\dots,v_k,v_{k+1},\dots,v_n)$ of $\mathbb{R}^n$. We let $A\in O(n)$ be the orthogonal matrix with columns $(v_1,\dots,v_n)$. Then the map

$$A\cdot \ : \ Gr(k,n) \ \longrightarrow \ Gr(k,n)$$

that sends a k-plane to its image under A is continuous. Indeed, the following square commutes:

$$\begin{tikzcd}
V_{k,n} \arrow[r, "{(v_1,\dots,v_k)\mapsto(Av_1,\dots,Av_k)}"] \arrow[d, "\mathrm{span}"'] & V_{k,n} \arrow[d, "\mathrm{span}"] \\
Gr(k,n) \arrow[r, "A\cdot"'] & Gr(k,n)
\end{tikzcd}$$

Since the upper horizontal map is continuous, so is the lower one, by the universal property of the quotient topology. We conclude that the map $A\cdot$ is a self-homeomorphism of the Grassmannian that takes the special k-plane $\mathbb{R}^k\oplus 0^{n-k}$ to the k-plane L. The set $A\cdot U$ is thus an open neighborhood of L that is homeomorphic to $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^k,\mathbb{R}^{n-k})$. This completes the proof that $Gr(k,n)$ is a compact manifold of dimension $k\cdot(n-k)$. $\square$

We write

$$\mathcal{P}_{k,n} \ = \ \{q\in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n,\mathbb{R}^n)\colon q^*=q^2=q,\ \mathrm{trace}(q)=k\}$$

the set of linear endomorphisms of $\mathbb{R}^n$ that are self-adjoint, idempotent and have trace k; we endow $\mathcal{P}_{k,n}$ with the subspace topology of the linear topology of $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n,\mathbb{R}^n)$. In the proof of the previous theorem we have already used the bijective map

$$p \ : \ Gr(k,n) \ \longrightarrow \ \mathcal{P}_{k,n}$$

that sends a k-plane to its orthogonal projection; the inverse map sends an element of $\mathcal{P}_{k,n}$ to its image. We showed in the proof that this map is continuous; since $Gr(k,n)$ is compact and $\mathcal{P}_{k,n}$ is Hausdorff, we can conclude:

Corollary 4. *The map $p: Gr(k,n)\longrightarrow\mathcal{P}_{k,n}$ that sends a k-plane to its orthogonal projection is a homeomorphism.*

We shall now show that the Grassmannians have a certain symmetry/duality property, in the sense that the Grassmannians $Gr(k,n)$ and $Gr(n-k,n)$ for complementary dimensions are homeomorphic.

Proposition 5. *For all $0\le k\le n$, the map*

$$(-)^\perp \ : \ Gr(k,n) \ \longrightarrow \ Gr(n-k,n)$$

that sends a k-plane L to its orthogonal complement $L^\perp$ is a homeomorphism.

Proof. Given Corollary 4, this is easy: the orthogonal projections of L and $L^\perp$ are complementary in the sense of the relation

$$p_L + p_{L^\perp} \ = \ \mathrm{Id}_{\mathbb{R}^n} \ .$$

So the following square of continuous maps commutes:

$$\begin{tikzcd}
Gr(k,n) \arrow[r, "\mathcal{P}_{k,n}", "\cong"'] \arrow[d, "L\mapsto L^\perp"'] & \mathcal{P}_{k,n} \arrow[d, "q\mapsto \mathrm{Id}_{\mathbb{R}^n}-q"] \\
Gr(n-k,n) \arrow[r, "\cong", "\mathcal{P}_{n-k,n}"'] & \mathcal{P}_{n-k,n}
\end{tikzcd}$$

The two horizontal maps are homeomorphisms by Corollary 4. The right vertical map is clearly continuous for the topology induced by the linear topology of $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^n)$, and so is its inverse (which is given by the same formula). So the right vertical map is a homeomorphism, and hence the left vertical map is as well. $\square$

We briefly mention the complex and quaternion Grassmannians, defined as

$$\begin{aligned} Gr^{\mathbb{C}}(k,n) &= \{L \subset \mathbb{C}^n \colon L \text{ is a } \mathbb{C}\text{-vector subspace with } \dim_{\mathbb{C}}(L) = k\} \quad \text{and} \\ Gr^{\mathbb{H}}(k,n) &= \{L \subset \mathbb{H}^n \colon L \text{ is an } \mathbb{H}\text{-vector subspace with } \dim_{\mathbb{H}}(L) = k\} \ . \end{aligned}$$

These spaces carry the quotient topology of the respective complex and quaternion Stiefel manifolds. In particular, for $n = 1$ we recover the projective spaces

$$Gr^{\mathbb{C}}(1,n) \ = \ \mathbb{C}P^{n-1} \qquad \text{and} \qquad Gr^{\mathbb{H}}(1,n) \ = \ \mathbb{H}P^{n-1} \ .$$

Very much like in the real case one shows that $Gr^{\mathbb{C}}(k,n)$ and $Gr^{\mathbb{H}}(k,n)$ are compact manifolds of dimensions $2k \cdot (n-k)$ and $4k \cdot (n-k)$, respectively.