Definition: An *m-manifold* is a Hausdorff space M such that every point of M has an open neighborhood homeomorphic to $\mathbb{R}^m$.
The number $m \geq 0$ is the *dimension* of M.

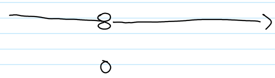
Rk: The empty space is an m-manifold for all $m \geq 0$.
For non-empty manifolds, the dimension is intrinsic and can be calculated from the local homology groups:
Let M be a manifold, $x \in M$. Let $U \subseteq M$ be an open neighborhood that admits a homeomorphism $\varphi: \mathbb{R}^m \to U$, $\varphi(0) = x$.
Then:

$$\begin{array}{ccccc} H_i(M, M \setminus \{x\}; \mathbb{Z}) & \xleftarrow[\text{excision}]{\cong} & H_i(U, U \setminus \{x\}; \mathbb{Z}) & \xleftarrow[\varphi_*]{\cong} & H_i(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}; \mathbb{Z}) \\ & & & & \downarrow \partial \ \cong \quad \mathbb{R}^m \text{ is contractible} \\ \left.\begin{array}{ll} \mathbb{Z} & \text{if } i = m \\ 0 & \text{if } i \neq m \end{array}\right\} & = & \tilde{H}_{i-1}(S^{m-1}; \mathbb{Z}) & \xrightarrow[\cong]{\text{incl.}} & \tilde{H}_{i-1}(\mathbb{R}^m \setminus \{0\}; \mathbb{Z}) \\ & & & \text{htp invariance} & \end{array}$$

$\Rightarrow$ The dimension of M is the dimension in which the local homology is concentrated.

Rk: The Hausdorff condition is included to avoid certain pathological examples, such as the "line with double origin":

$$X = \mathbb{R} \times \{0,1\} / \sim \quad \text{where} \quad (x,0) \sim (x,1) \quad \text{for all } x \neq 0.$$



Check: X is not Hausdorff.
X is locally homeomorphic to $\mathbb{R}$.

Example: Open subsets of $\mathbb{R}^m$ are m-manifolds.

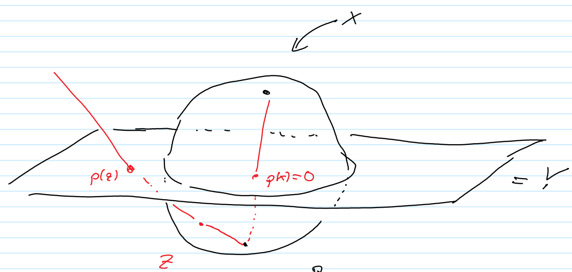
Example: Let M be a Hausdorff space such that every point has an open neighborhood that is an m-manifold.
Then M is an m-manifold. In particular, the disjoint union (with disjoint union topology) of two m-manifolds is an m-manifold.

Example: Let M be an m-manifold and N an n-manifold. Then $M \times N$ (with product topology) is an $(m+n)$-manifold.

Example: The n-sphere $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \dots + x_{n+1}^2 = 1\}$ is an n-manifold.
For $x = (x_1, \dots, x_{n+1}) \in S^n$ let $Y = \{y \in \mathbb{R}^{n+1} : \langle y, x\rangle = 0\}$ be the orthogonal complement.
The stereographic projection is a homeomorphism

$$p: \underbrace{S^n \setminus \{-x\}}_{\text{an open neighborhood of } x} \xrightarrow{\cong} Y \cong \mathbb{R}^n$$

$$p(z) = \frac{z - \langle z, x\rangle \cdot x}{1 + \langle z, x\rangle}$$



Example: The real projective space $\mathbb{R}P^n = S^n / \text{antipodal action}$ is an n-manifold.
Consider a point $\{x, -x\} \in \mathbb{R}P^n$, choose one of the points x.

$$U = \{z \in S^n : \langle z, x\rangle > 0\} = \text{"hemisphere around } x\text{"}$$

Then the composite

$$\mathbb{R}^n \cong U \hookrightarrow S^n \xrightarrow{\text{quotient}} \mathbb{R}P^n$$

is a homeomorphism onto an open neighborhood of $\{x, -x\}$.

Example: The complex projective space $\mathbb{C}P^n = \{L \leq \mathbb{C}^{n+1} : L \text{ is 1-dimensional } \mathbb{C}\text{-vector subspace}\}$ is a $2n$-manifold. Consider first $L_0 = [0 : 0 : \dots : 0 : 1] \in \mathbb{C}P^n$.
Then

$$\begin{array}{ccc} \mathbb{R}^{2n} = \mathbb{C}^n & \longrightarrow & \mathbb{C}P^n \\ (z_1, \dots, z_n) & \longmapsto & [z_1 : \dots : z_n : 1] \end{array}$$

is a homeomorphism onto an open neighborhood of L_0.
If $L \in \mathbb{C}P^n$ is any complex line in $\mathbb{C}^{n+1}$, let $v \in L$ be a non-zero vector, and choose an

$$\begin{aligned} \mathbb{R}^{2n} = \mathbb{C}^n &\longrightarrow \mathbb{C}P^n \\ (z_1, \dots, z_n) &\longmapsto [z_1 : \dots : z_n : 1] \end{aligned}$$ is a homeomorphism onto an open neighborhood of L_0.

If $L \in \mathbb{C}P^n$ is any complex line in $\mathbb{C}^{n+1}$, let $v \in L$ be a non-zero vector, and choose an invertible matrix $A \in GL_{n+1}(\mathbb{C})$ such that $A \cdot (0, \dots, 0, 1) = v$.

Then $A : \mathbb{C}P^n \longrightarrow \mathbb{C}P^n$, $A \cdot L = A(L)$ is self-homeomorphism of $\mathbb{C}P^n$ that takes $L_0 = \mathbb{C} \cdot (0, \dots, 0, 1)$ to $\mathbb{C} \cdot v = L$. Since $\mathbb{C}P^n$ is locally homeomorphic to $\mathbb{R}^{2n}$ around L_0, it is also locally homeomorphic to $\mathbb{R}^{2n}$ around L.

<u>Example:</u> The quaternionic projective space $\mathbb{H}P^n = \{ L \subseteq \mathbb{H}^{n+1} : L$ is a 1-dimensional left $\mathbb{H}$-sub vector space $\}$

Similarly as in the complex case, $\mathbb{H}P^n$ is a $4n$-manifold.