Topology II - Cohomology

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Reminder about homology:

$$\text{Top} \xrightarrow[\text{singular complex}]{\rho} (\text{simpicial sets}) \xrightarrow[\text{linearization}^{C(-,A)}]{} (\text{chain complex}) \xrightarrow[n-\text{th homology group}]{} \text{Ab}\,.$$

- For a space X, the singular complex $\rho(X)$ is the simplicial set with

$$\rho(X)_n = \text{map}^{\text{cpt}}(\nabla^n, X)$$

$$\nabla^n = \text{topological } n-\text{simplex} = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \quad : \quad x_n \geq 0, x_0 + ... + x_n = 1\}$$

- For a simplicial set Y and an abelian group, the linearization is the chain complex $C(Y; A)$ with

$$C_(Y; a) = A[Y_n] \quad A - \text{linarization of} Y_n \qquad (C_n(Y; A) = 0 \quad \text{for } n < 0)$$

- For a chain complex C and $n \in \mathbb{Z}$, the n-th homology group $H_n(C)$ is

$$\frac{\ker(d_n : C_n \to C_{n-1})}{\text{Im}(d_{n+1}) : C_{n+1} \to C_n}$$

0.1 Variation : Cohomology

Definition 0.1. A cochain complex C consists of abelian groups C^n for $n \in \mathbb{Z}$ and homomorphisms $d^n : C^n \to C^{n+1}$ such that

$$d^{n+1} \circ d^n = 0 : C^n \to C^{n+2}.$$

A morphism $f : C \to D$ of cochain complexes (cochain map) consists of homomorphisms $f^n : C^n \to D^n$ such that $d^n_D \circ f^n = f^{n+1} \circ d^n_C$

$$\begin{array}{ccc} C^n & \xrightarrow{f^n} & D^n \\ \downarrow{\scriptstyle d^n} & & \downarrow \\ C^{n+1} & \xrightarrow{f^{n+1}} & D^{n+1} \end{array}$$

The n-th cohomology group of a cochain complex C is

$$H^n C = \frac{\ker(d^n : C^n \to C^{n+1})}{\text{Im}(d^{n-1} : C^{n-1} \to C^n)}$$

A cochain homotopy between two morphisms $f, g : C \to D$ of cocain complexes consists of homomorphisms

$$s^n : C^n \to D^{n-1} \quad \text{such that} \quad d^{n-1} \circ s^n + s^{n+1} \circ d^n = ff^n - g^n$$

for all $n \in \mathbb{Z}$.

The main tools and properties carry over from chain complexes to cochain complexes, with essentially the same proofs, such as:

- a morphism $f : C \to D$ of cochain complexes induces a homomorphism $H^n f : H^n C \to H^n D$ for cohomology groups by

$$(H^n f)[x] = [f^n(x)], \quad x \in \ker(d^n : \mathbb{C}^n \to C^{n+1}).$$

- cochain homotopic morphisms $f, g : C \to D$ between cochain complexes induces the same map in cohomology, ie. $H^n f = H^n g$.
- every short exact sequence of cochain complexes

$$0 \xrightarrow{f} B \xrightarrow{g} C \to 0$$

gives rise to a long exact sequence of cohomology groups:

$$\dots \to H^n A \xrightarrow{H^n f} H^n B \xrightarrow{H^n g} H^n C \xrightarrow{\partial} H^{n+1}(A) \to \dots$$

where the connecting homomorphism ∂ is defined as follows: given $x \in C^n$ with $d^n(x) = 0$, choose $\tilde{x} \in B^n$ such that $g^n(\tilde{x}) = x$, then

$$g^{n+1}(d_B^n(\tilde{x})) = d_C^n(g^n(\tilde{x})) = d_C^n(x) = 0,$$

so there is a unique $y \in A^{n+1}$ such that $f^{n+1}(y) = d_B^{n+1}(\tilde{x})$. Set

$$\partial[x] = [y] \in H^{n+1}(A).$$

$$\begin{array}{ccccc} A \otimes H_n C & \xrightarrow{f \circ g \circ h} & H_n(A \otimes C) & \xrightarrow{H^n f} & \mathrm{Tor}(A, H_{n-1}(C)) \\ a \otimes [x] & \xmapsto{32.85008pt} & [a \otimes x] & & \end{array}$$

$$\begin{array}{ccccc} A \otimes H_n C & \xrightarrow{f \circ g} & H_n(A \otimes C) & \xrightarrow{H^n f} & \mathrm{Tor}(A, H_{n-1}(C)) \\ a \otimes [x] & \mapsto & [a \otimes x] & & \end{array}$$

$$\begin{array}{ccccc} A \otimes H_n C & \xrightarrow{f \circ g} & H_n(A \otimes C) & \xrightarrow{H^n f} & \mathrm{Tor}(A, H_{n-1}(C)) \\ a \otimes [x] & \mapsto & [a \otimes x] & & \end{array}$$