The boundary of shuffles

The purpose of this note is to establish the boundary formula for the shuffle map, see Proposition 7 below. Most likely, this document contains typos (but hopefully no serious mistakes); if you happen to find any typos, please let me know so that I can correct them.

Definition 1. Let $p, q \geq 0$ be a natural numbers. A (p,q)-*shuffle* is a permutation σ of the set $\{0, 1, \ldots, p+q-1\}$ such that

$$\sigma(0) < \sigma(1) < \cdots < \sigma(p-1) \quad \text{and} \quad \sigma(p) < \sigma(p+1) < \cdots < \sigma(p+q-1) \ .$$

A (p,q)-shuffle is thus determined by the set

$$A \ = \ \{\sigma(0), \sigma(1), \ldots, \sigma(p-1)\} \ ,$$

which is a p-element subset of $\{0, 1, \ldots, p+q-1\}$, and also by its complement

$$\bar{A} \ = \ \{0, \ldots, p+q-1\} \setminus A \ = \ \{\sigma(p), \sigma(p+1), \ldots, \sigma(p+q-1)\} \ ,$$

which is a q-element subset of $\{0, 1, \ldots, p+q-1\}$. In particular, the number of (p,q)-shuffles is $\binom{p+q}{p} = \binom{p+q}{q}$

Now we let $A = \{\mu_1 < \mu_2 < \cdots < \mu_p\}$ be any p-element subset of $\{0, 1, \ldots, p+q-1\}$. We define the weakly monotone surjection

$$s_A \ : \ [p+q] \ \longrightarrow \ [q] \qquad \text{by} \qquad s_A \ = \ s_{\mu_1} \cdot \ldots \cdot s_{\mu_p} \ .$$

Part (ii) of the following lemma shows that, loosely speaking, the set A can be recovered as the set of 'stationary points' of the monotone surjection s_A.

Lemma 2. *Let A be a p-element subset of $\{0, 1, \ldots, p+q-1\}$, and let $\bar{A} = \{0, 1, \ldots, p+q-1\} \setminus A$ denote its complement.*

(i) *The monotone surjection $s_A : [p+q] \longrightarrow [q]$ satisfies*

$$s_A(i) \ = \ |\{0, \ldots, i-1\} \setminus A|$$

for all $0 \leq i \leq p+q$.

(ii) *The relation*

$$A \ = \ \{i \in \{0, \ldots, p+q-1\} \ : \ s(i) = s(i+1)\}$$

holds.

(iii) *Then the monotone surjections $s_A : [p+q] \longrightarrow [q]$ and $s_{\bar{A}} : [p+q] \longrightarrow [p]$ satisfy*

$$s_A(i) + s_{\bar{A}}(i) \ = \ i$$

for all $i = 0, \ldots, p+q$.

Proof. (i) We argue by induction on p. For $p = 0$ we have $A = \emptyset$, $s_\emptyset = \mathrm{Id}_{[q]}$, and the formula hold.

Now we assume that $p \geq 1$. We let μ_p be the largest element of A, and we set $B = A \setminus \{\mu_p\}$.. Given $i \in [p+q]$, we distinguish two cases. If $i \leq \mu_p$, then

$$\begin{aligned} s_A(i) \ &= \ s_B(s_{\mu_p}(i)) \ = \ s_B(i) \\ &= \ |\{0, \ldots, i-1\} \setminus B\}| \ = \ |\{0, \ldots, i-1\} \setminus A\}| \ . \end{aligned}$$

If $i > \mu_p$, then A is contained in $\{0, \ldots, i-1\}$ and $B = A \setminus \{\mu_p\}$ is contained in $\{0, \ldots, i-2\}$. So

$$\begin{aligned} s_A(i) \ &= \ s_B(s_{\mu_p}(i)) \ = \ s_B(i-1) \\ &= \ |\{0, \ldots, i-2\} \setminus \{\mu_1, \ldots, \mu_{p-1}\}| \ = \ i - p \ = \ |\{0, \ldots, i-1\} \setminus A| \ . \end{aligned}$$

(ii) By part (i) the condition $s_A(i) = s_A(i+1)$ is equivalent to $\{0, \ldots, i-1\} \setminus A = \{0, \ldots, i\} \setminus A$; this, in turn, is equivalent to the condition $i \in A$.

(iii) The set $\{0, \dots, i-1\}$ is the disjoint union of the sets $\{0, \dots, i-1\} \setminus A$ and $\{0, \dots, i-1\} \setminus \bar{A}$. So
$$s_A(i) + s_{\bar{A}}(i) \;=\; |\{0, \dots, i-1\} \setminus A| + |\{0, \dots, i-1\} \setminus \bar{A}| \;=\; |\{0, \dots, i-1\}| \;=\; i \;. \qquad \square$$

While the notation $\bar{A}$ for the complement of a set A is very convenient, it is also slightly dangerous because it does not record the ambient set in which the complement in formed. If not already clear from the context, we will make sure to always clarify where the relevant complements are taken.

Definition 3 (Signs)**.** For a p-element subset A of $\{0, \dots, p+q-1\}$, we write $\mathrm{sgn}^{\mathrm{ft}}(A)$ for the sign of the (p, q)-shuffle α that satisfies $\{\alpha(0), \dots \alpha(p-1)\} = A$.

In other words, we view the set A as the 'front' part of the shuffle; the reader should beware that if we view A as the 'back' part of a (q, p)-shuffle, we arrive at a potentially different sign. Indeed, if β is the (q, p)-shuffle that satisfies $\{\beta(p), \dots \beta(p+q-1)\} = A$, then
$$\beta \;=\; \alpha \circ \chi_{p,q} \;,$$
where $\chi_{p,q}$ is the 'extreme' (p, q)-shuffle defined by
$$\chi_{p,q}(i) \;=\; \begin{cases} i+q & \text{for } 0 \le i \le p-1, \text{ and} \\ i-p & \text{for } p \le i \le p+q-1. \end{cases}$$
So
$$\mathrm{sgn}^{\mathrm{bk}}(A) \;=\; \mathrm{sgn}(\beta) \;=\; \mathrm{sgn}(\chi_{p,q}) \cdot \mathrm{sgn}(\alpha) \;=\; (-1)^{pq} \cdot \mathrm{sgn}^{\mathrm{ft}}(A) \;. \tag{4}$$

We can now rewrite the definition of the shuffle product in terms of p-element subsets of the set $\{0, \dots, p+q-1\}$. We write
$$\binom{p+q}{p} \;=\; \{A \subseteq \{0, \dots, p+q-1\} \;:\; |A| = p\}$$
for the set of these p-element subsets. We exploit the commutative triangle of bijections:
$$\begin{tikzcd} & (p,q)\text{-shuffles} \arrow[dl, "\sigma\mapsto\{\sigma(0),\dots,\sigma(p-1)\}"', "\cong"] \arrow[dr, "\sigma\mapsto\{\sigma(p),\dots,\sigma(p+q-1)\}", "\cong"'] & \\ \binom{p+q}{p} \arrow[rr, leftrightarrow, "\cong", "\text{complement}"'] & & \binom{p+q}{q} \end{tikzcd}$$
Let E and F be simplicial abelian groups, and $e \in E_p$ and $f \in F_q$. With the new notation, the shuffle product can now be rewritten as
$$\begin{aligned} e \nabla f \;&=\; \sum_{\sigma:(p,q)\text{-shuffle}} \mathrm{sgn}(\sigma) \cdot s^*_{\{\sigma(p),\dots,\sigma(p+q-1)\}}(e) \otimes s_{\{\sigma(0),\dots,\sigma(p-1)\}}{}^*(f) \\ &=\; \sum_{A \in \binom{p+q}{p}} \mathrm{sgn}^{\mathrm{ft}}(A) \cdot s^*_{\bar{A}}(e) \otimes s^*_A(f) \end{aligned}$$

We want to derive a Leibniz rule for $d(a \nabla b)$. To do that we will need to rewrite simplicial operators of the form $d^*_i \circ s^*_A = (s_A \cdot d_i)^*$. The next lemma provides the relevant relations.

Lemma 5. *Let A be a p-element subset of the set $\{0, \dots, p+q-1\}$.*

(i) *The monotone map $s_A \cdot d_0 : [p+q-1] \longrightarrow [q]$ is surjective if and only if $0 \in A$.*
(ii) *For $1 \le i \le p+q-1$, the monotone map $s_A \cdot d_i : [p+q-1] \longrightarrow [q]$ is surjective if and only if $i-1$ or i belongs to A.*
(iii) *The monotone map $s_A \cdot d_{p+q} : [p+q-1] \longrightarrow [q]$ is surjective if and only if $p+q-1 \in A$.*
(iv) *For each $0 \le i \le p+q$, at least one of the two monotone maps $s_A \cdot d_i$ and $s_{\bar{A}} \cdot d_i$ is surjective.*

Proof. (i) Because s_A is surjective and

$$\mathrm{image}(s_A d_0) \;=\; s_A(\mathrm{image}(d_0)) \;=\; s_A(\{1,\dots,p+q\})\ ,$$

the monotone map $s_A d_0$ is surjective if and only if $s_A(0) = s_A(1) = 0$. By Lemma 2, this is equivalent to $0 \in A$.

(ii) Because s_A is surjective we have

$$\begin{aligned}\mathrm{image}(s_A \cdot d_i) \;&=\; s_A(\mathrm{image}(d_i)) \;=\; s_A(\{0,\dots,i-1\} \cup \{i+1,\dots,p+q\})\\ &=\; s_A(\{0,\dots,i-1\}) \cup s_A(\{i+1,\dots,p+q\})\\ &=\; \{0,\dots,s_A(i-1)\} \cup \{s_A(i+1),\dots,q\} \qquad (6)\end{aligned}$$

Because $s_A(i-1) \leq s_A(i) \leq s_A(i+1)$, the last set coincides with $[q] = \{0,\dots,q\}$ if and only if $s_A(i-1) = s_A(i)$ or $s_A(i) = s_A(i+1)$.

The proof of (iii) is like the proof of (i).

(iv) We argue by contradiction and suppose that neither $s_A d_i$ nor $s_{\bar{A}} d_i$ is surjective. Then $\{i-1,i\} \cap A = \emptyset$ and $\{i-1,i\} \cap \bar{A} = \emptyset$ by parts (i), (ii) and (iii). But this contradicts the fact that $A \cup \bar{A} = \{0,\dots,p+q-1\}$. $\square$

Now we can prove the main result of this section.

Proposition 7. *The shuffle product satisfies the relation*

$$d(e \nabla f) \;=\; (de)\nabla f + (-1)^p \cdot e \nabla (df)$$

for all $e \in E_p$ and $f \in F_q$.

Proof. We split $d(e\nabla f)$ into a sum of three terms, as follows:

$$\begin{aligned}d(e\nabla f) \;&=\; \sum_{i=0}^{p+q} \sum_{A \in \binom{p+q}{p}} (-1)^i \cdot \mathrm{sgn}^{\mathrm{ft}}(A) \cdot d_i^*(s_{\bar{A}}^*(e) \otimes s_A^*(f))\\ &=\; \sum_{i=0}^{p+q} \sum_{A \in \binom{p+q}{p}} (-1)^i \cdot \mathrm{sgn}^{\mathrm{ft}}(A) \cdot (s_{\bar{A}} d_i)^*(e) \otimes (s_A d_i)^*(f)\\ &=\; (S0) + (S1) + (S2)\ .\end{aligned}$$

Here:

- (S0) is the sum over those terms (A,i) such that both $s_A d_i$ and $s_{\bar{A}} d_i$ are surjective;
- (S1) is the sum over those terms (A,i) such that $s_{\bar{A}} d_i$ is not surjective (and hence $s_A d_i$ is surjective);
- and (S2) is the sum over those terms (A,i) such that $s_A d_i$ is not surjective (and hence $s_{\bar{A}} d_i$ is surjective).

By Lemma 5 (iv), at least one of the maps $s_A d_i$ and $s_{\bar{A}} d_i$ is surjective, so this covers all summands.

We claim that **the sum (S0) is zero**. To see this, we consider a p-element subset A of $\{0,\dots,p+q-1\}$ such that both $s_A d_i : [p+q-1] \longrightarrow [q]$ and $s_{\bar{A}} d_i : [p+q-1] \longrightarrow [p]$ are surjective. By part (i) and (iii) of Lemma 5, we *cannot* have $i = 0$, nor $i = p+q$, and by part (ii) of Lemma 5, exactly one of the two numbers $i-1$ and i belong to A, and the other one belongs to the complement $\bar{A}$.

We define a new p-element subset B of $\{0,\dots,p+q-1\}$ as the symmetric difference of A and the set $\{i-1,i\}$, i.e.,

$$B \;=\; (A \cup \{i-1,i\}) \setminus (A \cap \{i-1,i\})\ .$$

Informally said: the sets A and B coincide outside of $\{i-1,i\}$, and the elements $i-1$ and i have been swapped between A and B. Then the monotone surjection $s_B : [p+q] \longrightarrow [q]$ satisfies $s_B(j) = s_A(j)$ for all $j \neq i$, and hence

$$s_A \cdot d_i \;=\; s_B \cdot d_i \qquad \text{and} \qquad s_{\bar{A}} \cdot d_i \;=\; s_{\bar{B}} \cdot d_i\ .$$

In particular, the set B also has the property that $s_B \cdot d_i$ and $s_{\bar{B}} \cdot d_i$ are surjective, so the pair (B, i) also contributes to the sum (S0). Moreover, the shuffle permutations that give rise to $(A, \bar{A})$ and $(B, \bar{B})$, respectively, differ by the transposition $(i-1, i)$, so $\mathrm{sgn}^{\mathrm{ft}}(A) = -\mathrm{sgn}^{\mathrm{ft}}(B)$. The upshot is that the terms in the sum (S0) contributed by (A, i) and (B, i) cancel. Since the passage $(A, i) \mapsto (B, i)$ given by the symmetric difference with the set $\{i-1, i\}$ is a fixed point free involution on the set of pairs that index the sum (S0), this sum (S0) is indeed zero.

Now we show that **the sum (S1) equals** $(de)\nabla f$. To see this, we consider a p-element subset A of $\{0, \ldots, p+q-1\}$ such that $s_{\bar{A}} d_i : [p+q-1] \longrightarrow [p]$ is not surjective. By Lemma 5, this is equivalent to $\{i-1, i\} \cap \bar{A} = \emptyset$, and it implies that

- $0 \in A$ in case $i = 0$,
- $\{i-1, i\} \subset A$ in case $1 \leq i \leq p+q-1$, and
- $p+q-1 \in A$ in case $i = p+q$.

We define an operation $\langle - | i \rangle$ from subsets of $\{0, \ldots, p+q-1\}$ to subsets of $\{0, \ldots, p+q-2\}$ by

$$\begin{aligned} \langle B | i \rangle &= ((B \cap \{0, \ldots, i-1\}) \cup ((B \cap \{i, \ldots, p+q-1\}) - 1)) \cap \{0, \ldots, p+q-2\} \\ &= \{k \in \{0, \ldots, p+q-2\} \ : \ k \in B \cap \{0, \ldots, i-1\} \text{ or } k+1 \in B \cap \{i, \ldots, p+q-1\}\} \end{aligned}$$

In other words: we leave elements of B in $\{0, \ldots, i-1\}$ untouched and we decrease all elements of B in $\{i, \ldots, p+q-1\}$ by 1; and should the process yield an element outside of $\{i, \ldots, p+q-2\}$ (which can only happen for $i = 0$ or $i = p+q$), then we discard it.

Because A contains $\{i-1, i\}$ (or contains 0 for $i = 0$, or contains $p+q-1$ for $i = p+q$), the set $\langle A | i \rangle$ has $p-1$ elements; because $\bar{A}$ does not intersect $\{i-1, i\}$, the set $\langle \bar{A} | i \rangle$ has q elements. Moreover, the sets $\langle A | i \rangle$ and $\langle \bar{A} | i \rangle$ are mutual complements of each other in $\{0, \ldots, p+q-2\}$. We claim that:

$$s_A \cdot d_i = s_{\langle A | i \rangle} \ , \tag{8}$$

$$s_{\bar{A}} \cdot d_i = d_{s_{\bar{A}}(i)} \cdot s_{\langle \bar{A} | i \rangle} \qquad \text{and} \tag{9}$$

$$\mathrm{sgn}^{\mathrm{ft}}(A) = (-1)^{s_A(i)} \cdot \mathrm{sgn}^{\mathrm{ft}}(\langle A | i \rangle) \ . \tag{10}$$

Proof of (8): We exploit that not only A and $\bar{A}$, but also $\langle A | i \rangle$ and $\langle \bar{A} | i \rangle$ are mutual complements of each other. Also, $\bar{A}$ contains neither $i-1$ nor i, so

$$\begin{aligned} s_A(d_i(k)) = &\begin{cases} s_A(k) & \text{for } k \leq i-1, \\ s_A(k+1) & \text{for } k \geq i; \end{cases} \\ = &\begin{cases} |\{0, \ldots, k-1\} \setminus A| & \text{for } k \leq i-1, \\ |\{0, \ldots, k\} \setminus A| & \text{for } k \geq i; \end{cases} \\ = &\ |\{0, \ldots, k-1\} \setminus \langle A | i \rangle| \ = \ s_{\langle A | i \rangle}(k) \ . \end{aligned}$$

Proof of (9):

$$s_{\bar A}(d_i(k)) \;=\; \begin{cases} s_{\bar A}(k) & \text{for } k \le i-1, \\ s_{\bar A}(k+1) & \text{for } k \ge i; \end{cases}$$
$$=\; \begin{cases} |\{0,\dots,k-1\} \setminus \bar A| & \text{for } k \le i-1, \\ |\{0,\dots,k\} \setminus \bar A| & \text{for } k \ge i; \end{cases}$$
$$=\; \begin{cases} |\{0,\dots,k-1\} \setminus \langle \bar A|i\rangle| & \text{for } k \le i-1, \\ |\{0,\dots,k-1\} \setminus \langle \bar A|i\rangle| + 1 & \text{for } k \ge i; \end{cases}$$
$$=\; \begin{cases} s_{\langle \bar A|i\rangle}(k) & \text{for } k \le i-1, \\ s_{\langle \bar A|i\rangle}(k)+1 & \text{for } k \ge i; \end{cases}$$
$$=\; d_{s_{\bar A}(i)}(s_{\langle \bar A|i\rangle}(k)) \;.$$

Proof of (10): We let α be the unique (p,q)-shuffle with $A = \{\alpha(0) < \cdots < \alpha(p-1)\}$. We have $\{i-1,i\} \subseteq A$, and we let l be the unique number such that

$$\alpha(l-1) \;=\; i-1 \qquad \text{and} \qquad \alpha(l) \;=\; i \;.$$

Then

$$s_A(i) \;=\; |\{0,\dots,i-1\} \setminus A| \;=\; |\{0,\dots,i-1\} \setminus \{\alpha(0),\dots,\alpha(l-1)\}| \;=\; i-l \;,$$

so $l = i - s_A(i) = s_{\bar A}(i)$. We define

$$\gamma \;=\; (p+q-1, p+q-2, \dots, i) \circ \alpha \circ (l, l+1, \dots, p+q-2, p+q-1) \;,$$

another permutation of the set $\{0,\dots,p+q-1\}$. Then

$$\gamma(p+q-1) \;=\; p+q-1$$

because $\alpha(l) = i$. So γ restricts to a permutation of the set $\{0,\dots,p+q-2\}$; we write $\bar\gamma$ for this restriction. We claim that $\bar\gamma$ is a $(p-1,q)$-shuffle. To see this we observe that

$$\bar\gamma(j) \;=\; \begin{cases} \alpha(j) & \text{for } 0 \le j \le l-1, \text{ and} \\ \alpha(j+1)-1 & \text{for } l \le j \le p-2. \end{cases} \tag{11}$$

Since α is monotone on $\{0,\dots,p-1\}$, this shows that $\bar\gamma$ is monotone on $\{0,\dots,p-2\}$. We set $k = i-l = s_A(i)$; then

$$|\{\alpha(p),\dots,\alpha(p+q-1)\} \cap \{0,\dots,i-1\}| \;=\; |\bar A \cap \{0,\dots,i-1\}| \;=\; |\{0,\dots,i-1\} \setminus A| \;=\; i-l \;=\; k \;,$$

so

$$\alpha(p+k-1) \le i-1 < i \le \alpha(p+k) \;.$$

So

$$\bar\gamma(j) \;=\; \begin{cases} \alpha(j+1) & \text{for } p-1 \le j \le p+k-2, \text{ and} \\ \alpha(j+1)-1 & \text{for } p+k-1 \le j \le p+q-2. \end{cases}$$

Since α is monotone on $\{p,\dots,p+q-1\}$, this shows that $\bar\gamma$ is monotone on $\{p-1,\dots,p+q-2\}$. So $\bar\gamma$ is indeed a $(p-1,q)$-shuffle. The relation (11) also proves that

$$\bar\gamma(\{0,\dots,p-2\}) \;=\; \{\alpha(0),\dots,\alpha(l-1),\alpha(l+1)-1,\dots,\alpha(p-1)-1\} \;=\; \langle A|i\rangle \;.$$

Altogether this shows that $\bar\gamma$ is the $(p-1,q)$-shuffle that defines $\mathrm{sgn}^{\mathrm{ft}}(\langle A|i\rangle)$. So

$$\mathrm{sgn}^{\mathrm{ft}}(\langle A|i\rangle) \;=\; \mathrm{sgn}(\gamma) \;=\; (-1)^{p+q-1-i} \cdot \mathrm{sgn}(\alpha) \cdot (-1)^{p+q-1-l}$$
$$=\; (-1)^{i-l} \cdot \mathrm{sgn}^{\mathrm{ft}}(A) \;=\; (-1)^{s_A(i)} \cdot \mathrm{sgn}^{\mathrm{ft}}(A) \;.$$

We **claim** that the map

$$\begin{array}{rccc}\Phi \ : \ \left\{(A,i) \in \binom{p+q}{p} \times [p+q] \ : \ \{i-1,i\} \cap \{0,\dots,p+q-1\} \subseteq A\right\} & \longrightarrow & \binom{p-1+q}{p-1} \times [p] \\ (A,i) & \longmapsto & (\langle A|i\rangle, |A \cap \{0,\dots,i-1\}|)\end{array}$$

is bijective.

To show that we observe: let $(A,i) \in \binom{p+q}{p} \times [p+q]$ satisfy $\{i-1,i\} \cap \{0,\dots,p+q-1\} \subseteq A$. Then

- if $i = 0$, then $0 \in A$ and $|A \cap \{0,\dots,i-1\}| = 0$;
- if $1 \le i \le p+q-1$, then $\{i-1,i\} \subset A$ and $1 \le |A \cap \{0,\dots,i-1\}| \le p-1$; and
- if $i = p+q$, then $p+q-1 \in A$ and $|A \cap \{0,\dots,i-1\}| \le p$.

So we may show that Φ restricts to three separate bijections, depending on these three cases. In the extreme cases $i = 0$ and $i = p+q$, the map Φ restricts to maps

$$\begin{array}{rccc}\Phi \ : \ \left\{A \in \binom{p+q}{p} \ : \ 0 \in A\right\} & \longrightarrow & \binom{p-1+q}{p-1} \\ A & \longmapsto & \langle A|0\rangle = (A \setminus \{0\}) - 1\end{array}$$

and

$$\begin{array}{rccc}\Phi \ : \ \left\{A \in \binom{p+q}{p} \ : \ p+q-1 \in A\right\} & \longrightarrow & \binom{p-1+q}{p-1} \\ A & \longmapsto & \langle A|p+q\rangle = A \cap \{0,\dots,p+q-2\} \ ,\end{array}$$

both of which are clearly bijective.

In the intermediate case we need to show that the map

$$\begin{array}{rccc}\Phi \ : \ \left\{(A,i) \in \binom{p+q}{p} \times \{1,\dots,p+q-1\} \ : \ \{i-1,i\} \subseteq A\right\} & \longrightarrow & \binom{p-1+q}{p-1} \times [p] \\ (A,i) & \longmapsto & (\langle A|i\rangle, |A \cap \{0,\dots,i-1\}|)\end{array}$$

is bijective.

Injectivity: Because $\{i-1,i\} \subseteq A$, the relation

$$\langle A|i\rangle \cap \{0,\dots,i-1\} \ = \ A \cap \{0,\dots,i-1\}$$

holds by construction, and the number $i-1$ belongs to $\langle A|i\rangle$. So the relations

$$|\langle A|i\rangle \cap \{0,\dots,i-2\}| \ < \ |\langle A|i\rangle \cap \{0,\dots,i-1\}| \ = \ |A \cap \{0,\dots,i-1\}|$$

hold. So we can recover i from the set $\langle A|i\rangle$ and the number $|A \cap \{0,\dots,i-1\}|$ as

$$i \ = \ \min\{k \in [p+q] \ : \ |\langle A|i\rangle \cap \{0,\dots,k-1\}| = |A \cap \{0,\dots,i-1\}|\} \ .$$

But i and $\langle A|i\rangle$ together determine the set A as

$$A \ = \ (\langle A|i\rangle \cap \{0,\dots,i-1\}) \cup ((\langle A|i\rangle \cap \{i-1,\dots,p+q-2\}) + 1) \ .$$

Surjectivity: We consider any pair (C,j) in $\binom{p-1+q}{p-1} \times [p]$ with $1 \le j \le p-1$. We define

$$i \ = \ \min\{k \in [p+q] \ : \ |C \cap \{0,\dots,k-1\}| = j\} \ .$$

We observe that $i-1 \in C$, for otherwise

$$|C \cap \{0,\dots,i-2\}| \ = \ |C \cap \{0,\dots,i-1\}| \ = \ j \ ,$$

contradicting the minimality of i. We define

$$A \ = \ (C \cap \{0,\dots,i-1\}) \cup ((C \cap \{i-1,\dots,p+q-2\}) + 1) \ ,$$

a p-element subset of $\{0,\dots,p+q-1\}$ that contains $i-1$ and i; moreover, $\langle A|i\rangle = C$ and

$$|A\cap\{0,\dots,i-1\}| \;=\; |C\cap\{0,\dots,i-1\}| \;=\; j\ .$$

So the pair (A,i) is indeed a preimage of (C,j).

Now we put the ingredients together and show that the sum (S1) equals $(de)\nabla f$:

$$\begin{aligned}
(S1) = &\sum_{i=0}^{p+q}\ \sum_{A\in\binom{p+q}{p}: s_{\bar A}d_i \text{ not epi}} (-1)^i\cdot \mathrm{sgn}^{\mathrm{ft}}(A)\cdot(s_{\bar A}d_i)^*(e)\otimes(s_Ad_i)^*(f)\\
= &\sum_{i=0}^{p+q}\ \sum_{A\in\binom{p+q}{p}:\{i-1,i\}\cap\bar A=\emptyset} (-1)^{s_{\bar A}(i)}\cdot \mathrm{sgn}^{\mathrm{ft}}(\langle A|i\rangle)\cdot(d_{s_{\bar A}(i)}\cdot s_{\langle\bar A|i\rangle})^*(e)\otimes s^*_{\langle A|i\rangle}(f)\\
= &\sum_{(C,j)\in\binom{p-1+q}{p-1}\times[p]} \mathrm{sgn}(C)^{\mathrm{ft}}\cdot(-1)^j\cdot(d_j\cdot s_{\bar C})^*(e)\otimes s^*_C(f)\\
= &\sum_{C\in\binom{p+q-1}{p-1}}\sum_{j=0}^{p} \mathrm{sgn}^{\mathrm{ft}}(C)\cdot(-1)^j\cdot s^*_{\bar C}(d^*_j(e))\otimes s^*_C(f)\\
= &\sum_{C\in\binom{p+q-1}{p-1}} \mathrm{sgn}^{\mathrm{ft}}(C)\cdot s^*_{\bar C}(de)\otimes s^*_C(f) \;=\; (de)\nabla f
\end{aligned}$$

The second equality uses the relation

$$(-1)^i\cdot\mathrm{sgn}^{\mathrm{ft}}(A) \underset{(10)}{=} (-1)^{i-s_A(i)}\cdot\mathrm{sgn}^{\mathrm{ft}}(\langle A|i\rangle) \;=\; (-1)^{s_{\bar A}(i)}\cdot\mathrm{sgn}^{\mathrm{ft}}(\langle A|i\rangle)\ .$$

The third equality exploits the relations (8) and (9), the bijection given by the map Φ, and the relation

$$s_{\bar A}(i) \;=\; |\{0,\dots,i-1\}\setminus\bar A| \;=\; |A\cap\{0,\dots,i-1\}|\ .$$

Finally, we show that **the sum (S2) equals** $(-1)^p\cdot e\nabla(df)$. One approach would be to adapt the argument for the sum (S1), interchanging the role of the first and the second tensor factor everywhere. But there is different argument that uses an important symmetry property of the shuffle map to deduces the relation (S2)$=(-1)^p\cdot e\nabla(df)$ from the relation for (S1), but applied to (q,p,F,E,f,e) instead of (p,q,E,F,e,f). We write

$$\tau\ :\ F_n\otimes E_n\ \cong\ E_n\otimes F_n\ ,\quad \tau(y\otimes x)\ x\otimes y$$

for the symmetry isomorphism of the tensor product of abelian groups. We claim that then for all $e\in E_p$ and $f\in F_q$, the relation

$$f\nabla e \;=\; (-1)^{pq}\cdot\tau(e\nabla f)$$

holds in $F_{q+p}\otimes E_{q+p}$. We postpone the proof of this symmetry relation unital after the proof of the boundary relation.

We obtain the relation

$$
\begin{aligned}
(S2) \;&=\; \sum_{i=0}^{p+q} \sum_{A\in\binom{p+q}{p}: s_A d_i \text{ not epi}} (-1)^i \cdot \operatorname{sgn}^{\mathrm{ft}}(A)\cdot d_i^*(s_{\bar{A}}^*(e)\otimes s_A^*(f))\\
&=\; \sum_{i=0}^{p+q} \sum_{B\in\binom{p+q}{q}: s_{\bar{B}} d_i \text{ not epi}} (-1)^i \cdot \operatorname{sgn}^{\mathrm{bk}}(B)\cdot d_i^*(s_B^*(e)\otimes s_{\bar{B}}^*(f))\\
\overset{(4)}{=}\;& (-1)^{pq}\cdot\tau\left(\sum_{i=0}^{q+p} \sum_{B\in\binom{q+p}{q}: s_{\bar{B}} d_i \text{ not epi}} (-1)^i \cdot \operatorname{sgn}^{\mathrm{ft}}(B)\cdot d_i^*(s_{\bar{B}}^*(f)\otimes s_B^*(e))\right)\\
&=\; (-1)^{pq}\cdot\tau\left((df)\nabla e\right)\\
&=\; (-1)^{pq}\cdot(-1)^{(q-1)p}\cdot e\nabla(df) \;=\; (-1)^p e\nabla(df)
\end{aligned}
$$

The second equality is a just variable substitution: instead of summing over p-element subsets A, we sum over the complement B. The third equality is the relationship (4) between the signs of the (p,q)-shuffle and the (q,p)-shuffle derived from viewing a set as the front or back part. The fourth equality is the previous calculation of (S1), but for the situation where the roles of p and q, of E and F, and of e and f have been reversed. □

It remains to show the symmetry property of the shuffle map.

Proposition 12. *Let E and F be simplicial abelian groups and $p,q\geq 0$. Then the relation*

$$f\nabla e \;=\; (-1)^{pq}\cdot\tau(e\nabla f)$$

holds in $F_{q+p}\otimes E_{q+p}$ for all $e\in E_p$ and $f\in F_q$.

Proof.

$$
\begin{aligned}
f\nabla e \;&=\; \sum_{B\in\binom{q+p}{q}} \operatorname{sgn}^{\mathrm{ft}}(B)\cdot s_{\bar{B}}^*(f)\otimes s_B^*(e)\\
&=\; \sum_{A\in\binom{q+p}{p}} \operatorname{sgn}^{\mathrm{bk}}(A)\cdot s_A^*(f)\otimes s_{\bar{A}}^*(e)\\
\overset{(4)}{=}\;& (-1)^{pq}\cdot\tau\left(\sum_{A\in\binom{p+q}{p}} \operatorname{sgn}^{\mathrm{ft}}(A)\cdot s_{\bar{A}}^*(e)\otimes s_A^*(f)\right) \;=\; (-1)^{pq}\cdot\tau\left(e\nabla f\right)\;.
\end{aligned}
$$

The second equality is a variable substitution, instead of summing over q-element subsets B, we sum over the complement A. The third equality is the relationship (4) between the signs of the (p,q)-shuffle and the (q,p)-shuffle derived from viewing a set as the front or back part. □