Cohomology with compact support

The proof of Poincaré duality that we will give uses a bootstrap argument: the given compact manifold is covered by finitely many euclidean open subsets, and then 'local' version of Poincaré duality are suitably patched together. For this kind of patching argument it is key to have a generalization of Poincaré duality to manifolds that are not necessarily compact.

As the example of the orientable 1-manifold $M = \mathbb{R}$ immediately shows, Poincaré duality does not naively generalize to non-compact manifolds: the group $H^1(\mathbb{R};\mathbb{Z})$ is trivial, and hence not isomorphic to $H_0(\mathbb{R};\mathbb{Z})$. The proper generalization of Poincaré duality involves a modified version of singular cohomology, called *cohomology with compact support*. In this section we will introduce this cohomology theory and establish some basic properties.

Construction 1 (Cohomology with compact support)**.** Let X be a topological space and A an abelian group. We recall that singular n-cochain $f \in C^n(\mathcal{S}(X);A)$ is a map

$$f \ : \ \mathcal{S}(X)_n = \ \mathrm{map}^{\mathrm{cts}}(\nabla^n, X) \ \longrightarrow \ A$$

from the set of singular n-simplices to the group A. The cochain f has *compact support* if there is a compact subset K of X such that $f(\varphi) = 0$ for all singular simplices $\varphi : \nabla^n \longrightarrow X$ whose image is contained in $X \setminus K$. Equivalently, there is a compact subset K of X such that f belongs to the kernel of the restriction map

$$C^n(\mathcal{S}(X);A) \ \longrightarrow \ C^n(\mathcal{S}(X\setminus K);A) \ ,$$

i.e., f lies in the relative cochain complex $C^*(\mathcal{S}(X), \mathcal{S}(X\setminus K);A)$. If this happens, we will say that f *is supported on* K.

We recall that the coboundary map in the singular cochain complex $C^*(\mathcal{S}(X);A)$ is given by

$$(df)(\varphi) \ = \ \sum_{i=0}^{n+1} (-1)^i \cdot f(\varphi \circ (d_i)_*) \ ,$$

where $\varphi : \nabla^{n+1} \longrightarrow X$ is any singular $(n+1)$-simplex, and $(d_i)_* : \nabla^n \longrightarrow \nabla^{n+1}$ is the affine linear embedding given by

$$(d_i)_*(x_0,\dots,x_n) \ = \ (x_0,\dots,x_{i-1},0,x_i\dots,x_n) \ .$$

The image of $\varphi \circ (d_i)_*$ is contained in the image of φ; so if f is supported on a compact subset K, then df is also supported on K. This shows that the compactly supported cochains form a subcomplex $C^*_{\mathrm{comp}}(X;A)$ of the singular cochain complex $C^*(\mathcal{S}(X);A)$.

Definition 2. The *compactly supported cohomology groups* of a space X with coefficients in an abelian group A are the cohomology groups of the cochain complex $C^*_{\mathrm{comp}}(X;A)$, i.e.,

$$H^n_{\mathrm{comp}}(X;A) \ = \ H^n(C^*_{\mathrm{comp}}(X;A)) \ .$$

If R is a ring, then the group $C^n_{\mathrm{comp}}(X;R)$ is closed under pointwise scalar multiplication, so $C^n_{\mathrm{comp}}(X;R)$ is a complex of R-modules, and the compactly supported cohomology groups inherit the structure of R-modules.

Example 3. If X is a *compact* space, then every singular cochain is supported on $K = X$, and hence every singular cochain is compactly supported. So in this case $C^*_{\mathrm{comp}}(X;A) = C^*(\mathcal{S}(X);A)$, and hence

$$H^n_{\mathrm{comp}}(X;A) \ = \ H^n(X;A) \ .$$

The euclidean space $\mathbb{R}^n$ is contractible, so for $n \geq 1$ the cohomology group $H^n(\mathbb{R}^n;A)$ is trivial. The following proposition in particular shows that, in contrast, the compactly supported cohomology group $H^n_{\mathrm{comp}}(\mathbb{R}^n;A)$ is isomorphic to

$$H^n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\};A) \ = \ H^n(\mathbb{R}^n|0;A) \ \cong \ A \ .$$

In the following we simplify the notation by writing $\mathbb{R}^n \setminus 0$ instead of $\mathbb{R}^n \setminus \{0\}$. Since a one-point space is compact, the relative cochain complex $C^*(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n \setminus 0); A)$ is a subcomplex of $C^*_{\mathrm{comp}}(\mathbb{R}^n; A)$.

Proposition 4. *For every $n \geq 0$ and every abelian group A, the inclusion*

$$C^*(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n \setminus 0); A) \ \longrightarrow \ C^*_{\mathrm{comp}}(\mathbb{R}^n; A)$$

is a quasi-isomorphism of cochain complexes. Hence

$$H^k_{\mathrm{comp}}(\mathbb{R}^n; A) \ \cong \ H^k(\mathbb{R}^n|0; A) \ \cong \ \begin{cases} A & \text{for } k = n, \text{ and} \\ 0 & \text{for } k \neq n. \end{cases}$$

Proof. To simplify the exposition, we omit the coefficient group A from the notation. We show that all cohomology groups of the quotient complex

$$E \ = \ C^*_{\mathrm{comp}}(\mathbb{R}^n)/C^*(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n \setminus 0))$$

are trivial. The long exact cohomology sequence then proves the proposition.

We let $f \in C^k_{\mathrm{comp}}(\mathbb{R}^n)$ be any compactly supported singular k-cochain that represents a cocycle in the quotient complex E. This means that the coboundary df lies in $C^{k+1}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n \setminus 0)$, i.e., df is supported on the set $\{0\}$.

By hypothesis, f is supported on some compact subset of $\mathbb{R}^n$. Compact subsets of $\mathbb{R}^n$ are bounded, so there is an $r \geq 0$ such that f is supported on $D^n_r = \{x \in \mathbb{R}^n \colon |x| \leq r\}$, the disc of radius r centered at the origin. We write $\mathrm{res}(f) \in C^k(\mathcal{S}(\mathbb{R}^n \setminus 0), \mathcal{S}(\mathbb{R}^n \setminus D^n_r))$ for the restriction of f from $\mathbb{R}^n$ to $\mathbb{R}^n \setminus 0$. The inclusion

$$\mathbb{R}^n \setminus D^n_r \ \longrightarrow \ \mathbb{R}^n \setminus 0$$

is a homotopy equivalence; so all relative cohomology groups of the pair $(\mathbb{R}^n \setminus 0, \mathbb{R}^n \setminus D^n_r)$ are trivial. Equivalently, the relative cochain complex

$$C^*(\mathcal{S}(\mathbb{R}^n \setminus 0), \mathcal{S}(\mathbb{R}^n \setminus D^n_r))$$

is acyclic. In particular, there is a relative cochain $g \in C^{k-1}(\mathcal{S}(\mathbb{R}^n \setminus 0), \mathcal{S}(\mathbb{R}^n \setminus D^n_r))$ such that $dg = \mathrm{res}(f)$.

We define a cochain $\tilde{g} : \mathcal{S}(\mathbb{R}^n)_{k-1} \longrightarrow A$ by 'extending g by zero'; more precisely, for a singular $(k-1)$-simplex $\varphi : \nabla^{k-1} \longrightarrow \mathbb{R}^n$ we set

$$\tilde{g}(\varphi) \ = \ \begin{cases} g(\varphi) & \text{if } \varphi(\nabla^{k-1}) \subset \mathbb{R}^n \setminus 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathrm{res}(\tilde{g}) = g$, by design; in particular, $\tilde{g}$ is also supported on D^n_r, and hence compactly supported. Moreover,

$$\mathrm{res}(d\tilde{g}) \ = \ d(\mathrm{res}(\tilde{g})) \ = \ dg \ = \ \mathrm{res}(f) \ .$$

This means that $\mathrm{res}(f - d\tilde{g}) = 0$, i.e., the cochain $f - d\tilde{g}$ is supported on 0, and it belongs to the complex $C^*(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n \setminus 0))$. Since f and $f - d\tilde{g}$ are visibly cohomologous, f represents the zero cohomology class in the quotient complex E. Since f was an arbitrary representative of a cohomology class of E, this shows that the cohomology groups of E are trivial. $\square$

The compactly supported cohomology groups are *not* functorial in arbitrary continuous maps. The issue is that for a continuous map $\psi : X \longrightarrow Y$ the induced cochain map $\psi^* : C^*(\mathcal{S}(Y); A) \longrightarrow C^*(\mathcal{S}(X); A)$ need not send compactly supported cochains to compactly supported cochains. Even worse: the compactly supported cohomology groups can neither be extended to a contravariant nor a covariant functor for arbitrary continuous maps in any way. Indeed, $\mathbb{R}^n$ is a continuous retract of $\mathbb{R}^{n+1}$; so if a functorial extension of compactly supported cohomology groups existed, then the non-trivial group $H^n_{\text{comp}}(\mathbb{R}^n; \mathbb{Z})$ would be a retract of the trivial group $H^n_{\text{comp}}(\mathbb{R}^{n+1}; A)$ – a contradiction. This example also shows that compactly generated cohomology groups are *not* invariants of the homotopy type.

Nevertheless, as we shall now discuss, the compactly supported cohomology groups are functorial for special classes of continuous maps. On the one hand, they are contravariantly functorial for *proper* maps; and on the other hand, they are covariantly functorial for open embeddings.

Definition 5. A continuous map $\psi : X \longrightarrow Y$ is *proper* if for every compact subset K of Y, the preimage $\psi^{-1}(K)$ is compact subset in the subspace topology of X.

Example 6. For every space X, the only map $X \longrightarrow *$ to a one-point space is proper if and only if X is compact. If K is compact and if X is any space, then the projection $X \times K \longrightarrow X$ to the first component is a proper map.

Proposition 7. *Let $\psi : X \longrightarrow Y$ be a proper continuous map. Then the cochain map $\psi^* : C^*(\mathcal{S}(Y); A) \longrightarrow C^*(\mathcal{S}(X); A)$ restricts to a cochain map*

$$\psi^* \ : \ C^*_{\text{comp}}(Y; A) \ \longrightarrow \ C^*_{\text{comp}}(X; A) \ ,$$

and hence it induces a homomorphism

$$\psi^* \ : \ H^*_{\text{comp}}(Y; A) \ \longrightarrow \ H^*_{\text{comp}}(X; A)$$

of compactly supported cohomology groups.

Proof. We let $f : \mathcal{S}(Y)_n \longrightarrow A$ be a singular cochain that is supported on a compact subspace K of Y. We claim that $\psi^*(f)$ is supported on $\psi^{-1}(K)$. Indeed, if $\varphi : \nabla^n \longrightarrow X$ is a singular simplex whose image is contained in $X \setminus \psi^{-1}(K)$, then the image of $\psi \circ \varphi : \nabla^n \longrightarrow Y$ is contained in $Y \setminus K$, and hence

$$\psi^*(f)(\varphi) \ = \ f(\psi \circ \varphi) \ = 0 \ .$$

Since f is proper, the subspace $\psi^{-1}(K)$ is compact, and so the singular cochain $\psi^*(f)$ is compactly supported. □