Honours Differential Equations

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Lecture 1

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What is a differential equation?

Definition

Given some independent variables $x_1, x_2, \ldots x_n$ *and some dependent variables* $y_1(x_j), y_2(x_j), \ldots y_p(x_j)$, *a differential equation is an algebraic equation relating them involving the derivatives* $\partial_j y_r\,, \partial_{ij} y_s \ldots$ *of the dependent variables:*

$$\mathcal{F}[y_r, \partial_j y_r\,, \ldots, x_j] = 0\,.$$

Examples:

$$y'(x) = 0\,,$$
$$\ddot{y}(t) - y(t) = 0\,,$$
$$(\partial_x y)^2 + \tanh x\,\partial_z y = e^{2x+z}\,.$$

Why study differential equations?

Newton said it all:

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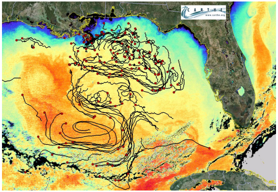
i.e.,

It is useful to solve differential equations (V I Arnold)

Why study differential equations?

Differential equations appear everywhere:

- $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t)$: trajectories $\mathbf{x}(t)$ in the velocity field $\mathbf{u}$,
- $m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ for $\mathbf{x}(t)$: classical mechanics,
- $\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p$ for $\mathbf{u}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$: fluid mechanics,
- $\partial_t u = \partial_{xx} u + u(1-u)$: Fisher–Kolmogorov equation in biology and chemistry,
- $\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_{SS} V + rS\partial_S V - rV = 0$ for $V(S, t)$: Black–Schole formula in finance.



Classification: ODEs vs PDEs

Given the two sets $\{x_1, x_2, \ldots x_n\}$ and $y_1(x_j), y_2(x_j), \ldots y_p(x_j)$

1. If $n = 1$: ordinary differential equation (ODE)
   - all derivatives become total derivatives
   - $y'(x) = \frac{dy}{dx}, \ldots, y^{(n)}(x) = \frac{d^n y}{dx^n}$
2. If $n > 1$: partial differential equation (PDE)

If there is more than one algebraic equation involving dependent variables and their derivatives: system of ODEs or PDEs

$$\begin{aligned}
\mathcal{F}_1[y_r, \partial_j y_r, \ldots x_j] &= 0\,, \\
\mathcal{F}_2[y_r, \partial_j y_r, \ldots x_j] &= 0\,, \\
&\ldots \\
\mathcal{F}_s[y_r, \partial_j y_r, \ldots x_j] &= 0\,.
\end{aligned}$$

Classification: linear vs non-linear

Consider for example the set of ODEs with a single dependent variable $y(x)$

We can view an ODE as the image of a map L acting on y giving rise to the ODE:

$$L[y] = \mathcal{F}(y,\, y',\ldots y^{(n)},\ldots x)$$

For example, $\ddot{y}(x) - y(x) = 0$ corresponds to $Ly = 0$ with $L \equiv \frac{d^2}{dx^2} - 1$.

Definition

We say an ODE is linear if it satisfies the property

$$L[a_1 y_1 + a_2 y_2] = a_1\, L[y_1] + a_2\, L[y_2] \qquad \forall\, a_1,\, a_2 \in \mathbb{R}$$

and for any pair of functions $y_1,\, y_2$

Linear vs non-linear: examples

Consider $L[y] = \frac{d^2y}{dx^2} = 0$.
Explicit calculation gives

$$L\left[a_1 y_1 + a_2 y_2\right] = \frac{d^2}{dx^2}\left[a_1 y_1 + a_2 y_2\right] = \frac{d^2}{dx^2}\left[a_1 y_1\right] + \frac{d^2}{dx^2}\left[a_2 y_2\right]$$
$$= a_1 \frac{d^2}{dx^2}\left[y_1\right] + a_2 \frac{d^2}{dx^2}\left[y_2\right] = a_1 L\left[y_1\right] + a_2 L\left[y_2\right],$$

where we used standard differentiation properties.
Thus, we conclude $L[y] = \frac{d^2y}{dx^2} = 0$ is a linear ODE.

Linear vs non-linear: examples

Consider $L[y] = \left(\frac{dy}{dx}\right)^2 = 0$.
Explicit calculation gives

$$L\left[a_1 y_1 + a_2 y_2\right] = \left(\frac{d}{dx}(a_1 y_1 + a_2 y_2)\right)^2 = \left(a_1 y_1' + a_2 y_2'\right)^2$$
$$= a_1^2 (y_1')^2 + a_2^2 (y_2')^2 + 2 a_1 a_2 y_1' y_2' \neq a_1 L\left[y_1\right] + a_2 L\left[y_2\right] .$$

Thus, we conclude $L[y] = \left(\frac{dy}{dx}\right)^2 = 0$ is a non-linear ODE.

Linear vs non-linear: examples

Identifying linear DEs

Linear ODEs or PDEs are those where the functional dependence on $y_i(x_s)$ and its derivatives is linear, i.e. raised to power one.

Most general linear *n*-th order ODE is

$$L[y] \equiv \frac{d^n y}{dx^n} + p_1(x)\,\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)\,y(x) = g(x)\,.$$

- Order: stands for the highest derivative present in the equation,
- $p_1(x), \ldots p_n(x)$, $g(x)$ are arbitrary,
- it is the dependence on $y(x), y'(x), \ldots y^{(n)}$ that determines the linearity of the equation.

Homogeneous vs non-homogeneous

Given the linear ODE

$$L[y] \equiv \frac{d^n y}{dx^n} + p_1(x)\,\frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)\,y(x) = g(x)\,.$$

- if $g(x) = 0$: homogeneous
- if $g(x) \neq 0$: non-homogeneous

2nd order linear ODEs : a reminder

Given a 2nd order linear ODE

$$y'' + p_1(x)\, y' + p_2(x)\, y = g(x)\,, \tag{1}$$

the general solution is given by

$$y_{\text{gen}}(x) = y_{\text{hom}}(x) + y_{\text{par}}(x)$$

where $y_{\text{hom}}(x)$ is the general solution to the homogeneous equation, i.e.

$$y''_{\text{hom}} + p_1(x)\, y'_{\text{hom}} + p_2(x)\, y_{\text{hom}} = 0\,,$$

and $y_{\text{par}}(x)$ is a particular solution to (1).

2nd order linear ODEs : a reminder

We can compute

- y_{hom} for constant coefficients ODEs,

$$y'' + by' + cy = 0,$$

  as a combination of two exponentials: $y(x) = \mathrm{e}^{kx}$.
- y_{par} for constant coefficients and simple (exp, polynomials, trig) rhs $g(x)$ using the method of undetermined coefficients,
- y_{par} for arbitrary $g(x)$ using the method of variations of parameters.

We start by generalising these methods to higher-order linear ODEs.

Honours Differential Equations

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Lecture 2

September 20, 2018

n-th order linear ODEs

Definition

The most general n-th order linear ODE is written as

$$L[y] = \frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)\,y = g(x)$$

Remark: the functions $p_j(x)$ do NOT have to be linear, i.e.

$$p_j(x) = x^3,\ \log x,\ \tfrac{x+1}{x^2+1}\,,\ \ldots$$

Questions:

1. what is its general solution?
2. when is it unique?
3. how do we find them?

n-th order linear ODEs

Theorem

Given an n-th order linear ODE

$$L[y] = \frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)\,y = g(x)$$

with $p_i(x)$ and $g(x)$ continuous $\forall\, x \in [\alpha,\, \beta]$, $\exists$ unique solution satisfying the initial conditions

$$y(x_0) = y_0,\; y'(x_0) = y_0',\quad \ldots \quad y^{(n-1)}(x_0) = y_0^{(n-1)}, \qquad x_0 \in [\alpha,\, \beta]$$

Remark

- extends to arbitrary n result seen for $n = 2$.

Remarks on uniqueness

Theorem uses initial conditions, not other types.
For example

$$y'' + y = 0 \quad \Rightarrow \quad y(t) = c_1 \cos t + c_2 \sin t$$

Imposing that $y(0) = 1$ and $y(\pi) = a$ gives

$$y(0) = 1 \Rightarrow c_1 = 1,$$
$$y(\pi) = a \Rightarrow c_1 = -a.$$

Hence,

- if $a \neq -1$: no solution exists
- if $a = -1$: infinite solutions exist

This is an example of a boundary value problem.

Homogenenous n-th order linear ODEs

Consider

$$L[y] = \frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)\,y = 0$$

Observation

- Assume you are given a set of solutions $\{y_j\}$, i.e. $L[y_j] = 0 \;\forall\, j$

$$\sum_j c_j y_j \text{ is also a solution: } L[\sum_j c_j y_j] = \sum_j c_j L[y_j] = 0$$

Solutions y_j to homogeneous ODEs can be summed and multiplied by (complex) numbers and remain solutions $\Rightarrow$ they behave like vectors.

In fact, the space of solutions of an homogeneous linear ODE is a vector space (see more in the Linear Algebra course in semester 2)

Homogenenous n-th order linear ODEs

Question: Can any solution be written as a linear combination of the set $\{y_j\}$?
Answer: For vectors, we know it is true if the set $\{y_j\}$ forms a basis. In particular, $\{y_j\}$ must be linearly independent.

Definition

The functions $\{y_j\}$ form a fundamental set of solutions if their Wronskian $W[y_1, \dots y_n] \neq 0$, where

$$W[y_1, \dots y_n] \equiv \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}.$$

Note: $W[y_1, y_2] = y_1 y_2' - y_2 y_1'$ (determinant).

Homogenenous n-th order linear ODEs

Properties:

1. If $W[y_1, \dots y_n](x_0) \neq 0$, then $W[y_1, \dots y_n](x) \neq 0$ for all $x \in [\alpha, \beta]$.
2. Any solution $y(x)$ can be written as a linear combination of any fundamental set of solutions $\{y_i(x)\}$.
3. The solutions $\{y_i(x)\}$ form a fundamental set iff they are are linearly independent, i.e.,
$c_1 y_1(x) + \cdots + c_n y_n(x) = 0, \forall x \;\; \Rightarrow \;\; (c_1, \cdots, c_n) = (0, \cdots, 0)$.

A fundamentals set forms a basis for the space of solutions.

Remark: $W[f_1, \cdots, f_n] \neq 0$ does not imply that $\{f_i\}$ are linearly independent when the f_i do not satisfy the ODE.

Non-homogenenous n-th order linear ODEs

Consider

$$L[y] = \frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)\,y = g(x)$$

Assume $Y_1(x)$ and $Y_2(x)$ solve this, then

$$L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = g(x) - g(x) = 0 \quad \Rightarrow \quad Y_1 - Y_2 = \sum_j c_j y_j$$

Thus, the general solution we are after is of the form

$$y_{\text{gen}} = \sum_j c_j y_j + y_{\text{par}} = y_{\text{hom}} + y_{\text{par}}$$

Summary: n-th order linear ODEs share the same features as 2nd order linear ODEs.
Question: how do we find these solutions?

2nd order with constant coefficients: a reminder

Consider

$$y'' + p\, y' + q\, y = 0\,, \qquad p, q \text{ constants} \tag{1}$$

Observation: if $y(x) = e^{kx} \Rightarrow y^{(n)} = k^n\, y$

Plugging this guess into (1):

$$e^{kx}\left(k^2 + pk + q\right) = 0 \;\; \forall\, x \;\; \Rightarrow \;\; k_{\pm} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

Cases:

1. if $\frac{p^2}{4} > q \Rightarrow k_{\pm}$ real & different $\Rightarrow y_{\text{hom}} = c_+\, e^{k_+ x} + c_-\, e^{k_- x}$
2. if $\frac{p^2}{4} < q \Rightarrow k_{\pm} = \alpha \pm i\beta$ complex conjugate
   $\left(\alpha = -p/2\,,\; \beta = \sqrt{q - p^2/4}\right)$
   $\Rightarrow y_{\text{hom}} = e^{\alpha x}\left(c_1\, e^{i\beta x} + c_2\, e^{-i\beta x}\right)$
   $\Rightarrow y_{\text{hom}} = e^{\alpha x}\left(d_1\, \cos \beta x + d_2 \sin \beta x\right)$
3. if $\frac{p^2}{4} = q \Rightarrow k_+ = k_- \Rightarrow y_{\text{hom}} = c_+\, e^{k_+ x}$ only one solution.

2nd order with constant coefficients: a reminder

Question: how do we find the missing solution?

Reduction of order (d'Alembert): consider

$$
\begin{aligned}
y_2(x) &= e^{-px/2}\, u(x) \\
&\Rightarrow y_2' = e^{-px/2}\left(u' - \frac{p}{2}u\right) \\
&\Rightarrow y_2'' = e^{-px/2}\left(u'' - p\,u' + \frac{p^2}{4}u\right)
\end{aligned}
$$

Plugging this back into (1), we obtain

$$e^{-px/2}\, u'' = 0 \Rightarrow u(x) = c_1 x + c_2 \Rightarrow y_2(x) = e^{-px/2}\,(c_1 x + c_2)$$

Observations:

1. $c_1 = 0$ reproduces the first solution
2. $c_1 \neq 0$ gives rise to a new solution [check it is linearly independent].
3. When $k_+ = k_- = -p/2$, $y_{\text{hom}} = e^{-px/2} M_1(x)$ where $M_1(x)$ is the most general polynomial of degree the order of the ODE minus 1

n-th order with constant coefficients

Consider

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n\, y = 0 \qquad a_j \text{ constants}$$

As before: introduce $y(x) = e^{kx}$

$$L[e^{kx}] = e^{kx} \left(a_0\, k^n + a_1\, k^{n-1} + \cdots + a_{n-1}\, k + a_n\right) = 0$$

Homogeneous ODE problem $\Leftrightarrow$ finding roots (k) of characteristic polynomial

$$\begin{aligned} Z(k) &\equiv \left(a_0\, k^n + a_1\, k^{n-1} + \cdots + a_{n-1}\, k + a_n\right) = 0 \\ &= a_0 (k - k_1) \ldots (k - k_n) = 0 \end{aligned}$$

using the fundamental theorem of algebra to factorise $Z(k)$.

n-th order with constant coefficients

The nature of the solutions depends on whether the roots $\{k_j\}$ are real or complex and different or repeated

Real & different roots: General solution

$$y_{\text{hom}} = \sum_{j=1}^{n} c_j \, e^{k_j x}$$

The functions $\{e^{k_j x}\}$ form a fundamental set because $k_i \neq k_j$ when $i \neq j$.

Example:

$$y^{(4)} + y^{(3)} - 7y'' - y' + 6y = 0$$

Consider $y = e^{kx} \Rightarrow k^4 + k^3 - 7k^2 - k + 6 = 0$ (char. eq.)
Observe $k = 1$ is a root and divide (synthetic division, Ruffini–Horner),

$$k^4 + k^3 - 7k^2 - k + 6 = (k-1)(k^3 + 2k^2 - 5k - 6)$$

Observe $k = -1$ is a root and divide,

$$\begin{aligned} k^4 + k^3 - 7k^2 - k + 6 &= (k-1)(k+1)(k^2 + k - 6) \\ &= (k-1)(k+1)(k-2)(k+3) = 0. \end{aligned}$$

General solution:

$$y_{\text{hom}} = c_1\, e^{1x} + c_2\, e^{-1x} + c_3\, e^{2x} + c_4\, e^{-3x}$$

n-th order with constant coefficients

Complex different roots:
If $Z(k) = 0$ has a complex root $k = \lambda + i\mu$, its complex conjugate $\lambda - i\mu$ must also be a root: complex roots come in pairs because $Z(k)$ is real valued.

We can write the general solution as a complex function

$$y_{\text{hom}} = c_1\, e^{(\lambda + i\mu)x} + c_2\, e^{(\lambda - i\mu)x} + \dots\,,$$

or as a real function

$$y_{\text{hom}} = e^{\lambda x}\,(d_1\, \cos \mu x + d_2\, \sin \mu x) + \dots$$

The connection between the two is obtained using Euler's formula

$$e^{i\alpha} = \cos \alpha + i \sin \alpha\,.$$

Example:

$$y^{(4)} - y = 0$$

Consider $y = e^{kx} \Rightarrow k^4 - 1 = 0$ (char. eq.)
Notice

$$k^4 - 1 = (k^2 - 1)(k^2 + 1) \Rightarrow k = \pm i\,, \pm 1$$

Since all roots are different, the general solution is

$$y_{\text{hom}} = a_1\, e^x + a_2\, e^{-x} + a_3 \cos x + a_4 \sin x\,.$$

Honours Differential Equations

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Lecture 3

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n-th order with constant coefficients

Repeated roots:

Definition

Let us refer to the number of times s a given root k_1 solves the characteristic equation $Z(k) = 0$ as having multiplicity s, i.e.

$$Z(k) = (k - k_1)^s \, Y(k) = 0 \, .$$

When $n = 2$, the multiplicity is either 0, 1 or 2.
We proved that when it is 2 (repeated root), the general solution involved the most general polynomial of degree 1.

In general, the degree of the polynomial must be $s - 1$.

n-th order with constant coefficients

Repeated real roots:

Assume k_1 is real, with multiplicity s and the remaining $n - s$ roots are different.

The general solution is given by

$$y_{\mathrm{hom}} = e^{k_1 x}\left(c_0 + c_1\, x + c_2\, x^2 + \cdots + c_{s-1} x^{s-1}\right) + \sum_{j \neq 1} a_j\, e^{k_j x}\,.$$

Repeated complex roots:

Assume $k_1 = \lambda + i\mu$ has multiplicity s $\Rightarrow \lambda - i\mu$ has multiplicity s

In terms of real solutions $\Rightarrow$ both $\cos \mu x$ and $\sin \mu x$ will be multiplied by polynomials of degree $s - 1$

The general solution is given by

$$\begin{aligned} y_{\mathrm{hom}} = e^{\lambda x}\Big[&\left(c_0 + c_1\, x + \ldots c_{s-1} x^{s-1}\right)\cos \mu x \\ &+ \left(d_0 + d_1\, x + \ldots d_{s-1} x^{s-1}\right)\sin \mu x\Big] + \ldots \end{aligned}$$

Notice both polynomials are independent.

Example: directly,

$$y^{(4)} + 2y'' + y = 0$$

Consider $y = e^{kx} \Rightarrow k^4 + 2k^2 + 1 = (k^2+1)^2 = 0$
Thus $k = \pm i$, both having multiplicity 2
General solution:

$$y_{\text{hom}} = (c_0 + c_1\,x)\,\cos x + (d_0 + d_1\,x)\,\sin x$$

Summary

1. The general solution to an n-th linear ODEs is given by

$$y_{\text{gen}} = y_{\text{hom}} + y_{\text{par}} = \sum_{j=1}^{n} c_j y_j + y_{\text{par}}$$

   where $W[y_1, \dots y_n] \neq 0$ to make the set of functions $\{y_j\}$ linearly independent

2. Homogeneous ODEs with constant coefficients: use $y = e^{kx}$
   The problem is mapped to finding the roots of the characteristic polynomial
   - roots can be real or complex and different or with multiplicity s
   - when the multiplicity is larger than 1, consider $y = e^{kx} M_{s-1}(x)$, where $M_{s-1}(x)$ is the most general polynomial of degree $s - 1$.

n-th order linear non-homogeneous ODEs

We want to solve particular solutions to equations

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(x) \qquad a_j \text{ constants}$$

We will discuss two methods:

1. Method of undetermined coefficients
2. Method of variation of parameters

Generalisation of methods seen for 2nd order ODEs.

Method of undetermined coefficients

- If $g(x)$ is a polynomial, exponential, trigonometric function or products of these, this method can be efficient
- $\exists$ cases that violate the general rule

Let us discuss the rule and the exceptions in an example:

$$y^{(3)} - 3y'' + 3y' - y = 4e^{ax} \qquad g(x) \equiv 4e^{ax}$$

First, solve the homogeneous problem

$$y = e^{kx} \Rightarrow (k-1)^3 = 0 \Rightarrow y_{\text{hom}} = e^{x}\left(c_1 + c_2\, x + c_3\, x^2\right)$$

Method of undetermined coefficients

Second, look for a particular solution of the form

$$y_{\text{par}} = A\, e^{ax}$$

Remark: we use the same function as $g(x)$, but we allow the constant multiplying it to be arbitrary
Question: does this trick work for any a?
Let us check: $y_{\text{par}}^{(n)} = a^n\, y_{\text{par}}$ + plugging in

$$A\, e^{ax}\,(a-1)^3 = 4e^{ax} \Rightarrow A = \frac{4}{(a-1)^3} \quad a \neq 1$$

Lesson: if the argument a in $g(x)$ is not a root of the characteristic equation, the method works.

Method of undetermined coefficients

Question: can we extend the method when $a = 1$?
Idea:

- the reason why the method fails is because $A\,e^x$ solves the homogeneous ODE.
- the same is true for $e^x(c_1 + c_2 x + c_3 x^2)$
- let us try

$$y_{\text{par}} = A\,x^3\,e^x$$

After some algebra, you discover

$$L[A\,x^3\,e^x] = 6A\,e^x = 4e^x \Rightarrow A = \frac{2}{3}$$

Method of undetermined coefficients

Summary: The general solution to

$$y^{(3)} - 3y'' + 3y' - y = 4e^{ax} \qquad g(x) \equiv 4e^{ax}$$

is

- when $a \neq 1$

$$y_{\text{gen}} = e^{x}\left(c_1 + c_2\,x + c_3\,x^2\right) + \frac{4}{(a-1)^3}\,e^{ax}$$

- when $a = 1$

$$y_{\text{gen}} = e^{x}\left(c_1 + c_2\,x + c_3\,x^2\right) + \frac{2}{3}x^3\,e^{x}$$

Method of undetermined coefficients

Extensions: $M_n(x)$ polynomial of degree n

$$y^{(3)} - 3y'' + 3y' - y = M_n(x)\, e^{ax} \qquad g(x) \equiv M_n(x)\, e^{ax}$$

- when $a \neq 1$

$$y_{\text{par}} = \left(A_n + A_{n-1}x + \cdots + A_1\, x^{n-1} + A_0\, x^n\right)\, e^{ax}$$

- when $a = 1$

$$y_{\text{gen}} = x^3\, \left(A_n + A_{n-1}x + \cdots + A_1\, x^{n-1} + A_0\, x^n\right)\, e^{x}$$

Method of undetermined coefficients: Example

$$y^{(3)} - 4y' = x + 3\cos x + e^{-2x}$$

1. Homogeneous solution: e^{kx}

$$k(k^2-4) = 0 \Rightarrow y_{\text{hom}} = c_1 + c_2\, e^{2x} + c_3\, e^{-2x}$$

2. What trial function should we use for y_{par}?
   - by linearity, we can break it into three pieces

     $$y_{\text{par}} = y_a + y_b + y_c$$

     one per each term in $g(x) = x + 3\cos x + e^{-2x}$
   - Consider $g(x) = x = x\, e^{0x}$. Since $k = 0$ is a root of characteristic equation with multiplicity 1

     $$y_a = x(Ax + B)$$

     Remark: the polynomial is one degree higher because of the multiplicity.

Method of undetermined coefficients: Example

- Consider $g(x) = 3\cos x$. There are no complex roots

$$y_b = C\cos x + D\sin x$$

Why? Because $\cos x$ and $\sin x$ both come from e^{ix}

- Consider $g(x) = e^{-2x}$. Since $k = -2$ is a root of multiplicity 1

$$y_c = E\,x\,e^{-2x}$$

Remark: the polynomial is one degree higher because of the multiplicity.

What remains to be done is to determine the constants $\{A, B, C, D, E\}$ by introducing our trial functions into the original ODE

$$y_{\text{par}} = -\frac{1}{8}x^2 - \frac{3}{5}\sin x + \frac{1}{8}x\,e^{-2x}$$

Method of variation of parameters

This is algebraically lengthy but it always works
Consider the non-homogeneous ODE

$$L[y] = g(x)\,.$$

Assume $y_{\text{hom}} = \sum_{j=1}^{n} c_j y_j(x)$ is the general solution to the associated homogeneous ODE.
Idea: Look for a particular solution of the form

$$y_{\text{par}} = \sum_{j=1}^{n} u_j(x) y_j(x)$$

i.e. replace the constants $\{c_j\}$ but a set of unknown functions $\{u_j(x)\}$
Intuition: n unknown functions but there is a single ODE.

Need constraints to avoid underdetermined system.

Method of variation of parameters

When computing derivatives of the guess $\sum_j u_j(x)y_j(x)$, we obtain

$$y' = \sum_j \left(u_j' y_j + u_j y_j'\right)$$

Idea: set the u_j' terms to zero, i.e.

$$\sum_j u_j' y_j = 0 \tag{1}$$

Remember $\{y_j\}$ are known
(1) is a linear ODE !! for the unknowns $\{u_j\}$
But we have n such functions: can we get more constraints?

Method of variation of parameters

If (1) holds, $y' = \sum_j u_j y_j'$, thus

$$y'' = \sum_j \left(u_j' y_j' + u_j y_j'' \right)$$

As before, set to zero the u_j' terms, ie.

$$\sum_j u_j' y_j' = 0 \tag{2}$$

This is a different linear ODE for the same set of unknowns $\{u_j\}$. This suggests a procedure that will give rise to a set of n linear ODEs that will determine $\{u_j\}$.

Method of variation of parameters

Procedure: iterate the previous manipulations $n - 1$ times
This gives rise to $n - 1$ linear ODEs

$$\sum_j u_j' y_j^{(m-1)} = 0 \qquad m = 1, 2, \dots, n-1$$

We are missing one equation, but ...
we have not used the ODE we want to solve !!
Compute

$$y^{(n)} = \sum_j \left(u_j y_j^{(n)} + u_j' y_j^{(n-1)} \right)$$

and introduce into

$$L[y] = y^{(n)} + p_1(x) y^{(n-1)} + \cdots + p_{n-1}(x) y' + p_n(x) y = g(x)$$

Method of variation of parameters

Let us substitute carefully

$$
\begin{aligned}
L[y] = \sum_j \left(u_j y_j^{(n)} + u_j' y_j^{(n-1)} \right) \\
+ p_1(x) \sum_j u_j y_j^{(n-1)} + \cdots + p_{n-1}(x) \sum_j u_j y_j' \\
+ p_n(x) \sum_j u_j y_j = g(x)
\end{aligned}
$$

Notice we can sum all terms involving no derivatives of u_j

$$
\begin{aligned}
L[y] &= \sum_j u_j' y_j^{(n-1)} + \sum_j u_j \left(y_j^{(n)} + p_1(x) y_j^{(n-1)} + \cdots + p_n(x) y_j \right) \\
&= \sum_j u_j' y_j^{(n-1)} + \sum_j u_j L[y_j] = \sum_j u_j' y_j^{(n-1)} + 0 = g(x)
\end{aligned}
$$

This gives the last linear ODE to solve for $\{u_j'\}$

Method of variation of parameters

To sum up, we have derived n linear constraints

$$\sum_j u_j' \, y_j^{(m-1)} = 0 \qquad m = 1, 2, \ldots, n-1$$

$$\sum_j u_j' \, y_j^{(n-1)} = g(x)$$

Using linear algebra, we solve for $\{u_j'\}$

Remark: the solution is unique because the matrix of coefficients has determinant $= W[y_1, \ldots y_n] \neq 0$ since we assumed linearly independent $\{y_j\}$.

Method of variation of parameters

Using Cramer's rule we can write a compact solution

$$u_j' = g(x)\frac{W_j[x]}{W[y_1, \dots y_n]}$$

where $W_j[x]$ stands for the determinant of the matrix where we replace the j-th column in the Wronskian by the column vector $(0, 0, \dots 1)$ Finally, we integrate the obtained 1st order ODE giving rise to an overall solution

$$y_{\mathrm{par}} = \sum_j y_j(x) \int_{x_0}^{x} g(s)\, \frac{W_j[s]}{W[y_1(s), \dots y_n(s)]}\, ds\,.$$

Remark: Notice the method works for non-constant coefficients

Method of variation of parameters: Example

Consider

$$y^{(3)} - y'' - y' + y = g(x)$$

The general solution $y_{\text{gen}} = y_{\text{hom}} + y_{\text{par}}$
Homogeneous solution obtained as usual: plug e^{kx}

$$k^3 - k^2 - k + 1 = (k-1)^2(k+1) = 0 \Rightarrow y_{\text{hom}} = e^x\,(c_1 + c_2\,x) + c_3\,e^{-x}$$

Let us introduce some notation (to match our abstract general discussion)

$$y_1 = e^x\,, \quad y_2 = x\,e^x\,, \quad y_3 = e^{-x}$$

Particular solution:

$$y_{\text{par}} = u_1(x)y_1(x) + u_2(x)y_2(x) + u_3(x)y_3(x)$$

Question: how do we determine $\{u_1,\, u_2,\, u_3\}$?

Method of variation of parameters: Example

We use the set of constraints derived in our discussion:

1. Our ODE is 3rd order $\Rightarrow 3-1=2$ constraints of the form

$$\sum_j u_j' y_j = u_1'\, e^x + u_2'\, xe^x + u_3'\, e^{-x} = 0$$

$$\sum_j u_j' y_j' = u_1'\, e^x + u_2'\, (x+1)e^x - u_3'\, e^{-x} = 0$$

2. One final constraint coming from the original ODE:

$$\sum_j u_j' y_j'' = u_1'\, e^x + u_2'\, (x+2)e^x + u_3'\, e^{-x} = g(x)$$

Three algebraic equations to solve for the unknowns u_1', u_2', u_3'.

Method of variation of parameters: Example

Use your favourite method to solve the linear system (or Cramer's rule)
Check:

$$u_1' = -g(x)\frac{1+2x}{4e^x}$$
$$u_2' = g(x)\frac{e^{-x}}{2}$$
$$u_3' = g(x)\frac{e^{x}}{4}$$

Thus, the particular solution is given by

$$y_{\text{par}} = -e^x \int g(x)\frac{1+2x}{4e^x}dx + x\,e^x \int g(x)\frac{e^{-x}}{2}dx + e^{-x}\int g(x)\frac{e^x}{4}dx\,.$$

Remark: We ignored the constants of integration because these can always be reabsorbed into the homogeneous solution.

Honours Differential Equations

Jacques Vanneste

Lecture 4

September 24, 2018

Laplace transform: motivation

Previous methods work, but they require to

1. Solve the homogeneous equation (find roots of a polynomial)
2. Find a particular solution: requires
   - to solve a linear system to find undetermined coefficients
   - to solve a linear system to find undetermined functions + integration
3. For specific initial conditions
   - Solve a linear system to fix the constants in the homogeneous solution

Summary: our methods work, but they involve many steps which can be algebraically expensive

Question: can we develop a different strategy to solve these linear ODEs and initial value problems?

Laplace transform: motivation

Idea: instead of working with $y(x)$, we shall map $y(x) \to F(s)$ through

$$F(s) = \int_{\alpha}^{\beta} K(s,x)\, y(x)\, dx \tag{1}$$

This is an example of an integral transform
The function $K(s,x)$ is generically called kernel
Goal: choose $K(s,x)$ appropriately, so that

1. $y(x)$ ODE $\Rightarrow$ algebraic equation for F(s),
2. (1) can be inverted.

Remarks:

1. we study the Laplace transform only;
2. the Fourier transform is a very useful alternative.

Laplace transform: definition

Definition

The Laplace transform *of* $f(x)$ *defined for* $x \in [0, \infty)$ *is*

$$F(s) = \mathcal{L}\{f\}(s) \equiv \int_0^\infty e^{-sx} f(x)\, dx = \lim_{T\to\infty} \int_0^T e^{-sx} f(x)\, dx \tag{2}$$

Examples:

1. $f(x) = 1$ (a constant in general)

$$\lim_{T\to\infty} \int_0^T e^{-sx}\, dx = \lim_{T\to\infty} \frac{1 - e^{-sT}}{s} = \begin{cases} \frac{1}{s} & s > 0 \\ \infty & s \le 0 \end{cases}$$

2. $f(x) = e^{ax}$ for a a real constant

$$\lim_{T\to\infty} \int_0^T e^{-sx}\, e^{ax} dx = \lim_{T\to\infty} \frac{e^{(a-s)T} - 1}{a - s} = \begin{cases} \frac{1}{s-a} & s > a \\ \infty & s \le a \end{cases}$$

Laplace transform: convergence

Remarks:

1. Because of the $T \to \infty$ limit, $\mathcal{L}\{f\}(s)$ may not exist
2. When it does, it is not defined in the same interval as $f(x)$:
   - $f(x) = e^{2x}$ is well defined for $\forall\, x \in [0,\ \infty)$, whereas $F(s)$ only exists for $s \in (2,\ \infty)$

Question: can we identify a set of conditions that guarantee the existence of $F(s)$?

Definition

A real- or complex-valued function $f(x)$ is said to be of exponential type, denoted by $f \in E$, if

- *on any interval $[0,\ T]$ where the function is defined, it is piecewise continuous, i.e. $f(x)$ is continuous except at some finite set of points*
- $|f(x)| \leq Ae^{Bx}\ \forall\, x \in [0,\ \infty)$ *for some constants A and B.*

Laplace transform: convergence

Theorem.

If $f(x) \in E$, then its Laplace transform exists for all s sufficiently large.

Proof.

Since $f(x)$ is piecewise continuous, the integral $F(s)$ exists $\forall\, T$.
The question is whether the limit $T \to \infty$ exists
Observe that since $f(x) \in E$, then

$$|\int_0^T e^{-sx} f(x)\, dx| \leq \int_0^T e^{-sx} |f(x)|\, dx \leq \int_0^T e^{-sx}\, Ae^{Bx}\, dx$$
$$\leq \frac{A}{s-B}$$

for $s > B$. Thus, $F(s)$ exists for $s > B$. $\square$

Laplace transform: further example

Consider $f(x) = \sin(ax)$ for $x \geq 0$.
Then

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-sx} \sin(ax)\, dx \\
&= -e^{-sx} \left.\frac{\cos(ax)}{a}\right|_0^\infty - \frac{s}{a} \int_0^\infty e^{-sx} \cos(ax)\, dx \\
&= \frac{1}{a} - \frac{s}{a} \int_0^\infty e^{-sx} \cos(ax)\, dx \quad \text{by parts \& assumed} \quad s > 0 \\
&= \frac{1}{a} - \frac{s^2}{a^2} \int_0^\infty e^{-sx} \sin(ax)\, dx \quad \text{by parts a second time}
\end{aligned}
$$

Thus,

$$
F(s)\left(1 + \frac{s^2}{a^2}\right) = \frac{1}{a} \Rightarrow F(s) = \frac{a}{s^2 + a^2}\,, \quad s > 0
$$

Laplace transforms: Linearity

The Laplace transform is a linear operation, i.e.

$$\mathcal{L}\{c_1 f_1(x) + c_2 f_2(x)\}(s) = c_1\,\mathcal{L}\{f_1(x)\} + c_2\,\mathcal{L}\{f_2(x)\}\,.$$

This follows from the linearity of standard integration

This is a key property for applications to linear ODEs.

Laplace transform & derivatives of functions

Theorem.

If $f(x)$ is continuous on $[0, \infty)$ and $f,\ f' \in E$ then

$$\mathcal{L}\{f'(x)\} = s\mathcal{L}\{f(x)\} - f(0)\,. \tag{3}$$

Proof.

For simplicity, assume $f'(x)$ is continuous. Using the identity

$$\left(e^{-sx}\, f(x)\right)' = -s\, e^{-sx} f(x) + e^{-sx} f'(x),$$

we can compute the $\mathcal{L}\{f'(x)\}$ using integration by parts

$$\begin{aligned}\mathcal{L}\{f'(x)\} &= \int_0^\infty e^{-sx}\, f'(x)\, dx = \left. e^{-sx}\, f(x)\right|_0^\infty + s\int_0^\infty e^{-sx}\, f(x)\, dx \\ &= -f(0) + s\mathcal{L}\{f(x)\}\end{aligned}$$

where we used $f \in E$ when evaluating the upper limit.

Laplace transform & derivatives of functions

Note: $\mathcal{L}\{f'(x)\}$

- does not involve derivatives,
- it involves initial data $f(0)$.

Higher-order derivatives:

Theorem.

If f, f', $\ldots f^{(n-1)}$ are continuous on $[0, \infty)$ and belong to E then

$$\mathcal{L}\{f^{(n)}(x)\} = s^n \mathcal{L}\{f(x)\} - s^{n-1} f(0) - \ldots s\, f^{(n-2)}(0) - f^{(n-1)}(0)\,.$$

Proof.

$$\begin{aligned}
\mathcal{L}\{f^{(n)}(x)\} &= s\mathcal{L}\{f^{(n-1)}(x)\} - f^{(n-1)}(0) \\
&= s^2 \mathcal{L}\{f^{(n-2)}(x)\} - sf^{(n-2)}(0) - f^{(n-1)}(0) \\
&= s^n \mathcal{L}\{f(x)\} - s^{n-1} f(0) - \ldots s\, f^{(n-2)}(0) - f^{(n-1)}(0).
\end{aligned}$$

$\square$

Laplace transforms and linear constant-coefficient ODEs

Solve the initial value problem:

$$y'' - y' - 2y = 0\,, \qquad y(0) = 1,\ y'(0) = 0$$

Compute the Laplace transform of the ODE: $\mathcal{L}\{y\} \equiv Y(s)$

$$\begin{aligned} \mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} &= 0\,, \\ s^2 Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) - 2Y(s) &= 0\,, \\ (s^2 - s - 2)Y(s) + (1 - s)y(0) - y'(0) &= 0\,. \end{aligned}$$

Using initial conditions

$$Y(s) = \frac{s-1}{(s-2)(s+1)}$$

Laplace transforms and linear constant-coefficient ODEs

Remarks:

- Y(s) is determined algebraically,
- Y(s) encodes the initial conditions
- Initial value problem is solved if we can compute $y = \mathcal{L}^{-1}\{Y(s)\}$

.

(Inversion formula: the inverse of a Laplace transform can be shown to equal

$$y(x) = \mathcal{L}^{-1}\{Y(s)\}(x) \equiv \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{sx}\, Y(s)\, ds$$

for large enough $\gamma > 0$.)
Need to compute integrals in the complex plane – cf 2nd semester Complex Variables – and typically involved.

Instead, invert the Laplace transform by inspection.

Laplace transforms and linear constant-coefficient ODEs

We can decompose

$$Y(s) = \frac{s-1}{(s-2)(s+1)}$$

into simple fractions

$$Y(s) = \frac{a}{s-2} + \frac{b}{s+1} \Rightarrow s-1 = (s+1)a + (s-2)b \Rightarrow a = 1/3,\ b = 2/3.$$

Thus,

$$Y(s) = \frac{1}{3}\frac{1}{s-2} + \frac{2}{3}\frac{1}{s+1}$$

Observe that $\mathcal{L}\{e^{ax}\} = \frac{1}{s-a}$
We can conclude that

$$y(x) = \frac{1}{3}e^{2x} + \frac{2}{3}e^{-x}\,.$$

Laplace transforms and linear constant-coefficient ODEs

The same discussion holds for any n-th order linear ODE with constant coefficients. For example, consider

$$ay'' + by' + cy = f(x)\,.$$

Compute its Laplace transform: $\mathcal{L}\{y\} \equiv Y(s)$, $\mathcal{L}\{f\} \equiv F(s)$

$$a\left(s^2 Y - sy(0) - y'(0)\right) + b\left(sY - y(0)\right) + cY = F$$
$$\Rightarrow Y = \frac{F}{as^2 + bs + c} + \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c}$$

1. Y(s) determined algebraically
2. Dependence on $y(0)$, $y'(0)$ taken into account
3. We need to learn efficient methods to compute $y = \mathcal{L}^{-1}\{Y(s)\}$ (next lecture)

Example

Consider the non-homogeneous initial value problem:

$$y'' + y = \sin(2x)\,, \qquad y(0) = 2\,,\ \ y'(0) = 1$$

Compute its Laplace transform:

$$s^2 Y - sy(0) - y'(0) + Y = \frac{2}{s^2+4}$$

$$\Rightarrow Y = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}$$

Decompose into simple fractions:

$$\frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4}$$

After some algebra:

$$Y(s) = \frac{2s}{s^2+1} + \frac{5}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{2}{s^2+4}$$

$$\Rightarrow y(x) = \frac{5}{3}\sin x - \frac{1}{3}\sin 2x + 2\cos x$$

Honours Differential Equations

Jacques Vanneste

Lecture 5

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Laplace transforms: standard formulas

$f(x)$	$F(s) = \mathcal{L}(f)(s)$
1	$1/s$
e^{at}	$1/(s-a)$
$\sin(ax)$	$a/(s^2+a^2)$
$\cos(ax)$	$s/(s^2+a^2)$
$\dots$	$\dots$

(See table 6.2.1 in Boyce & DiPrima for more.)
These formulas are complemented by properties of $\mathcal{L}$ which enrich the catalogue of simple $F(s)$.

Laplace transforms: properties

Theorem

Let $\mathcal{L}\{f(x)\}(s) = F(s)$*. For* $f \in E$*,*

1. *s-shift:* $\mathcal{L}\{e^{-cx} f(x)\}(s) = F(s+c)\,.$
2. *x-shift:* $\mathcal{L}\{f(x-c)\}(s) = e^{-sc} F(s)$ *if* $c \geq 0$ *and* $f(x) = 0$ *for* $x < 0\,.$
3. *s-derivative:* $\mathcal{L}\{x\, f(x)\}(s) = -F'(s)\,.$
4. *scaling:* $\mathcal{L}\{f(cx)\}(s) = \frac{1}{c} F\left(\frac{s}{c}\right),\ F(sc) = \frac{1}{c}\mathcal{L}\{f(x/c)\}$ *if* $c > 0$.

Laplace transforms: Properties

Proof.

1. It follows from

$$\int_0^\infty e^{-sx}\, e^{-cx}\, f(x)\, dx = \int_0^\infty e^{-(s+c)x}\, f(x)\, dx\,.$$

2. Let $u = x - c$. Then,

$$\begin{aligned}\int_0^\infty e^{-sx}\, f(x-c)\, dx &= \int_{-c}^\infty e^{-s(u+c)}\, f(u)\, du \\ &= \int_0^\infty e^{-sc}\, e^{-su}\, f(u)\, du\end{aligned}$$

Notice the limit $(-c, \infty)$ can be changed to $(0, \infty)$ by hypothesis.

□

Laplace transforms: Properties

Proof.

3. Notice that
$$\frac{d}{ds}\int_0^\infty e^{-sx}\, f(x)\, dx = \int_0^\infty -x\, e^{-sx}\, f(x)\, dx\,.$$
4. By change of variables. For example, $u = cx$, then
$$\int_0^\infty e^{-sx}\, f(cx)\, dx = \frac{1}{c}\int_0^\infty e^{-su/c}\, f(u)\, du = \frac{1}{c}F\left(\frac{s}{c}\right)\,.$$

$\square$

Examples

1. Laplace transform of xe^x.
   Remember $\mathcal{L}\{e^x\} = (s-1)^{-1}$. Applying s-derivative we learn that
   $$\mathcal{L}\{xe^x\}(s) = -\left(\frac{1}{s-1}\right)' = \frac{1}{(s-1)^2}$$
2. Laplace transform of $e^{3x}\sin x$.
   Remember $\mathcal{L}\{\sin x\} = (s^2+1)^{-1}$. Using the s-shift property,
   $$\mathcal{L}\{e^{3x}\sin x\} = \frac{1}{(s-3)^2+1}$$

Generalisations

1. Using s-shift,

$$\mathcal{L}\{e^{-cx}\cos(bx)\} = \frac{s+c}{(s+c)^2+b^2}$$
$$\mathcal{L}\{e^{-cx}\sin(bx)\} = \frac{b}{(s+c)^2+b^2}\,.$$

2. Extension of the s-derivative property:

$$\mathcal{L}\{x^n f(x)\} = \int_0^\infty e^{-sx} x^n f(x)\,dx$$
$$= (-1)^n \frac{d^n}{ds^n}\int_0^\infty e^{-sx} f(x)\,dx = (-1)^n F^{(n)}(s)\,.$$

Generalisations

Joining both results

$$\mathcal{L}\{x^n\, e^{ax} \cos bx\} = (-1)^n \frac{d^n}{ds^n} \left(\frac{s-a}{(s-a)^2 + b^2} \right)$$
$$\mathcal{L}\{x^n\, e^{ax} \sin bx\} = (-1)^n \frac{d^n}{ds^n} \left(\frac{b}{(s-a)^2 + b^2} \right) .$$

Remark: All these Laplace transforms are rational functions.

n-th linear ODEs & Laplace transforms

Consider an n-th order non-homogeneous ODE with constant coefficients

$$L[y] = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = f \in E$$

Compute its Laplace transform, using $\mathcal{L}\{y\}(s) = Y(s)$ and $\mathcal{L}\{f(x)\}(s) = F(s)$, then

$$Z(s)Y(s) = F(s) + Z_0(s)$$

where $Z(s)$ is the characteristic polynomial and $Z_0(s)$ is a polynomial of degree $\leq n-1$ and depending on initial conditions
Thus,

$$Y(s) = \frac{F(s)}{Z(s)} + \frac{Z_0(s)}{Z(s)}$$

When $f(x) = x^n e^{ax} (A \cos bx + B \sin bx)$, $F(s)$ is a rational function $\Rightarrow$ Y(s) is a rational function $\Rightarrow$ decompose into simple fractions to find $y(x) = \mathcal{L}^{-1}(Y(s))(x)$

Example

Solve the initial value problem

$$y'' - 2y' + 2y = 2e^x \qquad y(0) = 0\,,\ \ y'(0) = 1$$

Computing the Laplace transform we obtain

$$s^2 Y - 1 - 2sY + 2Y = 2\frac{1}{s-1} \Rightarrow Y = \frac{s+1}{(s-1)(s^2-2s+2)}$$

Decompose into simple fraction

$$\begin{aligned}\frac{s+1}{(s-1)(s^2-2s+2)} &= \frac{2}{s-1} + \frac{-2s+3}{s^2-2s+2} \\ &= \frac{2}{s-1} - 2\frac{s-1}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}\end{aligned}$$

We conclude $y(x) = 2e^x - 2e^x \cos x + e^x \sin x$.

Discontinuous functions

All our Laplace results hold for piecewise continuous functions
$\Rightarrow$ apply them to solve ODEs involving piecewise sources in non-homogeneous ODEs, i.e.

$$L[y] = g(x) \qquad g(x) \text{ piecewise continuous}$$

Piecewise continuous functions arise in applications:
Example:

- consider an electric circuit that is connected to an external motor in $[t_0, t_1]$
- we need a function that vanishes for $t \leq t_0$ and $t \geq t_1$, and equals the force provided by the motor only $t \in (t_0, t_1)$

Unit step function (Heaviside unit function)

Definition

The unit step function is defined as

$$u_0(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

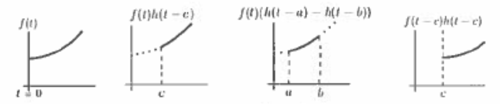
Notice this function has a discontinuity at $t = 0$
More generally, we can place the discontinuity at any $t = c$

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

Using the unit step function

Given a function $f(t)$ defined for $t \geq 0$,

- $$f(t)u_c(t) = \begin{cases} f(t) & \text{for } t \geq c, \\ 0 & \text{for } t < c, \end{cases}$$
- If $0 \leq a < b$,
$$f(t)(u_a(t) - u_b(t)) = \begin{cases} f(t) & \text{for } t \in [a, b), \\ 0 & \text{for } t \notin [a, b), \end{cases}$$
- If $g(t) = f(t - c)u_c(t)$, is a shift of $f(t)$ by $c > 0$ to the right, set to 0 for $t < c$.



Laplace transforms & the unit step function

Laplace transform of the Unit step function ($c \geq 0$)

$$\mathcal{L}\{u_c(t)\}(s) = \int_0^\infty e^{-st}\, u_c(t)\, dt = \int_c^\infty e^{-st}\, dt = \frac{e^{-sc}}{s} \quad s > 0\,.$$

Connection to the t-shift property of the Laplace transform
Remember *t-shift*

$$\mathcal{L}\{f(t-c)\}(s) = e^{-sc}\, F(s) \text{ if } c \geq 0 \text{ and } f(t) = 0 \text{ for } t < 0$$

Functions $g(t)$ satisfying the conditions above can be nicely described as $f(t-c)u_c(t)$. Thus

$$\mathcal{L}\{f(t-c)u_c(t)\}(s) = e^{-sc}\, F(s) \;\; c \geq 0$$

where $\mathcal{L}\{f(t)\}(s) = F(s)$.

Example

Inverse Laplace transform of $F(s) = \frac{1-e^{-2s}}{s^2}$.
First, decompose $F(s)$:

$$F(s) = \frac{1}{s^2} - \frac{e^{-2s}}{s^2}\,.$$

By linearity,

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\right] = t - \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\right]$$

where we used $\mathcal{L}\{t\}(s) = \frac{1}{s^2}$
By the t-shift property, we can conclude

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2}\right] = (t-2)u_2(t)\,.$$

Thus,

$$\mathcal{L}^{-1}[F(s)] = t - (t-2)u_2(t)\,.$$

Solving non-homogeneous ODEs

Solve the initial value problem for $t \geq 0$:

$$y'' - 3y' + 2y = f(t) \qquad y(0) = 0, \quad y'(0) = 0\,,$$

where

$$f(t) = \begin{cases} 1\,, & t \in [0,\,1),\ [2,3),\ [4,5) \\ 0\,, & t \in [1,\,2),\ [3,4),\ [5,\,\infty) \end{cases}$$

1. Compute Laplace transform: $\mathcal{L}\{y\} = Y(s)$ and $\mathcal{L}\{f\} = F(s)$

$$(s^2 - 3s + 2)\,Y(s) = F(s) \Rightarrow Y(s) = \frac{F(s)}{(s-1)(s-2)}$$

2. What is F(s)?
   Notice

$$f(t) = u_0(t) - u_1(t) + u_2(t) - u_3(t) + u_4(t) - u_5(t)$$

   Using linearity,

$$F(s) = \frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \frac{e^{-4s}}{s} - \frac{e^{-5s}}{s}$$

Solving non-homogeneous ODEs

Thus,

$$Y(s) = \frac{1}{s(s-1)(s-2)} \left(1 - e^{-s} + e^{-2s} - e^{-3s} + e^{-4s} - e^{-5s}\right) .$$

3. Find the inverse Laplace transform
   Use partial fraction decomposition,

$$\frac{1}{s(s-1)(s-2)} = \frac{1}{2}\frac{1}{s} - \frac{1}{s-1} + \frac{1}{2}\frac{1}{s-2} = \mathcal{L}\{\frac{1}{2} - e^{t} + \frac{1}{2}e^{2t}\}$$

   Using t-shift,

$$y(t) = \sum_{k=0}^{5} (-1)^k y_0(t-k)\, u_k(t) \qquad y_0(t) = \frac{1}{2} - e^{t} + \frac{1}{2}e^{2t} .$$

Honours Differential Equations

Jacques Vanneste

Lecture 6

September 28, 2018

ODEs with discontinuous forcing

Example: solve the initial-value problem

$$2y'' + y' + 2y = u_5(t) - u_{20}(t) = \begin{cases} 1\,, & 5 \le t < 20 \\ 0\,, & 0 \le t < 5 \text{ and } t \ge 20 \end{cases}$$

$$y(0) = y'(0) = 0\,.$$

Take the Laplace transform:

$$2s^2Y - 2sy(0) - 2y'(0) + sY - y(0) + 2Y = \frac{e^{-5s} - e^{-20s}}{s}$$

$$\Rightarrow Y = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}$$

Recall: $e^{-sc}\,G(s) = \mathcal{L}\{g(t-c)u_c(t)\}(s)$.

$$Y(s) = \left(e^{-5s} - e^{-20s}\right)\,G(s)$$

$$\Rightarrow y(t) = u_5(t)g(t-5) - u_{20}(t)g(t-20) \;\text{ with }\; g(t) = \mathcal{L}^{-1}\left[G(s)\right]$$

ODEs with discontinuous forcing

Decomposing into simple fractions :

$$G(s) = \frac{a}{s} + \frac{bs + c}{2s^2 + 2 + 2} = \frac{1/2}{s} - \frac{s + 1/2}{2s^2 + s + 2}$$

Trick: to deal with 2nd term, try to complete the square in the denominator

$$\frac{s + 1/2}{2s^2 + s + 2} = \frac{1}{2} \frac{(s + 1/4) + 1/4}{(s + 1/4)^2 + 15/16}$$

Using the results

$$\mathcal{L}\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2}, \qquad \mathcal{L}\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2 + b^2},$$

we derive

$$g(t) = \frac{1}{2} - \frac{1}{2} \left(e^{-t/4} \cos \frac{\sqrt{15}}{4} t + \frac{1}{\sqrt{15}} e^{-t/4} \sin \frac{\sqrt{15}}{4} t \right).$$

ODEs with discontinuous forcing

Solution

$$y(t) = u_5(t)g(t-5) - u_{20}(t)g(t-20)$$

$$g(t) = \frac{1}{2} - \frac{1}{2}\left(e^{-t/4}\cos\frac{\sqrt{15}}{4}t + \frac{1}{\sqrt{15}}e^{-t/4}\sin\frac{\sqrt{15}}{4}t\right)$$

1. $0 < t < 5$:
   ODE is $2y'' + y' + 2y = 0$ with $y(0) = y'(0) = 0$
   $\Rightarrow y(t) = 0$: no initial motion & no external force $=$ no motion
2. $5 < t < 20$:
   ODE is $2y'' + y' + 2y = 1$ with $y(5) = y'(5) = 0$
   $\Rightarrow \ y(t) = y_{\text{hom}}(t) + 1/2$, where $y_{\text{hom}}(t)$ describes a damped oscillation.

ODEs with discontinuous forcing

3. $t > 20$:
   ODE is $2y'' + y' + 2y = 0$ with initial conditions $y(20)$ and $y'(20)$ given by the limiting value of the previous interval solution

Remark: Notice y'' has discontinuities at the same points where the force function jumps

Lesson: Laplace transforms & properties of the unit step function allow us to solve ODEs with discontinuous forcing at once rather than piecewise.

Dirac distribution

Consider a non-homogeneous ODE

$$ay'' + by' + cy = g(t)$$

where $g(t) \neq 0$ only during an interval $t_0 - \tau < t < t_0 + \tau$.
Interested in the limit $\tau \to 0$, keeping the impulse

$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)\,dt = \int_{-\infty}^{\infty} g(t)\,dt$$

fixed.

Intuitive idea: model of very short forcing, large enough that it has a non-zero effect on $y(t)$.
Examples:

- a lightning stroke on a transmission line
- a hammer blow on a mechanical system

Dirac distribution

We can take

$$g(t) = d_\tau(t) = \begin{cases} \frac{1}{2\tau}\,, & -\tau < t < \tau \\ 0\,, & \text{otherwise} \end{cases}$$

By definition,

$$I(\tau) = 1 \quad \text{and} \quad \lim_{\tau \to 0} d_\tau(t) = 0. \tag{1}$$

Alternatively, we can take

$$g(t) = \frac{1}{\pi}\frac{\tau}{t^2 + \tau^2} \quad \text{or} \quad g(t) = \frac{e^{-t^2/(2\tau^2)}}{\sqrt{2\pi\tau^2}}$$

or any function satisfying (1).

We use the limiting object: $g(t)$ as $\tau \to 0$.

Dirac distribution

We use the following practical definition:

Definition

The Dirac delta function $\delta(t)$ is defined to be the "function" satisfying the conditions

$$\delta(t) = 0 \quad \textit{for} \quad t \neq 0 \quad \textit{and} \quad \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Shifted version:

$$\delta(t - t_0) = 0 \quad \text{for} \quad t \neq t_0 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1$$

Dirac distribution

Key property

For 'nice' functions $f(t)$,

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, f(t)\, dt = f(t_0).$$

This can be used for a rigorous definition of $\delta(t)$ (Schwartz).

Applying to $f(t) = e^{-st}$, we obtain:

Laplace transform

$$\mathcal{L}\{\delta(t - t_0)\}(s) = e^{-st_0}.$$

Regard $\delta(t)$ as $\lim_{t_0 \to 0} \delta(t - t_0)$, hence $\mathcal{L}\{\delta(t)\}(s) = 1$.

Dirac distribution

We can check the Laplace transform result for $g(t) = d_\tau(t)$:

$$\mathcal{L}\{d_\tau(t-t_0)\} = \frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} e^{-st}\,dt = \frac{e^{-st_0}}{2s\tau}\left(e^{s\tau} - e^{-s\tau}\right) = \frac{\sinh s\tau}{s\tau}\,e^{-st_0}$$

Taking the limit,

$$\mathcal{L}\{\delta(t-t_0)\} \equiv \lim_{\tau\to 0}\frac{\sinh s\tau}{s\tau}\,e^{-st_0} = e^{-st_0}\,.$$

Remark: recalling that $\mathcal{L}\{u_{t_0}(t)\}(s) = e^{-st_0}/s$ and that $\mathcal{L}\{f'(t)\}(s) = sF(s)$, we have the interpretation

$$(u_{t_0}(t))' = \delta(t-t_0).$$

Solving ODEs with Dirac forcing

Consider the initial value problem

$$y'' + y = \delta(t) \quad \text{with} \quad y(0) = y'(0) = 0$$

Taking the Laplace transform & using the initial conditions we derive

$$Y(s) = \frac{1}{s^2 + 1}\,.$$

Inverting, we find:

$$y(t) = \mathcal{L}^{-1}\left[Y(s)\right] = u_0(t)\sin t.$$

Solving ODEs with Dirac forcing

For the timorous, let us consider the problem:

$$y'' + y = f_a(t) = \begin{cases} \frac{1}{a}\,, & t \in [0, a) \\ 0\,, & \text{elsewhere} \end{cases} \qquad y(0) = y'(0) = 0$$

Notice that

$$f_a(t) = \frac{1}{a}\left(u_0(t) - u_a(t)\right)\,.$$

Computing the transform & using the initial conditions

$$Y(s) = \frac{1}{s^2+1}\frac{1-e^{-sa}}{as}.$$

Solving ODEs with Dirac forcing

Decomposing into simple fractions:

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{bs+c}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1}$$

Using

$$\mathcal{L}\{g(t-c)u_c(t)\}(s) = e^{-sc}\,G(s) \;\; c \geq 0$$

we can conclude

$$\begin{aligned} y(t) &= \frac{1}{a}u_0(t)\,(1-\cos t) - \frac{1}{a}u_a(t)\,(1-\cos(t-a)) \\ &= \begin{cases} 0\,, & (-\infty,0] \\ \frac{1-\cos t}{a}\,, & t\in(0,a) \\ \frac{\cos(t-a)-\cos t}{a}\,, & [a,\infty) \end{cases} \end{aligned}$$

Solving ODEs with Dirac forcing

In the limit $a \to 0^+$,

- the interval $t \in (0, a)$ vanishes, whereas the third interval tends to $t \in [0, \infty)$
- Taylor expanding

$$\lim_{a \to 0^+} \frac{\cos(t-a) - \cos t}{a} = \sin t$$

Thus, we conclude that

$$y(t) = u_0(t) \sin t.$$

Recover, after a painful computation, the result obtained using $\delta(t)$.

Example

Solve the initial value problem

$$2y'' + y' + 2y = \delta(t-5) \quad \text{with} \quad y(0) = y'(0) = 0\,.$$

Compute the Laplace transform & use initial conditions:

$$\left(2s^2 + s + 2\right) Y(s) = e^{-5t} \Rightarrow Y(s) = \frac{e^{-5s}}{2} \frac{1}{(s+1/4)^2 + 15/16}\,.$$

Since

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{(s+1/4)^2 + 15/16}\right] = \frac{4}{\sqrt{15}}\, e^{-t/4} \sin \frac{\sqrt{15}}{4} t\,,$$

we can write

$$Y(s) = \frac{1}{2} e^{-5s} G(s) \quad \text{with} \quad G(s) = \mathcal{L}\{g(t)\}$$

Example

Recall

$$\mathcal{L}\{g(t-c)u_c(t)\}(s) = e^{-sc}\,G(s) \;\; c \geq 0$$

where $\mathcal{L}\{g(t)\}(s) = G(s)$. Applying it to this case, we can conclude

$$y(t) = \frac{1}{2} u_5(t)\, g(t-5)$$

Interpretation: the solution vanishes for $0 < t < 5$, in the absence of forcing, but is non-zero after the kick at $t = 5$.

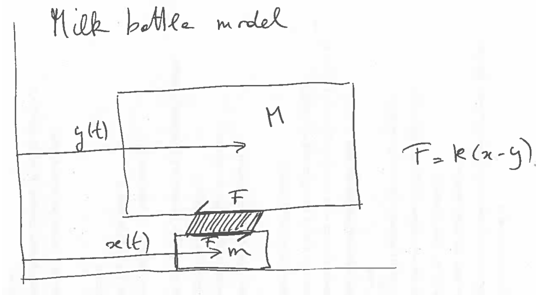
Honours Differential Equations

Jacques Vanneste

Lecture 7

October 1, 2018

Laplace transform and Dirac: an application



Small mass m at $x(t)$ and big mass M at $y(t)$ coupled by an elastic link. Newton's law gives

$$m\ddot{x} + k(x - y) = \delta(t),$$
$$M\ddot{y} + k(y - x) = 0.$$

Laplace transform:

$$ms^2X + k(X - Y) = 1,$$
$$Ms^2Y + k(Y - X) = 0.$$

Laplace transform and Dirac: an application

Solving for $X(s)$:

$$X(s) = \frac{1}{s^2(s^2+\omega^2)}\left(\frac{s^2}{m} + \frac{k}{Mm}\right), \quad \omega^2 = k\frac{M+m}{Mm} \approx \frac{k}{m}.$$

Partial fractions:

$$X(s) = \frac{1}{M+m}\frac{1}{s^2} + \frac{M}{m(M+m)}\frac{1}{s^2+\omega^2}.$$

Inverting:

$$x(t) = \frac{1}{M+m}\left(t + \frac{M}{m\omega}\sin(\omega t)\right).$$

Similarly,

$$y(t) = \frac{1}{M+m}\left(t - \frac{1}{\omega}\sin(\omega t)\right).$$

Note: $mx + My = t$ (centre of mass) and oscillation $x \gg y$ for $M \gg m$; $\dot{x} < 0$ for some t if $m < M$.

Convolution

Solve the initial value problem

$$y'' + \omega^2 y = \omega^2 f(t), \qquad y(0) = y'(0) = 0$$

where ω^2 is a constant and $f \in E$.
Taking the Laplace transform :

$$(s^2 + \omega^2)Y(s) = \omega^2 \mathcal{L}\{f\}, \Rightarrow Y(s) = \mathcal{L}\{f\} \frac{\omega^2}{s^2 + \omega^2}$$

Using the identity $\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}$, we find

$$Y(s) = \mathcal{L}\{y\} = (\mathcal{L}\{f\}) (\mathcal{L}\{\omega \sin \omega t\}).$$

Question : What is $y(t)$? Can we solve this problem for any $f(t)$?

Convolution

Definition

Given two functions f and g, we define their convolution as

$$(f \star g)(t) \equiv \int_0^t f(t_1)g(t - t_1)\,dt_1$$

Convolution is a product which satisfies

- Distributive : $f \star (g + h) = f \star g + f \star h$,
- Commutative : $f \star g = g \star f$,
- Associative : $f \star (g \star h) = (f \star g) \star h$.

But there are important differences, for example,

1. $f \star 1 \neq f$,
2. $f \star f \neq f^2$.

Convolution theorem

Theorem

If $f, g \in E$, then $f \star g \in E$ and $\mathcal{L}\{f \star g\} = (\mathcal{L}\{f\})(\mathcal{L}\{g\})$.

Proof.

1. If $f, g \in E$, then $|f(t)| \leq A_1\, e^{B_1 t}$ and $|g(t)| \leq A_2\, e^{B_2 t}$. Thus,

$$\begin{aligned} |\,(f \star g)\,| &\leq \int_0^t |f(t_1)||g(t-t_1)|\, dt_1 \\ &\leq \int_0^t A_1\, e^{B_1 t_1} A_2\, e^{B_2(t-t_1)}\, dt_1 \\ &= A_1 A_2 \frac{e^{B_1 t} - e^{B_2 t}}{B_1 - B_2} \Rightarrow f \star g \in E\,. \end{aligned}$$

$\square$

Convolution theorem

Proof.

2. By definition,

$$(\mathcal{L}\{f\})(\mathcal{L}\{g\}) = \int_0^\infty g(\eta)d\eta \int_0^\infty e^{-s(\xi+\eta)} f(\xi)d\xi \,.$$

Consider $\xi = t - \eta$ at fixed η, then

$$\begin{aligned}(\mathcal{L}\{f\})(\mathcal{L}\{g\}) &= \int_0^\infty g(\eta)d\eta \int_\eta^\infty e^{-st} f(t-\eta)dt \\ &= \int_0^\infty e^{-st}\, dt \int_0^t f(t-\eta)\, g(\eta)\, d\eta = \mathcal{L}\{f \star g\}\end{aligned}$$

where we reordered the integration on the $t - \eta$ plane.

$\square$

Convolution: application

Let us return to our original problem : we want to solve

$$\mathcal{L}\{y\} = (\mathcal{L}\{f\})\,(\mathcal{L}\{\omega \sin \omega t\})\,,$$

for any $f(t)$.

Using the uniqueness of the Laplace transform, we can infer

$$y(t) = f(t) \star \omega \sin \omega t = \omega \int_0^t f(t_1) \sin \omega (t - t_1)\, dt_1\,.$$

Question : Can we use the convolution of two functions to solve any non-homogeneous ODE?

Convolution: solving non-homogeneous ODEs

Consider the initial value problem

$$ay'' + by' + cy = g(t) \quad \text{with} \;\; y(0) = y_0\,, \;\; y'(0) = y_0'$$

Computing the Laplace transform, we derive

$$\begin{aligned}
\mathcal{L}\{y\}(s) &= \Phi(s) + \Psi(s) \\
\Phi(s) &= \frac{(as+b)y_0 + ay_0'}{as^2 + bs + c} \\
\Psi(s) &= \frac{\mathcal{L}\{g\}(s)}{as^2 + bs + c} \\
\Rightarrow y(t) &= \mathcal{L}^{-1}[\Phi(s)] + \mathcal{L}^{-1}[\Psi(s)] \equiv \phi(t) + \psi(t)\,.
\end{aligned}$$

1. If we define $H(s) = \frac{1}{as^2+bs+c}$, then by the convolution theorem

$$\psi(t) = \int_0^t h(t-\tau)\, g(\tau)\, d\tau \quad \text{where} \;\; h(t) = \mathcal{L}^{-1}[H(s)]$$

This encodes the information about the particular solution.

Convolution: solving non-homogeneous ODEs

$H(s)$ is the transfer function, Laplace transform of solution of the initial-value problem:

$$ah'' + bh' + ch = \delta(t), \quad h(0) = h'(0) = 0.$$

2. $\phi(t) = \mathcal{L}^{-1}[\Phi(s)]$: encodes the information about the solution to an homogeneous ODE with initial conditions $\phi(0) = y_0$ and $\phi'(0) = y_0'$

Example

Compute the inverse Laplace transform of $H(s)$

$$H(s) = \frac{1}{s^2(s^2 + a^2)}.$$

One strategy is to view $H(s) = F(s)G(s)$ so that

$$F(s) = \frac{1}{s^2} \Rightarrow f(t) = t$$
$$G(s) = \frac{1}{s^2 + a^2} \Rightarrow g(t) = \frac{1}{a} \sin at$$

In the last step, we apply the convolution theorem

$$h(t) = \mathcal{L}^{-1}[H(s)] = \int_0^t f(t-\tau)g(\tau)d\tau$$
$$= \frac{1}{a}\int_0^t (t-\tau) \sin a\tau \, d\tau = \frac{at - \sin at}{a^3}.$$

Example

Solve the initial value problem

$$y'' + 4y = g(t) \quad \text{with} \;\; y(0) = 3, \;\; y'(0) = -1.$$

Compute Laplace transform :

$$\begin{aligned} s^2 Y - 3s + 1 + 4Y &= G(s) \\ \Rightarrow Y(s) &= \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4} \\ &= 3\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}G(s). \end{aligned}$$

Thus, we identify

$$y(t) = 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t \sin 2(t-\tau)\, g(\tau)\, d\tau.$$

Honours Differential Equations

Jacques Vanneste

Lecture 8

October 4, 2018

Systems of ODEs

Question: Why to study systems of 1st-order ODEs?

- Any problem involving more than one variable is an example
  1. Motion of N particles,
  $$\frac{d^2x_i}{dt^2} = \sum_{j=1}^{N} F(x_i, x_j, t) \quad i = 1, \dots, N.$$
  2. Stock market: x_i price of i-th item depends on the prices of the other x_j $j \neq i$.
  3. Population dynamics: prey, predator. . . .
- Higher order ODEs reduce to systems of 1st-order ODEs.

Focus on linear systems, with constant coefficients:

- can be solved analytically,
- local approximations to nonlinear systems.

1st-order systems of ODEs

The most general 1st order ODE system in n variables is

$$x_i'(t) = F_i(x_j(t), t) \quad i, j = 1, \dots n.$$

Theorem

Let $F_i(x_j(t), t)$ and $\partial_{x_j} F_i(x_j(t), t)$ be continuous in the region R defined by $\alpha < t < \beta$, $\alpha_i < x_i < \beta_i$, $\forall\, i$.
Let $(x_i^0, t_0) \in R \Rightarrow \exists$ interval $|t - t_0| < h$ in which $\exists\; x_i = \phi_i(t)$ solving the initial condition $\phi_i(t_0) = x_i^0$.

If $\partial_t F_i = 0$ for all i, i.e., $F_i = F_i(x_j)$, the systems is autonomous.

1st-order systems of ODEs

Geometric interpretation:

- For fixed t and x_j, $\boldsymbol{F}(\boldsymbol{x}, t) = (F_1(x_j, t), \cdots, F_n(x_j, t)) \in \mathbb{R}^n$ is a vector,
- For fixed t, $\boldsymbol{F}(\cdot, t) = (F_1(\cdot, t), \cdots, F_n(\cdots, t)) : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field.
- Solving an autonomous system: find an (integral) curve $\boldsymbol{x}(t) = (x_1(t), \cdots, x_n(t))$ everywhere tangent to the vector field $\boldsymbol{F}$.

From n-th order to systems of 1st-order ODEs

Consider an arbitrary n-th order ODE

$$y^{(n)} = F(y, y', \dots, y^{(n-1)}, t).$$

1. Change dependent variables to $x_1, x_2, \cdots x_n$:

$$x_1 = y\,,\ x_2 = y'\,, \quad \dots \quad x_n = y^{(n-1)}.$$

2. Take derivatives of the new variables

$$\begin{aligned} &x_1' = x_2\,, \quad x_2' = x_3 \quad \dots \quad x_{n-1}' = x_n \\ &x_n' = y^{(n)} = F(y, y', \dots, y^{(n-1)}, t) = F(x_1, x_2, \dots, x_n, t). \end{aligned}$$

Transform the n-th order ODE into a system of n 1st-order ODEs, with vector field $(x_2, x_3, \cdots, x_n, F(x_1, x_2, \dots, x_n, t))$.

Systems of first-order linear ODEs

Definition

1. *A 1st order ODE system is linear if it has the form*

$$x_i' = \frac{dx_i}{dt} = \sum_{j=1}^{n} P_{ij}(t) x_j + g_i(t) \quad i = 1, \cdots, n.$$

   Otherwise, it is nonlinear.
2. *If $g_i(t) \equiv 0$, the system is homogeneous.*
   Otherwise, it is non-homogeneous.

Theorem

If $P_{ij}(t)$, $g_i(t)$ are continuous in $t \in [\alpha, \beta] \Rightarrow \exists$ unique solution $x_i = \phi_i(t)$ solving the given initial value problem, i.e. $\phi_i(t_0) = x_i^0$.

Homogeneous systems of first-order linear ODEs

Most general homogeneous system:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} P_{ij}(t)x_j \quad i = 1, \dots n,$$

or equivalently,

$$\frac{d\boldsymbol{x}}{dt} = P(t)\boldsymbol{x}, \quad \text{where} \quad P(t) = \begin{pmatrix} P_{11}(t) & \dots & P_{1n}(t) \\ . & \dots & . \\ P_{n1}(t) & \dots & P_{nn}(t) \end{pmatrix} \in M^{n\times n}.$$

Solutions correspond to vectors $\boldsymbol{x}(t)$
i.e. an array $(x_1(t), x_2(t), \dots x_n(t))$

Question: how many solutions are there?

Homogeneous systems of first-order linear ODEs

Note: If $\{\boldsymbol{x}^{(j)}\}$, $j = 1, \cdots, p$ is a set of p solutions,

$$\frac{d\boldsymbol{x}^{(j)}}{dt} = P(t)\boldsymbol{x}^{(j)} \Rightarrow \sum_{j=1}^{p} c_j \boldsymbol{x}^{(j)} \text{ is a solution}$$

Indeed,

$$\frac{d\left(\sum_{j=1}^{p} c_j \boldsymbol{x}^{(j)}\right)}{dt} = \sum_{j=1}^{p} c_j \frac{d\boldsymbol{x}^{(j)}}{dt} = \sum_{j=1}^{p} c_j P(t)\boldsymbol{x}^{(j)}$$

$$= P(t)\left(\sum_{j=1}^{p} c_j \boldsymbol{x}^{(j)}\right)$$

where we used the linearity satisfied by the product of matrices with vectors.

Homogeneous systems of first-order linear ODEs

- The space of solutions to the homogeneous linear system is a vector space,
- The dimension of this vector space is $n \Rightarrow$ there must be n linearly independent solutions.

Question: when is a set of n solutions linearly independent?

- Each solution is a vector,
- n vectors are linearly independent if the determinant of the squared matrix formed with each of them as a column is non-zero

Homogeneous systems of first-order linear ODEs

Given

$$\frac{d\boldsymbol{x}}{dt} = P(t)\boldsymbol{x}\,,$$

its general solution is given by the linear combination of any fundamental set of n solutions,

$$\boldsymbol{x}_{\text{gen}}(t) = \sum_{j=1}^{n} c_j\, \boldsymbol{x}^{(j)}(t)$$

$$\text{with } W[\boldsymbol{x}^{(j)}] = \begin{vmatrix} x_1^{(1)}(t) & \dots & x_1^{(n)}(t) \\ . & \dots & . \\ x_n^{(1)}(t) & \dots & x_n^{(n)}(t) \end{vmatrix} \neq 0.$$

The Wronskian W generalises that defined for higher-order ODEs.

Homogeneous systems of first-order linear ODEs

Remarks:

1. $W(t) \neq 0$ implies that $\{\boldsymbol{x}^{(j)}\}$ span $\mathbb{R}^n$.
2. if $W(t_0) \neq 0$ for some t_0, then $W(t) \neq 0$ for all $\alpha < t < \beta$. This can be shown as for higher-order ODEs, using that $\boldsymbol{x}(t) = 0$ is the unique solution with $\boldsymbol{x}(t_0) = 0$; alternatively, this is obvious from

Liouville's theorem

$$\dot{W} = \operatorname{tr} P\, W \quad \Rightarrow \quad W(t) = e^{\int_{t_0}^{t} \operatorname{tr} P(s)\, ds}\, W(t_0).$$

Proof: $\boldsymbol{x}(t+h) = (\mathbb{I} + hP(t))\,\boldsymbol{x}(t) + O(h^2)$, hence

$$\begin{aligned} W(t+h) &= \det\left(\mathbb{I} + h\,P(t)\right) W(t) + O(h^2) \\ &= W(t) + h \operatorname{tr} P(t)\, W(t) + O(h^2). \end{aligned}$$

Homogeneous systems with constant coefficients

Given

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x}\,, \tag{1}$$

where $A \in M^{n\times n}$ is constant.

Strategy: As with n-th order homogeneous ODEs, seek exponential solutions:

$$\boldsymbol{x} = e^{rt}\,\boldsymbol{\xi} \quad \Rightarrow \quad \frac{d\boldsymbol{x}}{dt} = r\boldsymbol{x}$$

Into (**??**),

$$(A - r\mathbb{I})\,\boldsymbol{\xi} = 0$$

Eigenvalue problem: a linear algebra problem.

Homogeneous systems with constant coefficients

Thus, solutions of

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x}\,,$$

have the form

$$\boldsymbol{x} = \sum_{j=1}^{n} c_j\, e^{r_j t} \boldsymbol{\xi}^{(j)},$$

where

1. $\det\,(A - r\mathbb{I}) = 0$: this determines $\{r_j\}$ as eigenvalues of A
2. $\boldsymbol{\xi}^{(j)}$ is the eigenvector associated to the eigenvalue r_j

$$(A - r_j\mathbb{I})\,\boldsymbol{\xi}^{(j)} = 0\,.$$

Remark: We have assumed all eigenvalues r_j are different.
When r_j has multiplicity ≥ 2, we need new methods (next week).

Example

Consider the linear ODE system

$$\frac{d\boldsymbol{x}}{dt} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \boldsymbol{x} \equiv A\boldsymbol{x} \,.$$

We look for solutions of the form $\boldsymbol{x} = \boldsymbol{\xi}\, e^{rt}$ where $\boldsymbol{\xi} = \begin{pmatrix} y \\ z \end{pmatrix}$ because the system involves a 2×2 matrix.

Introducing $\boldsymbol{\xi}\, e^{rt}$ into the ODEs gives

$$(A - r\mathbb{I})\, \boldsymbol{\xi} = 0.$$

This is an homogeneous linear algebra system of equations:

$$\begin{aligned} (1-r)y + z &= 0 \\ 4y + (1-r)z &= 0 \,. \end{aligned}$$

Example

For the solution not to be $y = z = 0$, we require the determinant of the matrix of coefficients to vanish

$$\det\,(A - r\mathbb{I}) = 0$$
$$\begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = (1-r)^2 - 4 = r^2 - 2r - 3 = (r-3)(r+1) = 0.$$

Thus, $r_1 = 3$ and $r_2 = -1$, giving rise to a solution of the form

$$\boldsymbol{x} = c_1\, e^{3t} \boldsymbol{\xi}^{(1)} + c_2\, e^{-t} \boldsymbol{\xi}^{(2)}\,.$$

We are left to determine the two eigenvectors.

Example

When $r = r_1 = 3$, the original system is equivalent to

$$\begin{aligned} (1-r_1)y + z &= 0 \quad \Rightarrow \quad -2y + z = 0 \\ 4y + (1-r_1)z &= 0 \quad \Rightarrow \quad 4y - 2z = 0\,. \end{aligned}$$

Only one equation is linearly independent
Thus, $z = 2y$.
The eigenvector equals

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

where y is any real number. The latter can be absorbed into c_1.

Example

When $r = r_2 = -1$, the original system is equivalent to

$$\begin{aligned} (1 - r_2)y + z = 0 \quad &\Rightarrow \quad 2y + z = 0 \\ 4y + (1 - r_2)z = 0 \quad &\Rightarrow \quad 4y + 2z = 0\,. \end{aligned}$$

Only one equation is linearly independent
Thus, z= -2y
The eigenvector equals

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

where y is any real number. The latter can be absorbed into c_2.

Example

Solutions:

$$\boldsymbol{x}^{(1)} = e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{x}^{(2)} = e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

They are linearly independent. Indeed

$$\begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{3t} \neq 0$$

Thus, the general solution is

$$\boldsymbol{x} = c_1 \, e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \, e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Honours Differential Equations

Jacques Vanneste

Lecture 9

October 5, 2018

Homogeneous systems with constant coefficients

We are discussing systems of the form

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x}.$$

There are several cases:

1. real and different eigenvalues,
2. complex and different eigenvalues,
3. repeated eigenvalues (next week)

Today: real, complex and different eigenvalues.

Real & different eigenvalues

By assumption,

$$r_i \neq r_j \quad \forall\, i, j = 1, \dots n \Rightarrow \text{ for each } r_i\,,\ \exists\, \boldsymbol{\xi}^{(i)}$$

so fundamental solutions take the form

$$\boldsymbol{x}^{(i)} = \boldsymbol{\xi}^{(i)}\, e^{r_i t}$$

This set of n solutions is linearly independent if

$$\det X(t) \neq 0 \quad \text{with} \quad X(t) = \begin{pmatrix} \xi_1^{(1)}\, e^{r_1 t} & \dots & \xi_1^{(n)}\, e^{r_n t} \\ . & \dots & . \\ \xi_n^{(1)}\, e^{r_1 t} & \dots & \xi_n^{(n)}\, e^{r_n t} \end{pmatrix}$$

$$\det X(t) = e^{\left(\sum_{j=1}^{n} r_j\right) t}\, W[\boldsymbol{\xi}^{(1)}, \dots \boldsymbol{\xi}^{(n)}] \neq 0$$

since the n eigenvectors $\boldsymbol{\xi}^{(j)}$ are linearly independent.

Real & different eigenvalues

Summary: given a linear 1st order ODE system

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x},$$

if the corresponding eigenvalue problem

$$(A - r\mathbb{I})\,\boldsymbol{\xi} = 0$$

has different eigenvalues r_i, the eigenvectors $\boldsymbol{\xi}^{(i)}$ are linearly independent, and the general solution reads

$$\boldsymbol{x}(t) = \sum_{j=1}^{n} c_j\, e^{r_j t} \boldsymbol{\xi}^{(j)}.$$

Example

Consider the linear ODE system

$$\frac{d\boldsymbol{x}}{dt} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \boldsymbol{x} \equiv A\boldsymbol{x}\,.$$

We look for solutions of the form $\boldsymbol{x} = \boldsymbol{\xi}\, e^{rt}$ where $\boldsymbol{\xi} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Introducing $\boldsymbol{\xi}\, e^{rt}$ in the ODE, we obtain

$$(A - r\mathbb{I})\,\boldsymbol{\xi} = 0\,.$$

This is the homogeneous linear system:

$$\begin{aligned} -rx + y + z &= 0 \\ x - ry + z &= 0 \\ x + y - rz &= 0\,. \end{aligned}$$

Example

Eigenvalues:

$$\det(A - r\mathbb{I}) = 0$$
$$= -r^3 + 3r + 2 = (r-2)(r+1)^2 = 0$$

Thus, $r_1 = 2$ has multiplicity one and $r_2 = -1$ has algebraic multiplicity 2, giving rise to a solution of the form

$$\boldsymbol{x} = c_1\, e^{2t}\boldsymbol{\xi}^{(1)} + c_2\, e^{-t}\boldsymbol{\xi}^{(2)} + c_2\, e^{-t}\boldsymbol{\xi}^{(3)},$$

One eigenvector

When $r = r_1 = 2$, the original system is equivalent to

$$\begin{aligned} -rx + y + z = 0 \quad &\Rightarrow \quad -2x + y + z = 0 \\ x - ry + z = 0 \quad &\Rightarrow \quad x - 2y + z = 0 \\ x + y - rz = 0 \quad &\Rightarrow \quad x + y - 2z = 0\,. \end{aligned}$$

The 3rd equation is equivalent to the sum of the first two.
Subtracting the 1st and 2nd $\Rightarrow x = y$
introducing into the 1st $\Rightarrow x = y = z$
The eigenvector equals

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

where x is any real number.

Two remaining eigenvectors

When $r = r_2 = -1$, the original system is equivalent to

$$\begin{aligned} -rx + y + z = 0 \quad &\Rightarrow \quad x + y + z = 0 \\ x - ry + z = 0 \quad &\Rightarrow \quad x + y + z = 0 \\ x + y - rz = 0 \quad &\Rightarrow \quad x + y + z = 0\,. \end{aligned}$$

Clearly, there is a unique linearly independent equation
Its solution is $z = -y - x$. The eigenvector(s) equals

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

where x, y are any real numbers.
Thus, there are two linearly independent eigenvectors.

$$\boldsymbol{\xi}^{(2)} = x \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \boldsymbol{\xi}^{(3)} = y \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Summary

The linear ODE system

$$\frac{d\boldsymbol{x}}{dt} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \boldsymbol{x} \equiv A\boldsymbol{x} \,,$$

has general solution

$$\boldsymbol{x}(t) = c_1\, e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2\, e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3\, e^{-t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \,.$$

Lesson: if a given eigenvalue r has multiplicity $s \geq 2$, the method still works if its geometric multiplicity (number of linearly independent eigenvectors ξ_r) equals s.

Dependence on parameters

Consider the linear ODE system

$$\frac{d\boldsymbol{x}}{dt} = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} \boldsymbol{x} \equiv A\boldsymbol{x}\,,$$

where $\alpha \in \mathbb{R}$.
We are interested in solving this linear system, exploring the change in the behaviour of the solutions depending on α.
Eigenvalues:

$$\begin{vmatrix} -1-r & -1 \\ -\alpha & -1-r \end{vmatrix} = r^2 + 2r + 1 - \alpha = 0 \Rightarrow r_\pm = -1 \pm \sqrt{\alpha}$$

Dependence on parameters

Eigenvectors: Consider $\boldsymbol{\xi} = \begin{pmatrix} x \\ y \end{pmatrix}$

Set of equations is:

$$\begin{aligned}(1+r)x + y &= 0 \\ \alpha\, x + (1+r)y &= 0\,.\end{aligned}$$

- when $1 + r_+ = \sqrt{\alpha} \Rightarrow y = -\sqrt{\alpha}\, x \Rightarrow \boldsymbol{\xi}_+ = x \begin{pmatrix} 1 \\ -\sqrt{\alpha} \end{pmatrix}$
- when $1 + r_- = -\sqrt{\alpha} \Rightarrow y = \sqrt{\alpha}\, x \Rightarrow \boldsymbol{\xi}_+ = x \begin{pmatrix} 1 \\ \sqrt{\alpha} \end{pmatrix}$

General solution:

$$\boldsymbol{x}(t) = c_1\, e^{r_+ t} \begin{pmatrix} 1 \\ -\sqrt{\alpha} \end{pmatrix} + c_2\, e^{r_- t} \begin{pmatrix} 1 \\ \sqrt{\alpha} \end{pmatrix}$$

Dependence on parameters

General solution:

$$\boldsymbol{x}(t) = c_1\, e^{r_+ t} \begin{pmatrix} 1 \\ -\sqrt{\alpha} \end{pmatrix} + c_2\, e^{r_- t} \begin{pmatrix} 1 \\ \sqrt{\alpha} \end{pmatrix}$$

Observe:

- if $0 < \alpha < 1 \;\Rightarrow\; r_\pm < 0$,
- if $\alpha = 1 \Rightarrow\; r_+ = 0\,,\; r_- < 0$,
- if $\alpha > 1 \;\Rightarrow\; r_+ > 0\,,\; r_- < 0$,
- if $\alpha = 0 \;\Rightarrow\; r_+ = r_-$ but there is a single linearly independent solution,
- if $\alpha < 0$, solutions are oscillatory.

Points in α-space where the behaviour of the solution to an ODE system changes qualitatively are referred to as bifurcation points. Dynamical systems theory studies these.

Complex eigenvalues

Consider a linear ODE system

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x}\,.$$

with matrix A real. Then if $r_1 = \lambda + i\mu$ is one of its eigenvalues eigenvalue, i.e.

$$(A - r_1\mathbb{I})\,\boldsymbol{\xi}_1 = 0$$

taking the complex conjugate,

$$(A - r_1^\star\mathbb{I})\,\boldsymbol{\xi}_1^\star = 0.$$

Thus, $r_1^\star = \lambda - i\mu$ is also an eigenvalue with eigenvector $\boldsymbol{\xi}_1^\star$.

Complex eigenvalues

Summary: We find two solutions

$$\boldsymbol{x}_1(t) = e^{r_1 t}\,\boldsymbol{\xi}_1\,, \qquad \boldsymbol{x}_2(t) = e^{r_1^\star t}\,\boldsymbol{\xi}_1^\star$$

If we are interested in real solutions, we can proceed as follows. Write $\boldsymbol{\xi}_1 = \boldsymbol{a} + i\boldsymbol{b}$ and compute $\boldsymbol{x}_1(t)$

$$\begin{aligned}\boldsymbol{x}_1(t) &= (\boldsymbol{a} + i\boldsymbol{b})e^{\lambda t}(\cos\mu t + i\sin\mu t)\\ &= e^{\lambda t}(\boldsymbol{a}\cos\mu t - \boldsymbol{b}\sin\mu t) + i\,e^{\lambda t}(\boldsymbol{a}\sin\mu t + \boldsymbol{b}\cos\mu t)\\ &\equiv \boldsymbol{u}(t) + i\,\boldsymbol{v}(t)\end{aligned}$$

The 2 real solutions are

$$\begin{aligned}\boldsymbol{u}(t) &= e^{\lambda t}(\boldsymbol{a}\cos\mu t - \boldsymbol{b}\sin\mu t)\\ \boldsymbol{v}(t) &= e^{\lambda t}(\boldsymbol{a}\sin\mu t + \boldsymbol{b}\cos\mu t)\,.\end{aligned}$$

General solution: $\boldsymbol{x}(t) = c_1\boldsymbol{u}(t) + c_2\boldsymbol{v}(t) + \ldots$

Example

Consider the linear ODE system

$$\frac{d\boldsymbol{x}}{dt} = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \boldsymbol{x} \equiv A\boldsymbol{x}\,.$$

We look for solutions of the form $\boldsymbol{x} = \boldsymbol{\xi}\, e^{rt}$ where $\boldsymbol{\xi} = \begin{pmatrix} x \\ y \end{pmatrix}$ because the system involves a 2×2 matrix.
When we introduce $\boldsymbol{\xi}\, e^{rt}$ in the ODE, we get the equation

$$(A - r\mathbb{I})\,\boldsymbol{\xi} = 0\,.$$

This is an homogeneous linear algebra system of equations:

$$\left(-\frac{1}{2} - r\right) x + y = 0$$
$$x + \left(\frac{1}{2} + r\right) y = 0\,.$$

Example

For the solution not to be $y = x = 0$, we require the determinant of the matrix of coefficients to vanish

$$\det\,(A - r\mathbb{I}) = 0$$
$$= \left(\frac{1}{2} + r\right)^2 + 1 = r^2 + r + \frac{5}{4} = 0 \Rightarrow r_\pm = -\frac{1}{2} \pm i\,.$$

Eigenvectors:

- $r = r_+ \Rightarrow y = ix \Rightarrow \boldsymbol{\xi}_+ \propto \begin{pmatrix} 1 \\ i \end{pmatrix}$
- $r = r_- \Rightarrow y = -ix \Rightarrow \boldsymbol{\xi}_- \propto \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Example

General solution: one complex solution equals

$$\boldsymbol{x}_+ = e^{-t/2}\,(\cos t + i \sin t)\begin{pmatrix}1\\ i\end{pmatrix}$$
$$= e^{-t/2}\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + i\,e^{-t/2}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}$$

Thus, we can identify

$$\boldsymbol{u}(t) = e^{-t/2}\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}, \quad \boldsymbol{v}(t) = e^{-t/2}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}.$$

General real solution: $\boldsymbol{x}(t) = c_1\boldsymbol{u}(t) + c_2\boldsymbol{v}(t)$.
Remark: $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent !!

$$W[\boldsymbol{u},\ \boldsymbol{v}] = e^{-t} \neq 0$$

Honours Differential Equations

Jacques Vanneste

Lecture 10

October 8, 2018

Recap

Given a linear system of 1st order ODEs

$$\frac{d\boldsymbol{x}}{dt} = A\,\boldsymbol{x}\,,$$

where A is a constant $n \times n$ matrix, we obtain n solutions

$$\boldsymbol{x}^{(i)} = \boldsymbol{\xi}^{(i)}\,e^{r_i t}, \quad i = 1, \ldots n,$$

where

$$(A - r_i \mathbb{I})\,\boldsymbol{\xi}^{(i)} = 0,$$
$$\det\,(A - r_i \mathbb{I}) = 0,$$

provided that

- the eigenvalues r_i are all different,
- if an eigenvalue has algebraic multiplicity $s \geq 2$, its geometric multiplicity is also s.

Recap

What if:

- One of the r_i has algebraic multiplicity $s \geq 2$ and geometric multiplicity $< s$?

We do not have n linearly independent solutions:
missing solutions.

Strategy: Use matrix algebra methods

1. Learn how to reproduce last week's solutions using these methods,
2. Apply these to the problem above.

Matrix algebra methods

Given a linear system of 1st order ODEs

$$\frac{d\boldsymbol{x}}{dt} = A\,\boldsymbol{x}\,,$$

with n fundamental solutions $\boldsymbol{x}^{(i)}$ $i = 1, 2, \ldots, n$

Definition

A fundamental matrix $\Psi(t)$ is an $n \times n$ matrix with fundamental solutions $\boldsymbol{x}^{(i)}$ as columns:

$$\Psi(t) = \begin{pmatrix} x_1^{(1)} & x_1^{(2)} & \ldots & x_1^{(n)} \\ x_2^{(1)} & x_2^{(2)} & \ldots & x_2^{(n)} \\ . & . & \ldots & . \\ x_{n-1}^{(1)} & x_{n-1}^{(2)} & \ldots & x_{n-1}^{(n)} \\ x_n^{(1)} & x_n^{(2)} & \ldots & x_n^{(n)} \end{pmatrix}.$$

Fundamental matrix

1. $\det\Psi(t) = W(t) \neq 0$ because $\boldsymbol{x}^{(i)}$ form a n fundamental set.
2. The general solution can be written as the matrix product

$$\boldsymbol{x}(t) = \sum_{j=1}^{n} c_j \boldsymbol{x}^{(j)} = \Psi(t)\boldsymbol{c}, \quad \text{where} \quad \boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ c_n \end{pmatrix}.$$

3. If the system satisfies the initial condition: $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, then the general solution equals

$$\boldsymbol{x}(t_0) = \Psi(t_0)\boldsymbol{c} = \boldsymbol{x}_0 \Rightarrow \boldsymbol{c} = \Psi^{-1}(t_0)\boldsymbol{x}_0$$
$$\boldsymbol{x}(t) = \Psi(t)\boldsymbol{c} = \Psi(t)\Psi^{-1}(t_0)\boldsymbol{x}_0\,.$$

Notice this last equation achieves the same goal as the Laplace transform, i.e. find the solution + include initial data $\Rightarrow$ requires to invert $\Psi(t)$

Fundamental matrix

Property

The fundamental matrix satisfies the matrix equation

$$\Psi' = A\Psi$$

Here,

$$\Psi' = \frac{d\Psi(t)}{dt} = \begin{pmatrix} \frac{dx_1^{(1)}}{dt} & \frac{dx_1^{(2)}}{dt} & \cdots & \frac{dx_1^{(n)}}{dt} \\ \frac{dx_2^{(1)}}{dt} & \frac{dx_2^{(2)}}{dt} & \cdots & \frac{dx_2^{(n)}}{dt} \\ . & . & \cdots & . \\ \frac{dx_{n-1}^{(1)}}{dt} & \frac{dx_{n-1}^{(2)}}{dt} & \cdots & \frac{dx_{n-1}^{(n)}}{dt} \\ \frac{dx_n^{(1)}}{dt} & \frac{dx_n^{(2)}}{dt} & \cdots & \frac{dx_n^{(n)}}{dt} \end{pmatrix}$$

The property is clear:

$$(\boldsymbol{x}^{(i)})' = A\boldsymbol{x}^{(i)} \quad \Rightarrow \quad (\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)})' = A(\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}).$$

Matrix exponential

Remark: linear ODE systems generalise

$$\frac{dx}{dt} = ax \Rightarrow x(t) = e^{at}\, x_0.$$

We can generalise the exponential function to matrices:

Definition

Define the exponential of a matrix A as

$$e^{A\,t} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = \mathbb{I} + At + \frac{1}{2!} A^2 t^2 + \cdots$$

Alternatively,

$$e^{A\,t} = \lim_{n\to\infty} \left(\mathbb{I} + \frac{1}{n} A \right)^n .$$

Matrix exponential

It can be shown that the previous definition converges.
Interesting properties for us:

1.
$$\frac{de^{A\,t}}{dt} = \sum_{n=1}^{\infty} \frac{A^n\, t^{n-1}}{(n-1)!} = A \sum_{n=1}^{\infty} \frac{A^{n-1}\, t^{n-1}}{(n-1)!} = \{n-1=m\}$$
$$= A \sum_{m=0}^{\infty} \frac{A^m\, t^m}{m!} = A \left(\mathbb{I} + \sum_{m=1}^{\infty} \frac{A^m\, t^m}{m!} \right) = A\, e^{At} \,.$$

2. $e^{A\,t}\big|_{t=0} = \mathbb{I}\,.$

Conclusion:

- $\boldsymbol{x}(t) = e^{A\,t}\,\boldsymbol{x}_0 \quad \Leftrightarrow \quad e^{A\,t} = \Psi(t)\Psi^{-1}(t_0)$,
- $e^{A\,t} = \Psi(t)$ for $\Psi(0) = \mathbb{I}$.

Relation to diagonalisation

We originally solved the linear ODE system

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x}\,,$$

in terms of solutions of the form $\boldsymbol{x}^{(i)} = e^{r_i\,t}\,\boldsymbol{\xi}^{(i)}$.

Consider the matrix T with the eigenvectors $\boldsymbol{\xi}^{(i)}$ as columns

$$T = \begin{pmatrix} \xi_1^{(1)} & \xi_1^{(2)} & \dots & \xi_1^{(n)} \\ \xi_2^{(1)} & \xi_2^{(2)} & \dots & \xi_2^{(n)} \\ . & . & \dots & . \\ . & . & \dots & . \\ \xi_n^{(1)} & \xi_n^{(2)} & \dots & \xi_n^{(n)} \end{pmatrix}.$$

Relation to diagonalisation

Observation: The matrix AT has columns equal to

$$A\xi^{(i)} = r_i\,\xi^{(i)}\,.$$

Thus, the matrix AT equals

$$AT = \begin{pmatrix} r_1\xi_1^{(1)} & r_2\xi_1^{(2)} & \dots & r_n\xi_1^{(n)} \\ r_1\xi_2^{(1)} & r_2\xi_2^{(2)} & \dots & r_n\xi_2^{(n)} \\ . & . & \dots & . \\ . & . & \dots & . \\ r_1\xi_n^{(1)} & r_2\xi_n^{(2)} & \dots & r_n\xi_n^{(n)} \end{pmatrix} = T\begin{pmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & r_n \end{pmatrix}$$

$$= T\mathrm{diag}\,(r_1,\ r_2\,,\dots r_n) \equiv TD$$

$$\Rightarrow D = T^{-1}AT$$

Relation to diagonalisation

$D = T^{-1}AT \Leftrightarrow$ change of basis.

To see this more explicitly, consider the change of variables

$$\boldsymbol{x} = T\,\boldsymbol{y} \Rightarrow T\,\frac{d\boldsymbol{y}}{dt} = AT\,\boldsymbol{y} \Rightarrow \frac{d\boldsymbol{y}}{dt} = T^{-1}AT\,\boldsymbol{y} = D\boldsymbol{y}\,.$$

In the new variables, the solution to the linear ODE system is trivial:

$$\boldsymbol{y}^{(i)} = e^{r_i t}\,\boldsymbol{e}^{(i)} = e^{r_i t}\begin{pmatrix} 0 \\ . \\ 1 \\ 0 \\ . \\ 0 \end{pmatrix} \quad 1 \text{ i-th component}$$

Thus, its fundamental matrix equals

$$Q(t) = e^{Dt} = \operatorname{diag}\left(e^{r_1 t},\, e^{r_2 t},\, \ldots,\, e^{r_n t}\right)$$

Relation to diagonalisation

The fundamental matrix in the original variables $\boldsymbol{x}$ is

$$\Psi(t) = T\,Q(t) = T\,\mathrm{diag}\left(e^{r_1 t},\, e^{r_2 t},\, \ldots, e^{r_n t}\right).$$

The exponential matrix is

$$e^{At} = \Psi(t)\Psi^{-1}(0) = TQT^{-1} = T\mathrm{diag}\left(e^{r_1 t},\, e^{r_2 t},\, \ldots, e^{r_n t}\right)\, T^{-1}.$$

Alternative derivation: note that $T^{-1}A^k T = \mathrm{diag}(r_1^k, \cdots, r_n^k)$,

$$T^{-1}e^{At}T = \sum_{k=0}^{\infty} \frac{1}{k!}\mathrm{diag}(r_1^k, \cdots, r_n^k) = \mathrm{diag}(e^{r_1 t}, \cdots, e^{r_n t}).$$

But, this works when A is diagonalisable.

example

Last week, we solved this linear system

$$\frac{d\boldsymbol{x}}{dt} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \boldsymbol{x} \equiv A\boldsymbol{x}\,.$$

Its eigenvalues are:

$$\det\,(A - r\mathbb{I}) = (r-3)(r+1) = 0\,.$$

After solving for the eigenvectors, the general solution is

$$\boldsymbol{x} = c_1\, e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\, e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}\,.$$

Example

Take the two eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ and construct the change of basis matrix

$$T = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \Rightarrow T^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix}$$

Notice that:

$$T^{-1}AT = T^{-1}\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

Thus, when we change variable from $\boldsymbol{x}$ to $\boldsymbol{y}$, $\boldsymbol{x} = T\boldsymbol{y}$, the linear ODE system becomes diagonal,

$$\frac{d\boldsymbol{y}}{dt} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}\boldsymbol{y} \equiv D\boldsymbol{y}\,.$$

Example

The fundamental matrix in the original basis equals

$$\Psi(t) = TQ = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$

From this matrix we can indeed recover the original solution

$$\boldsymbol{x}^{(1)} = e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{x}^{(2)} = e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Honours Differential Equations

Jacques Vanneste

Lecture 11

October 11, 2018

Recap

The general solution of linear ODE systems

$$\boldsymbol{x}' = A\boldsymbol{x},$$

takes the form

$$\boldsymbol{x}(t) = \sum_{j=1}^{n} c_j \, e^{r_j t} \, \boldsymbol{\xi}^{(j)}.$$

This is equivalent to diagonalising the matrix A by changing the coordinates, $\boldsymbol{x} = T\boldsymbol{y}$ so that

$$\frac{d\boldsymbol{y}}{dt} = D\boldsymbol{y} \qquad D = \mathrm{diag}\,(r_1,\, r_2, \dots, r_n)\,,$$

where T is the matrix with the eigenvectors $\boldsymbol{\xi}^{(i)}$ as columns. We can also write

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0, \quad \text{where} \;\; e^{At} = T\mathrm{diag}(e^{r_1 t}, \cdots, e^{r_n t})\,T^{-1}.$$

Recap

This connection emphasizes the importance of dealing with diagonalisable matrices A.

But:

- Eigenvalues with geometric multiplicity $<$ algebraic multiplicy lead to non-diagonialisable matrices.

Strategy:

- Find additional linearly independent solutions using our intuition,
- Complete answer provided by the Jordan normal forms (Honours Algebra course, 2nd semester).

Example

Consider the linear ODE system

$$\boldsymbol{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \boldsymbol{x} \equiv A\,\boldsymbol{x}\,.$$

Eigenvalues: $r = 2$ with multiplicity s=2

$$\begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = (r-2)^2 = 0$$

Eigenvectors: $\boldsymbol{\xi} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{aligned} (1-r)x - y = 0 &\Rightarrow x + y = 0 \\ x + (3-r)y = 0 &\Rightarrow x + y = 0 \Rightarrow x = -y \end{aligned}$$

Thus, $\boldsymbol{\xi} \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix}$: geometric multiplicity is $1 < 2$.

Example

The general solution requires two linearly independent solutions; we are missing one.

Idea:

- Similar situation for n-th order linear ODEs with roots of the characteristic equation with multiplicity $s \geq 2$,
- we considered solutions with e^{rt} was to $t\, e^{rt}$,
- can the same idea work here?

Example

Let us try

$$\boldsymbol{x} = t\, e^{2t}\, \boldsymbol{\xi} \Rightarrow \frac{d\boldsymbol{x}}{dt} = \boldsymbol{\xi}\, e^{2t} + 2t\, e^{2t}\, \boldsymbol{\xi}$$

Require this to be $A\boldsymbol{x}$:

$$A\boldsymbol{x} = t\, e^{2t}\, A\boldsymbol{\xi} = 2t\, e^{2t}\, \boldsymbol{\xi}$$

Lesson: It does not work (unless $\boldsymbol{\xi} = 0$)

- Should include an extra unknown vector $\boldsymbol{\eta}$ in our trial solution.

Example

Let us try

$$\boldsymbol{x}(t) = t\, e^{2t}\, \boldsymbol{\xi} + \boldsymbol{\eta}\, e^{2t} \Rightarrow \frac{d\boldsymbol{x}}{dt} = 2t\, e^{2t} \boldsymbol{\xi} + (\boldsymbol{\xi} + 2\boldsymbol{\eta})\ e^{2t}$$

Require this to be $A\boldsymbol{x}$:

$$A\boldsymbol{x} = A\left(t\, e^{2t}\, \boldsymbol{\xi} + e^{2t} \boldsymbol{\eta}\right) = 2t\, e^{2t}\, \boldsymbol{\xi} + e^{2t}\, A\boldsymbol{\eta}$$

Lesson: need $\boldsymbol{\eta}$ to satisfy

$$(A - 2\mathbb{I})\, \boldsymbol{\eta} = \boldsymbol{\xi}\,.$$

This equation has a non-trivial solution (because $\boldsymbol{\xi}$ is an eigenvector of A).
We say that $\boldsymbol{\eta}$ is a generalised eigenvector.

Example

Let $\boldsymbol{\eta} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$\begin{aligned} -x - y &= 1 \\ x + y &= -1 \Rightarrow y = -1 - x\,. \end{aligned}$$

Thus, the general solution for the vector $\boldsymbol{\eta}$ is

$$\boldsymbol{\eta} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + x \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + x\,\boldsymbol{\xi}.$$

Thus, our proposal for the second linearly independent solution is:

$$\boldsymbol{x}^{(2)} = t\,e^{2t}\,\boldsymbol{\xi} + e^{2t} \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k\,e^{2t}\,\boldsymbol{\xi}$$

Notice the last term is proportional to our first solution $e^{2t}\,\boldsymbol{\xi}$.
It can be ignored (or absorbed into a redefinition of our constants)

Example

Thus,

$$\boldsymbol{x}^{(2)} = t\, e^{2t}\, \boldsymbol{\xi} + e^{2t} \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

and the general solution to our linear ODE system is

$$\begin{aligned} \boldsymbol{x}(t) &= c_1 \boldsymbol{x}^{(1)}(t) + c_2 \boldsymbol{x}^{(2)}(t) \\ &= c_1\, e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \left[t\, e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] . \end{aligned}$$

Are both solutions linearly independent?

$$W[\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}] = \begin{vmatrix} e^{2t} & t\, e^{2t} \\ -e^{2t} & -e^{2t}(1+t) \end{vmatrix} = -e^{4t} \neq 0$$

They are $\Rightarrow$ problem solved.

Connection to matrix methods

Given the general solution, the fundamental matrix equals

$$\Psi(t) = \left(\boldsymbol{x}^{(1)},\, \boldsymbol{x}^{(2)}\right) = \begin{pmatrix} e^{2t} & t\,e^{2t} \\ -e^{2t} & -e^{2t}(1+t) \end{pmatrix}.$$

Check: since

$$\Psi(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \Rightarrow \Psi^{-1}(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

$$e^{A\,t} = \Psi(t)\,\Psi^{-1}(0) = e^{2t}\begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix}.$$

Question: what is the form of the matrix A in the basis $(\boldsymbol{\xi}, \boldsymbol{\eta})$?

Connection to matrix methods

Let us follow the construction in our previous lecture.
Build a matrix out of the two vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$:

$$T = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \Rightarrow T^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

Notice:

$$T^{-1}\, A\, T = T^{-1} \begin{pmatrix} 2 & 1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = J_2$$

J_2 is an upper triangular matrix of the form

$$J_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \text{Jordan form.}$$

Connection to matrix methods

To check the consistency of our approach,
consider the change of variables $\boldsymbol{x} = T\boldsymbol{y}$, then

$$\frac{d\boldsymbol{y}}{dt} = T^{-1}AT\,\boldsymbol{y} \equiv J_2\,\boldsymbol{y}$$

Using $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, the new linear ODE system is equivalent to:

$$\begin{aligned} y_2' &= 2y_2 \Rightarrow y_2(t) = c_2\,e^{2t} \\ y_1' &= 2y_1 + y_2 \Rightarrow y_1(t) = c_1\,e^{2t} + c_2\,t\,e^{2t}\,. \end{aligned}$$

Thus, its general solution can be written as

$$\boldsymbol{y}(t) = c_1\,e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\,e^{2t}\begin{pmatrix} t \\ 1 \end{pmatrix}\,.$$

Connection to matrix methods

The fundamental matrix of the new ODE system is:

$$Q(t) = \begin{pmatrix} e^{2t} & t\,e^{2t} \\ 0 & e^{2t} \end{pmatrix} \Rightarrow \Psi(t) = TQ(t) = \begin{pmatrix} e^{2t} & t\,e^{2t} \\ -e^{2t} & -e^{2t}(1+t) \end{pmatrix}.$$

Lesson: The matrix A is non-diagonalisable

- The best we can do is to bring it to an upper triangular form $\equiv$ Jordan form
$$J_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$
- Linear algebra result: all matrices can be reduced to blocks that are either diagonal or have Jordan form.

Connection to matrix methods

- A system in Jordan form is easy to integrate:

$$\frac{d\boldsymbol{y}}{dt} = J_\lambda \boldsymbol{y} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \boldsymbol{y}.$$

  Integrate from the bottom up:

$$\begin{aligned} y_2' &= \lambda y_2 \\ y_1' &= \lambda y_1 + y_2 \end{aligned}$$

  Compute the fundamental matrix in the original variables $\boldsymbol{x}$:
  $\Psi(t) = T\,Q(t)$ where $Q(t)$ is the fundamental matrix in the $\boldsymbol{y}$ variables and T the change of basis matrix.
- The exponential of Jordan blocks has also a memorable form

$$e^{J_\lambda t} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.$$

General method

Given a general 2×2 linear ODE system

$$\boldsymbol{x}' = A\,\boldsymbol{x}$$

with $\det(A - r\mathbb{I}) = (r - \lambda)^2 = 0$ & having a single eigenvector $\boldsymbol{\xi}_\lambda$

- One solution is $\boldsymbol{x}^{(1)} = e^{\lambda t}\,\boldsymbol{\xi}_\lambda$
- Second solution is of the form

$$\boldsymbol{x}^{(2)} = t\,e^{\lambda t}\,\boldsymbol{\xi}_\lambda + e^{\lambda t}\,\boldsymbol{\eta}$$

  where $(A - \lambda\mathbb{I})\,\boldsymbol{\eta} = \boldsymbol{\xi}_\lambda$.
- $\boldsymbol{\eta}$ is a generalised eigenvector.

General method

The methods explained extend to $n \times n$ linear ODE systems

For example consider a 3×3 system with eigenvalue λ having algebraic multiplicity 3 and geometric multiplicity 1 (single eigenvector $\boldsymbol{\xi}_\lambda$).

- One solution is $\boldsymbol{x}^{(1)} = e^{\lambda t}\,\boldsymbol{\xi}_\lambda$.
- Second solution is of the form

$$\boldsymbol{x}^{(2)} = t\,e^{\lambda t}\,\boldsymbol{\xi}_\lambda + e^{\lambda t}\,\boldsymbol{\eta}$$
$$\boldsymbol{\xi}_\lambda = (A - \lambda\mathbb{I})\,\boldsymbol{\eta}\,.$$

- Third solution is of the form

$$\boldsymbol{x}^{(3)} = \frac{t^2}{2}\,e^{\lambda t}\,\boldsymbol{\xi}_\lambda + t\,e^{\lambda t}\,\boldsymbol{\eta} + e^{\lambda t}\boldsymbol{\zeta}$$
$$\boldsymbol{\eta} = (A - \lambda\mathbb{I})\,\boldsymbol{\zeta}\,.$$

(See Assignment 3)
The method is algorithmic.

Honours Differential Equations

Jacques Vanneste

Lecture 12

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Non-homogeneous ODE systems

We want to solve linear ODE systems of the form

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t).$$

The general solution is of the form

$$\mathbf{x}(t) = \sum_i c_i \, \mathbf{x}^{(i)}(t) + \mathbf{x}_{\text{par}}(t),$$

where $\sum_i c_i \, \mathbf{x}^{(i)}(t)$ is the general solution of the homogeneous ODE system.

Non-homogeneous ODE systems

In what follows, we focus on systems of the form

$$\mathbf{x}' = A\,\mathbf{x} + \mathbf{g}(t)\,,$$

where A is an $n \times n$ constant matrix.

We shall discuss three different methods:

1. Diagonalisation (matrix methods),
2. Undetermined coefficients,
3. Variation of parameters.

Diagonalisation

Consider a non-homogeneous system

$$\mathbf{x}' = A\,\mathbf{x} + \mathbf{g}(t)\,.$$

Assume the corresponding homogeneous system is solved with eigenvalues r_i and eigenvectors $\boldsymbol{\xi}^{(i)}$
Introduce the change of variables $\mathbf{x} = T\,\mathbf{y}$:

$$T\mathbf{y}' = AT\,\mathbf{y} + \mathbf{g}(t) \Rightarrow \frac{d\mathbf{y}}{dt} = D\,\mathbf{y} + T^{-1}\mathbf{g} \equiv D\,\mathbf{y} + \mathbf{h}$$

System of n decoupled equations $\Rightarrow$ direct integration,

$$y_i' = r_i y_i + h_i \Rightarrow y_i(t) = c_i\,e^{r_i t} + e^{r_i t}\int_{t_0}^{t} e^{-r_i s} h_i(s) ds\,.$$

Diagonalisation

Check:

$$y_i' = r_i y_i + e^{r_i t}\, e^{-r_i t}\, h_i(t)\,.$$

General solution in the original variables equals

$$\mathbf{x}(t) = T\mathbf{y}(t).$$

Remarks:

- The particular solution is encoded in the $h_i(t)$ part.
- We assumed that A is diagonalisable (later, we will discuss the Jordan form case)
- The power of matrix methods is more apparent when dealing with non-homogeneous ODE systems.

Example

Consider

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}.$$

Since the system is linear and non-homogeneous, the general solution must be of the form

$$\mathbf{x}(t) = c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} + \mathbf{x}_{\text{par}}(t).$$

Homogeneous solution: eigenvalues & eigenvectors $\mathbf{x}_{\text{hom}} = e^{rt}\,\xi$

$$\begin{vmatrix} -2-r & 1 \\ 1 & -2-r \end{vmatrix} = r^2 + 4r + 3 = (r+3)(r+1) = 0$$

Example

When $r = -3$: if $\boldsymbol{\xi}_{-3} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow y = -x$. I will choose

$$\boldsymbol{\xi}_{-3} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(notice that $\boldsymbol{\xi}_{-3} \cdot \boldsymbol{\xi}_{-3} = 1$)
When $r = -1$: if $\boldsymbol{\xi}_{-1} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow y = x$. I will choose

$$\boldsymbol{\xi}_{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus, general homogeneous solution is

$$\mathbf{x}_{\text{hom}} = c_1\, e^{-3t}\, \boldsymbol{\xi}_{-3} + c_2\, e^{-t}\, \boldsymbol{\xi}_{-1}\,.$$

Example

To find the particular solution, we change variables $\mathbf{x} = T\mathbf{y}$ with

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Rightarrow T^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

The new variables satisfy the linear ODE:

$$\begin{aligned} \frac{d\mathbf{y}}{dt} &= \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{y} + T^{-1}\mathbf{g} \\ &= \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{y} + \frac{1}{\sqrt{2}}\begin{pmatrix} 2e^{-t} - 3t \\ 2e^{-t} + 3t \end{pmatrix} \end{aligned}$$

In terms of $\mathbf{y}^T = (y_1, y_2)$, this is equivalent to

$$y_1' + 3y_1 = \sqrt{2}\, e^{-t} - \frac{3}{\sqrt{2}}\, t \quad \text{(linear non-homogeneous 1st order)}$$

$$y_2' + y_2 = \sqrt{2}\, e^{-t} + \frac{3}{\sqrt{2}}\, t \quad \text{(linear non-homogeneous 1st order)}$$

Example

General solution must be of the form (using undetermined coefficients):

$$y_1(t) = c_1\, e^{-3t} + a\, e^{-t} + b\, t + c\,,$$
$$y_2(t) = c_2\, e^{-t} + f\, t\, e^{-t} + h\, t + m\,.$$

Algebra (& patience) determine:

$$y_1(t) = c_1\, e^{-3t} + \frac{\sqrt{2}}{2}\, e^{-t} + -\frac{3}{\sqrt{2}}\left(\frac{t}{3} - \frac{1}{9}\right)\,,$$
$$y_2(t) = c_2\, e^{-t} + \sqrt{2}\, t\, e^{-t} + \frac{3}{\sqrt{2}}\,(t-1)\,.$$

Example

We must write the solution in the original variables
Thus,

$$\mathbf{x}(t) = T\,\mathbf{y}(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} y_1 + y_2 \\ -y_1 + y_2 \end{pmatrix} (t)\,.$$

Expanding and grouping terms having the same t functional dependence, we get

$$\begin{aligned} \mathbf{x}(t) = {} & k_1\, e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2\, e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ & + \frac{1}{2}\, e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + t\, e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}\,. \end{aligned}$$

First line: general homogeneous solution
Second line: particular solution

Jordan form systems

We assumed A was diagonalisable, but the same method works when it can only be brought into Jordan form
Consider the change of variables $\mathbf{x} = T\,\mathbf{y}$ so that

$$\frac{d\mathbf{y}}{dt} = J\,\mathbf{y} + T^{-1}\mathbf{g} \equiv J\,\mathbf{y} + \mathbf{h}$$

These are not decoupled, but can be integrated in the appropriate order.
Example: consider

$$\mathbf{y}' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$\begin{aligned} y_1' &= \lambda\, y_1 + y_2 + h_1 \\ y_2' &= \lambda\, y_2 + h_2 \end{aligned}.$$

Method of undetermined coefficients

When the non-homogeneous system

$$\mathbf{x}' = A\,\mathbf{x} + \mathbf{g}(t)\,,$$

has a vector $\mathbf{g}(t)$ built out of polynomials, real or complex exponentials, you may consider the application of this method.

- It has the same rules as before: we assume a similar t dependence for $\mathbf{x}_{\text{par}}$ as in $\mathbf{g}(t)$
- One exception: if $\mathbf{g}(t) = \mathbf{u}\,e^{\lambda t}$ and λ is an eigenvalue of A with multiplicity 1, then

$$\mathbf{x}_{\text{par}} = t\,e^{\lambda t}\,\mathbf{a} + e^{\lambda t}\,\mathbf{b}$$

The term $e^{\lambda t}\,\mathbf{b}$ must be included to find a solution.
If the multiplicity is n, we must write $\mathbf{x}_{\text{par}} = e^{\lambda t}\sum_{i=0}^{n} t^i \mathbf{a}_i$.

Same example, different method

Consider, as before,

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} .$$

Since the eigenvalues of A equal $\lambda = -3,\ -1$ and the vector $\mathbf{g}(t)$ equals

$$\mathbf{g}(t) = e^{-t} \begin{pmatrix} 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

we look for a particular solution of the form

$$\mathbf{x}_{\text{par}}(t) = e^{-t} (\mathbf{a}\, t + \mathbf{b}) + \mathbf{c}\, t + \mathbf{d} .$$

The undetermined coefficients are the components of the vectors $\{\mathbf{a},\ \mathbf{b},\ \mathbf{c},\ \mathbf{d}\}$.

Same example, different method

To fix these coefficients, we introduce our guess into the non-homogeneous linear ODE. First,

$$\frac{d\mathbf{x}_{\text{par}}(t)}{dt} = e^{-t}\,(\mathbf{a} - \mathbf{a}\,t - \mathbf{b}) + \mathbf{c}\,.$$

Substituting into the linear ODE system,

$$(\mathbf{a} - \mathbf{b})\ e^{-t} - \mathbf{a}\,t\,e^{-t} + \mathbf{c} = A\mathbf{a}\,t\,e^{-t} + A\mathbf{b}\,e^{-t} + A\mathbf{c}\,t + A\mathbf{d} \\ + e^{-t}\begin{pmatrix} 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

Key observation: this set of equations must be satisfied $\forall\, t$.

Same example, different method

Thus, they are equivalent to 4 equations:

$$
\begin{aligned}
A\mathbf{a} &= -\mathbf{a} \quad (te^{-t}) \\
A\mathbf{b} &= \mathbf{a} - \mathbf{b} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (e^{-t}) \\
A\mathbf{c} &= -\begin{pmatrix} 0 \\ 3 \end{pmatrix} \quad (t) \\
A\mathbf{d} &= \mathbf{c} \quad (t^0)
\end{aligned}
$$

1st equation $\Rightarrow$ $\mathbf{a}$ is an eigenvector of A with eigenvalue $\lambda = -1$

$$\Rightarrow \mathbf{a} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$$

(from our previous analysis)

Same example, different method

Using this into the second equation: let $\mathbf{b} = (x, y)^{\mathrm{T}}$

Then, $A\mathbf{b} = \mathbf{a} - \mathbf{b} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ is equivalent to

$$-x + y = \alpha - 2$$
$$x - y = \alpha$$

Consistency requires $\alpha = 1$ & $y = -1 + x$. Thus,

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Third equation $A\mathbf{c} = -\begin{pmatrix} 0 \\ 3 \end{pmatrix} \Rightarrow \mathbf{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Fourth equation $A\mathbf{d} = \mathbf{c} \Rightarrow \mathbf{d} = -\begin{pmatrix} \frac{4}{3} \\ \frac{5}{3} \end{pmatrix}$

Same example, different method

Particular solution is:

$$\mathbf{x}_{\mathrm{par}}(t) = te^{-t}\begin{pmatrix}1\\1\end{pmatrix} - e^{-t}\begin{pmatrix}0\\1\end{pmatrix} + k\,e^{-t}\begin{pmatrix}1\\1\end{pmatrix} + t\begin{pmatrix}1\\2\end{pmatrix} - \frac{1}{3}\begin{pmatrix}4\\5\end{pmatrix}.$$

This agrees with our previous solution, obtained using diagonalisation, for $k = 1/2$.
Remember particular solutions are not unique.

Honours Differential Equations

Jacques Vanneste

Lecture 13

October 15, 2018

Non-homogeneous ODE systems

We are discussing non-homogeneous ODE systems

$$\frac{d\boldsymbol{x}}{dt} = P(t)\,\boldsymbol{x} + \boldsymbol{g}(t)\,.$$

When $P(t) = A$ is constant, we have discussed two methods to solve this problem:

- Diagonalisation: matrix algebra methods
- Undetermined coefficients

Today, we discuss variation of parameters. Its application is analogous to what we saw for higher-order differential equations.

Variation of parameters

Consider the system

$$\frac{d\boldsymbol{x}}{dt} = P(t)\,\boldsymbol{x} + \boldsymbol{g}(t)\,.$$

Assume the homogeneous problem

$$\frac{d\boldsymbol{x}_{\text{hom}}}{dt} = P(t)\,\boldsymbol{x}_{\text{hom}}\,,$$

is solved by

$$\boldsymbol{x}_{\text{hom}} = \Psi(t)\boldsymbol{c}\,,$$

for some constant $\boldsymbol{c}$.

The method consists in looking for solutions to the non-homogeneous problem of the form

$$\boldsymbol{x}(t) = \Psi(t)\boldsymbol{u}(t)\,.$$

That is, $\boldsymbol{c} \mapsto \boldsymbol{u}(t)$ (variation of parameters)

Variation of parameters

Question: how do we determine $\boldsymbol{u}(t)$?

Introducing $\boldsymbol{x} = \Psi(t)\boldsymbol{u}(t)$ into the ODE system:

$$\frac{d\boldsymbol{x}}{dt} = \Psi'(t)\boldsymbol{u}(t) + \Psi(t)\frac{d\boldsymbol{u}}{dt} = P(t)\Psi(t)\boldsymbol{u}(t) + \boldsymbol{g}(t)\,.$$

Remember $\Psi' = P(t)\Psi$, thus

$$\Psi\,\frac{d\boldsymbol{u}}{dt} = \boldsymbol{g}(t) \Rightarrow \frac{d\boldsymbol{u}}{dt} = \Psi^{-1}\boldsymbol{g}$$
$$\Rightarrow \boldsymbol{u}(t) = \int^{t} \Psi^{-1}(s)\boldsymbol{g}(s)\,ds + \boldsymbol{f}\,.$$

Thus, the general solution is

$$\boldsymbol{x}(t) = \Psi(t)\Psi^{-1}(t_0)\boldsymbol{x}_0 + \Psi(t)\int_{t_0}^{t} \Psi^{-1}(s)\boldsymbol{g}(s)\,ds\,.$$

Variation of parameters

$$\boldsymbol{x}(t) = \Psi(t)\Psi^{-1}(t_0)\boldsymbol{x}_0 + \Psi(t)\int_{t_0}^{t} \Psi^{-1}(s)\boldsymbol{g}(s)\, ds\,.$$

- We recover the general solution as general homogeneous + particular.
- We chose the constant $\boldsymbol{f}$ so that the particular solution vanishes at $t = t_0$ (this can always be done).
- The method applies even if $P(t)$ is *not constant*.

Example

Let us solve

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x} + \boldsymbol{g} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} .$$

again, using variation of parameters.
Our previous analysis gave the fundamental matrix:

$$\Psi(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}$$

With

$$\Psi(t)\frac{d\boldsymbol{u}}{dt} = \boldsymbol{g} .$$

Example

With $\boldsymbol{u} = (u_1,\ u_2)^{\mathrm{T}}$, the above is equivalent to

$$u_1' = e^{2t} - \frac{3}{2} t\, e^{3t}\,,$$
$$u_2' = 1 + \frac{3}{2} t\, e^{t}\,.$$

Both equations are 1st order $\Rightarrow$ direct integration !!

$$u_1(t) = \frac{1}{2} e^{2t} - \frac{1}{2} t\, e^{3t} + \frac{1}{6}\, e^{3t} + c_1$$
$$u_2(t) = t + \frac{3}{2} t\, e^{t} - \frac{3}{2} e^{t} + c_2\,.$$

Substituting into $\boldsymbol{x}(t) = \Psi(t)\boldsymbol{u}(t)$ reproduces the solution using this method.

Qualitative theory of ODEs

Consider a nonlinear *autonomous* system

$$\frac{dx}{dt} = F(x, y)\,,$$
$$\frac{dy}{dt} = G(x, y)\,.$$

Since F, G can be arbitrary, the problem is generically nonlinear.

Question: Can we interpret the solution geometrically?

- View (x, y) as a point in $\mathbb{R}^2$,
- (F, G) defines a vector field,
- $(x(t),\, y(t))$ parameterises a curve in $\mathbb{R}^2$,
- Integrating a solution $\Leftrightarrow$ finding a curve whose tangent at (x, y) is given by $\frac{dx}{dt} = \dot{x} = F(x, y)$ and $\frac{dy}{dt} = \dot{y} = G(x, y)$.

Phase plane

Let us introduce some terminology:

Definition

- *The (x, y) plane will be referred to as phase plane.*
- *Solutions to the ODE system $\boldsymbol{x}(t) = (x(t),\, y(t))^{\mathrm{T}}$ describe curves in the phase plane, often thought as trajectories of point moving with velocity $\frac{dx}{dt}$ and $\frac{dy}{dt}$.*

Phase plane

Further remarks:

- Solutions depend on initial conditions
  $\Rightarrow$ different initial conditions correspond to different trajectories.
- A given ODE system gives rise to as many trajectories as different initial conditions.
- But, there are as many initial conditions as points in the phase plane.
- Plotting a representative set of trajectories will be referred to as the phase portrait of the given ODE system.
- For autonomous systems, trajectories cannot intersect.

Strategy

We want to

- either integrate a non-linear ODE system,
- or give some qualitative description.

Concerning full analytical integration:

- only *some* ODE systems can be integrated,
- in some cases, we can derive implicit relations $x = x(y)$ describing our non-linear trajectories,
- generically, one resorts to numerical integration.

Strategy

Qualitative description:

- **Local description:** consider the dynamics in a patch near some $\boldsymbol{x}_0$,
  - not very interesting for most $\boldsymbol{x}_0$,
  - focus on critical points $\boldsymbol{x}_0$, where $F(\boldsymbol{x}_0) = G(\boldsymbol{x}_0) = 0$,
  - use linearisation (first-order Taylor expansion) and what we have learned about linear systems for $\boldsymbol{x} \approx \boldsymbol{x}_0$,
  - gives a local description of the phase portrait.
- **Global description**: understand behaviour in the entire phase plane.

Rectification

Definition

A point $\mathbf{x}_0 = (x_0\ y_0)$ is a critical point if $F(x_0, y_0) = G(x_0, y_0) = 0$.

Away from critical points, the local dynamics is simple:

Theorem

Let $\mathbf{x}_* = (x_*, y_*)$ be such that $(F(\mathbf{x}_*, \mathbf{y}_*), G(\mathbf{x}_*, \mathbf{y}_*)) \neq 0$, then, in a neighbourhood of $\mathbf{x}_*$ there is a smooth change of variable $(\tilde{x}, \tilde{y}) = H(x, y)$ under which the ODE system reduces to

$$\frac{d\tilde{x}}{dt} = 1,$$
$$\frac{d\tilde{y}}{dt} = 0.$$

Honours Differential Equations

Jacques Vanneste

Lecture 14

October 18, 2018

Motivating the linear approximation

Consider a generic 2×2 non-linear ODE system:

$$x' = F(x, y),$$
$$y' = G(x, y).$$

Given a point $(x_0,\ y_0)$ in phase space, we can approximate F, G by their Taylor expansions in some open neighbourhood

$$\begin{aligned} F(x,y) &= F(x_0,\ y_0) + \partial_x F(x_0,\ y_0)\,(x - x_0) + \partial_y F(x_0,\ y_0)\,(y - y_0) \\ &\quad + \eta_1(x,y), \\ G(x,y) &= G(x_0,\ y_0) + \partial_x G(x_0,\ y_0)\,(x - x_0) + \partial_y G(x_0,\ y_0)\,(y - y_0) \\ &\quad + \eta_2(x,y), \end{aligned}$$

where

$$\frac{\eta_1(x,y)}{\|\boldsymbol{x} - \boldsymbol{x}_0\|}, \frac{\eta_2(x,y)}{\|\boldsymbol{x} - \boldsymbol{x}_0\|} \to 0 \quad \text{as} \quad (x,y) \to (x_0,\ y_0)$$

Linear approximation

Recall: a critical point $(x_0\ y_0)$ is one satisfying $F(x_0, y_0) = G(x_0, y_0) = 0$.

To explore the evolution of the system around $(x_0\ y_0)$, it is natural to introduce new variables

$$u_1 \equiv x - x_0\,, \qquad u_2 \equiv y - y_0\,.$$

These satisfy:

$$\begin{pmatrix} \frac{du_1}{dt} \\ \frac{du_2}{dt} \end{pmatrix} = \begin{pmatrix} \partial_x F(x_0,\ y_0) & \partial_y F(x_0,\ y_0) \\ \partial_x G(x_0,\ y_0) & \partial_y G(x_0,\ y_0) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

Linear approximation

The linear approximation consists in dropping (η_1, η_2)
Thus, we are left with

$$\frac{d\boldsymbol{u}(t)}{dt} = A\,\boldsymbol{u}\,, \qquad A \equiv \begin{pmatrix} \partial_x F(x_0, y_0) & \partial_y F(x_0, y_0) \\ \partial_x G(x_0, y_0) & \partial_y G(x_0, y_0) \end{pmatrix}.$$

where $\boldsymbol{u}(t) = (u_1, u_2)^{\mathrm{T}}$. A is the Jacobian matrix.

Conclusion:

- Locally, around any critical point, nonlinear ODEs $\approx$ linear ODEs.

Local analysis: strategy

Given a nonlinear ODE system, we can

- identify all its critical points,
- solve the linear approximation around each of them,
- use nonlinear methods to connect linear behaviours and produce a global picture of the dynamics.

First we

- classify all possible linear behaviours,
- relate the linear dynamics to the stability of the critical point.

Example

Consider Newton's law for a pendulum with friction:

$$mL^2 \frac{d^2\theta}{dt^2} = -cL\frac{d\theta}{dt} - mgL\sin\theta \Leftrightarrow \frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega^2 \sin\theta = 0\,,$$

with $\gamma = \frac{c}{mL}\,, \quad \omega^2 = \frac{g}{L}$.

- Map the 2nd order ODE to a 1st order ODE system: $x = \theta$ and $y = \frac{d\theta}{dt}$

$$\frac{dx}{dt} = y\,, \qquad \frac{dy}{dt} = -\omega^2 \sin x - \gamma\, y\,.$$

- Critical points:

$$\frac{dx}{dt} = 0 \Rightarrow y = 0$$
$$\frac{dx}{dt} = 0 \Rightarrow x = n\pi\,, \quad n \in \mathbb{Z}$$

Example

Jacobian matrix:

$$F(x,y) = y \Rightarrow \partial_x F = 0\,, \quad \partial_y F = 1$$
$$G(x,y) = -\omega^2 \sin x - \gamma\, y \Rightarrow \partial_x G = -\omega^2 \cos x\,, \quad \partial_y G = -\gamma.$$

- Critical point $(x_0, y_0) = (2n\pi,\, 0)$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

where $u = x - 2n\pi$ and $v = y$.

Example

- Critical point $(x_0, y_0) = ((2n+1)\pi, 0)$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

where $u = x - (2n+1)\pi$ and $v = y$.

Notice how the information about the given critical point is encoded in a single sign matrix element.

Example

Solutions of the linear approximation: $(u, v) = e^{\lambda t} \boldsymbol{\xi}^{\mathrm{T}}$ leads to the eigenvalues λ.

- $(x_0, y_0) = (2n\pi, 0)$: $\lambda = (-\gamma \pm \sqrt{\gamma^2 - 4\omega^2})/2$,
- $(x_0, y_0) = ((2n+1)\pi, 0)$: $\lambda = (-\gamma \pm \sqrt{\gamma^2 + 4\omega^2})/2$.

Behaviour depends on sign of $\mathrm{Re}\,\lambda$ and differ for the two types of critical points: lower equilibrium is attracting, the upper equilibrium repelling.

Classification of critical points

- In the linear approximation,

$$\text{critical point} \Leftrightarrow \frac{d\boldsymbol{u}}{dt} = 0 = A\boldsymbol{u}.$$

  If $\det A \neq 0 \;\Rightarrow \boldsymbol{u} = 0$ is the unique critical point.
- Classification of critical points $\Leftrightarrow$ Classification of eigenvalues of A
  - This is because $\boldsymbol{u}(t) \equiv$ trajectory in phase space,
  - Different eigenvalues $\Leftrightarrow$ different trajectories,
  - Thus, different critical point behaviours $\Leftrightarrow$ different eigenvalues of A.

Real unequal eigenvalues

(i) negative $r_1 < r_2 < 0$.

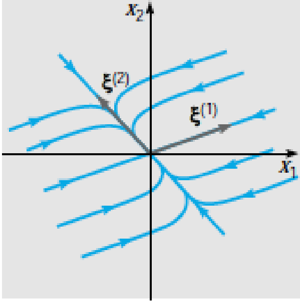
Solution is

$$\boldsymbol{x} = c_1\, e^{r_1 t}\, \boldsymbol{\xi}_1 + c_2\, e^{r_2 t}\, \boldsymbol{\xi}_2 = e^{r_2 t}\left[c_1\, e^{(r_1 - r_2)t} \boldsymbol{\xi}_1 + c_2 \boldsymbol{\xi}_2\right].$$

The solution always approaches the critical point.

- if $c_2 \neq 0$, $\boldsymbol{x} \to 0$ along $\boldsymbol{\xi}_2$ direction
- if $c_2 = 0$, $\boldsymbol{x} \to 0$ along $\boldsymbol{\xi}_1$ direction

Node (nodal sink)



Real unequal eigenvalues

(ii) $r_1 > r_2 > 0$.

Solution is

$$\boldsymbol{x} = c_1\, e^{r_1 t}\, \boldsymbol{\xi}_1 + c_2\, e^{r_2 t}\, \boldsymbol{\xi}_2 = e^{r_1 t} \left[c_1\, \boldsymbol{\xi}_1 + c_2\, e^{(r_2 - r_1)t} \boldsymbol{\xi}_2 \right]$$

The solution always gets away from the critical point.
This critical point is called Node (nodal source):

- Similar graph as before, but flipping arrows,
- If $c_1 \neq 0$, trajectories approach the one by $\boldsymbol{\xi}_1$ as $t \to \infty$,
- Only when $c_1 = 0$, they do it along $\boldsymbol{\xi}_2$.

Real unequal eigenvalues

(ii) different signs $r_1 > 0 > r_2$.

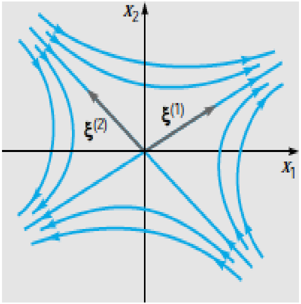
Solution is

$$\boldsymbol{x} = c_1\, e^{r_1 t}\, \boldsymbol{\xi}_1 + c_2\, e^{r_2 t}\, \boldsymbol{\xi}_2\,.$$

Hence $|\boldsymbol{x}| \to \infty$ as $t \to \pm\infty$ for most initial conditions.

- if $c_2 = 0 \Rightarrow |\boldsymbol{x}| \to \infty$ at $t \to \infty$ (it gets away along $\boldsymbol{\xi}_1$)
- if $c_1 = 0 \Rightarrow |\boldsymbol{x}| \to 0$ at $t \to \infty$ (it approaches along $\boldsymbol{\xi}_2$)
- if $c_1,\ c_2 \neq 0$, $|\boldsymbol{x}| \to \infty$ along $\boldsymbol{\xi}_1$ because $e^{r_1 t}$ dominates

Saddle point



Real equal eigenvalues

(iii) $r_1 = r_2 = r < 0$: double eigenvalues (different sign: just flip arrows).

Two cases to consider:

- $\exists$ two independent eigenvectors
- $\exists$ a single independent eigenvector

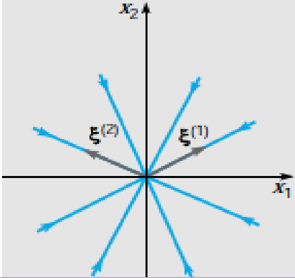
When two independent eigenvectors exist:

$$\boldsymbol{x} = c_1\, e^{rt}\, \xi_1 + c_2\, e^{rt}\, \xi_2\,.$$

Notice any ratio of x_2/x_1 is t independent $\Rightarrow$ straight line.

Trajectories approach the critical point

Proper node (star point)



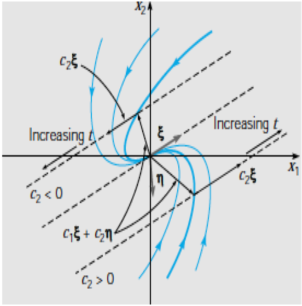
Real equal eigenvalues

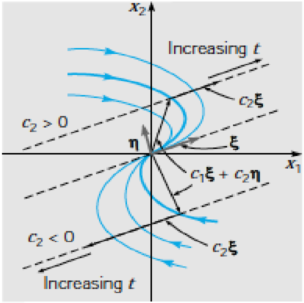
(iii) $r_1 = r_2 = r < 0$ with one independent eigenvector $\boldsymbol{\xi}$:

$$\boldsymbol{x} = c_1\, e^{rt}\, \boldsymbol{\xi} + c_2\, e^{rt}\, (t\boldsymbol{\xi} + \eta) = e^{rt}\left[c_1\, \boldsymbol{\xi} + c_2 \boldsymbol{\eta} + c_2\, t\, \boldsymbol{\xi}\right] .$$

Since $r < 0$, trajectories approach the critical point.
As $t \to \infty$, the trajectory is dominated by $\boldsymbol{\xi}$ (even if $c_2 = 0$).

Improper (or degenerate) node





Honours Differential Equations

Jacques Vanneste

Lecture 15

October 19, 2018

Classification of critical points

Classification
critical points $\Leftrightarrow$ matrix eigenvalues

When eigenvalues are real & different

- same sign: node
- different sign: saddle

When eigenvalues are real & equal

- two independent eigenvectors: proper node (star point)
- one independent eigenvector: improper node

Today: we will discuss critical points $\Leftrightarrow$ complex eigenvalues

Real equal eigenvalues

(iii) $r_1 = r_2 = r < 0$: double eigenvalues (different sign: just flip arrows).

Two cases to consider:

- $\exists$ two independent eigenvectors
- $\exists$ a single independent eigenvector

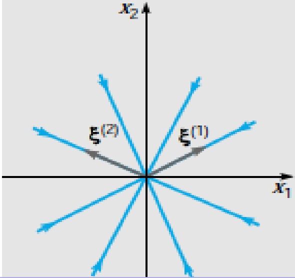
When two independent eigenvectors exist:

$$\mathbf{x} = c_1 \, e^{rt} \, \boldsymbol{\xi}_1 + c_2 \, e^{rt} \, \boldsymbol{\xi}_2 \, .$$

Notice any ratio of x_2/x_1 is t independent $\Rightarrow$ straight line.

Trajectories approach the critical point

Proper node (star point)



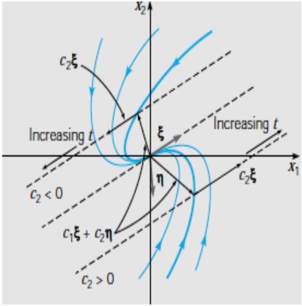
Real equal eigenvalues

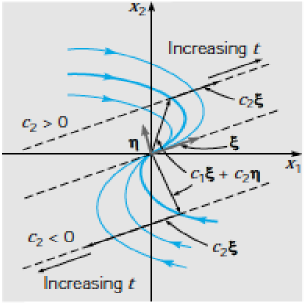
(iii) $r_1 = r_2 = r < 0$ with one independent eigenvector $\boldsymbol{\xi}$:

$$\mathbf{x} = c_1\, e^{rt}\, \boldsymbol{\xi} + c_2\, e^{rt}\, (t\boldsymbol{\xi} + \eta) = e^{rt}\, [c_1\, \boldsymbol{\xi} + c_2 \boldsymbol{\eta} + c_2\, t\, \boldsymbol{\xi}]\ .$$

Since $r < 0$, trajectories approach the critical point.
As $t \to \infty$, the trajectory is dominated by $\boldsymbol{\xi}$ (even if $c_2 = 0$).

Improper (or degenerate) node





Complex eigenvalues

(iv) complex eigenvalues $\lambda \pm i\mu$ ($\mu > 0$).

Eigenvectors are $\boldsymbol{\xi}$ and $\boldsymbol{\xi}^*$ and the transformation $\mathbf{x} = T\mathbf{y}$ leads to the diagonal matrix

$$T^{-1}AT = \begin{pmatrix} \lambda + i\mu & 0 \\ 0 & \lambda - i\mu \end{pmatrix}.$$

Make a further change of coordinates: $\mathbf{y} = P\mathbf{z}$, where

$$P = \begin{pmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{pmatrix}$$

corresponds to using $(\mathrm{Re}\,\boldsymbol{\xi}, \mathrm{Im}\,\boldsymbol{\xi})$ as a basis.
Then

$$\mathbf{z}' = P^{-1}T^{-1}ATP\,\mathbf{z} = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}\mathbf{z}.$$

Complex eigenvalues

For matrices of the form

$$A = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix},$$

$$\det(A - r\mathbb{I}) = (r-\lambda)^2 + \mu^2 = 0 \Rightarrow r = \lambda \pm i\mu$$

two linearly independent solutions are given by

$$e^{\lambda t}\begin{pmatrix} \cos(\mu t) \\ -\sin(\mu t) \end{pmatrix}, \quad e^{\lambda t}\begin{pmatrix} \sin(\mu t) \\ \cos(\mu t) \end{pmatrix},$$

Hence, the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\lambda t}\begin{pmatrix} c_1\cos(\mu t) + c_2\sin(\mu t) \\ -c_1\sin(\mu t) + c_2\cos(\mu t) \end{pmatrix} = Ce^{\lambda t}\begin{pmatrix} \cos(\phi - \mu t) \\ \sin(\phi - \mu t) \end{pmatrix},$$

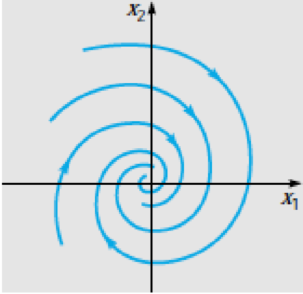
with C and ϕ arbitrary constants.

Complex eigenvalues

These solutions satisfy the identity :

$$x^2 + y^2 = C^2 e^{2\lambda t}.$$

- spirals towards origin if $\lambda < 0$: stable focus.
- diverges from origin if $\lambda > 0$: unstable focus.



In the original coordinates, 'elliptical' spiral.

Complex eigenvalues

An alternative way of reaching this conclusion.
Introduce polar coordinates in phase space

$$r^2 = x^2 + y^2 \,, \qquad \tan\phi = \frac{y}{x} \,.$$

(x, y) satisfy the ODE system

$$\begin{aligned} \dot{x} &= \lambda x + \mu y \\ \dot{y} &= \lambda y - \mu x \,. \end{aligned}$$

It follows

$$r\dot{r} = x\dot{x} + y\dot{y} = x(\lambda x + \mu y) + y(\lambda y - \mu x) = \lambda\, r^2 \,,$$

$$\frac{\dot{\phi}}{\cos^2\phi} = \frac{x\dot{y} - y\dot{x}}{x^2} \Rightarrow \dot{\phi} = -\mu$$

Complex eigenvalues

$$\dot{r} = \lambda r \Rightarrow r = c\, e^{\lambda t}$$
$$\dot{\phi} = -\mu \Rightarrow \phi = -\mu t + \phi_0\,.$$

- $\lambda > 0 \Rightarrow |\mathbf{x}| \to \infty$ for $t \to \infty$
- $\lambda < 0 \Rightarrow |\mathbf{x}| \to 0$ for $t \to \infty$
- ϕ decreases as t evolves (since $\mu > 0$) $\Rightarrow$ motion is clockwise

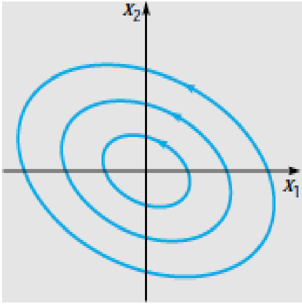
Conclusions are fully consistent with our spiral picture.

Purely imaginary eigenvalues

This corresponds to the particular case $\lambda = 0$

$$x^2 + y^2 = C^2 \,.$$

They correspond to ellipses in the original coordinates.



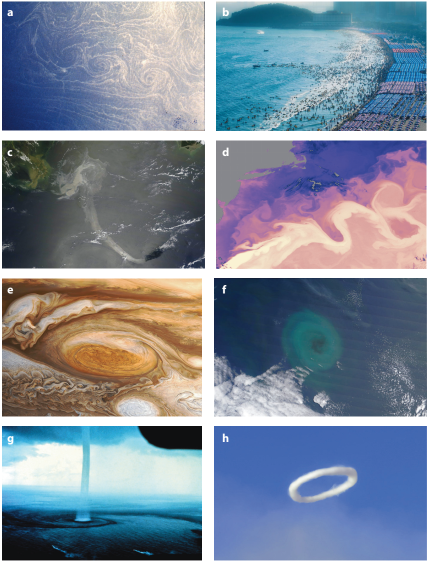
Summary

Critical points of different types have different behaviours as $t \to \infty$:

- $|\mathbf{x}| \to 0$ as $t \to \infty$
  1. real & negative eigenvalues: nodal sink
  2. complex eigenvalue with negative real part: stable focus
- bounded trajectory as $t \to \infty$: purely imaginary eigenvalue $\Rightarrow$ center
- $|\mathbf{x}| \to \infty$ as $t \to \infty$
  1. at least one eigenvalue is positive: saddle, nodal source
  2. complex eigenvalue with positive real part: unstable focus

This behaviour as $t \to \infty$ is related to the notion of stability.

Application: transport in atmosphere/ocean



Autonomous systems

Definition

An ODE system

$$\frac{dx}{dt} = F(x, y), \qquad \frac{dy}{dt} = G(x, y),$$

is referred to as an autonomous systems is F, G have no explicit time dependence.

This describes any physical system whose parameters, forces, etc … *do NOT depend on time*

Stability

We want to be mathematically more precise with the different mathematical behaviour satisfied by the different critical points.

Definition

A critical point $\mathbf{x}_0$ is stable if $\forall\,\epsilon$, $\exists\,\delta > 0$ such that every solution $\mathbf{x} = \phi(t)$ with $\|\phi(0) - \mathbf{x}_0\| < \delta$ at $t = 0$ satisfies

$$\|\phi(t) - \mathbf{x}_0\| < \epsilon, \quad \forall\, t > 0.$$

Definition

A critical point that is not stable is called unstable.

Stability

Definition

A critical point $\mathbf{x}_0$ *is asymptotically stable if it is stable and the solution* $\mathbf{x} = \phi(t)$ *is forced to approach* $\mathbf{x}_0$ *as* $t \to \infty$*, i.e.*

$$\lim_{t\to\infty} \phi(t) = \mathbf{x}_0 \,.$$

Stable and asymptotically stable are different concepts

Example: a center is stable, but not asymptotically stable.

Stability of linear systems

Eigenvalues	**Critical Points**	**Stability**
$r_1 > r_2 > 0$	Node (source)	unstable
$r_1 < r_2 < 0$	Node (sink)	asymp. stable
$r_2 < 0 < r_1$	saddle	unstable
$r_1 = r_2 > 0$	Proper/Improper node	unstable
$r_1 = r_2 < 0$	Proper/Improper node	asymp. stable
$r_1,\ r_2 = \lambda \pm i\mu\ (\lambda > 0)$	focus	unstable
$r_1,\ r_2 = \lambda \pm i\mu\ (\lambda < 0)$	focus	asymp. stable
$r_1 = i\mu\,, r_2 = -i\mu$	center	stable

Honours Differential Equations

Jacques Vanneste

Lecture 16

October 22, 2018

Global vs linear picture

- We classified all types of critical points and their stability properties

Questions:

- How can we put together a phase space portrait including the exact nonlinear ODE system?
- Can our linear calculations be modified at the nonlinear level? if so, when?

Today

- Example of stability analysis: damped pendulum
- Implicit trajectories as a nonlinear tool
- Almost linear system (qualitative discussion)

Pendulum with friction

By Newton's law ($\gamma = \frac{c}{mL}$, $\quad \omega^2 = \frac{g}{L}$):

$$mL^2 \frac{d^2\theta}{dt^2} = -cL\frac{d\theta}{dt} - mgL\sin\theta \Leftrightarrow \frac{d^2\theta}{dt^2} + \gamma\,\frac{d\theta}{dt} + \omega^2 \sin\theta = 0\,.$$

This is equivalent to the ODE system

$$\frac{dx}{dt} = y\,, \qquad \frac{dy}{dt} = -\omega^2 \sin x - \gamma\, y\,,$$

where $x = \theta$ and $y = \dot{\theta}$.

Critical points:

$$\begin{aligned}
\frac{dx}{dt} &= 0 \Rightarrow y = 0 \\
\frac{dx}{dt} &= 0 \Rightarrow x = n\pi\,, \quad n \in \mathbb{Z}
\end{aligned}$$

Pendulum with friction

Linear approximation:

- $(x_0, y_0) = (2n\pi,\ 0)$

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x - 2n\pi$ and $v = y$.

- Critical point $(x_0, y_0) = ((2n+1)\pi,\ 0)$

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x - (2n+1)\pi$ and $v = y$.

Pendulum with friction

We can jointly describe both types of critical points as follows:

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \epsilon\omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x - x_0$, $v = y$ and $\epsilon = \pm 1$.

Question: Let us study the stability of the two critical points
To do that, let us compute the eigenvalues of the constant matrix:

$$\lambda_\pm = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\epsilon\omega^2}}{2}.$$

Pendulum with friction

$$\lambda_{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\epsilon\omega^2}}{2}.$$

- When $\epsilon = 1 \Rightarrow \lambda_{\pm}$ real, different sign $\Rightarrow$ saddle point $\Rightarrow$ unstable
- When $\epsilon = -1$
  - if $\gamma = 0$ (no friction) $\Rightarrow$ center [periodic motion, stable, not asymptotically stable]
  - if $\gamma > 0 \Rightarrow$ asymptotically stable
    - if $\gamma^2 - 4\omega^2 > 0 \Rightarrow \lambda_{\pm} < 0$ and real $\Rightarrow$ node
    - if $\gamma^2 - 4\omega^2 = 0 \Rightarrow \lambda_{+} = \lambda_{-} < 0 \Rightarrow$ Proper/Improper node
    - if $\gamma^2 - 4\omega^2 < 0 \Rightarrow \lambda_{\pm}$ complex with negative real part $\Rightarrow$ spiral

Our linear stability analysis reproduces our intuition on the behaviour of pendulum critical points.

Nonlinear vs linear descriptions

Consider the ODE system:

$$\frac{dx}{dt} = 4 - 2y\,, \qquad \frac{dy}{dt} = 12 - 3x^2\,.$$

Critical points:: $(x_0, y_0) = (\pm 2, 2)$
Linear approximation: $u = x - x_0$, $v = y - y_0$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -6x_0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Eigenvalues:: $\lambda^2 - 12x_0 = 0$

- if $x_0 > 0 \Rightarrow \lambda_\pm$ real with different sign
  $\Rightarrow$ (2,2) is a saddle
- if $x_0 < 0 \Rightarrow \lambda_\pm$ purely imaginary
  $\Rightarrow$ (-2,2) is a center

Nonlinear vs linear descriptions

Question: Can we plot the nonlinear phase portrait for this ODE system?

Remark:

$$\left.\begin{array}{l} \frac{dx}{dt} = 4 - 2y \\ \frac{dy}{dt} = 12 - 3x^2 \end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{12 - 3x^2}{4 - 2y}$$

$$\Rightarrow (4 - 2y)dy = (12 - 3x^2)dx \Rightarrow 4y - y^2 = 12x - x^3 + c$$

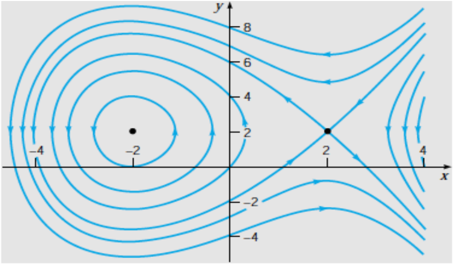
This is an exact (nonlinear) implicit description of the trajectories solving the ODE system (for any constant c)

- Plotting different values of $c \Leftrightarrow$ plotting different trajectories

Nonlinear vs linear descriptions

The nonlinear ODE system has an exact phase portrait given by

$$\frac{dx}{dt} = 4 - 2y\,, \qquad \frac{dy}{dt} = 12 - 3x^2\,,$$



- Notice how trajectories close to (-2,2) are indeed bounded
- (2,2) is indeed a saddle

Nonlinear vs linear descriptions

One lesson to take home: for those systems that admit implicit curves in the phase plane

$$\left.\begin{array}{l} \frac{dx}{dt} = F(x,y) \\ \frac{dy}{dt} = G(x,y) \end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{G(x,y)}{F(x,y)} \Rightarrow H(x,y) = c$$

as solutions to the ODE system, we can draw the exact trajectories

- This does not always happen!

Remark:

- The linear approximation eventually breaks down

Question:

- To which extent can we trust the information obtained in the linear approximation?

Almost linear systems

Consider the corrections to the linear approximation

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} \partial_x F(x_0,\, y_0) & \partial_y F(x_0,\, y_0) \\ \partial_x G(x_0,\, y_0) & \partial_y G(x_0,\, y_0) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

where $u = x - x_0$ and $v = y - y_0$, encoded in the vector $\boldsymbol{\eta}^t = (\eta_1,\, \eta_2)$.

Question: Does the type of critical point in the linear approximation change when we include $\boldsymbol{\eta} = (\eta_1,\, \eta_2)^{\mathrm{T}}$?

Theorem

Let r_1 and r_2 be the eigenvalues to the linear approximation corresponding to the almost linear system above, then the critical point $(0, 0)$ behaves as in the following table.

Stability properties of linear and almost linear systems

r_1, r_2	Linear System		Almost Linear System	
	Type	Stability	Type	Stability
$r_1 > r_2 > 0$	N	Unstable	N	Unstable
$r_1 < r_2 < 0$	N	Asymptotically stable	N	Asymptotically stable
$r_2 < 0 < r_1$	SP	Unstable	SP	Unstable
$r_1 = r_2 > 0$	PN or IN	Unstable	N or SpP	Unstable
$r_1 = r_2 < 0$	PN or IN	Asymptotically stable	N or SpP	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$				
$\lambda > 0$	SpP	Unstable	SpP	Unstable
$\lambda < 0$	SpP	Asymptotically stable	SpP	Asymptotically stable
$r_1 = i\mu, r_2 = -i\mu$	C	Stable	C or SpP	Indeterminate

- N: node PN: proper node IN: improper node
- SP: saddle point
- SpP: spiral (focus) point
- C: center

Center : linear vs non-linear

Consider ODE systems of the form :

$$\dot{x} = -y + x\,(x^2 + y^2)^n\,, \qquad \dot{y} = x + y\,(x^2 + y^2)^n$$

Critical points : $(x_0,\, y_0) = (0, 0)$
Linear system :

$$\begin{aligned}
F(x,y) &= -y + x\,(x^2+y^2)^n \\
&\Rightarrow \partial_x F = (x^2+y^2)^n + 2nx^2(x^2+y^2)^{n-1}\,, \\
&\quad\;\; \partial_y F = -1 + 2nyx(x^2+y^2)^{n-1} \\
G(x,y) &= x + y\,(x^2+y^2)^n \\
&\Rightarrow \partial_x G = 1 + 2nxy(x^2+y^2)^{n-1}\,, \\
&\quad\;\; \partial_y G = (x^2+y^2)^n + 2ny^2(x^2+y^2)^{n-1}\,.
\end{aligned}$$

Center: linear vs nonlinear

Evaluating at the critical point $(0,0)$, we obtain for $u = x$, $v = y$

$$\dot{u} = -v\,,\ \dot{v} = u \quad \Leftrightarrow \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

This corresponds to a center since eigenvalues $\lambda_{\pm} = \pm i$ are purely imaginary $\Leftrightarrow$ $\exists$ periodic trajectories (at least linearly)

Question: Does this interpretation remain true at the non-linear level?

Strategy: Change to polar coordinates and discuss the exact non-linear ODEs

Center: linear vs nonlinear

$$r^2 = x^2 + y^2 \,, \qquad \tan\phi = \frac{y}{x} \,.$$

$$r\dot{r} = x\dot{x} + y\dot{y} = x(-y + x\, r^{2n}) + y(x + y\, r^{2n}) = r^{2(n+1)} \,,$$

$$\frac{\dot{\phi}}{\cos^2\phi} = \frac{x\dot{y} - y\dot{x}}{x^2} \Rightarrow \dot{\phi} = 1$$

Integrating, we obtain :

$$\frac{1}{r^{2n}} - \frac{1}{r_0^{2n}} = -2nt \,, \qquad \phi = t + \phi_0 \,.$$

This describes a trajectory starting at (r_0, ϕ_0) :

- rotating anticlockwise
- as t increases, r increases
- in fact as $t \to \infty$, $r \to 0 \Rightarrow$ unstable

The linear centre trajectory is gone and it is replaced by an spiral.

Honours Differential Equations

Jacques Vanneste

Lecture 17

October 25, 2018

Today's lecture

We saw that the existence of linear centres is not guaranteed to survive at the nonlinear level.

Today we revisit the damped pendulum:

- to discuss how to properly draw the phase diagram in the linear approximation,
- to discuss the nonlinear phase portrait, stressing features that are more generic,
- introduce a prey-predator model.

Damped pendulum revisited

Pendulum with friction:

$$\frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + \omega^2 \sin\theta = 0\,,$$

- Friction is encoded in $\gamma = \frac{c}{mL}$ $(c > 0)$
- Newton's constant (gravity) is encoded in $\omega^2 = \frac{g}{L}$ $(g > 0)$
- L is the length of the pendulum

Map the 2nd order ODE to a 1st order ODE system:
$x = \theta$ and $y = \frac{d\theta}{dt}$

$$\frac{dx}{dt} = y \equiv F(x,y)\,, \qquad \frac{dy}{dt} = -\omega^2 \sin x - \gamma\, y \equiv G(x,y)\,.$$

Critical points & linear analysis

Critical points:

$$\frac{dx}{dt} = 0 \Rightarrow y = 0$$
$$\frac{dx}{dt} = 0 \Rightarrow x = n\pi\,, \quad n \in \mathbb{Z}$$

Linear approximation:

$$F(x,y) = y \Rightarrow \partial_x F = 0\,, \quad \partial_y F = 1$$
$$G(x,y) = -\omega^2 \sin x - \gamma\, y \Rightarrow \partial_x G = -\omega^2 \cos x\,, \quad \partial_y G = -\gamma$$

This gives rise to:

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \epsilon\omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x - x_0$, $v = y$ and $\epsilon = \pm 1$

- $\epsilon = -1$ when $x_0 = 2n\pi$
- $\epsilon = 1$ when $x_0 = (2n+1)\pi$

Critical points & linear analysis

Given the linear approximation, we compute its eigenvalues:

$$\begin{vmatrix} -\lambda & 1 \\ \epsilon\omega^2 & -\gamma - \lambda \end{vmatrix} = \lambda^2 + \gamma\lambda - \epsilon\omega^2 = 0$$
$$\Rightarrow \lambda_\pm = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\epsilon\omega^2}}{2}$$

Question: how do we draw a phase portrait?

Pick $\epsilon = -1$ and $\gamma^2 - 4\omega^2 < 0$: small damping $\Rightarrow$ spiral points

Stable critical points

Consider the critical point (0,0):

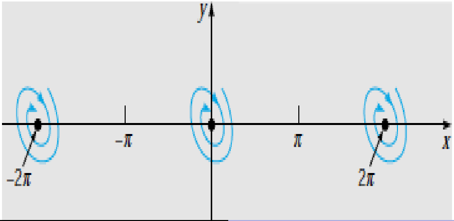
- to determine the arrow choose $x = 0$ (as a particular case) and evaluate the ODE system as that point

$$\frac{dx}{dt}(x = 0) = y\,, \quad \frac{dy}{dt}(x = 0) = -\gamma\, y\,.$$

  - if $y > 0$: x increases and y decreases
  - if $y < 0$: x decreases and y increases

  Thus, motion occurs clockwise

We can repeat the same analysis at the other critical points $(2n\pi, 0)$



Unstable critical points

Consider $\epsilon = 1$ and $x_0 = \pi$.

- Eigenvalues:

$$\lambda_{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\epsilon\omega^2}}{2} \Rightarrow \lambda_+ > 0,\ \lambda_- < 0$$

  This corresponds to a saddle point

- Linearly independent solutions:

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1\, e^{\lambda_+ t} \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} + c_2\, e^{\lambda_- t} \begin{pmatrix} 1 \\ \lambda_- \end{pmatrix}$$

  The only direction approaching the critical point at $t \to \infty$ is $c_1 = 0$

Unstable critical points

Focus on the $c_1 = 0$ direction

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_2 \, e^{\lambda_- t} \begin{pmatrix} 1 \\ \lambda_- \end{pmatrix}$$

The ratio of the solution components equals

$$\frac{v}{u} = \lambda_- < 0$$

Thus,

- if $c_2 > 0$: $u > 0, v < 0 \Rightarrow$ curve in 4th quadrant
- if $c_2 < 0$: $u < 0, v > 0 \Rightarrow$ curve in 2nd quadrant

Unstable critical points

Focus on the $c_2 = 0$ direction

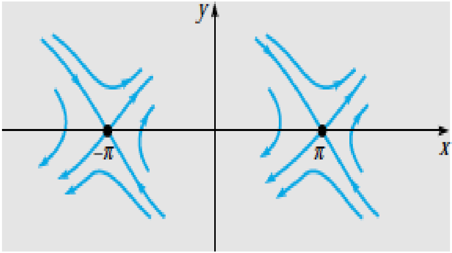
$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \, e^{\lambda_+ t} \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix}$$

The ratio of the solution components equals

$$\frac{v}{u} = \lambda_+ > 0$$

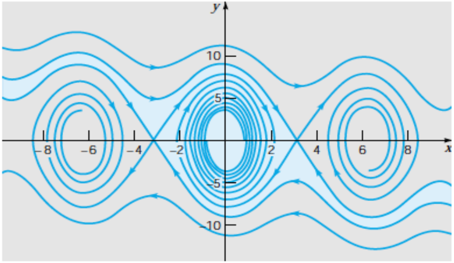
Thus,

- if $c_1 > 0$: $u > 0, v > 0 \Rightarrow$ curve in 1st quadrant
- if $c_1 < 0$: $u < 0, v < 0 \Rightarrow$ curve in 3rd quadrant



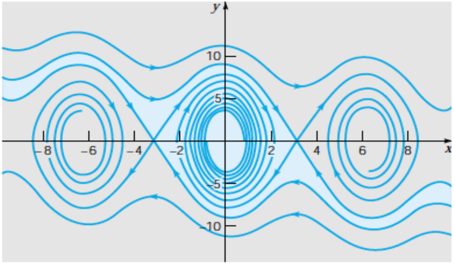
Nonlinear damped pendulum

When we plot different initial conditions and their associated trajectories, we obtain phase portraits of the form



- Critical points keep their nature and stability properties
- No matter how large $|y|$ (velocity) is, the existence of damping (friction) guarantees the solution will eventually stabilise around the stable pendulum point.
  - we will build on this comment next week to discuss nonlinear ODE systems in more generality

Nonlinear damped pendulum



- saddle points separate the entire phase space into regions satisfying the property that any trajectory in them asymptotes to a stable spiral point
  - the specific trajectory determining a change in such stable spiral point is called separatrix
- the set of all trajectories approaching a given asymptotically stable critical point is called basin of attraction
  - This is bounded by the separatrices through the neighbouring saddle points
  - For the origin, this is marked in the figure

Perturbed undamped pendulum (figures S S Abdullaev)

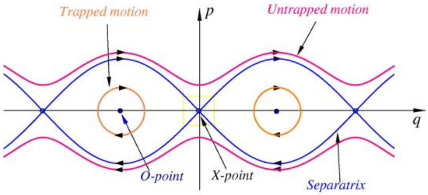


FIG. 4. Phase space (q, p) of the pendulum.

Complexity with periodic perturbation:

$$x' = y,$$
$$y' = -\omega^2 \sin x + \varepsilon \sin(x - t).$$

Poincaré, Kolmogorov, Arnold, Moser

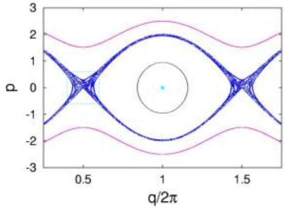


FIG. 7. Poincaré section of the pendulum under time-periodic perturbation.

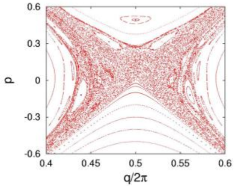


FIG. 8. Expanded view of the rectangular area near the X-point. The perturb

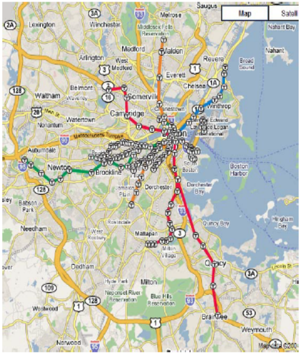
$\epsilon_a = 0.02$, a perturbation frequency $\Omega = 4.53236$, and a phase $\chi_a = 0$.

Nonlinear ODE systems

- In any nonlinear ODE system, it is important (and non-trivial.) to determine the basin of attraction of each asymptotically stable critical point
- Students interested in this topic are encouraged to enrol in courses dealing with dynamical systems.

What is modelling?

- The term modelling refers to the development of a mathematical representation of a physical situation.
- A model is like an idealization of reality: cf tube map vs real tube systems. (Mike Ashby, Cambridge University)
- A gross simplification, but one that captures essential elements.



"Physical situation"

"Model"

What is modelling?

- The map misrepresents distances and directions, but it elegantly displays the connectivity
- The quality or usefulness in a model is measured by its ability to capture governing key features of the problem.
- At worst, a model is a concise description of a body of data. At best, it captures the essentce of the problem, it illuminates the principles underlying the key observations,
- Models should it predict behaviour under conditions which have not yet been studied.

What is modelling?

- Any scientific theory is a model.
- Different models (theories) may have different regimes of validity
  For example, when we describe a table
  - we usually describe it in terms of its geometrical shape, colour, etc ...
  - surely, this is NOT accurate ... the table is made out of atoms, molecules ...it is not even continuous ... it just looks continuous to our eyes because we can not resolve the molecular scales.

Thus, different questions (different experiments) may require different models (descriptions)

Evolution of populations: Prey-Predator model

System: interaction between two species, prey and predator
Goal: to model the time evolution of both populations
Variables: denote by

- x the prey population
- y the predator population

Assumptions:

- If no predator, $\dot{x} = ax$ $(a > 0)$, where a is the rate of growth of the prey.
- If no prey, $\dot{y} = -cy$ $(y > 0)$, where c is the rate of death of the predator.
- Interactions between both populations are proportional to both populations, i.e. the number of their encounters is proportional to both populations.

Evolution of populations: Prey-Predator model

Under these assumptions, the proposed nonlinear ODE system describing the evolution of both populations reduces to:

$$\dot{x} = ax - \alpha xy \,,$$
$$\dot{y} = -cy + \gamma xy \,.$$

All parameters are positive, i.e. signs were taken into account in the ODEs explicitly.
These are the Lotka-Volterra equations (1925,1926).

- this is a simple model. (you do not have to agree with it)
- it is usually good to start with simpler models, analyse their consequences and eventually increase the complexity in your description

Honours Differential Equations

Jacques Vanneste

Lecture 18

October 26, 2018

Evolution of populations: Prey-Predator model

Let us analyse this nonlinear ODE system:

$$\dot{x} = ax - \alpha xy\,,$$
$$\dot{y} = -cy + \gamma xy\,.$$

Critical points:

$$\dot{x} = 0 \Rightarrow x = 0,\ \ y = \frac{a}{\alpha}$$
$$\dot{y} = 0 \Rightarrow y = 0,\ \ x = \frac{c}{\gamma}\,.$$

Thus, there is exist two critical points:

- the origin (absence of both populations) (0,0)
- non-trivial critical point $(c/\gamma,\ a/\alpha)$

Evolution of populations: Prey-Predator model

Linear analysis:

- $F(x,y) = ax - \alpha xy \Rightarrow \partial_x F = a - \alpha y, \quad \partial_y F = -\alpha x$
- $G(x,y) = -cy + \gamma xy \Rightarrow \partial_x G = \gamma y, \quad \partial_y G = -c + \gamma x$

Linear analysis: origin (0,0)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solution:

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\, e^{at} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\, e^{-ct} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus,

- (0,0) is a saddle point: it is an unstable critical point
- the only direction approaching (0,0) is along the y axis: if we introduce predators in the absence of prey, they will die.

Evolution of populations: Prey-Predator model

Linear analysis:

- $F(x, y) = ax - \alpha xy \Rightarrow \partial_x F = a - \alpha y, \quad \partial_y F = -\alpha x$
- $G(x, y) = -cy + \gamma xy \Rightarrow \partial_x G = \gamma y, \quad \partial_y G = -c + \gamma x$

Linear analysis: $(c/\gamma,\, a/\alpha)$

Introduce $u = x - c/\gamma$ and $v = y - a/\alpha$:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -c\frac{\alpha}{\gamma} \\ a\frac{\gamma}{\alpha} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Eigenvalues: $\lambda_\pm = \pm i\sqrt{ac} \Rightarrow$ critical point is a center

Dividing both linear equations:

$$\frac{dv}{du} = -\frac{\gamma(a/\alpha)}{c(\alpha/\gamma)}\frac{u}{v} \Rightarrow a\,\gamma^2\,u^2 + c\,\alpha^2\,v^2 = k^2 > 0 \text{ (ellipses)}$$

Evolution of populations: Prey-Predator model

Implicit (exact) trajectory: dividing both exact equations

$$\frac{dy}{dx} = \frac{y(-c+\gamma x)}{x(a-\alpha y)} \Rightarrow \left(\frac{a}{y} - \alpha\right) dy = \left(-\frac{c}{x} + \gamma\right) dx$$
$$\Rightarrow a \log y - \alpha\, y = -c \log x + \gamma x + C$$

Plots confirm the existence of periodic stable configurations

- if these models were accurate enough, they would predict the stable co-existence of both species in many areas of phase space

Evolution of populations: Prey-Predator model

Linear approximation to describe these periodic trajectories:

$$\dot{u} = -c\frac{\alpha}{\gamma}\,v\,, \qquad \dot{v} = a\frac{\gamma}{\alpha}\,u\,,$$
$$\Rightarrow \qquad \ddot{u} + ac\,u = \ddot{v} + ac\,v = 0\,.$$

General real solution:

$$u(t) = a_1 \cos\sqrt{ac}t + b_1 \sin\sqrt{ac}t\,,$$
$$v(t) = c_1 \cos\sqrt{ac}t + d_1 \sin\sqrt{ac}t\,.$$

For convenience, we can choose:

$$a_1 = A\cos\phi_0 \quad b_1 = A\sin\phi_0 \Rightarrow u(t) = A\cos\left(\sqrt{ac}t + \phi_0\right)\,,$$
$$c_1 = B\cos\phi_1 \quad d_1 = B\sin\phi_1 \Rightarrow v(t) = B\sin\left(\sqrt{ac}t + \phi_1\right)\,.$$

Evolution of populations: Prey-Predator model

Using the constraint $\dot{u} = -c\frac{\alpha}{\gamma}\, v$ determines

$$\phi_0 = \phi_1\,, \qquad A = \frac{\alpha}{\gamma}\sqrt{\frac{c}{a}}\, B\,.$$

The amplitude B is related to the constant k determining the ellipses:

$$a\,\gamma^2\, u^2 + c\,\alpha^2\, v^2 = k^2 \Rightarrow B = \frac{k}{\alpha\sqrt{c}}\,.$$

General solution:

$$x(t) = \frac{c}{\gamma} + \frac{k}{\gamma\sqrt{a}}\,\cos\left(\sqrt{ac}\,t + \phi_0\right)\,,$$

$$y(t) = \frac{a}{\alpha} + \frac{k}{\alpha\sqrt{c}}\,\sin\left(\sqrt{ac}\,t + \phi_0\right)\,.$$

Evolution of populations: Prey-Predator model

$$x(t) = \frac{c}{\gamma} + \frac{k}{\gamma\sqrt{a}} \cos\left(\sqrt{ac}\, t + \phi_0\right) ,$$
$$y(t) = \frac{a}{\alpha} + \frac{k}{\alpha\sqrt{c}} \sin\left(\sqrt{ac}\, t + \phi_0\right) .$$

- Period $T = \frac{2\pi}{\sqrt{ac}}$, independent of initial conditions.
- Both populations are periodic and out of phase by one quarter of a period (prey leads and predator lags)
- Amplitudes of the oscillations are $\frac{k}{\gamma\sqrt{a}}$ and $\frac{k}{\alpha\sqrt{c}}$: they depend on initial conditions (k)
- Averages over a cycle coincide with the critical point configuration:

$$\langle x \rangle = \frac{1}{T}\int_0^T x(t)\, dt = \frac{c}{\gamma} ,$$
$$\langle y \rangle = \frac{1}{T}\int_0^T y(t)\, dt = \frac{a}{\alpha} .$$

Intro to nonlinear methods

Given a nonlinear autonomous ODE system:

$$\dot{x} = F(x, y), \qquad \dot{y} = G(x, y)$$

we would like to know

- existence of critical points
- existence of closed trajectories at the nonlinear level
- intrinsic nonlinear behaviour that can not be seen in the linear approximation

If we can integrate the equations, we can explicitly explore these issues. But this clearly depends on the ODE under considerations.

Intro to nonlinear methods

What we want is to develop methods that allow to answer this type of questions without integrating the system

We will mainly discuss two results:

- Lyapunov's theory: deals with the nonlinear stability of certain critical points
- Poincaré-Bendixson theorem: deals with the existence of closed trajectories

Both theorems require math techniques (analysis) beyond the scope of this course, but

- we will make their results as plausible as possible
- we will apply them in several examples

Gaining some intuition from Newton's law

Consider Newton's law:

$$\ddot{x} = F(x)$$

where the force $F(x)$ does not depend explicitly on time.
This dynamical problem is equivalent to a nonlinear ODE system

$$\dot{x}_1 = x_2 \,, \quad \dot{x}_2 = F(x_1)$$

- Consider $V(x_1) \Rightarrow \dot{V} = \frac{dV}{dx_1}\,\dot{x}_1 = \frac{dV}{dx_1}\,x_2$
- Consider $T(x_2) \Rightarrow \dot{T} = \frac{dT}{dx_2}\,\dot{x}_2 = \frac{dT}{dx_2}\,F(x_1)$

Each separate function changes with time, but $\dot{T}$ and $\dot{V}$ suggest some combination of T and V may not.

Gaining some intuition from Newton's law

Consider $T(x_2) + V(x_1)$. Then,

$$\dot{T} + \dot{V} = \frac{dV}{dx_1} x_2 + \frac{dT}{dx_2} F(x_1) = 0 \quad \text{if} \quad \frac{dT}{dx_2} = x_2 \,, \quad F(x_1) = -\frac{dV}{dx_1}$$

Thus, when the force $F(x_1)$ comes from a potential $V(x_1)$, the quantity $\frac{1}{2}x_2^2 + V(x_1) = E$ (energy) is conserved, i.e. $\frac{dE}{dt} = 0$.

- $\frac{1}{2}x_2^2 + V(x_1) = E$ is an implicit trajectory for the nonlinear ODE, i.e. the existence of a conserved quantity allowed us to integrate the nonlinear ODE system implicitly
- Stable (unstable) critical points correspond to minima (maxima) of the potential $V(x_1)$ (see week-6 workshop)

Gaining some intuition from Newton's law

Consider a critical point $(x_0, 0)$ characterised by

$$\dot{x}_2 = -\frac{dV}{dx_1}(x_0) = 0 \Rightarrow x_0 \text{ local extremum of } V(x_1)$$

Linearising: $u = x_1 - x_0$ and $v = x_2$, we obtain

$$\dot{u} = v\,, \qquad \dot{v} = -\frac{d^2V}{dx_1^2}(x_0)u\,.$$

The eigenvalues λ of this linear system satisfy

$$\lambda^2 = -\frac{d^2V}{dx_1^2}(x_0)\,.$$

- If x_0 maximum $\Rightarrow \frac{d^2V}{dx_1^2}(x_0) < 0 \Rightarrow \lambda_\pm$ real and different signs $\Rightarrow$ saddle point $\Rightarrow$ unstable critical point.
- If x_0 minimum $\Rightarrow \frac{d^2V}{dx_1^2}(x_0) > 0 \Rightarrow \lambda_\pm$ purely imaginary $\Rightarrow$ center $\Rightarrow$ stable critical point. (linear approximation)

Example: pendulum with no friction

Take $\gamma = 0$ in our previous discussions (other parameters $\to 1$)

$$\dot{x} = y\,, \qquad \dot{y} = -\sin x\,.$$

The energy function $E(x, y)$ can be chosen to be:

$$E(x, y) = \frac{1}{2}y^2 + (1 - \cos x)$$

- Stable critical points: $y_0 = 0$ and $x_0 = 2n\pi \Rightarrow E(x_0, y_0) = 0$
- Unstable critical points:
  $y_0 = 0$ and $x_0 = (2n+1)\pi \Rightarrow E(x_0, y_0) = 2$

Example: pendulum with no friction

Question: How do trajectories look close to these critical points?

- Take (x, y) close to $(0, 0) \Rightarrow \cos x \sim 1 - \frac{1}{2}x^2 + \mathcal{O}(x^4)$

$$E \sim \frac{1}{2}y^2 + \frac{1}{2}x^2 \Rightarrow \frac{x^2}{2E} + \frac{y^2}{2E} \sim 1$$

  ellipses enclosing the origin

- Take (x, y) close to $(\pi, 0)$, i.e. $u = x - \pi$ and $v = y$ with $|u|,\ |v| \ll 1$
  $\cos x = \cos(u + \pi) = -\cos u \sim -\left(1 - \frac{1}{2}u^2\right) + \mathcal{O}(u^4)$

$$E - 2 \sim \frac{1}{2}v^2 - \frac{1}{2}u^2 \Rightarrow \frac{v^2}{2(E-2)} - \frac{u^2}{2(E-2)} \sim 1$$

  hyperbolas avoiding the critical point

Reinterpretation of conservation of energy

Given the energy function $E(x, y)$, its time derivative equals

$$\begin{aligned}\frac{dE}{dt} &= (\partial_x E)\,\dot{x} + (\partial_y E)\,\dot{y} \\ &= \boldsymbol{\nabla} E \cdot \boldsymbol{T}(x, y) = |\boldsymbol{\nabla} E||\boldsymbol{T}|\cos\phi\end{aligned}$$

- $\boldsymbol{\nabla} E = (\partial_x E,\ \partial_y E)$ is the gradient vector associated with the surface $E(x, y) =$ constant
- $\boldsymbol{T}(x, y) = (\dot{x},\ \dot{y})$ is the tangent vector to the trajectory at $(x(t),\ y(t))$
- ϕ is the angle between the gradient and the tangent vectors at the point $(x(t), y(t))$.

Reinterpretation of conservation of energy

Thus, conservation of energy means

$$\frac{dE}{dt} = \nabla E \cdot \boldsymbol{T}(x,y) = |\nabla E||\boldsymbol{T}|\cos\phi = 0$$

the gradient and tangent vectors are orthogonal, i.e. $\phi = \frac{\pi}{2}$
This makes sense:

- $E(x,y) =$ constant is a trajectory solving our nonlinear ODE system.
- $E(x,y) =$ constant defines a surface/curve $\Rightarrow \nabla E$ is orthogonal to the surface
- Orthogonality $\Rightarrow \nabla E \perp \boldsymbol{T} \Rightarrow$ conservation of energy

Honours Differential Equations

Jacques Vanneste

Lecture 10

October 8, 2018

Recap

Given a linear system of 1st order ODEs

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

where A is a constant $n \times n$ matrix, we obtain n solutions

$$\mathbf{x}^{(i)} = \boldsymbol{\xi}^{(i)} e^{r_i t}, \quad i = 1, \ldots n,$$

where

$$(A - r_i \mathrm{I})\boldsymbol{\xi}^{(i)} = 0,$$
$$\det(A - r_i \mathrm{I}) = 0,$$

provided that

- the eigenvalues r_i are all different,
- if an eigenvalue has algebraic multiplicity $s \geq 2$, its geometric multiplicity is also s.

Recap

What if:

- One of the r_i has algebraic multiplicity $s \geq 2$ and geometric multiplicity $< s$?

We do not have n linearly independent solutions: missing solutions.

Strategy: Use matrix algebra methods

1. Learn how to reproduce last week's solutions using these methods,
2. Apply these to the problem above.

Matrix algebra methods

Given a linear system of 1st order ODEs

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} ,$$

with n fundamental solutions $\mathbf{x}^{(i)}$ $i = 1, 2, \ldots, n$

Definition

A fundamental matrix $\Psi(t)$ is an $n \times n$ matrix with fundamental solutions $\mathbf{x}^{(i)}$ as columns:

$$\Psi(t) = \begin{pmatrix} x_1^{(1)} & x_1^{(2)} & \ldots & x_1^{(n)} \\ x_2^{(1)} & x_2^{(2)} & \ldots & x_2^{(n)} \\ . & . & \ldots & . \\ x_{n-1}^{(1)} & x_{n-1}^{(2)} & \ldots & x_{n-1}^{(n)} \\ x_n^{(1)} & x_n^{(2)} & \ldots & x_n^{(n)} \end{pmatrix} .$$

Fundamental matrix

1. $\det\Psi(t) = W(t) \neq 0$ because $\mathbf{x}^{(i)}$ form a n fundamental set.
2. The general solution can be written as the matrix product

$$\mathbf{x}(t) = \sum_{j=1}^{n} c_j \mathbf{x}^{(j)} = \Psi(t)\mathbf{c}, \quad \text{where} \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ c_n \end{pmatrix}.$$

3. If the system satisfies the initial condition: $\mathbf{x}(t_0) = \mathbf{x}_0$, then the general solution equals

$$\mathbf{x}(t_0) = \Psi(t_0)\mathbf{c} = \mathbf{x}_0 \Rightarrow \mathbf{c} = \Psi^{-1}(t_0)\mathbf{x}_0$$
$$\mathbf{x}(t) = \Psi(t)\mathbf{c} = \Psi(t)\Psi^{-1}(t_0)\mathbf{x}_0 .$$

Notice this last equation achieves the same goal as the Laplace transform, i.e. find the solution + include initial data $\Rightarrow$ requires to invert $\Psi(t)$

Fundamental matrix

Property

The fundamental matrix satisfies the matrix equation

$$\Psi' = A\Psi$$

Here,

$$\Psi' = \frac{d\Psi(t)}{dt} = \begin{pmatrix} \frac{dx_1^{(1)}}{dt} & \frac{dx_1^{(2)}}{dt} & \cdots & \frac{dx_1^{(n)}}{dt} \\ \frac{dx_2^{(1)}}{dt} & \frac{dx_2^{(2)}}{dt} & \cdots & \frac{dx_2^{(n)}}{dt} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{dx_{n-1}^{(1)}}{dt} & \frac{dx_{n-1}^{(2)}}{dt} & \cdots & \frac{dx_{n-1}^{(n)}}{dt} \\ \frac{dx_n^{(1)}}{dt} & \frac{dx_n^{(2)}}{dt} & \cdots & \frac{dx_n^{(n)}}{dt} \end{pmatrix}$$

The property is clear:

$$(\mathbf{x}^{(i)})' = A\mathbf{x}^{(i)} \quad \Rightarrow \quad (\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)})' = A(\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}).$$

Matrix exponential

Remark: linear ODE systems generalise

$$\frac{dx}{dt} = ax \Rightarrow x(t) = e^{at} x_0.$$

We can generalise the exponential function to matrices:

Definition

Define the exponential of a matrix A as

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = \mathrm{I} + At + \frac{1}{2!}A^2 t^2 + \cdots$$

Alternatively,

$$e^{At} = \lim_{n\to\infty} \left(\mathrm{I} + \frac{1}{n}A\right)^n.$$

Matrix exponential

It can be shown that the previous definition converges.
Interesting properties for us:

1.
$$\frac{de^{At}}{dt} = \sum_{n=1}^{\infty} \frac{A^n\, t^{n-1}}{(n-1)!} = A \sum_{n=1}^{\infty} \frac{A^{n-1}\, t^{n-1}}{(n-1)!} = \{n-1=m\}$$
$$= A \sum_{m=0}^{\infty} \frac{A^m\, t^m}{m!} = A \left(I + \sum_{m=1}^{\infty} \frac{A^m\, t^m}{m!} \right) = A\, e^{At}.$$

2. $e^{At}\big|_{t=0} = I.$

Conclusion:

- $x(t) = e^{At}\, x_0 \quad \Leftrightarrow \quad e^{At} = \Psi(t)\Psi^{-1}(t_0).$
- $e^{At} = \Psi(t)$ for $\Psi(0) = I.$

Relation to diagonalisation

We originally solved the linear ODE system

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x} \,,$$

in terms of solutions of the form $\boldsymbol{x}^{(i)} = e^{r_i t} \boldsymbol{\xi}^{(i)}$.

Consider the matrix T with the eigenvectors $\boldsymbol{\xi}^{(i)}$ as columns

$$T = \begin{pmatrix} \xi_1^{(1)} & \xi_1^{(2)} & \dots & \xi_1^{(n)} \\ \xi_2^{(1)} & \xi_2^{(2)} & \dots & \xi_2^{(n)} \\ . & . & \dots & . \\ . & . & \dots & . \\ \xi_n^{(1)} & \xi_n^{(2)} & \dots & \xi_n^{(n)} \end{pmatrix} .$$

Relation to diagonalisation

Observation: The matrix AT has columns equal to

$$A\xi^{(i)} = r_i\xi^{(i)}.$$

Thus, the matrix AT equals

$$AT = \begin{pmatrix} r_1\xi_1^{(1)} & r_2\xi_1^{(2)} & \dots & r_n\xi_1^{(n)} \\ r_1\xi_2^{(1)} & r_2\xi_2^{(2)} & \dots & r_n\xi_2^{(n)} \\ . & . & \dots & . \\ . & . & \dots & . \\ r_1\xi_n^{(1)} & r_2\xi_n^{(2)} & \dots & r_n\xi_n^{(n)} \end{pmatrix} = T\begin{pmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & r_n \end{pmatrix}$$

$$= T\mathrm{diag}(r_1, r_2, \dots, r_n) \equiv TD$$

$$\Rightarrow D = T^{-1}AT$$

Relation to diagonalisation

$D = T^{-1}AT \Leftrightarrow$ change of basis.

To see this more explicitly, consider the change of variables

$$\mathbf{x} = T\,\mathbf{y} \Rightarrow T\,\frac{d\mathbf{y}}{dt} = AT\,\mathbf{y} \Rightarrow \frac{d\mathbf{y}}{dt} = T^{-1}AT\,\mathbf{y} = D\,\mathbf{y}\,.$$

In the new variables, the solution to the linear ODE system is trivial:

$$\mathbf{y}^{(i)} = e^{r_i t}\,\mathbf{e}^{(i)} = e^{r_i t}\begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad 1 \text{ i-th component}$$

Thus, its fundamental matrix equals

$$Q(t) = e^{Dt} = \mathrm{diag}\left(e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t}\right)$$

Relation to diagonalisation

The fundamental matrix in the original variables x is

$$\Psi(t) = TQ(t) = T\,\mathrm{diag}\left(e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t}\right).$$

The exponential matrix is

$$e^{At} = \Psi(t)\Psi^{-1}(0) = TQT^{-1} = T\mathrm{diag}\left(e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t}\right) T^{-1}.$$

Alternative derivation: note that $T^{-1}A^kT = \mathrm{diag}(r_1^k, \cdots, r_n^k)$,

$$T^{-1}e^{At}T = \sum_{k=0}^{\infty} \frac{1}{k!}\mathrm{diag}(r_1^k, \cdots, r_n^k) = \mathrm{diag}(e^{r_1 t}, \cdots, e^{r_n t}).$$

But, this works when A is diagonalisable.

example

Last week, we solved this linear system

$$\frac{dx}{dt} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x \equiv Ax \, .$$

Its eigenvalues are:

$$\det\left(A - rI\right) = (r-3)(r+1) = 0 \, .$$

After solving for the eigenvectors, the general solution is

$$x = c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \, .$$

Example

Take the two eigenvectors $\xi^{(1)}$ and $\xi^{(2)}$ and construct the change of basis matrix

$$T = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \Rightarrow T^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix}$$

Notice that:

$$T^{-1}AT = T^{-1}\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

Thus, when we change variable from $\boldsymbol{x}$ to $\boldsymbol{y}$, $\boldsymbol{x} = T\boldsymbol{y}$, the linear ODE system becomes diagonal,

$$\frac{d\boldsymbol{y}}{dt} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}\boldsymbol{y} \equiv D\boldsymbol{y} \,.$$

Example

The fundamental matrix in the original basis equals

$$\Psi(t) = TQ = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$

From this matrix we can indeed recover the original solution

$$\boldsymbol{x}^{(1)} = e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{x}^{(2)} = e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Honours Differential Equations

Jacques Vanneste

Lecture 11

October 11, 2018

Recap

The general solution of linear ODE systems

$$\mathbf{x}' = A\mathbf{x},$$

takes the form

$$\mathbf{x}(t) = \sum_{j=1}^{n} c_j \, e^{r_j t} \boldsymbol{\xi}^{(j)}.$$

This is equivalent to diagonalising the matrix A by changing the coordinates, $\mathbf{x} = T\mathbf{y}$ so that

$$\frac{d\mathbf{y}}{dt} = D\mathbf{y} \qquad D = \mathrm{diag}\,(r_1, r_2, \ldots, r_n),$$

where T is the matrix with the eigenvectors $\boldsymbol{\xi}^{(i)}$ as columns.
We can also write

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0, \quad \text{where} \quad e^{At} = T\mathrm{diag}(e^{r_1 t}, \cdots, e^{r_n t})T^{-1}.$$

Recap

This connection emphasizes the importance of dealing with diagonalisable matrices A.

But:

- Eigenvalues with geometric multiplicity $<$ algebraic multiplicy lead to non-diagonialisable matrices.

Strategy:

- Find additional linearly independent solutions using our intuition,
- Complete answer provided by the Jordan normal forms (Honours Algebra course, 2nd semester).

Example

Consider the linear ODE system

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x} \equiv A\,\mathbf{x}\,.$$

Eigenvalues: $r = 2$ with multiplicity s=2

$$\begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = (r-2)^2 = 0$$

Eigenvectors: $\boldsymbol{\xi} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{aligned} &(1-r)x - y = 0 \Rightarrow x + y = 0 \\ &x + (3-r)y = 0 \Rightarrow x + y = 0 \Rightarrow x = -y \end{aligned}$$

Thus, $\boldsymbol{\xi} \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix}$: geometric multiplicity is $1 < 2$.

Example

The general solution requires two linearly independent solutions; we are missing one.

Idea:

- Similar situation for n-th order linear ODEs with roots of the characteristic equation with multiplicity $s \geq 2$,
- we considered solutions with e^{rt} was to $t\, e^{rt}$,
- can the same idea work here?

Example

Let us try

$$x = t\,e^{2t}\,\xi \Rightarrow \frac{dx}{dt} = \xi\,e^{2t} + 2t\,e^{2t}\,\xi$$

Require this to be Ax:

$$Ax = t\,e^{2t}\,A\xi = 2t\,e^{2t}\,\xi$$

Lesson: It does not work (unless $\xi = 0$)

- Should include an extra unknown vector η in our trial solution.

Example

Let us try

$$x(t) = t\, e^{2t}\,\xi + \eta\, e^{2t} \Rightarrow \frac{dx}{dt} = 2t\, e^{2t}\xi + (\xi + 2\eta)\; e^{2t}$$

Require this to be Ax:

$$Ax = A\left(t\, e^{2t}\xi + e^{2t}\eta\right) = 2t\, e^{2t}\,\xi + e^{2t}\, A\eta$$

Lesson: need η to satisfy

$$(A - 2\mathbb{I})\,\eta = \xi\,.$$

This equation has a non-trivial solution (because ξ is an eigenvector of A).
We say that η is a generalised eigenvector.

Example

Let $\eta = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$\begin{aligned} -x - y &= 1 \\ x + y &= -1 \Rightarrow y = -1 - x\,. \end{aligned}$$

Thus, the general solution for the vector η is

$$\eta = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + x \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + x\,\boldsymbol{\xi}.$$

Thus, our proposal for the second linearly independent solution is:

$$\mathbf{x}^{(2)} = t\, e^{2t}\, \boldsymbol{\xi} + e^{2t} \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k\, e^{2t}\, \boldsymbol{\xi}$$

Notice the last term is proportional to our first solution $e^{2t}\,\boldsymbol{\xi}$.
It can be ignored (or absorbed into a redefinition of our constants)

Example

Thus,

$$x^{(2)} = t\, e^{2t}\, \xi + e^{2t} \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

and the general solution to our linear ODE system is

$$\begin{aligned} x(t) &= c_1 x^{(1)}(t) + c_2 x^{(2)}(t) \\ &= c_1\, e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \left[t\, e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right]. \end{aligned}$$

Are both solutions linearly independent?

$$W[x^{(1)}, x^{(2)}] = \begin{vmatrix} e^{2t} & t\, e^{2t} \\ -e^{2t} & -e^{2t}(1+t) \end{vmatrix} = -e^{4t} \neq 0$$

They are $\Rightarrow$ problem solved.

Connection to matrix methods

Given the general solution, the fundamental matrix equals

$$\Psi(t) = \left(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}\right) = \begin{pmatrix} e^{2t} & t\,e^{2t} \\ -e^{2t} & -e^{2t}(1+t) \end{pmatrix}.$$

Check: since

$$\Psi(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \Rightarrow \Psi^{-1}(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

$$e^{At} = \Psi(t)\,\Psi^{-1}(0) = e^{2t} \begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix}.$$

Question: what is the form of the matrix A in the basis (ξ, η)?

Connection to matrix methods

Let us follow the construction in our previous lecture.
Build a matrix out of the two vectors ξ and η:

$$T = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \Rightarrow T^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

Notice:

$$T^{-1} A T = T^{-1} \begin{pmatrix} 2 & 1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = J_2$$

J_2 is an upper triangular matrix of the form

$$J_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \text{Jordan form.}$$

Connection to matrix methods

To check the consistency of our approach,
consider the change of variables $\mathbf{x} = T\mathbf{y}$, then

$$\frac{d\mathbf{y}}{dt} = T^{-1}AT\,\mathbf{y} \equiv J_2\,\mathbf{y}$$

Using $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, the new linear ODE system is equivalent to:

$$\begin{aligned} y_2' &= 2y_2 \Rightarrow y_2(t) = c_2\,e^{2t} \\ y_1' &= 2y_1 + y_2 \Rightarrow y_1(t) = c_1\,e^{2t} + c_2\,t\,e^{2t}\,. \end{aligned}$$

Thus, its general solution can be written as

$$\mathbf{y}(t) = c_1\,e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\,e^{2t}\begin{pmatrix} t \\ 1 \end{pmatrix}\,.$$

Connection to matrix methods

The fundamental matrix of the new ODE system is:

$$Q(t) = \begin{pmatrix} e^{2t} & t\,e^{2t} \\ 0 & e^{2t} \end{pmatrix} \Rightarrow \Psi(t) = TQ(t) = \begin{pmatrix} e^{2t} & t\,e^{2t} \\ -e^{2t} & -e^{2t}(1+t) \end{pmatrix}.$$

Lesson: The matrix A is non-diagonalisable

- The best we can do is to bring it to an upper triangular form $\equiv$ Jordan form

$$J_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

- Linear algebra result: all matrices can be reduced to blocks that are either diagonal or have Jordan form.

Connection to matrix methods

- A system in Jordan form is easy to integrate:

$$\frac{d\boldsymbol{y}}{dt} = J_\lambda \boldsymbol{y} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \boldsymbol{y}.$$

  Integrate from the bottom up:

$$\begin{aligned} y_2' &= \lambda y_2 \\ y_1' &= \lambda y_1 + y_2 \end{aligned}$$

  Compute the fundamental matrix in the original variables $\boldsymbol{x}$: $\Psi(t) = T\,Q(t)$ where $Q(t)$ is the fundamental matrix in the $\boldsymbol{y}$ variables and T the change of basis matrix.
- The exponential of Jordan blocks has also a memorable form

$$e^{J_\lambda t} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.$$

General method

Given a general 2×2 linear ODE system

$$\mathbf{x}' = A\,\mathbf{x}$$

with $\det(A - r\mathrm{I}) = (r - \lambda)^2 = 0$ & having a single eigenvector $\boldsymbol{\xi}_\lambda$

- One solution is $\mathbf{x}^{(1)} = e^{\lambda t}\,\boldsymbol{\xi}_\lambda$
- Second solution is of the form

$$\mathbf{x}^{(2)} = t\,e^{\lambda t}\,\boldsymbol{\xi}_\lambda + e^{\lambda t}\,\boldsymbol{\eta}$$

  where $(A - \lambda\mathrm{I})\,\boldsymbol{\eta} = \boldsymbol{\xi}_\lambda$.
- $\boldsymbol{\eta}$ is a generalised eigenvector.

General method

The methods explained extend to $n \times n$ linear ODE systems

For example consider a 3×3 system with eigenvalue λ having algebraic multiplicity 3 and geometric multiplicity 1 (single eigenvector $\boldsymbol{\xi}_\lambda$).

- One solution is $\boldsymbol{x}^{(1)} = e^{\lambda t}\,\boldsymbol{\xi}_\lambda$.
- Second solution is of the form
$$\begin{aligned} \boldsymbol{x}^{(2)} &= t\,e^{\lambda t}\,\boldsymbol{\xi}_\lambda + e^{\lambda t}\,\boldsymbol{\eta} \\ \boldsymbol{\xi}_\lambda &= (A - \lambda \mathrm{I})\,\boldsymbol{\eta}\,. \end{aligned}$$
- Third solution is of the form
$$\begin{aligned} \boldsymbol{x}^{(3)} &= \frac{t^2}{2}\,e^{\lambda t}\,\boldsymbol{\xi}_\lambda + t\,e^{\lambda t}\,\boldsymbol{\eta} + e^{\lambda t}\,\boldsymbol{\zeta} \\ \boldsymbol{\eta} &= (A - \lambda \mathrm{I})\,\boldsymbol{\zeta}\,. \end{aligned}$$

(See Assignment 3)
The method is algorithmic.

Honours Differential Equations

Jacques Vanneste

Lecture 12

October 12, 2018

Non-homogeneous ODE systems

We want to solve linear ODE systems of the form

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t).$$

The general solution is of the form

$$\mathbf{x}(t) = \sum_i c_i \mathbf{x}^{(i)}(t) + \mathbf{x}_{par}(t),$$

where $\sum_i c_i \mathbf{x}^{(i)}(t)$ is the general solution of the homogeneous ODE system.

Non-homogeneous ODE systems

In what follows, we focus on systems of the form

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)\,,$$

where A is an $n \times n$ constant matrix.

We shall discuss three different methods:

1. Diagonalisation (matrix methods),
2. Undetermined coefficients,
3. Variation of parameters.

Diagonalisation

Consider a non-homogeneous system

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)\,.$$

Assume the corresponding homogeneous system is solved with eigenvalues r_i and eigenvectors $\xi^{(i)}$

Introduce the change of variables $\mathbf{x} = T\mathbf{y}$:

$$T\mathbf{y}' = AT\,\mathbf{y} + \mathbf{g}(t) \Rightarrow \frac{d\mathbf{y}}{dt} = D\,\mathbf{y} + T^{-1}\mathbf{g} \equiv D\,\mathbf{y} + \mathbf{h}$$

System of n decoupled equations $\Rightarrow$ direct integration,

$$y_i' = r_i\,y_i + h_i \Rightarrow y_i(t) = c_i\,e^{r_i t} + e^{r_i t}\int_{t_0}^{t} e^{-r_i s}\,h_i(s)ds\,.$$

Diagonalisation

Check:

$$y_i' = r_i y_i + e^{r_i t} e^{-r_i t} h_i(t).$$

General solution in the original variables equals

$$\mathbf{x}(t) = T\mathbf{y}(t).$$

Remarks:

- The particular solution is encoded in the $h_i(t)$ part.
- We assumed that A is diagonalisable (later, we will discuss the Jordan form case)
- The power of matrix methods is more apparent when dealing with non-homogeneous ODE systems.

Example

Consider

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}.$$

Since the system is linear and non-homogeneous, the general solution must be of the form

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \mathbf{x}_{\text{par}}(t).$$

Homogeneous solution: eigenvalues & eigenvectors $\mathbf{x}_{\text{hom}} = e^{rt}\,\boldsymbol{\xi}$

$$\begin{vmatrix} -2-r & 1 \\ 1 & -2-r \end{vmatrix} = r^2 + 4r + 3 = (r+3)(r+1) = 0$$

Example

When $r = -3$: if $\xi_{-3} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow y = -x$. I will choose

$$\xi_{-3} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(notice that $\xi_{-3} \cdot \xi_{-3} = 1$)
When $r = -1$: if $\xi_{-1} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow y = x$. I will choose

$$\xi_{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus, general homogeneous solution is

$$\mathbf{x}_{hom} = c_1 \, e^{-3t} \, \xi_{-3} + c_2 \, e^{-t} \, \xi_{-1} \, .$$

Example

To find the particular solution, we change variables $\mathbf{x} = T\mathbf{y}$ with

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Rightarrow T^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

The new variables satisfy the linear ODE:

$$\begin{aligned} \frac{d\mathbf{y}}{dt} &= \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{y} + T^{-1}\mathbf{g} \\ &= \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{y} + \frac{1}{\sqrt{2}}\begin{pmatrix} 2e^{-t} - 3t \\ 2e^{-t} + 3t \end{pmatrix} \end{aligned}$$

In terms of $\mathbf{y}^T = (y_1, y_2)$, this is equivalent to

$$y_1' + 3y_1 = \sqrt{2}\, e^{-t} - \frac{3}{\sqrt{2}}t \quad \text{(linear non-homogeneous 1st order)}$$

$$y_2' + y_2 = \sqrt{2}\, e^{-t} + \frac{3}{\sqrt{2}}t \quad \text{(linear non-homogeneous 1st order)}$$

Example

General solution must be of the form (using undetermined coefficients):

$$
\begin{aligned}
y_1(t) &= c_1\, e^{-3t} + a\, e^{-t} + b\, t + c\,,\\
y_2(t) &= c_2\, e^{-t} + f\ t\, e^{-t} + h t + m\,.
\end{aligned}
$$

Algebra (& patience) determine:

$$
\begin{aligned}
y_1(t) &= c_1\, e^{-3t} + \frac{\sqrt{2}}{2}\, e^{-t} + -\frac{3}{\sqrt{2}}\left(\frac{t}{3} - \frac{1}{9}\right),\\
y_2(t) &= c_2\, e^{-t} + \sqrt{2}\, t\, e^{-t} + \frac{3}{\sqrt{2}}\,(t-1)\,.
\end{aligned}
$$

Example

We must write the solution in the original variables
Thus,

$$\mathbf{x}(t) = T\mathbf{y}(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} y_1 + y_2 \\ -y_1 + y_2 \end{pmatrix}(t).$$

Expanding and grouping terms having the same t functional dependence, we get

$$\mathbf{x}(t) = k_1 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$+ \frac{1}{2} e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + t e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

First line: general homogeneous solution
Second line: particular solution

Jordan form systems

We assumed A was diagonalisable, but the same method works when it can only be brought into Jordan form

Consider the change of variables $\mathbf{x} = T\,\mathbf{y}$ so that

$$\frac{d\mathbf{y}}{dt} = J\,\mathbf{y} + T^{-1}\mathbf{g} \equiv J\,\mathbf{y} + \mathbf{h}$$

These are not decoupled, but can be integrated in the appropriate order.

Example: consider

$$\mathbf{y}' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$\begin{aligned} y_1' &= \lambda\, y_1 + y_2 + h_1 \\ y_2' &= \lambda\, y_2 + h_2 \end{aligned} .$$

Method of undetermined coefficients

When the non-homogeneous system

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t),$$

has a vector $\mathbf{g}(t)$ built out of polynomials, real or complex exponentials, you may consider the application of this method.

- It has the same rules as before: we assume a similar t dependence for $\mathbf{x}_{\text{par}}$ as in $\mathbf{g}(t)$
- One exception: if $\mathbf{g}(t) = \mathbf{u}e^{\lambda t}$ and λ is an eigenvalue of A with multiplicity 1, then

$$\mathbf{x}_{\text{par}} = t\,e^{\lambda t}\,\mathbf{a} + e^{\lambda t}\,\mathbf{b}$$

The term $e^{\lambda t}\,\mathbf{b}$ must be included to find a solution.
If the multiplicity is n, we must write $\mathbf{x}_{\text{par}} = e^{\lambda t}\sum_{i=0}^{n} t^i \mathbf{a}_i$.

Same example, different method

Consider, as before,

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} .$$

Since the eigenvalues of A equal $\lambda = -3, -1$ and the vector $\mathbf{g}(t)$ equals

$$\mathbf{g}(t) = e^{-t} \begin{pmatrix} 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

we look for a particular solution of the form

$$\mathbf{x}_{\text{par}}(t) = e^{-t} (\mathbf{a}\, t + \mathbf{b}) + \mathbf{c}\, t + \mathbf{d} .$$

The undetermined coefficients are the components of the vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$.

Same example, different method

To fix these coefficients, we introduce our guess into the non-homogeneous linear ODE. First,

$$\frac{d\mathbf{x}_{\text{par}}(t)}{dt} = e^{-t}\left(\mathbf{a} - \mathbf{a}\,t - \mathbf{b}\right) + \mathbf{c}\,.$$

Substituting into the linear ODE system,

$$\begin{aligned}(\mathbf{a} - \mathbf{b})\,e^{-t} - \mathbf{a}\,t\,e^{-t} + \mathbf{c} = {} & A\mathbf{a}\,t\,e^{-t} + A\mathbf{b}\,e^{-t} + A\mathbf{c}\,t + A\mathbf{d} \\ & + e^{-t}\begin{pmatrix}2\\0\end{pmatrix} + t\begin{pmatrix}0\\3\end{pmatrix}\,.\end{aligned}$$

Key observation: this set of equations must be satisfied $\forall\, t$.

Same example, different method

Thus, they are equivalent to 4 equations:

$$A\mathbf{a} = -\mathbf{a} \quad (te^{-t})$$

$$A\mathbf{b} = \mathbf{a} - \mathbf{b} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (e^{-t})$$

$$A\mathbf{c} = -\begin{pmatrix} 0 \\ 3 \end{pmatrix} \quad (t)$$

$$A\mathbf{d} = \mathbf{c} \quad (t^0)$$

1st equation $\Rightarrow$ $\mathbf{a}$ is an eigenvector of A with eigenvalue $\lambda = -1$

$$\Rightarrow \mathbf{a} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$$

(from our previous analysis)

Same example, different method

Using this into the second equation: let $\mathbf{b} = (x, y)^T$

Then, $A\mathbf{b} = \mathbf{a} - \mathbf{b} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ is equivalent to

$$\begin{aligned} -x + y &= \alpha - 2 \\ x - y &= \alpha \end{aligned}$$

Consistency requires $\alpha = 1$ & $y = -1 + x$. Thus,

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Third equation $A\mathbf{c} = -\begin{pmatrix} 0 \\ 3 \end{pmatrix} \Rightarrow \mathbf{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Fourth equation $A\mathbf{d} = \mathbf{c} \Rightarrow \mathbf{d} = -\begin{pmatrix} \frac{4}{3} \\ \frac{5}{3} \end{pmatrix}$

Same example, different method

Particular solution is:

$$\mathbf{x}_{par}(t) = te^{-t}\begin{pmatrix}1\\1\end{pmatrix} - e^{-t}\begin{pmatrix}0\\1\end{pmatrix} + k\,e^{-t}\begin{pmatrix}1\\1\end{pmatrix} + t\begin{pmatrix}1\\2\end{pmatrix} - \frac{1}{3}\begin{pmatrix}4\\5\end{pmatrix}.$$

This agrees with our previous solution, obtained using diagonalisation, for $k = 1/2$.
Remember particular solutions are not unique.

Honours Differential Equations

Jacques Vanneste

Lecture 13

October 15, 2018

Non-homogeneous ODE systems

We are discussing non-homogeneous ODE systems

$$\frac{d\mathbf{x}}{dt} = P(t)\,\mathbf{x} + \mathbf{g}(t).$$

When $P(t) = A$ is constant, we have discussed two methods to solve this problem:

- Diagonalisation: matrix algebra methods
- Undetermined coefficients

Today, we discuss variation of parameters. Its application is analogous to what we saw for higher-order differential equations.

Variation of parameters

Consider the system

$$\frac{d\boldsymbol{x}}{dt} = P(t)\,\boldsymbol{x} + \boldsymbol{g}(t)\,.$$

Assume the homogeneous problem

$$\frac{d\boldsymbol{x}_{\text{hom}}}{dt} = P(t)\,\boldsymbol{x}_{\text{hom}}\,,$$

is solved by

$$\boldsymbol{x}_{\text{hom}} = \Psi(t)\boldsymbol{c}\,,$$

for some constant $\boldsymbol{c}$.

The method consists in looking for solutions to the non-homogeneous problem of the form

$$\boldsymbol{x}(t) = \Psi(t)\boldsymbol{u}(t)\,.$$

That is, $\boldsymbol{c} \mapsto \boldsymbol{u}(t)$ (variation of parameters)

Variation of parameters

Question: how do we determine $\boldsymbol{u}(t)$?

Introducing $\boldsymbol{x} = \Psi(t)\boldsymbol{u}(t)$ into the ODE system:

$$\frac{d\boldsymbol{x}}{dt} = \Psi'(t)\boldsymbol{u}(t) + \Psi(t)\frac{d\boldsymbol{u}}{dt} = P(t)\Psi(t)\boldsymbol{u}(t) + \boldsymbol{g}(t).$$

Remember $\Psi' = P(t)\Psi$, thus

$$\Psi\frac{d\boldsymbol{u}}{dt} = \boldsymbol{g}(t) \Rightarrow \frac{d\boldsymbol{u}}{dt} = \Psi^{-1}\boldsymbol{g}$$
$$\Rightarrow \boldsymbol{u}(t) = \int^t \Psi^{-1}(s)\boldsymbol{g}(s)\,ds + \boldsymbol{f}.$$

Thus, the general solution is

$$\boldsymbol{x}(t) = \Psi(t)\Psi^{-1}(t_0)\boldsymbol{x}_0 + \Psi(t)\int_{t_0}^t \Psi^{-1}(s)\boldsymbol{g}(s)\,ds.$$

Variation of parameters

$$x(t) = \Psi(t)\Psi^{-1}(t_0)x_0 + \Psi(t)\int_{t_0}^{t} \Psi^{-1}(s)\mathbf{g}(s)\, ds\,.$$

- We recover the general solution as general homogeneous + particular.
- We chose the constant $\mathbf{f}$ so that the particular solution vanishes at $t = t_0$ (this can always be done).
- The method applies even if $P(t)$ is *not constant*.

Example

Let us solve

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x} + \boldsymbol{g} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} .$$

again, using variation of parameters.
Our previous analysis gave the fundamental matrix:

$$\Psi(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}$$

With

$$\Psi(t)\frac{d\boldsymbol{u}}{dt} = \boldsymbol{g} .$$

Example

With $\boldsymbol{u} = (u_1, u_2)^{\mathrm{T}}$, the above is equivalent to

$$u_1' = e^{2t} - \frac{3}{2} t\, e^{3t}\,,$$
$$u_2' = 1 + \frac{3}{2} t\, e^{t}\,.$$

Both equations are 1st order $\Rightarrow$ direct integration !!

$$u_1(t) = \frac{1}{2} e^{2t} - \frac{1}{2} t\, e^{3t} + \frac{1}{6} e^{3t} + c_1$$
$$u_2(t) = t + \frac{3}{2} t\, e^{t} - \frac{3}{2} e^{t} + c_2\,.$$

Substituting into $\boldsymbol{x}(t) = \Psi(t)\boldsymbol{u}(t)$ reproduces the solution using this method.

Qualitative theory of ODEs

Consider a nonlinear *autonomous* system

$$\frac{dx}{dt} = F(x, y),$$
$$\frac{dy}{dt} = G(x, y).$$

Since F, G can be arbitrary, the problem is generically nonlinear.

Question: Can we interpret the solution geometrically?

- View (x, y) as a point in $\mathbb{R}^2$,
- (F, G) defines a vector field,
- $(x(t), y(t))$ parameterises a curve in $\mathbb{R}^2$,
- Integrating a solution $\Leftrightarrow$ finding a curve whose tangent at (x, y) is given by $\frac{dx}{dt} = \dot{x} = F(x, y)$ and $\frac{dy}{dt} = \dot{y} = G(x, y)$.

Phase plane

Let us introduce some terminology:

Definition

- *The* (x, y) *plane will be referred to as* *phase plane*.
- *Solutions to the ODE system* $\mathbf{x}(t) = (x(t), y(t))^{\mathrm{T}}$ *describe curves in the phase plane, often thought as* *trajectories* *of point moving with velocity* $\frac{dx}{dt}$ *and* $\frac{dy}{dt}$.

Phase plane

Further remarks:

- Solutions depend on initial conditions
  $\Rightarrow$ different initial conditions correspond to different trajectories.
- A given ODE system gives rise to as many trajectories as different initial conditions.
- But, there are as many initial conditions as points in the phase plane.
- Plotting a representative set of trajectories will be referred to as the phase portrait of the given ODE system.
- For autonomous systems, trajectories cannot intersect.

Strategy

We want to

- either integrate a non-linear ODE system,
- or give some qualitative description.

Concerning full analytical integration:

- only *some* ODE systems can be integrated,
- in some cases, we can derive implicit relations $x = x(y)$ describing our non-linear trajectories,
- generically, one resorts to numerical integration.

Strategy

Qualitative description:

- **Local description:** consider the dynamics in a patch near some x_0,
  - not very interesting for most x_0,
  - focus on critical points x_0, where $F(x_0) = G(x_0) = 0$,
  - use linearisation (first-order Taylor expansion) and what we have learned about linear systems for $x \approx x_0$,
  - gives a local description of the phase portrait.
- **Global description**: understand behaviour in the entire phase plane.

Rectification

Definition

A point $\mathbf{x}_0 = (x_0, y_0)$ is a critical point if $F(x_0, y_0) = G(x_0, y_0) = 0$.

Away from critical points, the local dynamics is simple:

Theorem

Let $\mathbf{x}_* = (x_*, y_*)$ be such that $(F(\mathbf{x}_*, \mathbf{y}_*), G(\mathbf{x}_*, \mathbf{y}_*)) \neq 0$, then, in a neighbourhood of $\mathbf{x}_*$ there is a smooth change of variable $(\tilde{x}, \tilde{y}) = H(x, y)$ under which the ODE system reduces to

$$\frac{d\tilde{x}}{dt} = 1,$$
$$\frac{d\tilde{y}}{dt} = 0.$$

Honours Differential Equations

Jacques Vanneste

Lecture 14

October 18, 2018

Motivating the linear approximation

Consider a generic 2×2 non-linear ODE system:

$$x' = F(x, y),$$
$$y' = G(x, y).$$

Given a point (x_0, y_0) in phase space, we can approximate F, G by their Taylor expansions in some open neighbourhood

$$F(x, y) = F(x_0, y_0) + \partial_x F(x_0, y_0)(x - x_0) + \partial_y F(x_0, y_0)(y - y_0) + \eta_1(x, y),$$
$$G(x, y) = G(x_0, y_0) + \partial_x G(x_0, y_0)(x - x_0) + \partial_y G(x_0, y_0)(y - y_0) + \eta_2(x, y),$$

where

$$\frac{\eta_1(x, y)}{\|\mathbf{x} - \mathbf{x}_0\|}, \frac{\eta_2(x, y)}{\|\mathbf{x} - \mathbf{x}_0\|} \to 0 \quad \text{as} \quad (x, y) \to (x_0, y_0)$$

Linear approximation

Recall: a critical point $(x_0\, y_0)$ is one satisfying $F(x_0, y_0) = G(x_0, y_0) = 0$.

To explore the evolution of the system around $(x_0\, y_0)$, it is natural to introduce new variables

$$u_1 \equiv x - x_0\,, \qquad u_2 \equiv y - y_0\,.$$

These satisfy:

$$\begin{pmatrix} \frac{du_1}{dt} \\ \frac{du_2}{dt} \end{pmatrix} = \begin{pmatrix} \partial_x F(x_0, y_0) & \partial_y F(x_0, y_0) \\ \partial_x G(x_0, y_0) & \partial_y G(x_0, y_0) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

Linear approximation

The linear approximation consists in dropping (η_1, η_2)
Thus, we are left with

$$\frac{d\boldsymbol{u}(t)}{dt} = A\,\boldsymbol{u}, \qquad A \equiv \begin{pmatrix} \partial_x F(x_0, y_0) & \partial_y F(x_0, y_0) \\ \partial_x G(x_0, y_0) & \partial_y G(x_0, y_0) \end{pmatrix}.$$

where $\boldsymbol{u}(t) = (u_1, u_2)^{\mathrm{T}}$. A is the Jacobian matrix.

Conclusion:

- Locally, around any critical point, nonlinear ODEs $\approx$ linear ODEs.

Local analysis: strategy

Given a nonlinear ODE system, we can

- identify all its critical points,
- solve the linear approximation around each of them,
- use nonlinear methods to connect linear behaviours and produce a global picture of the dynamics.

First we

- classify all possible linear behaviours,
- relate the linear dynamics to the stability of the critical point.

Example

Consider Newton's law for a pendulum with friction:

$$mL^2 \frac{d^2\theta}{dt^2} = -cL\frac{d\theta}{dt} - mgL\sin\theta \Leftrightarrow \frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega^2 \sin\theta = 0\,,$$

with $\gamma = \frac{c}{mL}\,, \quad \omega^2 = \frac{g}{L}$.

- Map the 2nd order ODE to a 1st order ODE system: $x = \theta$ and $y = \frac{d\theta}{dt}$

$$\frac{dx}{dt} = y\,, \qquad \frac{dy}{dt} = -\omega^2 \sin x - \gamma\, y\,.$$

- Critical points:

$$\frac{dx}{dt} = 0 \Rightarrow y = 0$$
$$\frac{dx}{dt} = 0 \Rightarrow x = n\pi\,, \quad n \in \mathbb{Z}$$

Example

Jacobian matrix:

$$F(x, y) = y \Rightarrow \partial_x F = 0\,, \quad \partial_y F = 1$$
$$G(x, y) = -\omega^2 \sin x - \gamma\, y \Rightarrow \partial_x G = -\omega^2 \cos x\,, \quad \partial_y G = -\gamma.$$

- Critical point $(x_0, y_0) = (2n\pi, 0)$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

where $u = x - 2n\pi$ and $v = y$.

Example

- Critical point $(x_0, y_0) = ((2n+1)\pi, 0)$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

where $u = x - (2n+1)\pi$ and $v = y$.

Notice how the information about the given critical point is encoded in a single sign matrix element.

Example

Solutions of the linear approximation: $(u, v) = e^{\lambda t}\xi^{\mathrm{T}}$ leads to the eigenvalues λ.

- $(x_0, y_0) = (2n\pi, 0)$: $\lambda = (-\gamma \pm \sqrt{\gamma^2 - 4\omega^2})/2$,
- $(x_0, y_0) = ((2n+1)\pi, 0)$: $\lambda = (-\gamma \pm \sqrt{\gamma^2 + 4\omega^2})/2$.

Behaviour depends on sign of $\mathrm{Re}\,\lambda$ and differ for the two types of critical points: lower equilibrium is attracting, the upper equilibrium repelling.

Classification of critical points

- In the linear approximation,

$$\text{critical point} \Leftrightarrow \frac{d\boldsymbol{u}}{dt} = 0 = A\boldsymbol{u}.$$

  If $\det A \neq 0 \;\Rightarrow \boldsymbol{u} = 0$ is the unique critical point.
- Classification of critical points $\Leftrightarrow$ Classification of eigenvalues of A
  - This is because $\boldsymbol{u}(t) \equiv$ trajectory in phase space,
  - Different eigenvalues $\Leftrightarrow$ different trajectories,
  - Thus, different critical point behaviours $\Leftrightarrow$ different eigenvalues of A.

Real unequal eigenvalues

(i) negative $r_1 < r_2 < 0$.

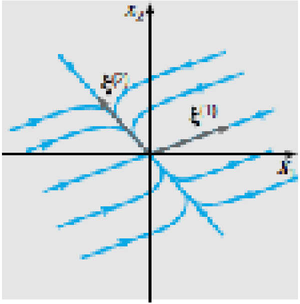
Solution is

$$x = c_1\, e^{r_1 t}\, \xi_1 + c_2\, e^{r_2 t}\, \xi_2 = e^{r_2 t}\left[c_1\, e^{(r_1 - r_2)t} \xi_1 + c_2 \xi_2\right].$$

The solution always approaches the critical point.

- if $c_2 \neq 0$, $x \to 0$ along ξ_2 direction
- if $c_2 = 0$, $x \to 0$ along ξ_1 direction

Node (nodal sink)



Real unequal eigenvalues

(ii) $r_1 > r_2 > 0$.

Solution is

$$x = c_1 e^{r_1 t} \xi_1 + c_2 e^{r_2 t} \xi_2 = e^{r_1 t} \left[c_1 \xi_1 + c_2 e^{(r_2 - r_1)t} \xi_2 \right]$$

The solution always gets away from the critical point.
This critical point is called Node (nodal source):

- Similar graph as before, but flipping arrows,
- If $c_1 \neq 0$, trajectories approach the one by ξ_1 as $t \to \infty$,
- Only when $c_1 = 0$, they do it along ξ_2.

Real unequal eigenvalues

(ii) different signs $r_1 > 0 > r_2$.

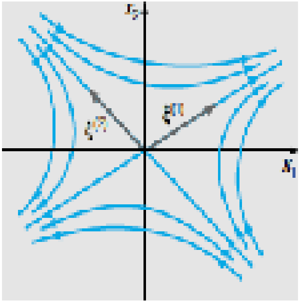
Solution is

$$x = c_1 e^{r_1 t} \xi_1 + c_2 e^{r_2 t} \xi_2 .$$

Hence $|x| \to \infty$ as $t \to \pm\infty$ for most initial conditions.

- if $c_2 = 0 \Rightarrow |x| \to \infty$ at $t \to \infty$ (it gets away along ξ_1)
- if $c_1 = 0 \Rightarrow |x| \to 0$ at $t \to \infty$ (it approaches along ξ_2)
- if $c_1, c_2 \neq 0$, $|x| \to \infty$ along ξ_1 because $e^{r_1 t}$ dominates

Saddle point



Real equal eigenvalues

(iii) $r_1 = r_2 = r < 0$: double eigenvalues (different sign: just flip arrows).

Two cases to consider:

- $\exists$ two independent eigenvectors
- $\exists$ a single independent eigenvector

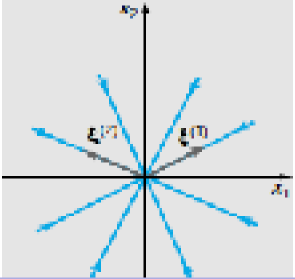
When two independent eigenvectors exist:

$$x = c_1 e^{rt} \xi_1 + c_2 e^{rt} \xi_2 .$$

Notice any ratio of x_2/x_1 is t independent $\Rightarrow$ straight line.

Trajectories approach the critical point

Proper node (star point)



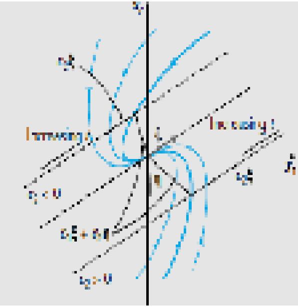
Real equal eigenvalues

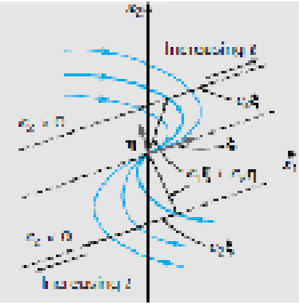
(iii) $r_1 = r_2 = r < 0$ with one independent eigenvector ξ:

$$\mathbf{x} = c_1\, e^{rt}\, \xi + c_2\, e^{rt}\, (t\xi + \eta) = e^{rt}\left[c_1\, \xi + c_2 \eta + c_2\, t\, \xi\right].$$

Since $r < 0$, trajectories approach the critical point.
As $t \to \infty$, the trajectory is dominated by ξ (even if $c_2 = 0$).

Improper (or degenerate) node





Honours Differential Equations

Jacques Vanneste

Lecture 15

October 19, 2018

Classification of critical points

Classification
critical points $\Leftrightarrow$ matrix eigenvalues

When eigenvalues are real & different

- same sign: node
- different sign: saddle

When eigenvalues are real & equal

- two independent eigenvectors: proper node (star point)
- one independent eigenvector: improper node

Today: we will discuss critical points $\Leftrightarrow$ complex eigenvalues

Real equal eigenvalues

(iii) $r_1 = r_2 = r < 0$: double eigenvalues (different sign: just flip arrows).

Two cases to consider:

- $\exists$ two independent eigenvectors
- $\exists$ a single independent eigenvector

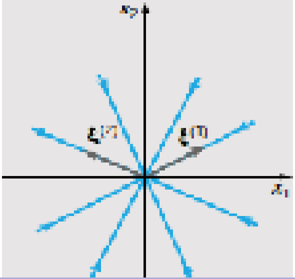
When two independent eigenvectors exist:

$$\mathbf{x} = c_1 e^{rt} \boldsymbol{\xi}_1 + c_2 e^{rt} \boldsymbol{\xi}_2 .$$

Notice any ratio of x_2/x_1 is t independent $\Rightarrow$ straight line.

Trajectories approach the critical point

Proper node (star point)



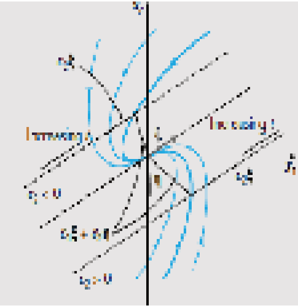
Real equal eigenvalues

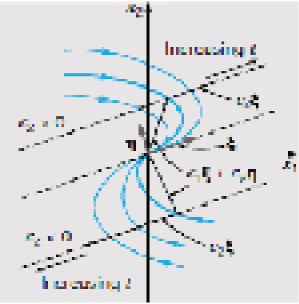
(iii) $r_1 = r_2 = r < 0$ with one independent eigenvector ξ:

$$\mathbf{x} = c_1 e^{rt}\xi + c_2 e^{rt}(t\xi + \eta) = e^{rt}\left[c_1\xi + c_2\eta + c_2 t\xi\right] .$$

Since $r < 0$, trajectories approach the critical point.
As $t \to \infty$, the trajectory is dominated by ξ (even if $c_2 = 0$).

Improper (or degenerate) node





Complex eigenvalues

(iv) complex eigenvalues $\lambda \pm i\mu$ ($\mu > 0$).

Eigenvectors are ξ and ξ^* and the transformation $\mathbf{x} = T\mathbf{y}$ leads to the diagonal matrix

$$T^{-1}AT = \begin{pmatrix} \lambda + i\mu & 0 \\ 0 & \lambda - i\mu \end{pmatrix}.$$

Make a further change of coordinates: $\mathbf{y} = P\mathbf{z}$, where

$$P = \begin{pmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{pmatrix}$$

corresponds to using $(\mathrm{Re}\,\xi, \mathrm{Im}\,\xi)$ as a basis.
Then

$$\mathbf{z}' = P^{-1}T^{-1}ATP\mathbf{z} = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} \mathbf{z}.$$

Complex eigenvalues

For matrices of the form

$$A = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix},$$

$$\det(A - rI) = (r-\lambda)^2 + \mu^2 = 0 \Rightarrow r = \lambda \pm i\mu$$

two linearly independent solutions are given by

$$e^{\lambda t}\begin{pmatrix} \cos(\mu t) \\ -\sin(\mu t) \end{pmatrix}, \quad e^{\lambda t}\begin{pmatrix} \sin(\mu t) \\ \cos(\mu t) \end{pmatrix},$$

Hence, the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\lambda t}\begin{pmatrix} c_1\cos(\mu t) + c_2\sin(\mu t) \\ -c_1\sin(\mu t) + c_2\cos(\mu t) \end{pmatrix} = Ce^{\lambda t}\begin{pmatrix} \cos(\phi - \mu t) \\ \sin(\phi - \mu t) \end{pmatrix},$$

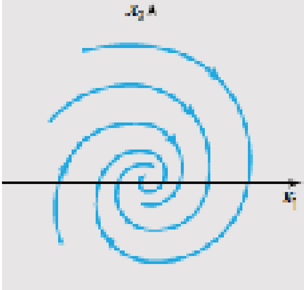
with C and ϕ arbitrary constants.

Complex eigenvalues

These solutions satisfy the identity :

$$x^2 + y^2 = C^2 e^{2\lambda t}.$$

- spirals towards origin if $\lambda < 0$: stable focus.
- diverges from origin if $\lambda > 0$: unstable focus.



In the original coordinates, 'elliptical' spiral.

Complex eigenvalues

An alternative way of reaching this conclusion.
Introduce polar coordinates in phase space

$$r^2 = x^2 + y^2 , \qquad \tan\phi = \frac{y}{x} .$$

(x, y) satisfy the ODE system

$$\begin{aligned} \dot{x} &= \lambda x + \mu y \\ \dot{y} &= \lambda y - \mu x . \end{aligned}$$

It follows

$$r\dot{r} = x\dot{x} + y\dot{y} = x(\lambda x + \mu y) + y(\lambda y - \mu x) = \lambda\, r^2 ,$$

$$\frac{\dot{\phi}}{\cos^2\phi} = \frac{x\dot{y} - y\dot{x}}{x^2} \Rightarrow \dot{\phi} = -\mu$$

Complex eigenvalues

$$\dot{r} = \lambda r \Rightarrow r = c\, e^{\lambda t}$$
$$\dot{\phi} = -\mu \Rightarrow \phi = -\mu t + \phi_0 \,.$$

- $\lambda > 0 \Rightarrow |\mathbf{x}| \to \infty$ for $t \to \infty$
- $\lambda < 0 \Rightarrow |\mathbf{x}| \to 0$ for $t \to \infty$
- ϕ decreases as t evolves (since $\mu > 0$) $\Rightarrow$ motion is clockwise

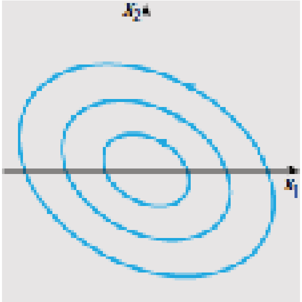
Conclusions are fully consistent with our spiral picture.

Purely imaginary eigenvalues

This corresponds to the particular case $\lambda = 0$

$$x^2 + y^2 = C^2 .$$

They correspond to ellipses in the original coordinates.



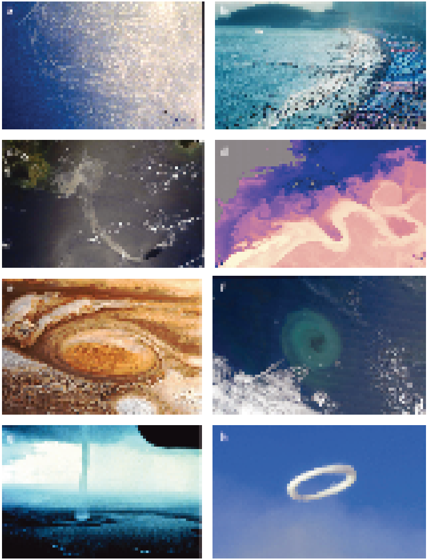
Summary

Critical points of different types have different behaviours as $t \to \infty$:

- $|\mathbf{x}| \to 0$ as $t \to \infty$
  1. real & negative eigenvalues: nodal sink
  2. complex eigenvalue with negative real part: stable focus
- bounded trajectory as $t \to \infty$: purely imaginary eigenvalue $\Rightarrow$ center
- $|\mathbf{x}| \to \infty$ as $t \to \infty$
  1. at least one eigenvalue is positive: saddle, nodal source
  2. complex eigenvalue with positive real part: unstable focus

This behaviour as $t \to \infty$ is related to the notion of stability.

Application: transport in atmosphere/ocean



Autonomous systems

Definition

An ODE system

$$\frac{dx}{dt} = F(x, y), \qquad \frac{dy}{dt} = G(x, y),$$

is referred to as an autonomous systems is F, G have no explicit time dependence.

This describes any physical system whose parameters, forces, etc ... *do NOT depend on time*

Stability

We want to be mathematically more precise with the different mathematical behaviour satisfied by the different critical points.

Definition

A critical point $\mathbf{x}_0$ *is* stable *if* $\forall\, \epsilon, \exists\, \delta > 0$ *such that every solution* $\mathbf{x} = \phi(t)$ *with* $\|\phi(0) - \mathbf{x}_0\| < \delta$ *at* $t = 0$ *satisfies*

$$\|\phi(t) - \mathbf{x}_0\| < \epsilon, \quad \forall\, t > 0.$$

Definition

A critical point that is not stable is called unstable.

Stability

Definition

A critical point $\mathbf{x}_0$ *is asymptotically stable if it is stable and the solution* $\mathbf{x} = \phi(t)$ *is forced to approach* $\mathbf{x}_0$ *as* $t \to \infty$, *i.e.*

$$\lim_{t \to \infty} \phi(t) = \mathbf{x}_0 \, .$$

Stable and asymptotically stable are different concepts

Example: a center is stable, but not asymptotically stable.

Stability of linear systems

Eigenvalues	Critical Points	Stability
$r_1 > r_2 > 0$	Node (source)	unstable
$r_1 < r_2 < 0$	Node (sink)	asymp. stable
$r_2 < 0 < r_1$	saddle	unstable
$r_1 = r_2 > 0$	Proper/Improper node	unstable
$r_1 = r_2 < 0$	Proper/Improper node	asymp. stable
$r_1, r_2 = \lambda \pm i\mu\ (\lambda > 0)$	focus	unstable
$r_1, r_2 = \lambda \pm i\mu\ (\lambda < 0)$	focus	asymp. stable
$r_1 = i\mu, r_2 = -i\mu$	center	stable

Honours Differential Equations

Jacques Vanneste

Lecture 16

October 22, 2018

Global vs linear picture

- We classified all types of critical points and their stability properties

Questions:

- How can we put together a phase space portrait including the exact nonlinear ODE system?
- Can our linear calculations be modified at the nonlinear level? if so, when?

Today

- Example of stability analysis: damped pendulum
- Implicit trajectories as a nonlinear tool
- Almost linear system (qualitative discussion)

Pendulum with friction

By Newton's law ($\gamma = \frac{c}{mL}$, $\quad \omega^2 = \frac{g}{L}$):

$$mL^2 \frac{d^2\theta}{dt^2} = -cL\frac{d\theta}{dt} - mgL\sin\theta \Leftrightarrow \frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega^2 \sin\theta = 0\,.$$

This is equivalent to the ODE system

$$\frac{dx}{dt} = y\,, \qquad \frac{dy}{dt} = -\omega^2 \sin x - \gamma y\,,$$

where $x = \theta$ and $y = \dot{\theta}$.

Critical points:

$$\begin{aligned} \frac{dx}{dt} &= 0 \Rightarrow y = 0 \\ \frac{dx}{dt} &= 0 \Rightarrow x = n\pi\,, \quad n \in \mathbb{Z} \end{aligned}$$

Pendulum with friction

Linear approximation:

- $(x_0, y_0) = (2n\pi, 0)$

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x - 2n\pi$ and $v = y$.

- Critical point $(x_0, y_0) = ((2n+1)\pi, 0)$

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x - (2n+1)\pi$ and $v = y$.

Pendulum with friction

We can jointly describe both types of critical points as follows:

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \epsilon\omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x - x_0$, $v = y$ and $\epsilon = \pm 1$.

Question: Let us study the stability of the two critical points
To do that, let us compute the eigenvalues of the constant matrix:

$$\lambda_{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\epsilon\omega^2}}{2} .$$

Pendulum with friction

$$\lambda_{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\epsilon\omega^2}}{2} \,.$$

- When $\epsilon = 1 \Rightarrow \lambda_{\pm}$ real, different sign $\Rightarrow$ saddle point $\Rightarrow$ unstable
- When $\epsilon = -1$
  - if $\gamma = 0$ (no friction) $\Rightarrow$ center [periodic motion, stable, not asymptotically stable]
  - if $\gamma > 0 \Rightarrow$ asymptotically stable
    - if $\gamma^2 - 4\omega^2 > 0 \Rightarrow \lambda_{\pm} < 0$ and real $\Rightarrow$ node
    - if $\gamma^2 - 4\omega^2 = 0 \Rightarrow \lambda_{+} = \lambda_{-} < 0 \Rightarrow$ Proper/Improper node
    - if $\gamma^2 - 4\omega^2 < 0 \Rightarrow \lambda_{\pm}$ complex with negative real part $\Rightarrow$ spiral

Our linear stability analysis reproduces our intuition on the behaviour of pendulum critical points.

Nonlinear vs linear descriptions

Consider the ODE system:

$$\frac{dx}{dt} = 4 - 2y\,, \qquad \frac{dy}{dt} = 12 - 3x^2\,.$$

Critical points:: $(x_0, y_0) = (\pm 2, 2)$
Linear approximation: $u = x - x_0$, $v = y - y_0$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -6x_0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Eigenvalues:: $\lambda^2 - 12x_0 = 0$

- if $x_0 > 0 \Rightarrow \lambda_\pm$ real with different sign
  $\Rightarrow$ (2,2) is a saddle
- if $x_0 < 0 \Rightarrow \lambda_\pm$ purely imaginary
  $\Rightarrow$ (-2,2) is a center

Nonlinear vs linear descriptions

Question: Can we plot the nonlinear phase portrait for this ODE system?

Remark:

$$\left.\begin{array}{l}\frac{dx}{dt} = 4 - 2y \\ \frac{dy}{dt} = 12 - 3x^2\end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{12 - 3x^2}{4 - 2y}$$

$$\Rightarrow (4 - 2y)dy = (12 - 3x^2)dx \Rightarrow 4y - y^2 = 12x - x^3 + c$$

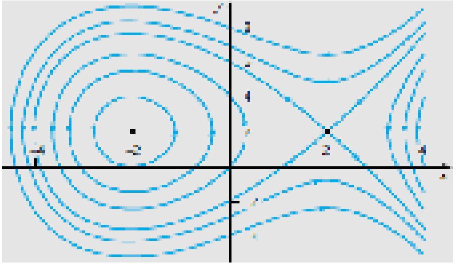
This is an exact (nonlinear) implicit description of the trajectories solving the ODE system (for any constant c)

- Plotting different values of $c \Leftrightarrow$ plotting different trajectories

Nonlinear vs linear descriptions

The nonlinear ODE system has an exact phase portrait given by

$$\frac{dx}{dt} = 4 - 2y\,, \qquad \frac{dy}{dt} = 12 - 3x^2\,,$$



- Notice how trajectories close to (-2,2) are indeed bounded
- (2,2) is indeed a saddle

Nonlinear vs linear descriptions

One lesson to take home: for those systems that admit implicit curves in the phase plane

$$\left.\begin{array}{l} \frac{dx}{dt} = F(x,y) \\ \frac{dy}{dt} = G(x,y) \end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{G(x,y)}{F(x,y)} \Rightarrow H(x,y) = c$$

as solutions to the ODE system, we can draw the exact trajectories

- This does not always happen!

Remark:

- The linear approximation eventually breaks down

Question:

- To which extent can we trust the information obtained in the linear approximation?

Almost linear systems

Consider the corrections to the linear approximation

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} \partial_x F(x_0, y_0) & \partial_y F(x_0, y_0) \\ \partial_x G(x_0, y_0) & \partial_y G(x_0, y_0) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

where $u = x - x_0$ and $v = y - y_0$, encoded in the vector $\eta^t = (\eta_1, \eta_2)$.

Question: Does the type of critical point in the linear approximation change when we include $\eta = (\eta_1, \eta_2)^{\mathrm{T}}$?

Theorem

Let r_1 and r_2 be the eigenvalues to the linear approximation corresponding to the almost linear system above, then the critical point $(0, 0)$ behaves as in the following table.

Stability properties of linear and almost linear systems

r_1, r_2	Linear System		Almost Linear System	
	Type	Stability	Type	Stability
$r_1 > r_2 > 0$	N	Unstable	N	Unstable
$r_1 < r_2 < 0$	N	Asymptotically stable	N	Asymptotically stable
$r_2 < 0 < r_1$	SP	Unstable	SP	Unstable
$r_1 = r_2 > 0$	PN or IN	Unstable	N or SpP	Unstable
$r_1 = r_2 < 0$	PN or IN	Asymptotically stable	N or SpP	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$				
$\lambda > 0$	SpP	Unstable	SpP	Unstable
$\lambda < 0$	SpP	Asymptotically stable	SpP	Asymptotically stable
$r_1 = i\mu, r_2 = -i\mu$	C	Stable	C or SpP	Indeterminate

- N: node PN: proper node IN: improper node
- SP: saddle point
- SpP: spiral (focus) point
- C: center

Center : linear vs non-linear

Consider ODE systems of the form :

$$\dot{x} = -y + x\,(x^2 + y^2)^n\,, \qquad \dot{y} = x + y\,(x^2 + y^2)^n$$

Critical points : $(x_0, y_0) = (0,0)$
Linear system :

$$\begin{aligned}
F(x,y) &= -y + x\,(x^2 + y^2)^n \\
&\Rightarrow \partial_x F = (x^2 + y^2)^n + 2nx^2(x^2 + y^2)^{n-1}\,, \\
&\quad\;\; \partial_y F = -1 + 2nyx(x^2 + y^2)^{n-1} \\
G(x,y) &= x + y\,(x^2 + y^2)^n \\
&\Rightarrow \partial_x G = 1 + 2nxy(x^2 + y^2)^{n-1}\,, \\
&\quad\;\; \partial_y G = (x^2 + y^2)^n + 2ny^2(x^2 + y^2)^{n-1}\,.
\end{aligned}$$

Center: linear vs nonlinear

Evaluating at the critical point $(0,0)$, we obtain for $u = x$, $v = y$

$$\dot{u} = -v\,,\ \ \dot{v} = u \quad \Leftrightarrow \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

This corresponds to a center since eigenvalues $\lambda_{\pm} = \pm i$ are purely imaginary $\Leftrightarrow \exists$ periodic trajectories (at least linearly)

Question: Does this interpretation remain true at the non-linear level?

Strategy: Change to polar coordinates and discuss the exact non-linear ODEs

Center: linear vs nonlinear

$$r^2 = x^2 + y^2\,, \qquad \tan\phi = \frac{y}{x}\,.$$

$$r\dot{r} = x\dot{x} + y\dot{y} = x(-y + x\,r^{2n}) + y(x + y\,r^{2n}) = r^{2(n+1)}\,,$$

$$\frac{\dot{\phi}}{\cos^2\phi} = \frac{x\dot{y} - y\dot{x}}{x^2} \Rightarrow \dot{\phi} = 1$$

Integrating, we obtain :

$$\frac{1}{r^{2n}} - \frac{1}{r_0^{2n}} = -2nt\,, \qquad \phi = t + \phi_0\,.$$

This describes a trajectory starting at (r_0, ϕ_0) :

- rotating anticlockwise
- as t increases, r increases
- in fact as $t \to \infty$, $r \to 0 \Rightarrow$ unstable

The linear centre trajectory is gone and it is replaced by an spiral.

Honours Differential Equations

Jacques Vanneste

Lecture 17

October 25, 2018

Today's lecture

We saw that the existence of linear centres is not guaranteed to survive at the nonlinear level.

Today we revisit the damped pendulum:

- to discuss how to properly draw the phase diagram in the linear approximation,
- to discuss the nonlinear phase portrait, stressing features that are more generic,
- introduce a prey-predator model.

Damped pendulum revisited

Pendulum with friction:

$$\frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + \omega^2 \sin\theta = 0 ,$$

- Friction is encoded in $\gamma = \frac{c}{mL}$ $(c > 0)$
- Newton's constant (gravity) is encoded in $\omega^2 = \frac{g}{L}$ $(g > 0)$
- L is the length of the pendulum

Map the 2nd order ODE to a 1st order ODE system:
$x = \theta$ and $y = \frac{d\theta}{dt}$

$$\frac{dx}{dt} = y \equiv F(x, y) , \qquad \frac{dy}{dt} = -\omega^2 \sin x - \gamma y \equiv G(x, y) .$$

Critical points & linear analysis

Critical points:

$$\frac{dx}{dt} = 0 \Rightarrow y = 0$$
$$\frac{dx}{dt} = 0 \Rightarrow x = n\pi\,, \quad n \in \mathbb{Z}$$

Linear approximation:

$$F(x,y) = y \Rightarrow \partial_x F = 0\,, \quad \partial_y F = 1$$
$$G(x,y) = -\omega^2 \sin x - \gamma\, y \Rightarrow \partial_x G = -\omega^2 \cos x\,, \quad \partial_y G = -\gamma$$

This gives rise to:

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \epsilon\omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x - x_0$, $v = y$ and $\epsilon = \pm 1$

- $\epsilon = -1$ when $x_0 = 2n\pi$
- $\epsilon = 1$ when $x_0 = (2n+1)\pi$

Critical points & linear analysis

Given the linear approximation, we compute its eigenvalues:

$$\begin{vmatrix} -\lambda & 1 \\ \epsilon\omega^2 & -\gamma-\lambda \end{vmatrix} = \lambda^2 + \gamma\lambda - \epsilon\omega^2 = 0$$

$$\Rightarrow \lambda_{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\epsilon\omega^2}}{2}$$

Question: how do we draw a phase portrait?

Pick $\epsilon = -1$ and $\gamma^2 - 4\omega^2 < 0$: small damping $\Rightarrow$ spiral points

Stable critical points

Consider the critical point $(0,0)$:

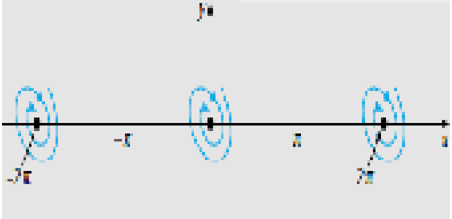
- to determine the arrow choose $x = 0$ (as a particular case) and evaluate the ODE system as that point

$$\frac{dx}{dt}(x=0) = y\,, \quad \frac{dy}{dt}(x=0) = -\gamma\, y\,.$$

  - if $y > 0$: x increases and y decreases
  - if $y < 0$: x decreases and y increases

  Thus, motion occurs clockwise

We can repeat the same analysis at the other critical points $(2n\pi, 0)$



Unstable critical points

Consider $\epsilon = 1$ and $x_0 = \pi$.

- Eigenvalues:

$$\lambda_{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\epsilon\omega^2}}{2} \Rightarrow \lambda_+ > 0,\ \lambda_- < 0$$

  This corresponds to a saddle point

- Linearly independent solutions:

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1\, e^{\lambda_+ t} \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} + c_2\, e^{\lambda_- t} \begin{pmatrix} 1 \\ \lambda_- \end{pmatrix}$$

  The only direction approaching the critical point at $t \to \infty$ is $c_1 = 0$

Unstable critical points

Focus on the $c_1 = 0$ direction

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_2 \, e^{\lambda_- t} \begin{pmatrix} 1 \\ \lambda_- \end{pmatrix}$$

The ratio of the solution components equals

$$\frac{v}{u} = \lambda_- < 0$$

Thus,

- if $c_2 > 0$: $u > 0, v < 0 \Rightarrow$ curve in 4th quadrant
- if $c_2 < 0$: $u < 0, v > 0 \Rightarrow$ curve in 2nd quadrant

Unstable critical points

Focus on the $c_2 = 0$ direction

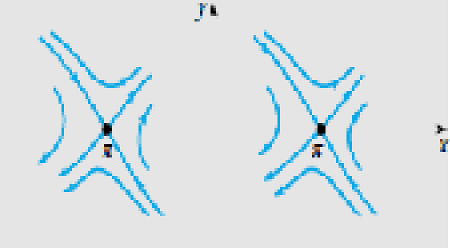
$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1\, e^{\lambda_+ t} \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix}$$

The ratio of the solution components equals

$$\frac{v}{u} = \lambda_+ > 0$$

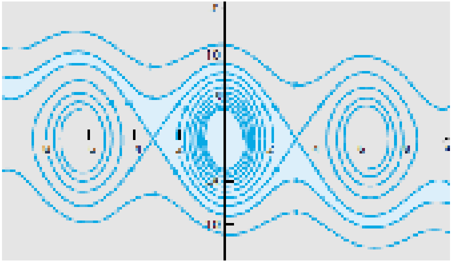
Thus,

- if $c_1 > 0$: $u > 0, v > 0 \Rightarrow$ curve in 1st quadrant
- if $c_1 < 0$: $u < 0, v < 0 \Rightarrow$ curve in 3rd quadrant



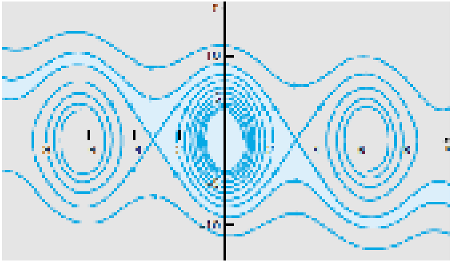
Nonlinear damped pendulum

When we plot different initial conditions and their associated trajectories, we obtain phase portraits of the form



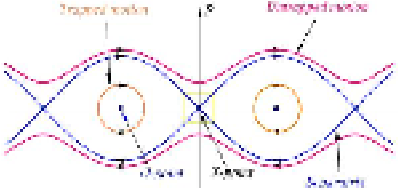
- Critical points keep their nature and stability properties
- No matter how large $|y|$ (velocity) is, the existence of damping (friction) guarantees the solution will eventually stabilise around the stable pendulum point.
    - we will build on this comment next week to discuss nonlinear ODE systems in more generality

Nonlinear damped pendulum



- saddle points separate the entire phase space into regions satisfying the property that any trajectory in them asymptotes to a stable spiral point
  - the specific trajectory determining a change in such stable spiral point is called separatrix
- the set of all trajectories approaching a given asymptotically stable critical point is called basin of attraction
  - This is bounded by the separatrices through the neighbouring saddle points
  - For the origin, this is marked in the figure

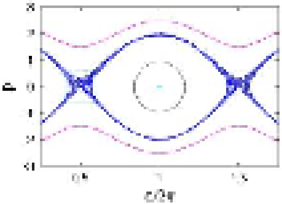
Perturbed undamped pendulum (figures S S Abdullaev)

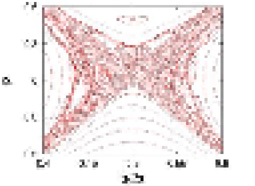


Complexity with periodic perturbation:

$$x' = y,$$
$$y' = -\omega^2 \sin x + \varepsilon \sin(x - t).$$

Poincaré, Kolmogorov, Arnold, Moser





Nonlinear ODE systems

- In any nonlinear ODE system, it is important (and non-trivial.) to determine the basin of attraction of each asymptotically stable critical point
- Students interested in this topic are encouraged to enrol in courses dealing with dynamical systems.

What is modelling?

- The term modelling refers to the development of a mathematical representation of a physical situation.
- A model is like an idealization of reality: cf tube map vs real tube systems. (Mike Ashby, Cambridge University)
- A gross simplification, but one that captures essential elements.

"Physical situation"

"Model"

What is modelling?

- The map misrepresents distances and directions, but it elegantly displays the connectivity
- The quality or usefulness in a model is measured by its ability to capture governing key features of the problem.
- At worst, a model is a concise description of a body of data. At best, it captures the essentce of the problem, it illuminates the principles underlying the key observations,
- Models should it predict behaviour under conditions which have not yet been studied.

What is modelling?

- Any scientific theory is a model.
- Different models (theories) may have different regimes of validity

  For example, when we describe a table
  - we usually describe it in terms of its geometrical shape, colour, etc ...
  - surely, this is NOT accurate ... the table is made out of atoms, molecules ...it is not even continuous ... it just looks continuous to our eyes because we can not resolve the molecular scales.

Thus, different questions (different experiments) may require different models (descriptions)

Evolution of populations: Prey-Predator model

System: interaction between two species, prey and predator
Goal: to model the time evolution of both populations
Variables: denote by

- x the prey population
- y the predator population

Assumptions:

- If no predator, $\dot{x} = ax$ $(a > 0)$, where a is the rate of growth of the prey.
- If no prey, $\dot{y} = -cy$ $(y > 0)$, where c is the rate of death of the predator.
- Interactions between both populations are proportional to both populations, i.e. the number of their encounters is proportional to both populations.

Evolution of populations: Prey-Predator model

Under these assumptions, the proposed nonlinear ODE system describing the evolution of both populations reduces to:

$$\dot{x} = ax - \alpha xy\,,$$
$$\dot{y} = -cy + \gamma xy\,.$$

All parameters are positive, i.e. signs were taken into account in the ODEs explicitly.
These are the Lotka-Volterra equations (1925,1926).

- this is a simple model. (you do not have to agree with it)
- it is usually good to start with simpler models, analyse their consequences and eventually increase the complexity in your description

Honours Differential Equations

Jacques Vanneste

Lecture 18

October 26, 2018

Evolution of populations: Prey-Predator model

Let us analyse this nonlinear ODE system:

$$\dot{x} = ax - \alpha xy\,,$$
$$\dot{y} = -cy + \gamma xy\,.$$

Critical points:

$$\dot{x} = 0 \Rightarrow x = 0,\;\; y = \frac{a}{\alpha}$$
$$\dot{y} = 0 \Rightarrow y = 0,\;\; x = \frac{c}{\gamma}\,.$$

Thus, there is exist two critical points:

- the origin (absence of both populations) $(0,0)$
- non-trivial critical point $(c/\gamma,\, a/\alpha)$

Evolution of populations: Prey-Predator model

Linear analysis:

- $F(x,y) = ax - \alpha xy \Rightarrow \partial_x F = a - \alpha y, \quad \partial_y F = -\alpha x$
- $G(x,y) = -cy + \gamma xy \Rightarrow \partial_x G = \gamma y, \quad \partial_y G = -c + \gamma x$

Linear analysis: origin (0,0)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solution:

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\, e^{at} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\, e^{-ct} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus,

- (0,0) is a saddle point: it is an unstable critical point
- the only direction approaching (0,0) is along the y axis: if we introduce predators in the absence of prey, they will die.

Evolution of populations: Prey-Predator model

Linear analysis:

- $F(x, y) = ax - \alpha xy \Rightarrow \partial_x F = a - \alpha y, \quad \partial_y F = -\alpha x$
- $G(x, y) = -cy + \gamma xy \Rightarrow \partial_x G = \gamma y, \quad \partial_y G = -c + \gamma x$

Linear analysis: $(c/\gamma, a/\alpha)$

Introduce $u = x - c/\gamma$ and $v = y - a/\alpha$:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -c\frac{\alpha}{\gamma} \\ a\frac{\gamma}{\alpha} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Eigenvalues: $\lambda_{\pm} = \pm i\sqrt{ac} \Rightarrow$ critical point is a center

Dividing both linear equations:

$$\frac{dv}{du} = -\frac{\gamma(a/\alpha)}{c(\alpha/\gamma)}\frac{u}{v} \Rightarrow a\,\gamma^2\,u^2 + c\,\alpha^2\,v^2 = k^2 > 0 \text{ (ellipses)}$$

Evolution of populations: Prey-Predator model

Implicit (exact) trajectory: dividing both exact equations

$$\frac{dy}{dx} = \frac{y(-c+\gamma x)}{x(a-\alpha y)} \Rightarrow \left(\frac{a}{y} - \alpha\right) dy = \left(-\frac{c}{x} + \gamma\right) dx$$

$$\Rightarrow a \log y - \alpha\, y = -c \log x + \gamma x + C$$

Plots confirm the existence of periodic stable configurations

- if these models were accurate enough, they would predict the stable co-existence of both species in many areas of phase space

Evolution of populations: Prey-Predator model

Linear approximation to describe these periodic trajectories:

$$\dot{u} = -c\frac{\alpha}{\gamma}\,v\,, \qquad \dot{v} = a\frac{\gamma}{\alpha}\,u\,,$$
$$\Rightarrow \qquad \ddot{u} + ac\,u = \ddot{v} + ac\,v = 0\,.$$

General real solution:

$$u(t) = a_1 \cos\sqrt{ac}t + b_1 \sin\sqrt{ac}t\,,$$
$$v(t) = c_1 \cos\sqrt{ac}t + d_1 \sin\sqrt{ac}t\,.$$

For convenience, we can choose:

$$a_1 = A\cos\phi_0 \quad b_1 = A\sin\phi_0 \Rightarrow u(t) = A\cos\left(\sqrt{ac}t + \phi_0\right)\,,$$
$$c_1 = B\cos\phi_1 \quad d_1 = B\sin\phi_1 \Rightarrow v(t) = B\sin\left(\sqrt{ac}t + \phi_1\right)\,.$$

Evolution of populations: Prey-Predator model

Using the constraint $\dot{u} = -c\frac{\alpha}{\gamma}\,v$ determines

$$\phi_0 = \phi_1\,, \qquad A = \frac{\alpha}{\gamma}\sqrt{\frac{c}{a}}\,B\,.$$

The amplitude B is related to the constant k determining the ellipses:

$$a\,\gamma^2\,u^2 + c\,\alpha^2\,v^2 = k^2 \Rightarrow B = \frac{k}{\alpha\sqrt{c}}\,.$$

General solution:

$$x(t) = \frac{c}{\gamma} + \frac{k}{\gamma\sqrt{a}}\,\cos\left(\sqrt{ac}\,t + \phi_0\right)\,,$$
$$y(t) = \frac{a}{\alpha} + \frac{k}{\alpha\sqrt{c}}\,\sin\left(\sqrt{ac}\,t + \phi_0\right)\,.$$

Evolution of populations: Prey-Predator model

$$x(t) = \frac{c}{\gamma} + \frac{k}{\gamma\sqrt{a}} \cos\left(\sqrt{ac}\, t + \phi_0\right) ,$$
$$y(t) = \frac{a}{\alpha} + \frac{k}{\alpha\sqrt{c}} \sin\left(\sqrt{ac}\, t + \phi_0\right) .$$

- Period $T = \frac{2\pi}{\sqrt{ac}}$, independent of initial conditions.
- Both populations are periodic and out of phase by one quarter of a period (prey leads and predator lags)
- Amplitudes of the oscillations are $\frac{k}{\gamma\sqrt{a}}$ and $\frac{k}{\alpha\sqrt{c}}$: they depend on initial conditions (k)
- Averages over a cycle coincide with the critical point configuration:

$$\langle x\rangle = \frac{1}{T}\int_0^T x(t)\, dt = \frac{c}{\gamma} ,$$
$$\langle y\rangle = \frac{1}{T}\int_0^T y(t)\, dt = \frac{a}{\alpha} .$$

Intro to nonlinear methods

Given a nonlinear autonomous ODE system:

$$\dot{x} = F(x, y), \qquad \dot{y} = G(x, y)$$

we would like to know

- existence of critical points
- existence of closed trajectories at the nonlinear level
- intrinsic nonlinear behaviour that can not be seen in the linear approximation

If we can integrate the equations, we can explicitly explore these issues. But this clearly depends on the ODE under considerations.

Intro to nonlinear methods

What we want is to develop methods that allow to answer this type of questions without integrating the system

We will mainly discuss two results:

- Lyapunov's theory: deals with the nonlinear stability of certain critical points
- Poincaré-Bendixson theorem: deals with the existence of closed trajectories

Both theorems require math techniques (analysis) beyond the scope of this course, but

- we will make their results as plausible as possible
- we will apply them in several examples

Gaining some intuition from Newton's law

Consider Newton's law:

$$\ddot{x} = F(x)$$

where the force $F(x)$ does not depend explicitly on time.
This dynamical problem is equivalent to a nonlinear ODE system

$$\dot{x}_1 = x_2 \,, \quad \dot{x}_2 = F(x_1)$$

- Consider $V(x_1) \Rightarrow \dot{V} = \frac{dV}{dx_1}\dot{x}_1 = \frac{dV}{dx_1}x_2$
- Consider $T(x_2) \Rightarrow \dot{T} = \frac{dT}{dx_2}\dot{x}_2 = \frac{dT}{dx_2}F(x_1)$

Each separate function changes with time, but $\dot{T}$ and $\dot{V}$ suggest some combination of T and V may not.

Gaining some intuition from Newton's law

Consider $T(x_2) + V(x_1)$. Then,

$$\dot{T} + \dot{V} = \frac{dV}{dx_1}\, x_2 + \frac{dT}{dx_2}\, F(x_1) = 0 \quad \text{if} \quad \frac{dT}{dx_2} = x_2\,, \quad F(x_1) = -\frac{dV}{dx_1}$$

Thus, when the force $F(x_1)$ comes from a potential $V(x_1)$, the quantity $\frac{1}{2}x_2^2 + V(x_1) = E$ (energy) is conserved, i.e. $\frac{dE}{dt} = 0$.

- $\frac{1}{2}x_2^2 + V(x_1) = E$ is an implicit trajectory for the nonlinear ODE, i.e. the existence of a conserved quantity allowed us to integrate the nonlinear ODE system implicitly
- Stable (unstable) critical points correspond to minima (maxima) of the potential $V(x_1)$ (see week-6 workshop)

Gaining some intuition from Newton's law

Consider a critical point $(x_0, 0)$ characterised by

$$\dot{x}_2 = -\frac{dV}{dx_1}(x_0) = 0 \Rightarrow x_0 \text{ local extremum of } V(x_1)$$

Linearising: $u = x_1 - x_0$ and $v = x_2$, we obtain

$$\dot{u} = v\,, \qquad \dot{v} = -\frac{d^2V}{dx_1^2}(x_0)u\,.$$

The eigenvalues λ of this linear system satisfy

$$\lambda^2 = -\frac{d^2V}{dx_1^2}(x_0)\,.$$

- If x_0 maximum $\Rightarrow \frac{d^2V}{dx_1^2}(x_0) < 0 \Rightarrow \lambda_\pm$ real and different signs $\Rightarrow$ saddle point $\Rightarrow$ unstable critical point.
- If x_0 minimum $\Rightarrow \frac{d^2V}{dx_1^2}(x_0) > 0 \Rightarrow \lambda_\pm$ purely imaginary $\Rightarrow$ center $\Rightarrow$ stable critical point. (linear approximation)

Example: pendulum with no friction

Take $\gamma = 0$ in our previous discussions (other parameters $\to 1$)

$$\dot{x} = y\,, \qquad \dot{y} = -\sin x\,.$$

The energy function $E(x, y)$ can be chosen to be:

$$E(x, y) = \frac{1}{2}y^2 + (1 - \cos x)$$

- Stable critical points: $y_0 = 0$ and $x_0 = 2n\pi \Rightarrow E(x_0, y_0) = 0$
- Unstable critical points:
  $y_0 = 0$ and $x_0 = (2n+1)\pi \Rightarrow E(x_0, y_0) = 2$

Example: pendulum with no friction

Question: How do trajectories look close to these critical points?

- Take (x, y) close to $(0, 0) \Rightarrow \cos x \sim 1 - \frac{1}{2}x^2 + O(x^4)$

$$E \sim \frac{1}{2}y^2 + \frac{1}{2}x^2 \Rightarrow \frac{x^2}{2E} + \frac{y^2}{2E} \sim 1$$

  ellipses enclosing the origin

- Take (x, y) close to $(\pi, 0)$, i.e. $u = x - \pi$ and $v = y$ with $|u|,\ |v| \ll 1$
  $\cos x = \cos(u + \pi) = -\cos u \sim -(1 - \frac{1}{2}u^2) + O(u^4)$

$$E - 2 \sim \frac{1}{2}v^2 - \frac{1}{2}u^2 \Rightarrow \frac{v^2}{2(E-2)} - \frac{u^2}{2(E-2)} \sim 1$$

  hyperbolas avoiding the critical point

Reinterpretation of conservation of energy

Given the energy function $E(x, y)$, its time derivative equals

$$\begin{aligned}\frac{dE}{dt} &= (\partial_x E)\,\dot{x} + (\partial_y E)\,\dot{y} \\ &= \nabla E \cdot \boldsymbol{T}(x, y) = |\nabla E||\boldsymbol{T}| \cos\phi\end{aligned}$$

- $\nabla E = (\partial_x E,\ \partial_y E)$ is the gradient vector associated with the surface $E(x, y) = \text{constant}$
- $\boldsymbol{T}(x, y) = (\dot{x},\ \dot{y})$ is the tangent vector to the trajectory at $(x(t), y(t))$
- ϕ is the angle between the gradient and the tangent vectors at the point $(x(t), y(t))$.

Reinterpretation of conservation of energy

Thus, conservation of energy means

$$\frac{dE}{dt} = \boldsymbol{\nabla} E \cdot \boldsymbol{T}(x, y) = |\boldsymbol{\nabla} E||\boldsymbol{T}| \cos\phi = 0$$

the gradient and tangent vectors are orthogonal, i.e. $\phi = \frac{\pi}{2}$
This makes sense:

- $E(x, y) =$ constant is a trajectory solving our nonlinear ODE system.
- $E(x, y) =$ constant defines a surface/curve $\Rightarrow$ $\boldsymbol{\nabla} E$ is orthogonal to the surface
- Orthogonality $\Rightarrow$ $\boldsymbol{\nabla} E \perp \boldsymbol{T}$ $\Rightarrow$ conservation of energy

Honours Differential Equations

Jacques Vanneste

Lecture 20

November 1, 2018

Existence of closed trajectories

When studying the linear approximation, we learnt

- centres may not survive nonlinear corrections
  $\Rightarrow$ when do closed (periodic) trajectories exist nonlinearly?
- are periodic trajectories always associated with nearby critical points?
- what is the long-time fate of trajectories that do not tend to ∞?

Notion of limit cycle

Consider the nonlinear ODE system:

$$\dot{x} = y - x(x^2 + y^2 - a^2)\,, \qquad \dot{y} = -x - y(x^2 + y^2 - a^2)\,.$$

Critical point: $(x_0, y_0) = (0, 0)$
Linear approximation:

$$\begin{aligned} F(x,y) &= y - x(x^2 + y^2 - a^2) \\ &\Rightarrow \quad \partial_x F(0,0) = a^2\,, \quad \partial_y F(0,0) = 1 \\ G(x,y) &= -x - y(x^2 + y^2 - a^2) \\ &\Rightarrow \quad \partial_x G(0,0) = -1\,, \quad \partial_y G(0,0) = a^2 \end{aligned}$$

This gives rise to the linear system approximation:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} a^2 & 1 \\ -1 & a^2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Notion of limit cycle

Eigenvalues: $\lambda_\pm = a^2 \pm i \Rightarrow$ unstable spiral
Thus, close to the origin, trajectories spiral away from the origin

Question: Does this behaviour persist in the entire phase space?

We need a nonlinear analysis $\Rightarrow$ let us use polar coordinates:

$$\dot{r} = (a^2 - r^2)\, r\,, \qquad \dot{\phi} = -1\,.$$

These can be integrated.

Notion of limit cycle

$$\dot{r} = (a^2 - r^2)\, r\,, \qquad \dot{\phi} = -1\,.$$

- $r = a\,, \quad \phi = -t + t_0 \Rightarrow$ periodic solution (rotating clockwise)
- when $r \neq a$, we obtain after some algebra

$$r(t) = \frac{a}{\sqrt{1 - B\, e^{-2a^2 t}}}\,, \quad \phi = -t + t_0\,.$$

If $t_0 = 0$ and we choose $r(0) = r_0$ and $\phi(0) = \phi_0$, then

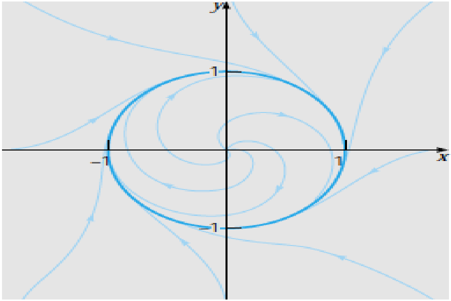
$$r(t) = \frac{a}{\sqrt{1 - \left(1 - \frac{a^2}{r_0^2}\right)\, e^{-2a^2 t}}}\,, \quad \phi = -\left(t - \phi_0\right)\,.$$

  - if $r_0 < a$, $r \to a$ as $t \to \infty \Leftrightarrow$ solutions approach the periodic one from inside
  - if $r_0 > a$, $r \to a$ as $t \to \infty \Leftrightarrow$ solutions approach the periodic one from outside

Notion of limit cycle

- Trajectories do spiral away from the origin, as predicted by the linear analysis
  But, they asymptote to the periodic solutions at $r = a$ (a=1 in the figure below)
- Any trajectory starting at $r_0 > a$ asymptotes the $r = a$ trajectory

This is an example of a limit cycle.



Notion of limit cycle

We could have reached the same conclusion without using the explicit trajectories

The exact radial ODE is:

$$\dot{r} = (a^2 - r^2)\, r\,.$$

Consider some initial condition $r(t = 0) = r_0$:

- if $r_0 < a$, then $\dot{r}(0) > 0 \Rightarrow$ trajectories starting inside the limit cycle asymptote to it, because its radial velocity grows, i.e. they spiral away from the origin in agreement with our linear approximation discussion
- if $r_0 > a$, then $\dot{r}(0) < 0 \Rightarrow$ trajectories starting outside of the limit cycle asymptote to it, because its radial velocity decreases, i.e. they spiral towards the limit cycle.

Taxonomy of limit cycles

Definition

Limit cycles are periodic solutions such that at least one other non-closed trajectory asymptotes to them as $t \to \infty$ or $t \to -\infty$ (or both).

- The limit cycle is asymptotically stable if all trajectories asymptote towards it.
  Notice limit cycles are not equilibrium configurations (orbital stability)
- If, for $t > 0$, trajectories from one side spiral towards the limit cycle and from the other side spiral away, the limit cycle is semistable
- If all trajectories spiral away from the limit cycle, it is unstable

The example we studied was asymptotically stable.

When do limit cycles/closed trajectories exist?

Given a nonlinear ODE system

$$\dot{x} = F(x, y), \qquad \dot{y} = G(x, y).$$

it is hard to integrate the trajectories. We want to determine conditions guaranteeing the (non-)existence of limit cycles.

Theorem

Let $F(x, y)$, $G(x, y)$ have continuous first partial derivatives in some domain D. A closed trajectory must necessarily enclose at least one critical point. If it encloses only one critical point, it can not be a saddle point.

When do limit cycles/closed trajectories exist?

Example:

$$\dot{r} = (a^2 - r^2)\, r\,, \qquad \dot{\phi} = -1\,.$$

The closed trajectory $r = a$ encloses a critical point $(0, 0)$ which is not a saddle point.

The theorem is not terribly constructive, but its negative version can be quite useful:

- if D contains no critical points $\Rightarrow$ no closed trajectories in D,
- if D contains a unique critical point and it is a saddle $\Rightarrow$ no closed trajectories in D.

When do limit cycles/closed trajectories exist?

Theorem

Let $F(x, y)$, $G(x, y)$ have continuous first partial derivatives in a simply connected domain D. If $\partial_x F + \partial_y G$ has the same sign in D $\Rightarrow$ there are no closed trajectories in D.

- simply connected $D \sim$ domain with no holes
- In our example

$$\dot{x} = y - x(x^2 + y^2 - a^2)\,, \qquad \dot{y} = -x - y(x^2 + y^2 - a^2)\,.$$

Thus, $\partial_x F = -3x^2 - y^2 + a^2$ and $\partial_y G = -x^2 - 3y^2 + a^2$:

$$\partial_x F + \partial_y G = 2(a^2 - 2r^2)$$

$$\partial_x F + \partial_y G > 0 \text{ for } 0 < r < \frac{a}{\sqrt{2}}$$

This is true since we have the exact trajectories in this case. In fact we know the statement holds till $r = a$. This shows the information provided by this theorem may not be optimal.

Poincaré-Bendixson theorem

Theorem

Let $F(x, y)$, $G(x, y)$ have continuous first partial derivatives in a domain D. Let D_1 be a bounded subdomain of D and let R consist of D_1 and its boundary. Suppose R has no critical points. If $\exists$ a trajectory $(x(t), y(t))$ staying in R $\forall\, t \geq t_0 \Rightarrow$ either the solution is periodic (closed trajectory) or it spirals towards one. Either way, $\exists$ a closed trajectory.

- If R contains a closed trajectory $\Rightarrow$ by 1st theorem, the trajectory encloses a critical point.
  However, this critical point can not belong to $R \Rightarrow$ R is not simply connected (it must have holes)
- In our example $\dot{x} = y - x(x^2 + y^2 - a^2)$ and $\dot{y} = -x - y(x^2 + y^2 - a^2)$, we could consider the region $\frac{a}{2} \leq r \leq 2a$: it contains no critical points and all trajectories stay inside it (check sign of velocities !!) $\Rightarrow$ $\exists$ closed trajectory

Poincaré-Bendixson theorem: example

Show that the system

$$\dot{x}_1 = x_2\,, \qquad \dot{x}_2 = -x_1 + x_2(1 - 3x_1^2 - 2x_2^2)$$

has a periodic solution.
Use radial polar coordinate $r^2 = x_1^2 + x_2^2$:

$$\begin{aligned} r\dot{r} &= x_1\dot{x}_1 + x_2\dot{x}_2 = x_2^2(1 - 3x_1^2 - 2x_2^2) \\ &\geq x_2^2(1 - 3x_1^2 - 3x_2^2) = x_2^2(1 - 3r^2) \geq 0 \quad \text{if} \quad r \leq \frac{1}{\sqrt{3}} \end{aligned}$$

Similarly,

$$r\dot{r} \leq x_2^2(1 - 2x_1^2 - 2x_2^2) \leq 0 \quad \text{if} \quad r \geq \frac{1}{\sqrt{2}}.$$

Thus the annulus $R : \frac{1}{\sqrt{3}} \leq r \leq \frac{1}{\sqrt{2}}$ is a trapping region (trajectories remain inside this region)
Since the only critical point is the origin, which does not belong to the annulus R, the Poincaré-Bendixson theorem guarantees the existence of a closed trajectory in R.

Poincaré-Bendixson theorem: example

Question: Prove that, for suitable a, there exists a trapping region of the form $R = \{(x_1, x_2) | x_1^2 + x_2^2 \leq a^2\}$ for the system

$$\begin{aligned}
\dot{x}_1 &= -\omega x_2 + x_1(1 - x_1^2 - x_2^2) - x_2(x_1^2 + x_2^2) \\
\dot{x}_2 &= \omega x_1 + x_2(1 - x_1^2 - x_2^2) + x_1(x_1^2 + x_2^2) - F,
\end{aligned}$$

where ω, F are constants. Show that the system has a periodic solution when $F = 0$ provided that $\omega \neq -1$. What happens if $\omega = -1$ (and $F = 0$)?

$$\begin{aligned}
r\dot{r} &= x_1\dot{x}_1 + x_2\dot{x}_2 = x_1^2(1 - x_1^2 - x_2^2) + x_2^2(1 - x_1^2 - x_2^2) - Fx_2 \\
&= r^2(1 - r^2) - Fx_2 \leq r^2(1 - r^2) + |F|r \leq 0
\end{aligned}$$

if r is large enough. Thus if a is large enough r, there is a trapping region.

Poincaré-Bendixson theorem: example

When $F = 0$, the origin $(0,0)$ is a critical point: $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$. To use Poincaré-Bendixson theorem we need to construct a region avoiding the origin. Notice

$$\begin{aligned} r\dot{r} &= x_1\dot{x}_1 + x_2\dot{x}_2 = x_1^2(1 - x_1^2 - x_2^2) + x_2^2(1 - x_1^2 - x_2^2) \\ &= r^2(1 - r^2) \geq 0 \text{ if } r \leq 1. \end{aligned}$$

Thus, we can choose R as an annulus excluding the origin. The theorem tells us there is a periodic solution.

- when $F = 0$, $\dot{r} = 0$ when $r = 1$. This shows that any solution which starts on $r = 1$ remains there.

Poincaré-Bendixson theorem: example

Even more, when $r = 1$ and $F = 0$ the original system simplifies to

$$\begin{aligned} \dot{x}_1 &= -\omega x_2 - x_2 \\ \dot{x}_2 &= \omega x_1 + x_1, \end{aligned}$$

and further to

$$\ddot{x}_1 = -x_1(\omega + 1)^2$$

which has periodic solutions provided that $\omega + 1 \neq 0$

- If $\omega = -1$ all points on the unit circle are critical points.

Honours Differential Equations

Jacques Vanneste

Lecture 21

November 2, 2018

Motivating the rest of the course

So far, we have focused on

- ordinary DEs, i.e., a single independent variable,
- initial value problems, specifying $\mathbf{x}(t_0)$,

In this setup, we discussed

- n-th order linear ODES $\Leftrightarrow$ 1st order linear systems
- introduction to non-linear methods (autonomous systems)

In the rest of the course, we will discuss

- boundary value problems,
- partial linear DEs, i.e. more than one independent variable,
- Sturm-Liouville: linear algebra in infinite dimension.

Boundary value problems

Solve

$$y'' + y = 0\,,$$

subject to the boundary conditions $y(0) = 1$ and $y(\pi) = a$.

The general (real) solution to the 2nd order ODE is:

$$y(x) = c_1 \cos x + c_2 \sin x\,.$$

Boundary conditions:

$$y(0) = c_1 = 1 \Rightarrow c_1 = 1$$
$$y(\pi) = -c_1 = a \Rightarrow c_1 = -a.$$

Thus,

- if $a \neq -1$ no solution,
- if $a = -1 \Rightarrow y(x) = \cos x + c_2 \sin x \Rightarrow$ infinitely many solutions.

Motivating boundary eigenvalue problems

A basis for vectors in $\mathbb{R}^n$ consists of n linearly independent vectors $\mathbf{e}_i$

Any vector $\mathbf{v}$ can be expanded as

$$\mathbf{v} = \sum_{i=1}^{n} v^i \, \mathbf{e}_i$$

Given a diagonalisable matrix A acting on $\mathbb{R}^n$, one particular such basis is given by the eigenvectors $\boldsymbol{\xi}_\lambda$:

$$A\boldsymbol{\xi}_\lambda = \lambda \boldsymbol{\xi}_\lambda \, .$$

- Any matrix $A \in M^{n\times n}$ can be identified as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Motivating boundary eigenvalue problems

Question: what about the space of functions?
Functions $y(x)$ can be thought of as vectors:

- $$af_1(x) + bf_2(x) \qquad a, b \in \mathbb{R} \qquad \text{is also a function.}$$
- we can define linear maps using derivatives:

  $$\frac{d}{dx} : y(x) \mapsto \frac{dy}{dx}(x)$$

  is linear. So are higher derivatives and their linear combinations, defining linear operators such as

  $$p(x)\frac{d^2}{dx^2} + q(x)\frac{d}{dx}.$$

Questions: can we define eigenvalue problems for linear operators? are the corresponding eigenvectors (= eigenfunctions) useful?

Boundary eigenvalue problems

Consider $\lambda \in \mathbb{R}$

$$y'' - \lambda\, y = 0 \Leftrightarrow \frac{d^2}{dx^2} y(x) = \lambda\, y(x)\,, \qquad y(0) = y(L) = 0\,.$$

- if $\lambda = 0 \Rightarrow y = c_1 x + c_2$. Using boundary conditions

$$y(0) = c_2 = 0$$
$$y(L) = c_1 L = 0 \Rightarrow c_1 = 0$$

  Only the trivial solution remains.

- If $\lambda = \mu^2 > 0 \Rightarrow y_\lambda = c_1 \cosh \mu x + c_2 \sinh \mu x$. Using boundary conditions

$$y(0) = c_1 = 0$$
$$y(L) = c_2 \sinh \mu L = 0 \Rightarrow c_2 = 0$$

  Only the trivial solution remains.

Boundary eigenvalue problems

Consider $\lambda \in \mathbb{R}$

$$y'' - \lambda\, y = 0 \Leftrightarrow \frac{d^2}{dx^2} y(x) = \lambda\, y(x)\,, \qquad y(0) = y(L) = 0\,.$$

- If $\lambda = -\mu^2 < 0$ then

$$y_\lambda(x) = c_1 \cos \mu x + c_2 \sin \mu x\,.$$

Using boundary conditions:

$$y(0) = 0 \Rightarrow c_1 = 0$$

$$y(L) = 0 \Rightarrow c_2 \sin \mu L = 0 \Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2 \;\; n = 1, 2, \ldots$$

Thus, the eigenfunctions are

$$y_n \propto \sin \frac{n\pi\, x}{L}.$$

Further questions (for the rest of the course)

- Is there any sense in which an element (function $y(x)$) in the subspace of functions satisfying the boundary conditions

$$y(0) = y(L) = 0$$

can be expanded in the basis of eigenfunctions $y_n(x)$,

$$y(x) = \sum_{n=1}^{\infty} c_n y_n(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} ?$$

- Is it possible to define some notion of orthogonality in the space of functions, using a scalar product such as

$$(f, g) = \int_a^b f^\star(x)\, g(x)\, dx,$$

- What is the equivalent for function of the symmetry condition $A^{\mathrm{T}} = A$ for matrices?

Periodic functions

Definition

A function $f(x)$ is periodic if $f(x+T) = f(x)\ \forall\, x$. We say its period is T. The smallest possible T is the fundamental period.

- Consider $f(x)$ and $g(x)$, functions with period T, then

$$a\, f(x) + b\, g(x)\,,$$

  is a new function with the same period T.
- Remember $\sin x$ and $\cos x$ have periods $T = 2\pi$.
  Thus, functions such as $\sin\frac{m\pi x}{L}, \cos\frac{m\pi x}{L}$ have periods $T = \frac{2L}{m}$, i.e. $x \sim x + T$, with $m = 1, 2, \ldots$
- $\sin\frac{m\pi x}{L}, \cos\frac{m\pi x}{L}$ are the eigenfunctions of

$$y'' = \lambda y, \quad \text{with} \quad y(x+2L) = y(x).$$

Orthogonality of functions for periodic functions

Consider the set of periodic functions $\{\sin\frac{n\pi x}{L},\, \cos\frac{m\pi x}{L}\}$ in the interval $x \in [-L,\, L]$ with $n, m = 1, 2, \ldots$
Using the identities

$$\begin{aligned}
\cos A \cos B &= \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right] , \\
\sin A \sin B &= \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] , \\
\sin A \cos B &= \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right] ,
\end{aligned}$$

we will show that defining the inner product of two such functions $u(x)$ and$v(x)$ as

$$(u(x),\ v(x)) \equiv \int_{-L}^{L} u(x)\, v(x)\, dx$$

the set $\{\sin\frac{n\pi x}{L},\, \cos\frac{m\pi x}{L}\}$ is mutually orthogonal, i.e. every pair of functions is orthogonal.

Orthogonality of functions for periodic functions

Proof.

Consider the pair $S_n(x) = \sin \frac{n\pi x}{L}$ and $S_m(x) = \sin \frac{m\pi x}{L}$:

$$\begin{aligned}
(S_m, S_n) &= \int_{-L}^{L} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L}\, dx \\
&= \frac{1}{2}\int_{-L}^{L} \left(\cos \frac{(m-n)\pi x}{L} - \cos \frac{(m+n)\pi x}{L}\right) dx \\
&= \frac{L}{2\pi}\left[\frac{\sin(m-n)\pi x/L}{m-n} - \frac{\sin(m+n)\pi x/L}{m+n}\right]_{-L}^{L} \\
&= 0 \text{ if } m \neq n
\end{aligned}$$

□

Orthogonality of functions for periodic functions

Proof.

If $m = n$, then notice that

$$\left(\sin\frac{n\pi x}{L}\right)^2 = \frac{1}{2}\left(1 - \cos\frac{2n\pi x}{L}\right).$$

Thus,

$$\begin{aligned}(S_n,\, S_n) &= \int_{-L}^{L}\left(\sin\frac{n\pi x}{L}\right)^2\, dx \\ &= \frac{1}{2}\left[x - \frac{\sin 2n\pi x/L}{2n\pi/L}\right]_{-L}^{L} \\ &= L\,.\end{aligned}$$

□

Orthogonality of functions for periodic functions

Proof.

We can write both results as

$$(S_m,\ S_n) = L\delta_{mn} \qquad m, n \neq 0$$

It is clear that if we define $C_n(x) = \cos\frac{n\pi x}{L}$, these satisfy

$$(C_m,\ C_n) = 0\,, \quad m \neq n$$

When $m = n$ since

$$\left(\cos\frac{n\pi x}{L}\right)^2 = \frac{1}{2}\left(1 + \cos\frac{2n\pi x}{L}\right)\,,$$

we still get $(C_n,\ C_n) = L \Rightarrow (C_m,\ C_n) = L\delta_{mn} \qquad m, n \neq 0$

$\square$

Orthogonality of functions for periodic functions

Proof.

Finally,

$$\begin{aligned}
(S_m,\ C_n) &= \int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx \\
&= \frac{1}{2}\int_{-L}^{L}\left[\sin\frac{(m+n)\pi x}{L} + \sin\frac{(m-n)\pi x}{L}\right]\,dx \\
&= -\frac{L}{2\pi}\left[\frac{\cos(m+n)\pi x/L}{(m+n)} + \frac{\cos(m-n)\pi x/L}{(m-n)}\right]_{-L}^{L} \\
&= 0
\end{aligned}$$

because $\cos x$ is an even function. Please notice the conclusion holds even if $n = m$:

$$(S_n,\ C_n) = \frac{1}{2}\int_{-L}^{L}\sin\frac{2n\pi x}{L}\,dx = -\frac{L}{2n\pi}\left[\cos\frac{2n\pi x}{L}\right]_{-L}^{L} = 0\,.$$

Orthogonality of functions for periodic functions

Thus, given the collection of periodic functions $\{\sin \frac{n\pi x}{L},\, \cos \frac{m\pi x}{L}\}$ in the interval $x \in [-L,\ L]$ with $n, m = 1, 2, \ldots,$ we have indeed shown that it defines a set of mutually orthogonal functions using the inner product

$$(u(x),\ v(x)) \equiv \int_{-L}^{L} u(x)\, v(x)\, dx$$

Coming next

Question: Is there any sense in which we can expand a periodic function $f(x)$ in terms of this orthogonal set of functions?

$$f(x) \sim \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \text{ ?}$$

- The study of Fourier series is precisely concerned about when this happens and if it does, what the properties of such expansion are.

Honours Differential Equations

Jacques Vanneste

Lecture 22

November 5, 2018

Fourier series

Definition

Given a periodic function $f(x)$ with period $2L$, it can be expressed as a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) ,$$

when $f(x)$ satisfies some conditions to be discussed.

- the coefficients $\{a_n, \ b_n\}$ are constants, the Fourier coefficients of $f(x)$. They are real if $f(x)$ is real.
- since the series involves an infinite number of terms, there is a natural question concerning the convergence of this series.

Euler-Fourier formulas

Assume the Fourier series of the $f(x)$ converges

Question: How do we determine its Fourier coefficients $\{a_n, b_n\}$?

Use orthogonality: given

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) ,$$

compute the inner product of both sides with $\cos \frac{n\pi x}{L}$, i.e. project the function $f(x)$ onto $\cos \frac{n\pi x}{L}$,

$$\begin{aligned} \left(f(x), \cos \frac{n\pi x}{L} \right) &= \left(\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right), \cos \frac{n\pi x}{L} \right) \\ &= L\, a_n . \end{aligned}$$

Euler-Fourier formulas

Thus,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\cos\frac{n\pi x}{L}\,dx \qquad n = 0, 1, 2, \ldots$$

- We used the orthogonality relations

$$(S_n, S_m) = (C_n, C_m) = L\delta_{mn}\,,\, m, n = 1, 2, \ldots$$
$$(S_n, C_m) = (C_m, S_n) = 0$$

- In fact, these relations can be extended to include $n = 0$. Define $C_0 = 1$ (the constant function equal to one). Notice,

$$(C_0, C_0) = \int_{-L}^{L} 1\,dx = 2L\,,$$
$$(C_0, C_m) = \int_{-L}^{L} \cos\frac{m\pi x}{L}\,dx = 0\,,$$
$$(C_0, S_m) = \int_{-L}^{L} \sin\frac{m\pi x}{L}\,dx = 0\,.$$

Euler-Fourier formulas

Thus, when we project along C_0, we obtain

$$(f(x), C_0) = \frac{a_0}{2}(C_0, C_0) = a_0\, L \Rightarrow a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx\,.$$

- This justifies/explains why there is a $\frac{1}{2}$ in the original definition, i.e. so that all a_n have the same formula, since $\cos\frac{n\pi x}{L}$ equals one for $n = 0$
- the constant term $\frac{a_0}{2}$ is the average value of the function over the period:

$$\frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx \equiv \langle f(x)\rangle\,.$$

Euler-Fourier formulas

Projecting onto $\sin \frac{n\pi x}{L}$, we can compute the remaining coefficients

$$\left(f(x), \sin \frac{n\pi x}{L}\right) = \left(\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L}\right), \sin \frac{n\pi x}{L}\right)$$
$$= L\, b_n$$

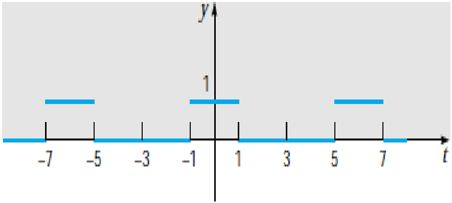
Thus,

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx\,.$$

The formulas computing the Fourier coefficients are referred to as Euler-Fourier formulas.

Example

$$f(x) = \begin{cases} 0 & -3 < x < -1 \\ 1 & -1 < x < 1 \\ 0 & 1 < x < 3 \end{cases} \qquad \text{with} \quad f(x+6) = f(x)$$



- Since the period is $T = 6 \Rightarrow 2L = 6 \Rightarrow L = 3$
- Thus,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{3} + b_n \sin \frac{n\pi x}{3} \right)$$

Example

- Using Euler-Fourier formulas:

$$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)dx = \frac{1}{3}\int_{-1}^{1} dx = \frac{2}{3}$$

$$a_n = \frac{1}{3}\int_{-3}^{3} f(x)\cos\frac{n\pi x}{3}dx = \frac{1}{3}\int_{-1}^{1}\cos\frac{n\pi x}{3}dx = \frac{2}{n\pi}\sin\frac{n\pi}{3}$$

$$b_n = \frac{1}{3}\int_{-3}^{3} f(x)\sin\frac{n\pi x}{3}dx = \frac{1}{3}\int_{-1}^{1}\sin\frac{n\pi x}{3}dx = 0\,.$$

Thus, the Fourier series of the function $f(x)$ is given by the infinite sum

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi}{3}\cos\frac{n\pi x}{3}\,.$$

Interpretation (application)

Given any periodic signal measured in any experiment, defining our previous function $f(x)$, we can view the Fourier series as

- decomposing the signal into an infinite (countable) number of waves
- each wave $\cos \frac{n\pi x}{L}$ and $\sin \frac{m\pi x}{L}$ has a discrete period
- the Fourier coefficients $\{a_n, b_n\}$ correspond to the amplitudes of these countably-many different waves
  - if we consider a single wave $S(t) = A \sin \omega t$, this has period $T = \frac{2\pi}{\omega}$ and the amplitude A corresponds to the size of the wave, i.e. the location of the maximum in the oscillation being described.

Connection to abstract algebra

There is a close analogy between the structure of Fourier series and vector algebra. The "dictionary" is as follows:

- the space of vectors is replaced by the set of periodic functions,
- the function $f(x)$ corresponds to an arbitrary vector $\boldsymbol{v}$,
- the orthogonal basis $\{\boldsymbol{e}_i\}$ correspond to $\{1, \cos\frac{n\pi x}{L}, \sin\frac{m\pi x}{L}\}$
- the vector components v^i, i.e. $\boldsymbol{v} = \sum_i v^i \boldsymbol{e}_i$, correspond to the Fourier coefficients $\{a_n, b_m\}$
  - both are computed using the inner product in the same way

$$v^i = (\boldsymbol{v}, \boldsymbol{e}_i)\ , \quad \text{vs} \quad a_n = (f(x), C_n)\ , \quad b_m = (f(x), S_m)$$

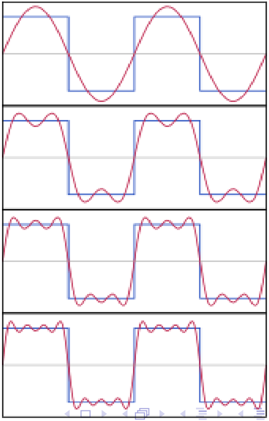
Thus, if the infinite sum converges, we will have discovered the existence of an infinite dimensional vector space, spanned by countably many basis functions (Hilbert space).

A signal vs its Fourier representation

Consider a truncation to a finite set of terms in the Fourier series

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) .$$

- the more discrete waves we add, the better the approximation
- the original signal has discontinuities (jumps)
- not clear whether the Fourier series will converge at these discontinuities



Fourier convergence theorem

Definition

A function $f(x)$ is said to be piecewise continuous on $x \in [a, b]$ if $[a, b]$ can be partitioned into a finite number of points $a = x_0 < x_1 < x_2 \ldots x_{n-1} < x_n = b$ such that

- *$f(x)$ continuous $x \in (x_{i-1}, x_i)$*
- *$f(x)$ has a finite limit as the endpoints of each subinterval (x_{i-1}, x_i) are approached from within the subinterval*
  For example, let $x_i = c$, then

$$f(c^+) \equiv \lim_{x \to c^+} f(x) \text{ finite} \quad \text{(limit from the right)}$$

$$f(c^-) \equiv \lim_{x \to c^-} f(x) \text{ finite} \quad \text{(limit from the left)}$$

Fourier convergence theorem

Theorem

Suppose $f(x)$ and $f'(x)$ are piecewise continuous for $x \in [-L, L]$
Suppose $f(x)$ is defined outside this interval so that it is periodic with period $2L$ (as we have done in this lecture).
Then, $f(x)$ has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) ,$$

with coefficients given by the Euler-Fourier formulas.
The series converges to $f(x)$ $\forall\, x$ where $f(x)$ is continuous and to $\frac{f(x^+)+f(x^-)}{2}$ at all points where $f(x)$ is discontinuous.

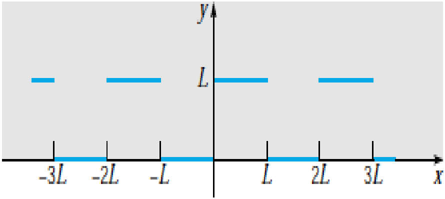
Fourier convergence theorem

Remarks:

- Functions like $\frac{1}{x^2}$ or $\log x$ have no convergent Fourier series because of the infinite divergences of the function at $x = 0$
- Functions having an infinite number of discontinuities are not guaranteed to have a convergent Fourier Series

Example:

$$f(x) = \begin{cases} 0 & -L < x < 0 \\ L & 0 < x < L \end{cases} \quad \text{with} \quad f(x + 2L) = f(x)$$



Example

- In the main interval $[-L, L]$ the function has discontinuities at $x = 0$ and $x = L$ (or $x = -L$) $\Rightarrow f(x)$ is piece-wise continuous $\Rightarrow$ theorem applies
- Using Euler-Fourier formulas:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \int_0^L dx = L\,,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \int_0^L \cos\frac{n\pi x}{L}\,dx = 0\,.$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \int_0^L \sin\frac{n\pi x}{L}\,dx$$

$$= \frac{L}{n\pi}(1 - \cos n\pi)$$

Since $\cos n\pi = (-1)^n$, we conclude only odd n contribute

Example

Thus,

$$f(x) = \frac{L}{2} + \frac{2L}{\pi} \sum_{m=1}^{\infty} \frac{\sin(2m-1)\pi x/L}{(2m-1)}$$

where we wrote $n = 2m - 1$ to sum over odd integers.

Remarks:

- Let x_n denote the points where the function has discontinuities, i.e. $x = 0, \pm nL$
- All terms in the series vanish at these points $\Rightarrow f(x_n) = \frac{L}{2}$,
  - this matches the prediction of the theorem,
- If we were to define a new function $\tilde{f}(x)$ equal to $f(x)$ everywhere except at the previous discontinuities where $\tilde{f}(x_n) = \frac{L}{2} \Rightarrow$ the Fourier series would converge everywhere.

Differentiation vs convergence

Consider the piece-wise continuous function:

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ -1 & -1 < x < 0 \end{cases}$$

Since the period is $T = 2 \Rightarrow L = 1$.
Its convergent Fourier series equals (check !!):

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\pi x}{2n-1}\,.$$

Question: Does term-by-term differentiation yield a convergent Fourier series?

Differentiation vs convergence

Formally differentiating term-by-term gives rise to:

$$f'(x)? = 4\sum_{n=1}^{\infty}\cos(2n-1)\pi x$$

- This is divergent !!
- Given the definition of $f(x)$, we can infer $f'(x) = 0 \;\; \forall\, x \neq 0$.
- In fact, $f'(0)$ is not defined at $x = 0$.
- Thus, the exact mathematical function $f'(x)$ does not satisfy the conditions guaranteeing the convergence of its Fourier series. This is consistent with what we found differentiating term by term.

Thus, convergence of $f'(x)$ as a Fourier series is not guaranteed even if $f(x)$ has a convergent Fourier series to begin with.

Honours Differential Equations

Jacques Vanneste

Lecture 23

November 8, 2018

Convergence of Fourier Series

We showed that the function

$$f(x) = \begin{cases} 0 & -L < x < 0 \\ L & 0 < x < L \end{cases} \qquad \text{with} \qquad f(x+2L) = f(x)$$

have the Fourier series

$$f(x) = \frac{L}{2} + \frac{2L}{\pi} \sum_{m=1}^{\infty} \frac{\sin(2m-1)\pi x/L}{(2m-1)}$$

To get some insight into its convergence, it is natural to define a finite truncation

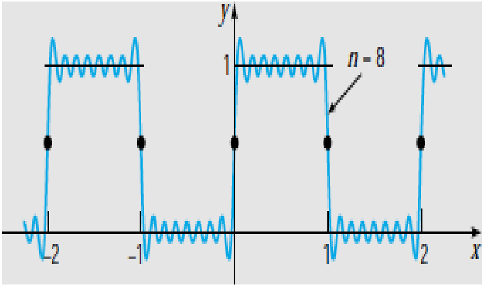
$$s_n(x) = \frac{L}{2} + \frac{2L}{\pi} \left(\sin \frac{\pi x}{L} + \cdots + \frac{\sin(2n-1)\pi x/L}{2n-1} \right)$$

and measure its error vs the real function by computing

$$|e_n(x)| = |s_n(x) - f(x)|.$$

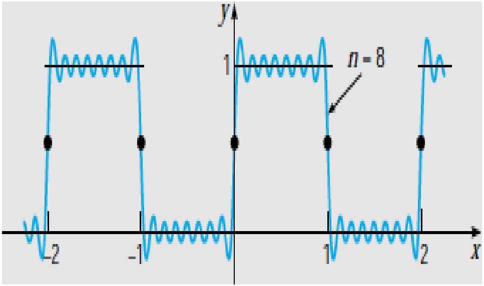
Gibbs phenomenon

When we plot $s_n(x)$ (n=8 in the picture)



- as *n* increases, the Fourier series approximates $f(x)$ better and better $\forall\, x$ where $f(x)$ is continuous.

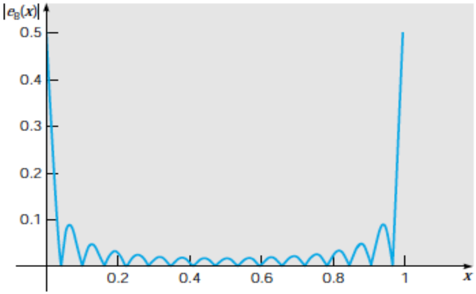
Gibbs phenomenon



- but the convergence to the mean value at the discontinuity points is not smooth
  - $s_n(x)$ overshoots the actual value of the function (the first/last peaks close to any discontinuity point),
  - overshooting persists at large n: Gibbs phenomenon,
  - see workshop sheet for more.

Error & convergence

When we plot $|e_n(x)|$ ($n = 8$ in the picture)



- $|e_n(x)|$ decreases as n increases $\forall\, x$ where $f(x)$ is continuous
- the rate of convergence depends on how quickly the Fourier coefficients $\{a_n, b_n\}$ decay with n
- $|e_n(x)|$ never decreases at the points where $f(x)$ is discontinuous.

Complex form of Fourier series

Using the identity

$$e^{ix} = \cos x + i \sin x$$

we can rewrite the Fourier series using complex exponentials:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{2} \left(e^{in\pi x/L} + e^{-in\pi x/L} \right) + \frac{b_n}{2i} \left(e^{in\pi x/L} - e^{-in\pi x/L} \right) \right]$$

$$= \sum_{n=-\infty}^{\infty} c_n \, e^{in\pi x/L}$$

where we defined

$$c_n = \frac{a_n - ib_n}{2} \quad (n > 0), \quad c_{-n} = \frac{a_n + ib_n}{2} \quad (n > 0)$$

$$c_0 = \frac{a_0}{2}$$

Notice the different range in the label n, which can take negative integer values.

Complex form of Fourier series

We could have directly worked in the basis of complex exponentials from the start, observing they satisfy the orthogonality relations

$$\left(e^{\frac{im\pi x}{L}},\, e^{\frac{in\pi x}{L}}\right) = \int_{-L}^{L} e^{\frac{-im\pi x}{L}}\, e^{\frac{in\pi x}{L}}\, dx = 2L\delta_{mn}$$

- Notice the sign in the exponentials is consistent with our definition of inner product for complex functions

$$(f,\, g) = \int_{\alpha}^{\beta} f^{\star}(x)\, g(x)\, dx\,.$$

Using this orthogonality, the Fourier coefficients c_n equal:

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-\frac{in\pi x}{L}}\, dx\,, \qquad n \in \mathbb{Z}$$

If the function is real, we get back the relation

$$c_{-n} = c_n^{\star}$$

Even & odd functions

Definition

A function $f(x)$ is even if $f(-x) = f(x)$.
A function $f(x)$ is odd if $f(-x) = -f(x)$.

Consider the integral

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)dx = -\int_{L}^{0} f(-s)ds + \int_{0}^{L} f(x)dx\,.$$

Thus,

- if $f(x)$ is even, then $f(-s) = f(s)$

$$\int_{-L}^{L} f(x)dx = -\int_{L}^{0} f(x)dx + \int_{0}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$$

- if $f(x)$ is odd, then $f(-s) = -f(s)$

$$\int_{-L}^{L} f(x)dx = +\int_{L}^{0} f(x)dx + \int_{0}^{L} f(x)dx = 0\,.$$

Even & odd functions: Fourier series

- Even functions only have cosine coefficient series
  - This is because $f(x) \sin \frac{m\pi x}{L}$ is odd $\Rightarrow b_m = 0$ and

$$a_m = \frac{2}{L} \int_0^L f(x) \cos \frac{m\pi x}{L} \, dx \, .$$

- Odd functions only have sine coefficient series
  - This is because $f(x) \cos \frac{m\pi x}{L}$ is odd $\Rightarrow a_m = 0$ and

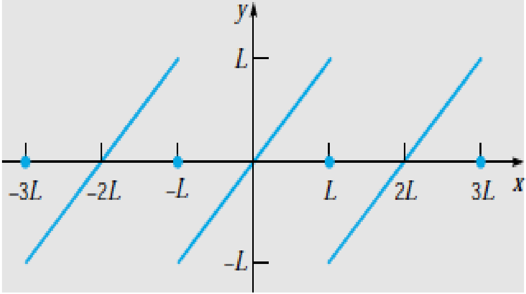
$$b_m = \frac{2}{L} \int_0^L f(x) \sin \frac{m\pi x}{L} \, dx \, .$$

Notice the factor of 2 appears because the range of integration is only over half of the period.

Example: sawtooth wave

Consider the sawtooth wave function $f(x) = x$ for $x \in (-L, L)$ satisfying

$$f(-L) = f(L) = 0 \quad \text{and} \quad f(x + 2L) = f(x)$$



Example: sawtooth wave

Notice $f(x) = x$ is odd since $f(-x) = -x = -f(x)$
Thus, $a_m = 0$. The remaining Fourier coefficients are:

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L x \sin\frac{n\pi x}{L} \\ &= \frac{2}{L}\left(\frac{L}{n\pi}\right)^2 \left[\sin\frac{n\pi x}{L} - \frac{n\pi x}{L}\cos\frac{n\pi x}{L}\right]_0^L \\ &= \frac{2L}{n\pi}(-1)^{n+1} \quad n = 1, 2, \dots \end{aligned}$$

Thus, the final sine series equals

$$f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}.$$

Example: sawtooth wave

$$f(x) = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{L} \, .$$

- Notice $f(\pm nL) = 0$, according to the series, which is indeed the mean value of the function, as claimed by the Fourier series convergence theorem.
- Since we defined $f(L) = f(-L) = 0$ in the first place, this makes the Fourier series convergent everywhere.

Extension of a function

- Due to symmetries, even/odd functions $f(x)$ only require information about half the interval $[0, L]$.
- Conversely, given some function $g(x)$ in $[0, L]$, is there a unique way of extending $g(x)$ over $[-L, L]$?
- Example: consider $f(x) = x$ for $x \in [0, L]$.
  Two natural ways of extending this function to the entire interval $[-L, L]$ are:
  - $f(x) = x$ for $x \in [-L, 0] \Rightarrow$ odd extension,
  - $f(x) = -x$ for $x \in [-L, 0] \Rightarrow$ even extension.

Extension of a function

This particular discussion applies more generally. Given some function $f(x)$ for $x \in [0, L]$

- its even extension can be defined as

$$h(x) = \begin{cases} f(x) & x \in [0, L] \\ f(-x) & x \in [-L, 0] \end{cases}$$

  This will have a cosine series

- its odd extension can be defined as

$$h(x) = \begin{cases} f(x) & x \in [0, L] \\ 0 & x = 0,\ L \\ -f(-x) & x \in [-L, 0] \end{cases}$$

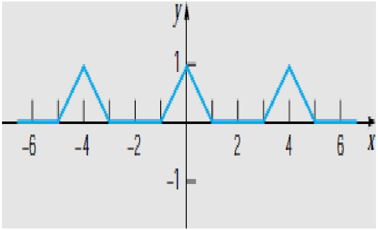
  This will have a sine series

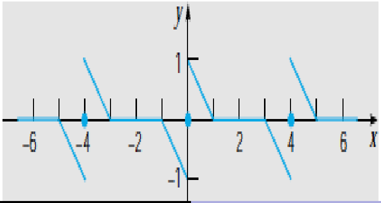
Extension of a function

Example: Consider the function

$$f(x) = \begin{cases} 1 - x & x \in [0, 1] \\ 0 & x \in (1, 2] \end{cases}$$

Its even and odd extensions are plotted below (in blue)





Parseval's theorem

Theorem

The square norm (f, f) *of periodic function with a convergent Fourier series*

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{in\pi x/L} \,,$$

satisfies

$$(f, f) = \int_{-L}^{L} |f(x)|^2 \, dx = 2L \sum_{n=-\infty}^{\infty} |c_n|^2$$
$$= L \left[\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} \left(|a_n|^2 + |b_n|^2 \right) \right]$$

This is Parseval's theorem applied to Fourier Series, a kind of infinite dimensional version of Pythagoras' theorem.

Parseval's theorem

Proof.

By definition and linearity,

$$
\begin{aligned}
(f,f) &= \sum_{n,m=-\infty}^{\infty} \int_{-L}^{L} c_n\, c_m^\star\, e^{i(n-m)\pi x/L}\, dx \\
&= \sum_{n,m=-\infty}^{\infty} c_n\, c_m^\star\, 2L\, \delta_{mn} \text{ by orthogonality} \\
&= 2L \sum_{n=-\infty}^{\infty} |c_n|^2\,.
\end{aligned}
$$

To prove the last bit, we just remember

$$
c_n = \frac{a_n - ib_n}{2} \quad (n>0)\,, \quad c_{-n} = \frac{a + ib_n}{2} \quad (n<0)
$$

$$
c_0 = \frac{a_0}{2}
$$

Parseval's theorem for Fourier Series

Proof.

Indeed,

$$
\begin{aligned}
2L \sum_{n=-\infty}^{\infty} |c_n|^2 &= 2L \left[|c_0|^2 + \sum_{n=1}^{\infty} \left(|c_n|^2 + |c_{-n}|^2 \right) \right] \\
&= 2L \left[\frac{|a_0|^2}{4} + \sum_{n=1}^{\infty} \left(\frac{|a_n|^2 + |b_n|^2}{4} + \frac{|a_n|^2 + |b_n|^2}{4} \right) \right] \\
&= L \left[\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} \left(|a_n|^2 + |b_n|^2 \right) \right] .
\end{aligned}
$$

□

Example/application

Consider the sawtooth wave function $f(x) = x$ for $x \in (-L,\, L)$
We proved

$$f(x) = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{L}\,.$$

Thus, applying Parseval's theorem, we get:

$$\int_{-L}^{L} x^2\, dx = \frac{2L^3}{3} = L \sum_{n=1}^{\infty} \frac{4L^2}{n^2\pi^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots$$

Parseval's theorem can be used to derive many beautiful identities of this type.

Honours Differential Equations

Jacques Vanneste

Lecture 24

November 9, 2018

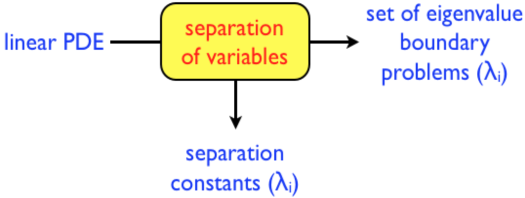
PDEs & separation of variables

We consider PDEs:

- linear & separable PDEs,
- solutions of the form: separation of variables

$$u(x_1, x_2, \dots x_n) = X_1(x_1) X_2(x_2) \dots X_n(x_n),$$

- subject to boundary & initial conditions in simple geometries.



Heat equation

The linear PDE:

$$\partial_t u = \alpha^2 \partial_x^2 u$$

describe heat conduction in 1 dimension.

- the function $u(x,t)$ measures the temperature of a thin rod of length L, i.e. $0 \leq x \leq L$ at time $t \geq 0$,
- the function $u(x,t)$ satisfies
  - initial condition: $u(x,0) = f(x)\,,\ 0 \leq x \leq L$,
  - boundary conditions: $u(0,t) = u(L,t) = 0\,,\ t > 0$,
- α is a real (positive) parameter.

Separation of variables in action

Assume solution takes separable form:

$$u(x,t) = X(x)T(t).$$

Introducing into the heat equation, we obtain

$$\alpha^2 X'' T = X \dot{T} \quad \Leftrightarrow \quad \frac{X''}{X}(x) = \frac{1}{\alpha^2}\frac{\dot{T}}{T}(t)$$

Question: how can a function of x be a function of t?

Answer: It cannot . . . unless both functions are constant:

$$\frac{X''}{X}(x) = \frac{1}{\alpha^2}\frac{\dot{T}}{T}(t) \equiv -\lambda$$

λ is an example of a separation constant.

Separation of variables in action

Thus, the PDE

$$\partial_t u = \alpha^2 \partial_x^2 u$$

reduces to the pair of ODE problems if $u(x,t) = X(x)T(t)$

$$X'' + \lambda X = 0\,, \qquad \dot{T} + \alpha^2 \lambda T = 0\,.$$

Question: What boundary/initial conditions do $X(x)$ and $T(t)$ satisfy ?

- Since $u(0,t) = u(L,t) = 0 \;\forall\, t > 0 \Rightarrow \; X(0) = X(L) = 0$

Thus, $X(x)$ satisfies the eigenvalue boundary problem

$$X'' + \lambda X = 0\,, \qquad X(0) = X(L) = 0$$

where the separation = eigenvalue.

Separation of variables in action

The eigenvalue boundary problem

$$X'' + \lambda X = 0 \,, \qquad X(0) = X(L) = 0$$

was solved previously:

- there exists an infinite (countable) set of eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad n = 1, 2, \ldots$$

- each λ_n has a single eigenfunction

$$X_n(x) \propto \sin \frac{n\pi x}{L}.$$

Separation of variables in action

The second ODE problem is 1st order:

$$\dot{T}_n + \alpha^2 \lambda_n \, T_n = 0 \Rightarrow \quad T_n \propto e^{-\alpha^2 \lambda_n \, t}$$

Thus, for each λ_n, there exists a different $T_n(t)$

- we find a countable infinity of solutions of the form

$$u_n(x,t) \propto X_n(x) \, T_n(t) \,,$$

  corresponding to each λ_n.

- Since the original PDE

$$\partial_t u = \alpha^2 \partial_x^2 u$$

  is linear, we conclude the most general solution obtained using this method must be the linear combination

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n \, e^{-n^2 \pi^2 \alpha^2 t / L^2} \, \sin \frac{n\pi x}{L} \,.$$

Connection to Fourier series

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n \, e^{-n^2\pi^2\alpha^2 t/L^2} \, \sin \frac{n\pi x}{L} \, .$$

The constants c_n are determined by the initial condition

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \, \sin \frac{n\pi x}{L} \, .$$

Thus, given a function $f(x)$, the coefficients c_n are fixed by the Euler-Fourier formulas:

$$c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \, ,$$

where we used the formulas derived for a sine series (factor of 2).

Separation of variables in action

Thus, the problem

$$\alpha^2 \partial_x^2 u = \partial_t u$$

satisfying

- initial condition: $u(x,0) = f(x)\,,\ 0 \le x \le L$
- boundary conditions: $u(0,t) = u(L,t) = 0\,,\ \ t > 0$

is solved by

$$u(x,t) = \sum_{n=1}^{\infty} c_n\, e^{-n^2\pi^2\alpha^2 t/L^2}\, \sin\frac{n\pi x}{L}\,,$$

$$\text{with}\ \ c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\,.$$

Remarks on convergence

Question: what about the convergence of the infinite sum ?

- for $t = 0$, if $f(x)$ is piece-wise continuous, then the Fourier convergence theorem guarantees convergence at $t = 0$
- The exponential dependence in t suggests the series will converge $\forall\ t > 0$
- This important issue will be addressed by the general Sturm-Liouville theory we will discuss during the last two weeks of the course.

Non-homogeneous boundary conditions

Consider a more realistic set of boundary conditions:

$$u(0,t) = T_1\,, \qquad u(L,t) = T_2 \qquad t > 0$$

(we keep the ends of the rod at some finite fixed temperature)
Trick: we map this problem to one with homogeneous boundary conditions

- Define $v(x)$ as the time independent function

$$v(x) = \lim_{t\to\infty} u(x,t)$$

- Since $v(x)$ satisfies $\partial_t v = 0$, the heat equation it will satisfy reduces to

$$v'' = 0 \qquad v(0) = T_1\,, \quad v(L) = T_2$$

Its general solution must be linear in x

$$v(x) = (T_2 - T_1)\,\frac{x}{L} + T_1\,.$$

Non-homogeneous boundary conditions

- Finally, we look for a solution to our original non-homogeneous boundary value problem of the form

$$u(x,t) = v(x) + \omega(x,t)\,.$$

The new function satisfies the same heat equation

$$\alpha^2 \partial_x^2 \omega = \partial_t \omega\,,$$

but with boundary conditions:

$$\begin{aligned}
\omega(0,t) &= u(0,t) - v(0) = 0\,,\\
\omega(L,t) &= u(L,t) - v(L) = 0\,,\\
\omega(x,0) &= u(x,0) - v(x) = f(x) - v(x)
\end{aligned}$$

Thus, $\omega(x)$ does satisfy an homogeneous set of boundary conditions but with a slightly different initial value function $f(x) - v(x)$.

Non-homogeneous boundary conditions

We conclude the general solution to our original non-homogeneous problem is:

$$u(x,t) = (T_2 - T_1)\,\frac{x}{L} + T_1 + \sum_{n=1}^{\infty} c_n\, e^{-n^2\pi^2\alpha^2 t/L^2}\, \sin\frac{n\pi x}{L}\,,$$

$$\text{with}\ \ c_n = \frac{2}{L}\int_0^L \left[f(x) - (T_2 - T_1)\,\frac{x}{L} - T_1\right] \sin\frac{n\pi x}{L}\, dx\,.$$

Different boundary conditions

Consider the same PDE equation:

$$\partial_t u = \alpha^2 \partial_x^2 u$$

but with boundary conditions

$$\partial_x u(0,t) = \partial_x u(L,t) = 0 \quad t > 0$$

- the temperature does not vary with the position at the end of both bars $\Leftrightarrow$ both ends are isolated

We shall proceed as before: $u(x,t) = X(x)T(t) \Rightarrow$

$$X'' + \lambda X = 0\,, \qquad X'(0) = X'(L) = 0\,,$$
$$\dot{T} + \alpha^2 \lambda T = 0\,.$$

Same eigenvalue problem (because of starting with the same PDE), different boundary value problem (due to different starting boundary conditions)

Solution to the eigenvalue boundary problem

$$X'' + \lambda X = 0\,, \qquad X'(0) = X'(L) = 0$$

- $\lambda = -\mu^2$ ($\lambda < 0$):

  $X(x) = a \sinh \mu x + b \cosh \mu x \Rightarrow X'(x) = a\mu \cosh \mu x + b\mu \sinh \mu x$

  Boundary conditions require:

  $$X'(0) = 0 \Rightarrow a = 0 \ (\mu \neq 0)$$
  $$X'(L) = 0 \Rightarrow b = 0 \ (\sinh y \neq 0 \ \ y \neq 0)$$

  Thus, solution is trivial
- $\lambda = 0 \Rightarrow X(x) = ax + b \Rightarrow X'(x) = a$ Boundary conditions require $a = 0$
  Constant solutions solve this problem,

  $$X_0(x) = b\,.$$

Solution to the eigenvalue boundary problem

- $\lambda = \mu^2$ $(\lambda > 0)$:

$$X(x) = a \sin \mu x + b \cos \mu x \Rightarrow X'(x) = a\mu \cos \mu x - b\mu \sin \mu x$$

Boundary conditions require:

$$X'(0) = 0 \Rightarrow a = 0 \quad (\mu \neq 0)$$
$$X'(L) = 0 \Rightarrow \sin \mu L = 0 \Rightarrow \mu L = n\pi$$
$$\Rightarrow \lambda_n = \frac{n^2\pi^2}{L^2} \quad n = 1, 2, \ldots$$

Once all possible compatible λ's are known, we can integrate the linear ODE for $T(t)$: $\dot{T} + \alpha^2 \lambda T = 0$

$$T_n(t) \propto e^{-\alpha^2 \lambda_n t}.$$

Solution to the eigenvalue boundary problem

For each λ_n, we find (including $n = 0$)

$$u_n(x,t) = c_n\, e^{-\alpha^2 \lambda_n t} \cos \frac{n\pi x}{L}$$

General solution: since PDE is linear, the general solution is a linear combination of the ones founds above, including the constant solution !!

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n\, e^{-n^2\pi^2\alpha^2 t/L^2} \cos \frac{n\pi x}{L}\,,$$

where the collection $\{c_n\}$ is again determined by the initial temperature function $f(x)$ using Fourier analysis

$$c_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L}\, dx \quad n = 1, 2, \ldots$$

Summary of the procedure

Given a linear PDE:

- Assume separation of variables: $u(x, t) = X(x)T(t)$.
- Introduce a separation of variable constant λ (depending on the number of independent variables, there may be more than one).
- Identify the eigenvalue boundary problems satisfied by these functions and separation constants: these depend on the original boundary/initial conditions
- Solve these problems: this step may require some quantisation of λ or some constraint on λ
- Write the most general solution as a linear combination of all solutions to the eigenvalue boundary problems
- Identify any undetermined coefficients using initial conditions.

Honours Differential Equations

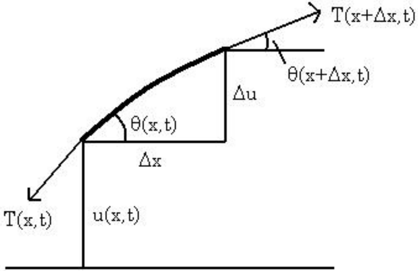
Jacques Vanneste

Lecture 25

November 12, 2018

Vibrating string

We consider the vibrations of a string (e,g., guitar).



- $u(x,t)$ is the string vertical displacement, assumed small,
- $T =$ const is the string tension,
- $\rho =$ const is density.

Newton's law for a short string section:

$$\begin{aligned}\rho\Delta x\partial_t^2 u(x,t) &= T[\sin(\theta(x+\Delta x,t)) - \sin(\theta(x,t))] \\ &= T\partial_x\theta(x,t)\Delta x + O(\Delta x^2) \\ &= T\partial_x^2 u(x,t)\Delta x + O(\Delta x^2)\end{aligned}$$

since $sin\theta \approx \theta \approx \tan\theta = \partial_x u$.

Vibrating string

In the limit $\Delta x \to 0$, we obtain wave equation in 1 dimension:

$$\partial_t^2 u = a^2 \partial_x^2 u,$$

with $a = \sqrt{T/\rho}$, the wave speed.

- 2d version, $\partial_t^2 u = a^2(\partial_x^2 u + \partial_y)u$ for vibrating membranes (drums),
- water waves in shallow water with $a = \sqrt{gH}$,
- electromagnetic waves with a the speed of light.

String with fixed ends and no initial velocity

Solve

$$\partial_t^2 u = a^2 \partial_x^2 u = \qquad 0 \le x \le L\,, \;\; t \ge 0$$

subject to the following conditions

- $u(0,t) = u(L,t) = 0$: ends of the string are fixed,
- $\partial_t u(x,t) = 0$: zero velocity to vibrate at $t = 0$,
- $u(x,0) = f(x)$: non-zero displacement at $t = 0$ (shape of string at $t = 0$).

Separation of variables

Use separation of variables: $u(x,t) = X(x)T(t)$

- Introduce into wave equation

$$a^2 X'' T = X \ddot{T}.$$

- Divide by $u(x,t)$ and rearrange a^2

$$\frac{X''}{X} = \frac{\ddot{T}}{a^2\, T} = -\lambda$$

where λ is the separation of variables constant.

Thus we are left with ODE problems:

$$X'' + \lambda X = 0\,,$$
$$\ddot{T} + a^2\lambda\, T = 0.$$

Boundary conditions

- Since $u(0,t) = 0 = X(0)T(t)$ $t \geq 0 \Rightarrow X(0) = 0$
- Since $u(L,t) = 0 = X(L)T(t)$ $t \geq 0 \Rightarrow X(L) = 0$
- Since $\partial_t u(x,0) = 0 = X(x)\dot{T}(0)$ $0 \leq x \leq L \Rightarrow \dot{T}(0) = 0$

Thus, using separation of variables, our problem is mapped to

$$\begin{aligned} X'' + \lambda X &= 0\,, \quad X(0) = X(L) = 0\,, \\ \ddot{T} + a^2\lambda\, T &= 0\,, \quad \dot{T}(0) = 0\,, \end{aligned}$$

with the further initial condition

$$u(x,0) = f(x) \quad 0 \leq x \leq L$$

which determines the initial profile of the wave/string/perturbation

Solving the eigenvalue boundary conditions problem

We already solved the problem

$$X'' + \lambda X = 0\,, \quad X(0) = X(L) = 0\,.$$

It requires

- $\lambda_n = \frac{n^2\pi^2}{L^2}$ with $n = 1, 2, 3, \ldots$
- with eigenfunctions

$$X_n(x) \propto \sin \frac{n\pi x}{L}.$$

Introducing the allowed values λ_n into the 2nd ODE

$$\ddot{T} + \left(\frac{n\pi a}{L}\right)^2 T = 0\,,$$

whose general solution is

$$T_n(t) = a_n \cos \frac{n\pi a}{L} t + b_n \sin \frac{n\pi a}{L} t\,.$$

Solving the eigenvalue boundary conditions problem

$$T_n(t) = a_n \cos \frac{n\pi a}{L} t + b_n \sin \frac{n\pi a}{L} t \,.$$

Using the boundary condition $\dot{T}(0) = 0$

$$\dot{T}_n(t) = \frac{n\pi a}{L} \left(-a_n \sin \frac{n\pi a}{L} t + b_n \cos \frac{n\pi a}{L} t \right)$$
$$\dot{T}_n(0) = 0 \Rightarrow b_n = 0.$$

Thus, for each $\lambda_n = \frac{n^2\pi^2}{L^2}$, we find a product solution of the form

$$u_n(x,t) = X_n(x) T_n(t) \propto \sin \frac{n\pi x}{L} \cos \frac{n\pi a}{L} t.$$

By linearity, the general solution can be written as a linear combination of the solutions associated with all $\lambda_n = \frac{n^2\pi^2}{L^2}$

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t).$$

Determining constants

Using the initial condition $u(x,0) = f(x)$ on the general solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \cos \frac{n\pi a}{L} t$$

and the orthogonality of the periodic functions

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}$$

$$c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \, .$$

- superposition of vibration at frequencies $f_n = n\pi a/(2\pi L)$.
- fundamental frequency + harmonics.

Interpreting the solution

Using trigonometry, we will interpret the solution as

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} \cos\frac{n\pi a}{L} t = \frac{1}{2}\left[h(x-at) + h(x+at)\right]$$

Notice that

$$\begin{aligned} h(x-at) &= \sum_{n=1}^{\infty} c_n \left(\sin\frac{n\pi x}{L}\cos\frac{n\pi a}{L}t - \cos\frac{n\pi x}{L}\sin\frac{n\pi a}{L}t\right) \\ &= \sum_{n=1}^{\infty} c_n \sin\frac{n\pi}{L}(x-at)\,, \\ h(x+at) &= \sum_{n=1}^{\infty} c_n \left(\sin\frac{n\pi x}{L}\cos\frac{n\pi a}{L}t + \cos\frac{n\pi x}{L}\sin\frac{n\pi a}{L}t\right) \\ &= \sum_{n=1}^{\infty} c_n \sin\frac{n\pi}{L}(x+at)\,. \end{aligned}$$

Interpreting the solution

Consider the solution

$$u(x,t) = \frac{1}{2}\left[h(x-at) + h(x+at)\right]$$

At $t = 0$ it satisfies

$$u(x,0) = h(x)$$

Thus, $h(x)$ controls the shape of the starting perturbation.
Assume $h(x)$ is a gaussian-like perturbation centered at $x = 0$.
As time flows,

$$u(x,t) = \frac{1}{2}\left[h(x-at) + h(x+at)\right]$$

the solution looks like the superposition of two shapes (lumps):

- One located at y=x-at: as t increases, x increases to describe the lump at $y = 0$. The perturbation propagates to the "right" (increases its location in x as t grows)
- One located at y=x+at: as t increases, x decreases to describe the lump at $y = 0$. The perturbation propagates to the "left" (decreases its location in x as t grows).

General solution & interpretation

Consider the general 1d wave equation

$$\partial_t^2 u = a^2 \partial_x^2 u.$$

Let us change coordinates:

$$\xi = x - at\,, \qquad \eta = x + at$$

Let us rewrite the PDE in the new coordinates:

$$u(x,t) = u(x(\xi,\eta), t(\xi,\eta)) = u(\xi,\eta)$$

Question: What about the partial derivatives?

$$\frac{\partial^2 u}{\partial x^2}, \; \frac{\partial^2 u}{\partial t^2}$$

General solution & interpretation

We need to use the chain rule

$$\frac{\partial u(\xi,\eta)}{\partial x} = \frac{\partial \xi}{\partial x}\partial_\xi u + \frac{\partial \eta}{\partial x}\partial_\eta u = \partial_\xi u + \partial_\eta u$$

Iterating the process

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\partial_\xi u + \partial_\eta u\right) \\
&= \partial_x \xi \partial^2_\xi u + \partial_x \eta \partial^2_{\eta\xi} u + \partial_x \xi \partial^2_{\xi\eta} u + \partial_x \eta \partial^2_\eta u \\
&= \partial^2_\xi u + \partial^2_\eta u + 2\partial^2_{\xi\eta} u
\end{aligned}$$

General solution & interpretation

We need to use the chain rule

$$\frac{\partial u(\xi,\eta)}{\partial t} = \frac{\partial \xi}{\partial t}\partial_\xi u + \frac{\partial \eta}{\partial t}\partial_\eta u = -a\left(\partial_\xi u - \partial_\eta u\right)$$

Iterating the process

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= -a\frac{\partial}{\partial t}\left(\partial_\xi u - \partial_\eta u\right)\\ &= a^2\partial_\xi^2 u + a^2\partial_\eta^2 u - 2a^2\partial_{\xi\eta}^2 u\end{aligned}$$

Altogether

$$a^2\partial_x^2 u = \partial_t^2 u \Leftrightarrow \frac{\partial^2 u}{\partial\xi\partial\eta} = 0 = \frac{\partial}{\partial\xi}\left(\frac{\partial u}{\partial\eta}\right)$$

$$\frac{\partial u}{\partial\eta} = g'(\eta) \Rightarrow \frac{\partial}{\partial\eta}\left(u - g(\eta)\right) = 0 \Rightarrow u = g(\eta) + f(\xi)$$

General solution & interpretation

Conclusion:

$$a^2\partial_x^2 u = \partial_t^2 u \Rightarrow u(x,t) = g(x+at) + f(x-at)$$

- If $f(x)$ represents a lump located at $x = 0 \Rightarrow f(x-at)$ describes a travelling lump at speed a moving to the right
- If $g(x)$ represents a lump located at $x = 0 \Rightarrow g(x+at)$ describes a travelling lump at speed a moving to the left

More general boundary conditions

Consider

$$\partial_t^2 u = a^2 \partial_x^2 u$$

subject to the boundary conditions

$$\begin{aligned} u(0,t) &= u(L,t) = 0 \\ u(x,0) &= f(x)\,, \quad \partial_t u(x,0) = g(x) \end{aligned}$$

If we look for solutions of the form $u(x,t) = X(x)T(t)$

- $X(x)$ will satisfy the same eigenvalue boundary problem
- Since $\dot{T}(0) \neq 0$ now, the most general solution will be of the form

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(c_n \sin \frac{n\pi a}{L} t + d_n \cos \frac{n\pi a}{L} t \right) .$$

More general boundary conditions

We can fix the two families of constant, using the initial conditions

- $u(x,0) = f(x)$:

$$\sum_{n=1}^{\infty} d_n \sin \frac{n\pi x}{L} = f(x) \;\Rightarrow\; d_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

- $\partial_t u(x,0) = g(x)$:

$$\sum_{n=1}^{\infty} \frac{n\pi a}{L} c_n \sin \frac{n\pi x}{L} = g(x) \;\Rightarrow\; \frac{n\pi a}{L} c_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} dx$$

More general boundary conditions

Remark: there are many more boundary conditions that we could consider, but the procedure to find the solutions would follow the same steps

- Assume separation of variables: $u(x,t) = X(x)T(t)$,
- Introduce a separation of variable constant λ (depending on the number of independent variables, there may be more than one),
- Identify the eigenvalue boundary problems defined by these functions and constants,
- Solve these problems: this step imposes quantisation of λ or some constraint on λ,
- Write the most general solution as a linear combination of all solutions to the eigenvalue boundary problems,
- Identify any undetermined coefficients using initial conditions.

Honours Differential Equations

Jacques Vanneste

Lecture 26

November 14, 2018

More general boundary conditions

Consider

$$\partial_t^2 u = a^2 \partial_x^2 u$$

subject to the boundary conditions

$$u(0,t) = u(L,t) = 0$$
$$u(x,0) = f(x)\,, \quad \partial_t u(x,0) = g(x)$$

If we look for solutions of the form $u(x,t) = X(x)T(t)$

- $X(x)$ will satisfy the same eigenvalue boundary problem
- Since $\dot{T}(0) \neq 0$ now, the most general solution will be of the form

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(c_n \sin\frac{n\pi a}{L}t + d_n \cos\frac{n\pi a}{L}t\right)\,.$$

Wave equation: more general boundary conditions

We can fix the two families of constant, using the initial conditions

- $u(x,0) = f(x)$:

$$\sum_{n=1}^{\infty} d_n \sin\frac{n\pi x}{L} = f(x) \;\Rightarrow\; d_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

- $\partial_t u(x,0) = g(x)$:

$$\sum_{n=1}^{\infty} \frac{n\pi a}{L} c_n \sin\frac{n\pi x}{L} = g(x) \;\Rightarrow\; \frac{n\pi a}{L}c_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

More general boundary conditions

General procedure:

- Assume separation of variables: $u(x,t) = X(x)T(t)$,
- Introduce one or more separation paramter λ,
- Solve eigenvalue problem(s): quantisation of λ,
- Write the most general solution as a linear combination of all solutions to the eigenvalue boundary problems,
- Identify any undetermined coefficients using initial conditions.

For example, 2D wave equation $\partial_t^2 u = a^2 \partial_x^2 u$ on $(x,y) \in [0,L]^2$ has solution

$$u(x,y,t) = \sum_{m,n} \sin\frac{m\pi x}{L} \sin\frac{n\pi y}{L} \left(c_n \cos\frac{\sqrt{m^2+n^2}\pi a t}{L} \right.$$
$$\left. + d_n \sin\frac{\sqrt{m^2+n^2}\pi a t}{L} \right).$$

Laplace equation

The Laplace equation

$$\nabla^2 u \equiv \partial_x^2 u + \partial_y^2 u = 0 \quad \text{in 2D}, \qquad \nabla^2 u \equiv \partial_x^2 u + \partial_y^2 u + \partial_z^2 u \quad \text{in 3D},$$

arises in numerous problems:

- steady solutions of the heat equation,
- equilibria of membranes, minimal surfaces (soap films),
- fluid dynamics (airfoils, (deep-)water waves,...),
- electromagnetism, gravity...

Links with:

- complex analysis (see 2nd semester course),
- probability (see Applied Stochastic Differential Equations).

Laplace equation: boundary conditions

Given the Laplace equation in 2 dimensions

$$\partial_x^2 u + \partial_y^2 u = 0\,,$$

there are different types of boundary conditions (b.c.) to consider

- Dirichlet b.c.: function $u(x, y)$ specified at the boundary
- Neumann b.c.: normal derivative of $u(x, y)$ specified at the boundary

We discuss two Dirichlet b.c. problems of different geometries.

Dirichlet problem for a rectangle

We solve

$$\partial_x^2 u + \partial_y^2 u = 0\,, \qquad 0 \le x \le a\,,\ \ 0 \le y \le b$$

subject to the boundary conditions (Dirichlet):

$$\begin{aligned} u(x,0) &= u(x,b) = 0\,, \\ u(0,y) &= 0\,, \quad u(a,y) = f(y). \end{aligned}$$

Assume separation of variables: $u(x,y) = X(x)Y(y)$
Plugging into Laplace's equation & dividing by $u(x,u)$

$$\frac{X''}{X}(x) = -\frac{\ddot{Y}}{Y}(y) = \lambda\,.$$

Thus, we are left with two eigenvalue boundary problems:

$$\begin{aligned} X'' - \lambda X = 0\,, &\qquad X(0) = 0 \\ \ddot{Y} + \lambda Y = 0\,, &\quad Y(0) = Y(b) = 0\,. \end{aligned}$$

Solving the eigenvalue boundary problem

We already solved the problem:

$$\ddot{Y} + \lambda Y = 0\,, \quad Y(0) = Y(b) = 0$$

Its general solution is given by

$$Y_n(y) \propto \sin \frac{n\pi y}{b}\,, \qquad \lambda_n = \frac{n^2\pi^2}{b^2}\,, \;\; n = 1, 2, \ldots$$

The remaining ODE becomes

$$X_n'' - \frac{n^2\pi^2}{b^2} X_n = 0 \quad \Rightarrow \quad X_n(x) = a_n \cosh \frac{n\pi x}{b} + c_n \sinh \frac{n\pi x}{b}$$

Since $X(0) = 0 \Rightarrow a_n = 0 \Rightarrow X_n \propto \sinh \frac{n\pi x}{b}$

Solving the eigenvalue boundary problem

Thus, for any λ_n, the solution is given by

$$u_n(x, y) \propto \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b} ,$$

Due to the linearity of the Laplace equation, the general solution is

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$$

To determine $\{c_n\}$ we use the final condition $u(a, y) = f(y)$ and the orthogonality of the sine functions

$$c_n \sinh \frac{n\pi a}{b} = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b} \, dy .$$

Dirichlet problem for a circle

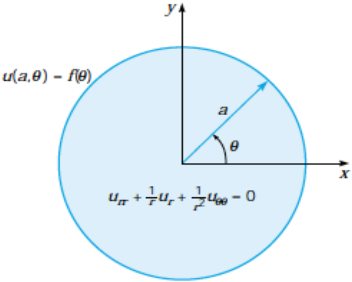
Let us consider a similar problem, but with a different geometry

Solve the Laplace equation

$$\partial_x^2 u + \partial_y^2 u = 0$$

subject to the boundary conditions

- $u(a, \theta) = f(\theta)$ with $x^2 + y^2 = a^2$ and $0 \leq \theta \leq 2\pi$
- $u(x, y)$ bounded for $\sqrt{x^2 + y^2} \leq a$.



Changing coordinates

- Boundary conditions refer to a disk ($r \le a$) and its boundary circle ($r = a$)
- We change coordinates to polar coordinates

$$x = r\cos\theta\,, \qquad y = r\sin\theta$$
$$r^2 = x^2 + y^2\,, \qquad \tan\theta = \frac{y}{x}$$

When we do this: $u(x,y) \to u(x(r,\theta), y(r,\theta)) = u(r,\theta)$

Using the chain rule,

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\,.$$

Changing coordinates

Similarly,

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}\,.$$

Since we are interested in second partial derivatives, we go one step further

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} &= \cos\theta\frac{\partial}{\partial r}\frac{\partial u}{\partial x} + \sin\theta\frac{\partial}{\partial r}\frac{\partial u}{\partial y} \\ &= \cos\theta\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial r}\right) \\ &+ \sin\theta\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial r}\right) \\ &= \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2}\end{aligned}$$

Changing coordinates

Similarly,

$$\frac{\partial^2 u}{\partial \theta^2} = -r\left(\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\right) + r^2\left(\sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2}\right)$$

from which we can derive multiplying by $1/r^2$

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\frac{\partial u}{\partial r} + \sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2}\,.$$

Changing coordinates

- Putting these pieces together, we conclude

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}=0\,,$$

- Next, we assume separation of variables

$$u(r,\theta)=R(r)\Theta(\theta)\Rightarrow R''\Theta+\frac{1}{r}R'\Theta+\frac{1}{r^2}R\ddot{\Theta}=0.$$

- Multiplying by r^2 and dividing by $u(r,\theta)=R(r)\Theta(\theta)$

$$r^2\frac{R''}{R}+r\frac{R'}{R}=-\frac{\ddot{\Theta}}{\Theta}=\lambda$$

where λ is the separation constant.

Solving eigenvalue boundary problem

Our original problem becomes

$$\begin{aligned} r^2 R'' + rR' &= \lambda R, \\ \ddot{\Theta} &= -\lambda \Theta. \end{aligned}$$

subject to periodicity and boundedness conditions.
Consider $\ddot{\Theta} + \lambda\Theta = 0$

- If $\lambda = -\mu^2$ $(\lambda < 0)$, then

$$\Theta(\theta) = c_1\, e^{\mu\theta} + c_2\, e^{-\mu\theta}\,.$$

  Since these solutions are not periodic under $\theta \to \theta + 2\pi$
  $\Rightarrow c_1 = c_2 = 0$

- If $\lambda = 0$, then

$$\Theta(\theta) = c_1\theta + c_2\,.$$

  Again $\Theta(\theta + 2\pi) \neq \Theta(\theta)$ unless $c_1 = 0$. Thus, constant $\Theta(\theta) = c_2$ is allowed.

Solving eigenvalue boundary problem

- If $\lambda = \mu^2$ ($\lambda > 0$), then

$$\Theta(\theta) = a_1 \sin \mu\theta + a_2 \cos \mu\theta$$

Periodicity requires $\Theta(\theta + 2\pi) = \Theta(\theta)$ and this is only achieved if $\mu = n$ with n=1,2,. . .

Thus, the first eigenvalue problem $\ddot{\Theta} + \lambda\Theta = 0$ is solved by

1. $\lambda = 0$, $\Theta_0(\theta) = C$ (constant)
2. $\lambda = n^2$, $\Theta_n(\theta) = e_n \cos n\theta + f_n \sin n\theta$.

- we are left to solve

$$r^2 R'' + rR' = \lambda R$$

for the two cases

Solving eigenvalue boundary problem

1. If $\lambda = 0 \Rightarrow r^2 R'' + rR' = 0$
   Define $R'(r) = Z(r)$, then

$$r^2 Z' + rZ = 0 \Rightarrow Z(r) = \frac{k}{r}$$
$$R' = \frac{k}{r} \Rightarrow R(r) = k \log r + \tilde{k}$$

   Since $\log r$ diverges at $r = 0 \Rightarrow k = 0$ (boundedness)
2. If $\lambda = n^2 \Rightarrow r^2 R'' + rR' - n^2 R = 0$. This is a 2nd order ODE with non-constant coefficients.

   Seek solution as a power:

$$R(r) = r^{\alpha}.$$

Solving eigenvalue boundary problem

$$r^2 R'' + rR' - n^2 R = 0 \quad \text{with} \quad R(r) = r^{\alpha}$$

Introducing

$$R' = \alpha\, r^{\alpha-1}\,, \qquad R'' = \alpha(\alpha-1)\, r^{\alpha-2}$$

into ODE

$$r^{\alpha}\left(\alpha(\alpha-1) + \alpha - n^2\right) = r^{\alpha}\left(\alpha^2 - n^2\right) = 0 \Rightarrow \alpha = \pm n$$

Thus, for any $\lambda_n = n^2$, we find the general solution to the 2nd order ODE is

$$R_n(r) = w_n\, r^{n} + q_n\, r^{-n}\,.$$

Since solutions must be bounded $\Rightarrow q_n = 0$ (to avoid divergence at $r = 0$).

Recapitulation

The solution to

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0\,,$$

under the hypothesis $u(r,\theta) = R(r)\Theta(\theta)$ requires:

$$r^2 R'' + rR' = \lambda R\,, \qquad \ddot{\Theta} = -\lambda\Theta$$

Periodicity and boundedness determined

- $\lambda = 0$ allows a constant solution

$$u_0(r,\theta) = \frac{c_0}{2}\,.$$

- $\lambda = n^2$ allows solutions of the form

$$u_n(r,\theta) = r^n\left(a_n \cos n\theta + b_n \sin n\theta\right)\,.$$

Linearity allows us to write the more general solution as

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n\left(e_n \cos n\theta + f_n \sin n\theta\right)\,.$$

Fixing coefficients

There is one boundary condition that remains to be satisfied:

$$u(a,\theta) = f(\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n \left(e_n \cos n\theta + f_n \sin n\theta\right) .$$

- Equality $\sim$ Fourier series
- $f(\theta + 2\pi) = f(\theta)$: period $T = 2L = 2\pi$
- Use Euler-Fourier formulas to determine all coefficients

$$a^n\, e_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\, \cos n\theta\, d\theta \;\Rightarrow\; e_n = \frac{1}{a^n\,\pi}\int_{-\pi}^{\pi} f(\theta)\, \cos n\theta\, d\theta\,,$$
$$a^n\, f_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\, \sin n\theta\, d\theta \;\Rightarrow\; f_n = \frac{1}{a^n\,\pi}\int_{-\pi}^{\pi} f(\theta)\, \sin n\theta\, d\theta\,.$$

Honours Differential Equations

Jacques Vanneste

Lecture 27

November 16, 2018

Towards Sturm–Liouville

In the last couple of weeks, we studied

- ODE boundary value problems,
- PDEs giving rise to ODE boundary value problems EMRN after separation of variables.

Recurrent features are:

- countable infinity of eigenfunctions,
- eigenfunction orthogonal in an inner product,
- arbitrary function represented as convergent series involving eigenfunctions.

Towards Sturm–Liouville

We found this structure, for eigenfunctions of the form $\sin(n\pi x/L)$.
We now ask:

- why this structure?
- is it more general?
- what is the relevance of the boundary conditions for the existence of this structure?

We answer these questions in the coming lectures:

- first, we revisit one example to motivate the general theory,
- second, we will develop the general theory for 2nd order ODEs.

Eigenvalue boundary problem revisited

We have encountered the eigenvalue boundary value problem

$$-X''(x) = \lambda\, X(x)\,, \qquad X(0) = X(L) = 0 \quad \text{with} \quad 0 \le x \le L$$

Let us rewrite it as (action of an operator on a function)

$$\mathcal{L}[X] = \lambda\, X\,, \qquad X(0) = X(L) = 0 \quad \text{with} \quad 0 \le x \le L$$

Solution: $\lambda_n = \frac{n^2\pi^2}{L^2}$, with $X_n \propto \sin\frac{n\pi x}{L}$

- Orthogonality: $(X_n, X_m) \propto \delta_{mn}$

Question: is the existence of an inner product between different eigenfunctions related to the specific form of the eigenvalue boundary problem (form of $\mathcal{L}$ + boundary conditions)?

Eigenvalue boundary problem revisited

Define the inner product of two complex functions in $0 \le x \le L$ as

$$(u, v) \equiv \int_0^L u(x) v^\star(x)\, dx$$

Compute $(\mathcal{L}[u], v)$:

$$\begin{aligned}
-(\mathcal{L}[u], v) &= \int_0^L u'' v^\star \, dx = \int_0^L \left(d(u'\, v^\star) - u' dv^\star \right) \\
&= u' v^\star \big|_0^L - \int_0^L u' v^{\star\prime} dx = u' v^\star \big|_0^L - \int_0^L \left(d(v^{\star\prime} u) - v^{\star\prime\prime} u dx \right) \\
&= \left(u' v^\star - v^{\star\prime} u \right) \big|_0^L + \int_0^L u v^{\star\prime\prime} \, dx \\
&= \left(u' v^\star - v^{\star\prime} u \right) \big|_0^L - (u, \mathcal{L}[v])
\end{aligned}$$

Eigenvalue boundary problem revisited

$$-(\mathcal{L}[u], v) + (u, \mathcal{L}[v]) = \left(u' v^\star - v^{\star\prime} u\right)\Big|_0^L$$

- This identity relates $\mathcal{L}$ to the the boundary conditions.
- If both u, v satisfy the homogeneous boundary conditions,

$$(\mathcal{L}[u], v) - (u, \mathcal{L}[v]) = 0 \qquad u(0) = u(L) = v(0) = v(L) = 0$$

- Consider $u = v = \Phi \equiv M(x) + i\,N(x)$. Assume Φ solves the eigenvalue boundary problem, i.e. $\mathcal{L}[\Phi] = \lambda\Phi$ for a complex $\lambda = \mu + i\nu$

$$\begin{aligned}(\mathcal{L}[\Phi], \Phi) &= (\Phi, \mathcal{L}[\Phi])\\ \lambda(\Phi, \Phi) &= \lambda^\star(\Phi, \Phi)\\ (\lambda - \lambda^\star)(\Phi, \Phi) &= 0\end{aligned}$$

Eigenvalue boundary problem revisited

$$(\lambda - \lambda^\star)(\Phi, \Phi) = 0$$
$$(\lambda - \lambda^\star)\int_0^L \Phi\Phi^\star\, dx = 0$$
$$(\lambda - \lambda^\star)\int_0^L (M^2 + N^2)\, dx = 0$$

We must conclude $\lambda = \lambda^\star \Rightarrow$ real eigenvalues.

- If $u = \Phi_1$ and $v = \Phi_2$ satisfy $\mathcal{L}[\Phi_i] = \lambda_i \Phi_i$ with $i = 1, 2$ & $\lambda_1 \neq \lambda_2$

$$(\mathcal{L}[\Phi_1], \Phi_2) = (\Phi_1, \mathcal{L}[\Phi_2])$$
$$\lambda_1(\Phi_1, \Phi_2) = \lambda_2(\Phi_1, \Phi_2)$$
$$(\lambda_1 - \lambda_2)(\Phi_1, \Phi_2) = 0 \Rightarrow (\Phi_1, \Phi_2) = 0$$

Eigenfunctions of different eigenvalues are orthogonal.

Eigenvalue boundary problem revisited

Without explicitly solving the eigenvalue boundary problem

$$\mathcal{L}[X] \equiv -X''(x) = \lambda\, X(x)\,, \qquad X(0) = X(L) = 0 \quad \text{with} \quad 0 \le x \le L$$

but using

- the form of $\mathcal{L}$,
- type of boundary conditions

we showed that

- $\lambda \in \mathbb{R}$,
- $(\Phi_n, \Phi_m) = 0$ if $n \neq m$.

What is the general for of $\mathcal{L}$ + b.c. for which this holds?

Finite dimensional version: matrices A whose eigenvalues are real and eigenvectors orthogonal, symmetric matrices.

Sturm-Liouville boundary problems

Consider some generalisation of the heat equation

$$r(x)\partial_t u(x,t) = \partial_x\left[p(x)\partial_x u(x,t)\right] - q(x)u(x,t) \quad 0 < x < 1$$

- $\{p(x), p'(x), q(x), r(x)\}$ are continuous on $x \in [0,1]$
- $p(x), r(x) > 0 \ \forall\ x \in [0,1]$

Separation $u(x,t) = X(x)T(t)$:

$$r(x)X(x)\dot{T}(t) = \left[p(x)X'(x)\right]' T(t) - q(x)X(x)T(t)$$

Dividing by $r(x)X(x)T(t)$:

$$\frac{\dot{T}}{T}(t) = \frac{[p(x)X'(x)]'}{r(x)X(x)} - \frac{q(x)}{r(x)} = -\lambda$$

Thus, we are left with 2 ODEs

$$\dot{T}(t) + \lambda T(t) = 0\,,$$

$$\left[p(x)X'(x)\right]' - q(x)X(x) + \lambda\, r(x)X(x) = 0\,.$$

Sturm–Liouville boundary problems

Define $X(x) \equiv y(x)$ and rewrite the 2nd ODE as

$$-\left[p(x)y'(x)\right]' + q(x)y(x) = \lambda\, r(x)y(x) \Leftrightarrow \mathcal{L}[y] = \lambda\, r(x)y(x).$$

Question: when do we have

$$(\mathcal{L}[u], v) = (u, \mathcal{L}[v]) \text{ ?}$$

By definition

$$(\mathcal{L}[u], v) = \int_0^1 \mathcal{L}[u]\, v^\star\, dx = \int_0^1 \left(-(pu')' + qu\right) v^\star\, dx$$

Let us study the first term.

Sturm–Liouville boundary problems

$$
\begin{aligned}
-\int_0^1 (pu')'v^\star dx &= -\int_0^1 v^\star \, d(pu') \\
&= -\, v^\star pu'\big|_0^1 + \int_0^1 pu'v^{\star\prime} \, dx \\
&= -\, v^\star pu'\big|_0^1 + \int_0^1 pv^{\star\prime} \, du \\
&= \left(puv^{\star\prime} - pu'v^\star\right)\big|_0^1 - \int_0^1 \left(pv^{\star\prime}\right)' \; u \, dx \\
&= -\left[p\left(u'v^\star - v^{\star\prime}u\right)\right]_0^1 - \int_0^1 \left(pv^{\star\prime}\right)' \; u \, dx \, .
\end{aligned}
$$

Sturm–Liouville boundary problems

Inserting the identity

$$-\int_0^1 (pu')' v^\star dx = -\left[p\left(u'v^\star - v^{\star\prime}u\right)\right]_0^1 - \int_0^1 \left(pv^{\star\prime}\right)' \, u\, dx$$

into

$$(\mathcal{L}[u], v) = \int_0^1 \mathcal{L}[u]\, v^\star \, dx = \int_0^1 \left(-(pu')' + qu\right) v^\star \, dx$$

we reach the conclusion

$$(\mathcal{L}[u], v) - (u, \mathcal{L}[v]) = -\left[p\left(u'v^\star - v^{\star\prime}u\right)\right]_0^1 .$$

- If the boundary conditions are such that the right-hand side vanishes, $(\mathcal{L}[u], v) = (u, \mathcal{L}[v]) = 0$ and the conclusions obtained for $\mathcal{L} = d^2/dx^2$ apply.

Sturm–Liouville boundary problems

Question: When does

$$\left[p\left(u'v^\star - v^{\star\prime}u\right)\right]_0^1 \quad \text{vanish ?}$$

- Consider boundary conditions of the type (on the space of real functions)

$$a_1 y(0) + a_2 y'(0) = 0\,, \qquad b_1 y(1) + b_2 y'(1) = 0$$

Evaluating our expression:

$$p(1)\left(u'(1)v(1) - u(1)v'(1)\right) - p(0)\left(u'(0)v(0) - u(0)v'(0)\right) =$$
$$p(1)\left(-\frac{b_1}{b_2}u(1)v(1) + \frac{b_1}{b_2}u(1)v(1)\right) -$$
$$-p(0)\left(-\frac{a_1}{a_2}u(0)v(0) + \frac{a_1}{a_2}u(0)v(0)\right) = 0$$

  - We assumed $b_2, a_2 \neq 0$. The same conclusion holds when either of them vanishes

Sturm–Liouville boundary problems

For specific operators $\mathcal{L}$, we can find other boundary conditions for which

$$(\mathcal{L}[u], v) = (u, \mathcal{L}[v])$$

- Consider periodic boundary conditions (on the space of real functions) when $p(x) = 1$

$$y(0) = y(1)\,, \qquad y'(0) = y'(1)$$

Our condition reduces to

$$\big(u'(1)v(1) - u(1)v'(1)\big) - \big(u'(0)v(0) - u(0)v'(0)\big) = 0.$$

Sturm–Liouville boundary problems

Definition

The eigenvalue ODE equation

$$\mathcal{L}[y] = \lambda\, r(x)y(x) \quad \textit{with} \quad \mathcal{L}[y] \equiv -\left(p(x)y'\right)' + q(x)y(x)$$

subject to the boundary conditions

$$a_1 y(0) + a_2 y'(0) = 0\,, \qquad b_1 y(1) + b_2 y'(1) = 0$$

defines a (regular) Sturm-Liouville boundary problem, where the functions $p(x), p'(x), q(x), r(x)$ are continuous and $p(x), r(x) > 0$ in the entire domain where the problem is defined.

By construction, given any Sturm-Liouville boundary problem, Lagrange's identity holds:

$$(\mathcal{L}[u], v) = (u, \mathcal{L}[v])$$

for any pair u, v satisfying the Sturm-Liouville boundary conditions.

Consequences

Given a Sturm–Liouville boundary problem and using the inner product

$$(u, v) \equiv \int_0^1 u(x)v^\star(x)\, dx\,,$$

we can extend the previous theorems to the general boundary problem discussed now.

Theorem

All eigenvalues of the Sturm–Liouville boundary problem are real.

Proof.

Let $u = v = \Phi \equiv M(x) + iN(x)$ and $\lambda = \mu + i\nu$
Using Lagrange's identity

$$(L[\Phi], \Phi) = (\Phi, L[\Phi])$$
$$(\lambda r\, \Phi, \Phi) = (\Phi, \lambda r\, \Phi)$$

Consequences

Proof.

$$\int_0^1 \lambda r(x)\Phi(x)\Phi^\star(x)\,dx = \int_0^1 \lambda^\star r(x)\Phi(x)\Phi^\star(x)\,dx$$

$$(\lambda - \lambda^\star)\int_0^1 r(x)\Phi(x)\Phi^\star(x)\,dx = 0$$

$$(\lambda - \lambda^\star)\int_0^1 r(x)\left[M^2(x) + N^2(x)\right]\,dx = 0 \Rightarrow \lambda = \lambda^\star$$

where in the last step we used $r(x) > 0$. □

This finally justifies why we never considered in previous lectures the possibility of complex eigenvalues.

Consequences

Theorem

Given Φ_1 and Φ_2 two eigenfunctions of a Sturm–Liouville boundary problem with respective eigenvalues $\lambda_1 \neq \lambda_2$, then

$$\int_0^1 r(x)\Phi_1(x)\Phi_2(x)\,dx = 0$$

That is, the pair is orthogonal with respect to the inner product defined by the Sturm–Liouville boundary problem.

Proof.

Choose $u = \Phi_1$ and $v = \Phi_2$ solving the Sturm–Liouville boundary problem into Lagrange's identity

$$(\mathcal{L}[\Phi_1], \Phi_2) = (\Phi_1, \mathcal{L}[\Phi_2])$$
$$(\lambda_1 r\,\Phi_1, \Phi_2) = (\Phi_1, \lambda_2 r\,\Phi_2)$$

Consequences

Proof.

$$(\lambda_1 r\,\Phi_1, \Phi_2) = (\Phi_1, \lambda_2 r\,\Phi_2)$$

$$(\lambda_1 - \lambda_2)\int_0^1 r(x)\Phi_1(x)\Phi_2(x)\,dx = 0$$

Since $\lambda_1 \neq \lambda_2$, we conclude

$$\int_0^1 r(x)\Phi_1(x)\Phi_2(x)\,dx = 0\,.$$

Notice we used our solutions are *real*. □

This suggests to define the modifed inner product

$$\langle \Phi_1,\, \Phi_2 \rangle \equiv \int_0^1 r(x)\,\Phi_1(x)\,\Phi_2(x)\,dx\,,$$

where the notation $\langle\ \rangle$ prevents confusion.

Honours Differential Equations

Jacques Vanneste

Lecture 28

November 19, 2018

Sturm–Liouville boundary problems

$$-\left[p(x)y'(x)\right]' + q(x)y(x) = \lambda\, r(x)y(x) \Leftrightarrow \mathcal{L}[y] = \lambda\, r(x)y(x).$$

Question: when do we have

$$(\mathcal{L}[u], v) = (u, \mathcal{L}[v])?$$

By definition

$$\begin{aligned}(\mathcal{L}[u], v) &= \int_0^1 \mathcal{L}[u]\, v^\star\, dx = \int_0^1 \left(-(pu')' + qu\right) v^\star\, dx \\ &= -\left[p\left(u'v^\star - v^{\star\prime}u\right)\right]_0^1 - \int_0^1 \left(pv^{\star\prime}\right)'\, u\, dx \\ &= -\left[p\left(u'v^\star - v^{\star\prime}u\right)\right]_0^1 + (u, \mathcal{L}[v]).\end{aligned}$$

Sturm–Liouville boundary problems

We reach the conclusion

$$(\mathcal{L}[u], v) - (u, \mathcal{L}[v]) = -\left[p\left(u' v^\star - v^{\star\prime} u\right)\right]_0^1 .$$

The right-hand side vanishes for boundary conditions of the type

$$a_1 y(0) + a_2 y'(0) = 0\,, \qquad b_1 y(1) + b_2 y'(1) = 0.$$

Check:

$$p(1)\left(u'(1)v(1) - u(1)v'(1)\right) - p(0)\left(u'(0)v(0) - u(0)v'(0)\right) =$$
$$p(1)\left(-\frac{b_1}{b_2}u(1)v(1) + \frac{b_1}{b_2}u(1)v(1)\right) -$$
$$-p(0)\left(-\frac{a_1}{a_2}u(0)v(0) + \frac{a_1}{a_2}u(0)v(0)\right) = 0$$

- We assumed $b_2, a_2 \neq 0$. The same conclusion holds when either of them vanishes

Sturm–Liouville boundary problems

For specific operators $\mathcal{L}$, we can find other boundary conditions for which

$$(\mathcal{L}[u], v) = (u, \mathcal{L}[v])$$

- Consider periodic boundary conditions (on the space of real functions) when $p(x) = 1$

$$y(0) = y(1)\,, \qquad y'(0) = y'(1)$$

Our condition reduces to

$$\big(u'(1)v(1) - u(1)v'(1)\big) - \big(u'(0)v(0) - u(0)v'(0)\big) = 0.$$

Sturm–Liouville boundary problems

Definition

The eigenvalue ODE equation

$$\mathcal{L}[y] = \lambda\, r(x)y(x) \quad \textit{with} \quad \mathcal{L}[y] \equiv -\left(p(x)y'\right)' + q(x)y(x)$$

subject to the boundary conditions

$$a_1 y(0) + a_2 y'(0) = 0\,, \qquad b_1 y(1) + b_2 y'(1) = 0$$

defines a (regular) Sturm-Liouville boundary problem, where the functions $p(x), p'(x), q(x), r(x)$ *are continuous and* $p(x), r(x) > 0$ *in the entire domain where the problem is defined.*

By construction, given any Sturm-Liouville boundary problem, Lagrange's identity holds:

$$(\mathcal{L}[u], v) = (u, \mathcal{L}[v])$$

for any pair u, v satisfying the Sturm-Liouville boundary conditions.

Consequences

Given a Sturm–Liouville boundary problem and using the inner product

$$(u, v) \equiv \int_0^1 u(x)v^\star(x)\, dx\,,$$

we can extend the previous theorems to the general boundary problem discussed now.

Theorem

All eigenvalues of the Sturm–Liouville boundary problem are real.

Proof.

Let $u = v = \Phi \equiv M(x) + iN(x)$ and $\lambda = \mu + i\nu$
Using Lagrange's identity

$$(L[\Phi], \Phi) = (\Phi, L[\Phi])$$
$$(\lambda r\, \Phi, \Phi) = (\Phi, \lambda r\, \Phi)$$

Consequences

Proof.

$$\int_0^1 \lambda r(x)\Phi(x)\Phi^\star(x)\,dx = \int_0^1 \lambda^\star r(x)\Phi(x)\Phi^\star(x)\,dx$$

$$(\lambda - \lambda^\star)\int_0^1 r(x)\Phi(x)\Phi^\star(x)\,dx = 0$$

$$(\lambda - \lambda^\star)\int_0^1 r(x)\left[M^2(x) + N^2(x)\right]\,dx = 0 \Rightarrow \lambda = \lambda^\star$$

where in the last step we used $r(x) > 0$. □

This finally justifies why we never considered in previous lectures the possibility of complex eigenvalues.

Consequences

Theorem

Given Φ_1 and Φ_2 two eigenfunctions of a Sturm–Liouville boundary problem with respective eigenvalues $\lambda_1 \neq \lambda_2$, then

$$\int_0^1 r(x)\Phi_1(x)\Phi_2(x)\,dx = 0$$

That is, the pair is orthogonal with respect to the inner product defined by the Sturm–Liouville boundary problem.

Proof.

Choose $u = \Phi_1$ and $v = \Phi_2$ solving the Sturm–Liouville boundary problem into Lagrange's identity

$$\begin{aligned}(\mathcal{L}[\Phi_1], \Phi_2) &= (\Phi_1, \mathcal{L}[\Phi_2]) \\ (\lambda_1 r\,\Phi_1, \Phi_2) &= (\Phi_1, \lambda_2 r\,\Phi_2)\end{aligned}$$

Consequences

Proof.

$$(\lambda_1 r\,\Phi_1, \Phi_2) = (\Phi_1, \lambda_2 r\,\Phi_2)$$

$$(\lambda_1 - \lambda_2)\int_0^1 r(x)\Phi_1(x)\Phi_2(x)\,dx = 0$$

Since $\lambda_1 \neq \lambda_2$, we conclude

$$\int_0^1 r(x)\Phi_1(x)\Phi_2(x)\,dx = 0\,.$$

Notice we used our solutions are *real*. □

This suggests to define the modifed inner product

$$\langle \Phi_1,\, \Phi_2 \rangle \equiv \int_0^1 r(x)\,\Phi_1(x)\,\Phi_2(x)\,dx\,,$$

where the notation $\langle\ \rangle$ prevents confusion.

Sturm-Liouville & completeness

Theorem

For each eigenvalue of a Sturm-Liouville boundary problem, there is a unique linearly independent eigenfunction. Furthermore, the eigenvalues form an infinite ordered sequence

$$\lambda_1 < \lambda_2 \cdots < \lambda_n < \ldots \lambda_n \to \infty \text{ as } n \to \infty$$

We will not prove this. But, we saw it in the cases we solved

$$L[X] = \lambda X \qquad X(0) = X(L) = 0$$
$$L[X] = -X''$$
$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n \propto \sin\frac{n\pi x}{L}$$

Sturm-Liouville & completeness

We established the existence of orthogonality among different eigenfunctions

$$\langle y_{n_1}, y_{n_2} \rangle = 0 \quad \text{if} \quad \lambda_{n_1} \neq \lambda_{n_2}$$

Usually, it is convenient to work with a set of orthonormal functions $\{\Phi_n\}$ satisfying

$$\langle \Phi_n, \Phi_m \rangle = \int_0^1 r(x)\, \Phi_n(x)\Phi_m(x)\, dx = \delta_{nm}$$

These are related to our solutions by a constant: $\Phi_n(x) = k_n y_n(x)$

Question: Can we expand any function in this set of orthonormal functions?

Sturm-Liouville & completeness

Formally, we can:

$$f(x) = \sum_n c_n \Phi_n(x)$$

Using the inner product, we can compute its components c_n:

$$\begin{aligned}
\langle f(x), \Phi_m \rangle &= \langle \sum_n c_n \Phi_n(x), \Phi_m(x) \rangle = \sum_n c_n \langle \Phi_n, \Phi_m \rangle \\
&= \sum_n c_n \delta_{nm} = c_m \\
c_m &= \int_0^1 r(x)\, f(x) \Phi_m(x)\, dx
\end{aligned}$$

As usual, the only question left is the convergence of this infinite sum.

Sturm-Liouville & completeness

Theorem

Let $\Phi_1(x), \dots \Phi_n(x), \dots$ be the orthonormal eigenfunctions of a Sturm-Liouville boundary problem

$$\left[p(x)y'\right]' - q(x)y + \lambda\, r(x)y = 0\,,$$
$$a_1 y(0) + a_2 y'(0) = 0\,, b_1 y(1) + b_2 y'(1) = 0$$

Let $f(x)$ and $f'(x)$ be piecewise continuous on $0 \le x \le 1$. Then the series $f(x) = \sum_n c_n \Phi_n(x)$ converges to $(f(x^-) + f(x^+))/2$ at each point in the open interval $0 < x < 1$.

- this is an extension of the Fourier convergence theorem,
- we will not prove it, but we will show that any truncation of the series minimises the mean square error.

Sturm-Liouville & completeness

Consider a function $f(x)$ and its expansion in an orthonormal basis

$$f(x) = \sum_n c_n \Phi_n(x) \quad \text{with} \quad c_m = \int_0^1 r(x)\, f(x) \Phi_m(x)\, dx$$

Consider a truncation of this series to finite N

$$S_N(x) = \sum_{n=1}^{N} c_n \Phi_n(x)$$

The mean square error involved in approximating $f(x)$ by $S_N(x)$ is

$$\epsilon_N = \int_0^1 r(x)\,(f(x) - S_N(x))^2\; dx\,.$$

Sturm-Liouville & completeness

Viewing the c_n as variables, we can compute

$$\frac{\partial \epsilon_N}{\partial c_m} = -2\int_0^1 r(x)\left(f(x) - \sum_n c_n \Phi_n(x)\right)\Phi_m(x)\,dx$$
$$= -2\int_0^1 r(x) f(x) \Phi_m(x)\,dx + 2\sum_n c_n \int_0^1 r(x)\Phi_n(x)\Phi_m(x)\,dx$$
$$= -2c_m + 2\sum_n c_n \delta_{nm} = 0$$

Thus, the Sturm-Liouville coefficients extremise the error.
In fact, they minimise it.

Example

Consider the 2nd order ODE $y'' + \lambda y = 0$ subject to boundary conditions

$$y(0) = 0 \quad \text{and} \quad y(1) + y'(1) = 0$$

Notice these define a Sturm-Liouville boundary problem

- because of that, we know $\lambda \in \mathbb{R}$
- $\lambda = 0$: general solution

  $$y(x) = c_1 x + c_2$$

  Using boundary conditions:

  $$\begin{aligned} y(0) &= c_2 = 0 \\ y(1) + y'(1) &= 2c_1 + c_2 = 0 \end{aligned}$$

  Only trivial solution, $c_1 = c_2 = 0$, exists.

Example

- $\lambda = -\mu$ ($\mu > 0$): general solution

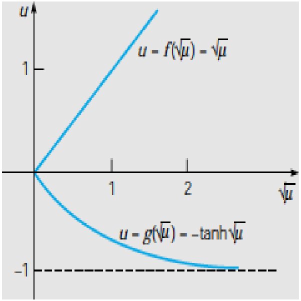
$$y(x) = c_1 \sinh \sqrt{\mu}x + c_2 \cosh \sqrt{\mu}x$$

Using boundary conditions:

$$y(0) = c_2 = 0$$
$$y(1) + y'(1) = c_1 \left(\sqrt{\mu} \cosh \sqrt{\mu} + \sinh \sqrt{\mu}\right) = 0$$

Does $\sqrt{\mu} = -\tanh \sqrt{\mu}$ have non-trivial solutions?

- only at $\mu = 0$, which is excluded by hypothesis
- only trivial solution exists



Example

- $\lambda = \mu$ ($\mu > 0$): general solution

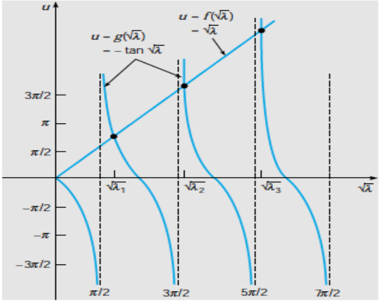
$$y(x) = c_1 \sin \sqrt{\mu} x + c_2 \cos \sqrt{\mu} x$$

Using boundary conditions:

$$y(0) = c_2 = 0 \Rightarrow y(x) = c_1 \sin \sqrt{\mu} x$$
$$y(1) + y'(1) = c_1 \left(\sqrt{\mu} \cos \sqrt{\mu} + \sin \sqrt{\mu} \right) = 0$$

Does $\sqrt{\mu} = -\tan \sqrt{\mu}$ have non-trivial solutions ?

- $\exists$ infinite countable λ_n (due to periodicity of $\tan x$)
- $\Phi_n(x) = k_n \sin \sqrt{\lambda_n} x$ $n = 1, 2, \ldots$



Example

Our eigenvalue boundary problem is solved by

$$\Phi_n = k_n \sin\sqrt{\lambda_n}x \quad \text{with} \quad \sqrt{\lambda_n} = -\tan\sqrt{\lambda_n}$$

and k_n fixed by requiring an orthonormal basis:

$$\begin{aligned} 1 &= \int_0^1 (\Phi_n(x))^2\,dx = k_n^2 \int_0^1 \left(\sin\sqrt{\lambda_n}x\right)^2\,dx \\ &= k_n^2 \left(\frac{x}{2} - \frac{\sin 2\sqrt{\lambda_n}x}{4\sqrt{\lambda_n}}\right)_0^1 \\ &= k_n^2 \frac{2\sqrt{\lambda_n} - \sin 2\sqrt{\lambda_n}}{4\sqrt{\lambda_n}} = k_n^2 \frac{1+\cos^2\sqrt{\lambda_n}}{2} \end{aligned}$$

Honours Differential Equations

Jacques Vanneste

Lecture 29

November 22, 2018

Parseval's theorem

We derive a general Parseval's identity for Sturm–Liouville problems:

$$\int_0^1 r(x)\,(f(x))^2\;dx = \sum_n (c_n)^2$$

where the set $\{c_n\}$ is determined by

$$c_n = \int_0^1 r(x) f(x) \Phi_n(x)\, dx\,.$$

The proof is a simple computation using the orthonormality of the $\Phi_n(x)$:

$$\begin{aligned}\int_0^1 r(x)\,(f(x))^2\;dx &= \int_0^1 r(x) \sum_m c_m \Phi_m(x) \sum_n c_n \Phi_n(x)\, dx \\ &= \sum_m \sum_n \int_0^1 c_m c_n \langle \Phi_m(x), \Phi_n(x)\rangle = \sum_n (c_n)^2.\end{aligned}$$

Self-adjoint problems

The general eigenvalue problem

$$\mathcal{L}[y] = \lambda\, r(x) y$$

for an *n*-th order operator

$$\mathcal{L}[y] = p_n(x)\frac{d^n y}{dx^n} + \cdots + p_1(x)\frac{dy}{dx} + p_0(x)y$$

subject to n linear homogeneous boundary conditions at the endpoints is self-adjoint provided that

$$(\mathcal{L}[u], v) = (u, \mathcal{L}[v]).$$

For example, for 4th order we can have

$$\mathcal{L}[y] = \left[p(x)y''\right]'' - \left[q(x)y'\right]' + s(x)y$$

plus suitable boundary conditions.

- Most of our results extend to these problems

Non-homogeneous boundary value problems

We are interested in solving the mathematical problem

$$\mathcal{L}[y] = \mu\, r(x)\, y + f(x)$$

with homogeneous boundary conditions, where

- μ is some fixed real number,
- $f(x)$ is given and renders the ODE as non-homogeneous.

Strategy:

- Solve the homogeneous Sturm-Liouville boundary problem

  $$\mathcal{L}[y] = \lambda\, r(x) y$$

  subject to the same boundary conditions. Note
  - the left hand side operator $\mathcal{L}[y]$ is the same,
  - the weighting function $r(x)$ is the same,
  - the eigenvalues λ will generically be different from μ.

Non-homogeneous boundary value problems

The eigenvalue problem

$$\mathcal{L}[y] = \lambda\, r(x) y$$

determines an orthonormal set of functions $\Phi_n(x)$ satisfying

$$\mathcal{L}[\Phi_n(x)] = \lambda_n\, r(x)\, \Phi_n(x)$$

according to Sturm-Liouville theory.

We now expand the solution to our non-homogeneous ODE into this basis:

$$y(x) = \sum_n b_n \Phi_n(x)$$

The unknowns are now the set $\{b_n\}$.

Non-homogeneous boundary value problems

- Into our equation:

$$\begin{aligned}\mathcal{L}[y] &= \mu\, r(x)\, y + f(x) \\ \sum_n b_n L[\Phi_n] &= r(x) \sum_n b_n \lambda_n \Phi_n(x) \\ &= r(x) \sum_n \mu\, b_n \Phi_n(x) + f(x)\end{aligned}$$

$$\sum_n b_n(\lambda_n - \mu)\Phi_n(x) = \frac{f(x)}{r(x)}.$$

- Expand the right hand side in the same basis

$$\frac{f(x)}{r(x)} = \sum_n c_n \Phi_n(x) \quad \text{with} \quad c_n = \int_0^1 r(x)\, \frac{f(x)}{r(x)}\, \Phi_n(x)\, dx$$

- Into the equation:

$$\sum_n \left[b_n(\lambda_n - \mu) - c_n\right] \Phi_n(x) = 0\,.$$

Non-homogeneous boundary value problems

$$\sum_n \left[b_n(\lambda_n - \mu) - c_n\right] \Phi_n(x) = 0\,.$$

Since $\Phi_n(x)$ are orthonormal, each coefficient must vanish:

- if $\mu \neq \lambda_n \;\; n = 1, 2, \ldots,$ then the solution is unique

$$b_n = \frac{c_n}{\lambda_n - \mu}\,.$$

- If $\exists\; m$ such that $\mu = \lambda_m$, then we have the condition

$$0\, b_m - c_m = 0$$

  - If $c_m = 0$, i.e. $(f(x)/r(x), \Phi_m(x)) = 0$,
    $\Rightarrow 0\, b_m - c_m = 0$ satisfied
    solution not unique : b_m undetermined,
  - If $c_m \neq 0 \Rightarrow$ no solution.

Example

Solve

$$y'' + 2y = -x, \quad y(0) = y(1) + y'(1) = 0.$$

- Using the method of undetermined coefficients, we can derive

$$y(x) = \frac{\sin\sqrt{2}x}{\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2}} - \frac{x}{2}.$$

- Let us use the method just discussed: we will solve

$$-y'' = 2y + x\,,$$

using the auxiliary Sturm–Liouville boundary value problem

$$-y'' = \lambda\, y \quad \text{with} \quad y(0) = y(1) + y'(1) = 0$$

Example

Solutions to the boundary problem

$$-y'' = \lambda\, y \quad \text{with} \quad y(0) = y(1) + y'(1) = 0$$

are given by orthonormal functions

$$\Phi_n(x) = k_n \sin\sqrt{\lambda_n} x\,, \quad \text{with} \quad \sin\sqrt{\lambda_n} + \sqrt{\lambda_n}\cos\sqrt{\lambda_n} = 0\,.$$

Since the set $\{\Phi_n(x)\}$ provides a complete basis of functions, we can expand the solution as

$$y(x) = \sum_n b_n \Phi_n(x)$$

Notice that at this stage the unknowns are the set $\{b_n\}$. Furthermore, we can expand the inhomogeneous function as

$$x = \sum_n c_n \Phi_n(x)$$

Example

Then

$$y(x) = \sum_n b_n \Phi_n(x) \Rightarrow y'' = \sum_n b_n \Phi_n'' = -\sum_n b_n \lambda_n \Phi_n$$

Plugging into the ODE :

$$\sum_n b_n \left(-\lambda_n + 2\right) \Phi_n(x) = -\sum_n c_n \Phi_n(x)$$

Since the set $\{\Phi_n(x)\}$ is a complete basis, we can conclude

$$b_n = \frac{c_n}{\lambda_n - 2}$$

Notice the solution is unique since $\lambda_n \neq 2$.

Non-homogeneous PDEs

Consider the boundary problem

$$r(x)\partial_t u(x,t) = \partial_x\left[p(x)\partial_x u(x,t)\right] - q(x)\,u(x,t) + F(x,t)$$

satisfying the boundary/initial conditions

$$\begin{aligned}
\partial_x u(0,t) - h_1\,u(0,t) &= 0 \\
\partial_x u(1,t) + h_2\,u(1,t) &= 0 \\
u(x,0) &= f(x)\,.
\end{aligned}$$

We will follow a similar strategy to the one discussed for ODEs.

Non-homogeneous PDEs

- Consider the Sturm-Liouville boundary problem associated with the homogeneous PDE

$$r(x)\partial_t u(x,t) = \partial_x \left[p(x)\partial_x u(x,t)\right] - q(x)\, u(x,t)$$

This means that $u(x,t) = X(x)T(t)$ where both functions satisfy

$$\dot{T} = -\lambda\, T$$

$$-\left(p(x)X'\right)' + qX = \lambda\, r\, X$$

satisfying $X'(0) - h_1 X(1) = X'(1) + h_2 X(1) = 0$.

Non-homogeneous PDEs

- Assume the Sturm-Liouville problem is solved
  - $\exists\ \{\lambda_n\}$ associated with orthonormal eigenfunctions $\{\Phi_n(x)\}$
- Look for a solution to the original non-homogeneous PDE by expanding in this basis

$$u(x,t) = \sum_n b_n(t)\Phi_n(x)$$

  The set of unknowns is now the set of $\{b_n(t)\}$
- Expand F(x,t) in the same basis. It is convenient to consider

$$\frac{F(x,t)}{r(x)} = \sum_n \gamma_n(t)\Phi_n(x)$$

$$\text{with}\ \ \gamma_n(t) = \int_0^1 r(x)\,\frac{F(x,t)}{r(x)}\,\Phi_n(x)dx$$

Non-homogeneous PDEs

- Substitute into the main equation

$$r(x)\sum_n \dot{b}_n \Phi_n(x) = \sum_n b_n(t)\left(\left[p(x)\Phi_n(x)'\right]' - q(x)\Phi_n(x)\right) + F(x,t)$$

Since $\Phi_n(x)$ satisfies the Sturm-Liouville boundary problem, it satisfies

$$-\left[p(x)\Phi_n(x)'\right]' + q(x)\Phi_n(x) = \lambda_n \, r(x)\, \Phi_n(x)$$

Substituting

$$r(x)\sum_n \dot{b}_n \Phi_n(x) = -r(x)\sum_n b_n(t)\lambda_n \Phi_n(x) + F(x,t)$$

$$\sum_n \dot{b}_n \Phi_n(x) = -\sum_n b_n(t)\lambda_n \Phi_n(x) + \frac{F(x,t)}{r(x)}$$

$$\sum_n \dot{b}_n \Phi_n(x) = -\sum_n b_n(t)\lambda_n \Phi_n(x) + \sum_n \gamma_n(t)\Phi_n(x)$$

Non-homogeneous PDEs

- Thus,

$$\sum_n \left[\dot{b}_n + \lambda_n b_n(t) - \gamma_n(t)\right] \Phi_n(x) = 0$$

  and using the orthonormality of the set $\{\Phi_n(x)\}$, we can conclude

$$\dot{b}_n + \lambda_n b_n(t) = \gamma_n(t) \qquad n = 1, 2, 3, \ldots$$

  This is an infinite countable linear set of non-homogeneous decoupled ODEs with constant coefficients.
  $\Rightarrow$ problem solved!

- Initial conditions determined by original initial condition:

$$u(x,0) = f(x) = \sum_n \alpha_n \Phi_n(x) \Rightarrow \alpha_n = \int_0^1 r(x)\, f(x)\, \Phi_n(x)\, dx$$

  Indeed, by our assumption,

$$u(x,0) = \sum_n b_n(0)\Phi_n(x) \Rightarrow b_n(0) = \alpha_n \,.$$

Non-homogeneous PDEs

- General solution

$$b_n(t) = \alpha_n \, e^{-\lambda_n t} + \int_0^t e^{-\lambda_n (t-s)} \, \gamma_n(s) \, ds$$

Thus, the general solution to our original non-homogeneous PDE boundary problem is

$$u(x,t) = \sum_n b_n(t) \Phi_n(x)$$

  - the set $\{b_n(t)\}$ is given above
  - the set $\{\Phi_n(x)\}$ solves the auxiliary Sturm-Liouville boundary problem

Honours Differential Equations

Jacques Vanneste

Lecture 30

November 23, 2018

Regular vs singular Sturm–Liouville problems

Regular Sturm-Liouville boundary problems were defined by ODEs

$$\mathcal{L}[y] = -\left(p(x)y'\right)' + q(x)y = \lambda\, r(x)y$$

where p, p', q, r are continuous and $p(x), r(x) > 0$ for $x \in [0, 1]$
The theory is based on boundary conditions that guarantee that

$$(\mathcal{L}[u], v) = (u, \mathcal{L}[v])$$

- We assumed $p(x) > 0$ for $x \in [0, 1]$,
- Today, consider $p(x)$ that can vanish at the endpoints of the domain of x: singular Sturm–Liouville problems.

3d Laplace equation in cylindrical coordinates

Consider the 3d Laplace equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0$$

in cylindrical coordinates :

$$x = \rho\cos\varphi\,, \qquad y = \rho\sin\varphi\,, \qquad z = z.$$

Laplace's equation becomes:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2} = 0.$$

Consider boundary conditions

$$\psi = 0 \ \text{ for } \rho = 1, \quad \psi = 0, \ \text{ for } z = 0, \quad \psi = \Psi(\rho,\varphi) \ \text{ for } z = 1.$$

3d Laplace equation in cylindrical coordinates

Using separation of variables : $\psi(\rho,\varphi,z) = R(\rho)\phi(\varphi)Z(z)$

$$\frac{1}{R\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2\phi}\frac{d^2\phi}{d\varphi^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0$$

we can identify 3 ODEs using 2 separation of variables χ, m

$$\frac{d^2 Z}{dz^2} = \chi^2 Z\,,$$

$$\frac{d^2\phi}{d\varphi^2} = -m^2\phi\,,$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(\chi^2 - \frac{m^2}{\rho^2}\right) R = 0\,.$$

3d Laplace equation in cylindrical coordinates

Using $\psi(z=0)=0$, $Z(0)=0$, hence

$$\frac{d^2 Z}{dz^2} = \chi^2 Z \;\Rightarrow\; Z(z) = a\sinh(\chi z),$$
$$\frac{d^2\phi}{d\varphi^2} = -m^2\phi \;\Rightarrow\; \phi(\varphi) = f\,\cos m\varphi + g\,\sin m\varphi.$$

Thus, if $m \in \mathbb{Z}$, solutions are periodic
The radial equation can be written in Sturm-Liouville form

$$-\left(\rho R'\right)' + \frac{m^2}{\rho} R = \chi^2 \rho\, R$$

Thus,

- $p(\rho) = r(\rho) = \rho$: vanish at the origin $\rho = 0$,
- $q(\rho) = \frac{m^2}{\rho}$: unbounded as $\rho \to 0$.

Singular Sturm-Liouville boundary problem.

Bessel's equation

Remember

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(\chi^2 - \frac{m^2}{\rho^2}\right) R = 0\,.$$

Consider the change of coordinates $x = \chi\rho$ and write $R(\rho) = y(x)$

$$y'' + \frac{1}{x}y'(x) + \left(1 - \frac{m^2}{x^2}\right) y(x) = 0$$

This defines Bessel's equation

- 2nd order linear ODE $\Rightarrow$ two linearly independent solutions
- non-constant coefficients: how do we find them?

Bessel's equation

Solutions are found as series expansion (Frobenius method, see SVCDE):

Bessel function of the 1st kind:

$$J_m(x) = \left(\frac{x}{2}\right)^m \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{k!(m+k)!}$$

(see also Week-11 workshop).

Bessel function of the 2nd kind:

$$Y_m(x) = \frac{2}{\pi} \log(x/2) J_m(x) + \sum_{k=0}^{\infty} \cdots x^{2k}.$$

Thus, the general solution to our ODE will be the linear combination

$$R_m(\rho) = c_1\, J_m(\chi\rho) + c_2\, Y_m(\chi\rho)\,.$$

Solving Bessel's equation ($m = 0$)

Define the Laplace's transform of the solution as

$$Y(s) = \mathcal{L}\{y(x)\}$$

Using the identity we proved in week 2 :

$$\mathcal{L}\{x^n f(x)\} = -\frac{d^n \mathcal{L}\{f(x)\}}{ds^n},$$

we derive

$$\mathcal{L}\{x\, y''\} = -\frac{d}{ds}\left[s^2 Y - sy(0) - y'(0)\right] = -2sY - s^2 Y' + y(0)$$
$$\mathcal{L}\{x\, y\} = -Y'.$$

Substituting into Bessel's equation, we get :

$$\left(1 + s^2\right) Y' + sY = 0.$$

This is an ODE for $Y(s)$!!

Solving Bessel's equation ($m = 0$)

This ODE is solved by

$$\frac{Y'}{Y} = -\frac{s}{s^2+1} \quad \Rightarrow \quad Y(s) = \frac{c}{(1+s^2)^{1/2}}\,,$$

Let me assume $s > 1$, so that we can expand the solution

$$\frac{c}{(1+s^2)^{1/2}} = \frac{c}{s}\sum_{k=0}^{\infty}(-1)^k s^{-2k}\,\frac{(2k-1)(2k-3)\ldots 1}{2^k\,k!}$$

$$= c\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}(k!)^2}\,\frac{(2k)!}{s^{2k+1}}$$

Last step : to compute the Laplace inverse

Solving Bessel's equation ($m = 0$)

We use linearity of the Laplace transform and the property

$$\mathcal{L}\{x^{2n}\} = \frac{(2n)!}{s^{2n+1}}$$

to conclude that one solution to Bessel's equation is given by

$$y(x) = c \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \equiv c\, J_0\,(x)$$

In terms of our original variable : $x = \chi\rho$

$$R(\rho) = c\, J_0\,(\chi\rho)$$

- Notice the solution is indeed an infinite series around $x = 0$

Solving Bessel's equation ($m = 0$)

We are missing a second linearly independent solution

- this is more difficult to derive
- Let us state without proof this second solution equals

$$Y_0(x) = \frac{2}{\pi}\left[\left(\gamma + \log\frac{x}{2}\right) J_0(x) + \sum_{n=1}^{\infty} \frac{(-1)^{m+1} H_m x^{2m}}{2^{2m}(m!)^2}\right]$$

$$H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$$

$$\gamma = \lim_{m\to\infty} (H_m - \log m)$$

- Thus, the general solution to our ODE will be the linear combination

$$R(\rho) = c_1 J_0(\chi\rho) + c_2 Y_0(\chi\rho).$$

Boundary conditions

We need to impose $R_m(1) = 0$ and finiteness of $R_m(\rho)$ as $\rho \to 0$.

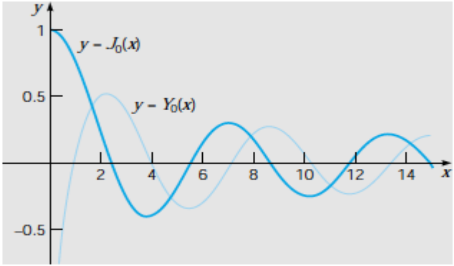
- $J_0(0) = 1$ and $J_m(0) = 0$ for $m \geq 1$,
- $Y_m(\rho) \to -\infty$ as $\rho \to 0$ (due to the logarithm).

Hence, we require

$$c_2 = 0 \quad \text{and} \quad J_m(\chi) = 0.$$

Plotting this Bessel functions J_m, we conclude there exists a countable infinity of eigenvalues,

$$\exists 0 < \chi_{m1} < \chi_{m2} < \cdots < \chi_{mn} < \ldots.$$



General solution

We now superpose the solutions

$$\psi_{mn} = (f_{mn}\cos(m\varphi) + g_{mn}\sin(m\varphi))\, J_m(\chi_{mn}\rho)\sinh(\chi_{mn}z)$$

to write the general solution as

$$\psi(\rho,\varphi,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\psi_{mn}(\rho,\varphi,z).$$

The constants f_{mn} and g_{mn} are found from the remaining boundary condition,

$$\Psi(\rho,\varphi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\psi_{mn}(\rho,\varphi,1).$$

by projection on $\cos(m\varphi)J_m(\chi_{mn}\rho)$ and $\sin(m\varphi)J_m(\chi_{mn}\rho)$.

Orthogonality

Orthogonality: the Bessel functions $R_{mn}(\rho) = J_n(\chi_{mn}\rho)$ satisfy

$$\mathcal{L}R_{mn} = -(\rho R'_{mn})' + \frac{m^2}{\rho}R_{mn} = \rho\xi^2 R_{mn}.$$

The has the Sturm-Liouville theory form, with

$$(\mathcal{L}[u], v) = (u, \mathcal{L}[v])$$

provided that

$$0 = p(x)\left[u'v - uv'\right]_0^1$$

- The functions R_{mn} satisfy standard Sturm-Liouville boundary conditions at $\rho = 1 \Rightarrow$ the contribution at $\rho = 1$ vanishes,
- At $\rho = 0$, we have

$$\lim_{\epsilon\to 0} p(\epsilon)\left[u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)\right] = 0$$

since $p(\rho) = \rho$ and u, v, u' and v' are bounded.

Orthogonality

We conclude

$$\int_0^1 \rho J_m(\chi_{mn}\rho) J_m(\chi_{mn'}\rho)\, d\rho = 0 \quad \text{for} \quad n \neq n'.$$

We can therefore identify the constants f_{mn} and g_{mn} by projection:

$$f_{mn} \propto \int_0^1 \int_{-\pi}^{\pi} \Psi(\rho,\varphi) \cos(m\varphi) J_m(\chi_{mn}\rho)\rho \, d\rho d\varphi,$$

$$g_{mn} \propto \int_0^1 \int_{-\pi}^{\pi} \Psi(\rho,\varphi) \sin(m\varphi) J_m(\chi_{mn}\rho)\rho \, d\rho d\varphi.$$

Honours Differential Equations

Jacques Vanneste

Lecture 31

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Wave equations in 2 dimensions

1. Rectangle

Solve

$$\partial_t^2 u = a^2 \left(\partial_x^2 + \partial_y^2\right) u$$

for $u(x, y, t)$ in the rectangle $0 \le x \le L$, $0 \le y \le M$ satisfying

$$u(x = 0, y) = u(x = L, y) = u(x, 0) = u(x, M) = 0.$$

This models the vibration of a rectangular drum.

Separation of variables: $u(x, y, t) = X(x)Y(y)T(t)$ gives

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{T''}{a^2 T} = -(\lambda + \mu)$$

with

$$\begin{aligned} X'' + \lambda X &= 0, \quad X(0) = X(L) = 0 \\ Y'' + \mu Y &= 0, \quad Y(0) = Y(M) = 0. \end{aligned}$$

Wave equations in 2 dimensions

In the usual way,

$$X = \sin(m\pi x/L), \quad \lambda = \lambda_m = m^2\pi^2/L^2, \quad m = 1, 2, \cdots,$$
$$Y = \sin(n\pi y/M), \quad \mu = \mu_n = n^2\pi^2/M^2, \quad n = 1, 2, \cdots.$$

Hence,

$$T'' + a^2(\lambda_m + \mu_n)T = 0,$$

with solution

$$T(t) = T_{mn}(t) = c_{mn}\cos(\omega_{mn}t) + d_{mn}\sin(\omega_{mn}t),$$

where $\omega_{mn} = a\pi\sqrt{m^2/L^2 + n^2/M^2}$ are the natural frequencies of the rectangle.

The general solution is given by superposition as

$$u(x, y, t) = \sum_{m,n} \left(c_{mn}\cos(\omega_{mn}t) + d_{mn}\sin(\omega_{mn}t)\right) \times \sin(m\pi x/L)\sin(n\pi y/M).$$

Wave equations in 2 dimensions

The constants c_{mn} and d_{mn} are determined by initial conditions:

$$u(x,y,0) = f(x,y) = \sum_{m,n} c_{mn} \sin(m\pi x/L) \sin(n\pi y/M),$$
$$\partial_t u(x,y,0) = g(x,y) = \sum_{m,n} d_{mn} \sin(m\pi x/L) \sin(n\pi y/M).$$

Projection (Fourier series) gives

$$c_{mn} = \frac{4}{LM} \int_0^L \int_0^M f(x,y) \sin(m\pi x/L) \sin(n\pi y/M)\, dxdy,$$
$$d_{mn} = \frac{4}{LM} \int_0^L \int_0^M g(x,y) \sin(m\pi x/L) \sin(n\pi y/M)\, dxdy.$$

Wave equations in 2 dimensions

2. Disc

Solve

$$\partial_t^2 u = a^2 \left(\partial_x^2 + \partial_y^2\right) u$$

when $u(x, y, t)$ is a bounded function defined over a disk of size one, i.e. $0 \leq x^2 + y^2 \leq 1$ satisfying

$$\begin{aligned} u(r = 1, \theta, t) &= 0 \\ \partial_t u(x, y, 0) &= 0 \\ u(r, \theta, 0) &= f(r, \theta). \end{aligned}$$

Using $x = r\cos\theta$ and $y = r\sin\theta$, the wave equation becomes

$$\partial_t^2 u = a^2 \left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right) u.$$

Wave equations in 2 dimensions

Separation of variables:

$$u(r,\theta,t) = R(r)\Theta(\theta)T(t)$$

gives

$$\frac{R'' + r^{-1}R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \frac{T''}{a^2\,T} = -\mu^2.$$

(we assumed the separation constant is negative)
This gives rise to the ODEs:

$$\begin{aligned} \frac{R''}{R} + \frac{R'}{rR} + \left(\mu^2 - \frac{m^2}{r^2}\right) R &= 0, \\ \Theta'' + m^2\Theta &= 0, \\ T'' + a^2\mu^2 T &= 0. \end{aligned}$$

Wave equations in 2 dimensions

General solution:

$$\begin{aligned}
T(t) &= k_1 \sin(\mu a t) + k_2 \cos(\mu a t), \\
\Theta(\theta) &= a_1 \cos(m\theta) + a_2 \sin(m\theta), \\
R(r) &= c_1 J_m(\mu r) + c_2 Y_m(\mu r)
\end{aligned}$$

where we have used that the 2nd ODE is Bessel's equation

Periodicity in θ requires $m = 1, 2, \cdots$.

Boundedness imposes that $a_2 = 0$.

$u(r = 1, \theta, t)$ imposes that $J_m(\mu) = 0$, i.e., $\mu = \mu_{m1}, \mu_{m2}, \cdots$, zeroes of the Bessel function J_m.

Initial condition $\partial_t u(r, \theta, 0) = 0$ imposes $k_1 = 0$.

Wave equations in 2 dimensions

Therefore the general solution is

$$u = \sum_m \sum_n (c_{mn} \cos(m\theta) + d_{mn} \cos(m\theta)) \cos(a\mu_{mn} t) J_m(\mu_{mn} r).$$

Recalling the orthogonality

$$\int_0^1 r J_m(\mu_{mn} r) J_m(\mu_{mn'}) \, dr = C\delta_{nn'}$$

we obtain c_{mn} and d_{mn} by projection

$$c_{mn} \propto \int_0^1 \int_0^{2\pi} f(r,\theta) \cos(m\theta) J_{mn}(\mu_{mn} r) r dr d\theta,$$
$$d_{mn} \propto \int_0^1 \int_0^{2\pi} f(r,\theta) \sin(m\theta) J_{mn}(\mu_{mn} r) r dr d\theta.$$

Summary of the course

The course has three main topics:

1. Linear ODE initial value-problems
   - higher-order ODEs,
   - systems of 1st-order ODEs,
2. nonlinear ODEs & dynamical systems,
3. Linear PDEs boundary problems & Sturm-Liouville.

Linear ODEs

The most general nth order linear ODE is:

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + P_n(t)y = G(t)$$

- homogeneous problem $(G(t) = 0)$ is solved by

$$y_h = \sum_{i=1}^{n} c_i y_i$$

  where $\{y_i\}$ form a fundamental set (linearly independent).
- non-homogeneous problem $(G(t) \neq 0)$ is solved by

$$y_g = y_h + y_p = \sum_{i=1}^{n} c_i y_i + y_p$$

  $y_p(x)$ particular solution to the non-homogeneous solution.
- Solution is unique if we are given an initial condition $\{y_0, y_0', \ldots y_0^{(n-1)}\}$

Linear ODEs

When $P_i(t)$ constants:

- we can compute $y_i = e^{k_i t}$ where k_i satisfy the characteristic equation
- if $k_i = k_j$, one needs to multiply $e^{k_i t}$ by a power of t whose degree depends on the multiplicity.
- This always determines the set $\{y_i\}$ to fix y_h
- To solve the non-homogeneous problem (y_p), we saw two methods
  - undetermined coefficients, for exponential $\cdot$ polynomials $G(t)$,
  - variation of parameters for arbitrary $G(t)$.

Linear ODEs & Laplace transforms

To solve the same problem including the initial conditions :
Laplace transforms

$$\mathcal{L}\{y(t)\} \equiv \int_0^\infty e^{-st}\, y(t) dt = Y(s)$$

- General properties & existence of this integral transform

To solve a linear ODE:

- Compute the Laplace transform of the original ODE
- Solve for $Y(s)$ algebraically
- Last step: inverse of Laplace transform
  - use previous properties to find $y(t) = \mathcal{L}^{-1}[Y(s)]$
- Laplace transforms allows us to solve linear ODEs where $G(t)$ was a generalised function (step function, delta function),
- convolution.

Linear ODE systems

- An n-th order homogeneous ODE linear system equivalent to a linear ODE system
$$\frac{d\boldsymbol{y}}{dt} = A\boldsymbol{y}$$
- Previous theorems on existence of solutions extend
- When matrix A is constant previous methods extend
  - Diagonalise the matrix A: find eigenvalues and eigenvectors
  - if A can not be diagonalised: reduce to Jordan form.
- When the system is non-homogeneous
$$\frac{d\boldsymbol{y}}{dt} = A\boldsymbol{y} + \boldsymbol{g}(t)$$
  we saw three methods to solve this problem
  - Diagonalization, undetermined coefficients and variation of parameters.

Nonlinear ODE systems

- Our previous linear ODE systems appear as linear approximations of nonlinear ODE systems around critical points
- Classification & stability properties of critical points in 2×2 systems
- Nonlinear corrections can change the nature of some critical points
- Lyapunov theory can assess the stability of critical points at the nonlinear level
- Limit cycles can exist at the nonlinear level: Poincaré-Bendixon theorem tries to identify when they exist, even without being able to integrate the nonlinear equations

Boundary value problems

- When moving from initial value problems to boundary value problem, linear ODEs may have a solution, or not. And if they do, the solution may not be unique,
- We introduced the notion of an eigenvalue boundary problem,
- Canonical eigenvalue problem:

$$-y'' = \lambda^2 y, \quad y(x+2L) = y(x).$$

- This allowed us to discover the existence of Fourier series: any periodic function can be expanded as a linear combination of cos and sin
  - we identified the conditions under which this expansion converges

Linear PDEs

- We solved linear PDEs by the method of separation of variables
- The solution to this problem involves eigenvalue boundary problems
- In particular examples, we explicitly saw the solutions to these eigenvalue boundary problems involved
  - some quantization condition on the separation of variables constant $\lambda \to \{\lambda_n\}$
  - infinite (countable) number of eigenvalues λ_n and eigenfunctions X_n

Sturm-Liouville theory

- We identified the types of 2nd order linear PDEs and boundary conditions under which
  - solution of eigenvalue boundary problem involves an infinite (countable) number of eigenvalues λ_n
  - to each λ_n there is a unique eigenfunction $X_n(x)$: $L[X_n] = \lambda_n r(x) X_n$
  - existence of orthogonality between eigenfunctions $(X_n, X_m) = \delta_{nm}$
  - any piece-wise function f(x) allows a convergent expansion in eigenfunctions

$$f(x) = \sum_x a_n X_n(x) \qquad a_n = \int_b^a r(x) f(x) X_n(x)\, dx\,.$$

- Thus, what we saw in Fourier series is a particular case of regular Sturm-Liouville theory
- We discussed examples on how to extend the Sturm-Liouville theory to singular eigenvalue boundary problems