# The topological Künneth theorem in homology

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Construction: (Exterior pairing) Let $R$ be a commutative ring and $X$ and $Y$ simplicial sets. The homological exterior pairing

$$\times : H_p(X;R) \otimes_R H_q(Y;R) \longrightarrow H_{p+q}(X\times Y;R)$$ is defined as the composite

$$H_p(X;R)\otimes_R H_q(Y;R) = H_p(C_*(X;R))\otimes_R H_q(C_*(Y;R)) \xrightarrow{\Phi} H_{p+q}(C_*(X;R)\otimes_R C_*(Y;R)) \xrightarrow{H_{p+q}(D)} H_{p+q}(C_*(X\times Y;R)) = H_{p+q}(X\times Y;R).$$

$$[x]\otimes[y] \longmapsto [x\otimes y]$$

For two spaces $A$ and $B$, the external homology pairing is the composite

$$\begin{tikzcd} H_p(A;R)\otimes_R H_q(B;R) = H_p(\mathcal{S}(A);R)\otimes_R H_q(\mathcal{S}(B);R) \arrow[r, "\times"] \arrow[rr, bend right, "\times"'] & H_{p+q}(\mathcal{S}(A)\times\mathcal{S}(B);R) \arrow[r, "H_{p+q}(c^{-1};R)", "\cong"'] & H_{p+q}(\mathcal{S}(A\times B);R) = H_{p+q}(A\times B;R) \end{tikzcd}$$

where $c : \mathcal{S}(A\times B) \xrightarrow{\cong} \mathcal{S}(A)\times\mathcal{S}(B)$, $c = (\mathcal{S}(p_1), \mathcal{S}(p_2))$, is the canonical isomorphism of simplicial sets, where $p_1 : A\times B \to A$, $p_2 : A\times B\to B$ are the projections.

Thm: Let $R$ be a field and let $X$ and $Y$ be spaces or simplicial sets. Then the external homology pairings provide a natural isomorphism of $R$-vector spaces

$$\bigoplus_{p+q=n} H_p(X;R)\otimes_R H_q(Y;R) \longrightarrow H_n(X\times Y;R)$$

Proof: Case of simplicial sets: the map in question is the composite of two isomorphisms:

$$\bigoplus_{p+q=n} H_p(C_*(X;R))\otimes_R H_q(C_*(Y;R)) \underset{\cong}{\xrightarrow{\Phi}} H_n(C_*(X;R)\otimes_R C_*(Y;R)) \underset{\cong}{\xrightarrow{H_n(D)}} H_n(C_*(X\times Y;R)) \qquad \square$$

algebraic Künneth theorem — Eilenberg-Zilber theorem.

Similar argument shows:

Thm Let $X$ and $Y$ be spaces or simplicial sets. Then the integral external homology pairings participate in a natural short exact sequence of abelian groups:

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(X;\mathbb{Z})\otimes H_q(Y;\mathbb{Z}) \xrightarrow{\times} H_n(X\times Y;\mathbb{Z}) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X,\mathbb{Z}), H_q(Y,\mathbb{Z})) \longrightarrow 0$$

The sequence splits (but the splitting cannot be chosen natural).