DMAN PRÉCIS

JOSÉ FIGUEROA-O'FARRILL

ABSTRACT. This is a précis of the contents of the Differentiable Manifolds (DMan) course. Its purpose is to help students taking the course revise for the exam, but also to help students taking any course having DMan as a pre-requisite to know what material is expected that they already know.

Smooth manifolds

An n-**dimensional smooth manifold** is a set M, a maximal **atlas** $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ where $M = \bigcup_{\alpha\in A} U_\alpha$,

$$\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$$

is a bijection onto an open subset of $\mathbb{R}^n$, such that for all $\alpha, \beta \in A$ for which $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, $\varphi_\alpha(U_{\alpha\beta}) \subset \mathbb{R}^n$ is an open set and

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_{\alpha\beta}) \to \varphi_\beta(U_{\alpha\beta})$$

is a diffeomorphism. One can also work with equivalence classes of atlases rather than a maximal atlas: two atlases being equivalent if and only if their union is an atlas. We often write M^n for an n-dimensional manifold.

A smooth manifold M is topologised by making the φ_α into homeomorphisms. Equivalently, a subset $U \subset M$ is **open** if and only if $\varphi_\alpha(U \cap U_\alpha) \subset \mathbb{R}^n$ is open for all $\alpha \in A$. We restrict our attention to manifolds where this topology is **Hausdorff** and **second-countable**. The latter condition simply says that the manifold admits a countable atlas. The former condition says that any two distinct points can be separated by disjoint open subsets. We say that a manifold is **compact** if every atlas has a finite sub-atlas.

Any open subset of $\mathbb{R}^n$ (including $\mathbb{R}^n$ itself) is an n-dimensional smooth manifold. This shows that the set $GL(n, \mathbb{R})$ of invertible $n \times n$ matrices is an n^2-dimensional manifold.

If $F : \mathbb{R}^n \to \mathbb{R}^m$ (possibly defined only in some open subset $U \subset \mathbb{R}^n$) is smooth and $0 \in \mathbb{R}^m$ is a **regular value** (i.e., the derivative map $DF_a : \mathbb{R}^n \to \mathbb{R}^m$ is surjective for all a with $F(a) = 0$), then

$$F^{-1}(0) = \{a \in \mathbb{R}^m \mid F(a) = 0\} \subset \mathbb{R}^n$$

is an $(n-m)$-dimensional manifold. This shows that the sphere

$$S^n := \left\{\mathbf{x} \in \mathbb{R}^{n+1} \,\middle|\, \|\mathbf{x}\| = 1\right\} \subset \mathbb{R}^{n+1}$$

is an n-dimensional smooth manifold. Similarly, we can show that

$$SL(n, \mathbb{R}) := \{A \in GL(n, \mathbb{R}) \mid \det A = 1\} \subset \mathbb{R}^{n^2}$$

and

$$O(n) := \left\{A \in GL(n, \mathbb{R}) \,\middle|\, AA^T = \mathbb{1}\right\} \subset \mathbb{R}^{n^2}$$

are smooth manifolds of dimensions $n^2 - 1$ and $n(n-1)/2$, respectively.

If M^m and N^n are smooth manifolds, the cartesian product $M \times N$ is an $(m+n)$-dimensional manifold. Indeed, if $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ is an atlas for M and $\{(V_i, \psi_i)\}_{i\in B}$ is an atlas for N, then $\{(U_\alpha \times V_i, \varphi_\alpha \times \psi_i)\}_{(\alpha,i)\in A\times B}$ is an atlas for $M \times N$.

Smooth maps

Let M^m and N^n be smooth manifolds. A map $F : M \to N$ is **smooth** if for each $a \in M$ and chart (U, φ) on M with $a \in U$ and chart (V, ψ) on N with $F(a) \in V$, the composition

$$\psi \circ F \circ \varphi^{-1} : \varphi(F^{-1}(V) \cap U) \to \mathbb{R}^n$$

is smooth as a map from an open subset of $\mathbb{R}^m$ to $\mathbb{R}^n$. Smooth maps are continuous relative to the manifold topology. A smooth map $F : M \to N$ is a **diffeomorphism** if it is bijective with a smooth inverse.

The set $C^\infty(M)$ of **smooth functions** $M \to \mathbb{R}$ is a fundamental object in the course. Sums and products of smooth functions are smooth, making $C^\infty(M)$ into a ring. A smooth map $F : M \to N$ defines a ring homomorphism $F^* : C^\infty(N) \to C^\infty(M)$ by $F^* f = f \circ F$ for all $f \in C^\infty(N)$. This homomorphism is called the **pull-back** by F.

A **Lie group** is a smooth manifold G which is also a group and such that group multiplication $\mu : G \times G \to G$ and group inversion $\iota : G \to G$ are smooth maps. If G and H are Lie groups, then a **Lie group homomorphism** is a smooth map $G \to H$ which is also a group homomorphism.

Vector bundles

A **rank-r real vector bundle** over an n-dimensional manifold M, is an $(n + r)$-dimensional manifold E with a smooth surjection $\pi : E \to M$ with **fibres** $\pi^{-1}(a) \cong \mathbb{R}^r$ for all $a \in M$ and subject to **local triviality**: M admits an open cover $\{U_\alpha\}_{\alpha \in A}$ relative to which there exist diffeomorphisms

$$\psi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{R}^r$$

with $\psi_\alpha|_{\pi^{-1}(a)} : \pi^{-1}(a) \xrightarrow{\cong} \{a\} \times \mathbb{R}^r$ an isomorphism of vector spaces and such that on $U_{\alpha\beta} \neq \emptyset$,

$$\psi_\beta \circ \psi_\alpha^{-1} : U_{\alpha\beta} \times \mathbb{R}^r \to U_{\alpha\beta} \times \mathbb{R}^r$$

is given by $(a, \mathbf{v}) \mapsto (a, g_{\alpha\beta}(a)\mathbf{v})$ for some **transition functions**

$$g_{\alpha\beta} : U_{\alpha\beta} \to GL(r, \mathbb{R}),$$

which satisfy the **cocycle conditions**

(1) $g_{\alpha\alpha}(a) = \mathbb{1}$ for all $a \in U_\alpha$,
(2) $g_{\alpha\beta}(a) g_{\beta\alpha}(a) = \mathbb{1}$ for all $a \in U_{\alpha\beta}$, and
(3) $g_{\alpha\beta}(a) g_{\beta\gamma}(a) g_{\gamma\alpha}(a) = \mathbb{1}$ for all $a \in U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$.

A vector bundle can be reconstructed from an open cover $\{U_\alpha\}$ and transition functions $\{g_{\alpha\beta} : U_{\alpha\beta} \to GL(r, \mathbb{R})\}$ satisfying the cocycle conditions.

The **trivial bundle** $E = M \times \mathbb{R}^k$ is a vector bundle. Every bundle is locally trivial.

A **bundle map** from a vector bundle $E \xrightarrow{p} M$ to a vector bundle $F \xrightarrow{q} N$ is a commuting square

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d, "p"'] & F \arrow[d, "q"] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

where $\Phi : E \to F$ and $\varphi : M \to N$ are smooth maps and such that for all $a \in M$, $\Phi|_{\pi^{-1}(a)} : E_a \to F_{\varphi(a)}$ is a linear map. We say that Φ **covers** φ. An important special case of a bundle map is when $M = N$ and $\varphi = \mathrm{id}_M$, which we write as a commuting triangle

$$\begin{tikzcd} E \arrow[rr, "\Phi"] \arrow[dr, "p"'] & & F \arrow[dl, "q"] \\ & M & \end{tikzcd}$$

or often simply as $\Phi : E \to F$. We also say that Φ *covers the identity*. An invertible bundle map whose inverse is also a bundle map is a **bundle isomorphism**.

A bundle $E \to M$ is **trivial** if there is bundle isomorphism $E \to M \times \mathbb{R}^k$.

The **tangent bundle** TM has typical fibre T_aM, the **tangent space at** a, defined in any of the following equivalent ways:

$$\begin{aligned} T_aM &= \{X_a : C^\infty(M) \to \mathbb{R} \mid X_a(fg) = f(a)X_a(g) + g(a)X_a(f)\} \\ &= \left\{c'(0) : C^\infty(M) \to \mathbb{R} \,\middle|\, c'(0)(f) = \frac{d}{dt}f(c(t))\Big|_{t=0},\ c : (-\varepsilon,\varepsilon) \xrightarrow{C^\infty} M,\ c(0) = a\right\}. \end{aligned}$$

If $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are two overlapping charts with local coordinates $(x^1, \dots, x^n)$ and $(y^1, \dots, y^n)$, respectively, the transition functions $g_{\alpha\beta}$ for the tangent bundle are given by the jacobian matrix of the change of coordinates.

The **cotangent bundle** T^*M has typical fibre T_a^*M, the **cotangent space at** a, defined as the dual vector space to the tangent space T_aM or, equivalently, as the quotient $C^\infty(M)/Z_a$, where $Z_a \subset C^\infty(M)$ are the functions whose derivative vanish at a. If $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are two overlapping charts with local coordinates $(x^1, \dots, x^n)$ and $(y^1, \dots, y^n)$, respectively, the transition functions $g_{\alpha\beta}$ for the cotangent bundle are given by the inverse transpose of the jacobian matrix of the change of coordinates.

Let $F : M \to N$ be a smooth map and let $a \in M$. Then the **derivative** DF_a **of** F **at** a is the linear map $DF_a : T_aM \to T_{F(a)}N$ defined by

$$DF_a(X_a)(f) = X_a(f \circ F)$$

for all $f \in C^\infty(N)$ and $X_a \in T_aM$. Equivalently, if $c : (-\varepsilon, \varepsilon) \to M$ is a smooth curve with $c(0) = a$ and $c'(0) \in T_aM$, then $DF_a c'(0) = (F \circ c)'(0) \in T_{F(a)}N$. The derivatives of F assemble to give a bundle map $DF : TM \to TN$ covering $F : M \to N$. Other typical notations for DF are TF and F_*. This latter notation is also called the **push-forward** by F. By contrast, a smooth map $F : M \to N$ does *not* induce a bundle map of the cotangent bundles.

A smooth map $F : M \to N$ is an **immersion** if $DF_a : T_aM \to T_{F(a)}N$ is injective for all $a \in M$ and it is a **submersion** if DF_a is surjective. An immersion $F : M \to N$ which is also a homeomorphism onto its image is an **embedding**. The image of an embedding is an **(embedded) submanifold**.

If $F : M^m \to N^n$ is a smooth map. A point $c \in N$ is a **regular value** of F if DF_a is surjective for all $a \in F^{-1}(c)$. If $c \in N$ is a regular value of F then $F^{-1}(c)$ is an embedded submanifold of M of dimension $m - n$ and for all $a \in F^{-1}(c)$, the tangent space $T_aF^{-1}(c) = \ker DF_a \subset T_aM$.

Sections

Let $E \xrightarrow{\pi} M$ be a vector bundle. A **(smooth) section** of E is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. All this means is that s assigns to every $a \in M$ a vector on the fibre E_a at that point. Let $\Gamma(E)$ denote the space of sections. It is a $C^\infty(M)$-module: that is, if $f \in C^\infty(M)$ and $s \in \Gamma(E)$, then $fs \in \Gamma(E)$, where $fs(a) = f(a)s(a)$. The **zero section** is the section $z : M \to E$ such that $z(a) = 0 \in E_a$. The image of any section $s : M \to E$ is an embedded submanifold of E as is any one of the fibres.

A bundle map $\Phi : E \to F$ (covering the identity) between vector bundles $E \to M$ and $F \to M$ induces a $C^\infty(M)$-linear map $\Phi_* : \Gamma(E) \to \Gamma(F)$ by $\Phi_*(s) = \Phi \circ s$. Conversely, any $C^\infty(M)$-linear map $\Gamma(E) \to \Gamma(F)$ is of the form Φ_* for some bundle map $\Phi : E \to F$.

Let $E \to M$ be a rank-k real vector bundle and (U, ψ) a local trivialisation, with $\psi : E|_U \xrightarrow{\cong} U \times \mathbb{R}^k$. Then we define local sections $s_1, \dots, s_k : U \to E|_U$ by $\psi(s_i(a)) = (a, e_i)$ for all $a \in U$, where $e_i \in \mathbb{R}^k$ are the elementary vectors. Since $(e_1, \dots, e_k)$ are a basis for $\mathbb{R}^k$, $(s_1(a), \dots, s_k(a))$ are a basis for E_a for all $a \in U$. We say that $(s_1, \dots, s_k)$ is a **local frame** for $E|_U$. A bundle is trivial if and only if it admits a **global frame**; that is, $s_1, \dots, s_k \in \Gamma(E)$ such that $s_1(a), \dots, s_k(a)$ is a basis for E_a for all $a \in M$.

Vector fields and one-forms

Sections of TM are called **vector fields** and we write $\mathscr{X}(M) := \Gamma(TM)$, whereas sections of T^*M are called **one-forms** and we write $\Omega^1(M) := \Gamma(T^*M)$. There is a $C^\infty(M)$-bilinear dual pairing

$$\langle -, - \rangle : \Omega^1(M) \times \mathscr{X}(M) \to C^\infty(M)$$

where if $X \in \mathscr{X}(M)$ and $\alpha \in \Omega^1(M)$, then for all $a \in M$,

$$\langle \alpha, X \rangle (a) = \alpha(a)(X(a)) \in \mathbb{R}.$$

Often we write $\langle \alpha, X \rangle$ simply as $\alpha(X)$. This pairing is non-degenerate, in the sense that if $\alpha, \beta \in \Omega^1(M)$ are such that $\alpha(X) = \beta(X)$ for all $X \in \mathscr{X}(M)$, then $\alpha = \beta$, and conversely if $X, Y \in \mathscr{X}(M)$ are such that $\alpha(X) = \alpha(Y)$ for all $\alpha \in \Omega^1(M)$, then $X = Y$.

If (U, φ) is a chart on M with local coordinates $(x^1, \ldots, x^n)$, a vector field X on U can be written as

$$X = \sum_{i=1}^n X^i(x) \frac{\partial}{\partial x^i} \qquad \text{for some} \qquad X^i \in C^\infty(U).$$

If $f \in C^\infty(M)$, we define its **derivative** $df \in \Omega^1(M)$ by $df(X) = X(f)$, for all $X \in \mathscr{X}(M)$, where $X(f)(a) = X_a(f)$ for all $a \in M$ or, in terms of local coordinates,

$$X(f) = \sum_{i=1}^n X^i(x) \frac{\partial f}{\partial x^i},$$

where, in a flagrant yet commonplace abuse of notation,

$$\frac{\partial f}{\partial x^i} := D_i(f \circ \varphi^{-1}),$$

where D_i means the partial derivative with respect to the ith coordinate on $\mathbb{R}^n$.

Vector fields are precisely the **derivations** of $C^\infty(M)$; that is, the $\mathbb{R}$-linear maps $X : C^\infty(M) \to C^\infty(M)$ such that

$$X(fg) = fX(g) + gX(f).$$

If $X, Y \in \mathscr{X}(M)$, we define their **Lie bracket** $[X, Y] \in \mathscr{X}(M)$ by

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

for all $f \in C^\infty(M)$. The Lie bracket is $\mathbb{R}$-bilinear, skewsymmetric and satisfies the **Jacobi identity**

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

and hence gives $\mathscr{X}(M)$ the structure of a **Lie algebra**. The Lie bracket is **not** $C^\infty(M)$-linear: indeed, if $f \in C^\infty(M)$, then

$$[X, fY] = X(f)Y + f[X, Y].$$

A **bump function centred at** $a \in M$ is a smooth function $\theta : M \to \mathbb{R}$ such that there are open subsets $V \subset U \subset M$ with $a \in V$, such that θ is identically 1 on V and with support

$$\operatorname{supp} \theta := \overline{\{a \in M \mid \theta(a) \neq 0\}} \subset U.$$

If (U, φ) is a chart on M we may extend a smooth function $f : U \to \mathbb{R}$ to an element in $C^\infty(M)$ by multiplying by a bump function θ with support inside U. This normally modifies the function in U, but for each $a \in U$, we can find a bump function centred at a so that the modified function coincides with f in a neighbourhood of a.

In particular, if $(x^1, \ldots, x^n)$ are local coordinates in U, then we may extend them to global functions on M and we can consider their derivatives dx^i. A one-form α on U can be written as

$$\alpha = \sum_{i=1}^n \alpha_i(x) dx^i$$

for some $\alpha_i \in C^\infty(U)$. The dual pairing between one-forms and vector fields is such that

$$\alpha(X) = \sum_i \alpha_i X^i \qquad \text{and hence} \qquad df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i.$$

A smooth map $F : M \to N$ defines a $C^\infty(M)$-linear map $F^* : \Omega^1(N) \to \Omega^1(M)$ by

$$(F^*\alpha)(a) = \alpha(F(a)) \circ DF_a,$$

called the **pull-back** of α by F. If $\alpha = df$, for some $f \in C^\infty(N)$, then

$$F^* df = dF^* f = d(f \circ F).$$

Vector fields geometrise ordinary differential equations. A vector field $X \in \mathcal{X}(M)$ defines a **local flow**: a one-parameter family $\{\phi_t : M \to M\}$ of diffeomorphisms, for $t \in (-\varepsilon, \varepsilon)$, such that whenever $s, t, s+t \in (-\varepsilon, \varepsilon)$, $\phi_t \circ \phi_s = \phi_{s+t}$ and such that for all $a \in M$, the smooth curve $c : (-\varepsilon, \varepsilon) \to M$ defined by $c(t) := \phi_t(a)$ is an **integral curve** of X; that is, $Dc_t\left(\frac{d}{dt}\right) = X_{c(t)}$. In particular, $X_a = c'(0)$.

The local flow of a vector field $X \in \mathcal{X}(M)$ allows us to define the **Lie derivative** L_X along the vector field. If $f \in C^\infty(M)$, $Y \in \mathcal{X}(M)$ and $\alpha \in \Omega^1(M)$, we have

$$\begin{aligned} L_X f &:= \tfrac{d}{dt}\phi_t^* f\Big|_{t=0} = \tfrac{d}{dt}(f \circ \phi_t)\Big|_{t=0} = X(f) \\ L_X Y &:= \tfrac{d}{dt}(\phi_{-t})_* Y\Big|_{t=0} = [X, Y] \\ L_X \alpha &:= \tfrac{d}{dt}\phi_t^* \alpha\Big|_{t=0}, \end{aligned}$$

so that for all $Y \in \mathcal{X}(M)$,

$$(L_X\alpha)(Y) = L_X(\alpha(Y)) - \alpha(L_X Y) = X(\alpha(Y)) - \alpha([X, Y]).$$

Linear algebra of vector bundles

Natural constructions on vector spaces, such as $\oplus$, $\otimes$, Hom,... can be applied to vector bundles. Let $E \to M$ and $F \to M$ be vector bundles of ranks k and ℓ, respectively. Without loss of generality we may assume that they are simultaneously trivialised relative to some open cover $\{U_\alpha\}_{\alpha \in A}$ of M with transition functions $g^E_{\alpha\beta}$ and $g^F_{\alpha\beta}$, respectively, on non-empty overlaps.

The **Whitney sum** $E \oplus F \to M$ is the rank-$(k+\ell)$ vector bundle with typical fibre $(E \oplus F)_a = E_a \oplus F_a$ and transition functions $g^{E\oplus F}_{\alpha\beta}(a) = \varrho_\oplus(g^E_{\alpha\beta}(a), g^F_{\alpha\beta}(a))$ for all $a \in U_{\alpha\beta}$, where

$$\varrho_\oplus : GL(k, \mathbb{R}) \times GL(\ell, \mathbb{R}) \to GL(k+\ell, \mathbb{R})$$

is the Lie group homomorphism defined by

$$\varrho_\oplus(A, B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in GL(k+\ell, \mathbb{R}).$$

The **tensor bundle** $E \otimes F \to M$ is the rank-$(k\ell)$ vector bundle with typical fibre $(E \otimes F)_a = E_a \otimes F_a$ and transition functions $g^{E\otimes F}_{\alpha\beta}(a) = \varrho_\otimes(g^E_{\alpha\beta}(a), g^F_{\alpha\beta}(a))$ for all $a \in U_{\alpha\beta}$, where

$$\varrho_\otimes : GL(k, \mathbb{R}) \times GL(\ell, \mathbb{R}) \to GL(k\ell, \mathbb{R})$$

is the Lie group homomorphism defined by

$$\varrho_\otimes(A, B) = A \otimes B \in GL(k\ell, \mathbb{R}),$$

where $(A \otimes B)(\mathbf{v} \otimes \mathbf{w}) = A\mathbf{v} \otimes B\mathbf{w}$.

The **Hom bundle** $\mathrm{Hom}(E, F) \to M$ is the rank-$(k\ell)$ vector bundle with typical fibre $\mathrm{Hom}(E, F)_a = \mathrm{Hom}(E_a, F_a)$ and transition functions $g^{\mathrm{Hom}(E,F)}_{\alpha\beta}(a) = \varrho_{\mathrm{Hom}}(g^E_{\alpha\beta}(a), g^F_{\alpha\beta}(a))$ for all $a \in U_{\alpha\beta}$, where

$$\varrho_{\mathrm{Hom}} : GL(k, \mathbb{R}) \times GL(\ell, \mathbb{R}) \to GL(k\ell, \mathbb{R})$$

is the Lie group homomorphism defined by

$$\varrho_{\mathrm{Hom}}(A,B)(\varphi) = B \circ \varphi \circ A^{-1},$$

for $\varphi : \mathbb{R}^k \to \mathbb{R}^\ell$ is a linear map.

The **dual bundle** $E^* \to M$ is the rank-k with typical fibre $(E^*)_a = (E_a)^*$ and transition functions $g^{E^*}_{\alpha\beta}(a) = \varrho_{\mathrm{dual}}(g_{\alpha\beta}(a))$, where

$$\varrho_{\mathrm{dual}} : GL(k,\mathbb{R}) \to GL(k,\mathbb{R})$$

is the Lie group homomorphism defined by

$$\varrho_{\mathrm{dual}}(A) = (A^T)^{-1}.$$

Iterating these constructions starting with the tangent bundle TM we can construct the (r,s)**-tensor bundle**

$$T^r_s(M) := \underbrace{TM \otimes \cdots \otimes TM}_{r \text{ times}} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{s \text{ times}}.$$

Sections of $T^r_s(M)$ are called (r,s)**-tensor fields** and they can be equivalently be thought of as $C^\infty(M)$-multilinear maps

$$\underbrace{\mathscr{X}(M) \times \cdots \times \mathscr{X}(M)}_{s \text{ times}} \times \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{r \text{ times}} \to C^\infty(M).$$

The Lie derivative extends naturally to tensor fields using the **product rule**: if $T \in T^r_s(M)$ and $X \in \mathscr{X}(M)$, then $L_X T \in T^r_s(M)$ is such that for all $Y_1, \dots, Y_s \in \mathscr{X}(M)$ and $\alpha_1, \dots, \alpha_r \in \Omega^1(M)$,

$$\begin{aligned}(L_X T)(Y_1, \dots, Y_s, \alpha_1, \dots, \alpha_r) &= X T(Y_1, \dots, Y_s, \alpha_1, \dots, \alpha_r) \\ &\quad - \sum_{i=1}^s T(Y_1, \dots, [X, Y_i], \dots, Y_s, \alpha_1, \dots, \alpha_r) \\ &\quad - \sum_{j=1}^r T(Y_1, \dots, Y_s, \alpha_1, \dots, L_X \alpha_j, \dots, \alpha_r).\end{aligned}$$

If $X, Y \in \mathscr{X}(M)$, then acting on $T^r_s(M)$, we have that $[L_X, L_Y] = L_{[X,Y]}$.

Differential forms

A **differential** p**-form** is a section of the p-th exterior power $\Lambda^p T^*M$ of the cotangent bundle. Equivalently, it is a $C^\infty(M)$-multilinear alternating map

$$\underbrace{\mathscr{X}(M) \times \cdots \times \mathscr{X}(M)}_{p \text{ times}} \to C^\infty(M).$$

The vector space of differential p-forms is denoted $\Omega^p(M)$, with the understanding that $\Omega^0(M) = C^\infty(M)$ and $\Omega^1(M)$ are the one-forms defined earlier. For each $p = 0, \dots, n = \dim M$, $\Omega^p(M)$ is a $C^\infty(M)$-module. The entirety of differential forms $\Omega^\bullet(M) = \bigoplus_{p=0}^n \Omega^p(M)$ is a **differential graded algebra** or **dga**. This means that $\Omega^\bullet(M)$ has a (graded, supercommutative, associative) **wedge product**

$$\wedge : \Omega^p(M) \times \Omega^q(M) \to \Omega^{p+q}(M)$$

obeying

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$$

for $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$. If $f \in \Omega^0(M) = C^\infty(M)$ and $\alpha \in \Omega^p(M)$, then $f \wedge \alpha = f\alpha$. It also has a **differential**

$$d : \Omega^p(M) \to \Omega^{p+1}(M)$$

satisfying $d^2 = 0$ and

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta,$$

where $\alpha \in \Omega^p(M)$. The differential $d : \Omega^0(M) \to \Omega^1(M)$ agrees with the derivative $f \mapsto df$ of a smooth function. If $\alpha \in \Omega^1(M)$, then $d\alpha \in \Omega^2(M)$ is given by

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]).$$

A smooth map $F : M \to N$ defines a dga-homomorphism $F^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$; that is, $C^\infty(M)$-linear maps $F^* : \Omega^p(N) \to \Omega^p(M)$ for each p, satisfying

$$F^*(\alpha \wedge \beta) = F^*\alpha \wedge F^*\beta \qquad \text{and} \qquad F^* d\alpha = dF^*\alpha.$$

If $X \in \mathscr{X}(M)$ we define the **contraction** $\imath_X : \Omega^p(M) \to \Omega^{p-1}$ by

$$(\imath_X\alpha)(X_1, \dots, X_{p-1}) = \alpha(X, X_1, \dots, X_{p-1}),$$

for $X_1, \dots, X_{p-1} \in \mathscr{X}(M)$ and $\alpha \in \Omega^p(M)$. It satisfies

$$\imath_X(\alpha \wedge \beta) = \imath_X\alpha \wedge \beta + (-1)^p \alpha \wedge \imath_X\beta,$$

for $\alpha \in \Omega^p(M)$. Notice that if $X, Y \in \mathscr{X}(M)$, then $\imath_X \circ \imath_Y = -\imath_Y \circ \imath_X$.

We define the **Lie derivative** of $\alpha \in \Omega^p(M)$ along $X \in \mathscr{X}(M)$ by

$$L_X\alpha := \frac{d}{dt}\phi_t^*\alpha\Big|_{t=0},$$

where $\{\phi_t\}$ is the local flow of X. The **Cartan formula** says that

$$L_X\alpha = d\imath_X\alpha + \imath_X d\alpha,$$

which imply that $[L_X, \imath_Y] = \imath_{[X,Y]}$ and $[L_X, L_Y] = L_{[X,Y]}$.

De Rham cohomology

A form $\alpha \in \Omega^p(M)$ is **closed** if $d\alpha = 0$ and **exact** if $\alpha = d\beta$ for some $\beta \in \Omega^{p-1}(M)$. Since $d^2 = 0$, exact forms are closed. The reverse inclusion is measured by the **de Rham cohomology** $H^\bullet_{dR}(M)$, defined by

$$H^p_{dR}(M) = \frac{\text{closed } p\text{-forms}}{\text{exact } p\text{-forms}}.$$

The cohomology class of a closed form α is denoted $[\alpha]$, with $[\alpha] = [\alpha + d\beta]$. The vector space $H^\bullet_{dR}(M) = \bigoplus_{p=0}^n H^p_{dR}(M)$ is a (graded, supercommutative) ring with multiplication

$$\wedge : H^p_{dR}(M) \to H^q_{dR}(M) \to H^{p+q}_{dR}(M),$$

given by $[\alpha] \wedge [\beta] = [\alpha \wedge \beta]$.

If $F : M \to N$ is a smooth map, the pull-back of differential forms defines a ring homomorphism

$$F^* : H^p_{dR}(N) \to H^p_{dR}(M)$$

by $F^*[\alpha] = [F^*\alpha]$.

Let $F : M \times [0,1] \to N$ be a smooth map and write $F_t(a) := F(a,t)$ so that $F_t : M \to N$ is a smooth map for all $t \in [0,1]$. Then at the level of de Rham cohomology, $F_0^* = F_1^*$. This is the **homotopy invariance** of the de Rham cohomology, which has two immediate consequences: the **Poincaré Lemma**

$$H^p_{dR}(M) \cong \begin{cases} \mathbb{R}, & p = 0 \\ 0, & p > 0. \end{cases}$$

and its generalisation

$$H^p_{dR}(M \times \mathbb{R}) \cong H^p_{dR}(M).$$

Integration

A smooth manifold M^n is **orientable** if there exists a nowhere vanishing n-form $\mu \in \Omega^n(M)$. Equivalently, M is orientable if it admits an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ such that the jacobian matrix of the transition functions $\varphi_\beta \circ \varphi_\alpha^{-1}$ has positive determinant.

Let M be a smooth manifold and $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be an open cover. A **partition of unity** subordinate to $\mathcal{U}$ is a collection $\{\rho_i : M \to [0,1]\}_{i\in I}$ of smooth functions whose supports $\operatorname{supp}\rho_i \subset U_\alpha$ for some $\alpha = \alpha(i)$, such that every $a \in M$ has a neighbourhood U which intersects finitely many of the $\{\operatorname{supp}\rho_i\}$ and such that $\sum_{i\in I}\rho_i(a) = 1$ for all $a \in M$ – this sum being finite at every point. Partitions of unity exist on manifolds admitting a countable atlas, which are the manifolds we are considering in this course.

Partitions of unity are used in defining **integration**: an $\mathbb{R}$-linear map

$$\int_M : \Omega^n_c(M) \to \mathbb{R}$$

where $\Omega^k_c(M)$ denotes the **compactly supported** k-forms on an orientable manifold M^n. **Stokes Theorem** says that for all $\omega \in \Omega^{n-1}_c(M)$,

$$\int_M d\omega = 0.$$

As a consequence of this result, if M^n is compact and orientable, then $H^n_{dR}(M) \neq 0$.

Integration does not detect submanifolds of positive codimension. This allows us to calculate integrals by using local coordinates valid in a dense open set (e.g., spherical polar coordinates on the sphere).

A diffeomorphism $F : M^n \to N^n$ is **orientation-preserving** if $\det DF_a > 0$ for all $a \in M$. If $F : M \to N$ is an orientation-preserving diffeomorphism and $\omega \in \Omega^n_c(N)$, then $F^*\omega \in \Omega^n_c(M)$ and

$$\int_M F^*\omega = \int_N \omega.$$

Let $\mathbb{R}^n_+ := \left\{(x^1,\dots,x^n) \in \mathbb{R}^n \mid x^n \geq 0\right\}$ and $\partial\mathbb{R}^n_+ := \left\{(x^1,\dots,x^{n-1},0) \in \mathbb{R}^n\right\}$. Replacing $\mathbb{R}^n$ for $\mathbb{R}^n_+$ in the definition of a manifold yields the notion of an n-dimensional **manifold with boundary**. The **boundary** of M^n with atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ is

$$\partial M := \{a \in M \mid \varphi_\alpha(a) \in \partial\mathbb{R}^n_+, \exists\alpha \in A\}.$$

It is an embedded submanifold of M.

If M is orientable, then so is ∂M. Let $(x^1,\dots,x^n)$ be local coordinates for M with $x^n \geq 0$ and where the boundary is $x^n = 0$ and suppose that the orientation on M agrees with $dx^1 \wedge \cdots \wedge dx^n$. Then the **induced orientation** on ∂M is $-dx^1 \wedge \cdots \wedge dx^{n-1}$, which agrees with the notion of *outward normal*.

Let M^n be an oriented manifold with boundary and $\omega \in \Omega^{n-1}_c(M)$. Then the **generalised Stokes Theorem** states that relative to the induced orientation on the boundary,

$$\int_M d\omega = \int_{\partial M} \omega.$$