Duality maps and connecting homomorphisms

We let M be a oriented n-manifold that is covered by two open subsets U and V. In the previous video we constructed a Mayer-Vietoris sequence involving the compactly supported cohomology groups of M, U, V and $U \cap V$. There is also a Mayer-Vietoris sequence relating the singular homology groups of M, U, V, and $U \cap V$; its connecting homomorphism is a map

$$\partial \;:\; H_k(M;\mathbb{Z}) \;\longrightarrow\; H_{k-1}(U\cap V;\mathbb{Z}) \ .$$

The purpose of this section is to show that the connecting homomorphisms for the two Mayer-Vietoris sequences are compatible under the duality maps, up to a sign. As a corollary, we will record that if the duality maps of U, V and $U \cap V$ are isomorphisms in all dimensions, then also the duality maps $D_M : H^i_{\mathrm{comp}}(M;\mathbb{Z}) \longrightarrow H_{n-i}(M;\mathbb{Z})$ for M are isomorphisms.

Proposition 1. *Let M be an n-manifold, and let U and V be open subsets such that $U \cup V = M$. Then the following square commutes up to the sign $(-1)^{i+1}$:*

$$\begin{tikzcd}
H^i_{\mathrm{comp}}(M) \arrow[r, "\partial"] \arrow[d, "D_M"'] & H^{i+1}_{\mathrm{comp}}(U\cap V) \arrow[d, "D_{U\cap V}"] \\
H_{n-i}(M) \arrow[r, "\partial"'] & H_{n-i-1}(U\cap V)
\end{tikzcd}$$

Proof. Every class in $H^i_{\mathrm{comp}}(M)$ is of the form $\lambda_C(\alpha)$ for some compact subset C of M and some class $\alpha \in H^i(M, M \setminus C)$. We write $C = K \cup L$ for some compact subset K of U and some compact subset L of V.

$$\begin{tikzcd}
H^i(M, M\setminus(K\cup L)) \arrow[r, "\partial"] \arrow[d, "\lambda_{K\cup L}"] \arrow[dd, bend right=60, "\mu_{K\cup L}\cap -"'] & H^{i+1}(M, M\setminus(K\cap L)) \arrow[r, "\text{excision}", "\cong"'] & H^{i+1}(U\cap V, (U\cap V)\setminus(K\cap L)) \arrow[d, "\lambda_{K\cap L}"'] \arrow[dd, bend left=60, "\mu_{K\cap L}\cap -"] \\
H^i_{\mathrm{comp}}(M) \arrow[rr, "\partial"] \arrow[d, "D_M"] & & H^{i+1}_{\mathrm{comp}}(U\cap V) \arrow[d, "D_{U\cap V}"'] \\
H_{n-i}(M) \arrow[rr, "\partial"'] & & H_{n-i-1}(U\cap V)
\end{tikzcd}$$

We are thus reduced to showing that the following square commutes up to the sign $(-1)^{i+1}$:

$$\begin{tikzcd}
H^i(M, M\setminus(K\cup L)) \arrow[r, "\partial"] \arrow[d, "\mu_{K\cup L}\cap -"'] & H^{i+1}(M, M\setminus(K\cap L)) \arrow[d, "\mu_{K\cap L}\cap -"] \\
H_{n-i}(M) \arrow[r, "\partial"'] & H_{n-i-1}(U\cap V)
\end{tikzcd}$$

The commutativity of this diagram is, unfortunately, far from tautological.

We will suppress the singular complex functor $\mathcal{S}(-)$ from the notation for the singular (co)chain complexes. We let $f \in C^i(M, M \setminus (K \cup L))$ be a cocycle that is supported on $K \cup L$. We write $f = f_K - f_L$ for cochains $f_K \in C^i(M, M \setminus K)$ and $f_L \in C^i(M, M \setminus L)$, i.e., so that f_K is supported on K, and f_L is supported on L; for example, we may set

$$f_K(\psi) \;=\; \begin{cases} f(\psi) & \text{if } \psi : \nabla^i \longrightarrow M \text{ does not have image in } M\setminus K, \\ 0 & \text{if } \psi : \nabla^i \longrightarrow M \text{ has image in } M \setminus K. \end{cases}$$

and then check that $f_L = f_K - f$ is indeed supported on L. One should beware that we will typically not be able to arrange that f_K and f_L are cocycles! But in any case,

$$df_K - df_L \ = \ d(f_K - f_L) \ = \ df \ = \ 0 \ ,$$

so the $(n+1)$-cocycle $df_K = df_L$ is supported on K and on L; one should beware that this does *not* imply that it is supported on $K \cap L$!

Cochains on M that are supported on $K \cap L$ are supported on K and on L, but not vice versa. The theorem of small simplices shows that the inclusion

$$C^*(M, M \setminus (K \cap L)) \ \longrightarrow \ C^*(M, (M \setminus K) + (M \setminus L))$$

with target the subcomplex of $C^*(M)$ spanned by $C^*(M, M \setminus K)$ and $C^*(M, M \setminus L)$ is a quasi-isomorphism. So there is singular cochain $g \in C^{i+1}(M, M \setminus (K \cap L))$ supported on $K \cap L$ that is cohomologous in $C^*(M, (M \setminus K) + (M \setminus L))$ to $df_K = df_L$. This cocycle represents the effect of the connecting homomorphism, i.e.,

$$\partial[f] \ = \ [g] \qquad \text{in } H^{i+1}(M, M \setminus (K \cap L)).$$

The manifold M is covered by the three open subset $U \setminus L$, $U \cap V$ and $(V \setminus K)$. By another application of the theorem of small simplices to this covering, we can represent the orientation class $\mu_{K \cup L} \in H_n(M, M \setminus (K \cup L))$ of $K \cup L$ by a singular n-chain of the form

$$\psi_{U \setminus L} \ + \ \psi_{U \cap V} \ + \ \psi_{V \setminus K}$$

with $\psi_{U \setminus L} \in C_n(U \setminus L)$, $\psi_{U \cap V} \in C_n(U \cap V)$ and $\psi_{V \setminus K} \in C_n(V \setminus K)$. The chains $\psi_{U \setminus L}$ and $\psi_{V \setminus K}$ lie in the complement of $K \cap L$, so for every point $x \in K \cap L$, these two chains vanish in the relative chain complex $C_*(M, M \setminus (K \cap L))$. Hence

$$\mu_x \ = \ r_x^{K \cup L}(\mu_{K \cup L}) \ = \ r_x^{K \cup L}[\psi_{U \setminus L} + \psi_{U \cap V} + \psi_{V \setminus K}] \ = \ r_x^{K \cap L}[\psi_{U \cap V}] \ .$$

Since this holds for all points in $K \cap L$, we conclude that the residue class of $\psi_{U \cap V}$ represents the orientation class of $K \cap L$, i.e.,

$$\mu_{K \cap L} \ = \ [\psi_{U \cap V}] \ \in H_n(M, M \setminus (K \cap L)) \ .$$

Similarly, the residue class of $\psi_{U-L} + \psi_{U \cap V}$ in $C_*(M, M \setminus K)$ represents the orientation class μ_K.

Chasing the cohomology class $[f]$ one way around the square thus yields

$$\mu_{K \cap L} \cap \partial[f] \ = \ \mu_{K \cap L} \cap [g] \ = \ \mu_{K \cap L} \cap [df_K] \ = \ [\psi_{U \cap V} \cap (df_K)] \ = \ [(d\psi_{U \cap V}) \cap f_K] \ . \tag{2}$$

The last equation uses the boundary formula for chain level cap products

$$(d\psi_{U \cap V}) \cap f_K - \psi_{U \cap V} \cap (df_K) \ = \ (-1)^i \cdot d(\psi_{U \cap V} \cap f_K) \ .$$

To work out the other way around the square, we start from the relation

$$\begin{aligned} \mu_{K \cup L} \cap [f] \ &= \ [(\psi_{U \setminus L} + \psi_{U \cap V} + \psi_{V \setminus K}) \cap f] \\ &= \ [\psi_{U \setminus L} \cap f \ + \ \psi_{U \cap V} \cap f \ + \ \psi_{V \setminus K} \cap f] \ . \end{aligned}$$

The $(n-i)$-chain $\psi_{U \setminus L} \cap f$ lies in $C_*(U)$, and the $(n-i)$-chain $\psi_{U \cap V} \cap f + \psi_{V \setminus K} \cap f$ lies in $C_*(V)$; so this representation of $\mu_{K \cup L} \cap [f]$ is by a cycle in the complex of small simplices (with respect to the covering of M by U and V). The value of the connecting homomorphism $\partial : H_{n-i}(M) \longrightarrow H_{n-i-1}(U \cap V)$ on this class can thus be calculated as

$$\begin{aligned} \partial(\mu_{K \cup L} \cap [f]) \ &= \ [d(\psi_{U \setminus L} \cap f)] \\ &= \ (-1)^i \cdot [d(\psi_{U \setminus L}) \cap f] \\ &= \ (-1)^i \cdot [d(\psi_{U \setminus L}) \cap f_K] \ . \end{aligned} \tag{3}$$

The second relation is the coboundary formula for the cap product, combined with the fact that $df = 0$. The third equation uses that $f = f_K - f_L$, and that $d(\psi_{U\setminus L}) \cap f_L = 0$ because f_L is supposed on L whereas $d(\psi_{U\setminus L})$ has image in the complement of L.

Because $\psi_{U\setminus L} + \psi_{U\cap V} + \psi_{V\setminus K}$ is a cycle, we have

$$d(\psi_{U\setminus L} + \psi_{U\cap V}) \ = \ -d(\psi_{V\setminus K}) \ .$$

Because $\psi_{V\setminus K}$ lies in $C_*(M \setminus K)$, the same is true for its boundary $d(\psi_{V\setminus K})$. So $d(\psi_{V\setminus K}) \cap f_K = 0$ because f_K is supported on K. Hence

$$d(\psi_{U\setminus L} + \psi_{U\cap V}) \cap f_K \ = \ -d(\psi_{V\setminus K}) \cap f_K \ = \ 0 \ . \tag{4}$$

We can now put all the pieces together to deduce the desired relation

$$\begin{aligned} \mu_{K\cap L} \cap \partial[f] \quad {}_{(2)}\!\!&= \quad [(d\psi_{U\cap V}) \cap f_K] \\ {}_{(4)}\!\!&= \ -[(d\psi_{U\setminus L}) \cap f_K] \\ {}_{(3)}\!\!&= \ (-1)^{i+1} \cdot \partial(\mu_{K\cup L} \cap [f]) \ . \end{aligned}$$

□

Now we can compare the Mayer-Vietoris sequences for the covering $M = U \cup V$ via the large diagram:

$$\begin{tikzcd}
\dots \arrow[r] & H^i_{\mathrm{comp}}(U\cap V) \arrow[r, "{(\iota^U_{U\cap V},\iota^V_{U\cap V})}"] \arrow[d, "D_{U\cap V}"'] & H^i_{\mathrm{comp}}(U) \oplus H^i_{\mathrm{comp}}(V) \arrow[r, "{\left(\begin{smallmatrix}\iota^X_U \\ -\iota^X_V\end{smallmatrix}\right)}"] \arrow[d, "D_U \oplus D_V"'] & H^i_{\mathrm{comp}}(M) \arrow[r, "\partial"] \arrow[d, "D_M"] & H^{i+1}_{\mathrm{comp}}(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & \dots \\
\dots \arrow[r] & H_{n-i}(U\cap V) \arrow[r, "{(\mathrm{incl}_*,\mathrm{incl}_*)}"'] & H_{n-i}(U) \oplus H_{n-i}(V) \arrow[r, "{\left(\begin{smallmatrix}\mathrm{incl}_* \\ -\mathrm{incl}_*\end{smallmatrix}\right)}"'] & H_{n-i}(M) \arrow[r, "\partial"'] & H_{n-i-1}(U\cap V) \arrow[r] & \dots
\end{tikzcd}$$

Here all (co)homology groups are to be taken with integer coefficients, and the understanding is that U, V and $U\cap V$ are endowed with the restricted orientations. The left and middle squares commute by naturality of the duality map for open embeddings. The right square commutes up to sign, by Proposition 1.

Now we assume that the duality maps for oriented manifolds U, V, and $U \cap V$ are isomorphisms in all dimensions. Replacing a homomorphism by its additive inverse has no effect on the kernel and image of the homomorphism. So the 5-lemma still applies, and it lets us conclude:

Corollary 5. *Let M be an oriented n-manifold, and let U and V be open subsets with $M = U \cup V$. If the duality maps for U, V and $U \cap V$ are isomorphisms in all dimensions, then also the duality map for M is an isomorphism in all dimensions.*