Countable ascending unions

Proposition 1. *Let M be an oriented n-manifold that is the union of an ascending sequence*

$$U_0 \subset U_1 \subset U_2 \subset \ldots$$

of nested open subsets. If the duality map $D_{U_k} : H^i_{\mathrm{comp}}(U_k;\mathbb{Z}) \longrightarrow H_{n-i}(U_k;\mathbb{Z})$ is an isomorphism for all $k \geq 0$, then the duality map $D_M : H^i_{\mathrm{comp}}(M;\mathbb{Z}) \longrightarrow H_{n-i}(M;\mathbb{Z})$ is an isomorphism.

Proof. The essence of the argument we are about to give is that 'compactly supported cohomology and singular homology commute with ascending unions of open subsets'. But rather than making this slogan precise, we will directly show that the duality map for M is bijective.

Surjectivity. We consider any singular $(n-i)$-cycle $x = \sum a_j \cdot \psi_j \in C_{n-i}(\mathcal{S}(M);\mathbb{Z})$, where $a_j \in \mathbb{Z}$ and $\psi_j : \nabla^{n-i} \longrightarrow M$ are singular simplices. Since the sum is finite, the union of all the images $\psi_j(\nabla^{n-i})$ is a compact subset of M, and hence contained in U_k for some $k \geq 0$. Then x is also a cycle in the subcomplex $C_{n-i}(\mathcal{S}(U_k);\mathbb{Z})$. Since the duality map for U_k is an isomorphism, there is a class $\alpha \in H^i_{\mathrm{comp}}(U_k;\mathbb{Z})$ such that $D_{U_k}(\alpha) = [x]$. Naturality of the duality maps for the open embedding $U_k \longrightarrow M$ thus yields

$$D_M(\iota^M_{U_K}(\alpha)) \;=\; \mathrm{incl}_*(D_{U_k}(\alpha)) \;=\; \mathrm{incl}_*[x] \;=\; [x]$$

in $H_{n-i}(M;\mathbb{Z})$. This shows that the duality map D_M is surjective.

Injectivity. We consider any class $\alpha \in H^i_{\mathrm{comp}}(M;\mathbb{Z})$ in the kernel of the duality map, and we represent it by a compactly supported i-cocycle $f \in C^i(M;\mathbb{Z})$. We let K be a compact subset of M on which f is supported. Then $K \subset U_k$ for some $k \geq 0$, and we write $\beta \in H^i_{\mathrm{comp}}(U_k;\mathbb{Z})$ for the class of the restriction of f to the open subset U_k. Then

$$\alpha \;=\; \iota^M_{U_k}(\beta)$$

by the very definition of the homomorphism $\iota^M_{U_k}$. We deduce the relation

$$\mathrm{incl}_*(D_{U_k}(\beta)) \;=\; D_M(\iota^M_{U_k}(\beta)) \;=\; D_M(\alpha) \;=\; 0\ .$$

We represent the homology class $D_{U_k}(\beta)$ by an $(n-i)$-cycle $z \in C_{n-i}(\mathcal{S}(U_k);\mathbb{Z})$. Because the class $D_{U_k}(y)$ maps to zero in $H_{n-i}(M;\mathbb{Z})$, it is the boundary of an $(n-i+1)$-chain in M. This chain is of the form $\sum a_j \cdot \psi_j \in C_{n-i}(\mathcal{S}(M);\mathbb{Z})$, where $a_j \in \mathbb{Z}$ and $\psi_j : \nabla^{n-i+1} \longrightarrow M$ are singular simplices. Since the sum is finite, the union of all the images $\psi_j(\nabla^{n-i+1})$ is a compact subset of M, and hence contained in U_l for some $l \geq 0$. By increasing l, if necessary, we can assume that $l \geq k$. This means that already the image of $D_{U_k}(\beta)$ in the group $H_{n-i}(U_l;\mathbb{Z})$ is zero, and so

$$D_{U_l}(\iota^{U_l}_{U_k}(\beta)) \;=\; \mathrm{incl}_*(D_{U_k}(\beta)) \;=\; 0\ .$$

Because the duality map for U_l is an isomorphism, we conclude that $\iota^{U_l}_{U_k}(\beta)$. Hence also

$$\alpha \;=\; \iota^M_{U_k}(\beta) \;=\; \iota^M_{U_l}(\iota^{U_l}_{U_k}(\beta)) \;=\; 0\ .$$

This shows that the duality map for M is injective, and it concludes the proof. $\square$