**The Alexander-Whitney map is a chain map**

We let $A$ and be $B$ be simplicial abelian groups. In degree $n \geq$, the *Alexander-Whitney map*

$$\mathrm{AW}_n \ : \ A_n \otimes B_n \ \longrightarrow \ \bigoplus_{p+q=n} A_p \otimes B_q$$

is given by

$$\mathrm{AW}_n(a \otimes b) \ = \ \sum_{p+q=n} d^*_{\mathrm{front}}(a) \otimes d^*_{\mathrm{back}}(b) \ ,$$

where $d_{\mathrm{front}} : [p] \longrightarrow [p+q]$ and $d_{\mathrm{back}} : [q] \longrightarrow [p+q]$ are the morphisms of simplicial sets defined by

$$d_{\mathrm{front}}(i) \ = \ i \qquad \text{and} \qquad d_{\mathrm{back}}(i) \ = \ p+i \ .$$

The notation is somewhat abusing because the morphisms $d_{\mathrm{front}}$ and $d_{\mathrm{back}}$ depend on the pair $(p,q)$, but this dependence is not reflected in the notation.

**Proposition 1.** *Let $A$ and be $B$ be simplicial abelian groups. Then for varying $n$, the Alexander-Whitney maps $\mathrm{AW}_n$ form a chain map from the chain complex $C(A \otimes B)$ to the chain complex $C(A) \otimes C(B)$.*

*Proof.* For the course of this proof we write $\partial$ for the differential in the chain complex of a simplicial abelian group, in order to avoid confusion with the simplicial face maps $d_i^*$. We must show the relation

$$\mathrm{AW}_{n-1}(\partial(a \otimes b)) \ = \ \partial(\mathrm{AW}_n(a \otimes b)) \tag{2}$$

in the group $\bigoplus_{p=0}^{n-1} A_p \otimes B_{n-p-1}$, for all $n \geq 0$, $a \in A_n$ and $b \in B_n$. This calculation is, in a sense, 'dual' to the coboudary formula $d(f \cup g) = d(f) \cup g + (-1)^{|f|} \cdot f \cup d(g)$ for the cup product.

For the course of the calculation, we remove the notational ambiguity from the front and back operators, and we write

$$d_{\mathrm{front},p,q} \ : \ [p] \ \longrightarrow \ [p+q] \qquad \text{and} \qquad d_{\mathrm{back},p,q} \ : \ [q] \ \longrightarrow \ [p+q] \ .$$

To prove the relation we fix a number $p$ in the range $0 \leq p \leq n-1$ and we show that both sides of (2) have the same projection to the summand $A_p \otimes B_{n-p-1}$. The component in $A_p \otimes B_{n-p-1}$ of $\partial(\mathrm{AW}_n(a \otimes b))$ is

$$\partial(d^*_{\mathrm{front},p+1,n-p-1}(a)) \otimes d^*_{\mathrm{back},p+1,n+p-1}(b) \ + \ (-1)^p \cdot d^*_{\mathrm{front},p,n-p}(a) \otimes \partial(d^*_{\mathrm{back},p,n-p}(b)) \ ;$$

To simplify the notation we drop the argument $a \otimes b$ from the following calculation; so it is to be read as a equality between homomorphisms from $A_n \otimes B_n$ to $A_p \otimes B_{n-p-1}$.

$$(\partial \circ d^*_{\text{front},p+1,n-p-1}) \otimes d^*_{\text{back},p+1,n-p-1} + (-1)^p \cdot d^*_{\text{front},p,n-p} \otimes (\partial \circ d^*_{\text{back},p,n-p})$$

$$= \sum_{i=0}^{p+1} (-1)^i \cdot (d^*_i \circ d^*_{\text{front},p+1,n-p-1}) \otimes d^*_{\text{back},p+1,n-p-1} + (-1)^p \cdot \sum_{j=0}^{n-p} (-1)^j \cdot d^*_{\text{front},p,n-p} \otimes (d^*_j \circ d^*_{\text{back},p,n-p})$$

(3)

$$= \left( \sum_{i=0}^{p} (-1)^i \cdot (d_{\text{front},p+1,n-p-1} \circ d_i)^* \otimes d^*_{\text{back},p+1,n-p-1} \right) + \left( \sum_{j=1}^{n-p} (-1)^{p+j} \cdot d^*_{\text{front},p,n-p} \otimes (d_{\text{back},p,n-p} \circ d_j)^* \right)$$

(4)

$$= \left( \sum_{i=0}^{p} (-1)^i \cdot (d_i \circ d_{\text{front},p,n-p-1})^* \otimes (d_i \circ d_{\text{back},p,n-p-1})^* \right)$$

$$+ \left( \sum_{j=1}^{n-p} (-1)^{p+j} \cdot (d_{p+j} \circ d_{\text{front},p,n-p-1})^* \otimes (d_{p+j} \circ d_{\text{back},p,n-p-1})^* \right)$$

$$= \sum_{i=0}^{n} (-1)^i \cdot (d_i \circ d_{\text{front},p,n-1-p})^* \otimes (d_i \circ d_{\text{back},p,n-p-1})^*$$

$$= \sum_{i=0}^{n} (-1)^i \cdot (d^*_{\text{front},p,n-p-1} \circ d^*_i) \otimes (d^*_{\text{back},p,n-p-1} \circ d^*_i) \quad .$$

The last term is the component in $A_p \otimes B_{n-p-1}$ of $\mathrm{AW}_{n-1} \circ \partial$

Various steps in this calculation need justification. Equation (3) uses the contravariant functoriality of simplicial abelian groups; and we exploit the relations

$$\begin{aligned} d_{\text{front},p+1,n-p-1} \circ d_{p+1} &= d_{\text{front},p,n-p} \ : \ [p] \longrightarrow [n] \quad \text{and} \\ d_{\text{back},p,n-p} \circ d_0 &= d_{\text{back},p+1,n-p-1} \ : \ [n-p-1] \longrightarrow [n] \end{aligned}$$

to cancel the summand indexed by $i = p+1$ in the left sum with the summand indexed by $j = 0$ in the right sum. Equation (4) uses the relations between the simplicial face operators and the front and back operators $d_{\text{front}} : [p] \longrightarrow [p+q]$ and $d_{\text{back}} : [q] \longrightarrow [p+q]$ that we have already seen when verifying the coboundary formula for the cup product, namely:

$$d_i \circ d_{\text{front},p,q} = \left\{ \begin{array}{ll} d_{\text{front},p+1,q} \circ d_i & \text{if } 0 \le i \le p, \\ d_{\text{front},p,q+1} & \text{if } p \le i \le p+q \end{array} \right\} : [p-1] \longrightarrow [p+q]$$

$$d_{p+j} \circ d_{\text{back},p,q-1} = \left\{ \begin{array}{ll} d_{\text{back},p+1,q-1} & \text{if } 0 \le i \le p, \\ d_{\text{back},p,q} \circ d_j & \text{if } p \le i \le p+q \end{array} \right\} : [q-1] \longrightarrow [p+q] \ .$$

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