Poincaré duality

We have now developed all the necessary tools to prove the Poincaré duality theorem for oriented manifolds.

Theorem 1 (Poincaré duality). *For every oriented n-manifold M and all $i \geq 0$, the duality map*

$$D_M \;:\; H^i_{\mathrm{comp}}(M;\mathbb{Z}) \;\longrightarrow\; H_{n-i}(M;\mathbb{Z})$$

is an isomorphism.

Proof. We have already established the following facts:
(a) The duality map is an isomorphism for $M = \mathbb{R}^n$.
(b) If $M = U \cup V$ for open subsets U and V of M, and if the duality maps are isomorphisms for U, V and $U \cap V$, then it is also an isomorphism for M.
(c) If $M = \bigcup_{k \geq 0} U_k$ is the ascending union of a nested squence of open subsets, and if the duality maps are isomorphisms for all U_k, then it is also an isomorphism for M.

Claim A: The duality map is an isomorphism whenever M is homeomorphic to an open subset of $\mathbb{R}^n$. The euclidean topology on $\mathbb{R}^n$ has a countable basis consisting of open metric balls. So every open subset V of $\mathbb{R}^n$ can be written as $V = \bigcup_{j \geq 0} B_j$ where each B_j is an open metric ball. Every finite intersection of open metric balls is either empty or homeomorphic to $\mathbb{R}^n$. So the duality map is an isomorphism for every B_j and all finite intersections $B_{j_1} \cap \cdots \cap B_{j_l}$ by item (a). Item (b) and induction show that the duality map for the manifold

$$U_k \;=\; B_0 \cup \cdots \cup B_k$$

is an isomorphism for every $k \geq 0$. Because V is the ascending union of the open subsets U_k, the duality map for V is an isomorphism, by item (c).

Claim B: Let K be a compact subset of M, and let U be open subset of M for which Poincaré duality holds; then there is another open subset V of M for which Poincaré duality holds and such that

$$K \cup U \subset V \ .$$

Because K is compact, it can be covered by finitely many open subsets $U_1, \ldots, U_k$ of M, each of which is homeomorphic to $\mathbb{R}^n$. All intersections of some of the sets U_i and U are then homeomorphic to open subsets of $\mathbb{R}^n$, and so Poincaré duality holds for all U_i, all intersections of some U_i's, and all intersections of some U_i's with U. So Claim A and item (b) show that Poincaré duality holds for the open set

$$V \;=\; U \cup U_1 \cup \cdots \cup U_k$$

that contains K and U.

Surjectivity of the duality map. We consider any homology class $x \in H_{n-i}(M;\mathbb{Z})$ and represent it by a singular $(n-i)$-cycle $\sum a_j \cdot \psi_j \in C_{n-i}(\mathcal{S}(M);\mathbb{Z})$, where $a_j \in \mathbb{Z}$ and $\psi_j : \nabla^{n-i} \longrightarrow M$ are singular simplices. Since the sum is finite, the union of all the images $\psi_j(\nabla^{n-i})$ is a compact subset of M. Claim B (with $U = \emptyset$) provides an open subset V of M that contains all the sets $\psi_j(\nabla^{n-i})$ and for which the duality map is an isomorphism. The $(n-i)$-cycle representing x then also defines a homology class $y \in H_{n-i}(V;\mathbb{Z})$ that maps to x under the inclusion $V \longrightarrow M$. Since Poincaré duality holds for V, there is a class $\alpha \in H^i_{\mathrm{comp}}(V;\mathbb{Z})$ such that $D_V(\alpha) = y$. But then

$$D_M(\iota_V^M(\alpha)) \;=\; \mathrm{incl}_*(D_V(\alpha)) \;=\; \mathrm{incl}_*(y) \;=\; x \ ,$$

so the duality map $D_M : H^i_{\mathrm{comp}}(M;\mathbb{Z}) \longrightarrow H_{n-i}(M;\mathbb{Z})$ is surjective.

Injectivity of the duality map. We consider any class α in the kernel of the duality map $D_M : H^i_{\mathrm{comp}}(M;\mathbb{Z}) \longrightarrow H_{n-i}(M;\mathbb{Z})$. We represent α by a cocycle $f \in C^i_{\mathrm{comp}}(M;\mathbb{Z})$, and we let K be a compact subset of M on which f is supported. Claim B provides an open neighborhood of K for which Poincaré duality holds; we call this neighborhood U. We write $\beta \in H^i_{\mathrm{comp}}(U;\mathbb{Z})$ for the class of the restriction of f to the open subset U. Then

$$\alpha \ = \ \iota_U^M(\beta) \ ,$$

by the defining property of ι_U^M. We deduce the relation

$$\mathrm{incl}_*(D_U(\beta)) \ = \ D_M(\iota_U^M(\beta)) \ = \ D_M(\alpha) \ = \ 0 \ .$$

We represent the homology class $D_U(\beta) \in H_{n-i}(U;\mathbb{Z})$ by an $(n-i)$-cycle in $C_{n-i}(\mathcal{S}(U);\mathbb{Z})$. Because the class $D_U(\beta)$ maps to zero in $H_{n-i}(M;\mathbb{Z})$, the representing cycle is the boundary of an $(n-i+1)$-chain in M, i.e., in the complex $C_*(\mathcal{S}(M);\mathbb{Z})$. This $(n-i+1)$-chain is of the form $\sum a_j \cdot \psi_j \in C_{n-i+1}(\mathcal{S}(M);\mathbb{Z})$, where $a_j \in \mathbb{Z}$ and $\psi_j : \nabla^{n-i+1} \longrightarrow M$ are singular simplices. Claim B provides an open subset V of M that contains U and all the images $\psi_j(\nabla^{n-i+1})$, and such that Poincaré duality holds for V. This means that already the image of $D_U(\beta)$ in the group $H_{n-i}(V;\mathbb{Z})$ is zero, and so

$$D_V(\iota_U^V(\beta)) \ = \ \mathrm{incl}_*(D_U(\beta)) \ = \ 0 \ .$$

Because the duality map for V is an isomorphism, we conclude that $\iota_U^V(\beta) = 0$. Hence also

$$\alpha \ = \ \iota_U^M(\beta) \ = \ \iota_V^M(\iota_U^V(\beta)) \ = \ 0 \ .$$

This shows that the duality map for M is injective, and it concludes the proof. $\square$