Topology II - Manifolds

Tor Gjone September 15, 2021

Definition 0.1. An <u>m-manifold</u> is a Hausdorff space M such that every point of M has an open neighborhood homeomorphic to $\mathbb{R}^m$. The number $m \geq 0$ is the <u>dimension</u> of M.

Remark 0.2. The empty space is an m-manifold for all $m \geq 0$. For non-empty manifolds, the dimension is intrinsic and can be calculated from the local homology groups:

Let M be a manifold, $x \in M$. Let $U \subset M$ be an open neighborhood of x that admits a homeomorphism $\phi : \mathbb{R}^m \to U$, such that $\phi(0) = x$.

Then:

$$\begin{tikzcd}
H_i(M, M \setminus \{x\}; \mathbb{Z}) & H_i(U, U \setminus \{x\}; \mathbb{Z}) \arrow[l, "\text{excision}"'] & H_i(\mathbb{R}^m, R^m \setminus \{0\}; \mathbb{Z}) \arrow[l, "\phi_*"'] \arrow[d, "\partial"] \\
\left.\begin{matrix} \mathbb{Z} & \text{if } i = m \\ 0 & \text{if } i \neq m \end{matrix}\right\} & \tilde{H}_{i-1}(S^{m-1}; \mathbb{Z}) \arrow[l, "\cong"'] & \tilde{H}_{i-1}(\mathbb{R}^m \setminus \{0\}; \mathbb{Z}) \arrow[l, "\text{inclusion}"']
\end{tikzcd}$$

So the dimensions of M is the dimension in which the local homology is concentrated.

Remark 0.3. The Hausdorff condition is included to avoid certain pathological examples, such as the "line with double origin":

$$X = \mathbb{R} \times \{0, 1\} / \sim$$

where $(x, 0) \sim (x, 1)$ for all $x \neq 0$.

Example 0.4. Open subsets of $\mathbb{R}^m$ are m-manifolds.

Example 0.5. Let M be a Hausdorff space such that every point has an open neighbourhood that is an m-manifold. Then M is an m-manifold. In particular, the disjoint union (with disjoint union topology) of two m-manifolds is an m-manifold.

Example 0.6. Let M be an m-manifold and N and n-manifold. Then $M \times N$ (with the product topology) is an $(m + n)$-manifold.

Example 0.7. The n-sphere $S^n = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} \ : \ x_1^2 + ... + x_{n+1}^2 = 1\}$ is an n-manifold.

For $x = (x_1, ..., x_{n+1}) \in S^n$ let $Y = \{y \in \mathbb{R}^{n+1} : \langle y, x \rangle = 0\}$ be the orthogonal complement. The <u>stereographic projection</u> is homeomorphism

$$p : S^n \setminus \{-x\} \xrightarrow{\cong} Y \cong \mathbb{R}^n,$$

defined by

$$p(z) = \frac{z - \langle z, x \cdot x \rangle}{1 + \langle z, x \rangle}.$$

Example 0.8. The real projective space $\mathbb{RP}^n = S^n/\textbf{antipodal map}$ is an n-manifold. Consider any point $\{x, -x\} \in \mathbb{RP}^n$, choose one of the points x. Let

$$U = \{z \in S^n \ : \ \langle z, x \rangle > 0\} = \text{"hemisphere around x"}.$$

Then the composite

$$\mathbb{R}^n \cong U \hookrightarrow S^n \xrightarrow{\text{quotient}} \mathbb{RP}^n$$

is a homeomorphism of open neighbourhoods of $\{x, -x\}$.

Example 0.9. The complex projective space $\mathbb{CP}^n = \{L \subset \mathbb{C}^{n+1} \ : \ L \text{ a 1-dim } \mathbb{C}\text{-subspace of } \mathbb{C}\}$ is a $2n$-manifold. Consider first $L_0 = [0 : ... : 0 : 1] \in \mathbb{CP}^n$. Then

$$\mathbb{R}^{2n} = \mathbb{C}^n \to \mathbb{CP}^n,$$

defined by

$$(z_1, ..., z_n) \mapsto [z_1 : ... : z_n : 1]$$

is a homeomorphism onto an open neighbourhood of L_0.

If $L \in \mathbb{CP}^n$ is any complex line in $\mathbb{C}^{n+1}$, let $v \in L$ be a non-zero vector, and choose an invertible matrix $A \in \mathrm{GL}_{n+1}(\mathbb{C})$ such that $A \cdot (0, ..., 0, 1) = v$. Then

$$A : \mathbb{CP}^n \to \mathbb{CP}^n; L \mapsto A \cdot L$$

is a self-homomorphism of $\mathbb{CP}^n$ that maps L_0 to $\mathbb{C} \cdot v = L$. Since $\mathbb{CP}^n$ is locally homeomorphic to $\mathbb{R}^{2n}$ around L_0, is is also locally homeomorphic to $\mathbb{R}^{2n}$ around L.

Example 0.10. The quaternic projective space $\mathbb{HP}^n = \{L \subset \mathbb{H}^{n+1} : L \text{ a 1-dim left } \mathbb{H}\text{-subspace of } \mathbb{H}\}$. Similarly as for the complex case, $\mathbb{HP}^n$ is a $4n$-manifold.