Notation: · coefficients for homology will always be $\mathbb{Z}$, omitted from the notation.

· If X is a space and $Y \subseteq X$ subspace, we write $H_n(X|Y) = H_n(X, X \setminus Y; \mathbb{Z})$ "local homology at Y".

Note: if $Y \subseteq U \subseteq X$ and U is a neighbourhood of Y, then excision provides an isomorphism

$$H_n(U|Y) = H_n(U, U \setminus Y) \xrightarrow{\cong} H_n(X, X \setminus Y) = H_n(X|Y).$$

For n-manifolds M and $x \in M$, $\quad H_n(M|x) = H_n(M, M\setminus\{x\}; \mathbb{Z}) \cong \mathbb{Z}$.

A <u>local orientation</u> of M at x is a generator of $H_n(M|x)$. There are exactly two local orientations.

Heuristically: an orientation of M is a "continuous choice" of local orientations.

<u>Construction</u>: <u>The orientation covering</u>.

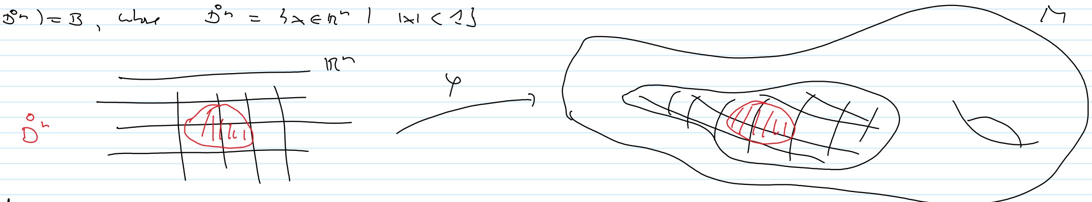
Let M be an n-manifold. Set

$$\tilde{M} = \{ (x, \mu) : x \in M, \mu \in H_n(M|x) \text{ a local orientation} \}.$$

The map $p : \tilde{M} \longrightarrow M$, $p(x,\mu) = x$ is surjective and every point has exactly two preimages.

We endow $\tilde{M}$ with a topology that makes p into a twofold covering.

A subset B of M is a <u>local ball</u> if B is open and there is a homeomorphism $\varphi : \mathbb{R}^n \longrightarrow M$ onto an open subset such that $\varphi(\mathring{D}^n) = B$, where $\mathring{D}^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$



Note: the inclusion $M \setminus B \longrightarrow M \setminus \{x\}$ is a homotopy equivalence for all $x \in B$.

So it induces an isomorphism $r_x^B \; H_n(M|B) \xrightarrow{\cong} H_n(M|x) \cong \mathbb{Z}$

We let $\mu \in H_n(M|B)$ be a generator and define

$$U(B,\mu) = \{ (x, r_x^B(\mu)) : x \in B \} \subseteq \tilde{M}.$$

<u>Theorem</u>: Let M be an n-manifold.

i) As (B,μ) varies over all local balls B and all generators of $H_n(M|B)$, the sets $U(B,\mu)$ form the basis of a topology on $\tilde{M}$.

ii) For this topology, the map $p : \tilde{M} \to M$, $p(x,\nu) = x$ is a twofold covering map, the <u>orientation covering</u> of M.

iii) $\tilde{M}$ is an n-manifold.

<u>Proof</u>: i) We show that $U(B,\mu) \cap U(B',\mu')$ is a union of basis sets.

Let $(x,\nu) \in U(B,\mu) \cap U(B',\mu')$. Then $x \in B \cap B'$ and

$\nu = r_x^B(\mu) = r_x^{B'}(\mu')$. We choose another local ball B'' with $x \in B'' \subseteq B \cap B'$!

We obtain a diagram of local homology groups

$$\begin{tikzcd}
H_n(M|B) \arrow[dr] \arrow[drr, bend left, "\cong!"] \arrow[drrr, bend left=30, "\cong"] & & & \\
& \underbrace{H_n(M|B\cap B')}_{\text{need not be a local ball!}} \arrow[r] & H_n(M|B'') \arrow[r, "\cong"] & H_n(M|x) \\
H_n(M|B') \arrow[ur] \arrow[urr, bend right, "\cong!"'] \arrow[urrr, bend right=30, "\cong"'] & & &
\end{tikzcd}$$

Since μ and μ' restrict to the same generator of $H_n(M|x)$, they restrict to the same generator of $H_n(M|B'')$.

$$\mu'' = r^B_{B''}(\mu) = r^{B'}_{B''}(\mu')$$

Hence: $\quad (x,\nu) \in U(B'',\mu'') \subseteq U(B,\mu) \cap U(B',\mu')$.

So the sets $U(B,\mu)$ form a basis for a topology on $\tilde{M}$.

ii) Because M is an n-manifold, the local balls form a basis of the topology of M.

Moreover $p^{-1}(B) = U(B,\mu) \cup U(B,-\mu)$ where $\pm\mu$ are the two generators of $H_n(M|B)$.

So p is continuous. Moreover, the restriction

$p|_{U(B,\mu)} : U(B,\mu) \longrightarrow B$ is a bijective continuous map. The map is also open (and hence a homeomorphism) because a basis of the subspace topology of $U(B,\mu)$ is given by the sets $U(B'',\mu'')$ for local balls $B'' \subseteq B$ and $\mu'' = r^B_{B''}(\mu)$. Because $p(U(B'',\mu'')) = B''$ is open in M, the restriction of p to $U(B,\mu)$ is an open map.

So $p^{-1}(B) \cong B \amalg B$ is homeomorphic in a way that matches p with the fold map $B \amalg B \longrightarrow B$.

So p is a twofold covering map.

iii) By design, every point $(x,u) \in \tilde{M}$ has a open neighborhood $U(B,\mu)$, which is homeomorphic to $B \cong D^n \cong \mathbb{R}^n$. So $\tilde{M}$ is locally euclidean of dimension n. Since M is Hausdorff and $p : \tilde{M} \longrightarrow M$ a covering, $\tilde{M}$ is Hausdorff. $\square$

<u>Definition</u>: An <u>orientation</u> of an n-manifold M is a continuous section $s : M \longrightarrow \tilde{M}$ of the orientation covering $p : \tilde{M} \longrightarrow M$. The manifold M is <u>orientable</u> if there exists an orientation of M.

<u>Remark</u>: Because manifolds are locally euclidean, their path components are open. So manifolds are the topological disjoint unions of their path components. For many purposes one can restrict to connected manifolds by considering each path component separately.

<u>Corollary</u>. A connected orientable manifold has exactly 2 orientations. An orientable manifold with n components has 2^n orientations.

<u>Proof</u>: If M is orientable, then $\tilde{M} \cong M \amalg M$, taking $p : \tilde{M} \longrightarrow M$ to the fold map. So there are exactly two continuous sections if M is connected. In general, you can independently choose an orientation of each path component. $\square$

<u>Note</u>: M connected: $p : \tilde{M} \longrightarrow M$ is product covering $\Leftrightarrow$ there is a continuous section to p

$\Leftrightarrow$ M is orientable.

<u>Corollary</u>: Let M be a connected n-manifold such that for some (hence any) $x \in M$, the group $\pi_1(M,x)$ does <u>not</u> have a subgroup of index 2. Then M is orientable. In particular, all simply connected manifolds are orientable.

<u>Proof</u>: Let $\tilde{x} \in \tilde{M}$ be any point over x. If M were not orientable, then $p : \tilde{M} \longrightarrow M$ is not a product covering, and $\tilde{M}$ would be connected. Then $p_* : \pi_1(\tilde{M},\tilde{x}) \longrightarrow \pi_1(M,x)$ is an injective group homomorphism whose image has index 2 in $\pi_1(M,x)$. This contradicts the hypothesis, so M is orientable. $\square$

<u>Example</u>: S^n is simply connected for $n \geq 2$, and hence orientable for $n \geq 2$.

For all $n \geq 1$, $\mathbb{C}P^n$ and $\mathbb{H}P^n$ are simply connected, and hence orientable.

<u>Example</u>: Let M be an n-manifold that also admits the structure of a topological group, i.e. group structure such that the multiplication and inverse map are continuous in the given topology. Then M is orientable. Examples: S^1, $O(n)$, $U(n)$, $Sp(n)$, $SO(n)$, $SU(n)$

<u>Proof</u>: Let $m : M \times M \longrightarrow M$ be the group structure and let $e \in M$ be the neutral element. We choose a local orientation $\mu_0 \in H_n(M|e)$. For any $x \in M$, the map

$m(x,-) : M \longrightarrow M$ is a homeomorphism with inverse $m(x^{-1},-) : M \to M$

that take e to x. So it induces an isomorphism of local homology groups

$$m(x,-)_* : H_n(M|e) \xrightarrow{\cong} H_n(M|x)$$
$$\mu_0 \longmapsto \mu_x$$

We define $s : M \longrightarrow \tilde{M}$ by $s(x) = \mu_x = m(x,-)_*(\mu_0)$; this is continuous, and hence an orientation of M.

Prop: Let M be an n-manifold.

i) The manifold $\widetilde{M}$ is orientable, and the map $\tau: \widetilde{M} \longrightarrow \widetilde{M}$, $\tau(x,\nu) = (x,-\nu)$ reverses the local orientations of $\widetilde{M}$.

ii) Suppose that $q: N \longrightarrow M$ is a 2-fold covering and N an orientable manifold. Suppose moreover that the non-identity deck transformation $\tau: N \longrightarrow N$ reverses the local orientations.

Then $q: N \longrightarrow M$ is isomorphic, as a covering, to $p: \widetilde{M} \longrightarrow M$.

Proof: i) Let $\tilde{x} = (x,\mu) \in \widetilde{M}$ be any point. Since $p: \widetilde{M} \longrightarrow M$ is a local homeomorphism, it induces an isomorphism

$$p_*: H_n(\widetilde{M}|\tilde{x}) \xrightarrow{\cong} H_n(M|x) = H_n(M|p(\tilde{x}))$$

We let $p_*^{-1}(\mu)$ be the "tautological" local orientation of $\widetilde{M}$ at $\tilde{x}$.

This defines a continuous (!) map
$$\widetilde{M} \longrightarrow \widetilde{\widetilde{M}}, \quad \tilde{x} = (x,\mu) \longmapsto (\tilde{x}, p_*^{-1}(\mu))$$
, hence an orientation of $\widetilde{M}$.

The deck transformation $\tau: \widetilde{M} \longrightarrow \widetilde{M}$, $\tau(x,\mu) = (x,-\mu)$ reverses the local orientations of $\widetilde{M}$:

$\tilde{x} = (x,\mu)$
$$\tau_*: H_n(\widetilde{M}|\tilde{x}) \longrightarrow H_n(\widetilde{M}|\tau(\tilde{x}))$$
$$(\tilde{x}, p_*^{-1}(\mu)) \longmapsto (\tau(\tilde{x}), \tau_*(p_*^{-1}(\mu)))$$
$$\underset{p\circ\tau = p}{=} ((x,-\mu), p_*^{-1}(\mu))$$
$$= ((x,-\mu), -p_*^{-1}(-\mu)) \neq ((x,-\mu), p_*^{-1}(-\mu))$$
$$= \text{tautological orientation at } \tau(\tilde{x}).$$

ii) Let $q: N \longrightarrow M$ be any 2-fold covering such that N is orientable and $\tau: N \longrightarrow N$ is orientation reversing. Let $\{\mu_y\}_{y\in N}$ be an orientation of N.

We define $f: N \longrightarrow \widetilde{M}$ by $f(y) = (q(y), \underbrace{q_*(\mu_y)}_{\text{generator of } H_n(M, q(y))}) \in \widetilde{M}$ $\quad q_*: H_n(N,y) \xrightarrow{\cong} H_n(M, q(y))$

The continuity of the local orientations $\{\mu_y\}$ implies the continuity of f (!). Because $\tau: N \longrightarrow N$ reverses the orientation, f is compatible with the non-trivial deck transformations:

$$f(\tau y) = (q(y), q_*(\mu_{\tau y})) \underset{?}{=} (q(y), q_*(-\tau_*(\mu_y)))$$
$$\underset{q\tau = q}{\overset{\tau: N\to N \text{ orientation reversing}}{=}} (q(y), -q_*(\mu_y)) = \tau(q(y), q_*(\mu_y))$$
$$= \tau(f(y))$$

So f is a continuous bijection between two-fold coverings over the same base M, so f is also open, and hence an isomorphism of coverings. $\square$

Example: The antipodal map $A: S^n \longrightarrow S^n$, $A(x) = -x$ has degree $(-1)^{n+1}$.

Let $e \in H_n(S^n;\mathbb{Z})$ be any generator, and orient S^n by the images of e in $H_n(S^n|x)$ for all $x \in S^n$.

So A is orientation reversing if and only if n is even.

The projection $q: S^n \longrightarrow \mathbb{R}P^n$, $q(x) = \mathbb{R}\cdot x$ is a 2-fold covering with orientable total space.

If n is even, the non-identity deck transformation A reverses the orientation. So for n even, $q: S^n \longrightarrow \mathbb{R}P^n$ is isomorphic to the orientation covering of $\mathbb{R}P^n$. So for n even, $\mathbb{R}P^n$ is <u>not</u> orientable.

Example: For n odd, $\mathbb{R}P^n$ is orientable. We construct an orientation of $\mathbb{R}P^n$ by choosing a generator $e \in H_n(S^n,\mathbb{Z})$ and defining the local orientation at $\mathbb{R}\cdot x \in \mathbb{R}P^n$ as the image of e under the isomorphism

$$H_n(S^n;\mathbb{Z}) \xrightarrow{\cong} H_n(S^n|x) \xrightarrow[\cong]{q} H_n(\mathbb{R}P^n|\mathbb{R}x)$$
$$e \longmapsto \mu_{\mathbb{R}x}$$

Because the antipodal map has degree $+1$, this yields the same class for x and $-x$.

In fact all the local orientations arise from one class in $H_n(\mathbb{R}P^n,\mathbb{Z})$, so they vary continuously in $\mathbb{R}x \in \mathbb{R}P^n$.