Relativistic Quantum Field Theory

Sec 1: Introduction

Motivation

Last year you learned something about Relativistic Quantum Mechanics (RQM), in particular, the Klein Gordon equation (KGE), and the Dirac equation (DE). These equations are the correct starting points for fully-relativistic quantum theories of spin-0 and spin-$\frac{1}{2}$ particles respectively, but there are problems with the interpretation of their solutions within the framework of RQM, both equations having "negative energy" solutions, for example. Furthermore, the naïve interpretation of the KG continuity equation leads to negative probabilities for negative energy particles — whatever that may mean. These problems can be solved (or at least circumvented) for free particles in RQM, but the addition of interactions leads to more severe complications.

As an example, consider a collision between an electron (e^-) and a positron (e^+): we observe experimentally that the particles may annihilate each other, and if they are energetic enough several e^+e^- pairs may be produced. We conclude that we need a relativistic quantum theory that can deal with an *arbitrary* number of particles.

One may reach the same conclusion using a naïve uncertainty principle argument. Start with the "energy-time uncertainty principle": $\Delta E\,\Delta t > \hbar$. For sufficiently small Δt, we have $\Delta E > mc^2$, and once again we conclude that we need a formalism that's capable of dealing with an arbitrary number of particles. Such a formalism is Quantum Field Theory (QFT).

The purpose of this course is to introduce you to the formalism and the physics of Quantum Field Theory. QFT provides us with the theoretical framework for understanding all of the fundamental interactions, with the possible exception of gravity. It has been staggeringly successful in its application to Quantum Electrodynamics (QED). Consider, for example, the calculation of the anomalous magnetic moment of the electron, characterised by $a \equiv (g-2)/2$ for the electron:

$$\begin{aligned} a_{\rm theor} &= 1159651941(128) \times 10^{-12} \\ a_{\rm exptl} &= 1159652188.4(4.3) \times 10^{-12} \end{aligned}$$

By the end of the course you should be able to perform lowest-order perturbation theory calculations in QED using Feynman diagram methods.

Relativistic Wave Equations

We begin by recalling the single-particle relativistic wave equations for free particles: massive spin-0, massless spin-1, and massive spin-$\frac{1}{2}$ free particles respectively. For detailed derivations, see the Revision Notes.

We shall work in 'natural' units, where $\hbar = c = 1$.

The Klein-Gordon Equation

$$\boxed{(\partial^2 + m^2)\phi(x) = 0} \qquad \text{or} \qquad \boxed{(\Box^2 + m^2)\phi(x) = 0}$$

has free-particle solutions of the form

$$\phi(x) = \exp(-ip \cdot x) = \exp(-ip^\mu x_\mu)$$

provided that $E^2 \equiv p_0^2 = |\underline{p}|^2 + m^2$. (Remember that $c = 1$ in natural units).

Note that if $\phi(x)$ is a solution, then so is $\phi^*(x) \equiv \exp(+ip \cdot x)$.

Electromagnetic Vector Potential Equation

$$\boxed{\partial^2 A^\mu - \partial^\mu \partial_\nu A^\nu = 0} \tag{1}$$

You probably won't recognise this as the equation for massless spin-1 particles. If you don't be patient, all will be revealed later.

Define the **electromagnetic field strength tensor**

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

the four-dimensional 'curl' of the 4-vector potential. It is also the commutator of the covariant derivative $D^\mu = \partial^\mu - ieA^\mu$,

$$[D^\mu, D^\nu]f = -ie(\partial^\mu A^\nu + A^\mu \partial^\nu - \partial^\nu A^\mu - A^\nu \partial^\mu)f = -ie(\partial^\mu A^\nu - \partial^\nu A^\mu)f = -ieF^{\mu\nu}f.$$

Why isn't this commutator a differential operator acting on the arbitrary function f?

$F^{\mu\nu}$ is a second-rank tensor whose components are the electric field $\underline{E}$, and the magnetic field $\underline{B}$. It can be represented by an array whose rows and columns are labelled by the 4 values of the indices $\{0, 1, 2, 3\}$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$

It is easy to see that equation (1) can be written as

$$\boxed{\partial_\mu F^{\mu\nu} = 0}\,.$$

The Dirac Equation We shall usually work with the covariant form of the Dirac equation:

$$\boxed{(i\gamma^\mu\,\partial_\mu - m)\;\psi(x) = 0} \qquad \text{or} \qquad \boxed{(i\not{\partial} - m)\;\psi(x) = 0}$$

where $\gamma^0 \equiv \beta$ and $\gamma^i \equiv \beta\alpha^i$, and the γ matrices satisfy the *Clifford algebra*

$$\boxed{\{\gamma^\mu,\,\gamma^\nu\} = 2\,g^{\mu\nu}}\,.$$

Positive energy free-particle solutions of the Dirac equation are of the form

$$\psi(x) = \exp(-ip\cdot x)\,u(p)$$

where the 4-component, free-particle spinor, $u(p)$, satisfies

$$(\gamma^\mu p_\mu - m)\,u \equiv (\not{p} - m)\,u = 0,$$

whilst negative energy (negative 4-momentum) solutions are of the form

$$\psi(x) = \exp(+ip\cdot x)\;v(p)$$

with

$$(\gamma^\mu p_\mu + m)\,v = (\not{p} + m)\,v = 0.$$

Covariance of Relativistic Wave Equations

Covariance of a relativistic equation means the *form* of the equation is the same in all inertial frames

Let us consider two inertial coordinate systems x^μ and x'^μ, which are connected by the Lorentz transformation Λ^μ_ν

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu. \tag{2}$$

(Convince yourself by doing some explicit examples that $x_\nu = \Lambda^\mu{}_\nu x'_\mu$).

Klein-Gordon Equation

Let us denote by $\phi(x)$ the scalar field as observed in the x^μ coordinate system and by $\phi'(x')$ the same field as observed at the same physical point, but in the x'^μ coordinate system. Then for a scalar field these two values are equal

$$\phi'(x') = \phi(x). \tag{3}$$

This relation defines the function $\phi'(x')$ in terms of the function $\phi(x)$. In particular, by using this and equation (2) we get $\phi'(x') = \phi(\Lambda^{-1}x')$.

Recall the Klein–Gordon equation from Lecture 1

$$\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} + m^2\right)\phi(x) = 0. \tag{4}$$

We note the chain rule for partial derivatives

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} = \Lambda^\nu{}_\mu \frac{\partial}{\partial x'^\nu}\ ; \tag{5}$$

thus writing equation (4) in the prime coordinates we obtain

$$\left(\Lambda^\nu{}_\mu \Lambda_{\bar\nu}{}^\mu \frac{\partial}{\partial x'^\nu}\frac{\partial}{\partial x'_{\bar\nu}} + m^2\right)\phi'(x') = 0, \tag{6}$$

where we used equation (3). Recalling that $\Lambda^\nu{}_\mu \Lambda_{\bar\nu}{}^\mu = g^\nu_{\bar\nu}$ we see that equations (4) and (6) are of the same form.

Electromagnetic Vector Potential Equation

In this case

$$A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x),$$

so that

$$F'^{\mu\nu}(x') = \Lambda^\mu{}_{\bar\mu}\Lambda^\nu{}_{\bar\nu}F^{\bar\mu\bar\nu}(x).$$

Recall the equation of motion for the field tensor

$$\frac{\partial}{\partial x^\mu}F^{\mu\nu}(x) = 0;$$

expressing this equation in terms of the prime coordinates, it becomes

$$\Lambda^{\bar{\bar\mu}}{}_\mu \frac{\partial}{\partial x'^{\bar{\bar\mu}}}\Lambda_{\bar\mu}{}^\mu \Lambda_{\bar\nu}{}^\nu F'^{\bar\mu\bar\nu}(x') = 0. \tag{7}$$

Noting that $\Lambda^{\bar{\bar\mu}}{}_\mu \Lambda_{\bar\mu}{}^\mu = g^{\bar{\bar\mu}}{}_{\bar\mu}$, equation (7) becomes

$$\Lambda_{\bar\nu}{}^\nu \frac{\partial}{\partial x'^{\bar\mu}}F'^{\bar\mu\bar{\bar\nu}}(x') = 0. \tag{8}$$

Multiplying both sides by $\Lambda^{\bar\nu}{}_\nu$ we then obtain the desired form

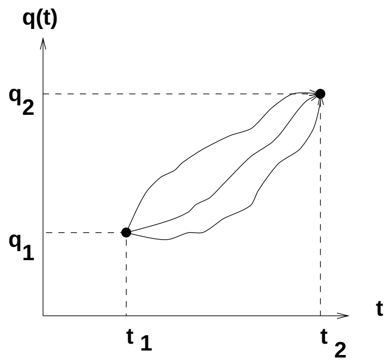
$$\frac{\partial}{\partial x'^{\bar\mu}}F^{\bar\mu\bar\nu}(x') = 0.$$

Lagrangian Field Theory

Classical Mechanics of Point Particles

Consider the motion in one space dimension of a particle of mass m in a conservative force field. Let q be the generalized coordinate of the particle, $\dot{q} \equiv dq/dt$ its velocity, and $L(q, \dot{q}) \equiv T(\dot{q}) - V(q)$, the Lagrangian. Then *Hamilton's Principle*, also known as the *Principle of Least Action*, states that the particle motion is determined by

$$\boxed{\delta S \equiv \delta \int_{t_1}^{t_2} dt\, L(q, \dot{q}) = 0}\,.$$



Possible trajectories between $q(t_1) = q_1$ and $q(t_2) = q_2$. The actual trajectory is that which minimizes the action S.

This principle simply tells us that the classical path, $q(t)$ of the particle traveling from $q(t_1) = q_1$ to $q(t_2) = q_2$ is such that the *action* S is stationary. In other words, small variations from this path: $q(t) \to q(t) + \delta q(t)$ leave S unchanged to first order in δq.

This is a simple problem in calculus of variations. We consider a small variation $\delta q(t)$ in the function $q(t)$, subject to the condition that the values of q at the end-points are unchanged

$$\delta q(t_1) = \delta q(t_2) = 0.$$

To first order, the variation in $L(q, \dot{q})$ is

$$\delta L = \frac{\partial L}{\partial q}\,\delta q + \frac{\partial L}{\partial \dot{q}}\,\delta \dot{q} \qquad \text{where} \qquad \delta \dot{q} \equiv \frac{d}{dt}\,\delta q;$$

thus the variation in the action is

$$\delta S = \int_{t_1}^{t_2} dt\, \left(\frac{\partial L}{\partial q}\,\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\,\delta q \right).$$

Integrating the second term by parts and noting that the integrated term

$$\left[\frac{\partial L}{\partial \dot{q}}\,\delta q \right]_{t_1}^{t_2}$$

vanishes at the limits, we obtain

$$\delta S = \int_{t_1}^{t_2} dt\, \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q(t).$$

For S to be stationary, this variation must vanish for an *arbitrary* small variation $\delta q(t)$, which is only possible if the integrand vanishes identically. Thus we obtain the so-called *Euler–Lagrange Equation*

$$\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) = 0}\,.$$

The generalization to 3 dimensions is straightforward

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \quad \text{for } i = 1, 2, 3}\,.$$

We can define *generalized momenta*, also known as *canonical momenta*,

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

and define the Hamiltonian function

$$\boxed{H(p, q) \equiv \sum_i p_i \dot{q}_i - L}\,.$$

From the LHS

$$dH = \sum_i \left(\frac{\partial H}{\partial q_i}\,\delta q_i + \frac{\partial H}{\partial p_i}\,\delta p_i\right)$$

and from the RHS

$$dH = \sum_i \left(\dot{q}_i \delta p_i + p_i \delta \dot{q}_i - \frac{\partial L}{\partial q_i}\,\delta q_i - \frac{\partial L}{\partial \dot{q}_i}\,\delta \dot{q}_i\right) = \sum_i \left(\dot{q}_i \delta p_i - \dot{p}_i \delta q_i\right),$$

where from the Euler–Lagrange equations $\dot{p}_i = \partial L/\partial q_i$. Comparing the LHS and RHS we obtain *Hamilton's equations*

$$\boxed{\frac{\partial H}{\partial p_i} = \dot{q}_i \quad \text{and} \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i}\,.$$

For functions $A(p, q)$ and $B(p, q)$ we define the *Poisson Bracket* as

$$\{A, B\} \equiv \sum_i \left(\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} - \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i}\right);$$

Hamilton's equations then can be expressed as

$$\boxed{\dot{q}_i = \{H, q_i\} \quad \text{and} \quad \dot{p}_i = \{H, p_i\}}\,.$$

Notice the similarity in form to the Heisenberg Operator equations of motion in quantum mechanics.

The generalization to a system of N particles is also straightforward. We now have a set of generalized coordinates, q_i^α, where α labels the different particles with masses m^α, say. We can simplify the notation by simply extending the range of the index i from $1, 2, 3$ to $1, 2, \ldots, 3N$ with the conventions that $m_1 = m_2 = m_3 \equiv m^1$, etc.

Exercise: Establish the equivalence to the more conventional approach by showing that the Euler–Lagrange equation for a single point particle is just Newton's equation of motion.

Continuous Systems

The analysis of the previous section is fine for systems with a finite number of degrees of freedom, such as a set of N massive particles connected by a massless string, executing transverse vibrations, as shown in Figure 1(a), but how do we deal with a continuous system, such as a massive string vibrating transversely as in Figure 1(b)? In one space dimension we have an amplitude ϕ which depends on a continuous 'label' x. Here, ϕ plays the role of q, while x is analogous to α (or i). As you might guess, the correct procedure is to replace

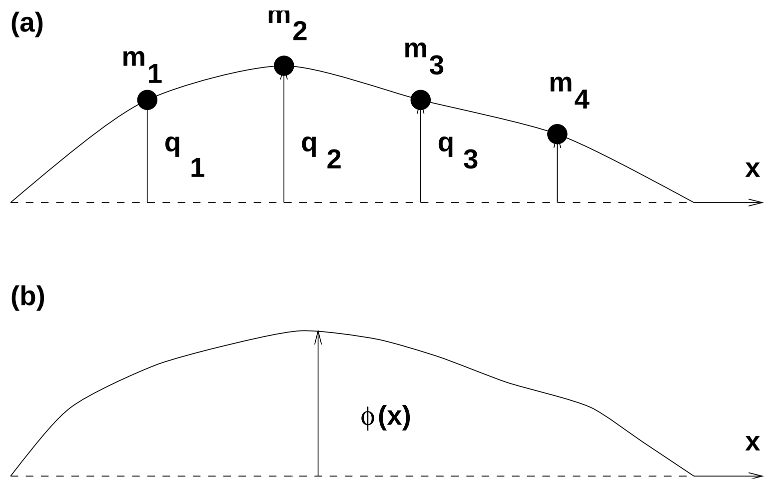


Figure 1: *Transverse oscillations of (a) a set of point masses connected by a massless string and (b) a massive string.*

summation over i by integration over x, or more generally, a three-dimensional integral over the spatial coordinates. The Lagrangian becomes not a sum but an integral over a *Lagrangian density* $\mathcal{L}$, which depends, in general, on the field, ϕ, and its derivatives with respect to space and time, which we will denote in the general case by $\partial_\mu\phi$. L is said to be a *functional* of ϕ and $\partial_\mu\phi$. Thus, for our string

$$L = \int_0^\ell dx\, \mathcal{L}(\phi, \dot{\phi}, \phi'),$$

where $\dot{\phi} = \partial\phi/\partial t$ and $\phi' = \partial\phi/\partial x$.

The action integral is

$$S = \int_{t_1}^{t_2} dt \int_0^\ell dx\, \mathcal{L}(\phi, \dot{\phi}, \phi').$$

If we consider small variations, $\delta\phi(x,t)$, which vanish at t_1 and t_2, and also at $x = 0$ and $x = \ell$, the fixed end points of the string, we find that

$$\delta S = \int_{t_1}^{t_2} dt \int_0^\ell dx \left(\frac{\partial \mathcal{L}}{\partial \phi}\, \delta\phi + \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\, \frac{\partial}{\partial t}\, \delta\phi + \frac{\partial \mathcal{L}}{\partial \phi'}\, \frac{\partial}{\partial x}\, \delta\phi \right).$$

Integrating the second term by parts with respect to t and the third term with respect to x yields

$$\delta S = \int_{t_1}^{t_2} dt \int_0^\ell dx \left[\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi'} \right) \right] \delta\phi(x,t).$$

Again, we can only satisfy Hamilton's Principle for arbitrary small variations of ϕ if the integrand vanishes identically, yielding the Euler–Lagrange equations

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\right) = 0}\,.$$

Classical Relativistic Fields

Let us now consider the more general case of a classical scalar field, $\phi(x)$. We restrict ourselves to considering theories which can be derived using Hamilton's Principle from an action integral involving a Lorentz-invariant functional $\mathcal{L}$ of the field ϕ and its first derivatives $\partial_\mu \phi$,

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi).$$

We define the action integral S_Ω for an arbitrary region Ω of space-time by

$$\boxed{S_\Omega = \int_\Omega d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)}\,.$$

Proceeding as before, we consider variations $\delta\phi(x)$ which vanish on the surface $\partial\Omega$ bounding Ω. The corresponding change in the action must vanish

$$\delta S_\Omega = \int_\Omega d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi}\,\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\delta\partial_\mu\phi\right] = 0$$

which, on noting that

$$\delta\partial_\mu\phi = \partial_\mu\,\delta\phi,$$

can be integrated by parts to give

$$\delta S_\Omega = \int_\Omega d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\right]\,\delta\phi = 0.$$

Requiring this to hold for arbitrary variations $\delta\phi$ and regions Ω leads to the Euler–Lagrange equations

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0}\,.$$

This is the equation of motion of the field ϕ.

The canonical 'momentum' corresponding to ϕ is defined by

$$\pi(x) \equiv \frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial \dot{\phi}(x)},$$

and the Hamiltonian is written as the volume integral over a Hamiltonian density

$$H = \int d^3x\, \mathcal{H}\Big(\pi(x), \phi(x)\Big), \qquad \mathcal{H} = \pi\dot{\phi} - \mathcal{L}.$$

More generally, for a system of n independent fields $\phi_r(x)$, $r = 1, 2, \ldots, n$ we obtain n field equations

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)}\right) = 0, \qquad r = 1, 2, \ldots, n}\,.$$

The Klein-Gordon Field

As our first example, we show that the Klein–Gordon equation can be obtained as the Euler–Lagrange equation of motion of a classical field $\phi(x)$ described by the Lagrangian density

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2}\,.$$

Proof.

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi;$$

giving the Euler–Lagrange equation

$$\partial_\mu \partial^\mu \phi + m^2\phi = 0,$$

which we can write as

$$(\Box^2 + m^2)\phi = 0.$$

The first term in $\mathcal{L}$ is usually referred to as the *kinetic term*, while the second term is known as the *mass term*.

The field or 'momentum density' conjugate to $\phi(x)$ is given by

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}(x),$$

yielding a Hamiltonian density

$$\mathcal{H} = \pi(x)\dot{\phi}(x) - \mathcal{L}(\phi, \partial_\mu \phi) = \tfrac{1}{2}\left[\pi^2(x) + (\nabla\phi)^2 + m^2\phi^2\right].$$

The Free Electromagnetic Field

Consider the Lagrangian density

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}\,F^{\mu\nu}}\,.$$

This can be written

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu).$$

Treating the components of the 4-vector potential A_ν as independent fields, we have

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \partial^\nu A^\mu;$$

giving as the Euler–Lagrange equations

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = -\partial_\mu \partial^\mu A^\nu + \partial^\nu(\partial_\mu A^\mu) = 0,$$

which we can rewrite as

$$\Box^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0,$$

recognizable as equivalent to Maxwell's equations in the absence of sources.

The canonically conjugate field is

$$\pi^\mu(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -F^{0\mu}(x).$$

Note. The antisymmetry of the Maxwell tensor implies that $\pi^0(x) \equiv 0$, which will lead to problems later when we try to quantize the electromagnetic field by imposing canonical commutation relations!

In the presence of sources, corresponding to the electromagnetic current 4-vector $j^\mu_{\text{em}}(x)$, we can derive the field equations from the Lagrangian density

$$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu}\, F^{\mu\nu} - j^\mu_{\text{em}}(x)\, A_\mu(x)}\,.$$

It is left as an exercise to show that the corresponding Euler–Lagrange equations are

$$\Box^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu_{\text{em}}.$$

Symmetries and Noether's Theorem

If the action is invariant under a group of symmetry transformations then we can construct corresponding conserved quantities. For example, we can show that invariance of the action, or equivalently of the equations of motion, under translations in time leads to energy conservation while spatial translational invariance gives rise to momentum conservation.

Consider the simplest case where $\mathcal{L}$ itself is invariant under the field variation

$$\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x).$$

Now the variation in $\mathcal{L}$ arising from variation of the fields is

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\,\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\delta(\partial_\mu\phi) = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi) = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\delta\phi\right),$$

where we have used the Euler–Lagrange equation in the second step.

Invariance implies that $\delta\mathcal{L} = 0$ and hence the continuity equation

$$\partial_\mu j^\mu(x) = 0$$

with

$$\boxed{j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\delta\phi}\,.$$

The corresponding "charges"

$$Q = \int d^3x\, j^0(x)$$

are constant in time: from the continuity equation

$$\dot{Q} = \int d^3x\,\frac{\partial}{\partial x^0}\,j^0(x) = -\int d^3x\,\frac{\partial}{\partial x^i}\,j^i(x).$$

Using the divergence theorem, we can rewrite the RHS as a surface integral and assuming that the fields, and hence the currents, vanish sufficiently fast at infinity as usual, we obtain

$$\dot{Q} = 0.$$

More generally, we can allow $\mathcal{L}$ to be invariant up to a 4-divergence

$$\mathcal{L} \to \mathcal{L} + \partial_\mu \Lambda^\mu \tag{9}$$

for some Λ^μ. It is straightforward to show that the Euler–Lagrange equations of motion are invariant under such a change. Setting

$$\delta\mathcal{L} = \partial_\mu \Lambda^\mu$$

the conserved current is

$$\boxed{j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - \Lambda^\mu}\,.$$

This result is an example of *Noether's Theorem*: for each continuous symmetry of $\mathcal{L}$, there is a conserved current.

We can also apply these considerations to space-time transformations, such as infinitesimal translations

$$x^\mu \to x'^\mu = x^\mu + a^\mu$$

which induce a transformation of the scalar fields

$$\phi(x) \to \phi(x + a) = \phi(x) + a^\mu \partial_\mu \phi(x).$$

The Lagrangian density is also a scalar and so must transform in the same way

$$\mathcal{L} \to \mathcal{L} + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu (\delta^\mu{}_\nu \mathcal{L}).$$

Comparing with equation 9 we see that

$$\partial_\mu \Lambda^\mu = a^\nu \partial_\mu (\delta^\mu{}_\nu \mathcal{L}).$$

We now have four separately conserved currents. For example, if we consider time translations so that $\nu = 0$ only, we have $\delta\phi = a^0 \partial_0 \phi$ and the conserved current is

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} a^0 \partial_0 \phi - a^0 \delta^\mu{}_0 \mathcal{L}.$$

We obtain similar conserved currents for the three spatial translations so that, scaling out the irrelevant parameters, we can write the four conserved currents as

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta^\mu{}_\nu,$$

one for each value of the index ν. We can also write this as

$$\boxed{T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \mathcal{L} g^{\mu\nu}}\,.$$

This is the so-called *stress-energy tensor* or *energy-momentum tensor* of the field ϕ. The conserved "charge" associated with the $\nu = 0$ component is just our old friend, the Hamiltonian

$$\int d^3x\, T^{00} = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^0 \phi - \mathcal{L} \right) = \int d^3x\, \mathcal{H} = H$$

and is associated with time translations.

The conserved charges associated with spatial translations are

$$P^i = \int d^3x\, T^{0i} = \int d^3x\, \pi \partial^i \phi = -\int d^3x\, \pi \partial_i \phi,$$

and we interpret them as the components of the momentum carried by the field.