Algebraic Geometry

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Lecture 8: non-rationality of smooth cubic curves

Claire Voisin (Notices of the AMS)

Claire Voisin (Lectures at Tata Institute)

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COLLOQUIUM TALK

4 PM, 1 OCTOBER 2018

SOME NEW RESULTS ON RATIONALITY

An algebraic variety is rational if it is birational to the projective space or affine space of the same dimension. In dimension 1 and 2 and over the complex numbers, smooth projective rational varieties have several characterizations and in particular it is known that they are the same as unirational or rationally connected varieties. Starting from dimension 3, it has been proved in the 70's that there are varieties which are unirational (that is rationally dominated by projective space) but not rational. I will describe in this talk further obstructions to rationality, which have been proved recently to be very effective and powerful, as a consequence of the degeneration argument that I introduced. The key notion is that of decomposition of the diagonal in the spirit of Bloch and Srinivas.

CHOW GROUPS AND BIRATIONAL INVARIANTS

The general subject of the lectures is the distinction between rational (or stably rational) varieties and unirational or rationally connected ones. By unirationality, the birational invariants from topology or K-theory that we are going to use are of torsion. We are going to discuss first the higher dimensional generalization of the so-called Artin-Mumford invariant, which is a topological obstruction to rationality. These invariants take the form of unramified cohomology and they are of a K-theoretic nature. In nonzero characteristic, some other obstructions appear, like algebraic forms.

LECTURE 2 | Obstructions to rationality: unramified cohomology
10:30 AM, 2 OCTOBER 2018

LECTURE 3 | Zero-cycles and decomposition of the diagonal
10:30 AM, 3 OCTOBER 2018

LECTURE 4 | The degeneration method and various improvements
10:30 AM, 4 OCTOBER 2018

LECTURE 5 | Cohomological decomposition of the diagonal in small dimension
10:30 AM, 5 OCTOBER 2018

1 – 5 OCTOBER 2018
MADHAVA HALL,
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CLAIRE VOISIN

Claire Voisin is an algebraic geometer recognized for her work on Hodge theory and algebraic cycles. She is known particularly for her construction of compact Kaehler manifolds not homeomorphic to complex projective manifolds, for her proof of the generic Green conjecture on syzygies of canonical curves, and for her contribution to the stable Lueroth problem.

Voisin was born in 1962 in the North suburbs of Paris and grew up there. She entered Ecole Normale Superieure in 1981 and she defended her PhD thesis in 1986 under the supervision of Arnaud Beauville. She then got a permanent position at CNRS, that she kept until 2016 where she became Professor at College de France (Algebraic geometry chair). She has been invited as distinguished visiting professor at IAS

Ellipse, hyperbola and parabola

Example

Let $\mathcal{C}$ be the ellipse in $\mathbb{R}^2$ given by

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1.$$

Then it can be parameterized by $(2\cos(t), 3\sin(t))$ for $t \in [0, 2\pi)$.

Example

Let $\mathcal{C}$ be the hyperbola in $\mathbb{R}^2$ given by

$$\frac{x^2}{5^2} - \frac{y^2}{7^2} = 1.$$

Then it can be parameterized by $(5\sinh(t), 7\cosh(t))$ for $t \in \mathbb{R}$.

- These are traditional parameterizations.
- They do not belong to Algebraic Geometry.

Rational parametrization

Example

Let $\mathcal{C}$ be the ellipse in $\mathbb{R}^2$ given by

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1.$$

Then it can be parameterized by $(\frac{2t^2-2}{t^2+1}, \frac{6t}{t^2+1})$ for $t \in \mathbb{R}$.

Example

Let $\mathcal{C}$ be the hyperbola in $\mathbb{R}^2$ given by

$$\frac{x^2}{5^2} - \frac{y^2}{7^2} = 1.$$

Then it can be parameterized by $(\frac{5+5t^2}{2t}, \frac{7-7t^2}{2t})$ for $t \in \mathbb{R}$.

- These are rational parameterizations.
- They belong to Algebraic Geometry.

Maps from $\mathbb{P}^1_{\mathbb{C}}$ to $\mathbb{P}^2_{\mathbb{C}}$

Example

Let $\mathcal{C}$ be the conic in $\mathbb{P}^2_{\mathbb{C}}$ given by

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} = z^2.$$

Let $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ be the map $[u : v] \mapsto [2u^2 - 2v^2 : 6uv : u^2 + v^2]$.
Then this map induces a bijection $\mathbb{P}^1_{\mathbb{C}} \to \mathcal{C}$.

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Let $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ be the map $[u : v] \mapsto [5v^2 + 5u^2 : 7v^2 - 7u^2 : 2uv]$.
Then this map induces a bijection $\mathbb{P}^1_{\mathbb{C}} \to \mathcal{C}$.

Pythagorean triples

Example (Pythagoras)

Let m, n, k be any integers. Then

$$\Big(k(m^2-n^2)\Big)^2+\Big(2kmn\Big)^2=\Big(k(m^2+n^2)\Big)^2,$$

which gives all integral solutions to $x^2+y^2=z^2$.

- Let $\mathcal{C}$ be a circle in $\mathbb{R}^2$ given by $x^2+y^2=1$.
- All points in $\mathcal{C}\setminus(1,0)$ with rational coordinates are given by

  $$\left(\frac{m^2-k^2}{m^2+k^2},\frac{2mk}{m^2+k^2}\right)$$

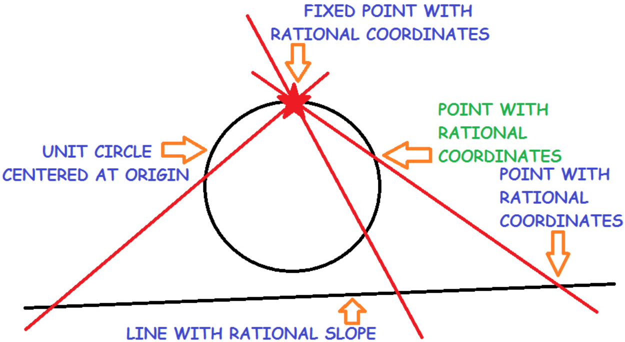
  for some integers m and k such that $(m,k)\neq(0,0)$.
- All points in $\mathcal{C}\setminus(1,0)$ with rational coordinates are given by

  $$\left(\frac{t^2-1}{t^2+1},\frac{2t}{t^2+1}\right)$$

  for some $t\in\mathbb{Q}$.

Stereographic projection

Pythagoras got his triples using stereographic projection:



This also gives a rational parametrization of the unit circle:

$$\left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1} \right)$$

for $t \in \mathbb{R}$. It gives us all points in the unit circle except $(1,0)$. The same idea can be applied to any smooth conic in $\mathbb{P}^2_{\mathbb{C}}$.

Diophantine equation

Theorem (Euler)

Let $\mathcal{C}_3$ be cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ given by

$$\boxed{x^3 + y^3 = z^3.}$$

Then $\mathcal{C}_3(\mathbb{Q}) = \{[1:0:1], [0:1:1], [1:-1:0]\}$.

Theorem (Andrew Wiles)

Let $\mathcal{C}_n$ be a curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree $n \geqslant 4$ given by

$$\boxed{x^n + y^n = z^n.}$$

Then $\mathcal{C}_n(\mathbb{Q}) \subset \{[1:0:1], [0:1:1], [1:-1:0]\}$.

Theorem (Gerd Faltings)

Let $\mathcal{C}_n$ be a smooth curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree $n \geqslant 4$.
If the curve $\mathcal{C}_n$ is defined over $\mathbb{Q}$, then $\mathcal{C}_n(\mathbb{Q})$ is finite.

Non-rational curves

The rings $\mathbb{Z}$ and $\mathbb{C}[t]$ are both UFD and PID.

Theorem

Let $x(t)$, $y(t)$, $z(t)$ be coprime polynomials in $\mathbb{C}[t]$ such that

$$\boxed{x^3(t) + y^3(t) = z^3(t).}$$

Then all $x(t)$, $y(t)$, $z(t)$ are constant.

- The proof of this theorem is easy and elementary.

Theorem

Let $x(t)$, $y(t)$, $z(t)$ be coprime polynomials in $\mathbb{C}[t]$ such that

$$\boxed{x^n(t) + y^n(t) = z^n(t)}$$

for some $n \geqslant 3$. Then $x(t)$, $y(t)$, $z(t)$ are constant.

- The proof of this theorem is also easy and elementary.

Infinite descent

Let $x(t)$, $y(t)$, $z(t)$ be coprime non-zero polynomials in $\mathbb{C}[t]$ such that

$$\boxed{x^3(t) + y^3(t) = z^3(t)}$$

and $x(t)$, $y(t)$, $z(t)$ are coprime polynomials in $\mathbb{C}[t]$.
Then $x(t)$, $y(t)$, and $z(t)$ are pairwise coprime in $\mathbb{C}[t]$.
Let d_x, d_y, d_z be the degrees of $x(t)$, $y(t)$, $z(t)$, respectively.
Put $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Then

$$\left(x(t) + y(t)\right)\left(x(t) + \omega y(t)\right)\left(x(t) + \omega^2 y(t)\right) = z^3(t),$$

and $x(t) + y(t)$, $x(t) + \omega y(t)$, $x(t) + \omega^2 y(t)$ are pairwise coprime.
Then there are polynomials $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ such that

$$\boxed{x(t) + y(t) = \alpha^3(t)}, \boxed{x(t) + \omega y(t) = \beta^3(t)}, \boxed{x(t) + \omega^2 y(t) = \gamma^3(t)}.$$

Then $-\omega\alpha^3(t) + (\omega + 1)\beta^3(t) = \gamma^3(t)$. Then

$$\boxed{\left(\sqrt[3]{-\omega}\,\alpha(t)\right)^3 + \left(\sqrt[3]{\omega + 1}\,\beta(t)\right)^3 = \gamma^3(t)}$$

and the degree of α is $\frac{d_z}{3}$. Now iterate.

Fermat cubic is non-rational

Theorem

Let $x(t)$ and $y(t)$ be rational functions in $\mathbb{C}(t)$ such that

$$\boxed{x^3(t) + y^3(t) = 1}.$$

Then both $x(t)$ and $y(t)$ are constant.

Proof.

We may assume that neither $x(t) = 0$ nor $y(t) = 0$.
There are coprime $a(t)$ and $b(t)$ in $\mathbb{C}[t]$ such that $x(t) = \frac{a(t)}{b(t)}$.
There are coprime $c(t)$ and $d(t)$ in $\mathbb{C}[t]$ such that $y(t) = \frac{c(t)}{d(t)}$.
Since $x^3(t) + y^3(t) = 1$, we have

$$\boxed{a^3(t)d^3(t) + c^3(t)b^3(t) = b^3(t)d^3(t).}$$

Then $b^3(t)|d^3(t)|b^3(t)$. Then $b(t) = \lambda d(t)$ for some $\lambda \in \mathbb{C}^*$.
This implies that $a(t)$, $b(t)$, $c(t)$ and $d(t)$ are constant. $\square$

Lemma about 4 squares

- Let $x(t)$ and $y(t)$ be coprime non-zero polynomials in $\mathbb{C}[t]$.
- Suppose there are $z_1(t)$, $z_2(t)$, $z_3(t)$, $z_4(t)$ in $\mathbb{C}[t]$ such that
$$a_i x(t) + b_i y(t) = z_i^2(t)$$
for four different points $[a_1 : b_1]$, $[a_2 : b_2]$, $[a_3 : b_3]$, $[a_4 : b_4]$ in $\mathbb{P}^1_{\mathbb{C}}$.
- Then both $x(t)$ and $y(t)$ are constant.

Indeed, we may assume that $a_1 \neq 0$. Then
$$\frac{a_1}{a_1 b_2 - a_2 b_1} z_2^2 - \frac{a_1}{a_1 b_3 - a_3 b_1} z_3^2 = \Big(\frac{a_2}{a_1 b_2 - a_2 b_1} + \frac{a_3}{a_3 b_1 - a_1 b_3} \Big) z_1^2.$$
Note that $z_1(t)$, $z_2(t)$, $z_3(t)$, $z_4(t)$ are pairwise coprime. Then
$$\sqrt{\frac{a_1}{a_1 b_2 - a_2 b_1}} z_2 + \sqrt{\frac{a_1}{a_1 b_3 - a_3 b_1}} z_3, \sqrt{\frac{a_1}{a_1 b_2 - a_2 b_1}} z_2 - \sqrt{\frac{a_1}{a_3 b_1 - a_1 b_3}} z_3,$$
are all squares in $\mathbb{C}[t]$. Similarly, we see that
$$\sqrt{\frac{a_1}{a_1 b_2 - a_2 b_1}} z_2 + \sqrt{\frac{a_1}{a_1 b_4 - a_4 b_1}} z_4, \sqrt{\frac{a_1}{a_1 b_2 - a_2 b_1}} z_2 - \sqrt{\frac{a_1}{a_1 b_4 - a_4 b_1}} z_4$$
are all squares in $\mathbb{C}[t]$. Now iterate.

Smooth plane cubic curves are non-rational

Theorem

Let $x(t)$ and $y(t)$ be rational functions in $\mathbb{C}(t)$ such that

$$\boxed{y^2(t) = x(t)\Big(x(t) - 1\Big)\Big(x(t) - \lambda\Big)}$$

where $\lambda \in \mathbb{C}$ and $0 \neq \lambda \neq 1$. Then $x(t)$ and $y(t)$ are constant.

Proof.

We may assume that neither $x(t) = 0$ nor $y(t) = 0$.
There are coprime $a(t)$ and $b(t)$ in $\mathbb{C}[t]$ such that $x(t) = \frac{a(t)}{b(t)}$.
There are coprime $c(t)$ and $d(t)$ in $\mathbb{C}[t]$ such that $y(t) = \frac{c(t)}{d(t)}$.
Since $y^2(t) = x(t)(x(t) - 1)(x(t) - \lambda)$, we have

$$b^3(t)c^2(t) = d^2(t)a(t)\Big(a(t) - b(t)\Big)\Big(a(t) - \lambda b(t)\Big).$$

Then $d^2(t) | b^3(t) | d^2(t)$. Then $d^2(t) = \mu b^3(t)$ for some $\mu \in \mathbb{C}^*$.
Then $a(t)$, $b(t)$, $c(t)$ are constant by lemma about 4 squares. $\square$

Non-constant maps from $\mathbb{P}^1_{\mathbb{C}}$ to $\mathbb{P}^2_{\mathbb{C}}$

Let $f(x, y, z)$ be a homogeneous polynomial of degree 3 such that

$$f(x, y, z) = 0$$

defines a smooth (irreducible) cubic curve in $\mathbb{P}^2_{\mathbb{C}}$.

Corollary

Let $x(t)$, $y(t)$, $z(t)$ be coprime polynomials in $\mathbb{C}[t]$ such that

$$\boxed{f\Big(x(t), y(t), z(t)\Big) = 0.}$$

Then $x(t)$, $y(t)$, $z(t)$ are in $\mathbb{C}$.

Let $\phi\colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ be a map

$$[a : b] \mapsto \big[\alpha(a, b) : \beta(a, b) : \gamma(a, b)\big]$$

where α, β, γ are coprime homogeneous polynomials of degree d.
If $d \geqslant 1$, then

$$\boxed{f\Big(\alpha(t_0, t_1), \beta(t_0, t_1), \gamma(t_0, t_1)\Big) \neq 0.}$$

Smooth plane cubic curves are bagels

Observe that $\mathbb{P}^1_{\mathbb{C}}$ is homeomorphic to a sphere.

- Let $\mathcal{C}$ be the curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by
$$zy^2 = x(x-z)(x-\lambda z)$$
for some $\lambda \in \mathbb{C}$ such that $\lambda \neq 0$ and $\lambda \neq 1$.
- Then $\mathcal{C}$ be a compact oriented two-dimensional real manifold.

Let $\psi\colon \mathcal{C} \to \mathbb{P}^1_{\mathbb{C}}$ be the map given by
$$\big[x:y:z\big] \mapsto \begin{cases} \big[x:z\big] \text{ if } \big[x:y:z\big] \neq \big[0:1:0\big], \\ [1:0] \text{ if } \big[x:y:z\big] = \big[0:1:0\big]. \end{cases}$$

Thus, for $[a:b] \in \mathbb{P}^1_{\mathbb{C}}$, one has
$$\psi^{-1}\big(P\big) = \begin{cases} \big[a:\sqrt{a(a-b)(a-\lambda b)}:b\big] \text{ if } \big[a:b\big] \neq \big[0:1\big], \\ [0:1:0] \text{ if } \big[a:b\big] = \big[0:1\big]. \end{cases}$$

Gluing two spheres with two cuts, we obtain

Corollary

The curve $\mathcal{C}$ is homeomorphic to a bagel.

Triangulation

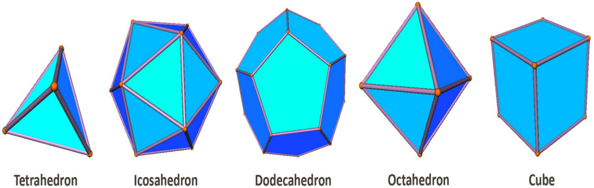
Do you feel comfortable with gluing two spheres with cuts?
If NOT, lets use some help from Topology.
Let S be a real compact oriented two-dimensional manifold.
Let us divide S into a union of curved triangles.

- Denote by $\mathbf{f}$ the number of triangles (faces).
- Denote by $\mathbf{e}$ the number of sides (edges).
- Denote by $\mathbf{v}$ the number of points (vertices).

Theorem (Euler)

If S is homeomorphic to a sphere with g handles. Then

$$\mathbf{v} - \mathbf{e} + \mathbf{f} = 2 - 2g.$$



Triangulated plane cubic curves

- Let $\mathcal{C}$ be a curve in $\mathbb{P}^2$ that is given by
$$zy^2 = x(x-z)(x-\lambda z)$$
for some $\lambda \in \mathbb{C}$ such that $\lambda \neq 0$ and $\lambda \neq 1$.
- Then $\mathcal{C}$ is a real compact oriented two-dimensional manifold.
- It is homeomorphic to a sphere with g handles attached.

We already know that g=0. Let us show this one more time. Triangulate the projective line $\mathbb{P}^1_{\mathbb{C}}$ such that the points

$$[0:1],\ [1:1],\ [\lambda:1],\ [1:0]$$

are among the vertices of our triangulation.

- Denote by **f** the number of faces.
- Denote by **e** the number of edges.
- Denote by **v** the number of vertices.

Lift this triangulation to $\mathcal{C}$ using the map ψ defined 2 slides ago. It has 2**f** faces, 2**e** edges, and $2(\mathbf{v}-4)+4$ vertices. Then

$$2g-2 = 2(\mathbf{v}-4)+4-2\mathbf{e}+2\mathbf{f} = 2(\mathbf{v}-\mathbf{e}+\mathbf{f})-4 = 4-4 = 0.$$

Riemann–Hurwitz formula

Let $\phi\colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ be a map

$$[a:b] \mapsto \big[\alpha(a,b) : \beta(a,b) : \gamma(a,b)\big]$$

where α, β, γ are coprime homogeneous polynomials of degree d.

Let $\mathcal{C}$ be a smooth cubic curve in $\mathbb{P}^2_{\mathbb{C}}$.

Suppose that the image of the map ϕ is the curve $\mathcal{C}$.

Then the preimage of general point $P \in \mathcal{C}$ consists of $\widehat{d} \leqslant d$ points.

There is a finite subset $\Sigma \subset \mathcal{C}$ such that

- $\phi^{-1}(P)$ consists of $\widehat{d}$ points for $P \in \mathcal{C}$ that is not in Σ,
- $\phi^{-1}(P)$ consists of less than $\widehat{d}$ points for every $P \in \Sigma$.

Triangulate $\mathcal{C}$ such that Σ is contained among the vertices.

- Denote by $\mathbf{f}$ the number of faces.
- Denote by $\mathbf{e}$ the number of edges.
- Denote by $\mathbf{v}$ the number of vertices.

Lift this triangulation to $\mathbb{P}^1_{\mathbb{C}}$ using the map ϕ.

It has $\widehat{d}\mathbf{f}$ faces, $\widehat{d}\mathbf{e}$ edges, and $\widehat{\mathbf{v}} \leqslant d'\mathbf{v}$ vertices. Then

$$2 = \widehat{\mathbf{v}} - \widehat{d}\mathbf{e} + \widehat{d}\mathbf{f} \leqslant \widehat{d}\mathbf{v} - \widehat{d}\mathbf{e} + \widehat{d}\mathbf{f} = 0.$$