Universal coefficient theorem for cohomology

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Reminder: UCT for homology, algebraic version: Let C be a chain complex of free abelian groups and let A be an abelian group.

Then there is a natural short exact sequence

$$0 \longrightarrow A \otimes H_n C \longrightarrow H_n(A \otimes C) \longrightarrow \mathrm{Tor}(A, H_{n-1}(C)) \longrightarrow 0$$
$$a \otimes [x] \longmapsto [a \otimes x]$$

Moreover, the sequence splits, i.e.

$$H_n(A \otimes C) \cong (A \otimes H_n C) \oplus \mathrm{Tor}(A, H_{n-1} C)$$

Specializing to $C = C(S(X); \mathbb{Z})$ for a space X gives a UCT for $H_n(X; A) \cong H_n(A \otimes C(S(X); \mathbb{Z}))$

The UCT for cohomology is an algebraic recipe to calculate $H^n(X; A)$ from $H_n(X; \mathbb{Z})$ and $H_{n-1}(X; \mathbb{Z})$.

So integral homology determines both homology and cohomology with arbitrary coefficient.

Strategy: Prove a UCT for $H^n(\mathrm{Hom}(C, A))$ for a chain complex C consisting of free abelian groups, then specialize to $C = C(S(X); \mathbb{Z})$ for a space X.

Construction: Let C be a chain complex and A an abelian group. We'll describe a natural homomorphism

$$\Phi : H^n(\mathrm{Hom}(C, A)) \longrightarrow \mathrm{Hom}(H_n C, A)$$

Let $f : C_n \to A$ be an n-cocycle in $\mathrm{Hom}(C, A)$, i.e. $d^n f = f \circ d_{n+1} = 0 : C_{n+1} \to A$.

We define $\Phi[f][x] = f(x)$. $\quad x \in C_n,\ d_n x = 0$

We must check that this is well-defined:

a) Let $y \in C_{n+1}$. Then $f(x + d_{n+1} y) = f(x) + \underbrace{f(d_{n+1}(y))}_{=0} = f(x)$, i.e. so $f(x)$ only depends on the homology class $[x] \in H_n C$ of x.

b) Let $g : C_{n-1} \to A$ be any homomorphism. Then $(f + d^{n-1}(g))(x) = (f + (g \circ d_n))(x) = f(x) + \underbrace{g(d_n x)}_{=0} = f(x)$. So $f(x)$ only depends on the class $[f] \in H^n(\mathrm{Hom}(C, A))$.

Prop: Let C be a chain complex of free abelian groups and A any abelian group. Then the homomorphism

$$\Phi : H^n(\mathrm{Hom}(C, A)) \longrightarrow \mathrm{Hom}(H_n C, A)$$

has an additive section. In particular, Φ is surjective.

Warning: The additive section cannot be arranged as a natural transformation in C and A.

Proof: We start from the short exact sequence of abelian groups $0 \to Z_n \xrightarrow{\mathrm{incl}} C_n \xrightarrow{d_n} B_{n-1} \to 0$, where $Z_n = \ker(d_n : C_n \to C_{n-1})$, $B_{n-1} = \mathrm{im}(d_n : C_n \to C_{n-1})$.

Since C_{n-1} is a free abelian group, so is its subgroup B_{n-1}; So we can choose an additive section $\sigma : B_{n-1} \to C_n$ of d_n, i.e. so that $d_n \circ \sigma = \mathrm{Id}_{B_{n-1}}$.

We observe that

$$d_n \circ (\mathrm{Id}_{C_n} - \sigma \circ d_n) = d_n - d_n \circ \sigma \circ d_n = 0.$$

So the homomorphism $\mathrm{Id}_{C_n} - \sigma \circ d_n : C_n \to C_n$ has image in the subgroup Z_n of n-cycles.

We set

$$r = \mathrm{Id}_{C_n} - \sigma \circ d_n : C_n \longrightarrow Z_n.$$

Construction of an additive section to Φ: Let $\gamma : H_n C \to A$ be any homomorphism.

Define $s(\gamma)$ to be the composite

$$C_n \xrightarrow{r} Z_n \xrightarrow{\mathrm{proj}} H_n C \xrightarrow{\gamma} A$$

$s(\gamma)$ is a cocycle in $\mathrm{Hom}(C, A)$:

$$d^n(s(\gamma)) = s(\gamma) \circ d_{n+1} : C_{n+1} \longrightarrow A$$

for $y \in C_{n+1}$, we have

$$d^n(s(\gamma))(y) = \gamma[r(d_{n+1}(y))] = \gamma[d_{n+1}(y) - \sigma(\underbrace{d_n(d_{n+1}(y))}_{=0})]$$
$$= \gamma[\underbrace{d_{n+1}(y)}_{=0}] = 0$$

So $d^n(s(\gamma)) = 0$, so $s(\gamma)$ represents a class

$$[s(\gamma)] \in H^n(\mathrm{Hom}(C, A))$$

- $s(\gamma)$ is additive in γ because composition of homomorphisms is additive in both variables.

- s defines a section to Φ:

$\Phi[s(\gamma)] = \gamma : H_n C \longrightarrow A$. Indeed, for all $x \in Z_n \subseteq C_n$, we have

$$\Phi[s(\gamma)][x] = s(\gamma)(x) = \gamma[x - \sigma(\underbrace{d_n x}_{=0})] = \gamma[x]$$ □

Review of Ext groups:

Let A and B be abelian groups. We choose an epimorphism $\varepsilon : F \longrightarrow B$ from a free abelian group F.

Let $K = \ker(\varepsilon)$ denote the kernel, and $i : K \longrightarrow F$ the inclusion. The Ext group is

$$\mathrm{Ext}(B,A) = \mathrm{coker}\left(\mathrm{Hom}(i,A) : \mathrm{Hom}(F,A) \longrightarrow \mathrm{Hom}(K,A)\right)$$

$$= \mathrm{Hom}(K,A) \Big/ \{ f : K \to A : f \text{ extends to a homomorphism } F \to A \}$$

Fact:
- The definition of $\mathrm{Ext}(B,A)$ is independent of the choice of $\varepsilon : F \to B$ up to preferred isomorphism.
- $\mathrm{Ext}(B,A)$ can be extended to a covariant functor in A, and to a contravariant functor in B
- $\mathrm{Ext}(B,A)$ is additive for direct sums in both variables.

Examples:
- Suppose that B is a free abelian group. Then we can take $F = B$, $\varepsilon = \mathrm{Id}_B$, so that $K = 0$. Hence $\mathrm{Ext}(B,A) = 0$ whenever B is free.
- Suppose $B = \mathbb{Z}/n\mathbb{Z}$. Then $\varepsilon : \mathbb{Z} \longrightarrow \mathbb{Z}/n$, $\varepsilon(x) = x + n\mathbb{Z}$ is a free resolution.

$$\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, A) = \mathrm{Hom}(n\mathbb{Z}, A) \Big/ \{ f : n\mathbb{Z} \to A : f \text{ has an additive extension to } \mathbb{Z} \}$$

$$\begin{array}{ccc} (g : n\mathbb{Z} \to A) & & \\ \downarrow & & \downarrow \cong \\ g(n) + nA & & A/nA \end{array}$$

- This determines $\mathrm{Ext}(B,A)$ for all finitely generated abelian groups B.

Rk: Ext and extension An extension of an abelian group B by an abelian group A is a short exact sequence of abelian groups

$$0 \to A \xrightarrow{i} E \xrightarrow{p} B \to 0 .$$

Such an extension is isomorphic to $0 \to A \xrightarrow{i'} E' \xrightarrow{p'} B \to 0$ if there is an isomorphism $f : E \xrightarrow{\cong} E'$ such that $f \circ i = i'$ and $p' \circ f = p$.

We define

$$\mathcal{E}xt(B,A) = \text{set of isomorphism classes of extensions of } B \text{ by } A.$$

I'll describe a map $\mathrm{Ext}(B,A) \longrightarrow \mathcal{E}xt(B,A)$ that is bijective.

Moreover, the addition in $\mathrm{Ext}(B,A)$ has an interpretation in terms of "Baer sum" of extensions.

$0 \to K \xrightarrow{i} F \xrightarrow{\varepsilon} B \to 0$

$$[f : K \to A] \longmapsto \left\{ 0 \to A \to \frac{F \oplus A}{\{(k, f(k)) : k \in K\}} \longrightarrow B \to 0 \right\}$$

$$a \mapsto [0, a]$$
$$[x, a] \longmapsto \varepsilon(x)$$

Thm: Let C be a chain complex of free abelian groups and A an abelian group. Then the kernel of the split epimorphism $\Phi : H^n(\mathrm{Hom}(C,A)) \longrightarrow \mathrm{Hom}(H_n C, A)$ is naturally isomorphic to $\mathrm{Ext}(H_{n-1}C, A)$.

Proof: Since C_{n-1} is free abelian, so is its subgroup $Z_{n-1} = \ker(d_{n-1} : C_{n-1} \to C_{n-2})$.
So the projection $\varepsilon : Z_{n-1} \to H_{n-1}(C)$, $x \mapsto [x]$, is a free resolution of $H_{n-1}C$, with $K = \ker(\varepsilon) = B_{n-1} = \mathrm{im}(d_n : C_n \to C_{n-1})$. So

$$\mathrm{Ext}(H_{n-1}C, A) = \frac{\mathrm{Hom}(B_{n-1}, A)}{\mathrm{Hom}(Z_{n-1}, A)}.$$

We define two chain complexes with trivial differentials B and Z made up from the boundaries $\{B_n\}_{n\in\mathbb{Z}}$ and cycles $\{Z_n\}_{n\in\mathbb{Z}}$, respectively. These participate in a short exact sequence of chain complexes

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$$0 \longrightarrow \underset{\text{here } d=0}{Z} \xrightarrow{\text{incl}} C \xrightarrow{\ d\ } \underset{\text{here } d=0}{B[1]} \longrightarrow 0 \qquad \text{shift}$$

Since all the groups B_n are free abelian (as subgroups of C_n), the short exact sequence splits dimensionwise

$$0 \longrightarrow Z_n \xrightarrow{\text{incl}} C_n \xrightarrow[\nabla \ldots]{d_n} B_{n-1} \longrightarrow 0$$

So $\quad Z_n \oplus B_{n-1} \xrightarrow{\cong} C_n$.

So $\mathrm{Hom}(-,A)$ yields a short exact sequence of cochain complexes

$$0 \longrightarrow \underset{\text{here } d=0}{\mathrm{Hom}(B[1],A)} \xrightarrow{\mathrm{Hom}(d,A)} \mathrm{Hom}(C,A) \xrightarrow{\text{restriction}} \underset{\text{here } d=0}{\mathrm{Hom}(Z,A)} \longrightarrow 0$$

We obtain a long exact sequence of cohomology groups:

$$\begin{tikzcd}
\cdots \arrow[r] & H^{n-1}(\mathrm{Hom}(Z,A)) \arrow[r, "\partial"] \arrow[d, equal] & H^n(\mathrm{Hom}(B[1],A)) \arrow[r] \arrow[d, equal] & H^n(\mathrm{Hom}(C,A)) \arrow[r] & H^n(\mathrm{Hom}(Z,A)) \arrow[r, "\partial"] \arrow[d, equal] & H^{n+1}(\mathrm{Hom}(B[1],A)) \arrow[r] \arrow[d, equal] & \cdots \\
& \mathrm{Hom}(Z_{n-1},A) \arrow[r, "\mathrm{Hom}(\text{incl},A)"'] & \mathrm{Hom}(B_{n-1},A) & & \mathrm{Hom}(Z_n,A) \arrow[r, "\mathrm{Hom}(\text{incl},A)"'] & \mathrm{Hom}(B_n,A) &
\end{tikzcd}$$

This decomposes into short exact sequence

$$0 \to \mathrm{Ext}(H_{n-1}C,A) = \frac{\mathrm{Hom}(B_{n-1},A)}{\mathrm{Hom}(Z_{n-1},A)} \longrightarrow H^n(\mathrm{Hom}(C,A)) \xrightarrow{(*)} \ker\left(\mathrm{Hom}(\text{incl},A)\right) \longrightarrow 0$$

It remains to construct an isomorphism $\ker\left(\mathrm{Hom}(\text{incl},A) : \mathrm{Hom}(Z_n,A) \to \mathrm{Hom}(B_n,A)\right)$.

that takes $(*)$ to the epimorphism Φ. $\qquad \cong \ \mathrm{Hom}(H_nC,A)$

General fact: Let $0 \to B \xrightarrow{i} Z \xrightarrow{p} H \to 0$ be a short exact sequence of abelian groups

(such as $0 \to B_n \xrightarrow{\text{incl}} Z_n \xrightarrow{\text{proj}} H_nC \to 0$), then the following sequence is also exact:

$$0 \longrightarrow \mathrm{Hom}(H,A) \xrightarrow{\mathrm{Hom}(p,A)} \mathrm{Hom}(Z,A) \xrightarrow{\mathrm{Hom}(i,A)} \mathrm{Hom}(B,A)$$

$\square$

Summary: Thm (algebraic UCT) For every chain complex of free abelian groups C and every abelian group A,

there is a natural short exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}C,A) \longrightarrow H^n(\mathrm{Hom}(C,A)) \xrightarrow{\Phi} \mathrm{Hom}(H_nC,A) \longrightarrow 0$$

The sequence splits (but not naturally).

We apply this to $C = C(S(X);\mathbb{Z})$ for some space X; we obtain:

Thm (topological UCT for cohomology) For every space X and abelian group A there is a split short exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X;\mathbb{Z}),A) \longrightarrow H^n(X;A) \longrightarrow \mathrm{Hom}(H_n(X;\mathbb{Z}),A) \longrightarrow 0$$

The sequence is natural for continuous maps in X.

In particular, $\quad H^n(X;A) \cong \mathrm{Hom}(H_n(X;\mathbb{Z}),A) \oplus \mathrm{Ext}(H_{n-1}(X;\mathbb{Z}),A).$

Application: Let $f: X \to Y$ be a continuous map such that $H_n(f;\mathbb{Z}) : H_n(X;\mathbb{Z}) \to H_n(Y;\mathbb{Z})$ is an isomorphism for all $n \geq 0$. Then $H^n(f;A) : H^n(Y;A) \to H^n(X;A)$ is an isomorphism for all $n \geq 0$ and all abelian groups A.

Proof: Compare the two UCT exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(Y;\mathbb{Z}),A) \arrow[r] \arrow[d, "\mathrm{Ext}(H_{n-1}(f;\mathbb{Z}),A)"', "\cong"] \arrow[dr, phantom, "\circlearrowleft \text{ naturality}"] & H^n(Y;A) \arrow[r, "\Phi"] \arrow[d, "H^n(f;A)"] \arrow[dr, phantom, "\circlearrowleft \text{ naturality of } \Phi"] & \mathrm{Hom}(H_n(Y;\mathbb{Z}),A) \arrow[r] \arrow[d, "\cong"', "\mathrm{Hom}(H_n(f;\mathbb{Z}),A)"] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X;\mathbb{Z}),A) \arrow[r] & H^n(X;A) \arrow[r, "\Phi"'] & \mathrm{Hom}(H_n(X;\mathbb{Z}),A) \arrow[r] & 0
\end{tikzcd}$$

The 5-lemma shows that $H^n(f, A)$ is an isomorphism.