Naturality of the duality map

In this section we verify that the duality maps for oriented manifold are natural for open embeddings. If U is an open subset of the n-manifold M, then U is itself an n-manifold. Moreover, if $\mu = \{\mu_x\}_{x\in M}$ is an orientation of M, then by simply forgetting the local orientations for points not in U, we obtain a 'restricted' orientation

$$\mu|_U \ = \ \{\mu_x\}_{x\in U}$$

for U. Compactly supported cohomology and singular homology are both covariantly functorial for the open inclusion $U \longrightarrow M$, and we now show that the duality maps are compatible with the functoriality.

Proposition 1. *Let (M,μ) be an oriented n-manifold, and let U be an open subset of M, endowed with the restricted orientation $\mu|_U$. Then the following square of group homomorphisms commutes for all $i \geq 0$:*

$$\begin{tikzcd}
H^i_{\mathrm{comp}}(U;\mathbb{Z}) \arrow[r, "D_U"] \arrow[d, "\iota^M_U"'] & H_{n-i}(U;\mathbb{Z}) \arrow[d, "\mathrm{incl}_*"] \\
H^i_{\mathrm{comp}}(M;\mathbb{Z}) \arrow[r, "D_M"'] & H_{n-i}(M;\mathbb{Z})
\end{tikzcd}$$

Proof. We start by recording that for all compact subsets K of U, the two orientation classes $(\mu|_U)_K$ and μ_K of K match up under the homomorphism

$$\mathrm{incl}_* \ : \ H_n(U, U\setminus K;\mathbb{Z}) \ \longrightarrow \ H_n(M, M\setminus K;\mathbb{Z})$$

induced by the inclusion $U \longrightarrow M$. Indeed, for every point $x \in K$ we have

$$r^M_x(\mathrm{incl}_*((\mu|_U)|_K)) \ = \ r^U_x(\mu|_U) \ = \ \mu_x$$

in the group $H^n(M|x;\mathbb{Z})$; the first equation is functoriality of relative singular homology, and the second equation is the definition of the restricted orientation $\mu|_U$. So the class $\mathrm{incl}_*(\mu|_U)$ enjoys the property that characterizes the class μ_K, and hence

$$\mathrm{incl}_*((\mu|_U)_K) \ = \ \mu_K \ . \tag{2}$$

To keep track of the following calculation, the reader might want to refer to the following diagram:

$$\begin{tikzcd}
H^i(U, U\setminus K;\mathbb{Z}) \arrow[r, "\lambda_K"'] \arrow[rr, bend left, "(\mu|_U)_K\cap -"] & H^i_{\mathrm{comp}}(U;\mathbb{Z}) \arrow[r, "D_U"'] \arrow[d, "\iota^M_U"'] & H_{n-i}(U;\mathbb{Z}) \arrow[d, "\mathrm{incl}_*"] \\
H^i(M, M\setminus K;\mathbb{Z}) \arrow[u, "\mathrm{incl}^*", "\cong"'] \arrow[r, "\lambda_K"] \arrow[rr, bend right, "\mu_K\cap -"'] & H^i_{\mathrm{comp}}(M;\mathbb{Z}) \arrow[r, "D_M"] & H_{n-i}(M;\mathbb{Z})
\end{tikzcd}$$

The left square commutes by the defining property of the homomorphism ι^M_U. The commutativity of the right square is what we aim to show.

Every class in $H^i_{\mathrm{comp}}(U;\mathbb{Z})$ is of the form $\lambda_K(\alpha)$ for some compact subset K of U and some relative cohomology class $\alpha \in H^i(U, U\setminus K;\mathbb{Z})$. By excision, $\alpha = \mathrm{incl}^*(\beta)$ for a unique class $\beta \in H^i(M, M\setminus K;\mathbb{Z})$.

So

$$\begin{aligned}
\mathrm{incl}_*(D_U(\lambda_K(\alpha))) &= \mathrm{incl}_*((\mu|_U)_K \cap \alpha) \\
&= \mathrm{incl}_*((\mu|_U)_K \cap \mathrm{incl}^*(\beta)) \\
&= \mathrm{incl}_*((\mu|_U)_K) \cap \beta \\
\overset{(2)}{=}&\ \mu_K \cap \beta \\
&= D_M(\lambda_K(\beta)) \\
&= D_M(\iota_U^M(\lambda_K(\mathrm{incl}^*(\beta)))) \\
&= D_M(\iota_U^M(\lambda_K(\alpha)))\ .
\end{aligned}$$

Since the classes $\lambda_K(\alpha)$ account for all classes in $H^i_{\mathrm{comp}}(U;\mathbb{Z})$, this proves the claim. $\square$