Poincaré duality with mod-2 coefficients

With mod-2 coefficients, Poincaré duality holds for all manifolds, orientable or not. The arguments are analogous to – but simpler than – the arguments for integral Poincaré duality in the oriented case; so we omit the proof. As an application of mod-2 Poincaré duality we show that every compact manifold of odd dimension has vanishing Euler characteristic.

We let M be an oriented n-manifold. In an earlier lecture we had constructed the mod-2 orientation classes

$$\nu_K \in H_n(M, M \setminus K; \mathbb{F}_2) = H_n(M|K; \mathbb{F}_2)$$

for compact subsets K of M. The class ν_K is uniquely characterized by the property that for all points $x \in K$, the restriction

$$r_x^K(\nu_K) \in H_n(M|x; \mathbb{F}_2)$$

is non-zero, and hence the generator of $H_n(M|x; \mathbb{F}_2) \cong \mathbb{F}_2$. If M itself is compact, then ν_M is the mod-2 fundamental class in $H_n(M; \mathbb{F}_2)$. In contrast to the integral case, the class ν_K exists without any orientability hypothesis, and no local orientations need to be chosen.

From here the construction of the mod-2 duality proceeds exactly as in the integral case: cap product with the class ν_K is a homomorphism

$$\nu_K \cap - \ : \ H^i(M, M \setminus K; \mathbb{F}_2) \longrightarrow H_{n-i}(M; \mathbb{F}_2) \ ,$$

and these homomorphisms satisfy the compatibilities necessary to assemble into a homomorphism from $H^i_{\mathrm{comp}}(M; \mathbb{F}_2)$. So there is a unique homomorphism

$$D_M \ : \ H^i_{\mathrm{comp}}(M; \mathbb{F}_2) \longrightarrow H_{n-i}(M; \mathbb{F}_2) \ ,$$

the *mod-2 duality map*, such that for all compact subsets K of M, the composite

$$H^i(M, M \setminus K; \mathbb{F}_2) \xrightarrow{\lambda_K} H^i_{\mathrm{comp}}(M; \mathbb{F}_2) \xrightarrow{D_M} H_{n-i}(M; \mathbb{F}_2)$$

is cap product with the mod-2 orientation class ν_K. If M itself happens to be compact, then $D_M(\alpha) = \nu_M \cap \alpha$.

Theorem 1 (Mod-2 Poincaré duality). *For every n-manifold M and all $i \geq 0$, the duality map*

$$D_M \ : \ H^i_{\mathrm{comp}}(M; \mathbb{F}_2) \longrightarrow H_{n-i}(M; \mathbb{F}_2)$$

is an isomorphism.

As already mentioned, the proof of mod-2 Poincaré duality is completely analogous to the proof of the intregral version; at some points, the arguments are even simpler because orientations play no role.

Corollary 2. *Let M be a compact n-manifold.*

(i) *For every $i \geq 0$, the $\mathbb{F}_2$-vector space $H_i(M; \mathbb{F}_2)$ is finite-dimensional.*
(ii) *If the dimension n is odd, then the Euler characteristic of M is zero.*

Proof. (i) We recall some generalities that are valid over any field k. For a k-vector space V, we write $V^* = \mathrm{Hom}_k(V, k)$ for the dual vector space. If V is finite-dimensional, then V is isomorphic (non-canonically) to V^*. If V is infinite dimensional, then the dual V^* has a strictly larger cardinality; in particular, V and V^* are then not isomorphic.

The contravariant functor sending V to V^* is exact; so for every chain complex C of k-vector spaces, and every integer i, the evaluation homomorphism

$$\Phi \ : \ H^i(\mathrm{Hom}_k(C, k)) \longrightarrow \mathrm{Hom}_k(H_i(C), k) = H_i(C)^* \ , \quad \Phi[f : C_i \longrightarrow k][x] = f(x)$$

is an isomorphism of k-vector spaces.

Now we turn to the situation at hand: we take $k = \mathbb{F}_2$ and we let $C = C_*(\mathcal{S}(M); \mathbb{F}_2)$ be the singular chain complex of M with coefficients in $\mathbb{F}_2$. Then the evaluation homomorphism specializes to isomorphisms

$$\Phi \ : \ H^i(M; \mathbb{F}_2) \ \cong \ H_i(M; \mathbb{F}_2)^* \ .$$

These evaluation isomorphisms in dimensions i and $n - i$ and two instances of Poincaré duality yield a sequence of isomorphisms of $\mathbb{F}_2$-vector spaces

$$H_i(M; \mathbb{F}_2) \ \cong \ H^{n-i}(M; \mathbb{F}_2) \ \cong \ H_{n-i}(M; \mathbb{F}_2)^* \ \cong \ H^i(M; \mathbb{F}_2)^* \ \cong \ H_i(M; \mathbb{F}_2)^{**}$$

between $H_i(M; \mathbb{F}_2)$ and its double-dual vector space. This can only happen if the vector space $H_i(M; \mathbb{F}_2)$ is finite-dimensional.

(ii) Part (i) guarantees that the Euler characteristic based on mod-2 homology

$$\chi(M) \ = \ \sum_{i \geq 0} (-1)^i \cdot \dim_{\mathbb{F}_2} H_i(M; \mathbb{F}_2)$$

is well-defined. Poincaré duality and evaluation provide two isomorphisms

$$H_i(M; \mathbb{F}_2) \ \cong \ H^{n-i}(M; \mathbb{F}_2) \ \cong \ H_{n-i}(M; \mathbb{F}_2)^* \ ;$$

so the vector spaces $H_i(M; \mathbb{F}_2)$ and $H_{n-i}(M; \mathbb{F}_2)$ have the same dimension. If n is odd, then the dimensions of $H_i(M; \mathbb{F}_2)$ and $H_{n-i}(M; \mathbb{F}_2)$ contribute with opposite signs to $\chi(M)$, so the contributions cancel in pairs. Hence $\chi(M) = 0$ whenever the dimension n is odd. $\square$

Remark 3. Part (i) of the previous theorem is only a shadow of the stronger statement that for every compact manifold M, all the integral homology groups $H_i(M; \mathbb{Z})$ are finitely generated. The proofs of this fact that I know of are all involved, so I will not prove the statement in this class. Hatcher gives a proof in Corollaries A.8 and A.9 in the Appendix of 'Algebraic Topology', based on the concept of 'euclidean neighborhood retracts'.

For those compact manifolds that admit a CW-structure (necessarily finite), the finite generation of $H_i(M; \mathbb{Z})$ follows from cellular homology; but one should beware that there are manifolds that do not admit a CW-structure, although examples are hard to come by. In contrast, *smooth manifold* do admit triangulations, and hence also CW-structure; but that fact, too, is not easy to prove.