Legendrian Knots

Tor Gjone (s1706798)

Year 4 Project
School of Mathematics
University of Edinburgh
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Chapter 1

Legendrian Knots

1.1 Contact Geometry

1.1.1 Definition

A contact structure on an oriented 3-manifold M is a certain kind of plane filed over M. More precisely it is a rank 2 sub-bundle $\xi \subset TM$, given locally by the kernel of a one-form α (ie. for all $x \in M$, $\exists U \subset M$, such that $\xi_x = \ker(\alpha_x)$ for all $x \in U$) satisfying

$$\alpha \wedge \mathrm{d}\alpha \neq 0. \quad \text{(ie. everywhere non-zero.)} \tag{1.1}$$

We will also require that $\alpha \wedge \mathrm{d}\alpha$ agrees with the orientation orientation of M. Intuitively condition (1.1), require the planes to be twisting, see figure (1.1). Because of this twisting, there can not be a surface in $S \subset M$, such that $\xi|S = TS$. A contact manifold is a pair (M, ξ), where M is an oriented 3-manifold and ξ is a contact structure on M.

1.1.2 Examples

Example 1.1.1. *Consider $M = \mathbb{R}^3$, with Cartesian coordinates (x, y, z). Let $\alpha = \mathrm{d}z - y\mathrm{d}x$, then $\alpha \wedge \mathrm{d}\alpha = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \neq 0$. Define*

$$\xi_{std} := \ker\alpha = \operatorname{span}\left\{\frac{d}{dy}, \frac{d}{dx} + y\frac{d}{dz}\right\}.$$

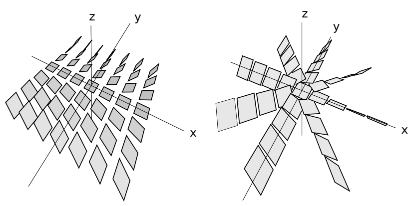

Figure 1.1: Contact structure ξ_{std} (left) and ξ_{sym} (right).

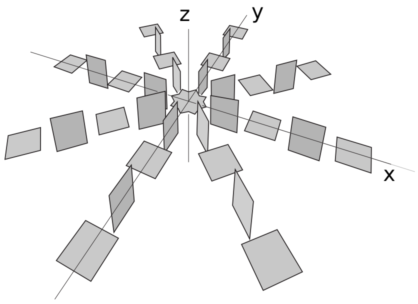

Figure 1.2: The overtwisted contact structure ξ_{ot}.

ξ_{std} is called the standard contact structure on $\mathbb{R}^3$. See fig (1.1).

There are a couple key features of this plane filed. Note that restricted to a plane parallel with the xz-plane all the planes are parallel, or sad in a different way the pane filed is invariant under translations orthogonal to the y direction. In particular the planes along the xz-plane are all horizontal, that is parallel with the xy-plane. On the other hand, when moving along the y-axis the planes twist in a left hand manner. Twisting a total of 90° going from the origin "to infinity" in the y-direction.

Example 1.1.2. *Again let $M = \mathbb{R}^3$ and let $\alpha = \mathrm{d}z + x\mathrm{d}y - y\mathrm{d}x$ (or $\alpha = \mathrm{d}z + r^2\mathrm{d}\theta$ in cylindrical coordinates.) Then $\alpha \wedge \mathrm{d}\alpha = 2\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \neq 0$, so α defines a contact structure. Define*

$$\xi_{sym} := \ker \alpha = \operatorname{span}\left\{ \frac{d}{dr}, \frac{d}{d\theta} - r^2 \frac{d}{dz} \right\}.$$

This is the symmetric version of ξ_{std}.

Here the planes are invariant under vertical translations (along the z-axis) and rotation about the z-axis.

Definition 1.1.3. *Let M and N, be two oriented 3-manifolds and suppose ξ_M and ξ_N are are contact structures on M and N respectively. A diffeomorphism $f : M \to N$ is called a contactomorphism, if f_* maps ξ_M to ξ_N.*

If such a contactomorphism exist ξ_M and ξ_N are sad to be contactomorphic.

The contact structures ξ_{std} and ξ_{sym} are contactomorphic, so in a way they are the same structure. Also suppose we defined ξ_{std} with the opposite sign (ie. $\alpha = \mathrm{d}z + y\mathrm{d}x$), then the planes would instead twist in a right handed manner, when moving in the y direction. Then this structure would be contactomorphic to ξ_{std}, by the diffeomorphism mapping y to $-y$.

Example 1.1.4. *Let $M = \mathbb{R}^3$ and $\alpha = \cos r\mathrm{d}z + r\sin r\mathrm{d}\theta$ and define the contact structure $\xi_{ot} = \ker \alpha$. See fig (1.2)*

Note how this structure in a way is very similar to ξ_{sym}, except that when moving along a ray perpendicular to the z-axis, the twisting is much quicker. So in instead of rotating a total of 90° when going to infinity, the planes are already twisted 90° when getting to a radius of $\pi/2$ and rotates an infinite number of times when going to infinity.

Definition 1.1.5. *Suppose ξ defines a contact structure on M. Then, we say ξ is overtwisted, if there exist an immersion $u : D(0,1) \to M$, where $D(0,1) = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ is the unit disk, such that*

$$\xi|_{\partial D} = TD|_{\partial D}.$$

(ie. ξ is tangent to D along the border of D.) Such a disk immersion is called an overtwisted disk.

If ξ is not overtwisted we say ξ is tight.

Clearly ξ_{ot} is overtwisted since $D = \{(r,\theta,z) \mid z = 0, 0 \leq r \leq 1\}$ is an overtwisted disk. On the other hand ξ_{std} and ξ_{sym} are tight. It turns out that tight contact structures are more interesting then the overtwisted ones, see Eliashberg (1989). In this paper we will mainly concentrate on $(\mathbb{R}^3, \xi_{std})$. However, we will mention one other example.

Example 1.1.6. *Let $M = S^3$ considered as the unit sphere in $\mathbb{R}^4$ (with Cartesian coordinates (x_1, y_1, x_2, y_2)), and let*

$$\alpha = i^*(x_1 \mathrm{d}y_1 - y_1 \mathrm{d}x_1 + x_2 \mathrm{d}y_2 - y_2 \mathrm{d}x_2),$$

where $i : S^3 \mathbb{R}^4$ is the inclusion map. One may check that $\alpha \wedge \mathrm{d}\alpha \neq 0$, and define $\xi = \ker \alpha$.

By removing one point p from S^3, the contact structure ξ on $S^3 \setminus \{p\}$ is contactomorphic to ξ_{sym} (and thus ξ_{std}) on $\mathbb{R}^3$.

In fact, if we associate $\mathbb{R}^4$ with $\mathbb{C}^2$, we can construct this contact structure from the complex structure on C^2. Let J denote complex multiplication in $\mathbb{C}^2$, ie. $Jx_i = y_i$ and $Jy_i = x_i$ for $i = 1, 2$. This complex structure includes a complex structure on the tangent space $J\frac{d}{dx_i} = \frac{d}{dy_i}$ and $J\frac{d}{dy_i} = \frac{d}{dx}$.

Clame 1.1.7. *Then ξ is the complex tangents of S^3. ie.*

$$\xi = TS^3 \cap J(TS^3),$$

that is the tangent space closed under complex multiplication.

Proof. Let $f : R^4 \to R^4$, defined by $f(x_1, y_1, x_2, y_2) = x_1^2 + y_1^2 + x_2^2 + y_2^2$, then $S^3 = f^{-1}(1)$. Then the tangent space of S^3 at (x_1, y_1, x_2, y_2)

$$T_{(x_1,y_1,x_2,y_2)}S^3 = \ker \mathrm{d}f_{(x_1,y_1,x_2,y_2)},$$

and

$$J(T_{(x_1,y_1,x_2,y_2)}S^3) = \ker \mathrm{d}f_{(x_1,y_1,x_2,y_2)} \circ J.$$

Since

$$\mathrm{d}f_{(x_1,y_1,x_2,y_2)} \circ J = 2x_1 \mathrm{d}y_1 - 2y_1 \mathrm{d}x_1 + 2x_2 \mathrm{d}y_2 - 2y_2 \mathrm{d}x_2$$

So $\alpha = (\mathrm{d}f \circ J)|_S^3$, which proves the clime. □

1.2 Legendrian Knots

1.2.1 Definition

Definition 1.2.1. *Let (M, ξ) be a contact manifold. A* Legendrian knot *$L \subset M$ is an embedding of S^1, such that*

$$T_x L = \xi_x, \quad \forall x \in L. \tag{1.2}$$

For the rest of this paper we will assume $(M, \xi) = (\mathbb{R}^3, \xi_{std})$. Most of the discussions and result generalize to general contact manifolds. But for simplicity we will stick with this particular case. This allows us to work in terms of the projections onto the coordinate planes, which will simplify some of the arguments significantly. In particular the proofs found in chapter (4), will naturally simplify to a combinatorial argument.

1.2.2 Projections

For this section it will convenient to have a parametrization for L. Let

$$\gamma : S^1 \to L;\ t \mapsto (x(t), y(t), z(t)),$$

be a C^1 (ones continuously differentiable) parametrization of L. Then we can be written equation (1.2),

$$\gamma'(t) \subset \xi_{\gamma(t)}, \quad \forall t \in S^1,$$

or equivalently, since $\xi = \xi_{std} = \ker \mathrm{d}z - y\mathrm{d}x$,

$$z'(t) - y(t)x'(t) = 0. \tag{1.3}$$

Definition 1.2.2. Front projection *is the map*

$$\Pi : \mathbb{R}^3 \to \mathbb{R}^2;\ (x, y, z) \mapsto (x, z),$$

Let $\phi_\Pi = \Pi \circ \gamma$. By equation (1.3), $z'(t) = y(t)x'(t)$, so if $x'(t) = 0$, $z'(t) = 0$. Therefore $\Pi(L)$ can not have any vertical tangents $\big($ie. $\phi'_\Pi(t) \notin \operatorname{span}\left\{\frac{d}{dz}\right\} \setminus \{0\}$ for all t. $\big)$ Also $(\phi_\Pi)'(0) = 0$, whenever $x'(t) = 0$, so ϕ_Π is on not an immersion. On the other hand, away from $x'(t) = 0$ (ie. $t \in I \setminus x'^{-1}(0)$), ϕ_Π is an immersion. If $x'(t) \neq 0$, by equation (1.3), we may recover $y(t)$ from the projection

$$y(t) = \frac{z'(t)}{x'(t)}.$$

A point in $\Pi(L)$ where $x'(t) = 0$ we will call a "cusp point". Here the tangent of L is parallel with the y-axis. See figure (1.3).

Lemma 1.2.3. *A diagram $D \subset \mathbb{R}^2$, is the Front projection of a Legendrian knot if and only if:*

1. *There are no vertical tangents.*

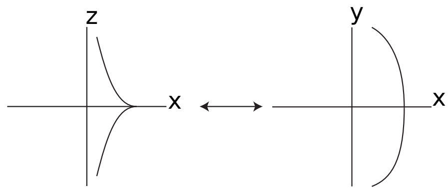

Figure 1.3: Front and Lagrangian projection of a part of a Legendrian knot where $x'(t) = 0$ (ie. a cusp point, in the front projection.)

2. *D is an immersion away from a finite collection of cusp, points.*
3. *At each crossing the slope of the overcrossing is less then the slope of the undercrossing.*

Example 1.2.4. *In figure (1.4) we can see the front projection of a Legendrian realization of the unkont, left and right handed trefoil knot and the figure of eight knot.*

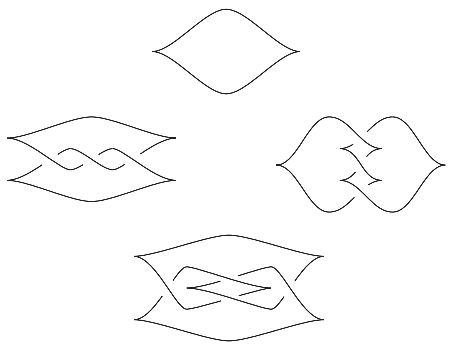

Figure 1.4: Front projection of Legendrian realization of the unknot, left and right handed trefoil knots and the figure of eight knot.

Definition 1.2.5. *The Lagrangian projection of L is the map*

$$\pi : \mathbb{R}^3 \to \mathbb{R}^2;\ (x, y, z) \to (x, y),$$

Let $\phi_\pi = \pi \circ \gamma$. In contrast with the front projection, ϕ_Π is always an immersion, since by equation (1.3), if $x'(t) = 0$ also $z'(t) = 0$ and thus $y'(t) \neq 0$, as ϕ is an immersion.

By integrating equation (1.3), we have

$$z(t) = z_0 + \int_0^t y(t')x'(t')\mathrm{d}t',$$

(Here we consider $t \in [0, 2\pi)$) so we may recover the z-component of L from the projection up to an overall factor z_0. Note that for L to be closed we require $z(2\pi) = z_0$, so

$$\int_S^1 y(t)x'(t)\mathrm{d}t = 0. \tag{1.4}$$

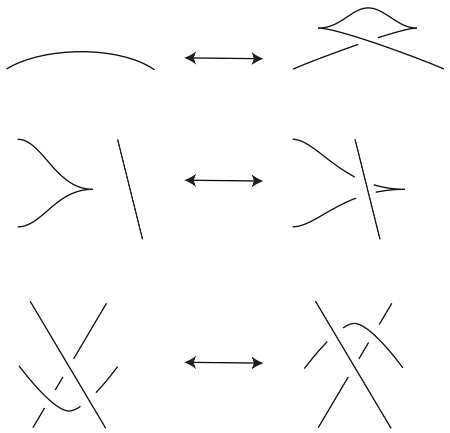

Figure 1.5: Legendrian Reidemeister moves in front projection.

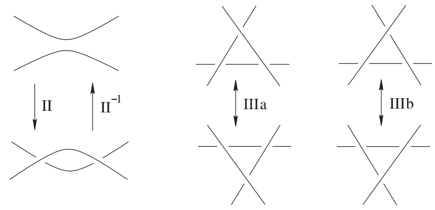

Figure 1.6: Legendrian Reidemeister moves in Lagrangian projection.

Lemma 1.2.6. *A diagram $D \in \mathbb{R}^2$ is the Lagrangian projection of a Legendrian knot if and only if*

1. *D is an immersion.*
2. *$\int_S^1 y(t)x'(t)\mathrm{d}t = 0$, where $(x, y) : S^1 \to \mathbb{R}^2$ is a parameterization of D.*
3. *$\int_{t_1}^{t_2} y(t)x'(t)\mathrm{d}t = 0$, if $(x(t_1), y(t_1)) = (x(t_2), y(t_2))$.*

This projection will be the more interesting for us. Though the constraints (ie. (1.4)), given by the Legendrian condition (1.2), is a little harder to work with.

1.3 Reidemeister moves

Like in the case of topological knots, there are also Legendrian Reidemeier moves for these projections, see figure (1.6)

1.4 Classical Invariants

Definition 1.4.1. *Let $\gamma : [0, 1] \to \mathbb{R}^2$ be a smooth curve. The gausian map of γ $g : [0, 1] \to S^1 \subset \mathbb{R}^2$ is given by*

$$g(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}.$$

Where $S^1 \subset \mathbb{R}^2$ is thought of as the unit circle.

Consider the quationt map $q : \mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$. There is a unique (up to translation by $\mathbb{Z}$) lift (by q) of g, $\widetilde{g}$, ie. a map $\widetilde{g} : [0, 1] \to \mathbb{R}$ such that $g = q \circ \widetilde{g}$.

rotation number of $r(\gamma) = g(1) - g(0) \in \mathbb{R}$.

Then the rotation number of γ is $r(\gamma) = \widetilde{g}(1) - \widetilde{g}(0)$.

Note that, if the path is closed, $r(\gamma) \in \mathbb{Z}$. Also the rotation number invariant under reparameterization, so we may write $r(\mathrm{im}(\gamma))$. (In particualt we will write $r(L) \in \mathbb{Z}$, for the rotation number of L)

Chapter 2

The associated A_∞algebra

2.1 A_∞-algebras

2.1.1 Definition

Definition 2.1.1. *Let k be a field and $\Gamma = \mathbb{Z}/c\mathbb{Z}$ be a cyclic group, for some $c \in \mathbb{Z}$. An $\underline{A_\infty\text{-algebra over } k}$ is a Γ-graded vector space*

$$A = \bigoplus_{p \in \Gamma} A^p$$

endowed with a family of graded k-linear maps

$$m_n : A^{\otimes n} \to A, n \geq 0,$$

of degree $2 - n$ (mod c) satisfying the A_∞relations:

$$\sum_{r+s+t=n} (-1)^{r+st} m_u(\mathbb{1}^{\otimes r} \otimes m_s \otimes \mathbb{1}^{\otimes t}) = 0. \tag{2.1}$$

Where the sum runs over all $r, s, t \geq 0$, such that $r + s + t = n$ and $u = r + 1 + t$.

Remark:

Often this definition is known as a <u>curved</u> A_∞-algebra, where m_0 is the <u>curvature</u> of A. If $m_0 = 0$ we say A is <u>uncurved</u>. If A is uncurved, the first A_∞relation (ie. $n = 1$), demands that $m_1 \circ m_1 = 0$ and we may define the <u>cohomology</u> $H^*(A, m) = \ker(m_1)/\operatorname{im}(m_1)$, a graded k-vector space. The second A_∞relation is

$$m_2(m_2 \otimes \mathbb{1} + \mathbb{1} \otimes m_2),$$

so m_2 defines a product on the <u>cohomology</u>, which by the third relation if associative.

In this notes we will, by an A_∞-algebra, refer to the more general case of a curved A_∞-algebra.

Notation

Let

$$TA := \bigoplus_{n\in\mathbb{Z}} A^{\otimes n}$$

denote the tensor-algebra generated by A, with the canonical product

$$w : TA \otimes TA \to TA;\ (v,w) \mapsto v \otimes w.$$

Then the family $m_n : A^{\otimes n} \to A$ may be writen as a single map $m : TA \to A$. Throughout this paper we will write an A_∞-algebra as a pair (A, m), where $m : TA \to A$.

If $a \in A$, them we will write $|a| \in \Gamma$ to denote the degree of a.

Definition 2.1.2. *An A_∞-algebra (A, m) is* <u>*finite*</u> *if there exists N such that*

$$m_n = 0,\ \textit{for all } n \geq N.$$

2.1.2 Morphisms

Definition 2.1.3. *Let (A, m^A) and (B, m^b) be A_∞-algebras. An* <u>*A_∞-homomorphism*</u> *$f : A \dashrightarrow B$ is a family of graded k-linear maps*

$$f_n : A^{\otimes n} \to B, n \geq 0$$

of degree $1 - n$ such that

$$\sum(-1)^{r+st} f_u(\mathbb{1}^{\otimes r} \otimes m_s^A \otimes \mathbb{1}^{\otimes t}) = \sum(-1)^s m_r^B(f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_r}), \qquad (2.2)$$

where the first sum runs over all decompositions $n = r + s + t$, like in (2.1), and the second sum runs over all $1 \leq r \leq n$ and all decompositions $n = i_1 + \dots + i_r$. The sign on the right hand side is given by

$$s = \sum_{j=1}^{r-1}(r - j)(i_j - 1).$$

f_0 is called the <u>curvature</u> of f and if $f_0 = 0$ we say f is <u>uncurved</u>. If (A, m^A), (B, m^B) and f are all uncurved, then the first A_∞-relation is

$$f_1 \circ m_1^A = m_1^B \circ f_1.$$

So f_1 induces a linear map between the cohomologies

$$f^* : H^*(A, m^A) \to H^*(B, m^B).$$

Note that the further conditions demands that the induced linear map, also conserves the other algebraic structure on the cohomology. In particular the second relation ($n = 2$), implies that

$$f^* \circ m_2^* = m_2(f^* \otimes f^*),$$

where m_2^* is the product induced by m_2. So the linear map is in particular a homomorphism of algebras.

Example 2.1.4. *If (A, m) is an A_∞-algebra, the identity $\mathrm{id}^A : A \dashrightarrow A$ given by*

$$\mathrm{id}_i^A = \begin{cases} \mathrm{id} & \text{if} \quad i = 1 \\ 0 & \text{if} \quad i \neq 1, \end{cases}$$

is an A_∞-homomorphism.

Definition 2.1.5. *If (A, m^A) and (B, m^B) are finite and $f : A \dashrightarrow B$ is an A_∞-homomorphism, we say f is finite if there exists N, such that $f_n = 0$ for all $n > N$.*

If $N = 1$ and $f_0 = 0$, we say f is strict.

Definition 2.1.6. *The composition of two A_∞-homomorphisms $f : B \dashrightarrow C$ and $g : A \dashrightarrow B$ is given by*

$$(f \circ g)_n := \sum (-1)^s f_r \circ (g_{i_1} \otimes ... \otimes g_{i_r}), \tag{2.3}$$

where the sum and sign are the same as in the defining identity.

Lemma 2.1.7. *The composition $f \circ g : A \dashrightarrow C$, defines a new A_∞-homomorphism and if both f and g are finite, so is*

Proof. Through a simple swap of sums it is relatively easy to show that $f \circ g$, indeed satisfy (2.2).

Suppose $N \in \mathbb{N}$, is such that $f_n = 0$ and $g_n = 0$ for all $n > N$, then if $n > N^2$, either $r > N$ or at least one $i > N$. So $(f \circ g)_n = 0$. □

Definition 2.1.8. *An A_∞homomorphism $f : A \dashrightarrow B$, is an isomorphism if there exists an A_∞-homomorphism $g : B \dashrightarrow A$, such that*

$$g \circ f = \mathrm{id}^A \quad \text{and} \quad f \circ g = \mathrm{id}^B .$$

Lemma 2.1.9. *Let (A, m^A) and (B, m^b) be uncurved A_∞-algebras. If $f : A \dashrightarrow B$ is an A_∞-isomorphism, then the induced linear map $f^* : H^*(A, m^A) \to H^*(B, m^B)$ is also an isomorphism of graded vector spaces.*

move

Lemma 2.1.10. *Let (A, m) be an A_∞-algera and $f : A \dashrightarrow A$ an A_∞-pre-homomorphism, such that f_1 is an isomorphism. Then there exists a unique A_∞-structure $\widetilde{m}$ such that f is an A_∞-homomorphism from (A, m) to $(A, \widetilde{m})$.*

Proof.

$$\widetilde{m}_k(f_1(a_1), ..., f_1(a_k)) = P(f_i, \widetilde{m}_{\leq k-1}\}, m_j).$$

□

Proof. By definition, $(f \circ g)_1 = f_1 \circ g_1$. So $(f \circ g)^* = f^* \circ g^*$. Hence $f^* \circ g^* = \mathrm{id}_{H^*(A,m^A)}$ and $g^* \circ f^* = \mathrm{id}_{H^*(B,m^B)}$ and thus f^* is invertible. Since f_1 has degree 0, so does f^*. $\square$

2.1.3 Tame morphisms

Definition 2.1.11. *Let (A, m^A) be an A_∞-algebra, where $A = k\langle a_1, ..., a_n\rangle$. An A_∞-automorphism $f : A \dashrightarrow A$ is called* <u>*elementary*</u> *if there exist $j \in \{1, ..., n\}$ and $u : TA \to k$, such that for all $m \in \mathbb{N}$*

$$u_m(..., a_j, ...) = 0 \ \text{ and } \ f_m(a_{i_1}, ..., a_{i_m}) = \mathrm{id}^A + a_j u(a_{i_1}, ..., a_{i_m}),$$

A finite composition of such A_∞-automorphisms is called <u>*tame automorphism*</u>*. A* <u>*tame A_∞-isomorphism*</u> *$f : A \dashrightarrow B$ is the composition of a tame automorphisms and a strict isomorphisms.*

2.1.4 Stabilization

Definition 2.1.12. *Let (A, m) be an A_∞algebra. The* <u>*i-th stabilization*</u> *of A, is a new A_∞-algebra $\left(S_i(A), m^{S_i(A)}\right)$, where*

$$S_i(A) = A \oplus k\langle e_1, e_2\rangle,$$

$|e_1| = i$, $|e_2| = i + 1$,

$$m_1(e_1) = e_2, \quad m_1(e_2) = 0, \quad m_n(..., e_j, ...) = 0 \quad \forall n \neq 1$$

and $m^{S_i(A)}(a_1, ..., a_n) = m_n(a_1, ..., a_n)$ for all $n \in \mathbb{N}$ and $a_1, ..., a_n \in A$.

Definition 2.1.13. *Let (A, m^A) and (B, m^B) be A_∞-algebras, then they are sad to have the same* <u>*stable type*</u> *if there exist $i_1, ..., i_k, j_1, ..., j_l \in \mathbb{Z}$ and a tame A_∞-isomorphism*

$$f : S_{i_1}(...(S_{i_k}(A, m^A))...) \to S_{j_1}(...(S_{j_l}(B, m^B))...).$$

Let $\sigma : S_i(A) \dashrightarrow A$ and $\tau : A \dashrightarrow S_i(A)$, such that $\sigma_k = \tau_k = 0$ for $k > 1$, and

$$\tau_1(a) = a, \sigma_1(a) = a, \ \text{ and } \ \sigma_1(e_j) = 0,$$

for all $a \in A$ and $j = 1, 2$. ie. τ is the inclusion of A in $S_i(A)$ and σ is the projection of $S_i(A)$ onto A.

This is not done !!!

Lemma 2.1.14. *There exists a graded linear map $h : TS_i(A) \to TS_i(A)$, such that*

$$\tau + \mathrm{id}_{S_i(A)} = h \circ m^{S_i(A)} + m \circ h$$

$\tau \circ \sigma$ is homotopic to id'_A as A_∞-hom.

2.2 Constructing the A_∞algebra

In this section we will to every Legendrian knot (or more acuratly the Legendrian projection) assosiate a finite A_∞-algebra. This contruction directly corresonds to the Chekanov-Eliashberg DGA constricted in (Chekanov (2002)).

Let L be a Legendrian knot in the standard contact structure on $\mathbb{R}^3$.

2.2.1 The vector space

Definition 2.2.1. *L is sad to be generic wrt. π, if the set of crossings in $\pi(L)$ is finite and consist of transversal double points. More precisely, if $c \in \pi(L)$ is a crossing. Then $\pi^{-1} \cap L = \{c^+, c^-\}$ and if $T^\pm$ is the tangent line of L at $c^\pm$, then $\pi(T^+) \cap \pi(T^-) = \{c\}$.*

For the rest if this paper will assume L is generic and let $Y = \pi(L)$. (Note that, any Legendrian knot is C^∞-close to a generic one, ie. there exist a smooth one parameter family of Legendrian knots $\{L_\alpha\}\alpha \in [0,1]$, such that $L = L_0$ and L_α is generic for $\alpha > 0$.)

Definition 2.2.2. *Let $\mathcal{C} = \{a_1, \ldots, a_n\}$ denote the set of double points in Y and $A = \mathbb{Z}_2 \langle \mathcal{C} \rangle$, be the $\mathbb{Z}_2$-vector space generated by $\mathcal{C}$.*

This vector space will be the base space, for the A_∞-algebra. The grading on the vector space will take values in the cyclic group $\Gamma = \mathbb{Z}/m(L)\mathbb{Z}$, where $m(L)$ is twice the rotation number of L. To define the grading on the vector space we will define the grading on each of crossing $c \in \mathcal{C}$, by the the rotation number of the path from c_+ to c_- along L.

Definition 2.2.3. *Let $x, y \in L$ be distinct, then there are two paths from x to y along L (that is up to re-parametrization, and such that the path does not go around L multiple times, ie. L is not contained in the image of the path).*

Let $\gamma_1, \gamma_2 : [0,1] \to L$, be parametrizations of these two paths. Then $\gamma_1 * (-\gamma_2)$ is a parametrization of L (where $*$ denotes concatenation), so $r(\gamma_1) - r(\gamma_2) = r(L)$.

Define $\theta : L \times L \to \mathbb{R}/m(L)\mathbb{Z}$, by

$$\theta(x, y) = 2r(\gamma_1) = 2r(\gamma_2) \mod m(L),$$

where $m(L) = 2r(L)$. (See figure (2.1))

Note that θ is anti-symmetric and satisfies, $\theta(x, y) + \theta(y, z) = \theta(x, z)$, for all $x, y, z \in L$.

Consider a crossing $c \in \mathcal{C}(Y)$, then since we assumed L is generic wrt. π, there exists unique $c_+, c_- \in L$, such that $\pi(c_+) = \pi(c_-) = c$ and $z(c_+) > z(c_-)$, where $z(c)$ denotes the z-component of c in $\mathbb{R}^3$. We define the grading of c, by

$$|c| = \lceil \theta(c^+, c^-) \rceil \in \mathbb{Z}/2r(L)\mathbb{Z}.$$

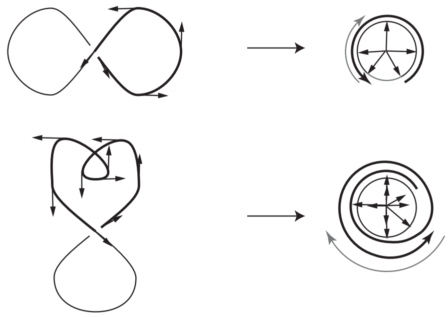

Figure 2.1: Gaussian map

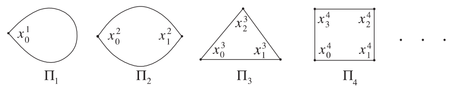

Figure 2.2: k sided (curved) polygon.

2.2.2 Admissible immersions

Before defining the A_∞-maps, we need to define some combinatorial objects, used to define them.

For $k \in \mathbb{N}$, let $\Pi_k \subset \mathbb{R}^2$ be a convex k-gon, like in figure (2.2), with vertices $x_0^k, ..., x_{k-1}^k$ numbered anticlockwise.

Definition 2.2.4. *Let $W_k(Y)$ be the set of orientation-preserving immersions $u : \Pi_k \to \mathbb{R}^2$, such that $u(\partial\Pi_k) \subset Y$, where $\partial\Pi_k$ denotes the boundary of Π_k.*

Let $\mathrm{Diff}^*(\Pi_k)$ be the group of orientation-preserving diffeomorphisms of Π_k fixing the vertices, ie. the set of re-parametrisations of Π_k. Then we define

$$\widetilde{W}_k(Y) = W_k / \mathrm{Diff}^*(\Pi_k),$$

the set of non-parametrized immersions.

Note that if $u \in W_k(Y)$, then $u(x_i^k) \in \mathcal{C}$ is a crossing in Y and inner angle is less then π.

Let $u \in \widetilde{W}_k(Y)$ and consider a small neighbourhood $U \subset \mathbb{R}^2$ of the crossing $u(x_i^k) \in \mathcal{C}$. Then Y divides U into four quadrants. See figure (2.3). Exactly one of this quadrants is then covered by a neighbourhood $V \subset \Pi_k$ of x_i^k.

Let $c_+, c_- \in L$, such that $\pi(c_+) = \pi(c_-) = u(x_i^k)$ and $z(c_+) > z(c_-)$. Consider a short path $\phi : I_\epsilon = (-\epsilon, \epsilon) \to L$, such that $\pi(\phi(0)) = u(x_i^k)$, $\pi(\phi(I_\epsilon)) \subset u(U \cap \partial\Pi_k)$ and going anticlockwise through the crossing (that is anticlockwise around the quadrant covered by u). Then either

$$\lim_{x \to 0^\pm} \phi(x) = c_\pm \quad \text{or} \quad \lim_{x \to 0^\pm} \phi(x) = c_\mp.$$

In the first case the z-component of ϕ increase going through the crossing and in the second case it decreases. This is clearly only dependent on the crossing and which quadrant is covered, so we may associate the signs with the quadrants as shown in figure (2.3).

Definition 2.2.5. *Let $u \in \widetilde{W}_k(Y)$, if a neighbourhood of x_i^k is mapped to a positive (resp. negative) quadrant, we say, x_i^k is positive (resp. negative) for u.*

An immersion $u \in \widetilde{W}_k(Y)$ is called admissible, if x_0^k is positive for u and x_i^k is negative for u, for all $i \geq 1$. Let

$$W_k^+(Y) = \{u \in \widetilde{W}_k(Y) \mid u \text{ is admissible}\}.$$

If $a \in \mathcal{C}$, we write

$$W_k^+(Y, a) = \{u \in W_k^+(Y) \mid u(x_0^k) = a\}.$$

Lemma 2.2.6. *$W_k^+(Y)$ is finite.*

The proof will be covered in the following chapter.

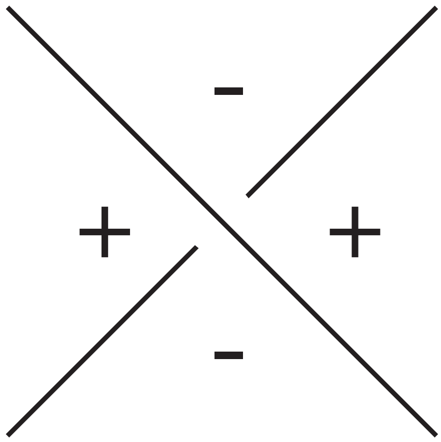

Figure 2.3: Quadrants near a crossing in $\pi(L)$.

2.2.3 The A_∞-maps m_k

Fix k, and let $a, b_1, ..., \in \mathcal{C}$, then denote

$$\mathcal{M}^a_{b_1 \dots b_k} = \{u \in W^+(Y, a) \mid u(x_i^k) = b_i \text{ for } i \geq 1\}.$$

Then define

$$m_k(b_1, ..., b_k) := \sum_{a \in \mathcal{C}} (\#_2 \mathcal{M}^a_{b_1 \dots b_k}) a,$$

where $\#_2$ denotes the number of elements modulo 2, which is well-defined, by lemma (2.2.6).

Theorem 2.2.7 (Chekanov 2002, Chekanov (2002))**.** *The maps m_k have degree $2 - k$ and satisfies the curved A_∞-relations, ie. $\forall n \geq 0$,*

$$\xi_n := \sum_{r+s+t=n} m_u(\mathbb{1}^{\otimes r} \otimes m_s \otimes \mathbb{1}^{\otimes t}) = 0, \tag{2.4}$$

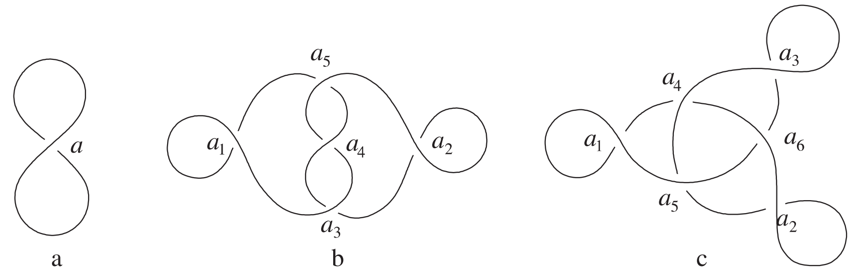

Figure 2.4: k sided polygon.

where $u = r + 1 + t$. Hence, by lemma (2.2.6), (A, m_i) defines a finite curved A_∞-algebra.

Theorem 2.2.8 (Chekanov 2002, Chekanov (2002))**.** *Let (A, m) and (A', m') be the A_∞-algebras associated to Legendrian knots L and L'. Then if L and L' differ by a Legendrian isotopy, (A, m) and (A', m') have the same sable type.*

The proofs of these theorems are given in chapter 4.

2.3 Examples

We will start by computing the A_∞algebra associated with the Legendrian projection of some very simple knots shown in figure 2.4.

Example 2.3.1. *Let Y_0 be the Lagrangian projection of the Legendrian unknot L_0, shown figure (2.4a). The winding number $\sigma(L_0) = 0$, so the grading takes values in $\Gamma = \mathbb{Z}$. There is only one crossing a, with degree $|a| = 1$. So $A = \mathbb{Z}_2\langle a\rangle$. The set of immersions $\widetilde{W}_k(Y_0)$ is empty for $k > 1$ and $\widetilde{W}_k(Y_0) = \{f_1, f_2\}$, where f_1 and f_2 are the immersions whose image are the two bounded components of $\mathbb{R}^2 \setminus Y_a$. Considering the sign of the quadrants of the crossing, we observe that both f_1 and f_2 are admissible. Hence $m_k = 0$ for all $k > 0$ and $m_0(1) = \#_2\{f_1, f_2\}a = 0$.*

Example 2.3.2. *Consider Y_r given by figure (2.4b), the Lagrangian projection of the Legendrian Right-handed trefoil knot L_r. The winding number $r = 0$ again, so $\Gamma = \mathbb{Z}$. There is 5 crossings, $A = \langle b_1, b_2, b_3, a_1, a_2\rangle$, and the grading is given by*

$$|a_i| = 1, \quad |b_i| = 0.$$

There is two admissible immersions with one vertex, there is also four with two vertices and two with four (see figure 2.4b). So m is given by

$$\begin{aligned} m_0(1) = m_1(b_1) = m_1(b_3) &= a_1 + a_2 \\ m_3(b_1, b_2, b_3) &= a_1 \\ m_3(b_3, b_2, b_1) &= a_2. \end{aligned}$$

And all other terms are zero.

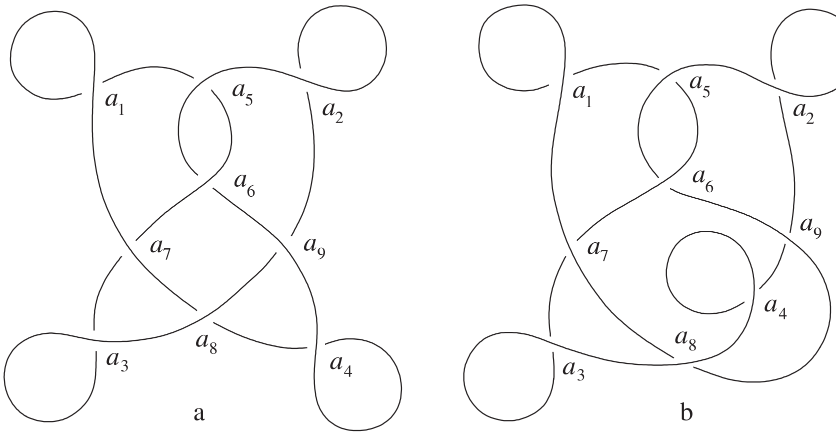

Figure 2.5: k sided polygon.

We will also have a look at two other slightly more complicated knots, shown in figure 2.5. Later we will see that with the invariant induced by the associated A_∞algebra one are able to distinguish this two knots, even though both knots has the same classical invariants.

Example 2.3.3. *Let Y_a and Y_b be given by figure (2.5). For both of the knots the winding number $r = 0$ and there is 9 crossings, enumerated as shown in the figure. Let's first consider diagram (a), the grading is given by*

$$|a_i| = 1, \quad |a_5| = 2, \quad |a_6| = -2 \text{ and } |a_j| = 0,$$

where $i = 1, ..., 4$ and $j = 7, ..., 9$. The A_∞maps m_i is given by

$$\begin{aligned} m_0(1) &= a_1 + a_2 + a_3 + a_4, \\ m_3(a_7, a_6, a_5) = m_1(a_7) &= a_1, \\ m_3(a_5, a_6, a_9) = m_1(a_2) &= a_2, \\ m_2(a_8, a_7) &= a_3, \\ m_2(a_9, a_8) &= a_4. \end{aligned}$$

And all other terms are zero. Now les's consider the second diagram. The grading is given by

$$|b_i| = 1 \text{ and } |b_j| = 0,$$

where $i = 1, ..., 4$ and $j = 5, ..., 9$. The A_∞maps are given by

$$\begin{aligned}
m_0(1) &= b_1 + b_2 + b_3 + b_4,\\
m_3(b_7, b_6, b_5) = m_1(b_7) &= b_1,\\
m_3(b_9, b_8, b_5) = m_1(b_5) &= b_1,\\
m_3(b_5, b_6, b_9) = m_1(b_9) &= b_2,\\
m_2(b_8, b_7) &= b_3,\\
m_2(b_9, b_8) &= b_4.
\end{aligned}$$

And all other terms are zero. So clearly this are different A_∞ *algebras, but that is not in itself a sufficent argument for to say that they are different knots.*

Chapter 3

The Chekanov invariant

3.1 Maurer-Cartan elements and the Chekanov-Poincaré polynomials

Consider a finite A_∞-algebra (A, m) over $\mathbb{Z}_2$. If A in uncurved, recall that we may define the cohomology

$$H^*(A, m_1) = \frac{\ker m_1}{\operatorname{im} m_1},$$

a Γ graded k-vector space.

Definition 3.1.1. *If A is uncurved and finite-dimensional then so is the cohomology, so we may define the Poincaré Polynomial*

$$P_A(\lambda) := \sum_{i \in \Gamma} \dim \left(H^i(A, m_1)\right) \cdot \lambda^i.$$

This is clearly a invariant of the cohomology of uncurved A_∞algebras. However is is not clear how this is useful as an invariant of Legendrian knots, as the associated A_∞algebra in general is curved.

Definition 3.1.2. *Let $a \in A$, then, for $n \in \mathbb{N}$, define $m^c : TA \to A$,by*

$$m_n^c(a_1, ..., a_n) := \sum_{i_0,...,i_n} m_u\big(\underbrace{c, ..., c}_{i_0}, a_1, \underbrace{c, ..., c}_{i_1}, a_2, c, ..., c, a_n, \underbrace{c, ..., c}_{i_n}\big), \tag{3.1}$$

where the sum runs over all combinations of $i_0, ..., i_n \geq 0$.

Note that, since A is finite, the sum is well-defined. Also note that from here on, we will write $c...$, to mean a sequence of c's. If we write $c, ..., c$, the dots might also contain more then just c's.

Lemma 3.1.3. *If $|c| = 1$, then $A^c := (A, m^c)$ defines a new finite curved A_∞-algebra*

Proof. First we need $|c| = 1$, in order for the maps to have the correct degree. With $|c| = 1$ this works out, since the degree increases by 1 for each occurrence of c and decreases by 1 for each extra entry in m_u. It is immediately clear from the definition that it is finite, so it suffuses to check that the A_∞-relations hold.

$$\begin{aligned}
&\sum_{r,s,t} m^c_u(a_1, ..., a_r, m^c_s(a_{r+1}, ..., a_{r+s}), a_{r+s+1}, ..., a_n) \\
&= \sum_{r,s,t} \sum_{i_0,...,i_s} m^c_u(a_1, ..., a_r, m_{u'}(\underbrace{c...}_{i_0}, a_{r+1}, c, ..., c, a_{r+s}, \underbrace{c...}_{i_s}) a_{r+s+1}, ..., a_n) \\
&= \sum_{r,s,t} \sum_{i_0,...,i_s} \sum_{j_0,...,j_u} m_{u''}(\underbrace{c...}_{i_0}, a_1, c, ...c, a_r, \underbrace{c...}_{j_r}, \\
&\qquad m_{u'}(\underbrace{c...}_{i_0}, a_{r+1}, c, ..., c, a_{r+s}, \underbrace{c...}_{i_{r+s}}), \underbrace{c, ...}_{j_{r+1}}, a_{r+s+1}, c, ..., c, a_n, \underbrace{c...}_{j_u})
\end{aligned}$$

In the first line we expand the inner m^c according to (3.1), and in the second line we expand the outer m^c. The indices r, s, t are as in the A_∞-relations (2.1) and the indices $i_0, ..., i_s$ and $j_0, ..., j_u$ are as in (3.1). Finally $u = r+t+1$, $u' = s+\sum i_k$ and $u'' = r+t+\sum j_k$. (Note that, for now on, this indices will be assumed without stated explicitly.)

Considering the sums in the last line, we can sort them according to the total number of c's (ie. $N = \sum_k i_k + \sum_k j_k$).

$$\sum_{r,s,t} \sum_{i_0,...,i_s} \sum_{j_0,...,j_u} \{...\} = \sum_{N\geq 0} \sum_{r,s,t} \sum_{i'_0,...,i'_s} \sum_{j'_0,...,j'_u} \{...\}.$$

Then for each N the inner sum is the left hand side of the N'th A_∞-relation in m applied to $\xi_N \in A^{\otimes N}$, the sum of all ways of putting N c's between the a's. ie.

$$\xi_N = \sum_{k_0,...,k_n} (\underbrace{c...}_{k_0}, a_1, c, ..., c, a_n, \underbrace{c...}_{k_n}), \qquad k_0 + ... + k_n = N.$$

Since the m satisfy the A_∞-relation, we are done. □

Definition 3.1.4. *An element $c \in A$ is called a* <u>*Maurer-Cartan element*</u> *(MC element) if $|c| = 1$ and*

$$\sum_{n\geq 0} m_n(c...) = 0.$$

Lemma 3.1.5. *(A, m^c) is uncurved if and only if c is a MC element.*

Proof. By definition,

$$m^c_0 = \sum_{n\geq 0} m_n(c...) = 0$$

□

Definition 3.1.6. *If A is a finite-dimensional finite curved A_∞-algebra, define*

$$I(A) := \{P_{A^c} \ : \ c \in A \textit{ is a MC element}\}.$$

We later will see that this set of Poincaré polynomials is invariant under stable tame isomorphism and thus Legendrian isotopy. Not that the isomorphism need not actually be tame, for this to be true.

Definition 3.1.7. *Let $c \in A$ and $f : A \dashrightarrow B$, then like in (3.1), define $f_n : A^{\otimes n} \to B$ by*

$$f_n^c(a_1, ..., a_n) := \sum_{i_0,...,i_n} f_u(\underbrace{c...}_{i_0}, a_1, c, ..., c, a_n, \underbrace{c...}_{i_n}). \tag{3.2}$$

Also, we write $f_*(c) := f_0^c = \sum_{n\geq 0} f_n(c...)$.

Lemma 3.1.8. *If $f : A \dashrightarrow B$ and $c \in A$ is a MC element, then so is $f_*(c) \in B$.*

Proof. By the defining relation of f (2.2) and linearity we have.

$$\begin{aligned} m_n^B(f_*(c), ..., f_*(c)) &= \sum_{i_1,...,i_n} m_n^B(f_{i_1}(c...), ..., f_{i_n}(c...)) = \sum_{r,s,t} f_u(c..., m_s(c...), c...) \\ &= \sum_{r,t} f_u(c..., \sum_s m_s(c...), c...) = 0 \end{aligned}$$

□

Lemma 3.1.9. $\{f_n^c\}_{n\geq 1}$ *defines a finite uncurved A_∞-homomorphism*

$$f : A^c \to B^{f_*(c)}.$$

Proof. We need to check that f, satisfy equation (2.2). We have

$$\begin{aligned} &\mathrm{LHS}(a_1, ..., a_n) \\ &= \sum_{r,s,t} f_u^c(a_1, ..., a_r, m_s^{A,c}(a_{r+1}, ..., a_{r+s}), a_{r+s+1}, ..., a_n) \\ &= \sum_{r,s,t} \sum_{i_0,...,i_s} \sum_{j_0,...,j_u} f_u\big(\underbrace{c...}_{j_0}, a_1, c, ..., c, a_r, \underbrace{c...}_{j_r}, \\ &\qquad\qquad m_s^A(\underbrace{c...}_{i_0}, a_{r+1}, c, ..., c, a_{r+s}, \underbrace{c...}_{i_s}), c..., a_{r+s+1}, c..., a_n\big) \end{aligned}$$

The first line is precisely what appear in the equation. In the following line we expand m^c and f^c. Like in the proof of lemma (3.1.3), we sort the terms according the total number of c's. We get the left hand side of the Nth A_∞-homomorphism relation in f and m, applied to $\xi_N \in A^{\otimes N}$ (where ξ is as in lemma (3.1.3)). Applying the relation we have

$$\begin{aligned} \mathrm{LHS}(a_1, ..., a_n) &= \sum_N \sum_{i_1,...,i_r} m_r^A(f_{i_1} \otimes ... \otimes f_n)(\xi) \\ &= \sum_N \sum_{i_1,...,i_r} \sum_{k_0,...,k_n} m_r^A(f_{i_1} \otimes ... \otimes f_{i_r})(\underbrace{c...}_{k_0}, a_1, c, ..., c, a_n, \underbrace{c...}_{k_n}) \end{aligned}$$

Now, let's also study the right hand side.

$$
\begin{aligned}
&\mathrm{RHS}(a_1, ..., a_n) \\
&= \sum_{i_1,...,i_r} m_r^{B,f_*(c)}(f^c_{i_1}(a_1, ..., a_{i_1}), ..., f^c_{i_r}(a_{n-i_r}, ..., a_n)) \\
&= \sum_{i_1,...,i_r} \sum_{k_0,...,k_s} m_r^B(f_{i_1}(\underbrace{c...}_{k_0}, a_1, c, ..., c, a_{i_1}, \underbrace{c...}_{k_{i_1}}), ..., f_{i_r}(\underbrace{c...}_{k_{s-i_r}}, a_{n-i_r}, c..., a_n, \underbrace{c...}_{k_s}))
\end{aligned}
$$

where $s = r + \sum i_*$. It is quite clear that these sums agree and thus we are done. $\square$

Lemma 3.1.10. *Let $f : B \dashrightarrow C$ and $g : A \to B$, then for any MC element $c \in A$, $(f \circ g)^c = f^{g_*(c)} \circ g^c$ and in particular $(f \circ g)_*(c) = f_*(g_*(c))$.*

Proof.

$$
\begin{aligned}
&(f_{g_*(c)} \circ g_c)_n(a_1, ..., a_n) \\
&= \sum_{i_1,...,i_r} f^{g_*(c)}((g_c)_{i_1}(a_1, ..., a_{i_1}), ..., (g_c)_{i_r}(a_{n-i_r}, ..., a_n)) \\
&= \sum_{i_1,...,i_r} \sum_{j_0,...,j_s} f(\underbrace{g_*(c)...}_{j_0 \text{ times}}, (g_c)_{i_1}(a_1, ..., a_{i_1}), g_*(c), ..., g_*(c), (g_c)_{i_r}(a_{n-i_r}, ..., a_n), \underbrace{g_*(c)...}_{j_s})
\end{aligned}
$$

Here the first equality follows from expanding the definition of the composition. So the sum in i runs over all i's such that $i_1 + ... + i_r = n$. The second equality, follows from the expanding of the definition of f^c. Note the notation $\alpha...$, means α is repeated some number of times.

$$
\begin{aligned}
&(f \circ g)^c_n(a_1, ..., a_n) \\
&= \sum_{i_1,...,i_n} (f \circ g)_n(\underbrace{c...}_{i_0}, a_1, c, ..., c, a_n, \underbrace{c...}_{i_n}) \\
&= \sum_{i_1,...,i_n} \sum_{j_1,...,j_s} f_s(\underbrace{\overbrace{c, ..., c}^{\text{apply } g_{j_0}}, c..., c...}_{i_0}, \overbrace{a_1, c, ..., c, a_*, c...}^{g_{j_*}}, \overbrace{c, ..., c}^{g_{j_*}}, ..., \overbrace{c..., a_n, c...}^{g_{j_*}}, ..., \overbrace{c, ..., c}^{g_{j_s}}) \\
&= \sum_{k_1,...,k_r} \sum_{l_0,...,l_s} f(\underbrace{g_*(c)...}_{j_0}, (g_c)_{i_1}(a_1, ..., a_{i_1}), g_*(c), ..., g_*(c), (g_c)_{i_r}(a_{n-i_r}, ..., a_n), \underbrace{g_*(c)...}_{j_s})
\end{aligned}
$$

In the first line we expand $(f \circ g)^c$ according to def. (3.1.7). And in the second line we expand the composition, where the underbraces indicate applying g to the indicated elements. On the last line we recognise the definition of g_c and g_* in the previous line and re-index accordingly. $\square$

Corollary 3.1.11. *If $f : A \dashrightarrow B$ is a finite uncurved A_∞-isomorphism, then for any MC element $c \in A$,*

$$(f^c)^* : H^*(A, m_1^{A,c}) \to H^*(B, m_1^{B,f_*(c)})$$

is an isomorphism.

Proof. We have

$$A^c \xrightarrow{f^c} B^{f_*(c)} \xrightarrow{g^{f_*(c)}} A^{g_*(f_*(c))} = A^c,$$

where the equality follows from the lemma above. Also, by the lemma above

$$g^{f_*(c)} \circ f^c = (g \circ f)^c = \mathrm{id}_A^c = \mathrm{id}\,.$$

Similarly $f^c \circ g^{f_*(c)} = \mathrm{id}_B$. So by lemma (2.1.9), the result follows. $\square$

Lemma 3.1.12. *If $f : A \dashrightarrow B$ is an A_∞-isomorphism, then $I(A) = I(B)$.*

Proof. If $c \in A$ is a MC element, $f_*(c) \in B$ is an MC element and

$$(f^c)^* : H^*(A, m_1^{A,c}) \to H^*(B, m_1^{B,f_*(c)})$$

is an isomorphism, and thus

$$P_{A^c} = P_{B^{f_*(c)}}.$$

Therefore $I(B) \subseteq I(A)$. By applying the same argument with $g^{(f_*(c))}$,
$I(A) \subseteq I(B)$ and the result follows. $\square$

Now, the only thing we have left in order to show that $I(A)$ is a Legendrian knot invariant. That is

Lemma 3.1.13. $I(A) = I(S_i(A))$.

Proof. Let $c \in A$ be a MC element. Then $c \oplus 0 \in A \oplus k\langle e_1, e_2\rangle$ is a MC element in $S_i(A)$. Since clearly $|c \oplus 0| = |c| = 1$ and

$$\sum_{n \geq 0} m_n^{S_i(A)}((c \oplus 0)...) = \sum_{n \geq 0} m_n^A(c...) = 0.$$

Also, it follows easily from the definition of the stabilization that $m_1^{S_i(A),c}(a) = m_n^{A,c}(a)$ for all $a \in A$ and $m_1^{S_i(A),c}(e_j) = m_1^{S_i(A)}(e_j)$. So, since $m_1^{S_i(A)}(e_1) = e_2$ and $m_1^{S_i(A)}(e_2) = 0$,

$$\operatorname{im} m_1^{S_i(A),c} = \operatorname{im} m_1^{A,c} \oplus \operatorname{im} e_2 \quad \text{and} \quad [e_1] = [0] \in H^*(S_i(A), m_1^{S_i(A),c}).$$

. So

$$H^*\Big(S_i(A), m_1^{S_i(A),c}\Big) = H^*(A, m_1^{A,c})$$

Thus $I(A) \subseteq I(S_I(A))$.

Now for the reverse inclusion, suppose $c \oplus (\alpha e_1 + \beta e_2) \in S_i(A)$ is a MC element for some $c \in A$, $\alpha, \beta \in k$. Then

$$0 = \sum_{n \geq 0} m_n^{S_i(A)}(c \oplus (\alpha e_1 + \beta e_2)...) = \left(\sum_{n \geq 0} m_n^A(c...)\right) \oplus \alpha e_2.$$

So $\alpha = 0$ and $c \in A$ is a MC element.

As above it is easy to check that

$$H^*\left(S_i(A), m_1^{S_i(A),c\oplus\beta e_2}\right) = H^*(A, m_1^{A,c}).$$

Hence $I(A) \supseteq I(S_I(A))$. $\square$

It then follows directly from two lemmas above that.

Theorem 3.1.14. *$I(A)$ an invariant of stable type and thus, by theorem (2.2.8), a Legendrian knot invariant.*

Note that the tame A_∞-isomorphism condition, in the definition of the stable type, is not necessary for this theorem to hold. Instead an finite A_∞-isomorphism would suffice.

3.2 Examples

Calculate $I(A)$ for the check-elias-knots.

Chapter 4

The Proofs

In this chapter we will takel the proof of the two most centereal theorems in this paper. Theorem (2.2.7) and (2.2.8). Since we are spesificly working with the standard contactstructure on $\mathbb{R}^3$ and more spesificly the lagrandian projection, the proofs will nauturaly reduce to a combinatorial agrument.

Throught this chapter we will consider a fixed but arbitrary Legendrian knot L and it's Lagrangian projection $Y = \pi(L)$.

Both of the proofs, including the lemmas presente in the first section is an adoptation of the the proofs given in Chekanov (2002). (Addapted to the language of A_∞-algebras.)

4.1 Preliminary Lemmas

Consider a crossing $c \in \mathcal{C}(Y)$, then like when we defined the grading, let $c_+, c_- \in L$, such $\pi(c_+) = \pi(c_-) = c$ and $z(c_+) > z(c_-)$.

Definition 4.1.1. *Definie the the height map*

$$H : \mathcal{C}(Y) \to \mathbb{R}^+,$$

by $H(c) = z(c_+) - z(c_-)$, for all $c \in \mathcal{C}$.

Lemma 4.1.2. *Let $u \in W_k(Y)$. Then*

$$\sum_{x \in Q_+} H(u(x)) - \sum_{x \in Q_-} H(u(x)) = \int_{\Pi_k} u^*(\mathrm{d}y \wedge \mathrm{d}x) > 0,$$

where Q_+ (resp. Q_-) is the set of positive (resp. negative) vertices for u. In particular, at least one of the vertices of Π_k is positive.

Proof. Consider the integral $I = \int_{u(\partial \pi_k)} y \mathrm{d}x$. Then, by Stoke's theorem, we have

$$I = \int_{\Pi_k} u^*(\mathrm{d}y \wedge \mathrm{d}x) > 0,$$

where the inequality is due to u beeing orientation preserving.

Let $X_i \subset \partial\Pi_i$ denote the edge connection x_i^k and x_{i+1}^k, where $0 \leq i \leq k-1$. Let $\gamma_i : X_i \to L$, such that

$$\pi \circ \gamma_i = u|_{X_i}.$$

Since $\partial\Pi_k = \bigsqcup_i X_i$, we have

$$I = \sum_i \int_{\gamma_i(X_i)} y \mathrm{d}x.$$

Also for all i, γ_i is a legendrian curve, ie. $\gamma_i'(d) \in \ker(\mathrm{d}z - y\mathrm{d}x)$. So

$$I = \sum_i \int_{\gamma_i(X_i)} \mathrm{d}z.$$

For each i, let $\sigma : [0,1] \to \mathbb{R}^3$ be a parameterisation of the vertical line segment connecting $z_+(u(x_i^k))$ with $z_-(u(_i^k))$ (ie. $\pi \circ \sigma_i$ is constent at $u(x_i^k)$,) oriented from z_- to z_+ if x_i^k if positive for u and from z_+ to z_- if x_i^k is negative for u.

By assosiating X_i with the unit interval, oriented from x_i^k to x_{i+1}^k we concatenate the curves into one closed curve, $\Gamma = \gamma_0 \star \sigma_0 \star ... \star \gamma_{k-1} \star \sigma_{k-1}$. Since Γ is closed, $\int_\Gamma \mathrm{d}z = 0$. Hence

$$I = \sum_i \int_{\gamma_i(X_i)} \mathrm{d}z - \int_\Gamma \mathrm{d}z = -\sum_i \int_{\sigma[0,1]} \mathrm{d}z \quad = LHS.$$

$\square$

The following corolary follows immideatly.

Corollary 4.1.3. *If $u \in W^+(Y)$ then $H(u(x_0^k)) \geq \sum_{i=1}^{k-1} H(u(x_i^k))$.*

proof of lemma (2.2.6)

Consider the complement of L in $\mathbb{R}^2$. Then for some indexing set I, let $\{U_i\}_{i\in I}$ denote the set of connected components, ie. U_i is connected for each $i \in I$ and

$$\bigcup_{i\in I} U_i = \mathbb{R}^2 \setminus L.$$

Since L is compact, clearly I is finite. Let $I = \{0, 1, ..., m\}$ and suppose U_0 is the unbounded component.

Let S_i be the area of U_i, then

$$\int_{\Pi_k} f^*(\mathrm{d}y \wedge \mathrm{d}x) = \sum_{i=1}^{n} n_i(f) S_i,$$

where the integer $n_i(f) \geq 0$ equal to the cardinality of $f^{-1}(z)$, for $z \in U_i$.

It follows from from the corollary above that $\sum_{i=1}^m n_i(f) S_i$ is bounded above by $\max_{c\in\mathcal{C}} H(c)$. Since $S_i > 0$, there are only finitely many ways of choosing the coefficients n_i. Also, for a fixed sequence $n_1, ..., n_m$, there are clearly only finitely

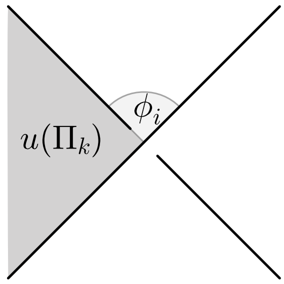

Figure 4.1: The exterior angle ϕ_i.

many admissible immersions f, such that $n_i(f) = n_i$. Hence, the total number of admissible immersions is finite. $\square$

Lemma 4.1.4. *Let* $u \in \widetilde{W}_k(Y)$*. Then*

$$\sum_{x \in Q_+} \deg(u(x)) - \sum_{x \in Q_-} \deg(u(x)) = 2 - |Q_-|,$$

where Q_+ *(resp.* Q_-*) is the set of positive (resp. negative) vertices with respect to* f*.*

Note that it follows from the particular case of $|Q_+| = 1$, that the A_∞-maps m_k has degree $2 - k$.

Proof. Like in the proof of (4.1.2), let $X_i \subset \partial \Pi_k$ be the edge connecting x_i^k and x_{i+1}^k. Consider the smooth immersions $\rho_i : X_i \to L$, such that $f|_{X_i} = \pi \circ \rho_i$. Let $y_i = \rho_i(x_i^k)$, $y'_{i+1} = \rho_i(x_{i+1}^k)$ and $y'_0 = y'_k$, for $0 \leq i \leq k-1$. Then let

$$C_1 = \sum_{i=0}^{k-1} \theta(y_i, y'_{i+1}) \quad \text{and} \quad C_2 = \sum_{i=0}^{k-1} \theta(y'_i, y_i),$$

where $\theta : L \times L \to \Gamma$ is the map from definition (2.2.3).

Then, by the additivity of θ, $C_1 + C_2 = 0 \bmod m(L)$. Also $C_1 = 2r(K)$, where $r(K)$ is the rotation number of the closed curve $K = u(\partial \Pi_k)$ (defined as the sum of the rotation numbers of its smooth pieces). Let ϕ_i denote the exterior angle of the crossing at $u(x_i^k)$, where exterior refers to the angle of a quadrant adjacent to the one covered by u, see figure (4.1).

Then $r(K) = 2 - \sum \phi_i$ and thus $C_2 = -C_1 = 2 \sum \phi$. Also,

$$|u(x_i^k)| = \lceil \theta(z_i^+, z_i^-) \rceil = \begin{cases} -\theta_i + \phi_i & \text{if} \quad x_i \in Q_+ \\ Q_i + (1 - \phi_) & \text{if} \quad x_i \in Q_-. \end{cases}$$

By defining the sign μ_i, given by $\mu_i = \mp 1$ if $x_i \in Q_\pm$, we can write

$$|u(x_i^k)| = \mu \theta_i - \mu_i + \frac{1 + \mu_i}{2}.$$

Then, using $\mu_i^2 = 1$,

$$LHS = -\sum \mu_i |f(x_i)| = \sum (\theta_i - \phi_i + \frac{1+\mu_i}{2}) = -(-2 + |Q_-|).$$

$\square$

4.2 Proof of theorem (2.2.7)

The proof here is essentially the same as that of Theorem 3.3 in chapter 7 [Chekanov (2002)]. The magare differece is found in the fist subsection, in which we reduce the algebraic problem to a combinatoiral one.

4.2.1 Plan of the proof

By the linearity of (2.4), it suffuses to show that it is satisfied when applied on the generators. That is

$$\xi_n(a_1, ..., a_n) = \sum_{r+s+t=n} m_u(a_1, ..., a_r, m_s(a_{r+1,...,r+s}), a_{r+s+1}, ..., a_n) = 0,$$

for all $a_1, ..., a_n \in \mathcal{C}$ (and $u = r + 1 + t$, like before.)

To show that relation hold, we will show every term in ξ_n occur an even number of times and thus cancel out, since we are working over $\mathbb{Z}/2\mathbb{Z}$.

First consider one term in the sum, ie. for a fixed r, s, t, such that $r+s+t = n$. Then

$$\begin{aligned}
m_u(a_1, ..., a_r, &m_s(a_{r+1}, ..., a_{r+s}), a_{r+s+1}, ..., a_n) \\
&= \sum_{a\in\mathcal{C}} (\#_2 \mathcal{M}^a_{a_{r+1}\dots a_{r+s}}) m_u(a_1, ..., a_r, a, a_{r+s+1}, ..., a_n) \\
&= \sum_{a\in\mathcal{C}} \sum_{b\in\mathcal{C}} (\#_2 \mathcal{M}^a_{a_{r+1}\dots a_{r+s}})(\#_2 \mathcal{M}^b_{a_1,...,a_r,a,a_{r+s+1},...,a_n}) b \\
&= \sum_{b\in\mathcal{C}} Z_{r,s,t}(b, a_1, ..., a_n) b,
\end{aligned}$$

where $Z_{r,s,t}(b, a_1, ..., a_n)$ is the number of triplets (u', a, u''), such that

- $u' \in W^+_{k'}(Y, b), \quad a \in \mathcal{C}, \quad u'' \in W^+_{k''}(Y, a)$;
- $\big(u'(x_1^{k'}), ..., u'(x_{u-1}^{k'})\big) = (a_1, ..., a_r, a, a_{r+s+1}, ..., a_n)$
- $\big(u''(x_1^{k''}), ..., u''(x_s^{k''})\big) = (a_{r+1}, ..., a_{r+s})$.

Here $k' = u + 1$ and $k'' = s + 1$. For an example see figure (4.2). Let $U_{r,s,t}(b, a_1, ..., a_n)$ denote the set of such triplets. Note that for $(r', s', t') \neq (r, s, t)$,

$$U_{r,s,t}(b, a_1, ..., a_n) \cap U_{r',s',t'}(b, a_1, ..., a_n) = \emptyset,$$

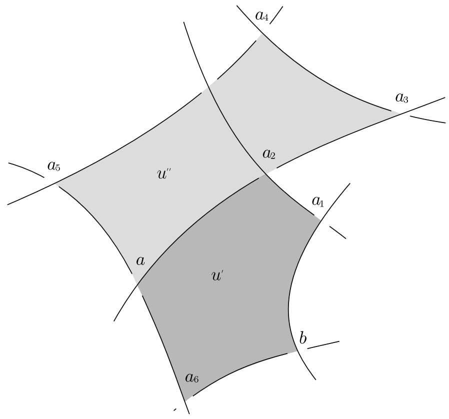

Figure 4.2: An example of a triplet $(u', a, u'') \in U_{1,4,1}(b, a_1, ..., a_6)$.

so we can conclude that

$$\xi_n(a_1, ..., a_n) = \sum_{b \in \mathcal{C}} Z(b, a_1, ..., a_n) b,$$

where $Z(b, a_1, ..., a_n)$ is the cardinality of

$$U(b, a_1, ..., a_n) = \bigsqcup_{r+s+t=n} U_{r,s,t}(b, a_1, ..., a_n).$$

(Note that r, s, t is not in general determined by (u', a, u''), ie. there might be (u', a, u'') might be in multiple $U_{r,s,t}$. So, by the **discrete** sum, we meed that the elements in $U(b, a_1, ..., a_n)$ are indexed by r, s, t.) Now in order to prove that ξ_n vanish, we will need to show that $Z(b, a_1, ..., a_n)$ is even. To prove this we will introduce a new space $V^+(b, a_1, ..., a_n)$ and construct maps

$$\varphi : U(b, a_1, ..., a_n) \to V^+(b, a_1, ..., a_n), \qquad \psi_1, \psi_2 : V^+(b, a_1, ..., a_n) \to U(b, a_1, ..., a_n),$$

such that for all $u \in V^+(b, a_1, ..., a_n)$ and $g \in U(b, a_1, ..., a_n)$;

$$\phi(\psi_1) = \phi(\psi_2) = u, \quad \psi_1(g) \neq \psi_2(g), \text{ and } \quad \psi_1(\phi(g) = g \text{ or } \psi_2(\phi(g) = g.$$

Then clearly $|U(b, a_1, ..., a_2)| = 2|V^+(b, a_1, ..., a_n)|$ and the result follows. [1]

4.2.2 Auxiliary definitions

The maps above indicates that U splits into two pars of equal size. So let

$$U(b, a_1, ..., a_n) = U^l(b, a_1, ..., a_n) \sqcup U^r(b, a_1, ..., a_n).$$

Given a triplet $(u', a, u'') \in U(b, a_1, ..., a_n)$, let $S' \subset \Pi_{k'}$ be a neighbourhood of $x^{k'}_{r+1}$ and $S'' \subset \Pi_{k''}$ a neighbourhood of $x^{k''}_0$. Then there are two possibly relative positions of $S_1 = u'(S')$ and $S_2 = u''(S'')$ near the crossing $u'(x^{k'}_{r+1}) = u''(x^{k''}_0) = a$,

[1]Here $|V|$, means the cardinality of a set V

Figure 4.3: Relative position of S_1 and S_2.

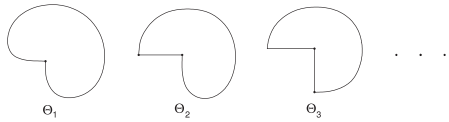

Figure 4.4: Concave k sided (curved) polygon.

see figure (4.3)

Definition 4.2.1. *We'll define the subsets $U^l(b, a_1, ..., a_n)$ and $U^r(b, a_1, ..., a_n)$ to be the sets consisting of triplets (u', a, u''), for which S_1 and S_2 have the relative position represented on the left and right side of figure 4.3 respectively.*

We will now define $V^+(b, a_1, ..., a_n)$. For each $k \in \mathbb{N}$, let $\Theta_k \subset \mathbb{R}^2$ be a k-sided (curved) polygon, such that the angle of exactly one vertex is grater then π. Denote this vertex by y_0^k and the rest $y_1^k, ..., y_{k-1}^k$ numbered anti-clockwise (see figure (4.4)).

Definition 4.2.2. *Let $V_k(Y)$ be the set of orientation-preserving immersions $u : \Theta_k \to \mathbb{R}^2$, such that $\hat{u}(\partial\Theta_k) \subset Y$.*

Let $\widetilde{V}_k(Y) = V_k(Y)/\operatorname{Diff}_+(\Theta_k)$, the set of non-parametrized immersions. Here $\operatorname{Diff}_0(\Theta_k)$ is the set of orientation-preserving diffeomorphisms of Θ_k.

Consider $\hat{u} \in \widetilde{V}_k(Y)$, then for $i > 0$, we define the sign of y_i^k in the same way as we did for the immersions of Π_k in section (2.2). The image of a neighbourhood of y_0^k in Θ_k covers three quadrants, we'll say y_0^k is positive (resp. negative) for u if two of these quadrants are positive (resp. negative).

Definition 4.2.3. *Let $\hat{u} \in \widetilde{V}_k(Y)$, then we'll say $\hat{u}$ is admissible if for exactly one $i \in \{0, ..., k-1\}$, the vertices y_i^k is positive.*

Let $V_{k,s}^+(Y) \subset \widetilde{V}_k(Y)$, be the set of admissible immersions, such that y_s^k is positive. Let $V^+(b, a_1, ..., a_{k-1}) \subset V_k^+(Y)$, such that $\hat{u} \in V^+(b, a_1, ..., a_{k-1})$ if and only if $u(y_0^k) = b$ and $u(y_i^k) = a_i$ for all $i \in \{1, ..., k-1\}$.

4.2.3 Constructing ϕ (Gluing)

Let $(u', a, u'') \in U_{r,s,t}(b, a_1, ..., a_n)$. The plan is to glue together the two polygons $\Pi_{k'}$ and $\Pi_{k''}$ (recall that $k' = u + 1 = r + t + 2$ and $k'' = s + 1$) into one concave polygon Θ_{n+1}, in such a way that the immersions u' and u'' combine into an immersion $\hat{u} \in V^+(Y)$.

Suppose $(u', a, u'') \in U^l(b, a_1, ..., a_n)$. Fix parametrisations $u_0' : \Pi_{k'} \to \mathbb{R}^2$ and $u_0'' : \Pi_{k''} \to \mathbb{R}^2$ representing u' and u'' respectively. Acording to (4.3), there exists maps $\sigma' : [0, 1] \to \partial\Pi_{k'}$ and $\sigma'' : [0, 1] \to \Pi_{k''}$ such that

$$\sigma'(0) = x_{r+1}^{k'},\ \sigma''(0) = x_0^{k''} \text{ and } u_0' \circ \sigma' = u_0'' \circ \sigma''.$$

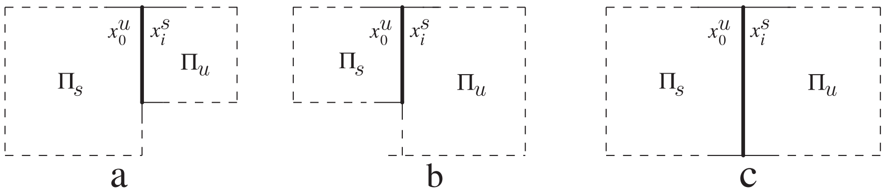

Figure 4.5: The thicker line in the middle, indicated the image of $u' \circ \sigma' = u'' \circ \sigma''$.

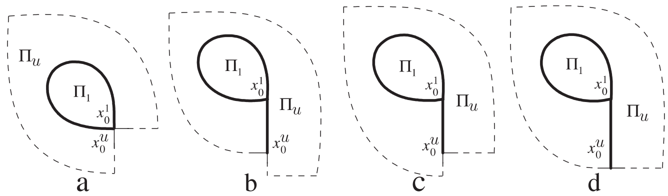

Figure 4.6:

Choose the maps σ' and σ'' such that the images $\sigma'([0,1])$, $\sigma''([0,1])$ are maximised. Then either $\sigma'(1)$, $\sigma''(1)$ or both is a vertex.

Case $k'' > 1$:

(ie. the terms not involving the curvature of the A_∞-structure m_0.) We'll first consider the case when $k'' > 1$. Then the above description looks like one of the three cases showed in figure (4.5)

In case (a) $\sigma'(1) = x_{r+2}^{k'}$ and $\sigma''(1) \neq x_s^{k''}$, in case (b) $\sigma'(1) \neq x_{r+2}^{k'}$ and $\sigma''(1) = x_s^{k''}$ and in case (c) both $\sigma'(1) = x_{r+2}^{k'}$ and $\sigma''(1) = x_s^{k''}$

Define

$$\Sigma = \Pi_{k'} \sqcup \Pi_{k''} / \sim_r,$$

where $\sigma'(t) \sim_r \sigma''(t)$ for all $t \in [0,1]$. Also define $\hat{u} : \Sigma \to \mathbb{R}^2$, by

$$\hat{u}|_{\Pi_{k'}} = u' \text{ and } \hat{u}|_{\Pi_{k''}} = u''.$$

In fact it follows from lemma (4.1.3) that case (c) is impossible. Indeed, by identifying Σ with Π_n, we have $\hat{u} \in \widetilde{W}_n(Y)$. But since both the positive vertices in the original immersions are removed by gluing them together with a negative one, all the vertices are negative, which is impossible, according to the lemma,

In case (a) and (b) showed in figure (4.5), we have $\Sigma \simeq \Theta_{n+1}$. Observe that exactly one of the vertices of Σ is positive for $\hat{u}$, namely the one coming from $x_0^{k'}$. So $\hat{u} \in V^+(Y)$. In particular $\hat{u} \in V^+(b, a_1, ..., a_n)$. Define $\phi(u', a,' u'') = \hat{u}$.

Case $k'' = 1$:

(ie. the terms coming from the curvature m_0.) Suppose $s = 1$. If $pp(1) \neq x_0^1$, we can proceed like above by gluing together $\Pi_{k'}$ and Π_1, like in figure (4.5a), and define $\phi(u', a, u'')$ in the same way.

So suppose $pp(1) = x_0^1$. There are four different cases we need to consider, see figure (4.6).

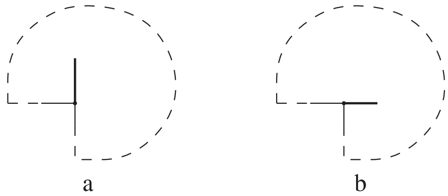

Figure 4.7:

If $\sigma'(1) = x^{k'}_{r+2}$, we can glue $\Pi_{k'}$ to Π_1 by identifying $\sigma'(t)$ with $pp(t)$, like in figure (4.6a). But if $\sigma'(1) \neq x^{k'}_{i+1}$, we need to do some more gluing, we need to glue $\Pi_{k'}$ to itself. Let $\sigma_\pm : [0,1] \to \partial\Pi_{k'}$, such that $\sigma_+(0) = x^{k'}_0$, $\sigma_-(0) = \sigma'(1)$, $u' \circ \sigma_+(t) = u' \circ \sigma_-(t)$, for all $t \in [0,1]$ and such that the images of σ_+ and σ_- are maximized. Define

$$\Sigma = \Pi_{k'} \sqcup \Pi_1 / \sim,$$

where $\sigma'(t) \sim pp(t)$ and $\sigma_+(t) \sim \sigma_-(t)$ for all $t \in [0,1]$.

There are three cases to consider, marked by (c), (b) and (d) in figure (4.6). Like with case (c) in figure (4.5), case (d), where $\sigma_+(1) = x^{k'}_{r+2}$ and $\sigma_-(1) = x^{k'}_r$, is impossible, due to lemma (**??**), since there are no positive vertices.

In either of the three cases (a)-(c), $\Sigma \simeq \Theta_{n+1}$. So like above, we can, by combining u' and u'' along σ', σ'' and $\sigma_\pm$, define a new immersion $\hat{u} \in \widetilde{V}_k(Y)$. Again, exactly one of the vertices of Θ_{n+1} (ie. the original one from $x^{k'}_0$) is positive for $\hat{u}$. Hence $\hat{u} \in V^+(Y)$. In particular $\hat{u} \in V^+(b, a_1, ..., a_n)$, so we may define $\phi(u', a, u'') = \hat{u}$.

Now consider an immersion $w \in U^r(b, a_1, ..., a_n)$, then the construction of $\phi(w)$ will be exactly the mirror image of the construction above.

4.2.4 Constructing ψ_1, ψ_2 (Cutting)

Let $\hat{u} \in V^+(b, a_1, ..., a_n)$. The idea is to cut Θ_{n+1} into a pair of polygons diffeomorphic to $\Pi_{k'}$ and $\Pi_{k''}$, such that $k' + k'' = n + 2$. Then the restriction of $\hat{u}$ to these polygons yield immersions u' and u'', such that $(u', a, u'') \in U(b, a_1, ..., a_n)$ for $a = u''x^{k''}_0$. We will see that there are two ways of doing this cut. One will define the map ψ_1 and the other the map ψ_2.

For $i = 1, 2$, let $\sigma_i : [0, \frac{1}{2}] \to \Theta_{n+1}$, such that $\sigma_i(0) = y^{n+1}_0$, $\sigma_i((0, \frac{1}{2}]) \subset \hat{u}^{-1}(Y) \setminus \partial\Theta_{n+1}$ and such that $\sigma_1((0, \frac{1}{2}]) \cap \sigma_2((0, \frac{1}{2}]) = \emptyset$. See figure (4.7). We extend σ_i to smooth immersion $\sigma_i : [0, 1] \to \Theta_{n+1}$, such that, $\sigma_i((0, \frac{1}{2}]) \subset \hat{u}^{-1}(Y) \setminus \partial\Theta_{n+1}$ and either $\sigma_i(1) \in \partial\Theta_{n+1}$ or $\sigma_i(1) \in \sigma_i([0, 1))$. The immersions σ_i are then defined uniquely up to reparametrizations.

There are four possible configurations of the image of σ_1 shown in figure (4.8).

In each of the four cases $\sigma_1([0, 1])$ divides θ_{n+1} into two parts, let's call them Σ_1 and Σ_2. In case (a), the positive vertex of Θ_{n+1} for $\hat{u}$ is a vertex of either Σ_1 or Σ_2, say Σ_1. In the cases (b)-(d), let Σ_1 be the polygon on the outside, ie. such that $\partial\Theta_{n+1} \subset \partial\Sigma_1$, and let Σ_2 be the polygon on the inside. We choose k' and k'', such that $\Pi_{k'}$ (resp. $\Pi_{k''}$) is diffeomorphic (by an orientation-preserving diffeomorphism) to Σ_1 (resp. Σ_2.)

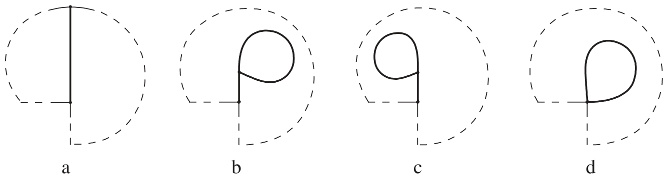

Figure 4.8: The image of σ_1 is represented by the thick line.

Define $u' = \hat{u}|_{\Sigma_1}$ and $u'' = \hat{u}|_{\Sigma_2}$, then (quotienting by the set of diffeomorphism) $u' \in \widetilde{W}_{k'}(Y)$ and $u'' \in \widetilde{W}_{k''}(Y)$. By lemma (4.1.2), both polygons most have at least one positive vertex for u' and u'', and since the total number of positive vertices is 2, we can conclude both u' and u'' are admissible. Hence $u' \in W^+_{k'}(Y, b)$ and $u'' \in W^+(Y, a)$, where $a = u''(x_0^{k''})$.

In the case (a)-(c), r, s, t are determined uniquely by (u', a, u''). But in case (d), there are two possible values for r, say $r_1 < r_2$, such that

$$(u', a, u'') \in U_{r_1,1,t_1}(b, a_1, ..., a_n) \quad \text{and} \quad (u', a, u'') \in U_{r_2,1,t_2}(b, a_1, ..., a_n)$$

($s = 1$ in case (d) and $t_i = n - r_i - 1$.)

Define $\psi_1(\hat{u}) = (u', a, u'')$, such that $(u', a, u'')U_{r_1,1,t_2}(b, a_1, ..., a_n)$ in case (d).

Following the same construction for r_2, case (a)-(c), gives us a new $(u', a, u'') \in U(b, a_1, ..., a_n)$. In case (d), on the other hand, we get the same triplet as for r_1. Using r_2, we define $\psi_2(\hat{u}) = (u', a, u'')$, such that in case (d) $(u', a, u'') \in U_{r_2,1,t_2}(b, a_1, ..., a_n)$ (ie. the larger value of r).

This defines ψ_1 and ψ_2, such that for all $\hat{u} \in V^+(b, a_1, ..., a_n)$, $\psi_1(\hat{u}) \neq \psi_2(\hat{u})$.

4.2.5 Completing the proof

We need to check that $\hat{u} = \phi(\psi_1(\hat{u})) = \phi(\psi_2(\hat{u}))$ for all $\hat{u} \in V^+(b, a_1, ..., a_n)$. This follows easily from the observation that, by decomposing Θ_{n+1} into two parts by either σ_1 or σ_2 follows by gluing them together, we get back the same polygon.

It remains to show that for each $\tau \in U(b, a_1, ..., a_n)$, either $\tau = \psi_1(\phi(\tau))$ or $\psi_2(\phi(\tau))$. That is we need to check that after gluing together $\Pi_{k'}$ and $\Pi_{k''}$ into a single Θ_{n+1} followed by cutting along either σ_1 or σ_2, we get back the same pair of polygons. This is quite easy to check. (Note that we also need to check that r, s and t match up, which follows since r, s, t was uniquely determined in case (a)-(c), in the cutting procedure, and in case (d) there was exactly two cases corresponding to ψ_1 and ψ_2.)

4.3 Proof of theorem (2.2.8)

In this section we will prove Theorem 2.2.8, claming that the stable type of the assosiaded A_∞-algebra is invarient under legendrian isotopy.

4.3.1 Legendrian Reidemeister Moves

Consider a generic Legendrien isotopy $\{L_t\}$ connecting L_0 and L_1, and let $Y_t = \pi(L_t)$ denote the Lagrangian projection. Throughout the deformation a finite number of bifurcations or Reidemeister moves (see Figure 4.9) may occur. If no bifurcations occurs for $t \in [x, y]$, it is clear by the definition of the associated A_∞-algebra that (A_{L_x}, m_{L_x}) and (A_{L_y}, m_{L_y}) are naturally isomorphic. So In order to prove that the stable type of the associated A_∞-algebra is preserved throughout the isotopy, it suffices to prove that it is preserved by the bifurcations.

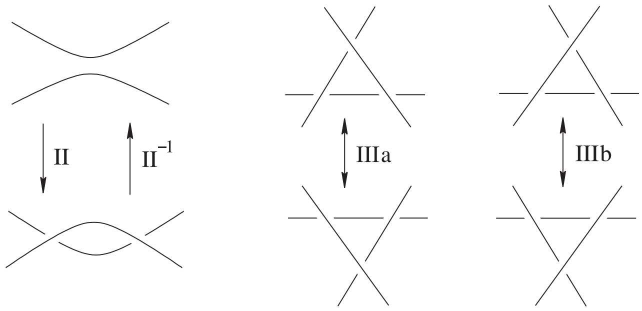

Figure 4.9: Legendrian Reidemeister Moves

Note that the third Reidemeister move has two version, both preserving the number of crossings, whereas the second does not. We will see that for in either of the two versions of the third move the A_∞-algebra changes by a tame isomorphism. In the case of the second move we will also need to stabilize, since the number of crossings has changed.

Suppose a bifurcation occurs at $t = t'$. Consider a small $\epsilon > 0$ and let $(A^\pm, m^\pm) = (A_{L_{t\pm\epsilon}}, m_{L_{t\pm\epsilon}})$. Without loss of generality, we will assume that for $t \in [t'-\epsilon, t'+\epsilon]$ the projection of the knot is unchanged outside a small neighbourhood of where the bifurcation occurs.

4.3.2 Move IIIa

Let $A^\pm = A = \mathbb{Z}_2\langle a, b, c, a_1, ..., a_l\rangle$ where a, b and c are as shown in figure (4.10) and $a_1, ..., a_l$ denote the crossing outside a neighbourhood of the bifurcation. Note that the grading of a, b and c are the same in both diagrams and thus A^+ and A^- are isomorphic as graded vector spaces. We claim that $m^+ = m^-$. To prove this we will, by continuity, construct a bijective map $R_k : W_k^+(Y_{t'\pm\epsilon}) \to W_k^+(Y_{t'\pm\epsilon})$ such that for each $f \in W_k^+(Y_{t'\pm\epsilon})$ and vertex x_i^k of Π_k,

$$R(f)(x_i^k) = f(x_i^k) \in \mathbb{Z}_2\langle a, b, c, a_1, ..., a_l\rangle.$$

It then follows immediately that $m^+ = m^-$.

Let $f^\pm \in W_3^+(Y_{t'\pm\epsilon})$ be the immersions whose image is the small triangles with vertices a, b, c and let $R(f_-) = f_+$. Then by Corollary (4.1.3),

$$H(a) > H(b) + H(c), \tag{4.1}$$

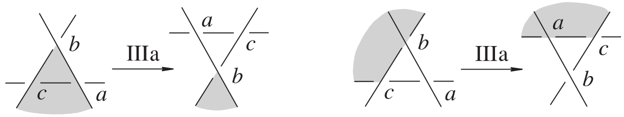

Figure 4.10:

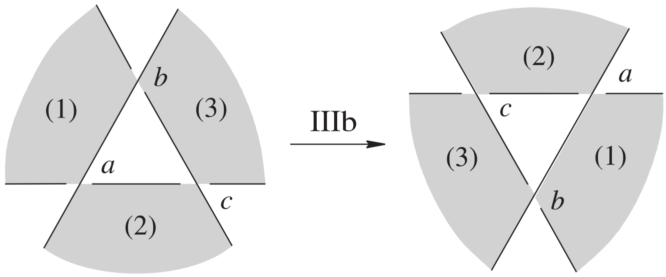

Figure 4.11: .

Lemma 4.3.1. *Let $f \in W_k^+(Y_{t'\pm\epsilon})\backslash\{f_\pm\}$, then neither of the segments $[a,b],[b,c],[a,c]$ is the image of an edge of Π_k under the immersion f.*

Proof. If $[a,b]$ (resp. $[a,c]$) is the image of an edge of Π_k, the vertex sent b (resp. c) is positive and that sent to a negative. Then Corollary 4.1.3 would imply $H(b) > H(a)$ (resp. $H(a) > H(c)$), which would contradict (4.1). If $[b,c]$ is the image of an edge, both b and c would be positive and thus f would not be admissible. □

Assume $f \in W_k^+(Y_{t'\pm\epsilon})\backslash\{f_-\}$ such that one of the vertices is mapped to a, b or c. Then, by lemma 4.3.1, f can be deformed, continuously in t, to an immersion $R(f) \in W_k^+(Y_{t'\pm\epsilon})$ as shown in figure 4.10.

By changing the direction of time, the very same construction defines a map $R' : W_k^+(Y_{t'\pm\epsilon}) \to W_k^+(Y_{t'\pm\epsilon})$ which is inverse to R.

4.3.3 Move IIIb

Let $A^\pm = A = \mathbb{Z}_2\langle a, b, c, a_1, ..., a_l\rangle$ where a, b and c are as shown in figure 4.11 and again $a_1, ..., a_l$ denote the crossing outside a neighbourhood of where the bifurcation occur. Again, the grading of a, b and c is unchanged and thus A^+ and A^- are isomorphic as graded vector spaces.

Let $f_\pm \in \widetilde{W}_3(Y)$ be the immersion whose images are the small triangles $T_\pm$ with vertices a, b, c, then by lemma (4.1.4), we find that $\deg(a) = \deg(bc)$.

Let $g : TA \to A$, such that $g_2(b,c) = a$, $g_1 = \mathrm{id}_A$ and g is otherwise zero. Then $g^2 = g \circ g = \mathrm{id}^A$, since

$$g^2(b,c) = g_2(g_1(b), g_1(c)) + g_1(g_2(b,c)) = a + a = 0,$$

and we claim,

$$\sum_{r,s,t} g_u(\mathbb{1}^{\otimes r} \otimes m_s^+ \otimes \mathbb{1}^{\otimes t}) = m_n^- \circ g \tag{4.2}$$

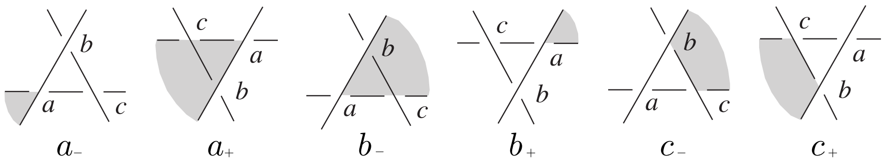

Figure 4.12: .

So g is a tame A_∞-isomorphism from (A, m^+) to (A, m^-).

Remark 4.3.2. In addition to (4.2), we also requite that the equation with plus and minus swapped is satisfied. Since this is the condition for $g = g^{-1} : (A, m^+) \to (A, m^-)$, to be an A_∞-homomorphism. However, this follows immediately, by the symmetry of the problem when reversing the direction of time.

Consider $f \in W_k^+(Y_{t'\pm\epsilon})$, then the triangle $T_\pm$ cannot be the image of f, since it a two positive vertices. Assume f maps an edge X_i of Π_k to one of the sides of $T_\pm$. Then a neighbourhood of X_i is mapped to one of the shaded areas in (4.11). If the shaded area is marked with (1) or (2), clearly $f \in W_k^+(Y_{t'\pm\epsilon}, a)$ (ie. $f(x_0^k) = a$,) since a is positive.

Lemma 4.3.3. *If the shaded area marked by (3), $f(x_0^k) = a_i$ for some $i \in \{1, ..., l\}$.*

Proof. Indeed, by corollary (4.1.3), $f(x_0^k) \neq f(x_i^k)$ for $i \neq 0$, so $f(x_0^k)$ cannot be b nor c. Also if $f(x_0^k) = a$, then by lemma (4.1.2),

$$H(a) - H(b) - H(c) > |f(\Pi_k)| > 0.$$

However, by applying lemma (4.1.2) to the $f_\pm$, $H(b)+H(c)-H(a) = |T_\pm| > 0$. □

In order to prove the claim (4.2) we will simplify the problem, by fixing a generator $v \in \mathcal{C}$ and consider the problem restricted to the one-dimensional subspace spanned by v. ie.

$$\mathrm{pr}_v \circ \left\{ \sum_{r,s,t} g_u(\mathbb{1}^{\otimes r} \otimes m_s^+ \otimes \mathbb{1}^{\otimes t}) \right\} = \mathrm{pr}_v \circ m_n^- \circ g. \tag{4.3}$$

Case $v \neq a$:

First, we'll assume $v \neq a$. Then by the definition of g_n, $\mathrm{pr}_v \circ g_n = 0$ if $n \neq 1$ and $g_1 = \mathrm{id}_A$, so the left hand side of the equation simplifies to $\mathrm{pr}_v \circ m_n^+$.

To prove this equation, there are 6 different kinds of fragments, we need to consider (Numbered $a_\pm, b_\pm$ and $c_\pm$ in figure (4.12)).

It turns out that the terms in $m^\pm$ coming from fragments that look like $a_\pm$ transforms differently form terms coming from $b_\pm$. In order to distinguish between these two cases we will define a new vectors space $\widetilde{A} = \mathbb{Z}_2\langle a', a'', b, c, a_1, ..., a_l\rangle$ and a family of maps

$$\widetilde{m}_k^\pm : \widetilde{A}^{\otimes k} \to \widetilde{A}.$$

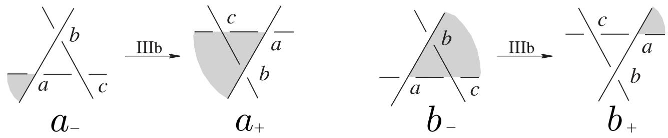

Figure 4.13:

(Note that we will not require that the maps satisfy the A_∞relations.) Let $\widetilde{\mathcal{C}} = \{a', a'', b, c, a_1, ..., a_l\}$ denote the set of generators.

Like in section (2.2.3), let $v_1, ..., v_k \in \widetilde{\mathcal{C}}$ and

$$\widetilde{\mathcal{M}}^v_{v_1,...,v_k}(Y_{t'\pm\epsilon}) = \{u \in W^+_k(Y_{t'\pm\epsilon}, v) \mid u(x^k_i) = v_i\},$$

where by $u(x^k_i) = a'$ (resp. $u(x^k_i) = a''$), we means that $u(x^k_i) = a$ and a neighbourhood of x^k_i in Π_k is mapped to the shaded area in (4.12), marked by $a_\pm$ (resp. $b_\pm$). For $v_1, ..., v_k \in \widetilde{\mathcal{C}}$, define

$$\widetilde{m}^\pm_k(v_1, ..., v_k) = \sum_{v\in\widetilde{\mathcal{C}}} (\#_2\widetilde{\mathcal{M}}^v_{v_1,...,v_k}(Y_{t'\pm\epsilon}))v.$$

Define $\sigma : TA \to \widetilde{A}$, by $\sigma_1(v) = v$ for all $v \in \{b, c, a_1, ..., a_n\}$,

$$\sigma_1(a) = a' + a''$$

and $\sigma_k = 0$ for $k > 1$. Also define $g', g'' : T\widetilde{A} \to \widetilde{A}$, by

$$g'_1 = g''_1 = \mathrm{id}_{\widetilde{A}}, \quad g'_2(b, c) = a', \quad g''_2(b, c) = a''$$

and both are otherwise 0. Since $(\widetilde{A}, m^\pm)$, is not an A_∞-algebra, it make non sense to speak of these maps as A_∞-homomorphisms. However we can still, consider their composition like definition (2.1.6). Then it is quite easy to check that

$$\sigma \circ g = g' \circ g'' \circ \sigma \quad \text{and} \quad \mathrm{pr}_v \circ m^- = \mathrm{pr}_v \circ \widetilde{m}^m \circ\ \sigma. \tag{4.4}$$

We will define two more families of maps $\overline{m}_k : A^{\otimes k} \to A$. Consider the subset $S_\pm(v) \subset W^+(Y_\pm, v)$ (her $W^+(Y_{t'\pm\epsilon}) = \bigcup_{k\geq 1} W^+_k(Y_{t'\pm\epsilon}, v)$,) consisting of immersions, which contain no fragment marked by $c_\pm$ in (4.12). For $v_1, ..., v_k \in \widetilde{\mathcal{C}}$, define

$$\overline{m}^\pm_k(v_1, ..., v_k) = \sum_{v\in\widetilde{\mathcal{C}}} \Big(\#_2\Big(\widetilde{\mathcal{M}}^v_{v_1,...,v_k}(Y_{t'\pm\epsilon}) \cap S^\pm(v)\Big)\Big)v. \tag{4.5}$$

Like in the case of move iiia, all immersions $u \in S_-(v)$ deforms continuously through $t = t'$, defining a bijection $S_-(v) \to S_+(v)$. See fig (4.13). So, actually $\overline{m}^+ = \overline{m}^-$.

Let $u \in S_+(v)$ (resp. $u \in S_-(v)$) and let $M_u \subset \{1, ..., k\}$, such that $u(x_i) = a$ and a neighbourhood of x_i is mapped to the shaded area marked by a_+ (resp. b_-) in fig. (4.13). (Note that might both no such fragments, in which case M_u is empty and there might be multiple fragments Π_k this same area.) Given $C \subset M_u$, define $u^C \in W^+_{n+|C|}(Y_+, v)$, by replacing the fragment near x_i, for each $i \in C$, as shown in figure (4.14).

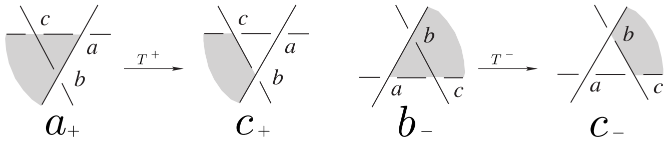

Figure 4.14:

Define the map $T^+_u : \mathcal{P}(M_u) \to \bigcup_{k\geq 1} W^+_k(Y_{t'\pm\epsilon}, v)$, by $C \mapsto u^C$. Then [2]

$$W^+(Y_{t'\pm\epsilon}, v) = \bigsqcup_{u\in S^-(v)} T^-_u \tag{4.6}$$

We clime that this implies

$$\mathrm{pr}_v \circ \widetilde{m}^+ = \mathrm{pr}_v \circ \overline{m}^+ \circ\ g' \quad \text{and} \quad \mathrm{pr}_v \circ \widetilde{m}^- = \mathrm{pr}_v \circ \overline{m}^- \circ\ g''. \tag{4.7}$$

Let $(v_1, ..., v_k) \in A^{\otimes k}$, write

$$(v_1, ..., v_k) = (v_{I_1}, b, c, v_{I_2}, ..., b, c, v_{I_p}),$$

where $v_{I_1} \in A^{\otimes I_1}$ is a tuple containing no consecutive b, c. Then if there exists $u' \in W^+_{k+1}(Y_{t'\pm\epsilon}, v)$, such that $(v_1, ..., v_k) = (u'(x^k_1), ..., u'(x^{k+1}_k))$. By eq. (4.6), $u' = T^+_u(C)$ for some $\in S_+(v)$ and $C \in \{1, ..., k\}$. (Actually, we most have $C = (I_1 + 1, I_2 + 2, ..., v_{I_p} + p)$). So

$$(v_{I_1}, g'(b, c), v_{I_2}, ..., g'(b, c), v_{I_p}) = \Big(u(x^{k-|C|}_1), ..., u(x^{k-|C|}_1)\Big).$$

Hence for every term in $\widetilde{m}^+(v_1, ..., v_k)$ there is a corresponding term in $(\overline{m}^+ \circ g')(v_1, ..., v_k)$. It is easy to check that also the converse hold and the argument for the second equation is exactly the same.

Finally, by combining the above relations, (4.4), (4.7) and $\overline{m}^+ = \overline{m}^-$, as well as the fact that $(g')^2 = \mathrm{id}$, we have

$$\begin{aligned} m^- \circ g &= \widetilde{m}^- \circ\ \sigma \circ\ g \\ &= \widetilde{m}^- \circ\ g'' \circ\ g' \circ\ \sigma \\ &= \overline{m}^- \circ\ g' \circ \sigma \\ &= \overline{m}^+ \circ\ g' \circ \sigma \\ &= \widetilde{m}^+ \circ \sigma \\ &= m^+. \end{aligned}$$

(Note, we have suppressed the pr_v at the beginning of each line in the above calculations. The equalities only hold in the subspace spanned by $v \neq a$.)

[2]Here $\mathcal{P}(M_u)$ denotes the power set.

Case $v = a$:

When $v = a$, g does no longer disepere from the LHS of equation (4.3), though, by the simplisity of g, it does simplify

$$\begin{aligned} LHS &:= \mathrm{pr}_v \circ \left\{ \sum_{r,s,t} g_u(\mathbb{1}^{\otimes r} \otimes m_s^+ \otimes \mathbb{1}^{\otimes t}) \right\} \\ &= g_1(\mathrm{pr}_a \circ m_n^+) + g_2(\mathrm{pr}_b, \mathrm{pr}_c \circ m_{n-1}^+) + g_2(\mathrm{pr}_b \circ m_{n-1}^+, \mathrm{pr}_c). \end{aligned}$$

By Lemma (4.1.2), we can, by decreasing ϵ if necessary, that $H(a) > H(b), H(c)$. Hence by corollary (4.1.3),

$$\mathrm{pr}_v \circ m^+(..., a, ...) = 0, \quad \text{for } v = a, b, c.$$

Therefore the RHS of equation (**??**), simplifies

$$RHS := \mathrm{pr}_a \circ m_n^- \circ\ g = \mathrm{pr}_a \circ m_n^-.$$

There are only two kinds of fragments that are relevant, the ones marked by (1) and (2) in figure (4.11). For $i = 1, 2$, let $S_k^{\pm,i} \subset W_k^+(Y_{t'\pm\epsilon}, a)$ denote the set of immersions that maps a neighbourhood of x_0^k to the shaded area marked by (i). Like above we will define pre-A_∞-structures $\widetilde{m}_n^{\pm,i} : A^{\otimes n} \to A$. Given $v_1, ..., v_n \in \mathcal{C}$, define

$$\widetilde{m}_k^{\pm,i}(v_1, ..., v_k) = \sum_v \left(\#_2 \left(\widetilde{\mathcal{M}}_{v_1,...,v_k}^v (Y_{t'\pm\epsilon}) \cap S_k^{\pm,i} \right) \right) v. \tag{4.8}$$

4.3.4 Move II

Let a, b be the two crossings in $Y_{t'\pm\epsilon}$, which vanish during the bifurcation. By slightly perturbing L_t and decreasing ϵ if necessary, we number the crossings of $Y_{t'\pm\epsilon}$ by $a, b, a_1, ..., a_l, b_1, ..., b_m$ in such a way that (through out the bifurcation. Ignoring a, b after they have vanished)

$$H(a_l) \geq ... \geq H(a_1) \geq H(a) > H(b) \geq H(b_1) \geq ... \geq H(b_m).$$

Here, $H(a)$ denotes the absolute difference between the z-values of the two parts of L_t, projection down to a.

Note that, by lemma (4.1.2), the difference between $H(a)$ and $H(b)$ equals the area of the curved 2-gon (let's call it D), which vanish at $t = t'$. Let $\bar{f} \in W_2^+(Y, a)$ be the immersion whose image is D. Then, since the area of D and thus also the difference between $H(a)$ and $H(b)$, corollary (4.1.3) implies that, there cannot be any immersion $u \in W_k^+(Y_{t'\pm\epsilon}, a)$ mapping a vertex to b, except $\bar{f}$. Therefore

$$\mathrm{pr}_a \circ m^-(..., b, ...) = 0, \quad \text{(except if both the ...'s are empty)}.$$

Hence $\mathrm{pr}_a \circ m^- = \delta_b^a + af$, where $\delta_b^a : TA \to A$ and $f : TA \to \mathbb{Z}_2$, are given by

$$(\delta_b^a)_1(b) = a, \quad (\delta_b^a)_1(v) = 0, \quad (\delta_b^a)_k = 0, \quad \forall v \neq b, k \geq 1,$$

and

$$f(..., v, ...) = 0, \quad \forall v \in \{a, b, a_1, ..., a_n\}.$$

Denote $(A, m) = (A^+, m^+)$, where $A = \mathbb{Z}_2\langle a_1, ..., a_l, b_1, ..., b_m\rangle$ and let $(A', m') = (S_i(A), m^{S_i(A)})$, where $i = |a|$. We want to show that (A^-, m^-) and (A, m') are tame-isomorphic.

We'll start by defining the A_∞-pre-homomorphism $s : A' \dashrightarrow A^-$, given by $s = \widetilde{s} + bf$, where $\widetilde{s} : A' \dashrightarrow A^-$ is the strict A_∞-homomorphism mapping e_1, e_2 to a and b respectively and fixes a_i and b_i.

Let $\widehat{m}$ be the unique A_∞-structure, from lemma (2.1.10), such that s is an A_∞-homomorphism. Then by construction (A^-, m^-) and $(A', \widehat{m})$ are tame isomorphic.

Let $A_{[i]} = \mathbb{Z}_2\langle a_1, ..., a_i, b_1, ..., b_m, e_1, e_2\rangle$, for each $i \in \{0, ..., l\}$ and let $\tau : A \dashrightarrow A'$ be the inclusion, from (2.1.14). Then

Lemma 4.3.4. *a)* $\mathrm{pr}_{[0]} \circ \widehat{m} = \mathrm{pr}_{[0]} \circ m'$, *where* $\mathrm{pr}_{[i]}$ *is the projection onto* $A_{[i]}$,

b) $m' \circ \tau = \widehat{m} \circ \tau$.

In order to show that $(A', \widehat{m})$ is also tame isomorphic to (A', m'), we will inductively construct a sequence of tame isomorphic A_∞-structures $(A', m^{[i]})$. Such that, $(A', m^{[0]}) = (A', \widehat{m})$, for each $i \in \{0, ..., l\}$,

$$\mathrm{pr}_{[i]} \circ m^{[i]} = \mathrm{pr}_{[i]} \circ m'.$$

Note that, for $i = l$, $\mathrm{pr}_{[l]} = \mathrm{id}_{A'}$, and thus $(A', m') = (A', m^{[l]})$, concluding the proof. Also note that the case of $i = 0$ is satisfied, by lemma (4.3.4), which we will prove after constructing the $m^{[i]}$'s.

Suppose we have already defined $m^{[0]}, ..., m^{[j-1]}$. To define $m^{[j]}$, we first define an A_∞-pre-homomorphism $g^j : A' \dashrightarrow A'$ (with g^1 an isomorphism) and define $m^{[j]}$ to be the A_∞-structure on A', such that g^j is an A_∞-homomorphism from $(A', m^{[j]})$ to $(A', m^{[j-1]})$.

Let $c^j, q^j : A' \dashrightarrow A'$ be A_∞-pre-homomorphisms, given by,

$$c^j = \mathrm{pr}_{a_j} \circ \left\{m' + m^{[j-1]}\right\} \quad \text{and} \quad q^j = c^j \circ h, \tag{4.9}$$

where $h : A' \dashrightarrow A'$ is the A_∞-homotopy, from lemma (2.1.14). Then define $g^j := \mathrm{id}_{A'} + q^j$.

It follows immediately from the definition of h, that

$$q^j(..., a_k, ...) = 0, \quad \forall k \geq j. \tag{4.10}$$

Hence g^j is tame and g^j_1 is an isomorphism.

Since g^j is an A_∞-homomorphism, the following A_∞-relations,

$$\sum_{r,s,t} g^j_u(\mathbb{1}^{\otimes r} \otimes m^{[j]}_s \otimes \mathbb{1}^{\otimes t}) = (m^{[j-1]} \circ g^j)_n. \tag{4.11}$$

By projecting eq. 4.11 onto $A_{[j-1]}$, we have

$$LHS = \mathrm{pr}_{[j-1]} \circ m_n^{[j]}, \quad \text{since } \mathrm{pr}_{[j-1]} \circ q^j = 0.$$

And, using the inductive hypothesise,

$$RHS = \mathrm{pr}_{[j-1]} \circ m'_n + \mathrm{pr}_{[j-1]} \circ (m' \circ q^j)_n,$$

where the last term disappears, by corollary (4.1.3).

It remains to prove that $\mathrm{pr}_{a_j} \circ m_n^{[j]} = \mathrm{pr}_{a_j} \circ m'_n$. Projecting eq. 4.11 onto a_j, we have

$$LHS = \mathrm{pr}_{a_j} \circ m_n^{[j]} + \sum_{r,s,t} q_u^j(\mathbb{1}^{\otimes r} \otimes m_s^{[j]} \otimes \mathbb{1}^{\otimes r}),$$

By eq. 4.10 and since $\mathrm{pr}_{[j-1]} \circ m_s^{[j]} = \mathrm{pr}_{[j-1]} \circ m'_s$,

$$q_u^j(\mathbb{1}^{\otimes r} \otimes m_s^{[j]} \otimes \mathbb{1}^{\otimes r}) = q_u^j(\mathbb{1}^{\otimes r} \otimes m'_s \otimes \mathbb{1}^{\otimes r})$$

And $RHS = \mathrm{pr}_{a_j} \circ (m^{[j-1]} \circ \ g^j)_n$.

Lemma 4.3.5. *For all $k \geq j$ and n,*

$$(\mathrm{pr}_{a_j} \circ m^{[j-1]})_n(..., a_k, ...) = 0.$$

Proof. By lemma 6.1, the claim holds for $j = 1$. We show that the equation holds any j by induction on j and n. Suppose the equation holds for $j - 1$. Then, by the explicit construction of $m^{[j]}$, in the proof of lemma (2.1.10),

$$m_n^{[j]} = \sum_{r,s<n,t} g_u^j(\mathbb{1}^{\otimes r} \otimes m_s^{[j]} \otimes \mathbb{1}^{\otimes t}) + (m_r^{[j-1]} \circ \ g^j)_n.$$

Applying the RHS on $(..., a_k, ...)$ (and projection onto a_j), the g^j in the last term acts as the identity, by equation 4.10, and thus the last term vanish by the inductive hypothesise. When $n = 1$, there is no first term (Note that we are ignoring the curvature, ie. $n = 0$, since we are applying it to at least a_k.), so this case is immediate. It then also follows, by simple induction on n, that the RHS, vanish for all n. □

By corollary (4.1.3), $\mathrm{pr}_{a_j} \circ m'(..., a_k, ...) = 0$. Therefore it follows from lemma (4.3.5) that also

$$\mathrm{pr}_{a_j} \circ c^j(..., a_k, ...) = 0. \tag{4.12}$$

By lemma (4.3.5), the g on the RHS acts as the identity, so we have (suppressing the pr_{a_j} at the beginning of each line)

$$\begin{aligned} m_n^{[j]} &= m_n^{[j-1]} + \sum_{r,s,t} q_u^j(\mathbb{1}^{\otimes r} \otimes m_n^{[j-1]} \otimes \mathbb{1}^{\otimes t}) \\ &= m_n' + c_n^j + \sum_{r,s,t} (c^j \circ h)_u(\mathbb{1}^{\otimes r} \otimes m_n^{[j-1]} \otimes \mathbb{1}^{\otimes t}), && \text{by eq. 4.9} \\ &= m_n' + c_n^j + \sum_{r,s,t} c^j(h_1 \otimes ... \otimes h_1)(\mathbb{1}^{\otimes r} \otimes m_n^{[j-1]} \otimes \mathbb{1}^{\otimes t}) \\ &= m_n' + c_n^j + \end{aligned}$$

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