CS221: Algorithms and Data Structures Lecture #1 Complexity Theory and Asymptotic Analysis

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2014W1

Today’s Outline

- Programming Project #1 and Forming Teams
- Brief Proof Reminder
- Asymptotic Analysis, Briefly
- Silicon Downs and the SD Cheat Sheet
- Asymptotic Analysis, Proofs and Programs
- Examples and Exercises

Prog Proj #1 & Teams

Proof by...

- Counterexample
  - show an example which does not fit with the theorem
  - QED (the theorem is ***dis***proven)
- Contradiction
  - assume the opposite of the theorem
  - derive a contradiction
  - QED (the theorem is proven)
- Induction
  - prove for one or more base cases (e.g., $n = 1$)
  - assume for one or more anonymous values (e.g., k)
  - prove for the next value (e.g., $k + 1$)
  - QED

Example Proof by Induction

A number is divisible by 3 iff the sum of its digits is divisible by three

Example Proof by Induction (Worked)

"A number is divisible by 3 iff the sum of its digits is divisible by three."

First, some definitions:

Consider a positive integer x to be made up of its n digits: $x_1x_2x_3..x_n$.

For convenience, let's say SD(x) = $\sum_{i=1}^{n} x_i$

Example Proof by Induction (Worked)

"A number is divisible by 3 iff the sum of its digits is divisible by three."

There are ***many*** ways to solve this, here's one.

We'll prove a somewhat *stronger* property, that for a non-negative integer x with any positive integral number of digits *n*, $SD(x) \equiv x \pmod 3$.

(That is, the remainder when we divide SD(x) by 3 is the same as the remainder when we divide x by 3.)

Example Proof by Induction (INSIGHT FIRST!)

How do we break a problem down?

We often break sums down by "pulling off" the first or last term:

$$SD(x) = \sum_{i=1}^{n} x_i = x_n + \sum_{i=1}^{n-1} x_i$$

We can "peel off" the rightmost or leftmost digit. Peeling off the rightmost is like dividing by 10 (dumping the remainder). Peeling off the leftmost is harder to describe.

Let's try peeling off the rightmost digit!

Example Proof by Induction (Worked)

Following our insight, we only need one base case.

Base case (where n = 1):

Consider any number x with one digit (0-9).

$$SD(x) = \sum_{i=1}^{1} x_i = x_1 = x.$$

So, it's trivially true that $SD(x) \equiv x \pmod{3}$.

Example Proof by Induction (Worked)

Following our insight, we'll break a problem of size n into a problem of size n-1. So, we need to assume our theorem holds for size n-1.

WLOG, let n be an arbitrary integer greater than 1.

Induction Hypothesis: Assume for any non-negative integer x with n-1 digits: $SD(x) \equiv x \pmod 3$.

Inductive step:

…

Example Proof by Induction (Worked)

Following our insight, we now just need to break the problem down as we figured out it breaks down…

IH: Assume for integer x with n-1 digits: $SD(x) \equiv x \pmod 3$.

Inductive step:

Consider an arbitrary number y with n digits.

We can think of y as being made up of its digits: $y_1y_2\ldots y_{n-1}y_n$. Clearly, $y_1y_2\ldots y_{n-1}$ (which we'll call z) is itself an n-1 digit number; so, the induction hypothesis applies:

$SD(z) \equiv z \pmod 3$.

Example Proof by Induction (Worked)

Inductive step continued:

Now, note that $y = z*10 + y_n$.

So: $y \equiv (z*10 + y_n) \pmod 3$

$\equiv (z*9 + z + y_n) \pmod 3$

$z*9$ is divisible by 3; so, it has no impact on the remainder of the quantity when divided by 3:

$\equiv (z + y_n) \pmod 3$

Example Proof by Induction (Worked)

Inductive step continued:

By the IH, we know SD(z) $\equiv$ z (mod 3).

So:

$$\begin{aligned}
&\equiv (z + y_n) && \pmod 3\\
&\equiv (SD(z) + y_n) && \pmod 3\\
&\equiv (y_1 + y_2 + \ldots + y_{n-1} + y_n) && \pmod 3\\
&\equiv SD(y) && \pmod 3
\end{aligned}$$

Can I really sub SD(z) for z inside the mod, even though they're only equal *mod 3*? Yes… they only differ by a multiple of 3, which cannot affect the "mod 3" sum.

QED!

Proof by Induction Pattern Reminder

First, find a way to break down the theorem interpreted for some (large-ish, arbitrary) value **`k`** in terms of the theorem interpreted for smaller values.

Next, prove any base cases needed to ensure that the "breakdown" done over and over eventually hits a base case.

Then, assume the theorem works for all the "smaller values" you needed in the breakdown (as long as **`k`** is larger than your base cases).

Finally, build up from those assumptions to the **`k`** case.

Induction Pattern to Prove P(n)

First, figure out how P(n) breaks down in terms of P(*something(s) smaller*).

P(n) is *theorem goes here*.

Theorem: P(n) is true for all n ≥ *smallest case*.

Proof: We proceed by induction on n.

Base Case(s) (P(·) is true for *whichever cases you need to "bottom out"*)**:**
Prove each base case via some (usually easy) proof techniques.

Inductive Step (if P(·) is true for *the "something smaller" case(s)*, then P(n) is true, *for all n not covered by the base case (usually: greater than the largest base case)*):

WLOG, let n be an arbitrary integer *not covered by the base case (usually: greater than the largest base case)*.

Assume P(·) is true for *the "something smaller" case(s)*. (The Induction Hypothesis (IH).)

Break P(n) down in terms of the smaller case(s).

The smaller cases are true, by the IH.

Build back up to show that P(n) is true.

This completes our induction proof. QED

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A Task to Solve and Analyze

Find a student’s name in a class given her student ID

Analysis of Algorithms

- Analysis of an algorithm gives insight into how long the program runs and how much memory it uses
  - time complexity
  - space complexity
- Analysis can provide insight into alternative algorithms
- Input size is indicated by a number *n* (sometimes there are multiple inputs)
- Running time is a function of *n* ($\mathbf{Z}^0 \rightarrow \mathbf{R}^0$) such as

  $T(n) = 4n + 5$

  $T(n) = 0.5\ n \log n - 2n + 7$

  $T(n) = 2^n + n^3 + 3n$
- But...

Asymptotic Analysis Hacks

- Eliminate low order terms
  - $4n + 5 \Rightarrow 4n$
  - $0.5\, n \log n - 2n + 7 \Rightarrow 0.5\, n \log n$
  - $2^n + n^3 + 3n \Rightarrow 2^n$
- Eliminate coefficients
  - $4n \Rightarrow n$
  - $0.5\, n \log n \Rightarrow n \log n$
  - $n \log (n^2) = 2\, n \log n \Rightarrow n \log n$

Rates of Growth

- Suppose a computer executes 10^{12} ops per second:

n =	10	100	1,000	10,000	10^{12}
n	10^{-11}s	10^{-10}s	10^{-9}s	10^{-8}s	1s
n lg n	10^{-11}s	10^{-9}s	10^{-8}s	10^{-7}s	40s
n^2	10^{-10}s	10^{-8}s	10^{-6}s	10^{-4}s	10^{12}s
n^3	10^{-9}s	10^{-6}s	10^{-3}s	1s	10^{24}s
2^n	10^{-9}s	10^{18}s	10^{289}s		

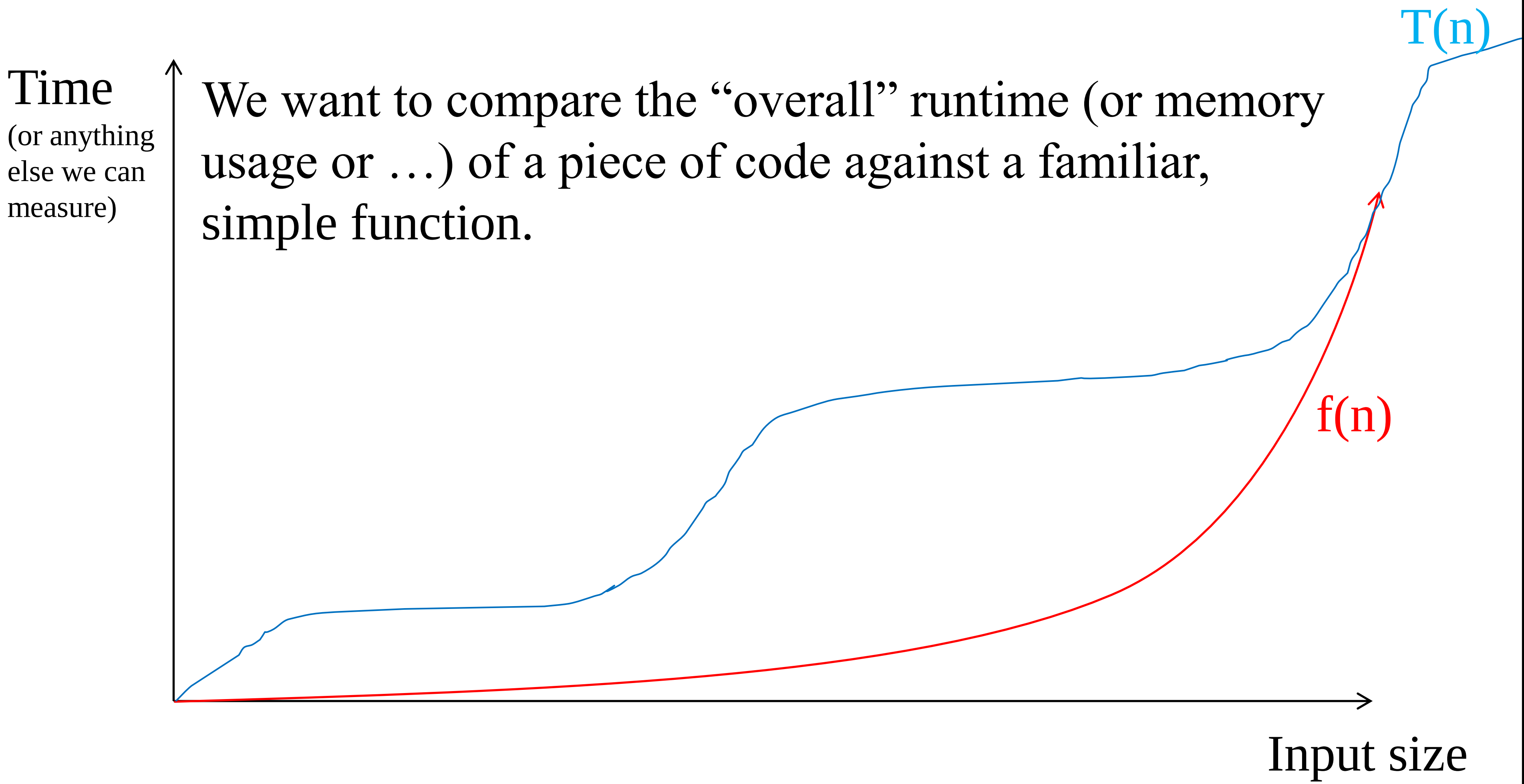
10^4s = 2.8 hrs

10^{18}s = 30 billion years

Order Notation

- $T(n) \in O(f(n))$ if there are constants c and n_0 such that $T(n) \leq c\, f(n)$ for all $n \geq n_0$

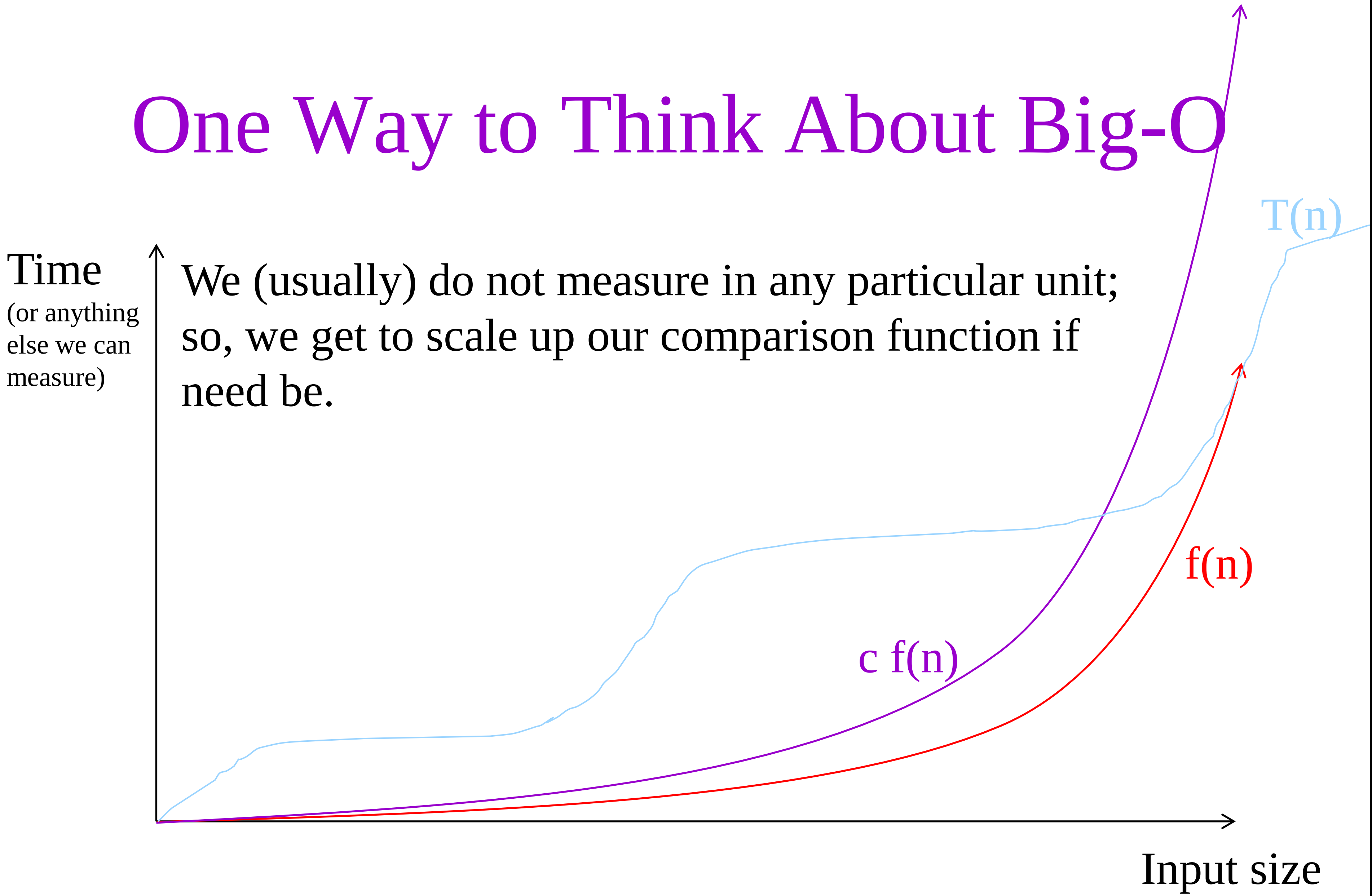
One Way to Think About Big-O



We want to compare the "overall" runtime (or memory usage or …) of a piece of code against a familiar, simple function.

T(n) ∈ O(f(n)) if there are constants c and n_0 such that $T(n) \le c\, f(n)$ for all $n \ge n_0$

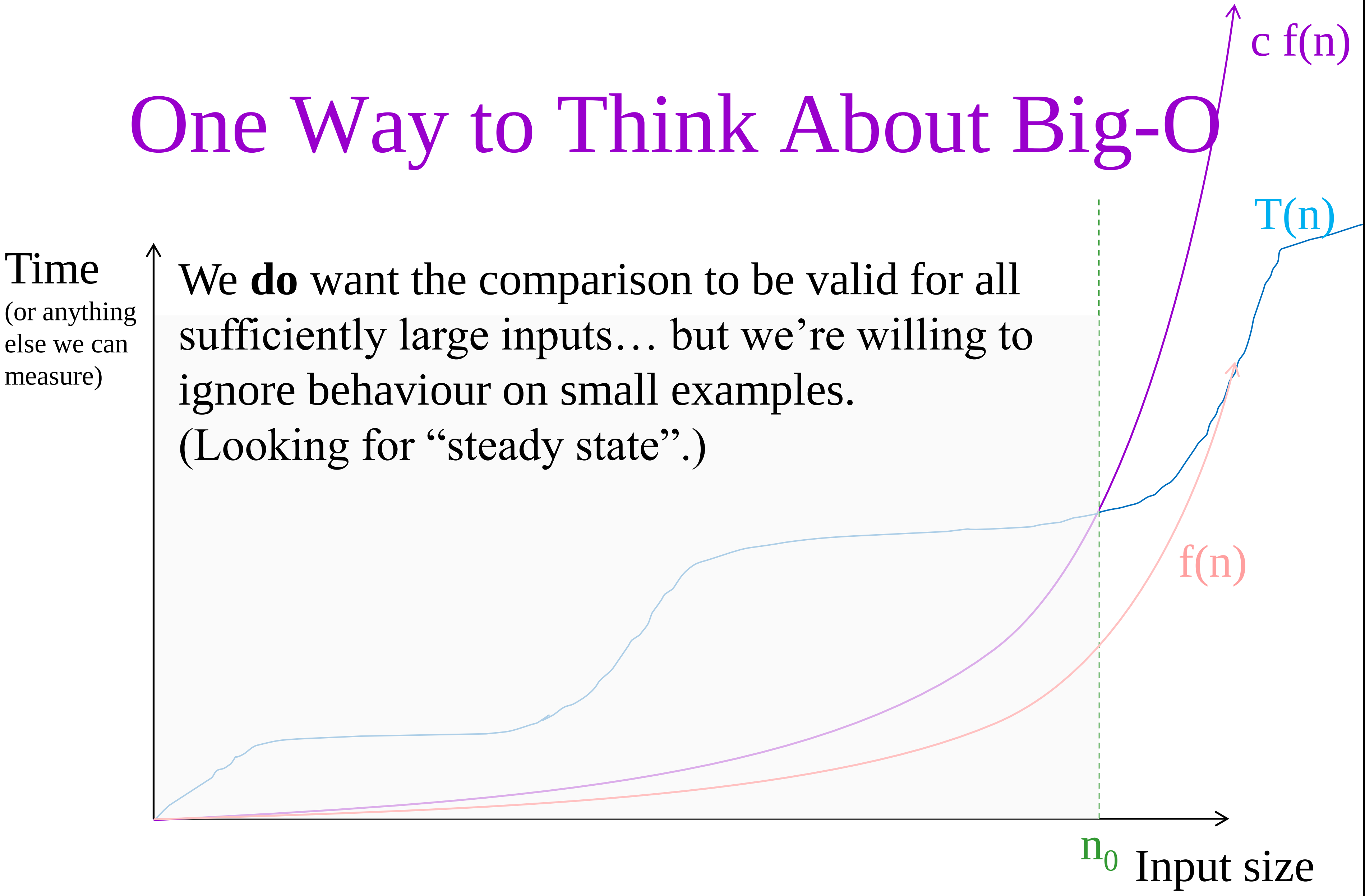
One Way to Think About Big-O



We (usually) do not measure in any particular unit; so, we get to scale up our comparison function if need be.

$T(n) \in O(f(n))$ if there are constants **c** and n_0 such that $T(n) \leq$ **c f(n)** for all $n \geq n_0$

One Way to Think About Big-O

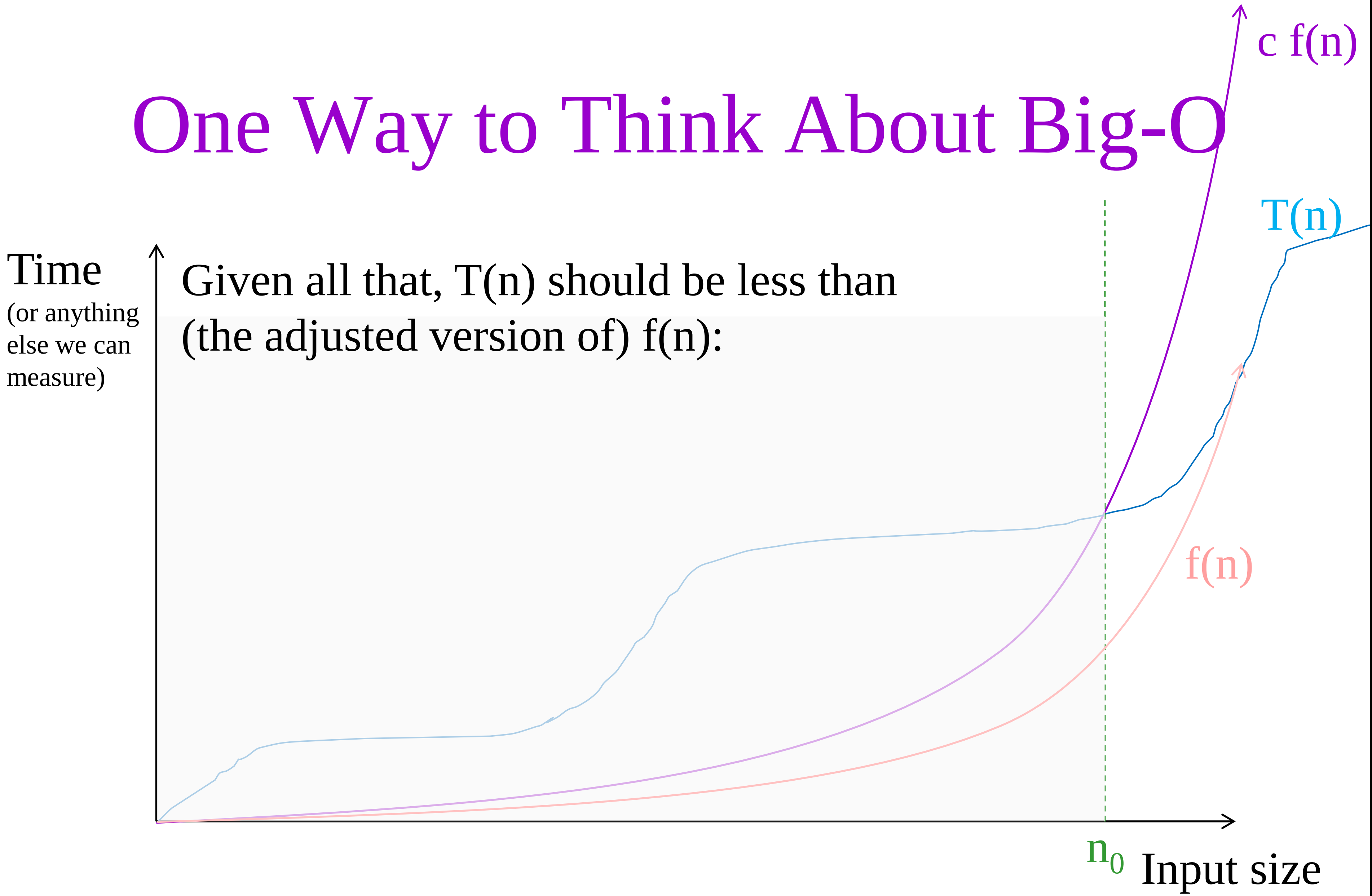


We **do** want the comparison to be valid for all sufficiently large inputs… but we're willing to ignore behaviour on small examples. (Looking for "steady state".)

T(n) ∈ O(f(n)) if there are constants c and $\mathbf{n_0}$ such that T(n) ≤ c f(n) **for all n ≥ $\mathbf{n_0}$**

One Way to Think About Big-O

Given all that, T(n) should be less than
(the adjusted version of) f(n):



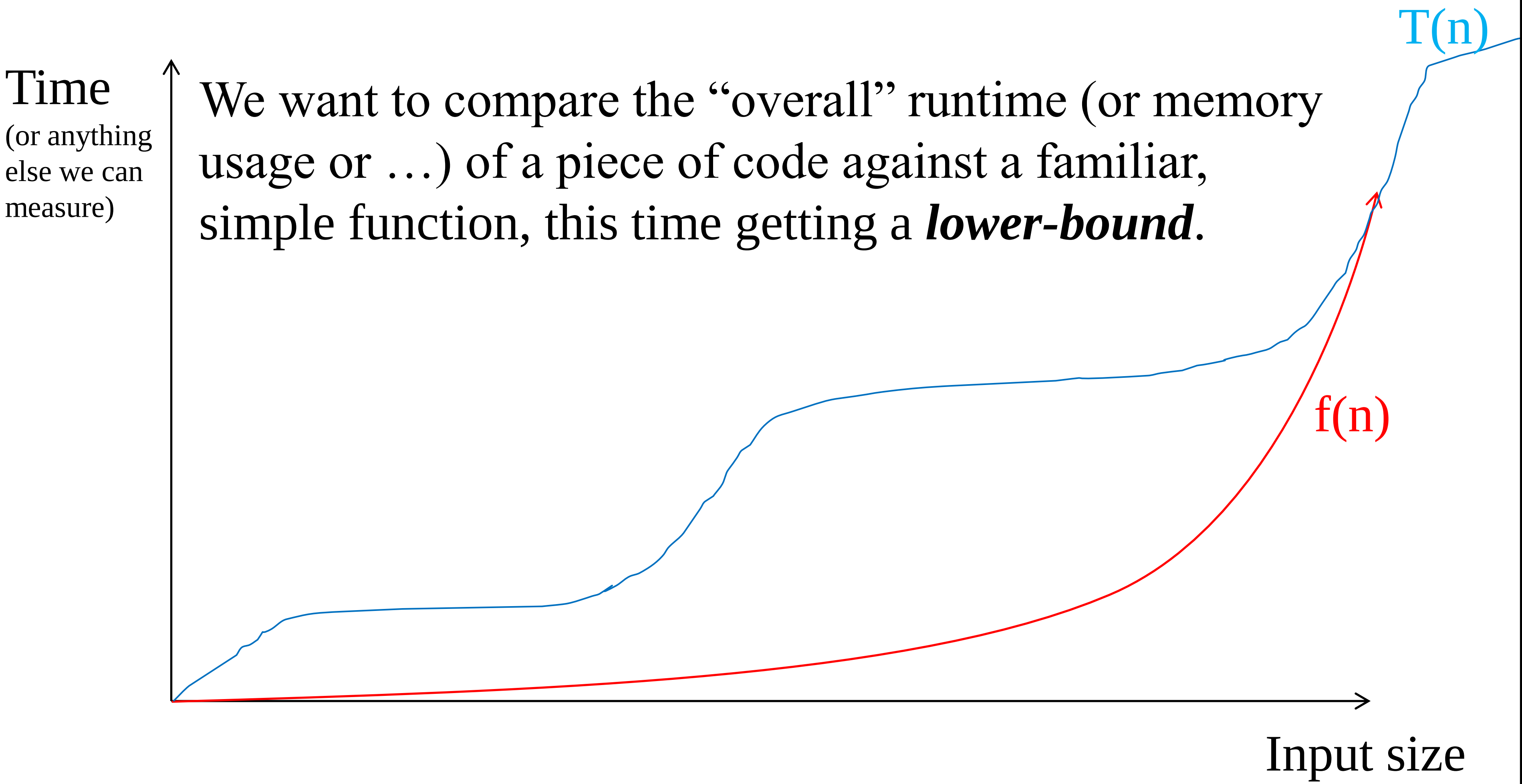
T(n) ∈ O(f(n)) if there are constants c and n_0 such that **T(n) ≤ c f(n)** for all $n \geq n_0$

Order Notation

- $T(n) \in O(f(n))$ if there are constants c and n_0 such that $T(n) \leq c\, f(n)$ for all $n \geq n_0$
- $T(n) \in \Omega\,(f(n))$ if $f(n) \in O(T(n))$
- $T(n) \in \Theta(f(n))$ if $T(n) \in O(f(n))$ and $T(n) \in \Omega\,(f(n))$

One Way to Think About Big-Ω

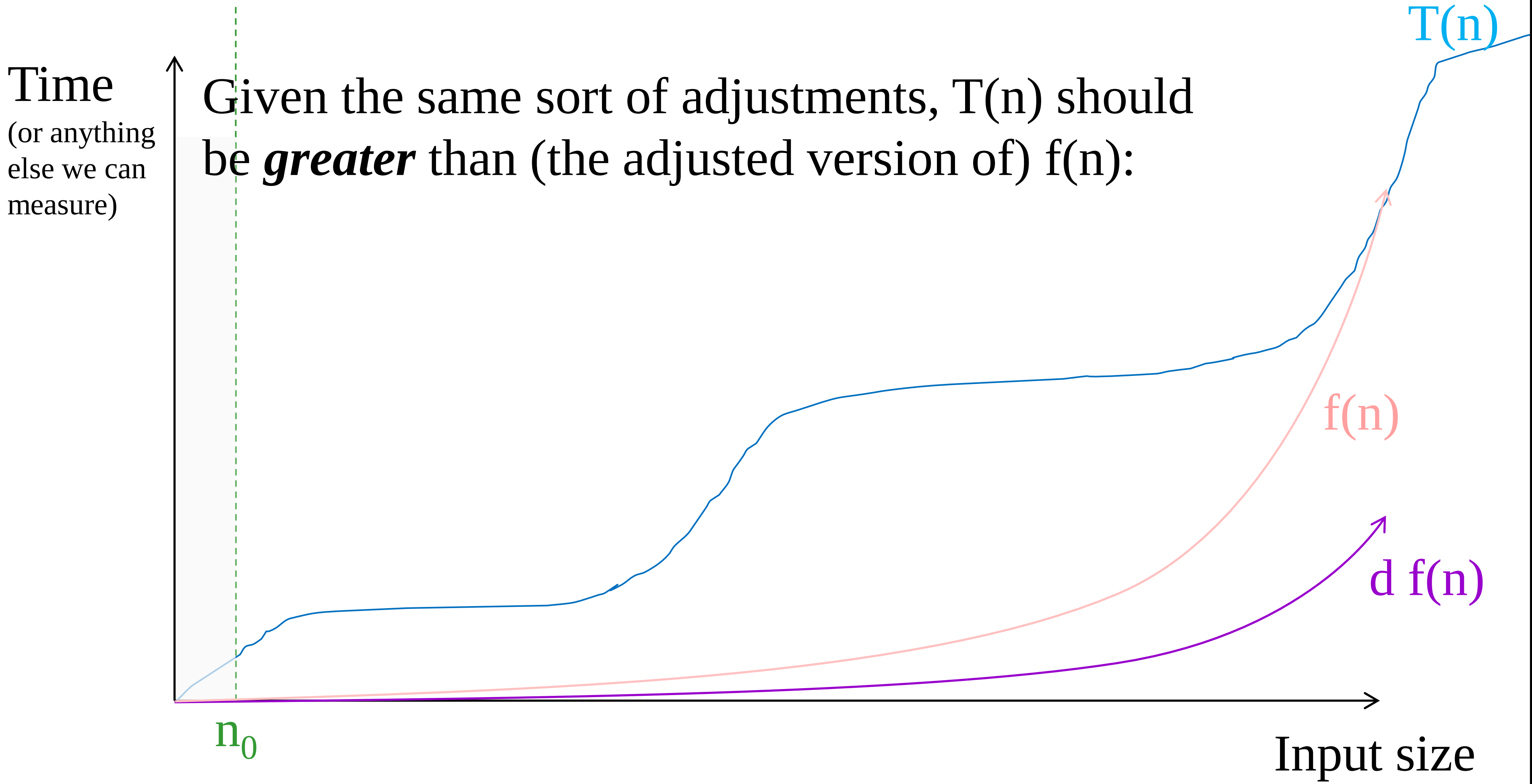
We want to compare the “overall” runtime (or memory usage or …) of a piece of code against a familiar, simple function, this time getting a ***lower-bound***.



T(n) ∈ **Ω(f(n))** if there are constants d and n_0 such that T(n) ≥ d f(n) for all n ≥ n_0

One Way to Think About Big-Ω

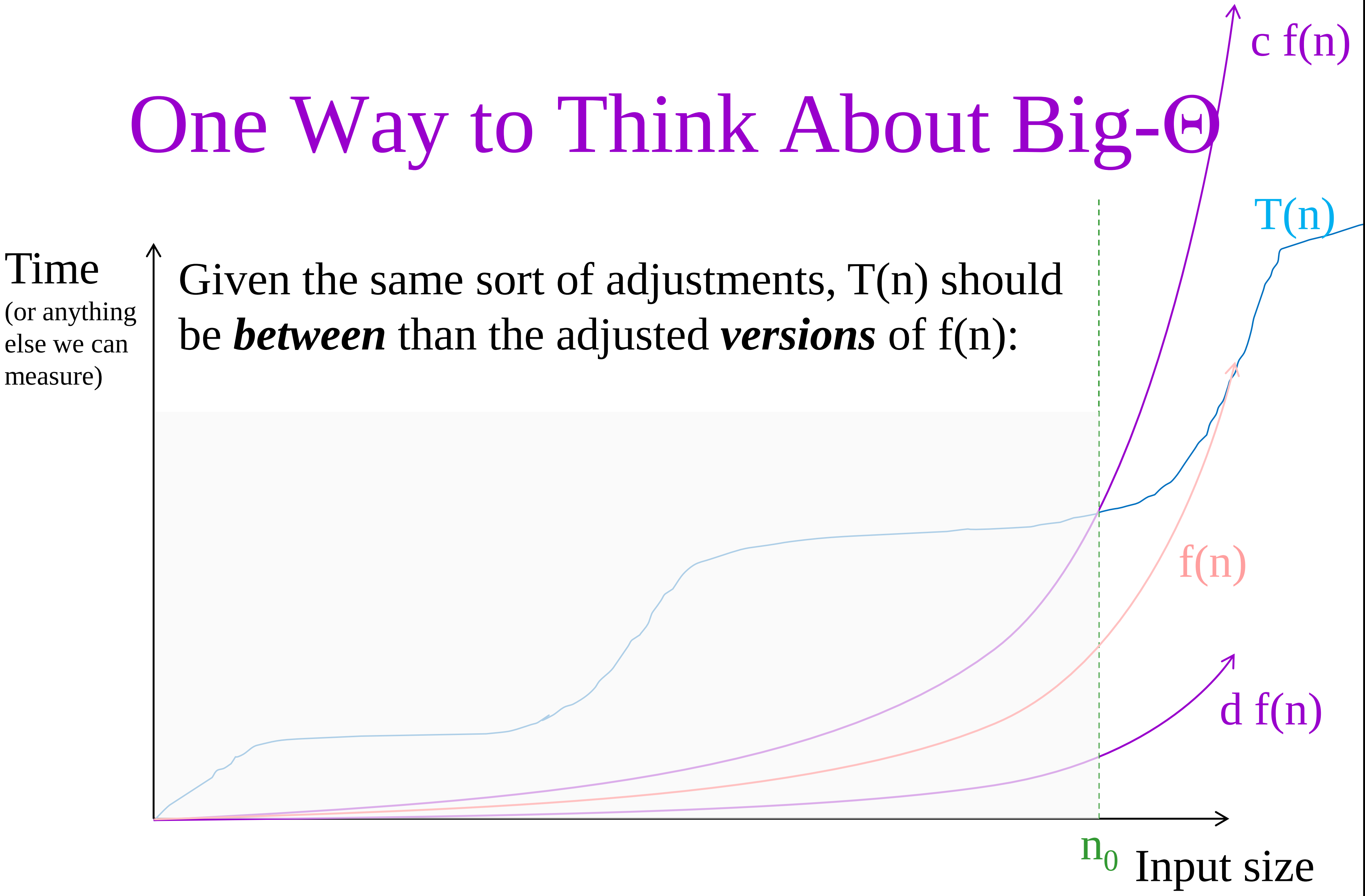
Given the same sort of adjustments, T(n) should be ***greater*** than (the adjusted version of) f(n):



$T(n) \in \Omega(f(n))$ if there are constants c and n_0 such that $\mathbf{T(n) \geq d\, f(n)}$ for all $n \geq n_0$

One Way to Think About Big-Θ

Given the same sort of adjustments, T(n) should be ***between*** than the adjusted ***versions*** of f(n):



T(n) ∈ Θ(f(n)) if ∃ c, d and n_0 such that **d f(n) ≤ T(n) ≤ c f(n)** for all $n \geq n_0$

Order Notation

- $T(n) \in O(f(n))$ if there are constants c and n_0 such that $T(n) \leq c\, f(n)$ for all $n \geq n_0$
- $T(n) \in \Omega\,(f(n))$ if $f(n) \in O(T(n))$
- $T(n) \in \Theta(f(n))$ if $T(n) \in O(f(n))$ and $T(n) \in \Omega\,(f(n))$

How would you prove one of these?

Prove O(..) by finding a good c and n_0 (often helps in scratch work to "solve for" c, keeping notes of constraints on n).

Then, assume $n \geq n_0$ and show that $T(n) \leq c\, f(n)$.

Prove Ω and Θ by breaking down in terms of O.

Examples

$10{,}000\, n^2 + 25\, n \in \Theta(n^2)$

$10^{-10}\, n^2 \in \Theta(n^2)$

$n \log n \in O(n^2)$

$n \log n \in \Omega(n)$

$n^3 + 4 \in O(n^4)$ but not $\Theta(n^4)$

$n^3 + 4 \in \Omega(n^2)$ but not $\Theta(n^2)$

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Silicon Downs

Post #1	Post #2
$n^3 + 2n^2$	$100n^2 + 1000$
$n^{0.1}$	$\log n$
$n + 100n^{0.1}$	$2n + 10 \log n$
$5n^5$	$n!$
$n^{-15}2^n/100$	$1000n^{15}$
$8^{2\lg n}$	$3n^7 + 7n$
mn^3	$2^m n$

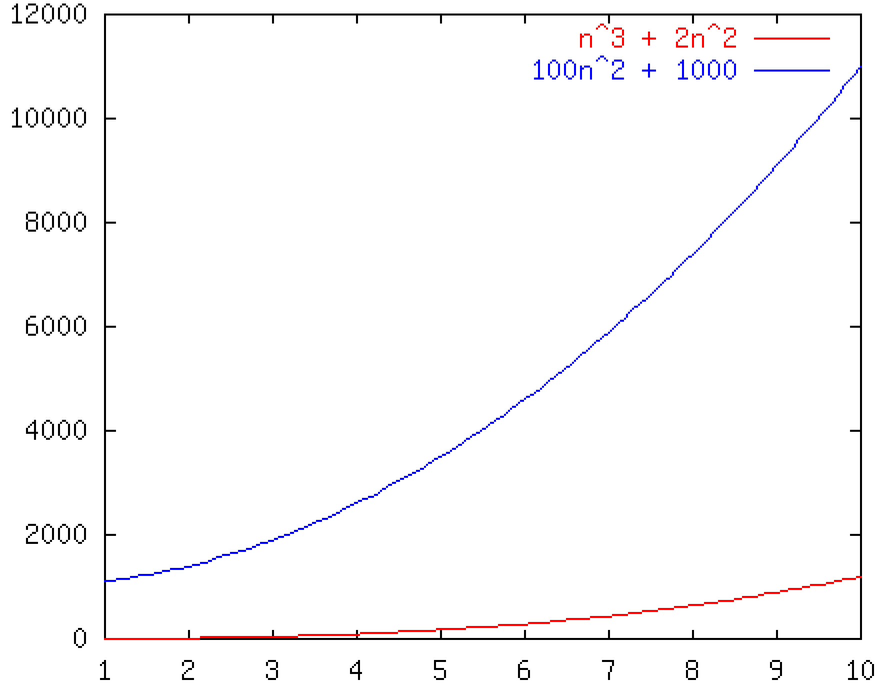
For each race, which "horse" is "faster". Note that faster means *smaller*, not larger!

a. Left
b. Right
c. Tied
d. It depends
e. I am opposed to algorithm racing.

a. Left
b. Right
c. Tied
d. It depends

Race I

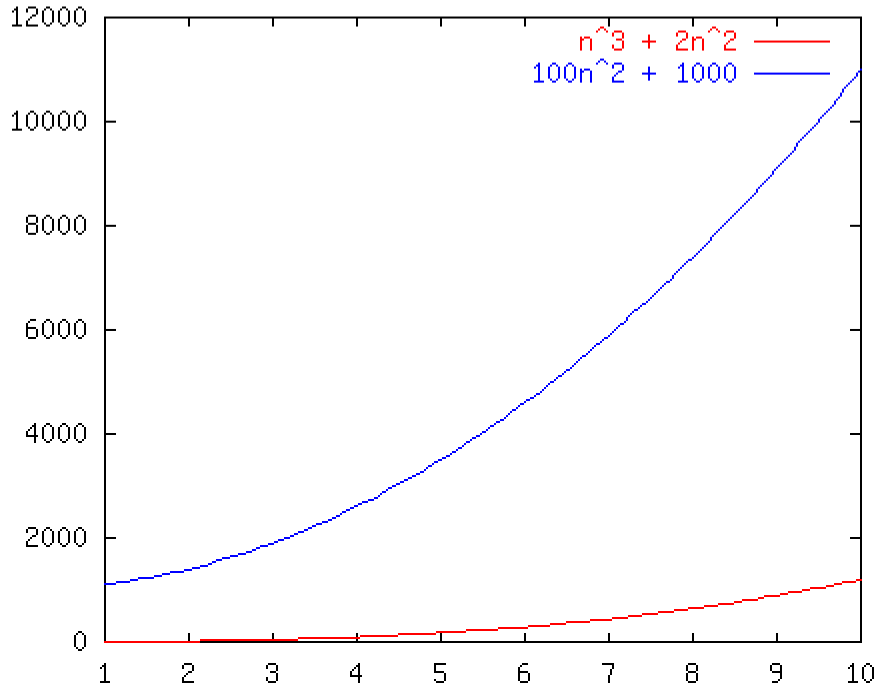
$n^3 + 2n^2$ vs. $100n^2 + 1000$

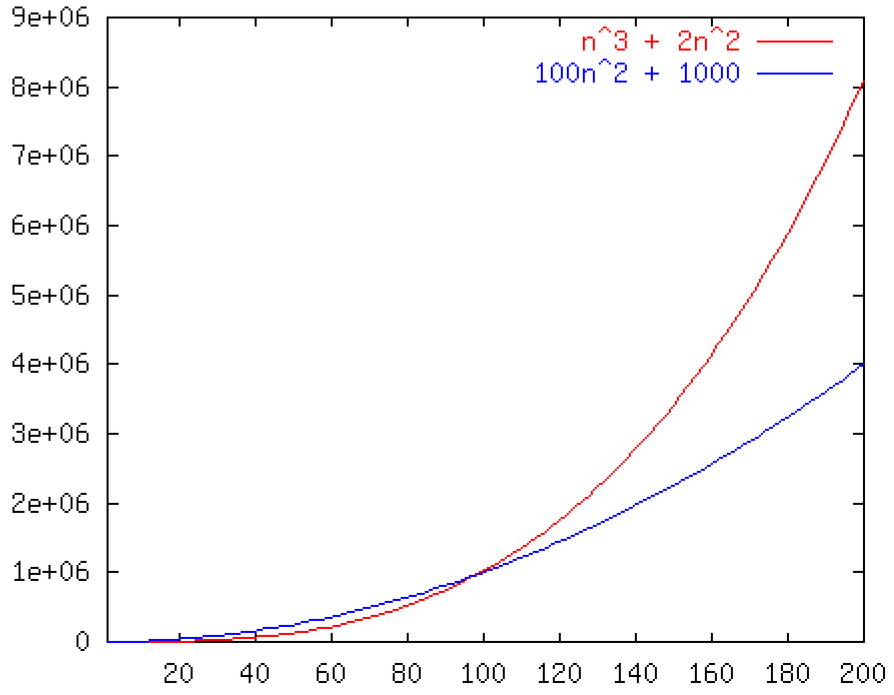


a. Left
b. Right
c. Tied
d. It depends

Race I

$n^3 + 2n^2$ vs. $100n^2 + 1000$

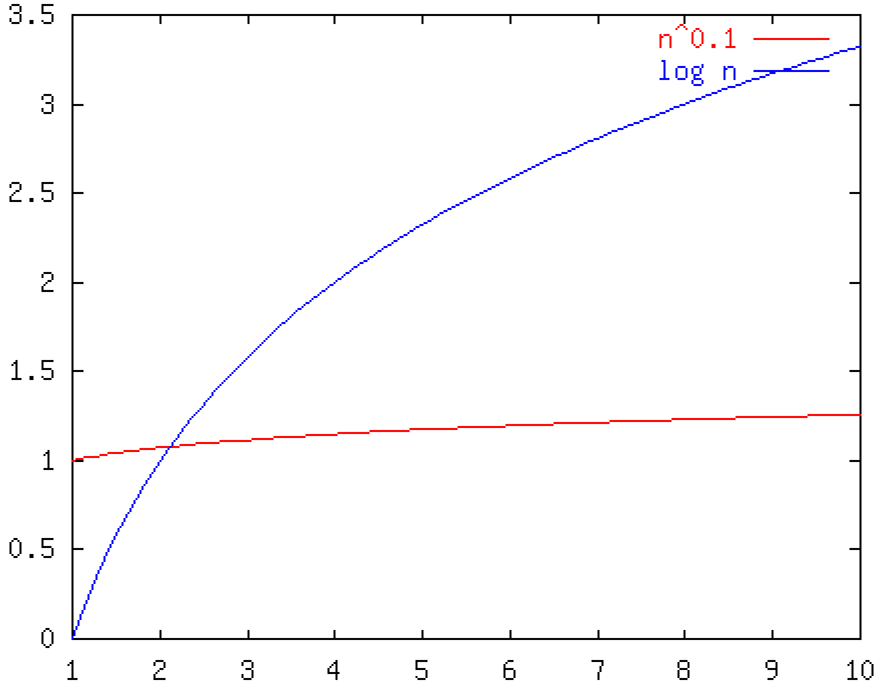




a. Left
b. Right
c. Tied
d. It depends

Race II

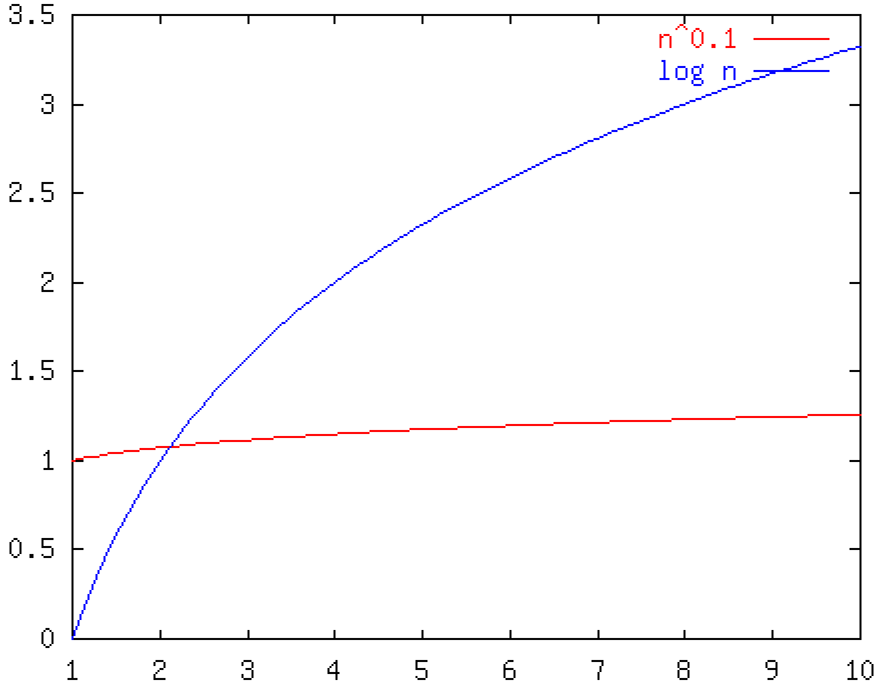
$n^{0.1}$ vs. log n

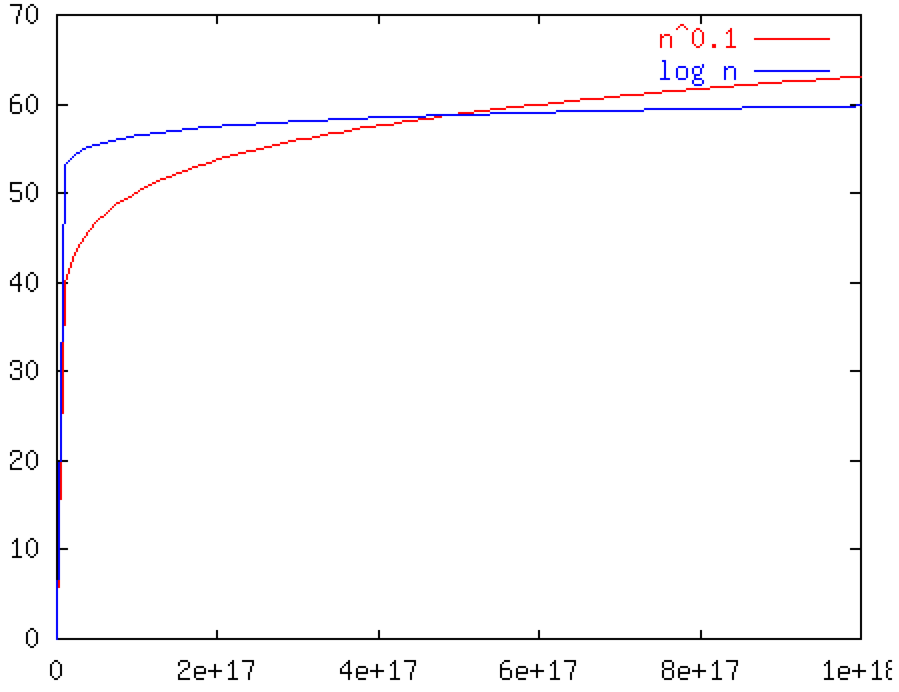


Race II

a. Left
b. Right
c. Tied
d. It depends

$n^{0.1}$ vs. **log n**

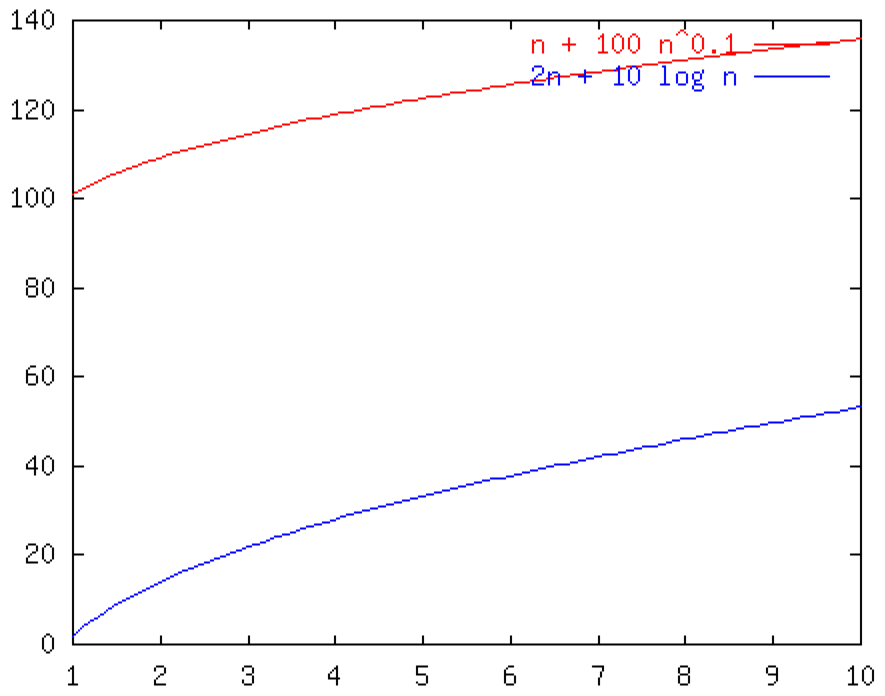




a. Left
b. Right
c. Tied
d. It depends

Race III

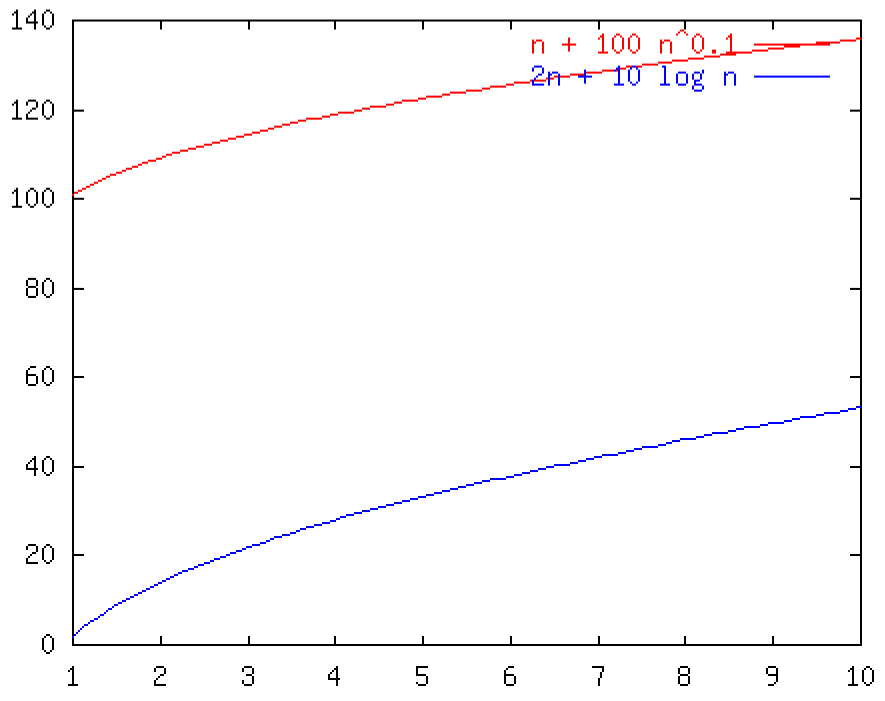
$n + 100n^{0.1}$ vs. $2n + 10 \log n$

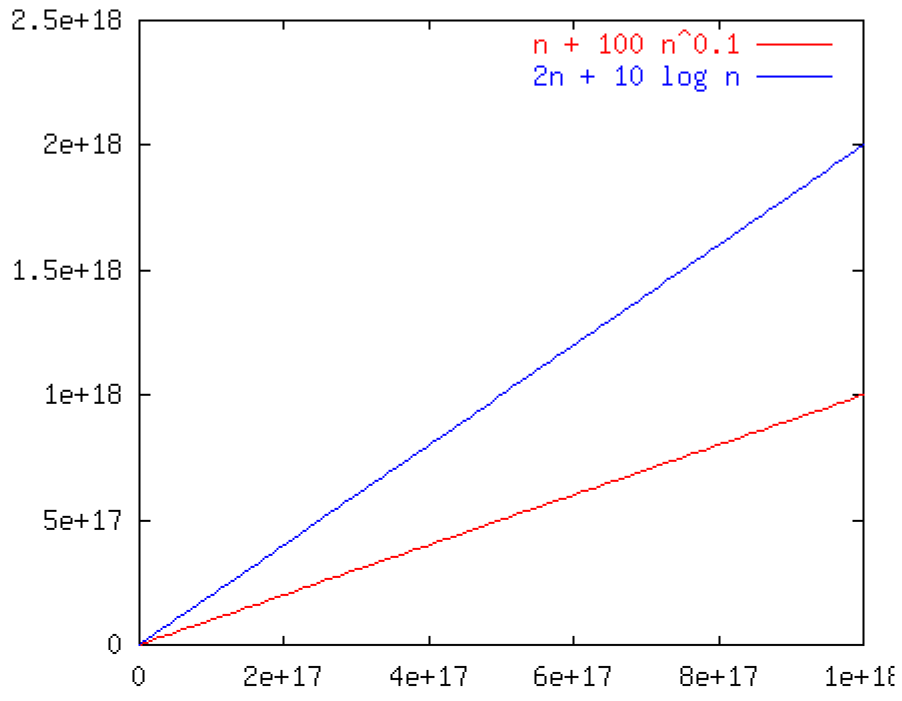


a. Left
b. Right
c. Tied
d. It depends

Race III

$n + 100n^{0.1}$ vs. $2n + 10 \log n$

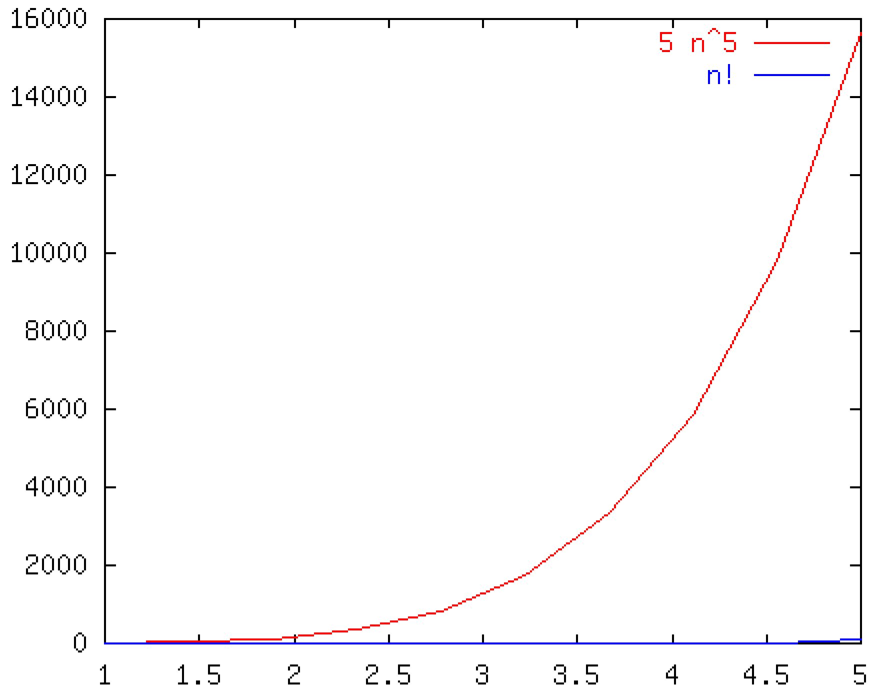




a. Left
b. Right
c. Tied
d. It depends

Race IV

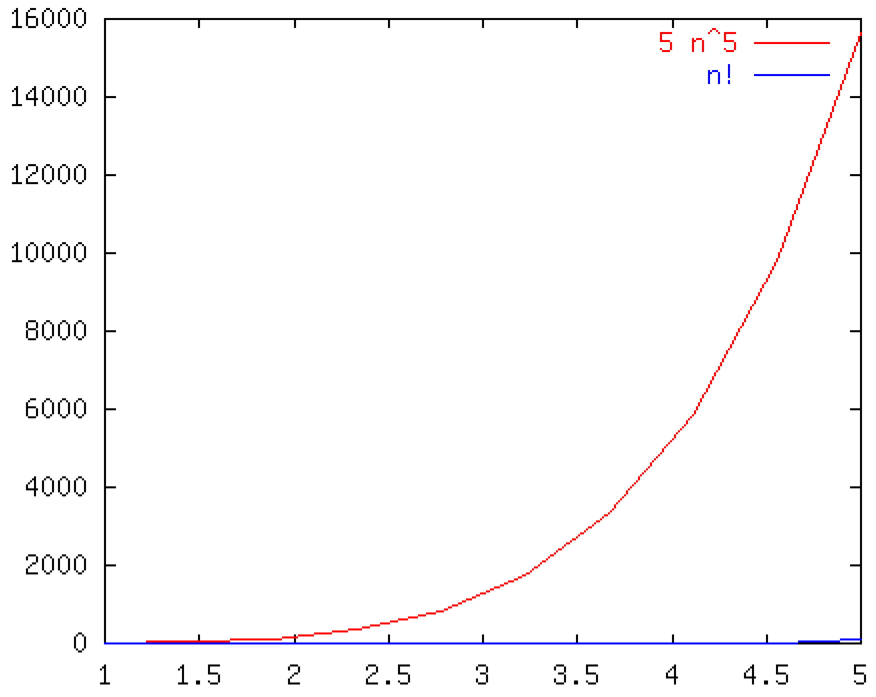
$5n^5$ vs. $n!$

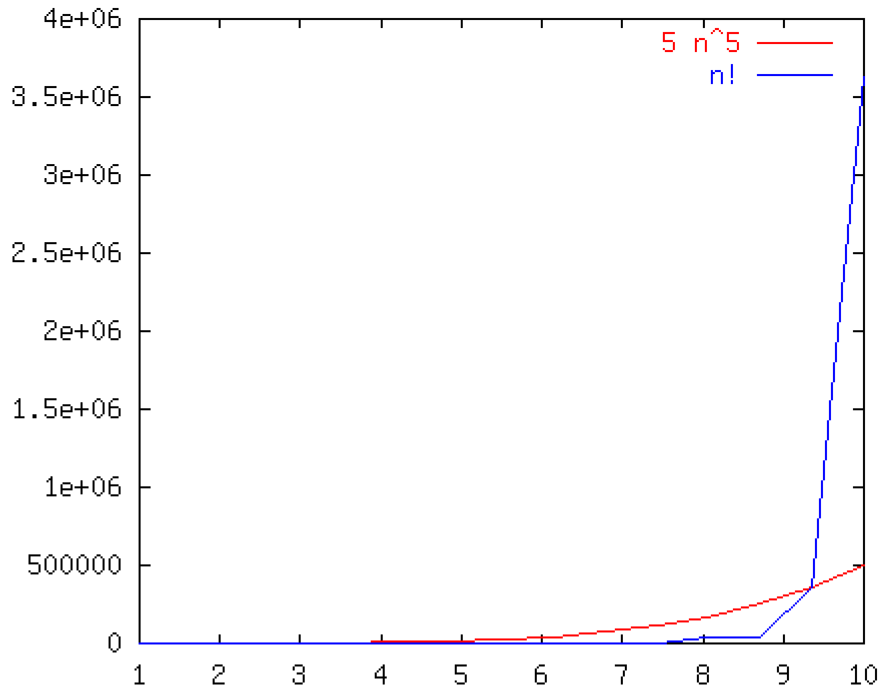


Race IV

a. Left
b. Right
c. Tied
d. It depends

$5n^5$ vs. $n!$

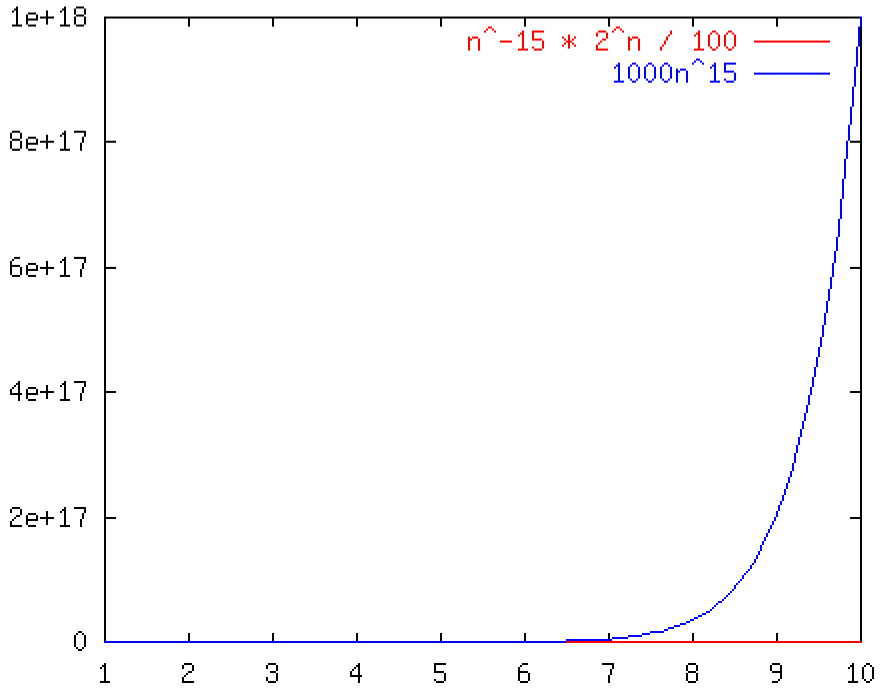




a. Left
b. Right
c. Tied
d. It depends

Race V

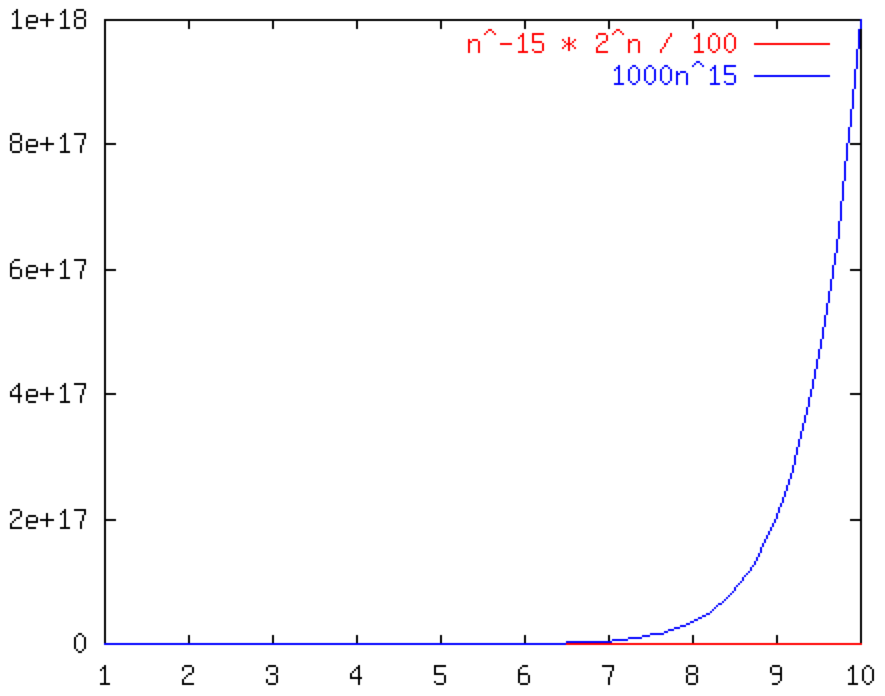
$n^{-15}2^n/100$ vs. $1000n^{15}$

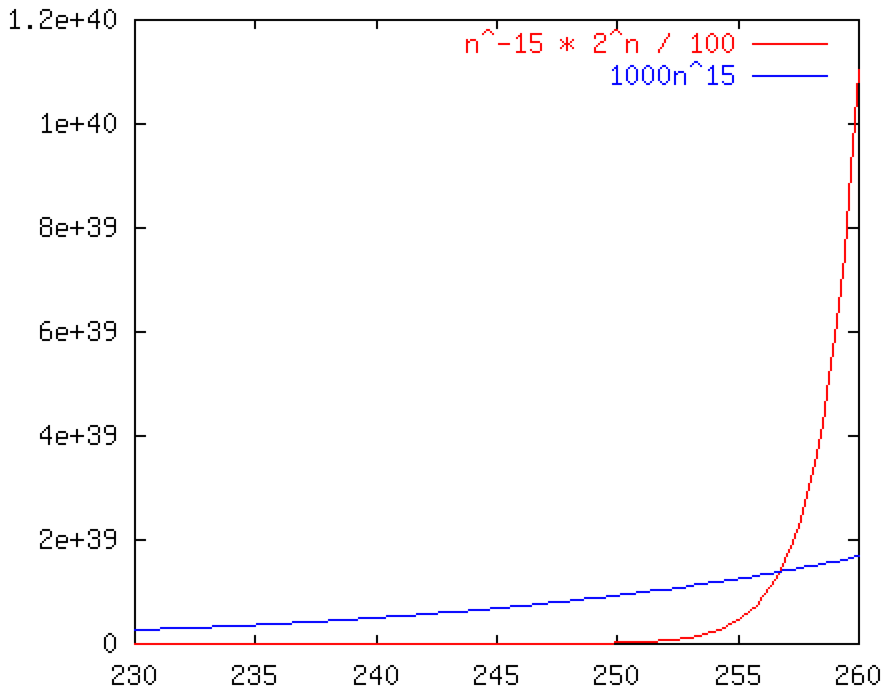


a. Left
b. Right
c. Tied
d. It depends

Race V

$n^{-15}2^n/100$ vs. $1000n^{15}$

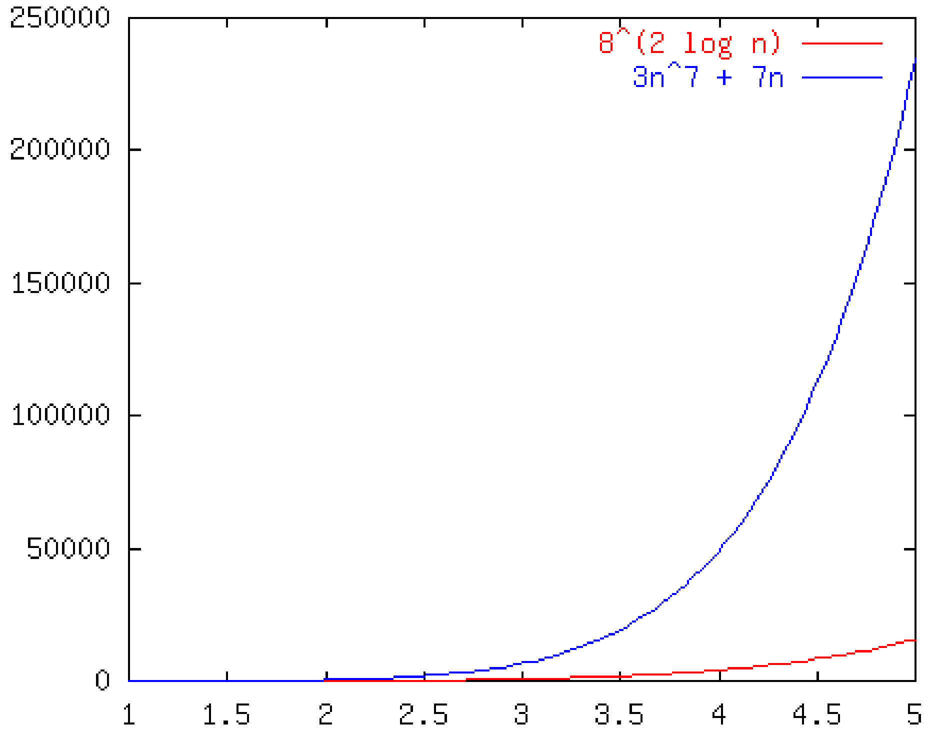




a. Left
b. Right
c. Tied
d. It depends

Race VI

$8^{2\lg(n)}$ vs. $3n^7 + 7n$



a. Left
b. Right
c. Tied
d. It depends

Race VII

mn^3 vs. $2^m n$

Silicon Downs

Post #1	Post #2	Winner
$n^3 + 2n^2$	$100n^2 + 1000$	$O(n^2)$
$n^{0.1}$	$\log n$	$O(\log n)$
$n + 100n^{0.1}$	$2n + 10 \log n$	**TIE** $O(n)$
$5n^5$	$n!$	$O(n^5)$
$n^{-15}2^n/100$	$1000n^{15}$	$O(n^{15})$
$8^{2\lg n}$	$3n^7 + 7n$	$O(n^6)$
mn^3	$2^m n$	**IT DEPENDS**

Mounties Find Silicon Downs Fixed

- The fix sheet (typical growth rates in order)

– constant:	$O(1)$	
– logarithmic:	$O(\log n)$	$(\log_k n, \log (n^2) \in O(\log n))$
– poly-log:	$O((\log n)^k)$	
– linear:	$O(n)$	
– log-linear:	$O(n \log n)$	note: even a tiny power "beats" a log
– superlinear:	$O(n^{1+c})$	(c is a constant > 0)
– quadratic:	$O(n^2)$	
– cubic:	$O(n^3)$	
– polynomial:	$O(n^k)$	(k is a constant) "tractable"
– exponential:	$O(c^n)$	(c is a constant > 1) "intractable"

The VERY Fixed Parts

- There’s also a notion of asymptotic “dominance”, which means one function as a fraction of another (asymptotically dominant) function goes to zero.
- Each line below dominates the one above it:
  - $O(1)$
  - $O(\log^k n)$, where $k > 0$
  - $O(n^c)$, where $0 < c < 1$
  - $O(n)$
  - $O(n (\log n)^k)$, where $k > 0$
  - $O(n^{1+c})$, where $0 < c < 1$
  - $O(n^k)$, where $k \geq 2$ (the rest of the polynomials)
  - $O(c^n)$, where $c > 1$

USE those cheat sheets!

- Which is faster, n^3 or $n^3 \log n$?

(Hint: try dividing one by the other.)

- Which is faster, n^3 or $n^{3.01}/\log n$?

(Ditto the hint above!)

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Terminology (Reminder)

Given an algorithm whose running time is T(n)

- $T(n) \in O(f(n))$ if there are constants c and n_0 such that $T(n) \leq c\, f(n)$ for all $n \geq n_0$
- $T(n) \in \Omega(f(n))$ if $f(n) \in O(T(n))$
- $T(n) \in \Theta(f(n))$ if $T(n) \in O(f(n))$ and $T(n) \in \Omega(f(n))$

Types of analysis

Orthogonal axes

- bound flavor
  - upper bound (O)
  - lower bound (Ω): especially useful for *problems*
  - asymptotically tight (Θ)
- analysis case
  - worst case (adversary)
  - average case
  - best case
  - “common” case
- analysis quality
  - loose bound (any true analysis)
  - tight bound (no better “meaningful” bound that is asymptotically different)

Analyzing Code

• C++ operations	- constant time
• consecutive stmts	- sum of times
• conditionals	- sum of branches, condition
• loops	- sum of iterations
• function calls	- cost of function body

We don’t know just how long each operation will take on each machine. So, we try to figure out which operations will:

- definitely run quickly (less than some constant limit).
- have runtimes dependent on inputs
- really *matter* to our algorithm

Analyzing Code

```
// Linear search
find(key, array)
  for i = 1 to length(array) do
    if array[i] == key
      return i
  return -1
```

Step 1: What’s the input size **n**?

Analyzing Code

```
// Linear search
find(key, array)
  for i = 1 to length(array) do
    if array[i] == key
      return i
  return -1
```

Step 2: What kind of analysis should we perform?
  Worst-case? Best-case? Average-case?

Expected-case, amortized, …

Analyzing Code

```
// Linear search
find(key, array)
  for i = 1 to length(array) do
    if array[i] == key
      return i
  return -1
```

Step 3: How much does each line cost? (Are lines the right unit?)

Analyzing Code

```
// Linear search
find(key, array)
  for i = 1 to length(array) do
    if array[i] == key
      return i
  return -1
```

Step 4: What's **`T(n)`** in its raw form?

Analyzing Code

```
// Linear search
find(key, array)
  for i = 1 to length(array) do
    if array[i] == key
      return i
  return -1
```

Step 5: Simplify **`T(n)`** and convert to order notation.
(Also, which order notation: O, Θ, Ω?)

Analyzing Code

```
// Linear search
find(key, array)
  for i = 1 to length(array) do
    if array[i] == key
      return i
  return -1
```

Step 6: Casually name-drop the appropriate terms in order to sound bracingly cool to colleagues: “Oh, linear search? That’s tractable, polynomial time. What polynomial? Linear, duh. See the name?! I hear it’s sub-linear on quantum computers, though. Wild, eh?”

Analyzing Code

```
// Linear search
find(key, array)
  for i = 1 to length(array) do
    if array[i] == key
      return i
  return -1
```

Step 7: **Prove** the asymptotic bound by finding constants $\mathbf{c}$ and $\mathbf{n_0}$ such that for all $\mathbf{n} \geq \mathbf{n_0}$, $\mathbf{T(n)} \leq \mathbf{cn}$.

You usually won't do this in practice.

Today’s Outline

- Programming Project #1 and Forming Teams
- Brief Proof Reminder
- Asymptotic Analysis, Briefly
- Silicon Downs and the SD Cheat Sheet
- Asymptotic Analysis, Proofs and Programs
- Examples and Exercises

More Examples Than You Can Shake a Stick At (#0)

```
// Linear search
find(key, array)
  for i = 1 to length(array) do
    if array[i] == key
      return i
  return -1
```

Here's a whack-load of examples for us to:

1. find a function **`T(n)`** describing its runtime
2. find **`T(n)`**'s asymptotic complexity
3. find **`c`** and **`n`**$_0$ to prove the complexity

METYCSSA (#1)

```
for i = 1 to n do
  for j = 1 to n do
    sum = sum + 1
```

Time complexity:

a. $O(n)$
b. $O(n \lg n)$
c. $O(n^2)$
d. $O(n^2 \lg n)$
e. None of these

METYCSSA (#2)

```
i = 1
while i < n do
  for j = i to n do
    sum = sum + 1
  i++
```

Time complexity:

a. $O(n)$
b. $O(n \lg n)$
c. $O(n^2)$
d. $O(n^2 \lg n)$
e. None of these

Three METYCSSA2 Approaches: Pure Math

```
i = 1                        takes "1" step
while i < n do               i varies 1 to n-1
  for j = i to n do          j varies i to n
    sum = sum + 1            takes "1" step
  i++                        takes "1" step
```

Now, we write a function T(n) that adds all of these up, summing over the iterations of the two loops:

$$T(n) = 1 + \sum_{i=1}^{n-1}\left(1 + \sum_{j=i}^{n} 1\right)$$

Three METYCSSA2 Approaches: Pure Math

Here's our function for the runtime of the code:

$$T(n) = 1 + \sum_{i=1}^{n-1}\left(1 + \sum_{j=i}^{n} 1\right)$$

Summing 1 for *j* from *i* to *n* is just going to be 1 added together (*n*-*i*+1) times, which is (*n*-*i*+1):

$$T(n) = 1 + \sum_{i=1}^{n-1} 1 + n - i + 1 = 1 + \sum_{i=1}^{n-1} n - i + 2$$

Three METYCSSA2 Approaches: Pure Math

Here's our function for the runtime of the code:

$$T(n) = 1 + \sum_{i=1}^{n-1} 1 + n - i + 1 = 1 + \sum_{i=1}^{n-1} n - i + 2$$

The *n* and 2 terms don't change as *i* changes. So, we can pull them out (and multiply by the number of times they're added):

$$T(n) = 1 + n(n-1) + 2(n-1) - \sum_{i=1}^{n-1} i$$

And, we know that $\sum_{i=1}^{k} i = k(k+1)/2$, so:

$$T(n) = 1 + n^2 - n + 2n - 2 - \frac{(n-1)n}{2}$$

Three METYCSSA2 Approaches: Pure Math

Here's our function for the runtime of the code:

$$T(n) = 1 + n^2 - n + 2n - 2 - \frac{(n-1)n}{2}$$

$$= n^2 + n - 1 - \frac{n^2}{2} + \frac{n}{2} = \frac{n^2}{2} + \frac{n}{2} - 1$$

So, $T(n) = \frac{n^2}{2} + \frac{n}{2} - 1$.

Drop low-order terms and the ½ coefficient, and we find:

$T(n) \in \Theta(n^2)$.

Yay!!!

Three METYCSSA2 Approaches: Faster Code/Slower Code

```
i = 1                       takes "1" step
while i < n do              i varies 1 to n-1
  for j = i to n do         j varies i to n
    sum = sum + 1           takes "1" step
  i++                       takes "1" step
```

This code is "too hard" to deal with. So, let's find *just* an upper bound.

In which case we get to change the code so in any way that makes it run no faster (even if it runs slower).

We'll let *j* go from 1 to *n* rather than *i* to *n*. Since $i \geq 1$, this is no *less* work than the code was already doing…

Three METYCSSA2 Approaches: Faster Code/Slower Code

```
i = 1                              takes "1" step
while i < n do                     goes n-1 times
  for j = 1 to n do                goes n times
    sum = sum + 1                  takes "1" step
  i++                              takes "1" step
```

Now, each iteration of each loop body takes the same amount of time as the next iteration, and we get:

$$T(n) = 1 + (n-1)(1+n) = 1 + n^2 - 1 = n^2$$

Clearly, $T(n) \in O(n^2)$!

BUT, that's just an upper-bound (big-O), since we changed the code, possibly making it run **slower**.

Three METYCSSA2 Approaches: Faster Code/Slower Code

```
i = 1                          takes "1" step
while i < n do                 i varies 1 to n-1
  for j = i to n do            j varies i to n
    sum = sum + 1              takes "1" step
  i++                          takes "1" step
```

Let’s do a lower-bound, in which case we can make the code run *faster* if we want. The trouble is that j starts at i. If it started at $n - 1$, we wouldn’t have to worry about i… but we’d get an $\Omega(n)$ bound, which is lower than we’d like.

We can’t start j at something nice like $n/2$ because i grows larger than $n/2$. So, let’s keep i from growing so large!

Three METYCSSA2 Approaches: Faster Code/Slower Code

```
i = 1                                takes "1" step
while i < $\boxed{n/2 + 1}$ do             goes n/2 times
  for j = i to n do                  j varies i to n
    sum = sum + 1                    takes "1" step
  i++                                takes "1" step
```

We used *n*/2 + 1 so the outer loop will go exactly *n*/2 times.

Now we can start *j* at *n*/2 + 1, knowing that *j* will never get that large, so we're certainly not making the code slower!

Three METYCSSA2 Approaches: Faster Code/Slower Code

```
i = 1                          takes "1" step
while i < n/2 + 1 do           goes n/2 times
  for j = n/2 + 1 to n do      goes n/2 times
    sum = sum + 1              takes "1" step
  i++                          takes "1" step
```

Again, the loop bodies take the same amount of time from one iteration to the next, and we get:

$$T(n) = 1 + \frac{n}{2}\left(\frac{n}{2} + 1\right) = 1 + \frac{n^2}{4} + \frac{n}{2}$$

Droping low-order terms and constant coefficients:

$$T(n) \in \Omega(n^2)$$

Yay!!!

Three METYCSSA2 Approaches: Pretty Pictures!

```
i = 1                              takes "1" step
while i < n do                     i varies 1 to n-1
  for j = i to n do                j varies i to n
    sum = sum + 1                  takes "1" step
  i++                              takes "1" step
```

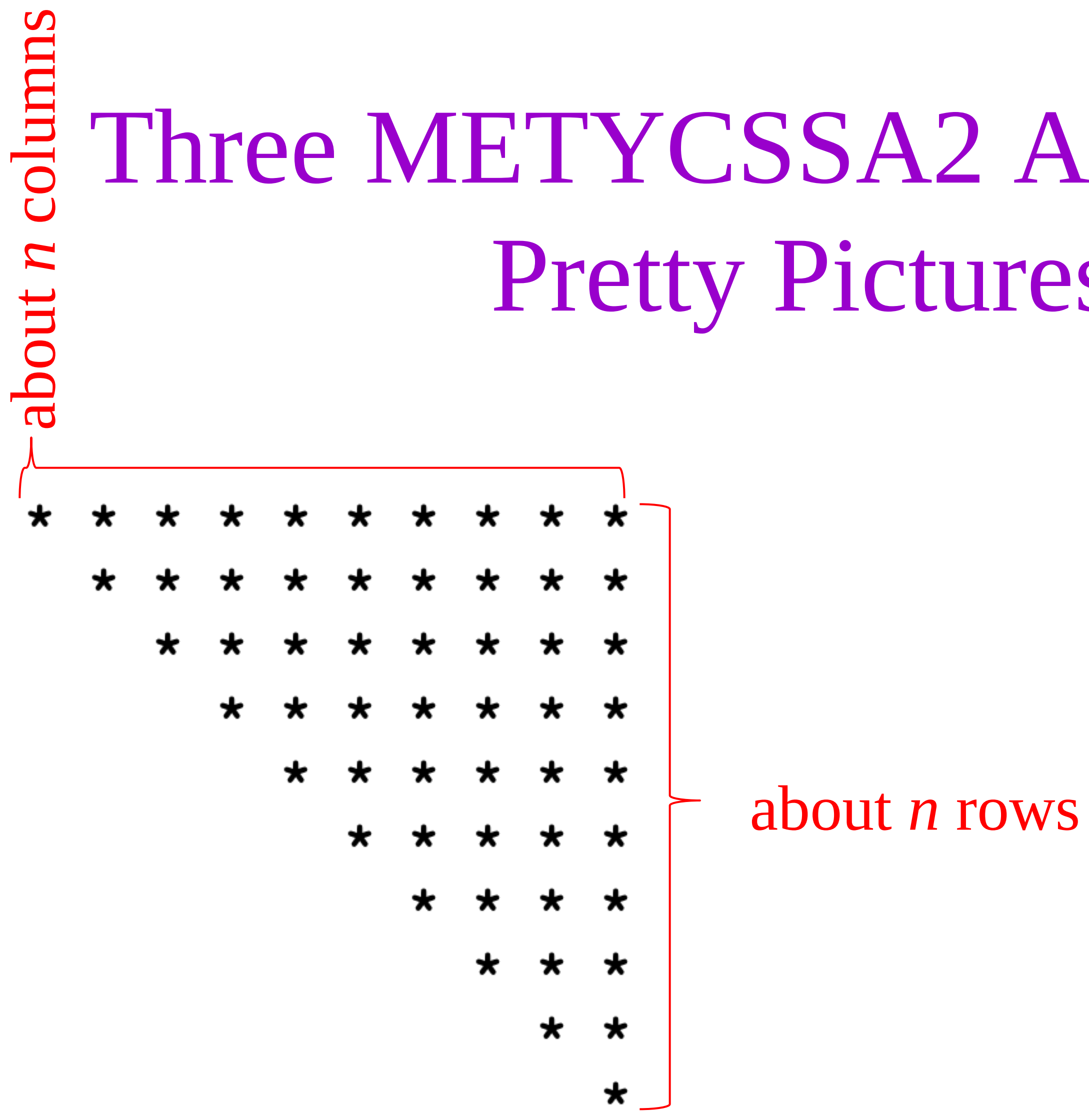
Imagine drawing one point for each time the gets executed. In the first iteration of the outer loop, you'd draw *n* points. In the second, *n*-1. Then *n*-2, *n*-3, and so on down to (about) 1. Let's draw that picture…

Three METYCSSA2 Approaches: Pretty Pictures!

```
* * * * * * * * * *
  * * * * * * * * *
    * * * * * * * *
      * * * * * * *
        * * * * * *
          * * * * *
            * * * *
              * * *
                * *
                  *
```

It’s a triangle, and its area is proportional to runtime

Three METYCSSA2 Approaches: Pretty Pictures!

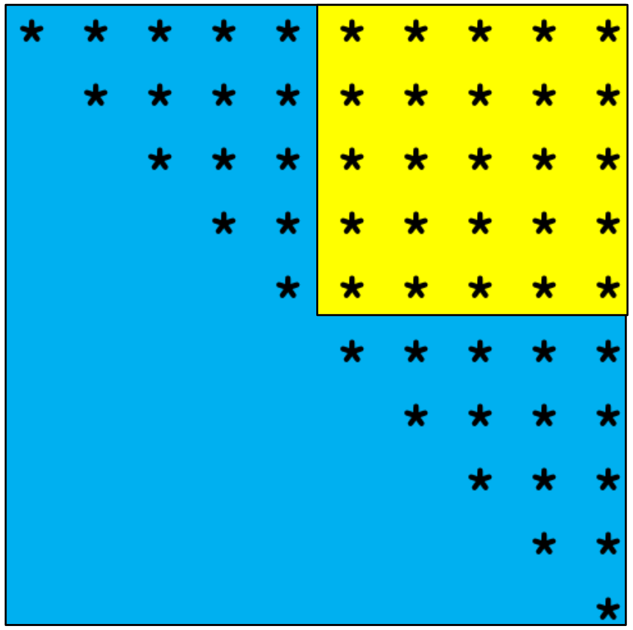


```
*  *  *  *  *  *  *  *  *  *
   *  *  *  *  *  *  *  *  *
      *  *  *  *  *  *  *  *
         *  *  *  *  *  *  *
            *  *  *  *  *  *
               *  *  *  *  *
                  *  *  *  *
                     *  *  *
                        *  *
                           *
```

It's a triangle, and its area is proportional to runtime:

$$T(n) = \frac{base * height}{2} = \frac{n^2}{2} \in \Theta(n^2)$$

Note: Pretty Pictures and Faster/Slower are the Same(ish)



Both the overestimate (upper-bound) and underestimate (lower-bound) are squares with sides proportional to n (area proportional to n^2). So, it's $\Theta(n^2)$

METYCSSA (#3)

```
i = 1
while i < n do
  for j = 1 to i do
    sum = sum + 1
  i += i
```

Time complexity:

a. O(n)
b. O(n lg n)
c. $O(n^2)$
d. $O(n^2 \lg n)$
e. None of these

Rule of thumb: Each nested loop adds one to n’s exponent.
But… that’s *just* a rule of thumb. **Look** at what code (like this!) does.

METYCSSA (#4)

- Conditional

  `if` C `then` S_1 `else` S_2

- Loops

  `while` C `do` S

METYCSSA (#5)

- Recursion almost always yields a *recurrence*
- Recursive max:

```
if length == 1: return arr[0]
else: return larger of arr[0] and max(arr[1..length-1])
```

$T(1) \le b$

$T(n) \le c + T(n - 1)$ *if* $n > 1$

- Analysis

$T(n) \le c + c + T(n - 2)$ (by substitution)

$T(n) \le c + c + c + T(n - 3)$ (by substitution, again)

$T(n) \le kc + T(n - k)$ (extrapolating $0 < k \le n$)

$T(n) \le (n - 1)c + T(1) = (n - 1)c + b$ (for $k = n - 1$)

- $T(n) \in$

METYCSSA (#6): Mergesort

- Mergesort algorithm
  - split list in half, sort first half, sort second half, merge together
- `T(1) <= b`

  `T(n) <= 2T(n/2) + cn` *if* `n > 1`
- Analysis

```
T(n) <= 2T(n/2) + cn
     <= 2(2T(n/4) + c(n/2)) + cn
      = 4T(n/4) + cn + cn
     <= 4(2T(n/8) + c(n/4)) + cn + cn
      = 8T(n/8) + cn + cn + cn
```

$\mathtt{<=\ 2^kT(n/2^k)\ +\ kcn}$ (extrapolating $1 < k \le n$)

$\mathtt{<=\ nT(1)\ +\ cn\ lg\ n}$ (for $2^k = n$ or $k = \lg n$)

- **T(n)** $\in$

METYCSSA (#7): Fibonacci

- Recursive Fibonacci:

```
int Fib(n)
   if (n == 0 or n == 1) return 1
  else return Fib(n - 1) + Fib(n - 2)
```

- *Lower* bound analysis
- `T(0), T(1) >= b`

  `T(n) >= T(n - 1) + T(n - 2) + c` *if* `n > 1`

- Analysis

  let ϕ be $(1 + \sqrt{5})/2$ which satisfies $\phi^2 = \phi + 1$

  show by induction on n that $T(n) \geq b\phi^{n-1}$

Example #7 continued

- Basis: $T(0) \geq b > b\phi^{-1}$ and $T(1) \geq b = b\phi^{0}$
- Inductive step: Assume $T(m) \geq b\phi^{m-1}$ for all $m < n$

$$
\begin{aligned}
T(n) &\geq T(n-1) + T(n-2) + c \\
&\geq b\phi^{n-2} + b\phi^{n-3} + c \\
&\geq b\phi^{n-3}(\phi + 1) + c \\
&= b\phi^{n-3}\phi^{2} + c \\
&\geq b\phi^{n-1}
\end{aligned}
$$

- $T(n) \in$
- Why? Same recursive call is made numerous times.

Example #7: Learning from Analysis

- To avoid recursive calls
  - store all basis values in a table
  - each time you calculate an answer, store it in the table
  - before performing any calculation for a value `n`
    - check if a valid answer for `n` is in the table
    - if so, return it
- This strategy is called "*memoization*" and is closely related to "*dynamic programming*"

- How much time does this version take?

Abstract Example (#8): It's Log!

Problem: find a tight bound on **`T(n) = lg(n!)`**

Time complexity:

a. $O(n)$
b. $O(n \lg n)$
c. $O(n^2)$
d. $O(n^2 \lg n)$
e. None of these

Log Aside

$\mathtt{log_a b}$ means "the exponent that turns **a** into **b**"

`lg x` means "$\mathtt{log_2 x}$" (our usual log in CS)

`log x` means "$\mathtt{log_{10} x}$" (the common log)

`ln x` means "$\mathtt{log_e x}$" (the natural log)

But… **`O(lg n) = O(log n) = O(ln n)`** because:

$$\mathtt{log_a b = log_c b \;/\; log_c a} \text{ (for } \mathtt{c > 1}\text{)}$$

so, there's just a constant factor between log bases

Asymptotic Analysis Summary

- Determine what characterizes a problem's size
- Express how much resources (time, memory, etc.) an algorithm requires as a function of input size using O(•), Ω(•), Θ(•)
  - worst case
  - best case
  - average case
  - common case
  - overall???

Some Well-Known Horses from the Downs

For general problems (not particular algorithms):

We can prove lower bounds on any solution.

We can give example algorithms to establish “upper bounds” for the best possible solution.

Searching an unsorted list using comparisons: provably $\Omega(n)$, linear search is $O(n)$.

Sorting a list using comparisons: provably $\Omega(n \lg n)$, mergesort is $O(n \lg n)$.

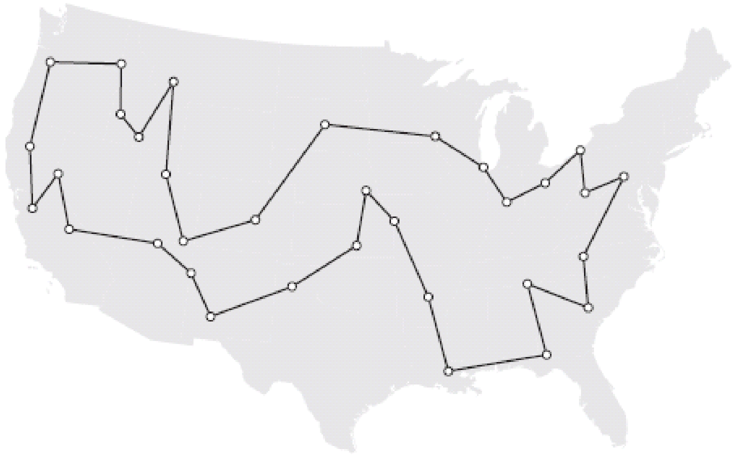
Aside: Who Cares About $\Omega(\lg (n!))$? Can You Beat $O(n \lg n)$ Search?

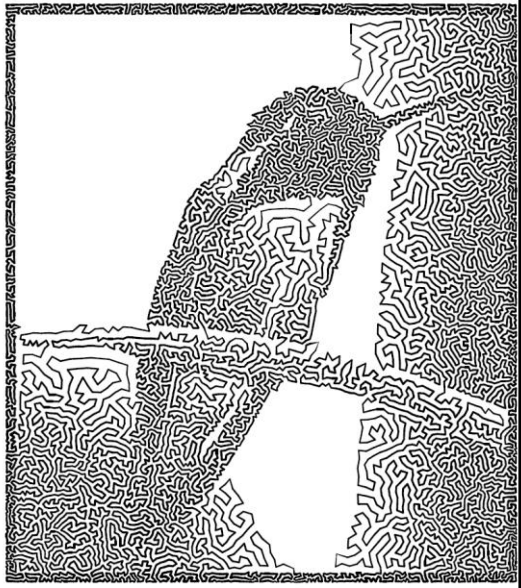
Chew these over:

1. How many values can you represent with n bits?
2. Comparing two values ($x < y$) gives you one bit of information.
3. There are n! possible ways to reorder a list. We could number them: 1, 2, …, n!
4. Sorting basically means choosing which of those reorderings/numbers you'll apply to your input.
5. How many comparisons does it take to pick among n! different values?

Some Well-Known Horses from the Downs

- Searching and Sorting: polynomial time, tractable
- Traveling Salesman Problem: non-deterministic polynomial… can check a guess in polynomial time, *maybe* exponential time to solve.





Are problems in NP really in P? **$1M** prize to prove yea or nay.

Some Well-Known Horses from the Downs

- Searching and Sorting numbers: P, tractable
- Traveling Salesman Problem: NP, intractable
- Halting Problem: uncomputable

Halting Problem: Does a given program halt on a given input. Clearly solvable in many (interesting) cases, but provably unsolvable in general.

(We can substitute "halt on" for almost anything else interesting: "print the value 7 on", "call a function named Buhler on", "access memory location 0xDEADBEEF on", …)

To Do

- Find a teammate for labs and assignments (not necessarily the same!)
- Keep up with the quizzes and programming projects!
- Read Epp 9.2-9.3 (3rd ed.) or 11.2-11.3 (4th ed.) and Koffman 2.6
- Prepare for upcoming labs

Coming Up

- Recursion and Induction
- Loop Invariants and Proving Program Correctness
- Call Stacks and Tail Recursion
- First written corrections due
- First Programming Project due