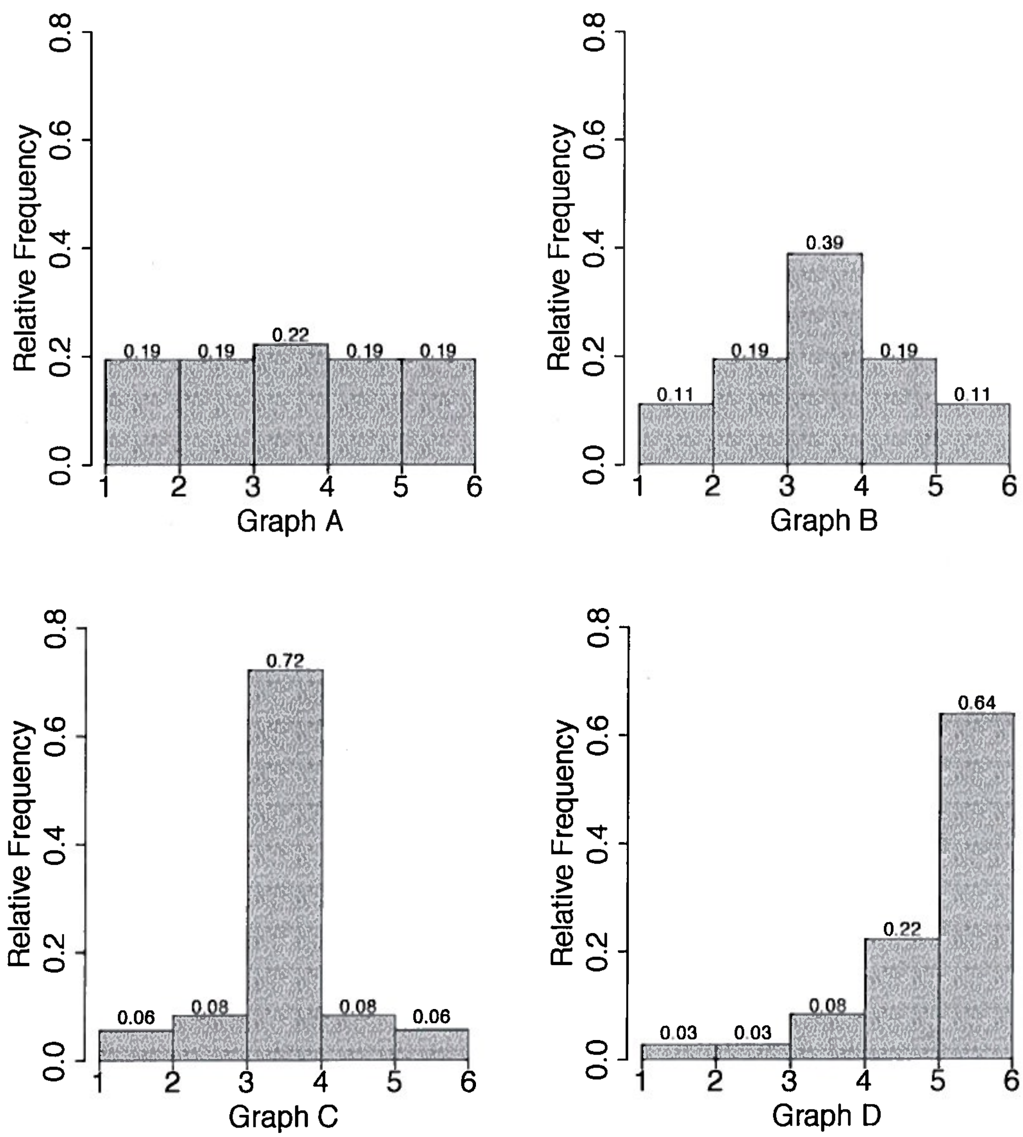
Solutions.

1. **Multiple Choice**. Please circle the most appropriate response.

   (a) Consider the four histograms below (Graphs A, B, C, D) with relative frequencies along the y-axis. The relative frequencies are also listed above each bar so it is easier to read. There are 36 observations in each graph. Which of the following has the largest standard deviation?

   I. Graph A
   II. Graph B
   III. Graph C
   IV. Graph D
   V. There is insufficient information



(b) Let A = {Draw a red card from a regular deck of 52 cards}, and B = {Draw an ace from a regular deck of 52 cards}. Then the events A and B are:

I. disjoint
II. independent
III. complements
IV. none of the above

(c) The number of defective parts produced each hour by a certain production line has the following probability distribution:

Number of defective parts (x)	0	1	2	3	4
$P(X = x)$	0.15	0.30	0.25	0.20	0.10

Suppose it is known that more than 2 defective parts were produced in a particular hour. What is the probability that the number of defective parts was fewer than 4?

I. 0.10
II. 0.36
III. 0.67
IV. 0.90
V. None of the above

(d) The length of a metal sheet is a random variable with mean 150m and standard deviation 2m. The mean (in m) and variance (in m^2) of the total length of five randomly chosen metal sheets will be:

I. Mean = 150, Variance = 10
II. Mean = 750, Variance = 10
III. Mean = 150, Variance = 20
IV. Mean = 750, Variance = 20
V. Mean = 750, Variance = 100

(e) Consider tossing a coin 3 times, and define the following events: A = {Toss 3 heads in a row} and B = {Toss a head, then a tail, then a head}. Choose one of the following answers.

I. $P(A) > P(B)$
II. $P(A) = P(B)$
III. $P(A) < P(B)$
IV. Not enough info to tell

Short Answer. Please show all your work. Be sure to define variables, state models used and check assumptions where appropriate.

2. A process for making a particular type of alloy yields up to 1 ton of alloy a day. The actual amount produced, Y is a random variable because of machine breakdowns and various slowdowns. Suppose Y has the following pdf

$$f(y) = \begin{cases} 2y & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The company is paid \$300 per ton of alloy, but there is also a fixed overhead cost of \$100 per day. Let U be the company's daily profit (in hundreds of dollars).

(a) Find the probability density function of U.

$U = 3Y - 1 \qquad 0 \leq y \leq 1 \Rightarrow -1 \leq u \leq 2$

$F_Y(y) = \int_0^y 2t\, dt = t^2 \Big|_0^y = y^2$

$G_U(u) = P(U \leq u) = P(3Y - 1 \leq u) = P\left(Y \leq \frac{u+1}{3}\right)$

$= F_Y\left(\frac{u+1}{3}\right)$

$G_U(u) = \left(\frac{u+1}{3}\right)^2$

$g_U(u) = \frac{d}{du}\left(\frac{u+1}{3}\right)^2 = 2\left(\frac{u+1}{3}\right) \cdot \frac{1}{3}$

$$g_U(u) = \begin{cases} \frac{2}{9}(u+1), & -1 \leq u \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

(b) What is the company's expected daily profit?

$$E(u) = \int_{-\infty}^{\infty} u\, g(u)\, du$$

$$= \int_{-1}^{2} u \cdot \frac{2}{9}(u+1)\, du$$

$$= \frac{2}{9}\int_{-1}^{2} u^2 + u \quad du$$

$$= \frac{2}{9}\left[\frac{u^3}{3} + \frac{u^2}{2}\right]_{-1}^{2} = \frac{2}{9}\left[\frac{8}{3} + \frac{4}{2} - \left(-\frac{1}{3} + \frac{1}{2}\right)\right]$$

$$= \frac{2}{9}\left[\frac{9}{2}\right] = 1$$

The expected daily profit is \$100

3. Suppose you are standing on the side of a road. Cars pass by at a rate of 4 per hour. The number of cars can be modelled with a Poisson distribution. What is the probability that at least two cars pass by in the next half hour?

Let X be the number of cars passing in next half hour.

$X \sim Pois\left(\frac{1}{2} \cdot 4 = 2\right)$

$$P(X \geqslant 2) = 1 - P(X < 2)$$
$$= 1 - [P(X=0) + P(X=1)]$$
$$= 1 - \frac{e^{-2} \cdot 2^0}{0!} - \frac{e^{-2} \cdot 2^1}{1!} = 1 - e^{-2} - 2e^{-2}$$
$$= 1 - 3e^{-2}$$
$$= 0.594$$

4. A factory produces metal sheets with thickness that is normally distributed with a mean of 10m and variance of 0.04m. Manufacturer requirements say that the sheets must be between 9.5 and 10.5m.

(a) What percentage of sheets produced will not meet the manufacturer's requirements?

Let X be thickness of metal sheets. $X \sim N(10, 0.04)$

meet requirements:

$$P(9.5 \le X \le 10.5) = P\left(\frac{9.5-10}{\sqrt{0.04}} \le \frac{X-\mu}{\sigma} \le \frac{10.5-10}{\sqrt{0.04}}\right)$$

$$= P(-2.5 \le Z \le 2.5) = P(Z \le 2.5) - P(Z \le -2.5)$$

$$= 2P(Z \le 2.5) - 1$$

$$= 2 \times 0.9938 - 1 = 0.9876$$

$\therefore$ percent not meeting requirements:

$1 - 0.9876 = 0.0124$

$\rightarrow$ 1.24% will not meet requirements.

(b) Suppose you randomly select a sheet and find that it does meet manufacturer's requirements, what is the probability that it has a thickness greater than 10.2m?

$$P(X > 10.2 \mid 9.5 \le X \le 10.5) = \frac{P(\{X > 10.2\} \text{ and } \{9.5 \le X \le 10.5\})}{P(9.5 \le X \le 10.5)}$$

$$= \frac{P(10.2 < X \le 10.5)}{P(9.5 \le X \le 10.5)}$$

$$P(10.2 < X \le 10.5) = P\left(\frac{10.2-10}{\sqrt{0.04}} < Z \le \frac{10.5-10}{\sqrt{0.04}}\right)$$

$$= P(1 < Z \le 2.5) = P(Z \le 2.5) - P(Z < 1)$$

$$= 0.9938 - 0.8413$$

$$= 0.1525$$

$$P(X > 10.2 \mid 9.5 \le X \le 10.5) = \frac{0.1525}{0.9876} = 0.1544$$

$\rightarrow$ 15.44%

(c) The manufacturer wants to make sure the sheets are meeting the manufacturer requirements. Testing of the sheets begins on a production run in this factory and independent random sheets are inspected. What is the probability the third sheet inspected will be the first sheet that does not meet manufacturer requirements?

Let X be # of the test at which manufacturer requirement is not met

$X \sim geo(p = 0.0124)$

$P(X=3) = 0.9876^2 \times 0.0124 = 0.01209$

(d) An independent random sample of 10 sheets are inspected. What is the expectation and standard deviation of the number of sheets that do not meet manufacturer requirements?

Let Y be the number of sheets not meeting requirements in 10 sheets.

$Y \sim Bin(n=10, p=0.0124)$

$E(Y) = np = 10 \times 0.0124 = 0.124$

$Var(Y) = np(1-p) = 10 \times 0.0124 \times 0.9876$
$= 0.12246$

$SD(Y) = \sqrt{0.1225} = 0.3499$

5. A computer manufacturer offers a one year warranty. If the computer fails for any reason during this period, it is replaced. The time to failure is modelled by an exponential distribution with a mean of 4 years.

(a) Find the median time to failure.

Let X be time time to failure. $X \sim exp(\frac{1}{4})$

$$F(x) = \int_0^x \frac{1}{4} e^{-\frac{1}{4}t} dt = \frac{1}{4} \cdot (-4) e^{-\frac{1}{4}t} \Big|_0^x = 1 - e^{-\frac{x}{4}}$$

set

$F(x) = 0.5$

$1 - e^{-x/4} = 0.5$

$e^{-x/4} = 0.5 \rightarrow x = 2.77$ years.

(b) What percentage of computers will fail within the warranty period?

$$P(X \leq 1) = F_X(1) = 1 - e^{-(1)(\frac{1}{4})} = 0.2212$$

$\rightarrow$ 22.12% of computers will fail within warranty.

(c) Suppose you randomly sample 100 computers. Assume failure times for computers are independent. What is the probability that the average time to failure for 100 computers is less than 3.5 years?

n is large $n = 100 \geq 20$

$\bar{X} \sim N(\mu = 4, \frac{4^2}{100})$ (Recall $Var(X) = \frac{1}{\lambda^2}$ if $X \sim exp(\lambda)$, $\lambda = \frac{1}{4}$ in this case.)

approx., by CLT

$$P(\bar{X} < 3.5) = P\left(Z < \frac{3.5-4}{4/10}\right) = P(Z < -1.25)$$

$= 1 - P(Z < 1.25)$

$= 1 - 0.8944$

$= 0.1056$

6. Two teams are competing to complete a certain project. Completion times of each team can be be modelled as exponential random variables with mean 4 days. Assume each team is independent of one another. What is the probability that the completion time of the project will be more than 5 days?

Let X_i be completion time for each team. $i = \{1, 2\}$

$X_i \sim \exp(\lambda = \frac{1}{4})$

$Y = \min(X_1, X_2)$ completion time of project.

$$
\begin{aligned}
P(Y \le y) &= 1 - P(Y > y) \\
&= 1 - P(X_1 > y, X_2 > y) \\
&= 1 - P(X_1 > y)\,P(X_2 > y) \quad \text{because } X_i\text{'s independent.} \\
&= 1 - [P(X > y)]^2 \\
&= 1 - [1 - F_X(y)]^2 \\
&= 1 - [1 - (1 - e^{-y/4})]^2 \\
&= 1 - (e^{-y/4})^2
\end{aligned}
$$

$$
\begin{aligned}
F(x) &= \int_0^x \frac{1}{4} e^{-t/4}\, dt \\
&= -e^{-t/4} \Big|_0^x \\
&= 1 - e^{-x/4}
\end{aligned}
$$

$$
\begin{aligned}
P(Y > 5) &= e^{-10/4} \\
&= 0.0821
\end{aligned}
$$