$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Ch. 3 - Probability

Notation:

Random Phenomenon: a situation where we know the possible outcomes ahead of time, but individual outcomes are only known after the situation occurs.
e.g. Roll a die, toss a coin, whether it will rain tomorrow, whether an item/unit meets certain specifications...

Sample Space (denoted S): set of all possible outcomes of a random phenomenon.
e.g. $S = \{\text{H, T}\}$, $S = \{1, 2, 3, 4, 5, 6\}$, toss 2 coins $S = \{\text{HH, TH, HT, TT}\}$
e.g. pair dice. sum dots on two faces.

Event: outcome or some outcomes from a random phenomenon. Usually denoted with capital letters.
e.g. $A = \{\text{H}\}$, $B = \{\text{Roll an even number}\}$

We use the notation $P(A)$ to denote the probability that an event A will occur.

- $0 \leq P(A) \leq 1$ (A probability is a number between 0 and 1)
- $P(A) = 0$ implies that event A is impossible
  $P(A) = 1$ implies that event A always occurs

NEVER CERTAIN

0 1/2 1

Example 1. A fair die is tossed once. Find the probability the number is odd.

Solution:
Let E be the event of obtaining an odd number.

$P(S) = 1$

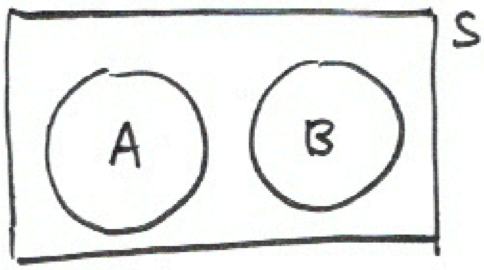
$E = \{1, 3, 5\}$ and $S = \{1, 2, 3, 4, 5, 6\}$

$$P(E) = \frac{\text{number of elements in E}}{\text{number of elements in sample space}} = \frac{3}{6} = \frac{1}{2}$$

Disjoint or Mutually Exclusive Events: when events have no outcomes in common they are said to be *disjoint*. (They cannot occur simultaneously)

e.g. If $A = \{\text{roll a 1}\}$ and $B = \{\text{roll an even number}\}$ then A and B are disjoint.
$P(A \text{ and } B \text{ occur simultaneously}) = 0$

We can visualize disjoint events using a venn diagram:



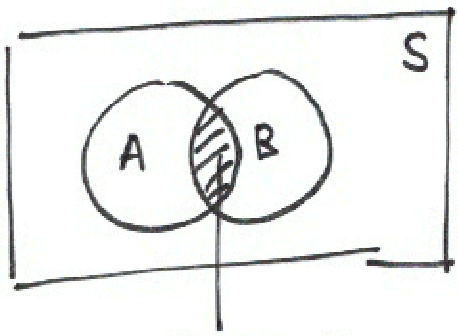
A and B are disjoint events.

Some Notation:

intersect

$P(A \text{ and } B) = P(A \cap B)$

A and B must occur together.

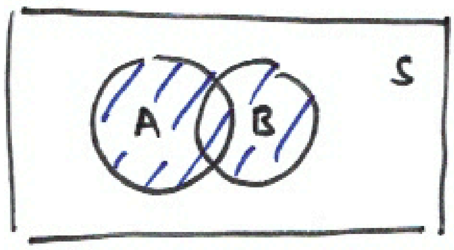


A and B.

P(event A occurs and event B occurs)

union

$P(A \text{ or } B) = P(A \cup B)$

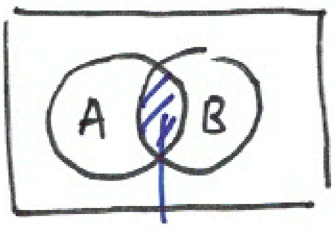


P(event A occurs or event B occurs or both occur)

Some Properties of Probability

1. General Addition Rule: $\underline{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

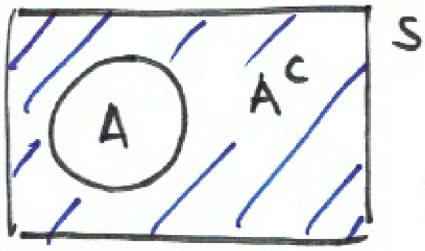
   Notice that if A and B and disjoint events [i.e. $P(A \cap B) = 0$] then $P(A \cup B) = P(A) + P(B)$



$A \cap B$

simply adding $P(A)$ and $P(B)$ we count $P(A \cap B)$ twice we compensate by subtracting $P(A \cap B)$

2. Complement Rule: $P(A^c) = 1 - P(A)$

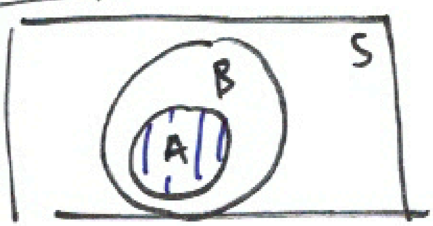


eg. $A = \{H\}$
$A^C = \{T\}$

eg. $B = \{\text{roll } 1\}$ $\quad B^C = \{2, 3, 4, 5, 6\}$

3. If $A \subseteq B$ then $P(A \cap B) = P(A)$

subset



since all outcomes of A are in B, the intersect is A.

eg. $A = \{\text{roll } 1\}$
$B = \{\text{roll odd } \#\} = \{1, 3, 5\}$
$A \cap B = \{\text{roll } 1\}$

4. If $A \subseteq B$ then $P(A) \leq P(B)$

if all outcomes in A are also in B.

$\Rightarrow P(A) \leq P(B)$

eg. $A = \{\text{roll } 1\}$ $\quad P(A) = \frac{1}{6}$

$B = \{\text{roll odd}\} = \{1, 3, 5\}$ $\quad P(B) = \frac{1}{2}$

Example 2. Police report that 78% of drivers stopped on suspicion of drunk driving are given a breath test, 36% are given a blood test and 22% receive both. What is the probability that a randomly selected drunk driving suspect is given:

Let $A = \{$suspect given breath test$\}$

$B = \{$blood test$\}$

(a) a test?

(b) exactly one of the 2 tests?

Solution: (worked out in class)

a) "test" ⇒ given blood, breath or both.

given $\begin{cases} P(A) = 0.78 \\ P(B) = 0.36 \\ P(A \cap B) = 0.22 \end{cases}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ general addition rule.

$= 0.78 + 0.36 - 0.22$

$= \cancel{0.98}\ 0.92.$

b) exactly one of two tests.

$P(A \text{ or } B, \text{ but NOT both}) = P(A \text{ or } B) - P(A \cap B)$

$= 0.92 - 0.22$

$= 0.7$

Conditional Probability

For any two events, A and B with $P(B) > 0$, the **conditional probability** of A given that B has occurred is defined by:

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}} \quad ①$$

Notice, if we rearrange the equation above we get the Multiplication Rule: $P(A \cap B) = P(B) \times P(A|B)$ [Also $P(A \cap B) = P(A) \times P(B|A)$]

Independence = not associated

Two events, A and B are **independent** if knowing that one occurs does not change the probability that the other occurs. That is:

$$② \quad \boxed{P(A) = P(A|B)} = P(A|B^c)$$

The probability of A is the same when we are given that B has occurred. Equivalently, A and B are independent if $P(B) = P(B|A)$.

Notice that if A and B are independent $[P(A|B) = P(A)]$ and if we rearrange the formula for the conditional probability of A given B above then we get:

$$\boxed{P(A \cap B) = P(B)P(A)}$$

Note: we do not use Venn diagrams to visualize independence. If asked to show 2 events are independent, you may use either applicable formula above. ②

$$① \quad P(A|B) = \frac{P(A \cap B)}{P(B)} \rightarrow P(B)\underbrace{P(A|B)}_{P(A)} = P(A \cap B)$$

$$\Rightarrow P(A \cap B) = P(A)P(B)$$

Example 3. Continuing on with example 2 above,

(a) ~~If we know that a~~ randomly selected drunk driving suspect has been given a breath test, what is the probability that they have also been given a blood test?

(b) Are the two tests independent?

Solution: $B = \{blood\}$, $A = \{breath\}$

a) $$P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{0.22}{0.78} = 0.28$$

given

b) $$P(B) = 0.36 \neq P(B \mid A) = 0.28$$

$\therefore$ NOT independent (associated)

If events A and B are <u>independent</u> then

(a) A^c and B are also independent

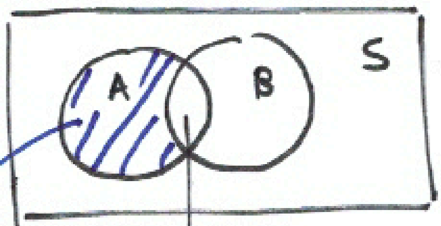
(b) A and B^c are also independent

(c) A^c and B^c are also independent

Want to show:

$P(A \cap B^c) = P(A)\,P(B^c)$

Proof for (a):



$P(A \cap B^c)$

$P(A \cap B)$

$$P(A) = P(A \cap B) + \underline{P(A \cap B^c)}$$

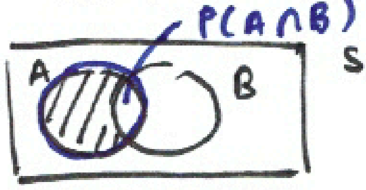
$$\begin{aligned}
P(A \cap B^c) &= P(A) - P(A \cap B) \\
&= P(A) - P(A)P(B) \quad \text{b/c } A, B \text{ indep. events.} \\
&= P(A)\,[1 - P(B)] \\
&= P(A)\,P(B^c) \quad \text{complement rule}
\end{aligned}$$

$\therefore$ A, B^c independent.

(You can use similar logic to show (b) and (c))

Example 4. Suppose A and B are independent events, where $P(A) = 0.6$ and $P(B) = 0.3$. Find $P(A \cap B^c)$.

<u>Solution:</u>
Since A and B are independent, then A and B^c are also independent (on a midterm or assignment you should prove this before applying the formula)
$P(A \cap B^c) = P(A) \times P(B^c) = 0.6 \times 0.7 = 0.42$



Alternate method:

$$\begin{aligned} \underline{\underline{P(A \cap B^c)}} &= P(A) - P(A \cap B) \\ &= P(A) - P(A)P(B) \qquad \text{[Since A and B are independent]} \\ &= 0.6 - (0.6 \times 0.3) \\ &= 0.42 \end{aligned}$$

Example 5. A and B are two events such that $P(A) = \frac{1}{3}$, $P(B) = \frac{2}{5}$ and $P(A \cup B) = \frac{17}{30}$. Find $P(A \cap B)$.

<u>Solution:</u>

$\cancel{P(A \cap B) = P(A)P(B)}$ only works if A, B are independent.

$P(A \cup B) = P(A) + P(B) - P(\underline{\underline{A \cap B}})$
gen. add. rule.

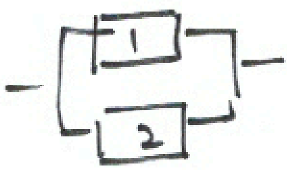
$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$
$$= \frac{1}{3} + \frac{2}{5} - \frac{17}{30} = \frac{1}{6}$$

Example 6. An electronic system has 2 components operating independently of one another (i.e. one component's failure or non-failure does not affect the second component's chance of failure/non-failure). Each component has a 0.75 probability of operating properly.

(a) If the components are arranged in parallel what is the probability that the entire system operates? ↳ entire system works if at least 1 operates.

(b) If the system is instead connected in series, what is the probability that the entire system operates?

Solution: Let A_i rep. event component i works. $(i = 1, 2)$

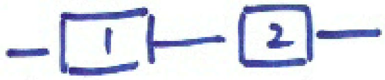


$P(A_i) = 0.75$

$$
\begin{aligned}
P(\text{system works}) &= P(A_1 \text{ or } A_2 \text{ or both work}) \\
&= P(A_1 \cup A_2) \\
\text{gen. add. rule} &= P(A_1) + P(A_2) - P(A_1 \cap A_2) \\
&= P(A_1) + P(A_2) - \underbrace{P(A_1)P(A_2)}_{\text{indep.}} \\
&= 0.75 + 0.75 - 0.75^2 \\
&= 0.9375
\end{aligned}
$$

Note: In some cases it may be easier to work using the complement property, where
P(system works) = 1 - P(system fails)

(b)



if one component fails entire system fails.

$$
\begin{aligned}
P(\text{system works}) &= P(A_1 \cap A_2) \\
&= P(A_1)\,P(A_2) \quad \underline{\underline{\text{indep.}}} \\
&= 0.75^2 = 0.5625
\end{aligned}
$$

Bayes Theorem

Example 7. Suppose an assembly plant receives voltage regulators from three different suppliers. Of the regulators, 50% are from supplier 1, 30% from supplier 2 and 20% from supplier 3. It is known that 2.5% of supplier 1's regulators are defective, whereas the corresponding percentages for suppliers' 2 and 3 are 2% and 1% respectively.

(a) What is the probability that a randomly selected regulator comes from supplier 1 and is defective?

(b) What is the probability that a randomly selected regulator is defective?

(c) If we select a regulator and find that it is defective, what is the probability it came from supplier 1? Supplier 2? Supplier 3?

Solution:

Let $A_i = \{\text{regulator comes from supplier } i\}$
$i = 1, 2, 3$

$P(A_1) = 0.5$, $P(A_2) = 0.3$, $P(A_3) = 0.2$

once supplier is selected, 2nd stage involves observing whether the selected regulator is defective.

Let $B = \{\text{defective}\}$ $B^c = \{\text{not defective}\}$

$P(B|A_1) = 0.025$, $P(B|A_2) = 0.02$, $P(B|A_3) = 0.01$

3 questions above can be translated to set notation:

a) $P(A_1 \cap B) = ?$ 0.0125

b) $P(B) = ?$ 0.0205

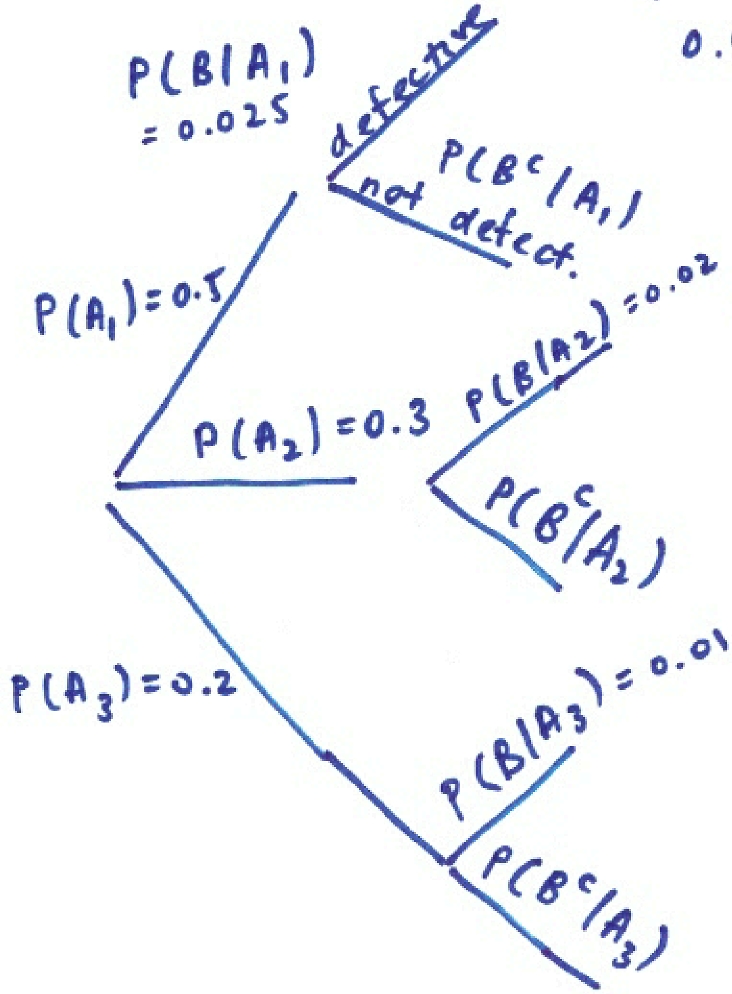
c) $P(A_1 | B) = ?$ 0.61

$P(A_2 | B) = ?$ 0.29

$P(A_3 | B) = ?$ 0.1

$P(B|A_1)P(A_1) = P(A_1 \cap B)$

$0.025 \times 0.5 = \underline{0.0125}$



$P(B|A_2)P(A_2) = P(B \cap A_2)$

$= 0.006$

$P(B|A_3)P(A_3) = P(B \cap A_3)$

$= 0.002$

$P(B) = 0.0125 + 0.006 + 0.002$

$= 0.0205$

c) $P(A_1|B) = \dfrac{P(A_1 \cap B)}{P(B)} = \dfrac{0.0125}{0.0205} = 0.61$

$P(A_2|B) = \dfrac{0.006}{0.0205} = 0.29$

$P(A_3|B) = 0.1$

Notice:

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3)$$

$$= P(B \mid A_1)\,P(A_1) + P(B \mid A_2)\,P(A_2) + P(B \mid A_3)\,P(A_3)$$

General form: Let $A_1, A_2, ..., A_n$ be disjoint events that together form the sample space S. Let B be any event from the same sample space, such that $P(B) > 0$. Then,

$$P(A_i|B) = \frac{P(A_i \cap B)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}.$$

n=3 in our example

Summary

1. Disjoint events, $P(A \cap B) = 0$
2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $\quad$ $P(A\cap B)=0$
   If A, B are disjoint then $P(A \cup B) = P(A) + P(B)$
3. $P(A|B) = \frac{P(A \cap B)}{P(B)}$ $\quad$ cond. prob.
4. $P(A \cap B) = P(B) \times P(A|B) = P(A) \times P(B|A)$
   If A, B are independent then $P(A \cap B) = P(A) \times P(B)$
5. $P(A) = P(A \cap B) + P(A \cap B^c)$