# Ch. 6 - Some Probability Models

First recall some concepts from Ch. 4.

**Discrete Random Variables**:

For discrete random variables, we have **probability mass functions** (pmf) (denoted $f(x)$), which gives the probability for each possible value $x$ of $X$. For example, a pmf can be in the form of an equation or in the form of a table

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(x)$ | 0.4 | 0.3 | 0.2 | 0.1 |

Recall, for discrete random variables, the meaning of the cdf is the same as for a continuous random variable. i.e. $F(x) = P(X \leq x)$.

Note that in the discrete case, $P(X < x) \neq P(X \leq x)$. Be careful when dealing with $P(a \leq X \leq b)$ because

$$P(a < X \leq b) = F(b) - F(a)$$

Why? unlike continuous r.V
$<$ and $\leq$ matters!

e.g. Let $X$ be the number of heads flipped in 10 tosses of a fair coin. $P(2 \leq X \leq 4) = ?$ Then

$$
\begin{aligned}
\underbrace{P(2 \leq X \leq 4)}_{\{2,3,4\}} &= \overbrace{P(X \leq 4)}^{\{0,1,2,3,4\}} - P(X < 2)] \ \{0,1\} \\
&= P(X \leq 4) - P(X \leq 1) \\
&= F(4) - F(1)
\end{aligned}
$$

Because we want to <u>include</u> 2 in this case.

*Example* 1. $F(0) = 0.58$, $F(1) = 0.72$, $F(2) = 0.76$, $F(3) = 0.81$, $F(4) = 0.88$, $F(5) = 0.94$
What is $P(X = 3)$?

$$
\begin{aligned}
P(X = 3) &= \overbrace{P(X \leq 3)}^{\{0,1,2,3\}} - \overbrace{P(X \leq 2)}^{\{0,1,2\}} \\
&= F(3) - F(2) \\
&= 0.81 - 0.76 = 0.05
\end{aligned}
$$

In this chapter, we will cover the following discrete r.v's:

1) Bernoulli trials
2) Geometric r.v.
3) Binomial r.v.
4) Poisson r.v.

# 1 Bernoulli Experiment

ex. coin flip

A **Bernoulli experiment** is a random experiment with the following features:

- The experiment consists of $n$ independent trials
- Each trial has only 2 possible outcomes (usually denoted as a **success** or **failure**)
- The probability of success is the same for all trials. We usually denote: $P(\text{success}) = p$ and $P(\text{failure}) = 1 - p = q$

We usually assign the 2 possible values 0 and 1. The pmf of a Bernoulli random variable is

$$P(X = x) = (1-p)^{1-x}p^x, \qquad x = 0, 1$$

with mean $E(X) = p$ and $Var(X) = p(1-p)$.

We can see easily by using a table:

| $x$ | 0 | 1 |
|---|---|---|
| $P(X = x)$ | $1-p$ | $p$ |

$E(X) = \sum_x x f(x)$

$$E(X) = 0 \times (1-p) + 1 \times p = p$$

$E(X^2) = \sum x^2 f(x)$

$$\begin{aligned} E(X^2) &= 0^2 \times (1-p) + 1^2 \times p = p \\ Var(X) &= E(X^2) - [E(X)]^2 \\ &= p - p^2 \\ &= p(1-p) \end{aligned}$$

## 2 Geometric Random Variables

Suppose we want to model how long it will take to achieve the first success in a series of Bernoulli trials. A **Geometric random variable** counts the number of independent trials needed until the first success occurs.
e.g. Let $X$ be the count of the number of coin tosses until you get a head.

↳ success

you must have $(x-1)$ tails

say you get a head on the $x$th toss.

We write

$$X \sim Geo(p)$$

$$P(X=3) = \overset{T}{(1-0.5)}\overset{T}{(1-0.5)}\overset{H}{0.5} = (1-0.5)^{3-1}\, 0.5$$

where $p$ is the probability of success.

$$P(X = x) = (1-p)^{x-1}p \qquad x = 1, 2, 3...$$

starts from 1 for this def. of $P(X=x)$

Mean and Variance of Geometric Random Variable

also called return period

$$E(X) = \frac{1}{p}$$

$$Var(X) = \frac{1-p}{p^2}$$

See pg. 107 and problem 6.6 in course text for proof.

CDF of a Geometric Random Variable

Suppose $X \sim Geo(p)$

Recall from Ch. 4, $F(x) = P(X \leq x) = \underbrace{\sum_{k \leq x}}\, f(k)$ $\leq$ or $<$

$$F(x) = P(X \leq x) = 1 - P(X > x)$$

$$\begin{aligned} P(X > x) &= \sum_{k=x+1}^{\infty} (1-p)^{k-1} p \\ &= (1-p)^x p + (1-p)^{x+1} p + (1-p)^{x+2} p + ... \\ &= (1-p)^x p \underbrace{[1 + (1-p) + (1-p)^2 + ...]} \end{aligned}$$

Recall from high school/calculus, the formula for the sum of a geometric series:

$$[1 + r + r^2 + ...] = \frac{1}{(1-r)} \leftarrow$$

where $0 < r < 1$. Here $r = (1-p)$ so we get,

$$\begin{aligned} P(X > x) &= (1-p)^x p [\frac{1}{(1 - \underbrace{(1-p)}_{r})}] \quad = (1-p)^x \cancel{p} \frac{1}{\cancel{p}} \\ &= (1-p)^x \\ F(x) &= 1 - (1-p)^x \end{aligned}$$

*Example* 2. 1% of manufactured cells at a certain battery plant must be scrapped due to internal shorts. Suppose that testing of cells for shorts begins on a production run in this plant and random cells are inspected.

(a) What is the probability the second cell inspected will be the first short discovered?

(b) What is the probability that at least 50 cells are tested without finding a short?

Solution:

a) $X$ = # of the test at which 1st short is discovered.

$X \sim geo(p)$ $\quad p = 0.01$

$P(X=x) = (1-p)^{x-1} p$

$P(X=2) = (1-0.01)^{2-1} \, 0.01$

$= \underset{\substack{\uparrow \\ \text{No short}}}{0.99^1} \times \underset{\substack{\uparrow \\ \text{short discovered.}}}{0.01} = 0.0099$

b) at least 50 cells are tested w/o finding a short

$P(X > 50) = 1 - \underline{P(X \le 50)} = 1 - \underbrace{\left[1-(1-0.01)^{50}\right]}_{\text{cdf of } X}$

$P(X \le x) = [1-(1-p)^{x}]$ cdf.

$= (1-0.01)^{50}$

$= 0.61$

# Last Class:

Bernoulli Experiment:

→ sequence of independent trials

→ two outcomes (S / F)

→ P(success) = p (constant)

P(failure) = 1 - p = q

Ex coin flip

sequence lottery wins / losses ~~lottery~~

$X \sim Ber(p)$

$E(X) = p \cdot 1 + (1-p) \cdot 0 = p$

$Var(X) = p(1-p)$

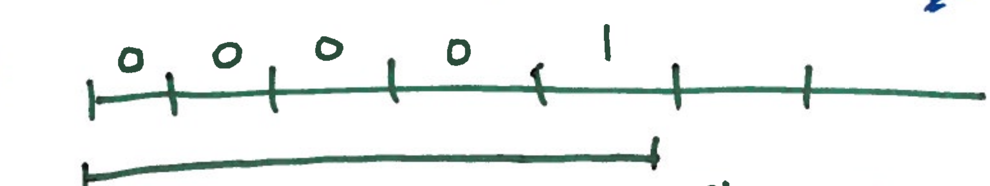
Bernoulli trials

→ time until first success? ①

\# trials

→ For a given # trials, ②

how many successes have we had?

① 


sequence of independent Bernoulli trials

$X$ = # trials until first success $\sim geo(p)$

$$P(X=x) = (1-p)^{x-1}\, p \qquad x = 1, 2, \cdots$$

$$E(X) = \frac{1}{p}$$

$$Var(X) = \frac{1-p}{p^2}$$

② Binomial → today.

# 3 Binomial Random Variables

The next model we will talk about, the Binomial Model, is also based on the idea of Bernoulli Trials.

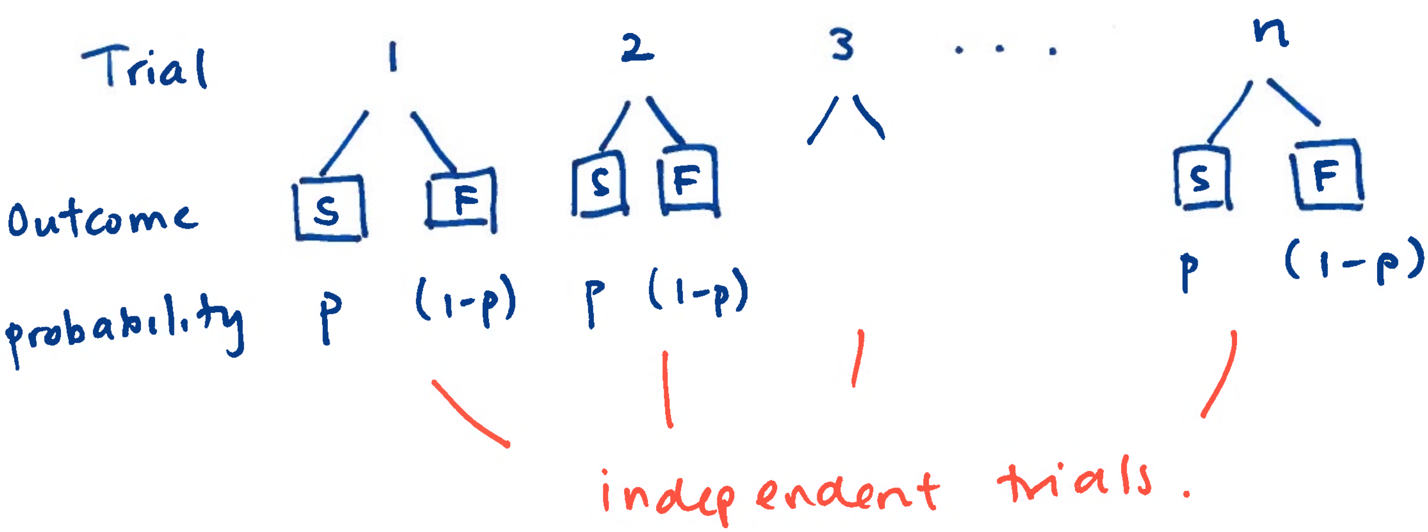
Binomial Model:

A **Binomial random variable** is the number of successes for $n$ independent trials (fixed number of trials). A Binomial model tells us the probability for a random variable that counts the number of successes in a fixed number of Bernoulli trials.
e.g. Let $X$ = the number of heads we see in 5 tosses of a fair coin. Then $X$ is a Binomial random variable. How come?

Notation: If $X$ is a Binomial random variable, we write

$$X \sim Bin(n, p)$$

where $n$ and $p$ are the parameters of the model ($n$ is the number of trials, $p$ is the probability of success).



Example. Let's think intuitively about the Binomial model with an example. Let $X$ be the number of heads in 5 tosses of a fair coin. What is the probability that we get exactly 2 heads in 5 tosses. (i.e. $P(X = 2)$)?

There are 10 different ways we can get 2 heads in 5 tosses:

- HHTTT     THTHT
- HTHTT     THTTH
- HTTHT     TTHHT
- HTTTH     TTHTH
- THHTT     TTTHH

Let's look at the first order only: HHTTT

$$\begin{aligned} P(H \cap H \cap T \cap T \cap T) &= P(H)P(H)P(T)P(T)P(T) \qquad \text{independent trials} \\ &= p \times p \times (1-p) \times (1-p) \times (1-p) \\ &= p^2(1-p)^3 \\ &= 0.5^2 \times (1-0.5)^3 = 0.03125 \end{aligned}$$

This is the probability of getting 2 heads and then 3 tails. All 10 of our above sequences have the same probability. Thus we multiply our above equation by the number of possible 2H combinations.

$$P(X = 2) = 10 \times p^2(1-p)^3 = 10 \times 0.03125 = 0.3125$$

Clearly for large numbers, counting the number of possible orders isn't practical. Each different order in which we can have $k$ successes in $n$ trials is called a **combination**. The total number of ways that can happen is written $\binom{n}{k}$

n choose k

factorial

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \qquad \text{where } n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

eg $5! = 5 \times 4 \times 3 \times 2 \times 1$

Note: $0! = 1$

e.g. $n = 6, k = 2$

$$\binom{6}{2} = \frac{6!}{(6-2)!2!} = \frac{6 \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{(\cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1})(2 \times 1)} = 15$$

This agrees with our coin example, where

$$\binom{5}{2} = \frac{5!}{(5-2)!2!} = \frac{5 \times 4 \times \cancel{3} \times \cancel{2} \times \cancel{1}}{(\cancel{3} \times \cancel{2} \times \cancel{1})(2 \times 1)} = 10$$

In general, the probability mass function for a binomial random variable is:

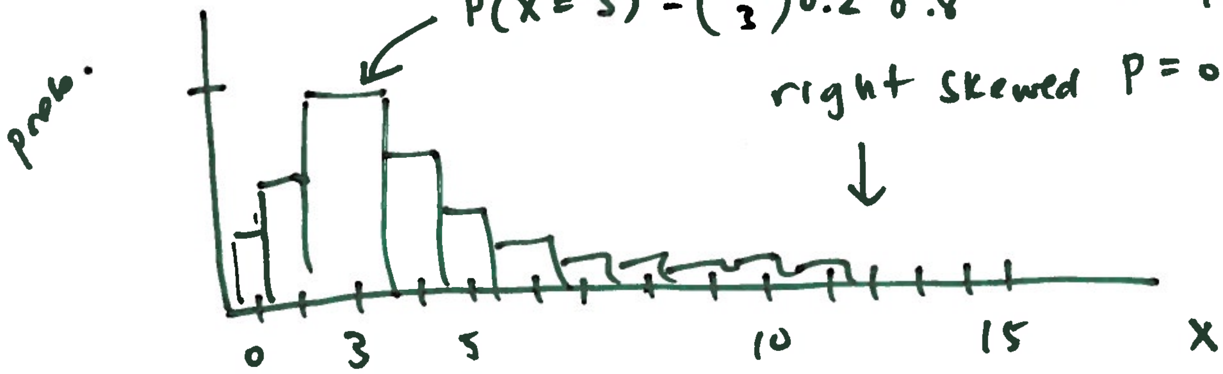
$$\boxed{P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, 2, \ldots, n}$$

# of possible outcomes w/ x successes out of n

prob. of each of the outcomes that have x successes out of n.

$P(X=3) = \binom{15}{3} 0.2^3 \, 0.8^{12}$ $\quad n = 15$

right skewed $\quad p = 0.2$



You should be able to recognize a Binomial situation (often on a midterm/exam it will not tell you it is Binomial)

Binomial situation:

- fixed number of independent trials ($n$)
- each trial only has 2 possible outcomes
- the probability of success, $p$, is the same for each trial

*Example* 3.
Suppose that 4% of computer chips manufactured by a certain company are defective. If you randomly inspect 5 chips, what is the probability you find

(a) exactly 1 defective chip?

(b) more than 1 defective chip?

(c) at least 2 non-defective chips?

Solution:

X = # defective computer chips out of 5.

$X \sim Bin(n=5, p=0.04)$

a) $P(X=1) = \binom{5}{1} 0.04^1 \times 0.96^{5-1}$

$= 5 \times 0.04^1 \times 0.96^4 = 0.1699$

b) $P(X>1) = P(X \geq 2) = P(X=2) + P(X=3) + P(X=4) + P(X=5)$

or $1 - P(X \leq 1) = 1 - [P(X=0) + P(X=1)]$

$= 1 - \binom{5}{0} 0.04^0 \, 0.96^5 - 0.1699$

$= 0.01476$

c) P(at least 2 non defective chips)

$= P($at most 3 defective chip$) = P(X \leq 3)$

$= 1 - P(X > 3)$

{2, 3, 4, 5) non defective

{3, 2, 1, 0} defective

$= 1 - P(X > 3)$

$= 1 - P(X=4) - P(X=5)$

$= 1 - \binom{5}{4} 0.04^4 \, 0.96^1 - \binom{5}{5} 0.04^5 \, 0.96^0$

$= 1 - \frac{5 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{1! \, \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} 0.04^4 \, 0.96^1 - \frac{\cancel{5!}}{0! \, \cancel{5!}} 0.04^5 \cdot 1$

$= 1 - 5 \times 0.04^4 \, 0.96 - 1 \cdot 0.04^5 \cdot 1$

$= 0.999988$

Last Class:

$X \sim Bin(n, p)$ # successes in n trials

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$E(X) = np$

$Var(X) = np(1-p)$

Mean and Variance of Binomial Random Variable

Suppose $Y_i \sim Ber(p)$, $i = 1, 2, ..., n$.
Let $X$ be the number of successes in $n$ independent trials,

$$\text{Binomial} \rightarrow X = Y_1 + Y_2 + ... + Y_n \leftarrow \text{Bernoulli}$$

Now we can find the mean and variance ($E(X)$ and $Var(X)$) of a Binomial random variable.
Recall $E(Y_i) = p$ and $Var(Y_i) = p(1-p)$. Then:

$$\begin{aligned}
E(X) &= E(Y_1 + Y_2 + ... + Y_n) \\
&= E(Y_1) + E(Y_2) + ... + E(Y_n) \\
&= p + p + ... + p \\
&= np \\
Var(X) &= Var(Y_1 + Y_2 + ... + Y_n) \\
&= Var(Y_1) + Var(Y_2) + ... + Var(Y_n) \quad \text{because } Y_i\text{'s are independent} \\
&= np(1-p)
\end{aligned}$$

*Example* 4. The random variable, $X$, has a Binomial distribution with mean 12 and variance 8. What is $P(X = 12)$?

Solution:

$X \sim Bin(n, p)$

$E(X) = np = 12$ — ① $\rightarrow n = \frac{12}{p}$.

$Var(X) = np(1-p) = 8$ — ②

Sub ① into ② $\quad Var(X) = \frac{12}{\cancel{p}} \cdot \cancel{p}(1-p) = 8$

$12(1-p) = 8$

$1 - p = \frac{8}{12}$

$$1 - \frac{2}{3} = p$$

$$p = \frac{1}{3}$$

$$n = \frac{12}{p} = \frac{12}{(1/3)} = 36$$

$$X \sim Bin\left(n = 36,\ p = \frac{1}{3}\right)$$

$$P(X = 12) = \binom{36}{12}\left(\frac{1}{3}\right)^{12}\left(1 - \frac{1}{3}\right)^{36-12}$$

$$= 0.140$$

Examples of Bernoulli experiments:

(1) The number of heads on repeated coin tosses
**Event**: Flip a H
**Trial**: Each coin toss
**Outcomes**: Head (success) or Tail (failure)

(2) Randomly testing selected items for defects:
**Event**: A defective item
**Trial**: Inspection of each item
**Outcomes**: Defective (success) or non-defective (failure)

Bernoulli experiments give rise to Binomial and Geometric random variables:

(1) **Binomial rv:** The number of heads (successes) out of $n$ coin tosses
**Geometric rv:** The number of tosses (trials) until you get the first head (success)

(2) **Binomial rv:** The number of defective items (successes) in $n$ items
**Geometric rv:** the number of items inspected (trials) until the first defective item (success) is found

# 4 Poisson Process

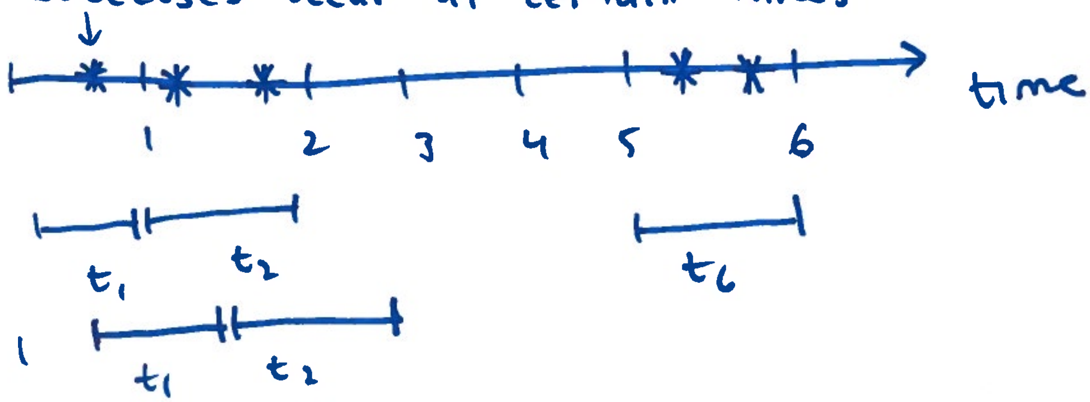
The **Poisson Process** is used to model a count of occurrences of events per unit of time/space. (e.g. number of traffic accidents occurring on a highway in a year, number of customers joining a line in an hour, number of defects on a $1 \times 1$m surface). The poisson process gives rise to a discrete random variable called a **Poisson random variable** and a continuous random variable called the **Exponential random variable**.

**Rate of the process** (denoted by $\lambda$) is the average number of occurrences of a certain event per unit time (or per unit of space).

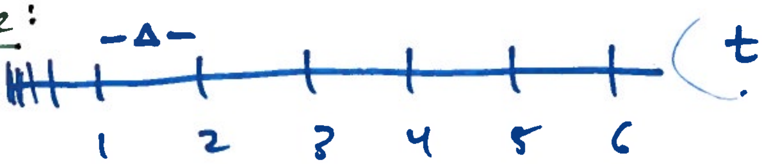
Characteristics of a Poisson Process:

- The number of occurrences of an event in any non-overlapping interval are independent.
- The number of occurrences of the event in an interval is proportional to the size of the interval.
- The probability of an event within a certain interval does not change over different intervals.
- Events cannot occur simultaneously.
- Events have a low probability of occurrence.

successes occur at certain times



Aside:




$n = 6$

$n = \frac{t}{\Delta}$ $\qquad$ $p = \lambda\Delta = \lambda \frac{t}{n}$

$\Delta \to 0$

$n \to \infty$

$$P(x \text{ arrivals}) = \binom{n}{x}\left(\frac{\lambda t}{n}\right)^x \left(1-\frac{\lambda t}{n}\right)^{n-x}$$

$n \to \infty$

Poisson Random Variable

If the above conditions are satisfied, the random variable $X =$ the number of occurrences in a given interval of time/space, has a Poisson distribution.

We write,

length of interval

$$X \sim Pois(\lambda t)$$

rate

where $\lambda$ is the rate of occurrences of $A$ per unit time and $t =$ the number of unit of time we are looking at.

$$P(X = x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}, \qquad x = 0, 1, 2, 3...$$

Mean and Variance of Poisson Random Variable

$$E(X) = \lambda t \qquad Var(X) = \lambda t$$

*Example* 5. Suppose that the average number of earthquakes with a reading over 8.0 on the Richter scale is 1 per year. What is the probability that there are no earthquakes over 8.0 in the next year? In the next 3 years?

Solution

Here $\lambda = 1$ per year and $t = 1$. Let $X$ be the number of earthquakes with a reading over 8.0 on the Richter scale in the next year. So,

$$X \sim Pois(\lambda t = 1 \times 1)$$

$$P(X = 0) = \frac{e^{-1}(1)^0}{0!} = 0.368$$

Let $Y$ be the number of earthquakes with a reading over 8.0 on the Richter scale in the next 3 years. So,

$t = 3$

$$Y \sim Pois(1 \times 3)$$

$$P(Y = 0) = \frac{e^{-3}(3)^0}{0!} = 0.0498$$

Exponential Random Variable

$T =$ the time between consecutive occurrences of the event. $T$, called the waiting time, is a continuous random variable and

$$T \sim Exp(\lambda)$$

with pdf and cdf

$$f(t) = \lambda e^{-\lambda t}, \qquad t > 0$$
$$F(t) = 1 - e^{-\lambda t}$$

Mean and Variance of Exponential Random Variable

$$E(T) = \frac{1}{\lambda} \qquad Var(T) = \frac{1}{\lambda^2}$$

*Example* 6. Continuing on from example 4 above: What is the probability that you wait less than 2 years until the next occurrence of an earthquake over 8.0?

$\lambda = 1$

Solution:

Let $T$ be the time until the next earthquake with reading over 8.0 on the Richter scale.

$$T \sim exp(\lambda = 1) \qquad T \sim exp(1)$$

$$E(X) = \frac{1}{\lambda} = 1$$

$$P(T < 2) = F(2)$$
$$= 1 - e^{-2}$$
$$= 0.865$$

$X = \#$ occurrences in the next 2 yrs.

Alt soln. Convert time question to one involving occurrence.

P( waiting time less than 2 yrs for the next occurrence)

= 1 - P(no occurrences in the next 2 yrs)

$= 1 - \frac{e^{-2} 2^{0}}{0!} = 0.865$ $\qquad X \sim Pois(\lambda t = 2)$

Last lecture we learnt the Poisson distribution $X \sim pois(\lambda t)$

pmf:

$$P(X = x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!} \qquad x = 0, 1, 2, 3 \ldots$$

$$E(X) = \lambda t$$

$$Var(X) = \lambda t$$

wait time $T$ between consecutive occurrences of the event of interest is $T \sim exp(\lambda)$

$$E(T) = \frac{1}{\lambda}$$

$$Var(T) = \frac{1}{\lambda^2}$$

# 5 Poisson Approximation to the Binomial

If $X \sim Bin(n, p)$ with large $n$ ($n \geq 20$) and small $p$ ($np < 5$), then we can use a Poisson random variable with $\lambda t = np$ to approximate Binomial.

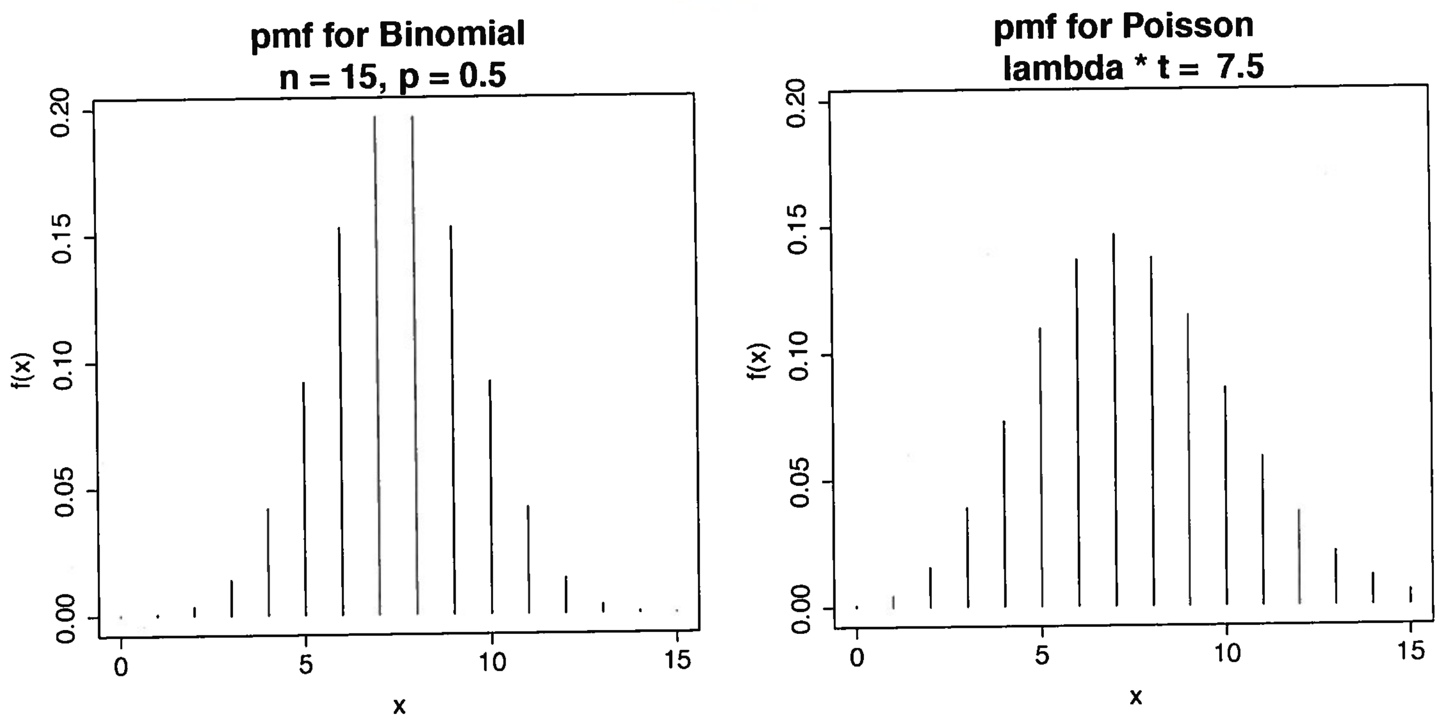



Figure 1: small $n$, $min\{np, n(p-1)\} = 7.5 > 5$

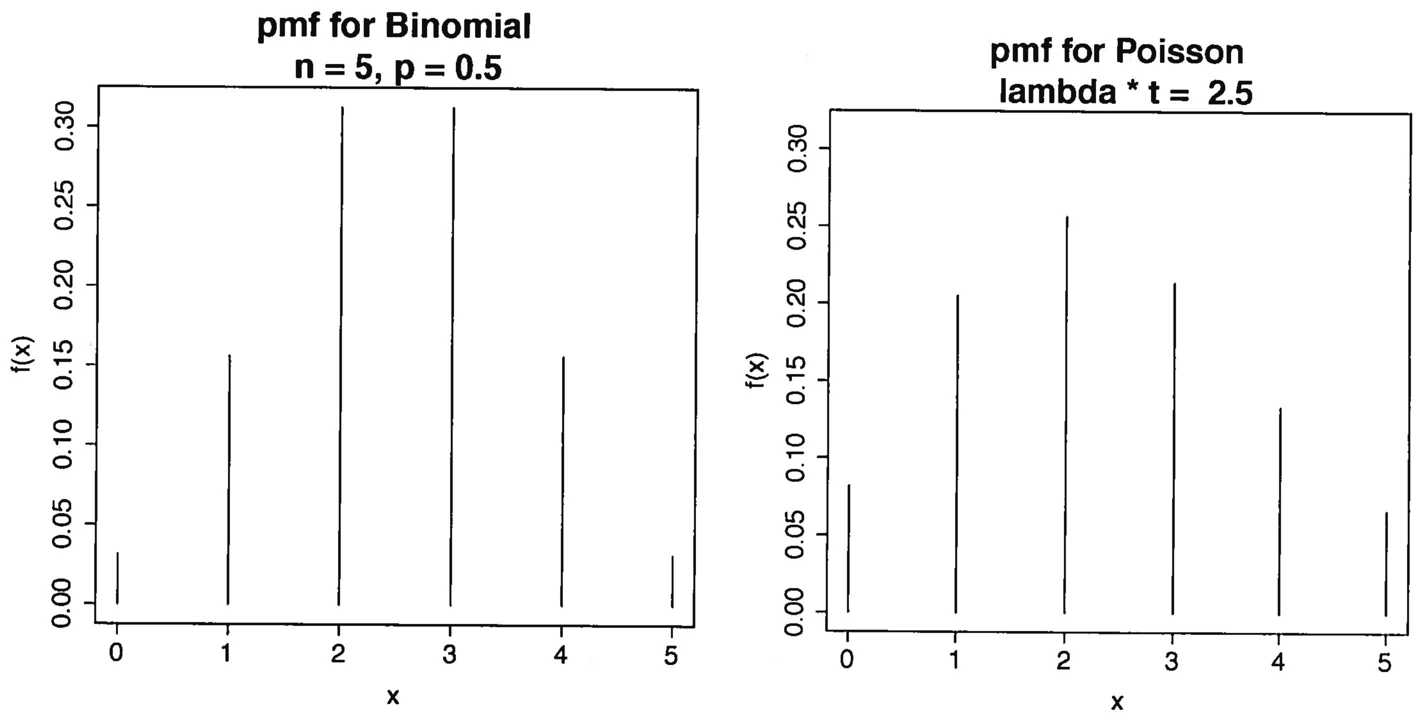



Figure 2: small $n, min\{np, n(p-1)\} = 2.5 < 5$

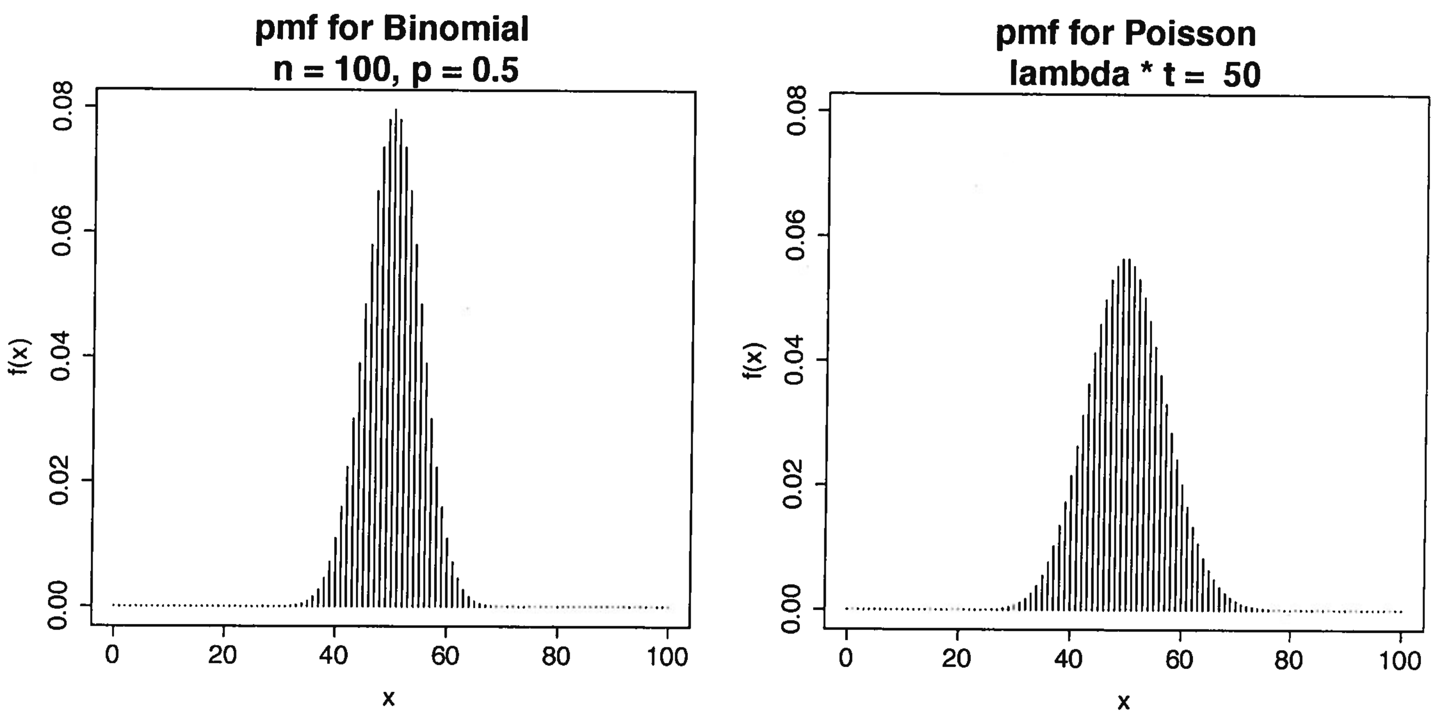
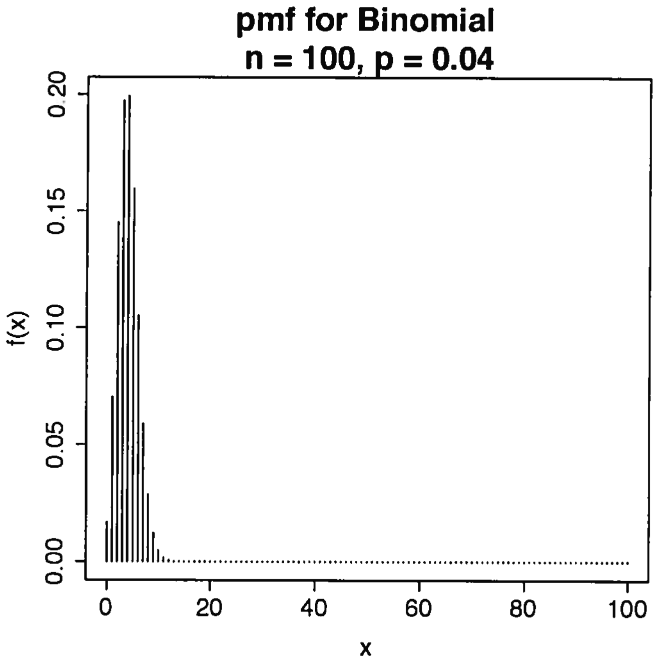



Figure 3: large $n, min\{np, n(p-1)\} = 50 > 5$



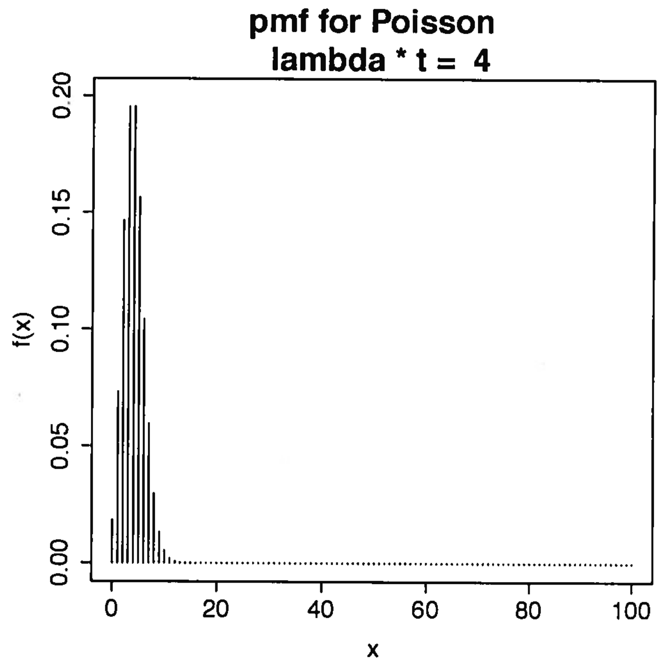



Figure 4: large $n, min\{np, n(p-1)\} = 4 < 5$
(meets our assumptions for using poisson approximation to binomial)

*Example* 7. If 1% of the output from a machine is defective, then what is the probability that 4 or more are defective in a random sample of 200?

$p = 0.01$     Fixed $n$,

$n = 200$     defective = success (2 outcomes)

independent

Let $X$ = # defective items in 200

$X \sim Bin(n = 200, p = 0.01)$

$$P(X \geq 4) = 1 - P(X < 4)$$
$$= 1 - P(X \leq 3)$$
$$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$$

$$= 1 - \left[ 1 \times 0.99^{200} + \binom{200}{1} 0.01^{1} \times 0.99^{199} + \binom{200}{2} 0.01^{2} \times 0.99^{198} + \binom{200}{3} 0.01^{3} \times 0.99^{197} \right]$$

Exact method

$$= 1 - 0.8580 = 0.142$$

Approximate method:

Since $n$ large $(n \geq 20)$

and $p$ small $(np = 200 \times 0.01 = 2 < 5)$

We can use Poisson to approximate Binomial:

$$X \sim Pois(\lambda t = 2)$$

$$\lambda t = np = 200 \times 0.01 = 2$$

$$P(X \geq 4) = 1 - P(X \leq 3)$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$$

$$= 1 - \left[ \frac{e^{-2} 2^{0}}{0!} + \frac{e^{-2} 2^{1}}{1!} + \frac{e^{-2} 2^{2}}{2!} + \frac{e^{-2} 2^{3}}{3!} \right]$$

$$= 1 - 0.8571 = 0.143$$

You may not save much time and effort using this method.