Delta Hedging : A Theoretical and Computational Approach

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1 Motivations

In modern financial markets, options and other derivatives expose traders to complex forms of risk due to their stochastic and non-linear behaviour. Among these risks, sensitivity to small fluctuations in the underlying asset's price, captured by the option's *delta*, raises a major challenge, especially for institutions writing large volumes of options, such as investment banks.
To address this exposure, traders have developed the technique of delta hedging, which consists in a dynamical adjustment of positions in the underlying asset to neutralize the directional risk of an option portfolio.
The hedging efficiency will depends on the model parameters, namely on volatility, which will determine the frequency we should adjust our portfolio to in order to be as close as possible to a zero delta exposure.

In this personal project, I will dive deeper into this central concept, through the study of a Delta Hedging strategy, both theoretically and computationally. The aim is to put light on the stakes we face in this kind of strategy, particularly on the influence of parameters as volatility. We will also discuss the need in considering other sensitivities than delta.

2 Setting

Consider a short position in a European call option on a non-dividend paying stock with the following caracteristics :

- Maturity $T = 1\ year$
- Strike $K = 99\ EUR$
- Risk-free interest rate $r = 6\%$
- Initial stock price $S_0 = 100\ EUR$
- Volatility $\sigma = 20\%$

We consider an underlying asset with a price S_t following a geometric brownian motion

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{P}}$$

under the real measure $\mathbb{P}$.

The Call option price $C(t, S_t)$ is assumed to be solution of the Black-Scholes PDE :

$$rC - \frac{\partial C}{\partial t} - rS_t \frac{\partial C}{\partial S} - \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} = 0$$

3 Portfolio replication

The goal is to create a portfolio of underlying assets, such as risky assets (options) and risk-free assets (cash), that evolves in a way that matches the value of the target derivative, in our case an option.

To do so mathematically, we want to determine ξ_t and η_t, respectively the portfolios' holdings in the risky asset S_t and in the riskless asset A_t, such that :

$$V_t = C(S_t, t) = \xi_t S_t + \eta_t A_t \quad \forall t \in [0, T]$$

where V_t and $C(S_t, t)$ denote respectively the portfolio's value and the price of our Call option.

4 Delta Hedging

4.1 Principle

The idea here is to simulate a strategy of *delta hedging* over one year to hedge our position of call seller. This strategy is a practice example of portfolio replication.

Delta hedging consists in rebalancing the portfolio regularly in time by computing the delta of the option price on a discrete scale of time. Indeed we can not consider a continuous time update in real life.

At each time step we have to update the value S_t of the underlying asset, to compute the corresponding $\Delta_t = \frac{\partial C(S_t,t)}{\partial S}$, which is the sensitivity of the option price with respect to the underlying, and to adjust our shares in the latter.

This technique allows us to offset the stochastic behaviour associated with small changes in option's price due to small movements in the underlying asset's price, as we will show in the following.

4.1.1 Mathematical argument

Assume we have sold a European call option with price $C(t, S_t)$.
To hedge the local risk, we buy Δ_t units of the underlying asset.

The value of this portfolio is therefore :

$$\Pi_t = -C(t, S_t) + \Delta_t S_t$$

Assuming the portfolio is self-financing :

$$d\Pi_t = -dC_t + \Delta_t dS_t$$

Applying Itô's lemma to $C(t, S_t)$, we get :

$$dC_t = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S_t^2 dt$$

and since $dS_t = \mu S_t dt + \sigma S_t dW_t$, we substitute :

$$d\Pi_t = -\left(\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \left(\Delta_t - \frac{\partial C}{\partial S} \right) dS_t$$

Choosing $\Delta_t = \frac{\partial C}{\partial S}$, the stochastic term vanishes :

$$d\Pi_t = -\left(\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} \right) dt$$

This residual drift corresponds to the instantaneous profit or loss of the hedging strategy and vanishes under the Black-Scholes PDE. Indeed assuming $C(t, S_t)$ satisfies the Black-Scholes PDE, we have :

$$d\Pi_t = \left(-rC + rS \frac{\partial C}{\partial S} \right) dt = r\left(-C + \Delta S\right) dt = r\Pi_t dt$$

So the portfolio growths at the risk-free interest rate.

Conclusion : Holding $\Delta_t = \frac{\partial C}{\partial S}$ shares in the underlying allows a perfect portfolio replication.

We hold $\xi_t = \Delta_t$ in the underlying as we have seen and put the remaining value in cash.

Denoting by V_t the portfolio's value, we then have :

$$V_t = \Delta_t S_t + \eta_t A_t$$

with

$$\eta_t = \frac{C_t - \Delta_t S_t}{A_t}$$

If we adjust the portfolio's holdings sufficiently often in time, this one is supposed to perfectly replicate the call payoff at maturity.

4.2 Volatility Match

In this first section, the implicite volatility appearing in the Black-Scholes PDE is taken equal to the underlying asset evolution one, fixed at $\sigma = 0.2$.

4.2.1 Underlying asset's evolution

Using Euler's discretization method, we can simulate trajectories of the underlying asset price, which satisfies :

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{P}}$$

under the real measure $\mathbb{P}$.

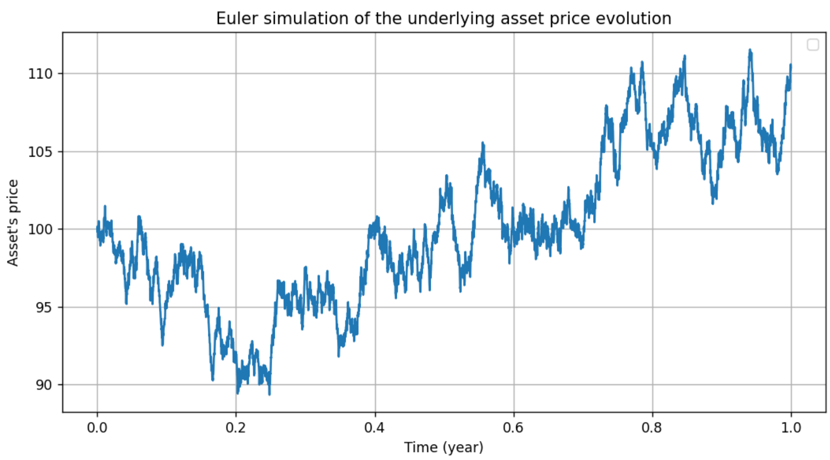
Approximating dS_t by $S_{t+\delta t} - S_t$, for small δt, Euler scheme leads to :

$$S_{t+\delta t} \approx S_t + S_t(r\Delta t + \sigma\sqrt{\delta t}Z)$$

where $Z \sim \mathcal{N}(0,1)$.

This formula will allow us to compute the new price of the underlying asset at each time step.

We can implement this formula in Python to obtain such a path for the underlying asset's price evolution over one year :



4.2.2 Option's price

For a European call option, the payoff at maturity is given by :

$$C(S_T, T) = \max(S_T - K, 0)$$

Furthermore, the Black-Scholes theory gives us an explicit formula to compute the option price.

$$\boxed{C(S_t, t) = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2)}$$

where

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

and Φ stands for the CDF function of the standard normal distribution.

Directly we see that the delta of the call is given by :

$$\Delta_t = \frac{\partial C(S_t, t)}{\partial S} = \Phi(d_1)$$

Concretely, this formula gives us the amount of underlying asset we have to possess at each instant to hedge our position.

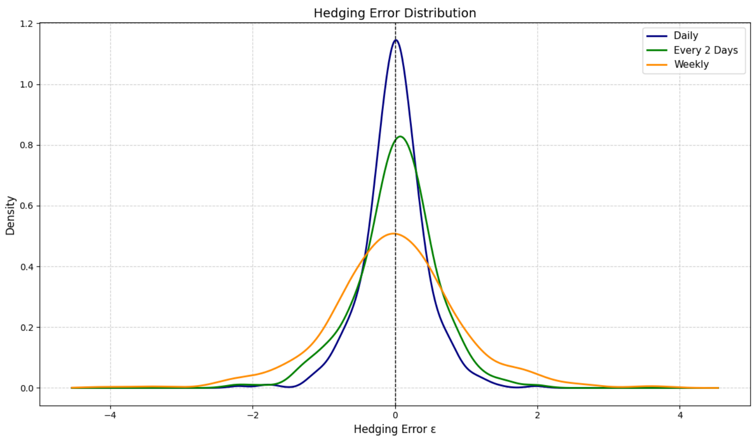
4.2.3 Hedging Error

To estimate our strategy efficiency and to understand the influence of parameters such as time scaling, we can compute the hedging error at maturity:

$$\varepsilon = V_T - \max(S_T - K, 0)$$

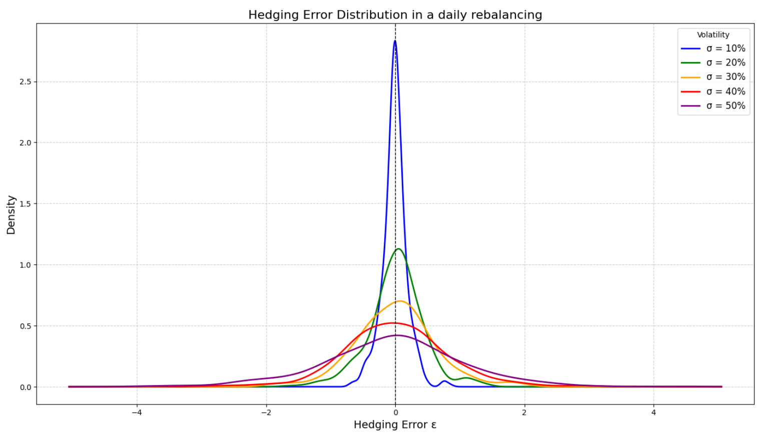
adjusting at each time step our portfolio with the formula $\Delta_t = \Phi(d_1)$.

Implementing this formula for a wide number of simulation, we can estimate the hedging error distribution.



Interpretation : This graphic clearly shows that as the rebalancing frequency increases, the distribution of errors becomes increasingly concentrated around zero. Daily adjustments allow better compensation for fluctuations in the underlying asset price, thereby reducing the difference between the value of the hedging portfolio and the option payoff at maturity.
Conversely, lower rebalancing frequencies, such as weekly, result in a wider spread of errors, indicating less effective hedging. This observation highlights the crucial role of rebalancing frequency in dynamic risk management, confirming that more frequent adjustments lead to more accurate and reliable hedging.

The need in a high frequency is all the more obvious as the volatility increase, since it makes the price fluctuations occur in a shorter time slot.
Here we illustrate this idea, comparing a daily rebalancing hedging for different volatility values.



4.3 Volatility Mismatch

Consider the following underlying price evolution under the real measure $\mathbb{P}$:

$$dS_t = rS_t dt + \sigma_t S_t dW_t^{\mathbb{P}}$$

where σ_t is the instantaneous realized volatility.

Nevertheless, the call option is priced with a constant volatility, fixed at the value of the implied volatility, such that :

$$rC_t - r\Delta_t S_t - \frac{\partial C_t}{\partial t} - \frac{1}{2}\sigma_{imp}^2 S_t^2 \Gamma_t = 0 \tag{1}$$

When we omit the directional changes in the underlying asset when hedging with a different volatility ($\sigma_{imp} \neq \sigma_t$), an error appear through a non-null $P\&L$ at maturity, which is basically the hedging error.

This error depends on several parameters, namely on the volatility mismatch, as we have seen before.

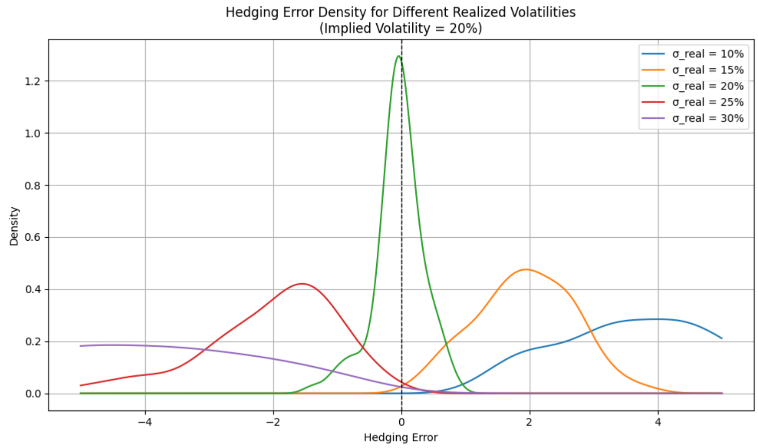
4.3.1 Computationally

Here we keep our implied volatility fixed at $\sigma_{imp} = 0.2$ as before ; however the instantaneous realized volatility will be modified to throw light on the mismatch impact on hedging efficiency.

To keep things not too difficult, we will only consider σ_t as a constant in time in this computational study, although different of σ_{imp}.
The general case is however considered in the theoretical study.

Even if it's an unrealistic approximation, it will raise the main stake here :
volatility mismatch.

Here is the graph representing the Hedging Error for σ_t going through $\{0.1\,;\,0.5\}$, with a daily rebalancing.



Interpretation :

- **Case $\sigma_\mathbf{t} = 20\%$:** This is the scenario we encountered before, where the volatility used for delta-hedging matches the true dynamics of the asset. The narrow density peak confirms the low variance of the hedging error, only due to the discontinuous time rebalancing.

- **Case $\sigma_\mathbf{t} < 20\%$:** Here, the actual underlying's price is less volatile than what was expected for hedging. The delta hedge is too aggressive, leading to an over-hedged portfolio. As a result, the terminal value of the hedge portfolio tends to overshoot the option payoff, producing systematically positive hedging errors. The distribution is shifted to the right.

- **Case $\sigma_\mathbf{t} > 20\%$:** In this case, the underlying's price is more volatile than expected. The hedging is too conservative, which causes the portfolio to underreact to large price movements. This leads to negative hedging errors, as the portfolio value at maturity tends to fall short of the option payoff. The error distribution is then shifted to the left.

Moreover, the variance of hedging errors increases with the magnitude of the mismatch, reflecting greater uncertainty and inefficiency in the hedging strategy. Consequently, the density flattens and spreads, illustrating a loss in precision and reliability of the hedge as the volatility mismatch grows.

4.3.2 Theoretically

Applying Itô's formula to $C(S_t, t)$, we get :

$$dC_t = \frac{\partial C}{\partial t} dt + \Delta_t dS_t + \frac{1}{2} \sigma_{\text{imp}}^2 S_t^2 \frac{\partial^2 C}{\partial S^2} dt \tag{2}$$

Combining (1) and (2), we get:

$$dC_t = \frac{\partial C}{\partial t} dt + \Delta_t r S_t dt + \Delta_t \sigma_t S_t dW_t + \frac{1}{2} \sigma_{\text{imp}}^2 S_t^2 \frac{\partial^2 C}{\partial S^2} dt$$

On the other hand, we have :

$$dV_t = \xi_t \, dS_t + \eta_t \, dA_t$$

with

$$\xi_t = \Delta_t \quad \text{and} \quad \eta_t = \frac{C_t - \Delta_t S_t}{A_t}$$

So

$$\begin{aligned} dV_t &= \Delta_t \, dS_t + r \eta_t A_t \, dt \\ &= \Delta_t \, dS_t + r C_t \, dt - r \Delta_t S_t \, dt \end{aligned}$$

As a result :

$$d(V_t - C_t) = dV_t - dC_t = \frac{1}{2} S_t^2 \Gamma_t \left(\sigma_{\text{imp}}^2 - \sigma_t^2 \right) dt$$

where $\Gamma_t = \frac{\partial^2 C}{\partial S^2}$.

Actualizing, we get the differential of the $P\&L$:

$$d(P\&L) = e^{-rt} \frac{1}{2} S_t^2 \Gamma_t \left(\sigma_{\text{imp}}^2 - \sigma_t^2 \right) dt$$

The total $P\&L$ over $[0, T]$ is given by :

$$\boxed{P\&L = \int_0^T e^{-rt} \frac{1}{2} S_t^2 \Gamma_t \left(\sigma_{\text{imp}}^2 - \sigma_t^2 \right) dt}$$

We notice that the non-null $P\&L$ at maturity depends on :

- The mismatch of volatilities : $\sigma_{\text{imp}}^2 - \sigma_t^2$
- The time to maturity itself : T
- The sensitivity of Δ_t with changes in the underlying price : Γ_t
- The risk free interest rate : r
- The underlying value : S_t

As soon as $\sigma_{imp} = \sigma_t$, it comes that the $P\&L$ is zero.

Interpretation : A higher gamma (typically for at-the-money options near maturity) amplifies the impact of volatility mismatch. The term S_t^2 shows that a larger asset price lead to a larger potential error, while the discount factor e^{-rt} reduces the influence of errors occurring later in the option's life. Overall, the hedging error becomes significant when there is a persistent volatility mismatch, especially during periods when the option is most sensitive to the underlying price.

Furthermore, we remark that for the $P\&L$ to vanish on average, one must have:

$$\mathbb{E}[\text{P\&L}] = \mathbb{E}\left[\int_0^T e^{-rt}\frac{1}{2}S_t^2\Gamma_t\left(\sigma_{\text{imp}}^2 - \sigma_t^2\right)dt\right] = 0$$

This can occur when the implied volatility used in the model is well-calibrated on average to match the actual volatility over time. Even though volatility varies, the average squared volatility over the period aligns with the implied volatility, leading to a zero expected $P\&L$.

Essentially, we get this balance when volatilities match on average.

5 Conclusion

This study highlights the effectiveness of delta hedging in mitigating directional risk in options portfolios.
When the realized volatility matches the implied volatility used for pricing and hedging, the hedging error remains tightly centered around zero, indicating efficient risk neutralization. In that case the variance only depends on the discrete time update of the portfolio. The accuracy of the hedging can be increased as soon as we tighten the rebalancing frequency.

However, in the presence of a volatility mismatch the hedging performance deteriorates, all the more that the gap between implied and realized volatility increase. The distribution of hedging errors becomes wider and uncentered, reflecting residual risk due to model ill-specification.

This underlines the importance of accurate volatility estimation in the practical implementation of dynamic hedging strategies.

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