Digital Option Pricing : Different methods

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1 Motivations

In modern quantitative finance, exotic derivatives have gained momentum because they meet specific risk management needs that standard options could not address, allowing for more flexible and customized financial solutions.

To broaden our vision of exotic options, we focus here on **digital options**, which represents an important class with discontinuous payoff that challenges standard pricing techniques. Digital options pay a fixed amount if the underlying asset overtakes a threshold, making it discontinuous with a jump.

The goal of this work is to explore and compare several numerical techniques for pricing such an option.

To capture the diverse computational challenges raised by this contract, we will implement and analyze three methods:

- **PDE closed-form solutions** : This method relies on solving the Black-Scholes partial differential equation analytically under certain assumptions to derive an explicit pricing formula for the option.
- **Monte Carlo simulation** : This approach involves simulating several independent paths of the underlying asset and estimating the option price by averaging the discounted payoffs.
- **Finite difference method** : This numerical technique discretizes the time and space domains of the PDE to approximate the option price iteratively using schemes such as Euler explicit, implicit, or Crank-Nicolson.

2 Black-Scholes PDE

To start the study with a theoretical approach, let us dive deeper into the PDE method under the Black-Scholes framework :

2.1 Derivation of the Black-Scholes PDE for the Digital Call Option

Let $\mathbb{Q}$ be the risk-neutral measure.

It follows that

$$C(t,S) \;=\; \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}\,\mathbf{1}_{\{S_T\geq K\}} \mid S_t\big]$$

is the risk neutral price of the digital call at time t, where K and T denote respectively the option strike and maturity.

Under $\mathbb{Q}$, the underlying asset price is assumed to be driven by a geometric Brownian motion,

$$dS_t = r\,S_t\,dt + \sigma\,S_t\,dW_t^{\mathbb{Q}}.$$

We define

$$f(t,S_t) \;=\; e^{-rt}\,C(t,S_t) = \mathbb{E}^{\mathbb{Q}}\big[e^{-rT}\,\mathbf{1}_{\{S_T\geq K\}} \mid S_t\big]$$

Since $f(t,S_t)$ is a $\mathbb{Q}$–martingale, **its drift term must vanish**.

Assuming the function $C(t,S_t)$ is sufficiently smooth, i.e of class $C^{1,2}(\mathbb{R})$, f is it self of class $C^{1,2}(\mathbb{R})$, and that makes sense to apply *Itô's formula*, as follows :

$$\begin{aligned} df &= \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial S}\big(rS_t\,dt + \sigma S_t\,dB_t\big) + \tfrac{1}{2}\frac{\partial^2 f}{\partial S^2}\,\sigma^2 S_t^2\,dt \\ &= \Big(\frac{\partial f}{\partial t} + rS_t\frac{\partial f}{\partial S} + \tfrac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial S^2}\Big)dt + \sigma S_t\frac{\partial f}{\partial S}\,dB_t \end{aligned}$$

However, we have to express the derivatives of f with respect to the ones of C, knowing $f(t,S) = e^{-rt}C(t,S)$.

$$\begin{cases} \dfrac{\partial f}{\partial t} = -r\,e^{-rt}C + e^{-rt}\dfrac{\partial C}{\partial t} \\ \dfrac{\partial f}{\partial S} = e^{-rt}\dfrac{\partial C}{\partial S} \\ \dfrac{\partial^2 f}{\partial S^2} = e^{-rt}\dfrac{\partial^2 C}{\partial S^2} \end{cases}$$

We can substitute these expressions into the drift part of df, which must be zero knowing f is a $\mathbb{Q}-martingale$:

$$e^{-rt}\Big[\frac{\partial C}{\partial t} - r\,C + rS\frac{\partial C}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\Big]dt = 0$$

Adding the terminal condition for the payoff at time $t = T$, we obtain the following PDE satisfied by $C(t, S_t)$:

$$\boxed{\begin{array}{c} \dfrac{\partial C}{\partial t} + rS\dfrac{\partial C}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 C}{\partial S^2} - r\,C = 0, \quad t \in [0, T] \\[2em] C(T, S_T) = \mathbf{1}_{\{S_T \geq K\}} \end{array}}$$

2.2 Solution of the PDE

We could have found the expression of $C(t, S_t)$ resolving the previous PDE with this terminal condition, which as a unique solution.

However, we can derive it in an easier way from the following expression:

$$C(t, x) \;=\; e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\big[\,\mathbf{1}_{\{S_T \geq K\}} \mid S_t = x\big]$$

Indeed,

$$\mathbb{E}^{\mathbb{Q}}\big[\,\mathbf{1}_{\{S_T \geq K\}} \mid S_t = x\big] = \mathbb{Q}(S_T \geq K | S_t = x)$$

Therefore,

$$C_d(t, x) = e^{-r(T-t)}\mathbb{Q}(S_T \geq K \mid S_t = x).$$

But we know that under the risk-neutral measure $\mathbb{Q}$, the asset price follows a geometric Brownian motion which means that $\log S_t$ is normally distributed, and more precisely :

$$\log S_T \sim \mathcal{N}\left(\log S_t + \left(r - \frac{1}{2}\sigma^2\right)(T-t),\, \sigma^2(T-t)\right).$$

Which is equivalent to write :

$$\log S_T \overset{d}{=} \sigma\sqrt{T-t}\,\cdot\, Z \,+\, \log S_t + \left(r - \frac{1}{2}\sigma^2\right)(T-t)$$

where $Z \sim \mathcal{N}(0, 1)$.

Hence,

$$\mathbb{Q}(S_T \geq K \mid S_t = x) = \mathbb{Q}(\log S_T \geq \log K) = \mathbb{Q}\left(Z \geq \frac{\log K - \log x - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

This yields :

$$\begin{aligned}
\mathbb{Q}(S_T \geq K \mid S_t = x) &= 1 - \mathbb{Q}\left(Z \leq \frac{\log K - \log x - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= 1 - \Phi\left(\frac{\log K - \log x - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \Phi\left(\frac{\log x - \log K + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \Phi\left(d_-(T-t)\right)
\end{aligned}$$

where we put :

$$d_-(T-t) := \frac{\log(x/K) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

Therefore, the value of the digital option is given by :

$$\boxed{C(t,x) = e^{-r(T-t)}\Phi\left(d_-(T-t)\right)}$$

3 Monte Carlo method

Now we can run a Monte Carlo simulation, for fixed parameters, in order to compare the price to the one given by the explicit formula.

We set the following parameters :

- $S_0 = 100$, the initial underlying's price
- $K = 100$, the strike price
- $\sigma = 0.2$, the volatility
- $T = 1$ year, the time to maturity
- $r = 0.05$, the constant risk-free interest rate

The idea is simply to simulate N paths, to compute the payoff for each of them, to actualize the mean.
The Monte-Carlo estimator of the option price is therefore :

$$C(t, S_t) = e^{-r(T-t)} \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\{S_T^{(i)} \geq K\}}$$

with :

- $S_T^{(i)}$ is the terminal underlying's price for path i
- N is the total number of path simulations

Implementing this estimator on Python, we get for $N = 10^6$:

$$\boxed{\begin{array}{l} \textit{Monte Carlo Price} : 0.532831 \\ \textit{Theoretical Price} : 0.532325 \\ \textit{Absolute error} : 5.063467 \cdot 10^{-4} \end{array}}$$

The result is relevant with a meaningless error for a sufficient number of generated paths.

4 Finite Difference method

We can now deal with the last method evoked in the introduction : the discrete schemes.

The PDE we have to deal with is non linear due to its non constant coefficients. We have to transform it with a change of variables to simplify the problem.

4.1 Linearization of the PDE

Define

$$\tau := T - t \quad | \quad x := \log(S)$$

and the function

$$\psi(\tau, x) := C(t, S) = C(T - \tau, e^x)$$

We now compute the necessary derivatives using the chain rule :

$$\frac{\partial C}{\partial t} = -\frac{\partial \psi}{\partial \tau}, \quad \frac{\partial C}{\partial S} = \frac{1}{S}\frac{\partial \psi}{\partial x}, \quad \frac{\partial^2 C}{\partial S^2} = \frac{1}{S^2}\left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial \psi}{\partial x}\right)$$

Substituting into the original PDE :

$$-\frac{\partial \psi}{\partial \tau} + r \cdot \frac{\partial \psi}{\partial x} + \frac{1}{2}\sigma^2\left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial \psi}{\partial x}\right) - r\psi = 0$$

Simplifying :

$$\frac{\partial \psi}{\partial \tau} = \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial \psi}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 \psi}{\partial x^2} - r\psi$$

Now define the function :

$$f(\tau, x) := e^{r\tau}\psi(\tau, x)$$

Then :

$$\frac{\partial f}{\partial \tau} = re^{r\tau}\psi + e^{r\tau}\frac{\partial \psi}{\partial \tau} = rf + \left(\left(r - \frac{1}{2}\sigma^2\right)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2} - rf\right)$$

$$\Rightarrow \frac{\partial f}{\partial \tau} = \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}$$

Finally, define :

$$\phi(\tau, x) := f(\tau, x)e^{-\alpha x}, \quad \text{where } \alpha := -\frac{r - \frac{1}{2}\sigma^2}{\sigma^2}$$

Then, the function ϕ satisfies the standard heat equation :

$$\frac{\partial \phi}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 \phi}{\partial x^2} \qquad (\mathcal{E})$$

This PDE has now constant coefficients.

4.2 Discretization of the PDE

Theoretically, the range of the asset value is :

$$S \in (0, \infty) \quad \Rightarrow \quad x = \log(S) \in (-\infty, \infty)$$

Since we cannot simulate over an infinite spatial domain, we truncate :

$$S_{\min} = \frac{K}{5}, \quad S_{\max} = 5K \quad \Rightarrow \quad x_{\min} = \log\left(\frac{K}{5}\right), \quad x_{\max} = \log(5K)$$

The time domain however remains :

$$\tau \in [0, T]$$

Now, we can discretize this equation using the Implicit Euler scheme.

Let $\phi_i^n \approx \phi(x_i, \tau_n)$, with:

$$x_i = x_{\min} + i\Delta x$$
$$\tau_n = n\Delta\tau$$

4.3 Boundary conditions

We have to set the initial and boundary conditions.

At $\tau = 0$:

$$C(x, 0) = \max(e^x - K, 0) \quad \Rightarrow \quad \phi(x, 0) = e^{-\alpha x} \max(e^x - K, 0)$$

When it comes to boundary conditions :

$$\phi(x_{\min}, \tau) = 0$$
$$\phi(x_{\max}, \tau) = e^{-\alpha x_{\max}}$$

4.4 Euler scheme

We approximate the derivatives at time level $n+1$ using implicit Euler :

$$\frac{\partial \phi}{\partial \tau} \approx \frac{\phi_i^{n+1} - \phi_i^n}{\Delta \tau}$$
$$\frac{\partial^2 \phi}{\partial x^2} \approx \frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{(\Delta x)^2}$$

Substituting into the PDE gives the scheme :

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta \tau} = \frac{1}{2}\sigma^2 \frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{(\Delta x)^2}$$

Multiplying through by $\Delta \tau$ and rearranging :

$$-\nu \phi_{i-1}^{n+1} + (1 + 2\nu)\phi_i^{n+1} - \nu \phi_{i+1}^{n+1} = \phi_i^n$$

where

$$\nu := \frac{\Delta \tau}{2} \frac{\sigma^2}{(\Delta x)^2}$$

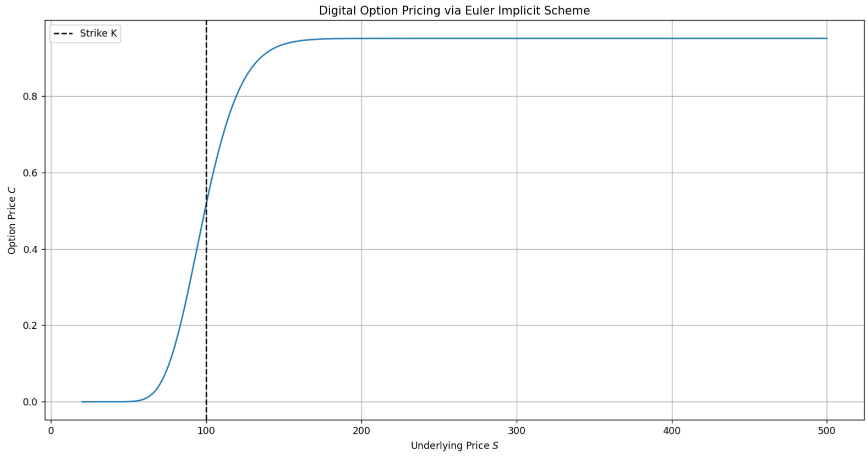
This results in a tridiagonal linear system at each time step :

$$\boxed{a\phi_{i-1}^{n+1} + b\phi_i^{n+1} + c\phi_{i+1}^{n+1} = \phi_i^n}$$

with :

$$a = -\nu$$
$$b = 1 + 2\nu$$
$$c = -\nu$$

We keep the same parameters as used in the Monte-Carlo simulation. Implementing the discretized equation and all the limit conditions, we get the graph of $C(x, 0)$ (i.e. the initial option price) with respect to the asset's price.



Interpretation : The graph depicts the price of a European digital call option at the initial time $t = 0$ as a function of the underlying asset price S, within the truncated domain $[K/5, 5K]$.
The digital call option pays a fixed amount 1 if the asset price at maturity exceeds the strike K, and zero otherwise. Consequently, for asset prices well below the strike ($S \ll K$), the option price remains close to zero since the probability of finishing in-the-money is very low.
Near the strike price $S \approx K$, the option price increases sharply, reflecting the jump-like payoff structure.
Futhermore, the value asymptotically approaches the discounted payoff e^{-rT} for large underlying prices $S \gg K$, as the option almost certainly finishes in-the-money.
This results exhibits a much steeper transition compared to a vanilla call option. The option's Delta is near zero far below the strike, but spikes sharply around the strike, before reaching zero again for greater S values.

Overall, the graph confirms the expected behavior of a digital option price and validates the numerical scheme's ability to capture the discontinuous payoff accurately within the transformed coordinate framework.

4.5 Crank-Nicolson scheme

Now we can make the same for the Crank-Nicolson scheme.
This scheme is obtained by taking the mean between the Implicit and Explicit Euler schemes.

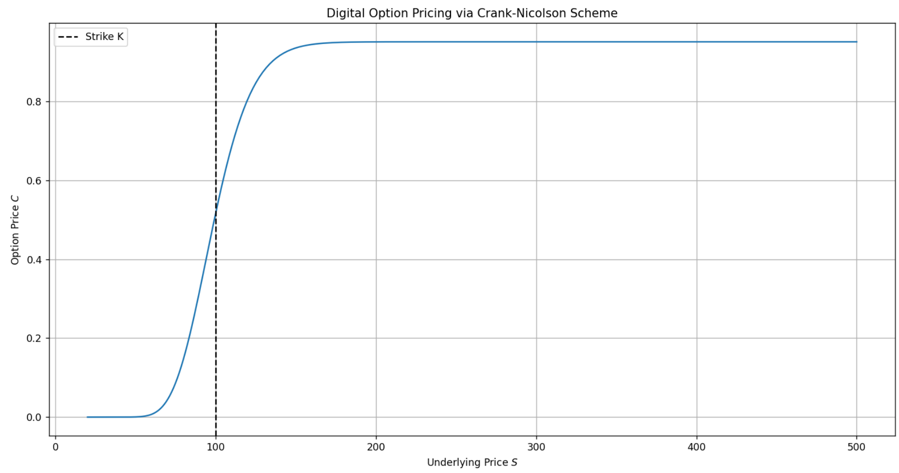
We have,

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta \tau} = \frac{1}{2}\sigma^2 \left(\frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{(\Delta x)^2} + \frac{\phi_{i+1}^{n} - 2\phi_i^{n} + \phi_{i-1}^{n}}{(\Delta x)^2} \right)$$

Which leads to :

$$\boxed{-\nu\, \phi_{i-1}^{n+1} + (1+2\nu)\, \phi_i^{n+1} - \nu\, \phi_{i+1}^{n+1} = \nu\, \phi_{i-1}^{n} + (1-2\nu)\, \phi_i^{n} + \nu\, \phi_{i+1}^{n}}$$

Implementing this scheme in Python, we get a similar graph of $C(x, 0)$ with respect to the asset's price.



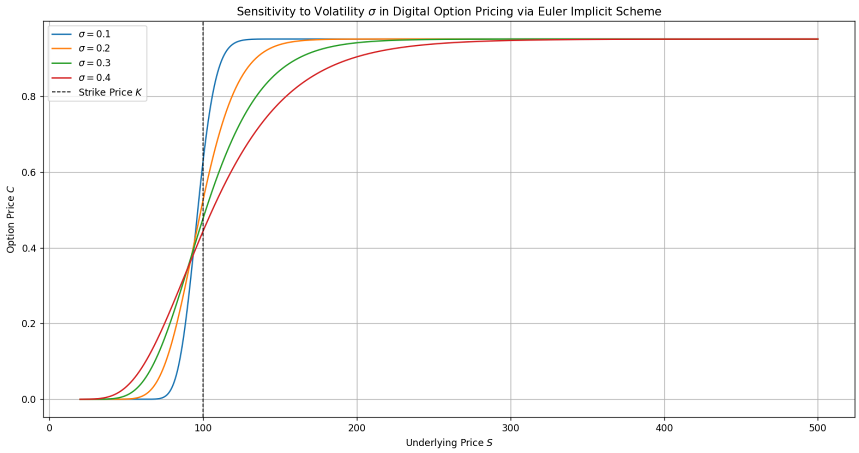
Interpretation : This graph seems indistinguishable from the previous, relevantly leading to the same interpretation.
In theory, the Crank-Nicolson scheme is second-order accurate in both time and space, whereas the implicit Euler scheme is only first-order accurate in time. Therefore, one would expect Crank-Nicolson to yield better numerical accuracy. However, for digital options, the payoff function is discontinuous at the strike, which severely limits the convergence order of any numerical scheme.
As a result, both methods tend to produce visually similar pricing curves for digital options, despite their theoretical differences in accuracy.

4.6 Sensibility to parameters Analysis

What we can do now is modify some parameters to assess their influence on the option price. We will run a sensibility analysis for the two main influence parameters : T and σ.

4.6.1 Volatility's influence

First, we fix parameters at the same values as before, just making the volatility σ of the option evolve.
Using the Implicit Euler scheme, we get :

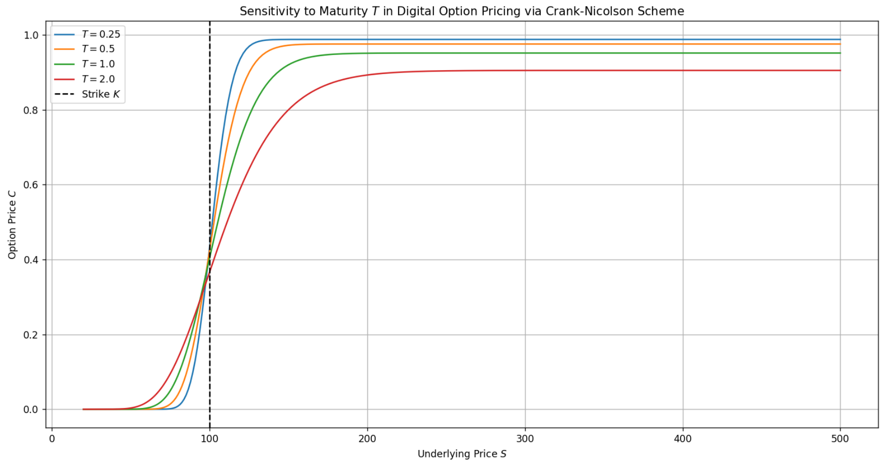


Interpretation : As the volatility parameter σ decreases, the price curve of the digital option becomes steeper and more closely tends to a step function centered on the strike price K.
This behavior can be explained by the fact that, in the limit as $\sigma \to 0$, the uncertainty in the underlying asset's future price vanishes. The option becomes essentially a deterministic instrument : it will either pay out 1 if the initial asset price is above the strike, or nothing if it is below.
Hence, the transition in the pricing function becomes sharper, approaching the idealized discontinuous payoff function of the digital option.

Conversely, for higher values of σ, the increased uncertainty smooths out the transition, resulting in a more gradual slope in the pricing curve.

4.6.2 Time to maturity influence

Now, we make the time to maturity T vary, as highlighted in this graph (using this time the Crank-Nicolson scheme) :

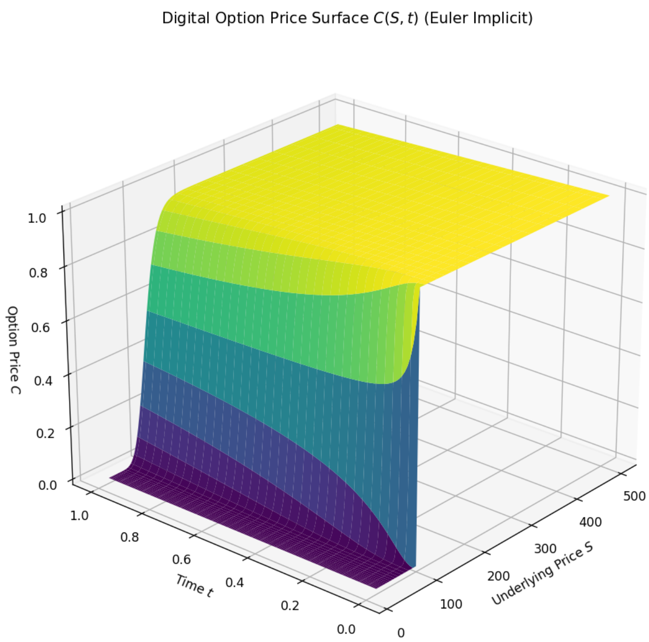


Interpretation : When the maturity T of the digital option decreases, the price function becomes more abrupt around the strike price K, tending to reproduce closely an indicator function.
This stands because a shorter time to maturity reduces the impact of future uncertainty on the option's value. With less time for the underlying asset to fluctuate, the probability distribution of its final value becomes more concentrated around its current value. As a result, the pricing function rapidly transitions from 0 to 1 near the strike, since the chance of ending in or out of the money becomes binary.

In the limit as $T \to 0$, the option value approaches a step function at the strike, reflecting the immediate payoff condition of the digital option.

4.7 Option's price surface

To highlight the influence of time in the payoff profil, we can plot the surface of $C(S,t)$ for t in a range $[0,1]$, as follows :



Interpretation : The surface plot of the digital option price reveals a key feature of diffusion processes : the smoothing effect of time.
At $t = 0$, the option payoff exhibits a discontinuity at the strike price K, clearly representing the step function.

As time increases, the option price surface becomes progressively smoother around the strike. This phenomenon is due to the probabilistic nature of option pricing. More time to maturity remaining means that the likelihood of the asset crossing the strike level is spread over a broader range of values, which results in a gradual transition in the option value.

In contrast, near maturity, the digital option behaves almost deterministically, with sharp transitions between zero and 1 payoff.

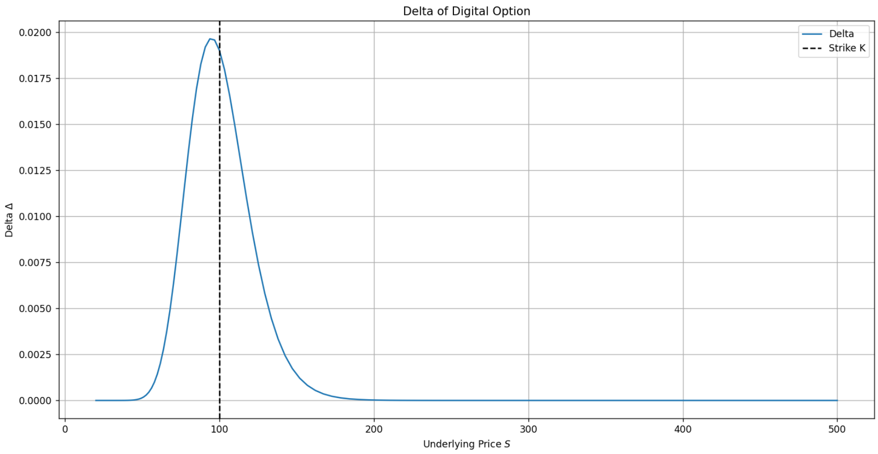
4.8 Option's Delta

Finally we can compute the delta of the option using the following approximation formula :

$$\Delta = \frac{\partial C}{\partial S} = \frac{\partial C}{\partial X} \cdot \frac{\partial X}{\partial S} = \frac{1}{S} \cdot \frac{\partial C}{\partial X}$$

$$\Delta_i \approx \frac{1}{S_i} \frac{C_{i+1} - C_{i-1}}{2\Delta X}$$

Implementing the formula on Python, we get the following graph :



Interpretation : The graph of the delta for a digital call option exhibits a sharp peak centered around the strike price K. This shape looks like a narrow Gaussian distribution and reflects the discontinuity of the digital payoff at maturity.

This behavior is consistent with the fact that the delta of a digital option approximates a Dirac delta function $\delta(S - K)$ in the limit as $\sigma \to 0$ or $T \to 0$.

5 Conclusion

5.1 Results

In this study, we analyzed digital options and their pricing trough different methods. The payoff function is an indicator function, reflecting the binary nature of digital options. However, our results demonstrated that as the maturity, time to expiration, and volatility increase, the option price exhibits a smoothing effect. This behavior aligns with the characteristics of diffusion processes, where the discontinuous payoff becomes progressively regularized over time. These findings highlight the fundamental impact of temporal and volatility parameters on the pricing dynamics of digital options.

Furthermore, we have seen that the price given by the MC method and the PDE method are basically extremely close, which aligns with the Option price for S_0 we deduce from the schemes.

5.2 Further insights

Digital options are valuable tools for modeling discontinuous payoffs and exploring sensitivities as we have done.

However, this derivative remains a theoretical object, which cannot be sold by investment banks. Indeed, the theoretical Delta of such an option is a Dirac distribution. This makes hedging extremely unstable, especially near maturity, and exposes the issuer to significant risk from small movements in the underlying asset. A trader supposed to hedge the position on such a product couldn't buy an infinite amount of the underlying at the strike.

To overcome this, traders typically offer smoothed alternatives that approximate digital payoffs but remain hedgeable. One common structure is the **call spread**, which consists in buying a call at strike $K - \delta$ and selling a call at $K + \delta$, with $\delta \ll 1$.
The payoff of such a structure is :

$$\text{Payoff}(S_T) = \max(S_T - (K - \delta), 0) - \max(S_T - (K + \delta), 0)$$

When properly scaled by $\frac{1}{2\delta}$, this replicates a sloped digital payoff that converges to a step function as $\delta \to 0$, but remains continuous and differentiable for pricing and hedging.
Such products retain the risk-profile of digital options while offering practical advantages for market makers.

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