SM212 Test 4 Practice Problems

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These practice problems are designed to help you review for the test (Sections 8.1, 8.2, 11.2, 11.3, 12.1, 12.3). They are intended to cover most of the basic topics in the sections we have looked at. However, these problems are not necessarily comprehensive and may NOT resemble the actual test. The best way to prepare for any exam is to practice as many problems as possible.

1. *Use eigenvalues and eigenvectors* to find the general solution of the system of IVPs:

   (a) $\begin{cases} \frac{dx}{dt} = 2x + y \\ \frac{dy}{dt} = -4x + 2y \\ x(0) = 2, \quad y(0) = 1. \end{cases}$

   (b) $\begin{cases} \frac{dx}{dt} = x + 2y \\ \frac{dy}{dt} = 4x + 3y \\ x(0) = 0, \quad y(0) = 1. \end{cases}$

   NOTE: To study for the Final Exam, you should also practice solving for these systems by *using Laplace transforms* instead. Compare your answers using both methods.

2. Consider the IVP: $y'' + 4y' - 8y = 0$, with initial conditions $y(0) = 1$, $y'(0) = 0$.

   (a) Use Euler's method with step size $h = 0.25$ to approximate $y(0.5)$ up to four decimal places.

   (b) Solve for the exact solution $y(x)$ of the given IVP. Then compare the exact value $y(0.5)$ with your approximation in part (a).

3. Consider the function $f(x) = \begin{cases} 0, & -2\pi < x < 0 \\ x, & 0 \le x \le 2\pi. \end{cases}$

   (a) Find the Fourier series expansion $FS(x)$ of f.

   (b) What values does the Fourier series $FS(x)$ converge to, when $x = -2\pi$, $x = 0$, $x = \pi$, and $x = -8$? Answer by filling in: $FS(-2\pi) =$ $FS(0) =$ $FS(\pi) =$ $FS(-8) =$

   (c) Sketch the graph of the Fourier series $FS(x)$ over an interval that extends to include *at least three periods*. Clearly label your axes, and the important convergence values on the graph.

4. Consider the function $f(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 < x < 2. \end{cases}$

   Use the even reflection of f to find the half-range Fourier Cosine expansion of f. Then sketch it with at least *three periods*. Clearly label your axes and the important convergence values on the graph.

5. (a) Find the half-range Fourier Sine expansion of the function $f(x) = \begin{cases} 100, & 0 \le x < 1 \\ 0, & 1 \le x \le 2. \end{cases}$

   (b) The temperature $u(x,t)$ on a thin bar satisfies the following conditions $\begin{cases} 4\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \\ u(0,t) = u(2,t) = 0, \quad t > 0, \\ u(x,0) = f(x), \quad 0 \le x \le 2, \end{cases}$
   where $f(x)$ is given in part (a). Clearly show all steps of the separation of variables process to solve the above heat equation for $u(x,t)$ and write it in summation notation. You may use your answer in (a).

(c) Use the *first two non-zero terms* of your answer in part (b) to find the approximate temperature at the middle of the bar at the time $t = 0.1$ seconds.

6. Use separation of variables to find all product solutions $u(x, y)$ for the following PDEs:

   (a) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 5u$

   (b) $yu_{xy} + u = 0$

   (c) Suppose a rod of length $L = \pi$ meters is made of a material with a thermal diffusivity constant $k = 4$. Assume that the endpoints of the rod are insulated, and that the initial distribution of heat in the rod is given by $u(x, 0) = x^2$. Clearly show all steps of the separation of variables process to solve the heat equation to find the function $u(x, t)$ that describes the temperature of the rod at time $t > 0$ and at a position $0 < x < \pi$.

Additional Practice: (on the USNA Math Department resources web page)

- Fall 2013 final exam: #13–16, 21, 23
- May 2014 final exam Part I: #17–20; Part II: # 6, 7, 8 (for #6, only find the complementary solution $\overrightarrow{X_c}(t)$ of the *homogeneous* system.)
- Fall 2014 final exam Multiple Choice: #17, 18, 19; Free Response: #5, 6 (for #5(a), only find the complementary solution $\overrightarrow{X_c}(t)$ of the *homogeneous* system.)

Test 4 Practice – ANSWER KEYS:

1. (a) $\overrightarrow{X}(t) = c_1 e^{2t}\left[\begin{pmatrix}0\\2\end{pmatrix}\cos(2t) - \begin{pmatrix}-1\\0\end{pmatrix}\sin(2t)\right] + c_2 e^{2t}\left[\begin{pmatrix}-1\\0\end{pmatrix}\cos(2t) + \begin{pmatrix}0\\2\end{pmatrix}\sin(2t)\right]$, use initial conditions to solve $c_1 = \frac{1}{2}, c_2 = -2$.

   (b) $\overrightarrow{X}(t) = c_1 e^{-t}\begin{pmatrix}-1\\1\end{pmatrix} + c_2 e^{5t}\begin{pmatrix}1\\2\end{pmatrix}$, use initial conditions to solve $c_1 = \frac{1}{3}, c_2 = \frac{1}{3}$.

2. (a) Euler $y(0.5) \approx 1.5$

   (b) exact solution $y(x) = \frac{1}{6}e^{-2(1+\sqrt{3})x}[(3+\sqrt{3})e^{4\sqrt{3}x} + 3 - \sqrt{3}]$, exact value $y(0.5) = 1.65368$. Part (a) is an underestimate.

3. (a) $FS(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{\pi n^2}((-1)^n - 1)\cos(\frac{nx}{2}) + \frac{2}{n}(-1)^{n+1}\sin(\frac{nx}{2})\right]$.

   (b) $FS(-2\pi) = \pi, \qquad FS(0) = 0, \qquad FS(\pi) = \pi, \qquad FS(-8) = 1.45\pi.$

4. $\frac{3}{4} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}[\cos(\frac{n\pi}{2}) - 1]\cos(\frac{n\pi x}{2})$

5. (a) $\sum_{n=1}^{\infty} \frac{200}{n\pi}[1 - \cos(\frac{n\pi}{2})]\sin(\frac{n\pi x}{2})$

   (b) $u(x, t) = \sum_{n=1}^{\infty} \frac{200}{n\pi}[1 - \cos(\frac{n\pi}{2})]\sin(\frac{n\pi x}{2}).e^{-n^2\pi^2 t}$

   (c) Expand the above $u(x, t)$ by plugging in $n = 1, 2, 3, \ldots$, then evaluated at $x = 1$ and $t = 0.1$, and add the first two nonzero terms, get $u(1, 0.1) \approx 23.724°$.

6. (a) $u(x, y) = Ce^{-\lambda x + (\lambda+5)y}$

   (b) $u(x, y) = Ce^{-\lambda x}.|y|^{1/\lambda}$ or $u(x, y) = Ce^{x/\lambda}.|y|^{-\lambda}$

   (c) $u(x, t) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}e^{-4n^2 t}\cos(nx)$.