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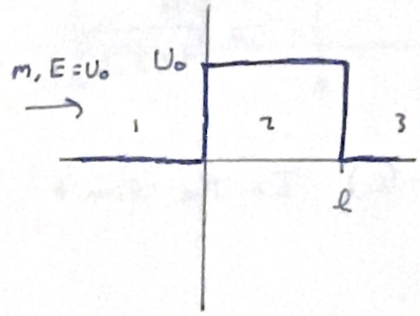
QM HW #5
2/20/19
Profumo

1.)

(a) $\left[-\frac{\hbar^2}{2m}\nabla^2+V\right]\psi = E\psi$



in region 1, V = 0, $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi \Rightarrow$

$\psi'' = \overbrace{\frac{-2U_0 m}{\hbar^2}}^{-\gamma^2}\psi$, $\psi = Ae^{i\gamma x}$ for incoming waves

$\psi = Be^{-i\gamma x}$ For reflected waves.

ok

Likewise, in region 3, $\psi = Ce^{i\gamma x}$

In region 2, $V = U_0$, so

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + U_0\psi = U_0\psi$$

$\psi'' = 0$, so $\psi = \alpha + \beta x$ ✓

Imposing boundary conditions:

$A + B = \alpha$, $\alpha + \beta L = Ce^{i\gamma\ell}$

Imposing continuity of ψ':

$i\gamma A - i\gamma B = \beta$, $i\gamma C e^{i\gamma\ell} = \beta$

Assuming we know A from incoming beam, 4 eqs, 4 unknowns.

$\Rightarrow$ Mathematica to the rescue!

$C = \frac{-2Ae^{-i\gamma\ell}}{i\gamma\ell-2}$, $B = \frac{Ai\gamma\ell}{i\gamma\ell-2}$, $\alpha = \frac{2A(i\gamma\ell-1)}{i\gamma\ell-2}$, $\beta = \frac{-2Ai\gamma}{i\gamma\ell-2}$

ok!

$\Rightarrow$

(cont)

1.)

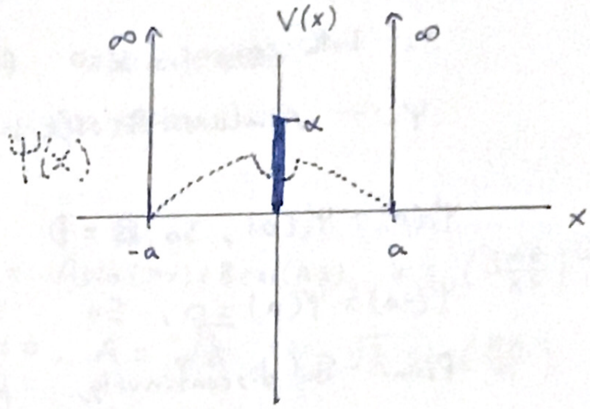
(b) The ratio of transmitted beams is $\frac{|C|^2}{|A|^2} = \frac{-2e^{-i\gamma L}}{i\gamma l-2} \cdot \frac{-2e^{i\gamma L}}{-i\gamma l-2}$

$= \frac{4}{4+\gamma^2 l^2}$, where $\gamma = \sqrt{\frac{2U_0 m}{\hbar^2}}$

(c) In the limit $l \to \infty$, there are no transmitted particles!

Perfect!

2.) $\alpha = \frac{\hbar^2}{10ma}$



(a) ✓

(b)

Since there is an extra dip compared to the normal infinite well, I would expect the energy to be higher

(c) $E_0 \leq \langle\Psi|H|\Psi\rangle$ for any Ψ

For the plain well, $\Psi_0 = \sqrt{\frac{1}{a}}\cos\left(\frac{\pi x}{2a}\right)$

So $\langle\Psi_0|H|\Psi_0\rangle = \int_{-a}^{a} \frac{\cos\left(\frac{\pi x}{2a}\right)}{a}\left(\frac{P^2}{2m} + \frac{\hbar^2}{10ma}\delta(x)\right)\cos\left(\frac{\pi x}{2a}\right) dx$

$= \int_{-a}^{a} + \frac{\pi^2}{4a^2}\cdot\frac{1}{a}\frac{\hbar^2}{2m}\cos^2\left(\frac{\pi x}{2a}\right) + \delta(x)\cos^2\left(\frac{\pi x}{2a}\right)\cdot\frac{\hbar^2}{10ma^2}\, dx$

$= \underbrace{\frac{\pi^2\hbar^2}{8ma^2}}_{\text{ground Energy}} + \underbrace{\frac{\hbar^2}{10ma^2}}_{\text{correction}} \Rightarrow \frac{E'}{E_0} = 1 + \underbrace{\frac{8}{10\pi^2}}$

$\approx 8.1\%$

✓

(d)

$\left(\frac{-P^2}{2m} + \frac{\hbar^2}{10ma}\delta(x)\right)\Psi(x) = E\Psi(x)$

Integrate around δ function :

$\lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} \frac{+\hbar^2}{2m}\Psi'' + \frac{\hbar^2}{10ma}\delta(x)\Psi\, dx = \int_{-\epsilon}^{\epsilon} \xrightarrow{\to 0 \text{ for } \epsilon\to 0} E\Psi\, dx \Rightarrow$

$\lim_{\epsilon\to 0} \frac{+\cancel{\hbar^2}}{\cancel{2m}}\Psi'\Big|_{-\epsilon}^{\epsilon} + \frac{\cancel{\hbar^2}}{5\cancel{10m}a}\Psi(0) = 0$

So $\Psi^{II\prime}(0) - \Psi^{I\prime}(0) = \frac{1}{5ak}\Psi(0)$

$\Rightarrow$

(d) cont.

In both regions, $V=0$ so we get sinusoidal solutions

$\psi_I = A\sin(kx) + B\cos(kx)$, $\psi^{II} = C\sin(kx) + D\cos(kx)$, $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$

$\psi_1(0) = \psi_2(0)$, so $B = D$

$\psi(-a) = \psi(a) = 0$, so $B = A\tan(ka) = D$

From δ discontinuity, $kC - kA = \frac{1}{5a}\tan(ka)\cdot A$

$$C = A\left(1 + \frac{1}{5ka}\tan(ka)\right)$$

We know that at $-a, a$

$$-A\sin(ka) + A\cancel{\tan(ka)}\cos(ka) = A\left(1+\frac{1}{5ka}\tan(ka)\right)\sin(\cancel{ka}) + A\cancel{\tan(ka)}\cos(ka)$$

$$-2 = \frac{1}{5ka}\tan ka \Rightarrow \tan(ka) = -10ka$$

So $ka = 1.632$ (from table)

thus

$$(1.632)^2 = \frac{a^2 \cdot 2mE}{\hbar^2} \Rightarrow E = \frac{2.663\hbar^2}{2ma^2}$$

compare to ground : $\frac{2.663}{2}\cdot\frac{8}{\pi^2} = 1.079$

So the true energy is $\simeq$ 8% higher than the ground energy !

(e)

I believe the delta function amplitude is "small" because it has a small effect on the energy.

As α becomes large, so too will the correction.

✓

3.) $$V(x) = \begin{cases} V_0 \delta(x), & -a < x < \infty \\ \infty, & x < -a \end{cases}$$

(a) At $t=0$, $\psi \in (-a, 0)$

In this region, there is 0 potential so $\psi = A\sin(kx) + B\cos(kx)$, $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$

From boundary conditions ($\psi(-a)=0$, $\psi(0)=0$) $B=0$, $A = \sqrt{\frac{2}{a}}$, $\psi_0 = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$

(b)

$\psi = 0$ for $x < -a$, so $\underline{\psi_k^{I}(-a) = 0}$, (1)

From continuity, $\underline{\psi_k^{I}(0) = \psi_k^{II}(0)}$ (2)

At $x=0$, integrate to find discontinuity in ψ'

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \frac{-\hbar^2}{2m} \psi'' + V_0 \delta(x)\psi \, dx = \int_{-\epsilon}^{\epsilon} E\psi \, dx$$

$$\lim_{\epsilon \to 0} \frac{-\hbar^2}{2m} \left(\psi' \Big|_{-\epsilon}^{\epsilon}\right) + V_0 \psi(0) = 0$$

Since $\psi^{I}(0) = \psi^{II}(0)$

$$\Rightarrow \underline{\left(\psi_k^{II\prime}(0) - \psi_k^{I\prime}(0)\right) = \frac{2mV_0}{\hbar^2} \psi(0)} \quad (3)$$

(c)

$$\frac{-\hbar^2}{2m} \psi'' = E\psi \Rightarrow$$

In region 1, we still have $\psi^{I} = A\sin(kx) + B\cos(kx)$

From b.c. (1), we know $-A\sin(ka) + B\cos(ka) = 0$, $B = \tan(ka) \cdot A$

Similarly, in region 2, $\psi^{II} = C\sin(kx) + D\cos(kx)$

From b.c. (2), $\underline{D = B = \tan(ka) \cdot A}$

From b.c. (3), $kC - kA = \frac{2mV_0}{\hbar^2} \cdot A \cdot \tan(ka)$

$$\underline{C = A\left(1 + \frac{2mV_0}{k\hbar^2} \tan(ka)\right)}$$

$\Rightarrow$

$$\boxed{\psi^{I} = A\sin(kx) + A\tan(ka)\cos(kx), \quad \psi^{II} = A\left(1 + \frac{2mV_0}{k\hbar^2} \tan(ka)\right) \sin(kx) + A\tan(ka)\cos(kx)}$$

(d) $\psi(x) = \int_{-\infty}^{\infty} F(k)\,\psi_k(x)\,dk$

Integrate both sides w.r.t. x after multiplying by $\psi_{k'}$, use orthogonality.

$$\int_{-\infty}^{\infty} \psi(x)\,\psi_{k'}(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k)\,\psi_k(x)\,\psi_{k'}(x)\,dx\,dk$$

$$= F(k')\int_{-\infty}^{\infty} \psi_{k'}^2(x)\,dx = \boxed{F(k') = \int_{-\infty}^{\infty} \psi(x)\,\psi_{k'}(x)\,dx} \quad (\text{at } t=0)$$

(e) Since $F(k)$ is expressed in terms of energy eigen functions, the time development is simply $e^{-iEt/\hbar}$

So $\psi(x,t) = \int_{-\infty}^{\infty} F(k')\,e^{-iEt/\hbar}\,\psi_{k'}(x)\,dk'$

At large times, the smallest energy values will contribute the most, so small k values govern time development as $t \to \infty$

4.) For potential $-iV$, $V << E$

$$\frac{-\hbar^2}{2m}\psi'' = (E+iV)\psi$$

$$\psi = A e^{\pm ikx}, \quad k = \sqrt{\frac{2m(E+iV)}{\hbar^2}}$$

For $E >> V$, $k \simeq \sqrt{\frac{2m}{\hbar^2}}\sqrt{E}\left(1+\frac{iV}{2E}\right)$

Probability current $j = \frac{\hbar}{2mi}\left(\psi^*\psi' - \psi\psi^{*\prime}\right) \rightarrow$ for Ae^{+ikx},

$$j = \frac{\hbar}{2mi}\left(Ae^{-ik^*x}\cdot A(+ik)e^{ikx} - Ae^{ikx}\cdot A(-ik^*)e^{-ik^*x}\right)$$

$$= \frac{\hbar A^2}{2m}\left(e^{-ix(k-k^*)}k + e^{ix(k-k^*)}k^*\right)$$

$$= \frac{\hbar A^2}{2m} e^{-ix\,Im(k)}\cdot Re(k)$$

$$= \frac{\cancel{\hbar} A^2}{2m}\sqrt{\frac{2mE}{\cancel{\hbar^2}}}\cdot e^{-x\cdot\sqrt{\frac{2mV^2}{\hbar^2 E}}}$$

$$= \boxed{A^2\sqrt{\frac{E}{2m}}\, e^{-x\cdot\sqrt{\frac{2mV^2}{\hbar^2 E}}}}$$

So the absorption coefficient is $\boxed{\sqrt{\frac{2mV^2}{\hbar^2 E}}}$

ok!