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Quantum HW #1
Due 1/16/19

WELL DONE!
10/10

1.) Using the dual correspondence definition:

$$c|\alpha\rangle \underset{DC}{\leftrightarrow} \langle\alpha|c^*$$

Consider $|\alpha\rangle\langle\beta|$ acting on $|\psi\rangle$, and take the dual correspondence $\Rightarrow$

$$(|\alpha\rangle\cdot\underbrace{(\langle\beta|\psi\rangle)}_{c})^\dagger = (c|\alpha\rangle)^\dagger \underset{DC}{\leftrightarrow} \langle\alpha|c^*$$

By the associative axiom of multiplication, we could also have written

$$(\underbrace{(|\alpha\rangle\langle\beta|)}_{\hat{X}}\cdot|\psi\rangle)^\dagger = \hat{X}|\psi\rangle \underset{DC}{\leftrightarrow} \langle\psi|\hat{X}^\dagger$$

Using: $c^* = \langle\beta|\psi\rangle^* = \langle\psi|\beta\rangle$, we know that

$$\langle\alpha|c^* = \langle\psi|\beta\rangle\langle\alpha| = \langle\psi|\hat{X}^\dagger$$

Thus, again by associativity, $\langle\psi|\cdot(|\beta\rangle\langle\alpha|)$ $\quad \hat{X}^\dagger = (|\alpha\rangle\langle\beta|)^\dagger = |\beta\rangle\langle\alpha|$

Q.E.D.

✓

2.) Starting with R.H.S.

$$\sum_{b',b''} \langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b''\rangle\langle b''|c'\rangle \qquad (i)$$

If $[\hat{A},\hat{B}]=0$, and if the system is non-degenerate, then $\hat{A}$ and $\hat{B}$ have a common eigenbasis. ✓

This implies: $\left\{ \sum_{b''} \langle a'|b''\rangle = \sum_{b''} \delta_{b',b''}\langle a'|b''\rangle \right\}$ This is an odd way of writing the relation, but it will be convienient ⇒

So $(i) = \sum_{b',b''} \langle c'|b'\rangle\langle b'|a'\rangle\, \delta_{b',b''}\langle a'|b''\rangle\langle b''|c'\rangle$

The δ collapses the sum over b'', leaving us with

$$(i) = \sum_{b'} \langle c'|b'\rangle\underbrace{\langle b'|a'\rangle\langle a'|b'\rangle}\langle b'|c'\rangle = L.H.S.$$

These $=1$ b/c $\{|a^{(i)}\rangle\} = \{|b^{(i)}\rangle\}$ But I leave them in this form for clarity.

Q.E.D.

(The proof holds for $[B,C]=0$ as well. Simply replace $\langle b''|c'\rangle$ with $\delta_{b',b''}\langle b''|c\rangle$ ...)

ok!

3.)

For $|S_z;+\rangle$,

$$\langle S_x \rangle = \langle S_z;+|\hat{S}_x|S_z;+\rangle$$

Expressed in the S_z basis, this gives

$$\langle S_x \rangle = \begin{pmatrix}1 & 0\end{pmatrix} \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 0$$

Which is what one would expect since measurement of S_x yields $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ with equal probability.

Now,

$$\langle S_x^2 \rangle = \langle S_z;+|\hat{S}_x\hat{S}_x|S_z;+\rangle \Rightarrow \text{again, choose } S_z \text{ basis}$$

$$= \begin{pmatrix}1 & 0\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix}1 & 0\end{pmatrix}\underbrace{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}_{\mathbb{1}}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar^2}{4}$$

Thus, $\langle S_x^2\rangle - \langle S_x\rangle^2$

$$= \frac{\hbar^2}{4} - (0)^2 = \frac{\hbar^2}{4} \checkmark$$

yes!

4.) $H = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}(|1\rangle\langle 2| + |2\rangle\langle 1|)$

Expressing H as a matrix in the 1,2 basis $\Rightarrow$

$$H \doteq \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}$$

To Find eigenvalues, $H|\psi\rangle = \lambda|\psi\rangle$, require $\begin{vmatrix} H_{11}-\lambda & H_{12} \\ H_{12} & H_{22}-\lambda \end{vmatrix} = 0$

for non-trivial solutions

The characteristic equation gives

$$\lambda^2 - (H_{11}+H_{22})\lambda + (H_{11}H_{22} - H_{12}^2) = 0$$

Plugging this in to the quadratic formula yields 2 eigenvalues

$$\lambda_{+,-} = \frac{1}{2}\left(H_{11} + H_{22} \pm \sqrt{H_{11}^2 + H_{22}^2 - 2H_{11}H_{22} + 4H_{12}^2}\right)$$

To find the corresponding eigenvectors, solve the two systems

$$H|\psi_{+,-}\rangle = \lambda_{+,-}|\psi_{+,-}\rangle \Rightarrow \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}\begin{pmatrix} \alpha_{1,2} \\ \beta_{1,2} \end{pmatrix} = \lambda_{+,-}\begin{pmatrix} \alpha_{1,2} \\ \beta_{1,2} \end{pmatrix}$$

After some messy algebra, we find (see attached)

$$\lambda_+: \quad |\psi_+\rangle \doteq \begin{pmatrix} \dfrac{-2\cdot H_{12}}{H_{11}-H_{22}-\gamma} \\ 1 \end{pmatrix}$$

where $\gamma = \sqrt{H_{11}^2 + H_{22}^2 - 2H_{11}H_{22} + 4H_{12}^2}$

$$\lambda_-: \quad |\psi_-\rangle \doteq \begin{pmatrix} \dfrac{-2\cdot H_{12}}{H_{11}-H_{22}+\gamma} \\ 1 \end{pmatrix}$$

4.) Messy Algebra

λ_+ :

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{2}\left(H_{11} + H_{22} + \gamma\right)\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where $\gamma = \sqrt{H_{11}^2 + H_{22}^2 - 2H_{11}H_{22} + 4H_{12}^2}$

$$\Rightarrow \begin{pmatrix} H_{11} - \frac{1}{2}(H_{11} + H_{22} + \gamma) & H_{12} \\ H_{12} & H_{22} - \frac{1}{2}(H_{11} + H_{22} + \gamma) \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

eq1. $\frac{1}{2}(H_{11} - H_{22} - \gamma)\alpha + H_{12}\beta = 0 \Rightarrow \beta = \frac{-\frac{1}{2}(H_{11} - H_{22} - \gamma)\alpha}{H_{12}}$

eq2. $H_{12}\alpha + \frac{1}{2}(H_{22} - H_{11} - \gamma)\beta = 0 \Rightarrow$ Plug in for β

$$H_{12}\alpha + \frac{-\alpha}{4H_{12}}\left(\cancel{H_{11}H_{22}} - \cancel{H_{22}^2} - \cancel{H_{22}\gamma} - \cancel{H_{11}^2} + \cancel{H_{11}H_{22}} + \cancel{H_{11}\gamma} - \cancel{H_{11}\gamma} + \cancel{H_{22}\gamma} + \gamma^2\right) = 0$$

$$\alpha\left[H_{12} - \frac{1}{4H_{12}}\left(2H_{11}H_{22} - H_{22}^2 - H_{11}^2 + \gamma^2\right)\right] = 0$$

$$\alpha\left[H_{12} - \frac{1}{4H_{12}}\left(\cancel{2H_{11}H_{22}} - \cancel{H_{22}^2} - \cancel{H_{11}^2} + \cancel{H_{11}^2} + \cancel{H_{22}^2} - \cancel{2H_{11}H_{22}} + 4H_{12}^2\right)\right] = 0$$

$\alpha\left[H_{12} - H_{12}\right] = 0 \Rightarrow$ Redundancy in solution.

Choose $\alpha = \frac{-2H_{12}}{H_{11} - H_{22} - \gamma}$, Then $\beta = 1$

For λ_-, it is simple to see $\alpha = \frac{-2H_{12}}{H_{11} - H_{22} + \gamma}$, $\beta = 1$

5.) We can construct a transformation matrix which connects the natural S_z basis to the natural S_x basis as follows

$$U_{z\to x} = \sum_k |x^{(k)}\rangle\langle z^{(k)}|$$

In the S_z basis, the matrix elements of $U_{z\to x}$ are represented as

$\langle z^{(k)}|U_{z\to x}|z^{(\ell)}\rangle$, which, by our construction of $U_{z\to x}$

is equivalent to $\langle z^{(k)}|x^{(\ell)}\rangle$

So. $$U_{z\to x} \doteq \begin{pmatrix} \langle +z|+x\rangle & \langle +z|-x\rangle \\ \langle -z|+x\rangle & \langle -z|-x\rangle \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} (1\ 0)\binom{1}{1} & (1\ 0)\binom{1}{-1} \\ (0\ 1)\binom{1}{1} & (0\ 1)\binom{1}{-1} \end{pmatrix} = \underline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}$$

Check unitarity:

$$U^\dagger U = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \mathbb{1} \checkmark$$

$\checkmark$

6.)

Initially, when α is measured, the eigenvalue a_1 is obtained. Therefore, we know the initial state to be $|\phi_1\rangle$ since $A|\phi_1\rangle = a_1|\phi_1\rangle$.

Now B is measured, collapsing the state into either $|\chi_1\rangle$ or $|\chi_2\rangle$ with probabilities given by:

$$P(|\chi_1\rangle) = |\langle\chi_1|\phi_1\rangle|^2 = \left|\langle\chi_1|\left(\frac{2|\chi_1\rangle + 3|\chi_2\rangle}{\sqrt{13}}\right)\right|^2 = \frac{4}{13}$$

$$P(|\chi_2\rangle) = |\langle\chi_2|\phi_1\rangle|^2 = \left|\langle\chi_2|\left(\frac{2|\chi_1\rangle + 3|\chi_2\rangle}{\sqrt{13}}\right)\right|^2 = \frac{9}{13}$$

Since $|\chi\rangle$ eigenstates are orthonormal.

If α is measured again, the probability of observing a_1 a second time is the probability of the state being in either $|\chi_1\rangle$ or $|\chi_2\rangle$ times the probability of that state collapsing into $|\phi_1\rangle$:

$$P(a_1 \text{ again}) = |\langle\phi_1|\chi_1\rangle|^2 \cdot \frac{4}{13} + |\langle\phi_1|\chi_2\rangle|^2 \cdot \frac{9}{13}$$

$$= \left(\frac{4}{13}\right)^2 + \left(\frac{9}{13}\right)^2 = \frac{97}{169} \simeq \underline{57\%}$$

ok!