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Quantum HW #2
Due: 1/23/19

$10/10$ 8

1.) For the Gaussian wave packet $\langle x'|\alpha\rangle = \frac{1}{\pi^{1/4}\sqrt{d}} \exp\left[ikx' - \frac{x'^2}{2d^2}\right]$,

$\int_{-\infty}^{\infty} \psi^* \hat{p} \psi \, dx \Rightarrow$

$$\langle p\rangle = \int_{-\infty}^{\infty} \frac{1}{\pi^{1/2} d} \exp\left[-ikx' - \frac{x'^2}{2d^2}\right] \left(-i\hbar \frac{\partial}{\partial x'}\right) \exp\left[ikx' - \frac{x'^2}{2d^2}\right] dx'$$

$$= \frac{1}{\pi^{1/2} d} \int_{-\infty}^{\infty} \exp\left[\frac{-x'^2}{2d^2}\right] (-i\hbar)\left(ik - \frac{x'}{d^2}\right) dx'$$

$$= \frac{1}{\pi^{1/2} d}\left[\int_{-\infty}^{\infty} \hbar k \exp\left[\frac{-x'^2}{2d^2}\right] dx' + i\hbar/d^2 \int_{-\infty}^{\infty} x' \exp\left[\frac{-x'^2}{2d^2}\right] dx'\right]$$

Use $\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \left(\frac{\pi}{\alpha}\right)^{1/2}$ with $\alpha = 1/d^2$ for the first integral. For the second, use the fact that an odd function integrated symmetrically about 0 gives 0.

$$\langle p\rangle = \frac{1}{\pi^{1/2} d}\left[\hbar k (\pi d^2)^{1/2} + 0\right] = \boxed{\hbar k}$$

$$\langle p^2\rangle = \int_{-\infty}^{\infty} \frac{1}{\pi^{1/2} d} \exp\left[-ikx' - \frac{x'^2}{2d^2}\right]\left(-i\hbar \frac{\partial}{\partial x'}\right)\left(-i\hbar \frac{\partial}{\partial x'}\right) \exp\left[ikx' - \frac{x'^2}{2d^2}\right] dx'$$

$$= \frac{1}{\pi^{1/2} d} \int_{-\infty}^{\infty} \exp\left[\frac{-x'^2}{2d^2}\right] (-\hbar^2)\left(ik - \frac{x'}{d^2}\right)^2 dx'$$

$$= \frac{1}{\pi^{1/2} d} \int_{-\infty}^{\infty} \exp\left[\frac{-x'^2}{2d^2}\right] \left(\hbar^2 k^2 + 2i\hbar^2 k \frac{x'}{d^2} - \frac{\hbar^2}{d^4} x'^2\right) dx'$$

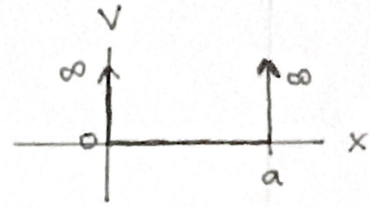
The first term is the same form as the first integral above and yields $\hbar^2 k^2$. The second term is of the same form as the second integral above and yields 0.
The third term is evaluated as follows:

$$\frac{\hbar^2}{\pi^{1/2} d^5} \int_{-\infty}^{\infty} \exp\left[\frac{-x'^2}{2d^2}\right] x'^2 dx' \Rightarrow \text{use } \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{\pi^{1/2}}{2\alpha^{3/2}}$$

$\alpha = 1/d^2$

$$\langle p^2\rangle = \boxed{\hbar^2 k^2 + \frac{\hbar^2}{2d^2}}$$

ok!



2.)

For a particle in a box, we know the wavefunction to be $\Psi = \sqrt{2/a}\,\sin\left(\frac{n\pi x}{a}\right)$ for $n = 1, 2, 3 \ldots$ for $0 < x < a$ and 0 elsewhere.

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^* \hat{x} \Psi \, dx = \frac{2}{a}\int_0^a x \sin^2\left(\frac{n\pi x}{a}\right) dx \Rightarrow \sin^2(x) = \frac{1-\cos(2x)}{2} \Rightarrow$$

$$= \frac{2}{a}\int_0^a \frac{x}{2} - \frac{x\cos\left(\frac{2n\pi x}{a}\right)}{2}\, dx \Rightarrow \text{integrate by parts}$$

$$(i) = \int_0^a x\cos\left(\frac{2n\pi x}{a}\right) dx \Rightarrow \quad u = x \quad v = \frac{a}{2n\pi}\sin\left(\frac{2n\pi x}{a}\right)$$

$$du = dx \quad dv = \cos\left(\frac{2n\pi x}{a}\right) dx$$

$$(i) = \frac{xa}{2n\pi}\cancelto{0}{\sin\left(\frac{2n\pi x}{a}\right)}\Big|_0^a - \int_0^a \frac{a}{2n\pi}\sin\left(\frac{2n\pi x}{a}\right) dx$$

$$= \frac{a^2}{4n^2\pi^2}\cancelto{0}{\cos\left(\frac{2n\pi x}{a}\right)}\Big|_0^a = 0$$

$$\text{Thus,} \quad \langle x \rangle = \frac{2}{a}\int_0^a \frac{x}{2}\, dx = \frac{2}{a}\left(\frac{x^2}{4}\right)\Big|_0^a = \boxed{\frac{a}{2}}$$

$$\langle x^2 \rangle = \frac{2}{a}\int_0^a x^2 \sin^2\left(\frac{n\pi x}{a}\right) dx \Rightarrow \text{Similarly,}$$

$$= \frac{2}{a}\int_0^a \frac{x^2}{2} - \frac{x^2\cos\left(\frac{2n\pi x}{a}\right)}{2}\, dx \Rightarrow \text{integrate by parts}$$

$$(ii) = \int_0^a x^2\cos\left(\frac{2n\pi x}{a}\right) dx \Rightarrow \quad u = x^2 \quad v = \frac{a}{2n\pi}\sin\left(\frac{2n\pi x}{a}\right)$$

$$du = 2x\,dx \quad dv = \cos\left(\frac{2n\pi x}{a}\right)$$

$$(ii) = \frac{x^2 a}{2n\pi}\cancelto{0}{\sin\left(\frac{2n\pi x}{a}\right)}\Big|_0^a - \int_0^a \frac{2a}{2n\pi}\sin\left(\frac{2n\pi x}{a}\right) x\, dx \Rightarrow \text{integrate by parts}$$

$$(iii) = \int_0^a x\sin\left(\frac{2n\pi x}{a}\right) \Rightarrow \quad u = x \quad v = \frac{-a}{2n\pi}\cos\left(\frac{2n\pi x}{a}\right)$$

$$du = dx \quad dv = \sin\left(\frac{2n\pi x}{a}\right)$$

$$(iii) = \frac{-ax}{2n\pi}\cos\left(\frac{2n\pi x}{a}\right)\Big|_0^a + \int_0^a \frac{a}{2n\pi}\cos\left(\frac{2n\pi x}{a}\right) dx$$

$$= \frac{-a^2}{2n\pi} + \frac{a^2}{2n^2\pi^2}\cancelto{0}{\sin\left(\frac{2n\pi x}{a}\right)}\Big|_0^a = \frac{-a^2}{2n\pi}$$

ok!

$$\text{Thus} \quad \langle x^2 \rangle = \frac{2}{a}\left[\int_0^a \frac{x^2}{2}\, dx - \frac{a^3}{4n^2\pi^2}\right] = \boxed{a^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right)}$$

Cont ⇒

$$\langle P\rangle = \frac{2}{a}\int_{-\infty}^{\infty} \sin\left(\frac{n\pi x}{a}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)\sin\left(\frac{n\pi x}{a}\right)dx$$

$$= \frac{2}{a}\int_0^a \frac{-i\hbar}{a} n\pi \sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{n\pi x}{a}\right)dx. \quad \text{let } u = \sin\left(\frac{n\pi x}{a}\right)$$

$$du = \frac{n\pi}{a}\cos\left(\frac{n\pi x}{a}\right)dx$$

$$\langle P\rangle = \frac{-2i\hbar}{a}\int u\,du = \frac{-i\hbar}{a}\sin^2\left(\frac{n\pi x}{a}\right)\Big|_0^a = \boxed{0}$$

$$\langle P^2\rangle = \frac{2}{a}\int_0^a \frac{+\hbar^2}{a^2} n^2\pi^2 \sin^2\left(\frac{n\pi x}{a}\right)dx$$

$$= \frac{2(n\pi\hbar)^2}{a^3}\int_0^a \frac{1}{2} - \frac{\cos(2n\pi x/a)}{2}\,dx = \frac{(n\pi\hbar)^2}{a^3}\left(x - \frac{a}{2n\pi}\cancelto{0}{\sin\left(\frac{2n\pi x}{a}\right)}\right)\Big|_0^a$$

$$\langle P^2\rangle = \boxed{\frac{n^2\pi^2\hbar^2}{a^2}}$$

$$\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 = a^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right) - \frac{a^2}{4}$$

$$= a^2\left(\frac{4n^2\pi^2 - 6 - 3n^2\pi^2}{12n^2\pi^2}\right) = a^2\left(\frac{1}{12} - \frac{1}{2n^2\pi^2}\right)$$

$$\langle(\Delta P)^2\rangle = \langle P^2\rangle - \langle P\rangle^2 = \frac{n^2\pi^2\hbar^2}{a^2}$$

$$\langle(\Delta x)^2\rangle\langle(\Delta P)^2\rangle = \frac{\cancel{a^2}n^2\pi^2\hbar^2}{\cancel{a^2}}\left(\frac{1}{12} - \frac{1}{2n^2\pi^2}\right)$$

$$= \boxed{\frac{n^2\pi^2\hbar^2}{12} - \frac{\hbar^2}{2}}$$

By the Heisenberg uncertainty relation, this quantity must be greater than or equal to $\hbar^2/4$.

$$\frac{n^2\pi^2\hbar^2}{12} - \frac{\hbar^2}{2} \geq \hbar^2/4 \Rightarrow \frac{\hbar^2\pi^2}{12}n^2 \geq \frac{3\hbar^2}{4}$$

Which is true for $n \geq 1$!

3.)

For $\Psi(x) = \begin{cases} 1/\sqrt{2a}, & |x|<a \\ 0, & \text{elsewhere} \end{cases}$

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^* \hat{x} \Psi \, dx$$

$$= \int_{-a}^{a} \frac{1}{2a} x \, dx = \frac{1}{2a}\left(\frac{x^2}{2}\right)\Big|_{-a}^{a} = \boxed{0}$$

$$\langle x^2 \rangle = \int_{-a}^{a} \frac{1}{2a} x^2 dx = \frac{1}{2a}\left(\frac{x^3}{3}\right)\Big|_{-a}^{a} = \boxed{\frac{a^2}{3}}$$

For $\langle p \rangle$, $\langle p^2 \rangle$, the derivative of Ψ w.r.t. x must be dealt with at $x = \pm a$, so write $\Psi = \frac{1}{\sqrt{2a}}\left(\Theta(x+a) - \Theta(x-a)\right)$

$$\langle p \rangle = \int_{-\infty}^{\infty} \frac{1}{2a}\left[\left(\Theta(x+a) - \Theta(x-a)\right)\left(-i\hbar \frac{\partial}{\partial x}\right)\left(\Theta(x+a) - \Theta(x-a)\right)\right] dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{2a}\left[\left(\Theta(x+a) - \Theta(x-a)\right)(-i\hbar)\left(\delta(x+a) - \delta(x-a)\right)\right] dx$$

$= \boxed{0}$ for non-zero a

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \frac{1}{2a}\left[\left(\Theta(x+a) - \Theta(x-a)\right)\left(-\hbar^2 \frac{\partial}{\partial x}\right)\left(\delta(x+a) - \delta(x-a)\right)\right] dx$$

but $\frac{\partial}{\partial x}\delta(x)$ is divergent so $\langle p^2 \rangle$ is as well.

$\langle (\Delta x)^2 \rangle \langle (\Delta p)^2 \rangle$ is divergent. ok!