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Quantum HW #7
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1.)

(a) $\vec{J}^2 = J_z^2 + J_x^2 + J_y^2$

$J_+ J_- = J_x^2 - iJ_xJ_y + iJ_yJ_x + J_y^2$

$= J_x^2 + J_y^2 + i[J_y, J_x] = J_x^2 + J_y^2 + i(-i\hbar)J_z$

Thus, $\underline{\vec{J}^2 = J_z^2 + J_+J_- - \hbar J_z}$ ✓

(b) Act on a state first with J_- and then with J_z

$J_z(J_-|\psi_{j,m}\rangle) = J_z(J_x - iJ_y)|\psi_{j,m}\rangle$

$= ([J_z, J_x] + J_xJ_z - i([J_z, J_y] + J_yJ_z))|\psi_{j,m}\rangle$

$= (i\hbar J_y + J_x m\hbar - \hbar J_x + J_y m\hbar)|\psi_{j,m}\rangle$

$= J_x(m\hbar - \hbar) - iJ_y(m\hbar - \hbar)|\psi_{j,m}\rangle$

$= \underbrace{\hbar(m-1)}_{J_z \text{ eigenvalue}}(J_-|\psi_{j,m}\rangle)$

Thus, J_- acting on $|\psi_{j,m}\rangle$ has the result of lowering m by 1.

$J_-|\psi_{j,m}\rangle = C_-|\psi_{j,m-1}\rangle$ ✓

2.)

i) $\langle L_x \rangle = \langle l,m | L_x | l,m \rangle$

Some constant

$= \frac{1}{2} \langle l,m | (L_+ + L_-) | l,m \rangle = \alpha \langle l,m | (|l,m+1\rangle + |l,m-1\rangle)$

Since states are orthogonal, $\underline{\langle L_x \rangle = 0}$! Similarly for $\langle L_y \rangle$

✓

ii) $\langle L_x^2 \rangle = \langle l,m | L_x^2 | l,m \rangle$

$= \frac{1}{4} \langle l,m | (\cancel{L_+^2} + L_+L_- + L_-L_+ + \cancel{L_-^2}) | l,m \rangle$

First and last terms die by orthogonality.

$= \frac{1}{4} \langle l,m | (C_{+,m-1} C_{-m} + C_{-,m+1} C_{+m}) | l,m \rangle$

$= \frac{1}{4} \left(\hbar^2 (l(l+1) - m(m-1))^{1/2} \cdot (l(l+1) - (m-1)(m-1+1))^{1/2} \right.$
$\left. + \hbar^2 (l(l+1) - m(m+1))^{1/2} \cdot (l(l+1) - (m+1)(m+1-1))^{1/2} \right)$

$= \frac{\hbar^2}{4} \left[\left((l(l+1))^2 - 2m(m-1)l(l+1) + m^2(m-1)^2 \right)^{1/2} \right.$
$\left. + \left((l(l+1))^2 - 2m(m+1)l(l+1) + m^2(m+1)^2 \right)^{1/2} \right]$

$= \frac{\hbar^2}{4} \left(l(l+1) + m(m-1) + l(l+1) + m(m+1) \right)$

ok!

$= \underline{\frac{\hbar^2}{2} \left(l(l+1) - m^2 \right)}$ ✓ Similarly for $\langle L_y^2 \rangle$

Imagine a precessing spinning top. The average L_x, L_y would be 0, but average L_x^2 and L_y^2 would be non-zero.

✓

3.)

(a)

$$\Psi(\vec{x}) = (x+y+3z)F(r)$$ } This looks a lot like one of the EM quals!

Rewrite $\Psi(\vec{x})$ in terms of spherical harmonics:

$Y_1^0 = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$, $Y_1^1 = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{i\phi}$, $Y_1^{-1} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{-i\phi}$

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

So $$\Psi(\vec{x}) = \left[r\sqrt{\frac{2\pi}{3}}(Y_1^{-1} - Y_1^1) + ir\sqrt{\frac{2\pi}{3}}(Y_1^{-1} + Y_1^1) + r\sqrt{\frac{4\pi}{3}}Y_1^0\right]F(r)$$

All the spherical harmonics have $l = 1$

So $\vec{L}^2$ acting on $\Psi(\vec{x})$ yields a $\underline{2\hbar^2}$ $\underline{\text{eigenvalue}}$

(b)

YUP!

The possibilities can be read easily from $\Psi(\vec{x})$

$\underline{l = 1 \text{ always}}$, $\underline{m = 0, \pm 1}$

(c) We can find $V(r)$ by looking at the total energy and the kinetic terms in the Hamiltonian.

$$V(r)\Psi = E\Psi - \left(\nabla_r^2 - \frac{L^2}{\hbar^2 r^2}\right)\Psi \cdot \left(\frac{-\hbar^2}{2m}\right)$$

$$= E\Psi - \left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\Psi}{\partial r} - \frac{2}{r^2}\Psi\right)\left(\frac{-\hbar^2}{2m}\right)$$

angular terms

$$V\cancel{Y}F(r) = E\cancel{Y}F(r) - \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3F'(r) + r^2F(r) - \frac{2}{r}F(r)\right]Y\,Z\left(\frac{-\hbar^2}{2m}\right)$$

$$\boxed{V = E + \frac{\hbar^2}{2mF(r)}\left[\frac{4F'(r)}{r} + F''(r)\right]}$$

ok!

4.) $H = -\vec{\mu}\cdot\vec{B} = \hbar\omega\sigma_z$, $\omega = \frac{eB}{2mc}$

$|\Psi(t=0)\rangle = |+y\rangle \overset{S_z}{\Rightarrow} \frac{1}{\sqrt{2}}\left(|+\rangle + i|-\rangle\right)$

The Hamiltonian will cause the state to precess. Since it is time independent, we can use $U = e^{-i\hbar\omega\sigma_z/\hbar\, t} = e^{-i\omega\sigma_z t}$

$|\Psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\omega\sigma_z t}\left(|+\rangle + i|-\rangle\right) = \underline{\frac{1}{\sqrt{2}}\left[e^{-2i\omega t}|+\rangle + i e^{2i\omega t}|-\rangle\right]}$

So $\langle S_x\rangle = \langle\Psi(t)|S_x|\Psi(t)\rangle$, $S_x \overset{S_z}{\rightarrow} \frac{\hbar}{2}\left(|+\rangle\langle -| + |-\rangle\langle +|\right)$

$\langle S_x\rangle = \frac{\hbar}{4}\left[i e^{2i\omega t}\langle +|+\rangle - e^{2i\omega t}\langle -|-\rangle\right]$

$= \underline{-\frac{\hbar}{2}\sin(2\omega t)}$

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PERFECT!!