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Quantum HW #3
Due: 1/30/19

Perfect!
10/10

1.) $|\Psi(0)\rangle = |\phi_1\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle + |u_2\rangle)$, $|\phi_2\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle - |u_2\rangle)$

Where $|u_1\rangle, |u_2\rangle$ are eigenstates of the Hamiltonian with eigenvalues E_1, E_2 and $|\phi_1\rangle, |\phi_2\rangle$ are eigenstates of observable A with eigenvalues a_1, a_2.

$$\langle A\rangle(t) = \langle\Psi(t)|A|\Psi(t)\rangle$$

$$|\Psi(t)\rangle = U(t,0)|\Psi(0)\rangle = e^{-i\hat{H}t/\hbar}|\Psi(0)\rangle$$

$$\Rightarrow \langle A\rangle(t) = \frac{1}{\sqrt{2}}\left(e^{iE_1t/\hbar}\langle u_1| + e^{iE_2t/\hbar}\langle u_2|\right)(A)\left(\frac{1}{\sqrt{2}}\left(e^{-iE_1t/\hbar}|u_1\rangle + e^{-iE_2t/\hbar}|u_2\rangle\right)\right)$$

$\Rightarrow$ rewrite in A eigenbasis $\Rightarrow$ $|u_1\rangle = \frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_2\rangle)$

$|u_2\rangle = \frac{1}{\sqrt{2}}(|\phi_1\rangle - |\phi_2\rangle)$

$$\langle A\rangle(t) = \frac{1}{2}\Big[\Big(e^{iE_1t/\hbar}\frac{1}{\sqrt{2}}(\langle\phi_1| + \langle\phi_2|) + e^{iE_2t/\hbar}\frac{1}{\sqrt{2}}(\langle\phi_1| - \langle\phi_2|)\Big)$$
$$(A)\Big(e^{-iE_1t/\hbar}\frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_2\rangle) + e^{-iE_2t/\hbar}\frac{1}{\sqrt{2}}(|\phi_1\rangle - |\phi_2\rangle)\Big)\Big]$$

$$= \frac{1}{4}\Big[a_1 + e^{i(E_1-E_2)t/\hbar}a_1 + a_2 - e^{i(E_1-E_2)t/\hbar}a_2 + e^{i(E_2-E_1)t/\hbar}a_1 + a_1 \ldots$$
$$- e^{i(E_2-E_1)t/\hbar}a_2 + a_2\Big]$$

$$= \frac{1}{4}\left[2a_1 + 2a_2 + 2(a_1 - a_2)\cos((E_1-E_2)t/\hbar)\right]$$

$$= \boxed{\frac{a_1+a_2}{2} + \frac{a_1-a_2}{2}\cos\left(\frac{(E_1-E_2)t}{\hbar}\right)}$$

YEAH!

2.) $\Psi(x,0) = c\, e^{-x^2/4\Delta_0^2}$

In free space, $\hat{H} = \frac{\hat{p}^2}{2m}$ and is time independent.

Thus, $U(t) = e^{-i\hat{H}(t-t_0)/\hbar} = e^{-i\hat{p}^2 t/(2m\hbar)}$

Let's transform our wavefunction into momentum space:

$$\phi_p(p,0) = \frac{c}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{\left(\frac{-x^2}{4\Delta_0^2} - \frac{ipx}{\hbar}\right)} dx \Rightarrow \text{complete the } \square$$

$$x^2 + \frac{4ipx\Delta_0^2}{\hbar} = \left(x - \frac{2ipx\Delta_0^2}{\hbar}\right)^2 + \frac{4\Delta_0^4 p^2}{\hbar^2} \Rightarrow$$

$$\phi_p(p,0) = \frac{c}{\sqrt{2\pi\hbar}} e^{-\frac{\Delta_0^2 p^2}{\hbar^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{4\Delta_0^2}\left(x - \frac{2ipx\Delta_0^2}{\hbar}\right)^2} dx$$

use $\int_{-\infty}^{\infty} e^{-\alpha(x+\beta)^2} dx = \sqrt{\frac{\pi}{\alpha}} \Rightarrow$

$$\phi_p(p,0) = \frac{c}{\sqrt{2\pi\hbar}} \sqrt{4\Delta_0^2\pi}\, e^{-\Delta_0^2 p^2/\hbar^2}$$

Now act with $U(t, t'=0)$ on $\phi_p(x, t'=0)$

$$\phi(p,t) = e^{-ip^2 t/(2m\hbar)} \cdot e^{-\Delta_0^2 p^2/\hbar^2} \cdot \frac{\sqrt{2}\,\Delta_0 c}{\sqrt{\hbar}} = \frac{\sqrt{2}\,\Delta_0 c}{\hbar} e^{-p^2\left(\frac{it}{2m\hbar} + \frac{\Delta_0^2}{\hbar}\right)}$$

$$|\phi(p,t)|^2 = \frac{2\Delta_0^2 c^2}{\hbar^2} e^{-p^2\left(\cancel{\frac{-it}{2m\hbar}} + \frac{\Delta_0^2}{\hbar^2} + \cancel{\frac{it}{2m\hbar}} + \frac{\Delta_0^2}{\hbar^2}\right)} = \frac{2\Delta_0^2 c^2}{\hbar^2} e^{-p^2\left(\frac{2\Delta_0^2}{\hbar^2}\right)}$$

This is a gaussian wave packet (surprise!) with $(\Delta p)^2 = \frac{\hbar^2}{4\Delta_0^2}$, $\Delta p = \frac{\hbar}{2\Delta_0}$

Returning to position space,

$$\Psi(x,t) = \frac{\sqrt{2}\,\Delta_0 c}{\hbar\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-p^2(it/(2m\hbar) + \Delta_0^2/\hbar^2)} e^{ipx/\hbar} dp \Rightarrow \text{complete the } \square$$

again (ew)

$$\Psi(x,t) = \frac{\Delta_0 c}{\sqrt{\hbar^3\pi}} e^{-\left(\frac{x^2}{4\hbar^2}\left(\frac{it}{2m\hbar} + \frac{\Delta_0^2}{\hbar^2}\right)^{-1}\right)} \int_{-\infty}^{\infty} e^{-\left(\frac{it}{2m\hbar} + \frac{\Delta_0^2}{\hbar^2}\right)\left(p - \frac{ix}{2\hbar}\left(\frac{it}{2m\hbar} + \frac{\Delta_0^2}{\hbar^2}\right)^{-1}\right)^2} dp$$

✓

over $\Rightarrow$

$$\Psi(x,t) = \frac{\Delta_0 C}{\sqrt{\hbar^3 \pi}}\, e^{-\left(\frac{x^2}{4\hbar^2}\left(\frac{it}{2m\hbar} + \frac{\Delta_0^2}{\hbar^2}\right)^{-1}\right)} \cdot \sqrt{\pi \left(\frac{it}{2m\hbar} + \frac{\Delta_0^2}{\hbar^2}\right)^{-1}}$$

$$|\Psi(x,t)|^2 = \frac{\Delta_0^2 C^2}{\hbar^3 \cancel{\pi}}\, e^{-\left(\frac{x^2}{4\hbar^2}\gamma^* + \frac{x^2}{4\hbar^2}\gamma\right)} \cdot (\cancel{\pi}\gamma^*)^{+1/2}(\cancel{\pi}\gamma)^{1/2} \Big\} \quad \gamma = \left(\frac{it}{2m\hbar} + \frac{\Delta_0^2}{\hbar^2}\right)^{-1}$$

$$= \frac{\Delta_0^2 C^2}{\hbar^3}\, e^{-\left[\frac{x^2}{4\hbar^2}(\gamma^* + \gamma)\right]} (|\gamma|^2)^{1/2}, \quad \text{So } (\Delta x(t))^2 \equiv \Delta_t^2 = \frac{2\hbar^2}{(\gamma^* + \gamma)} \Rightarrow$$

$$\gamma = \left(\frac{it}{2m\hbar} + \frac{\Delta_0^2}{\hbar^2}\right)^{-1} = \left(\frac{it\hbar + 2m\Delta_0^2}{2m\hbar^2}\right)^{-1} = \frac{2m\hbar^2}{it\hbar + 2m\Delta_0^2} \Rightarrow$$

$$\gamma^* + \gamma = \frac{2m\hbar^2}{-it\hbar + 2m\Delta_0^2} + \frac{2m\hbar^2}{it\hbar + 2m\Delta_0^2} = \frac{2m\hbar^2\left[(i\cancel{t\hbar} + 2m\Delta_0^2) + (-i\cancel{\hbar}t + 2m\Delta_0^2)\right]}{4m^2\Delta_0^4 + \hbar^2 t^2}$$

$$= \frac{8m^2\hbar^2\Delta_0^2}{4m^2\Delta_0^4 + \hbar^2 t^2} \Rightarrow \Delta_t^2 = \frac{2\hbar^2}{(\gamma^* + \gamma)} = \frac{8m^2\Delta_0^4\hbar^2 + 2\hbar^4 t^2}{8m^2\hbar^2\Delta_0^2} = \Delta_0^2 + \frac{\hbar^2 t^2}{4m^2\Delta_0^2}$$

But $(\Delta P)^2 = \frac{\hbar^2}{4\Delta_0^2} \Rightarrow \Delta_t^2 = \Delta_0^2 + \frac{t^2(\Delta P)^2}{m^2}$

Using $\Delta P = m\Delta V \Rightarrow$

$$\Delta_t^2 = \Delta_0^2 + t^2(\Delta V)^2 \checkmark$$

✓ Bravo!

3.)

(a) $H\Psi = i\hbar \frac{d\Psi}{dt}$

For $H(t) = \omega_0(t) S_z$, $\omega_0(t) = \begin{cases} 0, & t<0 \\ \omega_0 t/T, & 0\le t\le T \\ 0, & t>T \end{cases}$

$\Rightarrow \omega_0(t) S_z |\Psi(t)\rangle = i\hbar \frac{d|\Psi(t)\rangle}{dt}$

$\underset{S_z \text{ basis}}{\Rightarrow} \frac{1}{2}\hbar\omega_0(t)\sigma_z \begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix} = i\hbar \frac{d}{dt}\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix}$

(b) H is time dependent, but $[H(t_1), H(t_2)] = 0$

So $U(t) = \exp\left[-\frac{i}{\hbar}\int_0^t H(t')dt'\right]$.

$\int_0^t H(t')dt' = \int_0^t \omega_0(t') S_z dt' = S_z \int_0^t \frac{\omega_0}{T} t' dt' = \begin{cases} \frac{\omega_0}{T}\frac{t^2}{2} S_z, & 0\le t\le T \\ \frac{\omega_0 T}{2} S_z, & t>T \end{cases}$

$|\Psi(0)\rangle = |+y\rangle \overset{S_z}{\Rightarrow} \frac{1}{\sqrt{2}}(|+\rangle + i|-\rangle)$

So $|\Psi(t)\rangle = \begin{cases} \frac{1}{\sqrt{2}}\left(\exp\left[\frac{-i\omega_0 t^2}{4T}\right]|+\rangle + i\exp\left[\frac{i\omega_0 t^2}{4T}\right]|-\rangle\right), & 0\le t\le T \\ \frac{1}{\sqrt{2}}\left(\exp\left[\frac{-i\omega_0 T}{4}\right]|+\rangle + i\exp\left[\frac{i\omega_0 T}{4}\right]|-\rangle\right), & t>T \end{cases}$

$\alpha = \frac{1}{\sqrt{2}}$

$\beta = \frac{i}{\sqrt{2}}$, $\theta(t) = \begin{cases} -\frac{\omega_0 t^2}{4T}, & 0\le t\le T \\ -\frac{\omega_0 T}{4}, & t>T \end{cases}$

$\Rightarrow$

(c) The possible results of measuring S_y at time $t > T$ are $+\hbar/2$ and $-\hbar/2$. To find the probability of each, project the $+,-$ y states onto $\Psi(t) \Rightarrow$

$$P(+\tfrac{\hbar}{2}) = |\langle +y|\Psi(t)\rangle|^2 \xRightarrow{S_z \text{ basis}} \left| \frac{1}{\sqrt{2}}(1 \;\; -i)\frac{1}{\sqrt{2}} \begin{pmatrix} \exp[\frac{-i\omega_0 T}{4}] \\ i\exp[\frac{i\omega_0 T}{4}] \end{pmatrix} \right|^2$$

$$= \left| \frac{1}{2}\left(e^{-i\omega_0 T/4} + e^{i\omega_0 T/4} \right) \right|^2 = \cos^2\left(\frac{\omega_0 T}{4}\right)$$

$$P(-\tfrac{\hbar}{2}) = |\langle -y|\Psi(t)\rangle|^2 \xRightarrow{S_z} \left| \frac{1}{\sqrt{2}}(1 \;\; i)\frac{1}{\sqrt{2}} \begin{pmatrix} \exp[\frac{-i\omega_0 T}{4}] \\ i\exp[\frac{i\omega_0 T}{4}] \end{pmatrix} \right|^2$$

$$= \left| \frac{1}{2}\left(e^{-i\omega_0 T/4} - e^{i\omega_0 T/4} \right) \right|^2 = \sin^2\left(\frac{\omega_0 T}{4}\right)$$

For a unique result, $\Psi(t)$ must be a pure state,

So $\underbrace{\frac{\omega_0 T}{4} = \frac{\pi}{2} n}_{\text{either sin or cos term yields zero}}$. $\omega_0 T = 2\pi n$, $n = 0, \pm 1, \pm 2 \ldots$

oh!