Nolan Smyth
Quantum HW #4
Due: 2/6/19
Profumo

1.) $\Psi(x,0) = A e^{-x^2/a^2}$

(a) Find A by normalizing Ψ =>

$$\int_{-\infty}^{\infty} \Psi^* \Psi dx = 1 = |A|^2 \int_{-\infty}^{\infty} e^{-2x^2/a^2} dx$$

Using $\int_{-\infty}^{\infty} e^{-x^2\alpha} dx = \sqrt{\frac{\pi}{\alpha}}$,

$$|A|^2 \cdot \sqrt{\frac{\pi a^2}{2}} = 1 \Rightarrow A = \left(\frac{2}{\pi}\right)^{1/4} a^{-1/2}$$

Now find $\phi(p,0)$ from Fourier:

$$\phi(p,0) = \left(\frac{2}{\pi}\right)^{1/4} \cdot a^{-1/2} \cdot \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-x^2/a^2} e^{-ipx/\hbar} dx \Rightarrow \text{complete}^2$$

$$x^2 + \frac{ipxa^2}{\hbar} = \left(x + \frac{ipa^2}{2\hbar}\right)^2 + \frac{p^2a^4}{4\hbar^2} \Rightarrow$$

$$\phi(p,0) = \left(\frac{2}{\pi}\right)^{1/4} \frac{a^{-1/2}}{\sqrt{2\pi\hbar}} e^{-\frac{p^2a^2}{4\hbar^2}} \int_{-\infty}^{\infty} e^{-\left(x + \frac{ipa^2}{2\hbar}\right)^2/a^2} dx$$

Using $\int_{-\infty}^{\infty} e^{-\alpha(x+\beta)^2} = \sqrt{\frac{\pi}{\alpha}}$,

$$\phi(p,0) = \left(\frac{2}{\pi}\right)^{1/4} \frac{a^{-1/2}}{\sqrt{2\pi\hbar}} e^{-\frac{p^2a^2}{4\hbar^2}} \cdot \sqrt{\pi a^2}$$

$$= \boxed{\left(\frac{a}{\hbar}\right)^{1/2} \left(\frac{1}{2\pi}\right)^{1/4} e^{-p^2a^2/(4\hbar^2)}}$$

Good!

=>

cont.

1.)

(b) Since $\hat{H}$ has no time dependence, $\hat{U} = e^{-ip^2t/(2m\hbar)}$

$$\phi(p,t) = U|\phi(p,t)\rangle$$

$$= \left(\frac{a}{\hbar}\right)^{1/2}\left(\frac{1}{2\pi}\right)^{1/4} e^{\frac{-ip^2t}{2m\hbar}} e^{\frac{-p^2a^2}{4\hbar^2}} \Rightarrow \text{F.T. back to } \psi(x,t)$$

$$\psi(x,t) = \left(\frac{a}{\hbar}\right)^{1/2}\left(\frac{1}{2\pi}\right)^{1/4}\int_{-\infty}^{\infty} e^{-p^2\left(\frac{it}{2m\hbar}+\frac{a^2}{4\hbar^2}\right)} e^{\frac{+ipx}{\hbar}}\,dp \Rightarrow \text{complete}^2$$

* let $\gamma = \left(\frac{it}{2m\hbar}+\frac{a^2}{4\hbar^2}\right) \Rightarrow p^2 + \frac{ipx}{\hbar\gamma} = \left(p+\frac{ix}{2\hbar\gamma}\right)^2 - \frac{x^2}{4\hbar^2\gamma^2}$

$$\psi(x,t) = \left(\frac{a}{\hbar}\right)^{1/2}\left(\frac{1}{2\pi}\right)^{1/4} e^{\frac{-x^2}{4\hbar^2\gamma}}\int_{-\infty}^{\infty} e^{-\gamma\left(p+\frac{ix}{2\hbar\gamma}\right)^2}\,dp$$

using $\int_{-\infty}^{\infty} e^{-\alpha(x+\beta)^2}\,dx = \sqrt{\frac{\pi}{\alpha}}$,

$$\psi(x,t) = \left(\frac{a}{\hbar}\right)^{1/2}\left(\frac{1}{2\pi}\right)^{1/4} e^{-\frac{x^2}{4\hbar^2\gamma}}\sqrt{\frac{\pi}{\gamma}}$$

$$= \boxed{\left(\frac{a}{\gamma\hbar}\right)^{1/2}\left(\frac{\pi}{2}\right)^{1/4} e^{-x^2/(4\hbar^2\gamma)}}, \quad \gamma = \left(\frac{it}{2m\hbar}+\frac{a^2}{4\hbar^2}\right)$$

✓

Yup!

2.) $|z\rangle \equiv e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$. $|n\rangle$ is E. state of $N = a^\dagger a$

(a)

i) Check normalization:

$$\langle z|z\rangle = e^{-|z|^2} \sum_{m,n=0}^{\infty} \frac{z^{*m} z^n}{\sqrt{n!\,m!}} \overbrace{\langle m|n\rangle}^{N \text{ is hermitian, so this gives } \delta_{m,n}}$$

$$= \underbrace{e^{-|z|^2} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!}}_{= \text{ taylor expansion of } e^{|z|^2}} = e^{-|z|^2} \cdot e^{|z|^2} = \boxed{1}$$

ii)

$$a|z\rangle = \sqrt{n}\, e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n-1\rangle \Rightarrow |-1\rangle \text{ is not a physical state so sum from 1 to } \infty$$

$$= e^{-\frac{|z|^2}{2}} \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{(n-1)!}} |n-1\rangle = z\, e^{-\frac{|z|^2}{2}} \sum_{n=1}^{\infty} \frac{z^{n-1}}{\sqrt{(n-1)!}} |n-1\rangle$$

let $m = n-1$: $a|z\rangle = z\, e^{-|z|^2/2} \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!}} |m\rangle = \boxed{z|z\rangle}$

(b)

$$\langle N\rangle = \langle z|N|z\rangle = \langle z|a^\dagger a|z\rangle$$

$$= |z|^2 \langle z|z\rangle = |z|^2$$

$$\langle N^2\rangle = \langle z|a^\dagger a a^\dagger a|z\rangle = |z|^2 \langle z|a a^\dagger|z\rangle \Rightarrow \text{Commute}$$

$$= |z|^2 \langle z|a^\dagger a + 1|z\rangle = |z|^4 + |z|^2$$

So $(\Delta N)^2 = \langle N^2\rangle - \langle N\rangle^2 = \cancel{|z|^4} + |z|^2 - \cancel{|z|^4}$

$= \boxed{|z|^2}$

$\Rightarrow$

(b) cont.

In the limit $N \to \infty$, $\frac{\Delta N}{N} = \frac{|z|}{|z|^{\cancel{2}}} = \frac{1}{|z|}$

But $\frac{1}{|z|} = \frac{1}{\sqrt{\langle N \rangle}}$ which tends to 0 for $N \to \infty$!

(c).

At $t=0$, state is e. state of a w/ e. val z.

time evolve using $U = e^{-iE_n t/\hbar}$

$$|\psi(t)\rangle = U|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} e^{-iE_n t/\hbar} |n\rangle$$

Where $E_n = (n + \frac{1}{2})\hbar\omega$

So the probability of being in state $|z\rangle$ at time t is

$$|\langle z|\psi(t)\rangle|^2 = \left| e^{-|z|^2} \sum_n \frac{|z|^{2n}}{n!} e^{-i\omega(n+\frac{1}{2})t} \langle n|n\rangle \right|^2$$

$$= \left| e^{-|z|^2} \cdot e^{|z|^2 e^{-i\omega t}} \cdot e^{-i\omega t/2} \right|^2$$

$$= e^{-2|z|^2} e^{|z|^2 (e^{-i\omega t} + e^{i\omega t})}$$

$$= e^{-2|z|^2} e^{2|z|^2 \cos(\omega t)} = \boxed{e^{2|z|^2(\cos(\omega t) - 1)}}$$

JR ✓

10/10