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QM HW #6
Prof. Profumo
Due: 2/27/19

1.) $Z = \int d^3x' \, K(\vec{x}', t; \vec{x}', 0)|_{\beta = \frac{it}{\hbar}}$

$$K = \sum_{a'} \langle x'|a'\rangle\langle a'|x'\rangle e^{-E_{a'}\beta}$$

$$Z = \int d^3x' \sum_{a'} \langle x'|a'\rangle\langle a'|x'\rangle e^{-E_{a'}\beta}$$

$$\frac{\partial Z}{\partial \beta} = \int d^3x' \sum_{a'} \langle x'|a'\rangle\langle a'|x'\rangle e^{-E_{a'}\beta} \cdot (-E_{a'})$$

$$-\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \frac{\int d^3x' \sum_{a'} \langle x'|a'\rangle\langle a'|x'\rangle e^{-E_{a'}\beta} (+E_{a'})}{\int d^3x' \sum_{b'} \langle x'|b'\rangle\langle b'|x'\rangle e^{-E_{b'}\beta}}$$

for $\lim_{\beta\to 0} \frac{-1}{Z}\frac{\partial Z}{\partial \beta}$, the lowest $E_{a'}$ values will dominate the exponential and the brackets just give us $\mathbb{1}$

So $\lim_{\beta\to 0} \frac{-1}{Z}\frac{\partial Z}{\partial \beta} = E_{a'}|_{a'=0} = \boxed{E_0}$ ✓ ✓

For particle in a box, $\langle x|a'\rangle = \psi_x = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, 3 \ldots$

So $Z = \int_{-L}^{L} d^3x' \frac{2}{L} \sum_n \sin^2\left(\frac{n\pi x'}{L}\right) e^{-\beta\left(\frac{n^2\pi^2\hbar^2}{2mL^2}\right)}$ ⇒ Over full periods $\sin^2$ integral gives $L/2$

$$Z = e^{-\beta\left(\frac{n^2\pi^2\hbar^2}{2mL^2}\right)}$$

$$\lim_{\beta\to 0} -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \boxed{\frac{\pi^2\hbar^2}{2mL^2} = E_0}$$ ✓

On!

2.

(a) $\frac{A}{B} = AB^{-1}$ for matricies.

$$[a_0 - i(\vec{\sigma}\cdot\vec{a})]^{-1} = \frac{1}{\sum_n a_n^2}\begin{pmatrix} a_0 + ia_z & i(a_x - a_y) \\ i(a_x + y) & a_0 - ia_z \end{pmatrix}$$

$$= \frac{1}{\sum_n a_n^2}\left[a_0 \mathbb{1} + i(\vec{\sigma}\cdot\vec{a})\right]$$ (Thanks David)

So $U = \dfrac{[a_0 \mathbb{1} + i(\vec{\sigma}\cdot\vec{a})]^2}{\sum_n a_n^2}$

$$UU^{+} = \frac{[a_0\mathbb{1} + i(\vec{\sigma}\cdot\vec{a})]^2[a_0\mathbb{1} - i(\vec{\sigma}\cdot\vec{a})]^2}{(\sum_n a_n^2)^2}$$

$$= \frac{(a_0^4 + 2a_0^2\vec{a} + \vec{a}^4)}{(a_0^4 + 2a_0^2\vec{a} + \vec{a}^4)} = \mathbb{1}$$ ✓

(b) $$|U| = \left|(a_0 + i\vec{\sigma}\cdot\vec{a})\cdot\frac{1}{\sum_n a_n^2}(a_0 + i(\vec{\sigma}\cdot a)\right|$$

$$= \left|\frac{(a_0\mathbb{1} + i\vec{\sigma}\cdot\vec{a})^2}{(\sum_n a_n^2)^2}\right| = \left|\frac{a_0^2 + 2i\vec{\sigma}\cdot\vec{a} - \vec{a}^2}{(\sum_n a_n^2)^2}\right|$$

$$= \frac{(\sum_n a_n^2)^2}{(\sum_n a_n^2)^2} = 1$$ ✓

$\Rightarrow$

(b)

We know from lecture, (EVENTUALLY!)

$$e^{-i \frac{\vec{\sigma}\cdot\vec{n}}{2}\phi} = \mathbb{1}\cos(\phi/2) - i(\vec{\sigma}\cdot\hat{n})\sin(\phi/2)$$

compare this to U: $\dfrac{[a_0 \mathbb{1} + i(\vec{\sigma}\cdot\vec{a})]^2}{\sum_n a_n^2}$

So $\dfrac{a_0^2 - \vec{a}^2}{\sum_n a_n^2}$ is like $\cos(\phi/2)$, $\dfrac{2\, a_0 \vec{a}}{\sum_n a_n^2}$ is like $-\sin(\phi/2)$

So $\phi = \cos^{-1}\left(\dfrac{a_0^2 - \vec{a}^2}{\sum_n a_n^2}\right)\cdot 2$

axis of rotation is $\hat{n}$ which is analagous to $\underline{\vec{a}}$

(Sorry this one is a bit rushed)

B in $\underline{z}$ direction

3.) $H = A\vec{S}^{(e^-)} \cdot \vec{S}^{(e^+)} + \left(\frac{eB}{mc}\right)\left(S_z^{(e^-)} - S_z^{(e^+)}\right)$

(a) For $A \to 0$, $\frac{eB}{mc} \neq 0$

$H = \frac{eB}{mc}\left(S_z^{(e^-)} - S_z^{(e^+)}\right)$, Spin function: $\chi_+^{(e^-)}\chi_-^{(e^+)}$

$H\chi_+^{(e^-)}\chi_-^{(e^+)} = \frac{eB}{mc}\left(S_z^{(e^-)} - S_z^{(e^+)}\right)\chi_+^{(e^-)}\chi_-^{(e^+)}$

$= \frac{eB}{mc}\left(\frac{\hbar}{2}\chi_+^{(e^-)}\chi_-^{(e^+)} - \frac{-\hbar}{2}\chi_+^{(e^-)}\chi_-^{(e^+)}\right)$

$= \underbrace{\frac{\hbar eB}{mc}}_{\text{eigenvalue}}\chi_+^{(e^-)}\chi_-^{(e^+)}$ ✓ $\underline{\text{Yes}}$ it is an eigenfunction of H!

(b) $A \neq 0$, $\frac{eB}{mc} \to 0$

$H\chi_+^{(e^-)}\chi_-^{(e^+)} = A\vec{S}^{(e^-)} \cdot \vec{S}^{(e^+)}\chi_+^{(e^-)}\chi_-^{(e^+)}$

(Thanks Evan)

$= A\left[(S^-S^+)_x + (S^-S^+)_y + (S^-S^+)_z\right]\chi_+^{(e^-)}\chi_-^{(e^+)}$

$= \underline{A\frac{\hbar^2}{4}\left[2\chi_-^{(e^-)}\chi_+^{(e^+)} - \chi_+^{(e^-)}\chi_-^{(e^+)}\right]}$

$\underline{\text{Not}}$ an eigenfunction of H!