ECON408: Computational Methods in Macroeconomics

AR(1) Models and Ergodicity

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Overview

Motivation and Materials

- In this lecture, we will introduce our first stochastic process, the **AR(1)** process
- This is a simple, univariate process, but it is directly useful in many cases
- We will also introduce the concept of **ergodicity** to help us understand long-run behavior
- While this section is not directly introducing new economic models, it provides the backbone for our analysis of the wealth and income distribution

Deterministic Processes

- We have seen deterministic processes in previous lectures, e.g. the linear

$$X_{t+1} = aX_t + b$$

  → These are coupled with an initial condition X_0, which enables us to see the evolution of a variable

  → The state variable, X_t, could be a vector

  → The evolution could be non-linear $X_{t+1} = h(X_t)$, etc.

- But many states in the real world involve randomness

Materials

- Adapted from QuantEcon lectures coauthored with John Stachurski and Thomas J. Sargent
  - → AR1 Processes
  - → LLN and CLT

```julia
using LaTeXStrings, LinearAlgebra, Plots, Statistics
using Random, StatsPlots, Distributions
using Plots.PlotMeasures
default(;legendfontsize=16, linewidth=2, tickfontsize=12,
         bottom_margin=15mm)
```

Random Variables Review

Random Variables

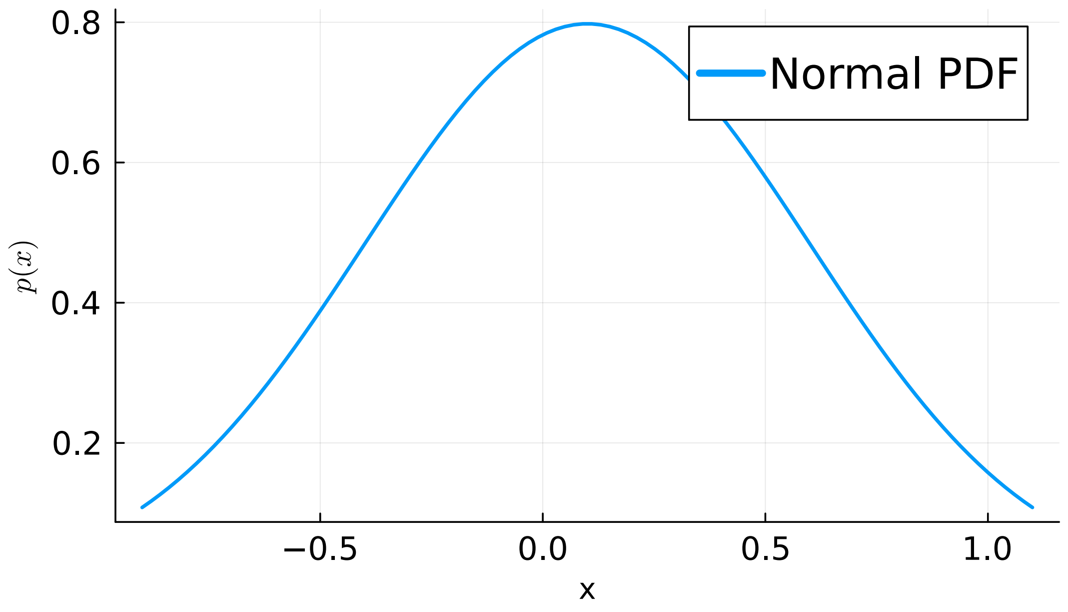
- **Random variables** are a collection of values with associated probabilities
- For example, a random variable Y could be the outcome of a coin flip
  - → Let $Y = 1$ if heads and $Y = 0$ if tails
  - → Assign probabilities $\mathbb{P}(Y = 1) = \mathbb{P}(Y = 0) = 0.5$
- or a **normal random variable** with mean μ and variance σ, denoted $Y \sim \mathcal{N}(\mu, \sigma^2)$ has density $p(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$

Discrete vs. Continuous Variables

- If discrete (e.g., $X \in \{x_1, \ldots, x_N\}$) , then
  - → The **probability mass function** (pmf) is the probability of each value $p \in \mathbb{R}^N$
  - → Such that $\sum_{i=1}^N p_i = 1$, and $p_i \geq 0$
  - → i.e. $p_i = \mathbb{P}(X = x_i)$
- If continuous, then the **probability density function** (pdf) is the probability of each value and can be represented by a function
  - → $p : \mathbb{R} \to \mathbb{R}$ if X is defined on $\mathbb{R}$
  - → $\int_{-\infty}^{\infty} p(x)dx = 1$, and $p(x) \geq 0$
  - → $\mathbb{P}(X = a) = 0$ in our examples, and $\mathbb{P}(X \in [a, b]) = \int_a^b p(x)dx$

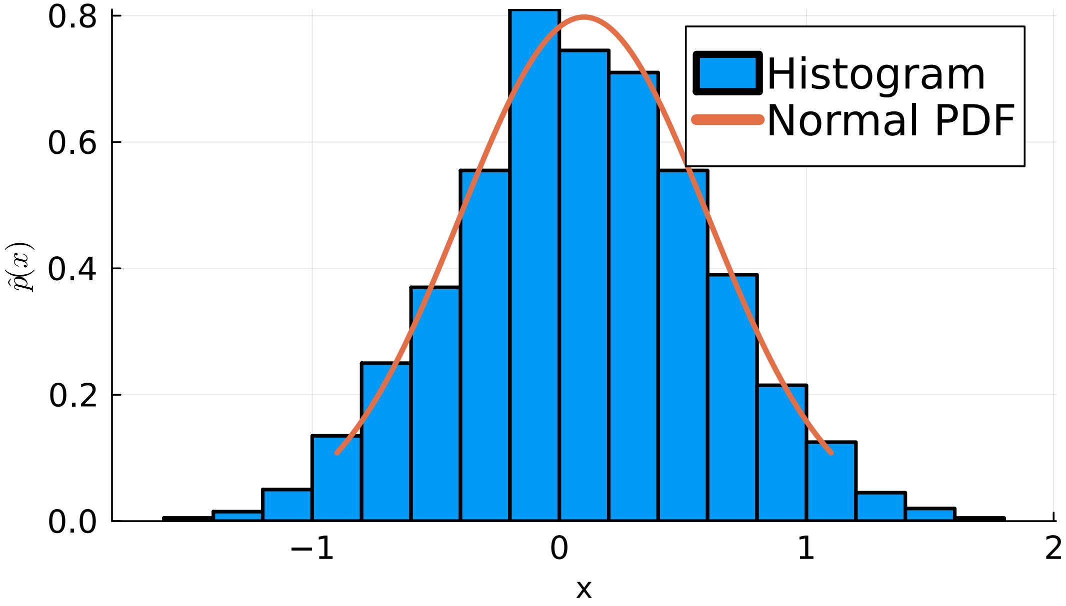
Normal Random Variables

```
mu = 0.1
sigma = 0.5
d = Normal(0.1, sigma) # SD not variance
x = range(mu - 2 * sigma,
          mu + 2 * sigma;
          length=100)
plot(x, pdf.(d, x); label="Normal PDF",
     xlabel="x", ylabel=L"p(x)",
     size=(600,400))
```



Comparing to a Histogram

```
n = 1000
x_draws = rand(d, n) # gets n samples
histogram(x_draws; label="Histogram",
          xlabel="x", ylabel=L"\hat{p}(x)",
          normalize=true, size=(600,400))
plot!(x, pdf.(d, x); label="Normal PDF",
      lw=3)
```



Normal Random Variables

- Normal random variables are special for many reasons (e.g., central limit theorems)
- They are the only continuous random variable with finite variance that is closed under linear combinations
  - → If $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$, then
  - → $aX + bY \sim \mathcal{N}(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2)$
  - → Also true with multivariate normal distributions
- Common transformation taking out mean and variance
  - → Could draw $Y \sim N(\mu, \sigma^2)$
  - → Or could draw $X \sim N(0, 1)$ and then $Y = \mu + \sigma X$

Expectations

- For discrete-valued random variables

$$\mathbb{E}[f(X)] = \sum_{i=1}^{N} f(x_i)p_i$$

- For continuous valued random variables

$$\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)p(x)dx$$

Moments

- The **mean** of a random variable is the first moment, $\mathbb{E}[X]$
- The **variance** of a random variable is the second moment, $\mathbb{E}[(X - \mathbb{E}[X])^2]$
  - $\rightarrow$ Note the recentering by the mean. Could also calculate as $\mathbb{E}[X^2] - \mathbb{E}[X]^2$
- Normal random variables are characterized by their first 2 moments

Law(s) of Large Numbers

- Let $X_1, X_2, \ldots$ be independent and identically distributed (iid) random variables with mean $\mu \equiv \mathbb{E}(X) < \infty$, then let

$$\bar{X}_n \equiv \frac{1}{n}\sum_{i=1}^{n} X_i$$

- One version is **Kolmogorov's Strong Law of Large Numbers**

$$\mathbb{P}\left(\lim_{n\to\infty} \bar{X}_n = \mu\right) = 1$$

→ i.e. the average of the random variables converges to the mean

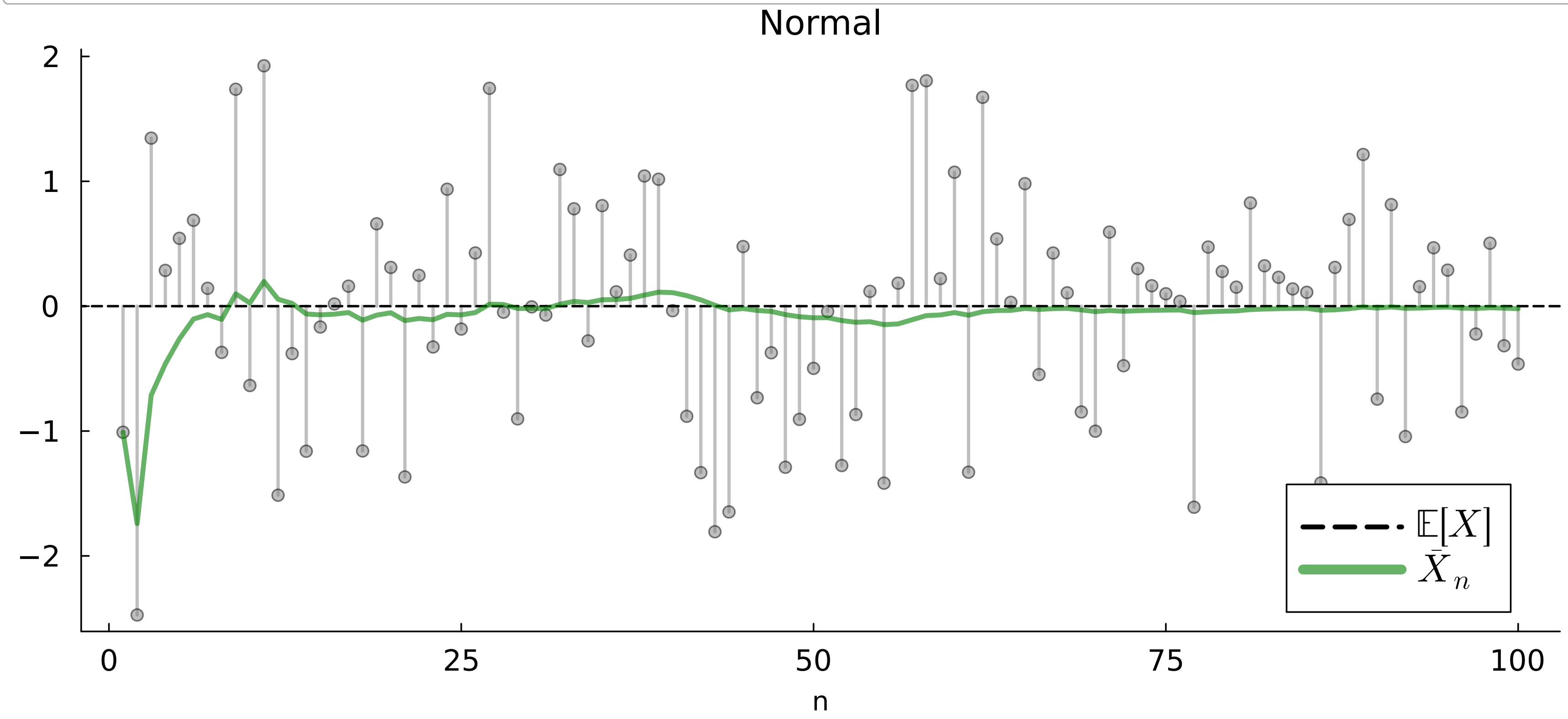
Sampling and Plotting the Mean

```julia
function ksl(distribution, n = 100)
    title = nameof(typeof(distribution))
    observations = rand(distribution, n)
    sample_means = cumsum(observations) ./ (1:n)
    mu = mean(distribution)
    plot(repeat((1:n)', 2), [zeros(1, n); observations']; title,  xlabel="n",
         label = "", color = :grey, alpha = 0.5)
    plot!(1:n, observations; color = :grey, markershape = :circle,
          alpha = 0.5, label = "", linewidth = 0)
    if !isnan(mu)
        hline!([mu], color = :black, linewidth = 1.5, linestyle = :dash,
               grid = false, label = L"\mathbb{E}[X]")
    end
    return plot!(1:n, sample_means, linewidth = 3, alpha = 0.6, color = :green, label = L"\bar{X}_n")
end
```

```
ksl (generic function with 2 methods)
```

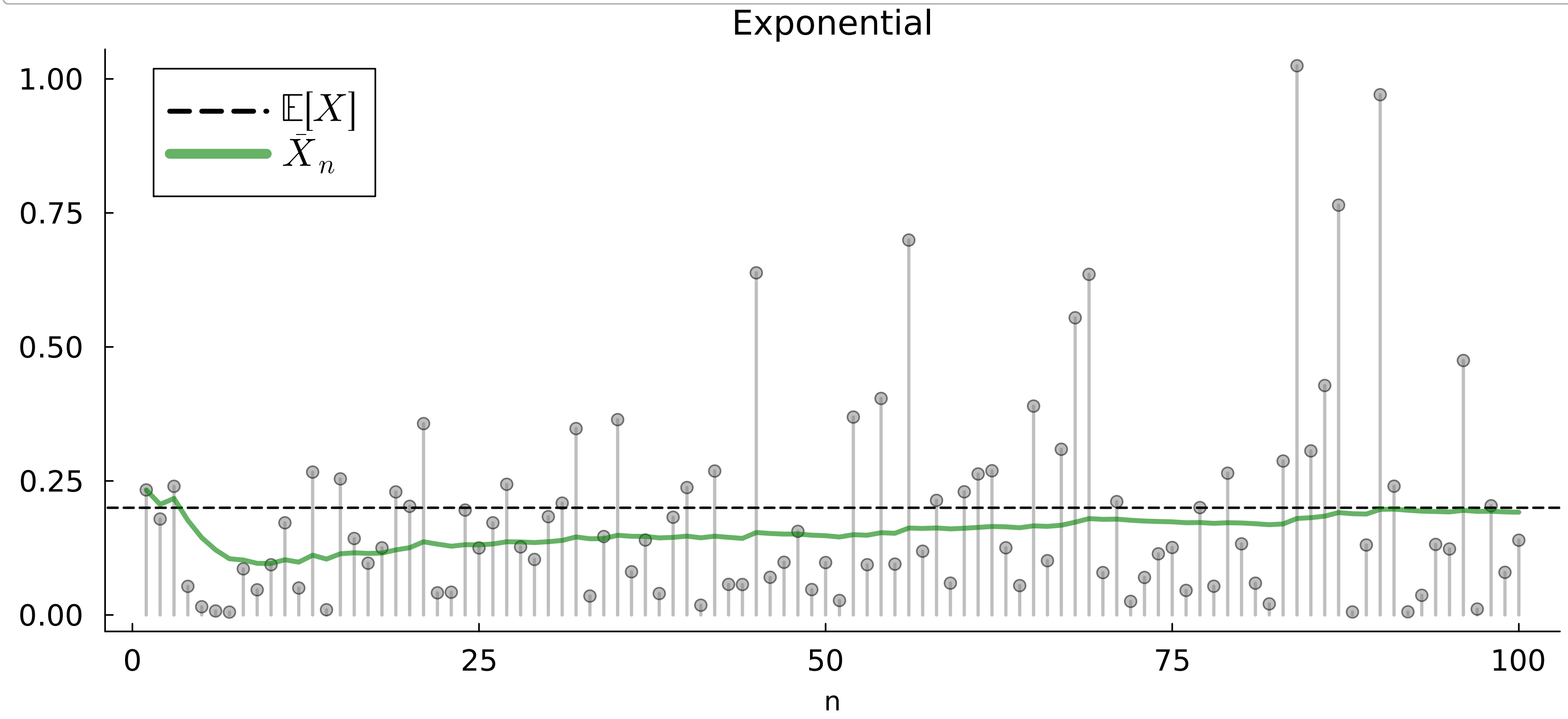
LLN with the Normal Distribution

```
dist = Normal(0.0, 1.0) # unit normal
ksl(dist)
```



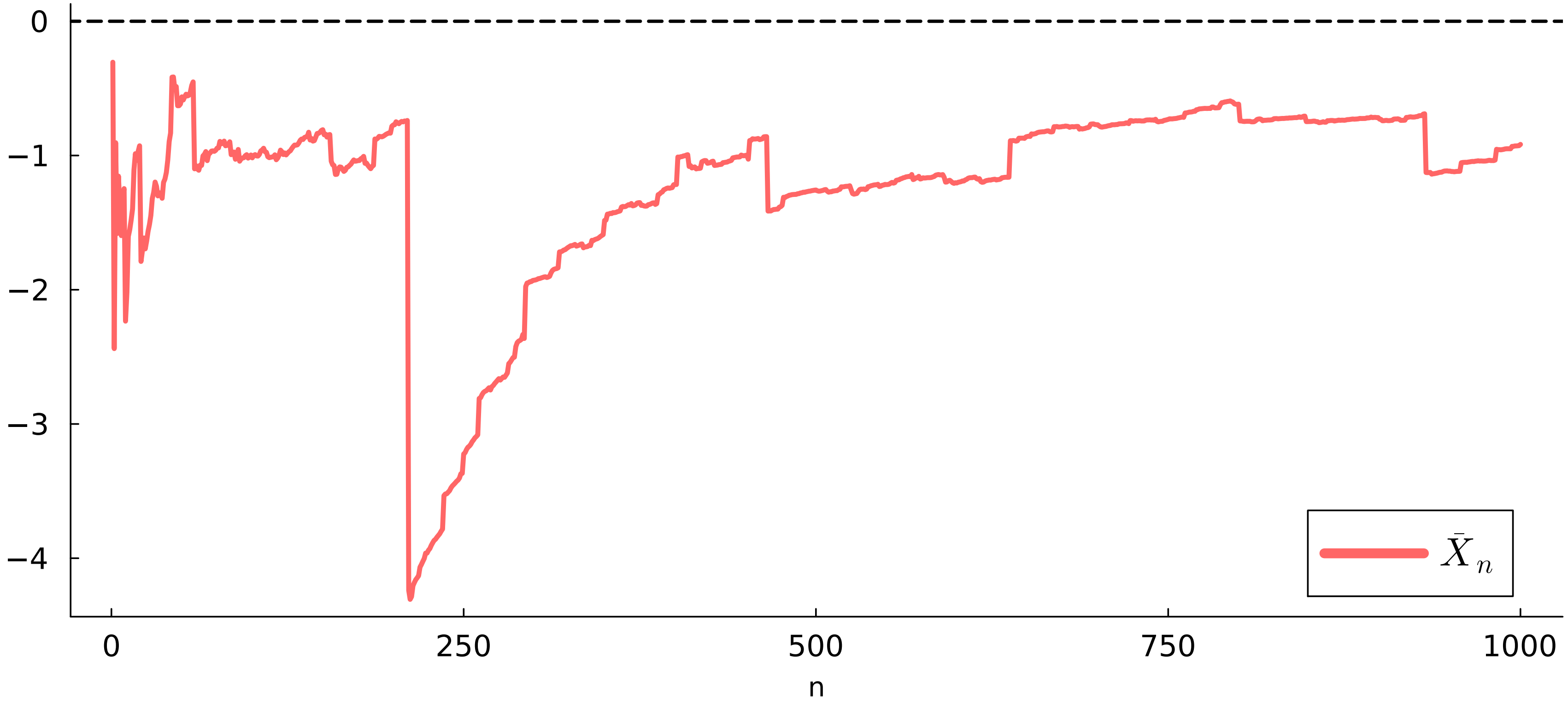
LLN with the Exponential

```
dist = Exponential(0.2)
ksl(dist)
```



LLN with the Cauchy

```julia
Random.seed!(0); # reproducible results
dist = Cauchy() # Doesn't have an expectation!
sample_mean = cumsum(rand(dist, n)) ./ (1:n)
plot(1:n, sample_mean, color = :red, alpha = 0.6, label =  L"\bar{X}_n",
       xlabel = "n", linewidth = 3)
hline!([0], color = :black, linestyle = :dash, label = "", grid = false)
```



Monte-Carlo Calculation of Expectations

- One application of this is the numerical calculation of expectations
- Let X be a random variable with density $p(x)$, and hence $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)p(x)dx$ (or $\sum_{i=1}^{N} f(x_i)p_i$ if discrete)
- These integrals are often difficult to calculate analytically, but if we can draw $X \sim p$, then we can approximate the expectation by

$$\mathbb{E}[f(X)] \approx \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

- Then by the LLN this converges to the true expectation as $n \to \infty$

Discrete Example

- Let X be a discrete random variable with N states and probabilities p_i
- Then $\mathbb{E}[f(X)] = \sum_{i=1}^{N} f(x_i)p_i$
- For example, the Binomial distribution and $f(x) = \log(x+1)$

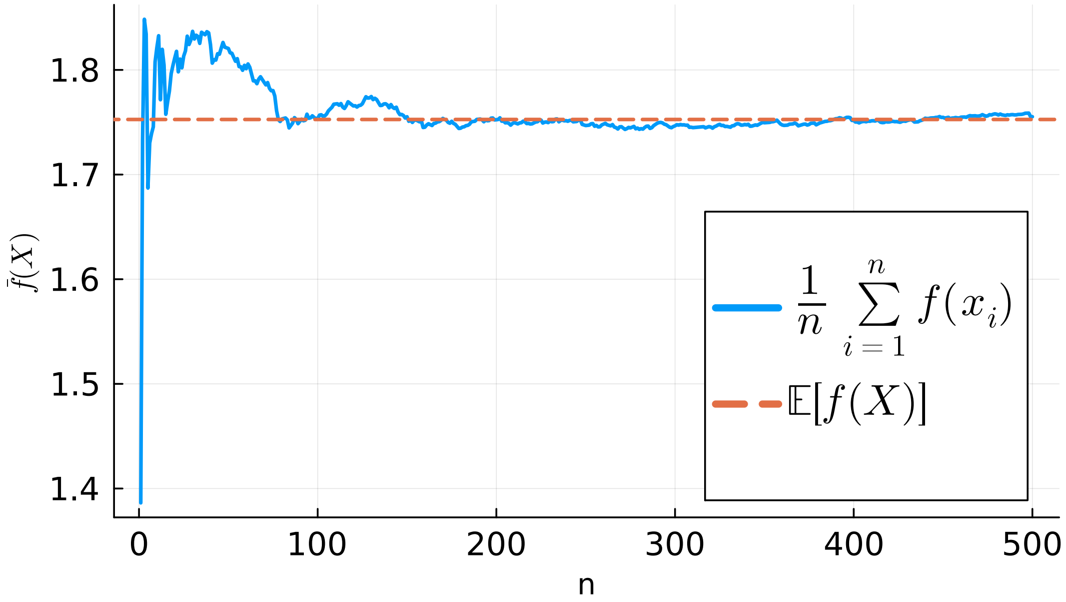
```
# number of trials and probability of success
dist = Binomial(10, 0.5)
plot(dist;label="Binomial PMF",
     size=(600,400))
vals = support(dist) # i.e. 0:10
p = pdf.(dist, vals)
# Calulate the expectation manually
@show mean(dist), dot(vals, p);
```

```
(mean(dist), dot(vals, p)) = (5.0, 5.000000000000008)
```

Using Monte-Carlo

```julia
N = 500
# expectation with PMF, then MC
f_expec = dot(log.(vals .+ 1), p)
x_draws = rand(dist, N)
f_x_draws = log.(x_draws .+ 1)
f_expec_mc = sum(f_x_draws) / N
@show f_expec, f_expec_mc
# Just calculate sums then divide by N
f_means = cumsum(f_x_draws)./(1:N)
plot(1:length(f_means), f_means;
     label=L"\frac{1}{n}\sum_{i=1}^n f(x_i)",
     xlabel="n", ylabel=L"\bar{f}(X)",
     size=(600,400))
hline!([f_expec];linestyle = :dash,
       label = L"\mathbb{E}[f(X)]")
```

```
(f_expec, f_expec_mc) = (1.7526393207741702,
1.7552834928857293)
```



Stochastic Processes

Stochastic Processes

- A **stochastic process** is a sequence of random variables
  - $\longrightarrow$ We will focus on **discrete time** stochastic processes, where the sequence is indexed by $t = 0, 1, 2, \ldots$
  - $\longrightarrow$ Could be discrete or continuous random variables
- Skipping through some formality, assume that they share the same values but probabilities may change
- Denote then as a sequence $\{X_t\}_{t=0}^{\infty}$

Joint, Marginal, and Conditional Distributions

- Can ask questions on the probability distributions of the process
- The **joint distribution** of $\{X_t\}_{t=0}^{\infty}$ or a subset
  - $\rightarrow$ In many cases things will be correlated over time or else no need to be a process
- The **marginal distribution** of X_t for any t
  - $\rightarrow$ This is a proper PDF, marginalized from the joint distribution of all values
- **Conditional distributions**, fixing some values
  - $\rightarrow$ e.g. X_{t+1} given X_t, X_{t-1}, etc. are known

AR(1) Process

$$X_{t+1} = aX_t + b + cW_{t+1}$$

- Just added randomness to the deterministic process from time t to $t+1$
- $W_{t+1} \sim \mathcal{N}(0,1)$ is IID "shocks" or "noise"
- Could have an initial condition for X_0 Or could have an initial distribution
  - → X_t is a random variable, and so can X_0
  - → "Degenerate random variable" if $P(X_0 = x) = 1$ for some x
  - → Assume $X_0 \sim \mathcal{N}(\mu_0, v_0)$, and $v_0 \to 0$ is the degenerate case

Markov Process

- Before we go further, lets discuss a broader class of these processes useful in economics
- A **Markov process** is a stochastic process where the conditional distribution of X_{t+1} given $X_t, X_{t-1}, \ldots$ is the same as the conditional distribution of X_{t+1} given X_t
  - → i.e. the future is independent of the past given the present
- Note that with the AR(1) model, if I know X_t then I can calculate the PDF of X_{t+1} directly without knowing the past
- This is "first-order" since only one lag is required, but could be higher order
  - → A finite number of lags can always be added to the state vector to make it first-order

Evolution of the AR(1) Process

- Both W_{t+1} and X_0 are assumed to be normally distributed
- As we discussed, linear combinations of normal random variables are normal
  - $\rightarrow$ So X_t is normal for all t by induction
- Furthermore, we have a formula for the recursion
  - $\rightarrow$ If $X_t \sim \mathcal{N}(\mu_t, v_t)$, then $X_{t+1} \sim \mathcal{N}(a\mu_t + b, a^2 v_t + c^2)$
  - $\rightarrow$ Hence, the evolution of the mean and variance follow a simple difference equation $\mu_{t+1} = a\mu_t + b$ and $v_{t+1} = a^2 v_t + c^2$
  - $\rightarrow$ Let $X_t \sim \psi_t \equiv \mathcal{N}(\mu_t, v_t)$

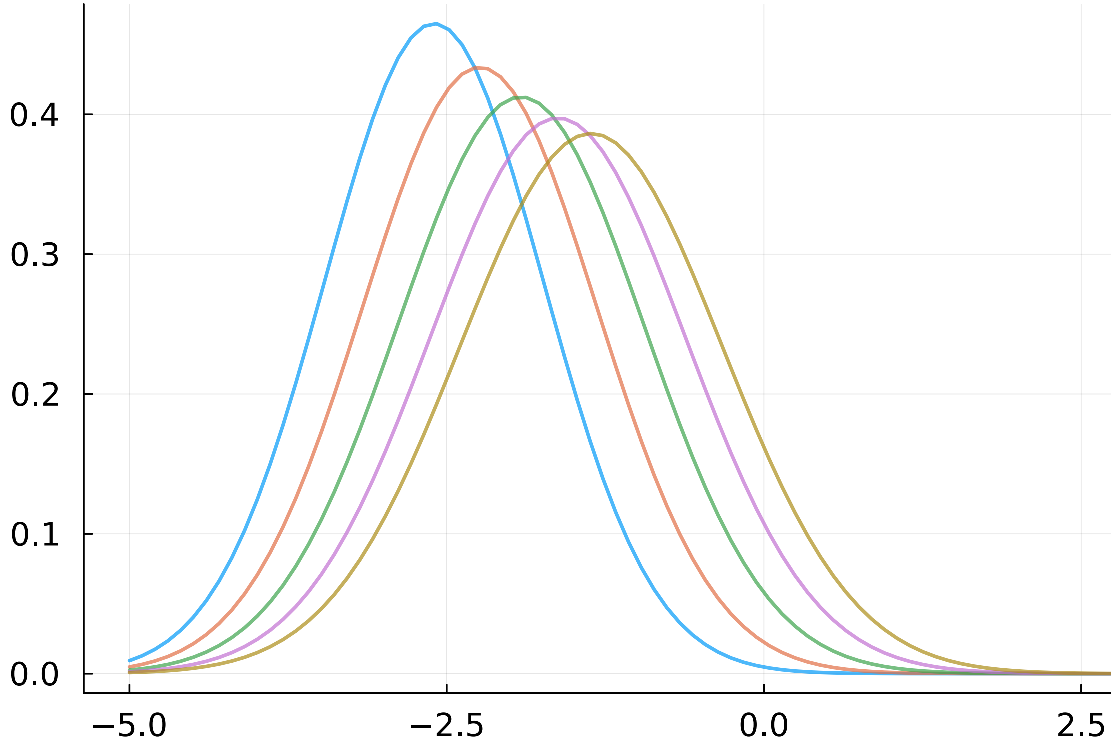
Visualizing the AR(1) Process

```
a = 0.9
b = 0.1
c = 0.5

# initial conditions mu_0, v_0
mu = -3.0
v = 0.6
```

```
0.6
```

Visualizing the AR(1) Process

```
sim_length = 5
x_grid = range(-5, 7, length = 120)

plt = plot()
for t in 1:sim_length
    mu = a * mu + b
    v = a^2 * v + c^2
    dist = Normal(mu, sqrt(v))
    plot!(plt, x_grid, pdf.(dist, x_grid),
    label = L"\psi_{%$t}", linealpha = 0.7)
end
plt
```



From a Degenerate Initial condition

```julia
mu = -3.0
v = 0.0
plt = plot()
for t in 1:sim_length
    mu = a * mu + b
    v = a^2 * v + c^2
    dist = Normal(mu, sqrt(v))
    plot!(plt, x_grid, pdf.(dist, x_grid),
    label = L"\psi_{%$t}", linealpha = 0.7)
end
plt
```