ECON408: Computational Methods in Macroeconomics

Deterministic Dynamics and Introduction to Growth Models

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Overview

Motivation and Materials

- In this lecture, we will introduce (non-linear) dynamics
  - $\longrightarrow$ This lets us explore stationarity and convergence
  - $\longrightarrow$ We will see an additional example of a **fixed point** and **convergence**
- The primary applications will be to simple models of growth, such as the **Solow** growth model.

Materials

- Adapted from QuantEcon lectures coauthored with John Stachurski and Thomas J. Sargent
  - → Julia by Example
  - → Dynamics in One Dimension

```
using LaTeXStrings, LinearAlgebra, Plots
default(;legendfontsize=16)
```

Difference Equations

(Nonlinear) Difference Equations

$$x_{t+1} = h(x_t)$$

- A **time homogeneous first order difference equation**
  - $\rightarrow$ $h : S \rightarrow S$ for some $S \subseteq \mathbb{R}$ in the univariate case
  - $\rightarrow$ S is called the **state space** and x is called the **state variable**.
  - $\rightarrow$ Time homogeneity: g is the same at each time t
  - $\rightarrow$ First order: depends on one lag (i.e., x_{t+1} and x_t but not x_{t-1})

Trajectories

- An initial condition x_0 is required to solve for the sequence $\{x_t\}_{t=0}^{\infty}$
- Given this, we can generate a **trajectory** recursively

$$
\begin{aligned}
x_1 &= h(x_0) \\
x_2 &= h(x_1) = h(h(x_0)) \\
x_{t+1} &= h(x_t) = h(h(\dots h(x_0))) \equiv h^t(x_0)
\end{aligned}
$$

- If not time homogeneous, we can write $x_{t+1} = h_t(x_t)$
- Stochastic if $x_{t+1} = h(x_t, \epsilon_{t+1})$ where ϵ_{t+1} is a random variable

Linear Difference Equations

$$x_{t+1} = ax_t + b$$

For constants a and b. Iterating,

$$\begin{aligned}
x_1 &= h(x_0) = ax_0 + b \\
x_2 &= h(h(x_0)) = a(ax_0 + b) + b = a^2 x_0 + ab + b \\
x_3 &= a(a^2 x_0 + ab + b) + b = a^3 x_0 + a^2 b + ab + b \\
\dots \\
x_t &= b \sum_{j=0}^{t-1} a^j + a^t x_0 = b \frac{1 - a^t}{1 - a} + a^t x_0
\end{aligned}$$

Convergence and Stability for Linear Difference Equations

- If $|a| < 1$, take limit to check for **global stability**, for all x_0

$$\lim_{t\to\infty} x_t = \lim_{t\to\infty} g^{t-1}(x_0) = \lim_{t\to\infty} \left(b\frac{1-a^t}{1-a} + a^t x_0 \right) = \frac{b}{1-a}$$

- If $a = 1$ then diverges unless $b = 0$ and $|a| > 1$ diverges for all b
- Linear difference equations are either globally stable or globally unstable
- Nonlinear difference equations may be **locally stable**
  - $\rightarrow$ For some $|x_0 - x^*| < \epsilon$ for some x^* and $\epsilon > 0$. Global if $\epsilon = \infty$

Nonlinear Difference Equations

- We can ask the same questions for nonlinear $h(\cdot)$
- Keep in mind the connection to the fixed points from the previous lecture

  $\rightarrow$ If $h(\cdot)$ has a unique fixed point from any initial condition, it tells us about the dynamics
- Connecting to **contraction mappings** etc. would help us be more formal, but we will stay intuitive here
- Let us investigate nonlinear dynamics with a classic example

Solow Growth Model

Models of Economic Growth

- There are different perspectives on what makes countries grow
  - → **Malthusian models:** population growth uses all available resources
  - → **Capital accumulation:** more capital leads to more output, tradeoff of consumption today to build more capital for tomorrow
  - → **Technological progress/innovation:** new ideas lead to more output, so the tradeoffs are between consumption today vs. researching technologies for the future
- The appropriate model depends on country and time-period
  - → Malthusian models are probably most relevant right up until about the time he came up with the idea

Exogenous vs. Endogenous

- In these, the tradeoffs are key
  - → Can be driven by some sort of decision driven by the agent's themselves (e.g., government plans, consumers saving, etc.) endogenously
  - → Or exogenously choose, not responsive to policy and incentives
  - → You always leave some things exogenous to isolate a key force
- What determines the longrun growth rate? Use **fixed points**!
  - → In models of capital accumulation, technology limits the longrun growth
  - → Models of innovation choice often called **endogenous growth models**

Solow Model Summary

- The **Solow** model describes aggregate growth from the perspective of accumulating physical capital
  - $\longrightarrow$ The tension is between **consumption** and **savings**
  - $\longrightarrow$ Production not directed towards consumption goods is used to build capital for future consumption
  - $\longrightarrow$ e.g. factories, robots, facilities, etc.
- Endogeneity
  - $\longrightarrow$ Technology and population growth are left fully exogenous
  - $\longrightarrow$ Capital accumulation occurs through an exogenously given savings rate
  - $\longrightarrow$ The neoclassical growth model endogenizing that rate

Technology

- In this economy, output is produced by combining labor and capital
- Labor N_t, which we assume is supplied **inelastically**
  - $\longrightarrow$ Assume it is proportional to the population
- Capital, K_t, which is accumulated over time
- In addition, **total factor productivity (TFP)**, z_t, is the technological level in the economy
- The physical output from operating the technology is:

$$Y_t = z_t F(K_t, N_t)$$

Properties of the Technology Function

- Note that land, etc. are NOT a factor of production
- We will assume $F(\cdot,\cdot)$ is **constant returns to scale**

$$F(\alpha K, \alpha N) = \alpha F(K, N) \quad \forall \alpha > 0$$

- Assume F has **diminishing marginal products**

$\rightarrow$ i.e. $\frac{\partial F(K,N)}{\partial K} > 0$, $\frac{\partial F(K,N)}{\partial N} > 0$, $\frac{\partial^2 F(K,N)}{\partial K^2} < 0$, $\frac{\partial^2 F(K,N)}{\partial N^2} < 0$

Constant Returns to Scale

- Define output per worker as $y_t = Y_t/N_t$ and capital per worker as $k_t = K_t/N_t$
- Take F, divide by N_t, use CRS, and define $f(\cdot)$

$$
\begin{aligned}
Y_t &= z_t F(K_t, N_t) \\
\frac{Y_t}{N_t} &= \frac{z_t F(K_t, N_t)}{N_t} \\
y_t &= z_t F\left(\frac{K_t}{N_t}, \frac{N_t}{N_t}\right) = z_t F(k_t, 1) \equiv z_t f(k_t)
\end{aligned}
$$

$\rightarrow$ f also has diminishing marginal products, $f'(k) > 0$, $f''(k) < 0$

Population Growth

- From some initial condition N_0 for population
- Assume that population grows at a constant rate g_N, i.e.

$$N_{t+1} = (1 + g_N)N_t$$

- Hence $N_{t+1}/N_t = 1 + g_N$ and $N_t = (1 + g_N)^t N_0$
- If $g_N < 0$ then shrinking population

Capital Accumulation

- Capital is accumulated by **investment**, X_t, with per capital $x_t \equiv X_t/N_t$
  - $\longrightarrow$ Macroeconomists should think in "allocations", not "dollars"!
- Output, Y_t from production can be used for consumption or investment
- Between periods, $\delta \in (0,1)$ proportion of capital depreciates
  - $\longrightarrow$ e.g. machines break down, buildings decay, etc.

$$C_t + X_t = Y_t \equiv \underbrace{z_t F(K_t, N_t)}_{\text{Total Output}}$$

$$\underbrace{K_{t+1}}_{\substack{\text{Next}\\ \text{periods}\\ \text{capital}}} = \underbrace{(1-\delta)}_{\substack{\text{depreciation}\\ \text{of capital}}} K_t + \underbrace{X_t}_{\substack{\text{investment}\\ \text{in new}\\ \text{capital}}} \quad , \delta \in (0,1)$$

Per Capita Capital Dynamics

- Recall that $N_{t+1}/N_t = 1 + g_N$ and $y_t = z_t f(k_t)$

$$\frac{K_{t+1}}{N_t} = (1-\delta)\frac{K_t}{N_t} + \frac{X_t}{N_t}$$
$$\frac{N_{t+1}}{N_{t+1}}\frac{K_{t+1}}{N_t} = \left(\frac{N_{t+1}}{N_t}\right)\left(\frac{K_{t+1}}{N_{t+1}}\right) = (1-\delta)\frac{K_t}{N_t} + \frac{X_t}{N_t}$$
$$k_{t+1}(1+g_N) = (1-\delta)k_t + x_t$$

- So, the per-capita dynamics of capital are

$$k_{t+1} = \frac{1}{1+g_N}\left[(1-\delta)k_t + x_t\right]$$

Savings

- In the Solow model, the savings rate is exogenously given as $s \in (0,1)$
  - $\rightarrow$ In the neoclassical growth model, it is endogenously determined based on consumer or planner preferences
- Hence, $x_t = sy_t = sz_t f(k_t)$. Combine with previous dynamics to get

$$k_{t+1} = \frac{1}{1+g_N}\left[(1-\delta)k_t + sz_t f(k_t)\right]$$

- Given assumptions on $f(\cdot)$, this is a nonlinear difference equation given an exogenous z_t process
- With this, we can analyze the dynamics of y_t, k_t, and c_t over time

Steady State

- Assume that $z_t = \bar{z}$ is constant over time
- Look for **steady state**: $k_{t+1} = k_t = \bar{k}$ and $y_{t+1} = y_t = \bar{y}$, etc.
  - $\rightarrow$ Note that this is a **fixed point** of the dynamics. May or may not exist

$$\bar{k} = \frac{1-\delta}{1+g}\bar{k} + \frac{s\bar{z}}{1+g} f(\bar{k})$$

$$\left(\frac{1+g}{1+g} - \frac{1-\delta}{1+g}\right)\bar{k} = \frac{s\bar{z}}{1+g} f(\bar{k})$$

$$\underbrace{(g+\delta)\bar{k}}_{\text{Growth-adjusted depreciation}} = \underbrace{s\bar{z}f(\bar{k})}_{\text{investment per capita}}$$

Example Production Function

- Consider production function of $f(k) = k^{\alpha}$ for $\alpha \in (0,1)$
- In this case, α will be interpretable as the **capital share** of income

$$\bar{k} = \left(\frac{s\bar{z}}{g+\delta}\right)^{\frac{1}{1-\alpha}}$$

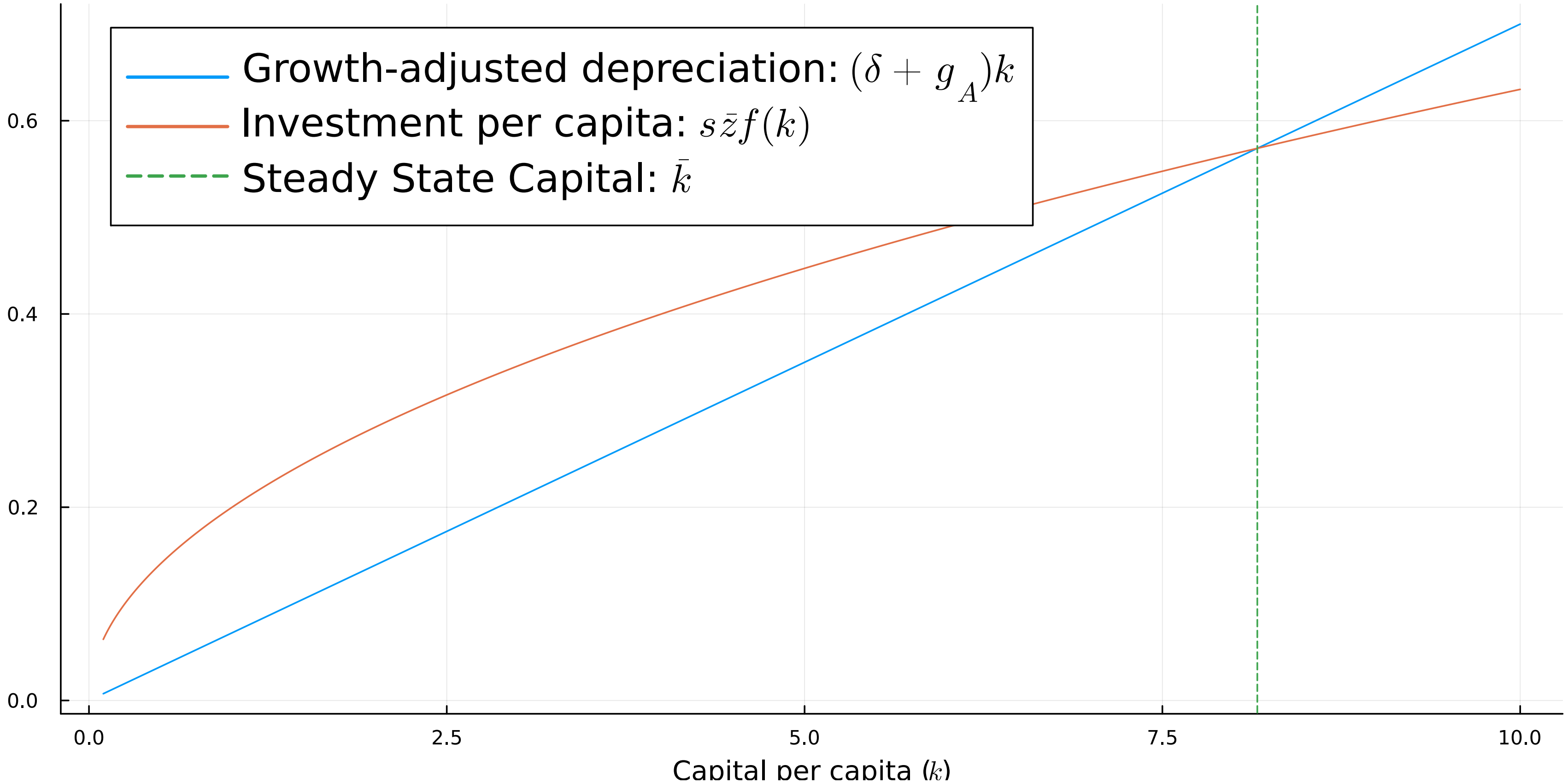
Visualizing the Steady State

```julia
g_A = 0.02
delta = 0.05
s = 0.2
z = 1.0
alpha = 0.5
k_bar = (s * z / (g_A + delta))^(1/(1-alpha))

k_values = 0.1:0.01:10.0
lhs = (g_A + delta) * k_values
rhs = s * z .* k_values.^alpha

plot(k_values, lhs, label=L"Growth-adjusted depreciation: $(\delta + g_A)k$",
     xlabel=L"Capital per capita ($k$)")
plot!(k_values, rhs, label=L"Investment per capita: $s \bar{z} f(k)$")
vline!([k_bar], label=L"Steady State Capital: $\bar{k}$", linestyle=:dash)
```

Visualizing the Steady State



Wages and Rental Rate of Capital

- Production could be run by a planner, or by a set of firms
- Consider (real) profit maximizing firms. Price normalized to 1
  - $\rightarrow$ Hire labor and capital at real rates w_t and r_t respectively

$$\max_{K_t, N_t} \left[z_t F(K_t, N_t) - w_t N_t - r_t K_t \right]$$

- The first order conditions are

$$z_t \frac{\partial F(K_t, N_t)}{\partial K_t} = r_t$$
$$z_t \frac{\partial F(K_t, N_t)}{\partial N_t} = w_t$$

Using Constant Returns to Scale

- Can show that for any CRS $F(K_t, N_t)$ that $\frac{\partial F(\gamma K_t, \gamma N_t)}{\partial K_t} = \frac{\partial F(K_t, N_t)}{\partial K_t}$
  - $\rightarrow$ Same for N_t derivative. ,
- Set $\gamma = 1/N_t$ and write the marginal products as ratios,

$$\frac{\partial F(K_t, N_t)}{\partial K_t} = f'(K_t/N_t) = f'(k_t)$$

$$\frac{\partial F(K_t, N_t)}{\partial N_t} = f(k_t) - k_t f'(k_t)$$

Wages and Rental Rate of Capital

- Finally, we can write

$$
\begin{aligned}
r_t &= z_t f'(k_t) \\
w_t &= z_t f(k_t) - z_t f'(k_t) k_t
\end{aligned}
$$

- For the Cobb-Douglas production function $F(K_t, N_t) = K_t^\alpha N_t^{1-\alpha}$, we have

$$
\begin{aligned}
f(k_t) &= k_t^\alpha \\
f'(k_t) &= \alpha k_t^{\alpha-1} \\
r_t &= \alpha z_t k_t^{\alpha-1} \\
w_t &= z_t k_t^\alpha - z_t \alpha k_t^{\alpha-1} k_t = (1-\alpha) z_t k_t^\alpha
\end{aligned}
$$

Shares of Income

- Recall that per-capita output is $y_t = z_t f(k_t)$
- $w_t = (1-\alpha)z_t k_t^\alpha$
  - $\longrightarrow$ Interpret $1-\alpha$ as the **labor share** of output, or income
- $r_t k_t = \alpha z_t k_t^\alpha$
  - $\longrightarrow$ Interpret α as the **capital share**
- Key to these expressions were competitive markets in hiring labor/capital
  - $\longrightarrow$ i.e., workers end up paid their marginal products

Solow Model Dynamics

Summary of Equations

- Exogenous z_t sequence. e.g., $z_{t+1}/z_t = 1 + g_z$ given some initial z_0
- Population growth $N_{t+1}/N_t = 1 + g_N$ given some initial N_0

$$k_{t+1} = \frac{1}{1+g_N}[(1-\delta)k_t + sz_t f(k_t)], \quad \text{given } k_0$$

→ Output per capita $y_t = z_t f(k_t)$

→ Consumption per capita $c_t = (1-s)y_t = (1-s)z_t f(k_t)$

- Wages $w_t = (1-\alpha)z_t k_t^{\alpha}$ and rental rate of capital $r_t = \alpha z_t k_t^{\alpha-1}$
- Steady state capital $\bar{k} = \left(\frac{s\bar{z}}{g_N+\delta}\right)^{\frac{1}{1-\alpha}}$ if $g_z = 1$ and $z_0 = \bar{z}$

45 Degree Diagram

- With a fixed point $k_{t+1} = h(k_t)$ note that a fixed point is when $\bar{k} = h(\bar{k})$
- We can plot the dynamics of the sequence comparing the functions to the 45 degree line where that occurs
- This diagram will help us interpret stability and convergence

Iteration

- First, lets write a general function to iterate a (univariate) map

```
function iterate_map(f, x0, T)
    x = zeros(T + 1)
    x[1] = x0
    for t in 2:(T + 1)
        x[t] = f(x[t - 1])
    end
    return x
end
```

```
iterate_map (generic function with 1 method)
```

Plotting the Dynamics

```julia
function plot45(f, xmin, xmax, x0, T; num_points = 100, label = L"h(k)",
                xlabel = "k", size = (600, 500))
    # Plot the function and the 45 degree line
    x_grid = range(xmin, xmax, num_points)
    plt = plot(x_grid, f.(x_grid); xlim = (xmin, xmax), ylim = (xmin, xmax),
               linecolor = :black, lw = 2, label, size)
    plot!(x_grid, x_grid; linecolor = :blue, lw = 2, label = nothing)

    # Iterate map and add ticks
    x = iterate_map(f, x0, T)
    xticks!(x, [L"%$(xlabel)_{%$i}" for i in 0:T])
    yticks!(x, [L"%$(xlabel)_{%$i}" for i in 0:T])

    # Plot arrows and dashes
    for i in 1:T
        plot!([x[i], x[i]], [x[i], x[i + 1]], arrow = :closed, linecolor = :black,
              alpha = 0.5, label = nothing)
```

```
plot45 (generic function with 1 method)
```

Solow Parameter Examples

```
p = (z_bar = 2, s = 0.3, alpha = 0.3,
     delta = 0.4, g_N = 0.0)
k_p(k; p) = (1 / (1 + p.g_N)) * (
   p.s * p.z_bar * k^p.alpha
   + (1 - p.delta) * k)
k_min = 0.0
k_max = 4.0
plot45(k -> k_p(k; p), k_min, k_max, 0, 6)
```