ECON408: Computational Methods in Macroeconomics

Deterministic Dynamics and Introduction to Growth Models

Jesse Perla

jesse.perla@ubc.ca

University of British Columbia

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Overview

Motivation and Materials

- In this lecture, we will introduce (non-linear) dynamics
  - $\longrightarrow$ This lets us explore stationarity and convergence
  - $\longrightarrow$ We will see an additional example of a **fixed point** and **convergence**
- The primary applications will be to simple models of growth, such as the **Solow** growth model.

Materials

- Adapted from QuantEcon lectures coauthored with John Stachurski and Thomas J. Sargent
  - → Julia by Example
  - → Dynamics in One Dimension

```julia
using LaTeXStrings, LinearAlgebra, Plots
default(;legendfontsize=16)
```

Difference Equations

(Nonlinear) Difference Equations

$$x_{t+1} = g(x_t)$$

- A **time homogeneous first order difference equation**
  - $\rightarrow$ $g : S \rightarrow S$ for some $S \subseteq \mathbb{R}$ in the univariate case
  - $\rightarrow$ S is called the **state space** and x is called the **state variable**.
  - $\rightarrow$ Time homogeneity: g is the same at each time t
  - $\rightarrow$ First order: depends on one lag (i.e., x_{t+1} and x_t but not x_{t-1})

Trajectories

- An initial condition x_0 is required to solve for the sequence $\{x_t\}_{t=0}^{\infty}$
- Given this, we can generate a **trajectory** recursively

$$
\begin{aligned}
x_1 &= g(x_0) \\
x_2 &= g(x_1) = g(g(x_0)) \\
x_{t+1} &= g(x_t) = g(g(\ldots g(x_0))) \equiv g^t(x_0)
\end{aligned}
$$

- If not time homogeneous, we can write $x_{t+1} = g_t(x_t)$
- Stochastic if $x_{t+1} = g(x_t, \epsilon_{t+1})$ where ϵ_{t+1} is a random variable

Linear Difference Equations

$$x_{t+1} = ax_t + b$$

For constants a and b. Iterating,

$$\begin{aligned}
x_1 &= g(x_0) = ax_0 + b \\
x_2 &= g(g(x_0)) = a(ax_0 + b) + b = a^2x_0 + ab + b \\
x_3 &= a(a^2x_0 + ab + b) + b = a^3x_0 + a^2b + ab + b \\
&\dots \\
x_t &= b\sum_{j=0}^{t-1} a^j + a^t x_0 = b\frac{1-a^t}{1-a} + a^t x_0
\end{aligned}$$

Convergence and Stability for Linear Difference Equations

- If $|a| < 1$, then take limit
- Then for any x_0 we have **global stability**

$$\lim_{t\to\infty} x_t = \lim_{t\to\infty} g^{t-1}(x_0) = \lim_{t\to\infty} \left(b\frac{1-a^t}{1-a} + a^t x_0 \right) = \frac{b}{1-a}$$

- Otherwise
  - $\rightarrow$ If $a = 1$ then diverges unless $b = 0$
  - $\rightarrow$ If $|a| > 1$ diverges for all b

Nonlinear Difference Equations

- We can ask the same questions for nonlinear $g(\cdot)$
- Keep in mind the connection to the fixed points from the previous lecture
  - $\rightarrow$ If $g(\cdot)$ has a unique fixed point from any initial condition, it tells us about the dynamics
- Connecting to **contraction mappings** etc. would help us be more formal, but we will stay intuitive here

Solow Growth Model

Models of Economic Growth

- There are different perspectives on what makes countries grow
  - → **Malthusian models:** population growth uses all available resources
  - → **Capital accumulation:** more capital leads to more output, tradeoff of consumption today to build more capital for tomorrow
  - → **Technological progress/innovation:** new ideas lead to more output, so the tradeoffs are between consumption today vs. researching technologies for the future
- The appropriate model depends on country and time-period
  - → Malthusian models are probably most relevant right up until about the time he came up with the idea

Exogenous vs. Endogenous

- In these, the tradeoffs are key
  - → Can be driven by some sort of decision driven by the agent's themselves (e.g., government plans, consumers saving, etc.) endogenously
  - → Or can be exogenously chosen as not responding to policy and incentives
  - → You always leave some things exogenous to isolate a key force
- Consequence of choices: what determines the longrun growth rate?
  - → Analyze with **fixed points**!
  - → In models of capital accumulation, technological limitations limit the longrun growth rate
  - → Often people refer to models of technological progress as **endogenous growth models** because the long-run growth rate is determined by innovation decisions rather than by limitations on capital accumulation