



ECON408: Computational Methods in Macroeconomics

Deterministic Dynamics and Introduction to Growth Models

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Overview

Motivation and Materials

- In this lecture, we will introduce (non-linear) dynamics
 - This lets us explore stationarity and convergence
 - We will see an additional example of a **fixed point** and **convergence**
- The primary applications will be to simple models of growth, such as the **Solow** growth model.

Materials

- Adapted from QuantEcon lectures coauthored with John Stachurski and Thomas J. Sargent
 - Julia by Example
 - Dynamics in One Dimension

```
1 using LaTeXStrings, LinearAlgebra, Plots, NLsolve, Roots
2 using Plots.PlotMeasures
3 default(;legendfontsize=16, linewidth=2, tickfontsize=12,
4           bottom_margin=15mm)
```



Difference Equations

(Nonlinear) Difference Equations

$$x_{t+1} = h(x_t)$$

- A **time homogeneous first order difference equation**
 - $h : S \rightarrow S$ for some $S \subseteq \mathbb{R}$ in the univariate case
 - S is called the **state space** and x is called the **state variable**.
 - Time homogeneity: h is the same at each time t
 - First order: depends on one lag (i.e., x_{t+1} and x_t but not x_{t-1})

Trajectories

- An initial condition x_0 is required to solve for the sequence $\{x_t\}_{t=0}^{\infty}$
- Given this, we can generate a **trajectory** recursively

$$x_1 = h(x_0)$$

$$x_2 = h(x_1) = h(h(x_0))$$

$$x_{t+1} = h(x_t) = h(h(\dots h(x_0))) \equiv h^t(x_0)$$

- If not time homogeneous, we can write $x_{t+1} = h_t(x_t)$
- Stochastic if $x_{t+1} = h(x_t, \epsilon_{t+1})$ where ϵ_{t+1} is a random variable

Linear Difference Equations

$$x_{t+1} = ax_t + b$$

For constants a and b . Iterating,

$$x_1 = h(x_0) = ax_0 + b$$

$$x_2 = h(h(x_0)) = a(ax_0 + b) + b = a^2x_0 + ab + b$$

$$x_3 = a(a^2x_0 + ab + b) + b = a^3x_0 + a^2b + ab + b$$

...

$$x_t = b \sum_{j=0}^{t-1} a^j + a^t x_0 = b \frac{1 - a^t}{1 - a} + a^t x_0$$

Convergence and Stability for Linear Difference Equations

- If $|a| < 1$, take limit to check for **global stability**, for all x_0

$$\lim_{t \rightarrow \infty} x_t = \lim_{t \rightarrow \infty} g^{t-1}(x_0) = \lim_{t \rightarrow \infty} \left(b \frac{1 - a^t}{1 - a} + a^t x_0 \right) = \frac{b}{1 - a}$$

- If $a = 1$ then diverges unless $b = 0$ and $|a| > 1$ diverges for all b
- Linear difference equations are either globally stable or globally unstable
- Nonlinear difference equations may be **locally stable**
 - For some $|x_0 - x^*| < \epsilon$ for some x^* and $\epsilon > 0$. Global if $\epsilon = \infty$

Nonlinear Difference Equations

- We can ask the same questions for nonlinear $h(\cdot)$
- Keep in mind the connection to the fixed points from the previous lecture
 - If $h(\cdot)$ has a unique fixed point from any initial condition, it tells us about the dynamics
- Connecting to **contraction mappings** etc. would help us be more formal, but we will stay intuitive here
- Let us investigate nonlinear dynamics with a classic example



Solow Growth Model

Models of Economic Growth

- There are different perspectives on what makes countries grow
 - **Malthusian models:** population growth uses all available resources
 - **Capital accumulation:** more capital leads to more output, tradeoff of consumption today to build more capital for tomorrow
 - **Technological progress/innovation:** new ideas lead to more output, so the tradeoffs are between consumption today vs. researching technologies for the future
- The appropriate model depends on country and time-period
 - Malthusian models are probably most relevant right up until about the time he came up with the idea

Exogenous vs. Endogenous

- In these, the tradeoffs are key
 - Can be driven by some sort of decision driven by the agent's themselves (e.g., government plans, consumers saving, etc.) endogenously
 - Or exogenously choose, not responsive to policy and incentives
 - You always leave some things exogenous to isolate a key force
- What determines the longrun growth rate? Use **fixed points**!
 - In models of capital accumulation, technology limits the longrun growth
 - Models of innovation choice often called **endogenous growth models**

Solow Model Summary

- The **Solow** model describes aggregate growth from the perspective of accumulating physical capital
 - The tension is between **consumption** and **savings**
 - Production not directed towards consumption goods is used to build capital for future consumption
 - e.g. factories, robots, facilities, etc.
- Endogeneity
 - Technology and population growth are left fully exogenous
 - Capital accumulation occurs through an exogenously given savings rate
 - The neoclassical growth model endogenizing that rate

Technology

- In this economy, output is produced by combining labor and capital
- Labor N_t , which we assume is supplied **inelastically**
 - Assume it is proportional to the population
- Capital, K_t , which is accumulated over time
- In addition, **total factor productivity (TFP)**, z_t , is the technological level in the economy
- The physical output from operating the technology is:

$$Y_t = z_t F(K_t, N_t)$$

Properties of the Technology Function

- Note that land, etc. are NOT a factor of production
- We will assume $F(\cdot, \cdot)$ is **constant returns to scale**

$$F(\alpha K, \alpha N) = \alpha F(K, N) \quad \forall \alpha > 0$$

- Assume F has **diminishing marginal products**

$$\rightarrow \text{i.e. } \frac{\partial F(K, N)}{\partial K} > 0, \frac{\partial F(K, N)}{\partial N} > 0, \frac{\partial^2 F(K, N)}{\partial K^2} < 0, \frac{\partial^2 F(K, N)}{\partial N^2} < 0$$

Constant Returns to Scale

- Define output per worker as $y_t = Y_t/N_t$ and capital per worker as $k_t = K_t/N_t$
- Take F , divide by N_t , use CRS, and define $f(\cdot)$

$$\begin{aligned} Y_t &= z_t F(K_t, N_t) \\ \frac{Y_t}{N_t} &= \frac{z_t F(K_t, N_t)}{N_t} \\ y_t &= z_t F\left(\frac{K_t}{N_t}, \frac{N_t}{N_t}\right) = z_t F(k_t, 1) \equiv z_t f(k_t) \end{aligned}$$

→ f also has diminishing marginal products, $f'(k) > 0$, $f''(k) < 0$

Population Growth

- From some initial condition N_0 for population
- Assume that population grows at a constant rate g_N , i.e.

$$N_{t+1} = (1 + g_N)N_t$$

- Hence $N_{t+1}/N_t = 1 + g_N$ and $N_t = (1 + g_N)^t N_0$
- If $g_N < 0$ then shrinking population

Capital Accumulation

- Capital is accumulated by **investment**, X_t , with per capital $x_t \equiv X_t/N_t$
 - Macroeconomists should think in “allocations”, not “dollars”!
- Output, Y_t from production can be used for consumption or investment
- Between periods, $\delta \in (0, 1)$ proportion of capital depreciates
 - e.g. machines break down, buildings decay, etc.

$$C_t + X_t = Y_t \equiv \underbrace{z_t F(K_t, N_t)}_{\text{Total Output}}$$

$$\underbrace{K_{t+1}}_{\text{Next periods capital}} = \underbrace{(1 - \delta) K_t}_{\text{depreciation of capital}} + \underbrace{X_t}_{\text{investment in new capital}}, \delta \in (0, 1)$$

Per Capita Capital Dynamics

- Recall that $N_{t+1}/N_t = 1 + g_N$ and $y_t = z_t f(k_t)$

$$\frac{K_{t+1}}{N_t} = (1 - \delta) \frac{K_t}{N_t} + \frac{X_t}{N_t}$$

$$\frac{N_{t+1}}{N_{t+1}} \frac{K_{t+1}}{N_t} = \left(\frac{N_{t+1}}{N_t} \right) \left(\frac{K_{t+1}}{N_{t+1}} \right) = (1 - \delta) \frac{K_t}{N_t} + \frac{X_t}{N_t}$$

$$k_{t+1}(1 + g_N) = (1 - \delta)k_t + x_t$$

- So, the per-capita dynamics of capital are

$$k_{t+1} = \frac{1}{1 + g_N} [(1 - \delta)k_t + x_t]$$

Savings

- In the Solow model, the savings rate is exogenously given as $s \in (0, 1)$
 - In the neoclassical growth model, it is endogenously determined based on consumer or planner preferences
- Hence, $x_t = sy_t = sz_t f(k_t)$. Combine with previous dynamics to get

$$k_{t+1} = \frac{1}{1 + g_N} [(1 - \delta)k_t + sz_t f(k_t)]$$

- Given assumptions on $f(\cdot)$, this is a nonlinear difference equation given an exogenous z_t process
- With this, we can analyze the dynamics of y_t , k_t , and c_t over time

Steady State

- Assume that $z_t = \bar{z}$ is constant over time
- Look for **steady state**: $k_{t+1} = k_t = \bar{k}$ and $y_{t+1} = y_t = \bar{y}$, etc.
 - Note that this is a **fixed point** of the dynamics. May or may not exist

$$\begin{aligned}\bar{k} &= \frac{1 - \delta}{1 + g_N} \bar{k} + \frac{s\bar{z}}{1 + g_N} f(\bar{k}) \\ \left(\frac{1 + g_N}{1 + g_N} - \frac{1 - \delta}{1 + g_N} \right) \bar{k} &= \frac{s\bar{z}}{1 + g_N} f(\bar{k}) \\ \underbrace{(g_N + \delta)\bar{k}}_{\text{Growth-adjusted depreciation}} &= \underbrace{s\bar{z}f(\bar{k})}_{\text{investment per capita}}\end{aligned}$$

Example Production Function

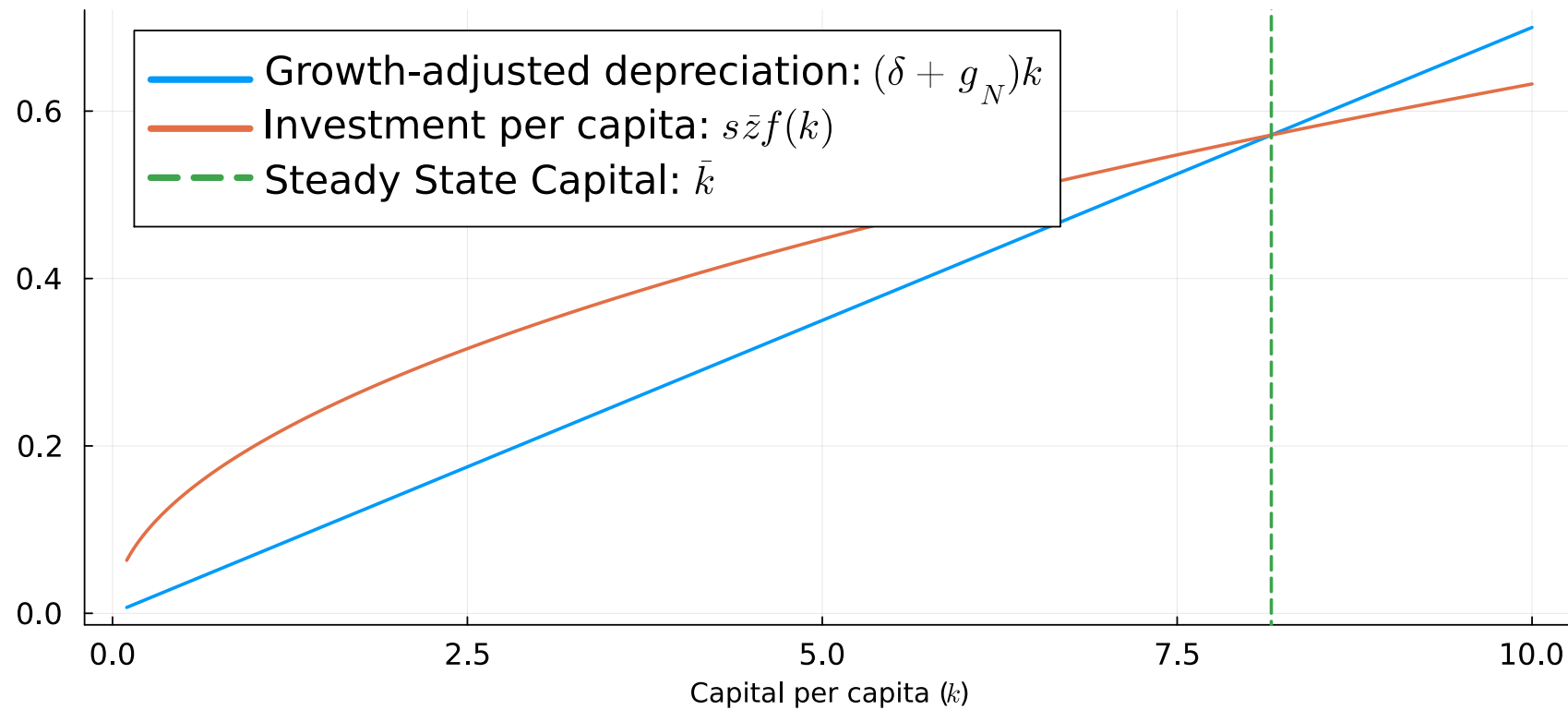
- Consider production function of $f(k) = k^\alpha$ for $\alpha \in (0, 1)$
- In this case, α will be interpretable as the **capital share** of income

$$\bar{k} = \left(\frac{s\bar{z}}{g_N + \delta} \right)^{\frac{1}{1-\alpha}}$$

Visualizing the Steady State

```
1 g_N = 0.02
2 delta = 0.05
3 s = 0.2
4 z = 1.0
5 alpha = 0.5
6 k_ss = (s * z / (g_N + delta))^(1/(1-alpha))
7
8 k_values = 0.1:0.01:10.0
9 lhs = (g_N + delta) * k_values
10 rhs = s * z .* k_values.^alpha
11
12 plot(k_values, lhs, label=L"Growth-adjusted depreciation:  $(\delta + g_N)k$ ",
13      xlabel=L"Capital per capita ( $k$ )")
14 plot!(k_values, rhs, label=L"Investment per capita:  $s \bar{z} f(k)$ ")
15 vline!([k_ss], label=L"Steady State Capital:  $\bar{k}$ ", linestyle=:dash)
```

Visualizing the Steady State



Wages and Rental Rate of Capital

- Production could be run by a planner, or by a set of firms
- Consider (real) profit maximizing firms. Price normalized to 1
→ Hire labor and capital at real rates w_t and r_t respectively

$$\max_{K_t, N_t} [z_t F(K_t, N_t) - w_t N_t - r_t K_t]$$

- The first order conditions are

$$z_t \frac{\partial F(K_t, N_t)}{\partial K_t} = r_t$$
$$z_t \frac{\partial F(K_t, N_t)}{\partial N_t} = w_t$$

Using Constant Returns to Scale

- Can show that for any CRS $F(K_t, N_t)$ that $\frac{\partial F(\gamma K_t, \gamma N_t)}{\partial K_t} = \frac{\partial F(K_t, N_t)}{\partial K_t}$
→ Same for N_t derivative. ,
- Set $\gamma = 1/N_t$ and write the marginal products as ratios,

$$\frac{\partial F(K_t, N_t)}{\partial K_t} = f'(K_t/N_t) = f'(k_t)$$

$$\frac{\partial F(K_t, N_t)}{\partial N_t} = f(k_t) - k_t f'(k_t)$$

Wages and Rental Rate of Capital

- Finally, we can write

$$\begin{aligned}r_t &= z_t f'(k_t) \\ w_t &= z_t f(k_t) - z_t f'(k_t) k_t\end{aligned}$$

- For the Cobb-Douglas production function $F(K_t, N_t) = K_t^\alpha N_t^{1-\alpha}$, we have

$$\begin{aligned}f(k_t) &= k_t^\alpha \\ f'(k_t) &= \alpha k_t^{\alpha-1} \\ r_t &= \alpha z_t k_t^{\alpha-1} \\ w_t &= z_t k_t^\alpha - z_t \alpha k_t^{\alpha-1} k_t = (1 - \alpha) z_t k_t^\alpha\end{aligned}$$

Shares of Income

- Recall that per-capita output is $y_t = z_t f(k_t)$
- $w_t = (1 - \alpha)z_t k_t^\alpha$
 - Interpret $1 - \alpha$ as the **labor share** of output, or income
- $r_t k_t = \alpha z_t k_t^\alpha$
 - Interpret α as the **capital share**
- Key to these expressions were competitive markets in hiring labor/capital
 - i.e., workers end up paid their marginal products

Solow Model Dynamics

Summary of Equations

- Exogenous z_t sequence. e.g., $z_{t+1}/z_t = 1 + g_z$ given some initial z_0
- Population growth $N_{t+1}/N_t = 1 + g_N$ given some initial N_0

$$k_{t+1} = \frac{1}{1 + g_N} [(1 - \delta)k_t + sz_t f(k_t)], \quad \text{given } k_0$$

- Output per capita $y_t = z_t f(k_t)$
- Consumption per capita $c_t = (1 - s)y_t = (1 - s)z_t f(k_t)$
- Wages $w_t = (1 - \alpha)z_t k_t^\alpha$ and rental rate of capital $r_t = \alpha z_t k_t^{\alpha-1}$
- Steady state capital $\bar{k} = \left(\frac{s\bar{z}}{g_N + \delta} \right)^{\frac{1}{1-\alpha}}$ if $g_z = 0$ and $z_0 = \bar{z}$

45 Degree Diagram

- With a fixed point $k_{t+1} = h(k_t)$ note that a fixed point is when $\bar{k} = h(\bar{k})$
- We can plot the dynamics of the sequence comparing the functions to the 45 degree line where that occurs
- This diagram will help us interpret stability and convergence

Iteration

- First, lets write a general function to iterate a (univariate) map

```
1 function iterate_map(f, x0, T)
2     x = zeros(T + 1)
3     x[1] = x0
4     for t in 2:(T + 1)
5         x[t] = f(x[t - 1])
6     end
7     return x
8 end
```

iterate_map (generic function with 1 method)

Plotting the Dynamics

```
1 function plot45(f, xmin, xmax, x0, T; num_points = 100, label = L"h(k)",
2               xlabel = "k", size = (600, 500))
3     # Plot the function and the 45 degree line
4     x_grid = range(xmin, xmax, num_points)
5     plt = plot(x_grid, f.(x_grid); xlim = (xmin, xmax), ylim = (xmin, xmax),
6              linecolor = :black, lw = 2, label, size)
7     plot!(x_grid, x_grid; linecolor = :blue, lw = 2, label = nothing)
8
9     # Iterate map and add ticks
10    x = iterate_map(f, x0, T)
11    if !isnothing(xlabel) && T > 1
12        xticks!(x, [L"%$(xlabel)_{%i}" for i in 0:T])
13        yticks!(x, [L"%$(xlabel)_{%i}" for i in 0:T])
14    end
15
16    # Plot arrows and dashes
17    for i in 1:T
```

plot45 (generic function with 1 method)

Fixed Points

```
1 # Implementation of our capital dynamics
2 h(k; p) = (1 / (1 + p.g_N)) * (
3     p.s * p.z_bar * k^p.alpha
4     + (1 - p.delta) * k)
5 k_bar(p) = (p.s * p.z_bar /
6     (p.g_N + p.delta))^(1/(1-p.alpha))
7 p = (;z_bar = 2.0, s = 0.3, alpha = 0.3,
8     delta = 0.4, g_N = 0.0)
9 @show k_bar(p)
10 @show h(k_bar(p); p)
11 @show h(0.0;p);
```

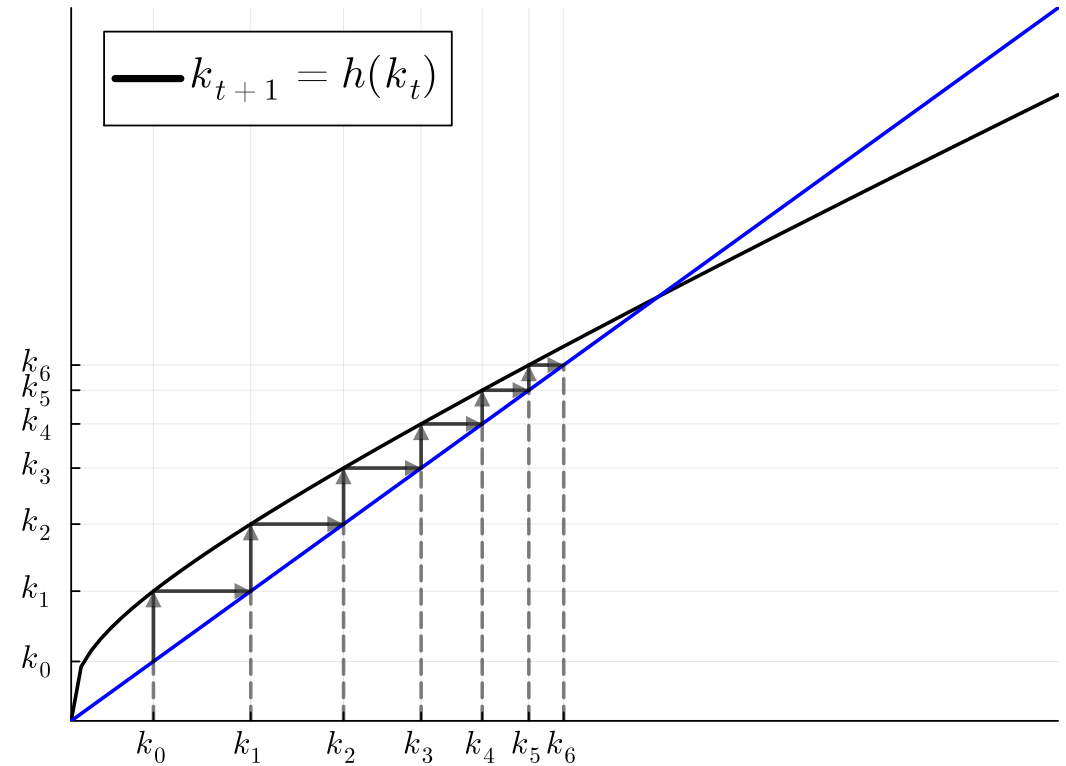
```
k_bar(p) = 1.7846741842265788
h(k_bar(p); p) = 1.7846741842265788
h(0.0; p) = 0.0
```

45 Degree Diagram for Solow

```

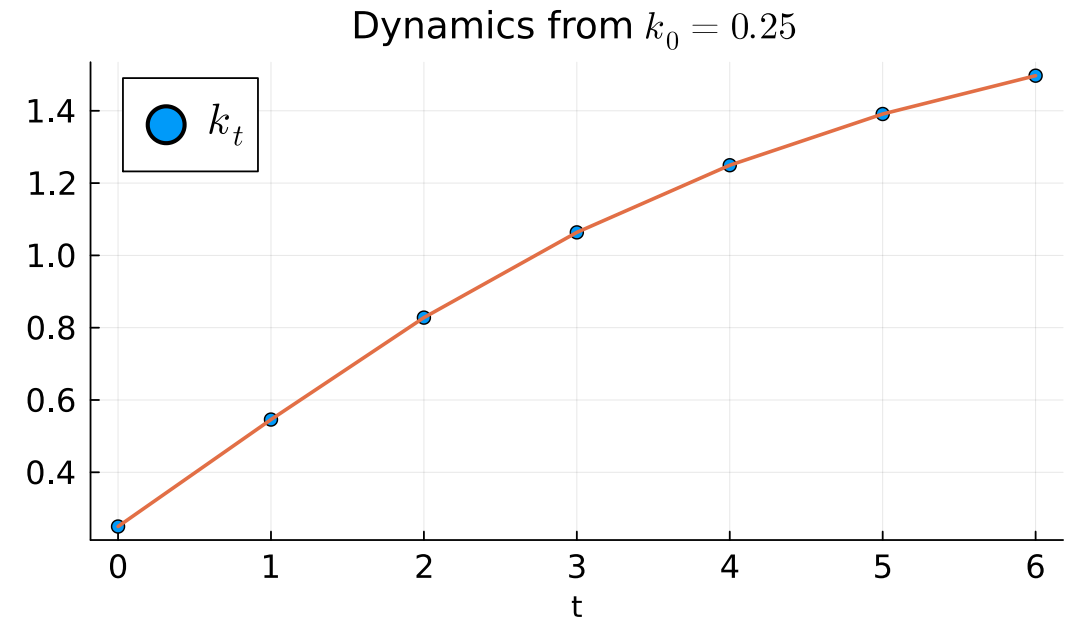
1 k_min = 0.0
2 k_max = 3.0
3 k_0 = 0.25
4 T = 6
5 plot45(k -> h(k; p), k_min, k_max, k_0,
6         T; label = L"k_{t+1} = h(k_t)")

```



Transition Dynamics

```
1 k_vals = iterate_map(k -> h(k; p), k_0, T)
2 plot(0:T, k_vals; label=[L"k_t" nothing],
3      title=L"Dynamics from $k_0 = %$k_0$",
4      seriestype = [:scatter, :line],
5      xlabel = "t", size=(600, 400))
```

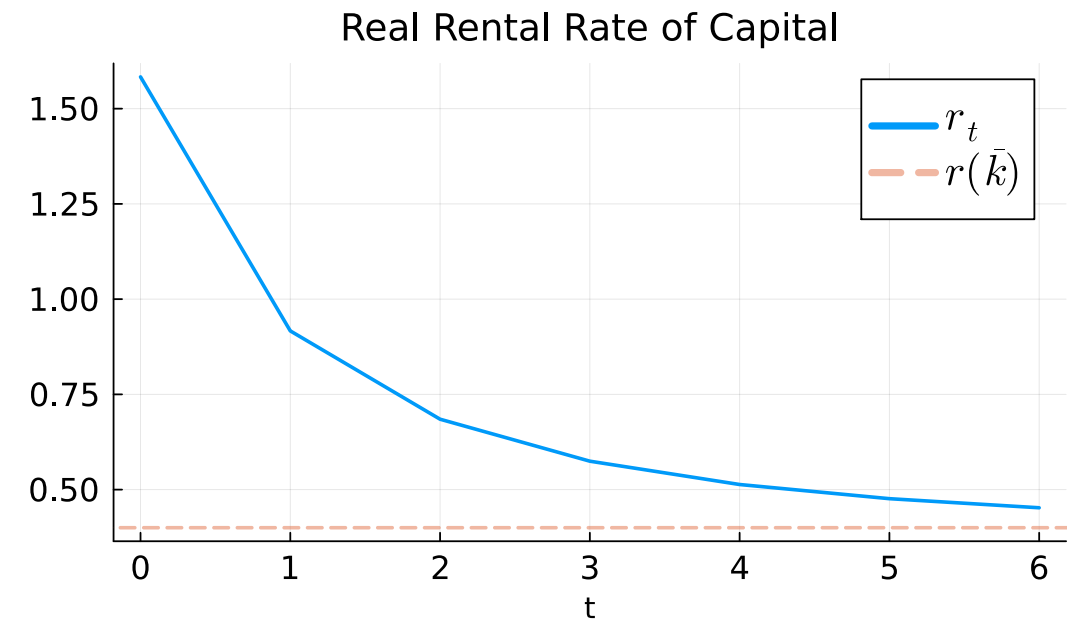


Rental Rate of Capital

- Why does it decrease?

```
1 r(k;p) = p.alpha * p.z_bar * k^(p.alpha - 1)
2 w(k;p) = (1 - p.alpha) * p.z_bar * k^(p.alpha)
3
4 @show k_0
5 plot(0:T, r.(k_vals; p); label = L"r_t",
6       title="Real Rental Rate of Capital",
7       xlabel = "t", size=(600, 400))
8 hline!([r(k_bar(p);p)];linestyle=:dash,
9         label=L"r(\bar{k})", alpha=0.5)
```

$k_0 = 0.25$

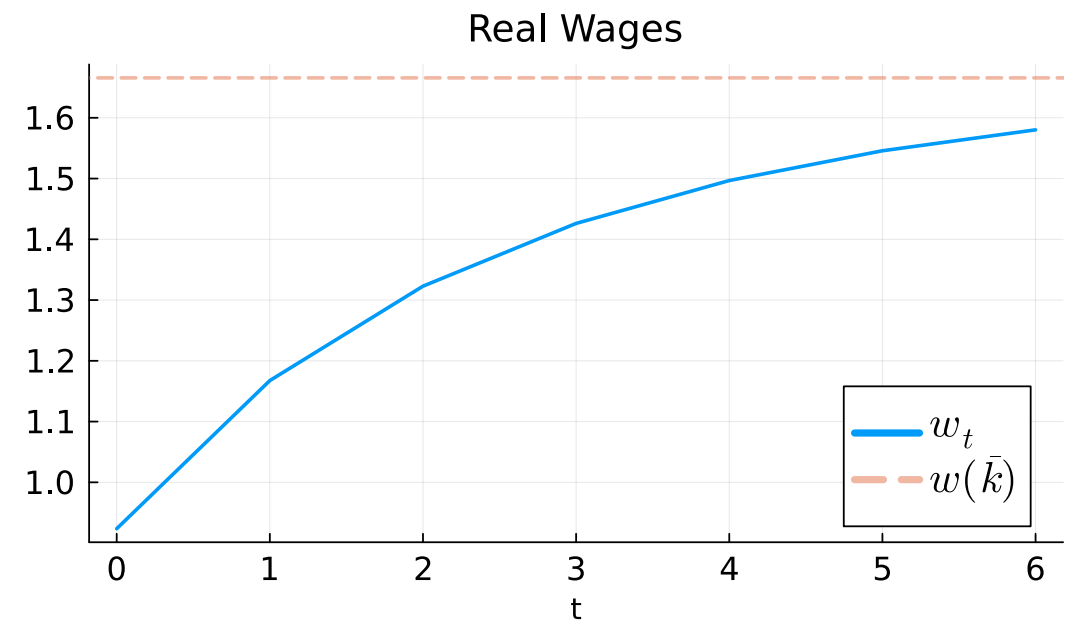


Wages

- Why does it increase?

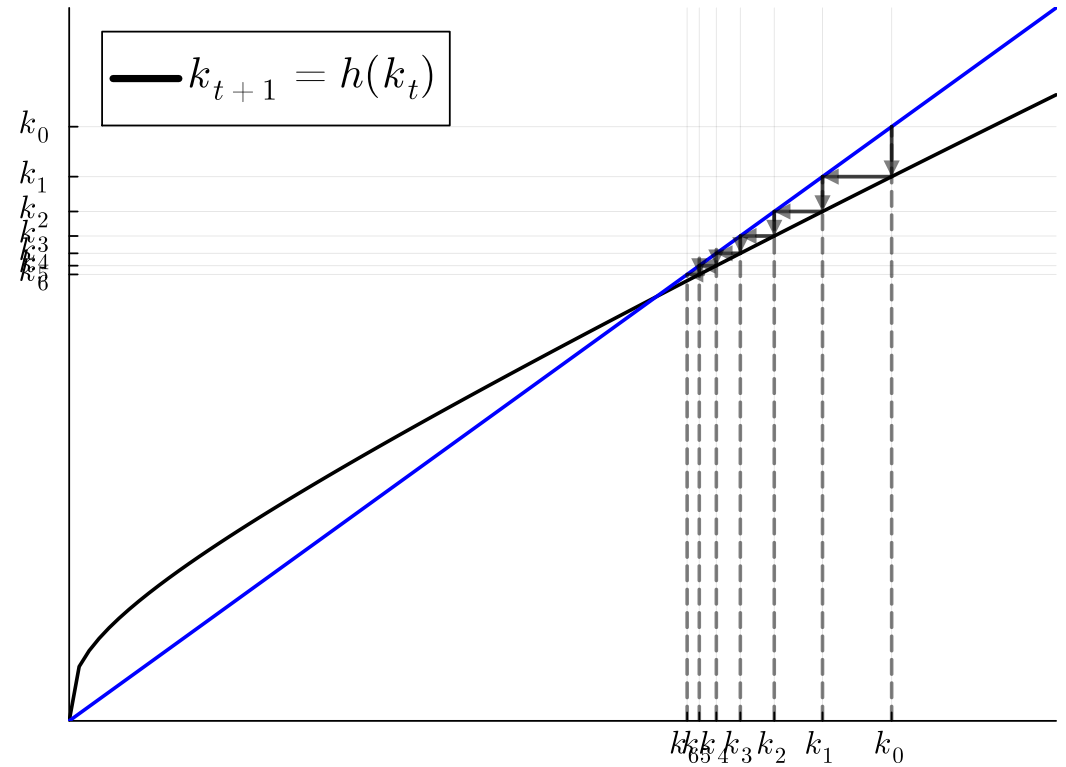
```
1 @show k_0
2 plot(0:T, w.(k_vals; p); label = L"w_t",
3      title="Real Wages",
4      xlabel = "t", size=(600, 400))
5 hline!([w(k_bar(p);p)];linestyle=:dash,
6        label=L"w(\bar{k})", alpha=0.5)
```

$k_0 = 0.25$



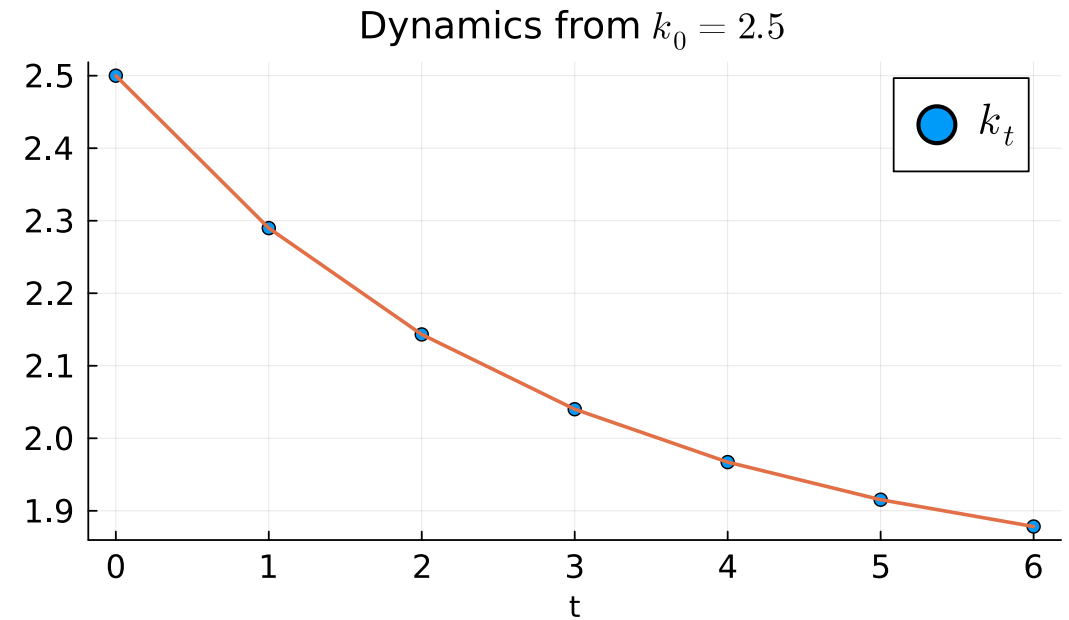
Above the Steady State?

```
1 k_0 = 2.5
2 plot45(k -> h(k; p), k_min, k_max, k_0,
3         T; label = L"k_{t+1} = h(k_t)")
```



Transition Dynamics

```
1 k_vals = iterate_map(k -> h(k; p), k_0, T)
2 plot(0:T, k_vals; label=[L"k_t" nothing],
3      title=L"Dynamics from $k_0 = %$k_0$",
4      seriestype = [:scatter, :line],
5      xlabel = "t", size=(600, 400))
```

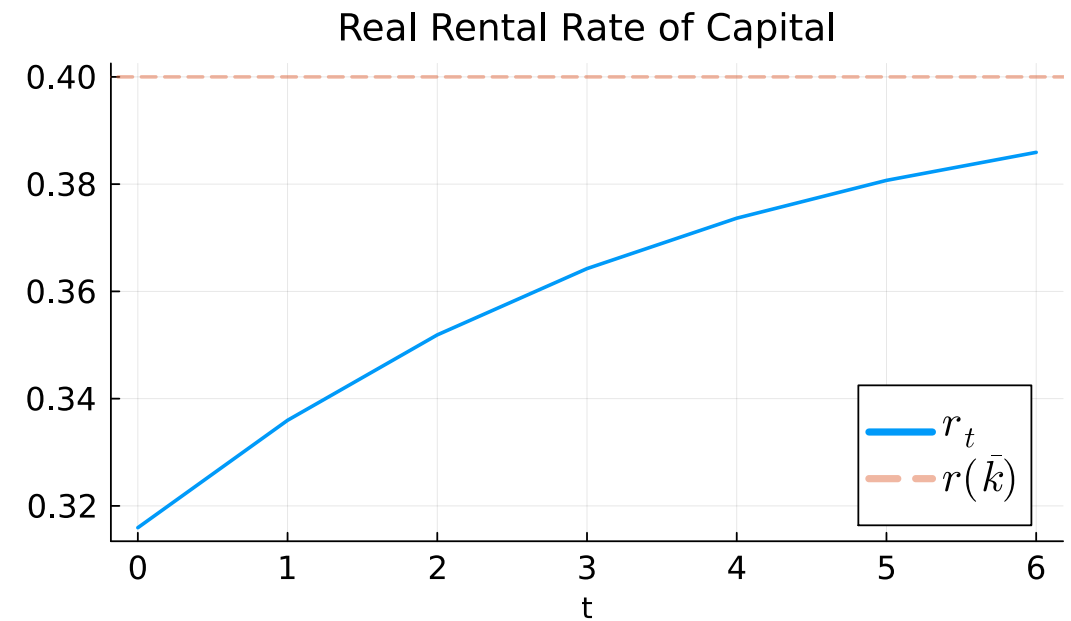


Rental Rate of Capital

- Why does it increase?

```
1 @show k_0
2 plot(0:T, r.(k_vals; p); label = L"r_t",
3      title="Real Rental Rate of Capital",
4      xlabel = "t", size=(600, 400))
5 hline!([r(k_bar(p);p)];linestyle=:dash,
6        label=L"r(\bar{k})", alpha=0.5)
```

$k_0 = 2.5$

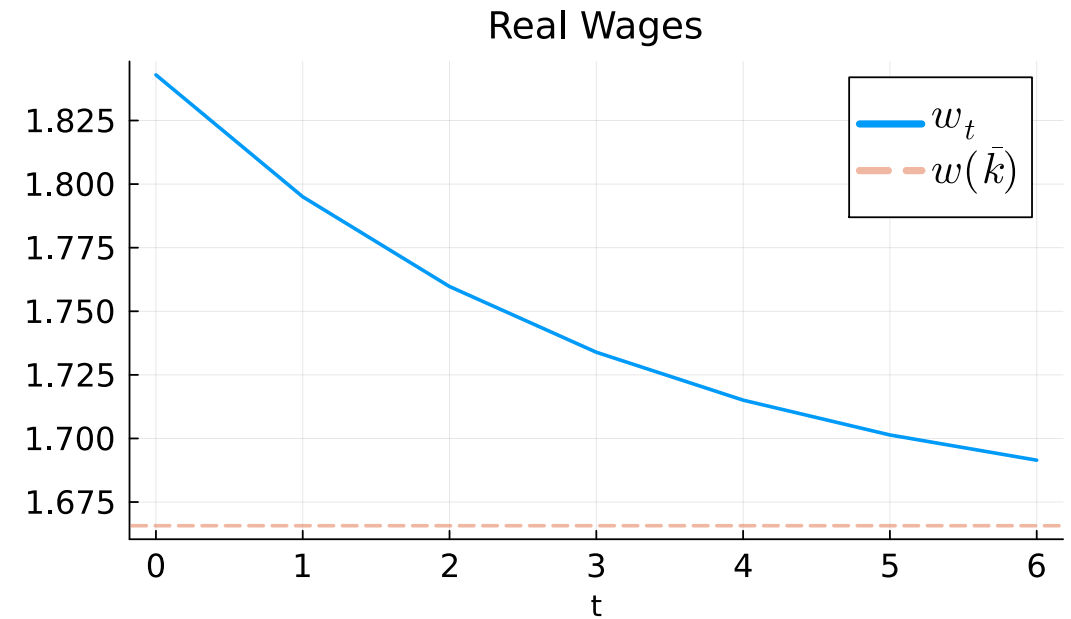


Wages

- Why does it decrease?

```
1 @show k_0
2 plot(0:T, w.(k_vals; p); label = L"w_t",
3      title="Real Wages",
4      xlabel = "t", size=(600, 400))
5 hline!([w(k_bar(p);p)];linestyle=:dash,
6        label=L"w(\bar{k})", alpha=0.5)
```

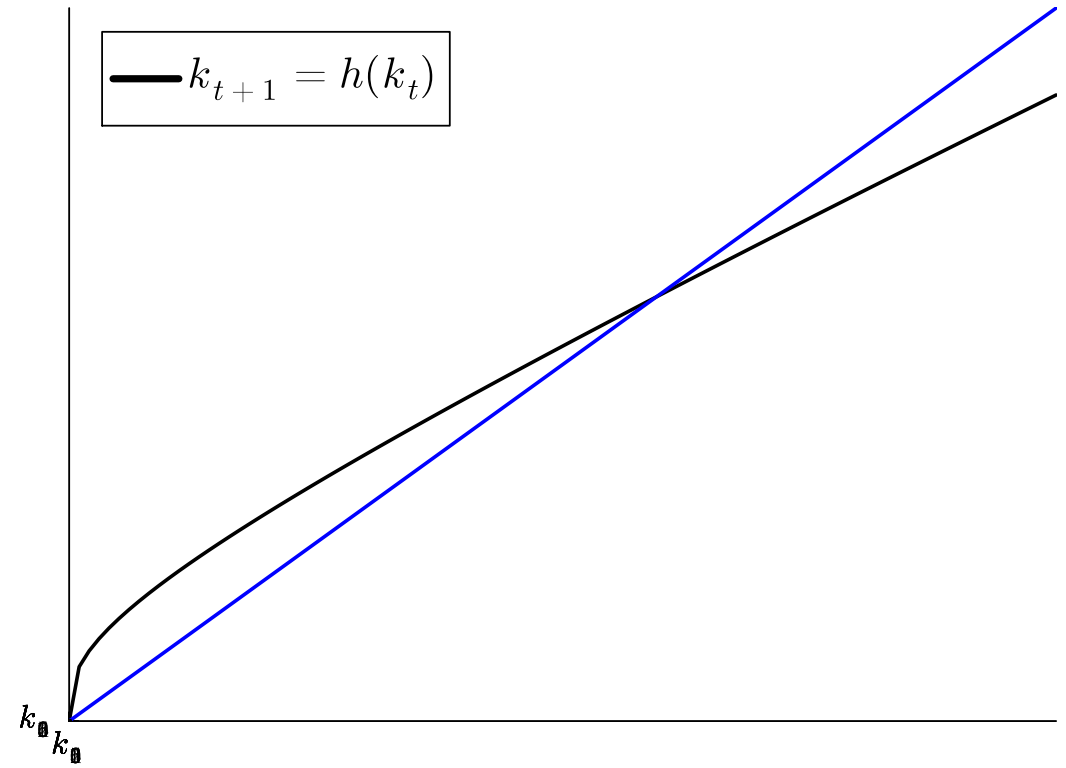
$k_0 = 2.5$



At Zero Capital

```
1 k_0 = 0.0
2 @show h(k_0; p)
3 plot45(k -> h(k; p), k_min, k_max, k_0,
4         T; label = L"k_{t+1} = h(k_t)")
```

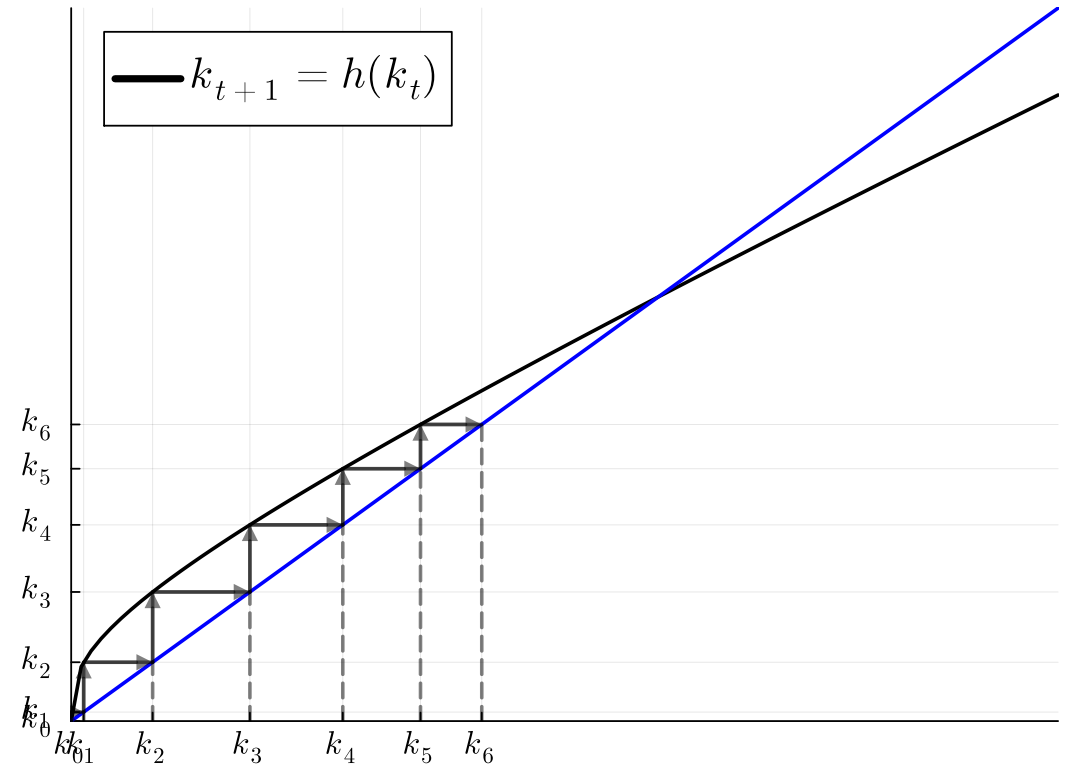
$$h(k_0; p) = 0.0$$



Perturb Zero Capital

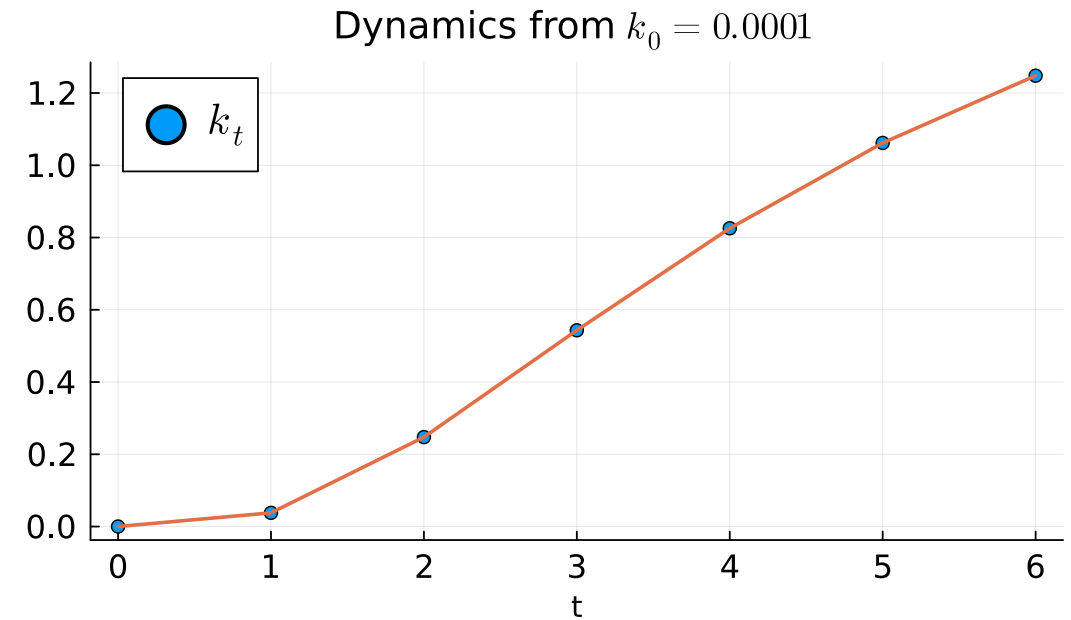
```
1 k_0 = 0.0001
2 @show h(k_0; p)
3 plot45(k -> h(k; p), k_min, k_max, k_0,
4         T; label = L"k_{t+1} = h(k_t)")
```

$h(k_0; p) = 0.03791744066881159$



Transition Dynamics

```
1 k_vals = iterate_map(k -> h(k; p), k_0, T)
2 plot(0:T, k_vals; label=[L"k_t" nothing],
3      title=L"Dynamics from $k_0 = %$k_0$",
4      seriestype = [:scatter, :line],
5      xlabel = "t", size=(600, 400))
```



Marginal Product of Capital

- Recall that the Marginal Product of Capital (MPK) is

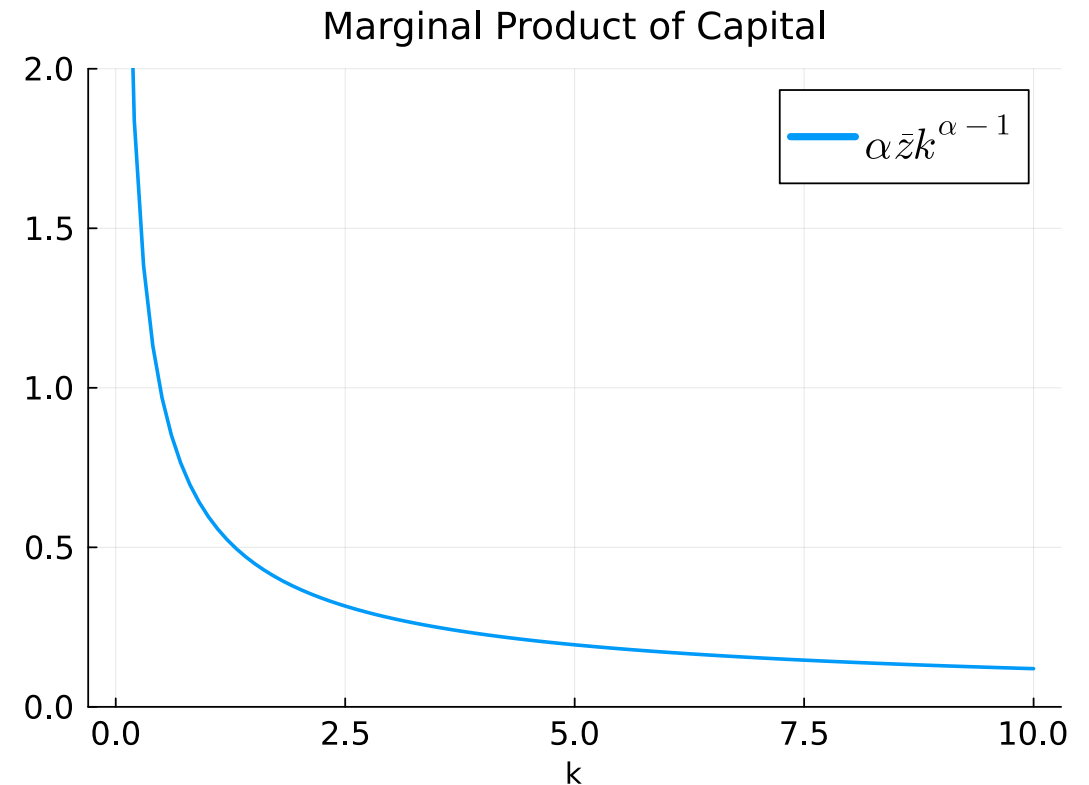
$$z_t \frac{\partial F(K_t, N_t)}{\partial K_t} = z_t f'(K_t/N_t) = \alpha z_t k_t^{\alpha-1}$$

- How does the MPK change as the economy grows?

Visualizing the MPK

- Strictly positive, monotonically decreasing, asymptote at $k = 0$

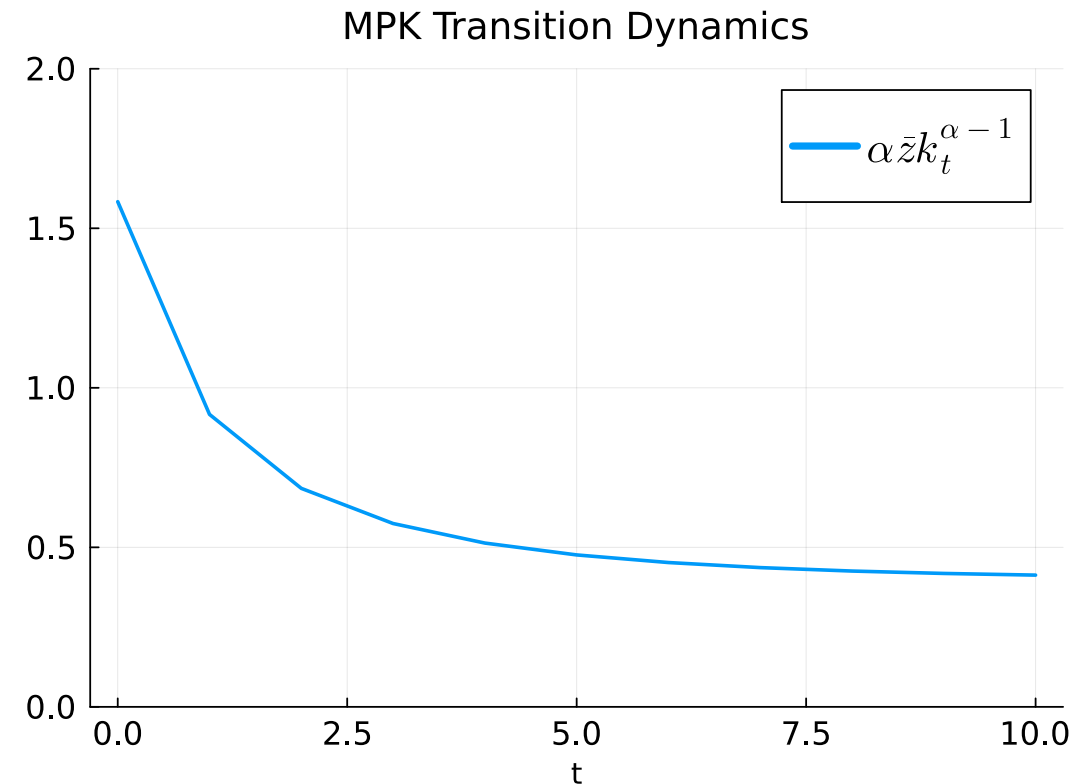
```
1 # No g_N here, that is in dynamics
2 MPK(k; p) = p.alpha * p.z_bar * k^(p.alpha - 1)
3 k_vals = range(0.0, 10.0; length=100)
4 plot(k_vals, MPK.(k_vals; p);
5       label = L"\alpha\bar{z}k^{\alpha-1}",
6       xlabel = "k",
7       title="Marginal Product of Capital",
8       size=(600, 500), ylim=(0.0, 2.0))
```



Visualizing the MPK Transition

- Converges to a point where capital can't accumulate without smaller s

```
1 k_0 = 0.25
2 T = 10
3 k_vals = iterate_map(k -> h(k; p), k_0, T)
4 plot(0:T, MPK.(k_vals; p);
5     label = L"\alpha\bar{z}k_t^{\alpha-1}",
6     xlabel = "t",
7     title="MPK Transition Dynamics",
8     size=(600, 500), ylim=(0.0, 2.0))
```



Gradients of $h(k)$

$$\nabla h(k; p) = \frac{1}{1 + g_N} (\alpha s z_t k^{1-\alpha} + 1 - \delta)$$

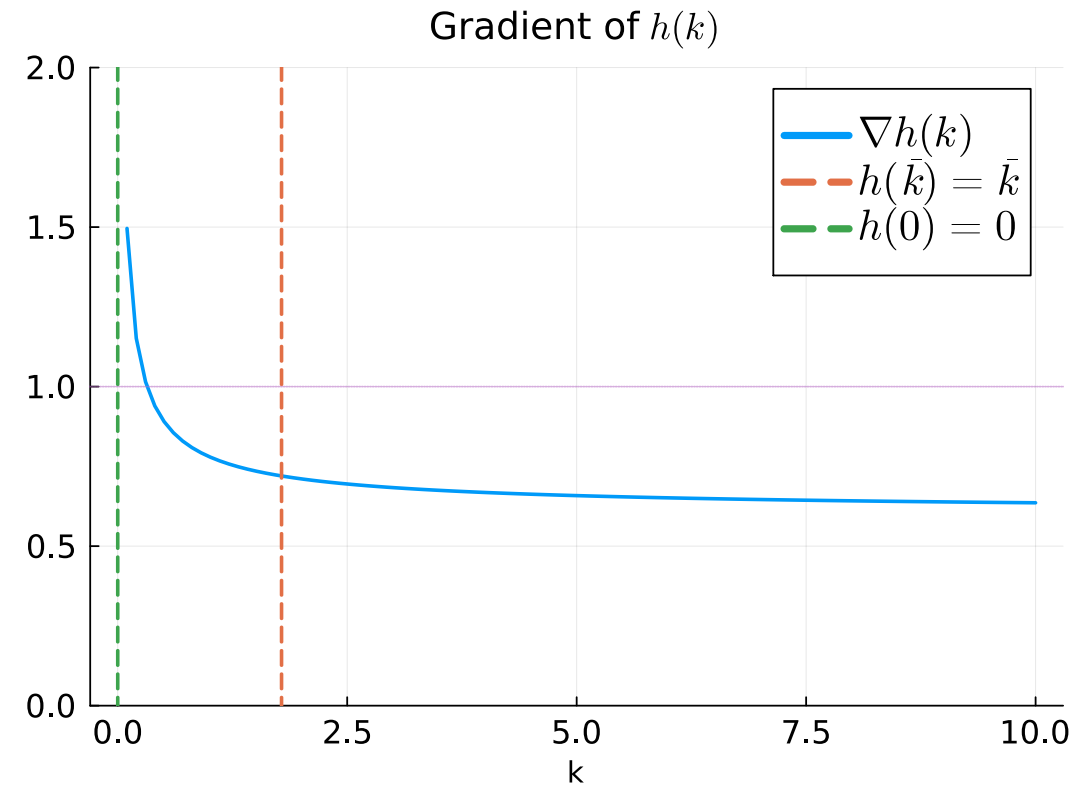
- Recall fixed points are $k^* = h(k^*)$
- Contraction mappings are a global property where points get closer
- Locally, we can ask the same question. Gradients help us understand whether points expand or contract

Plotting $\nabla h(k)$

```

1  h_k(k; p) = (1 / (1 + p.g_N)) * (
2      p.alpha * p.s * p.z_bar * k^(p.alpha-1)
3      + 1 - p.delta)
4  k_vals = range(0.0, 10.0; length=100)
5  plot(k_vals, h_k.(k_vals; p);
6      label = L"\nabla h(k)",
7      xlabel = "k",
8      title=L"Gradient of $h(k)$",
9      size=(600, 500), ylim=(0.0, 2.0))
10 vline!([k_bar(p)];label=L"h(\bar{k})=\bar{k}",
11        linestyle=:dash)
12 vline!([0.0];label=L"h(0)=0", linestyle=:dash)
13 hline!([1.0];linestyle=:dot, label=nothing,
14        alpha=0.5, lw=1)

```



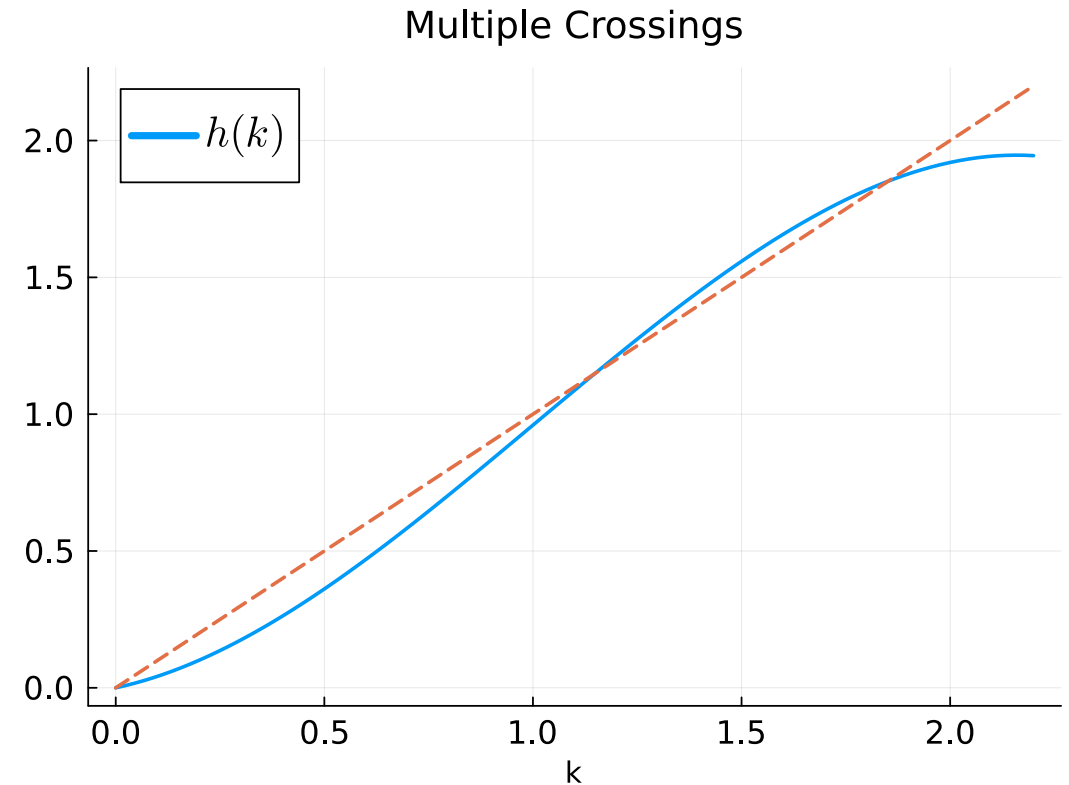
Local and Global Stability

- Consider a fixed point $\bar{k} = h(\bar{k})$
- \bar{k} is **local stable** if there exists an $\epsilon > 0$
 - For all $|k - \bar{k}| < \epsilon$, $\lim_{T \rightarrow \infty} h^T(k) = \bar{k}$
- i.e., starting close to the steady state it converges to steady state
 - **Global stability** if $\epsilon = \infty$
 - Given some continuity assumptions, can show that $\nabla h(\bar{k}) < 1$ is sufficient
- Solow: $k = 0$ blows up, so this is only locally stable for $\epsilon = \bar{k}$

Non-convexity in Production

- Let $f(k) = -\frac{1}{3}k^3 + k^2 + \frac{1}{3}k$, $h(k) = sf(k) + (1 - \delta)k$

```
1 function h_multi(k;s=0.95, delta=0.99)
2   return s*(-k^3/3 + k^2 + k/3) + (1 - delta)*k
3 end
4 k_min = 0.0
5 k_max = 2.2
6 k_vals = range(k_min, k_max; length=100)
7 plot(k_vals, h_multi.(k_vals);
8       label = L"h(k)",
9       xlabel = "k",
10      title="Multiple Crossings",
11      size=(600, 500))
12 plot!(k_vals, k_vals; label=nothing,
13        style=:dash)
```



Fixed Points

- There are now 3 fixed points
- In this case we can eyeball the graph for initial conditions

```
1 h_multi_vec(k_vec) = [h_multi(k_vec[1])] # pack/unpack as vector
2 k_star_1 = fixedpoint(h_multi_vec, [0.0]).zero[1]
3 k_star_2 = fixedpoint(h_multi_vec, [1.0]).zero[1]
4 k_star_3 = fixedpoint(h_multi_vec, [2.0]).zero[1]
5 @show k_star_1, k_star_2, k_star_3;
```

```
(k_star_1, k_star_2, k_star_3) = (0.0, 1.1483123394919963, 1.8516876604860064)
```

Detour into Roots of Equations

- A **root** or zero of a function $\hat{h}(\cdot)$ is a point k^* such that $\hat{h}(k^*) = 0$
- Fixed points $k^* = h(k^*)$ is a **root** of the equation $\hat{h}(k) \equiv h(k) - k$
- Alternative algorithms for univariate solvers can bracket a solution

```
1 sol = nlsolve(k -> h_multi_vec(k) .- k, [1.0]) # vectorized, NLSolve
2 @show sol.zero
3 @show find_zero(k -> h_multi(k) - k, (1.0, 1.5)); # bracketed, from Roots
```

```
sol.zero = [1.1483123394996155]
find_zero((k->begin
    #= In[23]:3 =#
    h_multi(k) - k
end), (1.0, 1.5)) = 1.1483123395307762
```


Steady States and Gradients

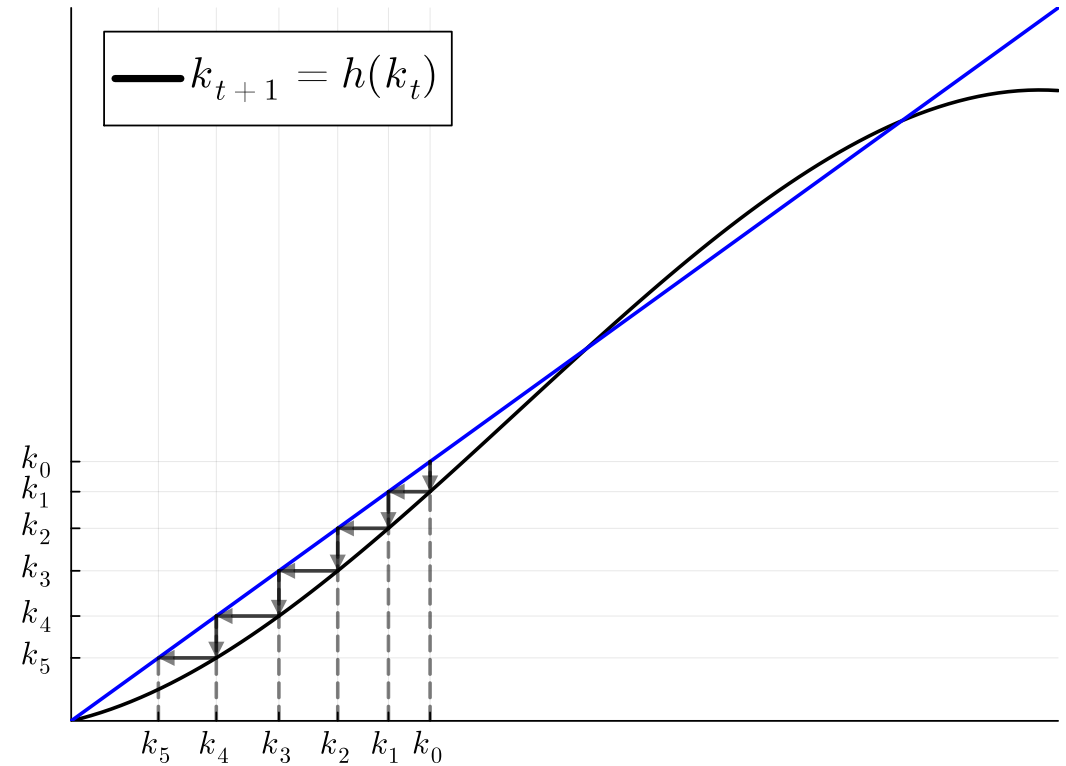
- There are now 3 fixed points, lets look at the gradients
- Remember that $\nabla h(\bar{k}) < 1$ is sufficient for stability

```
1 h_multi_k(k;s=0.95, delta=0.99) = s*(-k^2 + 2 * k + 1/3) + (1 - delta)
2 @show k_star_1, h_multi_k(k_star_1)
3 @show k_star_2, h_multi_k(k_star_2)
4 @show k_star_3, h_multi_k(k_star_3);
```

```
(k_star_1, h_multi_k(k_star_1)) = (0.0, 0.32666666666666666)
(k_star_2, h_multi_k(k_star_2)) = (1.1483123394919963, 1.2557699441233567)
(k_star_3, h_multi_k(k_star_3)) = (1.8516876604860064, 0.5875633891937458)
```

In Increasing Returns Region

```
1 k_0 = 0.8
2 T = 5
3 plot45(h_multi, k_min, k_max, k_0,
4         T; label = L"k_{t+1} = h(k_t)")
```

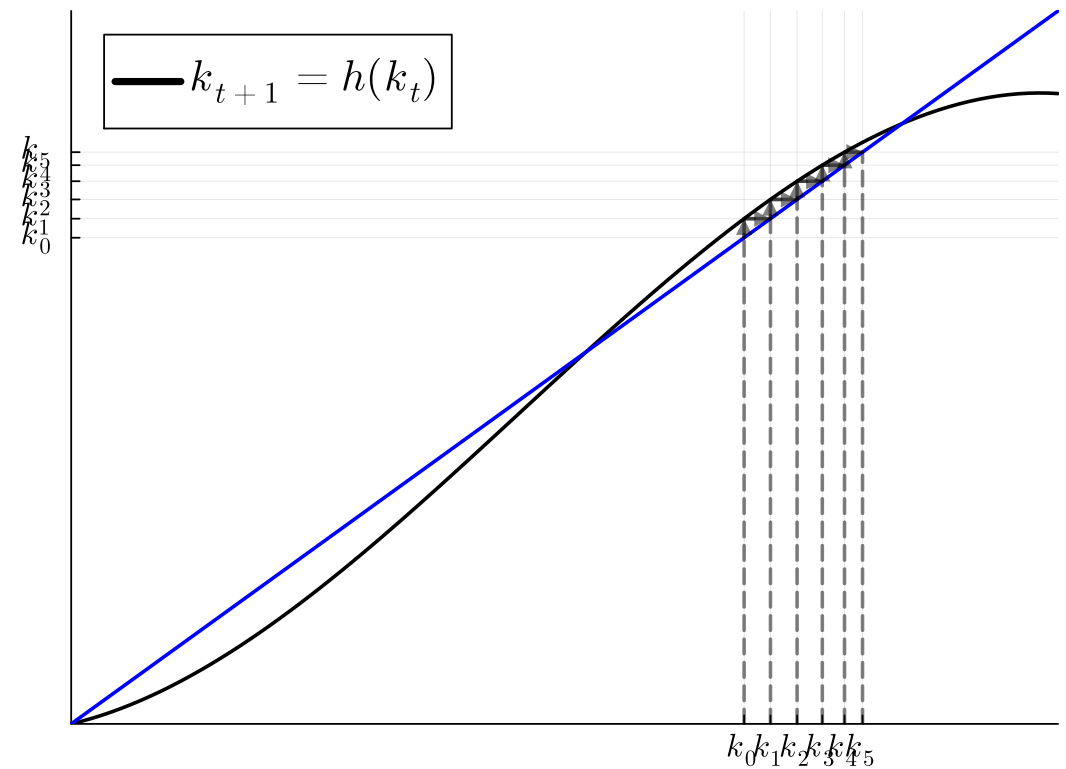


In Decreasing Returns Region

```

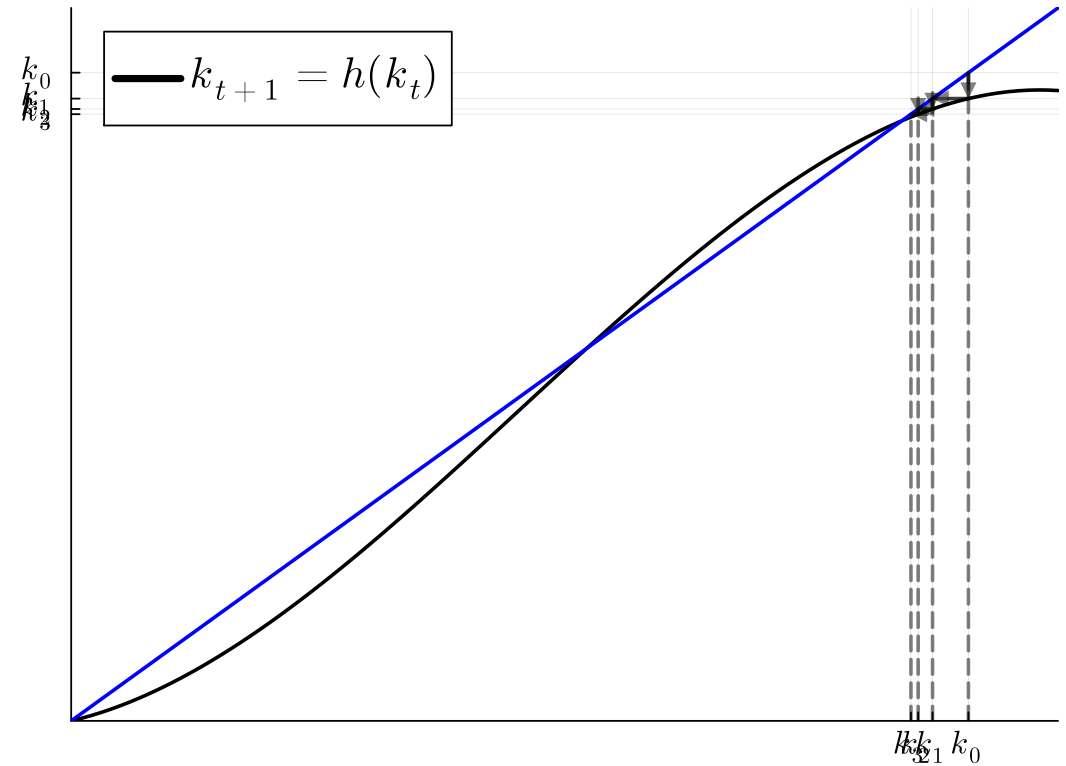
1  k_0 = 1.5
2  T = 5
3  plot45(h_multi, k_min, k_max, k_0,
4         T;label = L"k_{t+1} = h(k_t)")

```



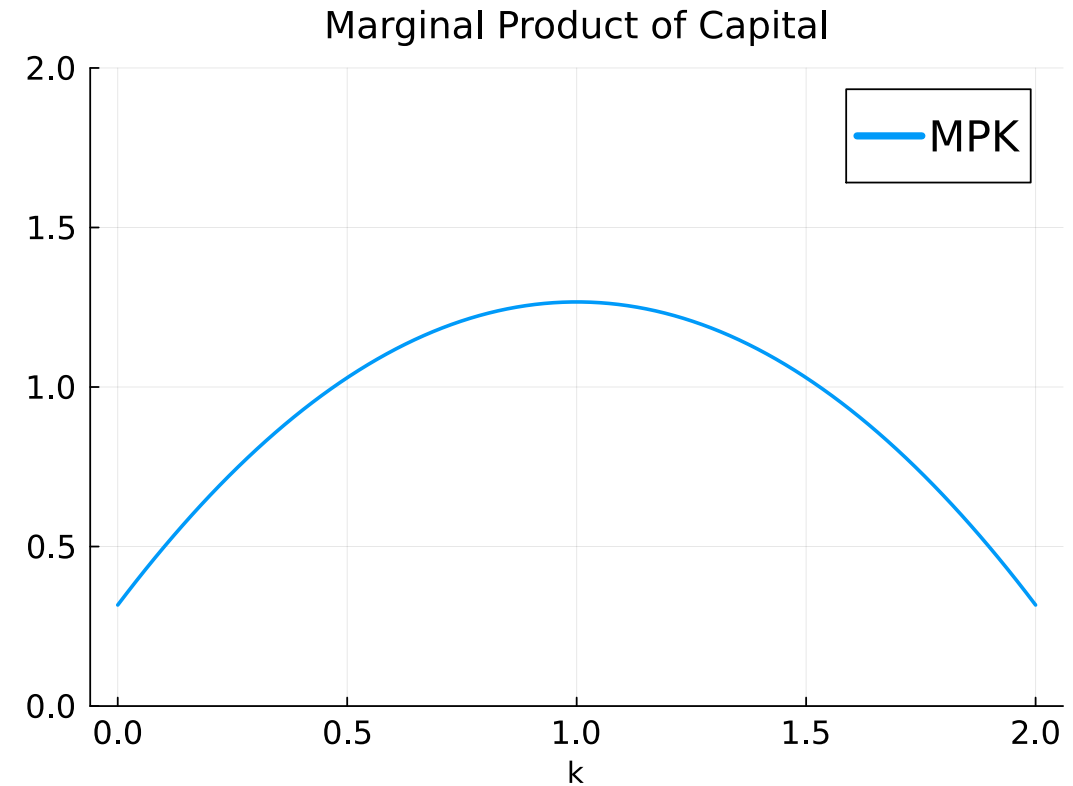
In Decreasing Returns Region

```
1 k_0 = 2.0
2 T = 3
3 plot45(h_multi, k_min, k_max, k_0,
4         T;label = L"k_{t+1} = h(k_t)")
```



Visualizing the MPK

```
1 function MPK_multi(k;s=0.95, delta=0.99)
2     return s*(-k^2 + 2 * k + 1/3)
3 end
4 k_vals = range(0.0, 2.0; length=100)
5 plot(k_vals, MPK_multi.(k_vals);
6     label = "MPK",
7     xlabel = "k",
8     title="Marginal Product of Capital",
9     size=(600, 500), ylim=(0.0, 2.0))
```



Malthusian Model

Population Growth and Fixed Factors

- With population growth ($g_N > 0$), investment leads to more capital accumulation through savings
- Wages and consumption increase in the Solow model
 - The key is that capital can expand
- What if resources are limited (and no substitutes)?

Malthusian Model

- Malthusian model is pretty accurate for most of human history
 - Population growth expands when food/shrinks with scarcity
 - Productivity can grow, but so can population
 - Land is in fixed supply
- Assume there is a subsistence consumption per capita

Population Growth

- Consumption per capital is $c_t \equiv Y_t/N_t$
 - i.e., no savings, all production to consumption
- Subsistence consumption per capita is c^*
- Population growth rate for some $\gamma \in (0, 1)$

$$g_N(Y_t, N_t) \equiv \left(\frac{c_t}{c^*} \right)^\gamma - 1$$

- Note: $c_t > c^* \implies g_N > 0$ and $c_t < c^* \implies g_N < 0$

Production

- Production is $Y_t = z_t F(L, N_t)$ where L is land
 - Same assumptions as before for $F(L, N_t) = L^\alpha N_t^{1-\alpha}$
 - Let $\ell_t \equiv L/N_t$ be land per capita
- Then following CRS logic, we see that consumption per capital is

$$y_t = c_t = z_t f(\ell_t) = z_t \ell_t^\alpha$$

Substitute into Population Growth

$$\begin{aligned}\frac{N_{t+1}}{N_t} &= 1 + g_N(N_t) = \left(\frac{c_t}{c^*}\right)^\gamma \\ &= \left(\frac{z_t \ell_t^\alpha}{c^*}\right)^\gamma = \left(\frac{z_t}{c^*}\right)^\gamma \ell_t^{\alpha\gamma} \\ &= \left(\frac{z_t}{c^*}\right)^\gamma L^{\alpha\gamma} N_t^{-\alpha\gamma} \\ N_{t+1} &= \left(\frac{z_t}{c^*}\right)^\gamma L^{\alpha\gamma} N_t^{1-\alpha\gamma}\end{aligned}$$

Steady State

- For a fixed $z = \bar{Z}$, assume \bar{N} and substitute

$$\bar{N} = \left(\frac{\bar{z}}{c^*} \right)^\gamma L^{\alpha\gamma} \bar{N}^{1-\alpha\gamma}$$

$$\bar{N}^{\alpha\gamma} = \left(\frac{\bar{z}}{c^*} \right)^\gamma L^{\alpha\gamma}$$

$$\bar{N} = \left(\frac{\bar{z}}{c^*} \right)^{\frac{\gamma}{\alpha\gamma}} L^{\frac{\alpha\gamma}{\alpha\gamma}} = \left(\frac{\bar{z}}{c^*} \right)^{\frac{1}{\alpha}} L$$

$$\bar{c} = \bar{z} \bar{\ell}^\alpha = \bar{z} \left(\frac{L}{\bar{N}} \right)^\alpha = \bar{z} \left(\left(\frac{c^*}{\bar{z}} \right)^{\frac{1}{\alpha}} \frac{L}{L} \right)^\alpha = c^*$$

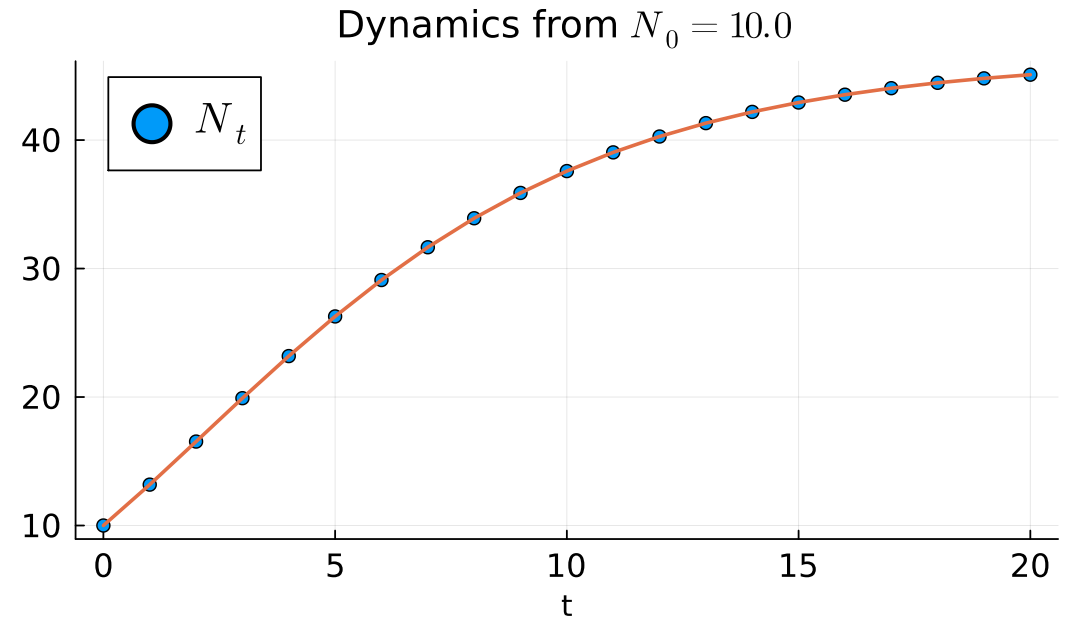
Implementation

```
1 h(N; p) = (p.z_bar / p.c_star)^p.gamma * p.L^(p.alpha)
2 N_bar(p) = (p.z_bar / p.c_star)^(1/p.alpha) * p.L
3 c(N; p) = p.z_bar * (p.L / N)^p.alpha
4 p = (;z_bar = 1.0, c_star = 0.1, alpha = 0.6,
5      gamma = 0.3, L = 1.0)
6 @show N_bar(p);
```

$N_{\text{bar}}(p) = 46.4158883361278$

Population Growth

```
1 N_0 = 10.0
2 T = 20
3 N_vals = iterate_map(N -> h(N; p), N_0, T)
4 plot(0:T, N_vals; label=[L"N_t" nothing],
5      title=L"Dynamics from $N_0 = %N_0$",
6      seriestype = [:scatter, :line],
7      xlabel = "t", size=(600, 400))
```

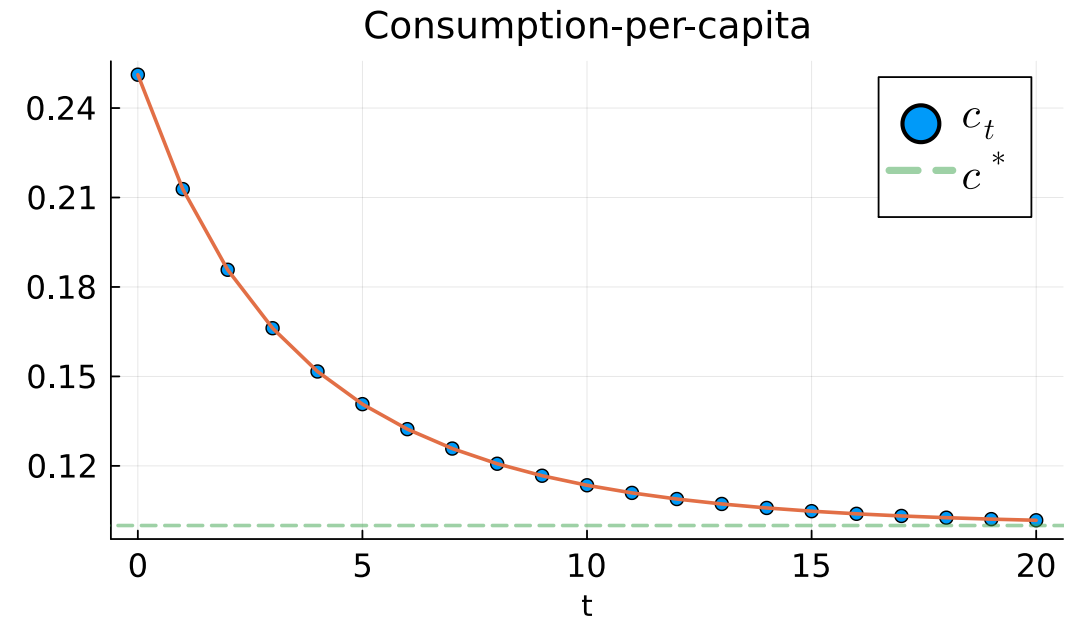


Consumption per Capita

```

1 c_vals = c.(N_vals; p)
2 plot(0:T, c_vals; label = [L"c_t" nothing],
3      title="Consumption-per-capita",
4      seriestype = [:scatter, :line],
5      xlabel = "t", size=(600, 400))
6 hline!([p.c_star];linestyle=:dash,
7        label=L"c^*", alpha=0.5)

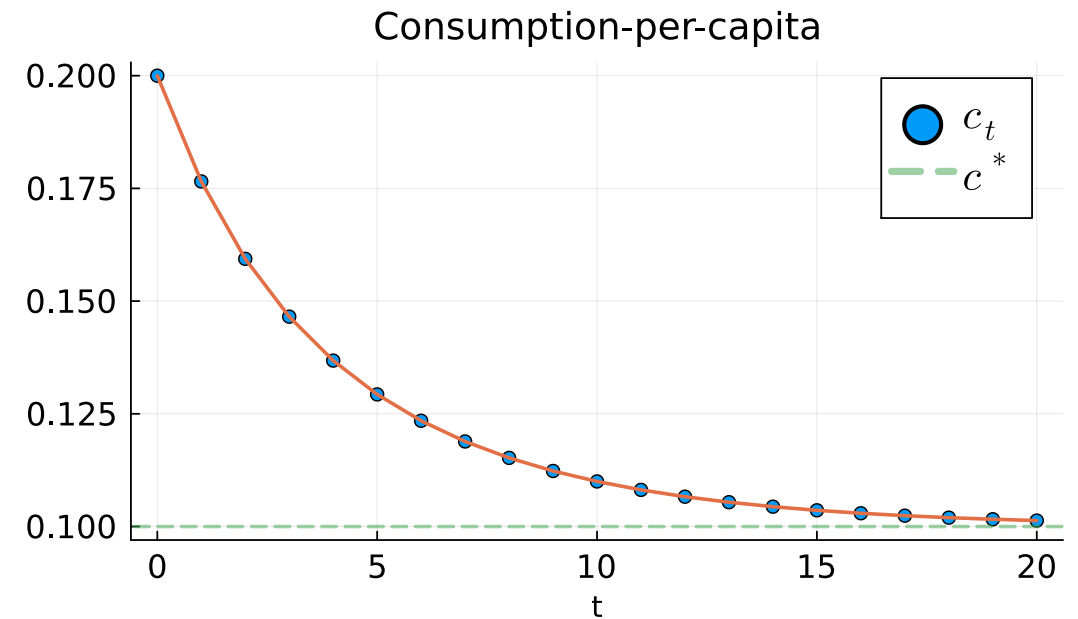
```



Technological Growth!

- Start at \bar{N} then double \bar{z}

```
1 N_0 = N_bar(p) # old steady state
2 p = merge(p, (; z_bar = 2.0)) # changes a field
3 N_vals = iterate_map(N -> h(N; p), N_0, T)
4 c_vals = c.(N_vals; p)
5 plot(0:T, c_vals; label=[L"c_t" nothing],
6      title="Consumption-per-capita",
7      seriestype = [:scatter, :line],
8      xlabel = "t", size=(600, 400))
9 hline!([p.c_star];linestyle=:dash,
10       label=L"c^*", alpha=0.5)
```



Pessimistic Perspective on Technology

- Population will expand until subsistence consumption is reached
- Technology growth only leads to a higher population, not to material welfare gains
- The key assumption here: Fixed factors and population growth
- Are there fixed factors with modern production technologies?