ECON408: Computational Methods in Macroeconomics

Optimal Consumption, Savings, and the Permanent Income Model

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Overview

Motivation

- Previously, the savings rate has been an exogenously given function of income
- In this section we will analyze optimal consumption and savings decisions in a simple model: the classic permanent income model of Milton Friedman and refined by Hall (1978)
  - → Given these decision processes, we could embed them into our previous models of income dynamics

Fort Knox

- Fort Knox holds about 13K tons of gold
  → Secured in granite, with a steel door 21 inches thick weighing 20 tons
- The data:
  → Fort Knox has been around since 1937 and has never been robbed
- Counterfactual: How much should we spend on protection?
- What if we only look at the historical data
  → If it has never been robbed, why does it need to be secured at all?
- "Structural" models let us consider a broader class of counterfactuals
  → Need to have a "deep" model of criminal incentives, detection ability, etc. to see the appropriate level of security
- This is used as a classic example of the Lucas Critique

Exogenous vs. Endogenous Savings

- Why do we need to bother with a model of savings and consumption? Couldn't we just estimate it from the data use it empirically?
- The challenge is that this only leads to a limited number of counterfactuals
  - → For example, we can simulate a panel of agents living in a fixed economy spanned by the data
- But what about numerical experiments where the environment changes??
  - → A tax cut when future taxes balance budget
  - → Wouldn't the savings rate change in response to these plans?
- This led early macro-economists to consider that the Marginal Propensity to Consume (MPC) might adjust based on information sets alone

Materials

- Adapted from QuantEcon lectures coauthored with John Stachurski and Thomas J. Sargent
  - → Optimal Savings I: The Permanent Income Model
    - Note using $F_t = -b_t$. i.e., financial assets rather than debt

```julia
using Distributions, Plots, LaTeXStrings, LinearAlgebra, Statistics
using Plots.PlotMeasures, QuantEcon, StatsPlots
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Preferences

Welfare and Preferences

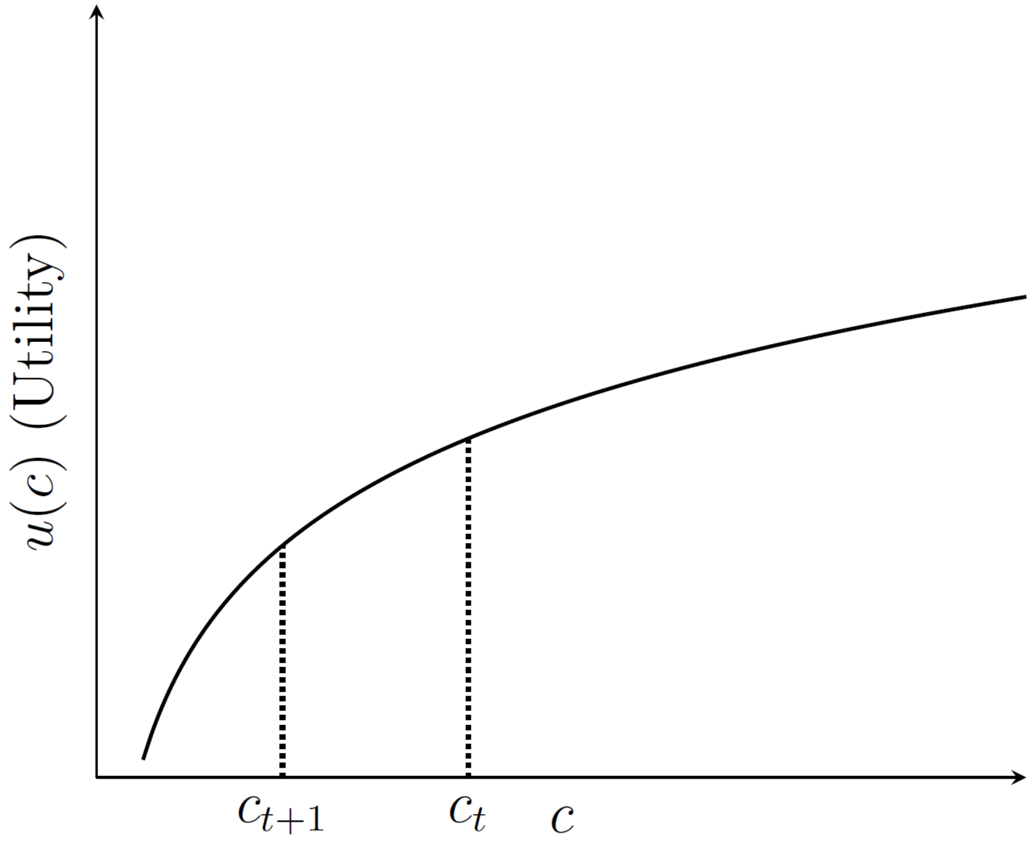
- To introduce an endogenous choice, consider how an agent would compare alternative bundles of consumption goods. Assume agent:
  - → Lives for $t = 0, \ldots \infty$ (see our first lectures importance)
  - → Gains period utility $u(c_t)$ from consumption c_t. Previously we assumed this was linear
  - → Discounts future $u(c_t)$ with discount factor $\beta \in (0, 1)$
- This leads to preferences that are additively separable and they compare $\{c_t\}_{t=0}^{\infty}$ streams of consumption at time t

$$\sum_{j=0}^{\infty} \beta^j u(c_{t+j})$$

Period Utility

- Previously we had assumed a linear utility function (which we called “risk neutral”)
- Consider utility which is strictly concave where:
  - → $u'(c) > 0$: More is better
  - → $u''(c) < 0$: Diminishing Marginal Utility
  - → $\lim_{c \to 0} u'(c) = \infty$: Infinite Marginal Utility at zero
- Examples include
  - → $u(c) = \log(c)$ and $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ for $\gamma > 0$
  - → $u(c) = \frac{a_2}{2}c^2 + a_1 c + a_0$ for $a_2 < 0$ as long as c is less than the “satiation point” where $u'(c) = 0$

Strictly Concave Utility



- Positive Marginal Utility of Consumption
- Diminishing Returns
- No (visible, at least) point of satiation

Uncertainty

- What if the agent does not know $\{c_t\}_{t=0}^{\infty}$ because it is random or uncertain?
- In that case, we can instead have the agent compare expected utility streams

$$\mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j u(c_{t+j})\right]$$

→ Where $\mathbb{E}_t[\cdot] \equiv \mathbb{E}[\cdot|I_t]$ with I_t the information set we make available at time t for forecasting in our model

→ This uses our model of expectation formation from the previous lecture

Risk Aversion vs. Inter-temporal Substitution

- If $u(c)$ is strictly concave the agent:
  - → **Risk Averse:** Prefers more deterministic consumption to those with a higher variance
  - → **Preferences for Consumption Smoothing:** Will substitute between time periods rather than smoother consumption over time rather than large fluctuations
- One challenge in macroeconomics with these preferences is that the $u(c)$ serves both purposes, which have different economic interpretations.
  - → To disentangle, can use recursive preferences such as Epstein-Zin which decouple these two concepts

Smoothing Incentives

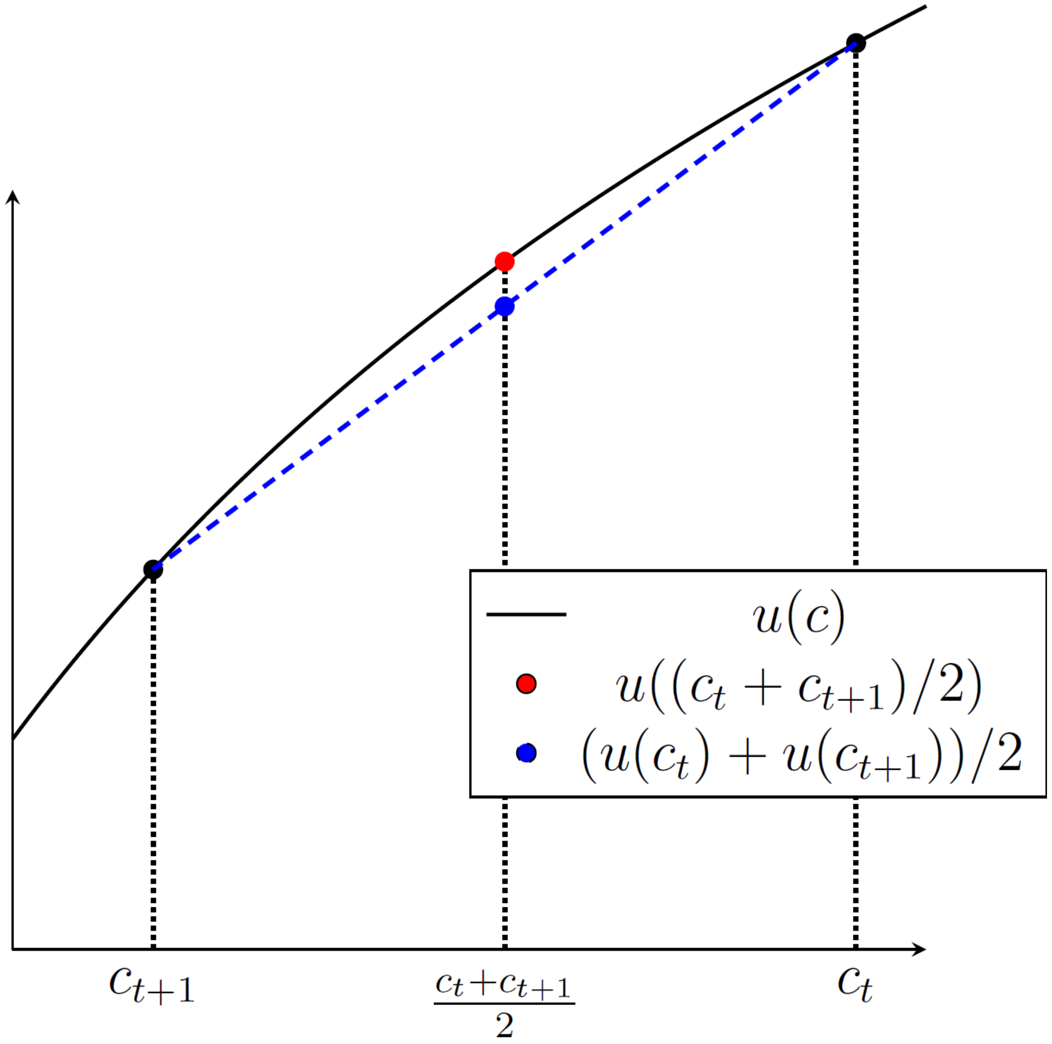
- Consider a simpler case where they live for two periods and don't discount the future: $V(c_1, c_2) \equiv u(c_1) + u(c_2)$
- Consider two possible bundles: $\{c_t, c_{t+1}\}$ and $\{\bar{c}, \bar{c}\}$ where $c_t + c_{t+1} = 2\bar{c}$
- If the agent is risk-neutral, we see that $V(c_t, c_{t+1}) = V(\bar{c}, \bar{c})$
- However, if the agent if risk-averse, then

$$V(c_t, c_{t+1}) < V(\bar{c}, \bar{c}) \quad \textbf{unless } c_t = c_{t+1} = \bar{c}$$

  → They strictly prefer smoother consumption over time

  → i.e., would forgo consumption on average to gain smoother consumption

Smoothing and Concavity



- Recall $\bar{c} \equiv (c_t + c_{t+1})/2$
- 2 periods, $\beta = 1$
- Same "price" for c_t and c_{t+1}
- Two possible bundles:
  1. $\{c_t, c_{t+1}\}$
  2. $\{\bar{c}, \bar{c}\}$
- Later, β and prices will simply distort this exact tradeoff

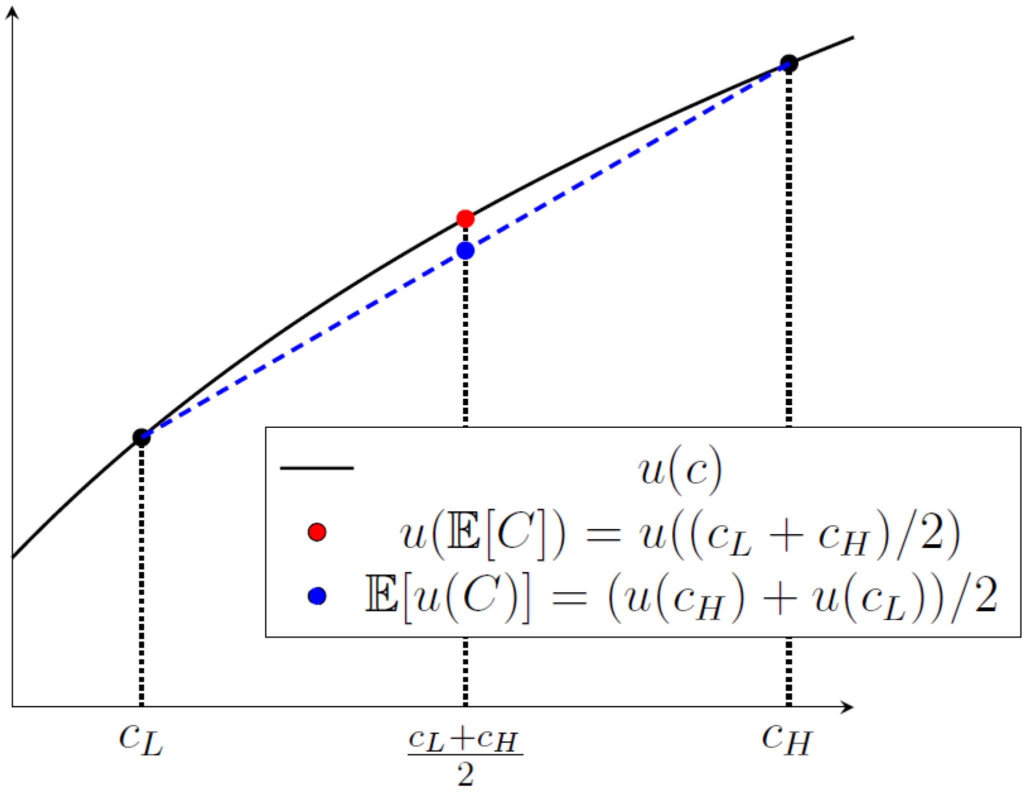
Risk-Aversion Intuition

- Consider a utility $u(c)$ and a lottery which is a random variable
  - → $C = \begin{cases} c_L & \text{with probability } \frac{1}{2} \\ c_H & \text{with probability } \frac{1}{2} \end{cases}$
  - → Let $(c_L + c_H)/2 = \bar{c}$
  - → We can form expected utility as $\mathbb{E}[u(C)] = \frac{1}{2}u(c_L) + \frac{1}{2}u(c_H)$
- Note if risk-neutral then $\mathbb{E}[C] = \frac{1}{2}c_L + \frac{1}{2}c_H = \bar{c} = u(\bar{c})$
- Then if an agent is risk-averse,

$$u(\mathbb{E}(C)) > \mathbb{E}[u(C)]$$

  - → i.e., would forgo consumption on average to avoid the risk

Risk Aversion and Concavity



- Interpretation as fair, risk-neutral prices for lotteries
- Then compare choice between lotteries:
    1. $\mathbb{E}[u(C)] \equiv \frac{1}{2}u(c_L) + \frac{1}{2}u(c_2)$
    2. $u(\mathbb{E}(C)) = u(\frac{1}{2}c_L + \frac{1}{2}c_H)$
- The strict concavity of $u(c)$ shows you are better off with the deterministic consumption

The Decision Problem

Permanent Income Model

- The classic permanent income model explored the impact of these economic forces on consumption and savings decisions. The agent
  - → has an exogenous, potentially stochastic, income stream $\{y_t\}_{t=0}^{\infty}$
  - → chooses a consumption policy $\{c_t\}_{t=0}^{\infty}$ to maximize expected welfare
  - → forecasts the random variables $\{c_t\}_{t=0}^{\infty}$ and $\{y_t\}_{t=0}^{\infty}$ streams using mathematical expectations
  - → has access to a risk-free bond market with interest rate R to either save or borrow, enabling them to smooth consumption over time or deal with uncertainty
  - → has financial assets at time t be F_t which also must be forecast

Period-By-Period Budgets

- Given income y_t, consumption c_t, and financial assets F_t, the agent's budget constraint is

$$F_{t+1} = R(F_t + \underbrace{y_t - c_t}_{\text{savings}})$$

  → where $R > 1$ is the **gross interest rate** on saving or borrowing, and $1 + r \equiv R$ would be the **net interest rate**

  → the interpretation is simple: take their bank account value (positive or negative), add or subtract savings that period, and then they gain interest on the new balance (or the the debt grows if negative)

- If this was the only constraint, you might have infinite borrowing each period

Lifetime Budget Constraint (LBC)

- Alternatively, if all of those accounting relationships must hold, substitute to form a single budget
- Given forecasts of c_{t+j} and y_{t+j}, the budget must fulfill

$$\underbrace{\mathbb{E}_t\left[\sum_{j=0}^{\infty}\frac{c_{t+j}}{R^j}\right]}_{\text{EPDV of consumption}} = \underbrace{\mathbb{E}_t\left[\sum_{j=0}^{\infty}\frac{y_{t+j}}{R^j}\right]}_{\text{EPDV of income}} + F_t$$

- The F_t is the current financial assets. Consider leaving in bank to pay for c_{t+j}
  - → Then R^{-j} enters because $\$1$ today grows to $\$1 \times R^j$ in j periods
  - → That allows you to buy $1 \times R^{-j}$ units of c_{t+j}

A Special Case of $R = 1/\beta$

- Where does R come from? The decisions of other agents in the economy
  - → Lenders are asked to give up 1 unit of consumption today for R units of consumption tomorrow
  - → Hence the R should reflect the degree of impatience of the lender
- An important case is when $\beta R = 1$
  - → As we will discuss later, this will arise in **equilibrium** as the natural rate of interest when agent's can smooth consumption fully
- The intuition is that the gross interest rate exactly offsets the impatience, as captured by the discount factor. Risk, etc. will enter later

Lifetime Budget Constraint when $\beta R = 1$

- In that case, we see that the budget constraint simplifies to

$$\underbrace{\mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j c_{t+j}\right]}_{\text{EPDV of consumption}} = \underbrace{\mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j y_{t+j}\right]}_{\text{EPDV of income}} + F_t$$

→ This should give you hope on tractability: if c_{t+j} and y_{t+j} follow simple stochastic processes (e.g., the LSS) then we can can calculate these EPDV

Decision Problem

- Economists (usually) formalize decisions as optimization problems
- Taking an exogenous gross interest rate $R > 1$ and $\{F_t, y_t, y_{t-1} \dots\}$

$$
\begin{aligned}
\max_{\{c_{t+j}, F_{t+j}\}_{j=0}^{\infty}} \; & \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j u(c_{t+j}) \right] \\
\text{s.t. } & F_{t+j+1} = R(F_{t+j} + y_{t+j} - c_{t+j}) \qquad \text{for all } j \geq 0 \\
& \text{no-ponzi scheme/transversality condition}
\end{aligned}
$$

- No-ponzi condition treated informally: prevents the agent from borrowing too quickly, equivalent to not dying in debt if they had a finite life

Decision Problem (Alternative)

- While we cannot in extensions with borrowing constraints, etc., here we can use the LBC

$$\max_{\{c_{t+j}\}_{j=0}^{\infty}} \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j u(c_{t+j}) \right]$$

$$\text{s.t.}\ \mathbb{E}_t \left[\sum_{j=0}^{\infty} R^{-j} (c_{t+j} - y_{t+j}) \right] = F_t$$

→ rearranged to show that the EPDV of savings = initial financial assets. Implicitly uses the no-ponzi scheme condition

Consumption Plans, Information Sets, and Forecasts

- c_{t+j} and F_{t+j+1} are random variables for all $j > 0$
- The agent is making a consumption plan for each realization of the random shocks
  → Without proof, with these preferences the plan is **time-consistent**: they will not want to change their plan, even after seeing y_{t+j} for $j > 0$
- To forecast the future, they are conditioning on their own decisions given the randomness inherent to the y_{t+j} process
  → Formally modeling information to use $\mathbb{E}_t[\cdot]$
- This seems like an intractable problem?

First-Order Conditions

- Without a full derivation, can show that the solution to this problem exists, for strictly concave $u(c)$ where, given a F_0 initial condition

$$
\begin{aligned}
u'(c_t) &= \beta R\, \mathbb{E}_t[u'(c_{t+1})], && \textbf{Euler equation} \\
F_{t+1} &= R(F_t + y_t - c_t), && \textbf{Budget Constraint} \\
0 &= \mathbb{E}_0 \left[\lim_{j\to\infty} \beta^j F_{t+j} \right], && \textbf{No-Ponzi Scheme}
\end{aligned}
$$

  → Or, equivalently in this case, the LBC must also hold

  → Note that we have switched from the $t+j$ to just t for current period

- Still challenging since we need to forecast optimal c_{t+1}

Motivating Derivation of the Euler Equation

- See here for a more complete derivation in the deterministic case
- Will derive Euler Equation for the simple case of 2-periods t and $t+1$, which ends up nesting the general case
- Budget $F_{t+1} = R(F_t + y_t - c_t)$ but assume $F_{t+2} = 0$ since they "die"
  - → Then $c_{t+1} = y_{t+1} + F_{t+1}$

$$c_{t+1} = y_{t+1} + R(F_t + y_t - c_t)$$

Decision Problem with 2 Periods

$$\max_{c_t} [u(c_t) + \beta \mathbb{E}_t[u(c_{t+1})]]$$
$$\text{s.t. } c_{t+1} = y_{t+1} + R(F_t + y_t - c_t)$$

- And substitute the budget constraint into the objective function

$$\max_{c_t} \left[u(c_t) + \beta \mathbb{E}_t[u(y_{t+1} + R(F_t + y_t - c_t))]\right]$$

- Take the FONC, which can be rearranged to the Euler equation

$$0 = u'(c_t) - \beta R \mathbb{E}_t[u'(y_{t+1} + R(F_t + y_t - c_t))]$$
$$u'(c_t) = \beta R \, \mathbb{E}_t[u'(c_{t+1})]$$

Interpreting the Euler Equation

- Euler Equations are ubiquitous intertemporal optimality conditions

$$u'(c_t) = \beta R\,\mathbb{E}_t[u'(c_{t+1})]$$

- Tradeoff of consuming less today is the marginal utility today
- The right-hand term is the benefit
  - → You gain the marginal utility (MU) of consuming a little more tomorrow
  - → Need to forecast MU tomorrow, considering risk aversion/smoothing
  - → A unit of utility tomorrow is only worth β times that of today
  - → However, you are compensated by the savings growing at interest rate R which increases the amount of units of consumption you can afford

Special Case of Deterministic Income

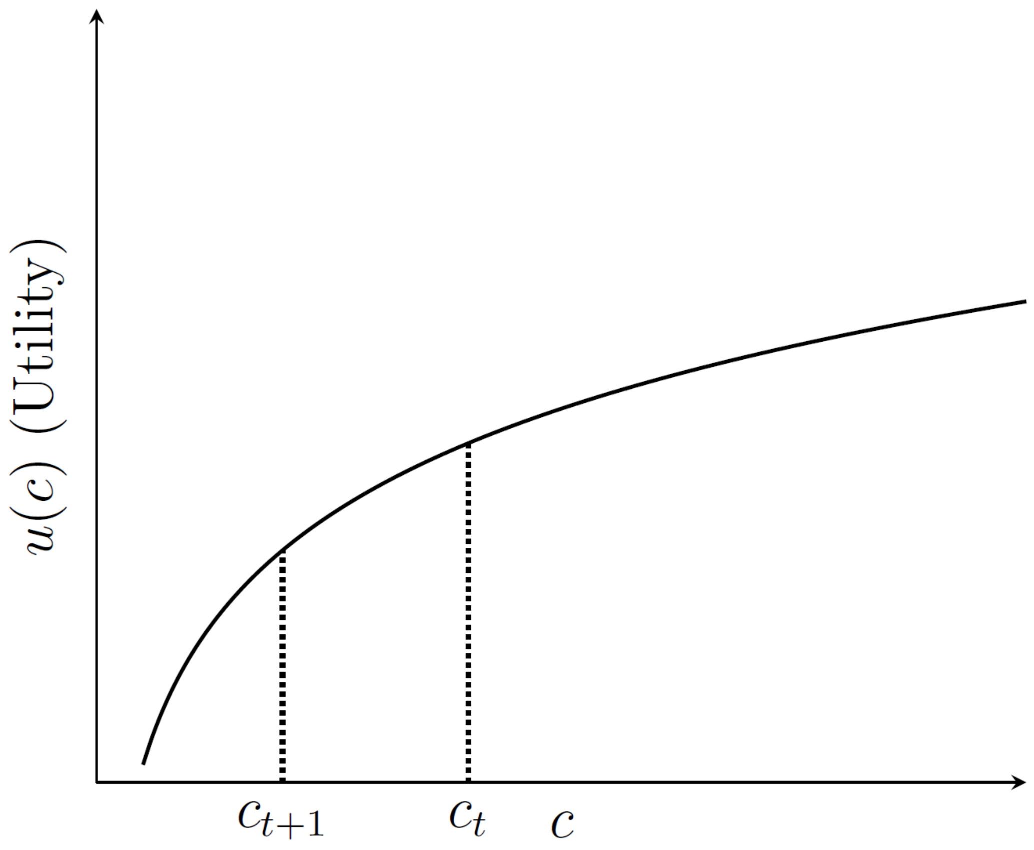
Special case of Deterministic Income and $\beta R = 1$

- If y_t is deterministic, then this problem no longer requires forcasts
- Furthermore, assume $\beta R = 1$ (i.e. interest exactly offsets impatience)

$$u'(c_t) = u'(c_{t+1})$$

- Moreover, given assumptions that $u'(c) > 0$ and $u''(c) < 0$, this implies that $c_t = c_{t+1}$
- This is the classic Permanent Income Result

Reminder: Strictly Concave Utility



- Positive Marginal Utility of Consumption
- Diminishing Returns
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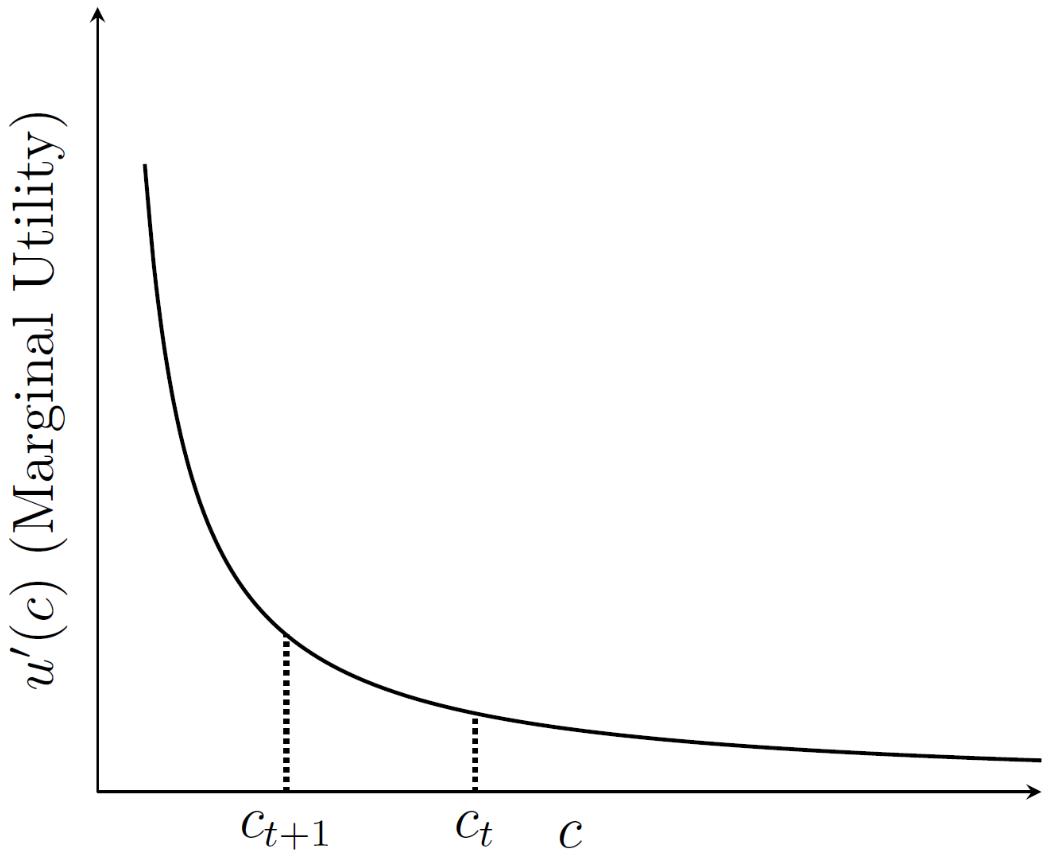
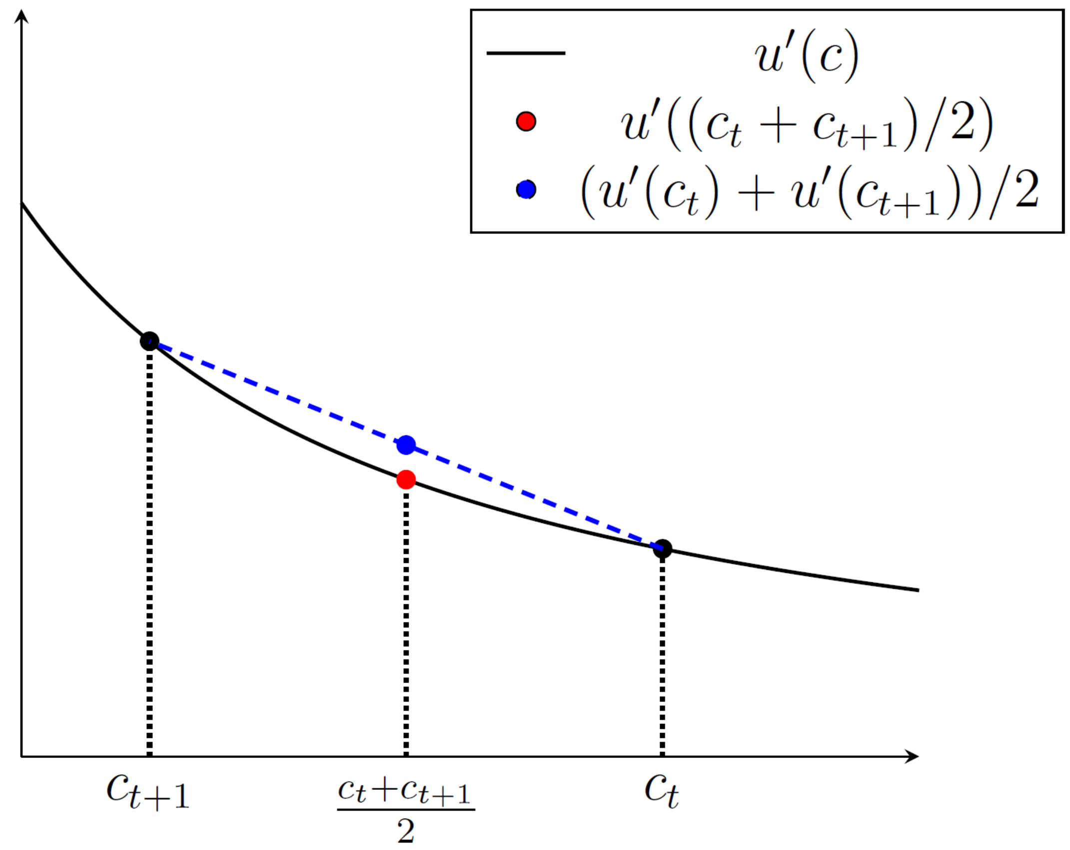
Marginal Utility



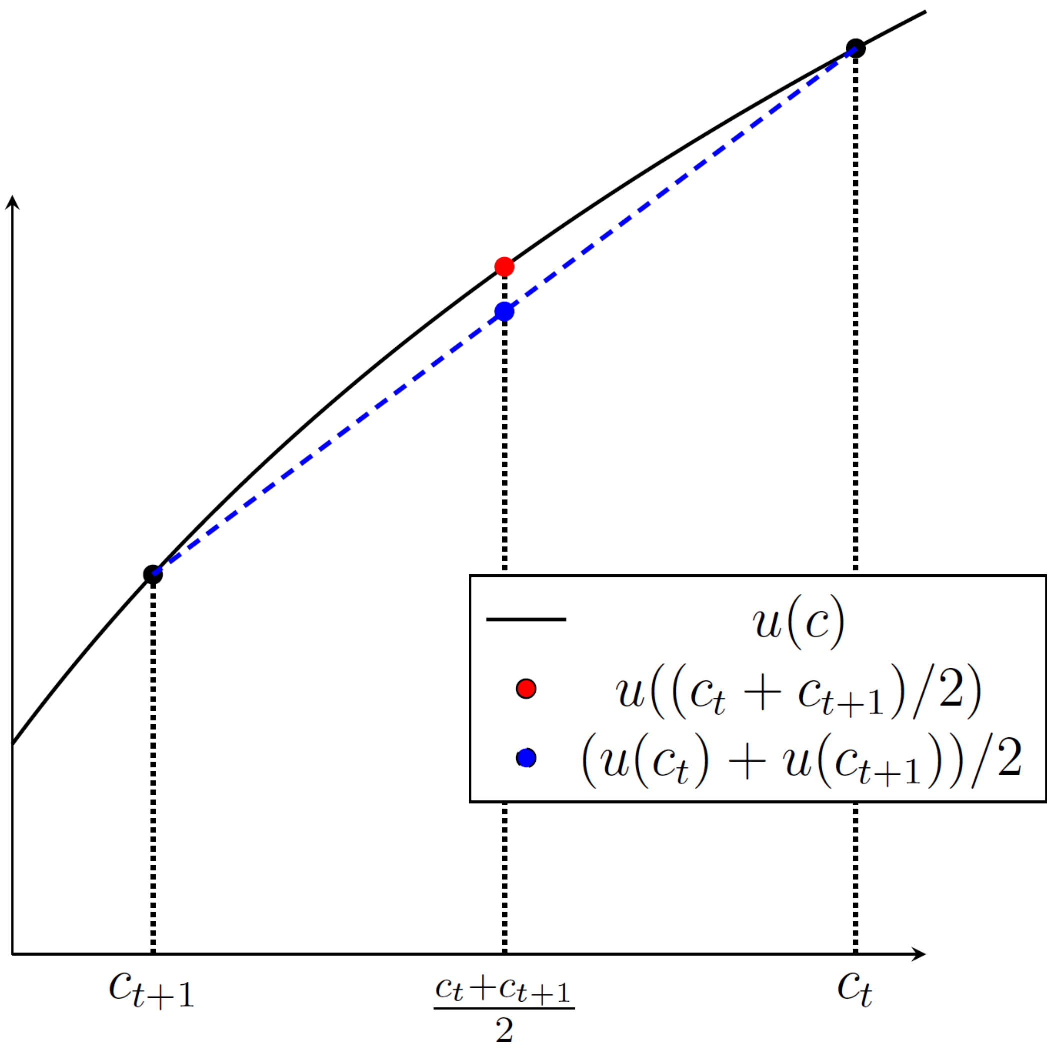
- $u'(c) > 0$ but decreasing $u''(c) < 0$
- $u'(c_1) = u'(c_2) \implies c_1 = c_2$
- If $u'(c_t) < u'(c_{t+1})$ then $c_t > c_{t+1}$
- The less they consume, the more valuable additional consumption in that period would be

Equating Marginal Utilities



- Euler here: $u'(c_t) = u'(c_{t+1})$ for all t
- Exact for simple deterministic, $\beta R = 1$ case
- By equating marginal utilities at all points, they gain a lower average marginal utility

Smoothing and Welfare



- The higher average marginal utility for the volatile consumption path corresponds to a lower average utility
  - → i.e. welfare here
- If "risk-neutral", then the agent is indifferent between the two paths
  - → We see that since the utility function would be linear itself

Permanent Income Result for Strictly Concave $u(c)$

- With $\beta R = 1$, use $c_t = \bar{c}$ with the LBC

$$\sum_{j=0}^{\infty} \beta^j c_{t+j} = \sum_{j=0}^{\infty} \beta^j y_{t+j} + F_t$$

$$\bar{c} \sum_{j=0}^{\infty} \beta^j = \sum_{j=0}^{\infty} \beta^j y_{t+j} + F_t$$

$$c_t = \bar{c} = \underbrace{(1-\beta)}_{\text{MPC}} \left[\underbrace{\sum_{j=0}^{\infty} \beta^j y_{t+j}}_{\text{Human Wealth}} + F_t \right]$$

→ The consumer has a constant MPC out of **total wealth**

Annuity Values

- If $R \equiv 1 + r = 1/\beta$ then notice that $1 - \beta = \frac{r}{1+r} = \frac{R-1}{R}$
- Consider F_t in financial assets and want to consume some and save the rest
- Assume no income, and consider if you consume $\frac{R-1}{R}$ proportion of F_t

$$F_{t+1} = R(F_t - c_t) = R(F_t - \frac{R-1}{R}F_t) = F_t$$

- So the annuity value is the amount I can take out and leave the bank account identical after interest
- Alternatively, the inverse of the price of an asset that payed out $\$1$ each period forever

Stochastic Income and Consumption

What about Stochastic Income?

- Leaving $\beta R = 1$ for the remainder of the slides
- Optimality: agents would **LOVE** to equate all marginal utilities

$$u'(c_t) = \mathbb{E}_t[u'(c_{t+1})]$$

  → Will do the best they can given information sets

- With enough financial instruments to hedge all risks, they might!

  → However, for an arbitrary $u(c)$ function this is hard to achieve in our environment, where they only have a single, risk-free asset

Quadratic Utility

- A special case of these preferences is $u(c) = \frac{a_2}{2}c^2 + a_1 c + a_0$ for $a_2 < 0$
  - → This is a **quadratic utility function**
  - → If $a_2 < 0$ this is strictly concave
  - → However, $u'(c) = a_2 c + a_1$ is always negative for large enough c
    - i.e, satiation point for the $c_{\max}$ where $u'(c_{\max}) = 0$
- Assume conditions such that $c_t \ll c_{\max}$, and this is strictly concave in the relevant range

Euler Equation for Quadratic Utility

- Since $u'(c) = a_2 c + a_1$ we can write the euler equation as

$$\begin{aligned} u'(c_t) &= \mathbb{E}_t[u'(c_{t+1})] \\ a_2 c_t + a_1 &= \mathbb{E}_t[a_2 c_{t+1} + a_1] \\ c_t &= \mathbb{E}_t[c_{t+1}] \end{aligned}$$

  → That is, the agent will choose consumption so that the **expected value** of consumption next period is equal to the current period

- With more general strictly concave preferences, often: $c_t \approx \mathbb{E}_t[c_{t+1}]$

Recall: Martingales

- Reminder: X_t is a martingale if $\mathbb{E}_t[X_{t+1}] = X_t$
- In other words, **consumption is a martingale** for any y_t stochastic process!
- Key feature of martingales: the history has **no predictive power** for the future
- This will also come up in macro-finance and asset pricing later

  → i.e., if the past had systematic and consistent predictive power, then there would be systematic and consistent profits to be made

  → If there were systematic profits to be made, wouldn't prices adjust as people tried to make those profits?

Consumption is a Martingale

- Similar logic: Consumers use all available information to smooth consumption
- With the financial assets we give them, a martingale is the closest they can get to fully smoothed consumption
- The agent will look at their permanent income (i.e. EPDV of human wealth + financial assets) and plan to keep it constant on average
- This highlights that the changes in consumption come from "surprise"
  - → If any of that surprise was in their information sets, then they would have already adjusted

Some Implications

- As discussed, suggests that agents will adjust based on their forecasts of future income, so they are harder to trick. Tax cut now with tax increase later may have little to know effect?
  - → Limits the effectiveness of fiscal policy
- Makes it harder to interpret the data
  - → A rapidly increasing income process might have little or no effect on consumption if it is forecast
- Policies which smooth consumption will increase consumer welfare
  - → e.g., social security, unemployment insurance, etc.
  - → Financial assets allowing more intertemporal substitution (e.g. bonds) or across states (e.g. insurance, or risky assets like bonds)

Changes in Consumption

- We know that $c_t = \mathbb{E}_t[c_{t+1}]$ and there is no systematic bias in forecasting
- Changes are driven by shocks. Without proof, can show

$$c_{t+1} - c_t = (1-\beta)\sum_{j=0}^{\infty} \beta^j \left[\mathbb{E}_{t+1}[y_{t+j+1}] - \mathbb{E}_t[y_{t+j+1}]\right]$$

→ Changes in consumption come only from information ($\mathbb{E}_{t+1}[\cdot]$ vs. $\mathbb{E}_t[\cdot]$)

→ "Surprises" were anything they couldn't forecast on average

→ By law of iterated expectations we see this is mean zero, consistent with the martingale property

Linear State Space Models

Linear State Space Models for Income

- While this theory applies to any stochastic process for income, consider a special case which is a Linear Gaussian State Space

$$
\begin{aligned}
x_{t+1} &= A x_t + C w_{t+1}, && \text{evolution equation} \\
y_t &= G x_t, && \text{observation equation}
\end{aligned}
$$

  - → $x_t \in \mathbb{R}^n, y \in \mathbb{R}$, and $w_{t+1} \sim \mathcal{N}(0, I) \in \mathbb{R}^m$
  - → $A \in \mathbb{R}^{n \times n}, C \in \mathbb{R}^{n \times m}, G \in \mathbb{R}^{1 \times n}$
- Key result if $\beta \in (0,1)$ and $\max\{|\text{ eigenvalue of } A|\} < 1/\beta$:

$$
\mathbb{E}_t\left[\sum_{j=0}^{\infty} \beta^j y_{t+j}\right] = G(I - \beta A)^{-1} x_t
$$

Optimal Consumption with the LSS

- Take the optimal consumption we derived earlier

$$
\begin{aligned}
c_t &= (1-\beta)\left[\mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j y_{t+j}\right] + F_t\right] \\
&= (1-\beta)\left[G(I-\beta A)^{-1}x_t + F_t\right]
\end{aligned}
$$

- We can use this in the evolution of the financial assets, with $R = 1/\beta$

$$
\begin{aligned}
F_{t+1} &= \beta^{-1}(F_t + y_t - c_t) \\
&= \beta^{-1}\left(F_t + Gx_t - (1-\beta)\left[G(I-\beta A)^{-1}x_t + F_t\right]\right) \\
&= F_t + G(I-\beta A)^{-1}(I-A)x_t, \qquad \text{after some algebra}
\end{aligned}
$$

Impulse Response Function (IRFs)

- A common tool in macro: look at the response of the system to a "shock"
- This is called the **impulse response function**
  - → Think of this as feeding in a one-time change to w_{t+1} and then seeing how that propagates for x_{t+j} and y_{t+j}
  - → These are especially easy in LSS models because you can solve the system feeding in zeros for all other shocks as the comparison
- Given this, we can also look at the present discounted value of the impulse response function, which will help us interpret the model

Impulse Response for a LSS

- The impulse is a w_1 shock, typically just zeros and ones depending on the experiment and then $w_{t+1} = 0$ for all $t > 1$
- Then for some x_0 initial condition, compare the evolution with this shock relative to one with $w_{t+1} = 0$ for all $t \geq 1$
- Denote the version with zero shocks throughout as $\bar{x}_t$, then

$$
\begin{aligned}
x_1 - \bar{x}_1 &= Ax_0 + Cw_1 - (Ax_0 + C \times 0) = Cw_1 \\
x_2 - \bar{x}_2 &= Ax_1 + C \times 0 - (A\bar{x}_1 + C \times 0) = ACw_1
\end{aligned}
$$

- More generally, for any $t > 0$
  - → $x_t - \bar{x}_t = A^{t-1}Cw_1$
  - → $y_t - \bar{y}_t = GA^{t-1}Cw_1$

EPDV of an Impulse Response

- Consider instead the expected present-discounted value of this "shock"

$$\sum_{j=0}^{\infty} \beta^j y_{t+j} - \sum_{j=0}^{\infty} \beta^j \bar{y}_{t+j} = \sum_{j=0}^{\infty} \beta^j G A^{t+j-1} C w_1 = G(I - \beta A)^{-1} C w_1$$

- Going back to the change in consumption, we can show that

$$\begin{aligned} c_{t+1} - c_t &= (1-\beta) \sum_{j=0}^{\infty} \beta^j \left[\mathbb{E}_{t+1}[y_{t+j+1}] - \mathbb{E}_t[y_{t+j+1}] \right] \\ &= (1-\beta) G (I - \beta A)^{-1} C w_{t+1} \end{aligned}$$

- Interpretation: change in c = MPC × EPDV of IRF to "shock"

Consolidating into a Single LSS

- The "state" of the agent is then summarized by x_t, F_t
- The key observations are the y_t and c_t, where the later is now the optimal decision
- Given that everything is still linear and Gaussian, we can combine these into a new LSS (note: could instead have used the c_{t+1} LOM)
  - → State: $\tilde{x}_t \equiv \begin{bmatrix} x_t & F_t \end{bmatrix}^\top$
  - → Observables: $\tilde{y}_t \equiv \begin{bmatrix} y_t & c_t \end{bmatrix}^\top$
- Then the evolution and observation equations just need to be stacked

Stacked LSS

- The stacked evolution equation (for $\mathbf{0}$ a vector of zeros)

$$\underbrace{\begin{bmatrix} x_{t+1} \\ F_{t+1} \end{bmatrix}}_{\tilde{x}_{t+1}} = \underbrace{\begin{bmatrix} A & \mathbf{0} \\ G(I-\beta A)^{-1}(I-A) & 1 \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} x_t \\ F_t \end{bmatrix}}_{\tilde{x}_t} + \underbrace{\begin{bmatrix} C \\ 0 \end{bmatrix}}_{\tilde{C}} w_{t+1}$$

$$\underbrace{\begin{bmatrix} y_t \\ c_t \end{bmatrix}}_{\tilde{y}_t} = \underbrace{\begin{bmatrix} G & 0 \\ (1-\beta)G(I-\beta A)^{-1} & 1-\beta \end{bmatrix}}_{\tilde{G}} \underbrace{\begin{bmatrix} x_t \\ F_t \end{bmatrix}}_{\tilde{x}_t}$$

→ Then use our LSS tools with $\tilde{x}_{t+1} = \tilde{A}\tilde{x}_t + \tilde{C}w_{t+1}$ and $\tilde{y}_t = \tilde{G}\tilde{x}_t$

Examples

IID Income LSS

- Consider the case where IID income: $y_t \sim N(\mu, \sigma^2)$
  - → Hence: $y_t = \mu + \sigma w_t$ and $y_{t+1} = \mu + \sigma w_{t+1}$ where $w_{t+1} \sim N(0,1)$
- One way to write as a state-space model is: $x_t \equiv \begin{bmatrix} w_t & 1 \end{bmatrix}^\top$

$$\underbrace{\begin{bmatrix} w_{t+1} \\ 1 \end{bmatrix}}_{x_{t+1}} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} w_t \\ 1 \end{bmatrix}}_{x_t} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{C} w_{t+1}$$

$$y_t = \underbrace{\begin{bmatrix} \sigma & \mu \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} w_t \\ 1 \end{bmatrix}}_{x_t}$$

  - → Note: $G(I - \beta A)^{-1} = \begin{bmatrix} \sigma & \frac{\mu}{1-\beta} \end{bmatrix}, G(I - \beta A)^{-1}(I - A) = \begin{bmatrix} \sigma & 0 \end{bmatrix}$

Stacked LSS for IID Income

$$\underbrace{\begin{bmatrix} w_{t+1} \\ 1 \\ F_{t+1} \end{bmatrix}}_{\tilde{x}_{t+1}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma & 0 & 1 \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} w_t \\ 1 \\ F_t \end{bmatrix}}_{\tilde{x}_t} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\tilde{C}} w_{t+1}$$

$$\underbrace{\begin{bmatrix} y_t \\ c_t \end{bmatrix}}_{\tilde{y}_t} = \underbrace{\begin{bmatrix} \sigma & \mu & 0 \\ (1-\beta)\sigma & \mu & 1-\beta \end{bmatrix}}_{\tilde{G}} \underbrace{\begin{bmatrix} w_t \\ 1 \\ F_t \end{bmatrix}}_{\tilde{x}_t}$$

Analysis of Savings

- From this, we see that

$$F_{t+1} = F_t + \sigma w_t$$

  → i.e., financial assets are a random walk, a martingale

- Note that if $F_0 = 0$ and $w_0 = 0$ (i.e., $y_t = \mu$)

$$F_t = \sigma w_0 + \sigma w_1 + \ldots + \sigma w_{t-1} = \sigma \sum_{j=0}^{t-1} w_j$$

  → i.e., financial assets = accumulated unanticipated income shocks

  → Lack of persistence in the income process means agent does not need for smoothing for predictable drifts

Analysis of Consumption

- The row of $\tilde{G}$ in the LSS row is $c_t = \mu + (1-\beta)\left[\sigma w_t + F_t\right]$
- And with $F_0 = 0$ and $w_0 = 0$ (i.e., $y_t = \mu$)

$$c_t = \mu + (1-\beta)\left[\sigma w_t + \sigma \sum_{j=0}^{t-1} w_j\right] = \mu + (1-\beta)\sigma \sum_{j=0}^{t} w_j$$

  → i.e., consumption is also a random walk and a martingale

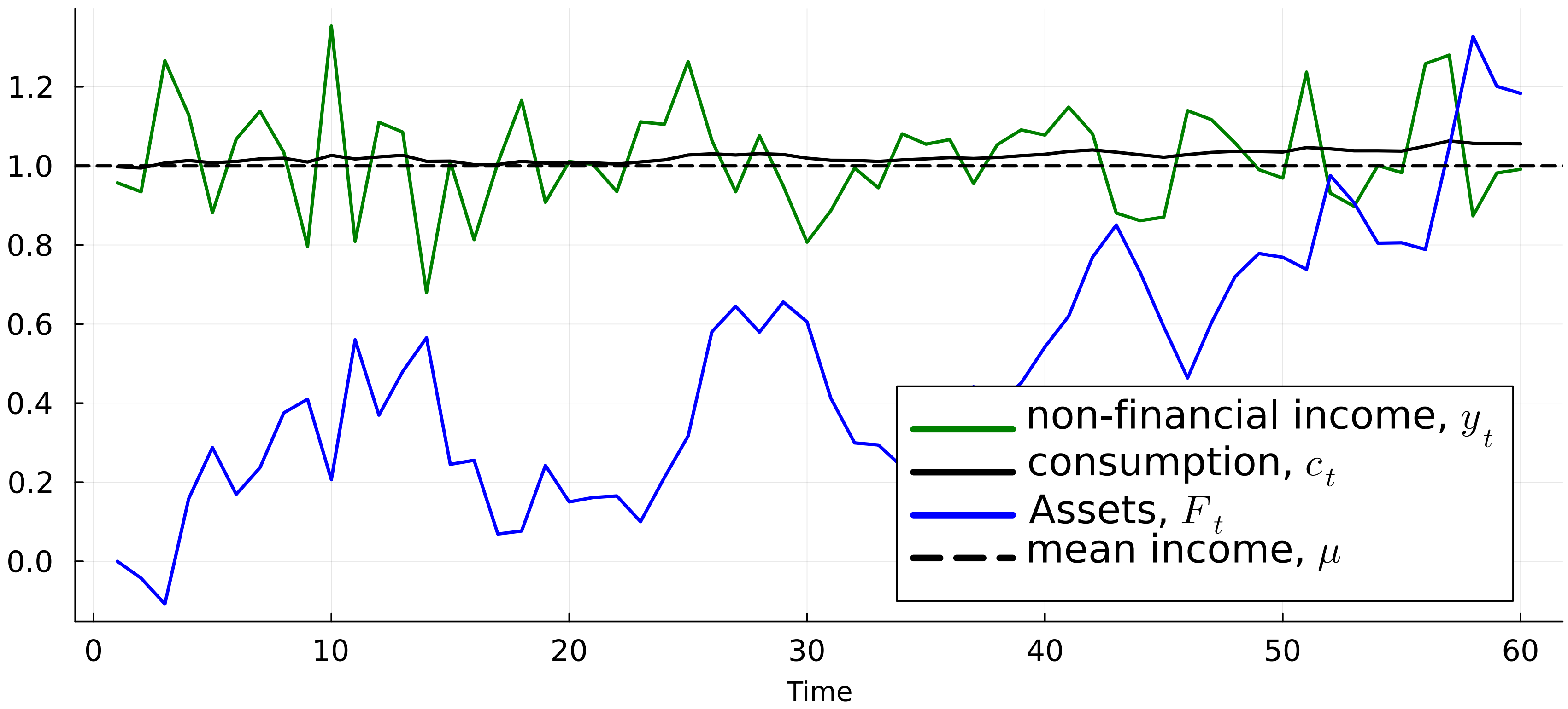
- Change in consumption is driven by IID shocks. MPC $\times$ annuitized value of the shock

$$c_{t+1} - c_t = (1-\beta)\sigma w_{t+1}$$

Direct Simulation

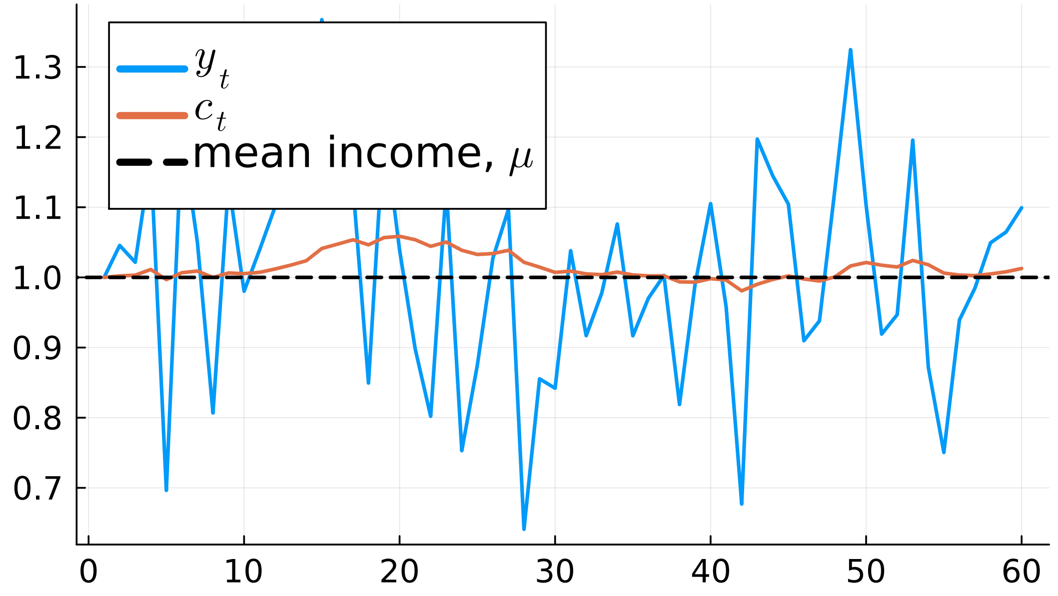
```julia
function simulate_iid_income(p, T; w = randn(T))
    w_sum = cumsum(w) #(w_1, w_1 + w_2, w_1 + w_2 + w_3, ... sum_{j=1}^T w_j))
    c = p.mu .+ (1 - p.beta) * p.sigma  * w_sum # (c_1, c_2, ... c_T)
    y = p.mu .+ p.sigma * w # (y_1, y_2, ... y_T)
    F = [0.0; p.sigma * w_sum[1:end-1]] #(F_1, F_2, ... F_T)
    return (;w, F, c, y)
end

p = (;beta= 1.0 / (1.0 + 0.05), mu = 1.0, sigma = 0.15)
T = 60
res = simulate_iid_income(p, T)
plot(1:T, res.y, color = :green, label = L"non-financial income, $y_t$",
     xlabel="Time", legend = :bottomright)
plot!(res.c, color = :black, label = L"consumption, $c_t$")
plot!(res.F, color = :blue, label = L"Assets, $F_t$")
hline!([p.mu], color = :black, linestyle = :dash, label = L"mean income, $\mu$")
```

Direct Simulation



LSS Formulation

```julia
A = [0 0; 0 1]
G = [p.sigma p.mu]
C = [1; 0]
H = G*inv(I-p.beta*A)
A_tilde = [A zeros(2,1); H*(I-A) 1]
C_tilde = [C; 0]
G_tilde = [G 0; (1-p.beta)*H 1-p.beta]
x_tilde_0 = [0.0, 1, 0.0] #[w_0, 1, F_0]
lss_pi = LSS(A_tilde, C_tilde, G_tilde;
             mu_0 = x_tilde_0)
x, y = simulate(lss_pi, T)
plot(1:T, y[1,:];label=L"y_t", size=(600,400))
plot!(1:T, y[2,:], label=L"c_t")
hline!([p.mu], color=:black, linestyle=:dash,
       label = L"mean income, $\mu$")
```



Permanent and Transitory Income

- Classic model analyzes income with **persistent** and **transitory** shocks
- Let x_{1t} be the permanent component and x_{2t} be the transitory component where

$$\begin{bmatrix} x_{1t+1} \\ x_{2t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} w_{1t+1} \\ w_{2t+1} \end{bmatrix}}_{w_{t+1}}$$

- And income itself is the sum of these two components

$$y_t = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{G} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}$$

Changes in Consumption

- Recall from previous slides: $c_{t+1} - c_t = (1-\beta)G(I-\beta A)^{-1}Cw_{t+1}$
- For this LSS note that $G(I-\beta A)^{-1} = \begin{bmatrix} \frac{1}{1-\beta} & 1 \end{bmatrix}$. Substitute:

$$c_{t+1} - c_t = \sigma_1 w_{1t+1} + (1-\beta)\sigma_2 w_{2t+1}$$

→ Savings can show to follow:

$$F_{t+1} - F_t = \sigma w_{2t+1}$$

Interpretation

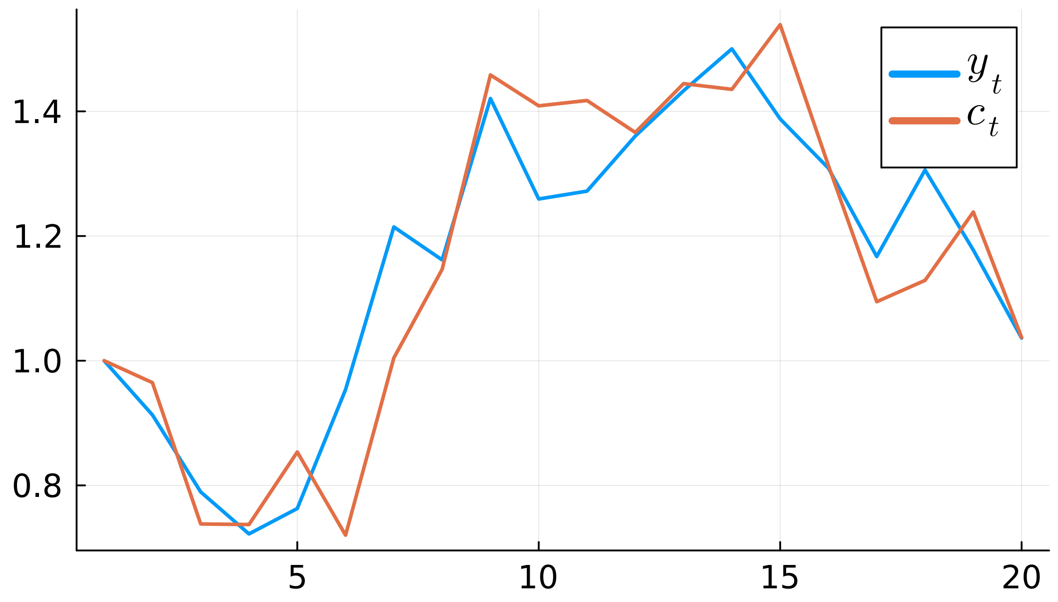
- After they see shock, the consumer:
  - → Consumes the entire permanent shock (i.e. MPC $= 1$)
  - → Consumers an annuitized fraction of transitory shock (i.e., MPC $= 1 - \beta$)
- Given this, next periods savings:
  - → Do not change from the permanent shocks
  - → Add the transitory shock (after consuming the $1 - \beta$ fraction, and paid $1/\beta$ gross interest)

Iteration and IRFs

```
function iterate_LSS(A, C, G, x_0, T; w = randn(size(C,2), T))
    x = zeros(length(x_0), T + 1)
    x[:, 1] = x_0
    for t in 2:(T + 1)
        x[:, t] = A * x[:, t - 1] + C * w[:, t - 1]
    end
    return x, G * x #x, y
end
function IRF(A, C, G, w_1, T)
  h_x = zeros(size(A,1), T+1)
  h_x[:,2] = C * w_1
  for t in 3:T+1
    h_x[:,t] = A * h_x[:,t-1]
  end
  return h_x, G * h_x #h_x, h_y
end
```

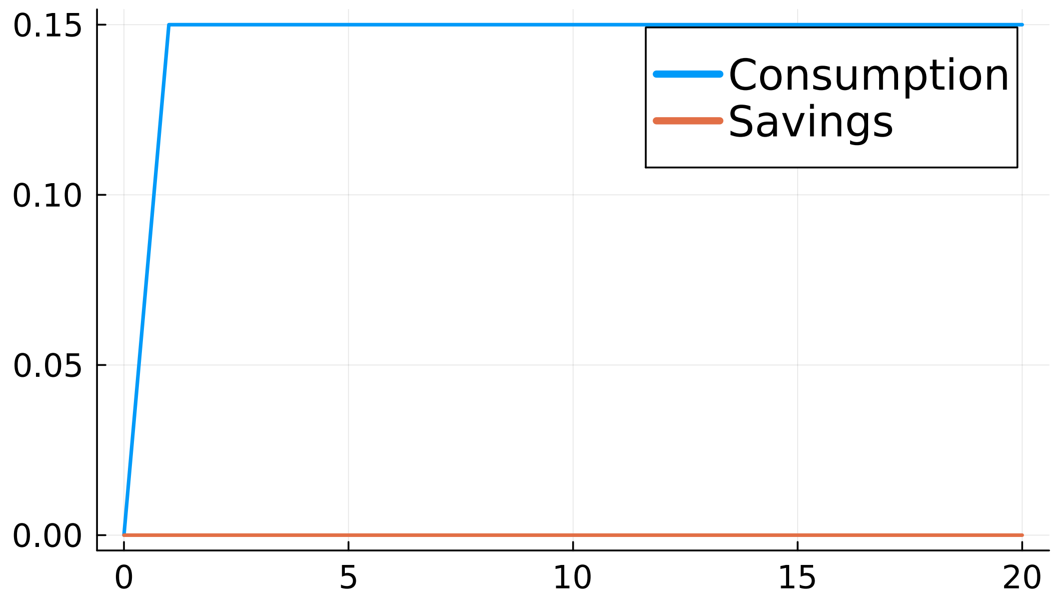
LSS Formulation

```julia
sigma_1, sigma_2 = 0.15, 0.15
beta = 1.0/(1.0 + 0.05)
A = [1 0; 0 0]
G = [1 1]
C = [sigma_1 0; 0 sigma_2]
H = G*inv(I-beta*A)
A_tilde = [A zeros(2,1); H*(I-A) 1]
C_tilde = [C; zeros(1,2)]
G_tilde = [G 0; (1-beta)*H 1-beta]
x_tilde_0 = [1.0, 0.0, 0.0]#[x_10, x_20, F_0]
lss_pi = LSS(A_tilde, C_tilde, G_tilde;
             mu_0 = x_tilde_0)
T = 20
x, y = simulate(lss_pi, T)
plot(1:T, y[1,:];label=L"y_t",size=(600,400))
plot!(1:T, y[2,:], label=L"c_t")
```



IRFs for Permanent Shock

```
w_1 = [1.0, 0.0]
h_x, h_y = IRF(A_tilde, C_tilde,
               G_tilde, w_1, T)
plot(0:T, h_y[2,:];label="Consumption",
     size=(600,400), legend = :topright)
plot!(0:T, h_x[3,:], label="Savings")
```



IRFs for Transitory Shock

```
w_1 = [0.0, 1.0]
h_x, h_y = IRF(A_tilde, C_tilde,
               G_tilde, w_1, T)
plot(0:T, h_y[2,:];label="Consumption",
     size=(600,400), legend = :topright)
plot!(0:T, h_x[3,:], label="Savings")
```