IT575 Computational Shape Modeling
Module - 3: Discrete Geometry

Aditya Tatu

Sampling a Surface

- Let M be a surface. A finite subset $P \in M$ is called a *sample set* of M, and the set P is often referred to as a *point cloud*.

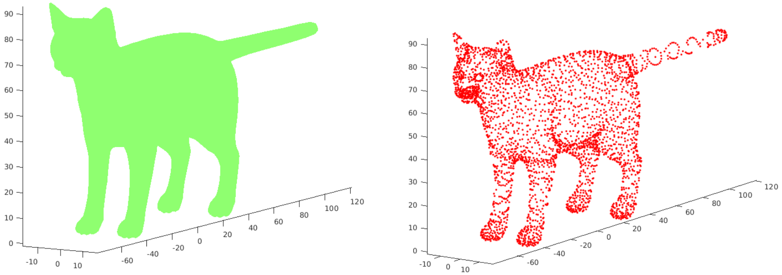

Figure : (left) Original Surface, (right) Point cloud representation.

Quality of Sampling

- Measure of uniformity: A sample-set P is said to be an *r-covering* of the surface M if

$$\cup_{p \in P} B_r(p) = M \quad \Leftrightarrow \quad \forall x \in M, \quad d_M(x, P) \leq r.$$

  - Issue: If P is an r-covering, is it a minimal set?
- A sample set P is said to be *r-separated* if $d_M(p, q) \geq r, \ \forall p, q \in P$, with $p \neq q$.

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Sampling strategy: Farthest Point Sampling

- Let $p_1 = x_1 \in M$, and $P = \{p_1\}$.
- How should the second point be selected?
- Let $p_2 = \arg\max_{x \in M} d_M(x, P)$, and $P = \{p_1, p_2\}$.
- P is an ______-separated set and an ______ -covering set.
- Similarly, $p_3 = \arg\max_{x \in M} d_M(x, P)$, and $P = P \cup \{p_3\}$.
- Let $P = \{p_1, \ldots, p_n\}$ be the sample set. It is a ______-separated and ______-covering sample set.
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Voronoi Tessellation

- Let P be a sample-set of the surface M. Every sample point represents a small nbhd of M.
- For $p_i \in P$, let $V_i(P) = \{x \in M \mid d_M(x, p_i) < d_M(x, p_j), j \neq i, p_j \in P\}$.
- $V_i(P)$ is called the *Voronoi region* of p_i with respect to the sample-set P, and the collection of Voronoi regions of M is called the *Voronoi decomposition* of M generated by P.
- If for $x \in M$, $d_M(x, p_i) = d_M(x, p_j)$, x is said to be on the *Voronoi edge* V_{ij} of the Voronoi regions V_i, V_j.
- If for $x \in M$, $d_M(x, p_i) = d_M(x, p_j) = d_M(x, p_k)$, x is said to be a *Voronoi vertex* V_{ijk} of the Voronoi regions V_i, V_j, V_k.
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- The decomposition of M into $\{V_i \mid p_i \in P\}$ such that $M = \cup_i V_i$, and $V_i \cap V_j = \emptyset$ is called a *tessellation or cell complex* of M.
  - If the tiles V_i are Voronoi regions, we call it a *Voronoi tessellation*.

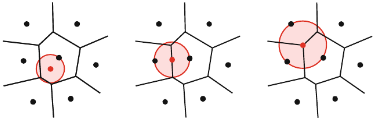

Figure : Voronoi cells. Image from: *Bronstein, Bronstein, Kimmel, Numerical Geometry of Nonrigid shapes, Springer 2008*, henceforth - BBK08

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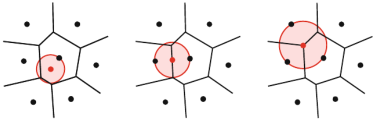

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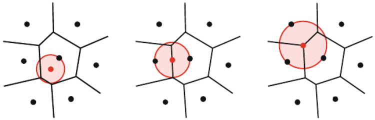

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  - Thus, $p_i = \arg\min_{x^* \in V_i} \int_{V_i} d_M^2(x, x^*)\ da$.
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Lloyd-Max algorithm

- Given a surface M, and an initial sample-set P.
- Construct voronoi tessellation of M generated by P.
- Update: $P =$ centroids of voronoi cells. Repeat till convergence.

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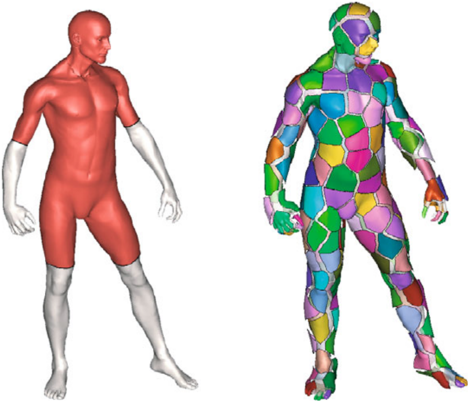

Figure : (left) Inadequate (right) Adequate sampling density. The voronoi regions can be deformed to unit disks in the example on the right. Image from BBK08.

Edge structure

- The sample-set P is a zero-dimensional approximation of the surface M.
- Connect two distinct sample points p_i and p_j if $d_M(p_i, p_j) < r$.
  - The set $N(p_k) = \{p_q \in P \mid d_M(p_k, p_q) < r\}$ is the *Neighborhood* of p_k.
  - Thus P with the above edges gives an *undirected binary graph*.
  - Approximate Edge lengths $l(p_i, p_j) \simeq ||p_i - p_j||_2$.
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  - Approximate Edge lengths $l(p_i, p_j) \simeq ||p_i - p_j||_2$.
  - The edges give a one-dimensional approximation of the surface M.

Edge structure

- The sample-set P is a zero-dimensional approximation of the surface M.
- Connect two distinct sample points p_i and p_j if $d_M(p_i, p_j) < r$.
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Two dimensional approximation

- Let p_i, p_j and p_k be three sample-sets whose Voronoi cells are adjacent to each other and thus yield a vertex V_{ijk}.
- Connect p_i with p_j by a curve of shortest length: Γ_{ij}. Similarly curves Γ_{jk} and Γ_{ki} connect points p_j with p_k and p_k with p_i, resp., forming a *geodesic triangle*.
- Degenerate cases: (a) Three *collinear* points, (b) Four *cocircular* points.
- Decomposition of M into these geodesic triangles is called *Delaunay tessellation*.

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Triangular meshes

- We can approximate the geodesic triangles by Euclidean/planar triangles, giving us a triangular mesh:

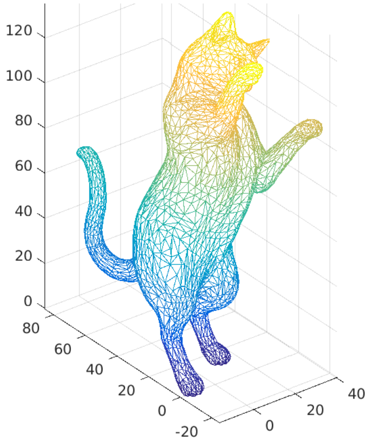

Figure : Triangular mesh surface approximation.

- Triangular meshes are by far the most preferred discretization.
  - Approximate surface locally via planar polygons - we prefer triangles.
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Datastructure for Mesh

- Data: Geometry & Topology.
  - Geometry: For a mesh with n vertices, the geometry consists of an *ordered* $n \times 3$ array of coordinates of the n vertices.
  - Topology: Connectivity information stored as an $m \times 3$ integer array, where m is the number of triangles in the mesh, and each row consisting of indices i, j, k such that the vertices v_i, v_j, v_k form a triangle.
  - A triangle is often referred to as a *face*. The surface represented by the triangular mesh is:

$$M' = \cup_{i=1}^{m} conv(x_{t_i}, y_{t_i}, z_{t_i})$$

  - Additional information may be stored, for example list of edges as a $k \times 2$ array consisting of the index pairs of vertices forming an edge.

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Degenerate cases

- Non-manifold meshes:

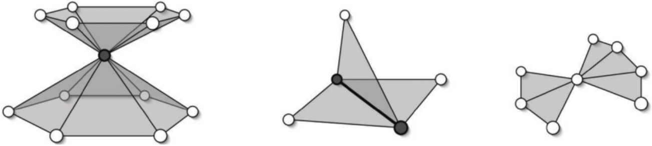

Figure : Examples of non-manifold meshes. Image from *Polygonal Mesh Processing, Botsch et al., A.K.Peters, 2010*, henceforth BKPAL10

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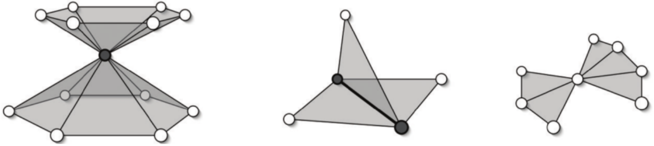

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Discrete Differential Geometry

Differential geometric properties

- A triangular mesh is a linear approximation of the original surface.
- Most of differential geometry depends on derivatives of different quantities.
- We would want to compute the values of the derivatives of given quantities/signal/function at the vertices.
- Given a manifold M, and a function $f : M \to \mathbb{R}$ on it, the directional derivative of f along $v \in T_pM$ at p is defined as $(D_v f)(p) = \lim_{t \to 0} \frac{f(\gamma(t)) - f(p)}{t}$, where $\gamma : (-\epsilon, \epsilon) \to M$ is a smooth curve on M such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v \in T_pM$.

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Approximating derivatives

- Let us first consider sampling of a 1 D signal as shown in the figure below.

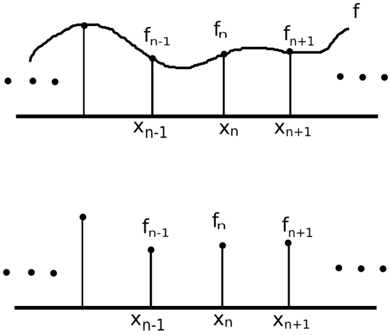

Figure : Sampling of a 1D signal.

- In the absence of the original signal f, we assume that the original signal lies in the span of the following *piece-wise linear* basis:

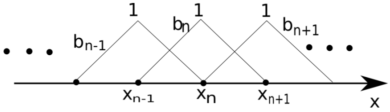

Figure : Piece-wise linear basis, note that $b_n(x) = 0, x \notin [x_{n-1}, x_{n+1}]$.

- Thus
  $\tilde{f}(x) = \ldots + a_{n-1}b_{n-1}(x) + a_n b_n(x) + a_{n+1}b_{n+1}(x) + \ldots$.
- Coefficients a_i's can be found by requiring that $\tilde{f}(x = x_n) = f_n$, the given sample values.
- For this particular choice of basis it is clear that $a_n = f_n, \forall n$, which gives for any $x \in [x_n, x_{n+1}]$,

$$\tilde{f}(x) = \frac{x_{n+1} - x}{x_{n+1} - x_n} f_n + \frac{x - x_n}{x_{n+1} - x_n} f_{n+1}.$$

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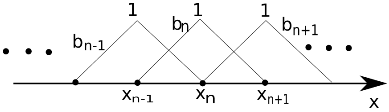

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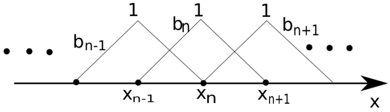

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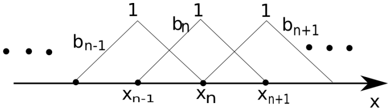

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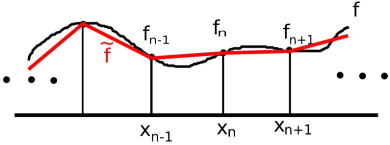

Figure : Original signal f and interpolated signal $\tilde{f}$.

- This allows us to compute derivatives of $\tilde{f}$ at any interior point $x \in (x_n, x_{n+1})$ as

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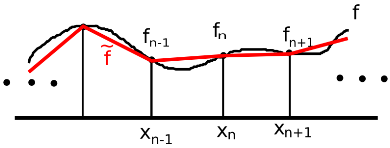

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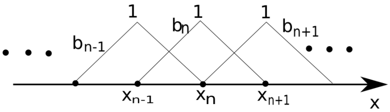

Figure : Piece-wise linear basis

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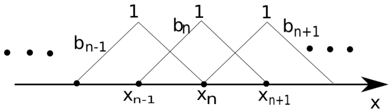

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  - Forward Difference: $D_+\tilde{f}(x_n) = \frac{f_{n+1} - f_n}{x_{n+1} - x_n}$.
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Gradients of functions on a mesh

- Linear basis on a triangle of the mesh:

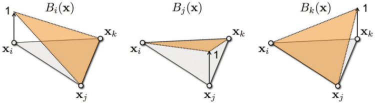

Figure : Three piece-wise linear basis per triangle. Image from BKPAL10.

- Any piece-wise linear function on the mesh with values at the vertices f_i can be written as

$$f(x) = f_i B_i(x) + f_j B_j(x) + f_k B_k(x)$$

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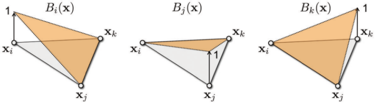

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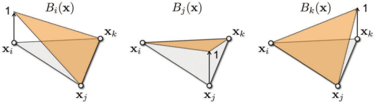

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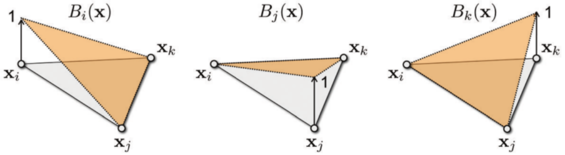

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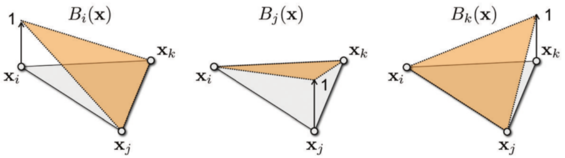

Figure : Three piece-wise linear basis per triangle. Image from BKPAL10.

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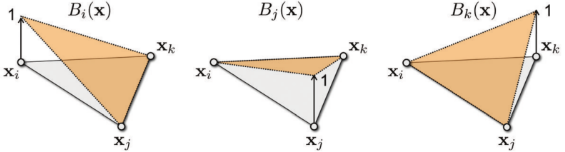

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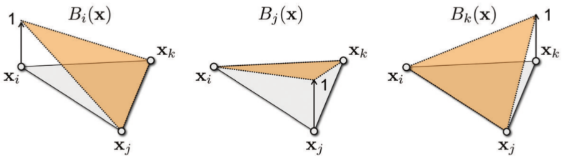

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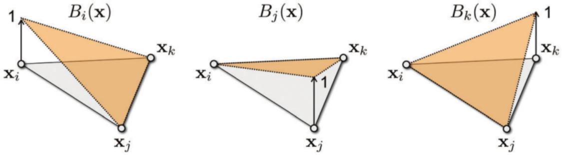

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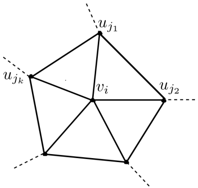
Surface Normal

- Surface normal for a triangle T with vertices (x_i, x_j, x_k), given in AC order seen from outside of the surface:
$N(T) = \frac{(x_j - x_i)\times(x_k - x_i)}{||(x_j - x_i)\times(x_k - x_i)||}$.
- On a vertex v: Weighted average of unit normals of all incident triangles.

$$N(v) = \frac{\sum_{T\in N_1(v)} a_T N(T)}{||\sum_{T\in N_1(v)} a_T N(T)||},$$

where a_T's are non-negative real scalars.
- *One-Ring*: Set of triangles incident to a given vertex v is called the one-ring of v and is denoted by $N_1(v)$.



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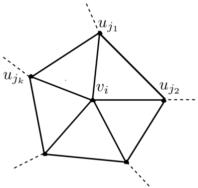

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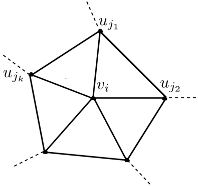

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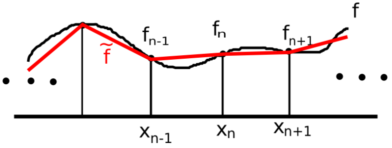

Figure : Original signal f and interpolated signal $\tilde{f}$.

- For now let's assume $x_n = \Delta n$, where Δ is a uniform sampling interval.
- $\frac{d^2\tilde{f}}{dx^2}(x) = 0, \;\; \forall x \in (x_n, x_{n+1})$.
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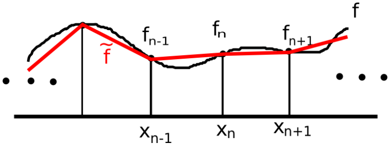

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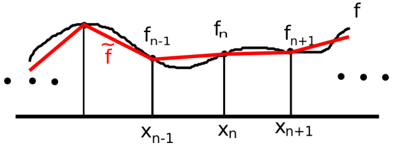

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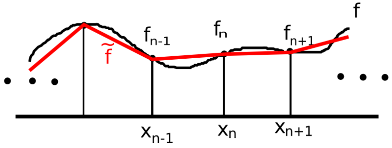

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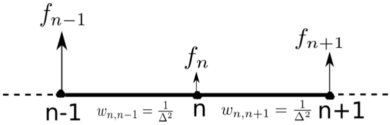

Figure : Signal domain interpreted as a graph.

- Denoting the Laplacian operator by L, we have

$$(L\tilde{f})_n = \sum_{j \in N(n)} w_{jn}(f_j - f_n), \quad w_{jn} = \frac{1}{\Delta^2}.$$

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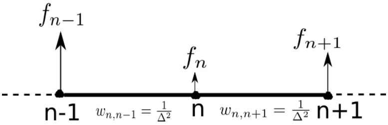

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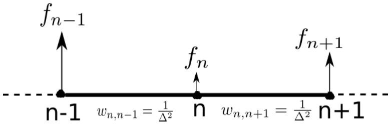

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where $w_{n,n} = -\sum_j w_{jn}$.

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- Thus $L = A - D$.
- In graph theory, $\tilde{L} = D - A$ is called the *Graph Laplacian*.

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Laplacian for meshes

- A mesh is a graph; note $\Delta = d(x_n, x_{n+1})$.

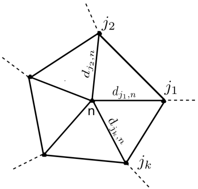

Figure : Mesh as a graph, $d_{jn} = ||x_j - x_n||^2$

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- If vertex i and j are edge-connected $A(i,j) = \frac{1}{d_{i,j}}$, else $A(i,j) = 0$.
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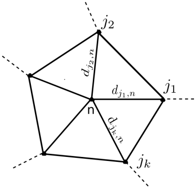

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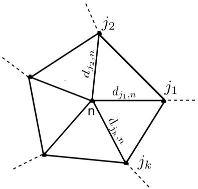

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- Properties of the measure in the continuous setting should be satisfied by the discrete estimate.
- Order of error: Finer sampling $\Rightarrow$ Error should go down.
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Laplace-Beltrami Operator (LBO)

- **Definition 1:** $\Delta := div \circ grad$.
- In $\mathbb{R}^n$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$, LBO = Laplacian.
- Properties/Characteristics of LBO:
  - We know that $A(\sigma) = \int\int \sqrt{EG - F^2}\, da$, for $\sigma : \Omega \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$.
  - For $\sigma(u, v) = (u, v, f(u, v))$, $EG - F^2 = 1 + f_u^2 + f_v^2$.
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Laplace-Beltrami Operator (LBO)

- **Definition 1:** $\Delta := div \circ grad$.
- In $\mathbb{R}^n$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$, LBO = Laplacian.
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- Let $g : \Omega \to \mathbb{R}$ be a perturbation added to f by a scalar amount s, so that the surface σ becomes $\tilde{\sigma}_s = (u, v, f(u, v) + sg(u, v))$, with $g|_{\partial\Omega} = 0$.
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  - Another notation for DD: $\frac{\partial A}{\partial g}(f)$.
  - As in finite dimensional cases, $\frac{\partial A}{\partial g}(f) = \langle \nabla A(f), g \rangle$.
  - It can be derived that

$$\nabla A(f) = -\frac{(1 + f_v^2) f_{uu} + (1 + f_u^2) f_{vv} - 2 f_u f_v f_{uv}}{(1 + f_u^2 + f_v^2)^{\frac{3}{2}}}.$$

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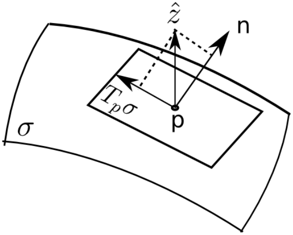

Figure : Area minimization surface evolution

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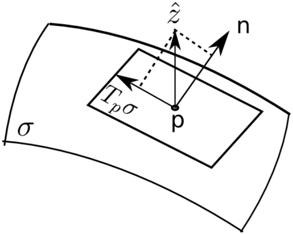

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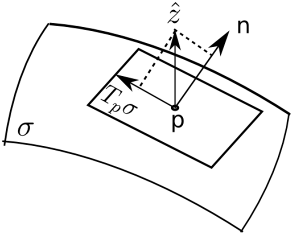

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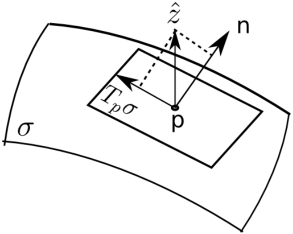

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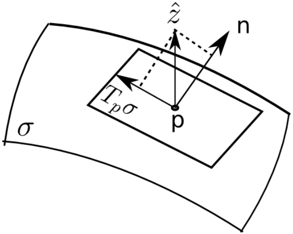

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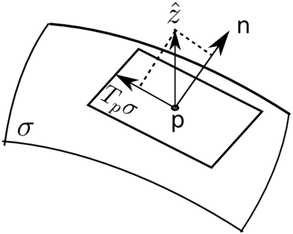

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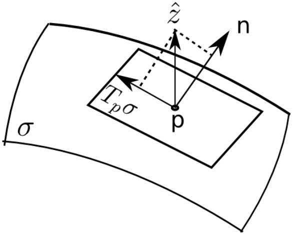

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What has this got to do with LBO?

- *Isothermal or Conformal parameterization*: If $F = 0$ and $E = G$ at every point of the surface, we say that the parameters are *isothermal* or *conformal*.
  - Assuming such parameters[1] u, v exist and $E = G = 1$, $H = L + N$.
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  - Conformal parameterization $\Rightarrow \Delta\sigma = Hn$.
- **Definition 2**[2]: LBO is the operator that assigns to each point $p \in M$ the *mean curvature vector* $H(p)n(p)$.

[1] Gauss(1822) proved existence of local conformal parameterization for any 2D surface, while global existence was shown by Lichtenstein (1916).

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Dirichlet Energy

- A parameterization σ of a surface maps points from $\mathbb{R}^2$ to points in $\mathbb{R}^3$.
- A measure of how rapidly σ varies:

$$E(\sigma) = \frac{1}{2}\int\int ||\frac{\partial\sigma}{\partial u}||^2 + ||\frac{\partial\sigma}{\partial v}||^2 \, du \, dv$$

- Typically written as $E(\sigma) = \frac{1}{2}\int\int ||\nabla\sigma||^2 \, du \, dv$ or $E(\sigma) = \frac{1}{2}\int\int Tr(J\sigma^t J\sigma) \, du \, dv$, and is called the *Dirichlet Energy*.
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Self-Adjoint operator

- Let $V := \{f : [0, 2\pi) \to \mathbb{R} \mid f \text{ is smooth and periodic}\}$, equipped with the standard L_2 inner product.
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$$\int_0^{2\pi} \Delta f(x)\, g(x) dx = [\nabla f(x) g(x)]_0^{2\pi} - \int_0^{2\pi} \nabla f(x) \nabla g(x)\ dx$$

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Eigenfunctions of discrete LBO

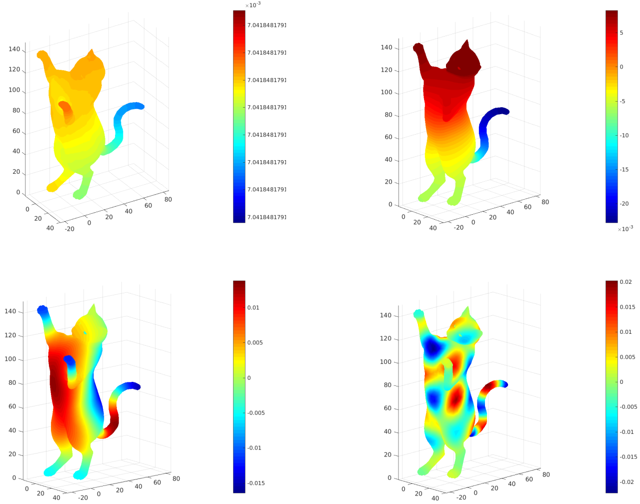

Figure : Can you interpret these eigenfunctions?

Deriving discrete LBO

- **Derivation 1.:** Minimizing Dirichlet energy[4].
- Let a surface be parameterized by the function f; the corresponding Dirichlet energy is

$$E_D(f) = \frac{1}{2} \int_\Omega ||\nabla f||^2 \, da$$

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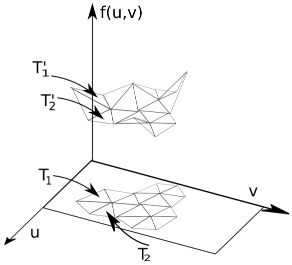

Figure : Triangles in local coordinates mapping to Triangles on the surface.

- For a discrete triangulated surface, we will assume that f is defined at vertices, while at all other points we will assume f to be linear.
  - Moreover, $E_D(f) = \sum_{i=1}^{m} E_D(f_i)$, where m is the total number of triangles, and f_i is a mapping between corresponding triangles i.

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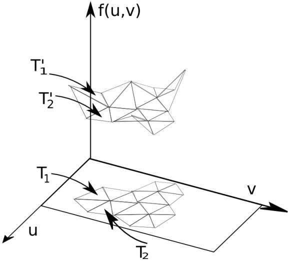

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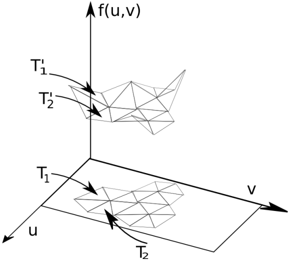

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- The linear map $f_i : \Delta_1 \to \Delta_2$ is defined by $f(v) = a, f(w) = b$.

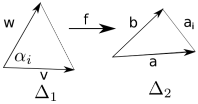

Figure : f_i between two corresponding triangles.

- The map f can also be written as $f = \psi \circ \phi^{-1}$, as shown in the following figure.

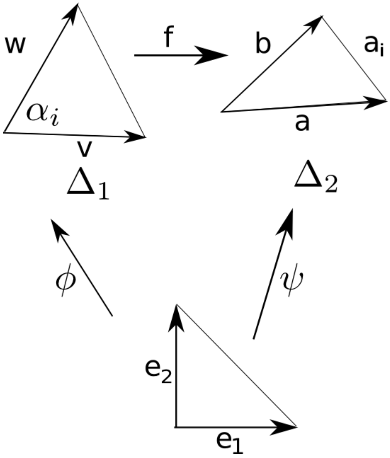

Figure : The map f via the standard triangle.

- Using Chain rule & Inverse Function theorem, $Jf = J\psi \circ J(\phi^{-1}) = J\psi \ (J\phi)^{-1}$.
  - Then,

$$\begin{aligned} Tr(Jf^t \ Jf) &= Tr((J\psi \ (J\phi)^{-1})^t J\psi \ (J\phi)^{-1}) \\ &= Tr(((J\phi)^{-1})^t J\psi^t J\psi \ (J\phi)^{-1}) \\ &= Tr(J\psi^t J\psi \ (J\phi)^{-1}((J\phi)^{-1})^t) \end{aligned}$$

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  - Let $d = det(J\phi)$ and $c = b - a$.

$$\begin{aligned} Tr(Jf^t \ Jf) = &\frac{1}{d} \left(\frac{\langle -w, v - w \rangle}{d} ||a||^2 + \frac{\langle w - v, -v \rangle}{d} ||b||^2 \right. \\ &\left. + \frac{\langle v, w \rangle}{d} ||c||^2 \right) \end{aligned}$$

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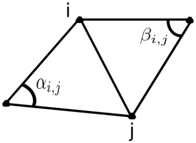

Figure : Weights for the Cotan Laplacian

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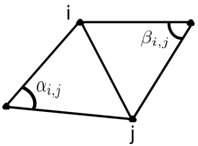

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Derivation 2: Weak formulation of the Poisson equation

- Compute solution to the Poisson's equation: $\Delta u = f$, i.e., given f, compute u.
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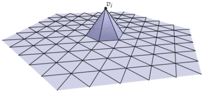

Figure : Piece-wise linear

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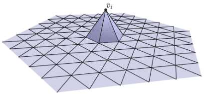

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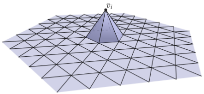

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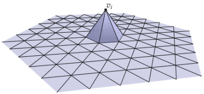

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    $\Rightarrow \langle \Delta \hat{u} - f, B_i \rangle = 0, \forall i$.

Derivation 2: Weak formulation of the Poisson equation

- Compute solution to the Poisson's equation: $\Delta u = f$, i.e., given f, compute u.
  - The domain is a mesh, and the space of functions we work with is $V := \{u = \sum_j u_j B_j\}$, where B_j's are the linear functions with value 1 at vertex j, else 0, as shown below.

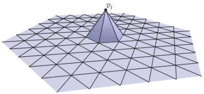

Figure : Piece-wise linear

  - Given f, can we compute $\hat{u} \in V$ as the best approximation to the actual solution u?
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- Inner product on the mesh Ω: $\langle f, g \rangle_\Omega = \int_\Omega f(x) g(x)\, da$.
- Solve: $\langle \Delta \hat{u}, B_i \rangle_\Omega = \langle f, B_i \rangle_\Omega \Rightarrow \sum_j u_j \langle \Delta B_j, B_i \rangle_\Omega = \langle f, B_i \rangle_\Omega$.
- Given that $\hat{u}$ (or B_j) $\in V$, how to define $\Delta \hat{u}$ (or ΔB_j).

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- Integration by parts:

$$\int_{\Omega} f \Delta g da = \int_{\partial\Omega} f \langle \nabla g, n \rangle \; dl - \int_{\Omega} \langle \nabla f, \nabla g \rangle da,$$

where n is the outward unit normal to Ω on the boundary $\partial\Omega$.

- Note that ∇f and ∇g are vector fields on the mesh Ω, and the second term on the right hand side can be written as $\langle \nabla f, \nabla g \rangle_{\Omega}$.
- Idea 2: If Ω is a closed mesh, $\partial\Omega = \emptyset$, giving us

$$\langle f, \Delta g \rangle_{\Omega} = -\langle \nabla f, \nabla g \rangle_{\Omega}.$$

- Thus rather than solving $\sum_j u_j \langle \Delta B_j, B_i \rangle_{\Omega} = \langle f, B_i \rangle_{\Omega}$, we solve $-\sum_j u_j \langle \nabla B_j, \nabla B_i \rangle_{\Omega} = \langle f, B_i \rangle_{\Omega}$.
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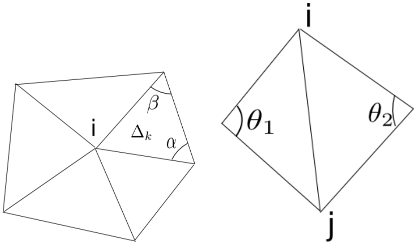

Figure : Laplacian at a point in the mesh

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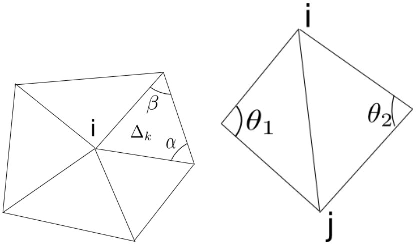

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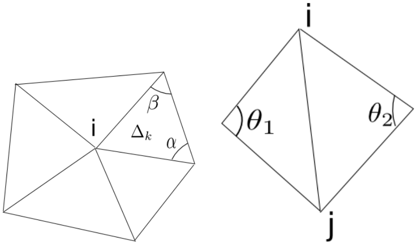

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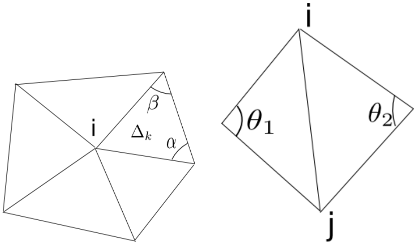

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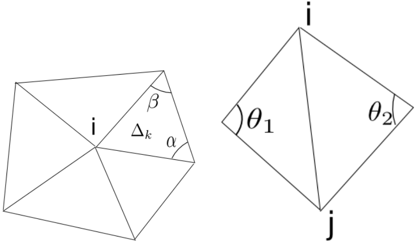

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- Thus $\langle \nabla B_i, \nabla B_i \rangle_\Omega = \frac{1}{2}\sum_{\Delta_i}(\cot\alpha_i + \cot\beta_i)$ and $\langle \nabla B_i, \nabla B_j \rangle_\Omega = -\frac{1}{2}(\cot\theta_1 + \cot\theta_2)$.
- The Discrete LBO is

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L_{ij} =& -\frac{1}{2}(\cot\alpha_j + \cot\beta_j), \quad \text{if } i \sim j. \\
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Cotan Laplacian[5]

- Using the definition of LBO: $\nabla = div \circ grad$:

$$\int_{A_i} \Delta f(x)\ da = \int_{A_i} div \nabla f(x)\ da = \int_{\partial A_i} \nabla f(x) \cdot n(x)\ dl,$$

where A_i refers to a local integration (averaging) area, typically the Voronoi cell of a vertex.

- This reduces to

$$\int_{A_i} \Delta f(x)\ da = \frac{1}{2} \sum_{j \in N(i)} (\cot \alpha_{ij} + \cot \beta_{ij})(f_j - f_i)$$

- Thus the LBO on function f is:

$$\Delta f(v_i) = \frac{1}{2A_i} \sum_{j \in N(i)} (\cot \alpha_{ij} + \cot \beta_{ij})(f_j - f_i)$$

[5] Meyer et.al., Discrete Differential Geometry Operators for Triangulated 2-Manifolds, Vis. & Math - 3, 2003

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Application 1: Heat Equation

- $\frac{\partial u}{\partial t} = \Delta u.$

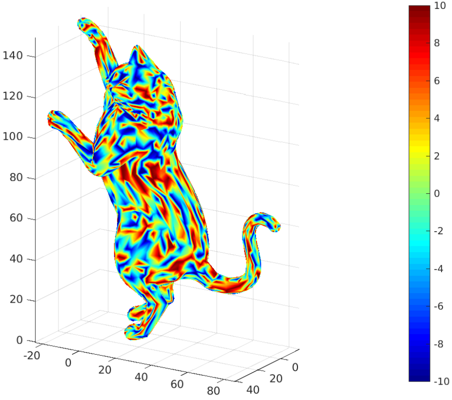

Figure : Heat diffusion on a cat.

Application 2: Compression

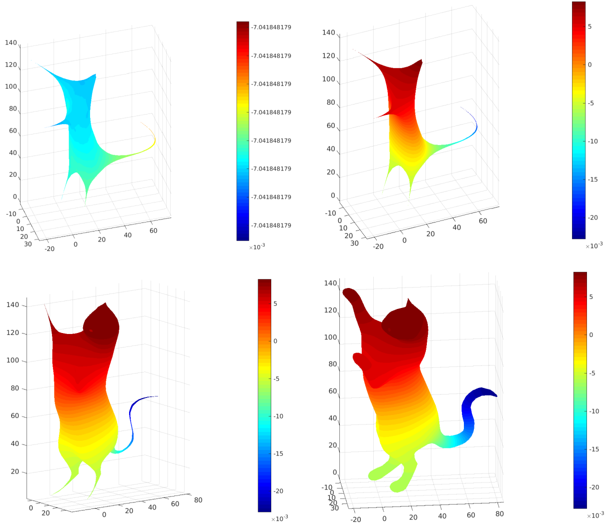

Figure : Mesh recovered using (top-left) 1, (top-right) 10, (bottom-left) 50 and (bottom-right) 200 coefficients, out of a total 3400 coefficients.