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Saddam Hussain[1]*, Muhammad Shoaib[2], Muhammad Tasleem[2], Saba Iram[2], Muhammad Faheem Amin[1], Noman Javed[3], Syed Adeel Ali Bukhari[4], Awais Muzafar[5]

[1]Department of Mathematics, Center for Advanced Studies in Pure and Applied Mathematics, Bahaudin Zakariya University Multan, Pakistan
[2]Department of Mathematics, The Islamia University of Bahawalpur, Pakistan
[3]Department of IT, University of West Scotland London
[4]Department of Public Administration, University of Karachi, Pakistan
[5]Department of Computer Science, Beijing Institute of Graphic Communication, China

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***Corresponding author:** Saddam Hussain
Department of Mathematics, Center for Advanced Studies in pure and Applied Mathematics Bahaudin Zakria University Multan, Pakistan

Abstract

This study investigates a static plane symmetric spacetime within the framework of f(Q)-Gravity, an advanced extension of gravitational theory offering new insights into spacetime geometry. We analyzed 12 distinct linear and nonlinear cases to identify homothetic vector fields (HVFs), which are crucial for understanding spacetime symmetries. Through direct integration, we discovered that in cases (ii), (vi), (viii), and (xii), HVFs reduce to four-dimensional Killing Vector Fields (KVFs). Conversely, cases (i), (iii), (iv), (v), (vii), (ix), (x), and (xi) exhibit HVFs with dimensions of five, seven, ten, and eleven. These results underscore the diversity in vector field dimensions, encouraging further exploration of the implications of f(Q)-Gravity.

Keywords: f(Q) gravity, Homothetic vector fields (HVFs), Killing vector fields (KVFs), Modified gravity.

1. INTRODUCTION

Modern theoretical physics relies heavily on symmetry, especially when examining spacetime structures. Because they provide a constant scaling factor, Homothetic Vector Fields (HVFs) are a significant symmetry that generalise Killing vector fields. The study of self-similar solutions in modified gravity theories, black hole physics, and cosmology depends heavily on these vector fields [1-3].

The study of HVFs has shed light on conserved quantities and the characteristics of self-similar spacetimes in classical General Relativity (GR) [4]. But in modified theories of gravity, such f(Q)-gravity, a function of the non-metricity scalar Q is used in place of the Ricci scalar R to introduce new geometric structures [5]. An alternate method of understanding cosmic acceleration, dark energy, and other unexplained gravitational phenomena is provided by this formulation [6]. The increasing interest in f(Q)-gravity makes it necessary to investigate the behaviour of classical symmetries such as HVFs in this context.

Plane Symmetry in Static Because spacetimes are used in cosmology and astrophysics, they have been thoroughly examined in the framework of GR and other modified gravity theories [7, 8]. Their study in f(Q)-gravity is still largely unexplored, nevertheless. Examining HVFs in this context may aid in comprehending spacetime's geometrical characteristics and yield new conserved quantities that are essential for both theoretical and observational research [9, 10].

This paper is to examine the presence and categorisation of homothetic vector fields in f(Q)-gravity based on static plane symmetric spacetime to calculate the HVF governing equations and contrast them with the General Relativity findings. To investigate the acquired vector fields' potential geometrical and physical ramifications in relation to changed gravity. Symmetric Static Planes The importance of spacetimes in cosmological models, exact solutions, and gravitational collapse situations has led to extensive research on them in classical and modified gravity theories [11-13]. These spacetimes are helpful for modelling thin-shell structures, gravitational fields, and specific astrophysical configurations because they show translational symmetry in two spatial directions.

The **line element** for a general Static Plane Symmetric Spacetime can be expressed as

$$(dz^2 + c^2(\mathrm{x})(\mathrm{dy} + \mathrm{B}^2(x)dx + A^2(x)dt = ds^2$$

Where A(x), B(x) and C(x) are metric functions to be determined through field equations. Very little work has been done on the symmetry qualities of Static Plane Symmetric Spacetimes within this framework, despite the increasing interest in f(Q)-gravity. Although Homothetic Vector Fields have been thoroughly examined in GR and other alternative gravity theories, nothing is known about how they function in f(Q)-gravity [13-17]. Exploring the physical and geometric implications of HVFs in f(Q)-gravity, particularly in relation to conserved quantities and self-similar solutions. Conservation laws, aiding in the study of both matter properties and spacetime geometry [17-20]. It is important to note that a vector field *K* is classified as a Killing vector field if it satisfies the Killing equation, given by

$$\underset{X}{L}\, g_{ab} = g_{\mathrm{ab},} X^c + g_{cb} X^c + g_{ac} X^\lambda = 0,$$

In equation (1) $\underset{X}{L}$ represents the metric tensor's Lie derivative along the vector field X This operation measures how the metric tensor changes when one flows along the direction specified by the vector field

2. Formulation and Solution of Field Equations for the Static Plane Symmetric Spacetime in f(Q)-Gravity

We consider a SPS Spacetime in the usual coordinates (t, x, y, z) given by (x^0, x^1, x^2, x^3) respectively with the line element (Stephani *et al.*, 2009)

$$ds^2 = -e^{2M} dt^2 + dx^2 + e^{2N}\left[dy^2 + dz^2\right], \qquad (2.1)$$

where $M = M(x)$ and $N = N(x)$ are nowhere zero functions of x only. The above spacetime (2.1) admits four linearly independent KVFs which are (Stephani *et al.*, 2009)

$$\frac{\partial}{\partial t},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\ \text{and}\ y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}. \qquad (2.2)$$

The KVFs extracted in this study are also related to some conserved quantity. For instance, the KVF $\left(\frac{\partial}{\partial t}\right)$ represents the conservation of energy for the physical system. Similarly, translational symmetries $\left(\frac{\partial}{\partial y}\right)$ and $\left(\frac{\partial}{\partial z}\right)$ act as the source to generate the conservation of linear momentum whereas the rotational KVF $\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$ helps us to deduce the conservation of angular momentum. The non-metricity scalar Q for the above spacetime (2.1) is

$$Q = 4M'N' + 2N'^2, \qquad (2.3)$$

where prime stands for the derivative with respect to x. Now, our goal is to find the static plane symmetric solutions in the f(Q)-Gravity, and utilize these solutions to categorize via HVFs. The field equations in f(Q)-Gravity are (Wang *et al.*, 2022)

$$f_Q G_{\mu\upsilon} + \frac{1}{2} g_{\mu\upsilon}\left(Qf_Q - f(Q)\right) + 2 f_{QQ} \nabla_\lambda p^\lambda_{\mu\upsilon} = kT_{\mu\upsilon}, \qquad (2.4)$$

where f_Q is the derivative of $f(Q)$ with respect to non-metricity scalar Q, $G_{\mu\upsilon}$ is the Einstein tensor, $g_{\mu\upsilon}$ is the metric tensor, Q is the non-metricity scalar, ∇ is the covariant derivative operator, $p^\lambda_{\mu\upsilon}$ represents the non-metricity conjugate, k is the coupling constant and $T_{\mu\upsilon}$ show the Energy Momentum Tensor (EMT).

The EMT for the perfect fluid admitting plane symmetry is given by

$$T_{\mu\upsilon} = (\rho + p)\mu_\mu \mu_\upsilon + g_{\mu\upsilon} p. \qquad (2.5)$$

In the equation (3.2.5), ρ is the density, p is the pressure and μ_υ represents the four-velocity vector. Using equation (2.2), (2.3) and (2.5) in equation (2.4), yield the following equations:

$$2f_{QQ}Q'N' + f_Q\left[-4N'^2 - 2N'' - 2M'N'\right] + \frac{f}{2} = \rho, \qquad (2.6)$$

$$\left[2N'^2 + 4M'N'\right] f_Q - \frac{f}{2} = p, \qquad (2.7)$$

$$\left[M'^2 + 2N'^2 + 3M'N' + M'' + N''\right] f_Q - f_{QQ}Q'\left(M' + N'\right) - \frac{f}{2} = p. \qquad (2.8)$$

In this chapter, we are implementing an approach in order to find solutions of equations (2.6) to (2.8). This approach involves constraints that restrict the components of the Spacetime and help to reduce the complexity in the above system of equations (2.6) to (2.8). Before implementing the above-mentioned approach, we utilize some algebra to get the following equation:

$$f_{QQ}Q'N'\left(M' + N'\right) + f_Q\left[M'N' - M'^2 - M'' - N''\right] = 0. \qquad (2.9)$$

Now, we extend the non-linear case and proceeds as:
Considering equation (2.9) is non-linear and require some restriction on Spacetime components, so we solve it for the following possibilities:
(i) M = constant and $N = N(x)$. (ii) $M = M(x)$, $N = N(x)$ and $M = N$.

The values of metric components by placing such restrictions on the Spacetime components, the values of non-metricity scalar Q and $f(Q)$ for the possibilities (i) and (ii) are given in the Table (1):
Solutions of equation (3.2.9) together with non-metricity scalar Q *and* $f(Q)$

Tab. 1

Case No.	Metric components	Value of Q	Value of $f(Q)$
(i)	$M =$ constant and $N = \ln(x)$.	$Q = 2x^{-2}$.	$f(Q) = \frac{\sqrt{2}d_1}{3}Q^{\frac{3}{2}} + c_2$.
(ii)	$M =$ constant and $N = x^2$.	$Q = 8x^2$.	$f(Q) = \frac{\sqrt{2}d_1}{3}Q^{\frac{3}{2}} + c_2$.
(iii)	$M =$ constant and $N = \ln\left(\sqrt{2(c_1x + c_2)}\right)$.	$Q = \frac{2c_1^2}{(2c_1x + 2c_2)^2}$.	$f(Q) = \frac{\sqrt{2}d_1}{3}Q^{\frac{3}{2}} + c_2$.
(iv)	$M =$ constant and $N = (c_1x + c_2)$.	$Q = 2c_1^2$.	$f(Q) = \frac{\sqrt{2}d_1}{3}Q^{\frac{3}{2}} + c_2$.
(v)	$M = N = \ln\left(\sqrt{2(c_1x + c_2)}\right)$.	$Q = \frac{3c_1^2}{2(c_1x + c_2)^2}$.	$f(Q) = \frac{\sqrt{2}d_1}{3}Q^{\frac{3}{2}} + c_2$.
(vi)	$M = N = \left(\frac{c_1x + c_2}{4}\right)^2$.	$Q = \frac{6c_1^2(c_1x + c_2)^2}{64}$.	$f(Q) = \frac{\sqrt{2}d_1}{3}Q^{\frac{3}{2}} + c_2$.

We assume $f(Q)$ to be linear and put in equation (2.9), then:

$$f(Q) = d_1Q + d_2, \qquad (2.10)$$

$d_1, d_2 \in R$. One of key advantages of choosing $f(Q)$ to be linear is that it reduces the complexity of equations of motion. Under the fact given by equation (2.10), equation (2.9) takes the form $d_1\left[M'N' - M'^2 - M'' - N''\right] = 0.$ As $d_1 \neq 0$, therefore

$$\left[M'N' - M'^2 - M'' - N''\right] = 0. \qquad (2.11)$$

Now, we implement certain constraints on the Spacetime components to get various solutions of equation (2.11). It is important to mention here that upon utilizing the observed values of Spacetime components into equation

(2.3), we have also computed the values of non-metricity scalar Q. For sake of simplicity, we list the results in the form of Table (2):

Solutions of equation (3.2.11) together with non-metricity scalar Q.

Tab. 2

Case No.	Metric components	Value of Q	Value of $f(Q)$
(vii)	$N=\text{constant}$ and $M=\ln(c_1x+c_2)$, where $c_1,c_2\in R(c_1\neq 0)$.	$Q=0.$	$f(Q)=c_1Q+c_2$, where $c_1,c_2\in R(c_1\neq 0)$.
(viii)	$M=\ln(c_1x+c_2)$ and $N=\left(\frac{c_1x^2}{2}+c_2x+c_3\right)$, where $c_1,c_2,c_3\in R(c_1,c_2\neq 0)$.	$Q=4c_1+2(c_1x+c_2)^2.$	$f(Q)=c_1Q+c_2$, where $c_1,c_2\in R(c_1\neq 0)$.
(ix)	$M=\text{constant}$ and $N=(c_1x+c_2)$, where $c_1,c_2\in R(c_1\neq 0)$.	$Q=2c_1^2.$	$f(Q)=c_1Q+c_2$, where $c_1,c_2\in R(c_1\neq 0)$.
(x)	$M=\ln(x^m)$ and $N=\ln\left(x^{\frac{m(m-1)}{m+1}}\right)$, where $m\in R(m\neq 0)$.	$Q=\frac{6m^4-4m^3-2m^2}{x^2(m+1)^2}.$	$f(Q)=m_1Q+m_2$, where $m_1,m_2\in R(m_1\neq 0)$.
(xi)	$M=(c_1x+c_2)$ and $N=(c_1x+c_2)$, where $c_1,c_2\in R(c_1\neq 0)$.	$Q=6c_1^2.$	$f(Q)=c_1Q+c_2$, where $c_1,c_2\in R(c_1\neq 0)$.
(xii)	$M=\ln\left(\frac{e^{c_1x}-c_1c_4}{c_1c_3}\right)$ and $N=(c_1x+c_2)$, where $c_1,c_2\in R(c_1\neq 0)$.	$Q=\frac{6c_1^2e^{c_1x}-2c_1^3}{e^{c_1x}-c_1c_3}.$	$f(Q)=c_1Q+c_2$, where $c_1,c_2\in R(c_1\neq 0)$.

We utilize the solutions given in the cases (i) to (xii) for finding the HVFs with the help of equation

$$L_\eta g_{im}=2\psi g_{im}, \qquad (2.12)$$

3. Homothetic Vector Fields of the Obtained Classes of Static Plane Symmetric Spacetime in f(Q)-Gravity

It is important to mention here in the process of finding HVFs, there exist a system of coupled partial differential equations (PDEs). We have used direct integration technique to solve such PDEs. We expand (2.12) by using $\psi=\alpha$ (constant $\neq 0$). Assuming $N=$ as a vector field and using (2.1), we derive the following system of equations:

$$M'X^1+X^0_{,0}=\alpha, \qquad (3.1)$$

$$X^1_{,0}-e^{2M}X^0_{,1}=0, \qquad (3.2)$$

$$e^{2N}X^2_{,0}-e^{2M}X^0_{,2}=0, \qquad (3.3)$$

$$e^{2N}X^3_{,0}-e^{2M}X^0_{,3}=0, \qquad (3.4)$$

$$X,_1^1 = \alpha, \tag{3.5}$$

$$e^{2N} X,_1^2 + X,_2^1 = 0, \tag{3.6}$$

$$e^{2N} X,_1^3 + X,_1^3 = 0, \tag{3.7}$$

$$N' X^1 + X,_2^2 = \alpha, \tag{3.8}$$

$$X,_3^2 + X,_2^3 = 0, \tag{3.9}$$

$$N' X^1 + X,_3^3 = \alpha. \tag{3.10}$$

Now, we solve above equations (3.1) to (3.10) for the cases given in Table (1) and Table (2):

Case (i)

In this case the metric coefficients are $M =$ constant and $N = \ln(x)$. The spacetime (3.1) takes the form as

$$ds^2 = -dt^2 + dx^2 + x^2\left[dy^2 + dz^2\right]. \tag{3.11}$$

From equation (3.1) and (3.2), we get

$$\left.\begin{aligned} X^0 &= \alpha t + A^1(x, y, z) \\ X^1 &= \alpha x + A^2(\mathrm{t}, y, z) \end{aligned}\right\}. \tag{3.12}$$

Where $A^1(x, y, z)$ and $A^2(t, y, z)$ are the function of integration to be determine. In view of above system of equations (3.12), by utilizing the value of X^1 in equation (3.3), we get

$$\frac{1}{x}(\alpha x + A^2(t, y, z)) + X_{,2}^2 = \alpha. \tag{3.13}$$

From equation (3.13), we get

$$X^2 = \int -\frac{1}{x} A^2(t, y, z) dy + A^3(t, x, z), \tag{3.14}$$

Where $A^3(t, x, z)$ is function of integration to be determined. Using equation (3.12) and equation (3.4), we get the value of X^3, which is

$$X^3 = \int -\frac{1}{x} A^2(t, y, z) dz + A^4(t, x, \mathrm{y}), \tag{3.15}$$.

Now we attained the following system from equations (3.12), (3.14) and (3.15).

$$\left.\begin{aligned} X^0 &= \alpha t + A^1(x, y, z) \\ X^1 &= \alpha x + A^2(t, y, \mathrm{z}) \\ X^2 &= -\int\left[\frac{1}{x} A^2(t, y, z) dy - A^3(t, x, z)\right] \\ X^3 &= -\int\left[\frac{1}{x} A^2(t, y, z) dz - A^4(t, x, y)\right] \end{aligned}\right\}. \tag{3.16}$$

After solving all equations and avoiding the lengthy calculations we get the final system which is

$$\left.\begin{aligned} X^0 &= \alpha t + d_4 \\ X^1 &= \alpha x \\ X^2 &= z d_7 + d_8 \\ X^3 &= -y d_7 + d_9 \end{aligned}\right\}. \tag{3.17}$$

The spacetime (3.11), admits five HVFs out of which one is proper HVF and four are the KVFs. The proper HVF after subtracting KVFs is

$$\left(t, x, 0, 0\right). \tag{3.18}$$

Case (ii)

The components of the line element are $A =$ constant and $B = x^2$. The spacetime (3.3.11) takes the accompanying structure

$$ds^2 = -dt^2 + dx^2 + e^{2x^2}\left[dy^2 + dz^2\right]. \tag{3.19}$$

From equation (3.1) and (3.2), we get

$$\left.\begin{aligned} X^0 &= \alpha t + A^1(x, y, z) \\ X^1 &= \alpha x + A^2(\mathrm{t}, y, z) \end{aligned}\right\}, \tag{3.20}$$

In view of system (3.20), using the value of X^1 in equations (3.3) & (3.4) we got

$$\left.\begin{aligned} X^2 &= \alpha y - 2\alpha x^2 y - 2xyA^2(t, y, z) + A^3(t, x, \mathrm{z}) \\ X^3 &= \alpha z - 2\alpha x^2 z - 2xzA^2(t, y, z) + A^4(t, x, \mathrm{y}) \end{aligned}\right\}, \tag{3.21}$$

Now, using equation (3.5) with above system (3.20) and (3.21), we get the following form

$$A_t^2(t, y, z) - A_x^1(x, y, z) = 0. \tag{3.22}$$

Integrating equation (3.22) twice over "t", we obtain

$$A^2(t, y, z) = tB^1(y, z) + B^2(y, z), \tag{3.23}$$

Differentiate equation (3.22) w.r.t "x" and then integrating twice, we have

$$A^1(x, y, z) = xB^1(y, z) + B^3(y, z), \tag{3.24}$$

Considering the values found in equations (3.20) and (3.21), the system (3.16) takes the following form

$$\left.\begin{aligned} X^0 &= \alpha t + xB^1(y, z) + B^3(y, z), \\ X^1 &= \alpha x + tB^1(y, z) + B^2(y, z), \\ X^2 &= \alpha y - 2x^2 y\alpha - 2xy(tB^1(y, z) + B^2(y, z)) + A^3(t, x, z), \\ X^3 &= \alpha z - 2x^2 z\alpha - 2xz(tB^1(y, z) + B^2(y, z)) + A^4(t, x, y), \end{aligned}\right\}. \tag{3.25}$$

After tedious calculations we get the new system:

$$\left.\begin{aligned} X^0 &= d_3 \\ X^1 &= 0 \\ X^2 &= zd_1 + td_2 \\ X^3 &= d_1 + d_4 \end{aligned}\right\}. \tag{3.26}$$

Case (iii)

The spacetime of this case is as follows when we put the values of $M =$ constant and $N = \ln\sqrt{2\left(c_1 x + c_2\right)}$.

$$ds^2 = -dt^2 + dx^2 + \left(2c_1 x + 2c_2\right)\left[dy^2 + dz^2\right]. \tag{3.27}$$

In compatibility of HVFs, one tackles the equations (3.1) to (3.10) alongside the spacetime (3.27). Following a long tedious process of calculations we have the accompanying HVFs.

$$\left.\begin{aligned} X^0 &= b_1 t + b_5, \\ X^1 &= b_1\left(\frac{c_1 x + c_2}{c_1}\right), \\ X^2 &= \frac{y}{2} b_1 - \frac{z}{2} b_2 + b_4, \\ X^3 &= -\frac{z}{2} b_1 + \frac{y}{2} b_2 + b_3. \end{aligned}\right\}. \qquad (3.28)$$

The spacetime (3.27) yields five HVFs out of which one is proper HVF and four are the KVFs. The proper HVF after subtracting KVFs is

$$\left(t, \frac{c_1 x + c_2}{2}, \frac{y}{2}, \frac{-z}{2}\right), \qquad (3.29)$$

where b_1, b_2, b_3, b_4 and $b_5 \in R.$

Case (iv)

The constraints in this case are $M =$ constant and $N = \left(c_1 x + c_2\right).$ The spacetime (3.1) becomes

$$ds^2 = -dt^2 + dx^2 + e^{2c_1 x + 2c_2}\left[dy^2 + dz^2\right]. \qquad (3.30)$$

One address equation (3.1) to (3.10) to track down HVFs. After extend computations we obtained $\alpha = 0,$ which implies HVF become KVFs which are given beneath

$$\left.\begin{aligned} X^0 &= b_7, \\ X^1 &= 2b_1 c_1 z - \frac{y e^{-2c_2}}{c_1} b_2 - \frac{e^{-2c_2}}{c_1} b_4, \\ X^2 &= -2b_1 c_1^2 yz + \frac{e^{-2xc_1 - 2c_2}\left(c_1^2 y^2 e^{2xc_1} - c_1^2 z^2 e^{2xc_1} - e^{2c_2}\right) b_2}{2c_1^2} - z e^{-c_2} b_3 \\ &+ y e^{-2c_2} b_4 + b_6, \\ X^3 &= \frac{e^{-2xc_1 - 2c_2}\left(c_1^2 y^2 e^{2xc_1 + 2c_2} - c_1^2 z^2 e^{2xc_1 + 2c_2} + 1\right) b_1}{2c_1^2} + yz e^{-2c_2} b_2 \\ &+ y e^{-2c_2} b_3 + b_5. \end{aligned}\right\}, \qquad (3.31)$$

where $b_1, b_{b_2,} b_3, b_4, b_5, b_6$ and $b_7 \in R.$

Case (v)

In this case $M = N = \ln\sqrt{2\left(c_1 x + c_2\right)}$ and the model of spacetime can be written as

$$ds^2 = -\left(2c_1 x + 2c_2\right) dt^2 + dx^2 + \left(2c_1 x + 2c_2\right)\left[dy^2 + dz^2\right]. \qquad (3.32)$$

Working on equations (3.1) to (3.10) with the utilization of space- time (3.32), we have

$$\left.\begin{aligned} X^0 &= \frac{t}{2}b_1 + \frac{y}{2}b_5 + \frac{z}{2}b_2 + b_7, \\ X^1 &= b_1\left(\frac{c_1x+c_2}{c_1}\right), \\ X^2 &= \frac{y}{2}b_1 + \frac{t}{2}b_5 - \frac{z}{2}b_3 + b_6, \\ X^3 &= \frac{z}{2}b_1 + \frac{y}{2}b_3 + \frac{t}{2}b_2 + b_4. \end{aligned}\right\}, \qquad (3.33)$$

where $b_1, b_{b_2,} b_3, b_4, b_5, b_6$ and $b_7 \in R.$ The spacetime (3.32) admits seven HVFs with one proper HVF and remaining six are the KVFs. The proper HVF after subtracting the KVFs is

$$\left(\frac{t}{2}, \frac{c_1x+c_2}{2}, \frac{y}{2}, \frac{z}{2}\right). \qquad (3.34)$$

Case (vii)

This form of the spacetime is formed when we enter the values of $M = \ln\left(c_1x+c_2\right)$ and $N =$ constant.

$$ds^2 = -\left(c_1x+c_2\right)^2 dt^2 + dx^2 + dy^2 + dz^2. \qquad (3.35)$$

Our next step is to find HVFs by solving the equations (3.1) to (3.10). In view of the above spacetime (3.35), we have the accompanying HVFs

$$\left.\begin{aligned} X^0 &= \frac{y}{\left(c_1x+c_2\right)}\left(b_{11}e^{c_1t} - b_{10}e^{-c_1t}\right) + \frac{z}{c_1x+c_2}\left(b_7e^{c_1t} - b_2e^{-c_1t}\right) \\ &-\frac{1}{c_1x+c_2}\left(b_4e^{c_1t} - b_5e^{-c_1t}\right) + b_6, \\ X^1 &= \left(\frac{c_1x+c_2}{c_1}\right)b_1 - y\left(b_{11}e^{c_1t} + b_{10}e^{-c_1t}\right) - z\left(b_7e^{c_1t} + b_2e^{-c_1t}\right) \\ &+b_4e^{c_1t} + b_5e^{-c_1t}, \\ X^2 &= b_1y - b_8z + \frac{c_1x+c_2}{c_1}\left(b_{11}e^{c_1t} + b_{10}e^{-c_1t}\right) + b_3, \\ X^4 &= b_1z + b_8y + \frac{c_1x+c_2}{c_1}\left(b_7e^{c_1t} + b_2e^{-c_1t}\right) + b_9. \end{aligned}\right\}, \qquad (3.36)$$

where $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}$ and $b_{11} \in R.$ The spacetime (3.35) concedes eleven HVFs with one proper HVF and remaining ten are KVFs. The proper HVF after deducting the KVFs is $\left(0, \dfrac{c_1x+c_2}{c_1}, y, z\right).$

Case (ix)

The constraints in this case are $A =$ constant and $N = \left(c_1x+c_2\right).$ The spacetime (3.2.1) becomes

$$ds^2 = -dt^2 + dx^2 + e^{2c_1x+2c_2}\left[dy^2 + dz^2\right]. \qquad (3.37)$$

At this point, we need to address the equations (3.1) to (3.10) for tracking down the HVFs. After stretch computations one has $\alpha = 0,$ which recommend HVFs become KVFs which are given in equation (3.2).

$$\left.\begin{aligned}
&X^0 = b_7,\\
&X^1 = 2b_1c_1z - \frac{ye^{-2c_2}}{c_1}b_2 - \frac{e^{-2c_2}}{c_1}b_4,\\
&X^2 = -2b_1c_1^2yz + \frac{e^{-2xc_1-2c_2}\left(c_1^2y^2e^{2xc_1} - c_1^2z^2e^{2xc_1} - e^{2c_2}\right)b_2}{2c_1^2}\\
&-ze^{-c_2}b_3 + ye^{-2c_2}b_4 + b_6,\\
&X^3 = \frac{e^{-2xc_1-2c_2}\left(c_1^2y^2e^{2xc_1+2c_2} - c_1^2z^2e^{2xc_1+2c_2} + 1\right)b_1}{2c_1^2} + yze^{-2c_2}b_2\\
&+ye^{-2c_2}b_3 + b_5.
\end{aligned}\right\}. \qquad (3.38)$$

Where $b_1, b_{b_2,} b_3, b_4, b_5, b_6$ and $b_7 \in R.$

Case (x)

This kind of spacetime is formed when we enter the values of $M = \ln\left(x^m\right)$ $N = \ln\left(x^{\frac{m(m-1)}{m+1}}\right).$

$$ds^2 = -x^{2m}dt^2 + dx^2 + x^{\frac{2m(m-1)}{m+1}}\left[dy^2 + dz^2\right]. \qquad (3.39)$$

Our objective is to find HVFs, for which we adopt equations (3.1) to (3.10), utilizing the spacetime (3.39). After proceeding a lengthy calculation, we have the following five HVFs

$$\left.\begin{aligned}
&X^0 = -\left(\frac{m+1}{m-1}\right)b_1 - b_4,\\
&X^1 = \frac{x(m+1)}{(m-1)^2}b_1,\\
&X^2 = -yb_1 + zb_2 - b_5,\\
&X^3 = -zb_1 - yb_2 - b_3.
\end{aligned}\right\}. \qquad (3.40)$$

Where $b_1, b_{b_2,} b_3, b_4$ and $b_5 \in R.$ In equation (3.40), one is proper HVF and remaining four are the KVFs. The proper HVF after subtracting KVFs is

$$\left[\frac{b_1(m+1)}{m-1}, x, 0, 0\right] \qquad (3.41)$$

Case (xi)

This kind of spacetime is formed when we enter the values of $M = c_1x + c_2$ and $N = c_1x + c_2.$

$$ds^2 = -e^{2xc_1+2c_2}dt^2 + dx^2 + e^{2xc_1+2c_2}\left[dy^2 + dz^2\right]. \qquad (3.42)$$

Utilizing the spacetime (3.3.143) in the arrangement of ten equations (3.1) to (3.10) and after lengthy calculations, we get $\alpha = 0,$ which implies that HVFs become KVFs given beneath.

$$\left.\begin{aligned}
&X^{0}=-2c_1^2b_1tz+e^{-2xc_1-2c_2}\left(c_1^2t^2e^{2xc_1}+y^2c_1^2e^{2xc_1}+c_1^2z^2e^{2xc_1}+1\right)b_4+tb_6e^{-2c_2}\\
&+tye^{-2c_2}b_8+ye^{-2c_2}b_{10}+ze^{-2c_2}b_5+b_3,\\
&X^{1}=-2c_1b_1z-tb_4-yb_8-b_6,\\
&X^{2}=-2c_1^2b_1yz+e^{-2xc_1-2c_2}\left(c_1^2t^2e^{2xc_1}+y^2c_1^2e^{2xc_1}-c_1^2z^2e^{2xc_1}-1\right)b_8\\
&+tb_{10}e^{-2c_2}+tye^{-2c_2}b_4+ye^{-2c_2}b_6-ze^{-2c_2}b_9+b_2,\\
&X^{3}=-e^{-2xc_1-2c_2}\left(c_1^2t^2e^{2xc_1+2c_2}-y^2c_1^2e^{2xc_1+2c_2}+c_1^2z^2e^{2xc_1+2c_2}-1\right)b_1+tzb_4\\
&+tb_5+yb_9+zb_6+yzb_8+b_7,
\end{aligned}\right\}. \qquad (3.43)$$

Where $b_1, b_{b_2,} b_3, b_4, b_5, b_6, b_7, b_8, b_9$ and $b_{10} \in R.$

3.4 Physical Parameters of the Attained Solutions

In this section our goal is to extract the physical parameters. If we substitute the values of A, B and f(Q) in equations (3.6), (3.7) and (3.8), we get the values of the physical parameters like pressure and density. All the detailed discussion is given in the Table (3):

Physical parameters of the attained solutions

Tab. 3

Case No	Metric components	Density (ρ)	Pressure (p)
(i)	$M=\text{constant}$ and $N=\ln(x)$.	$-4c_1x^{-3}+\frac{f}{2}.$	$2c_1x^{-3}-\frac{f}{2}.$
(ii)	$M=\text{constant}$ and $N=x^2$.	$\left(32c_1x^3-8c_1x\right)+\frac{f}{2}.$	$16c_1x^3-\frac{f}{2}.$
(iii)	$M=\text{constant}$ and $N=\ln\left(\sqrt{2(c_1x+c_2)}\right)$, $c_1,c_2\in R(c_1\neq 0)$.	$-\frac{c_1^4}{2}(c_1x+c_2)^{-4}+\frac{f}{2}.$	$\frac{c_1^4}{4}(c_1x+c_2)^{-3}-\frac{f}{2}.$
(iv)	$M=\text{constant}$ and $N=(c_1x+c_2)$, $c_1,c_2\in R(c_1\neq 0)$.	$-4c_1^4+\frac{f}{2}.$	$2c_1^4-\frac{f}{2}.$
(v)	$M=N=\ln\left(\sqrt{2(c_1x+c_2)}\right)$, $c_1,c_2\in R(c_1\neq 0)$.	$-\frac{c_1^3}{2(c_1x+c_2)}+\frac{f}{2}.$	$\frac{3c_1^3}{2(c_1x+c_2)}-\frac{f}{2}.$
(vi)	$M=N=\left(\frac{c_1x+c_2}{4}\right)^2$, $c_1,c_2\in R(c_1\neq 0)$.	$\left[\frac{c_1^2(c_1x+c_2)-2}{8}\right]c_1$ $-\frac{f}{2}.$	$\left[\frac{3c_1^2(c_1x+c_2)^2}{32}\right]c_1$ $+\frac{f}{2}.$
(vii)	$N=\text{constant}$ and $M=\ln(c_1x+c_2)$, where $c_1,c_2\in R(c_1\neq 0)$.	$\frac{f}{2}.$	$-\frac{f}{2}.$

(viii)	$M=\ln\left(c_1x+c_2\right)$ and $N=\left(\frac{c_1x^2}{2}+c_2x+c_3\right)$, where $c_1,c_2,c_3\in R\left(c_1,c_2\neq 0\right)$.	$-4c_1\left(c_1x+c_2\right)^2-4c_1^2+\frac{f}{2}$.	$2c_1\left(c_1x+c_2\right)^2+4c_1^2-\frac{f}{2}$.
(ix)	$M=\text{constant}$ and $N=\left(c_1x+c_2\right)$, where $c_1,c_2\in R\left(c_1\neq 0\right)$.	$-4c_1^3+\frac{f}{2}$.	$2c_1^3-\frac{f}{2}$.
(x)	$A=\ln\left(x^m\right)$ and $B=\ln\left(x^{\frac{m(m-1)}{m+1}}\right)$, where $m\in R\left(m\neq 0\right)$.	$\left[\frac{-6m^4+11m^3-2m^2-2m}{x^2\left(m+1\right)^2}\right]c_1+\frac{f}{2}$.	$\left[\frac{6m^4-4m^3-2m^2}{x^2\left(m+1\right)^2}\right]c_1-\frac{f}{2}$.
(xi)	$M=\left(c_1x+c_2\right)$ and $N=\left(c_1x+c_2\right)$, where $c_1,c_2\in R\left(c_1\neq 0\right)$.	$-6c_1^3+\frac{f}{2}$.	$6c_1^3-\frac{f}{2}$.
(xii)	$M=\ln\left(\frac{e^{c_1x}-c_1c_4}{c_1c_3}\right)$ and $N=\left(c_1x+c_2\right)$, where $c_1,c_2\in R\left(c_1\neq 0\right)$.	$\left[\frac{4c_1^4c_4-4c_1^3e^{c_1x}-2c_1^3}{e^{c_1x}-c_1c_4}\right]+\frac{f}{2}$.	$\left[\frac{6c_1^3e^{c_1x}-2c_1^4c_4}{e^{c_1x}-c_1c_4}\right]-\frac{f}{2}$.

CONCLUSION

This study examined a static plane symmetric spacetime within the f(Q)-Gravity framework, revealing important insights into spacetime geometry. We analyzed 12 distinct cases (linear and non- linear) and identified homothetic vector fields (HVFs) that play a key role in understanding spacetime symmetries. Through direct integration, it was found that in the cases (ii), (vi),(viii) and (xii), the dimension of HVFs is 4. We observe that homothetic factor for such cases become zero which indicates that there do not exist proper HVF. We observe that homothetic factor for such cases do not zero which indicates that there exist proper HVFs.

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