Optimization and Data Science

Lecture 3: Fourier Analysis of Data

Prof. Dr. Thomas Slawig

Kiel University - CAU Kiel
Dep. of Computer Science

Summer 2020

Contents

Contents

Fourier Analysis

- What is it?

  Transformation of data into the “frequency domain”
- Why are we studying this?

  One of the most popular ways to analyze data
- How does it work?

  Using complex numbers, then applying a matrix multiplication

  Very efficient implementations (FFT: Fast Fourier Transformation) available
- What if we can use it?

  Analyze frequencies in data

  Detect periodic structures in data

  Detect and delete high-frequency parts (that might be perturbations)

  Reduce data size

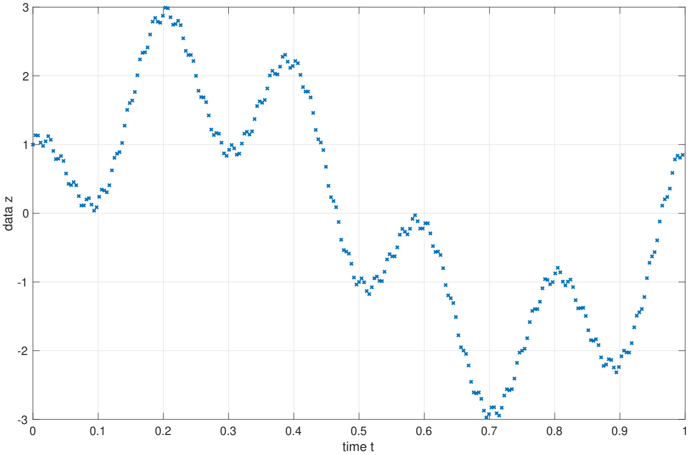
Contents

A different way to look at data: Example

- Given: dataset

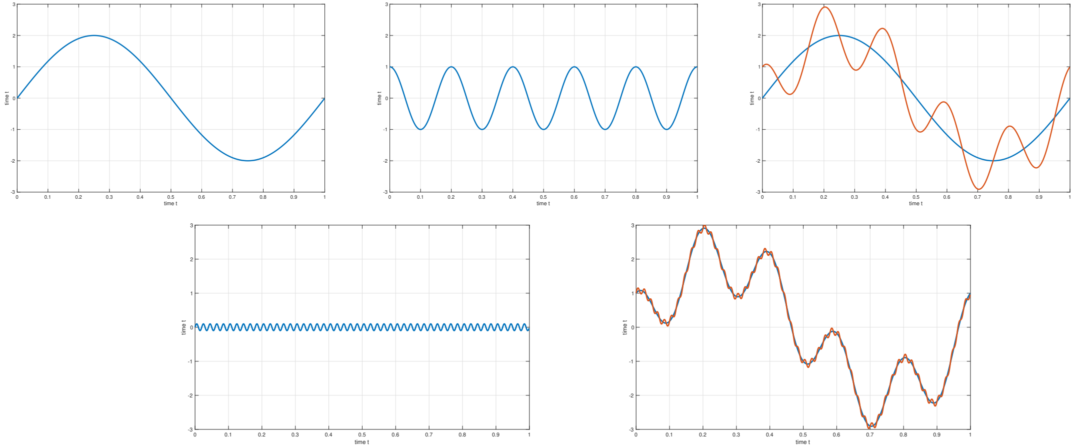
  $$(t_j, z_j)_{j=0,\dots,m-1},\ t_j, z_j \in \mathbb{R}.$$

- Example: t time, z measurements.
- We see "some" structure ...
- How to analyze this?



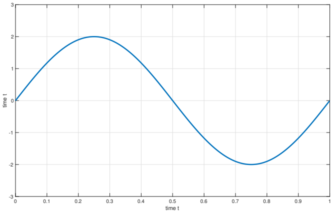
Example (continued)

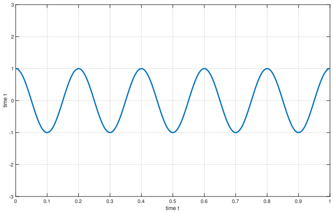
- There is a simple structure behind this dataset: Sum of 3 periodic datasets.

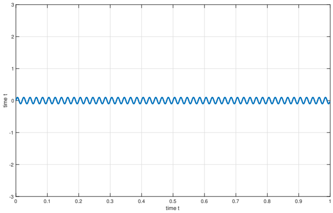


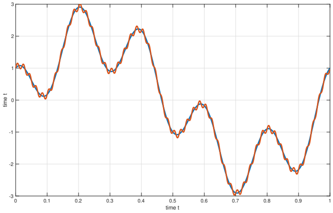
Example (continued)

- We need just 6 numbers instead of $m = 256$ in the original dataset. $\rightsquigarrow$ How???









Example: Image compression

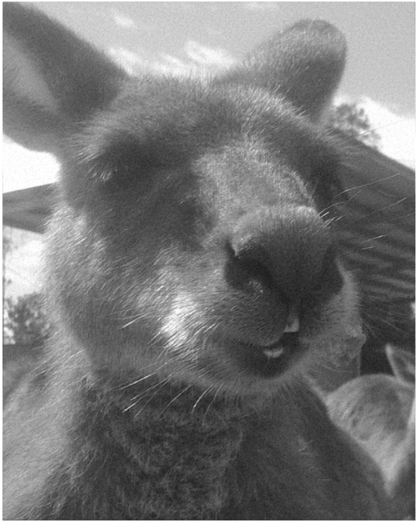
original size: 1060759

compressed, #fft coeff: 107264

Example: Image compression

original size: 1060759

compressed, #fft coeff: 50089

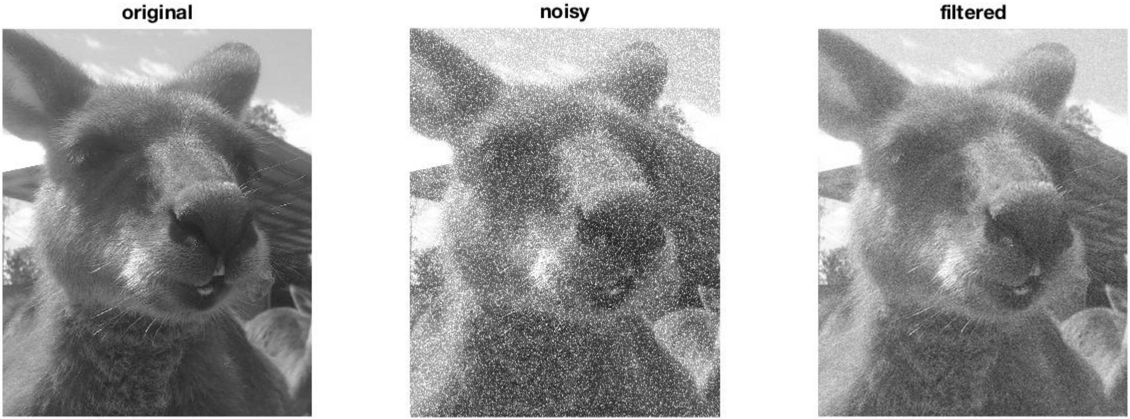


Example: Image compression

original size: 1060759

compressed, #fft coeff: 3559

Example: Denoising



Contents

Real-valued Fourier Analysis

Assume: data z_j is output of a (unknown) function $f : [0, L] \to \mathbb{R}, L > 0$, with equidistant input t_j:

$$t_j = j\frac{L}{m}, \quad z_j = f(t_j), \quad j = 0, \ldots, m-1.$$

Definition

For $f : [0, L] \to \mathbb{R}, L > 0$, we call

$$a_k = \frac{2}{L}\int_0^L f(t)\cos\frac{2\pi kt}{L}dt,$$
$$b_k = \frac{2}{L}\int_0^L f(t)\sin\frac{2\pi kt}{L}dt, \quad k \in \mathbb{N},$$

the **Fourier coefficients** of f.

Real-valued Fourier Analysis

Definition

Let $f : [0, L] \to \mathbb{R}, L > 0$, with Fourier coefficients a_k, b_k. The function $\hat{f} : \mathbb{R} \to \mathbb{R}$, defined as

$$\hat{f}(t) := \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \frac{2\pi kt}{L} + b_k \sin \frac{2\pi kt}{L} \right)$$

is called the **Fourier series** of f.

The Fourier series can be written as

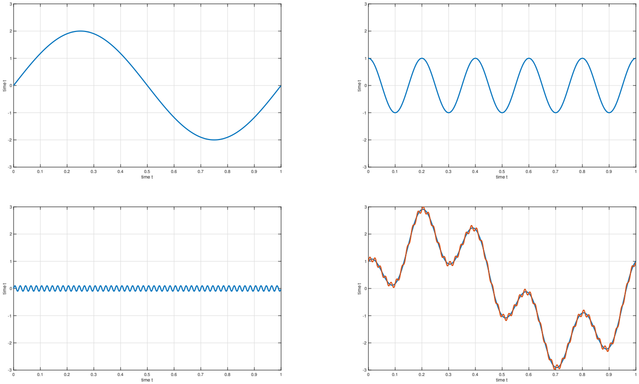
$$\hat{f}(t) = \lim_{n \to \infty} \hat{f}_n(t)$$

with

$$\hat{f}_n(t) := \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos \frac{2\pi kt}{L} + b_k \sin \frac{2\pi kt}{L} \right), n \in \mathbb{N}.$$

Example (from above)

- Here we have the Fourier coefficients $b_1 = 2, a_5 = 1, b_{50} = 0.1$.
- The Fourier coefficients are the amplitudes of the periodic parts in the data corresponding to the frequency k.



Real-valued Fourier Analysis

Theorem

If the integral of $f : [0, L] \to \mathbb{R}, L > 0$, *exists, then its Fourier series* $\hat{f}$ *converges to* f *in the sense that*

$$\lim_{n\to\infty} \|f - \hat{f}_n\|_{L^2(0,L)} = 0,$$

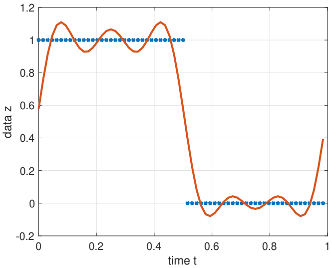
where

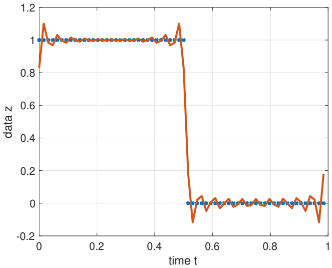
$$\|f\|_{L^2(0,L)} := \sqrt{\int_0^L |f(t)|^2 dt}.$$

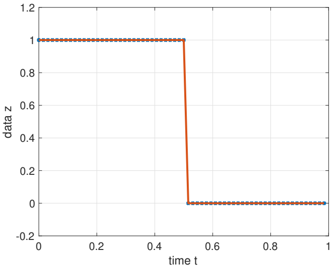
This type of convergence is called **convergence in the quadratic mean**.

- The function f does not have to be continuous to be approximable by the Fourier series.

Example: discontinuous function







Contents

What are complex numbers?

- Extension of real numbers, symbol $\mathbb{C}$
- $z \in \mathbb{C}$ an be identified with $(x, y) \in \mathbb{R}^2$
- There exist rules for multiplication (also exponential ...):

$$z_1, z_2 \in \mathbb{C} \Rightarrow z_1 \cdot z_2 \in \mathbb{C},$$

not possible in $\mathbb{R}^2$:

$$(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2, \quad (x_1, y_1) \cdot (x_2, y_2) \notin \mathbb{R}^2, \text{ not defined.}$$

Why are we studying complex numbers?

- Here: for the computation of Fourier coefficients.
- In general: complex numbers allow solution of every polynomial equation:

$$x^2 + 1 = 0 \text{ has no real solution, } x = \sqrt{-1} \notin \mathbb{R}.$$

- Fundamental theorem of algebra: Every polynomial equation of order $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} c_k z^k = 0, \quad c_k \in \mathbb{C},$$

  has n solutions (roots) $z_k \in \mathbb{C}, k = 1, \ldots, n$.
- We use a relation between trigonometric functions and the complex exponential.

How to define and use complex numbers?

- Solution: Define **imaginary unit** $i \in \mathbb{C}$ by

$$i^2 = i \cdot i := -1. \tag{1}$$

- Write complex number with **real part** $x \in \mathbb{R}$ and **imaginary part** $y \in \mathbb{R}$:

$$z := x + iy$$

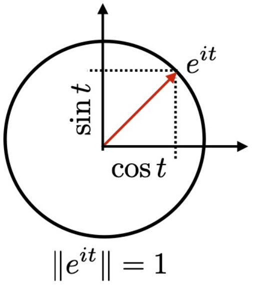
- $\mathbb{R} \subset \mathbb{C}$, real numbers = complex numbers with imaginary part $y = 0$.
- Multiplication in the "normal way" using (1):

$$(2+3i)(1-i) = 2 + 3i - 2i - 3i^2 = 2 + i + 3 = 5 + i.$$

- Exponential can be defined for complex numbers as well:

$$e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

- Euler's formula: $e^{it} = \cos t + i \sin t$.



Contents

Complex Fourier series

Definition

For $f : [0, L] \to \mathbb{R}, L > 0$, we call

$$c_k = \frac{1}{L} \int_0^L f(t) e^{-i \frac{2\pi kt}{L}} dt, \quad k \in \mathbb{Z},$$

the **(complex) Fourier coefficients** of f.
The function $\hat{f} : \mathbb{R} \to \mathbb{C}$, defined as

$$\hat{f}(t) := \sum_{k=-\infty}^{\infty} c_k e^{i \frac{2\pi kt}{L}} := \lim_{n \to \infty} \sum_{k=-n}^{n} c_k e^{i \frac{2\pi kt}{L}},$$

is called the **(complex) Fourier series** of f.

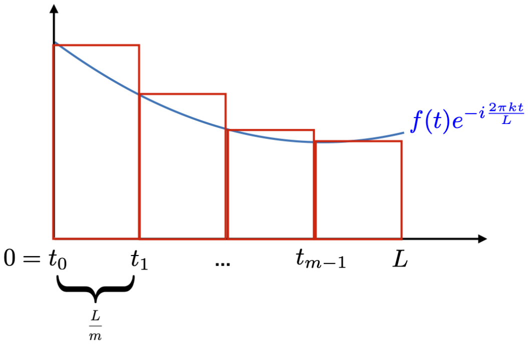
Computation of the coefficients: Discrete Fourier Transformation (DFT)

- We interpret the data $(z_j)_{j=0}^{m-1}$ as output of a function with equidistant input:

$$t_j = j\frac{L}{m}, \quad z_j = f(t_j), \quad j = 0, \dots, m-1.$$

- We approximate the integral for the complex Fourier coefficients by

$$\begin{aligned} c_k &= \frac{1}{L}\int_0^L f(t)e^{-i\frac{2\pi kt}{L}}\,dt \\ &\approx \frac{1}{L}\frac{L}{m}\sum_{j=0}^{m-1} f(t_j)e^{-i\frac{2\pi kt_j}{L}} \\ &= \frac{1}{m}\sum_{j=0}^{m-1} z_j e^{-i\frac{2\pi kj}{m}}, \quad k = 0, \dots, m-1. \end{aligned}$$



Picture is just a sketch, function complex-valued

Discrete Fourier Transformation

Definition

For $z = (z_j)_{j=0}^{m-1} \in \mathbb{C}^m$ we call the vector $c = (c_k)_{k=0}^{m-1} \in \mathbb{C}^m$ with

$$c_k = \frac{1}{m} \sum_{j=0}^{m-1} z_j e^{-i\frac{2\pi kj}{m}}, \quad k = 0, \ldots, m-1,$$

the **discrete Fourier transform** of z. The mapping $z \mapsto c$ is called **discrete Fourier transformation**.

The DFT mapping $z \mapsto c$ is given by a matrix-vector multiplication

$$c = \frac{1}{m} Mz \ \text{ with } M := \left(e^{-i\frac{2\pi kj}{m}}\right)_{k,j=0}^{m-1} \in \mathbb{C}^{m \times m}.$$

Inverse Discrete Fourier Transformation (Fourier Synthesis)

Theorem

For $z = (z_j)_{j=0}^{m-1} \in \mathbb{C}^m$ *and the Fourier coeffcients*

$$c_k = \frac{1}{m} \sum_{j=0}^{m-1} z_j e^{-i\frac{2\pi kj}{m}}, \quad k = 0, \ldots, m-1,$$

we get

$$\sum_{k=0}^{m-1} c_k e^{i\frac{2\pi kj}{m}} = z_j, \quad j = 0, \ldots, m-1.$$

The mapping $c \mapsto z$ *is called* **inverse discrete Fourier transformation (iDFT)** *or* **Fourier synthesis**.

DFT and inverse DFT as matrix-vector product

- The DFT mapping $z \mapsto c$ is given by a matrix-vector multiplication

$$c = \frac{1}{m} Mz \quad \text{with } M := \left(e^{-i\frac{2\pi kj}{m}} \right)_{k,j=0}^{m-1} \in \mathbb{C}^{m \times m}.$$

- The inverse DFT mapping $c \mapsto z$ is performed with the inverse matrix:

$$z = mM^{-1}c.$$

  Returns values $z_j = f(j\frac{L}{m})$ of function f at equidistant points from its Fourier coefficients.
- The inverse matrix is given by

$$M^{-1} = \frac{1}{m} \left(e^{i\frac{2\pi kj}{m}} \right)_{k,j=0}^{m-1} \in \mathbb{C}^{m \times m}.$$

- Note: Some literature/algorithms define the DFT coefficients c_k without factor $\frac{1}{m}$. Then the factor m has to be omitted in the inverse DFT.

Fourier Analysis: What is important

- Fourier analysis is an important and powerful tool to analyse, compress and denoise data in arbitrary dimensions.
- It performs a transformation of the data into the frequency domain.
- The mathematical analysis can be performed using real and complex numbers.
- The discrete algorithm is based on a transformation of the data considered as complex numbers.
- This transformation is a matrix-vector multiplication.