Solutions

Theory Sheet 2

Solution T-2.1: Conditional Probability

We start with:

$$P(F|E^c) = \frac{P(F, E^c)}{P(E^c)}$$

and use

$$E^c = \{v_1, v_2\}^c = \{v_3\}$$

to determine the numerator as

$$P(F, E^c) = P(\{v_1, v_3\} \cap \{v_3\}) = P(\{v_3\})$$

As a consequence:

$$P(F|E^c) = \frac{P(\{v_3\})}{P(E^c)} = \frac{P(\{v_3\})}{P(\{v_3\})} = 1$$

Solution T-2.2: Conditional Probability

Given an hypothetical sample space X of possible live spans, define the two events:

$$\begin{aligned} E &= \{x \in X | \text{man lives at least 70 years}\} \\ F &= \{x \in X | \text{man lives at least 80 years}\} \end{aligned}$$

We need to find the following conditional probability:

$$P(F|E) = \frac{P(F, E)}{P(E)}$$

The denominator is known $P(E) = \frac{4}{5}$.

As fas as the numerator is concerned, note that $F \subseteq E$ (all men who lived at least 80 years also lived at least 70 years). This implies $F \cap E = F$ and therefore:

$$P(F, E) = P(F) = \frac{1}{2}$$

Thus:

$$P(F|E) = \frac{P(F, E)}{P(E)} = \frac{P(F)}{P(E)} = \frac{1/2}{4/5} = \frac{5}{8}$$

Solution T-2.3: Conditional Probability

Let OK_i be the event that the i-th drawing of a piece from the box unveiled an acceptable tool. Thus, we have to determine

$$P(OK_1, OK_2, OK_3, OK_4, OK_5)$$

Using the multiplication rule $P(A, B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$ gives

$$\begin{aligned}
P(OK_1, OK_2, OK_3, OK_4, OK_5) &= P(OK_5, OK_4, OK_3, OK_2, OK_1) \\
P(OK_5, OK_4, OK_3, OK_2, OK_1) &= P(OK_5|OK_4, OK_3, OK_2, OK_1) \cdot \\
&\quad P(OK_4, OK_3, OK_2, OK_1) \\
&= P(OK_5|OK_4, OK_3, OK_2, OK_1) \cdot \\
&\quad P(OK_4|OK_3, OK_2, OK_1) \cdot \\
&\quad P(OK_3, OK_2, OK_1) \\
&= ... \\
&= P(OK_5|OK_4, OK_3, OK_2, OK_1) \cdot \\
&\quad P(OK_4|OK_3, OK_2, OK_1) \cdot \\
&\quad P(OK_3|OK_2, OK_1) \\
&\quad P(OK_2|OK_1) \cdot \\
&\quad P(OK_1)
\end{aligned}$$

With $P(OK_1) = 40/50$, $P(OK_2|OK_1) = 39/49$, $P(OK_3|OK_2, OK_1) = 38/48$, ... we get:

$$\begin{aligned}
P(OK_1, OK_2, OK_3, OK_4, OK_5) &= 40/50 \cdot 39/49 \cdot 38/48 \cdot 37/47 \cdot 36/46 \\
&= 0.31
\end{aligned}$$

A quite high acceptance rate of 31%.

Generalization:

$$P(OK_n, OK_{n-1}, OK_{n-2}, ..., OK_2, OK_1) = \prod_{i=0}^{n-1} \frac{N-M-i}{N-i}$$