Solutions

Theory Sheet 1

Solution T-1.1: Random variable

By the additivity of probability:

$$\begin{aligned} P(x > 17) &= P(\{x = 18\} \cup \{x = 19\} \cup \{x = 20\}) \\ &= P(\{x = 18\}) + P(\{x = 19\} + P(\{x = 20\}) \\ &= \frac{18}{210} + \frac{19}{210} + \frac{20}{210} = \frac{19}{70} \end{aligned} \tag{1}$$

Solution T-1.2: Probability

First note that, by additivity, $P(H) = P(\{v_2\} \cup \{v_4\}) = P(\{v_2\}) + P(\{v_4\})$.

Therefore, in order to compute $P(H)$, we need to compute $P(\{v_2\})$ and $P(\{v_4\})$.

$P(\{v_2\})$ is found using additivity on F:

$$\begin{aligned} \frac{5}{10} &= P(F) \\ &= P(\{v_1\} \cup \{v_2\}) \\ &= P(\{v_1\}) + P(\{v_2\}) \\ &= P(E) + P(\{v_2\}) \\ &= \frac{1}{10} + P(\{v_2\}) \end{aligned} \tag{2}$$

so that $P(\{v_2\}) = \frac{5}{10} - \frac{1}{10} = \frac{4}{10}$.

$P(\{v_4\})$ is found using the facts (a) that one minus the probability of an event is equal to the probability of its complement and (b) that $\{v_4\} = G^c$:

$$\begin{aligned} P(\{v_4\}) &= P(G^c) \\ &= 1 - P(G) \\ &= 1 - \frac{7}{10} = \frac{3}{10} \end{aligned}$$

As a consequence: $P(H) = P(\{v_2\}) + P(\{v_4\}) = \frac{4}{10} + \frac{3}{10} = \frac{7}{10}$

Solution T-1.3: Joint Probability

There are only two possible cases that give rise to the occurrence $x_1 + x_2 = 3$. These cases are $\mathbf{x} = [1,2]^t$ and $\mathbf{x} = [2,1]^t$. Therefore, since these two cases are disjoint events, we can use the additivity of probability:

$$\begin{aligned} P(x_1 + x_2 = 3) &= P(\{\mathbf{x} = [1,2]^t\} \cup \{\mathbf{x} = [2,1]^t\}) \\ &= P(\mathbf{x} = [1,2]^t) + P(\mathbf{x} = [2,1]^t) \\ &= \frac{1}{36}(1+2) + \frac{1}{36}(2+1) = \frac{1}{6} \end{aligned}$$

Solution T-1.4: Marginal Probability

The support of x_1 is $S_{x_1} = \{1, 2, 3\}$. We need to compute the probability of each element of the support of x_1:

$$\begin{aligned} P_{x_1}(0) &= \sum_{\{(x_1,x_2)\in S_x : x_1 = 0\}} P_x(x_1, x_2) \\ &= P_x(0,0) = \frac{1}{3} \\ P_{x_1}(1) &= \sum_{\{(x_1,x_2)\in S_x : x_1 = 1\}} P_x(x_1, x_2) \\ &= P_x(1,1) = \frac{1}{3} \\ P_{x_1}(2) &= \sum_{\{(x_1,x_2)\in S_x : x_1 = 2\}} P_x(x_1, x_2) \\ &= P_x(2,0) = \frac{1}{3} \end{aligned} \tag{3}$$

Thus, the probability mass function of x_1 is:

$$P_{x_1}(x) = \sum_{\{(x_1,x_2)\in S_x : x_1 = x\}} P_x(x_1, x_2) = \begin{cases} \frac{1}{3} & \text{if } x = 0 \\ \frac{1}{3} & \text{if } x = 1 \\ \frac{1}{3} & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

The support of x_2 is $S_{x_2} = \{0, 1\}$. We need to compute the probability of each element of the support of x_2:

$$\begin{aligned} P_{x_2}(0) &= \sum_{\{(x_1,x_2)\in S_x : x_2 = 0\}} P_x(x_1, x_2) \\ &= P_x(2,0) + P_x(0,0) = \frac{2}{3} \\ P_{x_2}(1) &= \sum_{\{(x_1,x_2)\in S_x : x_2 = 1\}} P_x(x_1, x_2) \\ &= P_x(1,1) = \frac{1}{3} \end{aligned} \tag{4}$$

Thus, the probability mass function of x_2 is:

$$P_{x_2}(x) = \sum_{\{(x_1,x_2)\in S_x : x_2 = x\}} P_x(x_1, x_2) = \begin{cases} \frac{2}{3} & \text{if } x = 0 \\ \frac{1}{3} & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$