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Solutions

Theory Sheet 3

Solution T-3.1: Dependent and Independent Variables

The probability of taking a faultless piece in the first draw is given by

$$P(x = 0) \quad = \quad \frac{N - M}{N}$$

The probability of taking a faultless piece in the second draw, given that a faultless piece was taken in the first draw, is given by

$$P(y = 0|x = 0) \quad = \quad \frac{N - M - 1}{N - 1}$$

The probability of taking a defect piece in the first draw is given by

$$P(x = 1) \quad = \quad \frac{M}{N}$$

The probability of taking a faultless piece in the second draw, given that a defect piece was taken in the first draw, is given by

$$P(y = 0|x = 1) \quad = \quad \frac{N - M}{N - 1}$$

The probability of taking a defect piece in the second draw, given that a faultless piece was taken in the first draw, is given by

$$P(y = 1|x = 0) \quad = \quad \frac{M}{N - 1}$$

The joint probability of taking a faultless piece in the first and second draw is calculated using the product rule

$$\begin{aligned} P(x = 0, y = 0) &= P(y = 0|x = 0) \cdot P(x = 0) \\ &= \frac{N - M - 1}{N - 1} \cdot \frac{N - M}{N} \end{aligned}$$

Using the same technique, we obtain

$$\begin{aligned} P(x=1,y=0) &= P(y=0|x=1)\cdot P(x=1) \\ &= \frac{N-M}{N-1}\cdot\frac{M}{N} \end{aligned}$$

and

$$\begin{aligned} P(x=0,y=1) &= P(y=1|x=0)\cdot P(x=0) \\ &= \frac{M}{N-1}\cdot\frac{N-M}{N} \\ &= P(x=1,y=0) \end{aligned}$$

The marginal probabilies are given by

$$\begin{aligned} P_x(x) &= \sum_{y\epsilon Y} P(x,y) \\ P_y(y) &= \sum_{x\epsilon X} P(x,y) \end{aligned}$$

Thus

$$\begin{aligned} P_x(x=0) &= \sum_{y\epsilon Y} P(x=0,y) \\ P_x(x=0) &= P(x=0,y=0)+P(x=0,y=1) \\ P_y(y=0) &= \sum_{x\epsilon X} P(x,y=0) \\ &= P(x=0,y=0)+P(x=1,y=0) \\ &= P(x=0,y=0)+P(x=0,y=1) \\ &= P_x(x=0) \end{aligned}$$

Inserting $P(x=0,y=0)$ and $P(x=0,y=1)$ gives

$$\begin{aligned} P_x(x=0) &= \frac{N-M-1}{N-1}\cdot\frac{N-M}{N}+ \\ &\quad \frac{M}{N-1}\cdot\frac{N-M}{N} \\ &= \frac{(N-M-1)\cdot(N-M)+M\cdot(N-M)}{N\cdot(N-1)} \\ &= \frac{N^2-MN-MN+M^2-N+M+MN-M^2}{N\cdot(N-1)} \\ &= \frac{N^2-MN-N+M}{N\cdot(N-1)} \\ &= \frac{(N-M)\cdot(N-1)}{N\cdot(N-1)} \\ &= \frac{N-M}{N} \end{aligned} \tag{1}$$

Since $P_x(x=0) = P_y(y=0)$ we get

$$\begin{aligned} P_x(x=0) \cdot P_y(y=0) &= \frac{N-M}{N} \cdot \frac{N-M}{N} \\ &\neq P(x=0, y=0) = \frac{N-M}{N} \cdot \frac{N-M-1}{N-1} \end{aligned}$$

Thus, the random variables x and y are dependent.

Solution T-3.2: Bayes Rule

In probabilistic terms, what we know about this problem can be formalized as follows:

$$\begin{aligned} P(red|urn1) &= \frac{1}{2} \\ P(red|urn2) &= \frac{3}{10} \\ P(urn1) &= \frac{1}{2} \\ P(urn2) &= \frac{1}{2} \end{aligned}$$

The unconditional probability of drawing a red ball can be derived using the law of total probability:

$$\begin{aligned} P(red) &= P(red, urn1) + P(red, urn2) \\ &= P(red|urn1)P(urn1) + P(red|urn2)P(urn2) \\ &= \frac{1}{2} \cdot \frac{1}{2} + \frac{3}{10} \cdot \frac{1}{2} \\ &= \frac{2}{5} \end{aligned}$$

Using Bayes' rule we obtain:

$$\begin{aligned} P(urn1|red) &= \frac{P(red|urn1) \cdot P(urn1)}{P(red)} \\ &= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{2}{5}} \\ &= \frac{5}{8} \end{aligned}$$

Solution T-3.3: Bayes Rule

What we know about this problem can be formalized as follows (rec. : recession, pred. predicted):

$$\begin{aligned}
P(\text{rec. pred.} \mid \text{rec. coming}) &= \frac{8}{10} \\
P(\text{rec. pred.} \mid \text{rec. not coming}) &= \frac{1}{10} \\
P(\text{rec. coming}) &= \frac{2}{10} \\
P(\text{rec. not coming}) &= 1 - P(\text{rec. coming}) = \frac{8}{10}
\end{aligned}$$

Using Bayes' rule we obtain:

$$P(\text{rec. coming} \mid \text{rec. pred.}) = \frac{P(\text{rec. pred.} \mid \text{rec. coming})P(\text{rec. coming})}{P(\text{rec. pred.})}$$

The unconditional probability of predicting a recession can be derived using the law of total probability:

$$\begin{aligned}
P(\text{rec. pred.}) &= P(\text{rec. pred., rec. coming}) + P(\text{rec. pred., rec. not coming}) \\
&= P(\text{rec. pred.} \mid \text{rec. coming})P(\text{rec. coming}) + \\
&\quad P(\text{rec. pred.} \mid \text{rec. not coming})P(\text{rec. not coming}) \\
&= \frac{8}{10} \cdot \frac{2}{10} + \frac{1}{10} \cdot \frac{8}{10} = \frac{6}{25}
\end{aligned}$$

Thus

$$\begin{aligned}
P(\text{rec. coming} \mid \text{rec. pred.}) &= \frac{P(\text{rec. pred.} \mid \text{rec. coming})P(\text{rec. coming})}{P(\text{rec. pred.})} \\
&= \frac{\frac{8}{10} \cdot \frac{2}{10}}{\frac{6}{25}} = \frac{2}{3}
\end{aligned}$$