Solutions

Theory and Practice Sheet 8

Solution T-8.1: MAP parameter estimation

The Maximum-A-Posteriori parameter estimation is given in Chapter 3.4 of the lecture

$$\hat{\Theta} = \arg\max_{\Theta} \left\{ p(\Theta) \cdot \prod_{k=1}^{n} p(x_k|\Theta) \right\}$$

The parameter to be estimated is c, i.e.

$$\hat{c} = \arg\max_{c} \left\{ p(c) \cdot \prod_{k=1}^{n} p(x_k|c) \right\} \tag{1}$$

Consequently, we need to determine the a-priori probability $p(c)$ and the a-posteriori probability $p(x_k|c)$.

The a-priori distribution of the constant gray value c is assumed to be a normal distribution with mean 0 and variance σ_c^2:

$$p(c) = \frac{1}{\sqrt{2\pi\sigma_c^2}} e^{-\frac{c^2}{2\sigma_c^2}}$$

The a-posteriori distribution, i.e. the probability of the sensor observation x_k given the constant gray value c, is determined by the sum of the given (i.e. fixed) constant c and the normally distributed noise random variable e with mean 0 and variance σ_e^2. The result is a normal distribution with mean c (since random variable e has mean 0) and variance σ_e^2 (i.e. the variance of the random variable e). Thus,

$$p(x_k|c) = \frac{1}{\sqrt{2\pi\sigma_e^2}} e^{-\frac{(x_k-c)^2}{2\sigma_e^2}}$$

Insertion of these distributions into Equation (1) gives

$$\hat{c} = \arg\max_c \left\{ \frac{1}{\sqrt{2\pi\sigma_c^2}} e^{-\frac{c^2}{2\sigma_c^2}} \cdot \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi\sigma_e^2}} e^{-\frac{(x_k-c)^2}{2\sigma_e^2}} \right\} \tag{2}$$

The term $1/\sqrt{2\pi\sigma_c^2}$ is independent of c and can be omitted in the maximization:

$$\hat{c} = \arg\max_c \left\{ e^{-\frac{c^2}{2\sigma_c^2}} \cdot \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi\sigma_e^2}} e^{-\frac{(x_k-c)^2}{2\sigma_e^2}} \right\} \tag{3}$$

The term $1/\sqrt{2\pi\sigma_e^2}$ is independent of k,

$$\hat{c} = \arg\max_c \left\{ e^{-\frac{c^2}{2\sigma_c^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_e^2}} \cdot \prod_{k=1}^{n} e^{-\frac{(x_k-c)^2}{2\sigma_e^2}} \right\} \tag{4}$$

and independent of c:

$$\hat{c} = \arg\max_c \left\{ e^{-\frac{c^2}{2\sigma_c^2}} \cdot \prod_{k=1}^{n} e^{-\frac{(x_k-c)^2}{2\sigma_e^2}} \right\} \tag{5}$$

In order to simplify the further calculations, we apply the ln function:

$$\begin{aligned} \hat{c} &= \arg\max_c \left\{ -\frac{c^2}{2\sigma_c^2} - \sum_{k=1}^{n} \frac{(x_k-c)^2}{2\sigma_e^2} \right\} \\ &= \arg\max_c \left\{ \frac{c^2}{\sigma_c^2} + \sum_{k=1}^{n} \frac{(x_k-c)^2}{\sigma_e^2} \right\} \end{aligned} \tag{6}$$

Optimization over c

$$\frac{d}{dc} \left\{ \frac{c^2}{\sigma_c^2} + \sum_{k=1}^{n} \frac{(x_k-c)^2}{\sigma_e^2} \right\} = \frac{2\hat{c}}{\sigma_c^2} - \frac{2}{\sigma_e^2} \sum_{k=1}^{n} (x_k - \hat{c}) \overset{!}{=} 0 \tag{7}$$

leads to

$$\begin{aligned} \frac{\hat{c}}{\sigma_c^2} - \frac{1}{\sigma_e^2} \left(\sum_{k=1}^{n} x_k - \hat{c} \sum_{k=1}^{n} 1 \right) &= 0 \\ \frac{\hat{c}}{\sigma_c^2} - \frac{1}{\sigma_e^2} \sum_{k=1}^{n} x_k + \frac{\hat{c}n}{\sigma_e^2} &= 0 \\ \hat{c} \left(\frac{1}{\sigma_c^2} + \frac{n}{\sigma_e^2} \right) &= \frac{1}{\sigma_e^2} \sum_{k=1}^{n} x_k \\ \hat{c} &= \frac{\sigma_c^2 \sigma_e^2}{\sigma_e^2 + n\sigma_c^2} \cdot \frac{1}{\sigma_e^2} \sum_{k=1}^{n} x_k \\ \hat{c}_{MAP} &= \frac{\sigma_c^2}{\frac{1}{n}\sigma_e^2 + \sigma_c^2} \cdot \frac{1}{n} \sum_{k=1}^{n} x_k \end{aligned}$$

The **Maximum-likelihood solution** for parameter c is obtained by maximizing the likelihood function:

$$\hat{c}_{MAP} = \arg\max_c p(D|c) \overset{i.i.d.}{=} \arg\max_c \left\{ \prod_{k=1}^{n} p(x_k|c) \right\}$$

The likelihood $p(x|c)$ is normally distributed with variance σ_e^2 and mean c (see above), i.e.

$$p(x|c) = \frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left[-\frac{1}{2} \left(\frac{x-c}{\sigma_e} \right)^2 \right]$$

Inserting the log-likelihood leads to

$$\begin{aligned} \hat{c}_{MAP} &= \arg\max_c \left\{ \sum_{k=1}^{n} \ln \left(\frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left[-\frac{1}{2} \left(\frac{x_k-c}{\sigma_e} \right)^2 \right] \right) \right\} \\ &= \arg\max_c \left\{ -\frac{1}{2} \sum_{k=1}^{n} \ln\left(\sqrt{2\pi\sigma_e^2} \right) - \frac{1}{2} \sum_{k=1}^{n} \left(\frac{x_k-c}{\sigma_e} \right)^2 \right\} \end{aligned}$$

Maximization

$$\begin{aligned} \frac{d}{dc} \left\{ -\frac{1}{2} \sum_{k=1}^{n} \ln\left(\sqrt{2\pi\sigma_e^2} \right) - \frac{1}{2} \sum_{k=1}^{n} \left(\frac{x_k-c}{\sigma_e} \right)^2 \right\} &\overset{!}{=} 0 \\ \sum_{k=1}^{n} \left(\frac{x_k-\hat{c}}{\sigma_e} \right) &= 0 \\ \sum_{k=1}^{n} x_k - \hat{c} \sum_{k=1}^{n} 1 &= 0 \\ \sum_{k=1}^{n} x_k &= \hat{c}n \\ \hat{c}_{ML} &= \frac{1}{n} \sum_{k=1}^{n} x_k \end{aligned}$$

Comparing this result with the MAP solution

$$\hat{c}_{MAP} = \frac{\sigma_c^2}{\frac{1}{n}\sigma_e^2 + \sigma_c^2} \cdot \frac{1}{n} \sum_{k=1}^{n} x_k$$

it can be seen that for $n \to \infty$ or if $\sigma_e^2 \to 0$ the MAP and the ML result are equal.

$$\hat{c}_{MAP} \overset{n\to\infty}{=} \frac{1}{n} \sum_{k=1}^{n} x_k = \hat{c}_{ML}$$